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Dirac Operators with δ-Shell Interactions on Lipschitz and $\mathcal{C}^1$ Surfaces

MASTER'S THESIS

to achieve the university degree of

Diplom-Ingenieur

Master's degree programme: Mathematics

submitted to

Graz University of Technology

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Institute of Applied Mathematics

Graz, January 2021

Abstract

In the recent past Dirac operators with δ-shell interactions on surfaces have been of special interest. The interest in these operators stems from theoretical physics since Dirac operators are used to describe particles with spin 1/2 and also comply with the theory of relativity. Additionally, δ-potentials are frequently applied in dealing with highly localized potentials. However, so far, most publications assumed smooth surfaces or at least $\mathcal{C}^2$ surfaces. The goal of this thesis is to study Dirac operators with δ-shell interactions on Lipschitz and $\mathcal{C}^1$ surfaces. In the present thesis we discover that for the case of $\mathcal{C}^1$ surfaces a majority of the results regarding self-adjointness, spectrum and the resolvent of Dirac operators with δ-shell interactions can be proven. Moreover, in a lot of cases the results can even be carried over to Lipschitz surfaces, particularly including the confinement case and the case of purely Lorentz scalar δ-shell interactions. The basis of the successful treatment of those operators comes from studying integral operators on Lipschitz surfaces. Furthermore, the theory of quasi boundary triples helps to connect integral operators and Dirac operators which allows us to make use of the obtained properties for integral operators.

Zusammenfassung

In der jüngeren Vergangenheit waren Dirac Operatoren mit δ-Interaktionen auf Flächen im Raum von besonderem Interesse. Dieses Interesse kommt aus der theoretischen Physik, da Dirac Operatoren Partikel mit Spin 1/2 beschreiben und gleichzeitig mit der Relativitätstheorie kompatibel sind. Des Weiteren werden δ-Potentiale verwendet, um stark lokalisierte Potentiale zu modellieren. Bisher wurden dabei hauptsächlich glatte Flächen oder zumindest $\mathcal{C}^2$ Flächen betrachtet. Ziel dieser Arbeit ist es, den Fall von Lipschitz bzw. $\mathcal{C}^1$ Flächen zu analysieren. Wir finden heraus, dass ein Großteil der aus dem $\mathcal{C}^2$ Fall bekannten Resultate bezüglich Selbstadjungiertheit, Spektrum und Resolvente sich ebenso für $\mathcal{C}^1$ Flächen zeigen lässt. Außerdem können in vielen Fällen diese Resultate auch für Lipschitz Flächen bewiesen werden. In diesen Fällen sind Lorentz-Skalare Potentiale und der Fall des eingesperrten Partikels inkludiert. Der Grundstein der erfolgreichen Behandlung von Dirac Operatoren mit δ-Interaktionen auf Flächen wird durch das Studium von Integraloperatoren auf Lipschitz Flächen gelegt. Dabei hilft uns die Theorie von Quasi Randtripel bei der Verbindung von Integraloperatoren und Dirac Operatoren.

Contents

Acknowledgement

First, I would like to thank my supervisor Prof. Dr. Jussi Behrndt for his valuable input during our talk about this thesis. Moreover, I would like to thank my co-supervisor Dr. Markus Holzmann for his continuous assistance and guidance during the writing process. His feedback and help was not only greatly appreciated, but also essential while elaborating this thesis.
I would also like to express my gratitude to my friends, my family and particularly my girlfriend Katharina for supporting me throughout my studies.

Introduction

The Dirac operator plays an important role in modelling particles with spin 1/2. Dirac successfully introduced this operator to describe particles in a quantum mechanical framework that also takes the theory of relativity into account. From this point of view the Dirac operator can be seen as a relativistic counterpart of the Schrödinger operator. He also deduced that $L^2(\mathbb{R}^3;\mathbb{C}^4)$ is the fitting Hilbert space in this realm. The Dirac operator in an external field has the general form

$$-i\sum_{l=1}^{3}\alpha_l\frac{\partial}{\partial x_l}+m\beta+V \tag{0.1}$$

in natural units, where the reduced Planck constant and the speed of light equal one, see [38]. Moreover, $\alpha_1,\alpha_2,\alpha_3,\beta\in\mathbb{C}^{4\times 4}$ are the self-adjoint Dirac matrices, their explicit form can be found in Definition 2.30, $m\in\mathbb{R}$ represents the mass of the particle and V denotes a potential which models the external field. In this thesis we focus on singular potentials of the form

$$V=(\eta I_4+\tau\beta)\delta_\Sigma \tag{0.2}$$

with $\eta,\tau\in\mathbb{R}$ and I_4 denoting the identity matrix in $\mathbb{C}^{4\times 4}$. Here, the potential δ_Σ represents a singular potential supported on a compact surface in $\mathbb{R}^3$ and the two interaction strengths η and τ correspond to electrostatic and Lorentz-scalar potentials, respectively. Such potentials are called δ-shell potentials and are used to approximate strongly localized potentials. They are a standard tool in the context of Schrödinger operators and are also valuable for Dirac operators, especially since [27, 36, 39] show that Dirac operators with δ-shell potentials can be interpreted as limits of Dirac operators with squeezed potentials. Here, it is worth mentioning that [36, 39] consider the one-dimensional Dirac operator and [27] treats the three-dimensional Dirac operator.

Dirac operators with δ-shell interactions have been studied in a wide range of publications, cf. [2, 3, 4, 6, 7, 9, 23], mostly focusing on the case of $\mathcal{C}^2$ surfaces. However, the case of non-smooth surfaces did not get much attention in the literature. Noteworthy is the paper of Arrizabalaga, Mas and Vega [2] where in Remark 3.4 it is mentioned that the $\mathcal{C}^2$ restriction could be weakened to $\mathcal{C}^1$. Moreover, [25, 34] discuss two-dimensional Dirac operators in domains with non-smooth boundaries. In [25] Dirac operators in polygons with boundary conditions are treated and in [34] Dirac operators with δ-shell interactions on piecewise $\mathcal{C}^2$ boundaries are considered.

The main goal in this thesis is to study three-dimensional Dirac operators with δ-shell interactions on Lipschitz and $\mathcal{C}^1$ surfaces. In order to do so we apply the theory of quasi boundary triples, which was introduced by Behrndt and Langer in [12, 13], as framework. This theory allows us to reduce the problem of self-adjointness mainly to the treatment of operators on boundaries which are in our case singular integral operators on surfaces. The construction of a quasi boundary triple includes the choice of fitting domains for Dirac operators. Roughly speaking, if we choose the domain of the Dirac operator such that it is contained in a Sobolev space of order s, then the domain of the singular integral operators is contained in a Sobolev space of order $s-1/2$. Since the singular integral operators are best understood, particularly for non-smooth surfaces, in the context of L^2 spaces analogous to [34] we choose $s = 1/2$. Hence, in contrast to the most common choice $s = 1$, as for instance in [7, 10, 22, 23], we are not able to use the classic trace theorem in our situation and therefore have to formulate an appropriate trace theorem. For proving the trace theorem we use the ideas from [8]. After rigorously defining Dirac operators with δ-shell interactions in (0.8) we once more discuss the aspect regarding the Sobolev regularity in the domains from a different point of view. To study the mentioned singular integral operators in L^2 we use the work of Coifman, McIntosh and Meyer [14] on Cauchy integrals on Lipschitz curves as well as the work of Axelsson, Grognard, Hogan and McIntosh [5] concerning harmonic analysis of Dirac operators on Lipschitz domains. Besides proving the self-adjointness the quasi boundary triple also helps us to examine the spectrum of Dirac operators with δ-shell interactions. Thereby, we mostly rely on methods which are well known from the $\mathcal{C}^2$ case and can be found e.g. in [22].

Next, let us introduce and discuss the main objects of this thesis in more detail. We assume $\Omega_+ \subset \mathbb{R}^3$ to be a Lipschitz domain with compact boundary, $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega_+}$ and $\Sigma := \partial\Omega_+ = \partial\Omega_-$. Then, the Dirac operator with δ-shell interactions is formally given by

$$A_{\eta,\tau} = -i(\alpha \cdot \nabla) + m\beta + (\eta I_4 + \tau\beta)\delta_\Sigma, \tag{0.3}$$

where $(\alpha \cdot \nabla) = \sum_{l=1}^3 \alpha_l \frac{\partial}{\partial x_l}$ and $\delta_\Sigma f = \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma)$. Here, $f_\pm$ denotes the restriction of f to $\Omega_\pm$. In the rigorous definition of $A_{\eta,\tau}$ as an operator in $L^2(\mathbb{R}^3;\mathbb{C}^4)$ the δ_Σ-interaction is modelled by jump conditions for functions in the domain of the operator $A_{\eta,\tau}$. In the following, we want to motivate these jump conditions. Interpreting $A_{\eta,\tau} f$ as a distribution yields for a test function $g \in \mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)$

$$\langle A_{\eta,\tau} f, g\rangle = \int_{\mathbb{R}^3} f \cdot i(\overline{\alpha} \cdot \nabla)g + mf \cdot \beta g \, dx + \int_\Sigma (\eta I_4 + \tau\beta)\frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma) \cdot g \, d\sigma(x). \tag{0.4}$$

On the contrary, since δ_Σ is only supported on Σ we have

$$A_{\eta,\tau} = -i(\alpha \cdot \nabla) + m\beta \quad \text{in } \Omega_\pm. \tag{0.5}$$

Hence, integration by parts gives us

$$\begin{aligned}\langle A_{\eta,\tau} f, g\rangle &= \int_{\Omega_-\cup\Omega_+} -i(\alpha\cdot\nabla)f\cdot g + m\beta f\cdot g\,dx \\ &= \int_{\mathbb{R}^3} f\cdot i(\overline{\alpha}\cdot\nabla)g + mf\cdot\beta g\,dx - \int_\Sigma i(\alpha\cdot\nu)(f_+|_\Sigma - f_-|_\Sigma)\cdot g\,d\sigma(x),\end{aligned} \tag{0.6}$$

where ν denotes the unit outward normal vector of Ω_+ and $\alpha\cdot\nu = \sum_{l=1}^3 \alpha_l\nu_l$. Comparing (0.4) and (0.6) for all $g\in\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)$ shows

$$i(\alpha\cdot\nu)(f_+|_\Sigma - f_-|_\Sigma) + (\eta I_4 + \tau\beta)\frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma) = 0 \text{ on } \Sigma. \tag{0.7}$$

Based on these heuristic considerations we rigorously define the Dirac operator with δ-shell interactions through

$$\begin{aligned}\operatorname{dom} A_{\eta,\tau} &:= \big\{f = f_+\oplus f_- \in H^{1/2}(\Omega_+;\mathbb{C}^4)\oplus H^{1/2}(\Omega_-;\mathbb{C}^4) : \\ &\qquad (\alpha\cdot\nabla)f_\pm \in L^2(\Omega_\pm;\mathbb{C}^4) \text{ and } f \text{ fulfills (0.7)}\big\} \subset L^2(\mathbb{R}^3;\mathbb{C}^4) \\ A_{\eta,\tau} f &:= (-i(\alpha\cdot\nabla)+m\beta)\,f_+ \oplus (-i(\alpha\cdot\nabla)+m\beta)\,f_- \quad \forall f\in\operatorname{dom} A_{\eta,\tau},\end{aligned} \tag{0.8}$$

where $H^{1/2}(\Omega_\pm;\mathbb{C}^4)$ denotes the Sobolev space of order 1/2 on $\Omega_\pm$. Historically, such operators were firstly considered in 1989 by Dittrich, Exner and Šeba in [16] where they studied Dirac operators with δ-shell interactions on spheres. There, they proved self-adjointness for all $\eta,\tau\in\mathbb{R}$. After a twenty-five year long period with little progress, Arrizabalaga, Mas and Vega successfully proved self-adjointness of $A_{\eta,\tau}$ for general $\mathcal{C}^2$ smooth surfaces and all $\eta^2-\tau^2\neq 4$ in [2, 3]. Such a configuration of the interaction strengths is called non-critical and the configuration $\eta^2-\tau^2 = 4$ is called critical. This distinction is made due to the fact that some theoretical tools fail to work in the critical case. Moreover, on the basis of the results from [11], where two-dimensional Dirac operators are considered, one can not expect the same spectral properties as well as any Sobolev regularity in $\operatorname{dom} A_{\eta,\tau}$ in the critical case. Due to these difficulties the critical case has been excluded in most publications concerning Dirac operators with δ-shell interactions, cf. [2, 3, 4, 6, 7, 23]. In general, the question of self-adjointness of $A_{\eta,\tau}$ in the critical case is still open, even for smooth domains. However, in the last few years this case got more attention and was examined in [9, 11, 32]. In the present thesis the critical case is excluded.

Before we summarize the main results of this thesis, let us come back to the discussion of the Sobolev space order in $\operatorname{dom} A_{\eta,\tau}$. For $\mathcal{C}^2$ surfaces one can show that in the non-critical case the domain $\operatorname{dom} A_{\eta,\tau}$ is contained in $H^1(\Omega_+;\mathbb{C}^4)\oplus H^1(\Omega_-;\mathbb{C}^4)$, see [22, Definition 4.2.1 and Theorem 4.2.3]. Thus, changing the order of the Sobolev spaces from 1/2 to 1 in (0.8) would not change the domain of $A_{\eta,\tau}$. However, due to [25, Theorem 1.2 (ii)] one can not expect this behaviour for interactions on non-smooth surfaces and therefore it is necessary to work with Sobolev spaces of lower order.

Now, we turn to the discussion of the main results of the present thesis.

Theorem 0.1. *Let Ω_+ be a Lipschitz domain. Then, there exists a constant $M \geq \frac{1}{4}$ such that for $\eta^2 - \tau^2 \notin \left[\frac{1}{M}, 16M\right]$ the following assertions hold:*

(i) The operator $A_{\eta,\tau}$ is self-adjoint.

(ii) $\sigma_{ess}(A_{\eta,\tau}) = (-\infty, -|m|] \cup [|m|, \infty)$.

(iii) $\sigma_{disc}(A_{\eta,\tau})$ is finite.

(iv) For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the difference $(A_{\eta,\tau} - \lambda)^{-4} - (A_{0,0} - \lambda)^{-4}$ belongs to the trace class, i.e. the singular values of $(A_{\eta,\tau} - \lambda)^{-4} - (A_{0,0} - \lambda)^{-4}$ are summable.

Moreover, if Ω_+ is a $\mathcal{C}^1$ domain, then $M = \frac{1}{4}$, i.e. the assertions hold for all $\eta^2 - \tau^2 \neq 4$.

The statements and proofs of Theorem 0.1 can be found in Theorem 5.8 and Theorem 5.14. Theorem 0.1 answers the question of self-adjointness of $A_{\eta,\tau}$ and gives us information about the spectrum of $A_{\eta,\tau}$ for a wide variety of interaction strengths. Furthermore, item *(iv)* of Theorem 0.1 serves as a valuable basis in scattering theory. We see that for $\mathcal{C}^1$ domains the self-adjointness can be shown for all non-critical interaction strengths. Hence, the results are similar to the results on smoother domains. In order to stress the significance of Theorem 0.1 we formulate two corollaries, where we consider two important special cases. Namely, the confinement case ($\eta^2 - \tau^2 = -4$) and the purely Lorentz-scalar case ($\eta = 0$). Let us start with the confinement case.

Corollary 0.2. *Let Ω_+ be a Lipschitz domain. If $\eta^2 - \tau^2 = -4$, then the operator $A_{\eta,\tau}$ is self-adjoint and the interface condition* (0.7) *can be decoupled into the two boundary conditions*

$$\left(\pm i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 + \tau\beta)\right) f_\pm|_\Sigma = 0 \text{ on } \Sigma. \tag{0.9}$$

The proof of Corollary 0.2 is stated in Theorem 5.10. In such a configuration of the interaction strengths, due to the decoupling of the interface condition, the boundary Σ becomes impermeable for particles. This phenomenon is also discussed in [3, Section 5], [22, Remark 4.2.2.] and [7, Lemma 3.1]. Dirac operators with boundary conditions as in (0.9) are treated in [10, 22] and are used to describe relativistic particles in domains as in the quark gluon confinement. Moreover, two-dimensional Dirac operators with such boundary conditions are called quantum-dot operators and are used to describe graphene, cf. [25, 34]. Corollary 0.2 directly implies the self-adjointness of Dirac operators with boundary conditions as in (0.9).
The upcoming corollary considers the case of purely Lorentz-scalar interactions.

Corollary 0.3. *Let Ω_+ be a Lipschitz domain and $\tau \in \mathbb{R}$. Then, $A_{0,\tau}$ is self adjoint and, in addition to the assertions in Theorem 0.1, the following statements hold:*

(i) $\lambda \in \sigma(A_{0,\tau})$ if and only if $-\lambda \in \sigma(A_{0,\tau})$.

(ii) The discrete eigenvalues of $A_{0,\tau}$ have even multiplicity.

(iii) If $\tau m \geq 0$, then $\sigma_{disc}(A_{0,\tau}) = \emptyset$.

This corollary is shown in Theorem 5.12 and deals with potentials that can be written in the form $V = \tau\beta\delta_\Sigma$ and therefore are invariant under Lorentz transformations. Such potentials are considered in [23] for interactions on smooth surfaces in $\mathbb{R}^3$ and in [34] for interactions on piecewise smooth closed curves in $\mathbb{R}^2$ in further detail. Our main takeaway from Corollary 0.3 is that the Lorentz-scalar case is fully covered by Theorem 0.1 for all $\tau \in \mathbb{R}$ without assuming additional smoothness of the surface.

Finally, we give a short overview on this thesis. In the first chapter we introduce necessary notations and theoretical tools which we apply in subsequent chapters. After that, we study Sobolev spaces in Chapter 2. There, Sobolev spaces for Dirac operators are to be highlighted. In this thesis they are the domains of Dirac operators and allow us to state an extended trace theorem. Afterwards, in Chapter 3, we focus on the significant topic of integral operators. We study singular integral operators on Lipschitz and $\mathcal{C}^1$ boundaries. There, the compactness result of Theorem 3.18 is especially important. It allows us to imply that Dirac operators with δ-shell interactions on $\mathcal{C}^1$ boundaries are self-adjoint for $\eta^2 - \tau^2 \neq 4$ which is a similar result as known for $\mathcal{C}^2$ domains. Chapter 4 deals with the free Dirac operator, i.e. the Dirac operator without a potential, and the construction of a quasi boundary triple for Dirac operators with interactions on the boundary. This construction succeeds due to preliminary work done in previous chapters. In the last chapter we use Fredholm theory in order to prove self-adjointness of $A_{\eta,\tau}$ for Lipschitz and $\mathcal{C}^1$ domains. Finally, we discuss special choices of the interaction strengths η and τ and deal with differences of powers of resolvents which are important with respect to scattering theory.

1 Preliminaries

In this chapter we set the stage for this thesis. First, we introduce important notations, then we deal with different types of operators. Thereafter, we introduce quasi boundary triples which are an important tool in finding self-adjoint extensions of symmetric operators. Last, we define Lipschitz domains and the non-tangential trace of functions on Lipschitz domains.

1.1 Notations

We fix frequently occurring notations which may be not clear at first sight or deviate from standard notations.
The natural number $n \geq 2$ always denotes the space dimension of the vector space $\mathbb{R}^n$.
With $C > 0$ we mean a generic constant which may change in-between lines.
For $z \in \mathbb{C} \setminus \mathbb{R}_+$ we choose the square root such that $\operatorname{Im} z > 0$.
The symbol $|\cdot|$ denotes the modulus of a scalar, the 2-norm of a column vector or the matrix norm induced by the 2-vector norm. Moreover, $|\cdot|_p$ for $1 \leq p \leq \infty$ denotes the p-norm of a column vector or the matrix norm induced by the p-vector norm. For two column vectors a, b of the same length k we set the dot product to be $a \cdot b = \sum_{l=1}^k a_l b_l$.
Furthermore, if $a = (a_1, a_2, \ldots, a_k)$ is a tuple with values in a complex vector space, $a \cdot b$ also represents the expression $\sum_{l=1}^k a_l b_l$.
Let V be a normed space. Then, we define V^* to be the dual of V and call the term ${}_{V^*}\langle f, v\rangle_V := f(v)$ the duality product of $f \in V^*$ and $v \in V$. If W is an additional normed space, we write $V \times W$ for the Cartesian product of the two normed spaces and endow this space with the norm $\|\cdot\|_{V\times W} = \sqrt{\|\cdot\|_V^2 + \|\cdot\|_W^2}$. Moreover, we write $\mathcal{L}(V, W)$ for the set of bounded linear operators mapping from V to W.
The expression $(\cdot, \cdot)_{\mathcal{H}}$ denotes the inner product of a Hilbert space $\mathcal{H}$. Here, we assume that $(\cdot, \cdot)_{\mathcal{H}}$ is antilinear in the first argument. If $\mathcal{G}$ is also a Hilbert space, then $\mathcal{G} \oplus \mathcal{H}$ expresses the outer orthogonal sum of $\mathcal{G}$ and $\mathcal{H}$.
If we do not mention a specific measure in context of measure and integral theory, we are talking about the Lebesgue measure.

1.2 Fredholm Operators

We discuss Fredholm operators and state the Fredholm alternative in Theorem 1.3. Furthermore, Theorem 1.2 shows that compact perturbations do not change the Fredhlom index. For a deeper investigation on Fredholm operators, see [28, Chapter 2].
In this and in the two upcoming sections we assume $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ to be Hilbert spaces.

Definition 1.1. Let $F : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear and bounded operator. We call F a Fredholm operator if

(i) $\operatorname{ran} F$ is closed in $\mathcal{H}_2$ as well as

(ii) $\dim(\ker F) < \infty$ and $\dim(\mathcal{H}_2/\operatorname{ran} F) < \infty$.

In this setting we define the Fredholm index as

$$\operatorname{index}(F) := \dim(\ker F) - \dim(\mathcal{H}_2/\operatorname{ran} F). \tag{1.1}$$

We make use of the next two theorems in this thesis.

Theorem 1.2. *[28, Theroem 2.26] Let $F : \mathcal{H}_1 \to \mathcal{H}_2$ be a Fredholm operator and $K : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear compact operator. Then, $F + K$ is also Fredholm and* $\operatorname{index}(F + K) = \operatorname{index}(F)$.

Another well known result corresponding to Fredholm operators is the Fredholm alternative. We state a form of the Fredholm alternative which is fitting for our applications.

Theorem 1.3. *[28, Theroem 2.27] Assume that $F : \mathcal{H}_1 \to \mathcal{H}_2$ is Fredholm with* $\operatorname{index}(F) = 0$. *There are two mutually exclusive possibilities:*

(i) The homogeneous equation $Fu = 0$ has only the trivial solution $u = 0$. Moreover, in this case the operator F is bijective.

(ii) The homogeneous equation $Fu = 0$ has exactly $p \in \mathbb{N}$ linearly independent solutions $u_1, u_2, \ldots, u_p$.

1.3 Closed Operators and Their Spectra

In this section we recall some basic definitions regarding closed operators and their spectra. Therefore, let us assume throughout this section that $S : \mathcal{H}_1 \supset \operatorname{dom} S \to \mathcal{H}_2$ is a closed operator, i.e. $\mathcal{G}(S) := \{(x, Sx) : x \in \operatorname{dom} S\}$ is closed in $\mathcal{H}_1 \times \mathcal{H}_2$.

Definition 1.4. The set

$$\rho(S) := \big\{\lambda \in \mathbb{C} : (S - \lambda)^{-1} \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)\big\} \tag{1.2}$$

is said to be the resolvent set of S and $\sigma(S) := \mathbb{C} \setminus \rho(S)$ is called the spectrum of S. Furthermore, we define the set of eigenvalues as the point spectrum $\sigma_p(S) \subset \sigma(S)$.

Next, we introduce the adjoint operator.

Definition 1.5. Let $\operatorname{dom} S$ be dense in $\mathcal{H}_1$. Then, we define the adjoint operator

$$\begin{aligned} \operatorname{dom} S^* &:= \left\{g \in \mathcal{H}_2 : \exists g^* \in \mathcal{H}_1 \text{ such that } (Sf, g)_{\mathcal{H}_2} = (f, g^*)_{\mathcal{H}_1} \ \forall f \in \operatorname{dom} S\right\} \\ S^* g &:= g^*. \end{aligned} \tag{1.3}$$

Moreover, if $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$ and $S \subset S^*$, we call S symmetric and if $S = S^*$, we call S self-adjoint.

In case of self-adjoint operators it is a well-known fact that the spectrum is real. Furthermore, in this setting we present two further spectral sets.

Definition 1.6. Let $S = S^*$ be a linear operator in the Hilbert space $\mathcal{H}$. Then,

$$\sigma_{\text{disc}}(S) := \{\lambda \in \sigma_p(S) : \dim(\ker(T - \lambda)) < \infty \text{ and } \lambda \text{ is isolated in } \sigma(S)\} \tag{1.4}$$

is the discrete spectrum of S and $\sigma_{\text{ess}}(S) := \sigma(S) \setminus \sigma_{\text{disc}}(S)$ is called the essential spectrum of S.

A relevant result concerning the essential spectrum reads as follows.

Theorem 1.7 ([35, Theorem 8.12])**.** *Let $A = A^*$ and $B = B^*$ be self-adjoint operators in $\mathcal{H}$. If*

$$(A - \mu)^{-1} - (B - \mu)^{-1} \tag{1.5}$$

is compact for a $\mu \in \rho(A) \cap \rho(B)$, then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$.

1.4 Schatten-von Neumann Ideals

In order to qualify compact operators, we use the so-called weak Schatten-von Neumann ideals. Therefore, we assume $K : \mathcal{H}_1 \to \mathcal{H}_2$ to be in the set of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, $\mathfrak{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$. Then, there exists a unique decreasing sequence of singular values $s_1(K) \geq s_2(K) \geq s_3(K) \geq \dots$. It is well known that

$$s_l(K) = \sqrt{s_l(KK^*)} = \sqrt{s_l(K^*K)} = s_l(K^*) \tag{1.6}$$

for $l \in N$. Here, $N = \{1, 2, \dots, k\}$ in case of $k \in \mathbb{N}$ singular values and $N = \mathbb{N}$ in case of infinetly many singular values. Before we introduce the Schatten-von Neumann ideals we define the trace class.

Definition 1.8. We call the set

$$\mathfrak{S}_1(\mathcal{H}_1, \mathcal{H}_2) := \left\{ K \in \mathfrak{S}_\infty(\mathcal{H}_1, \mathcal{H}_2) : \sum_{l \in N} s_l(K) < \infty \right\} \tag{1.7}$$

trace class. For $K \in \mathfrak{S}_1(\mathcal{H}_1, \mathcal{H}_2)$ we define the trace of K as

$$\operatorname{tr} K := \sum_{l \in N} s_l(K). \tag{1.8}$$

Definition 1.9. Let $0 < p < \infty$. Then, we define the weak Schatten-von Neumann ideal of order p

$$\mathfrak{S}_{p,\infty}(\mathcal{H}_1, \mathcal{H}_2) := \left\{ K \in \mathfrak{S}_\infty(\mathcal{H}_1, \mathcal{H}_2) : (l^{1/p} s_l(K))_{l \in N} \text{ is bounded} \right\}. \tag{1.9}$$

We summarize some important properties of these ideals in the next theorem.

Theorem 1.10. *Let $0 < p, q, r < \infty$. Then, the following assertions are true:*

(i) The inclusion $\mathfrak{S}_{p,\infty}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathfrak{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$ holds.

(ii) If $p < 1$, then $\mathfrak{S}_{p,\infty}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathfrak{S}_1(\mathcal{H}_1, \mathcal{H}_2)$.

(iii) If $p \leq q$, then $\mathfrak{S}_{p,\infty}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathfrak{S}_{q,\infty}(\mathcal{H}_1, \mathcal{H}_2)$.

(iv) If $S, T \in \mathfrak{S}_{p,\infty}(\mathcal{H}_1, \mathcal{H}_2)$, then $S + T \in \mathfrak{S}_{p,\infty}(\mathcal{H}_1, \mathcal{H}_2)$.

(v) If $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $S \in \mathfrak{S}_{p,\infty}(\mathcal{H}_2, \mathcal{H}_3)$ and $T \in \mathfrak{S}_{q,\infty}(\mathcal{H}_1, \mathcal{H}_2)$, then $ST \in \mathfrak{S}_{r,\infty}(\mathcal{H}_1, \mathcal{H}_3)$.

Proof. The first three assertions are trivial. Item *(iv)* and item *(v)* can be found in [33, Theorem 2.2.5. and Theorem 2.2.9], respectively. □

1.5 Quasi Boundary Triples

Quasi boundary triples are a useful tool in studying self-adjoint extensions of a symmetric operator and spectral properties of those extensions. They were introduced by Behrndt and Langer in [12, 13] as a generalization of boundary triples and are mostly applied on differential operators with boundary or interaction conditions. In this thesis we use the theory of quasi boundary triples to treat Dirac operators with singular interactions on the interface.
Throughout this section S denotes a densely defined closed symmetric operator in the Hilbert space $\mathcal{H}$.

Definition 1.11. [13, Definition 6.10] A triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ is said to be a quasi boundary triple for the operator S^* if $\mathcal{G}$ is a Hilbert space and there exists an operator T such that $\overline{T} = S^*$ and $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to \mathcal{G}$ are linear mappings satisfying

(i) $(Tf, g)_\mathcal{H} - (f, Tg)_\mathcal{H} = (\Gamma_1 f, \Gamma_0 g)_\mathcal{G} - (\Gamma_0 f, \Gamma_1 g)_\mathcal{G} \quad \forall f, g \in \operatorname{dom} T$,

(ii) $\operatorname{ran}(\Gamma_0, \Gamma_1)^T$ is dense in $\mathcal{G} \times \mathcal{G}$ and

(iii) $A_0 := T \upharpoonright \ker \Gamma_0$ is self-adjoint in $\mathcal{H}$.

For $\lambda \in \rho(A_0)$ it is easy to see that $\operatorname{dom} T$ can be decomposed into the direct sum

$$\operatorname{dom} T = \operatorname{dom} A_0 \dotplus \ker(T - \lambda) = \ker \Gamma_0 \dotplus \ker(T - \lambda). \tag{1.10}$$

In this case we can introduce two further operators.

Definition 1.12. Let $(\mathcal{G}, \Gamma_0, \Gamma_1)$ be a quasi boundary triple for $S^* = \overline{T}$. The two operator valued functions γ and M defined through

$$\begin{aligned} \rho(A_0) \ni \lambda &\mapsto (\Gamma_0 \upharpoonright \ker (T - \lambda))^{-1} \\ \text{and } \rho(A_0) \ni \lambda &\mapsto \Gamma_1 \gamma(\lambda) \end{aligned} \tag{1.11}$$

are called γ-field and Weyl function, respectively.

The γ-field and Weyl function are well defined due to (1.10). We note that $\gamma(\lambda)$ is a mapping from $\mathcal{G}_0 := \operatorname{ran} \Gamma_0$ to $\ker (T - \lambda)$ and $M(\lambda)$ maps from $\mathcal{G}_0$ to $\mathcal{G}_1 := \operatorname{ran} \Gamma_1$. The two upcoming theorems list properties of these two functions which are beneficial with respect to this thesis. The proofs can be found in [26, Propostion 2.11 and Lemma 2.12].

Theorem 1.13. *Let $(\mathcal{G}, \Gamma_0, \Gamma_1)$ be a quasi boundary triple for $S^* = \overline{T}$. Then, for $\lambda, \mu \in \rho(A_0)$ the following claims hold true:*

(i) *$\gamma(\lambda)$ is a densely defined operator from $\mathcal{G}$ to $\mathcal{H}$. Moreover, the adjoint of $\gamma(\overline{\lambda})$ satisfies the formula*

$$\gamma(\overline{\lambda})^* = \Gamma_1 (A_0 - \lambda)^{-1}. \tag{1.12}$$

(ii) *$M(\lambda)$ is a densely defined operator in $\mathcal{G}$, $M(\lambda) \subset M(\overline{\lambda})^*$ and*

$$(M(\lambda) - M(\overline{\mu}))\varphi = (\lambda - \overline{\mu})\gamma(\mu)^* \gamma(\lambda)\varphi \quad \forall \varphi \in \mathcal{G}_0. \tag{1.13}$$

Next, we focus on the differentiability properties of M and γ. Therefore, we introduce the following notation. We say that a function A mapping from an open set $O \subset \mathbb{C}$ to a separable Banach space V over $\mathbb{C}$ is holomorphic in $\lambda \in O$ if the limit

$$\lim_{\mu \to \lambda} \frac{A(\lambda) - A(\mu)}{\lambda - \mu} \tag{1.14}$$

exists.

Theorem 1.14. *Let $(\mathcal{G}, \Gamma_0, \Gamma_1)$ be a quasi boundary triple for $S^* = \overline{T}$. Then, the mappings $\rho(A_0) \ni \lambda \mapsto \gamma(\lambda)\varphi$ and $\rho(A_0) \ni \lambda \mapsto M(\lambda)\varphi$ are holomorphic for all $\varphi \in \mathcal{G}_0$. Furthermore, for $\lambda \in \rho(A_0)$ and $k \in \mathbb{N}$ the identities*

(i) $\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \gamma(\overline{\lambda})^* = k! \gamma(\overline{\lambda})^* (A_0 - \overline{\lambda})^{-k}$,

(ii) $\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} \gamma(\lambda)\varphi = k!(A_0 - \overline{\lambda})^{-k} \gamma(\lambda)\varphi \quad \forall \varphi \in \mathcal{G}_0$ *and*

(iii) $\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} M(\lambda)\varphi = k! \Gamma_1 (A_0 - \lambda)^{-k} \gamma(\lambda)\varphi \quad \forall \varphi \in \mathcal{G}_0$

are valid.

In order to study self-adjoint extensions of S, we introduce the operator

$$A_B := T \upharpoonright \ker(\Gamma_0 + B\Gamma_1), \tag{1.15}$$

where B denotes a linear operator in $\mathcal{G}$. In the following theorem we state a version of Krein's resolvent formula and investigate the relationship between A_B and B.

Theorem 1.15 ([6, Theorem 2.4.]). *Let $(\mathcal{G}, \Gamma_0, \Gamma_1)$ be a quasi boundary triple for $S^* = \overline{T}$ and B be a linear operator in $\mathcal{G}$. Then, for all $\lambda \in \rho(A_0)$ one has*

$$\ker(A_B - \lambda) = \{\gamma(\lambda)\varphi : \varphi \in \ker(I - BM(\lambda))\} \tag{1.16}$$

and, in particular, $\lambda \in \sigma_p(A_B)$ if and only if $-1 \in \sigma_p(BM(\lambda))$. Furthermore, if $\lambda \in \rho(A_0)$ is not an eigenvalue of A_B, then the following assertions hold:

(i) $g \in \operatorname{ran}(A_B - \lambda)$ if and only if $B\gamma(\overline{\lambda})^* g \in \operatorname{dom}(I + BM(\lambda))^{-1}$.

(ii) For all $g \in \operatorname{ran}(A_B - \lambda)$ we have

$$(A_B - \lambda)^{-1} g = (A_0 - \lambda)^{-1} g - \gamma(\lambda)(I + BM(\lambda))^{-1} B\gamma(\overline{\lambda})^* g. \tag{1.17}$$

If B is a bounded self-adjoint operator and $(I + BM(\lambda_\pm))^{-1} \in \mathcal{L}(\mathcal{G})$ for $\lambda_\pm \in \mathbb{C}^\pm$, then A_B is a self-adjoint operator in $\mathcal{H}$ and (1.17) *holds for all $\lambda \in \rho(A_0) \cap \rho(A_B)$ and all $g \in \mathcal{H}$.*

1.6 Lipschitz Domains

As already mentioned, Lipschitz domains play an essential role in this thesis. Therefore, we introduce them in this section. Let us start with some definitions.

Definition 1.16. Let $\zeta : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz continuous function. Then, Ω is a Lipschitz hypograph if it can be written in the following form

$$\Omega = \left\{ x = \begin{pmatrix} x' \\ x_n \end{pmatrix}, x' \in \mathbb{R}^{n-1} : x_n < \zeta(x') \right\}. \tag{1.18}$$

Definition 1.17. Let H be a hyperplane in $\mathbb{R}^n$, ν the unit normal on H, $\tau > 0$ and $r > 0$. The open cylinder with center $x_0 \in H$ is defined as

$$\mathcal{C}_{r,\tau}(x_0, \nu) := \{x \in \mathbb{R}^n : x = x_1 + t\nu, x_1 \in H, |x_1 - x_0| < r, t \in (-\tau, \tau)\}. \tag{1.19}$$

Definition 1.18. We say Ω has Lipschitz character at $x_0 \in \Sigma := \partial\Omega$ if there exists a hyperplane H, $x_0 \in H$, $r > 0$, $\tau > 0$ and a Lipschitz function $\widetilde{\zeta} : H \longrightarrow \mathbb{R}$, with Lipschitz constant L, such that

$$\mathcal{C}_{r,\tau}(x_0, \nu) \cap \Omega = \{x = x_1 + t\nu \in \mathcal{C}_{r,\tau}(x_0, \nu) : x_1 \in H, t \in (-\tau, \tau), t < \widetilde{\zeta}(x_1)\} \tag{1.20}$$

and $\widetilde{\zeta}(x_0) = 0$. Moreover, we call the rigid motion which maps $\mathcal{C}_{r,\tau}(x_0, \nu)$ to $\mathcal{C}_{r,\tau}(0, e_n)$ κ_{x_0} and define the boundary function $\zeta_{x_0} : \mathbb{R}^{n-1} \to \mathbb{R}$ through

$$\zeta_{x_0}(x') := \widetilde{\zeta}\left(\kappa_{x_0}^{-1}\begin{pmatrix} x' \\ 0 \end{pmatrix}\right) \quad \forall x' \in \mathbb{R}^{n-1}. \tag{1.21}$$

Definition 1.19. Let $\Omega \subsetneq \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ and $\Sigma = \partial\Omega$ be compact. We call Ω a Lipschitz domain if every $x_0 \in \Sigma$ has Lipschitz character.

Remark 1.20. If the boundary functions $\zeta_{x_0} : \mathbb{R}^{n-1} \to \mathbb{R}$ are $\mathcal{C}^k$ and all their derivatives up to order k are bounded, then we call Ω a $\mathcal{C}^k$ domain.

The surface measure σ of Lipschitz domains is the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ restricted to Σ. On $\mathcal{C}_{r,\tau}(x_0, \nu) \cap \Sigma$ we can compute the surface measure of $V \subset \mathcal{C}_{r,\tau}(x_0, \nu) \cap \Sigma$ with the formula

$$\sigma(V) = \int_{Q(\kappa_{x_0}(V))} \sqrt{1 + |\nabla\zeta_{x_0}(x')|^2}\, dx'. \tag{1.22}$$

Here, Q maps $x = \left((x')^T, x_n\right)^T \in \mathbb{R}^n$ to $x' \in \mathbb{R}^{n-1}$. Moreover, for $f \in L^1(\Sigma)$ we can calculate the integral via

$$\int_V f(x)\, d\sigma(x) = \int_{Q(\kappa_{x_0}(V))} f\left(\kappa_{x_0}^{-1}\begin{pmatrix} x' \\ \zeta_{x_0}(x') \end{pmatrix}\right) \sqrt{1 + |\nabla\zeta_{x_0}(x')|^2}\, dx'. \tag{1.23}$$

Another important property of the Lipschitz domains is the σ-a.e. existence of the unit outward normal vector, which we denote ν.

1.7 Non-Tangential Trace

The non-tangential trace is an important concept with respect to traces of functions in Lipschitz domains. First, we introduce the non-tangential neighbourhood as follows.

Definition 1.21. Let $\kappa > 0$ and $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then, for $x \in \Sigma$

$$\Gamma_{\kappa,\Omega}(x) := \{z \in \Omega : |x - z| < (1+\kappa)\operatorname{dist}(z, \Sigma)\} \tag{1.24}$$

denotes the non-tangential neighbourhood of x.

Now, we are able to define the non-tangential supremum as well as the non-tangential trace.

Definition 1.22. Let $\kappa > 0$, $r \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $u : \Omega \to \mathbb{C}^r$ be an arbitrary function. Then, the non-tangential supremum is defined as

$$(\mathcal{N}_{\kappa,\Omega} u)(x) := \sup_{z \in \Gamma_{\kappa,\Omega}(x)} |u(z)| \quad \text{for } x \in \Sigma. \tag{1.25}$$

Definition 1.23. Let $\kappa > 0$, $r \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $u : \Omega \to \mathbb{C}^r$ be an arbitrary function. Then, the non-tangential trace is defined as

$$u|_{\Sigma}^{n.t.}(x) := \lim_{\substack{z \to x \\ z \in \Gamma_{\kappa,\Omega}(x)}} u(z) \quad \text{for } x \in \Sigma \tag{1.26}$$

if the limit exists. Moreover, we say $u|_{\Sigma}^{n.t.}$ exists σ-a.e. if there exists a $\kappa > 0$ such that the limit in (1.26) exists for σ-a.e. $x \in \Sigma$.

Within this realm we have an extended version of the divergence theorem.

Theorem 1.24 ([29, Propositon 2.4.]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $U \in L^1_{loc}(\Omega, \mathbb{C}^n)$ such that* $\operatorname{div} U = \sum_{l=1}^n \partial_l U_l \in L^1(\Omega)$. *If, in addition, $\mathcal{N}_{\kappa,\Omega} U \in L^1(\Sigma)$ and $U|_{\Sigma}^{n.t.}$ exists σ-a.e., then*

$$\int_{\Sigma} \nu(x) \cdot U|_{\Sigma}^{n.t.}(x)\, d\sigma(x) = \int_{\Omega} \operatorname{div} U(x)\, dx. \tag{1.27}$$

2 Sobolev Spaces

The main object of this thesis are partial differential operators. In such a setting Sobolev spaces are indispensable. In the first part of this chapter we deal with classic Sobolev spaces. Afterwards, we discuss certain Sobolev spaces which are important in the context of this thesis.

2.1 Classic Sobolev Spaces

We present different types of Sobolev spaces on open domains. Here, we base our presentation on [28, Chapter 3] and refer to that book for a broader and more in-depth treatment of Sobolev spaces. We focus on definitions and statements which are helpful with respect to this work. Let us start with Sobolev spaces defined via weak derivatives.

Definition 2.1. Let $r \in \mathbb{N}_0$, $p \in [1, \infty]$ and $\Omega \subset \mathbb{R}^n$ be open. Then,

$$W_p^r(\Omega) := \{u \in L^p(\Omega) : \partial^a u \in L^p(\Omega) \ \forall a \in \mathbb{N}_0^n \text{ with } |a|_1 \leq r\} \tag{2.1}$$

and the corresponding norm is defined by

$$\|u\|_{W_p^r(\Omega)} := \left(\sum_{|a|_1 \leq r} \|\partial^a u\|_{L^p(\Omega)}^p \right)^{1/p} \tag{2.2}$$

in case of $p < \infty$ and

$$\|u\|_{W_p^r(\Omega)} := \sup_{|a|_1 \leq r} \|\partial^a u\|_{L^\infty(\Omega)} \tag{2.3}$$

in case of $p = \infty$.

Remark 2.2. The Sobolev spaces defined in Definition 2.1 are Banach spaces and if $p = 2$, $W_p^s(\Omega)$ is a Hilbert space. From now on, we omit the index p if $p = 2$.

There is also another way to introduce Sobolev spaces, namely with help of the Fourier transform which is defined through

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} \, dx \quad \forall \xi \in \mathbb{R}^n \tag{2.4}$$

for $f \in L^1(\mathbb{R}^n)$. Moreover, the Fourier transform can be extended to the dual of the Schwartz space $\mathcal{S}^*(\mathbb{R}^n)$ and defines a unitary mapping in $L^2(\mathbb{R}^n)$. We use the same notation for the Fourier transform defined by (2.4) and its extension. The statements about the Fourier transform can be found in section "Fourier Transforms" of [28, Chapter 3].

Definition 2.3. Let $s \in \mathbb{R}$. We define the Sobolev space of order s

$$H^s(\mathbb{R}^n) := \left\{ u \in \mathcal{S}^*(\mathbb{R}^n) : \left(1 + |\cdot|^2\right)^{s/2} \mathcal{F}u \in L^2(\mathbb{R}^n) \right\} \tag{2.5}$$

and endow it with the scalar product

$$(u, v)_{H^s(\mathbb{R}^n)} := \left(\left(1 + |\cdot|^2\right)^{s/2} \mathcal{F}u, \left(1 + |\cdot|^2\right)^{s/2} \mathcal{F}v \right)_{L^2(\mathbb{R}^n)} \quad \forall u, v \in H^s(\mathbb{R}^n). \tag{2.6}$$

Remark 2.4. It is worth mentioning that $H^{-s}(\mathbb{R}^n)$ is an isometric realization of $(H^s(\mathbb{R}^n))^*$, see [28, Eq. (3.22)].

For open subsets of $\mathbb{R}^n$ we present another class of Sobolev spaces.

Definition 2.5. Let $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be open. We define

$$H^s(\Omega) := \{u \in \mathcal{D}^*(\Omega) : \exists U \in H^s(\mathbb{R}^n) \text{ such that } U|_\Omega = u\} \tag{2.7}$$

and the corresponding norm

$$\|u\|_{H^s(\Omega)} := \inf_{\substack{U \in H^s(\mathbb{R}^n) \\ U|_\Omega = u}} \|U\|_{H^s(\Omega)}. \tag{2.8}$$

In many cases the following theorem shows that different approaches in defining Sobolev spaces yield the same spaces with equivalent norms. It reads as follows.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ either be a Lipschitz domain or $\Omega = \mathbb{R}^n$ and $r \in \mathbb{N}_0$. Then,*

$$W^r(\Omega) = H^r(\Omega) \tag{2.9}$$

and their norms are equivalent.

Proof. The proof for $\Omega = \mathbb{R}^n$ can be found in [28, Theorem 3.16]. If Ω is a Lipschitz domain, combining Theorem 3.18 and Theorem A.4 from [28] proves the statement. □

Theorem 2.6 comes in handy since, depending on the situation, both definitions prove to be useful. It is well known that $\mathcal{D}(\mathbb{R}^n)$ is densely contained in $H^s(\mathbb{R}^n)$, see e.g. section "Sobolev Spaces - First Definition" in [28, Chapter 3]. This property generally does not hold for open sets, but we have the following density statement for Lipschitz domains.

Theorem 2.7. *Let $s \geq 0$ and $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then, the set*

$$\mathcal{D}(\overline{\Omega}) := \{u : \exists U \in \mathcal{D}(\mathbb{R}^n) \text{ such that } U|_\Omega = u\} \tag{2.10}$$

is densely contained in $H^s(\Omega)$.

Proof. See [28, Theorem 3.21 (i)]. □

We introduce the spaces $H_0^s(\Omega)$ and $\mathring{W}^r(\Omega)$. They can be interpreted as Sobolev spaces containing functions which vanish at the boundary.

Definition 2.8. Let $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be open. Then,

$$H_0^s(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}. \tag{2.11}$$

Moreover, for $r \in \mathbb{N}_0$ we define the space

$$\mathring{W}^r(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^s(\Omega)}}. \tag{2.12}$$

Remark 2.9. Certainly, applying Theorem 2.6 gives us $H_0^r(\Omega) = \mathring{W}^r(\Omega)$ for Lipschitz domains and $r \in \mathbb{N}_0$.

The last classic Sobolev space on open domains we work with is defined as follows.

Definition 2.10. Let $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be open. Then,

$$\widetilde{H}^s(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^n)}}. \tag{2.13}$$

Remark 2.11. Although not explicitly mentioning it, all Sobolev spaces we introduced are complete, making them Banach spaces or even Hilbert spaces. This is a well-known fact and can be found in the sections "Sobolev Spaces - First Definition" and "Sobolev Spaces - Second Definition" in Chapter 3 of [28].

In case of Lipschitz domains there exist interesting relations between the different types of Sobolev spaces.

Theorem 2.12. *Let $s \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain. Then, the following assertions hold true:*

(i) The sets $\widetilde{H}^s(\Omega)$ and $\left\{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subset \overline{\Omega}\right\}$ are equal.

(ii) The mapping

$$\begin{aligned} \widetilde{H}^{-s}(\Omega) &\to (H^s(\Omega))^* \\ u &\mapsto F_u \end{aligned} \tag{2.14}$$

with $F_u(v) :={}_{H^{-s}(\mathbb{R}^n)} \langle \overline{u}, V \rangle_{H^s(\mathbb{R}^n)}$, where $V \in H^s(\mathbb{R}^n)$ and $V|_\Omega = v$, is isometric.

(iii) The mapping

$$\begin{aligned} H^{-s}(\Omega) &\to (\widetilde{H}^s(\Omega))^* \\ u &\mapsto G_u \end{aligned} \tag{2.15}$$

with $G_u(v) :={}_{H^{-s}(\mathbb{R}^n)} \langle \overline{U}, v \rangle_{H^s(\mathbb{R}^n)}$, *where* $v \in \widetilde{H}^s(\mathbb{R}^n)$ *and* $U \in H^{-s}(\mathbb{R}^n)$ *such that* $U|_\Omega = u$, *is isometric.*

Proof. Assertion *(i)* can be found in [28, Theorem 3.29 (ii)]. The remaining items follow from combining *(i)* and [28, Theorem 3.14]. □

In the last part of this section we study the embedding of $H_0^k(\Omega)$ in $H^s(\Omega)$ for bounded Lipschitz domains with $k \in \mathbb{N}$ and $s \in [0, 1)$. After reading [28, Theorem 3.27], we know that this embedding is compact. However, it turns out that we can improve this result in terms of Schatten-von Neummann ideals. To do so, we make some preliminary considerations. Here, the fractional Laplacian plays a major role.

Definition 2.13. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then, we define the Laplacian with Dirichlet boundary conditions as

$$\begin{aligned} \operatorname{dom} -\Delta_D &:= \left\{ u \in H_0^1(\Omega) : -\Delta u \in L^2(\Omega) \right\} \\ -\Delta_D u &:= -\Delta u \quad \forall u \in \operatorname{dom} \Delta_D. \end{aligned} \tag{2.16}$$

It is well known that $-\Delta_D$ is a positive definite self-adjoint operator in $L^2(\Omega)$. Moreover, $-\Delta_D$ has a purely discrete spectrum with eigenvalues $0 \le \lambda_D^1 \le \lambda_D^2 \le \dots$. These results can be found e.g. in [35, Section 12.3]. The spectral root of $-\Delta_D$ has the domain $H_0^1(\Omega)$ and is associated with the quadratic form $\boldsymbol{t}[\cdot, \cdot] = (\nabla(\cdot), \nabla(\cdot))_{L^2(\Omega, \mathbb{C}^n)}$, i.e. $\boldsymbol{t}[u, v] = \left((-\Delta_D)^{1/2} u, (-\Delta_D)^{1/2} v\right)_{L^2(\Omega)} = ((-\Delta_D) u, v)_{L^2(\Omega)}$ for all $u \in \operatorname{dom} -\Delta_D$ and $v \in \operatorname{dom} \boldsymbol{t} = \operatorname{dom} (-\Delta_D)^{1/2}$, see [35, Section 10.6.1 and Proposition 10.4].
The two subsequent lemmas deal with the mapping properties of powers of the Laplacian.

Lemma 2.14. *Let* $k \in \mathbb{N}$ *and* $\Omega \subset \mathbb{R}^n$ *be bounded Lipschitz domain. Then, the mapping*

$$\begin{aligned} T^k : H_0^k(\Omega) &\to L^2(\Omega) \\ u &\mapsto (-\Delta_D)^{k/2} u \end{aligned} \tag{2.17}$$

is bounded.

Proof. First, we notice $H_0^k(\Omega) \subset \operatorname{dom} (-\Delta_D)^{k/2}$. Hence, the definition in (2.17) is valid. If k is even, the statement is trivial. Now, we assume k is odd and see

$$(-\Delta_D)^{k/2} u = (-\Delta_D)^{1/2} \left((-\Delta_D)^{\frac{k-1}{2}} u \right) \quad \forall u \in H_0^k(\Omega). \tag{2.18}$$

Since $k-1$ is even, it follows by definition that

$$(-\Delta_D)^{\frac{k-1}{2}} u \in H_0^1(\Omega) \quad \text{and} \quad \left\|(-\Delta_D)^{\frac{k-1}{2}} u\right\|_{H^1(\Omega)} \leq C\|u\|_{H^k(\Omega)} \quad \forall u \in H_0^k(\Omega). \tag{2.19}$$

Applying (2.19) yields

$$\left\|(-\Delta_D)^k u\right\|_{L^2(\Omega)}^2 = \boldsymbol{t}\left[(-\Delta_D)^{\frac{k-1}{2}} u\right] \leq \left\|(-\Delta_D)^{\frac{k-1}{2}} u\right\|_{H^1(\Omega)}^2 \leq C\|u\|_{H^k(\Omega)} \tag{2.20}$$

for all u in $H_0^k(\Omega)$. Therefore, the statement is also true for odd $k \in \mathbb{N}$. □

Lemma 2.15. *Let $s \in [0,1]$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then, the mapping*

$$\begin{aligned} T^{-s} : L^2(\Omega) &\to H^s(\Omega) \\ u &\mapsto (-\Delta_D)^{-s/2} u \end{aligned} \tag{2.21}$$

is well defined and bounded.

Proof. The resolvent set of $-\Delta_D$ contains zero. This yields $0 \in \rho((-\Delta_D)^{s/2})$. Hence, $\operatorname{dom}(-\Delta_D)^{-s/2} = \operatorname{ran}(-\Delta_D)^{s/2} = L^2(\Omega)$. Moreover,

$$\operatorname{ran}(-\Delta_D)^{-s/2} = \operatorname{dom}(-\Delta_D)^{s/2} \subset H^s(\Omega). \tag{2.22}$$

Equation (2.22) is trivial for $s = 0$ and $s = 1$. For $s \in (0,1)$ it can be obtained from [31, eq. (2.13)]. The boundedness remains to be proven. The case $s = 0$ is trivial and in case of $s = 1$ the statement follows from the characterization of $(-\Delta_D)^{1/2}$ with quadratic forms. From now on, we assume $s \in (0,1)$. Then, using [1, Proposition 2.1.] gives us

$$\left\|(-\Delta_D)^{-s/2} u\right\|_{H^s(\Omega)} \leq C\left\|(-\Delta_D)^{s/2}(-\Delta_D)^{-s/2} u\right\|_{L^2(\Omega)} = C\|u\|_{L^2(\Omega)} \quad \forall u \in H^s(\Omega) \tag{2.23}$$

which concludes the proof. □

Theorem 2.16. *Let $k \in \mathbb{N}$, $s \in [0,1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Moreover, let $\iota_{k,s}$ be the embedding operator mapping from $H_0^k(\Omega)$ to $H^s(\Omega)$. Then, $\iota_{k,s} \in \mathfrak{S}_{\frac{n}{k-s},\infty}\left(H_0^k(\Omega), H^s(\Omega)\right)$.*

Proof. We start by rewriting $\iota_{k,s}$ as

$$\iota_{k,s} = T^{-s} A^{k,s} T^k \tag{2.24}$$

with

$$\begin{aligned} A^{k,s} : L^2(\Omega) &\to L^2(\Omega) \\ u &\mapsto (-\Delta_D)^{\frac{s-k}{2}} u \end{aligned} . \tag{2.25}$$

Due to the two previous lemmas, we know that T^k and T^{-s} are bounded operators. Thus, let us investigate $A^{k,s}$. Theorem 12.14 in [35] shows that there exists a $C > 0$ such that

$$\lambda_D^l \geq C l^{2/n} \quad \forall l \in \mathbb{N}. \tag{2.26}$$

Hence, there exists a constant $C > 0$ with

$$s_l((-\Delta_D)^{\frac{s-k}{2}}) \leq C l^{\frac{s-k}{n}} \quad \forall l \in \mathbb{N}. \tag{2.27}$$

This proves the statement. □

Corollary 2.17. *Let $k \in \mathbb{N}$, $s \in [0,1)$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and A be a bounded operator mapping from a Hilbert space $\mathcal{K}$ to the Sobolev space $H^s(\Omega)$. If $\operatorname{ran} A \subset H_0^k(\Omega)$, then $A \in \mathfrak{S}_{\frac{n}{k-s},\infty}(\mathcal{K}, H^s(\Omega))$.*

Proof. We introduce the operator $B : \mathcal{K} \to H_0^k(\Omega)$ with the values $Bu := Au$ and see that B can be factorized as the product

$$A = B\iota_{k,s}. \tag{2.28}$$

With Theorem 2.16 in mind, it suffices to show $B \in \mathcal{L}(\mathcal{K}, H_0^k(\Omega))$. Due to the closed graph theorem, it is sufficient to show that $\mathcal{K} \times \operatorname{ran} B$ is closed in $\mathcal{K} \times H_0^k(\Omega)$. Let $(u_l, Bu_l)_{l\in\mathbb{N}}$ be a convergent sequence in $\mathcal{K} \times H_0^k(\Omega)$ with limit $(u,w) \in \mathcal{K} \times H_0^k(\Omega)$. Then, u_l converges to u in $\mathcal{K}$ and Bu_l to w in $H_0^k(\Omega)$. The boundedness of A implies $Bu_l \to Bu$ in $H^s(\Omega)$. Moreover, $Bu_l \to w$ in $H_0^k(\Omega)$ implies $Bu_l \to w$ in $H^s(\Omega)$. Now, the uniqueness of the limit shows $Bu = w$. Therefore, $(u,w) \in \mathcal{K} \times \operatorname{ran} B$. Hence, $\mathcal{K} \times \operatorname{ran} B$ is closed and $B \in \mathcal{L}(\mathcal{K}, H_0^k(\Omega))$. □

2.2 Sobolev Spaces on the Boundary

We define Sobolev spaces on the boundary of Lipschitz domains. First, we treat Lipschitz hypographs, afterwards we use a partition of unity in order to give a meaningful definition on Lipschitz domains with compact boundary.
In view of Definition 2.18, we introduce

$$u_\zeta(x') := u\left(\begin{pmatrix} x' \\ \zeta(x') \end{pmatrix}\right) \quad \forall x' \in \mathbb{R}^{n-1} \tag{2.29}$$

for a function u defined on the boundary of a Lipschitz hypograph with boundary function ζ.

Definition 2.18. Let Ω be a Lipschitz hypograph with boundary Σ, boundary function ζ and $s \in [0,1]$. We define

$$H^s(\Sigma) := \{u \in L^2(\Sigma) : u_\zeta \in H^s(\mathbb{R}^{n-1})\} \tag{2.30}$$

and endow $H^s(\Sigma)$ with the scalar product

$$(u,v)_{H^s(\Sigma)} := (u_\zeta, v_\zeta)_{H^s(\mathbb{R}^{n-1})} \quad \forall u,v \in H^s(\Sigma).$$

Remark 2.19. If $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isometry and $\kappa(\Omega)$ a Lipschitz hypograph, then $H^s(\Sigma)$ can be defined analogously by defining u_ζ as $u\left(\kappa^{-1}\begin{pmatrix}\cdot\\ \zeta(\cdot)\end{pmatrix}\right)$.

Definition 2.20 (Partition of unity)**.** A partition of unity for an open set $O \subseteq \mathbb{R}^n$ is a finite or infinite sequence of functions $\varphi_1, \varphi_2, \dots \in \mathcal{C}^\infty(\mathbb{R}^n)$ which satisfies

(i) $\varphi_l(x) \geq 0 \quad \forall l$,

(ii) every point in O has a neighbourhood intersecting only finitely many $\operatorname{supp}\varphi_l$, and

(iii) $\sum_{l\geq 1} \varphi_l(x) = 1$ for all $x \in O$.

For non-open sets we denote a sequence of functions a partition of unity if it is a partition of unity for an open neighbourhood of O. If additionally $\mathcal{W}$ is an open covering of O such that for every l exists $W \in \mathcal{W}$ with $\operatorname{supp}\varphi_l \subset W$, we call $\varphi_1, \varphi_2, \dots$ a partition of unity subordinate to $\mathcal{W}$.

Theorem 2.21 ([28, Corollary 3.22 (ii)])**.** *Given any countable open cover* $\{W_1, W_2, \dots\}$ *of a set* $O \subset \mathbb{R}^n$*, there exists a partition of unity* $\varphi_1, \varphi_2, \dots$ *for* S *having the property that* $\operatorname{supp}\varphi_l \subset W_l$ *for each* $l \geq 1$*.*

Now, let Ω be a Lipschitz domain with a compact boundary $\Sigma = \partial\Omega$. Then, due to the definition of Lipschitz domains, cf. Definition 1.18 and Definition 1.19, there exist for every $x_0 \in \Sigma$ sets Ω_{x_0} and $\mathcal{C}_{r,\tau}(x_0,\nu)$ such that $\mathcal{C}_{r,\tau}(x_0,\nu)\cap\Omega = \mathcal{C}_{r,\tau}(x_0,\nu)\cap\Omega_{x_0}$, where Ω_{x_0} can be transformed into a Lipschitz hypograph by the rigid motion $\kappa_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$. This family of cylinders is an open cover of Σ and since Σ is compact, there exists a finite sub-cover of Σ, i.e. we can find finitely many open bounded sets $W_1, W_2, \dots W_p$ and $\Omega_1, \Omega_2, \dots \Omega_p$ with the following properties:

(i) $\bigcup_{l=1}^{p} W_l \supset \Sigma$.

(ii) $W_l \cap \Omega = W_l \cap \Omega_l$ for all $l \in \{1, 2, \dots p\}$.

Applying Theorem 2.21 yields a partition of unity $(\varphi_l)_{l\in\{1,2,\dots p\}}$ for Σ subordinate to $(W_l)_{l\in\{1,2,\dots,p\}}$ with $\operatorname{supp}\varphi_l \subset W_l$ for all $l \in \{1, 2, \dots, p\}$. We write Σ_l for the boundary of Ω_l. Moreover, for $u \in L^2(\Sigma)$ we define

$$u_l(x) := \begin{cases} \varphi_l(x)u(x) & \text{if } x \in \Sigma \cap W_l = \Sigma_l \cap W_l \\ 0 & \text{if } x \in \Sigma_l \setminus (\Sigma \cap W_l) = \Sigma_l \setminus W_l \end{cases} \tag{2.31}$$

for all $l \in \{1, 2, \dots, p\}$. After this preliminary work, we are ready to define Sobolev spaces on the boundary of Lipschitz domains.

Definition 2.22. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with boundary Σ and $s \in [0, 1]$. Then, we define the Hilbert space

$$H^s(\Sigma) := \left\{u \in L^2(\Sigma) : u_l \in H^s(\Sigma_l) \quad \forall l \in \{1, 2, \dots p\}\right\}. \tag{2.32}$$

The corresponding scalar product is defined by

$$(u, v)_{H^s(\Sigma)} := \sum_{l=1}^{p} (u_l, v_l)_{H^s(\Sigma_l)} \text{ for } u, v \in H^s(\Sigma). \tag{2.33}$$

Remark 2.23. With the help of [28, Theorem 3.20] and [28, Theorem 3.23], one can show that different choices of W_l and φ_l yield the same space with equivalent norms.

Remark 2.24. If $\Omega \subset \mathbb{R}^n$ is a $\mathcal{C}^k$ domain for $k \in \mathbb{N}$, one can define Sobolev spaces in the same way for $0 \leq s \leq k$.

At this moment, we are able to state an important result regarding the trace of functions in $H^s(\Omega)$.

Theorem 2.25. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain with boundary Σ and $\frac{1}{2} < s < \frac{3}{2}$. Then, the operator $\boldsymbol{t}_\Sigma : \mathcal{D}(\overline{\Omega}) \to \mathcal{D}(\Sigma)$ defined by*

$$\boldsymbol{t}_\Sigma u := u|_\Sigma \tag{2.34}$$

has a unique extension to a bounded linear operator

$$\boldsymbol{t}_\Sigma : H^s(\Omega) \to H^{s-1/2}(\Sigma). \tag{2.35}$$

Moreover, if $s \leq 1$, then $\boldsymbol{t}_\Sigma$ has a continuous right inverse.

Proof. See Theorem 3.37 and Theorem 3.38 in [28]. □

We use the following lemma to prove Theorem 2.27 which states an interesting compactness result.

Lemma 2.26. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz hypograph or a $\mathcal{C}^k$ hypograph with boundary Σ and $W \subset \mathbb{R}^n$ be a bounded Borel set. If Ω is a Lipschitz hypograph, let $0 \leq s < t \leq 1$, and if Ω is a $\mathcal{C}^k$ hypograph, let $0 \leq s < t \leq k$. Then, the inclusion*

$$\left\{u \in H^t(\Sigma) : u|_{\Sigma \setminus W} = 0 \, \sigma\text{-}a.e.\right\} \subset \left\{u \in H^s(\Sigma) : u|_{\Sigma \setminus W} = 0 \, \sigma\text{-}a.e.\right\} \tag{2.36}$$

is compact.

Proof. We choose a bounded sequence $(u^r)_{r\in\mathbb{N}}$ in $\left\{u \in H^t(\Sigma) : u|_{\Sigma\setminus W} = 0\right\}$. Then, the sequence $(u^r_\zeta)_{r\in\mathbb{N}}$ is also a bounded sequence in $H^t(\mathbb{R}^{n-1})$. Moreover, since W is bounded, there exists a compact set K such that $\operatorname{supp} u^r_\zeta \subset K$ for all $r \in \mathbb{N}$. Now, we can employ [28, Theorem 3.27 (i)] which guarantees the existence of a subsequence $(u^{r_q}_\zeta)_{q\in\mathbb{N}}$ which converges to some w_ζ in $H^s(\mathbb{R}^{n-1})$. Defining $w(x) := w_\zeta(x')$ for all $x = \big((x')^T, x_n\big)^T \in \Sigma$, yields that $(u^{r_q})_{q\in\mathbb{N}}$ converges to w in $H^s(\Sigma)$. Indeed, since $u^r|_{\Sigma\setminus W} = 0$ for all $r \in \mathbb{N}$, also $w|_{\Sigma\setminus W}$ has to be zero. This concludes the proof. □

Theorem 2.27. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz or a $\mathcal{C}^k$ domain with boundary Σ. If Ω is a Lipschitz domain, let $0 \le s < t \le 1$ and if Ω is a $\mathcal{C}^k$ domain, let $0 \le s < t \le k$. Then, the inclusion*

$$H^t(\Sigma) \subset H^s(\Sigma) \tag{2.37}$$

is compact.

Proof. Assume that $(u^r)_{r\in\mathbb{N}}$ is a bounded sequence in $H^t(\Sigma)$. We define u_j^r in the same way as in (2.31). Then, the sequences $(u_l^r)_{r\in\mathbb{N}}$ are bounded sequences in the set $\{u \in H^t(\Sigma_l) : u|_{\Sigma_l\setminus W} = 0\}$ for all $l = 1, \dots, p$ by definition. Employing Lemma 2.26 gives us the existence of a subsequence $(u^{l_q})_{q\in\mathbb{N}}$ such that $(u_l^{r_q})_{q\in\mathbb{N}}$ converges in $H^s(\Sigma_l)$ for all $l = 1, \dots, p$. Thus, the sequences $(u_l^{r_q})_{q\in\mathbb{N}}$ are Cauchy sequences in $H^s(\Sigma_l)$. Therefore, also $(u^{r_q})_{q\in\mathbb{N}}$ is a Cauchy sequence in $H^s(\Sigma)$. Since $H^s(\Sigma)$ is complete, $(u^{r_q})_{q\in\mathbb{N}}$ converges to a limit $u \in H^s(\Sigma)$. □

2.3 Sobolev Spaces for Laplace and Dirac Operators

We introduce two further spaces. These two spaces are Sobolev spaces where certain differential expressions possess a higher regularity than generally guaranteed. We use the Sobolev spaces for Dirac operators as the domain of certain Dirac operators. An important result concerning Sobolev spaces for Dirac operators is that the domain of the trace operator can be extended to the case $s = 1/2$. This is proven in Theorem 2.36. The main ingredient of the proof is the trace theorem regarding Sobolev spaces for Laplace operators. Therefore, we start by introducing Sobolev spaces for Laplace operators.

Definition 2.28. Let $s_1, s_2 \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ be open. We define $H^{s_1,s_2}(\Omega)$ through

$$H_\Delta^{s_1,s_2}(\Omega) := \{u \in H^{s_1}(\Omega) : \Delta u \in H^{s_2}(\Omega)\}. \tag{2.38}$$

Furthermore, we endow $H_\Delta^{s_1,s_2}(\Omega)$ with the norm

$$\|u\|_{H_\Delta^{s_1,s_2}(\Omega)} := \|u\|_{H^{s_1}(\Omega)} + \|\Delta u\|_{H^{s_2}(\Omega)}. \tag{2.39}$$

.

In such spaces it is possible to formulate an extended trace theorem.

Theorem 2.29. *Assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain with boundary Σ and fix an arbitrary $1 \ge \varepsilon > 0$. Then, the restriction of the boundary trace operator defined in Theorem 2.25 to the space $H_\Delta^{s,s-2+\varepsilon}(\Omega)$, originally considered for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, induces a well-defined linear continuous operator*

$$\boldsymbol{t}_\Sigma : H_\Delta^{s,s-2+\varepsilon}(\Omega) \longrightarrow H^{s-1/2}(\Sigma) \quad \forall s \in \left[\frac{1}{2}, \frac{3}{2}\right] \tag{2.40}$$

which continues to be compatible with the trace operator $\boldsymbol{t}_\Sigma$ *defined in Theorem 2.25, when* $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$. *Thus, the defined Dirichlet trace operator enjoys the following properties:*

(i) The Dirichlet boundary operator (2.40) *is compatible with the pointwise non-tangential trace in the sense that: if* $u \in H^{s,s-2+\varepsilon}_\Delta(\Omega)$ *for some* $s \in \left[\frac{1}{2}, \frac{3}{2}\right]$ *and* $1 \geq \varepsilon > 0$, *and if* $u|^{n.t.}_\Sigma$ *exists* σ*-a.e. on* Σ, *then*

$$u|^{n.t.}_\Sigma = \boldsymbol{t}_\Sigma u \in H^{s-1/2}(\Sigma). \tag{2.41}$$

(ii) For each $s \in \left[\frac{1}{2}, \frac{3}{2}\right]$ *and* $1 \geq \varepsilon > 0$ *the Dirichlet boundary trace operator satisfies*

$$\boldsymbol{t}_\Sigma(wu) = (w|_\Sigma)\boldsymbol{t}_\Sigma u \text{ at } \sigma\text{-a.e. point on } \Sigma \tag{2.42}$$

for all $u \in H^{s,s-2+\varepsilon}_\Delta(\Omega)$ *and all* $w \in \mathcal{D}(\overline{\Omega})$.

Proof. If Ω is bounded, the statement immediately follows from [8, Theorem 3.6]. Next, assume Ω is unbounded with compact boundary. We choose R_1 and R_2 such that $R_2 > R_1 > 0$ and $\Sigma \subset B(0, R_1)$. Note that $\Omega \cap B(0, R_2)$ is a bounded Lipschitz domain. Moreover, there exists a function $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ which satisfies $0 \leq g \leq 1$, $g = 1$ on $B(0, R_1)$ and $\operatorname{supp} g \subset B(0, R_2)$. Since $\varepsilon \leq 1$, there holds

$$\Delta(ug) = (\Delta u)g + 2\sum_{l=1}^{n} \frac{\partial u}{\partial x_l}\frac{\partial g}{\partial x_l} + u(\Delta g) \in H^{s-2+\varepsilon}(\Omega) \tag{2.43}$$

for $u \in H^{s,s-2+\varepsilon}_\Delta(\Omega)$. Hence, if $u \in H^{s,s-2+\varepsilon}_\Delta(\Omega)$, then $ug \in H^{s,s-2+\varepsilon}_\Delta(\Omega)$ and therefore also $ug \in H^{s,s-2+\varepsilon}_\Delta(\Omega \cap B(0, R_2))$. Now, all the claims follow by defining

$$\boldsymbol{t}_\Sigma u := \left(\boldsymbol{t}_{\partial(\Omega \cap B(0,R_2))} ug\right)|_\Sigma \quad \forall u \in H^{s,s-2+\varepsilon}_\Delta(\Omega) \tag{2.44}$$

and applying the case of bounded Lipschitz domains. In particular, the definition is independent of the special choice of g due to assertion *(ii)* for the case of bounded domains. □

After this short detour to Sobolev spaces for Laplace operators, let us introduce Dirac operators and Sobolev spaces for Dirac operators. We consider these spaces only for $n = 3$. Moreover, we work with vector-valued functions. The definitions of vector-valued Sobolev spaces should be clear, see for instance section "Vector-Valued Functions" in [28, Chapter 3]. Moreover, all the statements concerning Sobolev spaces we have stated so far stay true for vector-valued functions by applying the scalar-valued analogon elementwise.

Definition 2.30. We define α as follows

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^{4\times 4} \times \mathbb{C}^{4\times 4} \times \mathbb{C}^{4\times 4} \quad \text{with} \quad \alpha_l = \begin{pmatrix} 0 & \sigma_l \\ \sigma_l & 0 \end{pmatrix} \in \mathbb{C}^{4\times 4}. \tag{2.45}$$

The σ_l's denote the Pauli matrices, which are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.46}$$

Moreover, we set

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{2.47}$$

The matrices $\alpha_1, \alpha_2, \alpha_3$ and β are self-adjoint and satisfy the equations

$$\alpha_k \alpha_l + \alpha_l \alpha_k = 2\delta_{kl} I_4 \quad \text{and} \quad \alpha_l \beta + \beta \alpha_l = 0 \quad \text{for } k, l \in \{1, 2, 3\}. \tag{2.48}$$

Remark 2.31. If $v \in \mathbb{C}^3$, then

$$(\alpha \cdot v) := \sum_{l=1}^{3} \alpha_l v_l \quad \text{and} \quad (\alpha \cdot \nabla) := \sum_{l=1}^{3} \alpha_l \partial_l. \tag{2.49}$$

An easy computation shows $(\alpha \cdot v)^2 = I_4 \, |v|^2$ for $v \in \mathbb{R}^3$. In this case $(\alpha \cdot v)$ is self-adjoint.

Definition 2.32. We call the differential operator

$$-\, i(\alpha \cdot \nabla) + m\beta \tag{2.50}$$

a Dirac operator for $m \in \mathbb{R}$. For an open set $\Omega \subset \mathbb{R}^3$ and $s_1, s_2 \in \mathbb{R}$ we define the Sobolev space

$$H^{s_1,s_2}_{\alpha}(\Omega) := \{u \in H^{s_1}(\Omega; \mathbb{C}^4) : (\alpha \cdot \nabla)u \in H^{s_2}(\Omega; \mathbb{C}^4)\}. \tag{2.51}$$

Furthermore, we endow $H^{s_1,s_2}_{\alpha}(\Omega)$ with the norm

$$\|u\|_{H^{s_1,s_2}_{\alpha}(\Omega)} := \|u\|_{H^{s_1}(\Omega;\mathbb{C}^4)} + \|(\alpha \cdot \nabla)u\|_{H^{s_2}(\Omega;\mathbb{C}^4)}. \tag{2.52}$$

.

The two following lemmas state important facts about $H^{s_1,s_2}_{\alpha}(\Omega)$ and have corresponding counterparts concerning $H^{s_1,s_2}_{\Delta}(\Omega)$. Nevertheless, we only prove the $H^{s_1,s_2}_{\alpha}(\Omega)$ cases since these are important with respect to this work.

Lemma 2.33. *Let $s_1, s_2 \in \mathbb{R}$ and $u \in H^{s_1,s_2}_{\alpha}(\mathbb{R}^3)$. Then, there holds*

$$u \in H^{\max\{s_1,s_2+1\}}(\mathbb{R}^3; \mathbb{C}^4) \text{ as well as } \|u\|_{H^{\max\{s_1,s_2+1\}}(\mathbb{R}^3;\mathbb{C}^4)} \leq C\|u\|_{H^{s_1,s_2}_{\alpha}(\mathbb{R}^3)}. \tag{2.53}$$

Proof. If $s_1 = \max\{s_1, s_2 + 1\}$, the statement is trivial. Therefore, let us assume $s_1 < s_2 + 1$ and choose a $u \in H_\alpha^{s_1,s_2}(\mathbb{R}^3)$. We can write $\|(\alpha \cdot \nabla)u\|_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)}$ in the following way

$$\begin{aligned}\|(\alpha \cdot \nabla)u\|^2_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)} &= \int_{\mathbb{R}^3} (1 + |\xi|^2)^{s_2} \left|(\mathcal{F}(\alpha \cdot \nabla)u)(\xi)\right|^2 d\xi \\ &= \int_{\mathbb{R}^3} (1 + |\xi|^2)^{s_2} |\xi|^2 \left|(\mathcal{F}u)(\xi)\right|^2 d\xi.\end{aligned} \tag{2.54}$$

Simple approximations show the existence of a constant $C > 0$ which satisfies

$$(1 + |\xi|^2)^{s_2+1} \leq C \left((1 + |\xi|^2)^{s_1} + (1 + |\xi|^2)^{s_2}|\xi|^2\right) \quad \forall \xi \in \mathbb{R}^3. \tag{2.55}$$

Mulitplying (2.55) with $(\mathcal{F}u)(\xi)$ and integrating over $\mathbb{R}^3$ yields the statement. □

We already know $\mathcal{D}(\overline{\Omega})$ is densely contained in $H^s(\Omega)$. Theorem 2.34 states that this also holds for $H_\alpha^{s_1,s_2}(\Omega)$. The proof is based on the proof of [8, Lemma 2.13].

Theorem 2.34. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain and $s_1, s_2 \in \mathbb{R}$. Then, $\mathcal{D}(\overline{\Omega}; \mathbb{C}^4)$ is a dense subset of $H_\alpha^{s_1,s_2}(\Omega)$.*

Proof. If $s_1 - s_2 \geq 1$, the inclusion $H^{s_2}(\Omega; \mathbb{C}^4) \supset H^{s_1-1}(\Omega; \mathbb{C}^4)$ holds and therefore $H_\alpha^{s_1,s_2}(\Omega) = H^{s_1}(\Omega; \mathbb{C}^4)$. Since $\mathcal{D}(\overline{\Omega}; \mathbb{C}^4)$ is dense in $H^{s_1}(\Omega; \mathbb{C}^4)$, the statement is true. It remains to verify the statement in the case $s_1 - s_2 < 1$. Hence, let $s_1 - s_2 < 1$. We study the embedding

$$\begin{aligned}\iota : H_\alpha^{s_1,s_2}(\Omega) &\longrightarrow H^{s_1}(\Omega; \mathbb{C}^4) \oplus H^{s_2}(\Omega; \mathbb{C}^4) \\ \iota(u) &:= (u, (\alpha \cdot \nabla)u) \quad \forall u \in H_\alpha^{s_1,s_2}(\Omega).\end{aligned}$$

Note that $\operatorname{ran}(\iota) = \iota\left(H_\alpha^{s_1,s_2}(\Omega)\right)$ is a closed subspace of $H^{s_1}(\Omega; \mathbb{C}^4) \oplus H^{s_2}(\Omega; \mathbb{C}^4)$. Moreover, $\iota : H_\alpha^{s_1,s_2}(\Omega) \to \operatorname{ran}(\iota)$ is an isomorphism. We denote the inverse operator $\iota^{-1} : \operatorname{ran}(\iota) \to H_\alpha^{s_1,s_2}(\Omega)$. Let $\Lambda : H_\alpha^{s_1,s_2}(\Omega) \to \mathbb{C}$ be an arbitrary continuous linear functional. Then, $\Lambda \circ \iota^{-1}$ is a continuous linear functional on $\operatorname{ran}(\iota)$. Now, Hahn-Banach guarantees the existence of an extension of $\Lambda \circ \iota^{-1}$

$$\widehat{\Lambda} \in \left(H^{s_1}(\Omega; \mathbb{C}^4) \oplus H^{s_2}(\Omega; \mathbb{C}^4)\right)^* = \left(H^{s_1}(\Omega; \mathbb{C}^4)\right)^* \oplus \left(H^{s_2}(\Omega; \mathbb{C}^4)\right)^*. \tag{2.56}$$

According to Theorem 2.12 $(H^s(\Omega; \mathbb{C}^4))^* = \widetilde{H}^{-s}(\Omega; \mathbb{C}^4)$. Therefore,

$$\widehat{\Lambda} \in \widetilde{H}^{-s_1}(\Omega; \mathbb{C}^4) \oplus \widetilde{H}^{-s_2}(\Omega; \mathbb{C}^4) \tag{2.57}$$

and Λ can be represented by

$$\begin{aligned}\Lambda(u) &= \Lambda \circ \iota^{-1}(\iota u) = \widehat{\Lambda}(\iota u) \\ &= {}_{H^{-s_1}(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_1}, F\rangle_{H^{s_1}(\mathbb{R}^3;\mathbb{C}^4)} + {}_{H^{-s_2}(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_2}, G\rangle_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)},\end{aligned} \tag{2.58}$$

where $h_1 \in \widetilde{H}^{-s_1}(\Omega;\mathbb{C}^4)$ and $h_2 \in \widetilde{H}^{-s_2}(\Omega;\mathbb{C}^4)$. F and G are chosen such that $F|_\Omega = u$ and $G|_\Omega = (\alpha\cdot\nabla)u$. For the next step we choose an arbitrary $\varphi \in \mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)$. By setting $u = \varphi|_\Omega$, $F = \varphi$ and $G = (\alpha\cdot\nabla)\varphi$ we obtain

$$\begin{aligned}
\Lambda(\varphi|_\Omega) = \Lambda(u) &= {}_{H^{-s_1}(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_1}, F\rangle_{H^{s_1}(\mathbb{R}^3;\mathbb{C}^4)} + {}_{H^{-s_2}(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_2}, G\rangle_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)} \\
&= {}_{H^{-s_1}(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_1}, \varphi\rangle_{H^{s_1}(\mathbb{R}^3;\mathbb{C}^4)} + {}_{H^{-s_2}(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_2}, (\alpha\cdot\nabla)\varphi\rangle_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)} \\
&= {}_{\mathcal{D}^*(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_1}, \varphi\rangle_{\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)} + {}_{\mathcal{D}^*(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_2}, (\alpha\cdot\nabla)\varphi\rangle_{\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)} \\
&= {}_{\mathcal{D}^*(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_1}, \varphi\rangle_{\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)} - {}_{\mathcal{D}^*(\mathbb{R}^3;\mathbb{C}^4)}\left\langle \overline{(\alpha\cdot\nabla)h_2}, \varphi\right\rangle_{\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)} \\
&= {}_{\mathcal{D}^*(\mathbb{R}^3;\mathbb{C}^4)}\left\langle \overline{h_1 - (\alpha\cdot\nabla)h_2}, \varphi\right\rangle_{\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)}.
\end{aligned} \tag{2.59}$$

Let us assume

$$\Lambda(v) = 0 \quad \forall v \in \mathcal{D}(\overline{\Omega};\mathbb{C}^4). \tag{2.60}$$

If this implies

$$\Lambda(u) = 0 \quad \forall u \in H^{s_1,s_2}_\alpha(\Omega), \tag{2.61}$$

then $\mathcal{D}(\overline{\Omega};\mathbb{C}^4)$ is a dense subset of $H^{s_1,s_2}_\alpha(\Omega)$. Combining (2.59) and (2.60) yields

$$h_1 - (\alpha\cdot\nabla)h_2 = 0 \tag{2.62}$$

in the sense of distributions. Therefore, $(\alpha\cdot\nabla)h_2 \in H^{-s_1}(\mathbb{R}^3;\mathbb{C}^4)$. Furthermore, Lemma 2.33 shows $h_2 \in H^{-s_1+1}(\mathbb{R}^3;\mathbb{C}^4)$. Moreover, since $h_2 \in \widetilde{H}^{-s_2}(\Omega;\mathbb{C}^4)$, we get $\operatorname{supp} h_2 \subset \overline{\Omega}$ through Theorem 2.12 *(i)* and thus also $h_2 \in \widetilde{H}^{-s_1+1}(\Omega;\mathbb{C}^4)$. By definition $\mathcal{D}(\Omega;\mathbb{C}^4)$ is densely contained in $\widetilde{H}^{-s_1+1}(\Omega;\mathbb{C}^4)$. Hence, we can find a sequence $(\varphi_l)_{l\in\mathbb{N}}$ with $\varphi_l \in \mathcal{D}(\Omega;\mathbb{C}^4)$ such that

$$\widetilde{\varphi}_l \overset{l\to\infty}{\longrightarrow} h_2 \text{ in } H^{-s_1+1}(\mathbb{R}^3;\mathbb{C}^4). \tag{2.63}$$

Here, $\widetilde{\varphi}_l$ denotes the zero extension of φ_l. We observe

$$(\alpha\cdot\nabla)\left(\widetilde{\varphi}_l\right) = \widetilde{(\alpha\cdot\nabla)\varphi_l} \overset{l\to\infty}{\longrightarrow} (\alpha\cdot\nabla)h_2 = h_1 \text{ in } H^{-s_1}(\mathbb{R}^3;\mathbb{C}^4). \tag{2.64}$$

In addition, since $-s_2 < -s_1 + 1$,

$$\widetilde{\varphi}_l \overset{l\to\infty}{\longrightarrow} h_2 \text{ in } H^{-s_2}(\mathbb{R}^3;\mathbb{C}^4). \tag{2.65}$$

With previous considerations in mind, there holds for $u \in H_\alpha^{s_1,s_2}(\Omega)$

$$
\begin{aligned}
\Lambda(u) &= {}_{H^{-s_1}(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_1}, F\rangle_{H^{s_1}(\mathbb{R}^3;\mathbb{C}^4)} + {}_{H^{-s_2}(\mathbb{R}^3;\mathbb{C}^4)}\langle \overline{h_2}, G\rangle_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)} \\
&= \lim_{l\to\infty} \left\{ {}_{H^{-s_1}(\mathbb{R}^3;\mathbb{C}^4)}\left\langle \overline{\widetilde{(\alpha\cdot\nabla)\varphi_l}}, F\right\rangle_{H^{s_1}(\mathbb{R}^3;\mathbb{C}^4)} + {}_{H^{-s_2}(\mathbb{R}^3;\mathbb{C}^4)}\left\langle \overline{\widetilde{\varphi_l}}, G\right\rangle_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)} \right\} \\
&= \lim_{l\to\infty} \left\{ {}_{\mathcal{D}^*(\mathbb{R}^3;\mathbb{C}^4)}\left\langle F, \overline{\widetilde{(\alpha\cdot\nabla)\varphi_l}}\right\rangle_{\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)} + {}_{\mathcal{D}^*(\mathbb{R}^3;\mathbb{C}^4)}\left\langle G, \overline{\widetilde{\varphi_l}}\right\rangle_{\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)} \right\} \\
&= \lim_{l\to\infty} \left\{ {}_{\mathcal{D}^*(\Omega;\mathbb{C}^4)}\left\langle F|_\Omega, \overline{(\alpha\cdot\nabla)\varphi_l}\right\rangle_{\mathcal{D}(\Omega;\mathbb{C}^4)} + {}_{\mathcal{D}^*(\Omega;\mathbb{C}^4)}\langle G|_\Omega, \overline{\varphi_l}\rangle_{\mathcal{D}(\Omega;\mathbb{C}^4)} \right\} \\
&= \lim_{l\to\infty} \left\{ {}_{\mathcal{D}^*(\Omega;\mathbb{C}^4)}\left\langle u, \overline{(\alpha\cdot\nabla)\varphi_l}\right\rangle_{\mathcal{D}(\Omega;\mathbb{C}^4)} + {}_{\mathcal{D}^*(\Omega;\mathbb{C}^4)}\langle (\alpha\cdot\nabla)u, \overline{\varphi_l}\rangle_{\mathcal{D}(\Omega;\mathbb{C}^4)} \right\} \\
&= \lim_{l\to\infty} \left\{ -{}_{\mathcal{D}^*(\Omega;\mathbb{C}^4)}\langle (\alpha\cdot\nabla)u, \overline{\varphi_l}\rangle_{\mathcal{D}(\Omega;\mathbb{C}^4)} + {}_{\mathcal{D}^*(\Omega;\mathbb{C}^4)}\langle (\alpha\cdot\nabla)u, \overline{\varphi_l}\rangle_{\mathcal{D}(\Omega;\mathbb{C}^4)} \right\} = 0.
\end{aligned}
\tag{2.66}
$$

We see $\Lambda(u) = 0$ for all $u \in H_\alpha^{s_1,s_2}(\Omega)$, concluding the proof of the theorem. □

Lemma 2.35. *Let $s_1, s_2 \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Then, $H_\alpha^{s_1,s_2}(\Omega)$ is contained in $H_\Delta^{s_1,s_2-1}(\Omega;\mathbb{C}^4)$ holds and*

$$\|u\|_{H_\Delta^{s_1,s_2-1}(\Omega;\mathbb{C}^4)} \le \|u\|_{H_\alpha^{s_1,s_2}(\Omega)} \quad \forall u \in H_\alpha^{s_1,s_2}(\Omega). \tag{2.67}$$

Proof. Let $u \in H_\alpha^{s_1,s_2}(\Omega)$. Moreover, let $\varepsilon > 0$. The definition of $H_\alpha^{s_1,s_2}(\Omega)$ yields $(\alpha\cdot\nabla)u \in H^{s_2}(\Omega;\mathbb{C}^4)$. Thus, there exists $U \in H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)$ such that $U|_\Omega = (\alpha\cdot\nabla)u$ and $\|(\alpha\cdot\nabla)u\|_{H^{s_2}(\Omega;\mathbb{C}^4)} \ge \|U\|_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)} - \varepsilon$. It is easy to see that $(\alpha\cdot\nabla)U \in H^{s_2-1}(\mathbb{R}^3;\mathbb{C}^4)$ and $((\alpha\cdot\nabla)U)|_\Omega = (\alpha\cdot\nabla)(\alpha\cdot\nabla)u = \Delta u$. Hence, $\Delta u \in H^{s_2-1}(\Omega;\mathbb{C}^4)$. Furthermore, we can estimate the norm

$$
\begin{aligned}
\|\Delta u\|_{H^{s_2-1}(\Omega;\mathbb{C}^4)} &= \inf_{\substack{\Delta u = W|_\Omega,\\ W\in H^{s_2-1}(\mathbb{R}^3;\mathbb{C}^4)}} \|W\|_{H^{s_2-1}(\mathbb{R}^3;\mathbb{C}^4)} \le \|(\alpha\cdot\nabla)U\|_{H^{s_2-1}(\mathbb{R}^3;\mathbb{C}^4)} \\
&\le \|U\|_{H^{s_2}(\mathbb{R}^3;\mathbb{C}^4)} \le \|(\alpha\cdot\nabla)u\|_{H^{s_2}(\Omega;\mathbb{C}^4)} + \varepsilon.
\end{aligned}
\tag{2.68}
$$

Noticing that (2.68) holds for arbitrary small $\varepsilon > 0$ finishes the proof. □

Theorem 2.36. *Assume that $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain with boundary Σ and fix an arbitrary $1 \ge \varepsilon > 0$. Then, the restriction of the boundary trace operator defined in Theorem 2.25 to the space $H_\alpha^{s,s-1+\varepsilon}(\Omega)$, originally considered for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, induces a well-defined linear continuous operator*

$$\boldsymbol{t}_\Sigma : H_\alpha^{s,s-1+\varepsilon}(\Omega) \longrightarrow H^{s-1/2}(\Sigma;\mathbb{C}^4) \quad \forall s \in \left[\frac{1}{2}, \frac{3}{2}\right] \tag{2.69}$$

which continues to be compatible with the trace operator $\boldsymbol{t}_\Sigma$ defined in Theorem 2.25, when $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$. Thus, the defined Dirichlet trace operator enjoys the following properties:

(i) The Dirichlet boundary operator in (2.69) *is compatible with the pointwise non-tangential trace in the sense that: if* $u \in H_\alpha^{s,s-1+\varepsilon}(\Omega)$ *for some* $s \in \left[\frac{1}{2}, \frac{3}{2}\right]$ *and* $\varepsilon > 0$*, and if* $u|_\Sigma^{n.t.}$ *exists* σ*-a.e. on* Σ*, then*

$$u|_\Sigma^{n.t.} = \boldsymbol{t}_\Sigma u \in H^{s-1/2}(\Sigma; \mathbb{C}^4). \tag{2.70}$$

(ii) The Dirichlet boundary trace operator $\boldsymbol{t}_\Sigma$ *in* (2.69) *is the unique extension by continuity and density of the mapping* $\mathcal{D}(\overline{\Omega}; \mathbb{C}^4) \ni f \mapsto f|_\Sigma$.

(iii) For each $s \in \left[\frac{1}{2}, \frac{3}{2}\right]$ *and* $1 \geq \varepsilon > 0$ *the Dirichlet boundary trace operator satisfies*

$$\boldsymbol{t}_\Sigma(wu) = (w|_\Sigma)\boldsymbol{t}_\Sigma u \text{ at } \sigma\text{-a.e. point on } \Sigma \tag{2.71}$$

for all $u \in H_\alpha^{s,s-1+\varepsilon}(\Omega)$ *and all* $w \in \mathcal{D}(\overline{\Omega})$ *or* $w \in \mathcal{D}(\overline{\Omega}; \mathbb{C}^4 \times \mathbb{C}^4)$.

Proof. All claims follow directly from combining Theorem 2.29, Theorem 2.34 and Lemma 2.35. □

Corollary 2.37. *Let* $\Omega \subset \mathbb{R}^3$ *be a Lipschitz domain with boundary* Σ *and* ν *be the unit outward normal vector of* Ω*. Then,*

$$((\alpha \cdot \nu)\boldsymbol{t}_\Sigma u, \boldsymbol{t}_\Sigma v)_{L^2(\Sigma;\mathbb{C}^4)} = ((\alpha \cdot \nabla)u, v)_{L^2(\Omega;\mathbb{C}^4)} + (u, (\alpha \cdot \nabla)v)_{L^2(\Omega;\mathbb{C}^4)} \tag{2.72}$$

for all $u, v \in H_\alpha^{1/2,0}(\Omega)$.

Proof. The divergence theorem and product rule show the statement for $u, v \in \mathcal{D}(\overline{\Omega}; \mathbb{C}^4)$. Since $\mathcal{D}(\overline{\Omega}; \mathbb{C}^4)$ is densely contained in $H_\alpha^{1/2,0}(\Omega)$ by Theorem 2.34, the statement also holds for $u, v \in H_\alpha^{1/2,0}(\Omega)$ by continuity of the scalar product and the boundary trace operator $\boldsymbol{t}_\Sigma$. □

Corollary 2.38. *Let* $\Omega \subset \mathbb{R}^3$ *be a Lipschitz domain with boundary* Σ *and let us assume* $u \in H_\alpha^{1/2,0}(\Omega)$*. If* $\boldsymbol{t}_\Sigma u = 0$*, then* $u \in H_0^1(\Omega; \mathbb{C}^4)$.

Proof. We choose $u \in H_\alpha^{1/2,0}(\Omega)$ which satisfies $\boldsymbol{t}_\Sigma u = 0$ and denote with tilde the zero extensions of the respective functions. Applying Corollary 2.37 gives us

$$\begin{aligned}
&\left(\widetilde{(\alpha \cdot \nabla)u}, v\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} + (\widetilde{u}, (\alpha \cdot \nabla)v)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
&= ((\alpha \cdot \nabla)u, v)_{L^2(\Omega;\mathbb{C}^4)} + (u, (\alpha \cdot \nabla)v)_{L^2(\Omega;\mathbb{C}^4)} = ((\alpha \cdot \nu)\boldsymbol{t}_\Sigma u, \boldsymbol{t}_\Sigma v)_{L^2(\Sigma;\mathbb{C}^4)} \\
&= 0 \quad \forall v \in \mathcal{D}(\mathbb{R}^3; \mathbb{C}^4).
\end{aligned} \tag{2.73}$$

Therefore, $(\alpha \cdot \nabla)\widetilde{u} = \widetilde{(\alpha \cdot \nabla)u} \in L^2(\mathbb{R}^3; \mathbb{C}^4)$ which implies $\widetilde{u} \in H^1(\mathbb{R}^3; \mathbb{C}^4)$, cf. Lemma 2.33. Thus, $u \in H^1(\Omega; \mathbb{C}^4)$ and since $\boldsymbol{t}_\Sigma u = 0$, [28, Theorem 3.40] shows $u \in H_0^1(\Omega; \mathbb{C}^4)$. □

3 Integral Operators

In this chapter we treat integral operators with singular kernels. At first, we consider intgeral operators on $\mathbb{R}^{n-1}$ and subsets of $\mathbb{R}^{n-1}$. Afterwards, we use the obtained results in order to consider integral operators on the boundary of Lipschitz domains. Before we start with those two topics, we state a general helpful theorem, often referred to as the Schur test.

Theorem 3.1. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be complete and σ-finite measure spaces. Moreover, we assume that $t = t_1 t_2$, where t, t_1 and t_2 are $\overline{\mu \otimes \nu}$-measurable functions. If the two conditions*

$$\int_X |t_1(x,y)|^2 \, d\mu(x) \leq \gamma_1 < \infty \ \nu\text{-a.e. and} \int_Y |t_2(x,y)|^2 \, d\nu(y) \leq \gamma_2 < \infty \ \mu\text{-a.e.} \tag{3.1}$$

are fulfilled, then the integral

$$(Tf)(x) := \int_Y t(x,y) f(y) \, d\nu(y) \quad \text{exists for } \mu\text{-a.e. } x \in X \text{ and } f \in L^2(Y,\nu). \tag{3.2}$$

Furthermore, T is a bounded linear operator from $L^2(Y,\nu)$ to $L^2(X,\mu)$ with $\|T\|_{L^2(Y,\nu)\to L^2(X,\mu)} \leq \sqrt{\gamma_1\gamma_2}$.

Proof. Let us choose $f \in L^2(Y,\nu)$. Our goal is to prove the estimate

$$\int_X \left(\int_Y |t(x,y)| \, |f(y)| \, d\nu(y) \right)^2 d\mu(x) \leq \gamma_1\gamma_2 \|f\|^2_{L^2(Y,\nu)}. \tag{3.3}$$

Then, the rest is a consequence of Fubini, c.f. [17, Theorem 2.4]. We start by estimating

$$\begin{aligned} \left(\int_Y |t(x,y)| \, |f(y)| \, d\nu(y) \right)^2 &\leq \left(\int_Y |t_2(x,y)| \, |t_1(x,y)| \, |f(y)| \, d\nu(y) \right)^2 \\ &\leq \int_Y |t_2(x,y)|^2 \, d\nu(y) \int_Y |t_1(x,y)|^2 \, |f(y)|^2 \, d\nu(y) \\ &\leq \gamma_2 \int_Y |t_1(x,y)|^2 \, |f(y)|^2 \, d\nu(y) \quad \text{for } \mu\text{-a.e. } x \in X. \end{aligned} \tag{3.4}$$

We use (3.4) and apply Fubini in order to obtain

$$\begin{aligned}\int_X \left(\int_Y |t(x,y)|\,|f(y)|\; d\nu(y) \right)^2 d\mu(x) &\le \gamma_2 \int_X \int_Y |t_1(x,y)|^2\,|f(y)|^2\; d\nu(y)\,d\mu(x) \\ &= \gamma_2 \int_Y \left(\int_X |t_1(x,y)|^2\; d\mu(x) \right) |f(y)|^2\; d\nu(y) \qquad (3.5) \\ &\le \gamma_1\gamma_2 \int_Y |f(y)|^2\; d\nu(y).\end{aligned}$$

□

3.1 Integral Operators in $\mathbb{R}^{n-1}$

The groundwork for all results concerning integral operators in this thesis is layed in the framework of integral operators in $\mathbb{R}^{n-1}$. We start this essential section by defining integral kernels and integral operators.

Definition 3.2. Let $O \subset \mathbb{R}^{n-1}$ be open and k be a measurable function on $\overline{O} \times \overline{O}$. Then, we call k a kernel on $\overline{O}$ and define

$$(Kf)(x) := \int_{\overline{O}} k(x,y) f(y)\, dy \quad \text{for } x \in \overline{O} \text{ and } f \in L^2(\overline{O}) \tag{3.6}$$

if the integral exists.

The next theorem gives a sufficient condition under which Kf exists a.e. and K defines a bounded operator. Before that, we introduce a notation which helps classify singular kernels.

Definition 3.3. Let $O \subset \mathbb{R}^{n-1}$ be open $0 < a \le n-1$ and k be a kernel on $\overline{O}$. Then, we call k a kernel of order a if

$$k(x,y) = A(x,y)\,|x-y|^{-a} \quad \forall x,y \in \overline{O} \times \overline{O} \tag{3.7}$$

with $A \in L^\infty(\overline{O} \times \overline{O})$.

Theorem 3.4. *We assume k to be a kernel of order a on $\overline{O}$. If O is bounded and $a < n-1$, then the operator defined by* (3.6) *is well defined, linear, bounded and compact.*

Proof. We start with the well-definedness and boundedness. Therefore, we employ Theorem 3.1 with $t_1 = \sqrt{|k|}$ and $t_2 = \frac{k}{\sqrt{|k|}}$. Let $R > 0$ such that $R > \operatorname{diam}(\overline{O})$.

Then, we observe $\overline{O} \subset B(y, R)$ for all $y \in \overline{O}$ and by using spherical coordinates we can estimate

$$\begin{aligned}\int_{\overline{O}} |t_1(x,y)|^2 \, dx = \int_{\overline{O}} |k(x,y)| \, dx &\leq \|A\|_{L^\infty(\overline{O}\times\overline{O})} \int_{\overline{O}} |x-y|^{-a} \, dx \\ &\leq \|A\|_{L^\infty(\overline{O}\times\overline{O})} \int_{B(y,R)} |x-y|^{-a} \, dx \\ &\leq \|A\|_{L^\infty(\overline{O}\times\overline{O})} C \int_0^R r^{n-2-a} \, dr \\ &\leq \|A\|_{L^\infty(\overline{O}\times\overline{O})} C \frac{R^{n-1-a}}{n-1-a} < \infty \quad \forall y \in \overline{O}.\end{aligned} \tag{3.8}$$

Now, it is obvious that the same holds for t_2. Hence, $K : L^2(\overline{O}) \to L^2(\overline{O})$ is well defined and bounded. The compactness follows from [20, (3.11.) Proposition]. □

This is a satisfying result for the cases $a < n-1$. However, later on in this thesis we also have to deal with integral kernels of order $n-1$. Thus, we must investigate these types of integrals.

Definition 3.5. Let k be a kernel of the form as in (3.7). For $\varepsilon > 0$ we introduce the cut-off kernel

$$k_\varepsilon(x,y) := k(x,y)\mathbb{1}_{\{z\in\mathbb{R}^{n-1}:|z|>\varepsilon\}}(x-y). \tag{3.9}$$

The definition shows that k_ε has no singularities and therefore induces a well-defined linear bounded operator K_ε. Moreover, we set

$$(\widehat{K}f)(x) := \sup_{\varepsilon>0} |(K_\varepsilon f)(x)| \qquad \text{for } f \in L^2(\overline{O}) \text{ and } \forall x \in \overline{O}. \tag{3.10}$$

Theorem 3.6. *Let $F : \mathbb{R}^{n-1} \to \mathbb{C}$ as well as $g : \mathbb{R}^{n-1} \to \mathbb{R}$ be Lipschitz continuous functions and k have the special form*

$$k(x,y) = \frac{F(x) - F(y)}{(|x-y|^2 + (g(x)-g(y))^2)^{n/2}}. \tag{3.11}$$

Then,

$$(Kf)(x) := \lim_{\varepsilon \searrow 0} (K_\varepsilon f)(x) \tag{3.12}$$

exists a.e. on $\mathbb{R}^{n-1}$ and there exists a constant $C > 0$ independent of F such that $\|K\|_{L^2(\mathbb{R}^{n-1})\to L^2(\mathbb{R}^{n-1})} \leq C\|\nabla F\|_{L^\infty(\mathbb{R}^{n-1};\mathbb{C}^{n-1})}$.

Proof. Theoreme IX and Theoreme XI in [14] show

(a) $\lim_{\varepsilon\searrow 0}(K_\varepsilon f)(x) = (Kf)(x)$ exists for a.e. $x \in \mathbb{R}^{n-1}$,

(b) K is a bounded operator in $L^2(\mathbb{R}^{n-1})$ and

(c) $\left\|\widehat{K}f\right\|_{L^2(\mathbb{R}^{n-1})} \leq C\|f\|_{L^2(\mathbb{R}^{n-1})}$ for $f \in L^2(\mathbb{R}^{n-1})$.

Thus, it remains to prove the norm estimate. In order to prove the statement, we proceed in the same manner as in [19, Lemma 1.1] via the method of rotations. Let us fix $f \in L^2(\mathbb{R}^{n-1})$ and $\varepsilon > 0$. We see

$$\begin{aligned}(K_\varepsilon f)(x) &= \int_{|x-z|>\varepsilon} k(x,z)f(z)\,dz = \int_{|z|>\varepsilon} k(x,x+z)f(x+z)\,dz \\ &= \frac{1}{2}\int_{|z|>\varepsilon} \left(k(x,x+z)f(x+z) + k(x,x-z)f(x-z)\right)\,dz\end{aligned} \tag{3.13}$$

for $x \in \mathbb{R}^{n-1}$. Changing to spherical coordinates yields

$$\begin{aligned}&(K_\varepsilon f)(x) \\ &= \int_{\partial B(0,1)} \underbrace{\int_\varepsilon^\infty \frac{1}{2}(k(x,x+rw)f(x+rw) + k(x,x-rw)f(x-rw))r^{n-2}\,dr}_{=:(K_\varepsilon^w f)(x)}\,d\theta(w),\end{aligned} \tag{3.14}$$

where θ denotes the surface measure of the $(n-2)$-dimensional sphere. For a fixed w we can choose $v_1, v_2, \ldots, v_{n-2}$ such that $w, v_1, v_2, \ldots, v_{n-2}$ is an orthogonal basis of $\mathbb{R}^{n-1}$. Moreover, we can write $x \in \mathbb{R}^{n-1}$ in the form

$$x = tw + \underbrace{s_1v_1 + \cdots + s_{n-2}v_{n-2}}_{v} \text{ with } t, s_1, s_2, \ldots s_{n-2} \in (-\infty,\infty). \tag{3.15}$$

Next, we observe

$$\begin{aligned}(K_\varepsilon^w f)(x) &= \frac{1}{2}\int_\varepsilon^\infty (k(tw+v,(t+r)w+v)f((t+r)w+v) \\ &\qquad + k(tw+v,(t-r)w+v)f((t-r)w+v))r^{n-2}\,dr \\ &= \frac{1}{2}\int_{t+\varepsilon}^\infty k(tw+v,rw+v)f(rw+v)(r-t)^{n-2}\,dr \\ &\quad + \frac{1}{2}\int_{-\infty}^{t-\varepsilon} k(tw+v,rw+v)f(rw+v)(t-r)^{n-2}\,dr \\ &= \int_{|t-r|>\varepsilon} \frac{F(tw+v)-F(rw+v)}{\left(1+\left(\frac{g(tw+v)-g(rw+v)}{t-r}\right)^2\right)^{n/2}} \frac{f(rw+v)}{(t-r)^2}\,dr.\end{aligned} \tag{3.16}$$

Let $(K_\varepsilon^{w,v}f)(t) := (K_\varepsilon^w f)(tw+v)$. Then, [14, Lemme 4.] and [15, Chapitre IV, Proposition 3.] show

(a) $\lim_{\varepsilon\searrow 0}(K_\varepsilon^{w,v}f)(t) = (K^{w,v}f)(t)$ exists for a.e. $t \in \mathbb{R}$,

(b) $K^{w,v}$ is a bounded operator in $L^2(\mathbb{R})$ with $\|K^{w,v}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le C\|\nabla F\|_{L^\infty(\mathbb{R}^{n-1};\mathbb{C}^{n-1})}$ and

(c) $\left\|\widehat{K}^{w,v}h\right\|_{L^2(\mathbb{R})} \le C\|h\|_{L^2(\mathbb{R})}\left(1+\|\nabla F\|_{L^\infty(\mathbb{R}^{n-1};\mathbb{C}^{n-1})}\right)$ for $h \in L^2(\mathbb{R})$,

where $C > 0$ is independent of F, w and v. With these properties in mind, we apply Fubini, the dominated convergence theorem and a change of coordinates in order to get

$$
\begin{aligned}
\|Kf\|^2_{L^2(\mathbb{R}^{n-1})} &= \lim_{\varepsilon\searrow 0}\|K_\varepsilon f\|^2_{L^2(\mathbb{R}^{n-1})} = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^{n-1}} |K_\varepsilon f(x)|^2\,dx \\
&= \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^{n-1}}\left|\int_{|x-z|>\varepsilon} k(x,z)f(z)\,dz\right|^2 dx \\
&= \lim_{\varepsilon\searrow 0}\frac{1}{2}\int_{\mathbb{R}^{n-1}}\left|\int_{\partial B(0,1)} K^w_\varepsilon(x)\,d\theta(w)\right|^2 dx \\
&\le C\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^{n-1}}\int_{\partial B(0,1)} |K^w_\varepsilon(x)|^2\,d\theta(w)\,dx \\
&= C\lim_{\varepsilon\searrow 0}\int_{\partial B(0,1)}\int_{\mathbb{R}^{n-2}}\int_{-\infty}^{\infty} |K^{w,v}_\varepsilon(t)|^2\,dt\,ds\,d\theta(w) \qquad (3.17)\\
&= C\int_{\partial B(0,1)}\int_{\mathbb{R}^{n-2}}\lim_{\varepsilon\searrow 0}\|K^{w,v}_\varepsilon f\|^2_{L^2(\mathbb{R})}\,ds\,d\theta(w) \\
&= \int_{\partial B(0,1)}\int_{\mathbb{R}^{n-2}}\|K^{w,v} f\|^2_{L^2(\mathbb{R})}\,ds\,d\theta(w) \\
&\le C\|\nabla F\|^2_{L^\infty(\mathbb{R}^{n-1};\mathbb{C}^{n-1})}\int_{\partial B(0,1)}\int_{\mathbb{R}^{n-1}}\left\|f\left((\cdot)w+\sum_{l=1}^{n-2}s_l v_l\right)\right\|^2_{L^2(\mathbb{R})} ds\,d\theta(w) \\
&= C\|\nabla F\|^2_{L^\infty(\mathbb{R}^{n-1};\mathbb{C}^{n-1})}\int_{\partial B(0,1)}\|f\|^2_{L^2(\mathbb{R}^{n-1})}\,d\theta(w) \\
&\le C\|\nabla F\|^2_{L^\infty(\mathbb{R}^{n-1};\mathbb{C}^{n-1})}\|f\|^2_{L^2(\mathbb{R}^{n-1})}.
\end{aligned}
$$

□

We apply the previous theorem in Section 3.3 on integral operators on boundaries of Lipschitz domains. In order to succeed with this approach, we need Theorem 3.7. In view of Theorem 3.7 we call a function $\Upsilon : \mathbb{R}^n \to \mathbb{R}$ positive homogeneous of degree $r \in \mathbb{R}$ if $\Upsilon(tx) = t^r\Upsilon(x)$ for all $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$.

Theorem 3.7. *Let k be the kernel from* (3.11), *$j : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz continuous function and $f \in L^2(\mathbb{R}^{n-1})$. Moreover, we assume $\widetilde{k} : \mathbb{R}^n \to \mathbb{R}$ to be a function which is antisymmetric, differentiable, positive homogeneous of degree $-n+1$ and satisfies*

$$k(x,y) = \widetilde{k}\left(\begin{pmatrix} x-y \\ g(x)-g(y)\end{pmatrix}\right) \qquad \forall x \ne y \in \mathbb{R}^{n-1}. \qquad (3.18)$$

Then, if

$$(K'_\varepsilon f)(x) = \int_{|x-y|^2+(j(x)-j(y))^2>\varepsilon^2} k(x,y)f(y)\,dy \quad \forall x \in \mathbb{R}^{n-1}, \tag{3.19}$$

there holds

$$(Kf)(x) = \lim_{\varepsilon\searrow 0}(K_\varepsilon f)(x) = \lim_{\varepsilon\searrow 0}(K'_\varepsilon f)(x) =: (K'f)(x) \text{ for a.e. } x \in \mathbb{R}^{n-1}, \tag{3.20}$$

with K_ε as defined in Definition 3.5.

Proof. The ideas of the proof can be found in [30, Appendix C]. First, we prove the statement for $f \in \mathcal{D}(\mathbb{R}^{n-1})$. Due to density arguments, we drop the assumption at the end. Thus, we choose $f \in \mathcal{D}(\mathbb{R}^{n-1})$ and $x \in \mathbb{R}^{n-1}$ such that $(Kf)(x)$ exists. Since f is Lipschitz continuous, we see that $k(x,y)(f(y)-f(x))$ has order $n-2$ and therefore

$$\begin{aligned}
&\lim_{\varepsilon\searrow 0}\int_{R>|x-y|>\varepsilon} k(x,y)(f(y)-f(x))\,dy \\
&\qquad = \lim_{\varepsilon\searrow 0}\int_{\substack{|x-y|^2+(j(x)-j(y))^2>\varepsilon^2\\ R>|x-y|}} k(x,y)(f(y)-f(x))\,dy \\
&\qquad = \int_{R>|x-y|} k(x,y)(f(y)-f(x))\,dy
\end{aligned} \tag{3.21}$$

for a fixed $R>0$. Hence, it suffices to show

$$\lim_{\varepsilon\searrow 0}\int_{R>|x-y|>\varepsilon} k(x,y)\,dy = \lim_{\varepsilon\searrow 0}\int_{\substack{|x-y|^2+(j(x)-j(y))^2>\varepsilon^2\\ R>|x-y|}} k(x,y)\,dy. \tag{3.22}$$

The functions j and g are Lipschitz continuous, therefore due to Rademacher's theorem, c.f. [18, Chapter 3, Theorem 2], j and g are a.e. differentiable in $\mathbb{R}^{n-1}$. Since we only prove equality a.e., we can assume that g and j are differentiable at x. Thus,

$$\begin{aligned}
j(y) &= j(x) + \nabla j(x)\cdot(y-x) + o(|y-x|) \text{ and} \\
g(y) &= g(x) + \nabla g(x)\cdot(y-x) + o(|y-x|)
\end{aligned} \tag{3.23}$$

with $\lim_{\varepsilon\searrow 0} o(\varepsilon)/\varepsilon = 0$. If

$$\begin{aligned}
y \in A_\varepsilon = &\left\{y : |x-y|^2 + (j(x)-j(y))^2 > \varepsilon^2\right\} \\
&\qquad \setminus \left\{y : |x-y|^2 + (\nabla j(x)\cdot(x-y))^2 > \varepsilon^2\right\},
\end{aligned} \tag{3.24}$$

then $y \in \overline{B(x,\varepsilon)}$. Moreover, we observe for $y \in A_\varepsilon$

$$\begin{aligned}
|x-y|^2 + (j(x)-j(y))^2 = |x-y|^2 + (\nabla j(x)\cdot(x-y))^2 + o(\varepsilon^2) > \varepsilon^2 \\
\text{as well as } |x-y|^2 + (\nabla j(x)\cdot(x-y))^2 \le \varepsilon^2.
\end{aligned} \tag{3.25}$$

Therefore, $A_\varepsilon \subset \left\{ y : \frac{\varepsilon - o(\varepsilon)}{\sqrt{1+(\nabla j(x)\cdot w(y))^2}} < r(y) \leq \frac{\varepsilon}{\sqrt{1+(\nabla j(x)\cdot w(y))^2}} \right\}$ with the spherical coordinates $r(y) := |x-y|$ and $w(y) := \frac{x-y}{|x-y|}$. We are able to estimate the Lebesgue measure of A_ε by

$$\begin{aligned} |A_\varepsilon| &\leq \int_{\partial B(0,1)} \int_{\frac{\varepsilon - o(\varepsilon)}{\sqrt{1+(\nabla j(x)\cdot w(y))^2}}}^{\frac{\varepsilon}{\sqrt{1+(\nabla j(x)\cdot w(y))^2}}} r^{n-2}\, dr\, d\theta(w) \\ &= \int_{\partial B(0,1)} \frac{\varepsilon^{n-1} - (\varepsilon - o(\varepsilon))^{n-1}}{(n-1)\left(1 + (\nabla j(x)\cdot w(y))^2\right)^{(n-1)/2}}\, d\theta(w) \\ &= \int_{\partial B(0,1)} o(\varepsilon^{n-1})\, d\theta(w) \leq o(\varepsilon^{n-1}). \end{aligned} \tag{3.26}$$

A similar argument yields

$$\begin{aligned} y \in B_\varepsilon := &\left\{ y : |x-y|^2 + (\nabla j(x)\cdot(x-y))^2 > \varepsilon^2 \right\} \\ &\setminus \left\{ y : |x-y|^2 + (j(x) - j(y))^2 > \varepsilon^2 \right\} = o(\varepsilon^{n-1}). \end{aligned} \tag{3.27}$$

Thus, $|A_\varepsilon \triangle B_\varepsilon| = o(\varepsilon^{n-1})$ and $|k(x,y)| \leq C\,|x-y|^{-n+1}$ imply

$$\lim_{\varepsilon \searrow 0} \left(\int_{\substack{|x-y|^2 + (j(x)-j(y))^2 > \varepsilon^2 \\ R > |x-y|}} k(x,y)\, dy - \int_{\substack{|x-y|^2 + (\nabla j(x)\cdot(x-y))^2 > \varepsilon^2 \\ R > |x-y|}} k(x,y)\, dy \right) = 0. \tag{3.28}$$

Let

$$C_\varepsilon := \left\{ y : |x-y|^2 + (\nabla j(x)\cdot(x-y))^2 > \varepsilon^2 \right\}. \tag{3.29}$$

Then, $\mathbb{R}^{n-1} \setminus \overline{B(x,\varepsilon)} \subset C_\varepsilon$ and

$$C_\varepsilon \setminus \left(\mathbb{R}^{n-1} \setminus \overline{B(x,\varepsilon)} \right) = C_\varepsilon \cap \overline{B(x,\varepsilon)} \subset \left\{ y : \frac{\varepsilon}{1+L_j} \leq |x-y| \leq \varepsilon \right\}, \tag{3.30}$$

where $L_j \geq 0$ denotes the Lipschitz constant of j. We apply the antisymmetry and the positive homogeneity of $\widetilde{k}$ in order to obtain

$$\begin{aligned}
&\int_{C_\varepsilon\cap\overline{B(x,\varepsilon)}} k(x,y)\,dy = \int_{C_\varepsilon\cap\overline{B(x,\varepsilon)}} \tilde{k}\left(\begin{pmatrix} x-y \\ g(x)-g(y)\end{pmatrix}\right)dy \\
&\leq \int_{\frac{\varepsilon}{1+L_j}\leq|x-y|\leq\varepsilon} \tilde{k}\left(\begin{pmatrix} x-y \\ g(x)-g(y)\end{pmatrix}\right)dy \\
&= \frac{1}{2}\left(\int_{\frac{\varepsilon}{1+L_j}\leq|z|\leq\varepsilon} \tilde{k}\left(\begin{pmatrix} z \\ g(x)-g(x+z)\end{pmatrix}\right)dz\right. \\
&\qquad\qquad \left. + \int_{\frac{\varepsilon}{1+L_j}\leq|z|\leq\varepsilon} \tilde{k}\left(\begin{pmatrix} -z \\ g(x)-g(x-z)\end{pmatrix}\right)dz\right) \qquad (3.31) \\
&= \frac{1}{2}\int_{\frac{\varepsilon}{1+L_j}\leq|z|\leq\varepsilon} \tilde{k}\left(\begin{pmatrix} z \\ g(x)-g(x+z)\end{pmatrix}\right) - \tilde{k}\left(\begin{pmatrix} z \\ -g(x)+g(x-z)\end{pmatrix}\right)dz \\
&\leq C\varepsilon^{-n+1}\int_{\frac{\varepsilon}{1+L_j}\leq|z|\leq\varepsilon} \left|\tilde{k}\left(\begin{pmatrix} \frac{z}{|z|} \\ \frac{g(x)-g(x+z)}{|z|}\end{pmatrix}\right) - \tilde{k}\left(\begin{pmatrix} \frac{z}{|z|} \\ \frac{-g(x)+g(x-z)}{|z|}\end{pmatrix}\right)\right|dz \\
&\leq C\varepsilon^{-n+1}\sup_{|p|=1,s\in\mathbb{R}}\left|\nabla\widetilde{k}\left(\begin{pmatrix} p \\ s\end{pmatrix}\right)\right|\int_{\frac{\varepsilon}{1+L_j}\leq|z|\leq\varepsilon}\frac{|g(x)-g(x+z)+g(x)-g(x-z)|}{|z|}dz \\
&= C\varepsilon^{-n+1}\int_{\frac{\varepsilon}{1+L_j}\leq|z|\leq\varepsilon}\frac{|\nabla g(x)\cdot z+\nabla g(x)\cdot(-z)+o(|z|)|}{|z|}dz \leq \frac{o(\varepsilon)}{\varepsilon}\overset{\varepsilon\to 0}{\to} 0.
\end{aligned}$$

Combining the results (3.28) and (3.31) yields

$$\begin{aligned}
\lim_{\varepsilon\searrow 0}\int_{\substack{|x-y|^2+(j(x)-j(y))^2>\varepsilon^2 \\ R>|x-y|}} k(x,y)\,dy &= \lim_{\varepsilon\searrow 0}\int_{\substack{|x-y|^2+(\nabla j(x)\cdot(x-y))^2>\varepsilon^2 \\ R>|x-y|}} k(x,y)\,dy \\
&= \lim_{\varepsilon\searrow 0}\int_{R>|x-y|>\varepsilon} k(x,y)\,dy.
\end{aligned} \qquad (3.32)$$

Hence, the statement holds for $f\in\mathcal{D}(\mathbb{R}^{n-1})$.
Now, let us drop the assumption $f\in\mathcal{D}(\mathbb{R}^{n-1})$. It is easy to see

$$\begin{aligned}
\left\{y:|x-y|^2+(j(x)-j(y))^2>\varepsilon^2\right\} &= \left\{y:|x-y|^2>\frac{\varepsilon^2}{1+L_j^2}\right\} \\
&\setminus\left\{y:|x-y|^2>\frac{\varepsilon^2}{1+L_j^2},\ |x-y|^2+(j(x)-j(y))^2\leq\varepsilon^2\right\} \quad \forall x\in\mathbb{R}^{n-1}.
\end{aligned} \qquad (3.33)$$

Therefore,

$$|(K'_\varepsilon f)(x)| \leq \left|(K_{\frac{\varepsilon}{\sqrt{1+L_j^2}}}f)(x)\right| + L_F\varepsilon^{-n+1}\int_{|x-y|\leq\varepsilon}|f(y)|\,dy \qquad (3.34)$$

for all $x \in \mathbb{R}^{n-1}$ with L_F denoting the Lipschitz constant of F, implying

$$(\widehat{K}'f)(x) \leq (\widehat{K}f)(x) + L_F\,|B(0,1)|\,(Mf)(x) \quad \forall x \in \mathbb{R}^{n-1} \tag{3.35}$$

with the maximal operator defined by $(Mf)(x) := \sup_{\varepsilon>0} \frac{1}{|B(x,\varepsilon)|} \int_{|x-y|\leq\varepsilon} |f(y)|\,dy$ for $x \in \mathbb{R}^{n-1}$. We know from [37, Chapter 1, Theorem 1.] and from the proof of Theorem 3.6 that M and $\widehat{K}$ define bounded operators in $L^2(\mathbb{R}^{n-1})$, respectively. Due to the density of $\mathcal{D}(\mathbb{R}^{n-1})$ in $L^2(\mathbb{R}^{n-1})$, the boundedness of M, K and $\widehat{K}$, (a) from the beginning of the proof of Theorem 3.6 and [17, Chapter 6, 2.7 Korollar], there exists a sequence $(f_l)_{l\in\mathbb{N}}$ of smooth functions and a measurable set E_f which fulfil the conditions

(a) $|\mathbb{R}^{n-1} \setminus E_f| = 0$,

(b) $(M(f-f_l))(x) \xrightarrow{l\to\infty} 0 \quad \forall x \in E_f$,

(c) $(\widehat{K}(f-f_l))(x) \xrightarrow{l\to\infty} 0 \quad \forall x \in E_f$,

(d) $(K(f-f_l))(x) \xrightarrow{l\to\infty} 0 \quad \forall x \in E_f$ and

(e) $(K_\varepsilon f_l)(x) \xrightarrow{\varepsilon\to 0} (Kf_l)(x) \quad \forall x \in E_f,\ l \in \mathbb{N}$.

Let us fix $x \in E_f$ and $\eta > 0$. We use the stated properties (a)-(e) to conclude the existence of a $j_\eta \in \mathbb{N}$ such that

$$\begin{aligned} L_F\,|B(0,1)|\,(M(f-f_{l_\eta}))(x) \leq \frac{\eta}{4}, \quad (\widehat{K}(f-f_{l_\eta}))(x) \leq \frac{\eta}{4} \text{ and} \\ (K(f-f_{l_\eta}))(x) \leq \frac{\eta}{4}. \end{aligned} \tag{3.36}$$

Furthermore, we choose $\varepsilon_\eta > 0$ such that

$$\left|(K'_\varepsilon f_{l_\eta})(x) - (Kf_{l_\eta})(x)\right| \leq \frac{\eta}{4} \quad \forall \varepsilon \leq \varepsilon_\eta. \tag{3.37}$$

Employing the inequalities (3.35), (3.36) and (3.37) gives us

$$\begin{aligned} &|(K'_\varepsilon f)(x) - (Kf)(x)| \\ &\quad \leq \left|(K'_\varepsilon(f-f_{l_\eta})(x)\right| + \left|(K'_\varepsilon f_{l_\eta})(x) - (Kf_{l_\eta})(x)\right| + \left|(K(f-f_{l_\eta}))(x)\right| \\ &\quad \leq L_F\,|B(0,1)|\,\left|M(f-f_{l_\eta})(x)\right| + \left|\widehat{K}(f-f_{l_\eta})(x)\right| \\ &\qquad\qquad + \left|(K'_\varepsilon f_{l_\eta})(x) - (Kf_{l_\eta})(x)\right| + \left|(K(f-f_{l_\eta}))(x)\right| \\ &\quad \leq \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} = \eta. \end{aligned} \tag{3.38}$$

This shows (3.20) holds true for all $f \in L^2(\mathbb{R}^{n-1})$. □

Corollary 3.8. *Let α be defined as in Definition 2.30, $j : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz continuous function, $B \in \mathbb{R}^{3\times 3}$ an orthogonal matrix and the integral kernel $k_{\alpha,B}$ be*

$$k_{\alpha,B}(x,y) := \frac{\left(\alpha \cdot B \begin{pmatrix} x-y \\ j(x)-j(y) \end{pmatrix}\right)}{\left(|x-y|^2 + (j(x)-j(y))^2\right)^{3/2}} \quad \forall x \neq y \in \mathbb{R}^2. \tag{3.39}$$

Then, the induced integral operator $K_{\alpha,B}$, which is defined analogously to the scalar-valued case, see Definition 3.5 and (3.12)*, is a bounded linear operator mapping from $L^2(\mathbb{R}^2;\mathbb{C}^4)$ to $L^2(\mathbb{R}^2;\mathbb{C}^4)$ with $\|K_{\alpha,B}\|_{L^2(\mathbb{R}^2;\mathbb{C}^4)\to L^2(\mathbb{R}^2;\mathbb{C}^4)} \leq C\|\nabla j\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}$ for a $C>0$ independent of j. Moreover, there holds $K'_{\alpha,B} = K_{\alpha,B}$, i.e.*

$$\lim_{\varepsilon\searrow 0} \int_{|x-y|^2+(j(x)-j(y))^2>\varepsilon^2} k_{\alpha,B}(x,y) f(y)\,dy = \lim_{\varepsilon\searrow 0}\int_{|x-y|>\varepsilon} k_{\alpha,B}(x,y)f(y)\,dy \tag{3.40}$$

for a.e. $x \in \mathbb{R}^2$ and for all $f \in L^2(\mathbb{R}^2;\mathbb{C}^4)$.

Proof. The proof follows by applying Theorem 3.6 and Theorem 3.7 elementwise. □

The next theorem states a crucial compactness result regarding singular integrals connected to Dirac operators and helps us prove the self-adjointness of certain Dirac operators. The proof is inspired by [19, Theorem 1.2. (c)] and [2, Remark 3.6.].

Corollary 3.9. *We make the same assumptions as in the previous corollary. Additionally, let $j \in \mathcal{C}_0^1(\mathbb{R}^2)$ and $n(z) := \begin{pmatrix} -\nabla j(z) \\ 1 \end{pmatrix}$ for $z \in \mathbb{R}^2$. Then, for all $R>0$ the operator induced by the kernel*

$$a(x,y) = (\alpha \cdot Bn(x))k_{\alpha,B}(x,y) + k_{\alpha,B}(x,y)(\alpha\cdot Bn(y)) \quad \forall x\neq y \in \mathbb{R}^2 \tag{3.41}$$

is a bounded and compact linear operator in $L^2(B(0,R);\mathbb{C}^4)$.

Proof. Let $R>0$. The boundedness is trivial after applying Corollary 3.8. Simple algebraic calculations lead us to the identity

$$(\alpha\cdot v)(\alpha\cdot w) = -(\alpha\cdot v)(\alpha \cdot w) + 2(v\cdot w)I_4 \quad \forall v,w\in\mathbb{R}^3. \tag{3.42}$$

With (3.42) and the orthogonality of B in mind, we can write the kernel of A in the following way

$$\begin{aligned} a(x,y) = &-\frac{\left(\alpha\cdot B\begin{pmatrix} x-y \\ j(x)-j(y)\end{pmatrix}\right)}{|x-y|^2+(j(x)-j(y))^2)^{3/2}}(\alpha\cdot B(n(x)-n(y))) \\ &+ 2\frac{j(x)-j(y)-\nabla j(x)\cdot(x-y)}{(|x-y|^2+(j(x)-j(y)^2)^{3/2}} I_4 \quad \forall x\neq y\in\mathbb{R}^2. \end{aligned} \tag{3.43}$$

Now, let $(j_l)_{l\in\mathbb{N}}$ be a sequence of functions with $j_l \in \mathcal{D}(\mathbb{R}^2)$ and

$$j_l(x) \overset{l\to\infty}{\longrightarrow} j(x) \quad \text{and} \quad \nabla j_l(x) \overset{l\to\infty}{\longrightarrow} \nabla j(x) \text{ uniformly for all } x \in \mathbb{R}^2. \tag{3.44}$$

We define

$$\begin{aligned} k^l_{\alpha,B}(x,y) &:= \frac{\left(\alpha \cdot B \begin{pmatrix} x-y \\ j_l(x)-j_l(y) \end{pmatrix}\right)}{|x-y|^2 + (j(x)-j(y))^2)^{3/2}} \quad \forall x \neq y \in \mathbb{R}^2 \\ \text{and } n_l(x) &= B \begin{pmatrix} -\nabla j_l(x) \\ 1 \end{pmatrix} \quad \forall x \in \mathbb{R}^2 \end{aligned} \tag{3.45}$$

as well as

$$\begin{aligned} a_l(x,y) := &- \frac{\left(\alpha \cdot B \begin{pmatrix} x-y \\ j_l(x)-j_l(y) \end{pmatrix}\right)}{|x-y|^2 + (j(x)-j(y))^2)^{3/2}} (\alpha \cdot B(n_l(x)-n_l(y))) \\ &+ 2 \frac{j_l(x)-j_l(y)-\nabla j_l(x)\cdot(x-y)}{(|y-y|^2 + (j(x)-j(y)^2)^{3/2}} I_4 \quad \forall x \neq y \in \mathbb{R}^2. \end{aligned} \tag{3.46}$$

Due to $j_l \in \mathcal{D}(\mathbb{R}^2)$, the a_l's have order one. Using Theorem 3.4 elementwise shows that the operator A_l defined by a_l is compact in $L^2(B(0,R);\mathbb{C}^4)$. Furthermore, Theorem 3.6 implies $\left\|K^l_{\alpha,B} - K_{\alpha,B}\right\|_{L^2(\mathbb{R}^2;\mathbb{C}^4)\to L^2(\mathbb{R}^2;\mathbb{C}^4)} \leq C\|\nabla(j_l - j)\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)}$. Since $\nabla j_l \to \nabla j$ uniformly, also $n_l \to n$ uniformly. These considerations lead to

$$\begin{aligned} &\|A - A_l\|_{L^2(\mathbb{R}^2;\mathbb{C}^4)\to L^2(\mathbb{R}^2;\mathbb{C}^4)} \\ &= \left\|(\alpha\cdot n)K_{\alpha,B} + K_{\alpha,B}(\alpha\cdot n) - (\alpha\cdot n_l)K^l_{\alpha,B} - K^l_{\alpha,B}(\alpha\cdot n_l)\right\|_{L^2(\mathbb{R}^2;\mathbb{C}^4)\to L^2(\mathbb{R}^2;\mathbb{C}^4)} \\ &\leq C\Big((\|n_l\|_{L^\infty(\mathbb{R}^2,\mathbb{R}^3)}\left\|K_{\alpha,B} - K^l_{\alpha,B}\right\|_{L^2(\mathbb{R}^2;\mathbb{C}^4)\to L^2(\mathbb{R}^2;\mathbb{C}^4)} \\ &\qquad\qquad + \|n - n_l\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^3)}\|K_{\alpha,B}\|_{L^2(\mathbb{R}^2;\mathbb{C}^4)\to L^2(\mathbb{R}^2;\mathbb{C}^4)}\Big) \\ &\leq C\|\nabla(j - j_l)\|_{L^\infty(\mathbb{R}^2,\mathbb{R}^2)} \overset{l\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.47}$$

Hence, also $\|A - A_l\|_{L^2(\overline{O};\mathbb{C}^4)\to L^2(\overline{O};\mathbb{C}^4)} \overset{l\to\infty}{\longrightarrow} 0$ which proves the compactness of A in $L^2(B(0,R);\mathbb{C}^4)$. □

3.2 Integral Operators on Lipschitz Boundaries

Before we can investigate integral operators corresponding to Drirac operators, we need to transfer the results of the previous section to Lipschitz boundaries. Theorem 3.10 helps us when making this transfer. In advance, we have to recall and fix some notations.

From now on $\Omega = \Omega_+ \subset \mathbb{R}^n$ always denotes a Lipschitz domain with boundary Σ. Moreover, we set $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega_+}$. We recall some notations from Section 2.2. The open bounded sets $W_1, W_2, \dots W_p$ are chosen such that $\Sigma \subset \bigcup_{l=1}^{p} W_l$. Furthermore, for $l = 1, 2, \dots, p$ there holds $\Omega \cap W_l = \Omega_l \cap W_l$, where Ω_l is a set which can be transformed by a rigid motion κ_l into a Lipschitz hypograph with boundary function ζ_l, i.e. $\kappa_l(\Omega_l)$ is a Lipschitz hypograph and κ_l consists of a translation and a rotation. From the equation above and by setting $\Sigma_l = \partial\Omega_l$ we also have $\Sigma_l \cap W_l = \Sigma_l \cap \Sigma$. The functions $\varphi_1, \varphi_2, \dots \varphi_p$ are a partition of unity for Σ subordinate to $\{W_1, W_2, \dots, W_p\}$. For $f \in L^2(\Sigma)$ we define $f_{l,\zeta_l} = (f_l)_{\zeta_l} \in L^2(\mathbb{R}^{n-1})$ analogously to (2.29) and (2.31).

Theorem 3.10. *Let $\Sigma \subset \mathbb{R}^n$ be a boundary of a Lipschitz domain. Furthermore, let $\zeta_1, \zeta_2, \dots, \zeta_p$ be the boundary functions and $\varphi_1, \varphi_2, \dots, \varphi_p$ be the partition of unity for Σ, as discussed above. Moreover, we assume κ_l to be the map which transforms Ω_l to the Lipschitz hypograph $\kappa_l(\Omega_l)$, $r \in \mathbb{N}$ and $R > 0$ to be chosen in such a way that $\{x' \in \mathbb{R}^{n-1} : ((x')^T, \zeta_l(x'))^T \in \kappa_l(W_l)\} \subset B(0, R)$ for $l = 1, 2, \dots, p$. Then, for a measurable function $k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^{r\times r}$ which is continuous for all $x \neq y \in \mathbb{R}^n$ the following assertions hold:*

(i) If for $l = 1, 2, \dots p$

$$(K_l f)(x') := \lim_{\varepsilon\searrow 0} \int_{\substack{|x'-y'|^2+(\zeta_l(x')-\zeta_l(y'))^2>\varepsilon^2 \\ |y'|<R}} k\left(\kappa_l^{-1}\begin{pmatrix} x' \\ \zeta_l(x')\end{pmatrix}, \kappa_l^{-1}\begin{pmatrix} y' \\ \zeta_l(y')\end{pmatrix}\right) f(y')\, dy' \tag{3.48}$$

exists for a.e. x' in $B(0, R)$ and for all $f \in L^2(B(0, R); \mathbb{C}^r)$, then

$$(\mathcal{K}f)(x) := \lim_{\varepsilon\searrow 0} \int_{|x-y|>\varepsilon} k(x, y) f(y)\, d\sigma(y) \text{ exists for } \sigma\text{-a.e. } x \in \Sigma \tag{3.49}$$

for all $f \in L^2(\Sigma; \mathbb{C}^r)$.

(ii) If assertion (i) is satisfied and K_l is bounded in $L^2(B(0, R), \mathbb{C}^r)$ for $l = 1, 2, \dots p$, then $\mathcal{K}$ is bounded in $L^2(\Sigma; \mathbb{C}^r)$.

(iii) If assertion (i) is satisfied and K_l is compact in $L^2(B(0, R), \mathbb{C}^r)$ for $l = 1, 2, \dots p$, then $\mathcal{K}$ is compact in $L^2(\Sigma; \mathbb{C}^r)$.

Proof. We start with assertion *(i)* by fixing $f \in L^2(\Sigma; \mathbb{C}^r)$. Via partition of unity we can write

$$\lim_{\varepsilon\searrow 0} \int_{|x-y|>\varepsilon} k(x, y) f(y)\, d\sigma(y) = \sum_{l=1}^{p} \lim_{\varepsilon\searrow 0} \int_{\substack{|x-y|>\varepsilon \\ y\in\Sigma_l\cap W_l}} k(x, y) f_l(y)\, d\sigma(y) \text{ for } x \in \Sigma \tag{3.50}$$

if the integrals on the right-hand side exist. Let us fix an arbitrary $l \in \{1, 2, \dots k\}$. For $x \in \Sigma \setminus \Sigma_l$ we can bound $k(x, y) f_l(y)$ with $b_l\, |f_l(y)|$, where

$$b_l := \sup_{(x,y)\in \operatorname{supp}\varphi_l \times (\Sigma\cap W_l^C)} |k(x, y)| < \infty. \tag{3.51}$$

Thus, for all $x \in \Sigma \setminus \Sigma_l$ the integral as well as the limit

$$\lim_{\varepsilon\searrow 0} \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma_l\cap W_l}} k(x, y) f_l(y)\, d\sigma(y) = \int_{y\in\Sigma_l\cap W_l} k(x, y) f_l(y)\, d\sigma(y) \tag{3.52}$$

exist. Now, let us assume $x \in \Sigma_l$. In this situation we can represent x through $x = x(x') = \kappa_l^{-1}\left(\begin{pmatrix} x' \\ \zeta_l(x') \end{pmatrix}\right)$ with $x' \in \mathbb{R}^{n-1}$. We use the integral formula (1.23) in order to obtain for all $\varepsilon > 0$

$$\begin{aligned} \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma_l\cap W_l}} k(x, y) f_l(y)\, d\sigma(y) &= \int_{\substack{|\kappa_l(x)-y|>\varepsilon\\ y\in\kappa_l(\Sigma_l\cap W_l)}} k(x, \kappa_l^{-1}(y)) f_l(\kappa_l^{-1}(y))\, d\sigma(y) \\ &= \int_{\substack{|x'-y'|^2+(\zeta_l(x')-\zeta_l(y'))^2>\varepsilon^2\\ |y'|<R}} k\left(x(x'), y(y')\right) f_{l,\zeta_l}(y') \sqrt{|\nabla\zeta_l(y')|^2 + 1}\, dy'. \end{aligned} \tag{3.53}$$

We notice $f_{l,\zeta_l}\sqrt{|\nabla\zeta_l(\cdot)|^2 + 1} \in L^2(B(0, R); \mathbb{C}^4)$. Taking the limit and applying assumption *(i)* shows us that (3.53) exists for a.e. $x' \in B(0, R)$. Hence, (3.52) also exists for σ-a.e $x = x(x') \in \Sigma_l \cap W_l$. Combining these considerations for all $l \in \{1, 2, \dots, p\}$ yields item *(i)*.

Next, we prove assertion *(ii)* and *(iii)*. We can do this simultaneously. Again, we choose an arbitrary $l \in \{1, 2, \dots, p\}$ and define $\mathcal{K}_l$ and $\mathcal{K}_{l,j}$ through

$$(\mathcal{K}_l f)(x) := \lim_{\varepsilon\searrow 0} \int_{\substack{|x-y|>\varepsilon\\ y\in\Sigma_l\cap W_l}} k(x, y) f_l(y)\, d\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \Sigma \tag{3.54}$$

$$\mathcal{K}_{l,j} f := \begin{cases} (\mathcal{K}_l f)|_{\Sigma_l\cap W_l} & \text{for } l = j \\ (\mathcal{K}_l f)|_{(\Sigma_j\setminus\Sigma_l)\cap\Sigma} & \text{for } l \neq j \end{cases}. \tag{3.55}$$

If we can prove $\mathcal{K}_{l,j}$ is bounded or compact, then $\mathcal{K}$ is bounded or compact, respectively. Let us start with $j = l$. We introduce two further operators, specifically

$$\begin{aligned} T_l : L^2(\Sigma; \mathbb{C}^r) &\to L^2(B(0, R); \mathbb{C}^r) \\ f &\mapsto f_{l,\zeta_l}\sqrt{|\nabla\zeta_l(\cdot)|^2 + 1} \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} Q_l : L^2(B(0, R); \mathbb{C}^r) &\to L^2(\Sigma_l \cap W_l; \mathbb{C}^r) \\ f &\mapsto f\left(\begin{pmatrix} (\kappa_l(\cdot))_1 \\ (\kappa_l(\cdot))_2 \\ \dots \\ (\kappa_l(\cdot))_{n-1} \end{pmatrix}\right). \end{aligned} \tag{3.57}$$

These operators are bounded. Moreover, by looking at (3.53) we notice

$$\mathcal{K}_{l,l} = Q_l K_l T_l. \tag{3.58}$$

This shows that the boundedness or compactness of $\mathcal{K}_{l,l}$ follows from the boundedness or compactness of K_l, respectively. We turn to the case $j \neq l$. Therefore, we assume $x \in (\Sigma_j \setminus \Sigma_l) \cap \Sigma \subset \Sigma_j \cap W_j$. Then, $x = x(x') = \kappa_j^{-1}\left(\begin{pmatrix} x' \\ \zeta_j(x') \end{pmatrix}\right)$ with $x' \in B(0,R)$. With this in mind, we can write

$$\begin{aligned}
&(\mathcal{K}_l f)(x(x')) \\
&= \lim_{\varepsilon \searrow 0} \int_{\substack{|x(x')-y(y')|>\varepsilon \\ |y'|<R}} \underbrace{\chi_{(\Sigma_j \setminus \Sigma_l) \cap \Sigma})(x(x'))k(x(x'),y(y'))\chi_{\operatorname{supp}\varphi_l}(y(y'))}_{:=k'_{l,j}(x',y')} \\
&\qquad\qquad f_{l,\zeta_l}(y')\sqrt{|\nabla\zeta_l(x')|^2+1}\,dy'.
\end{aligned} \tag{3.59}$$

A similar argumentation as in proof of *(i)* shows that $k'_{l,j}$ is bounded on the set $\overline{B(0,R)} \times \overline{B(0,R)}$. Thus, Theorem 3.4 proves that the operator

$$K'_{l,j} : L^2(B(0,R);\mathbb{C}^r) \to L^2(B(0,R);\mathbb{C}^r) \tag{3.60}$$

induced by $k'_{l,j}$ is bounded and compact. Furthermore, the limit in (3.59) exists and equals $(K'_{l,j}T_l f)(x(\widetilde{x}'))$. The operator $\mathcal{K}_{l,j}$ can be represented as

$$\mathcal{K}_{l,j} f = P_{l,j} K'_{l,j} T_l f. \tag{3.61}$$

Hereby, $P_{l,j}$ is the bounded operator

$$\begin{aligned}
P_{l,j} : L^2(B(0,R);\mathbb{C}^r) &\to L^2((\Sigma_j \setminus \Sigma_l) \cap \Sigma;\mathbb{C}^r) \\
f &\mapsto f\left(\begin{pmatrix} (\kappa_j(\cdot))_1 \\ (\kappa_j(\cdot))_2 \\ \dots \\ (\kappa_j(\cdot))_{n-1} \end{pmatrix}\right).
\end{aligned} \tag{3.62}$$

Therefore, $\mathcal{K}_{l,j}$ is compact for $j \neq l$. □

3.3 Integral Operators Corresponding to the Dirac Operator

After this preliminary work, we are ready to treat those integral operators which are important with respect to Dirac operators. Hereby, we work with the fundamental solution of the equation

$$-i(\alpha \cdot \nabla) + (m\beta - \lambda)I_4 = \delta_0 I_4, \tag{3.63}$$

where $m \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. We denote the fundamental solution G_λ. The explicit form is given by

$$G_\lambda(z) = \left(\lambda I_4 + m\beta + \left(1 - i\sqrt{\lambda^2 - m^2}\,|z|\right)\frac{i(\alpha \cdot z)}{|z|^2}\right)\frac{e^{i\sqrt{\lambda^2-m^2}|z|}}{4\pi\,|z|} \tag{3.64}$$

for $z \in \mathbb{R}^3 \setminus \{0\}$. The formula (3.64) can be found in [38, eq. (1.263)].
We start with a simple theorem.

Theorem 3.11. *Let $\lambda \in \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, \infty))$. The operator defined by*

$$G_\lambda * f \textit{ for } f \in L^2(\mathbb{R}^3; \mathbb{C}^4) \tag{3.65}$$

is a bounded linear operator in $L^2(\mathbb{R}^3; \mathbb{C}^4)$.

Proof. We prove this theorem by applying Theorem 3.1. In order to do so, let us choose $t_1(x, y) = G_\lambda(x-y)\,|x-y|\,e^{\frac{\mu}{2}|x-y|}$ and $t_2(x,y) = \frac{1}{|x-y|}e^{-\frac{\mu}{2}|x-y|}$. Thereby, μ is defined as $\mu := \operatorname{Im}\sqrt{\lambda^2 - m^2} > 0$. We observe that there exits a constant $C > 0$ such that

$$|G_\lambda(z)| \le C\frac{1}{|z|^2}e^{-\mu|z|} \quad \forall z \neq 0 \in \mathbb{R}^3. \tag{3.66}$$

Rough estimations yield

$$\begin{aligned}\int_{\mathbb{R}^3} |t_1(x,y)|^2\,dx &\le C\int_{\mathbb{R}^3} |G_\lambda(x-y)|^2\,|x-y|^2\,e^{\mu|x-y|}\,dx \\ &\le C\int_{\mathbb{R}^3}\frac{e^{-\mu|z|}}{|z|^2}\,dz \le C\int_0^\infty e^{-\mu r}\,dr < \infty.\end{aligned} \tag{3.67}$$

The same holds for t_2 which concludes the proof.

□

Next, we consider the convolution on Lipschitz boundaries.

Theorem 3.12. *Let $\lambda \in \mathbb{C} \setminus ((-\infty, -|m|] \cup [|m|, \infty))$ and Σ be the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^3$. The operators defined by*

$$\begin{aligned}(\Phi_\lambda f)(x) &:= \int_\Sigma G_\lambda(x-y)f(y)\,d\sigma(y) \textit{ for a.e. } x \in \mathbb{R}^3 \textit{ and } f \in L^2(\Sigma;\mathbb{C}^4) \\ \textit{and } (\Phi_\lambda^* g)(x) &:= \int_{\mathbb{R}^3} G_{\overline{\lambda}}(x-y)g(y)\,dy \textit{ for } \sigma\textit{-a.e } x \in \Sigma \textit{ and } g \in L^2(\mathbb{R}^3;\mathbb{C}^4)\end{aligned} \tag{3.68}$$

are bounded linear operators which map from $L^2(\Sigma;\mathbb{C}^4)$ to $L^2(\mathbb{R}^3;\mathbb{C}^4)$ and from $L^2(\mathbb{R}^3;\mathbb{C}^4)$ to $L^2(\Sigma;\mathbb{C}^4)$, respectively. Moreover, Φ_λ^* is the adjoint of Φ_λ.

Proof. Again, we apply Theorem 3.1. We set $t_1(x,y) = G_\lambda(x-y)\,|x-y|^{3/4}\,e^{\frac{\mu}{2}|x-y|}$ and $t_2(x,y) = \frac{1}{|x-y|^{3/4}}e^{-\frac{\mu}{2}|x-y|}$ with $\mu = \mathrm{Im}\,\sqrt{\lambda^2-m^2} > 0$. As in (3.66), we find $C>0$ such that

$$|G_\lambda(z)| \leq C\frac{1}{|z|^2}e^{-\mu|z|} \quad \forall z \neq 0 \in \mathbb{R}^3. \tag{3.69}$$

We compute

$$\begin{aligned}\int_{\mathbb{R}^3} |t_1(x,y)|^2\,dx &\leq C\int_{\mathbb{R}^3} |G_\lambda(x-y)|^2\,|x-y|^{3/2}\,e^{-\mu|x-y|}\,dx \\ &\leq C\int_{\mathbb{R}^3} \frac{e^{-\mu|z|}}{|z|^{5/2}}dz \leq C\int_0^\infty \frac{e^{-\mu r}}{\sqrt{r}}dr < \infty.\end{aligned} \tag{3.70}$$

We assume Σ_l, ζ_l and κ_l to be defined as in the beginning of this section and set $y(y') = \kappa_l^{-1}\left(\begin{pmatrix} y' \\ \zeta_l(y')\end{pmatrix}\right)$ for $y' \in \mathbb{R}^2$. Then, we obtain

$$\begin{aligned}\int_\Sigma |t_2(x,y)|^2\,d\sigma(y) &= \int_\Sigma \frac{e^{-\mu|x-y|}}{|x-y|^{3/2}}\,d\sigma(y) \leq \sum_{l=1}^p \int_{\Sigma_l} \frac{e^{-\mu|x-y|}}{|x-y|^{3/2}}\,d\sigma(y) \\ &\leq \sum_{l=1}^p \int_{\mathbb{R}^2} \frac{e^{-\mu|x-y(y')|}}{|x-y(y')^T|^{3/2}}\sqrt{1+|\nabla\zeta_l(y')|^2}\,dy' \\ &\leq C\int_0^\infty \frac{e^{-\mu r}}{\sqrt{r}}\,dr < \infty.\end{aligned} \tag{3.71}$$

Thus, the conditions of Theorem 3.1 are satisfied. This proves the existence and boundedness of Φ_λ. For Φ_λ^* this can be justified analogously by considering $G_{\overline{\lambda}}$ instead of G_λ. It remains to prove the statement about the adjoint operator. Here, we apply Fubini and see that Φ_λ^* from (3.68) satisfies

$$\begin{aligned}(f,\Phi_\lambda^* g)_{L^2(\Sigma;\mathbb{C}^4)} &= \int_\Sigma \left(f(x), \int_{\mathbb{R}^3} G_{\overline{\lambda}}(x-y)g(y)\,dy\right)_{\mathbb{C}^4} d\sigma(x) \\ &= \int_{\mathbb{R}^3}\int_\Sigma (f(x), G_{\overline{\lambda}}(x-y)g(y))_{\mathbb{C}^4}\,d\sigma(x)\,dy \\ &= \int_{\mathbb{R}^3}\int_\Sigma (G_\lambda(y-x)f(x), g(y))_{\mathbb{C}^4}\,d\sigma(x)\,dy \\ &= \int_{\mathbb{R}^3}\left(\int_\Sigma G_\lambda(y-x)f(x)\,d\sigma(x), g(y)\right)_{\mathbb{C}^4} dy \\ &= (\Phi_\lambda f, g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)}\end{aligned} \tag{3.72}$$

for all $f \in L^2(\Sigma;\mathbb{C}^4)$ and $g \in L^2(\mathbb{R}^3;\mathbb{C}^4)$. Hence, Φ_λ^* is the adjoint of Φ_λ. □

We are able to show an even stronger result. Namely, Φ_λ is also bounded as a mapping from $L^2(\Sigma;\mathbb{C}^4)$ to $H^{1/2}(\Omega_+;\mathbb{C}^4)\oplus H^{1/2}(\Omega_-;\mathbb{C}^4)$. In order to prove this result, we introduce the fundamental solution of the Helmholtz equation

$$\rho_r(z) := \frac{e^{ir|z|}}{4\pi\,|z|} \quad \forall z\in\mathbb{R}^3\setminus\{0\}, \tag{3.73}$$

where $r\in\mathbb{C}$.

Lemma 3.13. *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain and $r\in\mathbb{C}$. Then, the single layer potential of the Helmholtz equation*

$$\begin{aligned} S_r : L^2(\Sigma) &\mapsto H^{3/2}(\Omega)\\ f &\to \rho_r * f, \end{aligned} \tag{3.74}$$

where $\rho_r * f$ is defined as

$$(\rho_r * f)(x) := \int_\Sigma \rho_r(x-y)f(y)\,d\sigma(y) \quad \forall x\in\Omega, \tag{3.75}$$

is bounded.

Proof. Let $f\in L^2(\Sigma)$. There holds

$$\begin{aligned} (S_r f)(x) &= \int_\Sigma \rho_0(x-y)f(y)\,d\sigma(y) + \int_\Sigma \frac{e^{ir|x-y|}-1}{4\pi\,|x-y|}f(y)\,d\sigma(y)\\ &= (S_0 f)(x) + \int_\Sigma \underbrace{\frac{e^{ir|x-y|}-1}{4\pi\,|x-y|}}_{=\varpi(|x-y|)} f(y)\,d\sigma(y) \quad \forall x\in\Omega \end{aligned} \tag{3.76}$$

with

$$\varpi(t) := \frac{e^{-irt}-1}{4\pi t} \quad \forall t\in\mathbb{R}. \tag{3.77}$$

The equation (2.127) in [21] states that S_0 is a bounded mapping from $L^2(\Sigma)$ to $H^{3/2}(\Omega)$. We see $\varpi\in\mathcal{C}^\infty(\mathbb{R})$,

$$\begin{aligned} \nabla\varpi(|z|) &= \varpi'(|z|)\frac{z}{|z|} \text{ and}\\ \nabla^2\varpi(|z|) &= \varpi''(|z|)\frac{zz^T}{|z|^2} + \varpi'(|z|)\frac{I_3\,|z|^2 - zz^T}{|z|^3} \end{aligned} \tag{3.78}$$

for all $z\in\mathbb{R}^3\setminus\{0\}$. Applying the dominated convergence theorem yields

$$(\nabla(S_r-S_0)f)(x) = (\nabla\varpi(|\cdot|)*f)(x) \text{ and } (\nabla^2(S_r-S_0)f)(x) = (\nabla^2\varpi(|\cdot|)*f)(x) \tag{3.79}$$

for $x\in\Omega$. Furthermore, (3.78) shows $|\nabla\varpi(|z|)| = \mathcal{O}(1)$ and $|\nabla^2\varpi(|z|)| = \mathcal{O}(1/\,|z|)$ for $z\to 0$. It is also easy to see that $|\varpi(|z|)| = \mathcal{O}(1)$ for $z\to 0$. Thus, using the boundedness of Ω and standard considerations lead to $\|\varpi(|\cdot|)*f\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Sigma)}$. This proves the desired result. □

Lemma 3.14. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $\Omega_1 \subset \mathbb{R}^n$ be an open set such that $\Omega \cap \Omega_1$ is also a Lipschitz domain. Moreover, let $\varepsilon > 0$, $F \subset F + \overline{B(0,\varepsilon)} \subset \Omega_1$ be closed and $s \geq 0$. If $f \in L^2(\Omega) \cap H^s(\Omega \cap \Omega_1)$ and $\operatorname{supp} f \subseteq F$, then $f \in H^s(\Omega)$ and there exists a $C > 0$ independent of f such that*

$$\|f\|_{H^s(\Omega)} \leq C\|f\|_{H^s(\Omega\cap\Omega_1)}. \tag{3.80}$$

Proof. Due to [28, Theorem 3.6], there exists a function $\chi_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$ with the properties

$$\begin{aligned} &\chi_\varepsilon(x) = 1 && \text{if } x \in F, \\ &0 \leq \chi_\varepsilon(x) \leq 1 \text{ and } |\partial^a \chi_\varepsilon(x)| \leq C\varepsilon^{-|a|} && \text{if } 0 \leq \operatorname{dist}(x,F) < \varepsilon \text{ and } a \in \mathbb{N}_0^n, \\ &\chi_\varepsilon(x) = 0 && \text{if } \operatorname{dist}(x,F) \geq \varepsilon. \end{aligned} \tag{3.81}$$

To prove the statement of Lemma 3.14, it is sufficient to show the boundedness of

$$\begin{aligned} M_\varepsilon : H^s(\Omega \cap \Omega_1) &\to H^s(\Omega) \\ u &\mapsto \chi_\varepsilon \widetilde{u}. \end{aligned} \tag{3.82}$$

Here, $\widetilde{u}$ denotes the zero extension of u. At first, we assume $s \in \mathbb{N}_0$. Furthermore, let $u \in H^s(\Omega \cap \Omega_1)$ and

$$g_a := \begin{cases} \partial^a(u\chi_\varepsilon) & \text{on } \Omega \cap \Omega_1 \\ 0 & \text{on } \Omega \setminus \Omega_1 \end{cases} \tag{3.83}$$

for all mutltiindices $a \in \mathbb{N}_0^n$ with $|a|_1 \leq s$. We observe $(\partial^a \chi_\varepsilon)h \in \mathcal{D}(\Omega \cap \Omega_1)$ for

$h \in \mathcal{D}(\Omega)$. Therefore, we obtain for all $h \in \mathcal{D}(\Omega)$

$$
\begin{aligned}
\int_\Omega g_a(x)h(x)\,dx &= \int_{\Omega\cap\Omega_1} g_a(x)h(x)\,dx = \int_{\Omega\cap\Omega_1} \partial^a(u\chi_\varepsilon)(x)h(x)\,dx \\
&= \sum_{b\le a}\binom{a}{b}\int_{\Omega\cap\Omega_1}(\partial^b u)(x)(\partial^{a-b}\chi_\varepsilon)(x)h(x))\,dx \\
&= \sum_{b\le a}\binom{a}{b}(-1)^{|b|_1}\int_{\Omega\cap\Omega_1} u(x)\partial^b((\partial^{a-b}\chi_\varepsilon)h)(x)\,dx \\
&= \sum_{b\le a}\sum_{c\le b}\binom{a}{b}(-1)^{|b|_1}\binom{b}{c}\int_{\Omega\cap\Omega_1} u(x)(\partial^{a-c}\chi_\varepsilon)(x)\partial^c h(x)\,dx \qquad (3.84)\\
&= \sum_{c\le a}\int_{\Omega\cap\Omega_1} u(x)(\partial^{a-c}\chi_\varepsilon)(x)\partial^c h(x)\,dx \sum_{c\le b\le a}\binom{a}{b}(-1)^{|b|_1}\binom{b}{c} \\
&= \sum_{c\le a}\int_{\Omega\cap\Omega_1} u(x)(\partial^{a-c}\chi_\varepsilon)(x)\partial^c h(x)\,dx \binom{a}{c}(-1)^{|c|_1}\sum_{c\le b\le a}\binom{a-c}{b-c}(-1)^{|b-c|_1} \\
&= \sum_{c\le a}\int_{\Omega\cap\Omega_1} u(x)(\partial^{a-c}\chi_\varepsilon)(x)\partial^c h(x)\,dx(-1)^{|c|_1}\underbrace{\binom{a}{c}\sum_{0\le b\le a-c}\binom{a-c}{b}(-1)^{|b|_1}}_{=\delta_{a,c}} \\
&= (-1)^{|a|_1}\int_{\Omega\cap\Omega_1} u(x)\chi_\varepsilon(x)(\partial^a h)(x))\,dx = (-1)^{|a|_1}\int_\Omega \widetilde{u}(x)\chi_\varepsilon(x)(\partial^a h)(x))\,dx.
\end{aligned}
$$

Thus, $\partial^a(\widetilde{u}\chi_\varepsilon) = g_a \in L^2(\Omega)$ and

$$\|\partial^a(\widetilde{u}\chi_\varepsilon)\|_{L^2(\Omega)} = \|\partial^a(u\chi_\varepsilon)\|_{L^2(\Omega\cap\Omega_1)} \le C\|u\|_{H^s(\Omega\cap\Omega_1)}. \tag{3.85}$$

This shows the boundedness for $s \in \mathbb{N}_0$. Applying the interpolation theorem, see [28, Theorem B.8], one can show the boundedness for $s \ge 0$. □

Theorem 3.15. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain, $\lambda \in \mathbb{C}\setminus((-\infty,-m]\cup[m,\infty))$ and Φ_λ be as defined in* (3.68)*. Then,* $\operatorname{ran}\Phi_\lambda \subset H^{1/2}(\Omega_+;\mathbb{C}^4)\oplus H^{1/2}(\Omega_-;\mathbb{C}^4)$ *and*

$$\|\Phi_\lambda f\|_{H^{1/2}(\Omega_+;\mathbb{C}^4)\oplus H^{1/2}(\Omega_-;\mathbb{C}^4)} \le C\|f\|_{L^2(\Sigma;\mathbb{C}^4)} \quad \forall f \in L^2(\Sigma;\mathbb{C}^4). \tag{3.86}$$

Proof. We assume w.l.o.g. that Ω_+ is bounded and Ω_- is unbounded. In the first part of the proof we show $\|\Phi_\lambda f\|_{H^{1/2}(\Omega_+;\mathbb{C}^4)} \le C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}$ for all $f \in L^2(\Sigma;\mathbb{C}^4)$. Let us choose $f \in L^2(\Sigma;\mathbb{C}^4)$ and set $r = \sqrt{\lambda^2-m^2}$. We note

$$G_\lambda(z) = (-i(\alpha\cdot\nabla)+m\beta+\lambda I_4)\frac{e^{i\sqrt{\lambda^2-m^2}|z|}}{4\pi\,|z|} = (-i(\alpha\cdot\nabla)+m\beta+\lambda I_4)\rho_r(z) \quad \forall z \in \mathbb{R}^3\setminus\{0\}. \tag{3.87}$$

By applying the dominated convergence theorem we get

$$\begin{aligned}(\Phi_\lambda f)(x) = (G_\lambda * f)(x) &= \left((-i(\alpha\cdot\nabla) + m\beta + \lambda I_4)\rho_r) * f\right)(x)\\ &= \left((-i(\alpha\cdot\nabla) + m\beta + \lambda I_4)S_r f\right)(x) \quad \forall x\in\Omega_+.\end{aligned} \tag{3.88}$$

The triangle inequality and Lemma 3.13 show

$$\begin{aligned}\|\Phi_\lambda f\|_{H^{1/2}(\Omega_+;\mathbb{C}^4)} &\le C\left(\|(-i(\alpha\cdot\nabla) + m\beta)S_r f\|_{H^{1/2}(\Omega_+;\mathbb{C}^4)} + \|S_r f\|_{H^{1/2}(\Omega_+;\mathbb{C}^4)}\right)\\ &\le C\|S_r f\|_{H^{3/2}(\Omega_+;\mathbb{C}^4)} \le C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}.\end{aligned} \tag{3.89}$$

This concludes the first part of the proof.
For the second part we choose $R_2 > R_1 > 0$ such that $\Omega_+ \subsetneq B(0,R_1)$. We consider $\Phi_\lambda f$ on $\mathbb{R}^3 \setminus \overline{B(0,R_1)}$ at first. Elementary calculations show

$$\left|\frac{\partial G_\lambda}{\partial z_j}(z)\right| \le C\left(|z|^{-1} + |z|^{-2} + |z|^{-3}\right)e^{-\mu|z|} \quad \forall z\in\mathbb{R}^3\setminus\{0\}, \tag{3.90}$$

where $\mu = \operatorname{Im} r$ and $j\in\{1,2,3\}$. We use the dominated convergence theorem and the Cauchy-Schwarz inequality to estimate

$$\begin{aligned}\left|\frac{\partial \Phi_\lambda f}{\partial x_j}(x)\right|^2 = \left|\int_\Sigma \frac{\partial G_\lambda}{\partial x_j}(x-y)f(y)\,d\sigma(y)\right|^2 &\le \int_\Sigma \left|\frac{\partial G_\lambda}{\partial x_j}(x-y)\right|^2 d\sigma(y)\int_\Sigma |f(y)|^2\,d\sigma(y)\\ &\le C\left(\sum_{l=1}^3 (\operatorname{dist}(\partial B(0,R_1),\Omega_+))^{-l}\right)^2 \|f\|^2_{L^2(\Sigma;\mathbb{C}^4)}\int_\Sigma e^{-2\mu|x-y|}\,d\sigma(y)\\ &\le C\|f\|^2_{L^2(\Sigma;\mathbb{C}^4)}\int_\Sigma e^{-2\mu|x-y|}\,d\sigma(y) \quad \forall x\in\mathbb{R}^3\setminus\overline{B(0,R_1)}.\end{aligned} \tag{3.91}$$

Integrating and applying Fubini yields

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\overline{B(0,R_1)}}\left|\frac{\partial \Phi_\lambda f}{\partial x_j}(x)\right|^2 dx &\le C\|f\|^2_{L^2(\Sigma;\mathbb{C}^4)}\int_{\mathbb{R}^3\setminus\overline{B(0,R_1)}}\int_\Sigma e^{-2\mu|x-y|}d\sigma(y)\,dx\\ &\le C\|f\|^2_{L^2(\Sigma;\mathbb{C}^4)}\int_{\mathbb{R}^3}\int_\Sigma e^{-2\mu|x-y|}\,d\sigma(y)\,dx\\ &= C\|f\|^2_{L^2(\Sigma;\mathbb{C}^4)}\int_\Sigma\int_{\mathbb{R}^3} e^{-2\mu|x-y|}\,dx\,d\sigma(y)\\ &= C\|f\|^2_{L^2(\Sigma;\mathbb{C}^4)}\int_\Sigma\int_{\mathbb{R}^3} e^{-2\mu|z|}\,dz\,d\sigma(y) \le C\|f\|^2_{L^2(\Sigma;\mathbb{C}^4)}.\end{aligned} \tag{3.92}$$

Consequently,

$$\frac{\partial \Phi_\lambda f}{\partial x_j} \in L^2(\mathbb{R}^3\setminus\overline{B(0,R_1)};\mathbb{C}^4) \text{ and } \left\|\frac{\partial \Phi_\lambda f}{\partial x_j}\right\|_{L^2(\mathbb{R}^3\setminus\overline{B(0,R_1)};\mathbb{C}^4)} \le C\|f\|_{L^2(\Sigma;\mathbb{C}^4)} \tag{3.93}$$

for $j \in \{1,2,3\}$. We already know from Theorem 3.12 $\|\Phi_\lambda f\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \leq C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}$. Hence, our considerations imply

$$\Phi_\lambda f \in H^1(\mathbb{R}^3 \setminus \overline{B(0,R_1)};\mathbb{C}^4) \text{ and } \|\Phi_\lambda f\|_{H^1(\mathbb{R}^3\setminus\overline{B(0,R_1)};\mathbb{C}^4)} \leq C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}. \tag{3.94}$$

Next, we note that

$$\Phi_\lambda f \in H^{1/2}(\Omega_- \cap B(0,R_2);\mathbb{C}^4) \text{ and } \|\Phi_\lambda f\|_{H^{1/2}(\Omega_-\cap B(0,R_2);\mathbb{C}^4)} \leq C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}. \tag{3.95}$$

The proof is analogous to the first part of the proof if we consider the bounded Lipschitz domain $\Omega_- \cap B(0,R_2)$ instead of Ω_+.
In the last step we use the obtained results in order to prove (3.86). Therefore, we choose $\frac{R_2-R_1}{4} > \varepsilon > 0$ and a function $g \in \mathcal{D}(\mathbb{R}^3)$ such that

$$0 \leq g \leq 1,\ g(x) = 1 \text{ for } x \in \overline{B(0,R_1+2\varepsilon)} \text{ and } \operatorname{supp} g \subseteq B(0,R_2-2\varepsilon). \tag{3.96}$$

Moreover, $\operatorname{supp}(1-g) \subseteq \mathbb{R}^3 \setminus \overline{B(0,R_1+2\varepsilon)}$, $\Phi_\lambda f \in H^{1/2}(\Omega_- \cap B(0,R_2);\mathbb{C}^4)$ and also $\Phi_\lambda f \in H^{1/2}(\mathbb{R}^3 \setminus \overline{B(0,R_1)};\mathbb{C}^4)$. Now, by setting $\Omega = \Omega_-$, $\Omega_1 = B(0,R_2)$ and $F = \overline{B(0,R_2-2\varepsilon)}$, as well as $\Omega = \Omega_-$, $\Omega_1 = \mathbb{R}^3\setminus B(0,R_1)$ and $F = \mathbb{R}^3\setminus B(0,R_1+2\varepsilon)$ in Lemma 3.14 we get $g\Phi_\lambda f \in H^{1/2}(\Omega_-;\mathbb{C}^4)$ and $(1-g)\Phi_\lambda f \in H^{1/2}(\Omega_-;\mathbb{C}^4)$, respectively. Thus,

$$\Phi_\lambda f = g\Phi_\lambda f + (1-g)\Phi_\lambda f \in H^{1/2}(\Omega_-;\mathbb{C}^4). \tag{3.97}$$

We use Lemma 3.14 and [28, Theorem 3.20] in order to estimate the norm

$$\begin{aligned}
&\|\Phi_\lambda f\|_{H^{1/2}(\Omega_-;\mathbb{C}^4)} = \|g\Phi_\lambda f + (1-g)\Phi_\lambda f\|_{H^{1/2}(\Omega_-;\mathbb{C}^4)} \\
&\leq \|g\Phi_\lambda f\|_{H^{1/2}(\Omega_-;\mathbb{C}^4)} + \|(1-g)\Phi_\lambda f\|_{H^{1/2}(\Omega_-;\mathbb{C}^4)} \\
&\leq C\left(\|g\Phi_\lambda f\|_{H^{1/2}(\Omega_-\cap B(0,R_2);\mathbb{C}^4)} + \|(1-g)\Phi_\lambda f\|_{H^{1/2}(\mathbb{R}^3\setminus\overline{B(0,R_1)};\mathbb{C}^4)}\right) \\
&\leq C\left(\|g\|_{W^1_\infty(\mathbb{R}^3)}\|\Phi_\lambda f\|_{H^{1/2}(\Omega_-\cap B(0,R_2);\mathbb{C}^4)} + \left(1+\|g\|_{W^1_\infty(\mathbb{R}^3)}\right)\|\Phi_\lambda f\|_{H^{1/2}(\mathbb{R}^3\setminus\overline{B(0,R_1)};\mathbb{C}^4)}\right) \\
&\leq C\left(\|\Phi_\lambda f\|_{H^{1/2}(\Omega_-\cap B(0,R_2);\mathbb{C}^4)} + \|\Phi_\lambda f\|_{H^{1/2}(\mathbb{R}^3\setminus\overline{B(0,R_1)};\mathbb{C}^4)}\right) \\
&\leq C\left(\|\Phi_\lambda f\|_{H^{1/2}(\Omega_-\cap B(0,R_2);\mathbb{C}^4)} + \|\Phi_\lambda f\|_{H^{1}(\mathbb{R}^3\setminus\overline{B(0,R_1)};\mathbb{C}^4)}\right) \leq C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}.
\end{aligned} \tag{3.98}$$

Combining (3.89) and (3.98) yields (3.86). □

Theorem 3.16. *Let $\lambda \in \mathbb{C} \setminus ((-\infty,-|m|] \cup [|m|,\infty))$ and $\Sigma \subset \mathbb{R}^3$ be the boundary of a Lipschitz domain $\Omega \subset \mathbb{R}^3$. The values of the boundary operators defined by*

$$\begin{aligned}
(\mathcal{C}_\lambda f)(x) &:= \lim_{\varepsilon\searrow 0}\int_{|x-y|>\varepsilon} G_\lambda(x-y)f(y)\,d\sigma(y) \text{ and} \\
(\mathcal{P}^\pm_\lambda f)(x) &:= \lim_{\Omega_\pm\ni z\xrightarrow{n.t.}x}(\Phi_\lambda f)(z) \text{ for } x \in \Sigma \text{ and } f \in L^2(\Sigma;\mathbb{C}^4)
\end{aligned}$$

exist σ-a.e. on Σ. Moreover, the operators are bounded from $L^2(\Sigma;\mathbb{C}^4)$ to $L^2(\Sigma;\mathbb{C}^4)$ and there exists a $\kappa > 0$ such that the non-tangential supremum, see Definition 1.22, satisfies $\left\|\mathcal{N}_{\kappa,\Omega_\pm}\Phi_\lambda f\right\|_{L^2(\Sigma;\mathbb{C}^4)} \leq C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}$ for all $f \in L^2(\Sigma;\mathbb{C}^4)$. Additionally, the following assertions hold:

(i) $\mathcal{P}_\lambda^\pm = \mp\frac{i}{2}(\alpha\cdot\nu) + \mathcal{C}_\lambda$.

(ii) For every bounded subset S of $\mathbb{C}\setminus((-\infty,-|m|]\cup[|m|,\infty))$ the operators $\mathcal{C}_\lambda$ and $\mathcal{P}_\lambda^\pm$ are uniformly bounded with respect to λ.

(iii) The difference $\mathcal{K}_\alpha - \mathcal{C}_\lambda$, where

$$k_\alpha(x,y) := \frac{i(\alpha\cdot z)}{|z|^3} \text{ for } z\in\mathbb{R}^3\setminus\{0\} \text{ and} \tag{3.99}$$

$$(\mathcal{K}_\alpha f)(x) := \lim_{\varepsilon\searrow 0}\int_{|x-y|>\varepsilon} k_\alpha(x-y)\,d\sigma(y) \text{ for } \sigma\text{-a.e. } x\in\Sigma \text{ and } f\in L^2(\Sigma;\mathbb{C}^4), \tag{3.100}$$

is compact in $L^2(\Sigma;\mathbb{C}^4)$.

(iv) $-4(\mathcal{C}_\lambda(\alpha\cdot\nu))^2 = I_4$.

Proof. The operator $\mathcal{K}_\alpha$ is well defined σ-a.e. on Σ due to Corollary 3.8 and Theorem 3.10. We define the operator

$$(\Psi_\alpha f)(x) := \int_\Sigma k_\alpha(x-y)f(y)\,d\sigma(y) \quad \forall x\in\mathbb{R}^3\setminus\Sigma. \tag{3.101}$$

Then, [5, Proposition 4.3. and Theorem 4.4.] state that there exists a $\kappa > 0$ such that $\left\|\mathcal{N}_{\kappa,\Omega_\pm}\Psi_\alpha f\right\|_{L^2(\Sigma;\mathbb{C}^4)} \leq C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}$ for all $f\in L^2(\Sigma;\mathbb{C}^4)$, the non-tangential limits of $\Psi_\alpha f$ exist σ-a.e. on Σ and

$$\lim_{\Omega_\pm\ni z\xrightarrow{n.t.}x}(\Psi_\alpha f)(z) = \mp\frac{i}{2}(\alpha\cdot\nu(x))f(x) + (\mathcal{K}_\alpha f)(x) \quad \text{for } \sigma\text{-a.e. } x\in\Sigma. \tag{3.102}$$

Now, we want to transfer the properties of $\mathcal{K}_\alpha$ and Ψ_α to $\mathcal{C}_\lambda$ and Φ_λ, respectively. Therefore, we study the difference of their kernels

$$\omega_\lambda(z) := G_\lambda(z) - k_\alpha(z) \quad \forall z\in\mathbb{R}^3\setminus\{0\}. \tag{3.103}$$

We fix a bounded set $S\subset\mathbb{C}\setminus((-\infty,-|m|]\cup[|m|,\infty))$ and choose $R>0$ such that $B(0,R)\subset\Sigma$. Then, there exists a constant $C>0$ such that

$$|\omega_\lambda(z)| \leq C\frac{1}{|z|} \quad \forall z\neq 0\in B(0,2R) \tag{3.104}$$

for all $\lambda \in S$. Applying Theorem 3.1 by setting $t_1 = \omega_\lambda \sqrt{|\cdot|}$ and $t_2 = \frac{1}{\sqrt{|\cdot|}}$ yields

$$\omega * f \text{ exists } \sigma\text{-a.e. on } \Sigma \text{ and } \|\omega * f\|_{L^2(\Sigma;\mathbb{C}^4)} \leq C\|f\|_{L^2(\Sigma;\mathbb{C}^4)} \tag{3.105}$$

for all $f \in L^2(\Sigma;\mathbb{C}^4)$ and $\lambda \in S$. Let $x \in \Sigma$ such that $(\omega * f)(x)$ exists and $\omega_\varepsilon := \chi_{|z|>\varepsilon}\omega$. Then,

$$\begin{gathered} \omega_\varepsilon(x-y)f(y) \xrightarrow{\varepsilon\to 0} \omega(x-y)f(y) \quad \forall y \in \Sigma \setminus \{x\}, \\ |\omega_\varepsilon(x-y)f(y)| \leq |\omega(x-y)f(y)| \text{ and } |\omega(x-\cdot)f(\,\cdot\,)| \in L^1(\Sigma). \end{gathered} \tag{3.106}$$

Hence, dominated convergence guarantees $(\omega_\varepsilon * f)(x) \xrightarrow{\varepsilon\to 0} (\omega * f)(x)$. Therefore,

$$\lim_{\varepsilon\searrow 0} \int_{|x-y|>\varepsilon} \omega(x-y)f(y) = (\omega * f)(x) \text{ for } \sigma\text{-a.e. } x \in \Sigma. \tag{3.107}$$

Next, we consider the non-tangential limit of $\omega * f$. For this purpose we choose $x \in \Sigma$ such that $\left(\frac{1}{|\cdot|} * |f|\right)(x)$ and $(\omega * f)(x)$ exist. Here, $\kappa > 0$ denotes the constant from the non-tangential neighbourhood $\Gamma_{\kappa,\Omega_\pm}(x)$ of x, see Definition 1.21. For all $y \in \Sigma$ and $z \in \Gamma_{\kappa,\Omega_\pm}(x)$ the estimates

$$|y-x| \leq |y-z| + |z-x| \leq |y-z| + (1+\kappa)\text{dist}(z,\Sigma) \leq (2+\kappa)\,|z-y| \tag{3.108}$$

$$\text{and } |\omega(z-y)f(y)| \leq C\frac{1}{|z-y|}\,|f(y)| \leq C\frac{1}{|x-y|}\,|f(y)| \tag{3.109}$$

hold. Thus, $C\frac{1}{|x-\cdot|}\,|f|$ is an integrable majorant for $\omega(z-\cdot)f$. Furthermore, there holds $\omega(z-y)f(y) \to \omega(x-y)f(y)$ for $z \to x$ and $y \neq x$. Applying dominated convergence yields

$$\begin{aligned} \lim_{\Omega_\pm \ni z \xrightarrow{n.t.} x} (\omega * f)(z) &= \lim_{\varepsilon\searrow 0} \int_{|x-y|>\varepsilon} \omega(x-y)f(y) \\ &= (\omega * f)(x) \text{ for } \sigma\text{-a.e. } x \in \Sigma. \end{aligned} \tag{3.110}$$

Combining (3.110), (3.103) and (3.102) proves assertion *(i)*. Next, we show the boundedness of the non-tangenial supremum. The equations (3.108) and (3.109) give us

$$(\mathcal{N}_{\kappa,\Omega_\pm}(\omega * f))(x) \leq C\left(\frac{1}{|\cdot|} * |f|\right)(x) \text{ for } \sigma\text{-a.e. } x \in \Sigma. \tag{3.111}$$

Therefore, Theorem 3.1 yields $\left\|\mathcal{N}_{\kappa,\Omega_\pm}(\omega * f)\right\|_{L^2(\Sigma;\mathbb{C}^4)} \leq C\|f\|_{L^2(\Sigma;\mathbb{C}^4)}$ for $f \in L^2(\Sigma;\mathbb{C}^4)$. Furthermore, *(ii)* follows from (3.105) and (3.110).
Item *(iii)* and *(iv)* are left to be proven. We start with *(iii)* by realizing that $\omega_\lambda\left(\kappa_l^{-1}\begin{pmatrix} x'-y' \\ \zeta_l(x')-\zeta_l(y')\end{pmatrix}\right)$ with $x' \neq y' \in \mathbb{R}^2$ defines a kernel of order -1. Consequently, Theorem 3.4 and Theorem 3.10 *(ii)* imply $\omega_\lambda * (\cdot) : L^2(\Sigma;\mathbb{C}^4) \to L^2(\Sigma;\mathbb{C}^4)$ is compact, proving assertion *(iii)*.

It remains to show item *(iv)*. This part of the proof is based on the proof of Lemma 3.3.(ii) in [2]. W.l.o.g. we assume that Ω_+ is bounded and Ω_- is unbounded. First, we prove for $f \in \mathcal{C}^\infty(\Omega_+)$, such that the non-tangential limit of f exists, $\mathcal{N}_{\kappa,\Omega_+} f \in L^2(\Sigma)$, $-i(\alpha \cdot \nabla)f + \beta f = \lambda f$, and $(\alpha \cdot \nabla) f \in L^2(\Omega_+)$, the reproducing formula

$$f(x) = \int_\Sigma G_\lambda(x-y) i(\alpha \cdot \nu(y)) f(y)\, d\sigma(y) \quad \forall x \in \Omega_+. \tag{3.112}$$

Let us fix $x \in \Omega_+$ and choose $\varepsilon > 0$ such that $B(x,\varepsilon) \subsetneq \Omega_+$. To simplify notation, we define $\Omega_{+,\varepsilon} := \Omega_+ \setminus B(x,\varepsilon)$ and $\Sigma_{+,\varepsilon} = \partial\Omega_{+,\varepsilon}$. We apply Theorem 1.24 and obtain

$$\begin{aligned}
&\int_{\Sigma_{+,\varepsilon}} G_\lambda(x-y) i(\alpha \cdot \nu(y)) f(y)\, d\sigma(y) = \int_{\Sigma_{+,\varepsilon}} \sum_{j=0}^{3} G_\lambda(x-y) i\alpha_j(y) f(y) \nu_j \, d\sigma(y) \\
&= \int_{\Omega_{+,\varepsilon}} \sum_{j=0}^{3} \frac{\partial\left(G_\lambda(x-y) i\alpha_j(y) f(y)\right)}{\partial y_j}\, dy \\
&= \int_{\Omega_{+,\varepsilon}} \left(\sum_{j=0}^{3} \frac{\partial G_\lambda(x-y)}{y_j} i\alpha_j f(y)\, dy + G_\lambda(x-y) i\alpha_j \frac{\partial f}{\partial y_j}(y) \right) dy \\
&= \int_{\Omega_{+,\varepsilon}} \left(\sum_{j=0}^{3} \frac{\partial G_\lambda(x-y)}{\partial y_j} i\alpha_j f(y) \right) + G_\lambda(x-y) i(\alpha \cdot \nabla) f(y)\, dy \\
&= \int_{\Omega_{+,\varepsilon}} G_\lambda(x-y)(\lambda I_4 - m\beta) f(y) - G_\lambda(x-y)(\lambda I_4 - m\beta) f(y)\, dy = 0.
\end{aligned} \tag{3.113}$$

Therefore,

$$\int_\Sigma G_\lambda(x-y) i(\alpha \cdot \nu(y)) f(y)\, d\sigma(y) = -\int_{\partial B(x,\varepsilon)} G_\lambda(x-y) i(\alpha \cdot \nu(y)) f(y)\, d\sigma(y). \tag{3.114}$$

Transforming to spherical coordinates, defining $w(\theta,\phi) := -\begin{pmatrix} \sin(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) \\ cos(\theta) \end{pmatrix}$ with

$\theta \in [0,\pi]$ and $\phi \in [0,2\pi)$, and applying the dominated convergence theorem yields

$$\begin{aligned}
&\int_{\partial B(x,\varepsilon)} G_\lambda(x-y) i(\alpha\cdot\nu(y)) f(y)\, d\sigma(y) \\
&= \int_0^\pi \int_0^{2\pi} G_\lambda(\varepsilon w(\theta,\phi)) i(\alpha\cdot w(\theta,\phi)) f(x-\varepsilon w(\theta,\phi)) \sin(\theta)\varepsilon^2 \, d\phi\, d\theta \\
&= \int_0^\pi \int_0^{2\pi} \left(\mathcal{O}(1)\varepsilon + i\alpha\cdot w(\theta,\phi) \frac{e^{i\sqrt{\lambda^2-m^2}\varepsilon}}{4\pi} \right) i(\alpha\cdot w(\theta,\phi)) f(x-\varepsilon w(\theta,\phi)) \sin(\theta)\, d\phi\, d\theta \\
&\xrightarrow{\varepsilon\to 0} \int_0^\pi \int_0^{2\pi} i(\alpha\cdot w(\theta,\phi)) \frac{1}{4\pi} i(\alpha\cdot w(\theta,\phi)) f(x)\sin(\theta)\, d\phi\, d\theta = -f(x).
\end{aligned} \tag{3.115}$$

Thus, by inserting in (3.114) and taking the limit we get (3.112). Now, let us choose $g \in L^2(\Sigma;\mathbb{C}^4)$. Then, $\Phi_\lambda i(\alpha\cdot\nu)g$ satisfies the necessary conditions regarding the reproducing formula. Hence,

$$(\Phi_\lambda i(\alpha\cdot\nu)g)(x) = (\Phi_\lambda i(\alpha\cdot\nu)\mathcal{P}_\lambda^+ i(\alpha\cdot\nu)g)(x) \quad \forall x \in \Omega_+ \tag{3.116}$$

which implies

$$\mathcal{P}_\lambda^+ i(\alpha\cdot\nu)g = \mathcal{P}_\lambda^+ i(\alpha\cdot\nu)\mathcal{P}_\lambda^+ i(\alpha\cdot\nu)g. \tag{3.117}$$

We use *(i)* and (3.117) in order to show

$$\begin{aligned}
\frac{1}{2}g + \mathcal{C}_\lambda i(\alpha\cdot\nu)g &= -\frac{1}{2} i(\alpha\cdot\nu) i(\alpha\cdot\nu)g + \mathcal{C}_\lambda i(\alpha\cdot\nu)g = \mathcal{P}_\lambda^+ i(\alpha\cdot\nu)g \\
&= \mathcal{P}_\lambda^+ i(\alpha\cdot\nu)\mathcal{P}_\lambda^+ i(\alpha\cdot\nu)g = \frac{1}{2}\mathcal{P}_\lambda^+ i(\alpha\cdot\nu)g + \mathcal{C}_\lambda i(\alpha\cdot\nu)\mathcal{P}_\lambda^+ i(\alpha\cdot\nu)g \\
&= \frac{1}{4}g + \frac{1}{2}\mathcal{C}_\lambda i(\alpha\cdot\nu)g + \frac{1}{2}\mathcal{C}_\lambda i(\alpha\cdot\nu)g + (\mathcal{C}_\lambda i(\alpha\cdot\nu)g)^2.
\end{aligned}$$

Hence, *(iv)* holds true. □

We state two further results which come in handy when proving the self-adjointness of certain Dirac operators.

Corollary 3.17. *The operator $\mathcal{K}_\alpha$, which is defined in* (3.100)*, is self-adjoint.*

Proof. One can use similar statements as in Theorem 3.10 and (c) in the beginning of the proof of Theorem 3.6 in order to show $\widehat{\mathcal{K}_\alpha}$, defined through

$$\widehat{\mathcal{K}_\alpha} f(x) := \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} k_\alpha(x-y) f(y)\, d\sigma(y) \right| \text{ for } x \in \Sigma \text{ and } f \in L^2(\Sigma;\mathbb{C}^4), \tag{3.118}$$

is a bounded mapping from $L^2(\Sigma;\mathbb{C}^4)$ to $L^2(\Sigma;\mathbb{C}^4)$. Due to the boundedness of $\widehat{\mathcal{K}_\alpha}$ and the dominated convergence theorem

$$\int_{|(\cdot)-y|>\varepsilon} k((\cdot)-y)f(y)\,d\sigma(y) \xrightarrow{\varepsilon\to 0} \mathcal{K}_\alpha f \text{ in } L^2(\Sigma) \quad \forall f\in L^2(\Sigma;\mathbb{C}^4). \tag{3.119}$$

Now, let us fix f and g in $L^2(\Sigma;\mathbb{C}^4)$. We obtain

$$\begin{aligned}
(\mathcal{K}_\alpha f, g)_{L^2(\Sigma;\mathbb{C}^4)} &= \int_\Sigma \lim_{\varepsilon\searrow 0}\left(\int_{|x-y|>\varepsilon} (k_\alpha(x-y)f(y), g(x))_{\mathbb{C}^4}\,d\sigma(y)\right) d\sigma(x)\\
&= \lim_{\varepsilon\searrow 0}\int_\Sigma\int_{|x-y|>\varepsilon} (f(y), k_\alpha(y-x)g(x))_{\mathbb{C}^4}\,d\sigma(y)\,d\sigma(x)\\
&= \lim_{\varepsilon\searrow 0}\int_\Sigma\int_\Sigma \chi_{|x-y|>\varepsilon}(y)(f(y), k_\alpha(y-x)g(x))_{\mathbb{C}^4}\,d\sigma(y)\,d\sigma(x)\\
&= \lim_{\varepsilon\searrow 0}\int_\Sigma\int_\Sigma \chi_{|x-y|>\varepsilon}(x)(f(y), k_\alpha(y-x)g(x))_{\mathbb{C}^4}\,d\sigma(x)\,d\sigma(y)\\
&= \int_\Sigma\left(\lim_{\varepsilon\searrow 0}\int_\Sigma \chi_{|x-y|>\varepsilon}(x)(f(y), k_\alpha(y-x)g(x))_{\mathbb{C}^4}\,d\sigma(x)\right) d\sigma(y)\\
&= (f, \mathcal{K}_\alpha g)_{L^2(\Sigma;\mathbb{C}^4)}
\end{aligned} \tag{3.120}$$

which concludes the proof. □

Last, we are able to formulate a consequential theorem which is based on Corollary 3.9. The result plays an important role in proving the self-adjointness of Dirac operators on $\mathcal{C}^1$ surfaces.

Theorem 3.18. *If Σ is a $\mathcal{C}^1$ boundary and $\lambda\in\mathbb{C}\setminus((-\infty,-|m|]\cup[|m|,\infty))$, then*

$$(\alpha\cdot\nu)\mathcal{C}_\lambda + \mathcal{C}_\lambda(\alpha\cdot\nu) \tag{3.121}$$

is a compact operator in $L^2(\Sigma;\mathbb{C}^4)$.

Proof. Corollary 3.9 and Theorem 3.10 imply the compactness of the linear operator $(\alpha\cdot\nu)\mathcal{K}_\alpha + \mathcal{K}_\alpha(\alpha\cdot\nu)$. From Theorem 3.16 *(iii)* we know that the difference of $\mathcal{C}_\lambda - \mathcal{K}_\alpha$ is compact. Consequently, also $(\alpha\cdot\nu)\mathcal{C}_\lambda + \mathcal{C}_\lambda(\alpha\cdot\nu)$ is compact. □

4 The Free Dirac Operator and a Quasi Boundary Triple for the Dirac Operator

In this chapter we use the obtained results in order to construct a quasi boundary triple for the Dirac operator. However, prior to that we study the free Dirac operator which describes a free spin 1/2 particle and constitutes the basis in constructing the quasi boundary triple.

4.1 The Free Dirac Operator

In this section we introduce the free Dirac operator A_0, prove its self-adjointness and study its spectrum.

Definition 4.1. Let $m \in \mathbb{R}$. Then, the free Dirac operator is defined by

$$\begin{aligned} \operatorname{dom} A_0 &:= H^1(\mathbb{R}^3; \mathbb{C}^4) \\ A_0 f &:= -i(\alpha \cdot \nabla) f + m\beta f. \end{aligned} \tag{4.1}$$

We define an auxiliary multiplication operator which proves to be valuable in studying the free Dirac operator.

Definition 4.2. Let $m \in \mathbb{R}$. Then, we define the operator

$$\begin{aligned} \operatorname{dom} \mathcal{M} &:= \left\{ f \in L^2(\mathbb{R}^3; \mathbb{C}^4) : (1 + |\cdot|^2)^{1/2} f \in L^2(\mathbb{R}^3; \mathbb{C}^4) \right\} \\ \mathcal{M} f &:= 2\pi(\alpha \cdot (\cdot)) f + m\beta f. \end{aligned} \tag{4.2}$$

Lemma 4.3. *The two operators A_0 and $\mathcal{M}$ are unitary equivalent, i.e. $\mathcal{F}^{-1}\mathcal{M}\mathcal{F} = A_0$, where $\mathcal{F}$ denotes the Fourier transform given by* (2.4).

Proof. Let us check the equality of the domains at first. Let $f \in \operatorname{dom} A_0 = H^1(\mathbb{R}^3; \mathbb{C}^4)$. Equivalently,

$$(1 + |\cdot|^2)^{1/2} \mathcal{F} f \in L^2(\mathbb{R}^3; \mathbb{C}^4) \Leftrightarrow \mathcal{F} f \in \operatorname{dom} \mathcal{M} \Leftrightarrow f \in \operatorname{dom} \mathcal{F}^{-1}\mathcal{M}\mathcal{F}. \tag{4.3}$$

It remains to prove $A_0 f = \mathcal{F}^{-1}\mathcal{M}\mathcal{F}f$, which is equivalent to $\mathcal{F}A_0 f = \mathcal{M}\mathcal{F}f$, for all $f \in \operatorname{dom} A_0 = \operatorname{dom} \mathcal{F}^{-1}\mathcal{M}\mathcal{F}$. Through applying the differentiation rule regarding the Fourier transform, cf. [28, eq. (3.17)], we get

$$\begin{aligned}(\mathcal{F}A_0 f)(\xi) &= \mathcal{F}(-i(\alpha \cdot \nabla)f + m\beta f)(\xi) \\ &= (2\pi(\alpha \cdot (\cdot))\mathcal{F}f + m\beta\mathcal{F}f)(\xi) = (\mathcal{M}\mathcal{F}f)(\xi) \quad \forall \xi \in \mathbb{R}^3\end{aligned} \tag{4.4}$$

and $f \in \operatorname{dom} A_0$ which concludes the proof. □

Due to Lemma 4.3, we can carry over the self-adjointness and the spectral properties of $\mathcal{M}$ to A_0. Hence, we examine $\mathcal{M}$ in the next theorem.

Theorem 4.4. *The operator $\mathcal{M}$ is self-adjoint in $L^2(\mathbb{R}^3;\mathbb{C}^4)$ and*

$$\sigma(\mathcal{M}) = \sigma_{\text{ess}}(\mathcal{M}) = (-\infty, -|m|] \cup [|m|, \infty)\,. \tag{4.5}$$

Proof. This proof is based on [40, Satz 20.1], where the statement is proven for $m > 0$. We begin with the self-adjointness of $\mathcal{M}$. The symmetry of $\mathcal{M}$ immediately follows from the self-adjointness of β and $(\alpha \cdot \xi)$ for all $\xi \in \mathbb{R}^3$. Thus, we have to prove $\mathcal{M}^* \subset \mathcal{M}$. Let $g \in \operatorname{dom} \mathcal{M}^*$, then for all $f \in \operatorname{dom} \mathcal{M}$ holds

$$\begin{aligned}(\mathcal{M}f, g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} &= \int_{\mathbb{R}^3} (2\pi(\alpha \cdot \xi)f(\xi) + \beta f(\xi), g(\xi))_{\mathbb{C}^4}\, d\xi \\ &= \int_{\mathbb{R}^3} (f(\xi), 2\pi(\alpha \cdot \xi)g(\xi) + \beta g(\xi))_{\mathbb{C}^4}\, d\xi \\ &= (f, \mathcal{M}^* g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = \int_{\mathbb{R}^3} (f(\xi), (\mathcal{M}^* g)(\xi))_{\mathbb{C}^4}\, d\xi.\end{aligned} \tag{4.6}$$

Since $\mathcal{D}(\mathbb{R}^3) \subset \operatorname{dom} \mathcal{M}$, we can apply [28, Theorem 3.7] and see

$$2\pi(\alpha \cdot (\cdot))g + m\beta g = \mathcal{M}^* g \in L^2(\mathbb{R}^3;\mathbb{C}^4) \tag{4.7}$$

which leads to $(\alpha \cdot (\cdot))g \in L^2(\mathbb{R}^3;\mathbb{C}^4)$. Moreover, Remark 2.31 yields

$$\begin{aligned}\int_{\mathbb{R}^3} |\xi|^2 |g(\xi)|^2\, d\xi &= \int_{\mathbb{R}^3} ((\alpha \cdot \xi)g(\xi), (\alpha \cdot \xi)g(\xi))_{\mathbb{C}^4} d\xi \\ &= \int_{\mathbb{R}^3} \left((\alpha \cdot \xi)^2 g(\xi), g(\xi)\right)_{\mathbb{C}^4} d\xi \\ &= \int_{\mathbb{R}^3} |(\alpha \cdot \xi)g(\xi)|^2\, d\xi < \infty.\end{aligned} \tag{4.8}$$

Hence, $g \in \operatorname{dom} \mathcal{M}$ and $\mathcal{M}^* g = \mathcal{M}g$. This shows $\mathcal{M} = \mathcal{M}^*$. Now, we study the spectrum of M. We see

$$2\pi(\alpha \cdot \xi) + m\beta = 2\pi \begin{pmatrix} \frac{m}{2\pi} & 0 & \xi_3 & \xi_1 - i\xi_2 \\ 0 & \frac{m}{2\pi} & \xi_1 + i\xi_2 & -\xi_3 \\ \xi_3 & \xi_1 - i\xi_2 & -\frac{m}{2\pi} & 0 \\ \xi_1 + i\xi_2 & -\xi_3 & 0 & -\frac{m}{2\pi} \end{pmatrix} \tag{4.9}$$

is a self-adjoint matrix for all $\xi \in \mathbb{R}^3$. We compute the eigenvalues and obtain

$$\lambda_1(\xi) = \lambda_2(\xi) = \sqrt{m^2 + 4\pi^2 |\xi|^2} \quad \text{as well as} \quad \lambda_3(\xi) = \lambda_4(\xi) = -\sqrt{m^2 + 4\pi^2 |\xi|^2}. \tag{4.10}$$

The corresponding normalized orthogonal eigenvectors are

$$\begin{aligned} u_1(\xi) &= p(\xi) \begin{pmatrix} 0 \\ 1 \\ -\frac{2\pi(\xi_1 - i\xi_2)}{m+\lambda_1(\xi)} \\ \frac{2\pi\xi_3}{m+\lambda_1(\xi)} \end{pmatrix}, u_2(\xi) = p(\xi) \begin{pmatrix} 1 \\ 0 \\ -\frac{2\pi\xi_3}{m+\lambda_1(\xi)} \\ -\frac{2\pi(\xi_1 + i\xi_2)}{m+\lambda_1(\xi)} \end{pmatrix} \\ u_3(\xi) &= p(\xi) \begin{pmatrix} -\frac{2\pi(\xi_1 - i\xi_2)}{m+\lambda_1(\xi)} \\ \frac{2\pi\xi_3}{m+\lambda_1(\xi)} \\ 0 \\ 1 \end{pmatrix} \quad \text{and } u_4(\xi) = p(\xi) \begin{pmatrix} -\frac{2\pi\xi_3}{m+\lambda_1(\xi)} \\ -\frac{2\pi(\xi_1 + i\xi_2)}{m+\lambda_1(\xi)} \\ 1 \\ 0 \end{pmatrix} \end{aligned} \tag{4.11}$$

with the normalizing factor $p(\xi) = 1/\sqrt{\frac{4\pi^2|\xi|^2}{(m+\lambda_1(\xi))^2} + 1}$. Now, let us introduce the unitary matrix $U(\xi) := (u_1(\xi)|u_2(\xi)|u_3(\xi)|u_4(\xi)) \in \mathbb{C}^{4\times 4}$ and the diagonal matrix $D(\xi) := \operatorname{diag}(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi), \lambda_4(\xi))$. Moreover, we define the mappings

$$\begin{aligned} \mathcal{M}_D : L^2(\mathbb{R}^3; \mathbb{C}^4) \supset \operatorname{dom} \mathcal{M} &\to L^2(\mathbb{R}^3; \mathbb{C}^4), \qquad \mathcal{M}_U : L^2(\mathbb{R}^3; \mathbb{C}^4) \to L^2(\mathbb{R}^3; \mathbb{C}^4) \\ f &\mapsto Df \qquad\qquad\qquad\qquad\qquad\qquad f \mapsto Uf \\ \text{and} \qquad \mathcal{M}_{U^*} : L^2(\mathbb{R}^3; \mathbb{C}^4) &\to L^2(\mathbb{R}^3; \mathbb{C}^4) \\ f &\mapsto U^* f. \end{aligned} \tag{4.12}$$

Trivial observations show that $\mathcal{M}_U$ is a unitary operator in $L^2(\mathbb{R}^3; \mathbb{C}^4)$ which satisfies $(\mathcal{M}_U)^* = \mathcal{M}_{U^*}$, $\mathcal{M}$ is unitary equivalent to $\mathcal{M}_D$, i.e. $\mathcal{M} = \mathcal{M}_U \mathcal{M}_D \mathcal{M}_{U^*}$, and $\sigma(\mathcal{M}_D) = \sigma_{\text{ess}}(\mathcal{M}_D) = (-\infty, -|m|] \cup [|m|, \infty)$. The unitary equivalence of the operators yields $\sigma(\mathcal{M}) = \sigma_{\text{ess}}(\mathcal{M}) = (-\infty, -|m|] \cup [|m|, \infty)$. □

As a consequence of the two previous statements, we are in the position to describe the spectrum of the free Dirac operator.

Corollary 4.5. *The free Dirac operator A_0 is self-adjoint and*

$$\sigma(A_0) = \sigma_{\text{ess}}(A_0) = (-\infty, -|m|] \cup [|m|, \infty). \tag{4.13}$$

Proof. Combining Lemma 4.3 and Theorem 4.4 proves the statement. □

In Theorem 4.6 we show a resolvent formula for the free Dirac operator in terms of the fundamental solution G_λ, which is given by (3.64).

Theorem 4.6. *The resolvent of A_0 is a bounded linear operator from $L^2(\mathbb{R}^3;\mathbb{C}^4)$ to $H^1(\mathbb{R}^3;\mathbb{C}^4)$ which is given by*

$$\big((A_0-\lambda)^{-1}f\big)(x)=\int_{\mathbb{R}^3} G_\lambda(x-y)f(y)\,dy \text{ for } f\in L^2(\mathbb{R}^3;\mathbb{C}^4) \tag{4.14}$$

for $\lambda\in\rho(A_0)=\mathbb{C}\setminus((-\infty,-|m|]\cup[|m|,\infty))$.

Proof. Foremost, we assume $g\in\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)$. One sees by applying the monotone convergence theorem, simple algebraic transformations, the divergence theorem and similar arguments as in Theorem 3.16 *(iv)*

$$\begin{aligned}\int_{\mathbb{R}^3} G_\lambda(x-y)(A_0-\lambda)g(y)\,dy &= \int_{\mathbb{R}^3} G_\lambda(x-y)((-i(\alpha\cdot\nabla)+m\beta-\lambda I_4)g)(y)\,dy\\ &=\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^3\setminus B(x,\varepsilon)} G_\lambda(x-y)((-i(\alpha\cdot\nabla)+m\beta-\lambda I_4)g)(y)\,dy\\ &=\lim_{\varepsilon\searrow 0}\int_{\partial B(x,\varepsilon)} G_\lambda(x-y)(-i(\alpha\cdot\nu(y))g(y)\,d\sigma(y) \qquad (4.15)\\ &+\int_{\mathbb{R}^3\setminus B(x,\varepsilon)} \underbrace{\Big(i\sum_{l=1}^3\frac{\partial}{\partial y_l}G_\lambda(x-y)\alpha_l+G_\lambda(x-y)(m\beta-\lambda I_4)\Big)}_{=0} g(y)\,dy\\ &=\lim_{\varepsilon\searrow 0}\int_{\partial B(x,\varepsilon)} G_\lambda(x-y)(-i(\alpha\cdot\nu(y))g(y)\,d\sigma(y)=g(x)\quad\forall x\in\mathbb{R}^3.\end{aligned}$$

Thus, $G_\lambda*((A_0-\lambda)g)=g$ for all $g\in\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)$. Next, let us choose a function $g\in\operatorname{dom}A_0=H^1(\mathbb{R}^3;\mathbb{C}^4)$. Since $\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)$ is dense in $H^1(\mathbb{R}^3;\mathbb{C}^4)$, there exists a sequence $(g_l)_{l\in\mathbb{N}}$ such that $g_l\in\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)$ for all $l\in\mathbb{N}$ and g_l converges to g in $H^1(\mathbb{R}^3;\mathbb{C}^4)$. Thus, $(A_0-\lambda)g_l$ converges to $(A_0-\lambda)g$ in $L^2(\mathbb{R}^3;\mathbb{C}^4)$. The $L^2(\mathbb{R}^3;\mathbb{C}^4)$-continuity of the mapping defined by $L^2(\mathbb{R}^3;\mathbb{C}^4)\ni f\mapsto G_\lambda*f$, see Theorem 3.11, leads to

$$G_\lambda*((A_0-\lambda)g)=g\quad\forall g\in\operatorname{dom}A_0. \tag{4.16}$$

Moreover, since $(A_0-\lambda)^{-1}f\in\operatorname{dom}A_0$ for all $f\in L^2(\mathbb{R}^3;\mathbb{C}^4)$, we obtain

$$G_\lambda*f=G_\lambda*\big((A_0-\lambda)(A_0-\lambda)^{-1}f\big)=(A_0-\lambda)^{-1}f\quad\forall f\in L^2(\mathbb{R}^3;\mathbb{C}^4). \tag{4.17}$$

This proves the resolvent formula (4.14). In order to show the boundedness from $L^2(\mathbb{R}^3;\mathbb{C}^4)$ to $H^1(\mathbb{R}^3;\mathbb{C}^4)$, we prove the closedness of $(A_0-\lambda)^{-1}$ as a mapping from $L^2(\mathbb{R}^3;\mathbb{C}^4)$ to $H^1(\mathbb{R}^3;\mathbb{C}^4)$. Therefore, we assume $L^2(\mathbb{R}^3;\mathbb{C}^4)\ni f_l\to f$ in $L^2(\mathbb{R}^3;\mathbb{C}^4)$ and $H^1(\mathbb{R}^3;\mathbb{C}^4)\ni(A_0-\lambda I_4)^{-1}f_l\to g$ in $H^1(\mathbb{R}^3;\mathbb{C}^4)$. The resolvent $(A_0-\lambda)^{-1}$ is bounded as a mapping in $L^2(\mathbb{R}^3;\mathbb{C}^4)$ which shows

$$g=\lim_{l\to\infty}(A_0-\lambda)^{-1}f_l=(A_0-\lambda)^{-1}\lim_{l\to\infty}f_l=(A_0-\lambda)^{-1}f\quad\text{ in } L^2(\mathbb{R}^3;\mathbb{C}^4).$$

Hence, $g = (A_0 - \lambda)^{-1} f \in H^1(\mathbb{R}^3; \mathbb{C}^4)$ and therefore

$$(A_0 - \lambda)^{-1} : L^2(\mathbb{R}^3; \mathbb{C}^4) \to H^1(\mathbb{R}^3; \mathbb{C}^4) \tag{4.18}$$

is closed and also bounded. □

4.2 A Quasi Boundary Triple Corresponding to the Dirac Operator

In this section we construct a quasi boundary triple for the Dirac operator which we use to study Dirac operators with singular interactions in Chapter 5. We start this section by introducing new operators.

Definition 4.7. Let $\Omega = \Omega_+ \subset \mathbb{R}^3$ be a Lipschitz domain, Ω_- be the Lipschitz domain $\mathbb{R}^3 \setminus \overline{\Omega}$ and $m \in \mathbb{R}$. The linear operator $T_{\max}$ is defined by

$$\begin{aligned} \operatorname{dom} T_{\max} &:= \left\{ f \in L^2(\mathbb{R}^3; \mathbb{C}^4) : -i(\alpha \cdot \nabla) f_\pm \in L^2(\Omega_\pm; \mathbb{C}^4) \right\} = H^{0,0}_\alpha(\Omega_+) \oplus H^{0,0}_\alpha(\Omega_-) \\ T_{\max} f &:= (-i(\alpha \cdot \nabla) f_+ + m\beta f_+) \oplus (-i(\alpha \cdot \nabla) f_- + m\beta f_-) \quad \forall f \in \operatorname{dom} T_{\max}, \end{aligned} \tag{4.19}$$

where $f_\pm$ denotes the restriction of f to the domain $\Omega_\pm$.

Remark 4.8. By taking a closer look at the definition of $T_{\max}$ and $H^{0,0}_\alpha(\Omega_\pm)$ we realize that the graph norm of $T_{\max}$ is equivalent to the $H^{0,0}_\alpha(\Omega_+) \oplus H^{0,0}_\alpha(\Omega_-)$-norm.

Definition 4.9. Let $T := T_{\max} \restriction H^{1/2,0}_\alpha(\Omega_+) \oplus H^{1/2,0}_\alpha(\Omega_-)$, i.e.

$$\begin{aligned} \operatorname{dom} T &= H^{1/2,0}_\alpha(\Omega_+) \oplus H^{1/2,0}_\alpha(\Omega_-) \\ Tf &= (-i(\alpha \cdot \nabla) + m\beta)\, f_+ \oplus (-i(\alpha \cdot \nabla) + m\beta)\, f_- \quad \forall f \in \operatorname{dom} T. \end{aligned} \tag{4.20}$$

In this setting we are able to use the trace operator due to Theorem 2.36. Thus, we can introduce the two boundary mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} T \to L^2(\Sigma; \mathbb{C}^4)$ given by

$$\Gamma_0 f := i(\alpha \cdot \nu)\, (\boldsymbol{t}_\Sigma f_+ - \boldsymbol{t}_\Sigma f_-) \text{ and } \Gamma_1 f := \frac{1}{2}\, (\boldsymbol{t}_\Sigma f_+ + \boldsymbol{t}_\Sigma f_-) \text{ for } f \in \operatorname{dom} T. \tag{4.21}$$

Remark 4.10. We defined T in a different way as Holzmann in [22, eq. (4.1)] where $\mathcal{C}^2$ domains were considered and the domain of T was chosen as $H^1(\Omega_+; \mathbb{C}^4) \oplus H^1(\Omega_-; \mathbb{C}^4)$. However, for Lipschitz domains one can not expect the existence of self-adjoint extensions because of [25]. There, it is shown in Theorem 1.2. that in non-convex sectors the two-dimensional Dirac operator with boundary conditions does not admit a self-adjoint extension in Sobolev spaces of order one.

Our next goal is to show that $(L^2(\Sigma;\mathbb{C}^4),\Gamma_0,\Gamma_1)$ is a quasi boundary triple for S^*, where $S := T \upharpoonright (\ker\Gamma_0 \cap \ker\Gamma_1)$. Lemma 4.11 shows the triple $(L^2(\Sigma;\mathbb{C}^4),\Gamma_0,\Gamma_1)$ satisfies (i) and (ii) in Definition 1.11.

Lemma 4.11. *The three operators T, Γ_0 and Γ_1 fulfil Green's identity*

$$(Tf,g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} - (f,Tg)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = (\Gamma_1 f,\Gamma_0 g)_{L^2(\Sigma;\mathbb{C}^4)} - (\Gamma_0 f,\Gamma_1 g)_{L^2(\Sigma;\mathbb{C}^4)} \tag{4.22}$$

for all $f,g \in \operatorname{dom} T$. Furthermore,

$$\operatorname{ran}(\Gamma_0,\Gamma_1)^T \supset H^{1/2}(\Sigma;\mathbb{C}^4) \times H^{1/2}_\alpha(\Sigma) \tag{4.23}$$

with $H^{1/2}_\alpha(\Sigma) := \{h \in L^2(\Sigma;\mathbb{C}^4) : (\alpha\cdot\nu)h \in H^{1/2}(\Sigma;\mathbb{C}^4)\}$.

Proof. First, we prove Green's identity. Corollary 2.37 yields

$$\begin{aligned}
(Tf,g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} =& i((\alpha\cdot\nabla)f_+,g_+)_{L^2(\Omega_+;\mathbb{C}^4)} + (m\beta f_+,g_+)_{L^2(\Omega_+;\mathbb{C}^4)}\\
&+ i((\alpha\cdot\nabla)f_-,g_-)_{L^2(\Omega_-;\mathbb{C}^4)} + (m\beta f_-,g_-)_{L^2(\Omega_-;\mathbb{C}^4)}\\
=& - i(f_+,(\alpha\cdot\nabla)g_+)_{L^2(\Omega_+;\mathbb{C}^4)} + i((\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_+,\boldsymbol{t}_\Sigma g_+)_{L^2(\Sigma;\mathbb{C}^4)}\\
&+ (f_+,m\beta g_+)_{L^2(\Omega_+;\mathbb{C}^4)}\\
&- i(f_-,(\alpha\cdot\nabla)g_-)_{L^2(\Omega_-;\mathbb{C}^4)} - i((\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_-,\boldsymbol{t}_\Sigma g_-)_{L^2(\Sigma;\mathbb{C}^4)}\\
&+ (f_-,m\beta g_-)_{L^2(\Omega_-;\mathbb{C}^4)}\\
=&(f,Tg)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} + i((\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_+,\boldsymbol{t}_\Sigma g_+)_{L^2(\Sigma;\mathbb{C}^4)}\\
&- i((\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_-,\boldsymbol{t}_\Sigma g_-)_{L^2(\Sigma;\mathbb{C}^4)}
\end{aligned} \tag{4.24}$$

for all $f,g \in \operatorname{dom} T$. Moreover, simple calculations and the self-adjointness of $(\alpha\cdot\nu)$ show

$$\begin{aligned}
&(\Gamma_1 f,\Gamma_0 g)_{L^2(\Sigma;\mathbb{C}^4)} - (\Gamma_0 f,\Gamma_1 g)_{L^2(\Sigma;\mathbb{C}^4)}\\
&\quad= \frac{1}{2}\Big((\boldsymbol{t}_\Sigma f_+, i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma g_+)_{L^2(\Sigma;\mathbb{C}^4)} - (\boldsymbol{t}_\Sigma f_+, i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma g_-)_{L^2(\Sigma;\mathbb{C}^4)}\\
&\qquad+ (\boldsymbol{t}_\Sigma f_-, i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma g_+)_{L^2(\Sigma;\mathbb{C}^4)} - (\boldsymbol{t}_\Sigma f_-, i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma g_-)_{L^2(\Sigma;\mathbb{C}^4)}\Big)\\
&\qquad- \frac{1}{2}\Big((i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_+,\boldsymbol{t}_\Sigma g_+)_{L^2(\Sigma;\mathbb{C}^4)} + (i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_+,\boldsymbol{t}_\Sigma g_-)_{L^2(\Sigma;\mathbb{C}^4)}\\
&\qquad- (i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_-,\boldsymbol{t}_\Sigma g_+)_{L^2(\Sigma;\mathbb{C}^4)} - (i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_-,\boldsymbol{t}_\Sigma g_-)_{L^2(\Sigma;\mathbb{C}^4)}\Big)\\
&\quad= i((\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_+,\boldsymbol{t}_\Sigma g_+)_{L^2(\Sigma;\mathbb{C}^4)} - i((\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_-,\boldsymbol{t}_\Sigma g_-)_{L^2(\Sigma;\mathbb{C}^4)}
\end{aligned} \tag{4.25}$$

for all $f,g \in \operatorname{dom} T$. Thus, (4.22) holds true.
Now, let us prove (4.23). We choose $\psi \in H^{1/2}(\Sigma;\mathbb{C}^4)$ and $\varphi \in H^{1/2}_\alpha(\Sigma)$. Therefore, $(\alpha\cdot\nu)\varphi \in H^{1/2}(\Sigma;\mathbb{C}^4)$ and Theorem 2.25 imply the existence of $f \in H^1(\Omega_+;\mathbb{C}^4)$ with

$$\boldsymbol{t}_\Sigma f = -i(\alpha\cdot\nu)\varphi. \tag{4.26}$$

Moreover, using Theorem 2.25 and [28, Theorem A.4] yields the existence of a function $g \in H^1(\mathbb{R}^3; \mathbb{C}^4)$ such that

$$\boldsymbol{t}_\Sigma g = \psi - \frac{1}{2}\boldsymbol{t}_\Sigma f. \tag{4.27}$$

Defining $u := (f + g_+) \oplus g_- \in H^1(\Omega_+; \mathbb{C}^4) \oplus H^1(\Omega_-; \mathbb{C}^4) \subset \operatorname{dom} T$ leads to

$$\begin{aligned}\Gamma_0 u =& i(\alpha\cdot\nu)\left(\boldsymbol{t}_\Sigma u_+ - \boldsymbol{t}_\Sigma u_-\right) = i(\alpha\cdot\nu)\left(\boldsymbol{t}_\Sigma (f+g_+) - \boldsymbol{t}_\Sigma g_-\right) = i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma f = \varphi \text{ and}\\ \Gamma_1 u =& \frac{1}{2}\left(\boldsymbol{t}_\Sigma u_+ + \boldsymbol{t}_\Sigma u_-\right) = \frac{1}{2}\left(\boldsymbol{t}_\Sigma (f+g_+) + \boldsymbol{t}_\Sigma g_-\right) = \frac{1}{2}\Big(\boldsymbol{t}_\Sigma f + \psi - \frac{1}{2}\boldsymbol{t}_\Sigma f + \psi - \frac{1}{2}\boldsymbol{t}_\Sigma f\Big)\\ =& \psi\end{aligned} \tag{4.28}$$

which concludes the proof. □

Obviously, (i) in Definition 1.11 is valid. Since $H^{1/2}(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$, one can see by the construction of $H^{1/2}(\Sigma; \mathbb{C}^4)$ that $H^{1/2}(\Sigma; \mathbb{C}^4)$ is also densely contained in $L^2(\Sigma; \mathbb{C}^4)$. Furthermore, $H^{1/2}_\alpha(\Sigma)$ is also densely contained in $L^2(\Sigma; \mathbb{C}^4)$ due to the properties of $(\alpha \cdot \nu)$. All in all, (ii) in Definition 1.11 is fulfilled.

Lemma 4.12. *The identity*

$$\ker \Gamma_0 = H^1(\mathbb{R}^3; \mathbb{C}^4) \tag{4.29}$$

holds.

Proof. The inclusion $H^1(\mathbb{R}^3; \mathbb{C}^4) \subset \ker \Gamma_0$ is obvious. In order to prove the inclusion $\ker \Gamma_0 \subset H^1(\mathbb{R}^3; \mathbb{C}^4)$, we choose $f \in \ker \Gamma_0$. We employ Green's identity and observe

$$(Tf, g)_{L^2(\mathbb{R}^3)} - (f, Tg)_{L^2(\mathbb{R}^3)} = 0 \quad \forall g \in \mathcal{D}(\mathbb{R}^3). \tag{4.30}$$

This is equivalent to

$${}_{\mathcal{D}^*(\mathbb{R}^3)}\langle (\alpha\cdot\nabla) f_+ \oplus (\alpha\cdot\nabla) f_-, g\rangle_{\mathcal{D}(\mathbb{R}^3;\mathbb{C}^4)} + {}_{\mathcal{D}^*(\mathbb{R}^3)}\langle f, (\overline{\alpha}\cdot\nabla) g\rangle_{\mathcal{D}(\mathbb{R}^3)} = 0 \tag{4.31}$$

for all $g \in \mathcal{D}(\mathbb{R}^3; \mathbb{C}^4)$. Therefore, $(\alpha\cdot\nabla) f = (\alpha\cdot\nabla) f_+ \oplus (\alpha\cdot\nabla) f_- \in L^2(\mathbb{R}^3; \mathbb{C}^4)$. Now, Lemma 2.33 shows $f \in H^1(\mathbb{R}^3; \mathbb{C}^4)$. □

Due to Lemma 4.12, we observe $T \upharpoonright \ker \Gamma_0 = A_0$ where A_0, the free Dirac operator, is self-adjoint. This implies (iii) in Definition 1.11. Through Green's identity one notices $S := T \upharpoonright (\ker \Gamma_0 \cap \ker \Gamma_1)$ is a symmetric operator. Simple considerations and Corollary 2.38 lead to $\operatorname{dom} S = H^1_0(\Omega_+; \mathbb{C}^4) \oplus H^1_0(\Omega_-; \mathbb{C}^4)$. Moreover, due to Lemma 2.33, it is easy to see that the graph norm of S is equivalent to the norm corresponding to $\operatorname{dom} S = H^1_0(\Omega_+; \mathbb{C}^4) \oplus H^1_0(\Omega_-; \mathbb{C}^4)$. Therefore, S is closed. To conclude that $(L^2(\Sigma; \mathbb{C}^4), \Gamma_0, \Gamma_1)$ is a quasi boundary triple, we still need to show that S is densely defined, closed and $S^* = \overline{T}$.

Lemma 4.13. *The linear operator* $S := T \upharpoonright (\ker \Gamma_0 \cap \ker \Gamma_1)$ *is densely defined, closed and symmetric in* $L^2(\mathbb{R}^3; \mathbb{C}^4)$*. Moreover,* $S^* = \overline{T} = T_{\max}$*, with* $T_{\max}$ *and* T *defined in Definition 4.7 and Definition 4.9, respectively.*

Proof. We already know that S is closed and symmetric from the above comments. The inclusion $\mathcal{D}(\Omega_+; \mathbb{C}^4) \oplus \mathcal{D}(\Omega_-; \mathbb{C}^4) \subset H_0^1(\Omega_+; \mathbb{C}^4) \oplus H_0^1(\Omega_-; \mathbb{C}^4) = \operatorname{dom} S$ shows that S is densely defined in $L^2(\mathbb{R}^3; \mathbb{C}^4)$. It remains to prove $S^* = T_{\max}$. Let us start with $S^* \subset T_{\max}$. We choose $f \in \operatorname{dom} S^*$. Then,

$$(Sg, f)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = (g, S^* f)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \quad \forall g \in \operatorname{dom} S = H_0^1(\Omega_+; \mathbb{C}^4) \oplus H_0^1(\Omega_-; \mathbb{C}^4). \tag{4.32}$$

Consequently,

$$((Sg)_\pm, f_\pm)_{L^2(\Omega_\pm;\mathbb{C}^4)} = (g_\pm, (S^* f)_\pm)_{L^2(\Omega_\pm;\mathbb{C}^4)} \quad \forall g \in H_0^1(\Omega_\pm; \mathbb{C}^4) \tag{4.33}$$

which leads to

$${}_{\mathcal{D}^*(\Omega_\pm;\mathbb{C}^4)}\langle (\overline{\alpha} \cdot \nabla) g_\pm, f_\pm \rangle_{\in \mathcal{D}(\Omega_\pm;\mathbb{C}^4)} = {}_{\mathcal{D}(\Omega_\pm;\mathbb{C}^4)} \langle g_\pm, -i(S^* f)_\pm + im\beta f_\pm \rangle_{\mathcal{D}(\Omega_\pm;\mathbb{C}^4)} \tag{4.34}$$

for all $g \in \mathcal{D}(\Omega_\pm; \mathbb{C}^4)$. This yields $(\alpha \cdot \nabla) f_\pm = i(S^* f)_\pm - im\beta f_\pm \in L^2(\Omega_\pm; \mathbb{C}^4)$. Therefore, $f \in \operatorname{dom} T_{\max}$ and $(S^* f)_\pm = -i(\alpha \cdot \nabla) f_\pm + m\beta f_\pm = (T_{\max} f)_\pm$. Hence, $S^* f = T_{\max} f$. Now, let us take $f \in \operatorname{dom} T_{\max}$. Then,

$$((\alpha \cdot \nabla)_\pm f, g_\pm)_{L^2(\Omega\pm;\mathbb{C}^4)} = -(f_\pm, (\alpha \cdot \nabla)_\pm g)_\pm)_{L^2(\Omega_\pm;\mathbb{C}^4)} \quad \forall g \in \mathcal{D}(\Omega_\pm; \mathbb{C}^4) \tag{4.35}$$

and by density also

$$((\alpha \cdot \nabla) f_\pm, g_\pm)_{L^2(\Omega\pm;\mathbb{C}^4)} = -(f_\pm, (\alpha \cdot \nabla) g_\pm)_{L^2(\Omega_\pm;\mathbb{C}^4)} \quad \forall g \in H_0^1(\Omega_\pm; \mathbb{C}^4). \tag{4.36}$$

Elementary transformations lead to

$$\begin{aligned}(T_{\max} f, g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} &= (f, T_{\max} g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\ &= (f, Sg)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \quad \forall g \in H_0^1(\Omega_+; \mathbb{C}^4) \oplus H_0^1(\Omega_-; \mathbb{C}^4) = \operatorname{dom} S\end{aligned} \tag{4.37}$$

which shows $S^* \supset T_{\max}$. In order to prove $\overline{T} = T_{\max}$, we apply Theorem 2.34 to obtain

$$\begin{aligned}&\overline{H_\alpha^{1/2,0}(\Omega_+) \oplus H_\alpha^{1/2,0}(\Omega_-)}^{\|\cdot\|_{H_\alpha^{0,0}(\Omega_+) \oplus H_\alpha^{0,0}(\Omega_-)}} \\ &\quad = \overline{\mathcal{C}_0^\infty(\overline{\Omega_+}; \mathbb{C}^4) \oplus \mathcal{C}_0^\infty(\overline{\Omega_-}; \mathbb{C}^4)}^{\|\cdot\|_{H_\alpha^{0,0}(\Omega_+) \oplus H_\alpha^{0,0}(\Omega_-)}} = H_\alpha^{0,0}(\Omega_+) \oplus H_\alpha^{0,0}(\Omega_-).\end{aligned} \tag{4.38}$$

Together with $\operatorname{dom} T = H_\alpha^{1/2,0}(\Omega_+) \oplus H_\alpha^{1/2,0}(\Omega_-)$ and Remark 4.8 equation (4.38) verifies $\overline{T} = T_{\max}$. □

Finally, we are able to prove that $(L^2(\Sigma; \mathbb{C}^4), \Gamma_0, \Gamma_1)$ is indeed a quasi boundary triple.

Theorem 4.14. *The triple* $(L^2(\Sigma;\mathbb{C}^4),\Gamma_0,\Gamma_1)$ *is a quasi boundary triple for*

$$T_{\max} = S^* = \overline{T}. \tag{4.39}$$

Proof. With Lemma 4.11, Lemma 4.12, Lemma 4.13 and the properties of the free Dirac operator A_0 in mind, we see that the requirements of Definition 1.11 are met. □

After constructing the quasi boundary triple $(L^2(\Sigma;\mathbb{C}^4),\Gamma_0,\Gamma_1)$, we study two operator-valued functions, namely the γ-field and the Weyl function M. Recalling Definition 1.12, we see if $\lambda \in \rho(A_0)$, then

$$\operatorname{dom} T = \operatorname{dom} A_0 \dot{+} \ker (T-\lambda) = \ker \Gamma_0 \dot{+} \ker (T-\lambda) \tag{4.40}$$

as well as

$$\gamma(\lambda) := (\Gamma_0 \upharpoonright \ker (T-\lambda))^{-1} \text{ and } M(\lambda) := \Gamma_1\gamma(\lambda) = \Gamma_1 \left(\Gamma_0 \upharpoonright \ker (T-\lambda)\right)^{-1}. \tag{4.41}$$

Theorem 4.15 shows $\gamma(\lambda)$ and $M(\lambda)$ are bounded operators which are closely related to Φ_λ and $\mathcal{C}_\lambda$, respectively. The definitions of Φ_λ and $\mathcal{C}_\lambda$ can be found in Theorem 3.12 and Theorem 3.16.

Theorem 4.15. *Let* $(L^2(\Sigma;\mathbb{C}^4),\Gamma_0,\Gamma_1)$ *be the quasi boundary triple for* $S^* = T_{\max}$, $\lambda \in \rho(A_0) = \mathbb{C}\backslash((-\infty,-|m|] \cup [|m|,\infty))$ *and* G_λ *be the integral kernel of the resolvent of the free Dirac operator. Then, the following statements hold:*

(i) The values of the γ*-field coincide with* Φ_λ*. Furthermore, the adjoint* $\gamma(\lambda)^* = \Phi_\lambda^*$ *is bounded from* $L^2(\mathbb{R}^3;\mathbb{C}^4)$ *to* $H^{1/2}(\Sigma;\mathbb{C}^4)$*.*

(ii) The values of the Weyl function M *coincide with* $\mathcal{C}_\lambda$*.*

Proof. We start proving $\gamma(\lambda) = \Phi_\lambda$ by choosing $f \in L^2(\Sigma;\mathbb{C}^4)$ and observe that $g := \Phi_\lambda f \in \operatorname{dom} T = \operatorname{dom} \Gamma_0$ due to Theorem 3.15 and (3.63). Theorem 2.36 *(i)* and Theorem 3.16 *(i)* give us

$$\Gamma_0 g = i(\alpha\cdot\nu)(\boldsymbol{t}_\Sigma g_+ - \boldsymbol{t}_\Sigma g_-) = i(\alpha\cdot\nu)(\mathcal{P}_\lambda^+ f - \mathcal{P}_\lambda^- f) = i(\alpha\cdot\nu)(-i)(\alpha\cdot\nu)f = f. \tag{4.42}$$

Thus, $\operatorname{ran}\Gamma_0 = L^2(\Sigma;\mathbb{C}^4)$ and consequently $\operatorname{dom}\gamma(\lambda) = L^2(\Sigma;\mathbb{C}^4)$. Furthermore, due to

$$(-i(\alpha\cdot\nabla) + m\beta - \lambda I_4)\, G_\lambda(z) = 0 \quad \forall z \in \mathbb{R}^3 \setminus \{0\}, \tag{4.43}$$

we conclude $g = \Phi_\lambda f \in \ker(T-\lambda)$ for $f \in L^2(\Sigma;\mathbb{C}^4)$. Therefore, we can apply $\gamma(\lambda)$ in equation (4.42) and obtain

$$\Phi_\lambda f = g = \gamma(\lambda)\Gamma_0 g = \gamma(\lambda) f \quad \forall f \in L^2(\Sigma;\mathbb{C}^4). \tag{4.44}$$

Hence, $\gamma(\lambda) = \Phi_\lambda$ and $\gamma(\lambda)^* = \Phi_\lambda^*$.

We note that $\gamma(\lambda)^* = \Gamma_1(A_0 - \overline{\lambda})^{-1}$, see Theorem 1.13 *(i)*. Thus, $\gamma(\lambda)^*$ is a bounded mapping from $L^2(\mathbb{R}^3; \mathbb{C}^4)$ to $H^{1/2}(\Sigma; \mathbb{C}^4)$.
Last, we prove *(ii)*. Applying Theorem 2.36 *(i)* and Theorem 3.16 *(i)* again yields

$$\begin{aligned} M(\lambda)f = \Gamma_1\gamma(\lambda)f &= \frac{1}{2}\big(\boldsymbol{t}_\Sigma\left(\gamma(\lambda)f\right)_+ + \left(\boldsymbol{t}_\Sigma\gamma(\lambda)f\right)_-\big) \\ &= \frac{1}{2}\big(\boldsymbol{t}_\Sigma\left(\Phi_\lambda f\right)_+ + \boldsymbol{t}_\Sigma\left(\Phi_\lambda f\right)_-\big) \\ &= \frac{1}{2}\big(\mathcal{P}_\lambda^+ f + \mathcal{P}_\lambda^- f\big) = \mathcal{C}_\lambda f \quad \forall f \in L^2(\Sigma; \mathbb{C}^4). \end{aligned} \tag{4.45}$$

. □

5 Dirac Operators with δ-Shell Interactions on the Boundary

In this chapter we discuss the main object of this thesis. This is the Dirac operator with δ-shell interactions, which is formally given by

$$A_{\eta,\tau} = -i(\alpha \cdot \nabla) + m\beta + (\eta I_4 + \tau\beta)\delta_\Sigma \tag{5.1}$$

with $\delta_\Sigma f = \frac{1}{2}(f_+|_\Sigma + f_-|_\Sigma)$. It describes a Dirac operator with δ-shell interactions supported on Lipschitz surfaces. The first section of this chapter deals with the self-adjointness of $A_{\eta,\tau}$ and concludes with Theorem 5.8, which states sufficient conditions for the self-adjointness, gives us a resolvent formula for $A_{\eta,\tau}$ and lists properties of the spectrum of $A_{\eta,\tau}$. Then, in Section 5.2 we examine two special cases which are covered by Theorem 5.8 without assuming additional smoothness of the interface. Finally, in the last section we study the qualitative properties of powers of resolvents. As discussed in the introduction the rigorous definition of $A_{\eta,\tau}$ looks as follows.

Definition 5.1. Let $\eta \in \mathbb{R}$ and $\tau \in \mathbb{R}$. Then, we define the operator

$$A_{\eta,\tau} := T \restriction \ker(\Gamma_0 + (\eta I_4 + \tau\beta)\Gamma_1) \tag{5.2}$$

with T, Γ_0 and Γ_1 from Definition 4.9. Moreover, we can rewrite the definition in the explicit form

$$\begin{aligned} \operatorname{dom} A_{\eta,\tau} := \Big\{ f = f_+ \oplus f_- \in H^{1/2}(\Omega_+;\mathbb{C}^4) \oplus H^{1/2}(\Omega_-;\mathbb{C}^4) : (\alpha\cdot\nabla) f_\pm \in L^2(\Omega_\pm;\mathbb{C}^4) \\ \text{and } f \text{ fulfils } i(\alpha\cdot\nu)(\boldsymbol{t}_\Sigma f_+ - \boldsymbol{t}_\Sigma f_-) + (\eta I_4 + \tau\beta)\frac{1}{2}(\boldsymbol{t}_\Sigma f_+ + \boldsymbol{t}_\Sigma f_-) = 0 \Big\} \\ A_{\eta,\tau} f := (-i(\alpha\cdot\nabla) + m\beta)\, f_+ \oplus (-i(\alpha\cdot\nabla) + m\beta)\, f_- \quad \forall f \in \operatorname{dom} A_{\eta,\tau}. \end{aligned} \tag{5.3}$$

5.1 Self-Adjointness and Spectral Properties of $A_{\eta,\tau}$

Our goal is to show that $A_{\eta,\tau}$ is self-adjoint and to study its spectral properties. Using Green's identity and the self-adjointness of $\eta I_4 + \tau\beta$, we see

$$\begin{aligned} (A_{\eta,\tau} f, g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} - (f, A_{\eta,\tau} g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} &= (\Gamma_1 f, \Gamma_0 g)_{L^2(\Sigma;\mathbb{C}^4)} - (\Gamma_0 f, \Gamma_1 g)_{L^2(\Sigma;\mathbb{C}^4)} \\ &= -(\Gamma_1 f, (\eta I_4 + \tau\beta)\Gamma_1 g)_{L^2(\Sigma;\mathbb{C}^4)} + ((\eta I_4 + \tau\beta)\Gamma_1 f, \Gamma_1 g)_{L^2(\Sigma;\mathbb{C}^4)} = 0 \end{aligned} \tag{5.4}$$

for all $f, g \in \operatorname{dom} A_{\eta,\tau}$. Hence, $A_{\eta,\tau}$ is symmetric. For the purpose of studying the self-adjointness of $A_{\eta,\tau}$ it is crucial to study the operator $M(\lambda)^2 - kI_4$ with $M(\lambda)$ introduced in (4.41).

Definition 5.2. Let A_0 be the free Dirac operator introduced in Definition 4.1. Then, we define the set

$$\mathcal{F}_0 := \left\{ k \in \mathbb{C} : \operatorname{index}(M(\lambda)^2 - kI_4) = 0 \quad \forall \lambda \in \rho(A_0) \right\}. \tag{5.5}$$

Remark 5.3. In order to show $k \in \mathcal{F}_0$, it suffices to prove $M(\lambda_0)^2 - kI_4$ has Fredholm index zero for one $\lambda_0 \in \rho(A_0)$ since the difference

$$M(\lambda_0)^2 - M(\lambda)^2 = (M(\lambda_0) - M(\lambda))(M(\lambda_0) + M(\lambda)) \tag{5.6}$$

is compact and due to Theorem 1.2, compact perturbations do not change the Fredholm index. The compactness follows from $M(\lambda) = \mathcal{C}_\lambda$, Theorem 3.16 *(iii)* and Theorem 1.2. In the same way, it suffices to show $\mathcal{K}_\alpha^2 - kI_4$ has Fredholm index zero with $\mathcal{K}_\alpha$ defined by (3.100).

The following lemma gives us a simple yet helpful criteria for k being in $\mathcal{F}_0$.

Lemma 5.4. *For any $\lambda_0 \in \rho(A_0)$ holds $\rho(M(\lambda_0)^2) \cup \rho(\mathcal{K}_\alpha^2) \subset \mathcal{F}_0$. Here, $\mathcal{K}_\alpha$ denotes the operator defined by* (3.100).

Proof. Let k be in $\rho(M(\lambda_0)^2)$. We know $\operatorname{dom} M(\lambda_0)^2 = L^2(\Sigma; \mathbb{C}^4)$. Hence, the operator $M(\lambda_0)^2 - kI_4$ is isomorphic in $L^2(\Sigma; \mathbb{C}^4)$. Moreover, $\operatorname{index}(M(\lambda_0)^2 - kI_4) = 0$. Consequently, Remark 5.3 yields $k \in \mathcal{F}_0$. The proof for $k \in \rho(\mathcal{K}_\alpha^2)$ works out analogously. □

We apply Lemma 5.4 in the next two lemmas.

Lemma 5.5. *Let*

$$M_- := \inf_{\lambda \in \rho(A_0)} \left\| M(\lambda)^2 \right\|_{L^2(\Sigma;\mathbb{C}^4) \to L^2(\Sigma;\mathbb{C}^4)}. \tag{5.7}$$

Then, $\mathbb{C} \setminus \left[\frac{1}{16 M_-}, M_- \right] \subset \mathcal{F}_0$.

Proof. At first, due to Corollary 3.17, we notice that $\mathcal{K}_\alpha^2$ is self-adjoint and bounded from below by zero. Therefore, $\mathbb{C} \setminus [0, \infty) \subset \rho(\mathcal{K}_\alpha^2)$. Using Lemma 5.4 we get the inclusion $\mathbb{C} \setminus [0, \infty) \subset \mathcal{F}_0$. Due to $(M(\lambda)(\alpha \cdot \nu))^2 = -\frac{1}{4} I_4$, see item *(iv)* of Theorem 3.16 and item *(ii)* of Theorem 4.15,

$$\begin{aligned} &\left\| M(\lambda)^2 \right\|^2_{L^2(\Sigma;\mathbb{C}^4) \to L^2(\Sigma;\mathbb{C}^4)} \\ &\quad = \left\| M(\lambda)^2 \right\|_{L^2(\Sigma;\mathbb{C}^4) \to L^2(\Sigma;\mathbb{C}^4)} \left\| ((\alpha \cdot \nu) M(\lambda)(\alpha \cdot \nu))^2 \right\|_{L^2(\Sigma;\mathbb{C}^4) \to L^2(\Sigma;\mathbb{C}^4)} \\ &\quad \geq \left\| M(\lambda)^2 ((\alpha \cdot \nu) M(\lambda)(\alpha \cdot \nu))^2 \right\|_{L^2(\Sigma;\mathbb{C}^4) \to L^2(\Sigma;\mathbb{C}^4)} \\ &\quad = \frac{1}{4} \| M(\lambda)(\alpha \cdot \nu) M(\lambda)(\alpha \cdot \nu) \|_{L^2(\Sigma;\mathbb{C}^4) \to L^2(\Sigma;\mathbb{C}^4)} = \frac{1}{16} \quad \forall \lambda \in \rho(A_0). \end{aligned} \tag{5.8}$$

This shows $M_- \geq \frac{1}{4}$ and $M_- \geq \frac{1}{16M_-}$. Now, let $k > M_-$. The definition of M_- implies that there exists a $\lambda_0 \in \rho(A_0)$ such that $k > \|M(\lambda_0)^2\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)}$. Hence,

$$M(\lambda_0)^2 - kI_4 = -k\left(I_4 - \frac{M(\lambda_0)^2}{k}\right) \tag{5.9}$$

is invertible since

$$\left\|\frac{M(\lambda_0)^2}{k}\right\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} < 1. \tag{5.10}$$

Therefore, $k \in \rho(M(\lambda_0)^2)$ and by applying Lemma 5.4 we get $k \in \mathcal{F}_0$. It remains to consider the case $0 < k < \frac{1}{16M_-}$. We observe

$$\begin{aligned}\left\|(M(\lambda)^2)^{-1}\right\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} &= 16\left\|((\alpha\cdot\nu)(M(\lambda)(\alpha\cdot\nu))^2\right\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} \\ &= 16\left\|M(\lambda)^2\right\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)}.\end{aligned} \tag{5.11}$$

Again, there exists a λ_0 such that $0 < k < \frac{1}{16\|M(\lambda_0)^2\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)}}$. We notice

$$M(\lambda_0)^2 - kI_4 = M(\lambda_0)^2(I_4 - k(M(\lambda_0)^2)^{-1}) \tag{5.12}$$

is invertible due to $\|k((M(\lambda_0)^2)^{-1}\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} < 1$. Hence, $k \in \rho(M(\lambda_0)^2)$ and Lemma 5.4 yield $k \in \mathcal{F}_0$. □

For $\mathcal{C}^1$ domains we can formulate a better result. Here, we use the compactness of $(\alpha\cdot\nu)\mathcal{C}_\lambda + \mathcal{C}_\lambda(\alpha\cdot\nu)$, which is proven in Theorem 3.18, as the main ingredient of the proof. The statement reads as follows.

Lemma 5.6. *If Ω is a $\mathcal{C}^1$ domain, then $\mathbb{C}\setminus\left\{\frac{1}{4}\right\} \subset \mathcal{F}_0$.*

Proof. We choose $\lambda \in \rho(A_0)$ and $k \in \mathbb{C}$ such that $k \neq \frac{1}{4}$. Theorem 4.15 *(ii)*, Theorem 3.16 *(iv)* and Theorem 3.18 yield the compactness of

$$M(\lambda)^2 - \frac{1}{4}I = \mathcal{C}_\lambda^2 - \frac{1}{4}I = \mathcal{C}_\lambda^2 + (\mathcal{C}_\lambda(\alpha\cdot\nu))^2 = \mathcal{C}_\lambda(\alpha\cdot\nu)\left((\alpha\cdot\nu)\mathcal{C}_\lambda + \mathcal{C}_\lambda(\alpha\cdot\nu)\right). \tag{5.13}$$

Furthermore, we can write

$$M(\lambda)^2 - k = M(\lambda)^2 - \frac{1}{4}I + \frac{1-4k}{4}I_4 \tag{5.14}$$

with the isomorphic operator $\frac{1-4k}{4}I_4$. Thus, due to Theorem 1.2, $M(\lambda)^2 - k$ has Fredholm index zero. □

Lemma 5.7. *If $\eta^2 = \tau^2$ or $\frac{1}{\eta^2-\tau^2} \in \mathcal{F}_0 \cap \mathbb{R}$, then $I_4 + (\eta I_4 + \tau\beta)M(\lambda)$ is isomorphic for $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

Proof. Let us start with the injectivity. Assume there exists an $f \neq 0 \in L^2(\Sigma;\mathbb{C}^4)$ such that $f + (\eta I_4 + \tau\beta)M(\lambda)f = 0$. We claim that in this case $(\lambda, \gamma(\lambda)f)$ is an eigenpair of $A_{\eta,\tau}$. The definition of $\gamma(\lambda)$, see (4.41), shows $\gamma(\lambda)f \in \ker(T-\lambda)$ and $\Gamma_0\gamma(\lambda)f = f$. Therefore,

$$\Gamma_0\gamma(\lambda)f + (\eta I_4 + \tau\beta)\Gamma_1\gamma(\lambda)f = f + (\eta I_4 + \tau\beta)M(\lambda)f = 0. \tag{5.15}$$

Hence, $(\lambda, \gamma(\lambda)f)$ is indeed an eigenpair of $A_{\eta,\tau}$ which contradicts $\lambda \in \mathbb{C}\setminus\mathbb{R}$. For the surjectivity we study the product

$$(I_4 + (\eta I_4 + \tau\beta)M(\lambda))(I_4 - (\eta I_4 + \tau\beta)M(\lambda)). \tag{5.16}$$

First, we rewrite $I_4 + (\eta I_4 + \tau\beta)M(\lambda)$ in the form $I_4 + M(\lambda)(\eta I_4 - \tau\beta) + K_1$, where $K_1 := \tau(\beta M(\lambda) + M(\lambda)\beta)$ is compact in $L^2(\Sigma;\mathbb{C}^4)$. The compactness of K_1 can be seen from combining Theorem 3.4, Theorem 3.10 *(iii)* and $\alpha_j\beta = -\alpha_j\beta$ for $j = 1,2,3$. Through

$$\begin{aligned} F &:= (I_4 + (\eta I_4 + \tau\beta)M(\lambda))(I_4 - (\eta I_4 + \tau\beta)M(\lambda)) \\ &= (I_4 + M(\lambda)(\eta I_4 - \tau\beta) + K_1)(I_4 - (\eta I_4 + \tau\beta)M(\lambda)) \\ &= I_4 - M(\lambda)(\eta I_4 - \tau\beta)(\eta I_4 + \tau\beta)M(\lambda) + \underbrace{K_1(I_4 - (\eta I_4 + \tau\beta)M(\lambda))}_{=:K_2} \\ &= I_4 - (\eta^2 - \tau^2)M(\lambda)^2 + K_2 \end{aligned} \tag{5.17}$$

one immediately sees $\operatorname{index}(F) = 0$ if $\eta^2 - \tau^2 = 0$. Otherwise, F equals

$$(\eta^2 - \tau^2)\left(\frac{1}{\eta^2 - \tau^2} I_4 - M(\lambda)^2\right) + K_2. \tag{5.18}$$

Due to $\frac{1}{\eta^2-\tau^2} \in \mathcal{F}_0$, the expression $\frac{1}{\eta^2-\tau^2} I_4 + M(\lambda)^2$ has Fredholm index zero. Applying Theorem 1.2 shows us $\operatorname{index}(F) = 0$. We already know $I_4 + (\eta I_4 + \tau\beta)M(\lambda)$ is injective. The same holds for $I_4 - (\eta I_4 + \tau\beta)M(\lambda)$, otherwise it would contradict the symmetry of $A_{-\eta,-\tau}$. Therefore, F is injective. Now, the Fredholm alternative, see Theorem 1.3, implies the surjectivity of F. Hence, also $I_4 + (\eta I_4 + \tau\beta)M(\lambda)$ is surjective. □

Finally, we are able to state the main theorem of this chapter. It states sufficient conditions on η and τ such that $A_{\eta,\tau}$ is self-adjoint. Moreover, properties of the spectrum are listed. In view of the next theorem, we define

$$M_+ := \sup_{\lambda \in (-|m|,|m|)} \|M(\lambda)\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)}. \tag{5.19}$$

We observe $M_+ < \infty$ due to Theorem 3.16 *(ii)*.

Theorem 5.8. *Let* $\eta^2 = \tau^2$ *or* $\frac{1}{\eta^2-\tau^2} \in \mathcal{F}_0 \cap \mathbb{R}$, A_0 *be the free Dirac operator introduced in Definition 4.1, and* $A_{\eta,\tau}$ *be given by* (5.2). *Then, the operator* $A_{\eta,\tau}$ *is self-adjoint and for all* $\lambda \in \rho(A_0) \cap \rho(A_{\eta,\tau})$ *the resolvent is given by*

$$(A_{\eta,\tau} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(I_4 + (\eta I_4 + \tau\beta)M(\lambda))^{-1}(\eta I_4 + \tau\beta)\gamma(\overline{\lambda})^*, \quad (5.20)$$

where the γ*-field and the Weyl function* M *are defined by* (4.41). *Moreover, there holds*

(i) $\sigma_{\text{ess}}(A_{\eta,\tau}) = (-\infty, -|m|] \cup [|m|, \infty)$,

(ii) $\sigma_{\text{disc}}(A_{\eta,\tau})$ *is finite,*

(iii) $\max\{|\eta+\tau|, |\eta-\tau|\} < \frac{1}{M_+} \Rightarrow \sigma_{\text{disc}}(A_{\eta,\tau}) = \emptyset$,

(iv) $\min\{|\eta+\tau|, |\eta-\tau|\} > 4M_+ \Rightarrow \sigma_{\text{disc}}(A_{\eta,\tau}) = \emptyset$ *and*

(v) $\lambda \in \sigma_p(A_{\eta,\tau}) \Leftrightarrow -1 \in \sigma_p((\eta I_4 + \tau\beta)M(\lambda))$.

Proof. The statements about self-adjointness, the resolvent formula (5.20) and assertion *(v)* are direct consequences of Lemma 5.7 and Theorem 1.15.
Proof of *(i)*: Item *(i)* of Theorem 4.15 states that $\gamma(\overline{\lambda})^*$ is bounded as a mapping from $L^2(\mathbb{R}^3;\mathbb{C}^4)$ to $H^{1/2}(\Sigma;\mathbb{C}^4)$. Thus, Theorem 2.27 proves the compactness of $\gamma(\overline{\lambda})^*$ as an operator from $L^2(\mathbb{R}^3;\mathbb{C}^4)$ to $L^2(\Sigma;\mathbb{C}^4)$. Therefore, $(A_{\eta,\tau}-\lambda)^{-1} - (A_0-\lambda)^{-1}$ is compact, implying

$$\sigma_{\text{ess}}(A_{\eta,\tau}) = \sigma_{\text{ess}}(A_0) = \sigma(A_0) = (-\infty, -|m|] \cup [|m|, \infty). \quad (5.21)$$

Proof of *(ii)*: In order to prove *(ii)*, we proceed in the same way as in [10, Theorem 5.4 (ii)]. First, we note

$$\sigma_{\text{disc}}(A_{\eta,\tau}) = \sigma_{\text{disc}}(A_{\eta,\tau}) \cap (-|m|, |m|) \text{ is finite if } \sigma_{\text{disc}}(A_{\eta,\tau}^2) \cap \big[0, m^2\big) \text{ is finite}. \quad (5.22)$$

Therefore, we prove that $\sigma_{\text{disc}}(A_{\eta,\tau}^2)$ is finite. Let $\mathfrak{a}$ be the sesquilinear form defined by

$$\mathfrak{a}[u,v] := (A_{\eta,\tau}u, A_{\eta,\tau}v)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \quad \forall u,v \in \operatorname{dom}\mathfrak{a} := \operatorname{dom} A_{\eta,\tau}. \quad (5.23)$$

One trivially sees that $\mathfrak{a}$ is a densely defined sesquilinear form and bounded from below by zero. Moreover, we notice that as a consequence of the closedness of $A_{\eta,\tau}$, $\mathfrak{a}$ is also closed. Hence, [24, Chapter 6, Theorem 2.1.] shows that $\mathfrak{a}$ is the form associated with the operator $A_{\eta,\tau}^2$. Our goal is to construct a second form $\mathfrak{b}$ such that

$$\mathfrak{a}[u] := \mathfrak{a}[u,u] \geq \mathfrak{b}[u,u] =: \mathfrak{b}[u] \quad \forall u \in \operatorname{dom}\mathfrak{a} \subset \operatorname{dom}\mathfrak{b}. \quad (5.24)$$

If we can show that the operator B associated with $\mathfrak{b}$ has only finite discrete spectrum below the threshold m^2, then also $A_{\eta,\tau}^2$ has only finite discrete spectrum below m^2, c.f. [35, Corollary 12.3]. This would prove the desired statement.

Let us start constructing $\mathfrak{b}$. First, we assume w.l.o.g. that Ω_+ is bounded and choose $R_2 > R_1 > 0$ such that $\Sigma \subset B(0, R_1)$. Now, let $g_1 \in \mathcal{C}^\infty(\mathbb{R}^3)$ have the properties

$$0 \leq g_1 \leq 1 \text{ in } \mathbb{R}^3,\; g_1 = 1 \text{ in } B\left(0, R_1\right) \text{ and } \operatorname{supp} g_1 \subset B\left(0, R_2\right). \tag{5.25}$$

Moreover, we define $g_2 := \sqrt{1 - g_1^2}$ and continue the proof in the same way as in [10, Theorem 5.4 (ii)]. We notice $g_j u \in \operatorname{dom} \mathfrak{a}$ for all $u \in \operatorname{dom} \mathfrak{a}$ and $j = 1, 2$ and see

$$A_{\eta,\tau}(g_j u) = g_j A_{\eta,\tau} u - i((\alpha \cdot \nabla) g_j) u \quad \text{ for } j = 1, 2. \tag{5.26}$$

Applying (5.26) yields

$$\begin{aligned}
\mathfrak{a}[g_j u] =& (g_j A_{\eta,\tau} u - i((\alpha \cdot \nabla) g_j) u, (g_j A_{\eta,\tau} u - i((\alpha \cdot \nabla) g_j) u)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
=& (g_j A_{\eta,\tau} u, g_j A_{\eta,\tau} u)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} + (((\alpha \cdot \nabla) g_j) u), ((\alpha \cdot \nabla) g_j) u)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
& - 2\operatorname{Re} i(g_j A_{\eta,\tau} u, ((\alpha \cdot \nabla) g_j) u)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
=& \left(g_j^2 A_{\eta,\tau} u, A_{\eta,\tau} u\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} + \left(\left|\nabla g_j\right|^2 u, u\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
& - 2\operatorname{Re} i(A_{\eta,\tau} u, ((\alpha \cdot \nabla) g_j) u)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \quad \forall u \in \operatorname{dom} \mathfrak{a} \text{ and } j = 1, 2.
\end{aligned} \tag{5.27}$$

Furthermore, we observe

$$(\alpha \cdot \nabla)(g_1^2 + g_2^2) = 0 \text{ and define } V := (\left|\nabla g_1\right|^2 + \left|\nabla g_2\right|^2). \tag{5.28}$$

Hence, for all $u \in \operatorname{dom} \mathfrak{a}$ one has

$$\begin{aligned}
\mathfrak{a}[u] =& (A_{\eta,\tau} u, A_{\eta,\tau} u)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = \left((g_1^2 + g_2^2) A_{\eta,\tau} u, A_{\eta,\tau} u\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
=& \mathfrak{a}[g_1 u] + \mathfrak{a}[g_2 u] - \left((\left|\nabla g_1\right|^2 + \left|\nabla g_2\right|^2) u, u\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
& + 2i \operatorname{Re}\left(A_{\eta,\tau} u, \left((\alpha \cdot \nabla)(g_1^2 + g_2^2)\right) u\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
=& \mathfrak{a}[g_1 u] + \mathfrak{a}[g_2 u] - (V u, u)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
=& \mathfrak{a}[g_1 u] + \mathfrak{a}[g_2 u] - \left(V (g_1^2 + g_2^2) u, u\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\
=& \mathfrak{a}[g_1 u] - (V g_1 u, g_1 u)_{L^2(B(0,R_2);\mathbb{C}^4)} + \mathfrak{a}[g_2 u] \\
& - (V g_2 u, g_2 u)_{L^2(\mathbb{R}^3 \setminus B(0,R_1);\mathbb{C}^4)}.
\end{aligned} \tag{5.29}$$

We construct two further forms. Namely,

$$\begin{aligned}
\operatorname{dom} \mathfrak{b}_{\text{int}} &:= \left\{u \in L^2(B(0, R_2); \mathbb{C}^4) : \widetilde{u} \in \operatorname{dom} \mathfrak{a}\right\} \\
\mathfrak{b}_{\text{int}}[u, v] &:= \mathfrak{a}[\widetilde{u}, \widetilde{v}] - (V u, v)_{L^2(B(0,R_2);\mathbb{C}^4)} \quad \forall u, v \in \operatorname{dom} \mathfrak{b}_{\text{int}}
\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
\operatorname{dom} \mathfrak{b}_{\text{ext}} &:= H_0^1(\mathbb{R}^3 \setminus B(0, R_1); \mathbb{C}^4) \\
\mathfrak{b}_{\text{ext}}[u, v] &:= \mathfrak{a}[\widetilde{u}, \widetilde{v}] - (V u, v)_{L^2(\mathbb{R}^3 \setminus B(0,R_1);\mathbb{C}^4)} \quad \forall u, v \in \operatorname{dom} \mathfrak{b}_{\text{ext}},
\end{aligned} \tag{5.31}$$

where $\widetilde{(\cdot)}$ denotes the zero extensions of the considered functions. Through a closer look we see

$$\begin{aligned}\mathfrak{b}_{\text{ext}}[u,v] &= ((-i(\alpha\cdot\nabla)+m\beta)u,(-i(\alpha\cdot\nabla)+m\beta)v)_{L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} \\ &\quad -(Vu,v)_{L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} \\ &= \big((-i(\alpha\cdot\nabla)+m\beta)^2u,v\big)_{L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} - (Vu,v)_{L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} \\ &= \big((-\Delta+m^2)u,v\big)_{L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} - (Vu,v)_{L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} \\ &= (\nabla u,\nabla v)_{L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4\otimes\mathbb{C}^3)} + \big((m^2-V)u,v\big)_{L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)}\end{aligned} \tag{5.32}$$

for all $u,v\in\mathcal{D}(\mathbb{R}^3\backslash B(0,R_1);\mathbb{C}^4)$ and by density also for all $u,v\in H_0^1(\mathbb{R}^3\backslash B(0,R_1);\mathbb{C}^4)$. Moreover, V has compact support in $B(0,R_2)\setminus B(0,R_1)$. With the same techniques as in the proof of [11, Proposition 3.9.], one can show that the operator associated with $\mathfrak{b}_{\text{ext}}$, B_{ext}, has finite discrete spectrum below m^2. Next, we consider the form $\mathfrak{b}_{\text{int}}$. The form is obviously sesquilinear and bounded from below by $m_{\mathfrak{b}_{\text{int}}} := -\max V$. In order to prove the closedness of $\mathfrak{b}_{\text{int}}$, we choose a sequence $(u_l)_{l\in\mathbb{N}}$ in $\operatorname{dom}\mathfrak{b}_{\text{int}}$ such that

$$\lim_{l,k\to\infty}\mathfrak{b}_{\text{int}}[u_l-u_k]+(1-m_{\mathfrak{b}_{\text{int}}})\|u_l-u_k\|^2_{L^2(B(0,R_2);\mathbb{C}^4)}=0. \tag{5.33}$$

Thus, $(\widetilde{u_l})_{l\in\mathbb{N}}$ is a sequence in $\operatorname{dom}\mathfrak{a}$ with

$$\lim_{l,k\to\infty}\mathfrak{a}[\widetilde{u_l}-\widetilde{u_k}]=0 \text{ as well as } \lim_{l,k\to\infty}\|\widetilde{u_l}-\widetilde{u_k}\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)}=0. \tag{5.34}$$

Using the closedness of $\mathfrak{a}$ yields the existence of w in $\operatorname{dom}\mathfrak{a}$ with

$$\lim_{l\to\infty}\mathfrak{a}[\widetilde{u_l}-w]+\|\widetilde{u_l}-w\|^2_{L^2(\mathbb{R}^3;\mathbb{C}^4)}=0. \tag{5.35}$$

Since $\widetilde{u_l}=0$ a.e. in $\mathbb{R}^3\setminus B(0,R_2)$, we see through the L^2 convergence that $w=0$ a.e. on $\mathbb{R}^3\setminus B(0,R_2)$. Combining these results and defining $u:=w|_{B(0,R_2)}$ leads to

$$u\in\operatorname{dom}\mathfrak{b}_{\text{int}} \text{ and } \lim_{l\to\infty}\mathfrak{b}_{\text{int}}[u_l-u]+(1-m_{\mathfrak{b}_{\text{int}}})\|u_l-u\|^2_{L^2(B(0,R_2);\mathbb{C}^4)}=0. \tag{5.36}$$

Therefore, $\mathfrak{b}_{\text{int}}$ is closed. Now, $\operatorname{dom}\mathfrak{b}_{\text{int}}\subset H^{1/2}(\Omega_+;\mathbb{C}^4)\oplus H^{1/2}(\Omega_-\cap B(0,R_2);\mathbb{C}^4)$ and [28, Theorem 3.27] show the compactness of the resolvent of B_{int}. Thus, B_{int} only has discrete spectrum which only accumulates at infinity. Hence, the discrete spectrum below m^2 of B_{int} is finite. All in all, $\sigma_{\text{disc}}(B_{\text{int}}\oplus B_{\text{ext}})\subset\sigma_{\text{disc}}(B_{\text{int}})\cup\sigma_{\text{disc}}(B_{\text{ext}})$ proves that the operator $B_{\text{int}}\oplus B_{\text{ext}}$, which is the operator corresponding to $\mathfrak{b}_{\text{int}}\oplus\mathfrak{b}_{\text{ext}}$, has only finitely many discrete eigenvalues below m^2. We define the operator

$$\begin{aligned}U: L^2(\mathbb{R}^3;\mathbb{C}^4)&\to L^2(B(0,R_2);\mathbb{C}^4)\oplus L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)\\ u&\mapsto g_1u\oplus g_2u.\end{aligned} \tag{5.37}$$

One can check that $\operatorname{ran} U$ is a closed subspace in $L^2(B(0,R_2);\mathbb{C}^4)\oplus L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)$ and that U is a unitary operator as a mapping from $L^2(\mathbb{R}^3;\mathbb{C}^4)$ to $\operatorname{ran} U$. We introduce an auxiliary form

$$\mathfrak{b}_U[u,v] := \mathfrak{b}_{\text{int}} \oplus \mathfrak{b}_{\text{ext}}[u,v] \text{ for } u,v \in \operatorname{dom} \mathfrak{b}_U := \operatorname{ran} U \cap (\operatorname{dom} \mathfrak{b}_{\text{int}} \oplus \operatorname{dom} \mathfrak{b}_{\text{ext}}). \tag{5.38}$$

This form is sesquilinear and bounded from below by construction. Next, we check that $\mathfrak{b}_U$ is densely defined in $\operatorname{ran} U$. Therefore, we choose $u \in \operatorname{ran} U$. Then, there exists a sequence $(u_l)_{l\in\mathbb{N}}$ in $L^2(B(0,R_2);\mathbb{C}^4) \oplus L^2(\mathbb{R}^3 \setminus B(0,R_1);\mathbb{C}^4)$ converging to u. Defining $w_l := UU^*u_l \in \operatorname{ran} U$ and recalling $U^*Uu = u$ yields

$$\begin{aligned} \|w_l - u\|_{\operatorname{ran} U} &= \|U^*Uu_l - U^*Uu\|_{L^2(B(0,R_2);\mathbb{C}^4)\oplus L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} \\ &\le \|u_l - u\|_{L^2(B(0,R_2);\mathbb{C}^4)\oplus L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} \xrightarrow{l\to\infty} 0. \end{aligned} \tag{5.39}$$

This shows that $\mathfrak{b}_U$ is densely defined. Since convergence in the norm induced by

$$\|\cdot\|_{L^2(B(0,R_2);\mathbb{C}^4)\oplus L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)} + (1 - m_{\mathfrak{b}_{\text{int}}} - m_{\mathfrak{b}_{\text{ext}}})\mathfrak{b}_{\text{int}} \oplus \mathfrak{b}_{\text{ext}}[\cdot], \tag{5.40}$$

where $m_{\mathfrak{b}_{\text{ext}}}$ denotes the lower bound of the sesquilinear form $\mathfrak{b}_{\text{ext}}$, implies convergence in the $\|\cdot\|_{L^2(B(0,R_2);\mathbb{C}^4)\oplus L^2(\mathbb{R}^3\setminus B(0,R_1);\mathbb{C}^4)}$-norm, the closedness of $\mathfrak{b}_U$ follows from the closedness of $\mathfrak{b}_{\text{int}} \oplus \mathfrak{b}_{\text{ext}}$ and $\operatorname{ran} U$. Hence, $\mathfrak{b}_U$ induces a self-adjoint operator B_U in the Hilbert space $\operatorname{ran} U$. Furthermore, [35, Corollary 12.3] states that B_U also only has finite discrete spectrum below m^2. The last quadratic form we need in this proof is defined by

$$\mathfrak{b}[u,v] := \mathfrak{b}_U[Uu, Uv] \text{ for } u,v \in \operatorname{dom} \mathfrak{b} := U^*(\operatorname{dom} \mathfrak{b}_U). \tag{5.41}$$

The operator B corresponding to $\mathfrak{b}$ has the form $B = U^*B_UU$. Since U is a unitary operator from $L^2(\mathbb{R}^3;\mathbb{C}^4)$ to $\operatorname{ran} U$, the discrete spectra of B and B_U coincide, implying B has only finitely many discrete eigenvalues below m^2. We obtain from (5.29)

$$\mathfrak{a}[u] = \mathfrak{b}[u] \quad \forall u \in \operatorname{dom} \mathfrak{a} \cap \operatorname{dom} \mathfrak{b}. \tag{5.42}$$

It remains to prove $\operatorname{dom} \mathfrak{a} \subset \operatorname{dom} \mathfrak{b}$. Let $u \in \operatorname{dom} \mathfrak{a} \subset H^{1/2,0}_\alpha(\Omega_+) \oplus H^{1/2,0}_\alpha(\Omega_-)$. We immediately see $g_1 u \in \operatorname{dom} \mathfrak{b}_{\text{int}}$. Moreover, $g_2 u \in H^{1/2,0}_\alpha(\mathbb{R}^3 \setminus B(0,R_1))$ and $\boldsymbol{t}_{\partial(\mathbb{R}^3\setminus B(0,R_1))} g_2 u = 0$. Thus, $g_2 u \in H^1_0(\mathbb{R}^3 \setminus B(0,R_1)) = \operatorname{dom} \mathfrak{b}_{\text{ext}}$ by applying Corollary 2.38. We observe

$$U^*(w_1 \oplus w_2) = g_1\widetilde{w_1} + g_2\widetilde{w_2} \quad \forall (w_1 \oplus w_2) \in L^2(B(0,R_2);\mathbb{C}^4) \oplus L^2(\mathbb{R}^3 \setminus B(0,R_1);\mathbb{C}^4). \tag{5.43}$$

Therefore, $u = U^*(g_1 u \oplus g_2 u) \in \operatorname{dom} \mathfrak{b}$ which proves $\operatorname{dom} \mathfrak{a} \subset \operatorname{dom} \mathfrak{b}$ and thereby also concludes the proof of *(ii)*.
In the remaining two parts of the proof we assume $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(A_{\eta,\tau}) = (-|m|, |m|)$.

Proof of *(iii)*: One easily sees $\eta I_4 + \tau\beta$ is a diagonal matrix with diagonal entries $\eta + \tau$ and $\eta - \tau$. Therefore, if $\max\{|\eta+\tau|, |\eta-\tau|\} < \frac{1}{M_+}$, then $|\eta I_4 + \tau\beta| < \frac{1}{M_+}$ and

$$\|(\eta I_4 + \tau\beta)M(\lambda)\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} < 1 \quad \text{for } \lambda \in (-|m|, |m|). \tag{5.44}$$

Hence, $-1 \in \rho((\eta I_4 + \tau\beta)M(\lambda))$. Moreover, we obtain $\lambda \notin \sigma_{\mathrm{disc}}(A_{\eta,\tau})$ from Theorem 1.15.

Proof of *(iv)*: We immediately realize that under the given assumptions $(\eta I_4 + \tau\beta)M(\lambda)$ is invertible and $\|(M(\lambda))^{-1}\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} = 4\|M(\lambda)\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} \leq 4M_+$. Furthermore,

$$I_4 - (\eta I_4 + \tau\beta)M(\lambda) = (\eta I_4 + \tau\beta)M(\lambda)\left((M(\lambda))^{-1}(\eta I_4 + \tau\beta)^{-1} - I_4\right). \tag{5.45}$$

The norm of $(\eta I_4 + \tau\beta)^{-1}$ equals $\frac{1}{\min\{|\eta+\tau|,|\eta-\tau|\}} < \frac{1}{4M_+}$. This leads to

$$\left\|(M(\lambda))^{-1}(\eta I_4 + \tau\beta)^{-1}\right\|_{L^2(\Sigma;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} < 1. \tag{5.46}$$

Thus, $-1 \in \rho((\eta I_4 + \tau\beta)M(\lambda))$. Applying Theorem 1.15 shows $\lambda \notin \sigma_{\mathrm{disc}}(A_{\eta,\tau})$. □

Remark 5.9. In case of a Lipschitz boundary Σ we have

$$\eta^2 - \tau^2 \notin \left[\frac{1}{M_-}, 16M_-\right] \Rightarrow \eta^2 = \tau^2 \text{ or } \frac{1}{\eta^2 - \tau^2} \in \mathcal{F}_0 \cap \mathbb{R} \Rightarrow A_{\eta,\tau} \text{ is self-adjoint.} \tag{5.47}$$

If the boundary Σ has even $\mathcal{C}^1$ regularity, there holds

$$\eta^2 - \tau^2 \neq 4 \Rightarrow \eta^2 = \tau^2 \text{ or } \frac{1}{\eta^2 - \tau^2} \in \mathcal{F}_0 \cap \mathbb{R} \Rightarrow A_{\eta,\tau} \text{ is self-adjoint.} \tag{5.48}$$

Certainly, our results let us conjecture that (5.48) is also true for Lipschitz domains. Unfortunately, we were not able to carry the proof of the important compactness result regarding $M(\lambda)^2 - \frac{1}{4}I_4$, cf. Theorem 3.18, over to Lipschitz domains. It is also worth mentioning that the critical case $\eta^2 - \tau^2 = 4$ is still open, even for smooth domains. However, for two-dimensional Dirac operators with singular interactions on smooth curves self-adjointness was proved in [11].

5.2 Particular Choices of the Interaction Strengths η and τ

In order to emphasize the results of Theorem 5.8, we take a closer look at two special cases which are included in Theorem 5.8. These special cases are covered by Theorem 5.8 for Lipschitz boundaries without additional regularity of the boundary. We consider the confinement case, where $\eta^2 - \tau^2 = -4$, and the case of purely Lorentz-scalar δ-shell interactions, i.e. $\eta = 0$.

Theorem 5.10. *If $\eta^2 - \tau^2 = -4$, then the operator $A_{\eta,\tau}$, which is defined by (5.2), is self-adjoint and the domain can be decoupled into*

$$\operatorname{dom} A_{\eta,\tau} = \left\{ f \in \operatorname{dom} T : \left(\pm i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 + \tau\beta) \right) \boldsymbol{t}_\Sigma f_\pm = 0 \right\}. \tag{5.49}$$

Proof. The self-adjointness is a direct consequence of Theorem 5.8 and Remark 5.9. It remains to prove the representation of $\operatorname{dom} A_{\eta,\tau}$ depicted in (5.49). Let us denote the right-hand side of (5.49) $M_{\eta,\tau}$. The inclusion $M_{\eta,\tau} \subset \operatorname{dom} A_{\eta,\tau}$ is obvious just by looking at the definitions of Γ_0 and Γ_1. We start proving $\operatorname{dom} A_{\eta,\tau} \subset M_{\eta,\tau}$ by choosing a function $f \in \operatorname{dom} A_{\eta,\tau}$. Recalling the definitions of Γ_0 and Γ_1, cf. (4.21), we see

$$\begin{aligned} &\Gamma_0 f + (\eta I_4 + \tau\beta)\Gamma_1 f \\ &\quad = \left(i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 + \tau\beta) \right) \boldsymbol{t}_\Sigma f_+ + \left(-i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 + \tau\beta) \right) \boldsymbol{t}_\Sigma f_- = 0. \end{aligned} \tag{5.50}$$

We multiply with $-i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 - \tau\beta)$ and obtain

$$\left(1 + \frac{\eta^2 - \tau^2}{4} \right) \boldsymbol{t}_\Sigma f_+ + \left(\left(-1 + \frac{\eta^2 - \tau^2}{4} \right) I_4 - i(\alpha \cdot \nu)(\eta I_4 + \tau\beta) \right) \boldsymbol{t}_\Sigma f_- = 0. \tag{5.51}$$

Inserting our assumption $\eta^2 - \tau^2 = -4$ and multiplying with $\frac{i}{2}(\alpha \cdot \nu)$ yields

$$\left(-i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 + \tau\beta) \right) \boldsymbol{t}_\Sigma f_- = 0. \tag{5.52}$$

Analogously, multiplying with $i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 - \tau\beta)$ gives us

$$\left(i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 + \tau\beta) \right) \boldsymbol{t}_\Sigma f_+ = 0 \tag{5.53}$$

and therefore $f \in M_{\eta,\tau}$. Hence, $\operatorname{dom} A_{\eta,\tau} = M_{\eta,\tau}$. □

Remark 5.11. The phenomenon stated in Theorem 5.10 is also studied in [3, Section 5], [22, Remark 4.2.2.] and [7, Lemma 3.1]. It is a remarkable fact of the case $\eta^2 - \tau^2 = -4$ since it implies that the operator $A_{\eta,\tau}$ can be decoupled into two independent operators defined on the domains $\left\{ f_\pm \in H_\alpha^{1/2,0}(\Omega_\pm) : \left(\pm i(\alpha \cdot \nu) + \frac{1}{2}(\eta I_4 + \tau\beta) \right) \boldsymbol{t}_\Sigma f_\pm = 0 \right\}$. The corresponding physical phenomenon is called confinement and means that the boundary Σ is impermeable for particles.

Now, we consider the case of purely Lorentz-scalar δ-shell interactions. The results listed below can be found e.g. in [22, Corollary 4.2.6.] or [23, Theorem 2.3], where these kinds of operators are studied in further detail for domains with smooth boundaries. The proof of Theorem 5.12 does not change for Lipschitz domains. However, we state the proof for the sake of completeness.

Theorem 5.12. *Let $\tau \in \mathbb{R}$ and $A_{0,\tau}$ be the self-adjoint operator defined by* (4.42)*. Then, in addition to the assertions in Theorem 5.8, the following assertions hold:*

(i) $\lambda \in \sigma(A_{0,\tau})$ *if and only if* $-\lambda \in \sigma(A_{0,\tau})$.

(ii) The discrete eigenvalues of $A_{0,\tau}$ have even multiplicity.

(iii) If $\tau m \geq 0$, *then* $\sigma_{\text{disc}}(A_{0,\tau}) = \emptyset$.

Proof. The ideas of the proof stem from [23, Theorem 2.3]. We start with assertion *(i)* and introduce the so-called charge conjugation operator

$$\mathcal{C} : L^2(\mathbb{R}^3; \mathbb{C}^4) \to L^2(\mathbb{R}^3; \mathbb{C}^4) \tag{5.54}$$
$$f \mapsto i\beta\alpha_2\overline{f}. \tag{5.55}$$

Keeping $\overline{\alpha_2} = -\alpha_2$ and $\overline{\beta} = \beta$ in mind, we obtain

$$\mathcal{C}^2 f = i\beta\alpha_2\overline{i\beta\alpha_2\overline{f}} = -\beta\alpha_2\beta\alpha_2 f = f. \tag{5.56}$$

We claim $f \in \operatorname{dom} A_{0,\tau}$ if and only if $\mathcal{C}f \in \operatorname{dom} A_{0,\tau}$. It is easy to check $f \in \operatorname{dom} T$ if and only if $\mathcal{C}f \in \operatorname{dom} T$. Therefore, let us investigate the boundary conditions. Using $(\alpha \cdot \nu)\alpha_2 = \overline{\alpha_2(\alpha \cdot \nu)}$ and elementary algebraic operations yield

$$\begin{aligned}\Gamma_0\mathcal{C}f + \tau\beta\Gamma_1\mathcal{C}f &= i(\alpha\cdot\nu)i\beta\alpha_2\overline{(\boldsymbol{t}_\Sigma f_+ - \boldsymbol{t}_\Sigma f_-)} + \tau\beta\frac{1}{2}i\beta\alpha_2\overline{(\boldsymbol{t}_\Sigma f_+ + \boldsymbol{t}_\Sigma f_-)}\\ &= -\overline{i\beta\alpha_2 i(\alpha\cdot\nu)\left(\boldsymbol{t}_\Sigma f_+ - \boldsymbol{t}_\Sigma f_-\right)} - \frac{1}{2}\overline{i\beta\alpha_2\tau\beta\left(\boldsymbol{t}_\Sigma f_+ + \boldsymbol{t}_\Sigma f_-\right)}\\ &= -\overline{i\alpha_2\beta\left(\Gamma_0 f + \tau\beta\Gamma_1 f\right)} \quad \forall f \in \operatorname{dom} T.\end{aligned} \tag{5.57}$$

This proves the claimed result. Next, we see, again through similar algebraic considerations,

$$\begin{aligned}(A_{0,\tau}\mathcal{C}f)_\pm &= -i(\alpha\cdot\nabla)i\beta\alpha_2\overline{f_\pm} + m\beta i\beta\alpha_2\overline{f_\pm} = -i\beta\alpha_2\left(\overline{-i(\alpha\cdot\nabla)f_\pm + m\beta f_\pm}\right)\\ &= -(\mathcal{C}A_{0,\tau}f)_\pm \quad \forall f \in \operatorname{dom} A_{0,\tau}.\end{aligned} \tag{5.58}$$

Summing up the properties of $\mathcal{C}$, we have

(a) $\mathcal{C}^2 = I$,

(b) $f \in \operatorname{dom} A_{0,\tau}$ if and only if $\mathcal{C}f \in \operatorname{dom} A_{0,\tau}$ and

(c) $A_{0,\tau}\mathcal{C}f = -\mathcal{C}A_{0,\tau}f$ for $f \in \operatorname{dom} A_{0,\tau}$.

This implies $\lambda \in \sigma_{\text{disc}}(A_{0,\tau})$ if and only if $-\lambda \in \sigma_{\text{disc}}(A_{0,\tau})$. Moreover, we already know that the essential spectrum of $A_{0,\tau}$ is symmetric and therefore item *(i)* is true.
For the second assertion we introduce another helpful operator

$$\begin{aligned}\mathcal{T} : L^2(\mathbb{R}^3; \mathbb{C}^4) &\to L^2(\mathbb{R}^3; \mathbb{C}^4)\\ f &\mapsto -i\gamma_5\alpha_2\overline{f}\end{aligned} \tag{5.59}$$

called the time reversal operator with $\gamma_5 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$. The matrix γ_5 satisfies the relations $\gamma_5\beta = -\beta\gamma_5$ and $\gamma_5\alpha_k = \alpha_k\gamma_5$. In an analogous manner as for $\mathcal{C}$, one can show

(a) $\mathcal{T}^2 = -I$,

(b) $f \in \operatorname{dom} A_{0,\tau}$ if and only if $\mathcal{T}f \in \operatorname{dom} A_{0,\tau}$ and

(c) $A_{0,\tau}\mathcal{T}f = \mathcal{T}A_{0,\tau}f$ for $f \in \operatorname{dom} A_{0,\tau}$.

Hence, (f,λ) is an eigenpair of $A_{\eta,\tau}$ if and only if $(\mathcal{T}f,\lambda)$ is an eigenpair of $A_{\eta,\tau}$. As a consequence of

$$\begin{aligned}(\mathcal{T}f, g)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} &= \int_{\mathbb{R}^3} \Big(-i\gamma_5\alpha_2\overline{f(x)}, g(x)\Big)_{\mathbb{C}^4}\, dx = \int_{\mathbb{R}^3} if(x)\cdot\alpha_2\gamma_5 g(x)\, dx \\ &= \int_{\mathbb{R}^3} (i\gamma_5\alpha_2 g(x))\cdot f(x)\, dx = \int_{\mathbb{R}^3} \Big(\overline{i\gamma_5\alpha_2\overline{g(x)}}\Big)\cdot f(x)\, dx \\ &= \int_{\mathbb{R}^3} \Big(i\gamma_5\alpha_2\overline{g(x)}, f(x)\Big)_{\mathbb{C}^4}\, dx = -(\mathcal{T}g, f)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \quad \forall f,g \in L^2(\mathbb{R}^3;\mathbb{C}^4)\end{aligned} \tag{5.60}$$

f and $\mathcal{T}f$ are orthogonal. Now, let λ be a discrete eigenvalue of $A_{0,\tau}$. We prove assertion *(ii)* through constructing a basis of the eigenspace $\ker(A_{0,\tau}-\lambda)$. Therefore, we choose an eigenfunction f_1 corresponding to the eigenvalue λ. Then, $\mathcal{T}f_1$ is also an eigenfunction which is orthogonal to f_1. Now, two cases can occur. Either the multiplicity of the eigenvalue λ is two or the multiplicity is larger than two. In the first case we are done. Otherwise, there exists another eigenfunction f_2 which is orthogonal to $\operatorname{span}\{f_1, \mathcal{T}f_1\}$. We use (5.60) to see

$$\begin{aligned}(\mathcal{T}f_2, f_1)_{L^2(\Sigma;\mathbb{C}^4)} &= -(\mathcal{T}f_1, f_2)_{L^2(\Sigma;\mathbb{C}^4)} = 0 \\ \text{and } (\mathcal{T}f_2, \mathcal{T}f_1)_{L^2(\Sigma;\mathbb{C}^4)} &= -\big(\mathcal{T}^2 f_1, f_2\big)_{L^2(\Sigma;\mathbb{C}^4)} = 0.\end{aligned} \tag{5.61}$$

Therefore, $\mathcal{T}f_2$ is orthogonal to $\operatorname{span}\{f_1, \mathcal{T}f_1, f_2\}$. Again, two cases can occur and we proceed iteratively. Since $\lambda \in \sigma_{\text{disc}}(A_{0,\tau})$, the multiplicity is finite and we can proceed with this procedure until we have constructed an orthogonal basis for the eigenspace of λ. Then, the eigenspace has even dimension by construction of the basis.
Last, we prove item *(iii)*. For this purpose we examine the expression $\|A_{0,\tau}f\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)}$, where $f = f_+ \oplus f_- \in \operatorname{dom} A_{0,\tau}$. We observe

$$\begin{aligned}&\|A_{0,\tau}f\|^2_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \\ &\quad \geq 2\operatorname{Re}\,((-i(\alpha\cdot\nabla)f_+)\oplus(-i(\alpha\cdot\nabla)f_-), m\beta f)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} + m^2\|f\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)}.\end{aligned} \tag{5.62}$$

We notice if $\operatorname{Re}\,((-i(\alpha\cdot\nabla)f_+)\oplus(-i(\alpha\cdot\nabla)f_-), m\beta f)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} \geq 0$, then an eigenvalue $\lambda \in (-|m|, |m|)$ can not exist. This would yield $\sigma_{\text{disc}}(A_{0,\tau}) = \emptyset$ since Theorem 5.8 *(i)* shows $\sigma_{\mathrm{ess}}(A_{0,\tau}) = (-\infty, -|m|] \cup [|m|, \infty)$. In the following lines we split the expression, employ integration by parts and use (4.25) as well as the boundary conditions to get

$$\begin{aligned}
&2\mathrm{Re}\,\big((-i(\alpha\cdot\nabla)f_+) \oplus (-i(\alpha\cdot\nabla)f_-), m\beta f\big)_{L^2(\mathbb{R}^3;\mathbb{C}^4)}\\
&\quad= (-i(\alpha\cdot\nabla)f_+, m\beta f_+)_{L^2(\Omega_+;\mathbb{C}^4)} + (-i(\alpha\cdot\nabla)f_-, m\beta f_-)_{L^2(\Omega_-;\mathbb{C}^4)}\\
&\quad\ + (m\beta f_+, -i(\alpha\cdot\nabla)f_+)_{L^2(\Omega_+;\mathbb{C}^4)} + (m\beta f_-, -i(\alpha\cdot\nabla)f_-)_{L^2(\Omega_-;\mathbb{C}^4)}\\
&\quad= (-i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_+, m\boldsymbol{t}_\Sigma\beta f_+)_{L^2(\Sigma;\mathbb{C}^4)} + (i(\alpha\cdot\nu)\boldsymbol{t}_\Sigma f_-, m\boldsymbol{t}_\Sigma\beta f_-)_{L^2(\Sigma;\mathbb{C}^4)}\\
&\quad= (\Gamma_1 f, \Gamma_0 m\beta f)_{L^2(\Sigma;\mathbb{C}^4)} - (\Gamma_0 f, \Gamma_1 m\beta f)_{L^2(\Sigma;\mathbb{C}^4)}\\
&\quad= -(\beta\Gamma_1 f, \Gamma_0 m f)_{L^2(\Sigma;\mathbb{C}^4)} - (\Gamma_0 f, \beta\Gamma_1 m f)_{L^2(\Sigma;\mathbb{C}^4)}\\
&\quad= -(\beta\Gamma_1 f, -\tau\beta m\Gamma_1 f)_{L^2(\Sigma;\mathbb{C}^4)} - (-\tau\beta\Gamma_1 f, \beta\Gamma_1 m f)_{L^2(\Sigma;\mathbb{C}^4)}\\
&\quad= 2\tau m\|\Gamma_1 f\|^2_{L^2(\Sigma;\mathbb{C}^4)} \geq 0.
\end{aligned} \tag{5.63}$$

□

5.3 Differences of Powers of Resolvents

In this section we study the qualitative properties of differences of the form

$$(A_{\eta,\tau} - \lambda)^{-l} - (A_0 - \lambda)^{-l}, \tag{5.64}$$

where $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and $l \in \mathbb{N}$. This is particularly important in scattering theory, because by showing that $(A_{\eta,\tau} - \lambda)^{-l} - (A_0 - \lambda)^{-l}$ belongs to the trace class for a $l \in \mathbb{N}$, i.e. the singular values of $(A_{\eta,\tau} - \lambda)^{-l} - (A_0 - \lambda)^{-l}$ are summable, one can prove that the wave operators of $\{A_{\eta,\tau}, A_0\}$ are complete, see [24, Chapter Ten Theorem 4.12 and Remark 4.13.].
The next two statements are based on Proposition 4.1.9. and Theorem 4.2.7 in [22]. The proofs are similar, but here we only assume Lipschitz continuity of the boundary Σ and therefore we obtain slightly lesser regularity in terms of Schatten-von Neumann ideals.

Lemma 5.13. *Let the γ-field and the Weyl function M be given by* (4.41)*. Then, the following statements hold true:*

(i) The operator-valued functions $\lambda \mapsto \gamma(\lambda)$ and $\lambda \mapsto \gamma(\overline{\lambda})^*$ are holomorphic in $\rho(A_0)$ and for $k \in \mathbb{N}_0$ and $\varepsilon \in (0,1)$

$$\begin{aligned}
&\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\gamma(\lambda) \in \mathfrak{S}_{\frac{6}{2k+1-\varepsilon},\infty}(L^2(\Sigma;\mathbb{C}^4), L^2(\mathbb{R}^3;\mathbb{C}^4)) \ \text{ and}\\
&\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\gamma(\overline{\lambda})^* \in \mathfrak{S}_{\frac{6}{2k+1-\varepsilon},\infty}(L^2(\mathbb{R}^3;\mathbb{C}^4), L^2(\Sigma;\mathbb{C}^4)).
\end{aligned} \tag{5.65}$$

In particular, $\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\gamma(\lambda)$ and $\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\gamma(\lambda)^*$ are compact.

(ii) The operator-valued function $\lambda \mapsto M(\lambda)$ is holomorphic in $\rho(A_0)$ and for $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k} M(\lambda) \in \mathfrak{S}_{\frac{3}{k-\varepsilon},\infty}(L^2(\Sigma;\mathbb{C}^4), L^2(\Sigma;\mathbb{C}^4)). \tag{5.66}$$

Proof. We start the proof by showing that $\gamma(\lambda)$,$\gamma(\overline{\lambda})^*$ and $M(\lambda)$ are holomorphic. First, we notice due to Theorem 4.15 that they are bounded and everywhere defined operators. It suffices to show the statement for $\gamma(\overline{\lambda})^* = \Gamma_1(A_0-\lambda)^{-1}$. For the other two operator-valued functions the statement follows by taking the adjoint and item *(ii)* from Theorem 1.13. We set $\frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma(\overline{\lambda})^* = \Gamma_1(A_0-\lambda)^{-2}$ and see that for $|\lambda-\mu|\,\|A_0-\lambda\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^1(\mathbb{R}^3;\mathbb{C}^4)} < 1$ holds

$$\begin{aligned}
&\left\|\frac{\gamma(\overline{\lambda})^*-\gamma(\overline{\mu})^*}{\lambda-\mu} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma(\overline{\lambda})^*\right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to L^2(\Sigma;\mathbb{C}^4)} \\
&\qquad\le \left\|\frac{\gamma(\overline{\lambda})^*-\gamma(\overline{\mu})^*}{\lambda-\mu} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\gamma(\overline{\lambda})^*\right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^{1/2}(\Sigma;\mathbb{C}^4)} \\
&\qquad\le \left\|\frac{(A_0-\lambda)^{-1}-(A_0-\mu)^{-1}}{\lambda-\mu} - (A_0-\lambda)^{-2}\right\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^1(\mathbb{R}^3;\mathbb{C}^4)}.
\end{aligned} \tag{5.67}$$

Next, we represent $(A_0-\mu)^{-1}$ as a Neumann series

$$(A_0-\mu)^{-1} = (A_0-\lambda)^{-1}\sum_{l=0}^{\infty}(\mu-\lambda)^l(A_0-\lambda)^{-l}. \tag{5.68}$$

Applying this representation we see

$$\begin{aligned}
\frac{(A_0-\lambda)^{-1}-(A_0-\mu)^{-1}}{\lambda-\mu} - (A_0-\lambda)^{-2} &= -\sum_{l=2}^{\infty}(\mu-\lambda)^{l-1}(A_0-\lambda)^{-l-1} \\
&= -(\mu-\lambda)(A_0-\lambda)^{-3}\sum_{l=0}^{\infty}(\mu-\lambda)^l(A_0-\lambda)^{-l}.
\end{aligned} \tag{5.69}$$

This expression can be bounded by

$$\frac{|\lambda-\mu|\,\|(A_0-\lambda)^{-1}\|^3_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^1(\mathbb{R}^3;\mathbb{C}^4)}}{1-|\lambda-\mu|\,\|(A_0-\lambda)^{-1}\|_{L^2(\mathbb{R}^3;\mathbb{C}^4)\to H^1(\mathbb{R}^3;\mathbb{C}^4)}} \xrightarrow{\mu\to\lambda} 0 \tag{5.70}$$

which shows that also (5.67) converges to zero for $\mu \to \lambda$. Hence, $\lambda \mapsto \gamma(\overline{\lambda})^*$ is holomorphic. Now, we prove the statements in item *(i)*. Again, it suffices to show the statement regarding the adjoint of $\gamma(\overline{\lambda})$. Then, the second claim follows by taking the adjoint. Item *(ii)* in Theorem 1.14 shows

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\gamma(\overline{\lambda})^* = k!\,\Gamma_1(A_0-\lambda)^{-k-1}. \tag{5.71}$$

Using the Fourier transform we obtain that $u \in \operatorname{dom} A_0^{k+1}$ is equivalent to

$$\begin{aligned}\infty > &\int_{\mathbb{R}^3} \left|((\alpha\cdot\xi)+m\beta)^{k+1}(\mathcal{F}u)(\xi)\right|^2 + |(\mathcal{F}u)(\xi)|^2\, d\xi \\ &= \int_{\mathbb{R}^3} (|\xi|^2+m^2)^{k+1}\,|(\mathcal{F}u)(\xi)|^2 + |(\mathcal{F}u)(\xi)|^2\, d\xi \approx \|u\|^2_{H^{k+1}(\mathbb{R}^3;\mathbb{C}^4)}.\end{aligned} \tag{5.72}$$

Thus, $\operatorname{ran}(A_0-\lambda)^{-k-1} = \operatorname{dom} A_0^{k+1} = H^{k+1}(\mathbb{R}^3;\mathbb{C}^4)$. Let us choose $R>0$ such that $\Sigma \subset B(0,R)$. Moreover, let $g \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \le g \le 1$, $g=1$ in $B(0,R)$ and $\operatorname{supp} g \subset B(0,2R)$. Then,

$$\begin{aligned} Q : H^{k+1}(\mathbb{R}^3;\mathbb{C}^4) &\to H_0^{k+1}(B(0,2R);\mathbb{C}^4)\\ u &\mapsto (gu)\,|_{B(0,2R)}\end{aligned} \tag{5.73}$$

is well defined and bounded. Therefore,

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\gamma(\overline{\lambda})^* = k!\,\Gamma_1 Q(A_0-\lambda)^{-k-1}. \tag{5.74}$$

Applying Corollary 2.17 yields

$$\begin{aligned} Q(A_0-\lambda)^{-k-1} &\in \mathfrak{S}_{\frac{3}{k+1-\frac{1+\varepsilon}{2}}}(L^2(\mathbb{R}^3;\mathbb{C}^4), H^{\frac{1+\epsilon}{2}}(B(0,2R);\mathbb{C}^4)\\ &= \mathfrak{S}_{\frac{6}{2k+1-\varepsilon},\infty}(L^2(\mathbb{R}^3;\mathbb{C}^4), H^{\frac{1+\epsilon}{2}}(B(0,2R);\mathbb{C}^4).\end{aligned} \tag{5.75}$$

We also know from the trace theorem, see Theorem 2.25, and from the continuous embedding of $H^{\frac{\varepsilon}{2}}(\Sigma;\mathbb{C}^4)$ in $L^2(\Sigma;\mathbb{C}^4)$ that Γ_1 is bounded from $H^{\frac{1+\varepsilon}{2}}(B(0,2R);\mathbb{C}^4)$ to $L^2(\Sigma;\mathbb{C}^4)$. This leads to $\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}\gamma(\overline{\lambda})^* \in \mathfrak{S}_{\frac{6}{2k+1-\varepsilon},\infty}(L^2(\mathbb{R}^3;\mathbb{C}^4), L^2(\Sigma;\mathbb{C}^4))$, implying *(i)*. Next, let us prove assertion *(ii)*. Since $\mathcal{G}_0$ from Theorem 1.14 equals $L^2(\Sigma;\mathbb{C}^4)$ in our case, *(iii)* from Theorem 1.14 yields

$$\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}M(\lambda) = k!\,\Gamma_1(A_0-\lambda)^{-k}\gamma(\lambda) = k\left(\frac{\mathrm{d}^{k-1}}{\mathrm{d}\lambda^{k-1}}\gamma(\overline{\lambda})^*\right)\gamma(\lambda). \tag{5.76}$$

We apply *(i)* and Theorem 1.10 *(iii)* in order to get item *(ii)*. □

Theorem 5.14. *Let $\eta^2 = \tau^2$ or $\frac{1}{\eta^2-\tau^2} \in \mathcal{F}_0 \cap \mathbb{R}$, A_0 be the free Dirac operator introduced in Definition 4.1, and $A_{\eta,\tau}$ be given by* (5.2). *Then, for $l \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$*

$$(A_{\eta,\tau}-\lambda)^{-l} - (A_0-\lambda)^{-l} \in \mathfrak{S}_{\frac{3}{l-\epsilon},\infty}\left(L^2(\mathbb{R}^3;\mathbb{C}^4), L^2(\mathbb{R}^3;\mathbb{C}^4)\right) \tag{5.77}$$

holds true. In particular, $(A_{\eta,\tau}-\lambda)^{-l} - (A_0-\lambda)^{-l}$ belongs to the trace class ideal if $l \ge 4$.

Proof. We start by rewriting (5.77) with the help of the product rule, cf. [26, eq. (2.2.7)], in the following way

$$
\begin{aligned}
(A_{\eta,\tau}-\lambda)^{-l}-(A_0-\lambda)^{-l} &= \frac{1}{(l-1)!}\frac{\mathrm{d}^{l-1}}{\mathrm{d}\lambda^{l-1}}\left((A_{\eta,\tau}-\lambda)^{-1}-(A_0-\lambda)^{-1}\right)\\
&= \frac{1}{(l-1)!}\frac{\mathrm{d}^{l-1}}{\mathrm{d}\lambda^{l-1}}\left(\gamma(\lambda)\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}(\eta I_4+\tau\beta)\gamma(\overline{\lambda})^*\right) \qquad (5.78)\\
&= \frac{1}{(l-1)!}\sum_{\substack{q,r,s\in\mathbb{N}_0\\ q+r+s=l-1}}\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}\gamma(\lambda)\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}(\eta I_4+\tau\beta)\frac{\mathrm{d}^s}{\mathrm{d}\lambda^s}\gamma(\overline{\lambda})^*.
\end{aligned}
$$

At first, we take a closer look at the terms with $r=0$. We already know from Lemma 5.7 that the operator $\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}$ is bounded in $L^2(\Sigma;\mathbb{C}^4)$. Moreover,

$$
\begin{aligned}
&\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}\gamma(\lambda)\in\mathfrak{S}_{\frac{6}{2q+1-\varepsilon},\infty}\left(L^2(\Sigma;\mathbb{C}^4),L^2(\mathbb{R}^3;\mathbb{C}^4)\right)\ \text{and}\\
&\frac{\mathrm{d}^s}{\mathrm{d}\lambda^s}\gamma(\overline{\lambda})^*\in\mathfrak{S}_{\frac{6}{2s+1-\varepsilon},\infty}\left(L^2(\mathbb{R}^3;\mathbb{C}^4),L^2(\Sigma;\mathbb{C}^4)\right).
\end{aligned} \qquad (5.79)
$$

Hence,

$$
\begin{aligned}
&\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}\gamma(\lambda)\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\\
&\qquad(\eta I_4+\tau\beta)\frac{\mathrm{d}^s}{\mathrm{d}\lambda^s}\gamma(\overline{\lambda})^*\in\mathfrak{S}_{\frac{3}{l-\varepsilon},\infty}\left(L^2(\mathbb{R}^3;\mathbb{C}^4),L^2(\mathbb{R}^3;\mathbb{C}^4)\right)
\end{aligned} \qquad (5.80)
$$

for $r=0$. Next, we prove the case $r\geq 1$. We claim

$$
\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\in\mathfrak{S}_{\frac{3}{r-\widetilde{\varepsilon}},\infty}\left(L^2(\Sigma;\mathbb{C}^4),L^2(\Sigma;\mathbb{C}^4)\right)\quad\forall\,\widetilde{\varepsilon}\in(0,1). \qquad (5.81)
$$

Let us prove the claim with induction. Therefore, we assume $r=1$ at first. Then, the inversion rule, cf. [26, eq. 2.2.8], yields

$$
\begin{aligned}
&\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\\
&\qquad=-\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}(\eta I_4+\tau\beta)\frac{\mathrm{d}}{\mathrm{d}\lambda}M(\lambda)\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}.
\end{aligned} \qquad (5.82)
$$

Thus, $\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\in\mathfrak{S}_{\frac{3}{r-\widetilde{\varepsilon}},\infty}(L^2(\Sigma;\mathbb{C}^4),L^2(\Sigma;\mathbb{C}^4))$ for all $\widetilde{\varepsilon}\in(0,1)$ according to Lemma 5.13. Now, let $r\in\mathbb{N}$. We use the inversion rule and the product rule in order to see

$$\begin{aligned}
&\frac{\mathrm{d}^{r+1}}{\mathrm{d}\lambda^{r+1}}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\\
&=-\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\Bigg(((I_4+(\eta I_4+\tau\beta)M(\lambda))^{-1}\\
&\qquad\qquad(\eta I_4+\tau\beta)\frac{\mathrm{d}}{\mathrm{d}\lambda}M(\lambda)\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\Bigg)\\
&=-\sum_{\substack{h,k,m\in\mathbb{N}_0\\ h+k+m=r}}\Bigg(\frac{\mathrm{d}^h}{\mathrm{d}\lambda^h}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\\
&\qquad\qquad(\eta I_4+\tau\beta)\frac{\mathrm{d}^{k+1}}{\mathrm{d}\lambda^{k+1}}M(\lambda)\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\Bigg).
\end{aligned}\tag{5.83}$$

Using the induction assumption we get

$$\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\in\mathfrak{S}_{\frac{6}{2j-\frac{2}{3}\widetilde{\varepsilon}},\infty}\left(L^2(\Sigma;\mathbb{C}^4),L^2(\Sigma;\mathbb{C}^4)\right)\tag{5.84}$$

for all $\widetilde{\varepsilon}\in(0,1)$ and $j\in\{1,\dots r\}$. Moreover, Lemma 5.13 *(ii)* yields

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}\lambda^{k+1}}M(\lambda)\in\mathfrak{S}_{\frac{6}{2(k+1)-\frac{2}{3}\widetilde{\varepsilon}},\infty}\left(L^2(\Sigma;\mathbb{C}^4),L^2(\Sigma;\mathbb{C}^4)\right)\tag{5.85}$$

for all $\widetilde{\varepsilon}\in(0,1)$ and $k\in\mathbb{N}_0$. Applying (5.84), (5.85), Theorem 1.10 *(iii)* and the boundedness of $(I_4+(\eta I_4+\tau\beta)M(\lambda))^{-1}$ shows us

$$\frac{\mathrm{d}^{r+1}}{\mathrm{d}\lambda^{r+1}}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\in\mathfrak{S}_{\frac{3}{r+1-\widetilde{\varepsilon}},\infty}\left(L^2(\Sigma;\mathbb{C}^4),L^2(\Sigma;\mathbb{C}^4)\right)\tag{5.86}$$

for all $\widetilde{\varepsilon}\in(0,1)$. Thus, the claim (5.81) holds true. We return to the expression

$$\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}\gamma(\lambda)\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\frac{\mathrm{d}^s}{\mathrm{d}\lambda^s}\gamma(\overline{\lambda})^*\tag{5.87}$$

for $r\geq 1$. Again, Lemma 5.13 *(i)* proves $\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}\gamma(\lambda)\in\mathfrak{S}_{\frac{6}{2q+1-\frac{2}{3}\varepsilon},\infty}(L^2(\Sigma;\mathbb{C}^4),L^2(\mathbb{R}^3;\mathbb{C}^4))$ as well as $\frac{\mathrm{d}^s}{\mathrm{d}\lambda^s}\gamma(\overline{\lambda})^*\in\mathfrak{S}_{\frac{6}{2s+1-\frac{2}{3}\varepsilon},\infty}(L^2(\mathbb{R}^3;\mathbb{C}^4),L^2(\Sigma;\mathbb{C}^4))$. Furthermore, we choose $\widetilde{\varepsilon}=\frac{2}{3}\varepsilon$. Then, Theorem 1.10 *(iii)* gives us

$$\begin{aligned}
&\frac{\mathrm{d}^q}{\mathrm{d}\lambda^q}\gamma(\lambda)\frac{\mathrm{d}^r}{\mathrm{d}\lambda^r}\left(I_4+(\eta I_4+\tau\beta)M(\lambda)\right)^{-1}\\
&\qquad(\eta I_4+\tau\beta)\frac{\mathrm{d}^s}{\mathrm{d}\lambda^s}\gamma(\overline{\lambda})^*\in\mathfrak{S}_{\frac{3}{l-\varepsilon},\infty}\left(L^2(\mathbb{R}^3;\mathbb{C}^4),L^2(\mathbb{R}^3;\mathbb{C}^4)\right)
\end{aligned}\tag{5.88}$$

for all $q,r,s\in\mathbb{N}_0$ with $q+r+s=l-1$. Summing up these terms concludes the proof. □

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