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Processes with Free Increments

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Abstract

In free probability one studies not necessarily commutative random variables. We give an introduction to this field of study by defining non-commutative probability spaces and distributions in this setting. To further understand how these distributions behave under addition and multiplication of random variables we study free independence which is an analogue to the classical notion of independence. This notion gives also rise to a description of factorization of mixed moments in terms of non-crossing partitions and free cumulants. Further we introduce analytical tools such as the Cauchy transform and the conditional expectation to describe the behavior of convolutions. These concepts allow us to describe additive and multiplicative processes with free increments. For these processes we show the existence of a Feller Markov kernel and the subordination property which we finally use to proof analogues of the Levy-Khintchine formulas for different kinds of free analogues of additive and multiplicative Levy processes.

Kurzfassung

In der freien Wahrscheinlichkeitsrechnung untersucht man nicht notwendigerweise kommutative Zufallsvariablen. Wir geben eine Einführung in dieses Studiengebiet, indem wir nicht-kommutative Wahrscheinlichkeitsräume und Verteilungen definieren. Um zu verstehen, wie sich diese Verteilungen unter Addition und Multiplikation von Zufallsvariablen verhalten, untersuchen wir die freie Unabhängigkeit, die ein Analogon zum klassischen Begriff der Unabhängigkeit ist. Dieser Begriff führt auch zu einer Beschreibung der Faktorisierung von gemischten Momenten in Form von nicht kreuzenden Partitionen und freien Kumulanten. Weiterhin führen wir analytische Werkzeuge wie die Cauchy-Transformation und den bedingten Erwartungswert ein, um das Verhalten von Faltungen zu beschreiben. Diese Konzepte erlauben es uns, additive und multiplikative Prozesse mit freien Inkrementen zu beschreiben. Für diese Prozesse zeigen wir die Existenz eines Feller-Markov-Kerns und die Subordinationseigenschaft, die wir schließlich verwenden, um Levy-Khintchine-Formeln für verschiedene freie Analoga von additiven und multiplikativen Levy-Prozessen zu beweisen.

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Contents

1. Introduction

In this thesis we will look at non-commutative analogs of processes with independent increments which are a prominent class of stochastic processes. Other than their classical counterpart increments of these processes can not only be stochastically independent, but also algebraically. Therefore they require a new notion of independence which is called freeness. Using this property we will then derive different tools, which will allow the calculation of distributions in form of probability measures and their free convolutions. Later we will introduce time-dependence and prove the existence of the analog of the Markov-Feller kernel which we will finally use to describe processes with free increments and prove analogs of the Levy-Khintchine formulas. The main source for this paper is [Bia98].

1.1. Processes with independent increments

We will start by stating important definitions and results from the classical theory. As stated above we discuss analogs of processes with independent increments. So we start by looking at their classical definition.

Definition 1.1 (Processes with independent increments).
A function $t \mapsto X_t$ which maps a time point $t \in \mathbb{R}^+$ to a random variable X_t on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a process.
Further X_t is said to have ***independent increments*** *if for any increasing sequence $t_1, ..., t_n \in \mathbb{R}^+$ of arbitrary length $n \in \mathbb{N}$ $X_{t_1}, (X_{t_2} - X_{t_1}), ..., (X_{t_n} - X_{t_{n-1}})$ are independent.*

Another important property when working with stochastic processes is the Markov property.

Definition 1.2 (Markov property).
A ***filtration*** *$\mathcal{F}_t$ is a family of sub-σ-algebras of $\mathcal{F}$ such that $s_1 \leq s_2 \implies \mathcal{F}_{s_1} \subseteq \mathcal{F}_{s_2}$. A process X_t is called adapted to the filtration $\mathcal{F}_t$ if for all $t \in \mathbb{R}^+$ X_t is $\mathcal{F}_t$-measurable.*
Such a process X_t has the ***Markov property*** *if for every Borel set A and for every $s, t \in \mathbb{R}^+$ it holds that $\mathbb{P}[X_{s+t} \in A|\mathcal{F}_s] = \mathbb{P}[X_{s+t} \in A|X_s]$*

Remark. *In other words the only relevant information on the history of a Markov process is the value of the processes at the current time.*

For the rest of this section we will always assume that the probability space comes with a filtration $\mathcal{F}_t$. We can also express this property by using bounded Borel measurable functions.

Proposition 1.3 (Markov property).
Assume that for any bounded Borel measurable functions φ

$$\mathbb{E}\left[\varphi(X_{s+t})|\mathcal{F}_s\right] = \Phi(X_s)$$

for some Φ bounded and Borel measurable. Then X_t satisfies the Markov property.

Next we will look at the distribution of such processes. In the classical theory we can describe them as time dependent probability measures as follows:

Definition 1.4.
*We denote the **distribution** of X_t by μ_t and the distribution of $X_t - X_s$ by $\mu_{s,t}$*

Remark.
Note that distributions of the processes as defined above depend on the underlying probability measure of the probability space.

Proposition 1.5.

$$\forall r < s < t : \mu_s * \mu_{s,t} = \mu_t \text{ and } \mu_{r,s} * \mu_{s,t} = \mu_{r,t}$$

Remark. *This proposition also shows that the transition probabilities $\mu_{s,t}$ are indeed sufficient to describe μ_t for all $t \in \mathbb{R}$.*

1.2. Basics of Free Probability

Having looked at important concepts in the classical theory we now try to find similar formulations such that the same concepts can be applied to more general space like non-commutative algebras. These ideas were first introduced by D. Voiculescu 1986. For a thorough introduction we refer to [NS06] or [DVN92].
We start by defining a non-commutative analog to a probability space.

1.2.1. Probability spaces

Definition 1.6 (Non-commutative probability space).
Let
$\mathcal{A}$ *algebra over $\mathbb{C}$ with unit $1_{\mathcal{A}}$*
φ *$\varphi : \mathcal{A} \to \mathbb{C}$ linear with $\varphi(1_{\mathcal{A}}) = 1$*
*Then $(\mathcal{A}, \varphi)$ is called a **non-commutative probability space** containing the **non-commutative random variables** $a \in \mathcal{A}$.*

Remark.
Throughout this thesis, we will often drop the "non-commutative" in these definitions if it is clear that these spaces are considered.

To see the extent of this definition we will look at some examples.

Example (Classical probability space from a free probabilistic view point)**.** $\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) := \{X \text{ random variable} \mid \exists C \in \mathbb{R} : \mathbb{P}[|X| \leq C] = 1\}$ *forms an algebra and contains* 1 *(i.e. the random variable with* $\mathbb{P}[X = 1] = 1$*). And obviously the expectation* $\mathbb{E} : \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ *is linear and satisfies* $\mathbb{E}(1) = 1$*.*
Thus $(\mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ *is a probability space as defined above (even though in this case the algebra is commutative).*

Example (Matrix probability spaces)**.**
Since $\mathbb{C}^{n\times n}$ *forms an algebra containing the identity matrix, we can choose* $\mathcal{A} = \mathbb{C}^{n\times n}$*. For the state we need to fix a linear functional* φ *such that*

$$\varphi\left(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right) = 1$$

Two popular choices for the linear function would be

1. *vacuum state:* $\nu(A) := a_{1,1}$
2. *normalized trace* $\tau(A) = \frac{1}{n}\sum_{i=1}^n a_{i,i}$

where $A = (a_{i,j})_{i,j=1}^n \in \mathbb{C}^{n\times n}$

Example (Random Matrix probability space)**.**
Now we combine that examples above to

$$\mathcal{A} = \mathbb{C}^{n\times n} \otimes \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) = \left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} : a_{ij} \in \mathcal{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \right\}$$

For the state we can just choose a mixture of the states of the individual sub probability spaces.

$$\tilde{\tau}(A) = \frac{1}{n}\sum_{i=1}^n \mathbb{E}(a_{i,i})$$

i.e. $\tilde{\tau} = \tau \otimes \mathbb{E}$

To introduce the concept of positivity we look at the connection of these random variables to their adjoints.

Remark.
In the classical theory an almost surely non-negative random variable has non-negative expectation. In particular remember the partial order $X \geq 0 \iff \mathbb{P}[X \geq 0] = 1$*.*
To define such an partial order on the probability space of example 1.2.1 we could use positive semi-definiteness. In other words $A \geq 0 \iff \exists B \in \mathbb{C}^{n\times n}\colon A = B^*B$
We denote the cone of non-negative matrices by $\mathbb{C}_+^{n\times n} = \{A^*A | A \in \mathbb{C}^{n\times n}\}$*.*
So the property $X \geq 0 \implies \mathbb{E}(X) \geq 0$ *corresponds to*
$\forall A \in C^{n\times n}\colon \varphi(A^*A) \geq 0$

Since we want to apply this to spaces more general than a random matrix space, we have to define an adjoint operator on more general algebras. In this context these are typically called involution.

Definition 1.7 (Involution, $*$-probability space)**.**
*An anti-linear operator $* : \mathcal{A} \to \mathcal{A}$ is called **involution** if for all $a, b \in \mathcal{A}$:*

- $(a^*)^* = a$
- $(ab)^* = b^* a^*$
- $(a+b)^* = a^* + b^*$
- *for all $\lambda \in \mathbb{C}$: $(\lambda a)^* = \bar{\lambda} a^*$*
- *$\varphi(a) \in \mathbb{R}$ for $a = a^*$*
- $\varphi(a^* a) \geq 0$

*If $(\mathcal{A}, \varphi)$ contains an involution it is called **$*$-probability space** and φ is called **state**.*

Note that since $\varphi : \mathcal{A} \to \mathbb{C}$, the condition $\varphi(a^* a) \geq 0$ also means that $\varphi(a^* a) \in \mathbb{R}$. Using this involution we can also start to characterize the random variables we are working with.

Definition 1.8. *$a \in \mathcal{A}$ is called:*

- ***self-adjoint** if $a^* = a$*
- ***unitary** if $a^* a = a a^* = 1_{\mathcal{A}}$*
- ***normal** if $a^* a = a a^*$*

In a $*$-probability space the expectation translates the involution into complex-conjugation.

Proposition 1.9. $\varphi(a^*) = \overline{\varphi(a)}$

Proof. We use the same approach as to prove the statement for complex numbers.
Let $x \in \mathcal{A}$.
We define $\alpha = \frac{a+a^*}{2}$ and $\beta = \frac{a-a^*}{2i}$. Then it is easy to check that $a = \alpha + i\beta$ and $a^* = \alpha - i\beta$
Note that α and β are self-adjoint and thus $\varphi(\alpha)$ and $\varphi(\beta)$ are real.
Finally φ, $\varphi(a^*) = \varphi(\alpha) - i\varphi(\beta) = \overline{\varphi(\alpha) + i\varphi(\beta)} = \overline{\varphi(a)}$ □

With the properties we have seen so far we can already define the distribution of a random variable in an algebraic sense. But contrary to classical probability theory the following definition just gives us a description of the moments and not yet a probability measure.

Definition 1.10 (Distribution in algebraic sense)**.**
The distribution of $a \in \mathcal{A}$ can be defined in terms of a linear functional

$$\mu_a : \quad \mathbb{C}\langle X, Y \rangle \to \mathbb{C}$$
$$P \mapsto \varphi(P(a, a^*))$$

where $\mathbb{C}\langle X, Y \rangle$ is the space of non-commutative polynomials in the indeterminates X and Y.

The non-commutative polynomials give us all possible ways of multiplying a and a^* to terms of an arbitrary finite length. These are exactly the moments of the algebra generated by $\{a, a^*, 1_\mathcal{A}\}$. Summing up these terms does not change anything since φ is linear. The problem is that this definition is not practical for analytical approaches. Since we already characterized the random variables to some degree, we will look at some properties of the state.

Definition 1.11 (normal, tracial, faithful state)**.**
The state $\varphi : \mathcal{A} \to \mathbb{C}$ is called:

1. ***normal*** *if it is continuous with respect to the weak *-topology*
2. ***tracial*** *if for all $a, b \in \mathcal{A}$: $\varphi(ab) = \varphi(ba)$*
3. ***faithful*** *if for all $a \in \mathcal{A}$: $\varphi(a^*a) = 0 \implies a = 0 \cdot 1_\mathcal{A}$*

Remark.
If a state φ is tracial, it is often denoted by τ.

Proposition 1.12. *$\langle a, b\rangle_\varphi = \varphi(b^*a)$ for $a, b \in \mathcal{A}$ defines an inner product and $\|a\|_\varphi^2 = \varphi(a^*a)$ is a norm on $\mathcal{A}$.*

Proof.

1. Linearity
   For $\lambda \in \mathbb{C}$: $\langle \lambda a, b\rangle = \varphi(b^*(\lambda a)) = \varphi(\lambda b^* a) = \lambda\varphi(b^*a) = \lambda\langle a, b\rangle$
   $\langle a + b, c\rangle = \varphi(c^*(a + b)) = \varphi(c^*a) + \varphi(c^*b) = \langle a, c\rangle + \langle b, c\rangle$
2. Conjugate symmetry
   $\langle a, b\rangle = \varphi(b^*a) = \overline{\varphi((b^*a)^*)} = \overline{\varphi(a^*b)} = \overline{\langle b, a\rangle}$
3. Positive semi-definite
   Holds since $\varphi(a^*a) \geq 0$ holds in $*$-probability spaces.
4. Point-separating
   This is exactly faithfulness of the state: $\varphi(a^*a) = 0 \iff a = 0$

□

Since we have now shown that a $*$-probability space with tracial state comes with an inherent norm, we want to discuss closed algebras. This leads to Banach and C* algebras.

Definition 1.13 (Banach-algebra)**.**
*A **Banach algebra** $\mathcal{A}$ is an associative algebra over $\mathbb{C}$ with norm $\|\cdot\|$ which satisfies*

$$\forall a, b \in A\colon \|ab\| \leq \|a\| \cdot \|b\|$$

and is complete.

Definition 1.14 (C*-algebra)**.**
A Banach-algebra $\mathcal{A}$ with norm $\|\cdot\| : \mathcal{A} \to \mathbb{R}_0^+$ and involution $* : \mathcal{A} \to \mathcal{A}$ which additionally satisfies

$$\forall a \in \mathcal{A}\text{: } \|a^*a\| = \|a^*\| \cdot \|a\|$$

is called a ***C*-algebra***

These C*-algebras can actually be faithfully represented as subalgebras the space of bounded linear operators on some Hilbert space.

Definition 1.15 ($*$-representation)**.** *Let $\mathcal{A}$ be a C*-algebra and $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space. A map $\pi : \mathcal{A} \to \mathcal{B}(H)$ is called* ***$*$-representation*** *if:*

- $\forall a, b \in \mathcal{A}$: $\pi(a + b) = \pi(a) + \pi(b)$
- $\forall a, b \in \mathcal{A}$: $\pi(ab) = \pi(a)\pi(b)$
- $\pi(1_{\mathcal{A}}) = 1_{\mathcal{B}(H)}$
- $\forall a \in \mathcal{A}$: $\pi(a^*) = \pi(a)^*$ *where $*$ is the involution in $\mathcal{A}$ and the adjoint in $\mathcal{B}(H)$.*
- $\{\pi(x)\xi | x \in \mathcal{A}$ *and* $\xi \in H\}$ *is dense in H*

Theorem 1.16 (Gelfand-Naimark-Segal construction)**.** *Let $\mathcal{A}$ be a C*-algebra with unit $1_{\mathcal{A}}$ and $\tau : \mathcal{A} \to \mathbb{C}$ a positive linear functional. There exists an Hilbert space H_f and a $*$-representation π_τ between $\mathcal{A}$ and $\mathcal{B}(H)$. Further there exists a $\xi \in H_f$ such that for all $a \in \mathcal{A}$: $\tau(a) = \langle a\xi, \xi\rangle$. This ξ is called the* ***cyclic vector***
For every other $*$-representation $\pi : \mathcal{A} \to \mathcal{H}$ with $\tau(a) = \langle a\xi', \xi'\rangle_{\mathcal{H}}$ there exists $U \in \mathcal{B}(H_\tau)$ unitary such that $\pi'(\cdot) = U\pi_\tau(\cdot)U^*$. This property is called ***unitarily equivalence****.*

Proof. Define $(a, b) := \tau(b^*a)$ for $a, b \in \mathcal{A}$. It is obvious that this satisfies all properties of an inner product except for strict positive definiteness.
Let $\mathcal{N} = \{a \in \mathcal{A} | \tau(a^*a) = 0\}$ with is clearly a closed subspace of $\mathcal{A}$.
Using the Cauchy-Schwarz inequality we see that $\tau(b^*a)^2 \leq \tau(a^*a)\tau(b^*b)$. And thus for $n \in \mathcal{N}$ and $a \in \mathcal{A}$:

$$\begin{aligned} \tau\left((an)^*(an)\right)^2 &= \tau(n^*a^*an)^2 \\ &\leq \tau(n^*n)\tau((a^*an)^*(a^*an)) = 0 \end{aligned}$$

With similar arguments as above we can show for all $n \in \mathcal{N}$ and all $a \in \mathcal{A}$ that

- $\tau(na) = 0$
- $\tau(an) = 0$

This shows that $\mathcal{N}$ is a left-ideal of $\mathcal{A}$.

For $\mathcal{A}/\mathcal{N}$ we define the inner product

$$(x + \mathcal{N}, y + \mathcal{N})_\tau := \tau(y^*x)$$

which is well defined since for $a, b \in \mathcal{A}$ and $n, m \in \mathcal{N}$

$$\tau((a+n)^*(b+m)) = \tau(a^*b) + \tau(a^*m) + \tau(n^*b) + \tau(n^*m) = \tau(a^*b)$$

Denote by H_τ the closure of $\mathcal{A}/\mathcal{N}$ with respect to this inner product.
For fixed $a \in \mathcal{A}$ the map $\pi_a : \mathcal{A}/\mathcal{N} \to \mathcal{A}/\mathcal{N}$ is defined for $x+\mathcal{N} \in \mathcal{A}/\mathcal{N}$ by $\pi_a(x+\mathcal{N}) = ax$.
Since $\mathcal{A}/\mathcal{N}$ is dense in H_τ and π_a is linear and bounded (therefore continuous) we can extend π_a to H_τ making it the mapping $\pi_a : H_\tau \to H_\tau$.

Now seeing it as a mapping in $\mathcal{A}$ we get $\pi : \mathcal{A} \to \mathcal{B}(H_\tau)$ which is a $*$-representation since $\{\pi_a(1_\mathcal{A} + \mathcal{N})\xi | a \in \mathcal{A} \text{ and } \xi \in H_\tau\}$ is dense in H_τ.

Choosing $\xi = 1_\mathcal{A} + \mathcal{N} \in \mathcal{A}/\mathcal{N} \subseteq H_\tau$ we get for $a \in \mathcal{A}$

$$\begin{aligned}
(\pi(a)\xi, \xi)_\tau &\overset{\xi \in \mathcal{A}/\mathcal{N}}{=} \tau(\xi^* a \xi) \\
&= \tau((1_\mathcal{A} + \mathcal{N})^* a (1_\mathcal{A} + \mathcal{N})) \\
&= \tau(1_\mathcal{A}^* a 1_\mathcal{A}) + \tau(1_\mathcal{A}^* a \mathcal{N}) + \tau(\mathcal{N}^* a 1_\mathcal{A}) + \tau(\mathcal{N}^* a \mathcal{N}) \\
&= \tau(a)
\end{aligned}$$

Taking another $*$-representation $\pi' : \mathcal{A} \to \mathcal{H}$ with cyclic vector χ, we define U on $\pi(\mathcal{A})\xi$ by $U\pi(a)\xi = \pi'(a)\chi$ for $a \in \mathcal{A}$. This is a well-defined isometry since

$$\|\pi'(a)\chi\|^2 = \langle \pi'(a)\chi, \pi'(a)\chi \rangle = \langle \pi'(a^*a)\chi, \chi \rangle = \tau(a^*a) = \langle \pi(a^*a)\xi, \xi \rangle_\tau = \|\pi(a)\xi\|_\tau^2$$

Since $\pi(\mathcal{A})\xi$ is dense in H_τ, U can be extended to H_τ and satisfies

$$U\pi(a)\pi(b)\xi = U\pi(ab)\xi = \pi'(ab)\chi = \pi'(a)\pi'(b)\chi = \pi'(a)U\pi(b)\xi$$

Since $b \in \mathcal{A}$ was arbitrary we see $U\pi(a) = \pi'(a)U$ □

The Gelfand-Naimark-Segal construction has proven that the space $(\mathcal{A}, \tau)$ can be represented by bounded linear operators if $\mathcal{A}$ is a C* algebra. Therefore we assume from now on that $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\tau(a) = \langle a\xi, \xi \rangle$.

Definition 1.17 (von Neumann algebra)**.**
A C^*-algebra N is called a ***von Neumann algebra*** *if N is closed under taking limits in the weak-*-topology.*
That is for all $(T_n) \subset N$: $\lim_{n\to\infty} \langle T_n\xi, \eta \rangle = \langle T\xi, \eta \rangle$ for all $\xi, \eta \in H \implies T \in N$

As we will see later, these spaces allow for a description of the distribution of special random variables using probability measures. We will now summarize the properties we want for the space we will be using in the future in the following definition.

Definition 1.18. *If $\mathcal{A}$ is a von Neumann algebra containing the identity operator and τ is normal, tracial and faithful we call $(\mathcal{A}, \tau)$ a* ***W*-probability space****.*
Further by theorem 1.16 we see: $\tau(a) = \langle a\xi, \xi \rangle$

Proposition 1.19.
For every $T \in \mathcal{A}$ self-adjoint:

$$\exists^1 \mu\colon \tau\big(T^k\big) = \int_{\mathbb{R}} \lambda^k d\mu(\lambda)$$

For every $U \in \mathcal{A}$ unitary:

$$\exists^1 \nu\colon \tau\big(U^k (U^*)^l\big) = \int_{\mathbb{T}} \lambda^k \bar{\lambda}^l d\nu(\lambda)$$

Theorem 1.20 (Spectral Theorem for normal operators)**.** *For every normal linear bounded operator T there exists a unique spectral measure $E_T : \mathcal{F} \to \mathcal{L}(H)$ with compact support in a Borel-σ-algebra of $\mathbb{C}$ defined by*

$$T = \int_{\sigma(T)} t dE_T(t)$$

The map $f \mapsto f(T) = \int_{\sigma(T)} f(t) dE_T(t)$ defined by

$$\langle f(T)x, y\rangle = \int_{\sigma(T)} f(t) d\langle E_T x, y\rangle$$

defines a unique functional calculus.

Proof. Since $\mathcal{A}$ is a W^*-algebra all random variables $a \in \mathcal{A}$ are bounded linear operators and both self-adjoint and unitary operators are normal. So by the spectral theorem we see for any Borel-measurable bounded function f:

$$f(A) = \int_{\sigma(A)} f(t) dE_A(t)$$

. By definition of integrals on normed spaces we further deduce that

$$\tau(f(A)) = \tau\left(\int_{\sigma(A)} f(t) dE_A(t)\right) = \int_{\sigma(A)} f(t) d\tau(E_A)(t)$$

We define $\mu = \tau(E_A)$. By definition of the spectral measure and the fact that τ is a linear functional this is a measure.
To show that $\mu(\Omega) = 1$ we use that $E_A(\sigma(A)) = 1_{\mathcal{A}}$. Thus

$$\begin{aligned} 1 = \tau(1_{\mathcal{A}}) &= \tau\left(\int_{\sigma(A)} 1 dE_A(t)\right) \\ &= \tau\left(\int_{\mathbb{C}} 1 dE_A(t)\right) \\ &= \int_{\mathbb{C}} 1 d\tau(E_A)(t) \\ &= \int_{\mathbb{C}} 1 d\mu(t) = \mu(\mathbb{C}) \end{aligned}$$

To show that $E_A \in \mathcal{A}$ is more complicated. So we refer to [Tak01] for the full proof. □

Definition 1.21. *Since we are interested in distributions for which the moments can be represented as an integral over a probability measure, we will mostly work with self-adjoint or normal elements of $\mathcal{A}$. To refer to these random variables we will use:*

$$\mathcal{A}_{sa} = \{a \in \mathcal{A}\colon a^* = a\}$$

$$\mathcal{A}_n = \{a \in \mathcal{A}\colon a^* a = a a^*\}$$

1.2.2. Freeness

Having defined distributions of random variables, we can now think about how these distribution interact with each other. For this the concept of independence, or freeness as it is called in free probability theory, is very important. One feature why independence is so important in the classical theory is that is allows the factorization of mixed moments into polynomials of moments of only one single random variable or random variable inside the same measurable sub-space. This is the property we also want for our setting, but since we also have to account for algebraic interaction this gets more complicated.

Definition 1.22 (freeness)**.**
*Subalgebras $\mathcal{A}_1, ..., \mathcal{A}_n$ of $\mathcal{A}$ are called **free** if for all $k \in \mathbb{N}$, for all multi-indices $\iota : \{1, ..., k\} \to \{1, ..., n\}$ and for all $a_1 \in \mathcal{A}_1, ..., a_n \in \mathcal{A}_n$ such that $\tau(a_i) = 0$ for all i: $\tau(a_{\iota(1)}...a_{\iota(k)}) = 0$.*

*Let $(B_i)_{i \in I}$ be subalgebras of $\mathcal{A}$ for some index set I. Then $(B_i)_{i \in I}$ are called **free** if for any finite $J \subset I$ the subalgebras $(B_j)_{j \in J}$ are free.*

*Like in the classical case random variables are **free** if the subalgebras they generate are free. We denote the algebra generated by $a \in \mathcal{A}$ by $\mathrm{Alg}(a)$.*

For this definition it is easy to prove that it actually results in the factorization we wanted.

Proposition 1.23. *Let $\mathcal{B}_n \subseteq \mathcal{A}$ be free with $a_j \in \mathcal{B}_{\iota(j)}$ for $\iota : \{1, ..., k\} \to \{1, ..., n\}$. Then every mixed moment in the random variables $a_1, ..., a_k$ factorizes into moments in $\mathcal{B}_1, ..., \mathcal{B}_n$*

Proof.
For this proof we will use arguments similar to [DVN92]
We may assume that for all j $\iota(j) \neq \iota(j+1)$, because otherwise we can look at $a_1, ..., a_j a_{j+1}, ..., a_k$ which is again the setting of the above proposition.

The statement is clear for $k = 1$.
Define $\tilde{a}_j = a_j - \varphi(a_j)1_{\mathcal{A}}$. Then we get that for all j: $\varphi(\tilde{a}_j) = 0$ and thus by freeness and $\iota(j) \neq \iota(j+1)$ we get

$$\begin{aligned} 0 &= \varphi(\tilde{a}_1...\tilde{a}_k) \\ &= \varphi(a_1...a_k) + \mathcal{R} \end{aligned}$$

where $\mathcal{R}$ is a term of moments in $a_1, ..., a_k$ of at most length $k-1$ as they contain at least one of $\varphi(a_j)1_{\mathcal{A}}$.
But this means that $\varphi(a_1...a_k) = -\mathcal{R}$. Using the induction assumption on $\mathcal{R}$ we get the statement. □

The proposition above merely implies existence of such a factorization. Of course we would be interesting in how exactly this factorization looks like. We will start by looking at a few examples.

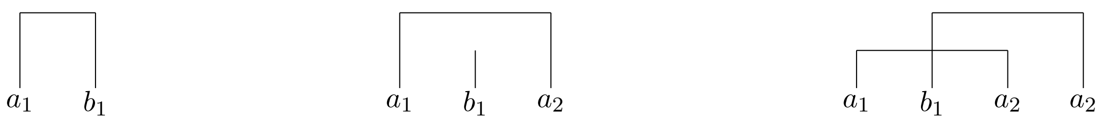
Example 1.24.
Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be free and $a_1, a_2 \in \mathcal{B}$, $b_1, b_2 \in \mathcal{C}$.

1.
$$\begin{aligned} 0 &\overset{free}{=} \varphi\big[(a_1 - \varphi(a_1)1_{\mathcal{A}})(b_1 - \varphi(b_1)1_{\mathcal{A}})\big] \\ &= \varphi\big[a_1b_1 - \varphi(a_1)b_1 - \varphi(b_1)a_1 + \varphi(a_1)\varphi(b_1)1_{\mathcal{A}}\big] \\ &= \varphi(a_1b_1) - \varphi(a_1)\varphi(b_1) \\ \Longrightarrow &\ \varphi(a_1b_1) = \varphi(a_1)\varphi(b_1) \end{aligned}$$

2.
$$\begin{aligned} 0 &\overset{free}{=} \varphi\big[(a_1 - \varphi(a_1)1_{\mathcal{A}})(b_1 - \varphi(b_1)1_{\mathcal{A}})(a_2 - \varphi(a_2)1_{\mathcal{A}})\big] \\ &= \varphi(a_1b_1a_2) - \varphi(a_1)\varphi(b_1a_2) - \varphi(b_1)\varphi(a_1a_2) - \varphi(a_2)\varphi(a_1b_2) \\ &\quad + 2\varphi(a_1)\varphi(b_1)\varphi(a_2) \\ &= \varphi(a_1b_1a_2) - \varphi(b_1)\varphi(a_1a_2) \\ \Longrightarrow &\ \varphi(a_1b_1a_2) = \varphi(b_1)\varphi(a_1a_2) \end{aligned}$$

3.
$$\begin{aligned} &\varphi\big[(a_1 - \varphi(a_1)1_{\mathcal{A}})(b_1 - \varphi(b_1)1_{\mathcal{A}})(a_2 - \varphi(a_2)1_{\mathcal{A}})(b_2 - \varphi(b_2)1_{\mathcal{A}})\big] \\ &= \varphi(a_1b_1a_2b_2) - \varphi(a_1)\varphi(a_2)\varphi(b_1b_2) - \varphi(b_1)\varphi(b_2)\varphi(a_1a_2) \\ &\quad + \varphi(a_1)\varphi(b_1)\varphi(a_2)\varphi(b_2) \\ &\Longrightarrow \ \varphi(a_1b_1a_2b_2) = \varphi(a_1)\varphi(a_2)\varphi(b_1b_2) + \varphi(b_1)\varphi(b_2)\varphi(a_1a_2) - \varphi(a_1)\varphi(b_1)\varphi(a_2)\varphi(b_2) \end{aligned}$$

One may notice that the first and second example lead to easy factorizations, whereas the third one gets us a more complicated one. The reason for this can by explained with non-crossing partitions and therefore leads to a combinatorial approach to the notion of freeness.
We can see every moment as a partition where two positions of random variables are in the same block if and only if they are in the same algebra. Drawing these partitions for the above examples leads to



So we can see that the complicated factorization comes from a moment corresponding to a partition with a crossing.

2. Combinatorial approach to freeness

As we have seen, the factorization of moments seems to be connected to non-crossing partitions. This combinatorial description of freeness was discovered by R. Speicher. For a detailed introduction we refer to [NS06].

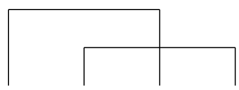
2.1. Non-crossing partitions

Definition 2.1 (non-crossing partitions)**.** *A partition π of a totally ordered set S* ***has a crossing*** *if*

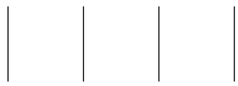
$$\exists i < j < k < l : i, k \in B_1 \text{ and } j, l \in B_2$$

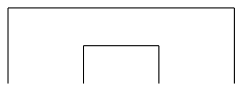
where B_1 and B_2 are blocks of π.
π is called ***non-crossing*** *if it has no crossings.*
$NC(S)$ denotes the set of non-crossing partitions.
Further we will abbreviate $NC(n) = NC(\{1, ..., n\})$.

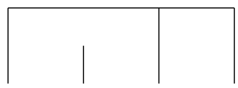
Example.
This definition fits what we have seen in the factorizations we have considered in example 1.24.
Let $S = \{1, 2, 3, 4\}$. Then in the definition of a crossing the only possibility to choose (i, j, k, l) is $(1, 2, 3, 4)$. So the only possibility to produce a crossing is $\{\{1, 3\}, \{2, 4\}\}$ which looks like



Other possibilities would be

$\{\{1\}, \{2\}, \{3\}, \{4\}\}$ 

$\{\{1, 4\}, \{2, 3\}\}$ 

$\{\{1, 3, 4\}, \{2\}\}$ 

But for our approach the following characterization is more practical.

Proposition 2.2. $\pi \in NC(S) \iff \exists V \in \pi \, block : V is\ interval\ and\ \pi \backslash V$ *non-crossing.*

Proof.

" $\implies$ ": We show this by induction with respect to the size of the set S. Assume $|S| = n$

$n = 1$: trivial

$n-1 \to n$: Assume that the statement holds for sets T such that $|T| \leq n-1$. We now choose a Block of $\pi \in NC(S)$. This Block is either an interval or has a gap. If it is an interval we have proven the statement. If it has a gap G then, since π is non-crossing, elements from within G can not be in the same block as elements from outside G and $|G| \leq n-1$. So we can apply the induction assumption to G and its sub-partition.

" $\impliedby$ ": easy to check

□

These properties match what we have seen in the example. We can consider a block of these partitions as dependent random variables (i.e. elements of the same subalgebra). A partition keeps track of the position of these random variables in the moment and the non-crossing property captures the non-commutativity of the probability space. To further illustrate this we will look at the proof of the central limit theorem of both the classical and free sense. But first we will define a new notation which will simplify working with non-crossing partitions.

Definition 2.3 (Multi-index Notation)**.** *A **multi-index** $a = (a(1), ..., a(n))$ is a function $a : \{1, ..., k\} \to I$ for some index set I.*
*The **kernel** $\ker(a)$ of a multi-index a is a partition of $\{1, ..., k\}$ such that p and q are in the same block if and only if $a(p) = a(q)$*

Remark. *We will also use the shortened form $[k] := \{1, ..., k\}$*

Theorem 2.4 (Central limit theorem)**.**
Take $(a_i)_{i\in\mathbb{N}} \in \mathcal{A}$ such that:

- $\tau(a_i^n) = \tau(a_j^n)$ *for all $i, j, n \in \mathbb{N}$ (so they are identically distributed)*
- $\tau(a_i) = 0, \tau(a_i^2) = 1$ *for all $i \in \mathbb{N}$*

and define $S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} a_i$

a) If $a_i \in \mathcal{A}$ are classically independent (so we also assume they are commutative), then

$$\lim_{n\to\infty} \tau(S_n^k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^k e^{-\frac{t^2}{2}} dt = \begin{cases} 0 & \text{if } k \text{ odd} \\ (k-1)\cdot(k-3)\cdot ... \cdot 5\cdot 3\cdot 1 & \text{else} \end{cases}$$

These are moments of the standard normal distribution.

b) If $a_i \in \mathcal{A}$ are free, then

$$\lim_{n\to\infty} \tau(S_n^k) = \begin{cases} 0 & \text{if } k \text{ odd} \\ \frac{1}{m+1}\binom{2m}{m} & \text{if } k = 2m \end{cases}$$

Proof. We will use similar arguments as in [Spe91]

$$\begin{aligned} \tau((a_1 + ... + a_n)^k) &= \sum_{\iota:[k]\to[n]} \tau(a_{\iota(1)}...a_{\iota(k)}) \\ &= \sum_{\pi\in\mathcal{P}(k)} \sum_{\substack{\iota:[k]\to[n] \\ \ker(\iota)=\pi}} \tau(a_{\iota(1)}...a_{\iota(k)}) \end{aligned}$$

And since the a_i are identically distributed, the term $\tau(a_{\iota(1)}...a_{\iota(k)})$ only depends on π. So we will call it $g(\pi)$ which results in

$$\begin{aligned} \tau((a_1 + ... + a_n)^k) &= \sum_{\pi\in\mathcal{P}(k)} \sum_{\substack{\iota:[k]\to[n] \\ \ker(\iota)=\pi}} g(\pi) \\ &= \sum_{\pi\in\mathcal{P}(k)} \#\{\iota : [k] \to [n] : \ker(\iota) = \pi\} g(\pi) \\ &= \sum_{\pi\in\mathcal{P}(k)} (n \cdot (n-1) \cdot ... \cdot (n - \#\pi + 1)) \cdot g(\pi) \\ &\sim \sum_{\pi\in\mathcal{P}(k)} n^{\#\pi} \cdot g(\pi) \text{ for } n \to \infty \end{aligned}$$

Now suppose that π contains a singleton, i.e. a block consisting of only one element. If $a_{\iota(s)}$ is the singleton then we can define $b_1 = a_{\iota(1)}...a_{\iota(s-1)}$ and $b_2 = a_{\iota(s+1)}...a_{\iota(k)}$. Since $a_{\iota(s)}$ is independent (or free) of all other $a_{\iota(l)}$ as it is a singleton, it is also independent (or free) of b_1 and b_2. Thus by independence (or for freeness like shown in example 1.24)

$$g(\pi) = \tau(a_{\iota(1)}...a_{\iota(k)}) = \tau(b_1 a_{\iota(s)} b_2) = \tau(a_{\iota(s)})\tau(b_1 b_2)$$

But since $\tau(a_i) = 0$ for all i the term vanishes

So the only relevant terms satisfy $\#V \geq 2$ for all blocks V of the partition π. For such partitions $\pi \in \mathcal{P}(k)$ it also holds that $\#\pi \leq \frac{k}{2}$. Hence $\tau(S_n^k) \sim \frac{1}{n^{\frac{k}{2}}} \sum_{\substack{\pi\in\mathcal{P}(k) \\ \#\pi\leq\frac{k}{2}}} g(\pi) n^{\#\pi} \xrightarrow{n\to\infty} \sum_{\substack{\pi\in\mathcal{P}(k) \\ \#\pi\leq\frac{k}{2}}} g(\pi) \lim_{n\to\infty}$

And since $\lim_{n\to\infty} \frac{n^{\#\pi}}{n^{\frac{k}{2}}} = \begin{cases} 0 & \#\pi < \frac{k}{2} \\ 1 & \#\pi = \frac{k}{2} \end{cases}$.

So the only relevant partitions have exactly $\frac{k}{2}$ blocks and each block has at least 2 elements. These are exactly the the pair partitions. We call the set of pair partitions $\mathcal{P}_2(k)$ Thus we already see that $\tau(S_n^k) \to 0$ for k odd.
Note that every step so far works for both cases. Now we start to differentiate:

1. Classical independence
   Since every random variable in a moment is contained exactly twice we know that $g(\pi) = 1$ for all $\pi \in \mathcal{P}_2(k)$. And thus $\tau(S_n^k) \to \#\mathcal{P}_2(k)$. This can be counted easily resulting in $\#\mathcal{P}_2(k) = (k-1)(k-2)...5 \cdot 3 \cdot 1$

2. Free independence
   Let $k = 2m$ and $\pi \in \mathcal{P}_2(2m)$.
   If π is crossing we can strip interval-blocks (by the formula $\tau(a_1 b a_2) = \tau(b)\tau(a_1 a_2)$ they can be extracted from the state, but since $\tau(a_i^2) = 1$ they make no difference)

until neighbors are always from different algebras. But by definition of freeness such moments are 0.
If π is non-crossing by the equivalence shown in proposition 2.2 we can continue the scheme described above until nothing is left and thus showing $g(\pi) = 1$.
So the only thing left is to calculate $C_m = \#\{\pi \in \mathcal{P}_2(2m) : \pi \text{ is non-crossing}\}$ which satisfies the recursion $C_m = \sum_{k=1}^{m} C_{k-1}C_{m-n}$ (for every possible splitting point we split in left and right) and thus we have proven that C_m are the Catalan numbers $C_m = \frac{1}{m+1}\binom{2m}{m}$

□

2.2. Non-crossing cumulants

Non-crossing cumulants are function which will help describe the factorization of moments with the help of partitions. First we will prove the easy factorization if we have moments corresponding to non-crossing partitions.

Theorem 2.5. *Let ι a multi-index with $\ker(\iota) = \{V_1, ..., V_m\} \subseteq NC(k)$, $\mathcal{A}_1, ..., \mathcal{A}_n \subseteq \mathcal{A}$ free subalgebras of $\mathcal{A}$ and $a_j \in \mathcal{A}_{\iota(j)}$ random variables. Then*

$$\tau(a_1 \cdot ... \cdot a_k) = \tau\left(\overrightarrow{\prod_{k\in V_1}} a_k\right) \cdot ... \cdot \tau\left(\overrightarrow{\prod_{k\in V_m}} a_k\right)$$

where $\overrightarrow{\prod}$ is used to emphasize order in the product.

Proof. We may assume that neighboring elements in $a_1...a_k$ are not from the same subalgebra, because otherwise $a_i a_{i+1} = \bar{a}_i \in \mathcal{A}_{\iota(i)}$.
Choose a block V of $\ker(\iota)$ and suppose $V = \{l_1, ..., l_n\}$ then we get
$\mathbb{E} = \tau(a_1...a_k) = \tau(b_1 a_{l_1} b_2 a_{l_2} ... b_n a_{l_n} b_{n+1})$ where, because $\ker(\iota) \subseteq NC(k)$, all b_j are free from each and from the subalgebra corresponding to the chosen block V.
Since b_1 and b_n are free from everything else we can use the already shown factorization to get $\mathbb{E} = \tau(b_1)\tau(b_{n+1})\tau(a_{l_1} b_2 a_{l_2} ... b_n a_{l_n})$
b_2 is also free from every other appearing term so we can write $\tau(a_{l_1} b_2 a_{l_2} ... b_n a_{l_n}) = \tau(a_{l_1} b_2 c) \overset{\text{factorization}}{=} \tau(b_2)\tau(a_{l_1} c) = \tau(b_2)\tau(a_{l_1} a_{l_2} ... b_n a_{l_n})$
We can continue this procedure for all b_j with the same arguments. Afterwards we are left with $\mathbb{E} = \tau(\overrightarrow{\prod}_{k\in V} a_k) \cdot \prod_{i=1}^{n} \tau(b_i)$
If now one of the b_i contains random variables a_j which are contained in more than one block of $\ker(\iota)$ then we can apply the same procedure to $\tau(b_i)$. At the end we arrive at the desired result. □

Definition 2.6. *For $n \geq 1$ let $\psi^{(n)}$ be n-linear forms on $(\mathcal{A}, \tau)$.*
For $\pi \in NC(n)$ define $\psi[\pi]$ by $\psi[\pi](a_1, ..., a_n) = \prod_{V\in\pi} \psi^{(|V|)}(a_V)$ where $a_V = (a_{j_1}, ..., a_{j_K})$ if $V = \{j_1, ..., j_K\}$ is a block of partition π.
If $\psi^{(n)}(a_1, ..., a_n) = \tau(a_1 \cdot ... \cdot a_n)$ then we denote the corresponding n-linear map $\psi[\pi]$ by $\tau[\pi]$.

Remark. *If the dimension of the linear form is clear we just write* ψ*.*

Proposition 2.7. *On a* W^**-probability-space* $(\mathcal{A}, \tau)$ *the linear form defined above satisfies* $|\tau[\pi](a_1, ..., a_n)| \leq \prod_{j=1}^{n} \|a_j\|$

Proof. Let $\chi : \mathcal{A} \to B(H)$ be an isomorphism and $\tau(a) = \langle \Omega, \chi(a)\Omega \rangle$ for all $a \in \mathcal{A}$. With this we have

$$|\tau[\pi](a_1, ..., a_n)| = \left| \prod_{V \in \pi} \tau(a_V) \right| = \prod_{V \in \pi} |\tau(a_V)|$$

Let $V = \{j_1, ..., j_K\} \in \pi$. Hence $\tau(a_V) = \tau(a_{j_1} \cdot ... \cdot a_{j_K})$

$$\begin{aligned} |\tau(a_{j_1} \cdot ... \cdot a_{j_K})|^2 &= |\langle \Omega, \chi(a_{j_1}) \cdot ... \cdot \chi(a_{j_K}) \cdot \Omega \rangle|^2 \\ &\overset{\text{Cauchy-Schwarz}}{\leq} \langle \Omega, \Omega \rangle \cdot \langle \chi(a_{j_1}) \cdot ... \cdot \chi(a_{j_K}) \cdot \Omega, \chi(a_{j_1}) \cdot ... \cdot \chi(a_{j_K}) \cdot \Omega \rangle \\ &= \|\Omega\|^2 \cdot \|\chi(a_{j_1}) \cdot ... \cdot \chi(a_{j_K}) \cdot \Omega\|^2 \\ &\leq \|\chi(a_{j_1})\|^2 \cdot ... \cdot \|\chi(a_{j_K})\|^2 \\ &= \|a_{j_1}\|^2 \cdot ... \cdot \|a_{j_K}\|^2 \end{aligned}$$

□

Definition 2.8 (non-crossing cumulants)**.** ***Non-crossing cumulants*** *are* n *linear maps given by*

$$R(a_1, ..., a_n) = \sum_{\pi \in NC(n)} \mu(\pi)\tau[\pi](a_1, ..., a_n)$$

where $\mu(\pi) = \mu(\pi, 1_{\mathcal{A}})$ *is the* ***Möbius function*** *on* $NC(n)$

Definition 2.9. *The following holds*

$$\tau(a_1, ..., a_n) = \sum_{\pi \in NC(n)} R[\pi](a_1, ..., a_n)$$

and more generally

$$\tau[\pi](a_1, ..., a_n) = \sum_{\sigma \in NC(n), \sigma \leq \pi} R[\sigma](a_1, ..., a_n)$$

where $R[\pi]$ *are n-linear forms corresponding to* R

We will now show that these functions indeed describe the factorization of moments.

Proposition 2.10. *Let* $(B_i)_{i \in I}$ *be free with* $a_j \in B_{i_j}$*. Then*

$$\exists j, k : i_j \neq i_k \implies R(a_1, ..., a_n) = 0$$

Proof. This statement was proven in [NS06] in Theorem 11.16. □

Remark. *For an alternative proof of this statement based on Weisner's lemma we refer to [Zwi12] theorem 24.*

Proposition 2.11. *Let B_1, B_2 be free and $a_1, ..., a_n \in \mathcal{A}$ like described in the statement. Let S_1, S_2 be two subsets s.t. $S_1 = \{j \in \{1, ..., n\} | a_j \in B_1\}$ and $S_2 = \{j \in \{1, ..., n\} | a_j \in B_2\}$. Let now π_1 be a partition of S_1 and π_2 a partition of S_2. Then $\pi_1 \cup \pi_2$ is a partition of $S_1 \cup S_2 = \{1, ..., n\}$. Assume that $\pi_1 \cup \pi_2$ is non-crossing and let ψ_1, ψ_2 be families of multi-linear functionals. We define*

$$\psi_1 \cup \psi_2[\pi_1 \cup \pi_2](a_1, ..., a_n) = \prod_{V \in \pi_1} \psi_1(a_V) \prod_{V \in \pi_2} \psi_2(a_V)$$

Let $\pi \in NC(S_1)$ and π^c the maximal non-crossing partition of S_2 such that $\pi \cup \pi^c$ is non-crossing. Then by proposition 2.10 the only terms contributing to

$$\tau(a_1 ... a_n) = \sum_{\pi \in NC(n)} R[\pi](a_1, ..., a_n)$$

are $\pi_1 \cup \pi_2$ where $\pi_i \in NC(S_i)$ and thus

$$\begin{aligned} \tau(a_1 ... a_n) &= \sum_{\pi_1 \in NC(S_1)} \sum_{\substack{\pi_2 \in NC(S_2) \\ \pi_1 \cup \pi_2 \in NC(n)}} R \cup R[\pi_1 \cup \pi_2](a_1, ..., a_n) \\ &= \sum_{\pi_1 \in NC(S_1)} \sum_{\substack{\pi_2 \in NC(S_2) \\ \pi_2 \leq \pi_1^c}} R \cup R[\pi_1 \cup \pi_2](a_1, ..., a_n) \end{aligned}$$

Then by applying 2.9 we get
$\tau(a_1 ... a_n) = \sum_{\pi_1 \in NC(S_1)} R \cup \tau[\pi_1 \cup \pi_1^c](a_1, ..., a_n)$

Remark. *This formula shows explicitly how mixed moments of free subalgebras can be decomposed such that it only depends on pure moments of each of the subalgebras as seen in Prop 1.23.*

3. Non-commutative convolutions

In this chapter we want to deduce rules how the manipulation of free random variables influences their distribution. For this we will look at different analytical transformations. Another analytical tool which is very important when working with processes is the conditional expectation. We will start by deriving a non-commutative analog.

3.1. Conditional expectation

In the classical theory the conditional expectations can be interpreted as an L^2-projection.

Theorem 3.1 (Existence of the Conditional Expectation)**.**
Let $\mathcal{B} \subset \mathcal{A}$ be a von Neumann subalgebra of $\mathcal{A}$. Then there exists a projection operator $\tau(\cdot|\mathcal{B})$ satisfying the $\mathcal{B}$-$\mathcal{B}$-bi-module property

$$\forall b_1, b_2 \in \mathcal{B}\ \forall a \in \mathcal{A}\text{: } \tau(b_1 a b_2|\mathcal{B}) = b_1 \tau(a|\mathcal{B}) b_2.$$

$\|x\|_1 := \tau(|x|)$ defines a norm where $|x| := (x^*x)^{\frac{1}{2}}$.
$L^1(\mathcal{A})$ is the completion of $\mathcal{A}$ with respect to this norm and $L^\infty(\mathcal{A})$ the completions with respect to the operator norm.
Since $\mathcal{A}$ is a von Neumann algebra, $L^\infty(\mathcal{A}) = \mathcal{A}$.

Lemma 3.2. *The dual space of $L^1(\mathcal{A})$ is $L^\infty(\mathcal{A}) = \mathcal{A}$*

Proof. This proof can be found in [Tak01].
The main idea is to show that $\psi:\ L^1(\mathcal{A}) \to M^*$ defined by $\psi(a)(\cdot) = \tau(a\cdot)$. is a linear isomorphism □

Lemma 3.3. *For $a, c \in \mathcal{A}$:*
$\tau(ab) = \tau(cb)$ for all $b \in \mathcal{A} \implies a = c$

Proof. Assume for all $b \in \mathcal{A}$ that $\tau(ab) = \tau(cb)$.
By linearity we get $\tau((a-c)b) = 0$ for all $b \in \mathcal{A}$.
In particular the choice $b = (a-c)^* \in \mathcal{A}$ yields by faithfulness $a - c = 0$ □

Proof of the Existence of the conditional Expectation.
Since $\mathcal{B} \subseteq \mathcal{A}$ is a subalgebra, we get that also the completions $L^1(\mathcal{A})$ and $L^1(\mathcal{B})$ are also sub spaces.
Because $L^1(\mathcal{A})$ and $L^1(\mathcal{B})$ are complete, there exists an isometric embedding $\psi : L^1(\mathcal{B}) \to L^1(\mathcal{A})$.
Then the dual map of ψ

$$\psi^* : L^1(\mathcal{A})^* = L^\infty(\mathcal{A}) = \mathcal{A} \to L^1(\mathcal{B})^* = L^\infty(\mathcal{B}) = \mathcal{B}$$

is defined as $\psi^*(\eta(x)) = \eta(\psi(x))$.
Since ψ is injective, ψ^* is surjective.
Now we can show the $\mathcal{B}$-$\mathcal{B}$-bi-module property. For this we take $x \in \mathcal{A}$, $y \in \mathcal{B}$ and $E = \psi^*$.
Then $\tau(E(x)y) = \langle E(x), y\rangle \overset{\text{Dual Map}}{=} \langle x, \psi(y)\rangle \overset{\text{projection}}{=} \langle x, y\rangle = \tau(xy)$
Since τ is tracial, $\tau(E(x)y) = \tau(xy)$
(This holds for all $y \in \mathcal{B}$ and thus $\|E\| = 1$)
Let now $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$.
Then $\tau(E(b_1 a)b_2) = \tau(b_1 a b_2) \overset{\text{tracial}}{=} \tau(a b_2 b_1) = \tau(E(a)b_2 b_1) \overset{\text{tracial}}{=} \tau(b_1 E(a) b_2)$
Since this holds for all $b_2 \in \mathcal{B}$ we get by lemma 3.3 that $E(b_1 a) = b_1 E(a)$.
One can show the property right multiplicativity analogously.
Finally we have to show that E is idempotent (and therefore a projection) Let again $b_1, b_2 \in \mathcal{B}$. Then since $b_1 \in \mathcal{B} \subseteq \mathcal{A}$, we get by the above that $\tau(b_1 b_2) = \tau(E(b_1)b_2)$.
Since this holds for any $b_2 \in \mathcal{B}$ this implies by lemma 3.3 that $b_1 = E(b_1)$. Thus E is idempotent. □

Having shown the existence of the non-commutative conditional expectation, we will now show some of the basic properties.

Theorem 3.4 (Properties of the Conditional Expectation)**.**
Let $a \in \mathcal{A}$ and $\mathcal{B} \subset \mathcal{A}$. Then

1. $\tau(a^*|\mathcal{B}) = \tau(a|\mathcal{B})^*$
2. $\tau(a^*a|\mathcal{B})$ *is positive*
   i.e. $\exists b \in \mathcal{B}$ *such that* $\tau(a^*a|\mathcal{B}) = b^*b$
3. $\tau(a|\mathcal{B})^*\tau(a|\mathcal{B}) \leq \tau(a^*a|\mathcal{B})$
4. $\|\tau(a|\mathcal{B})\| \leq \|a\|$

Proof.

1. We have shown in the existence of the conditional expectation that $\tau(ab) = \tau(\tau(a|\mathcal{B})b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$
   Therefore we have $\tau(a^*b) = \tau(\tau(a^*|\mathcal{B})b)$
   Further more we get $\tau(a^*b) = \overline{\tau(b^*a)} = \overline{\tau(b^*\tau(a|\mathcal{B}))} = \tau(\tau(a|\mathcal{B})^*b)$
   Since these equalities hold for every $b \in \mathcal{B}$ we infer by Lemma 3.3 that $\tau(a^*|\mathcal{B}) = \tau(a|\mathcal{B})^*$
2. In [Tom57] it was proven that every projection of norm one from one C^*-algebra onto a C^*-subalgebra is positive. Therefore this is especially also true for W^*-algebra.
3. Let $b \in \mathcal{B} \subseteq \mathcal{A}$ and $a \in \mathcal{A}$. Then $(a-b)^*(a-b) \geq 0$ and thus because of (2) the following holds.

$$
\begin{aligned}
0 &\leq \tau((a-b)^*(a-b)|\mathcal{B}) \\
&= \tau(a^*a - b^*a - a^*b + b^*b|\mathcal{B}) \\
&= \tau(a^*a|\mathcal{B}) - b^*\tau(a|\mathcal{B}) - \tau(a|\mathcal{B})^*b + b^*b
\end{aligned}
$$

If we now choose $b = \tau(a|\mathcal{B}) \in \mathcal{B}$ then we obtain $0 \leq \tau(a^*a|\mathcal{B}) - \tau(a|\mathcal{B})^*\tau(a|\mathcal{B})$

4. Since $a^*a \leq \|a\|^2 1_{\mathcal{A}}$ we see using (3)

$$\tau(a|\mathcal{B})^*\tau(a|\mathcal{B}) \leq \tau(a^*a|\mathcal{B}) \leq \|a\|^2\tau(1_{\mathcal{A}}|\mathcal{B}) = \|a\|^2 1_{\mathcal{A}}$$

and thus $\|\tau(a|\mathcal{B})\|^2 \leq \|a\|^2$

□

Proposition 3.5.
Let $X \in \mathcal{A}$ be normal with spectrum $\sigma(X)$. Then $\sigma(\tau(X|B)) \subset \mathrm{Conv}(\sigma(X))$ where Conv *denotes the convex hull.*

Lemma 3.6. *Let $T : H \to H$ and define the numerical range $W(T) := \{\langle T\xi, \xi\rangle | \xi \in H, \|\xi\| = 1\}$.*
Then $\sigma(T) \subseteq \overline{W(T)}$.
Further if T is normal $\overline{W(T)} = \mathrm{Conv}(\sigma(T))$.

Proof. By Lemma 3.6 we get that $\mathrm{Conv}(\sigma(X)) = \overline{W(X)}$ since X is normal and $\sigma(\tau(X|B)) \subseteq \overline{W(\tau(X|B))}$.
Since

$$W(\tau(X|B)) \subseteq C(\sigma(X)) \implies \overline{W(\tau(X|B))} \subseteq C(\sigma(X)) \implies \sigma(\tau(X|B)) \subseteq \sigma(X)$$

it is left to show that $W(\tau(X|B)) \subseteq C(\sigma(X))$.
Since X is normal we can write its spectral decomposition as

$$X = \int_{\sigma(X)} zP(dz)$$

and since $\tau(\cdot|B)$ is bounded and linear

$$\tau(X|B) = \int_{\sigma(X)} z\tau(P(dz)|B)$$

and thus

$$\langle\tau(X|B)\xi, \xi\rangle = \int_{\sigma(X)} z\langle\tau(P(dz)|B)\xi, \xi\rangle$$

Since $\tau(1) = 1$ and $P(\mathbb{C}) = 1$, by definition of the spectral integral $\mu(A) = \langle\tau(P(A)|B)\xi, \xi\rangle$ is a probability measure and thus the claim follows. □

Remark.
Let $X \in \tilde{\mathcal{A}}_{sa}$. Then $W(X) \subseteq \mathbb{R}$ and hence the spectrum of X is real.

Definition 3.7.
Let $X \in \tilde{\mathcal{A}}_{sa}$. Then the analytic map $\Re_X(\xi) = (\xi - X)^{-1}$ is called the ***resolvent map*** *which is well defined for $\xi \in \mathbb{C} \setminus \mathbb{R}$*

Lemma 3.8. *Let T be invertible. Then $(T^*)^{-1} = (T^{-1})^*$*

Proof. let $u, v \in H$. Then $\langle T^*(T^{-1})^*u, v\rangle = \langle (T^{-1})^*u, Tv\rangle = \langle u, T^{-1}Tv\rangle = \langle u, v\rangle$
Since this holds for arbitrary $u, v \in H$ we get $T^*(T^{-1})^* = 1_{\mathcal{B}(H)}$ and thus $(T^*)^{-1} = (T^{-1})^*$ □

Proposition 3.9. $\Re_X(\xi)^* = \Re_X(\bar{\xi})$

Proof. $\Re_X(\xi)^* = \left((\xi - X)^{-1}\right)^* \overset{\text{Lemma 3.8}}{=} ((\xi - X)^*)^{-1} \overset{X \text{ self-adjoint}}{=} (\bar{\xi} - X)^{-1} = \Re_X(\bar{\xi})$ □

Remark.
From proposition 3.5 it follows that $\sigma(\mathrm{Im}(\tau(\Re_X(\xi)|B)))$ is bounded away from zero for $\xi \in \mathbb{C} \setminus \mathbb{R}$

3.2. Free additive convolution

Given two probability measure μ and ν on $\mathbb{R}$, there exists a probability space $(\mathcal{A}, \tau)$ containing the self-adjoint random variables S and T such that $S \sim \mu$ and $T \sim \nu$ and S and T are free.
As we have seen in the factorization of moments the distribution of $S + T$ only depends only on μ and ν. This dependency can be described by the free convolution operator $\boxplus$. So the distribution of $S + T$ is the probability measure $\mu \boxplus \nu$.

Remark.
Because of the commutativity of freeness (i.e. if S if free from T then T is free from S) free convolution is commutative. Even for other types of independence (e.g. monotone independence) we still have that the distribution of $S + T$ only depends on μ and ν, but we may lose the commutativity of independence, in which case the respective convolution is non-commutative.

To compute the free convolution of measures explicitly we will use an analytical tool called the Cauchy transform. For a more detailed overview of this one may look at [SM17]

Definition 3.10 (Cauchy Transform)**.**
The ***Cauchy transform*** *of μ*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{1}{\xi - t} d\mu(t)$$

is analytic on $\mathbb{C} \setminus \mathbb{R}$.

Remark. *Note that* $\frac{1}{\xi - t} = \sum_{n \geq 0} \frac{t^n}{\xi^{n+1}}$ *and thus*

$$G_\mu(\xi) = \sum_{n \geq 0} \frac{1}{\xi^{n+1}} \int_{\mathbb{R}} t^n d\mu(t)$$

which is a generating series for the sequence of moments $m_n = \int_{\mathbb{R}} t^n d\mu(t)$

Proposition 3.11.
The Cauchy Transform satisfies the following properties:

1. $G_\mu(\overline{\xi}) = \overline{G_\mu(\xi)}$
2. $G_\mu(\mathbb{C}^+) \subset \mathbb{C}^-$

Remark. *The integral used in the definition of the Cauchy transform is always convergent since* $|\xi - t|^{-1} \leq |\operatorname{Im}(\xi)|^{-1}$ *and* μ *is a probability measure.*

Proof.

1. trivial
2. Assume that $\operatorname{Im}(w) \neq 0$ and $|z - w| < \frac{|\operatorname{Im}(w)|}{2}$ for some complex numbers z and w. Then for $t \in \mathbb{R}$ we get

$$\left| \frac{z - w}{t - w} \right| < \frac{|\operatorname{Im}(w)|}{2} \frac{1}{|\operatorname{Im}(w)|} = \frac{1}{2}$$

and thus the geometric series $\sum_{n=0}^{\infty} \left(\frac{z-w}{t-w} \right)^n$ converges uniformly to $\frac{t-w}{t-z}$. So we get

$$(z - t)^{-1} = -\sum_{n=0}^{\infty} (t - w)^{-(n+1)} (z - w)^n$$

Hence

$$G(z) = -\sum_{n=0}^{\infty} \left[\int_{\mathbb{R}} (t - w)^{-(n+1)} d\mu(t) \right] (z - w)^n$$

is analytic on $|z - w| < \frac{|\operatorname{Im}(w)|}{2}$
If $\operatorname{Im}(z) > 0$ then for $t \in \mathbb{R}$: $\operatorname{Im}((z - t)^{-1}) < 0$ and thus also $\operatorname{Im}(G(z)) < 0$. Thus $G(\mathbb{C}^+) \subseteq \mathbb{C}^-$.

□

Remark. *The proof above also shows that* G *is analytic.*

Lemma 3.12.

$$\lim_{y \to \infty} iyG(iy) = 1 \text{ and } \sup_{y > 0, x \in \mathbb{R}} y|G(x + iy)| = 1$$

Proof.

$$\begin{aligned}
\frac{1}{iy-t} &= \frac{iy+t}{iy+t}\frac{1}{iy-t} \\
&= -\frac{iy+t}{y^2+t^2} = -\frac{t}{y^2+t^2} - i\frac{y}{y^2+t^2} \\
y\,\mathrm{Im}(G(iy)) &= \int_{\mathbb{R}} y\,\mathrm{Im}\left(\frac{1}{iy-t}\right) d\mu(t) \\
& \int_{\mathbb{R}} -\frac{y^2}{y^2+t^2} d\mu(t) \\
& -\int_{\mathbb{R}} \frac{1}{1+(t/y)^2} d\mu(t) \\
\xrightarrow{y\to\infty} & -\int_{\mathbb{R}} 1 d\mu(t) = -1
\end{aligned}$$

We could take the limit in the last step by dominated convergence since $\frac{1}{1+(t/y)^2} \leq 1$. Doing the same for the real part results in

$$y\,\mathrm{Re}(G(iy)) = -\int_{\mathbb{R}} \frac{yt}{y^2+t^2} d\mu(t)$$

again we can apply dominated convergence since $\left|\frac{yt}{y^2+t^2}\right| \leq \frac{1}{2}$ and $\left|\frac{yt}{y^2+t^2}\right| \xrightarrow{y\to\infty} 0$ resulting in

$$y\,\mathrm{Re}(G(iy)) \xrightarrow{y\to\infty} 0$$

This gives the first part of the statement.

For the supremum we take $y > 0$ and $z = x + iy$. Then

$$y|G(y)| \leq \int_{\mathbb{R}} \frac{y}{|z-y|} d\mu(t) = \int_{\mathbb{R}} \frac{y}{\sqrt{(x-t)^2+y^2}} d\mu(t) \leq 1$$

Thus we have shown that $\sup_{y>0,x\in\mathbb{R}} y|G(x+iy)| \leq 1$, but in the first part we have seen that 1 is actually reached. □

Definition 3.13 (Poisson kernel)**.** *We recall the Poisson kernel* $P(t) = \frac{1}{\pi}\frac{1}{1+t^2}$ *and* $P_\epsilon(t) = \frac{1}{\epsilon}\frac{\epsilon}{t^2+\epsilon^2}$ *for* $\epsilon > 0$

Definition 3.14 (convolution)**.** *For* $f \in L^1(\mathbb{R}, \mu)$

$$[f * \mu](t) = \int_{-\infty}^{\infty} f(t-s) d\mu(s)$$

Remark. *Since the Poisson kernel* P *is bounded* $P_\epsilon * \nu$ *is well defined for any* ν *and* $\epsilon > 0$.
P_ϵ *is the density of the Cauchy distribution in the classical sense with parameter* ϵ*. We will call this distribution* $\delta_{-i\epsilon}$

Remark. *The probability measure* $\delta_{-i\epsilon} * \mu$ *has density function* $[P_\epsilon * \mu](x) = -\frac{1}{\pi}\,\mathrm{Im}(G(x+i\epsilon))$.
Since $\delta_{-i\epsilon} * \mu \xrightarrow{\epsilon\to 0} \mu$*, we can recover the measure* μ *by using the* ***Stieltjes inversion formula***

Theorem 3.15 (Stieltjes inversion formula)**.** *Let* $G : \mathbb{C}^+ \to \mathbb{C}^-$ *be analytic with* $\limsup_{y\to\infty} |G(iy)| = c < \infty$*. Then*

$$\exists^1 v : G(z) = \int_{\mathbb{R}} \frac{1}{\xi - t} dv(t) \text{ and } v(\mathbb{R}) = c$$

For a detailed proof of this we refer to [SM17].

Theorem 3.16. *Let* μ *be a probability measure with Cauchy transform* G*. Then for* $a < b$*:*

$$-\lim_{y\to 0^+} \frac{1}{\pi} \int_a^b \operatorname{Im}(G(x+iy))dx = \mu((a,b)) + \frac{1}{2}\mu(\{a,b\})$$

Further if μ_1 *and* μ_2 *are probability measures on* $\mathbb{R}$ *with respective Cauchy transform* G_{μ_1} *and* G_{μ_2}*, then* $G_{\mu_1} = G_{\mu_2} \implies \mu_1 = \mu_2$

Proof. $\operatorname{Im}(G(x+iy)) = \int_{\mathbb{R}} \frac{-y}{(x-t)^2+y^2} d\mu(t)$. So

$$\begin{aligned}
\int_a^b \operatorname{Im}(G(x+iy))dx &= \int_{\mathbb{R}} \int_a^b \frac{-y}{(x-t)^2+y^2} dx d\mu(t) \\
&\overset{\tilde{x}=\frac{x-t}{y}}{=} -\int_{\mathbb{R}} \int_{(a-t)/y}^{(b-t)/y} \frac{1}{1+\tilde{x}^2} d\tilde{x} d\mu(t) \\
&= -\int_{\mathbb{R}} \left[\tan^{-1}\left(\frac{b-t}{y}\right) - \tan^{-1}\left(\frac{a-t}{y}\right)\right] d\mu(t)
\end{aligned}$$

Define $f(y,t) = \tan^{-1}\left(\frac{b-t}{y}\right) - \tan^{-1}\left(\frac{a-t}{y}\right)$ and

$$f(t) = \begin{cases} 0 & t \notin [a,b] \\ \frac{\pi}{2} & t \in \{a,b\} \\ \pi & t \in (a,b) \end{cases}$$

Note that $\lim_{y\to 0^+} f(y,t) = f(t)$ and $|f(y,t)| \leq \pi$, thus by dominated convergence

$$\begin{aligned}
\lim_{y\to 0^+} \int_a^b \operatorname{Im}(G(x+iy))dx &= -\lim_{y\to 0^+} \int_{\mathbb{R}} f(y,t)d\mu(t) \\
&= -\int_{\mathbb{R}} f(t)d\mu(t) \\
&= -\pi\left(\mu((a,b)) + \tfrac{1}{2}\mu(\{a,b\})\right)
\end{aligned}$$

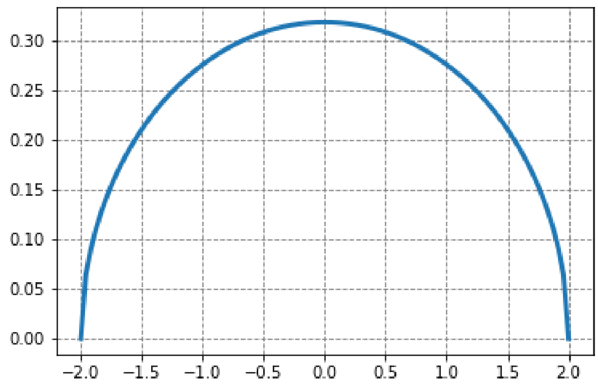
Assume that G_{μ_1} and G_{μ_2} are Cauchy transforms of two probability measures satisfying $G_{\mu_1} = G_{\mu_2}$. By the results above, this already implies that $\mu_1((a,b)) = \mu_2((a,b))$ for all a, b which are not atoms of μ_1 or μ_2. Since μ_1 and μ_2 have only countably many atoms, we can write any interval in the form $(a,b) = \bigcup_{n=1}^{\infty}(a+\epsilon_n, b-\epsilon_n)$ for a decreasing sequence $\epsilon_n \to 0^+$ such that all $a+\epsilon_n$ and $b-\epsilon_n$ are not atoms of μ_1 or μ_2. Then

$$\mu_1((a,b)) = \lim_{n\to\infty} \mu_1((a+\epsilon_n, b-\epsilon_n)) = \lim_{n\to\infty} \mu_2((a+\epsilon_n, b-\epsilon_n)) = \mu_2((a,b))$$

□

We will now look at an example. For that purpose we choose the semi-circle distribution which is an important and interesting example because as we have seen it is the analog to the normal distribution as it arises as the limit distribution of the central limit theorem.

Example. *The density the semi-circle distribution is* $d\mu(t) = \frac{\sqrt{4-t^2}}{2\pi}\mathbb{1}_{[-2,2]}(t)dt$. *The following plot explains the name of the distribution:*



Definition 3.17. *We can also encode the moments* m_n *in the ordinary moment-generating function which is more convenient than the exponential moment-generating function from classical probability theory.*

$$M(z) = \sum_{n\geq 0} m_n z^n$$

Comparison with the Cauchy transform in remark 3.2 shows that $M(z) = \frac{1}{z}G_\mu\left(\frac{1}{z}\right)$

As we have seen the moments of the semi-circle distribution are given by

$$m_n = \int_{-2}^{2} t^n d\mu(t) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ C_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ *are the Catalan numbers which are characterized by the recurrence* $\sum_{m+n=k} C_m C_n = C_{k+1}$.
The moment generating function of this distribution is thus
$M(z) = 1 + C_1 z^2 + C_2 z^4 + \cdots$
and by applying the formula above we see that

$$\begin{aligned} M(z)^2 &= \textstyle\sum_{m,n\geq 0} C_m C_n z^{2(m+n)} \\ &= \textstyle\sum_{k\geq 0}\left(\sum_{m+n=k} C_m C_n\right) z^{2k} \\ &= \textstyle\sum_{k\geq 0} C_{k+1} z^{2k} \\ &= \frac{1}{z^2}\textstyle\sum_{k\geq 0} C_{k+1} z^{2(k+1)} \end{aligned}$$

Hence we get that $z^2 M(z)^2 = M(z) - 1$ *and by replacing* $M(z)$ *by* $\frac{1}{z}G\left(\frac{1}{z}\right)$ *we obtain the following equation for the Cauchy transform*

$$zG(z) = 1 + G(z)^2$$

Solving this quadratic equations yields the following two branches:

$$G(z) = \frac{z \pm \sqrt{z^2-4}}{2}$$

To choose the sign we remember that by Lemma 3.12 the Cauchy transform satisfies $\lim_{y\to\infty} iyG(iy) = 1$.
Using that for $y > 0$ $\sqrt{(iy)^2 - 4} = i\sqrt{y^2+4}$ *we see that*

$$\lim_{y\to\infty} iyG_+(iy) = \lim_{y\to\infty} iy\frac{iy + i\sqrt{y^2+4}}{2} = \infty$$

$$\lim_{y\to\infty} iyG_-(iy) = \lim_{y\to\infty} iy\frac{iy - i\sqrt{y^2+4}}{2} = 1$$

So we have shown that the Cauchy transform of a semi-circle law is
$G(z) = \frac{z-\sqrt{z^2-4}}{2}$

Definition 3.18. *Let* $\alpha, \beta > 0$ *and* $\Theta_{\alpha,\beta} = \{z = x + iy \mid y < 0,\ \alpha y < x < -\alpha y,\ |z| \leq \beta\}$*. Then*

$$\forall \alpha > 0 \exists \beta > 0 : \exists K_\mu : \Theta_{\alpha,\beta} \to \Gamma_{\gamma,\lambda}$$

where K_μ *is the right inverse of* G_μ *and*
$\Gamma_{\gamma,\lambda} = \{z = x + iy \mid y > 0,\ -\gamma y < x < \gamma y,\ |z| \geq \lambda\}$

- $R_\mu(z) = K_\mu(z) - \frac{1}{z}$
- $F_\mu(\xi) = \frac{1}{G_\mu(\xi)}$
- $\varphi_\mu(z) = R_\mu(\frac{1}{z}) = F_\mu^{-1}(z) - z$

Then $\varphi_{\mu\boxplus\nu} = \varphi_\mu + \varphi_\nu$ on a domain of the form $\Gamma_{\gamma,\lambda}$. The restriction of $\varphi_{\mu\boxplus\nu}$ to such a domain determines $\mu \boxplus \nu$.

For a detailed discussion of the properties of these transformations we refer to [SM17] were they have been considered for measures with compact support, measures with finite variance and arbitrary measures respectively.

3.3. Free multiplicative convolution

Consider μ and ν probability measure on $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and $U \sim \mu$, $V \sim \nu$ unitary and free. Again we know that by factorisation of moments that the distribution of UV only depends on μ and ν and we denote the operator by $\boxtimes$.
For the calculation of the multiplicative free convolution, we need to look at several analytical transformations.

Definition 3.19.
The ψ-function is defined as

$$\psi_\mu(z) = \int_{\mathbb{T}} \frac{z\xi}{1 - z\xi} d\mu(\xi) = M(z) - 1$$

This power series converges on $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ with $\psi_\mu(0) = 0$.

Let $M_* = \{\mu | \int_{\mathbb{T}} \xi d\mu(\xi) \neq 0\}$. Then for $\mu \in M_*$ $\frac{\psi_\mu}{1+\psi_\mu}$ has a right inverse, called $\tilde{\chi}_\mu$, defined in a neighborhood of 0 with $\tilde{\chi}_\mu(0) = 0$.

Definition 3.20 (Σ transform)**.** *Let $\mu \in M_*$. Then $\Sigma_\mu(z) = \frac{1}{z}\tilde{\chi}_\mu(z)$ is called the Σ* ***transformation*** *of μ.*

Proposition 3.21.
Let $\mu, \nu \in M_*$. Then we have that $\mu \boxtimes \nu \in M_*$ and that $\Sigma_{\mu\boxtimes\nu} = \Sigma_\mu \Sigma_\nu$ in a neighborhood of 0 where all transformations are defined.

We can also consider a probability measure μ on $\mathbb{R}_+$ different from δ_0. Define $\psi_\mu(z) = \int_{\mathbb{R}_+} \frac{z\xi}{1-z\xi} d\mu(\xi)$. On $\mathbb{C} \setminus \mathbb{R}_+$ which is analytic on $\mathbb{C} \setminus \mathbb{R}_+$ and satisfies $\psi_\mu(\overline{z}) = \overline{\psi_\mu(z)}$ for $z \in \mathbb{C} \setminus \mathbb{R}_+$.
Further $\frac{\psi_\mu}{1+\psi_\mu}$ is univalent on $i\mathbb{C}^+$ and its image contains a neighborhood of $(\mu(\{0\})-1, 0)$.
Let $\tilde{\chi}_\mu$ be its right inverse and define $\Sigma_\mu(z) = \frac{1}{z}\tilde{\chi}_\mu(z)$ which is well defined on $\frac{\psi_\mu}{1+\psi_\mu}(i\mathbb{C}^+)$.

Let now $S \sim \mu$ and $T \sim \nu$ be free. The distribution of $S^{\frac{1}{2}}TS^{\frac{1}{2}}$ is $\mu \boxtimes \nu$ and one has $\Sigma_{\mu\boxtimes\nu} = \Sigma_\mu \Sigma_\nu$ in some neighborhood of the interval $(-\epsilon, 0)$ for some $\epsilon > 0$.
Also note that $S^{\frac{1}{2}}TS^{\frac{1}{2}}$ has the same distribution as $T^{\frac{1}{2}}ST^{\frac{1}{2}}$.

4. The Feller-Markov Kernel

Definition 4.1. *A **Markov kernel** on measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ is a map $\kappa : \mathcal{Y} \times X \to [0,1]$ satisfying*

- $\forall x \in X : \kappa(\cdot, x)$ *is a probability measure on* $(y, \mathcal{Y})$
- $\forall B \in \mathcal{Y} : x \mapsto \kappa(B, x)$ *is a* $\mathcal{X}$*-measurable function.*

Definition 4.2. *If $x \mapsto \kappa(x, dy)$ is weakly continuous and $\kappa(x, \cdot) \xrightarrow{x\to\infty} 0$ weakly $\kappa(x, du)$ is called **Feller**.*

4.1. Feller-Markov Kernel for bounded random variables

To prove the analog for bounded non-commutative random variables, we first need some lemmas.

Lemma 4.3. *Let $Y \in \tilde{\mathcal{A}}_{sa}$ and $\mathcal{B} \subset \mathcal{A}$ free from Y. Take $X \in \tilde{B}_{sa}$ and assume $X \sim \mu$ and $Y \sim \nu$ and that X and Y are bounded.*
On the domain $D = \{\xi \in \mathbb{C} \Big| |\xi| > \|X\| + 100\|Y\|\}$ there exists F such that for all $\xi \in D$:

- *F is analytic*
- $F(\bar{\xi}) = \overline{F(\xi)}$
- $\xi \in \mathbb{C}^+ \implies F(\xi) \in \mathbb{C}^+$
- $\tau(\Re_{X+Y}|\mathcal{B}) = \Re_X(F(\xi))$

Proof.

$$\begin{aligned} \Re_X(\xi)\big(1 - Y\Re_X(\xi)\big)^{-1} &= (\xi - X)^{-1}\left(1 - Y(\xi - X)^{-1}\right)^{-1} \\ &= \left(\left(1 - Y(\xi - X)^{-1}\right)(\xi - X)\right)^{-1} \\ &= (\xi - X - Y)^{-1} = (\xi - (X + Y))^{-1} \\ &= \Re_{X+Y}(\xi) \end{aligned}$$

First we show that $\|\Re_X(\xi)\| \le (|\xi| - \|X\|)^{-1}$ for $|\xi| \ge \|X\|$
$(\xi - X)^{-1} = \int_{\sigma(X)} \frac{1}{\xi - z} dE(z)$.
Since $|z| \ge \|X\| \ge \sup_{y\in\sigma(X)} |y|$ we get

$$\begin{aligned} \|(\xi - X)^{-1}\| &= \| \int_{\sigma(X)} \frac{1}{|\xi - z|} dE(z)\| \\ &\le \| \int_{\sigma(X)} \frac{1}{\inf_{y\in\sigma(X)} |\xi - y|} dE(z)\| \\ &= \frac{1}{\inf_{y\in\sigma(X)} |\xi - y|} \end{aligned}$$

Since $\inf_{y\in\sigma(X)} |\xi - y| \geq \inf_{y\in\sigma(X)} ||\xi| - |y||$, $\sup_{\sigma(X)} = \|X\|$ and $|\xi| \geq \|X\|$ we obtain $\|\Re_X(\xi)\| \leq (|\xi| - \|X\|)^{-1}$

If $|\xi| > \|X\| + \|Y\|$ we also have that $(|\xi| - \|X\|)^{-1} < \|Y\|^{-1}$ hence
$\|Y\Re_X(\xi)\| \leq \|Y\| \cdot \|\Re_X(\xi)\| \leq \|Y\|(|\xi| - \|X\|)^{-1} < 1$
So we can expand $\Re_{X+Y}(\xi)$ into an norm converging series

$$\Re_{X+Y}(\xi) = \sum_{k=0}^{\infty} \Re_X(\xi) \left[Y\Re_X(\xi)\right]^k$$

Applying the conditional expectation given $\mathcal{B}$ yields

$$\begin{aligned}
\tau\left(\Re_{X+Y}(\xi)|\mathcal{B}\right) &= \tau\left(\textstyle\sum_{k=0}^{\infty} \Re_X(\xi)\left[Y\Re_X(\xi)\right]^k |\mathcal{B}\right) \\
&= \tau\left(\Re_X(\xi) + \textstyle\sum_{k=1}^{\infty} \Re_X(\xi)\left[Y\Re_X(\xi)\right]^k |\mathcal{B}\right) \\
&= \tau\left(\Re_X(\xi) + \Re_X(\xi)\left(\textstyle\sum_{k=0}^{\infty} Y\left[\Re_X(\xi)Y\right]^k\right)\Re_X(\xi)|\mathcal{B}\right) \\
&\overset{\Re_X(\xi)\in\mathcal{B}}{=} \Re_X(\xi) + \Re_X(\xi)\left(\textstyle\sum_{k=0}^{\infty} \tau(Y\left[\Re_X(\xi)Y\right]^k |\mathcal{B})\right)\Re_X(\xi)
\end{aligned}$$

Taking $H \in \mathcal{B}$, we now want to find an expression of $\tau(Y\left(\Re_X(\xi)Y\right)^k H)$ using 2.11
For any $k \geq 0$ let $S_1(k) = T_1(2k+2) = \{1, 3, 5, ..., 2k+1\}$ and $S_2(k) = T_2(2k+4) = \{2, 4, ..., 2k+2\}, T_2(2) = \emptyset$.
Y is free from $\mathcal{B}$, so it is also free from $X, H \in \mathcal{B}$. Using this we get for all integers $k \geq 0$

$$\tau\left(Y\left(\Re_X(\xi)Y\right)^k H\right) = \sum_{\pi_1\in NC(S_1(k))} R \cup \tau[\pi_1 \cup \pi_1^c](Y, \Re_X(\xi), Y, ..., \Re_X(\xi), Y, H)$$

where the complement π_1^c is taken in the partition $S_1(k) \cup S_2(k)$. Because of the special form of $S_1(k) \cup S_2(k)$ the map $\pi_1 \mapsto \pi_1^c$ is bijective on $NC(S_1(k))$ taking values in $NC(S_2(k))$ thus we arrive at

$$\tau\left(Y\left(\Re_X(\xi)Y\right)^k H\right) = \sum_{\pi_2\in NC(S_2(k))} R \cup \tau[\pi_2^c \cup \pi_2](Y, \Re_X(\xi), Y, ..., \Re_X(\xi), Y, H)$$

Let $r > 0$ be an integer and $(l) = (l_{-1}, l_0, ..., l_r) \in \{0\} \times S_2(k)^{r+1}$ such that $0 = l_{-1} < l_0 < ... < l_r = 2k+2$.
For $0 \leq j \leq r$ we define
$S_{2,j}^{k,(l)} := \{l_{j-1}+2, ..., l_j - 2\} \subset S_2(k)$ with $S_{2,j}^{k,(l)} = \emptyset$ if $l_j - l_{j-1} = 2$
$S_{1,j}^{k,(l)} := \{l_{j-1}+1, ..., l_j - 1\} \subset S_1(k)$
Let $\pi_2 \in NC(S_2(k))$ with $\{l_0, l_1, ..., l_r\} \in \pi_2$ and denote by $\sigma_0, ..., \sigma_r$ the partitions induced by restriction of π_2 to $S_{2,1}^{k,(l)}, ..., S_{2,r}^{k,(l)}$. Then $\sigma_0, ..., \sigma_r$ are non-crossing. On the other hand for every set of non-crossing partitions of the sets $S_{2,1}^{k,(l)}, ..., S_{2,r}^{k,(l)}$ we can construct a non-crossing partition on $S_2(k)$ containing the block $\{l_0, l_1, ..., l_r\}$.
The same holds for π_2^c for restrictions to the sets $S_{1,1}^{k,(l)}, ..., S_{1,r}^{k,(l)}$. We denote the non-crossing partitions which emerge by this restriction by σ_j^c and note that these are complimentary to σ_j in $S_{1,j}^{k,(l)} \cup S_{2,j}^{k,(l)}$.

Furthermore $S_{1,j}^{k,(l)} \cup S_{2,j}^{k,(l)}$ is isomorphic to the partition $T_1(l_j - l_{j-1}) \cup T_2(l_j - l_{j-1})$.
Using the decomposition described above we get

$$\tau\left(Y\left(\Re_X(\xi)Y\right)^k H\right)$$
$$= \sum_{r=0}^{k} \sum_{\substack{l_0<\ldots<l_r\\ l_r=2k+2}} \sum_{\substack{\pi_2\in NC(S_2(k))\\ \{l_0,\ldots,l_r\}\in\pi_2}} R\cup\tau[\pi_2^c\cup\pi_2](Y,\Re_X(\xi),Y,\ldots,Y,H)$$
$$= \sum_{r=0}^{k} \sum_{\substack{l_0<\ldots<l_r\\ l_r=2k+2}} \sum_{\substack{(\sigma_0,\ldots,\sigma_r)\\ \sigma_j\in NC(T_2(l_j-l_{j-1}))}} \prod_{j=0}^{r} R\cup\tau[\sigma_j^c\cup\sigma_j](Y,\Re_X(\xi),\ldots,Y)\times$$
$$\times\tau\left(\Re_X(\xi)^r H\right)$$
$$= \tau\left(\sum_{r=0}^{k} \sum_{\substack{l_0<\ldots<l_r\\ l_r=2k+2}} \sum_{\substack{(\sigma_0,\ldots,\sigma_r)\\ \sigma_j\in NC(T_2(l_j-l_{j-1}))}} \prod_{j=0}^{r} R\cup\tau[\sigma_j^c\cup\sigma_j](Y,\Re_X(\xi),\ldots,Y)\times \right.$$
$$\left. \times\Re_X(\xi)^r H\right)$$

In the above the compliment is taken in the partition $T_1(l_j - l_{j-1}) \cup T_2(l_j - l_{j-1})$.
Also note that if $l_j - l_{j-1} = 2$ then $R\cup\tau[\sigma_j^c\cup\sigma_j](Y) = R(Y) = \tau(Y)$ Since this holds for all $H \in \mathcal{B}$ we arrive at

$$\tau\left(Y\left(\Re_X(\xi)Y\right)^k \middle| \mathcal{B}\right) =$$

$$\sum_{r=0}^{k}\left[\sum_{\substack{l_0<\ldots<l_r\\ l_r=2k+2}} \sum_{\substack{(\sigma_0,\ldots,\sigma_r)\\ \sigma_j\in NC(T_2(l_j-l_{j-1}))}} \prod_{j=0}^{r} R\cup\tau[\sigma_j^c\cup\sigma_j](Y,\Re_X(\xi),\ldots,Y)\right]\Re_X(\xi)^r$$

Using the inequality $|R(a_1,\ldots,a_n)| \leq 4^{2n}\prod_{j=1}^{n}\|a_j\|$, which we have shown in section 2, we can find the upper bound

$$\left\|\left[\prod_{j=0}^{r} R\cup\tau[\sigma_j^c\cup\sigma_j](Y,\Re_X(\xi),\ldots,Y)\right]\Re_X(\xi)^r\right\| \leq 4^{2k+2}\|Y\|^{k+1}\|\Re_X(\xi)\|^k$$

Hence

$$\sum_{k=0}^{\infty}\sum_{r=0}^{k}\left[\sum_{\substack{l_0<\ldots<l_r\\ l_r=2k+2}} \sum_{\substack{(\sigma_0,\ldots,\sigma_r)\\ \sigma_j\in NC(T_2(l_j-l_{j-1}))}} \prod_{j=0}^{r} R\cup\tau[\sigma_j^c\cup\sigma_j]\right]\Re_X(\xi)^r$$
$$\leq \sum_{k=0}^{\infty}\sum_{r=0}^{k}\left[\sum_{\pi_2\in NC(S_2(k))} 4^{2k+2}\|Y\|^{k+1}\|\Re_X(\xi)\|^k\right]$$
$$\leq \sum_{k=0}^{\infty} 4^{3k+3}\|Y\|^{k+1}\|\Re_X(\xi)\|^k$$

This series is convergent and uniformly bounded for $|\xi| > \|X\| + 100\|Y\|$
By exchanging the order of summation we obtain

$$\sum_{k=0}^{\infty}\tau\left(Y\left(\Re_X(\xi)Y\right)^k \middle| \mathcal{B}\right)$$
$$= \sum_{k=0}^{\infty}\left[\sum_{l=0}^{\infty}\sum_{\pi\in NC(T_2(2l))} R\cup\tau[\pi^c\cup\pi](Y,\Re_X(\xi),\ldots,\Re_X(\xi),Y)\right]^{r+1}\Re_X(\xi)^r$$

where the compliment π^c is taken over the partition $T_1(2l) \cup T_2(2l)$
Using the above estimate again we see that

$$\delta(\xi) := \sum_{l=0}^{\infty} \sum_{\pi \in NC(T_2(2l))} R \cup \tau[\pi^c \cup \pi](Y, \Re_X(\xi), ..., \Re_X(\xi), Y)$$

defines a uniformly convergent series on $\{\xi \in \mathbb{C} \Big| |\xi| > \|X\| + 100\|Y\|\}$ and the limit function is again bounded and analytic in this set. Thus

$$\sum_{r=0}^{\infty} \delta(\xi)^{r+1} \Re_X(\xi)^r = \delta(\xi)\big(1 - \delta(\xi)\Re_X(\xi)\big)^{-1}$$

Plugging in this result leads to

$$\begin{aligned}\tau(\Re_{X+Y}(\xi)|\mathcal{B}) &= \Re_X(\xi) + \Re_X(\xi)\delta(\xi)\Big(1 - \delta(\xi)\Re_X(\xi)\Big)^{-1}\Re_X(\xi) \\ &= \Re_X(\xi - \delta(\xi))\end{aligned}$$

$F(\xi) := \xi - \delta(\xi)$ is analytic in a neighborhood of ∞.
Thus we have shown that $\tau(\Re_{X+Y}(\xi)) = \Re_X(F(\xi))$ holds in a neighborhood of ∞.
Because of the adjoint preserving property of the conditional expectation $F(\bar{\xi}) = \overline{F(\xi)}$.
If $\xi \in \mathbb{C}^+$, then $\sigma(\Re_{X+Y}(\xi)) \subseteq \mathbb{C}^-$ is a compact, thus also for $\sigma(\tau(\Re_{X+Y}(\xi)|B)) \subseteq \mathbb{C}^-$ is compact. Thus it has to hold that $F(\xi) \in \mathbb{C}^-$ if F is defined on ξ. □

Remark. *A shorter alternative of the proof above can be found by following the arguments of Corollary 3.2, 3.5 and 3.7 in [LS20].*

Lemma 4.4. *For all $\xi \in \mathbb{C} \setminus \mathbb{R}$ $h(\xi) := \tau(\Re_{X+Y}(\xi)|B)$ is a normal operator*

Proof. By the above proposition, we already know that the statement is true for ξ in a neighborhood of ∞. Remember that $\Re_{X+Y}(\xi)$ is analytic in ξ. Then since the conditional expectation is a contraction, h is analytic in ξ on $\mathbb{C} \setminus \mathbb{R}$ and satisfies $h(\xi)^* = h(\bar{\xi})$.
$\left\{\xi \in \mathbb{C} \setminus \mathbb{R} : \frac{\partial^k}{\partial \xi^k}\frac{\partial^l}{\partial \bar{\xi}^l}\left[h(\xi)h(\bar{\xi}) - h(\bar{\xi})h(\xi)\right] = 0 \text{ for all } k, l \geq 0\right\}$ is open and closed and (by proposition 4.3) contains a neighborhood of ∞. Thus the set has to be equal to $\mathbb{C} \setminus \mathbb{R}$ and h is normal on $\mathbb{C} \setminus \mathbb{R}$. □

Theorem 4.5 (existence of the Feller Markov kernel for bounded operators)**.** *Consider two random variables $Y \in \tilde{\mathcal{A}}_{sa}$ and $X \in \tilde{B}_s a$ where the subalgebra $\tilde{B}_s a$ is free from Y and denote their distribution by μ and ν.*
Then there exists a ***Feller Markov kernel*** *$\mathcal{K} = \kappa(x, du)$ on $\mathbb{R} \times \mathbb{R}$ and an analytic function F on $\mathbb{C} \setminus \mathbb{R}$ such that*

1. *$\tau(f(X + Y)|\mathcal{B}) = \mathcal{K}f(X)$*
   where $\mathcal{K}f(x) = \int_{\mathbb{R}} f(u)k(x, du)$ and f Borell bounded

2. *$F(\bar{\xi}) = \overline{F(\xi)}$, $F(\mathbb{C}^+) \subset \mathbb{C}^+$, $Im(F(\xi)) \geq Im(\xi)$ and $\frac{F(iy)}{iy} \xrightarrow[y \in \mathbb{R}]{y \to \infty} 1$*

3. *for all* $\xi \in \mathbb{C} \setminus \mathbb{R}$*:* $\int_{\mathbb{R}} (\xi - u)^{-1} \kappa(x, du) = (F(\xi) - x)^{-1}$

4. *for all* $\xi \in \mathbb{C} \setminus \mathbb{R}$*:* $G_\mu(F(\xi)) = G_{\mu \boxplus \nu}(\xi)$

Remark. *Property 4 of the above theorem is called the* ***subordination property****. The subordination property was also established for the special case of classical random walks on free products in [Woe86].*

Proof. $\Re_{X+Y}(\xi)$ is normal and using the Cauchy transform we see $\sigma(\Re_{X+Y}(\xi)) = \frac{1}{\xi - \sigma(X+Y)}$. Since $X + Y$ is bounded its spectrum must be a compact subset of some circle with diameter $(0, -\frac{i}{\Im(\xi)})$ not containing 0.
Thus by 3.5 the spectrum $\sigma(h(\xi)) \subset (0, -\frac{i}{\Im(\xi)})$ and thus is bounded away from 0. Therefore we can invert h in $\mathcal{A}$ and $\rho(\xi) = h(\xi)^{-1} + X$ is normal and is analytic in ξ. Further we see that for every $\xi \in \mathbb{C} \setminus \mathbb{R}$

$$\rho(\xi)^* = \rho(\overline{\xi})$$

Since $\sigma(\rho(\xi)) \subseteq \{z \in \mathbb{C} | \Im(z) \geq \Im(\xi)\}$ by using lemma 4.3 we get $\rho(\xi) = F(\xi) 1_{\mathcal{A}}$ for ξ in some neighborhood of ∞ and thus by analytic continuation for all ξ.

Let $u, v \in H$ be orthogonal. Then $\langle \rho(\xi) u, v \rangle = 0$ for all ξ in a neighborhood of ∞ and thus we can again extend this statement to all ξ in $\mathbb{C} \setminus \mathbb{R}$. Since u, v were chosen arbitrarily, this holds also for all feasible choices of u, v. Thus $\rho(\xi)$ is a multiple of 1 for all ξ in $\mathbb{C} \setminus \mathbb{R}$.

So there exists an analytic function F on $\mathbb{C} \setminus \mathbb{R}$ satisfying

- $\forall \xi \in \mathbb{C} \setminus \mathbb{R}$: $\tau(\Re_{X+Y}(\xi) | B) = \Re_X(\xi)$
- $\forall \xi \in \mathbb{C}^+$: $\Im(F(\xi)) \geq \Im(\xi)$
- $F(\overline{\xi}) = \overline{F(\xi)}$

We define for $k, l \geq 0$ with $k + l \geq 1$

$$K_{\xi,k,l}(x) = (\xi - x)^k (\overline{\xi} - x)^l = \frac{\partial^k}{\partial \xi^l} \frac{\partial^l}{\partial \overline{\xi}^l} \left[(\overline{\xi} - \xi)^{-1} \Big((\xi - x)^{-1} - (\overline{\xi} - x)^{-1} \Big) \right]$$

By Stone-Weierstrass the linear span of these functions is dense in $C_0(\mathbb{R})$.

Taking the derivative of $\tau(\Re_{X+Y}(\xi)) = \Re_X(F(\xi))$ and $\tau(\Re_{X+Y}(\overline{\xi})) = \Re_X(F(\overline{\xi}))$ shows that $\tau(\cdot | B)$ maps $K_{\xi,k,l}(X + Y)$ to the C^*-algebra generated by X (denoted by $C^*(X)$) and thus by continuity $\tau(C^*(X + Y) | B) \subseteq C^*(X)$. Since $\tau(\cdot | B)$ is completely positive and identity preserving, there exists $\mathcal{K} = k(x, du)$ on $Spec(X) \times \mathbb{R}$ such that for f bounded and continuous

$$\tau(f(X + Y) | B) = \mathcal{K} f(X)$$

So $\forall x \in Spec(X)$: $\int_{\mathbb{R}}(\xi-u)^{-1}k(x,du) = (F(\xi)-x)^{-1}$ and thus $(F(\xi)-x)^{-1}$ is the Cauchy transform of some probability measure. By proposition 5.2 in [BV93] $\frac{F(\xi)}{\xi} \xrightarrow{\xi\to\infty} 1$ inside some angle $\Gamma_{\alpha,\beta}$. Thus also for any $x \in \mathbb{R}$ it holds that $\frac{F(\xi)-x}{\xi} \xrightarrow{\xi\to\infty} 1$. Therefore

$$\exists k(x,du) \text{ probability measure: } \int_{\mathbb{R}}(\xi-u)^{-1}k(x,du) = (F(\xi)-x)^{-1}$$

.

Taking the trace τ of $\tau(\Re_{X+Y}(\xi)|B) = \Re_X(F(\xi))$ yields

$$G_{\mu\boxplus\nu}(\xi) = \tau(\Re_{X+Y}(\xi)) = \tau(\tau(\Re_{X+Y}(\xi)|B)) = \tau(\Re_X(F(\xi))) = G_\mu(F(\xi))$$

By Proposition 5.4 in [BV93] there exists $\Gamma_{\gamma,\lambda}$ and $\Theta_{\alpha,\beta}$ for which

- $G_\mu = G_{\mu\boxplus\nu}$ on $\Gamma_{\gamma,\lambda}$
- $G_\mu(\Gamma_{\gamma,\lambda}) \supset \Theta_{\alpha,\beta}$
- $G_{\mu\boxplus\nu}(\Gamma_{\gamma,\lambda}) \supset \Theta_{\alpha,\beta}$

The constant can be chosen such that the above holds uniformly. Since $\frac{F(\xi)}{\xi} \to 1$ holds in $\Gamma_{\gamma,\lambda}$, the equation $G_\mu \circ F = G_{\mu\boxplus\nu}$ fixes F completely. □

Remark. *Using analogous computations, one can verify that for any complex polynomial P there exists a complex polynomial of degree less than that of P such that $\tau(P(X+Y)|B) = Q(X)$. From this we can quickly deduce that a Feller-Markov kernel exists on $Spec(X)\times\mathbb{R}$ but it difficult to calculate the kernel from its actions on polynomials alone.*

4.2. Feller-Markov Kernel for unbounded random variables

Now we want to look into the case of unbounded operators. So we consider possibly unbounded operators X and Y together with their spectral projections p_n and q_n for $n \in \mathbb{N}$ in the interval $[-n,n]$. Let μ_n and ν_n be the distributions of p_nXp_n and q_nYq_n. So $\mu_n \xrightarrow{n\to\infty} \mu$ and $\nu_n \xrightarrow{n\to\infty} \nu$ weakly.

Take a positive $H \in \mathcal{B} \subseteq \mathcal{A}$. Using subordination as shown in section 4.1 we deduce

$$\tau(\Re_{p_nXp_n+q_nYq_n}(\xi)H) = \tau(\Re_{p_nXp_n}(F_n(\xi))H)$$

for F_n analytic and for all $\xi \in \mathbb{C}\setminus\mathbb{R}$ and all $n \in \mathbb{N}$

Theorem 4.6. *For sequence of probability measures μ_n on $\mathbb{R}$ the following statements are equivalent*

1. *$\mu_n \to \mu$ with respect to the weak-$*$-topology*

2. $\varphi_{\mu_n} \overset{uniformly}{\to} \varphi$ *in compact subset of* $\Gamma_{\alpha,\beta}$ *for some* $\alpha, \beta > 0$ *and* $\varphi_{\mu_n}(z) = o(z)$ *uniformly in* n *as* $|z| \to \infty$ *and* $z \in \Gamma_{\alpha,\beta}$*.*

If these statements are satisfied φ *coincides with* φ_μ *in* $\Gamma_{\alpha,\beta}$

Remark. *We will prove this statement for compactly supported measures in proposition 5.10. For a detailed proof of the statement for unbounded measures we refer to [BV93].*

Then we get a domain $\Gamma_{\alpha,\beta}$ in which we have

$$F_n = K_{\mu_n} \circ G_{\mu_n \boxplus v_n} \xrightarrow{n\to\infty} F = K_\mu \circ G_{\mu \boxplus v}$$

uniformly on every compact subset.

We define $\tau^H(\cdot) := \frac{\tau(H^{\frac{1}{2}} \cdot H^{\frac{1}{2}})}{\tau(H)}$ on $\mathcal{A}$. Let $\mu_n^H, \mu^H, \varphi_n^H, \varphi^H$ be the distributions with respect to this state of $p_n X p_n, X, p_n X p_n + q_n Y q_n$ and $X + Y$ respectively.

Lemma 4.7. *Let* $T \in \tilde{\mathcal{A}}_{sa}$ *and* $t \in \mathbb{R}$*. Then*

$$1 - \mu_T((-\infty, t)) = \max\{\tau(p) : p = p^* = p^2 \in \mathcal{A}, pTp \geq tp\}$$

Proof. The projection $p = e_T([t, \infty))$, where e_T is the spectral measure of T, satisfies the condition $\tau(p) = 1 - \mu_T((-\infty, t))$ and $pTp \geq tp$.
Let q be a projection in $\mathcal{A}$ such that $\tau(q) > 1 - \mu_T((-\infty, t))$. Thus $\tau(q) + \tau(e_T((-\infty, t)1) > 1$ which implies that there exists a non-zero projection r such that $r \leq q$ and $r \leq e_T((-\infty, t))$ and thus

$$r(qTq - tq)r = r(e_T((-\infty, t))Te_T((-\infty, t)) - te_T((-\infty, t)))r < 0$$

and therefore q doe not satisfy the inequality $qTq \geq tq$ □

Proposition 4.8. *Let* p *a non-zero projection in* $\mathcal{A}$ *and* $T \in \mathcal{A}$ *a self-adjoint operator. Then*

$$\tau(p)\mu_{pTp}((-\infty, \cdot)) \leq \mu_T((-\infty, \cdot)) \leq \tau(p)\mu_{pTp}((-\infty, \cdot)) + \tau(1 - p)$$

Proof. We will use the notation $\mathcal{A}_p := p\mathcal{A}p$ and $\tau_p = \tau(p)^{-1}\, \tau|_{\mathcal{A}_p}$
Fix $t \in \mathbb{R}$ and let $q \in \mathcal{A}_p$ be a projection such that $\tau_p(q) = 1 - \mu_{pTp}((-\infty, t))$ and $q(pTp)q \geq tq$. Since $q \leq p$ we have $qTq \geq tq$ and

$$\tau(q) = \tau(p) - \tau(p)\mu_{pTp}((-\infty, t)) = 1 - [\tau(p)\mu_{pTp}((-\infty, t)) + \tau(1 - p)]$$

The first equality of the statement follows by application of lemma 4.7.
To prove the second one we choose $q \in \mathcal{A}$ such that $qTq \geq tq$ and $\tau(q) = 1 - \mu_T((-\infty, t))$. Define $r = q \wedge p$ and note that $1 - \tau(r) \leq (1 - \tau(p)) + (1 - \tau(q))$ or equivalently

$$1 - \tau_p(r) = 1 - \frac{\tau(r)}{\tau(p)} \leq \frac{1 - \tau(q)}{\tau(p)} = \frac{\mu_T((-\infty, t))}{\tau(p)}$$

Since we have $r(pTp)r \geq tr$ the last inequality implies

$$\mu_{pTp}((-\infty, t)) \leq \frac{\mu_T((-\infty, t))}{\tau)(p)}$$

□

By using this proposition we see that $\mu_n^H \xrightarrow{n\to\infty} \mu^H$ and $\varphi_n^H \xrightarrow{n\to\infty} \varphi^H$. ($\tau_p$ is not necessarily tracial but nevertheless the result applies.)

Thus we have $\tau(\Re_{p_nXp_n+q_nYq_n}(\xi)H) \xrightarrow{n\to\infty} \tau(\Re_{X+Y}(\xi)H)$ and $\tau(\Re_{p_nXp_n}(F_n(\xi))H) \xrightarrow{n\to\infty} \tau(\Re_X(F(\xi))H)$ for all ξ is some domain $\Gamma_{\alpha',\beta'}$. Thus $\tau(\Re_{X+Y}(\xi)|B) = \Re_X(F(\xi))$ holds in this domain and by the bimodule property we deduce for $n \geq 1$ that

$$\tau(p_n\Re_{X+Y}(\xi)p_n + (1-p_n)|B) = p_n\Re_X(F(\xi))p_n + (1-p_n)$$

$\tau(p_n\Re_{X+Y}(\xi)p_n+(1-p_n)|B)$ is analytic in ξ. With the same arguments as for bounded random variables we see that $\rho_n(\xi) = \tau(p_n\Re_{X+Y}(\xi)p_n + (1-p_n)|B)$ is invertible and analytic in ξ and $\rho^{-1}(\xi)+p_nXp_n = p_nF(\xi)p_n+1-p_n$. We can then extend F analytically to $\mathbb{C}\setminus\mathbb{R}$. Since the equation above is satisfied for all n, letting $n\to\infty$ yields

$$\tau(\Re_{X+Y}|B) = \Re_X(F(\xi))$$

The rest of the statements for the existence of the Feller-Markov kernel for unbounded random variables follow from analogous arguments in the bounded case.

4.3. Feller Markov Kernel for free multiplicative convolution

Finally we want to look at the Feller-Markov kernels for multiplicative processes.

Theorem 4.9 (existence of the Feller Markov kernel for unitary operators)**.** *Let $U \sim \mu$ and $V \sim \nu$ be unitary with $U \in \mathcal{B} \subseteq \mathcal{A}$ and V is free from $\mathcal{B}$. There exists $\mathcal{K} = k(\xi, d\omega)$ Feller Markov kernel on $\mathbb{T}\times\mathbb{T}$ and F analytic on $\mathbb{D}$, such that for f bounded and Borel-measurable and $z \in \mathbb{D}$*

1. $\tau(f(UV)|\mathcal{B}) = \mathcal{K}f(U)$
2. $|F(z)| \leq |z|$
3. $\int_{\mathbb{T}} \frac{z\omega}{1-z\omega}k(\xi, d\omega) = \frac{F(z)\xi}{1-F(z)\xi}$
4. $\psi_\mu(F(z)) = \psi_{\mu\boxtimes\nu}(z)$

F is uniquely determined by (2) *and* (4) *if μ satisfies $\int_{\mathbb{T}} \xi d\mu(\xi) \neq 0$.*

Proof. We will only sketch the proof. Take $H \in \mathcal{B}$ and $z \in \mathbb{D}$. Then

$$\tau(zUV(1-zUV)^{-1}H) = \sum_{k=1}^{\infty} \tau(z^k(UV)^kH)$$

Similar to the arguments of theorem 4.5 we get the existence of F in a neighborhood of 0 such that

$$\tau(zUV(1-zUV)^{-1}H) = \tau(F(z)U(1-F(z)U)^{-1}H)$$

holds for all $H \in \mathcal{B}$ and thus $\tau(zUV(1-zUV)^{-1}|\mathcal{B}) = F(z)U(1-F(z)U)^{-1}$ in some neighborhood of 0. Since $\sigma(zUV(1-zUV)^{-1})$ is contained in the ball with radius $\frac{|z|}{1+|z|}$ we get by proposition 3.5 that $\sigma(\tau(zUV(1-zUV)^{-1}|\mathcal{B}))$ is also contained in this ball and therefore $(1-(1+\tau(zUV(1-zUV)^{-1}|\mathcal{B}))^{-1})U^{-1}$ is bounded and $\sigma((1-(1+\tau(zUV(1-zUV)^{-1}|\mathcal{B}))^{-1})U^{-1})$ is contained in the ball with radius $|z|$. Further $(1-(1+\tau(zUV(1-zUV)^{-1}|\mathcal{B}))^{-1})U^{-1}$ coincides with $F(z)$ in some neighborhood of 0 and can thus be extended to coincide with $F(z)$ on $\mathbb{D}$. Since F satisfies $\mathbb{D}$ with $|F(z)| \leq |z|$ and

$$\tau(zUV(1-zUV)^{-1}H) = \tau(F(z)U(1-F(z)U)^{-1}H)$$

we see that the statements $(1), (2), (3)$ hold. For property (4) we apply the trace.
If μ satisfies $\int_{\mathbb{T}} \xi d\mu(\xi) \neq 0$ then ψ_μ is invertible in some neighborhood of 0 and thus F can be uniquely determined by $\psi_\mu \circ F = \psi_{\mu \boxtimes \nu}$ in $\mathbb{D}$ □

Theorem 4.10 (existence of the Feller Markov kernel for positive operators)**.** *Let $S \sim \mu$ and $T \sim \nu$ be self-adjoint with where $S \in \mathcal{B} \subseteq \mathcal{A}$ and T is free from $\mathcal{B}$. Further assume that μ and ν are not the Dirac measure at 0.*

There exists $\mathcal{K} = k(u, d\nu)$ Feller Markov kernel on $\mathbb{R}_+ \times \mathbb{R}_+$ and F analytic on $\mathbb{C} \setminus \mathbb{R}_+$, such that for f bounded and Borel-measurable and $\xi \in \mathbb{C}^+$

1. $\tau\left(f\left(S^{\frac{1}{2}}TS^{\frac{1}{2}}\right)|\mathcal{B}\right) = \mathcal{K}f(S)$
2. $F(\xi) \in \mathbb{C}^+$, $F(\bar{\xi}) = \overline{F(\xi)}$, $\operatorname{Arg}(F(\xi)) \geq \operatorname{Arg}(\xi)$
3. $\int_{\mathbb{R}_+} \frac{\xi v}{1-\xi v} k(u, d\nu) = \frac{F(\xi)u}{1-F(\xi)u}$
4. $\psi_\mu(F(\xi)) = \psi_{\mu \boxtimes \nu}(\xi)$

(2) *and* (4) *completely determine F*

Proof. First assume S, T are bounded and take $H \in \mathcal{B}$ and $\xi \in \mathbb{C} \setminus \mathbb{R}_+$ with norm small enough. Then

$$\tau(\xi S^{\frac{1}{2}}TS^{\frac{1}{2}}(1-\xi S^{\frac{1}{2}}TS^{\frac{1}{2}})^{-1}H) = \sum_{k=1}^{\infty} \tau(\xi^k (S^{\frac{1}{2}}TS^{\frac{1}{2}})^k H)$$

By similar arguments as in the above proof we get the existence of F defined in some neighborhood of 0 which satisfies for all $H \in \mathcal{B}$

$$\tau(\xi S^{\frac{1}{2}}TS^{\frac{1}{2}}(1-\xi S^{\frac{1}{2}}TS^{\frac{1}{2}})^{-1}H) = \tau(F(\xi)S(1-F(\xi)S)^{-1}H)$$

Thus $\tau(\xi S^{\frac{1}{2}}TS^{\frac{1}{2}}(1-\xi S^{\frac{1}{2}}TS^{\frac{1}{2}})^{-1}|\mathcal{B}) = F(\xi)S(1-F(\xi)S)^{-1}$ for ξ as chosen above.

Take now $\xi \in \mathbb{C}^+$ with argument on $(0, \pi)$. Then $\sigma(\xi S^{\frac{1}{2}} T S^{\frac{1}{2}} (1 - \xi S^{\frac{1}{2}} T S^{\frac{1}{2}})^{-1}) \subseteq (\mathbb{C}^- \cup \mathbb{R}) \cap C$ where C is the circle passing through the points $(-1, 0)$ and $(-\frac{1}{2} - i \tan \frac{\theta}{2})$. Further $\xi \in \mathbb{C}^+$ with argument on $(0, \pi)$ is bounded away from -1. Thus $1 - \Big(1 + \tau(\xi S^{\frac{1}{2}} T S^{\frac{1}{2}} (1 = \xi S^{\frac{1}{2}} T S^{\frac{1}{2}})^{-1} | \mathcal{B})^{-1}\Big)^{-1}$ is bounded and $\sigma(1 - \Big(1 + \tau(\xi S^{\frac{1}{2}} T S^{\frac{1}{2}} (1 = \xi S^{\frac{1}{2}} T S^{\frac{1}{2}})^{-1} | \mathcal{B})^{-$ $(\mathbb{C}^+ \cup (-\infty, 0)) \cap \{z \in \mathbb{C} | \operatorname{Arg}(z) > \operatorname{Arg}(\xi)\}$. Further $1 - \Big(1 + \tau(\xi S^{\frac{1}{2}} T S^{\frac{1}{2}} (1 = \xi S^{\frac{1}{2}} T S^{\frac{1}{2}})^{-1} | \mathcal{B})^{-1}\Big)^{-1} =$ $F(\xi) S$ in some neighborhood of 0 and we can again analytically extend F. $\square$

5. Processes with free increments

Now we want to apply the tools we have established to processes with free increments.

Definition 5.1 (processes with free increments)**.**

- *A **free additive increment process** is a map $X : \mathbb{R}^+ \to \tilde{\mathcal{A}}_{sa}$ such that for any $t_1 < ... < t_n \in \mathbb{R}^+$: $X_{t_1}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are free.*
- *A **unitary process with (left) multiplicative free increments** is a map $U : \mathbb{R}^+ \to \{a \in \mathcal{A} | a^* a = aa^* = 1_{\mathcal{A}}\}$ such that for any $t_1 < ... < t_n \in \mathbb{R}^+$: $U_{t_1}, U_{t_2} U_{t_1}^{-1}, ..., U_{t_n} U_{t_{n-1}}^{-1}$ are free.*
- *A **positive process with multiplicative free increments** is a map $S : \mathbb{R}^+ \to \{a \in \mathcal{A} | \exists b \in \mathcal{A}$: $a = b^* b$ and 0 is not in the discrete part of the spectrum of $a\}$ such that for any $t_1 < ... < t_n \in \mathbb{R}^+$: $S_{t_1}, S_{t_1}^{-\frac{1}{2}} S_{t_2} S_{t_1}^{-\frac{1}{2}}, ..., S_{t_{n-1}}^{-\frac{1}{2}} S_{t_n} S_{t_{n-1}}^{-\frac{1}{2}}$ are free.*

Remark. *There are several other definitions of positive processes with multiplicative free increments. Two other examples would that $\{S_{t_1}\} \cup \{S_{t_{k-1}}^{-1} S_{t_k}\}_{k=2}^n$ and $\{S_{t_1}\} \cup \{S_{t_k} S_{t_{k-1}}^{-1}\}_{k=2}^n$ respectively form a free family.*

Let $(X_t)_{t \in \mathbb{R}^+}$ be a free additive increment process with $X_t \sim \mu_t$ and $X_t - X_s \sim \mu_{s,t}$. Then for $r < s < t \in \mathbb{R}^+$

$$\mu_s \boxplus \mu_{s,t} = \mu_t \text{ and } \mu_{r,s} \boxplus \mu_{s,t} = \mu_{r,t}$$

On the other hand given μ_t, $\mu_{s,t}$ satisfying the above there exists a free additive increment process $(X_t)_{t \in \mathbb{R}^+}$ such that $X_t \sim \mu_t$ and $X_t - X_s \sim \mu_{s,t}$. This is an implication of the free product construction.

By Theorem 4.5 we further get a Feller Markov kernel $\mathcal{K}_{s,t}$ for each of the measures μ_t, $\mu_{s,t}$

5.1. Dilations

For such processes we now want to define a Markov property. This analog is called dilation.

Definition 5.2 (Dilation)**.** *Let D be a C^*-algebra with state ω_0. Assume $\Pi_{s,t} : D \to D$ is a family of completely positive, identity preserving contractions satisfying for all $s < t < u$: $\Pi_{s,t} \circ \Pi_{t,u} = \Pi_{s,u}$.*
*A **dilation** of (D, ω_0) is the data $(\mathcal{A}, \tau, A_t, j_t, \tau_t, t \in \mathbb{R}^+)$ where*

1. $(\mathcal{A}, \tau)$ *is a non-commutative probability space and* τ *is tracial*

2. $(\mathcal{A}_t)_{t\ in\mathbb{R}^+}$ *is an increasing family of subalgebras.*

3. $\tau_t : \mathcal{A} \to \mathcal{A}_t$ *is the conditional expectation*

4. $j_t : D \to A$ *are morphisms such that* $j_t(D) \subseteq A_t$

5. $\omega_0 = \tau \circ j_0$ *("initial state property")*

6. *for all* $s < t \in \mathbb{R}^+$ *one has* $\tau_s \circ j_t = j_s \circ \Pi_{s,t}$ *("Markov property")*

Theorem 5.3. *Let* $(X_t)_{t\in\mathbb{R}^+}$ *be a free additive increment process. We define the morphism* $j_t : C_0(\mathbb{R}) \to \mathcal{A}$ *by* $j_t(f) = f(X_t)$*. Define* $\mathcal{A}_t$ *as the von Neumann algebra generated by the process up to time* t *and denote* $\tau_t = \tau(\cdot|\mathcal{A}_t)$*. Then* $(\mathcal{A}, \tau, \mathcal{A}_t, j_t, \tau_t, t \in \mathbb{R}^+)$ *is a dilation of* $(C_0(\mathbb{R}), \mu_0, \mathcal{K})$

Proof. Everything except the Markov property is easy to verify.
Let f be a bounded Borel function on $\mathbb{R}$. By applying theorem 4.5 to $\mathcal{A}_s$ and the random variables X_s and $X_t - X_s$ we get $\tau_s(f(X_t)) = \mathcal{K}_{s,t}f(X_s)$ for all $s < t$.

□

Remark. *Since* $\mathcal{K}_{s,t} \circ \mathcal{K}_{t,u} = \mathcal{K}_{s,u}$ *they form a Markov transition function, there also exists a process on the classical sense satisfying the Markov property and having initial distribution* μ_0 *and transition described by* $\mathcal{K} = (\mathcal{K}_{s,t})_{s<t\in\mathbb{R}^+}$*. Since this process as the same distribution as the one described above we see*

$$\tau(\phi_1(X_{t_1})...\phi_n(X_{t_n})) = \mathbb{E}[\phi_1(Z_{t_1})...\phi_n(Z_{t_n})]$$

Remark. *Analogous statements also hold for free multiplicative processes.*

5.2. Martingales

Using these definitions we can now start to investigate properties of such processes. First we want to look at the existence of families of martingales.

Definition 5.4 (Martingale)**.** *Let* $(\mathcal{A}_t)_{t\in\mathbb{R}^+}$ *a increasing family of von Neumann subalgebras of* $\mathcal{A}$*. A martingale of* $(\mathcal{A}, \tau, \mathcal{A}_t)$ *map* $M : I \to \mathcal{A}$ *where is is some interval in* $\mathbb{R}^+$ *such that*

- *for all t:* $M_t \in \mathcal{A}_t$
- *for all* $s < t$*:* $\tau(M_t|\mathcal{A}_s) = M_s$

Let now $(X_t)_{t\in\mathbb{R}^+} \sim \mu_t$ be a free additive increment process. Further let R_{μ_t} be the R-transform of μ_t. In [BV93] it has been shown that for I is compact and $t \to \mu_t$ is weakly continuous there exists Λ such that all K_{μ_t} are defined on Λ and $F_{s,t}(K_{\mu_t}(z)) = K_{\mu_s}(z)$ for all $z \in \Lambda$ and $s < t \in I$.
In the following we will assume that $\mathcal{A}_t$ is the von Neumann algebra generated by $\{X_s|s \leq t\}$

Proposition 5.5. *$t \mapsto (1 - zX_t + zR_{\mu_t}(z))^{-1}$ is a martingale of $(\mathcal{A}, \tau, \mathcal{A}_t)$ on I for $z \in \Lambda$.*

Proof. First note that

$$\begin{aligned} (1 - zX_t + zR_{\mu_t}(z))^{-1} &= z^{-1}\left(\tfrac{1}{z} - X_t + R_{\mu_t}(z)\right)^{-1} \\ &= z^{-1}\left(K_{\mu_t}(z) - X_t\right)^{-1} \\ &= z^{-1}\Re_{X_t}(K_{\mu_t}(z)) \end{aligned}$$

so it is sufficient to show that $\Re_{X_t}(K_{\mu_t}(z))$ is a martingale, and by theorem 4.5 we get for all $s < t$ that

$$\tau(\Re_{X_t}(K_{\mu_t}(z))|\mathcal{A}_s) = \Re_{X_s}(F_{s,t}(K_{\mu_t}(z)) = \Re_{X_s}(K_{\mu_s}(z))$$

□

An similar statement is true for multiplicative processes. Let U_t be a unitary processes with multiplicative free increments with marginal distribution μ_s. Further let Σ_{μ_t} be the Σ-transform of μ_t and I be an interval such that for $t \in I$ Σ_{μ_t} have a common domain Λ where $F_{s,t}(\Sigma_{\mu_t}(z)) = z\Sigma_{\mu_s}(z)$

Proposition 5.6. *$t \mapsto (1 - z\Sigma_{\mu_t}(z)U_t)^{-1}$ is a martingale on I for $z \in \Lambda$.*

Remark. *An analogous statement is true for positive processes with multiplicative free increments.*

5.3. Levy processes

Having introduced martingales we can now also talk about stationary processes.

Definition 5.7 (Levy processes)**.**

- *Let X_t be a free additive increment process with corresponding distributions $\mu_{s,t}$ and Markov transition function $\mathcal{K}_{s,t}$ on $\mathbb{R}$. Then*
  - *X_t is called* ***free additive Levy process of the first kind*** *if $\mu_{s,t} = \mu_{s+t,t+u}$ for all $u \geq 0$.*
  - *X_t is called* ***free additive Levy process of the second kind*** *if $\mathcal{K}_{s,t} = \mathcal{K}_{s+t,t+u}$ for all $u \geq 0$.*
- *Let U_t be a unitary process with multiplicative free increments with corresponding distributions $\mu_{s,t}$ (i.e. $\mu_{s,t}$ is the distribution of $U_tU_s^{-1}$) and Markov transition function $\mathcal{K}_{s,t}$ on $\mathbb{T}$. Then*
  - *U_t is called* ***free unitary multiplicative Levy process of the first kind*** *if $\mu_{s,t} = \mu_{s+t,t+u}$ for all $u \geq 0$.*
  - *U_t is called* ***free unitary multiplicative Levy process of the second kind*** *if $\mathcal{K}_{s,t} = \mathcal{K}_{s+t,t+u}$ for all $u \geq 0$.*

- *Let S_t be a positive process with multiplicative free increments with corresponding distributions $\mu_{s,t}$ (i.e. $\mu_{s,t}$ is the distribution of $U_t U_s^{-1}$) and Markov transition function $\mathcal{K}_{s,t}$ on $\mathbb{T}$. Then*
  - *U_t is called* ***free positive multiplicative Levy process of the first kind*** *if $\mu_{s,t} = \mu_{s+t,t+u}$ for all $u \geq 0$.*
  - *U_t is called* ***free positive multiplicative Levy process of the second kind*** *if $\mathcal{K}_{s,t} = \mathcal{K}_{s+t,t+u}$ for all $u \geq 0$.*

Proposition 5.8. *Let X_t be a free additive Levy process of the first kind with associated measures $\mu_{s,t}$. Then the measures $v_t = \mu_{0,t}$ form a weakly continuous free additive convolution semigroup.*

Proof. Since a free additive Levy process of the first kind is a free increment process we note that $\mu_{r,s} \boxplus \mu_{s,t} = \mu_{r,t}$ for all $r \leq s \leq t \in \mathbb{R}$. Further it holds that $\mu_{s,t} = \mu_{s+t,t+u}$ for all $u \geq 0$, so we can see that $\mu_{0,t-s} = \mu_{s,t}$ for all $s \leq t$ and thus the above equality becomes $\mu_{0,s-r} \boxplus \mu_{0,t-s} = \mu_{0,t-r} = v_{s-r} \boxplus v_{t-s} = v_{t-r}$. Denoting $a = s - r$ and $b = t - s$ and noting $a + b = s - r + t - s = t - r$ results in $v_a \boxplus v_b = v_{a+b}$ for all $a, b \geq 0$ and thus showing that v_t form a semigroup. ☐

For weak-continuity we will use similar arguments as [BV92].

Lemma 5.9. *For every $a \in \mathbb{R}$ and every μ with supp$(\mu) \subset [-a, a]$ G_μ is one-to-one on U and $G_\mu(U) \supset V$.*
Thus φ_μ is defined on V and $K_\mu(V) \subset U$ where U is some neighborhood of ∞ and V some neighborhood of 0.

Proof. Let μ satisfy $\text{supp}(\mu) \subset [-a, a] \subset \mathbb{R}$ and define $h(z) := G\left(\frac{1}{z}\right)$. Then

$$\frac{\partial}{\partial z} h(z) = \int_{-a}^{a} \frac{1}{(1 - tz)^2} d\mu(t)$$

Further there exists an $\epsilon > 0$ independent of μ such that $\text{Re}\left(\frac{\partial}{\partial z} h(z)\right) > 0$ for $|z| < \epsilon$. Thus h is one-to-one for $|z| < \epsilon$ and

$$h(z) = z + \int_{-a}^{a} \frac{tz^2}{1 - tz} d\mu(t)$$

This shows that for a sufficiently small ϵ $|h(z) - z| < \frac{|z|}{2}$ for $|z| < \epsilon$. For such an ϵ we get $\{h(z) : |z| < \epsilon\} \supset \{z : |z| < \frac{\epsilon}{2}\}$ and the lemma follows for $U = \{z : |z| > \epsilon\} \cup \{\infty\}$ and $V = \{z : |z| < \frac{\epsilon}{2}\}$ ☐

Proposition 5.10. *Let $(\mu_n)_{n \in \mathbb{N}}$ be compactly supported measures on $\mathbb{R}$. Then the following are equivalent:*

1. *All supp(μ_n) are contained in a compact interval and $\mu_n \to \mu$ in the weak-$*$-topology.*

2. $\varphi_{\mu_n} \to \varphi$ *uniformly in some neighborhood of* 0.

If these conditions are satisfied then $\varphi = \varphi_\mu$

Proof. In this proof we use similar argument as [BV92]

(1) $\implies$ (2): Lemma 5.9 shows the existence of sets U and V such that $K_{\mu_n}(V) \subset U$ for all $n \in \mathbb{N}$. Since K_μ is continuous, the family $\{K_{\mu_n} : n \geq 1\}$, and hence also $\{\varphi_{\mu_n} : n \geq 1\}$, is a pre-compact subset of the space of continuous functions. Thus it is sufficient to show that the Taylor coefficients of φ_{μ_n} converge to the coefficients of φ_μ. This holds since there exists polynomials P_n such that for $\sum_{n=1}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} b_n z^n$

$$a_n = b_1^{-n!} P_n(b_1, ..., b_n)$$

and thus since K_{μ_n} is bounded by U for $z \in V$ the coefficients of the Taylor series converge.

(2) $\implies$ (1): It is sufficient to show that $\{\mu_n : n \geq 1\}$ are supported in a compact interval. Convergence follows from weak-$*$-compactness of the collection of measures supported in a fixed compact interval and from the first part of the proof.
We define $g_n(z) := \frac{1}{K_{\mu_n}(z)}$ and thus $\frac{\partial}{\partial z} g_n(z) = \frac{1 - z^2 \frac{\partial}{\partial z}\varphi_{\mu_n}(z)}{(1 + z\varphi_{\mu_n}(z))^2}$.
Hence there exists an $\epsilon > 0$ such that $\mathrm{Re}\left(\frac{\partial}{\partial z} g_n(z)\right) > 0$ for $|z| < \epsilon$ for all $n \in \mathbb{N}$.
Further $g_n(z) = z - z^2 \frac{\varphi_{\mu_n}(z)}{1 + z\varphi_{\mu_n}(z)}$ and thus for ϵ small enough g_n has an inverse defined for $|z| < \frac{\epsilon}{2}$ and therefore $\mathrm{supp}(\mu_n) \subset \left[-\frac{2}{\epsilon}, \frac{2}{\epsilon}\right]$ for all n.
The inverse exists since analytic function g in a convex set with $\mathrm{Re}(g') > 0$ for all z is one-to-one.

□

Lemma 5.11. *Let* $G : \mathbb{C}^+ \to \mathbb{C}^-$ *be analytic. Then the following statements are equivalent.*

1. $\exists \mu$ *probability measure on* $\mathbb{R}$ *such that* $G_\mu = G$
2. *for all* $\alpha > 0$ *we have* $\lim_{|x| \to \infty, x \in \Gamma_\alpha} zG(z) = 1$
3. $\lim_{y \to \infty} iyG(iy) = 1$

Proof.
(2) $\implies$ (3):
is obvious

(1) $\implies$ (3):
Has already been shown in proposition 3.12

(3) $\implies$ (1):

It is easy to check by looking at the real and imaginary part of $G(iy)$ that $\lim_{y\to\infty} iyG(iy) = 1$ implies that $\limsup_{y\to\infty} y|G(iy)| = 1$. Thus we get the statement by the Stieltjes inversion formula as discussed in theorem 3.15

(1) $\implies$ (2):
easy calculations show that $\left|\frac{t}{z-t}\right| \leq (\alpha^2+1)^{\frac{1}{2}}$ for all $z \in \Gamma_\alpha$ and $t \in \mathbb{R}$
So for fixed $T > 0$ and $z = x + iy \in \Gamma_\alpha$:

$$\begin{aligned} |zG(z)-1| &= \left|\int_{-\infty}^{\infty} \frac{t}{z-t} d\mu(t)\right| \\ &\leq \int_{-T}^{T} \left|\frac{t}{z-t}\right| d\mu(t) + (\alpha^2+1)^{\frac{1}{2}} \mu\left(\{t : |t| \geq T\}\right) \\ &\leq \int_{-T}^{T} \frac{|t|}{y} d\mu(t) + (\alpha^2+1)^{\frac{1}{2}} \mu\left(\{t : |t| \geq T\}\right) \\ &\leq \frac{T}{y} \mu\left((-T,T)\right) + (\alpha^2+1)^{\frac{1}{2}} \mu\left(\{t : |t| \geq T\}\right) \end{aligned}$$

and hence

$$\limsup_{|z|\to\infty, z\in\Gamma_\alpha} |zG(z)-1| \leq (\alpha^2+1)^{\frac{1}{2}} \mu\left(\{t : |t| \geq T\}\right)$$

and it is easy to observe that $\lim_{T\to\infty} (\alpha^2+1)^{\frac{1}{2}} \mu\left(\{t : |t| \geq T\}\right) = 0$ □

5.4. Levy-Khintchine formulas

Finally we want to use the tools we have discussed so far to prove analogs of the Levy-Khintchine formulas and hence complete the characterization of processes with free increments.

In the following we will denote the right inverse of the transform ψ_μ of some probability measure μ on the image $\frac{\psi_\mu}{1+\psi_\mu}(i\mathbb{C}^+)$ by $\tilde{\chi}_\mu$.

5.4.1. Additive Levy processes

Theorem 5.12. *1. Let X_t be a free additive Levy process of the second kind with associated measures $\mu_{s,t}$ and Markov transition function $\mathcal{K}_{s,t}$ on $\mathbb{R}$. Then $\mathcal{L}_t = \mathcal{K}_{0,t}$ form a Feller Markov semigroup.*
Let $F_{s,t}$ the analytic functions as seen in theorem 4.5 and define $F_t = F_{0,t}$ where $F_0(z) = z$. Then F_t form a semigroup under decomposition and there exists $\varphi : \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$ analytic such that $\lim_{\xi\to\infty, \xi\in\Gamma_{\alpha,\beta}} \frac{\varphi(\xi)}{\xi} = 0$ for every $\Gamma_{\alpha,\beta}$. Further F_t satisfy $\frac{\partial F_t}{\partial t} + \varphi(F_t) = 0$

2. Conversely let $\varphi : \mathbb{C}^+ \to \mathbb{C}^-$ be analytic such that $\lim_{\xi\to\infty, \xi\in\Gamma_{\alpha,\beta}} \frac{\varphi(\xi)}{\xi} = 0$ in some $\Gamma_{\alpha,\beta}$ and let F_t be a semigroup obtained by solving the differential equation $\frac{\partial F_t}{\partial t} + \varphi(F_t) = 0$ with $F_0(z) = z$. Then the following are equivalent

- *There exists a free additive Levy-process of the second kind with initial distribution μ_0, with associated semigroup of maps F_t*

- *For all $t > 0$ $\varphi \circ F_t^{-1} \circ F_{\mu_0}^{-1}$ has an analytic continuation to $\mathbb{C}^+$ with values on $\mathbb{C}^-$*

if and only if for all

In this case we will call φ the **Levy function**

Proof. 1. Since $t \mapsto \mu_t$ is weakly continuous by prop 5.10 we see that $t \mapsto F_t$ is continuous for the topology of uniform convergence on compact subsets of $\mathbb{C}^+$ and for each domain $\Gamma_{\alpha,\beta}$ there exists a number ϵ such that the functions $F_{-t} = F_t^{-1}$ for $0 \le t \le \epsilon$ are well defined on $\Gamma_{\alpha,\beta}$ and thus the maps F_t for $-\epsilon \le t\epsilon$ form a local group of analytic transformations in the sense of H. Cartan. Theorem 10 in [Car79] gives us for each $\alpha, \beta > 0$ an analytic $\varphi_{\alpha,\beta}$ on $\Gamma_{\alpha,\beta}$ such that F_t satisfy $\frac{\partial F_t}{\partial t} + \varphi(F_t) = 0$ for $t \in]-\epsilon, \epsilon[$ Since $\varphi_{\alpha,\beta}(\xi) = \frac{\partial F_t}{\partial t}\big|_{t=0}$, we can construct a map φ on $\mathbb{C}^+$ such that $\varphi|_{\Gamma_{\alpha,\beta}} = \varphi_{\alpha,\beta}$. Since F_t form a semigroup $\frac{\partial F_t}{\partial t} + \varphi(F_t) = 0$ holds for all $t \ge 0$ on $\mathbb{C}^+$. By Theorem 4.5 we get $\mathrm{Im}(F_t(\xi)) \ge \mathrm{Im}(\xi)$ for all $\xi \in \mathbb{C}^+$. So since $\varphi(\xi) = -\lim_{t\to 0} \frac{F_t(\xi)-\xi}{t}$ we get $\varphi(\mathbb{C}^+) \subseteq \mathbb{C}^- \cup \mathbb{R}$. Thus φ is a Nevanlinna function and has therefore the representation

$$\varphi(\xi) = a + b\xi + \int_{\mathbb{R}} \frac{1 + u\xi}{\xi - u} dv(u)$$

for some $a, b \in \mathbb{R}$ and $v \ge 0$ measure on $\mathbb{R}$. In particular one has $\lim_{\xi\to\infty, \xi\in\Gamma_{\alpha,\beta}} \frac{\varphi(\xi)}{\xi} = b$ in every domain $\Gamma_{\alpha,\beta}$. Using this and the differential equation, it is then easy to see that $\lim_{\xi\to\infty,\xi\in\Gamma_{\alpha,\beta}} \frac{F_t(\xi)}{\xi} = e^{bt}$ and hence by Theorem 4.5 we must have b = 0.

2. Since $\varphi(\mathbb{C}^+) \subset \mathbb{C}^-$ the differential equation has a solution for all $t \ge 0$ given by the flow of the analytic vector field $-\varphi(\xi)\frac{\partial}{\partial \xi}$. By well known existence theorems for differential equations with analytic coefficients, we can see that $F_t(\mathbb{C}^+) \subseteq \mathbb{C}^+$ and is one-to-one on $\mathbb{C}^+$.
Further we have that $\lim_{\xi\to\infty,\xi\in\Gamma_{\alpha',\beta'}} \frac{F_t\xi)}{\xi} = 1$ in some domain $\Gamma_{\alpha',\beta'} \subset \Gamma_{\alpha,\beta}$. F_t is invertible on $F_t(\mathbb{C}^+)$ and satisfies for $t \in \mathbb{R}_+$ $\frac{\partial F_s^{-1}}{\partial s} - \varphi(F_s^{-1}) = 0$ on $\bigcap_{s\le t} F_s(\mathbb{C}^+)$.
Let μ_0 be a probability measure on $\mathbb{R}$. By Proposition 5.4 in [BV93] we get for all $t \ge 0$ the existence of $\Gamma_{\alpha,\beta}$ such that $F_s^{-1} \circ F_{\mu_0}^{-1}$ are defined and analytic on $\Gamma_{\alpha,\beta}$.
Let now φ be Levy function of a free additive Levy process of the second kind with respective distributions μ_t and $\mu_{s,t}$. Then for every $s < t \in \mathbb{R}_+$

$$F_{\mu_t}^{-1}(z) - F_{\mu_s}^{-1}(z) = F_t^{-1} \circ F_{\mu_0}^{-1}(z) - F_s^{-1} \circ F_{\mu_0}^{-1}(z) = \varphi_{\mu_{x,t}}(z)$$

in some domain $\Gamma_{\alpha,\beta}$. Further if s, t are in some compact interval $\Gamma_{\alpha,\beta}$ can be chosen such that

$$\lim_{z\to\infty, z\in\Gamma_{\alpha,\beta}} \frac{F_s^{-1} \circ F_{\mu_0^{-1}(z)}}{z} = 1$$

uniformly for s in this interval. Choose $\Gamma_{\alpha,\beta}$ such that for $z \in \Gamma_{\alpha,\beta}$

$$n\left(F_{s+\frac{t}{n}}^{-1} \circ F_{\mu_0}^{-1}(z) - F_s^{-1} \circ F_{\mu_0}^{-1}(z)\right) = \int_s^{s+\frac{t}{n}} \varphi \circ F_u^{-1} \circ F_{\mu_0}^{-1}(z) du = \varphi_{\mu_{s,s+\frac{t}{n}}}^{\boxplus n}(z)$$

and thus for $n \to \infty$

$$\varphi^{\boxplus n}_{\mu_{s,s+\frac{t}{n}}}(z) \to t\varphi \circ F_s^{-1} \circ F_s^{-1}$$

uniformly on every compact subset of $\Gamma_{\alpha,\beta}$.
Further the functions $\varphi \circ F_u^{-1} \circ F_{\mu_0}^{-1} = o(z)$ uniformly for u in a neighborhood of s. By proposition 5.10 the limit $t\varphi \circ F_s^{-1} \circ F_s^{-1}$ has the form φ_{v_t} where v_t is a probability measure which form a free convolution group. Thus by theorem 5.10 in [BV93] $\varphi \circ F_s^{-1} \circ F_{\mu_0}^{-1}$ can by analytically extended to $\mathbb{C}^+$ with values in $\mathbb{C}^-$ for all s.

Again by theorem 5.10 of [BV93] φ is the φ-transform of some freely infinitely divisible measure. Thus $\epsilon\varphi \circ F_s^{-1} \circ F_{\mu_0}^{-1}$ is the φ-transform of some measure for all $\epsilon > 0$ and we can approximate the integral

$$F_t^{-1} \circ F_{\mu_0}^{-1} - F_s^{-1} \circ F_{\mu_0}^{-1} = \int_s^t \varphi \circ F_u^{-1} \circ F_{\mu_0}^{-1} du$$

by Riemann sums of the form

$$\sum_i (u_{i+1} - u_i)\varphi \circ F_{u_i}^{-1} \circ F_{\mu_0}^{-1}$$

which are uniformly convergent on every compact subset of $\Gamma_{\alpha,\beta}$ and $o(z)$ uniformly in the same domain.
Since $(u_{i+1} - u_i)\varphi \circ F_{u_i}^{-1} \circ F_{\mu_0}^{-1}$ are φ-transforms so is their sum and by proposition 5.10 $F_t^{-1} \circ F_{\mu_0}^{-1} - F_s^{-1} \circ F_{\mu_0}^{-1}$ is the φ-transform of some probability measure $\mu_{s,t}$. It is easy to check that this measure $\mu_{s,t}$ satisfies the desired conditions.

□

Denote by $\mathcal{L}_1^+$ the space of additive Levy function of the first kind and by $\mathcal{L}_2^+$ the space of additive Levy function of the second kind. (1) of the theorem above shows that $\mathcal{L}_2^+ \subseteq \mathcal{L}_1^+$. In the statement (2) however we have seen some restrictions as to when a Levy process of the first kind is a Levy process of the second kind. For a detailed introduction of the Levy-Khintchine formulas we refer to [BV92] and [BV93].

5.4.2. Unitary multiplicative Levy processes

Theorem 5.13. *1. Let U_t be a free unitary multiplicative Levy process of the second kind with associated analytical maps $F_{s,t}$. The maps $F_t = F_{0,t}$ with $F_0(z) = z$ form a semigroup and $t \mapsto |F_t(z)|$ is a decreasing for all $z \in \mathbb{D}$. Further there exists $u : \mathbb{D} \to \{z \in \mathbb{C} | \operatorname{Re}(z) \geq 0\}$ analytic such that $\frac{\partial F_t}{\partial t} + F_t u(F_t) = 0$*

2. Let $u : \mathbb{D} \to \{z \in \mathbb{C} | \operatorname{Re}(z) \geq 0\}$ be analytic and F_t the solution of $\frac{\partial F_t}{\partial t} + F_t u(F_t) = 0$ where $F_0(z) = z$. Then the following are equivalent

- *There exists a free multiplicative Levy process of the second kind with initial distribution μ_0 and associated semigroup of maps F_t*

- *For all $t > 0$ the function $u \circ F_t^{-1} \circ \tilde{\chi}_{\mu_0}^{-1}$ has an analytic extension to $\mathbb{D}$ taking values with non-negative real part*

5.4.3. Positive multiplicative Levy processes

Theorem 5.14. *1. Let S_t be a free positive multiplicative Levy process with associated analytic maps $F_{s,t}$. The maps $F_t = F_{0,t}$ for $t \geq 0$ form a semigroup of analytic mas on $\mathbb{C} \setminus \mathbb{R}_+$ such that $t \mapsto \operatorname{Arg} F_t(z)$ is an increasing map for $z \in \mathbb{C}^+$. There exists an analytic function v on $\mathbb{C} \setminus \mathbb{R}_+$, $\mathbb{C}^- \cup \mathbb{R}$, such that $v(\bar{z}) = \bar{v}(z)$ for $z \in \mathbb{C}^+$, $v(\mathbb{C}^+) \subset \mathbb{C}^- \cup \mathbb{R}$ and the maps F_t for $t \geq 0$ satisfy the differential equation $\frac{\partial F_t}{\partial t} + F_t v(F_t) = 0$*

2. Let v be an analytic function of $\mathbb{C} \setminus \mathbb{R}_+$ such that $v(\mathbb{C}^+) \subset \mathbb{C}^- \cup \mathbb{R}$ and $v(\bar{z}) = \bar{v}(z)$ for all $z \in \mathbb{C}^+$ and let F_t for all $t \geq 0$ be the solution of the differential equation $\frac{\partial F_t}{\partial t} + F_t v(F_t) = 0$ with $F_0(z) = z$. Then the following are equivalent

- *There exists a free positive multiplicative Levy process of the second kind with initial distribution μ_0 with associated semigroup of maps F_t*
- *For every $t > 0$ the function $v \circ F_t^{-1} \circ \tilde{\chi}_{\mu_0}^{-1}$ has an analytic continuation to $\mathbb{C} \setminus \mathbb{R}_+$ such that $v(\bar{z}) = \bar{v}(z)$ and $v(\mathbb{C}^+) \subset \mathbb{C}^- \cup \mathbb{R}$*

Proof. Again we can use similar arguments as in the proof for the Levy-Khintchine formula for free additive Levy processes of the second kind. □

We can now describe the distribution of Levy processes of the second kind by their initial distribution and their Levy function. We have also seen that $\mathcal{L}_2 \subsetneq \mathcal{L}_1$. Contrary to the classical theory we have proven the Levy-Khintchine formulas not with respect to their Levy measures but their Levy functions instead. To complete the picture we will finally state the Levy-Khintchine formulas with respect to the Levy measure for free additive and multiplicative infinitely divisible probability measures.

Theorem 5.15. *A measure μ on $\mathbb{R}$ is freely infinitely divisible if and only if φ_μ has an analytical extension to $\varphi_\mu : \mathbb{C}^+ \to \mathbb{C}^- \cup \mathbb{R}$ and $\lim_{y\to\infty} \frac{\varphi(iy)}{y} = 0$. This function has the representation*

$$\varphi(z) = \alpha + \int_{-\infty}^{\infty} \frac{1+tz}{z-t} dv(t)$$

where v is some finite measure on $\mathbb{R}$ and $\alpha \in \mathbb{R}$.

Theorem 5.16. *A measure μ on $\mathbb{T}$ is freely infinitely divisible if and only if $\Sigma_\mu(z) = e^{u(z)}$ where $u : \mathbb{D} \to \{z \in \mathbb{C} | \operatorname{Re}(z) \geq 0\}$ is an analytic. This function has the representation*

$$u(z) = i\alpha + \int_{\mathbb{T}} \frac{1+\xi z}{1-\xi z} dv(\xi)$$

where v is some finite measure on $\mathbb{T}$ and $\alpha \in \mathbb{R}$.

A. Glossary

B. Bibliography

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