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Model Updating and Damage Detection for Bridge Integrity Management

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ABSTRACT: The integrity management of bridges is crucial for ensuring public safety and economic stability. In practice, Structural Health Monitoring data recorded during bridge operation is increasingly used to guide maintenance decisions. However, incorporating structural damage information more effectively can lead to optimal strategies for integrity management. In this study, we employ Bayesian Model Updating to develop a more reliable structural model. The updated finite element model is then used to train a variational autoencoder-based surrogate model for damage detection, localization, and severity estimation. The variational autoencoder model establishes a link between damage-related features and the modal properties derived from SHM data. Damage information supports maintenance decision-making through a predefined decision rule.
KEY WORDS: Structural health monitoring, damage detection, surrogate models, structural integrity management, maintenance decision.

1 INTRODUCTION

Bridges and viaducts are fundamental components of transportation networks, ensuring connectivity and economic stability. However, their structural integrity is continuously challenged by aging, increasing traffic loads, and environmental stressors. Effective bridge health management can benefit from continuous monitoring and strategies to detect potential damage and mitigate the risks before they compromise safety.

Structural Health Monitoring (SHM) has emerged as a crucial tool for assessing bridge conditions in real-time, providing early detection of structural anomalies. The design and implementation of SHM systems for bridge integrity management were proposed in the study by Limongelli et al. [1]. SHM system provides continuous information about structural properties such as natural frequencies, damping ratios, and mode shapes. However, directly labeling the obtained modal properties as belonging to either damaged or undamaged states from data collected on real-world structures is challenging. This difficulty arises because the changes in modal features can also result from various factors not related to damage, such as environmental conditions, operational variability, or sensor noise. While some studies explored damage detection and localization through the modal properties [2], a fundamental part of damage detection strategies entails the use of physics-based models, which provide a basis for understanding the overall structural behavior under varying conditions. By integrating SHM data into physics-based formulations, the models are updated to represent the actual bridge conditions, enhancing structural integrity management, improving maintenance planning, and decision-making. However, the computational cost of updating a finite element model in real-time can be very high. Surrogate models provide a computationally efficient alternative to complex physics-based simulations.

In this paper, an approach based on Bayesian Model Updating (BMU) using Transitional Monte Carlo Markov Chain is implemented to update the structural model of a bridge using measured data. This approach refines the bridge model through the incorporation of modal properties extracted from SHM data, by reducing the discrepancy between measured and calculated modal properties. Thanks to the systematic updates of the structural parameters, the model accurately represents the bridge's current state. The high-fidelity and calibrated FE model is then used for training a surrogate model. Namely, the FE model is used to simulate several damage scenarios and generate the relevant response of the bridge, thus providing the necessary training data for the surrogate models. Several surrogate modeling approaches have been explored in the literature, with the most used ones including Kriging models, artificial neural network (ANN)-based surrogate models, and reduced order models [2], [3], [4], [5]. In this paper, a Variational Autoencoder (VAE) architecture is adopted to effectively capture complex, high-dimensional patterns in the structural response data. Unlike the other autoencoders, VAE provides a probabilistic latent representation, allowing better generalization, which is particularly valuable for long-term SHM tasks [6]. Furthermore, the use of fully connected layers in classifiers and regression blocks enables the estimation of damage severity and location directly from the latent space.

2 METHODOLOGY

The framework proposed in this paper integrates SHM information, a Bayesian finite element (FE) model updating approach, and surrogate modelling techniques to efficiently localize and quantify damage. A BMU framework is first employed to calibrate a high-fidelity FE model using SHM data, refining the model parameters to closely reflect the real structural behavior. Using the calibrated FE model, various damage scenarios are simulated to generate labeled datasets of

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modal responses. These datasets are then used to train a Variational AutoEncoder (VAE)-based surrogate model, which learns a latent representation of the relationship between modal features and damage states. Subsequently, the trained VAE model is utilized to obtain fast and scalable predictions of damage scenarios during online monitoring, bypassing the computational burden of running FE simulations in real time. This integrated approach bridges the gap between accurate physics-based modeling and the practical demands of efficient damage diagnosis in SHM systems. The overall architecture of this framework, including the surrogate model construction and its application for decision support, is illustrated in Figure 1.

After being trained, the surrogate model is capable of mapping newly acquired experimental modal features to damage location and severity, providing damage scenario indicators that can be used for decision support. This hybrid approach combines physics-based model updating for data generation with data-driven surrogate modeling for inference. It ensures a computationally efficient yet robust damage detection system for bridges.

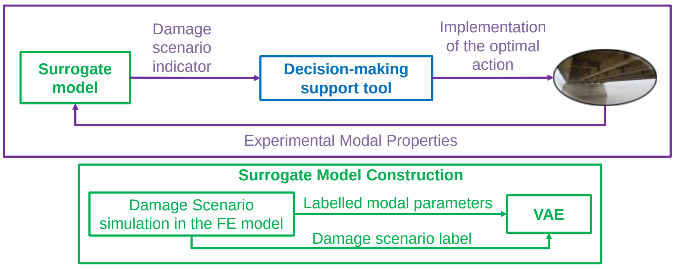

Figure 1: General workflow for online damage detection.

2.1 *FE calibration and simulation of damage scenarios*

Bayesian Model Updating (BMU) is employed to calibrate the FE model of the bridge using measured modal properties. The overall BMU workflow is indicated in Figure 2 that illustrates the step-by-step process of refining model parameters, from prior assumptions to the convergence of posterior distributions. The process aims to reduce discrepancies between experimental and simulated dynamic characteristics by updating uncertain model parameters, thereby enhancing the accuracy and predictive capabilities of the model. In this study, BMU is performed using Transitional Markov Chain Monte Carlo (TMCMC). TMCMC is a sampling-based Bayesian inference method that allows efficient estimation of the posterior distribution of model parameters, even in high-dimensional or nonlinear problems [7], [8].

2.1.1 Parameter Selection and Prior Definition

Parameters with high sensitivity to modal responses are selected for updating, specifically, the vertical stiffnesses of the girders. Each parameter θ is assigned a prior distribution $\pi(\theta)$, representing the initial uncertainty in its value based on engineering knowledge.

2.1.2 Likelihood Function Construction

The likelihood function $L(D|\theta)$ of data D quantifies the agreement between simulated and measured modal data, including both natural frequencies and mode shapes. Mode shape similarity is evaluated using the Modal Assurance Criterion (MAC). The likelihood is defined as indicated in Eq. 1.

$$L(D|\theta) = \exp\left(-\frac{1}{2}\sum_i w_i\left(\frac{f_{m,i}-f_{s,i}}{\sigma_i}\right)^2 - \frac{1}{2}\sum_j w_j \log(1-\mathrm{MAC}_j)\right) \quad (1)$$

where $f_{m,i}$ and $f_{s,i}$ are measured and simulated frequencies, σ_i represents uncertainty, and w_i, w_j are weighting factors.

2.1.3 Transitional Sampling via TMCMC

TMCMC introduces a sequence of intermediate, tempered distributions shown in Eq. 2.

$$\pi_\beta(\theta|D) \propto \pi(\theta)L(D|\theta)^\beta \quad (2)$$

where $\beta \in [0,1]$ gradually increases from 0 (prior only) to 1 (full posterior). At each state, samples are reweighted and resampled based on their likelihood, allowing efficient exploration of the parameter space. The process continues until the convergence is achieved.

2.1.4 Posterior Sampling and Model Updating

During the TMCMC process, the FE model is continuously evaluated as parameter samples are drawn and updated through each intermediate distribution. At every step, the simulated modal properties are compared with experimental data to assess the quality of the current model approximation. The iterative approach allows progressive refinement of the model, ensuring that the final set of posterior samples yields a calibrated model that reliably captures the dominant dynamic behavior of the structure. Despite the minor residual discrepancies (e.g. in higher modes), the updated model serves as a high-fidelity basis for generating synthetic damage scenarios, which provide the labeled data needed to train the surrogate model described in the following sections.

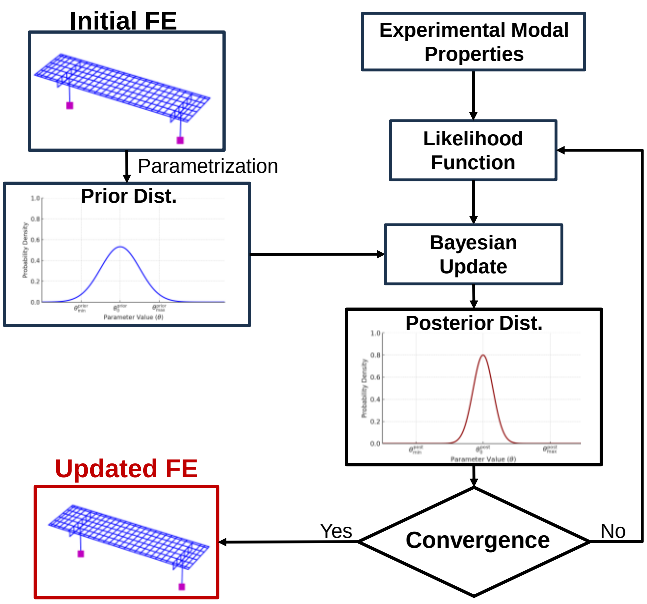

Figure 2: Bayesian Model Update diagram

In this study, vertical stiffnesses were chosen as updating parameters due to their high sensitivity to vertical and torsional modes identified in the experimental data. A total of 8 stiffness

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parameters (one per each girder) were assigned uniform prior distributions with ±50% bounds from nominal values. TMCMC was implemented with 50 intermediate β steps and 1000 samples per step. Convergence was evaluated using the coefficient of variation of the likelihood, with a threshold of 5% at each stage.

2.2 Surrogate model

The calibrated FE model is used to generate labelled dataset for several damage scenarios, which constitute the training set for the surrogate model. In this work, a Variational Autoencoder (VAE) is employed to learn a latent representation of modal properties across the various damage scenarios. The model is trained on synthetic modal data representing damage states of increasing severity and is developed for real-time estimation of damage location and severity based on updated modal properties. While this structure supports unsupervised feature learning, it has incorporated supervised outputs, providing reconstructed modal properties and predicted damage features.

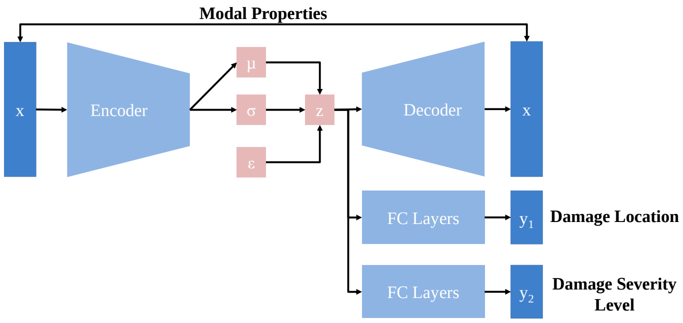

Figure 3: VAE architecture

The VAE consists of an encoder, which maps the modal properties (**x**) to a latent space *(μ, σ, ε,* z), and a decoder, which reconstructs the modal properties while ensuring regularization through a Kullback-Leibler (KL) divergence term. This term, commonly used in VAE architecture, encourages the latent variables to follow a normal distribution and helps the model learn meaningful and general features [9]. In addition to the conventional VAE architecture, two fully connected layers (FC Layers) are incorporated as a part of supervised feature learning: one serves as a classifier for identifying the location of damage y_1, and the other functions as a regressor to estimate damage severity levels y_2, as illustrated in Figure 3.

The input to train the surrogate model comprises modal frequencies and normalized mode shapes, extracted from modal analysis. The encoder consists of three fully connected layers with Rectified Linear Unit (ReLU) as a nonlinear activation function mapping the input to a latent space of dimension 32. The ReLU is widely used for its simplicity and effectiveness in preventing vanishing gradients. Two separate fully connected layers use the mean and log variance of the latent distribution to establish a relationship between the latent features and the damage locations and severity levels. The decoder follows a symmetric structure to reconstruct the input. Additionally, two parallel output layers predict damage locations and severity. Similar to the approach proposed by Yessoufou and Zhu [10], who employed a convolutional neural network-LSTM with distinct loss functions for damage classification and severity estimation, the proposed architecture treats damage location as a classification problem supervised with cross-entropy loss, while damage severity estimation is formulated as a regression task that predicts severity levels between 0 and 1 and is optimized using mean squared error.

The model is trained using a weighted loss function combining:

1. Reconstruction Loss: Mean Absolute Error (MAE) between input and reconstructed modal properties.
2. KL Divergence Loss: Enforcing latent space regularization.
3. Classification Loss: Cross-entropy loss for damage location prediction.
4. Regression Loss: Mean Squared Error (MSE) for damage severity estimation.

A cyclical KL annealing strategy is implemented, gradually increasing the weight of the KL term to improve latent space disentanglement [11]. To enhance the training performance, several incremental analyses were conducted, based on which the Adam optimizer was selected [12]. Additionally, the initial learning rate was set to 0.001 and configured to adaptively decrease throughout different phases of training to maintain stable convergence and improved generalization.

2.3 Decision-Making approach

A concept for a decision-making approach is proposed in Figure 4, drawing inspiration from existing SHM-informed response protocols proposed by Çelebi [13].

At the core of this approach lies a threshold-based logic that interprets the results produced by the VAE model. The VAE model identifies the most likely damaged locations and estimates the damage severity. For each identified component as the location of damage, the damage severity is evaluated individually through a decision-making layer that maps severity levels to specific actions. These outputs are contextualized through a decision-making layer that maps each damage severity level to a specific action. Namely, the outputs of the VAE (relevant to damage location and severity) are evaluated against predefined thresholds. The exceedance of a threshold triggers a specific action (continued monitoring, issuing a warning, or initiating a repair procedure). These layered interpretations add practical value to the detection results and allow for automatic mapping of evolving damage states into operational decisions. Exemplary actions are depicted in Figure 4. The VAE model provides two key outputs, which are the damage location and the damage severity. Each damaged component is associated with an evaluated damage severity (k_{red}) and a warning indicator. Green indicates normal condition, yellow suggests the need for inspection, and red prompts immediate repair or closing bridge suggestions, depending on severity. The goal of this approach is to support a straightforward integration of SHM-informed, rule-based maintenance strategies into bridge integrity management, ensuring that timely and proportional interventions are triggered as the condition of the structure evolves. The definition of the threshold is a critical aspect of this approach and must be carried out based on reliability analysis for specific limit states defined for the bridge.

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3 CASE STUDY

The procedure described in the previous section has been applied to a continuously monitored bridge located in northern Italy. The bridge consists of 15 spans, 11 of which are instrumented with acceleration sensors. Each monitored span is equipped with 5 to 6 acceleration sensors on the deck, strategically placed to capture the bridge's dynamic response under operational conditions.

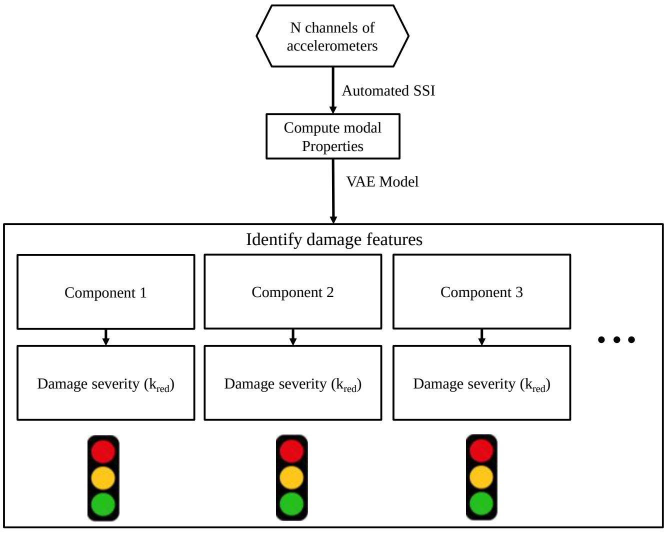

Figure 4: Decision-making framework based on damage level threshold

To identify the modal parameters from recorded responses, an online automatic Stochastic Subspace Identification (SSI) method has been developed. To ensure robust tracking of modal properties over time, a post-processing step involving modal clustering is employed. The identified modal properties are clustered using a hierarchical clustering algorithm based on a predefined Modal Assurance Criterion (MAC) and frequency similarity threshold. This process helps distinguish consistent modes from spurious ones, reducing uncertainties in the estimated modal parameters

The approach follows the clustering methodology detailed in previous works by Magalhães et al. [14] which has demonstrated its effectiveness on SSI-based modal tracking in bridge monitoring applications.

A detailed finite element model of the bridge was built using the OpenSees software [15], and calibrated by applying the Bayesian model updating process. During the BMU, vertical bending stiffnesses were selected as updating parameters, based on their higher sensitivity. This choice was made since the experimental mode shapes of the selected bridge are predominantly in the vertical direction, including vertical and torsional modes. These stiffness parameters were iteratively updated using the Transitional Markov Chain Monte Carlo (TMCMC) algorithm described in section 2.1. The resulting frequencies and MAC values before and after the BMU are indicated in Table 1. The mode shapes obtained from experimental data and the updated FE model are shown in Figure 5 and Figure 6, respectively.

After updating, the FE model showed improved agreement with the experimental modal properties. The first and second modes reached MAC values of 99.8% and 94.2%, respectively. However, the third mode retained a relatively low MAC value of 36.1%, which indicates limited consistency. This discrepancy is attributed to reduced sensitivity of vertical stiffness to higher terms not captured by the selected parameters.

Despite this, the updated FE model provides a sufficiently accurate representation of the bridge's dominant dynamic behavior, and it is used exclusively to generate synthetic damage scenarios for training the surrogate model. Since both training and test datasets are generated from the calibrated model, the surrogate model's performance reflects the behavior encoded in the updated FE model, while remaining independent of direct comparisons with experimental data.

Table 1: Modal properties comparison

	Experimental	FE Model before BMU	FE Model after BMU	
Mode Number	Frequency (Hz)	Frequency (Hz)	Frequency (Hz)	MAC value (%)
1	1.56	1.70	1.75	99.8
2	2.50	1.82	2.42	94.2
3	3.63	3.42	3.76	36.1

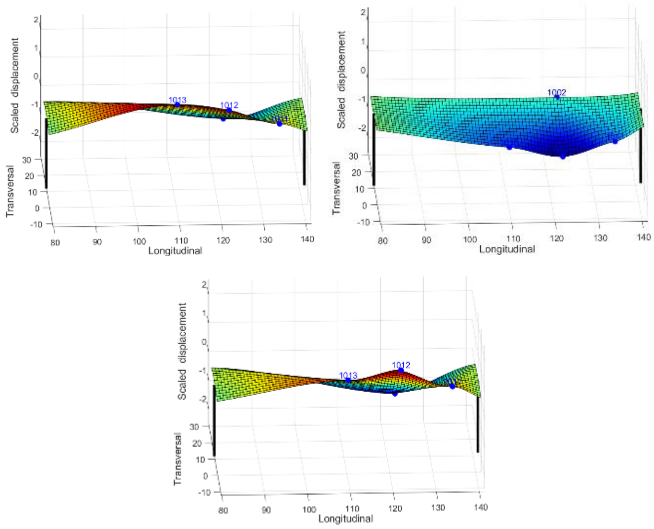

Figure 5: Experimental mode shapes

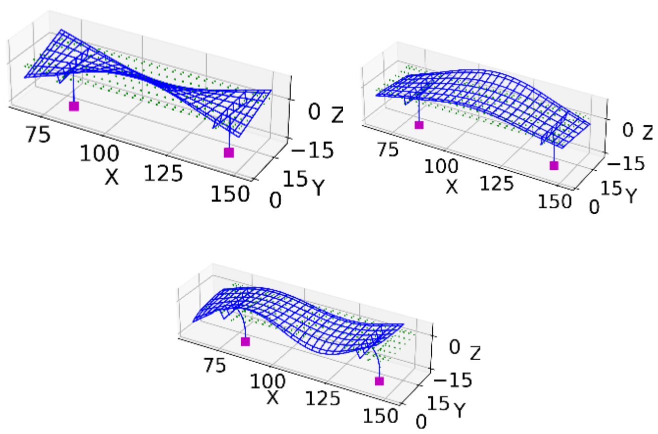

Figure 6: Updated FE mode shapes

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3.1 Construction of the surrogate model and damage identification

The proposed damage identification framework is applied to a single span to demonstrate its effectiveness in damage detection and integrity assessment. The Variational Autoencoder (VAE) model was trained and applied using numerically simulated data representing various damage scenarios with different damage locations and severity.
To generate the training dataset for the VAE model, damage was simulated in the FE model through the reduction in vertical rotational stiffness. For each damage scenario, the FE modal properties were obtained by dividing the span into 20 parts and applying the reductions to the corresponding vertical rotational stiffnesses.

Damage severity levels were defined as reductions ranging from 10% to 70% in the corresponding vertical rotational stiffness values. For each segment and each severity level, a separate damage scenario was created. In total, 140 damage scenarios were generated (20 segments × 7 severities). Table 2 summarizes the segments and the associated severity levels considered in the training data generation. To enhance the robustness of model learning, random noise was artificially added to the modal properties during the training data generation process. During the training phase, the surrogate model was trained using the labeled modal properties (natural frequencies and normalized mode shapes) corresponding to the various considered damage scenarios. The structural model was modified for each damage scenario, and a modal analysis was performed to obtain corresponding modal properties, enabling the VAE to learn patterns associated with different damage levels and locations. After the generation of the training dataset, it was divided into 80% and 20% portions, with 80% used to train the model, remaining 20% used to test the model, providing an unseen dataset to objectively evaluate the model's learning performance

It is acknowledged that the damage scenarios used in this study are synthetically generated and not validated against experimental damage. While the applied stiffness reduction levels serve to explore the sensitivity and robustness of the surrogate model, such values may not reflect the typical damage progression in real-world structures. These scenarios are intended to span a wide range of conditions, including rare or extreme cases.

Table 2: Summary of the damage scenarios

Damage Scenario	Segment No	Reduction Factors
DS1	1	From 0.1 to 0.7
DS2	2	From 0.1 to 0.7
DS3	3	From 0.1 to 0.7
…	…	…
DS20	20	From 0.1 to 0.7

3.2 Damage identification

The capability of the surrogate model to identify damage was tested using unseen test data, that is, samples of modal parameters corresponding to the considered damage scenarios, not used in the training phase. Results are represented by the confusion matrix in Figure 7. The confusion matrix compares the true and predicted damage locations, where diagonal elements represent correct predictions and off-diagonal elements indicate misclassification in the test datasets.

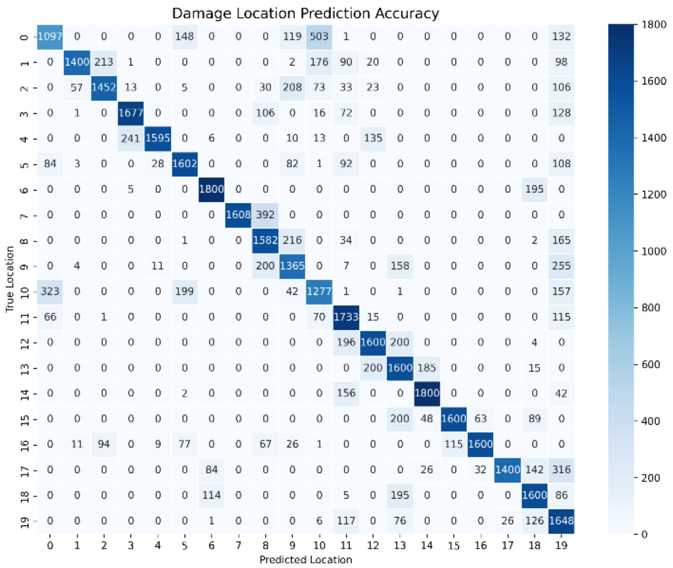

Figure 7: Confusion matrix for damage locations

In Figure 7, predicted and simulated (ground truth) damage locations are represented along the *x* and *y* axes by the element indices in the FE model. The diagonal elements of the matrix indicate the number of samples for which the damage location was correctly identified.

It is important to note that both training and testing datasets for the surrogate model were generated from the updated FE model, which was calibrated using experimental model properties. Although the updated model still presents some discrepancies, particularly in higher modes, the surrogate model operates entirely within the dynamic response space defined by the updated model. To enhance the robustness of the algorithm, artificial noise was introduced into synthetic modal data during both training and testing. This ensures that the model is not overfitted to idealized cases and can generalize across realistic measurement uncertainty, while maintaining consistency with the physical behavior captured by the updated FE model.

The model was tested on scenarios involving progressive damage evolution, effectively capturing and tracking the increasing severity over time. The data for the evolving damage severity was gathered from the unseen test dataset to indicate the model's performance in this context. The results are presented in Figure 8 where the vertical axis represents stiffness reduction factors. The predicted damage severity follows this predefined discretization to ensure the consistency between training and testing data. Additionally, Figure 8 illustrates the damage detection results over an evolving damage scenario, highlighting how the proposed framework translates predicted damage severity levels into actionable maintenance decisions. Each step in Figure 8 corresponds to a synthetic damage state generated by reducing the stiffness in the model. These steps represent hypothetical damage progression sequences, used to demonstrate the ability of the VAE model to track increasing severity.

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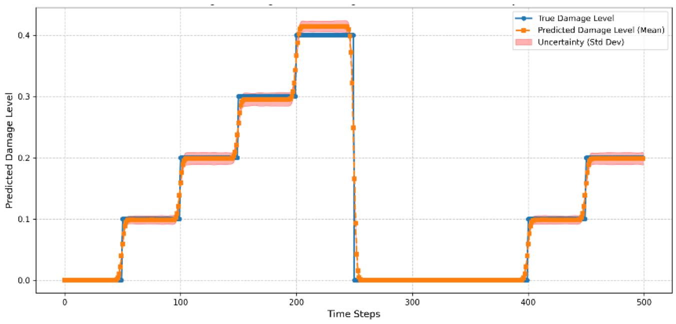

Figure 8: Damage level identification results

3.3 Decision Making

For the considered case study, the thresholds that trigger different management actions for each bridge component have been defined in terms of the stiffness reduction factor, k_{red}, and are reported in Table 3. To these thresholds correspond the damage scenarios considered for the training of the VAE model.

Table 3: Thresholds for stiffness reduction

k_{red}	Maintenance management action
0.1	normal condition
0.2	structural inspection
0.3	component repair
0.4	cautionary bridge closing

It is also worth mentioning that the maintenance actions proposed here are not intended to restore the load-bearing capacity but rather to compensate for localized stiffness reductions that may affect the bridge's dynamic behavior and long-term serviceability. The decision framework relies on stiffness reduction as a measurable proxy for damage progression, which triggers maintenance interventions aimed at preserving structural performance and reducing the risk of further deterioration. This approach reflects a conservative, condition-based strategy focused on sustaining system stiffness and structural continuity, even before reaching strength-based limit states.

4 CONCLUSION

This paper presents an online damage identification approach based on a Variational Autoencoder surrogate model. The proposed methodology combines model-based data generation with surrogate modelling to enhance the efficiency of the real-time data-driven damage identification without reducing accuracy.

To support timely maintenance decisions, a concept for a structured decision-making framework is proposed. The framework maps the structural condition into specific management actions. This structured, rule-based approach enables scalable, real-time decision support under varying operational scenarios.

Future work will explore the integration of stochastic deterioration models into the Finite Element model to refine long-term maintenance strategies, providing a more effective approach to bridge infrastructure management. The investigation of threshold values consistent with pre-defined limit states of the bridge will be a further research step.

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