Lorenzo Ciardo

The Mathematics of Computational Complexity

In a world where data processing, decision making, and – to a certain extent – high-level reasoning are being increasingly delegated to the machine, understanding the limits of efficient computation by describing the border line between tractability and hardness of computational tasks is becoming a scientific challenge of fundamental importance. The new developments of computational complexity theory have provided us with the mathematical tools to approach this challenge for a wide class of problems. Intriguingly, these mathematical tools are now proving powerful enough to explain phenomena observed beyond the classical theory of computation – in the realms of quantum physics.

The concept of computation is deeply rooted in human history. Since ancient times, we have devised methods to calculate, predict and reason systematically. And we have realised early on that some computational tasks seem to be much easier than others. Sorting a list of numbers in ascending order is straightforward, solving a system of linear equations is also manageable – but finding the optimal route to visit multiple cities while minimising the travel distance can be extremely difficult. What is the reason behind these differences? Is it simply because, for some problems, we have been clever enough to discover good procedures, while for others we haven't? Or is there a deeper reason, something intrinsic to the nature of computation and the structure of the problems themselves rather than to our ingenuity?

The advent of computational complexity theory gave a decisive answer to these questions in the 1970s. The difficulty of certain problems is not merely a reflection of our current ignorance, for some computational tasks are inherently harder than others. We can exercise our creativity to solve tractable problems more efficiently, or to find approximate solutions to difficult problems, but there exist intrinsic bounds regarding what can and cannot be achieved via computation – through both the algorithms we already know, and those that are yet to be found. A primary challenge for computer scientists is then to explore and describe the inherent complexity landscape of computational problems. How does the transition between "easy" and "hard" problems happen? Is there a sharp boundary between the two classes? Or are there problems of intermediate complexity, neither easy nor completely intractable? Assuming the widely believed "P≠NP" conjecture, such intermediate problems do exist. Yet, strikingly, most natural problems studied by computer scientists seem to fall on one side or the other. Either they admit an efficient (polynomial-time) algorithm – in which case, they are said to belong to the complexity class P – or they are at least as difficult as certain prototypical hard problems for which no efficient algorithm is believed to exist – and are thus named NP-hard.

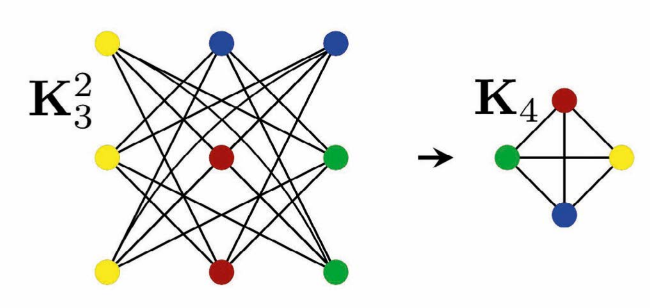

Figure 1: Constraint satisfaction problems can be described through the formalism of homomorphisms between relational structures. The figure illustrates one type of such homomorphisms, corresponding to the graph-colouring problem.

Source: Author's own illustration.

Over the decades, researchers have identified many such classes of "natural" problems, that provably witness a P vs. NP-hard dichotomy. In particular, these include a very general type of problems involving constraint satisfaction, where the goal is to assign a finite number of labels or values to a given set of variables so that certain rules are satisfied. While this classification of constraint satisfaction problems is itself a landmark achievement of computer science, an equally important outcome has been the mathematical machinery developed along the way. Indeed, as complexity theory evolved, researchers began to speak less about computers and more about structures and symmetries. The language of Turing machines and algorithms gradually gave way to that of algebra, geometry and logic. Today, many complexity theorists are virtually indistinguishable from pure mathematicians – working on theories from harmonic analysis, topology, or universal algebra to understand the limits of computation.

Figure 2: By associating tensor spaces to the algorithms for the solution of constraint satisfaction problems, it is possible to obtain information on the algorithms' power via multilinear algebra. Source: Ciardo, L. and Živný, S., 2023. Approximate graph colouring and the hollow shadow. In Proceedings of the 55th Annual ACM Symposium on Theory of Computing (pp. 623-631).

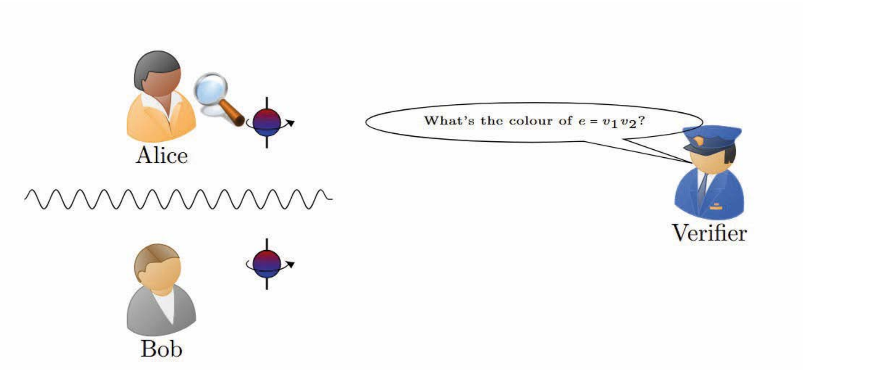

Figure 3: Illustration of quantum strategies in non-local games. Non-local games provide a common framework for studying both the complexity of computational problems and the emergence of non-classical phenomena such as non-locality and contextuality in quantum systems. Source: Author's own illustration.

Lorenzo Ciardo is an assistant professor at TU Graz, which he joined in 2025 to become part of the newly established Institute of Algorithms and Theory. Previously, he spent 4.5 years at the University of Oxford working as a senior research associate, after obtaining a PhD in mathematics at the University of Oslo in 2020. His research activity lies at the intersection of computational complexity, quantum information and discrete mathematics. A long-term goal of his work is to understand how far the algebraic and geometric methods from classical computational complexity can reach into quantum theory.

As often happens in mathematics, the increase in abstraction has been accompanied by a widening of the scope of the tools developed. Techniques originally introduced to explore the complexity landscape of constraint satisfaction problems have quickly started to illuminate fields far beyond classical complexity theory. A fascinating example of this phenomenon comes from the area of quantum physics. It is currently becoming clear that questions about the separation between quantum and classical models in physics and information theory, as well as the properties of quantum entanglement – such as those raised in the famous Bohr–Einstein debates on the nature of quantum mechanics – can be approached via a similar mathematical formalism to the one developed in the context of constraint satisfaction problems. This connection promises to bring a cross-contamination of techniques between quantum physics and constraint satisfaction theory, and it is currently becoming a remarkably active area of research at the intersection of computational complexity and quantum information.