Michael Müller, BSc

Gravitational bound states in quantum gravity

- Dark matter as a purely gravitational phenomenon -

MASTER'S THESIS

to achieve the university degree of

Master of Science

Master's degree programme: Physics

submitted to

Graz University of Technology

Supervisor

Axel Maas, Univ.-Prof. Dr. habil. Dipl.-Phys.

Institute of Physics, University of Graz

Graz, October 2020

Abstract

In this work the quantized version of the gravitational geon as a bound state of gravitational waves/gravitons, is studied as a possible purely gravitational dark matter candidate. The quantized theory is formulated in a non-perturbative way, on the basis of pure Einstein-Hilbert gravity. With the assumption that the metric describes the fundamental gravitational degrees of freedom, the Ricci scalar is identified as a suitable invariant bound state operator that can represent the geon. The two-point correlation function of the Ricci scalar is then studied as the geon propagator. It is discussed how the Fröhlich-Morchio-Strocchi (FMS) mechanism, usually employed in Brout-Englert-Higgs physics, can be applied to the problem of the geon propagator to reduce it to quantities that can be evaluated with perturbative methods. The application of the FMS mechanism goes hand in hand with the introduction of a vacuum expectation value (VEV) of the metric field, which is to be interpreted as the non-vanishing expectation value of the metric field in a gauge-fixed setup. The calculation of the geon propagator with either a Minkowski or a de Sitter spacetime as the VEV is then discussed. An explicit result for the geon is only provided for the case of Minkowski space, where it is found that under the employed approximations the geon is not a suitable dark matter candidate.

Kurzfassung

In dieser Arbeit untersuchen wir die quantisierte Version des gravitativen Geons, welches einen Bindungszustand von Gravitationswellen/Gravitonen darstellt, als einen möglichen Kandidaten für rein gravitative Dunkle Materie. Ausgehend von Einstein-Hilbert Gravitation, wird die quantisierte Theorie als nicht-perturbatives Problem formuliert. Unter der Annahme, dass die Metrik die fundamentalen Freiheitsgrade der Theorie darstellt, identifizieren wir den Ricci-Skalar als geeigneten Bindungszustands-Operator für das Geon. Die 2-Punkt Korrelationsfunktion des Ricci-Skalars wird als Propagator für das Geon behandelt. Wir diskutieren, wie der Fröhlich-Morchio-Strocchi (FMS) Mechanismus, welcher üblicherweise in der Brout-Englert-Higgs Physik eingesetzt wird, auf das Problem des Geon-Propagators angewandt werden kann, um denselben auf Objekte zurückzuführen, welche mit perturbativen Methoden berechnet werden können. Die Anwendung des FMS Mechanismus geht Hand in Hand mit der Einführung eines Vakuumerwartungswertes (VEV) des metrischen Feldes, welcher als nicht verschwindender Erwartungswert des metrischen Feldes in einer eichfixierten Behandlung interpretiert werden kann. Wir diskutieren die Berechnung des Geon-Propagators mit entweder einer Minkowski oder einer de Sitter Raumzeit als VEV. Ein explizites Ergebnis für den Geon-Propagator wird nur für den Fall einer Minkowski Raumzeit angegeben, wobei sich herausstellt, dass unter den verwendeten Näherungen das Geon keinen geeigneten Kandidaten für Dunkle Materie darstellt.

Acknowledgements

Here, I would like to thank the people that made this thesis possible.

First, I want to thank my supervisor Prof. Axel Maas for many clarifying discussions and useful pointers that helped me through the jungle of numerous new concepts that I encountered in this work.

I also want to thank my fellow master student Markus Markl, with whom I spent many an hour in discussion over topics regarding our respective theses and who kindly proofread this thesis.

Furthermore, I want to thank René Sondenheimer and Prof. Reinhard Alkofer for several fruitful discussions regarding the topics of my thesis, as well as Joshua Hoffer for countless debates on physics and various other topics that ensured very entertaining breaks.

Finally, I would like to thank my family for their company and support during my studies and especially my parents Gerlinde and Manfred Müller for encouraging me to pursue my chosen path, and for providing me with the opportunity to do so in a very carefree way.

Contents

1. Introduction

The gravitational interaction is responsible for the large-scale structure of the universe and according to current knowledge it is described by the theory of general relativity (GR). In the classical regime this theory is well understood and has successfully described phenomena such as the bending of light in a gravitational field, the orbit of planet Mercury and gravitational waves in particular. In the regime of quantum physics the standard model of particle physics (SM) successfully describes the electroweak and the strong interaction, yet a consistent quantum theory of gravitation is still missing. The successful description of gravitation within the rules of quantum physics is a major contemporary quest and this work is concerned with it as well, based on new ideas put forth in Ref. [1]. The motivation for studying a quantized theory of gravity in this work, lies in the question, whether quantum gravity could account for the phenomenon of dark matter (DM), which is an unresolved problem within the frameworks of both the SM and classical general relativity. The current models for the structure formation in the early universe as well as detailed analysis of the gravitational properties of galaxies and galaxy clusters all point towards a missing mass problem. One of the considered resolutions of this problem is the DM hypothesis, which states that the missing mass is due to an additional type or types of particle d.o.f.'s that have not been considered in our current models. The origin of these particle d.o.f.'s and their properties have been discussed in numerous different models, many of which consider additional matter fields, which go beyond the SM (see for example Ref. [2]). The common core to all these considerations is that DM should fulfill the following properties:

- It needs to be very abundant. According to the ΛCDM model DM accounts for approx. 26% of the energy content of the universe.
- It needs to interact at least gravitationally and possibly very weakly with SM matter.
- It can feature self-interaction, although stringent bounds can be put on the interaction strength, based on the small-scale behavior of DM and the considered models for the production mechanism [3].

It is an exciting question to ask, whether we have not overlooked something in the theory of gravitation that could account for the DM and remedy the need to introduce additional particles into the SM. An especially interesting property of GR that becomes relevant in this discussion is its non-linearity, which allows for gravitational fields to interact with themselves, without any matter present. This represents a gross difference to the theory of electromagnetic radiation, in which radiation fields of different origin pass through each other and while featuring interference due to the wave character of radiation, the different waves do not exchange energy between each other. The reason why gravity is different with respect to this is the mass-energy equivalence, which states that mass and energy are merely two sides of the same medal and is a consequence of the special theory of relativity [4]. If we consider two gravitational fields in close proximity, their energy content allows them to gravitate and thereby interact. The effect of gravitational self-interaction has been studied in the context of the missing-mass problem before and it was shown that even classically it could lessen the discrepancy between the observed galaxy rotation curves and the theoretical models, which represents one of the main arguments for DM [5]. However, this would then be an intrinsic property of gravity, not related to an additional particle d.o.f., and according to current models it does not get rid of the need for DM completely. In this work another consequence of the self-interaction of gravity is considered that could actually give rise to a particle d.o.f. with the properties of DM.

Since gravitational waves, which are merely oscillating gravitational fields, can interact with each other, one may consider the possibility that under certain conditions the waves get bound to each other due to their gravitational interaction, thus creating an object with well-defined spatial boundaries and mass. If suitably massive and abundant this could provide a purely gravitational d.o.f. that could act as DM. Such a bound state object was first considered in the context of electromagnetic waves forming a bound state through gravitational interaction in [6] and the bound state object was termed electromagnetic geon. This initial study demonstrated that certain types of electromagnetic geons could indeed be relatively stable. The concept was later on generalized to bound states of gravitational waves treated in GR and detailed analyses have been conducted to investigate the stability of such objects [7, 8]. However, while initially results were promising, the gravitational geon as a classical object has been argued to not represent a stable object due to inconsistencies between the final result and the initial hypothesis [9, 10, 11]. Hence the geon in classical GR does not represent a well defined bound state.

A similar problem occurred also in electrodynamics when physicists first tried to model the bound state of an electron with a positively charged nucleus, based on the classical electromagnetic interaction. The bound system turned out to decay on the order of 10^{-11} seconds due to the accelerated electron continuously emitting electromagnetic radiation. This issue was resolved of course by the advent of quantum mechanics, which allowed for a well posed definition of the hydrogen bound state that does indeed represent a stable object. Since the application of quantum mechanics to the problem of the hydrogen bound state was able to produce a stable object, one may then also consider the treatment of a gravitational bound state in the framework of quantum mechanics, to find out if this also gives rise to a stable entity.

Considerations of the same nature already provide promising results for non-Abelian gauge field theories, which are much closer to the theory of gravitation than QED. Yang-Mills type theories also allow for self-interaction between the excitations of the gauge field and it was shown that in the purely classical treatment of the theory a construction with analogous assumptions to the gravitational geon cannot exist [12] [1]. These theories have been studied in their quantized version in detail and the quantum object corresponding to the gauge field geon was investigated in the case of quantum chromodynamics (QCD), which represents a special case of Yang-Mills theory. The geon-like bound state, coined glueball, turned out to be a stable and massive entity [13].

Therefore it will be interesting to consider the gravitational geon as a quantum object and the formulation of the bound state problem for quantized gravitational waves, termed gravitons, in the frame work of quantum gravity (QG) will be the main goal of this thesis. It should be noted that similarly to quantum electrodynamics, where the perturbative theory is not sufficient to describe the hydrogen atom, we will also require a fully non-perturbative description of the quantum theory of gravity to discuss the graviton bound state. However, if the quantum theory fulfills certain assumptions regarding the state-space, perturbative methods can be utilized to obtain quantitative results, as was discussed for the case of Brout-Englert-Higgs physics in [14]. This work is organized as follows:

- In Section 2 we will first discuss the setup of the classical theory of gravity that will be used as a starting point for the formulation of the quantized theory. Then the quantization analogously to a gauge theory with the Feynman path integral as a quantization prescription, will be discussed. Furthermore, the interpretation of physical objects and the representation of the geon in the quantized setting, as well as the necessity of gauge-fixing, will be addressed.

- In Section 3 we discuss, how the Fröhlich-Morchio-Strocchi (FMS) mechanism can be applied to the quantized theory of gravitation in order to study gravitational geons, which are inherently non-perturbative bound state objects, with perturbative methods.

- In Section 4 we continue the discussion of geons, and present the explicit calculation of the tree-level geon propagator in a flat universe. It is demonstrated that in the used approximations the flat space geon propagator vanishes and thus, no geon exists in a flat universe.

- In Section 5 we apply the same method as in 4 to the geon in a de Sitter type universe with a non-zero cosmological constant, however, we encounter problems with this approach which presently prevent us from obtaining the corresponding propagator.

- In Section 6 we summarize the findings of this investigation and present further steps that could be take to study the properties of the de Sitter type geon.

- In the Appendices, we collect some technical results.

[1] However, the reason for the non-existence of this bound state is very different to the case of the gravitational geon and has to do with the fact that in contrast to gravity, Yang-Mills type gauge theories allow for both attractive and repulsive interactions and the latter causes the instability of the gauge field geon.

All the tensor calculations in this work were conducted with the **xAact** package for Mathematica. Calculations that had to be done in a specific coordinate frame were conducted with the **xCoba** extension of **xAact**.

2. Setup of the theory

Similarly to QED the gravitational interaction is of long range and therefore exerts influence also on macroscopic effects. In fact, the dynamics of the large scale structure of the universe is predominantly governed by the gravitational interaction. It is therefore not surprising that a very successful classical theory exists for the phenomenon of gravitation in the form of general relativity. In analogy to Maxwell's electromagnetism which was the starting point for the development of QED, the classical theory of general relativity will form the outset for the formulation of a quantum theory of gravitation. For this reason it is important to discuss the framework of classical GR, as many mathematical and physical concepts will also become important for the discussion of the quantized theory, and this is what will be done in Section 2.1.

2.1. The classical theory

In this section we follow closely [15, 16, 1, 17].

Classical general relativity is a field theory, defined in terms of events and their neighboring relations, which together form the construct of spacetime. Fields can be assigned an amplitude at every event and thus they are considered to be functions of the events. In order to give functions of events a mathematically easily manageable meaning, manifolds and coordinate frames are introduced.

2.1.1. Spacetime structure and geometry

Manifolds and coordinate systems

Spacetime is a four-dimensional set of events, i.e. we need in general four numbers to label the events $\{\mathscr{P}, \mathscr{Q}, \dots\}$. Of these four numbers, one denotes time and the other three represent three dimensional space. The world lines of such events and their crossings then define the neighboring relations and thus the spacetime structure. For mathematical simplicity we take the limit to a continuum of events and their respective world lines, which is described by a continuously differentiable manifold $\mathcal{M}$. On this manifold we may introduce coordinate systems/frames as maps that assign each event a quadruple $x^\alpha(\mathscr{P}) \in \mathbb{R}^4$. In this work Greek indices always range from 0 to 3 and from now on a coordinate system will always be identified by the collection of the four coordinate functions $\{x^\alpha(\mathscr{P})\}$ or more compact $\{x^\alpha\}$. Generally more than one map and thus multiple coordinate systems are required to describe the entirety of the manifold (Minkowski space of special relativity is an exception). Another important feature of such coordinate maps is that they are not unique, and an infinite group of equivalent coordinate systems does exist, which are related by infinitely differentiable, smooth maps called diffeomorphisms or simply coordinate transformations. The corresponding transformation law for the coordinate components between two coordinate systems $\{x^\alpha\}$ and $\{x'^\alpha\}$ is given by

$$x'^\alpha(\mathscr{P}) = \sum_\beta J^\alpha{}_\beta(\mathscr{P})\, x^\beta(\mathscr{P}) = \sum_\beta \left.\frac{\partial x'^\alpha}{\partial x^\beta}\right|_{\mathscr{P}} x^\beta(\mathscr{P}) \tag{2.1}$$

The formulation of this transformation is made in the passive picture.

Tangent and cotangent vectors

Special relativity relies heavily upon the concept of vectors and tensors to express physical laws; vectors on a manifold $\mathcal{M}$ are introduced as linear operators that can be identified with directional derivatives along an arbitrary curve on $\mathcal{M}$ and are thus also referred to as tangent vectors. The collection of all possible tangent vectors at an event $\mathscr{P}$ form a vector space that is referred to as tangent space $T_{\mathscr{P}}\mathcal{M}$. As discussed in Ref. [15], the coordinate basis $\left\{\hat{\partial}_\alpha\right\}$ naturally arises as a basis of $T_{\mathscr{P}}\mathcal{M}$, once a coordinate frame is chosen and an arbitrary vector $\hat{v} \in T_{\mathscr{P}}\mathcal{M}$ can be expressed as $\hat{v} = \sum_\alpha v^\alpha \hat{\partial}_\alpha$. In this work we adhere to common practice and identify any tangent vector $\hat{v}$ merely by its components v^α, always assuming that we are using the coordinate bases for the tangent space.

Besides the tangent space there also exists the cotangent space $T^*_{\mathscr{P}}\mathcal{M}$, which is the collection of all linear maps of tangent vectors to real numbers, i.e. the scalar product $\langle \hat{w}, \hat{v}\rangle \in \mathbb{R}$ with $\hat{w} \in T^*_{\mathscr{P}}\mathcal{M}$ and $\hat{v} \in T_{\mathscr{P}}\mathcal{M}$. The coordinate basis of $T_{\mathscr{P}}\mathcal{M}$ automatically induces basis vectors on $T^*_{\mathscr{P}}\mathcal{M}$, and thus we also identify the cotangent vectors with their components. While the tangent vector components v^α will be denoted by superscripts which are referred to as contravariant indices, the indices of cotangent vector components w_α will be subscripts and referred to as covariant indices. With this convention the scalar product is written as follows:

$$\langle \hat{w}, \hat{v}\rangle = w_\alpha v^\alpha, \tag{2.2}$$

where we have made use of the Einstein sum convention, which will be employed throughout this work. If we switch between two coordinate systems the coordinate basis of the (co)tangent space will be a different one and thus the (co)tangent vector components have to transform as well. Based on the transformation law 2.1 for the coordinate functions, the transformation for (co)tangent vectors follows accordingly:

$$\begin{aligned} v'^\alpha &= J^\alpha{}_\beta v^\beta && \text{for} \quad v^\alpha \in T_{\mathscr{P}}\mathcal{M} \\ w'_\alpha &= \left(J^{-1}\right)_\alpha{}^\beta w_\beta && \text{for} \quad w_\alpha \in T^*_{\mathscr{P}}\mathcal{M} \end{aligned} \tag{2.3}$$

Tensors - the metric tensor

Tensors form the generalization of (co)tangent vectors, which are linear maps, to multilinear maps. These can be expressed as outer products of (co)tangent vectors and are again represented by their components $T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l}$ with respect to the tensor product basis of (co)tangent vectors. Under a change of coordinates the tensor components transform like covariant or contravariant components on every index respectively:

$$T'^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} = J^{\alpha_1}{}_{\mu_1}\ldots J^{\alpha_k}{}_{\mu_k}\left(J^{-1}\right)_{\beta_1}{}^{\nu_1}\ldots\left(J^{-1}\right)_{\beta_l}{}^{\nu_l}T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l} \tag{2.4}$$

So far the tangent and cotangent vectors and tensors were only ever considered at a single event $\mathscr{P}$, but in general a (co)tangent space can be attached to every event. Furthermore, one can define (co)tangent vector fields which represent a specific element of the vector space $T_{\mathscr{P}}\mathcal{M}$ or $T^*_{\mathscr{P}}\mathcal{M}$ at all events $\mathscr{P} \in \mathcal{M}$. The generalization of this are then tensor fields and the metric tensor field is one representative of this class that is used to make the neighboring relations between events $\mathscr{P} \in \mathcal{M}$ in the coordinate description more concrete. The metric tensor field or just metric $g_{\mu\nu}(\mathscr{P})$ is a symmetric tensor field:

$$g_{\mu\nu}(\mathscr{P}) = g_{\nu\mu}(\mathscr{P}) \tag{2.5}$$

It acts as a symmetric, non-degenerate map of two tangent vectors to a real number and thus it defines the inner product on the tangent space $T_{\mathscr{P}}\mathcal{M}$, which may be denoted as

$$g_{\mu\nu}(\mathscr{P})\, u^{\mu}(\mathscr{P})\, v^{\nu}(\mathscr{P}) \in \mathbb{R}. \tag{2.6}$$

In particular this can be used to calculate the magnitude of the infinitesimal displacement vector between two events with the coordinates x^{μ} and $x^{\mu} + dx^{\mu}$. This corresponds to the square of the infinitesimal interval ds between the two events and thus gives rise to the notion of distance. The corresponding displacement vector is denoted dx^{μ} and the magnitude obtained from the inner product then becomes:

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{2.7}$$

An important property of the metric is its signature, i.e. if we choose a cotangent space basis that diagonalizes the component matrix $[g_{\mu\nu}]$ (this will in general not coincide with the coordinate induced basis), the number and position of positive and negative eigenvalues. For any physical spacetime, with x^0 denoting the time component and $\vec{x} = \left\{x^1, x^2, x^3\right\}$ denoting the spatial coordinates, the signature must be of type $(-+++)$ or $(+---)$, which is referred to as Lorentzian signature. In this work the signature $(-+++)$ will be adopted[1]. Being a tensor field the metric transforms as $g'_{\mu\nu} = \left(J^{-1}\right)_{\mu}{}^{\alpha}\left(J^{-1}\right)_{\nu}{}^{\beta}g_{\alpha\beta}$ under coordinate transformations, yet its signature is invariant under such transformations. Due to the Lorentzian signature the inner product induced by the metric actually becomes indefinite and the three usual cases can be

[1] A list of signatures used in the cited works can be found in Appendix A

distinguished:

$$\begin{array}{lll} g_{\mu\nu}v^\mu v^\nu > 0, & v^\mu & \text{is spacelike} \\ g_{\mu\nu}v^\mu v^\nu < 0, & v^\mu & \text{is timelike} \\ g_{\mu\nu}v^\mu v^\nu = 0, & v^\mu & \text{is null} \end{array} \tag{2.8}$$

For the component matrix $[g_{\mu\nu}]$ of the metric, we can also compute the determinant g and an inverse $[g_{\mu\nu}]^{-1}$. The latter can actually be interpreted as the component matrix of another tensor and will generally be denoted as $[g^{\mu\nu}]$:

$$g = \det\left([g_{\mu\nu}]\right) \tag{2.9}$$

$$g_{\mu\nu}g^{\nu\alpha} = g^{\alpha\nu}g_{\nu\mu} = \delta_\mu{}^\alpha \tag{2.10}$$

The sign of the metric determinant[2] for a Lorentzian signature is always negative.

If we act with the metric on just one tangent vector $\hat{v} \in T_{\mathscr{P}}\mathcal{M}$, we will end up with $g_{\mathscr{P}}(\cdot, \hat{v})$, which is a cotangent vector in $T^*_{\mathscr{P}}\mathcal{M}$. This linear map is unique for every tangent vector and can thus be used to map tangent vectors into cotangent vectors. Since the metric has an inverse, we can also perform a map from cotangent to tangent vectors. In component notation we thus obtain:

$$g_{\mu\nu}v^\nu = v_\mu \tag{2.11}$$

$$g^{\mu\nu}w_\nu = w^\mu \tag{2.12}$$

For this reason the metric is often referred to as index shifter.

Covariant derivatives and curvature

So far we have discussed tangent vectors that are attached to a specific event $\mathscr{P}$ in spacetime as well as vector fields which assign specific elements of the tangent vector spaces to every spacetime event. However, the vector spaces at different events are still fundamentally disjoint, this is resolved by the introduction of parallel transport, which uses the covariant derivative to identify vectors in (co)tangent spaces at different events with each other. The covariant derivative at $\mathscr{P}$ is defined as follows:

$$\nabla_\alpha v^\mu(\mathscr{P}) = \left(\partial_\alpha v^\mu + \Gamma^\mu{}_{\alpha\beta}v^\beta\right)\Big|_{\mathscr{P}} \tag{2.13}$$

While neither $\partial_\alpha v^\mu$ nor $\Gamma^\mu{}_{\alpha\beta}v^\beta$ represent a proper tensor, as they do not transform according to (2.4), the covariant derivative acting on a tangent vector actually does give rise to another tensor. The coefficients $\Gamma^\mu{}_{\alpha\beta}$ can be determined via the metric. Since all manifolds relevant for GR are equipped with a metric and a corresponding inner product, we may require that if we parallel transport a vector, the inner product of the vector with itself, i.e. its norm, remains the same. This requirement is referred to as **metric postulate** and it is something that is generally required for GR and we will also

[2] The metric determinant actually represents an additional type of object, called a tensor density, which transforms as $g(x'^\mu) = \left|\frac{\partial x'^\mu}{\partial x^\alpha}\right|^{-2} g(x^\alpha) = \det(J^\mu{}_\alpha)^{-2}\, g(x^\alpha)$, see [18].

assume its validity in this work. The conservation of the inner product under parallel transport translates into the condition $\nabla_\alpha g_{\mu\nu} = 0$, where the action of the covariant derivative on a general tensor is given by

$$\nabla_\alpha T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \partial_\alpha T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} + \sum_{\imath=1}^{k} \Gamma^{\mu_i}{}_{\alpha\rho} T^{\mu_1 \dots \rho \dots \mu_k}{}_{\nu_1 \dots \nu_l} - \sum_{\jmath=1}^{k} \Gamma^{\rho}{}_{\alpha\nu_j} T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \rho \dots \nu_l}. \tag{2.14}$$

From $\nabla_\alpha g_{\mu\nu} = 0$ a unique form of the coefficients $\Gamma^\mu{}_{\alpha\beta}$ can be derived:

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} \left(\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\alpha\rho} - \partial_\rho g_{\alpha\beta} \right) \tag{2.15}$$

The coefficients $\Gamma^\mu{}_{\alpha\beta}$ under the metric postulate are generally termed **Christoffel symbols** or affine connection and they are fully determined by the metric of the spacetime.

Another postulate of GR is that spacetime is torsion free, which surmounts to the condition that the commutator of two covariant derivatives acting on a scalar vanishes:

$$\left[\nabla_\mu, \nabla_\nu \right] f = 0 \tag{2.16}$$

Now finally all the tools are available to discuss the curvature of spacetime. Curvature manifests itself in many different ways, one of these is: A tangent vector which is parallel transported on an infinitesimal, closed curve such that it ends up at the initial position again, will not coincide with the initial tangent vector in a curved spacetime. Since covariant derivatives are involved in the parallel transport, this can be translated into the failure of two successive covariant derivatives to commute, when acting on said tangent vector:

$$\left[\nabla_\mu, \nabla_\nu \right] v^\alpha = \nabla_\mu, \nabla_\nu v^\alpha - \nabla_\nu, \nabla_\mu v^\alpha = R_{\mu\nu\beta}{}^\alpha v^\beta \neq 0 \tag{2.17}$$

The tensor $R_{\mu\nu\beta}{}^\alpha$ representing the action of the commutator on the tangent vector is called **Riemann tensor** and it can be represented purely in terms of the Christoffel symbols (and since we are employing the metric postulate purely in terms of the metric):

$$R_{\mu\nu\rho}{}^\sigma = \partial_\nu \Gamma^\sigma{}_{\mu\rho} - \partial_\mu \Gamma^\sigma{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho} \Gamma^\sigma{}_{\alpha\nu} - \Gamma^\alpha{}_{\nu\rho} \Gamma^\sigma{}_{\alpha\mu} \tag{2.18}$$

The Riemann tensor is directly related to the curvature of the manifold. Namely, by computing the trace of the Riemann tensor on its second and fourth index, we obtain the **Ricci tensor**

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho. \tag{2.19}$$

The trace of the Ricci tensor, which is computed by raising one index and then setting them equal, then gives the **Ricci scalar**, which can be identified with the scalar curvature of the manifold:

$$\mathcal{R} = g^{\alpha\nu} R_{\mu\nu} \big|_{\alpha=\mu} = R_\mu{}^\alpha \big|_{\alpha=\mu} = R_\mu{}^\mu = g^{\mu\nu} R_{\mu\nu} \tag{2.20}$$

At this point it is important that several conventions have been introduced that can and generally will differ from those in other treatments, the conventions introduced here are taken from Ref. [15]. An additional list of conventions can be found in App. A.

2.1.2. Einstein-Hilbert action

According to Einstein's equivalence principle, gravitation can be interpreted as the manifestation of a curved spacetime in the motion of particles. I.e. every matter particle moves on geodesics and the geodesics are determined by the curvature of spacetime. The important step was then to relate the action of gravity on matter to the concept of matter as the source of gravity or curvature in spacetime. This was achieved by Einstein through promoting the metric to a dynamical field, satisfying the following equation of motion:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{2.21}$$

Where $T_{\mu\nu}$ denotes the energy-momentum tensor which describes the matter content of the theory and Λ is the cosmological constant, which can be interpreted as the dark energy content in the universe and is (at least in GR) completely independent from the matter content.

In both classical and quantum mechanics the action plays a central role. In classical mechanics the action can be used to derive the classical e.o.m. through the stationary action principle. In quantum mechanics the action can be used as a weight factor in the Feynman path integral approach, which will be discussed in Section 2.2. Since the Einstein equation represents an equation of motion for the metric, coupled to some matter content, there must exist a corresponding action as well. The action for the pure geometric theory, i.e. $T_{\mu\nu} = 0\ \forall \mu,\nu$, is represented by the Einstein-Hilbert action:

$$S_{EH}\left[g_{\mu\nu}\right] = \frac{1}{2\kappa^2}\int d^4x\sqrt{|g|}(\mathcal{R} - 2\Lambda) \tag{2.22}$$

$$\kappa = \sqrt{8\pi G_N} \tag{2.23}$$

$$\Lambda = \kappa^2 \rho_{\text{vac}} \tag{2.24}$$

The currently accepted values for the Newton constant G_N and the vacuum energy density ρ_{vac}, and therefore by extension for the cosmological constant and κ (in natural units: $\hbar = c = 1$), are [19]:

$$\begin{array}{llll} G_N & = 6.95 \cdot 10^{-39}\text{GeV}^{-2} & \rightarrow \quad \kappa & = 4.17 \cdot 10^{-19}\text{GeV}^{-1} \\ \rho_{\text{vac}} & = 2.52 \cdot 10^{-47}\text{GeV}^{4} & \rightarrow \quad \Lambda & = 4.38 \cdot 10^{-84}\text{GeV}^{2} \end{array} \tag{2.25}$$

If we also wanted to include matter, we would have to add the specific matter action S_m to the Einstein-Hilbert action and this would give us the energy-momentum tensor in the e.o.m.. However, in this work we are only concerned with the gravitational sector and thus we only need to consider the Einstein-Hilbert action together with the resulting vacuum Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.26}$$

The explicit expression for the Ricci scalar and tensor in terms of the metric is included in App. B.

2.2. The quantum theory

In this section we follow closely [20, 21, 15]. Now that the framework of classical GR has been discussed, we can use it as a starting point to setup the quantum theory. It needs to be noted that while in the above presentation the metric is used as the fundamental d.o.f. of the theory and curvature is the manifestation of gravity, there are several completely equivalent formulations of classical GR, which rely upon different geometric objects. Such approaches are formulations of GR which do encode the effect of gravity not in spacetime curvature, but rather in spacetime torsion or its non-metricity [22], or they use the vierbein field as the fundamental d.o.f. instead of the metric [17]. But while these descriptions are all equivalent in the classical setup, once we turn to the quantization, they will lead to very different quantum theories. In this work the above framework using curvature and the metric will be used [3].

As argued in Ref. [1] a more appropriate choice than the metric for the fundamental field describing the gravitational interaction would be the vierbein field as it allows for a well-defined notion of spin even in a curved spacetime. The vierbein field is defined as the transformation matrix for the metric from the coordinate frame to an orthonormal Lorentz frame in which the metric becomes the one of special relativity η_{ab}:

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab} \tag{2.27}$$

The thereby generated Minkowski structure in tangent space allows to define global symmetries just as in special relativity, which then lead to the classification of spin. Thus, if fields with non-zero spin are considered in curved spacetime, the vierbeins should be used. However, in this work we will only be concerned with the simplest d.o.f. of the theory, which have spin-0 and thus we may use the metric instead of the vierbeins as a simplification as major differences between the two approaches are expected only for objects with non-zero spin. Since we will only work at tree-level in this treatment, the two formulations should lead to qualitatively similar results.

To quantize the classical theory of GR, we invoke the Feynman path integral method which is discussed in detail in the following section.

2.2.1. The gravitational path integral

In quantum field theory we are concerned with the calculation of expectation values of operators of fields. The path integral (PI) is a method of generating such expectation values. What the PI does is, it takes the fields that the theory is concerned with and averages over all possible configurations with a functional measure which is provided by the action of the theory. Thus for the case of gravity, where the fundamental field is supposed to be the metric, we would be averaging over the space of all possible metrics $\{g_{\mu\nu}\} = \mathcal{Q}$, weighted by the Einstein-Hilbert action. Thus we can formally define the

[3] If one starts out with the gauging of the Poincaré group, curvature might also occur together with torsion in the description of gravity, but for the simple spin-0 objects that are considered in this work the effect of torsion vanishes [23].

integration measure of the gravitational PI as:

$$\mathcal{D}\mu\left(g_{\mu\nu}\right) = \mathcal{D}g_{\mu\nu}e^{iS_{EH}[g_{\mu\nu}]} \tag{2.28}$$

The notion of such measure can be made sense of in the form of a Haar measure, however, it is as of now not at all clear, if such a measure exists as a well-defined quantity for the space of all metrics [20]. Based on this measure we may write down the partition function of the theory as

$$\mathcal{Z} = \int_{\mathcal{Q}} \mathcal{D}g_{\mu\nu}e^{iS_{EH}[g_{\mu\nu}]}. \tag{2.29}$$

The partition function is the central object of the theory and its well-definiteness is a major assumption of this treatment. Correlation functions of operators can then be obtained, by averaging over the operator with $\mathcal{Z}$ as normalization:

$$\langle \mathcal{O}\left[g_{\mu\nu}\right] \rangle = \frac{1}{\mathcal{Z}} \int_{\mathcal{Q}} \mathcal{D}g_{\mu\nu}\mathcal{O}\left[g_{\mu\nu}\right] e^{iS_{EH}[g_{\mu\nu}]} \tag{2.30}$$

Since we are looking at pure gravity without matter, we only have the metric as a fundamental field and thus the operators $\mathcal{O}\left[g_{\mu\nu}\right]$ can exclusively contain the metric.

Another important property of the above construction is the existence of a local symmetry group which leaves the action as well as the measure in the PI invariant, i.e. the diffeomorphism group of general relativity. As discussed in Section 2.1, the physical content of the metric is invariant under the coordinate transformation $J\left(x\right)$, which takes x^μ to $x^\mu + \xi^\mu\left(x\right)$, where $\xi^\mu\left(x\right)$ is an arbitrary vector field:

$$g'_{\mu\nu} = \left(J^{-1}\right)_\mu{}^\alpha \left(J^{-1}\right)_\nu{}^\beta g_{\alpha\beta} \tag{2.31}$$

This allows for an important separation of the elements in $\mathcal{Q}$ into a set $\mathcal{Q}'$ containing one representative of every physically distinct metric configuration, and the corresponding diffeomorphism orbits. The latter constitute the generally infinite sets of physically equivalent metrics that can be obtained from any element of $\mathcal{Q}'$ through diffeomorphisms [4]. Under the assumption that this PI is well-defined without fixing a coordinate system (which would select a specific set $\mathcal{Q}'$), this has an important consequence for the expectation values (2.30). The PI averages over all diffeomorphism orbits and because the action and the measure are diffeomorphism invariant, the weight for all these configurations is the same. Since all elements of an orbit contribute with the same weight, their average will amount to zero (similarly to averaging over the normal vector field of a sphere). For any operator $\mathcal{O}\left[g_{\mu\nu}\right]$ that contains non-invariant expressions under the diffeomorphism group, this implies $\langle \mathcal{O}\left[g_{\mu\nu}\right] \rangle = 0$ [1, 14].

In particular this constitutes that $\langle g_{\mu\nu} \rangle = 0$, i.e. in quantum gravity spacetime structure can generally not be inferred via the metric, but it is necessary to construct operators that have no open indices and are invariant under diffeomorphisms. Such operators are

[4] $\mathcal{Q}'$ is not a unique set, but merely an arbitrary equivalence class of $\mathcal{Q}$ with respect to which we can define the diffeomorphism orbits.

necessarily composite in this context, as the fundamental field is not diffeomorphism invariant and an example could be

$$\mathcal{K} = \frac{\int_{\mathcal{M}} d^4x \sqrt{|g|}\mathcal{R}(x)}{\int_{\mathcal{M}} d^4x \sqrt{|g|}}, \tag{2.32}$$

which corresponds to the integrated, normalized curvature of spacetime. However, this is a non-local operator which cannot be identified with a particle. So as to describe particles and in our case a bound state of gravitational excitations, we need local operators [24]. It follows that we also need to identify a suitable invariant operator that characterizes the geon as a particle.

The (geon) propagator

To study the particle properties of the geon, we need to calculate the correlation functions related to this particle. In particular, the two-point correlation function, generally referred to as the propagator, is the most fundamental quantity associated with a particle. More complicated correlation functions allow to study also the interaction between the geon and other particles of the theory, especially its decay and formation. However, for simplicity, we only study the propagator of the geon in this work, in order to make assertions regarding its mass.

Geons are supposed to be bound states of fluctuations in spacetime, i.e. gravitational waves, thus the quantum mechanical equivalent would be a bound state operator of the field $g_{\mu\nu}$, which describes such fluctuations. The simplest local, composite operator that could represent such a bound state, would be a scalar one. One readily obtained scalar in GR is the Ricci scalar $\mathcal{R}(\mathscr{P})$ [5], with $\mathscr{P} \in \mathcal{M}$. Viewed as a function of events, this is a manifestly diffeomorphism invariant object. Since this is a scalar object, which are generally associated with spin-0 particles, this automatically cannot be the graviton in this theory, which would require a spin-2 structure.[6] Therefore, this is the simplest possible operator of our theory describing a purely gravitational excitation besides the graviton in our theory. It will be this operator that we study as a geon candidate. The propagator of this object would then be provided by the correlation function

$$D(\mathscr{P}, \mathscr{Q}) = \langle \mathcal{R}(\mathscr{P})\mathcal{R}(\mathscr{Q}) \rangle. \tag{2.33}$$

This represents a manifestly diffeomorphism invariant operator and thus the PI can (but does not have to) provide us with a non-vanishing expectation value. However, although the definition of the Ricci scalar and the propagator as functions of events is geometrically well-defined, it is unwieldy. As discussed in Ref. [1], motivated by the limit to flat-space quantum field theory, this event-dependence could be identified with a dependence on the diffeomorphism invariant geodesic distance between the two events $r(\mathscr{P}, \mathscr{Q})$. To be precise, the shortest geodesic distance, which is calculated from

[5] The explicit form of $\mathcal{R}(\mathscr{P})$ expressed in the metric of a given coordinate frame is given in (B.1)

[6] For a discussion on how to construct a suitable graviton, see Ref. [1].

the metric. Since the latter is fluctuating in the quantum theory, we need to use the expectation value

$$r\left(\mathscr{P},\mathscr{Q}\right)=\left\langle \min_{z(t)}\int_{0\to\mathscr{P}}^{1\to\mathscr{Q}}\left(\left(g_{\mu\nu}(t)n^{\mu}(t)n^{\nu}(t)\right)^{1/2}dt\right\rangle, \tag{2.34}$$

where n^μ denote the tangent vectors of the geodesic $z(t)$ connecting $\mathscr{P}$ and $\mathscr{Q}$, which is parametrized by t. Thus the propagator can be written as $D\left(r\left(\mathscr{P},\mathscr{Q}\right)\right)$, which has the advantage that the geodesic distance can be calculated with coordinate methods and still provides a diffeomorphism invariant characterization of the events.[7] Due to the minimization condition we will automatically consider the shortest geodesic distance. While it is formally possible to define the expectation value (2.34), unless some numerical method is chosen, it is non-trivial to obtain an explicit expression for it. One possible way to treat this, is discussed in Section 3.

2.2.2. Gauge/frame-fixing

Now that we have defined the PI and the geon propagator, we need to calculate it and although we assume that the PI is well-defined as it stands, we cannot do any calculations with it. The reason for this is that we are not able to specify a certain coordinate system with which we can express the event-dependence and calculate the action. To introduce a coordinate frame, we can perform the analog of gauge-fixing in non-Abelian gauge theories. For GR the gauge-fixing procedure selects a specific coordinate-frame for every spacetime, which is why it will also be referred to as frame-fixing in the following. This works with the standard Faddeev-Popov procedure, where a functional δ-function is inserted into the PI which selects a certain representative of the $g_{\mu\nu}\in\mathcal{Q}$ for every physical distinct case and thus limits the PI to a specific $\mathcal{Q}'$. This will leave any diffeomorphism invariant expectation value unchanged and only affect diffeomorphism variant quantities. In particular this selects a specific coordinate frame for every contribution to the PI. The selection of the elements in $\mathcal{Q}'$ is implemented through a coordinate/gauge condition $C_\mu\left[g_{\mu\nu}\right]$, such that for every physically distinct configuration only one $g_{\mu\nu}$ in the diffeomorphism orbit is considered, for which $C_\mu\left[g_{\mu\nu}\right]=0$ is fulfilled. We will discuss the nature of this coordinate condition in Section 3 and leave it general for the following discussion. However, $C_\mu\left[g_{\mu\nu}\right]$ carries an index, as we need in total four conditions to fix a coordinate frame for a four dimensional spacetime. In order to apply the Faddeev-Popov (FP) procedure, we need to know the action of an infinitesimal coordinate transformation on our metric field. For the transformation $x'^\mu=x^\mu+\xi_f^\mu(x)$, with $\xi_f^\mu(x)$ infinitesimal we get:

$$\left(J^{-1}\right)_\alpha{}^\beta=\frac{\partial x^\beta}{\partial x'^\alpha}=\delta_\alpha{}^\beta-\partial_\alpha\xi_f^\beta \tag{2.35}$$

$$g^f_{\mu\nu}=g_{\mu\nu}-g_{\alpha\nu}\partial_\mu\xi_f^\alpha-g_{\mu\alpha}\partial_\nu\xi_f^\alpha+O\left(\xi^2\right) \tag{2.36}$$

[7] In general there are also other invariant lengths that could be used instead of the geodesic distance, such as Synge's world function, and in Section 5 we will make use of this, however, they are all uniquely connected. Hence, we are free to choose which one to use.

Now we can introduce the object

$$\Delta^{-1}\left[g_{\mu\nu}\right] = \int \mathcal{D}f \delta\left(C_\mu\left[g^f_{\mu\nu}\right]\right) \tag{2.37}$$

for one physically distinct field configuration $g_{\mu\nu}$, where we integrate over the whole diffeomorphism group with the invariant Haar-measure $\mathcal{D}f$. The inverse of this object is given by $\Delta\left[g_{\mu\nu}\right]$, which is diffeomorphism invariant.

$$1 = \Delta\left[g_{\mu\nu}\right] \int \mathcal{D}f \delta\left(C_\mu\left[g^f_{\mu\nu}\right]\right) \tag{2.38}$$

Inserting this into the PI and invoking the diffeomorphism invariance of the measure and action, this allows for the following rewriting:

$$\mathcal{Z} = \int_{\mathcal{Q}} \mathcal{D}g_{\mu\nu} e^{iS_{EH}[g_{\mu\nu}]} \tag{2.39}$$

$$= \int_{\mathcal{Q}} \mathcal{D}g_{\mu\nu} \Delta\left[g_{\mu\nu}\right] \int \mathcal{D}f \delta\left(C_\mu\left[g^f_{\mu\nu}\right]\right) e^{iS_{EH}[g_{\mu\nu}]} \tag{2.40}$$

$$= \left(\int \mathcal{D}f\right) \int_{\mathcal{Q}} \mathcal{D}g_{\mu\nu} \Delta\left[g_{\mu\nu}\right] \delta\left(C_\mu\left[g_{\mu\nu}\right]\right) e^{iS_{EH}[g_{\mu\nu}]} \tag{2.41}$$

If we calculate any expectation value, the integral over the diffeomorphism group cancels out and we are left with a modified PI, which due to the δ-function only considers the configurations in $\mathcal{Q}'$, which is defined by $C_\mu\left[g_{\mu\nu}\right] = 0$. Thus we have to consider only physically inequivalent contributions in a specific coordinate system.

The $\Delta\left[g_{\mu\nu}\right]$ is called the Faddeev-Popov determinant and it can be shown to have the form

$$\Delta\left[g_{\mu\nu}\right] = \det M_{\mu\nu}(x,y) \tag{2.42}$$

$$M_{\mu\nu}(x,y) = \int d^4z \sqrt{|g|} \frac{\delta C_\mu\left[g_{\mu\nu}\right](x)}{\delta g_{\rho\sigma}(z)} \frac{\delta g^f_{\rho\sigma}(z)}{\delta \xi^\nu(y)} \tag{2.43}$$

$$= \frac{\delta C_\mu\left[g_{\mu\nu}\right](x)}{\delta g_{\rho\sigma}(y)} \left(g_{\nu\sigma}\partial_\rho + g_{\nu\rho}\partial_\sigma\right) \tag{2.44}$$

Using Grassmann-valued scalar fields (ghosts) the FP determinant can be written as

$$\det M_{\mu\nu}(x,y) \approx \int \mathcal{D}\bar{c}\mathcal{D}c \exp\left(-i \int d^4x d^4y \bar{c}^\mu(x) M_{\mu\nu}(x,y) c^\nu(y)\right). \tag{2.45}$$

And the partition sum becomes

$$\mathcal{Z} = \int \mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}g_{\mu\nu} \delta\left(C_\mu\left[g_{\mu\nu}\right]\right) e^{iS_{EH}[g_{\mu\nu}] - iS_{GH}[g_{\mu\nu},\bar{c},c]}, \tag{2.46}$$

where $S_{GH}\left[g_{\mu\nu}, \bar{c}, c\right]$ denotes the ghost action, which is just the additional structure due to (2.45), and we have already omitted the diffeomorphism group integration as it will always cancel in the calculation of correlation functions (due to the normalization).

Gauge-fixing action and general comments

An additional modification can be made to the above expression, by rewriting the occurring δ-function with the identity

$$\delta\left(C_\mu\left[g_{\mu\nu}\right]\right) = \lim_{\zeta\to 0} \exp\left(-i\frac{1}{2\zeta}\int d^4x\sqrt{|g|}g^{\mu\nu}C_\mu\left[g_{\mu\nu}\right]C_\nu\left[g_{\mu\nu}\right]\right) \tag{2.47}$$

$$= \lim_{\zeta\to 0} e^{-iS_{GF}[g_{\mu\nu},\zeta]}. \tag{2.48}$$

Thus we obtain the following expression for the PI:

$$\mathcal{Z} = \lim_{\zeta\to 0}\int \mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}g_{\mu\nu}e^{iS_{EH}[g_{\mu\nu}]-iS_{GH}[g_{\mu\nu},\bar{c},c]-iS_{GF}[g_{\mu\nu},\zeta]} \tag{2.49}$$

In general it can be shown, that it is not necessary to take the limit of $\zeta\to 0$, but any value of ζ can be allowed [25]. However, a non-zero value for ζ will smear out the δ-function and instead of a specific coordinate frame for every physically inequivalent contribution, we get "averaged" coordinate frames. Since there is no obvious way to interpret such an averaged coordinate system, this will not be done here.

We will only be concerned with the tree-level expression for the geon bound state and thus we do not need to consider the ghosts as they would only factor into the loop-corrections. Therefore, we skip the ghost action in the following treatment under the awareness that it has to be included in any calculation going beyond tree-level.

Another important consequence of the FP construction is that the argument presented in Sec. 2.2.1 for the vanishing of diffeomorphism variant expectation values does not hold any longer and we can obtain $\langle g_{\mu\nu}\rangle \neq 0$. This allows us to introduce a coordinate dependent characterization of spacetime via the metric field again. Thus fixing a coordinate system restores the classical picture of having a spacetime to which we can assign a non-zero metric in an arbitrary coordinate frame. However, this coordinate system is still subject to quantum fluctuations, entailing fluctuating distances, which is quite different from what we would expect from the classical notion of coordinate system, and only the expectation value has a definite value. Hence, for actual calculations it will be important to somehow transfer such fluctuating coordinate frames into expressions that we can handle with conventional coordinate methods. This will be discussed in Sec. 3.

Well-definiteness of the path integral

Some aspects of the well-definiteness of the PI have already been addressed in the previous sections, however, at this point it is important to mention an additional subtlety in the PI formulation.

It has to be pointed that the current version of the PI is formulated in an inherently non-perturbative way, as otherwise we could not calculate the bound state operator of the geon, which is automatically a non-pert. object. Thus, while it is generally accepted knowledge that EH gravity as a perturbative quantum theory is not well-defined, this argument does not automatically hold for this discussion. There could be a non-pert.

effect that could remedy the non-renormalizability of the perturbative treatment and make the non-pert. PI well-defined, such as asymptotic safety. Under the assumption that such an effect will kick in eventually to make the theory well-posed, we will continue with the EH gravity formalism for our calculations. Albeit investigations of asymptotic safety indicate that additional terms are required in the EH action, such additional structure will not be considered here, with the awareness that the used action might be incomplete. The possible modifications of the action will not change the general structure of the calculations outlined in the following chapters.

Further problems of the non-perturbative setup are discussed in [1], but these are not relevant for the following treatment of the geon.

3. Coordinate frames and the Fröhlich-Morchio-Strocchi mechanism

In this chapter we turn back to the problem of introducing suitable coordinate frames into the formalism of quantum gravity, as these are essential for evaluating the Ricci scalar which is the central component of the geon propagator. This presentation is closely based on the discussion in Ref. [1]. At this point it becomes important to consider the large-scale behavior of gravity, which is well described by classical GR. We can calculate solutions to the classical equations of motion (2.26), which provides us with a non-vanishing metric field $g^c_{\mu\nu}$ in a certain coordinate frame, depending on the choice of the cosmological constant Λ. Without the consideration of matter the three simplest, cosmologically relevant solutions, which are homogeneous and isotropic in space and time, are Minkowski, de Sitter (dS) and Anti-de Sitter (AdS) space. More complicated solutions could represent the true nature of spacetime, however, current cosmological observations suggest that the macroscopic universe is very close to a flat universe, with measurement uncertainities still permitting a slightly positive curvature [26]. Therefore only the Minkowski and de Sitter spacetimes are considered in this work, as the simplest possible representatives of the classically observed universe. These two spacetimes distinguish themselves through the fact that they are maximally symmetric, which is a property that greatly simplifies calculations and in particular, the expression of the propagator as a function of the geodesic distance only works in a straightforward manner for maximally symmetric spacetimes.

The continued refusal of quantum gravitational effects to be subject to measurement tells us that on currently accessible energy scales the universe is well described by classical quantities. So as to introduce a coordinate system that can be used for claculations, it is thence common to many approaches to quantum gravity and QFT in curved spacetime, to introduce a classical background metric with respect to which the quantum fluctuations of the metric field can be considered. The advantage of this is that the same methods that are used in standard QFT on a fixed spacetime can be adapted [21]. Albeit the independence on the chosen background metric can be established in certain approaches, see Ref. [21], this is still a rather ad-hoc construction. Yet, this construction of splitting off a classical part and using it as a background has certain similarities to what is done in the general gauge-fixing procedure of Brout-Englert-Higgs (BEH) physics. There it is possible to choose certain gauges in which the Higgs field acquires a vacuum expectation value (VEV) which represents a background canvas ontop of which additional quantum fluctuations can act. Also in this case the background or VEV breaks the full gauge invariance of the theory as it is part of a specifically chosen gauge. This raises the question, whether a useful background metric can be introduced analogously to the VEV of the Higgs field, by making a certain gauge/coordinate choice. Consequently the background metric would naturally arise from fixing a certain coordinate frame in the PI. This approach is discussed in Ref. [1] and it is shown that it could possibly serve as a justification not only for describing quantum fluctuations of the metric on a general background, but also for why the flat Minkowski metric can serve as an approximate background for the SM QFT's, which do not account for the quantum effects of gravity. This construction for the case of pure gravity is discussed in the sections below.

The argument goes as follows, since we observe an essentially classical spacetime describing the large-scale behavior of the universe, we can fix the coordinate system in the PI such that the expectation value of the metric field is non-vanishing and in particular such that it corresponds to a metric $g^c_{\mu\nu}$, which can be assigned to the classical spacetime. To be concrete, assume that we can infer the classical structure of the universe via some diffeomorphism invariant observable, such as the averaged classical curvature $\mathcal{K}^c$ of the universe. Then the expectation value of this quantity would have to reproduce the same result, i.e. $\langle\mathcal{K}\rangle = \mathcal{K}^c$.[1] It is then possible to select a coordinate condition $C_\mu\left[g_{\mu\nu}\right]$ such that $C_\mu\left[g^c_{\mu\nu}\right] = 0$, i.e. such that a specific coordinate representation $g^c_{\mu\nu}$ of the classical metric associated with the expectation value $\mathcal{K}^c$ does fulfill the condition. Since the frame-fixed PI now only considers an equivalence class of the diffeomorphism group, it follows that $\langle g_{\mu\nu}\rangle \neq 0$ is possible, and it has to be equal to the representative of the metric associated with the spacetime characterized by $\mathcal{K}^c$, which is in the equivalence class. In short, what this frame-fixing achieves, is that $\langle g_{\mu\nu}\rangle = g^c_{\mu\nu}$. Therefore it is possible to fix the coordinate system in the PI such that we obtain a classical metric as expectation value. The non-zero expectation value of the metric field, which is generated by the frame-fixing procedure, is from now on referred to as VEV in analogy to BEH physics.

We can use this VEV to define a background with respect to which we can then discuss the quantum fluctuations in the frame-fixed setting. Just as in BEH physics, the configurations that contribute to the frame-fixed PI are split into contributions from the expectation value and the quantum fluctuations around it. One possible way to do this, is the linear split

$$g_{\mu\nu} \equiv g^c_{\mu\nu} + \gamma_{\mu\nu}, \tag{3.1}$$

where $\langle g_{\mu\nu}\rangle = g^c_{\mu\nu}$ and $\gamma_{\mu\nu}$ denotes the quantum fluctuations.[2] One problem with this linear split is that quantum fluctuations could alter the signature of the full metric as compared to $g^c_{\mu\nu}$, if the fluctuation tensor becomes large enough.[3] However, it is argued in Ref. [1] that similarly to BEH physics, if $\langle g_{\mu\nu}\rangle$ is fixed to $g^c_{\mu\nu}$ and we use this VEV in the split as above, this results in fluctuations around the VEV that are generally small.[4]

[1] The index c is placed on every object related to the classical spacetime and its position, i.e. super- or subscript does not encode any information.

[2] An important property of the split is that while $g_{\mu\nu}$ and $g^c_{\mu\nu}$ are metrics by construction, the fluctuation tensor does not have to be one.

[3] Since all elements in $\mathcal{Q}$ should have Lorentzian signature this would lead to inconsistencies.

[4] Small in the sense that the components of $\gamma_{\mu\nu}$ become small with respect to $g^c_{\mu\nu}$.

Hence, the quantum fluctuations cannot change the signature and to first order in $\gamma_{\mu\nu}$ the linear and exponential prescriptions become equivalent.

As already mentioned, the split (3.1) allows us to consider the quantum fluctuations on a fixed background provided by the frame-fixed expectation value of $g_{\mu\nu}$, which is what we were looking for all along. With this construction we can now use the coordinate system corresponding to $\langle g_{\mu\nu}\rangle = g^c_{\mu\nu}$ and discuss fluctuations with respect to this. A consequence of this split is that the expectation value of the fluctuations $\gamma_{\mu\nu}$ automatically needs to vanish, i.e.

$$\langle g_{\mu\nu}\rangle = \langle g^c_{\mu\nu} + \gamma_{\mu\nu}\rangle = g^c_{\mu\nu} \Leftrightarrow \langle \gamma_{\mu\nu}\rangle = 0. \tag{3.2}$$

For the frame-fixed PI we can also rewrite the measure and the action based on the split. Using the translational invariance of the PI measure, which we already used for the FP procedure, we can write

$$\mathcal{D}g_{\mu\nu} = \mathcal{D}\left(g^c_{\mu\nu} + \gamma_{\mu\nu}\right) = \mathcal{D}\gamma_{\mu\nu}, \tag{3.3}$$

since $g^c_{\mu\nu}$ only represents a shift of the fluctuation field [27].

This frame-fixed setup in combination with the linear split, does not only allow us to use a classical coordinate system, but it also provides a very powerful tool in the form of the FMS mechanism, which can be used to extract properties of non-perturbative objects such as the geon propagator with perturbative methods. This will be discussed in the following section.

3.1. The FMS mechanism

As soon as we also perform this split in the action and the operators, it is possible to order the contributions to the PI into a power series of $\gamma_{\mu\nu}$, which corresponds to an expansion similar to the one used in BEH physics [14, 1]. However, before we can discuss the details of this expansion, it is necessary to discuss the role of the inverse metric in the considered expressions, which is something that we have neglected so far. Both the action as well as the observable operators transform as singlets under the diffeomorphism group, which means that they must not feature any open indices. In the purely gravitational theory the only object that we can use to contract the open indices of the fundamental metric field, is the inverse metric, thereby the action and every relevant operator of this theory will contain the inverse metric. The difficulty arises due to the PI measures ignorance toward the inverse metric, since it is not independet of the metric and we consider only the latter as the integration variable. It is thus inevitable that the inverse metric is expressed as an explicit function of the metric. The standard definition of the components of the inverse based on a matrix only contains submatrices of $g_{\mu\nu}$ and not the metric explicitely [18]. A much more useful identification of the inverse is possible, if we use the metric split in the frame-fixed treatment. We can then obtain the inverse via the Woodbury matrix identity:

$$g^{-1} = (g^c + \gamma)^{-1} = g_c^{-1} + \sum_{k=1}^{\infty} \left(-g_c^{-1}\gamma\right)^k g_c^{-1}, \tag{3.4}$$

where the matrix/tensor indices have been suppressed for simplicity. The exact derivation of this formula is discussed in App. C.1. Therefore, it is possible to rewrite every expression containing the metric field and its inverse in terms of a generally infinite power series of the frame-fixed metric VEV, its inverse and the fluctuation tensor. For convenience, we can always split-off the lowest order term, that can be expressed purely in g^c and g_c^{-1}.

$$\mathcal{F}\left[g^k\right] = \mathcal{F}\left[g_c^k, \left(g_c^{-1}\right)^k\right] + \sum_{m,n\neq 0, m+n=k} \mathcal{F}\left[g_c^m, \gamma^n, \left(g_c^{-1}\right)^{m+n}\right] \tag{3.5}$$

If the dependence of $\mathcal{F}[g]$ on $g_{\mu\nu}$ is analytic, this expression is a power series [1].

Since after the split the PI measure does only depend on the fluctuation tensor $\gamma_{\mu\nu}$, it is possible to pull out the $g^c_{\mu\nu}$ of the expectation values. This allows us to map the correlation functions of a general diffeomorphism invariant operator to a series of diffeomorphism variant expectation values of the fluctuation tensor $\gamma_{\mu\nu}$, which are non-vanishing due to the frame-fixing procedure, and whose open indices are contracted with the $g^c_{\mu\nu}$ in a proper manner. This duality between diffeomorphism/gauge dependent and invariant correlation functions is termed FMS expansion or FMS mechanism in BEH physics [24, 14, 1] and it shall be referred to as such in this context as well.

Now, the frame-fixing together with the linear split (3.1) allows us to consider the fluctuation tensor-field to be small, thus the FMS expansion (3.5) of any expression $\mathcal{F}[g]$ as well as (3.4), become asymptotic series in $\gamma_{\mu\nu}$ and it is possible to extract information from the lowest order contributions to (3.5) (under the assumption that the higher-order terms will only provide small corrections to the result) [14, 1]. Furthermore, the correlation functions in $\gamma_{\mu\nu}$ can be treated in perturbation theory, as the fluctuation tensor-field is assumed to be small. This allows us to extract general properties of the genuinely non-perturbative bound state operator (2.33) in our theory with perturbative methods and we are finally ready to discuss the geon.

3.1.1. Geon propagator in the FMS expansion

In this section we discuss the lowest order contributions to the geon propagator in the FMS expansion. As was discussed in Sec. 2.2.1 the geon propagator (2.33) could be expressed as a function of the mean geodesic distance (2.34). Since $r(\mathscr{P},\mathscr{Q})$ is an expression of the type $\langle\mathcal{F}\left[g^1\right]\rangle$, we can now also apply the expansion in the quantum fluctuations to the calculation of the mean geodesic distance, which will provide us with the classical geodesic distance $r^c(\mathscr{P},\mathscr{Q})$ w.r.t. the VEV of the metric at lowest order and higher order correction terms in γ, collected in $\rho(\mathscr{P},\mathscr{Q})$ [1].

$$r(\mathscr{P},\mathscr{Q}) \equiv r^c(\mathscr{P},\mathscr{Q}) + \rho(\mathscr{P},\mathscr{Q}) \tag{3.6}$$

$$r^c(\mathscr{P},\mathscr{Q}) = \int_{0\to\mathscr{P}}^{1\to\mathscr{Q}} \left(\left(g^c_{\mu\nu}(t) n^\mu_c(t) n^\nu_c(t)\right)^{1/2} dt \tag{3.7}$$

We will use the FMS expansion of the geon propagator together with the expansion of the mean geodesic distance to calculate the geon propagator to second order in $\gamma_{\mu\nu}$ so as to study its properties. Performing the expansion to second order and using the linearity of the PI, gives the following expression for the propagator, where the superscript of the Ricci scalar denotes the order in $\gamma_{\mu\nu}$ of the respective object:

$$\begin{aligned}\langle\mathcal{R}(\mathscr{P})\mathcal{R}(\mathscr{Q})\rangle = &\langle\mathcal{R}^{(0)}(\mathscr{P})\mathcal{R}^{(0)}(\mathscr{Q})\rangle + \langle\mathcal{R}^{(0)}(\mathscr{P})\mathcal{R}^{(1)}(\mathscr{Q})\rangle + \langle\mathcal{R}^{(1)}(\mathscr{P})\mathcal{R}^{(0)}(\mathscr{Q})\rangle \\ &+ \langle\mathcal{R}^{(1)}(\mathscr{P})\mathcal{R}^{(1)}(\mathscr{Q})\rangle + \langle\mathcal{R}^{(0)}(\mathscr{P})\mathcal{R}^{(2)}(\mathscr{Q})\rangle + \langle\mathcal{R}^{(2)}(\mathscr{P})\mathcal{R}^{(0)}(\mathscr{Q})\rangle + O\left(\gamma^3\right)\end{aligned} \tag{3.8}$$

Before we also consider the expansion of $r(\mathscr{P},\mathscr{Q})$, we notice that for the terms of type $\langle\mathcal{R}^{(0)}(\mathscr{P})\mathcal{R}^{(n)}(\mathscr{Q})\rangle$ the quantum field w.r.t. the frame-fixed measure (3.3) occurs only at one event/position. These do not represent propagators in the frame fixed setting, but rather disconnected n-point correlation functions in $\gamma_{\mu\nu}$ and if we study the connected part of the propagator only, we may neglect these contributions.[5] Thus to second order we only have two relevant terms for the connected geon propagator:

$$\langle\mathcal{R}(\mathscr{P})\mathcal{R}(\mathscr{Q})\rangle_{\text{connected}} = \mathcal{R}^{(0)}(\mathscr{P})\mathcal{R}^{(0)}(\mathscr{Q}) + \langle\mathcal{R}^{(1)}(\mathscr{P})\mathcal{R}^{(1)}(\mathscr{Q})\rangle_{\text{connected}} + O\left(\gamma^3\right) \tag{3.9}$$

In order to use (2.34) as the argument of the expressions above, consistency would require to also expand this to second order, but for the second correlation function in 3.9 we are already at second order so the only consistent argument can be $r^c(\mathscr{P},\mathscr{Q})$. The zeroth order term of (3.9) would in principle need up to second order corrections of $r^c(\mathscr{P},\mathscr{Q})$, but in this work we restrict ourselves to maximally symmetric spacetimes for the VEV, which distinguish themselves through a globally constant curvature $\mathcal{R}^{(0)}(\mathscr{P}) = \mathcal{R}^{(0)}(\mathscr{Q}) = \mathcal{R}^c = \mathcal{K}^c$, consequently the term doesn't depend on the geodesic distance at all. As this term is then completely classical it can be neglected in the connected propagator. This provides us with the final expression for the geon propagator in the FMS expansion to second order:

$$\langle\mathcal{R}(\mathscr{P})\mathcal{R}(\mathscr{Q})\rangle_{\text{connected}} = D^{(2)}\left(r^c(\mathscr{P},\mathscr{Q})\right) + O\left(\gamma^3\right), \text{ with} \tag{3.10}$$

$$\langle\mathcal{R}^{(1)}(\mathscr{P})\mathcal{R}^{(1)}(\mathscr{Q})\rangle_{\text{connected}} = D^{(2)}\left(r^c(\mathscr{P},\mathscr{Q})\right) + O\left(\gamma^3\right) \tag{3.11}$$

Thus to the considered order the only non-trivial object that has to be calculated with the frame-fixed PI is the propagator $D^{(2)}\left(r^c(\mathscr{P},\mathscr{Q})\right)$. This is still a complicated expression in the $\gamma_{\mu\nu}$, but due to the FMS expansion it only contains second order terms in the quantum field and it is therefore possible to reduce it to derivatives of the propagator $\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle$, as will be shown in Sec. 3.2.3. As a sideremark, $\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle$ is generally denoted as the graviton propagator. Although the correlation function is not diffeomorphism invariant and only non-vanishing in the frame-fixed setup, we may use this unphysical correlation function as an auxiliary quantity, since the FMS expansion allows us to relate it to the manifestly diffeomorphism invariant geon propagator.

3.2. Application of the FMS mechanism

This section will be concerned with the explicit construction of a frame-fixing condition suitable for the FMS approach as well as the derivation of the Einstein-Hilbert

[5] $\langle\mathcal{R}^{(0)}(\mathscr{P})\mathcal{R}^{(1)}(\mathscr{Q})\rangle = \mathcal{R}^{(0)}(\mathscr{P})\langle\mathcal{R}^{(1)}(\mathscr{Q})\rangle$ also vanishes trivially due to the chosen frame fixing (3.2).

action in the frame-fixed setup, which will then be used to calculate the graviton propagator $\langle \gamma_{\mu\nu}\gamma_{\rho'\sigma'} \rangle$ as an auxiliary object. The actual calculation of $\langle \gamma_{\mu\nu}\gamma_{\rho'\sigma'} \rangle$ and $D^{(2)}\left(r^c\left(\mathscr{P}, \mathscr{Q}\right)\right)$ will then be discussed for Minkowski and de Sitter space in Sec. 4 and 5 respectively.

3.2.1. Discussion of useful coordinate condition

As discussed in the beginning of Sec. 3, we are looking for a frame-fixing condition $C_\mu\left[g_{\mu\nu}\right]$ such that $C_\mu\left[g^c_{\mu\nu}\right] = 0$ is fulfilled. In particular, it would be an advantage, if a condition could be identified, which is compatible with one of the standard coordinate representations of Minkowski and dS space, since these allow for a diagonal $g^c_{\mu\nu}$, which reduces the complexity. On the other hand selecting a condition that is not compatible with the standard coordinate systems, would mean that a coordinate transformation would be required to obtain a representation of $g^c_{\mu\nu}$ in compliance with $C_\mu\left[g^c_{\mu\nu}\right] = 0$, which will generally lead to complicated coordinate functions and metrics.

The harmonic coordinate condition (often referred to as de Donder condition), which is the standard condition used in many treatments of the Einstein equations [15], turns out to be not suitable for this approach, however, it forms the basis for a more general class of conditions that will be considered.[6] For other gauges we will see later on, we get systems of coupled PDE's for whom such a statement is inherently no-trivial and in general no existence proof can be provided.). The condition originates from the requirement that the coordinate functions x^μ are harmonic, i.e. satisfy $\Box x^\mu = \nabla_\alpha \nabla^\alpha x^\mu = 0$ [7]. This can be rexpressed in a condition for the metric [15], which is what we are looking for:

$$g^{\mu\nu}\Gamma^\rho{}_{\mu\nu} = -g^{\mu\alpha}g^{\rho\beta}\partial_\mu g_{\alpha\beta} + \frac{1}{2}g^{\rho\mu}g^{\alpha\beta}\partial_\mu g_{\alpha\beta} = 0 \tag{3.12}$$

While this condition is trivially fulfilled for the cartesian Minkowski frame, no suitable standard coordinate system could be identified for the de Sitter space, which is why it is not employed here.[8] Based on the de Donder condition a more general two-parameter family of coordinate conditions can be formulated:

$$C^\rho\left[g_{\mu\nu}, \kappa, \xi\right] = \kappa g^{\mu\nu}\partial^\rho g_{\mu\nu} - \xi g^{\rho\alpha}g^{\mu\beta}\partial_\mu g_{\alpha\beta} = 0 \tag{3.13}$$

Upon insertion of the coordinate representations of $g^c_{\mu\nu}$, only the reduced family $C^\rho\left[g^c_{\mu\nu}, \kappa = 0, \xi\right]$ possesses compatible $g^c_{\mu\nu}$ for both the Minkowski and dS case. For simplicity ξ is chosen to be 1, which provides us with the following four constrains for

[6] This coordinate condition is usually the preferred choice as the PDE's for the respective coord. transformation can be shown to have a solution that is unique (this doesn't mean that the gauge condition actually strictly singles out one equivalence class $\mathcal{Q}'$, but rather that for each gauge copy of an element in $\mathcal{Q}'$ there exists a unique transformation to a harmonic coordinate frame.

[7] The x^μ in this case really just denotes the 4 coordinate functions and does not represent a vector, thus this are really four scalar equations instead of one tensor equation and $\Box$ is the standard covariant d'Alembertian operator **for scalars**.

[8] The coordinate systems that were considered in this discusses are 2.26.1-2.26.6 in Ref. [28] in the de Sitter case and the standard cartesian and spherical coordinate systems for Minkowski space.

the metric:

$$C^\rho \left[g_{\mu\nu}, \kappa = 0, \xi = 1\right] = \partial_\alpha g^{\alpha\rho} = -g^{\rho\alpha} g^{\mu\beta} \partial_\mu g_{\alpha\beta} = 0 \tag{3.14}$$

$$H_\alpha \left[g_{\mu\nu}\right] = g^{\mu\beta} \partial_\mu g_{\alpha\beta} = \partial^\beta g_{\alpha\beta} = 0 \tag{3.15}$$

From (3.14) to (3.15) the open index on the inverse metric was contracted with a metric to simplify the expression. The following two coordinate systems for Minkowski and de Sitter space, fulfill the condition $H_\alpha \left[g^c_{\mu\nu}\right] = 0$, where the line element is chosen as representation of the metric in the coordinate system, as it is defined through $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$:

- Minkowski, in cartesian coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{3.16}$$

- de Sitter, in cartesian Robertson-Walker coordinates

$$ds^2 = -dt^2 + e^{2Ht} \left[dx^2 + dy^2 + dz^2\right], \tag{3.17}$$

  with $H = \sqrt{\frac{\Lambda}{3}}$.

In addition, one could also exchange the spatial part of the line element with the line element of spherical coordinates in both cases, and it would still be compatible with $H_\alpha \left[g^c_{\mu\nu}\right] = 0$. The condition (3.14) was first considered by J.H. Haywood in Ref. [29] in a classical linearized gravity setting and it will from now on be referred to as **Haywood condition/gauge**. Furthermore, it has also been indirectly considered in quantum gravity calculations in the context of more general coordinate condition classes, along the lines of (3.13), see Ref. [30, 31, 32].

From now on, whenever we need to invoke a coordinatization of Minkowski or de Sitter space, the coordinate frames corresponding to (3.16) and (3.17) will be used.

3.2.2. Gauge/Frame Identities

Now we can also study the constraints which the coordinate/gauge condition (3.15) puts on the fluctuation tensor field $\gamma_{\mu\nu}$. It needs to be stressed that the split (3.1) is only made after the frame-fixing procedure has been performed, but since the remaining contributions to the PI all fulfill $H_\mu \left[g_{\mu\nu}\right] = 0$ this indirectly also puts constraints on the $\gamma_{\mu\nu}$ which are summed over in the frame-fixed PI, after the split has been introduced. These constraints will be referred to as gauge/frame identities as these will only be true for fields in the frame-fixed setup.

When we talk about the split in the coordinate setup, we use a little hat above the quantities, which occur in the split of the inverse metric, before the Woodbury matrix identity is employed, see App. C.1:

$$g_{\mu\nu} = g^c_{\mu\nu} + \gamma_{\mu\nu} \tag{3.18}$$

$$g^{\mu\nu} = \hat{g}^{\mu\nu}_c + \hat{\gamma}^{\mu\nu} \tag{3.19}$$

The reason for this is to emphasize that the quantities that occur in the inverse split are not obtained by shifting the covariant indices of the tensors in the metric split to contravariant indices with the full metric. Rather what happens, is that we choose $\hat{g}_c^{\mu\nu}$ such that it represents the inverse of $g^c_{\mu\nu}$ and $\hat{\gamma}^{\mu\nu}$ is chosen such that the split is consistent with identity $g^{\mu\nu}g_{\nu\alpha} = \delta^\mu{}_\alpha$. In fact we can only get an explicit connection of $\hat{\gamma}^{\mu\nu}$ with $\gamma_{\mu\nu}$ via the infinite series from the Woodbury identity as discussed above.

Based on this we can first write the Haywood condition (3.15) in the split with $\hat{\gamma}^{\mu\nu}$ occurring explicitly[9], which is then an exact rewriting and only afterwards we shall employ (C.4) to get a condition that is approximately valid to first order purely in $\gamma_{\mu\nu}$:

$$H_\alpha\left[g_{\mu\nu}\right] = g^{\mu\nu}\partial_\mu g_{\nu\alpha} = \left(\hat{g}_c^{\mu\nu} + \hat{\gamma}^{\mu\nu}\right)\partial_\mu\left(g^c_{\nu\alpha} + \gamma_{\nu\alpha}\right) \tag{3.20}$$

$$= \hat{g}_c^{\mu\nu}\partial_\mu g^c_{\nu\alpha} + g^{\mu\nu}\partial_\mu\gamma_{\nu\alpha} + \hat{\gamma}^{\mu\nu}\partial_\mu g^c_{\nu\alpha} \tag{3.21}$$

$$g^{\mu\nu}\partial_\mu\gamma_{\nu\alpha} = H_\alpha\left[g_{\mu\nu}\right] - H_\alpha\left[g^c_{\mu\nu}\right] - \hat{\gamma}^{\mu\nu}\partial_\mu g^c_{\nu\alpha} \tag{3.22}$$

Now for the frame-fixed PI $H_\alpha\left[g_{\mu\nu}\right]$ and $H_\alpha\left[g^c_{\mu\nu}\right]$ will vanish by construction and we still use the full metric $g_{\mu\nu}$ to shift indices. Therefore we get the following identity:

$$\partial^\mu\gamma_{\mu\nu} = -\hat{\gamma}^{\mu\nu}\partial_\mu g^c_{\nu\alpha} \tag{3.23}$$

If we now expand the $\hat{\gamma}^{\mu\nu}$ and only consider contributions up to first order, we obtain the simplified relation:

$$\hat{g}_c^{\mu\nu}\partial_\mu\gamma_{\nu\alpha} = \left(\hat{g}_c^{\mu\rho}\hat{g}_c^{\nu\sigma}\partial_\mu g^c_{\nu\alpha}\right)\gamma_{\rho\sigma} + O\left(\gamma^2\right) \tag{3.24}$$

This allows us to identify the non-covariant divergences of $\gamma_{\mu\nu}$ with $\gamma_{\mu\nu}$ itself to first order. The relation (3.24) can be used to derive another identity that is actually valid to second order in $\gamma_{\mu\nu}$ and which will become important when we discuss the FMS expansion of the Einstein-Hilbert action and the frame-fixing action to second order.

$$g^{\rho\sigma}\left(\partial^\mu\gamma_{\mu\rho}\right)\left(\partial^\nu\gamma_{\nu\sigma}\right) = \left(\partial_\mu g^c_{\alpha\rho}\right)\left(\partial_\nu g^c_{\beta\sigma}\right)\left(\hat{g}_c^{\rho\sigma} + \hat{\gamma}^{\rho\sigma}\right)\hat{\gamma}^{\mu\alpha}\hat{\gamma}^{\nu\beta} \tag{3.25}$$

And therefore, if we expand $\hat{\gamma}^{\mu\nu}$, we obtain a condition that is valid to second order in $\gamma_{\mu\nu}$:

$$\hat{g}_c^{\rho\sigma}\left(\hat{g}_c^{\alpha\mu}\partial_\alpha\gamma_{\mu\rho}\right)\left(\hat{g}_c^{\beta\nu}\partial_\beta\gamma_{\nu\sigma}\right) = \left(\left(\partial_\lambda g^c_{\epsilon\rho}\right)\left(\partial_\tau g^c_{\kappa\sigma}\right)\hat{g}_c^{\rho\sigma}\hat{g}_c^{\lambda\mu}\hat{g}_c^{\epsilon\nu}\hat{g}_c^{\tau\alpha}\hat{g}_c^{\kappa\beta}\right)\gamma_{\mu\nu}\gamma_{\alpha\beta} + O\left(\gamma^3\right) \tag{3.26}$$

We will use this result to simplify the second order part of the Einstein-Hilbert action in the FMS expansion.

[9]It is important to not overlook that the inverse metric $g^{\mu\nu}$ implicitly acts on the partial derivative and lifts the index. This is why also $\hat{\gamma}^{\mu\nu}$ will occur in the expression.

3.2.3. FMS expansion of the Einstein-Hilbert action and Green's function equation for the graviton propagators

In this section we treat the FMS expansion of the Einstein-Hilbert action to second order, which will provide us with the means to calculate the tree-level propagator for the graviton, i.e. $\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle_{tl}$. Furthermore, we will get an explicit expression for the non-vanishing second order part of the geon propagator in the FMS expansion, which will show how the geon propagator can be decomposed with the $\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle$. Since both the Einstein-Hilbert action and the geon bound-state operator contain the curvature scalar, with the latter actually being the Ricci scalar, we will discuss the FMS expansion of this object first. Many expressions that occur in this context are formally similar to what one obtains in the background-field approach to quantum gravity and we can therefore compare our results for the FMS expansion to these as well as employ similar techniques. The derivation presented below is therefore adapted from the calculations in Ref. [20], which are done in Euclidean signature.

Ricci tensor and scalar in the FMS expansion

This section is based on [20]. In principle we could of course just take the explicit form of the Ricci scalar in terms of the metric, see (B.1), perform the metric split and then order everything in powers of $\gamma_{\mu\nu}$. This approach was indeed first pursued and the results are stated in App. B, with the final result being (B.10) and (B.11) for the first and second order contribution. However, these expressions can actually be written in a more compact form, by collecting the terms, where only derivatives of the VEV occur, in "classical" Christoffel symbols (3.27) and defining the "classical" covariant derivative (3.28) based on these:

$$\Gamma^{\alpha}_{c\,\mu\nu} = \frac{1}{2}\hat{g}_c^{\alpha\beta}\left(\partial_\mu g^c_{\beta\nu} + \partial_\nu g^c_{\mu\beta} - \partial_\beta g^c_{\mu\nu}\right) \tag{3.27}$$

$$\nabla^c_\mu v^\rho = \partial_\mu v^\rho + \Gamma^\rho_{c\mu\nu} v^\nu \tag{3.28}$$

$$\nabla^c_\rho g^c_{\mu\nu} = 0 \tag{3.29}$$

Furthermore, using the identity (3.30) we can map specific combinations of partial derivatives of the γ-field to covariant derivatives of the γ-field. This combination of partial derivatives is the one occurring in the full Christoffel symbols of $g_{\mu\nu}$, after the split has been performed. Since the Ricci scalar is the trace part of the Ricci tensor, which is expressed in terms of Christoffel symbols (2.19), these are just the combinations that will occur in the end. Based on (3.30) we define another auxiliary object (3.31), which corresponds to something similar to a "Christoffel" symbol for the $\gamma_{\mu\nu}$, with the partial derivatives replaced by (3.28):

$$\partial_\mu\gamma_{\tau\nu} + \partial_\nu\gamma_{\mu\tau} - \partial_\tau\gamma_{\mu\nu} = \nabla^c_\mu\gamma_{\tau\nu} + \nabla^c_\nu\gamma_{\mu\tau} - \nabla^c_\tau\gamma_{\mu\nu} + 2\Gamma^\rho_{c\mu\nu}\gamma_{\rho\tau} \tag{3.30}$$

$$\Theta^\alpha{}_{\mu\nu} = \frac{1}{2}\hat{g}_c^{\alpha\tau}\left(\nabla^c_\mu\gamma_{\tau\nu} + \nabla^c_\nu\gamma_{\mu\tau} - \nabla^c_\tau\gamma_{\mu\nu}\right) = \Theta^\alpha{}_{\nu\mu} \tag{3.31}$$

For the FMS expansion we consider the Ricci tensor at first and afterwards we take the trace to obtain the curvature scalar.

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\mu \Gamma^\alpha{}_{\alpha\nu} + \Gamma^\beta{}_{\mu\nu}\Gamma^\alpha{}_{\beta\alpha} - \Gamma^\beta{}_{\alpha\nu}\Gamma^\alpha{}_{\beta\mu} \tag{3.32}$$
$$= R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + O\left(\gamma^3\right) \tag{3.33}$$

After the split the Christoffel symbols can be rearranged decomposed into the contributions at every order, using (3.28), (3.27), (3.30) and (3.31):

$$\Gamma^\alpha{}_{\mu\nu} = \Gamma^\alpha_{c\,\mu\nu} + \Theta^\alpha{}_{\mu\nu} - \hat{g}_c^{\alpha\rho}\gamma_{\rho\sigma}\Theta^\sigma{}_{\mu\nu} + O\left(\gamma^3\right) \tag{3.34}$$

This decomposition can then be inserted into the definition of the Ricci tensor and we immediately get the contribution to each order, with the zeroth order part trivially reducing to the classical Ricci tensor:

$$R^{(0)}_{\mu\nu} = R^c_{\mu\nu} \tag{3.35}$$
$$R^{(1)}_{\mu\nu} = \nabla^c_\alpha \Theta^\alpha_{\mu\nu} - \nabla^c_\mu \Theta^\alpha{}_{\alpha\nu} \tag{3.36}$$
$$R^{(2)}_{\mu\nu} = -\nabla^c_\alpha \left(\hat{g}_c^{\alpha\tau}\gamma_{\tau\rho}\Theta^\rho{}_{\mu\nu}\right) + \nabla^c_\mu \left(\hat{g}_c^{\alpha\tau}\gamma_{\tau\rho}\Theta^\rho{}_{\alpha\nu}\right) + \Theta^\beta{}_{\mu\nu}\Theta^\alpha{}_{\alpha\beta} - \Theta^\beta{}_{\alpha\nu}\Theta^\alpha{}_{\beta\mu} \tag{3.37}$$

With these results we can compute the trace of the Ricci tensor, which the Ricci scalar:

$$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} = \left(\hat{g}_c^{\mu\nu} + \hat{\gamma}^{\mu\nu}\right)R_{\mu\nu} \tag{3.38}$$
$$= \left(\hat{g}_c^{\mu\nu} - \hat{g}_c^{\mu\alpha}\hat{g}_c^{\nu\beta}\gamma_{\alpha\beta} + \hat{g}_c^{\mu\alpha}\hat{g}_c^{\beta\rho}\hat{g}_c^{\nu\sigma}\gamma_{\alpha\beta}\gamma_{\rho\sigma} + O\left(\gamma^3\right)\right)R_{\mu\nu}$$
$$= \mathcal{R}^c + \mathcal{R}^{(1)} + \mathcal{R}^{(2)} + O\left(\gamma^3\right) \tag{3.39}$$
$$\mathcal{R}^c = \hat{g}_c^{\mu\nu}R^c_{\mu\nu} \tag{3.40}$$
$$\mathcal{R}^{(1)} = \hat{g}_c^{\mu\nu}R^{(1)}_{\mu\nu} - \hat{g}_c^{\mu\alpha}\hat{g}_c^{\nu\beta}\gamma_{\alpha\beta}R^c_{\mu\nu} \tag{3.41}$$
$$\mathcal{R}^{(2)} = \hat{g}_c^{\mu\nu}R^{(2)}_{\mu\nu} - \hat{g}_c^{\mu\alpha}\hat{g}_c^{\nu\beta}\gamma_{\alpha\beta}R^{(1)}_{\mu\nu} + \hat{g}_c^{\mu\alpha}\hat{g}_c^{\beta\rho}\hat{g}_c^{\nu\sigma}\gamma_{\alpha\beta}\gamma_{\rho\sigma}R^c_{\mu\nu} \tag{3.42}$$

The explicit expressions for the terms $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ are the subject of the following discussion. The terms that occur in the expressions derived above are such that every contravariant index occurs on the inverse of the VEV, i.e. $\hat{g}_c^{\mu\nu}$, and therefore it is expedient to introduce a new convention. Up to now the full metric was defined as the index shifter and this is still the case, i.e. only the full metric can truly map a covariant into a contravariant expression in our spacetime, however, after the split the full metric doesn't occur anymore and as a convention it is useful to associate every contravariant index as with an implicit contraction of a covariant object with the frame-fixed VEV, that is $t^\mu = \hat{g}_c^{\mu\nu}t_\nu$. This allows us to write the above expressions in a much more compact form, albeit much care is required when translating between objects that are expressed with the convention that $g^c_{\mu\nu}$ is the "index shifter" and expressions that contain $g_{\mu\nu}$ as the true index shifter. From now on we shall follow the new convention for the rest of this work and whenever we revert to the original definition it will be explicitly pointed out.

Through inserting (3.31) for $\Theta^{\alpha}{}_{\mu\nu}$ in the expanded Ricci tensor, i.e. in (3.36) and (3.37), one obtains the contributions to the FMS expansion of the Ricci scalar purely in terms of $\gamma_{\mu\nu}$, with occurrences of $\hat{g}_c^{\mu\nu}$ implicitly taken into account through the index shifting convention described above:

$$
\begin{aligned}
\mathcal{R}^{(1)} &= \left(\nabla_c^{\mu}\nabla_c^{\nu} - \hat{g}_c^{\mu\nu}\Box_c - R_c^{\mu\nu}\right)\gamma_{\mu\nu} \text{ , with } \Box_c = \hat{g}_c^{\rho\sigma}\nabla_\rho^c\nabla_\sigma^c \qquad (3.43)\\
\mathcal{R}^{(2)} &= \gamma_{\alpha\beta}\nabla_c^{\alpha}\nabla_c^{\beta}\gamma^{\rho}{}_{\rho} + \gamma_{\alpha\beta}\Box_c\gamma^{\alpha\beta} + \nabla_c^{\alpha}\gamma_{\alpha\rho}\nabla_c^{\rho}\gamma^{\mu}{}_{\mu} + \frac{3}{4}\nabla_c^{\alpha}\gamma_{\rho}{}^{\mu}\nabla_\alpha^c\gamma^{\rho}{}_{\mu}\\
&\quad - \frac{1}{4}\nabla_c^{\rho}\gamma^{\mu}{}_{\mu}\nabla_\rho^c\gamma^{\alpha}{}_{\alpha} - \frac{1}{2}\nabla_c^{\alpha}\gamma_{\rho}{}^{\mu}\nabla_c^{\rho}\gamma_{\alpha\mu} - \nabla_c^{\alpha}\gamma_{\alpha\rho}\nabla_c^{\mu}\gamma^{\rho}{}_{\mu}\\
&\quad - 2\gamma_{\alpha\beta}\nabla_c^{\alpha}\nabla_c^{\rho}\gamma^{\beta}{}_{\rho} + \gamma_{\alpha\beta}\hat{g}_c^{\alpha\lambda}R^c_{\rho\lambda\epsilon}{}^{\beta}\gamma^{\epsilon\rho} \qquad (3.44)
\end{aligned}
$$

The Riemann tensor $R^c_{\rho\lambda\epsilon}{}^{\beta}$ that is occurring in the explicit version of $\mathcal{R}^{(2)}$ is defined via the usual relation (2.17), for the "classical" covariant derivative (3.28) and it encodes the curvature of the manifold related to the VEV in the usual way.[10]

Geon propagator - explicit version

With the results above we can now write down the explicit version of the geon propagator in the FMS expansion to second order, in particular, what we need to write down, is $D^{(2)}\left(r^c\left(\mathscr{P},\mathscr{Q}\right)\right)$, see (3.11). With the explicit result for $\mathcal{R}^{(1)}$ in (3.43), and the convention that objects that are evaluated at point $\mathscr{Q}$ are denoted with primed indices (or merely a prime in the case of scalars), as compared to objects w.r.t. point $\mathscr{P}$, we

[10] The term containing the Riemann tensor is part of the second order contribution since two of the covariant derivatives have been exchanged and it could be removed again, by reverting this, however, it will be useful to keep it this way.

obtain:

$$D^{(2)}\left(r^{c}\left(\mathscr{P},\mathscr{Q}\right)\right)=\langle\left(\nabla_{c}^{\mu}\nabla_{c}^{\nu}-\hat{g}_{c}^{\mu\nu}\Box_{c}-R_{c}^{\mu\nu}\right)\gamma_{\mu\nu}\left(\nabla_{c}^{\rho'}\nabla_{c}^{\sigma'}-\hat{g}_{c}^{\rho'\sigma'}\Box_{c}'-R_{c}^{\rho'\sigma'}\right)\gamma_{\rho'\sigma'}\rangle \tag{3.45}$$

$$=\left(\nabla_{c}^{\mu}\nabla_{c}^{\nu}-\hat{g}_{c}^{\mu\nu}\Box_{c}-R_{c}^{\mu\nu}\right)\left(\nabla_{c}^{\rho'}\nabla_{c}^{\sigma'}-\hat{g}_{c}^{\rho'\sigma'}\Box_{c}'-R_{c}^{\rho'\sigma'}\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle \tag{3.46}$$

$$\begin{aligned}
&=\nabla_{c}^{\mu}\nabla_{c}^{\nu}\nabla_{c}^{\rho'}\nabla_{c}^{\sigma'}\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad-\left(\nabla_{c}^{\mu}\nabla_{c}^{\nu}\hat{g}_{c}^{\rho'\sigma'}\Box_{c}'+\hat{g}_{c}^{\mu\nu}\Box_{c}\nabla_{c}^{\rho'}\nabla_{c}^{\sigma'}\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad-\left(\nabla_{c}^{\mu}\nabla_{c}^{\nu}R_{c}^{\rho'\sigma'}+R_{c}^{\mu\nu}\nabla_{c}^{\rho'}\nabla_{c}^{\sigma'}\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad+\hat{g}_{c}^{\mu\nu}\Box_{c}\hat{g}_{c}^{\rho'\sigma'}\Box_{c}'\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad+\left(\hat{g}_{c}^{\mu\nu}\Box_{c}R_{c}^{\rho'\sigma'}+R_{c}^{\mu\nu}\hat{g}_{c}^{\rho'\sigma'}\Box_{c}'\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad+R_{c}^{\mu\nu}R_{c}^{\rho'\sigma'}\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle
\end{aligned} \tag{3.47}$$

$$\begin{aligned}
&=\nabla_{c}^{\mu}\nabla_{c}^{\nu}\nabla_{c}^{\rho'}\nabla_{c}^{\sigma'}\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad-\left(\nabla_{c}^{\mu}\nabla_{c}^{\nu}\hat{g}_{c}^{\rho'\sigma'}\Box_{c}'+\hat{g}_{c}^{\mu\nu}\Box_{c}\nabla_{c}^{\rho'}\nabla_{c}^{\sigma'}\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad-\Lambda\left(\nabla_{c}^{\mu}\nabla_{c}^{\nu}\hat{g}_{c}^{\rho'\sigma'}+\hat{g}_{c}^{\mu\nu}\nabla_{c}^{\rho'}\nabla_{c}^{\sigma'}\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad+\hat{g}_{c}^{\mu\nu}\Box_{c}\hat{g}_{c}^{\rho'\sigma'}\Box_{c}'\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad+\Lambda\left(\hat{g}_{c}^{\mu\nu}\Box_{c}\hat{g}_{c}^{\rho'\sigma'}+\hat{g}_{c}^{\mu\nu}\hat{g}_{c}^{\rho'\sigma'}\Box_{c}'\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle\\
&\quad+\Lambda^{2}\hat{g}_{c}^{\mu\nu}\hat{g}_{c}^{\rho'\sigma'}\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle
\end{aligned} \tag{3.48}$$

where for (3.46) we used that the derivative and the frame-fixed PI in $\gamma_{\mu\nu}$ (3.3) commute [25]. To obtain (3.48) from the general expression (3.47) we used the fact that we restrict our calculations to maximally symmetric spacetimes as the VEV, therefore we can employ relations (A.5) and (A.6) to replace $R^{c}_{\mu\nu}$ by $g^{c}_{\mu\nu}\Lambda$.

As we can see, the second order contribution to the geon propagator in the FMS expansion can be decomposed into nine terms containing an operator acting on the correlation function $\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle$. This makes it clear why the correlation function $\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle$ constitutes the central object for the investigation of the geon propagator with the FMS mechanism. Another point that is quite evident now, is that the correlation function $\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle$ represents a bitensorial object, i.e. a tensor(-field) which is defined w.r.t. to two events in spacetime, see App. D.1. This is merely the generalization of biscalars (geodesic distance) and bivectors (e.g. the photon propagator in QED [33]) which do already occur in standard QFT and their definition and manipulation is discussed e.g. in Ref. [34].

Einstein-Hilbert action in FMS expansion

In order to compute the unphysical graviton propagator, occuring in (3.48) we need the FMS expansion of the Einstein-Hilbert action. We alredy worked out the expansion of the Ricci scalar above, however, so as to write down the Einstein-Hilbert action to the second order in the FMS expansion, one important component is still missing, namely the expansion of the square root of the metric determinant $\sqrt{|g|}$ which occurs in the volume element of the spacetime integral in the action (2.24). If $|g^c|$ denotes the absolute value of the determinant of the VEV, then the expansion of $\sqrt{|g|}$ in terms of $\gamma_{\mu\nu}$ can be written as follows [20]:

$$\begin{aligned}\sqrt{|g|} &= \sqrt{|g^c|}\Bigg(1 + \frac{1}{2}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu} - \frac{1}{4}\hat{g}_c^{\mu\nu}\gamma_{\nu\rho}\hat{g}_c^{\rho\sigma}\gamma_{\sigma\mu} \\ &\qquad + \frac{1}{8}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\hat{g}_c^{\rho\sigma}\gamma_{\rho\sigma} + O\left(\gamma^3\right)\Bigg) &(3.49)\\ &= d^{(0)}\left(1 + d^{(1)} + d^{(2)} + O\left(\gamma^3\right)\right) &(3.50)\end{aligned}$$

Therefore, the full integrand of the EH action to second order in $\gamma_{\mu\nu}$ reads:

$$\begin{aligned}\sqrt{|g|}\,(\mathcal{R} - 2\Lambda) &= d^{(0)}\left(1 + d^{(1)} + d^{(2)} + O\left(\gamma^3\right)\right)\left(\mathcal{R}^c + \mathcal{R}^{(1)} + \mathcal{R}^{(2)} - 2\Lambda + O\left(\gamma^3\right)\right) &(3.51)\\ &= \sqrt{|g^c|}\Bigg((\mathcal{R}^c - 2\Lambda) + \left[\frac{1}{2}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\left(\mathcal{R}^c - 2\Lambda\right) + \mathcal{R}^{(1)}\right] \\ &\quad + \left[\mathcal{R}^{(2)} + \frac{1}{2}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\mathcal{R}^{(1)} + \left(-\frac{1}{4}\hat{g}_c^{\mu\nu}\gamma_{\nu\rho}\hat{g}_c^{\rho\sigma}\gamma_{\sigma\mu} + \frac{1}{8}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\hat{g}_c^{\rho\sigma}\gamma_{\rho\sigma}\right)(\mathcal{R}^c - 2\Lambda)\right] \\ &\quad + O\left(\gamma^3\right)\Bigg) &(3.52)\end{aligned}$$

Another trick can be employed to further simplify the integrand, namely the covariant Gauss-Stokes theorem [18], which is discussed in further detail in App. C.2. Under the assumption that $\gamma_{\mu\nu}$ vanishes asymptotically, we can neglect covariant divergence terms under the integral sign. This is not a general property of the $\gamma_{\mu\nu}$ as we only require that $|\gamma_{\mu\nu}|/\min_{\rho,\sigma}|\langle g_{\rho\sigma}\rangle| \ll 1\ \forall\mu,\nu$ and can therefore only be approximately true. In this approximation already the expressions for the Ricci tensor simplify significantly:

$$\begin{aligned}R^{(1)}_{\mu\nu} &= \nabla^c_\alpha\Theta^\alpha{}_{\mu\nu} - \nabla^c_\mu\Theta^\alpha{}_{\alpha\nu} \ \simeq -\nabla^c_\mu\Theta^\alpha{}_{\alpha\nu} &(3.53)\\ R^{(2)}_{\mu\nu} &= -\nabla^c_\alpha\left(\hat{g}_c^{\alpha\tau}\gamma_{\tau\rho}\Theta^\rho{}_{\mu\nu}\right) + \nabla^c_\mu\left(\hat{g}_c^{\alpha\tau}\gamma_{\tau\rho}\Theta^\rho{}_{\alpha\nu}\right) + \Theta^\beta{}_{\mu\nu}\Theta^\alpha{}_{\alpha\beta} - \Theta^\beta{}_{\alpha\nu}\Theta^\alpha{}_{\beta\mu} \\ &\simeq \nabla^c_\mu\left(\hat{g}_c^{\alpha\tau}\gamma_{\tau\rho}\Theta^\rho{}_{\alpha\nu}\right) + \Theta^\beta{}_{\mu\nu}\Theta^\alpha{}_{\alpha\beta} - \Theta^\beta{}_{\alpha\nu}\Theta^\alpha{}_{\beta\mu} &(3.54)\end{aligned}$$

However, one needs to be aware of these simplified versions of the contributions to the Ricci tensor only being applicable, if $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ occur under the integral sign and are not contracted with an additional factor of $\gamma_{\mu\nu}$. In the case of the latter, one would not obtain a total derivative and the full expressions (3.36) and (3.37) would have to be taken into account. This is the case for the second contribution to $\mathcal{R}^{(2)}$ for example, where we still have to use (3.36) instead of (3.53). When (3.53) and (3.54) occur contracted with $\hat{g}_c^{\mu\nu}$ in the terms in the Ricci scalar, the remaining terms containing ∇^c explicitly also reduce to covariant divergences and therefore vanish, which results in the following contributions to the Ricci scalar:

$$
\begin{aligned}
\mathcal{R}^{(1)} &\simeq -\hat{g}_c^{\mu\alpha}\hat{g}_c^{\nu\beta}\gamma_{\alpha\beta}R^c_{\mu\nu} = -\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\Lambda && (3.55)\\
\mathcal{R}^{(2)} &\simeq \frac{1}{4}\left(-\gamma_\rho{}^\mu\Box_c\gamma^\rho{}_\mu + \gamma^\mu{}_\mu\Box_c\gamma^\alpha{}_\alpha - 2\gamma_{\rho\mu}\nabla_c^\rho\nabla_c^\mu\gamma^\alpha{}_\alpha\right.\\
&\quad \left.+2\gamma_{\alpha\mu}\nabla_c^\rho\nabla_c^\alpha\gamma_\rho{}^\mu + 4\hat{g}_c^{\alpha\lambda}\gamma_{\alpha\beta}\gamma_{\lambda\epsilon}R_c^{\beta\epsilon}\right) - \hat{g}_c^{\mu\alpha}\hat{g}_c^{\nu\beta}\gamma_{\alpha\beta}R^{(1)}_{\mu\nu} && (3.56)\\
&\simeq \frac{1}{4}\left(\gamma_{\alpha\beta}\Box_c\gamma^{\alpha\beta} + \gamma^\mu{}_\mu\Box_c\gamma^\alpha{}_\alpha - 2\gamma_{\alpha\beta}\nabla_c^\rho\nabla_c^\alpha\gamma_\rho{}^\beta + 4\hat{g}_c^{\alpha\lambda}\gamma_{\alpha\beta}R^c_{\lambda\epsilon}\gamma^{\beta\epsilon}\right) && (3.57)\\
&= \frac{1}{4}\left(\gamma_{\alpha\beta}\Box_c\gamma^{\alpha\beta} + \gamma^\mu{}_\mu\Box_c\gamma^\alpha{}_\alpha - 2\gamma_{\alpha\beta}\nabla_c^\rho\nabla_c^\alpha\gamma_\rho{}^\beta + 4\Lambda\gamma_{\alpha\beta}\gamma^{\alpha\beta}\right) && (3.58)
\end{aligned}
$$

We again used the relations (A.5) and (A.6) for maximally symmetric cases to express $R^c_{\mu\nu}$. And we used the covariant Gauss-Stokes theorem (see App. C.2) in combination with partial integration to shift the occurring derivatives such that they act only on one of the γ-fields in (3.58).

With this additional simplifications taken into account we can insert the corresponding terms into (3.52) and obtain the FMS expansion of the EH action to second order. We provide the contributions to each order n in $\gamma_{\mu\nu}$ again as separate expressions $S^{(n)}_{EH}$:

$$
\begin{aligned}
S_{EH} &= S^{(0)}_{EH} + S^{(1)}_{EH} + S^{(2)}_{EH} + O\left(\gamma^3\right) && (3.59)\\
S^{(0)}_{EH} &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\left(\mathcal{R}^c - 2\Lambda\right) = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}2\Lambda = S^c_{EH} && (3.60)\\
S^{(1)}_{EH} &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\left[\frac{1}{2}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\left(\mathcal{R}^c - 2\Lambda\right) + \mathcal{R}^{(1)}\right]\\
&\simeq \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\left[\frac{1}{2}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\left(2\Lambda\right) - \hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\Lambda\right] = 0 && (3.61)
\end{aligned}
$$

The first two terms become very simple as the zeroth order contribution trivially reduces to the classical action for the VEV and the first order action vanishes identically in the considered approximations. The only non-trivial contribution hence originates form the second order part. For the occurring $\mathcal{R}^{(1)}$ we have to use (3.43) instead of (3.55), since we multiply it with an additional $\gamma_{\mu\nu}$, while for $\mathcal{R}^{(2)}$ the approximation (3.58) is valid.

$$\begin{aligned}
S_{EH}^{(2)} &= \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\Bigg[\mathcal{R}^{(2)} + \frac{1}{2}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\mathcal{R}^{(1)} \\
&\quad + \left(-\frac{1}{4}\hat{g}_c^{\mu\nu}\gamma_{\nu\rho}\hat{g}_c^{\rho\sigma}\gamma_{\sigma\mu} + \frac{1}{8}\hat{g}_c^{\mu\nu}\gamma_{\mu\nu}\hat{g}_c^{\rho\sigma}\gamma_{\rho\sigma}\right)(\mathcal{R}^c - 2\Lambda)\Bigg] \\
&\simeq \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\frac{1}{2}\Bigg[\frac{1}{2}\left(\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu} - \hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Box_c - \hat{g}_c^{\beta\mu}\nabla_c^{\nu}\nabla_c^{\alpha} \\
&\quad + \hat{g}_c^{\alpha\beta}\nabla_c^{\mu}\nabla_c^{\nu} + \left(\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu} - \frac{1}{2}\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Lambda\Bigg]\gamma_{\mu\nu} \qquad (3.62) \\
&= \frac{1}{4\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\mathcal{D}^{\alpha\beta\mu\nu}\gamma_{\mu\nu} \qquad (3.63)
\end{aligned}$$

In the last step we introduced the **Hessian**, which is an object that collects all derivative operators and contractions acting on the two γ-fields appearing in the second order action:

$$\mathcal{D}^{\alpha\beta\mu\nu} \equiv \frac{1}{2}\left(\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu} - \hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Box_c - \hat{g}_c^{\beta\mu}\nabla_c^{\nu}\nabla_c^{\alpha} + \hat{g}_c^{\alpha\beta}\nabla_c^{\mu}\nabla_c^{\nu} + \left(\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu} - \frac{1}{2}\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Lambda \qquad (3.64)$$

For completeness we provide also the general Hessian (3.65) which would be obtained in the case of a non-maximally-symmetric spacetime, albeit we will be working with (3.64) from now on.

$$\begin{aligned}
\mathcal{D}_{\text{gen.}}^{\alpha\beta\mu\nu} &\equiv \frac{1}{2}\left(\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu} - \hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Box_c - \hat{g}_c^{\beta\mu}\nabla_c^{\nu}\nabla_c^{\alpha} + \hat{g}_c^{\alpha\beta}\nabla_c^{\mu}\nabla_c^{\nu} + \left(\hat{g}_c^{\alpha\mu}R_c^{\beta\nu} + \hat{g}_c^{\beta\mu}R_c^{\alpha\nu}\right) \\
&\quad - R_c^{\alpha\beta}\hat{g}_c^{\mu\nu} - \frac{1}{2}\left(\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu} - \frac{1}{2}\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)(\mathcal{R}^c - 2\Lambda) \qquad (3.65)
\end{aligned}$$

In both cases we can utilize the symmetry of the γ-tensor fields to modify the Hessian such that it fulfills the following symmetry relations:

$$\mathcal{D}_{\text{symm.}}^{\alpha\beta\mu\nu} = \mathcal{D}_{\text{symm.}}^{(\alpha\beta)(\mu\nu)} = \mathcal{D}_{\text{symm.}}^{(\mu\nu)(\alpha\beta)} \qquad (3.66)$$

We achieve this by manipulating the contraction $\gamma_{\alpha\beta}\mathcal{D}^{\alpha\beta\mu\nu}\gamma_{\mu\nu}$. This calculation is discussed in detail in App. C.3, but what comes out of it is the symmetrized Hessian:

$$S_{EH}^{(2)} = \frac{1}{4\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\mathcal{D}^{\alpha\beta\mu\nu}\gamma_{\mu\nu} = \frac{1}{4\kappa^2}\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\mathcal{D}_{\text{symm.}}^{\alpha\beta\mu\nu}\gamma_{\mu\nu}\ , \text{ with} \qquad (3.67)$$

$$\mathcal{D}_{\text{symm.}}^{\alpha\beta\mu\nu} \equiv \frac{1}{2}\left(\mathcal{A}_1^{\alpha\beta\mu\nu} - \hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Box_c - \mathcal{A}_2^{\alpha\beta\mu\nu} + \mathcal{A}_3^{\alpha\beta\mu\nu} + \left(\mathcal{A}_1^{\alpha\beta\mu\nu} - \frac{1}{2}\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Lambda \qquad (3.68)$$

The three auxiliary objects $\mathcal{A}_j^{\alpha\beta\mu\nu}$ are defined as follows:

$$\mathcal{A}_1^{\alpha\beta\mu\nu} = \frac{1}{2}\left(\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu} + \hat{g}_c^{\alpha\nu}\hat{g}_c^{\beta\mu}\right) \tag{3.69}$$

$$\begin{aligned}\mathcal{A}_2^{\alpha\beta\mu\nu} = \frac{1}{8}\Big(&\hat{g}_c^{\beta\mu}\{\nabla_c^\nu,\nabla_c^\alpha\} + \hat{g}_c^{\beta\nu}\{\nabla_c^\mu,\nabla_c^\alpha\} + \hat{g}_c^{\alpha\mu}\{\nabla_c^\nu,\nabla_c^\beta\} + \hat{g}_c^{\alpha\nu}\{\nabla_c^\mu,\nabla_c^\beta\} \\ &+\frac{\Lambda}{6}\left(4\mathcal{A}_1^{\alpha\beta\mu\nu} - \hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Big)\end{aligned} \tag{3.70}$$

$$\mathcal{A}_3^{\alpha\beta\mu\nu} = \frac{1}{4}\left(\hat{g}_c^{\alpha\beta}\{\nabla_c^\mu,\nabla_c^\nu\} + \hat{g}_c^{\mu\nu}\{\nabla_c^\alpha,\nabla_c^\beta\}\right) \tag{3.71}$$

There remains one open point regarding the action which we have not addressed yet, i.e. the consequences of the frame-fixing condition. There are two possible scenarios, that were already discussed in Sec. 2.2.2:

- Either keep the δ-function in the functional integral and assume that thereby all contributing configurations inherently fulfill the frame-fixing condition, therefore we can utilize the frame identities discussed in Sec. 3.2.2 to simplify the action.
- Or we express the δ-function with the help of an additional frame-fixing Lagrangian and take the limit of the gauge parameter $\zeta \to 0$ in the end of the calculation.

For both cases we need to discuss the modifications to the FMS expansion of the action. Since we use relations in the first approach that hold only in the frame-fixed setting, we refer to the such obtained action as frame-adapted action, whereas for the second approach we add an additional frame-fixing action and thus the corresponding full action will be denoted rame-extended action. Since the only non-trivial contribution to the action is $S_{EH}^{(2)}$, we only need to discuss the modifications for this term and in particular it will suffice to consider only the changes to the Hessian $\mathcal{D}_{\text{symm.}}^{\alpha\beta\mu\nu}$.

Frame-adapted action

So as to simplify the Hessian, we want to employ the frame identities that were obtained in Sec. 3.2.2. Since we have introduced the "classical" covariant derivative (3.28) to write the occurring expressions in a more compact form, it will be useful to also consider the covariant versions of the frame identities, i.e. we need to replace the partial derivatives with "classical" covariant derivatives by adding the correct combination of Γ_c. The relevant identities for the manipulation of the Hessian are derived in App. C.4 and listed below:

$$\nabla_c^\mu\gamma_{\mu\nu} = \mathcal{T}^{\rho\sigma}{}_\nu\gamma_{\rho\sigma} + O\left(\gamma^2\right) \tag{3.72}$$

$$\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\alpha\nabla_c^\nu\gamma_{\mu\nu} = \gamma_{\alpha\beta}\mathcal{J}_1^{\alpha\beta\mu\nu}\gamma_{\mu\nu} + O\left(\gamma^3\right) \tag{3.73}$$

$$\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu} = \gamma_{\alpha\beta}\mathcal{J}_2^{\alpha\beta\mu\nu}\gamma_{\mu\nu} + O\left(\gamma^3\right) \tag{3.74}$$

$$\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu\gamma_{\mu\nu} = \gamma_{\alpha\beta}\left(\mathcal{J}_3^{\alpha\beta\mu\nu} + \hat{g}_c^{\alpha\beta}\Gamma^c{}_\rho^{\mu\nu}\nabla_c^\rho\right)\gamma_{\mu\nu} + O\left(\gamma^3\right) \tag{3.75}$$

$$\gamma_{\alpha\beta}\hat{g}_c^{\mu\nu}\nabla_c^\alpha\nabla_c^\beta\gamma_{\mu\nu} = \gamma_{\alpha\beta}\left(\mathcal{J}_4^{\mu\nu\alpha\beta} - \hat{g}_c^{\mu\nu}\Gamma^c{}_\rho^{\alpha\beta}\nabla_c^\rho\right)\gamma_{\mu\nu} + O\left(\gamma^3\right), \tag{3.76}$$

with the following definitions of the introduced auxiliary objects:

$$\begin{aligned}
\mathcal{T}^{\rho\sigma}{}_{\nu} &\equiv \Gamma^{c\,\rho\sigma}_{\nu} - \Gamma^{\rho\mu}_{c}{}_{\mu}\delta_{\nu}{}^{\sigma} && (3.77)\\
\mathcal{J}_1^{\alpha\beta\mu\nu} &\equiv \Gamma^{\nu\alpha\beta}_{c}\Gamma^{\mu\lambda}_{c}{}_{\lambda} + \Gamma^{\beta\mu\nu}_{c}\Gamma^{\alpha\lambda}_{c}{}_{\lambda} - \Gamma^{\rho\alpha\beta}_{c}\Gamma^{c\mu\nu}_{\rho} - \hat{g}^{\beta\nu}_{c}\Gamma^{\alpha\lambda}_{c}{}_{\lambda}\Gamma^{\mu\tau}_{c}{}_{\tau} && (3.78)\\
\mathcal{J}_2^{\alpha\beta\mu\nu} &\equiv \mathcal{J}_1^{\alpha\beta\mu\nu} - \frac{\Lambda}{3}\left(\hat{g}^{\alpha\beta}_{c}\hat{g}^{\mu\nu}_{c} - \hat{g}^{\alpha\mu}_{c}\hat{g}^{\beta\nu}_{c}\right) + \Lambda\hat{g}^{\alpha\nu}_{c}\hat{g}^{\beta\mu}_{c} && (3.79)\\
\mathcal{J}_3^{\alpha\beta\mu\nu} &\equiv \hat{g}^{\alpha\beta}_{c}\left(\nabla^{\rho}_{c}\Gamma^{c\mu\nu}_{\rho} - \nabla^{\nu}_{c}\Gamma^{\mu\rho}_{c}{}_{\rho} - \Gamma^{\rho\sigma}_{c}{}_{\sigma}\Gamma^{c\mu\nu}_{\rho} + \Gamma^{\mu\rho}_{c}{}_{\rho}\Gamma^{\nu\sigma}_{c}{}_{\sigma}\right) && (3.80)\\
\mathcal{J}_4^{\mu\nu\alpha\beta} &\equiv \mathcal{J}_3^{\mu\nu\alpha\beta} - \hat{g}^{\mu\nu}_{c}\nabla^{\rho}_{c}\Gamma^{c\alpha\beta}_{\rho} && (3.81)
\end{aligned}$$

It should be noted that the introduced auxiliary objects are not proper tensors in the sense that they follow the transformation law (2.4), this means that we can treat them as tensors in a chosen frame, however, if we consider a different frame the corresponding auxiliary objects $\mathcal{J}_j'^{\alpha\beta\mu\nu}$ and $\mathcal{T}'^{\rho\sigma}{}_{\nu}$ are not obtained via (2.4), but rather newly defined altogether. This is the same as for the Christoffel symbols [15].

With the identities introduced above, it is possible to rewrite the auxiliary objects $\mathcal{A}_2^{\alpha\beta\mu\nu}$ and $\mathcal{A}_3^{\alpha\beta\mu\nu}$, in a frame that fulfills (3.15). For $\mathcal{A}_2^{\alpha\beta\mu\nu}$ we can use (3.73) and (3.74), as well as (A.1):

$$\begin{aligned}
\mathcal{A}^{\alpha\beta\mu\nu}_{2,\text{ f.a.}} &= \frac{1}{2}\left(\mathcal{J}_2^{(\alpha\beta)(\mu\nu)} + \mathcal{J}_1^{(\alpha\beta)(\mu\nu)}\right) + \frac{\Lambda}{48}\left(4\mathcal{A}_1^{\alpha\beta\mu\nu} - \hat{g}^{\alpha\beta}_{c}\hat{g}^{\mu\nu}_{c}\right)\\
&\overset{(3.79)}{=} \frac{1}{2}\left(\mathcal{J}_1^{(\mu\nu)(\alpha\beta)} + \mathcal{J}_1^{(\alpha\beta)(\mu\nu)}\right) + \frac{9}{48}\Lambda\left(4\mathcal{A}_1^{\alpha\beta\mu\nu} - \hat{g}^{\alpha\beta}_{c}\hat{g}^{\mu\nu}_{c}\right) \qquad (3.82)
\end{aligned}$$

We have already arranged all terms such that the symmetry properties (C.15) of $\mathcal{A}_2^{\alpha\beta\mu\nu}$ are manifest of the r.h.s. of (3.82). $\mathcal{A}_3^{\alpha\beta\mu\nu}$ can then be re-expressed with (3.75), (3.76) and (A.1):

$$\begin{aligned}
\mathcal{A}^{\alpha\beta\mu\nu}_{3,\text{ f.a.}} &= \frac{1}{2}\left(\mathcal{J}_3^{\alpha\beta(\mu\nu)} + \mathcal{J}_4^{\mu\nu(\alpha\beta)} + \hat{g}^{\alpha\beta}_{c}\Gamma^{c(\mu\nu)}_{\rho}\nabla^{\rho}_{c} - \hat{g}^{\mu\nu}_{c}\Gamma^{c(\alpha\beta)}_{\rho}\nabla^{\rho}_{c}\right)\\
&\overset{(3.81)}{=} \frac{1}{2}\left(\mathcal{J}_3^{(\alpha\beta)(\mu\nu)} + \mathcal{J}_3^{(\mu\nu)(\alpha\beta)}\right) - \frac{1}{2}\hat{g}^{\mu\nu}_{c}\nabla^{\rho}_{c}\Gamma^{c(\alpha\beta)}_{\rho} + \frac{1}{2}\left(\hat{g}^{\alpha\beta}_{c}\Gamma^{c(\mu\nu)}_{\rho}\nabla^{\rho}_{c} - \hat{g}^{\mu\nu}_{c}\Gamma^{c(\alpha\beta)}_{\rho}\nabla^{\rho}_{c}\right)
\end{aligned} \qquad (3.83)$$

Now we can replace the respective terms in (3.68), which reduces the non-minimal second derivative terms to first derivative terms and additional tensor structure, which doesn't contain any derivatives:

$$\mathcal{D}^{\alpha\beta\mu\nu}_{\text{f.a.}} \equiv \frac{1}{2}\left(\mathcal{A}_1^{\alpha\beta\mu\nu} - \hat{g}^{\alpha\beta}_{c}\hat{g}^{\mu\nu}_{c}\right)\Box_c - \mathcal{A}^{\alpha\beta\mu\nu}_{2,\text{ f.a.}} + \mathcal{A}^{\alpha\beta\mu\nu}_{3,\text{ f.a.}} + \left(\mathcal{A}_1^{\alpha\beta\mu\nu} - \frac{1}{2}\hat{g}^{\alpha\beta}_{c}\hat{g}^{\mu\nu}_{c}\right)\Lambda \qquad (3.84)$$

where the subscript **f.a.** indicates that this is the Hessian for the frame-adapted action.

Frame-extended action

In the second approach we just need to apply the FMS expansion to the frame-fixing action, which will provide us with the additional terms for the frame-extended action. In the following we again just consider the Hessian, however we need to keep in mind that the frame-fixing action is tethered to a limiting procedure for the gauge-parameter ζ, which has to be conducted at the end of the calculation. We start with the frame-fixing action as defined in (2.47) and (2.48) and insert the Haywood condition (3.15). We also rescale ζ such that we can pull out an overall factor of $\left(2\kappa^2\right)^{-1}$ as this will make it simpler to incorporate the additional frame-fixing contribution into the EH action.

$$S_{GF}\left[g_{\mu\nu},\zeta\right] = \frac{1}{4\kappa^2}\frac{1}{2\zeta}\int d^4x\sqrt{|g|}g^{\mu\nu}H_\mu H_\nu \tag{3.85}$$

$$= \frac{1}{4\kappa^2}\frac{1}{2\zeta}\int d^4x\sqrt{|g|}g^{\mu\nu}g^{\alpha\beta}\left(\partial_\alpha g_{\beta\mu}\right)g^{\rho\sigma}\left(\partial_\rho g_{\sigma\nu}\right) \tag{3.86}$$

Performing the linear split and considering the FMS expansion to second order leads to the following expression:

$$\begin{aligned} S_{GF}\left[g^c_{\mu\nu},\gamma_{\mu\nu},\zeta\right] = \frac{1}{4\kappa^2}\frac{1}{2\zeta}\int d^4x d^{(0)}\left(1+d^{(1)}+d^{(2)}+O\left(\gamma^3\right)\right)\left(\hat{g}^{\mu\nu}_c+\hat{\gamma}^{\mu\nu}\right)\times \\ \times\left(H^{(0)}_\mu+H^{(1)}_\mu+H^{(2)}_\mu+O\left(\gamma^3\right)\right)\left(H^{(0)}_\nu+H^{(1)}_\nu+H^{(2)}_\nu+O\left(\gamma^3\right)\right) \end{aligned} \tag{3.87}$$

The only term up to $O\left(\gamma^3\right)$ that does not contain $H^{(0)}_\mu$, is the combination $H^{(1)}_\mu H^{(1)}_\nu$. But since we constructed $H_\mu\left[g_{\mu\nu}\right]$ such that $H^{(0)}_\mu = H_\mu\left[g^c_{\mu\nu}\right] = 0$ is fulfilled, all terms that contain $H^{(0)}$ vanish. Furthermore, any term in the series of the metric determinant or the inverse metric that is of an order $> O\left(\gamma^0\right)$ will - in combination with $H^{(1)}_\mu H^{(1)}_\nu$ - result in a term of order $O\left(\gamma^3\right)$ or higher and is thus irrelevant for the current treatment. Hence, the only non-vanishing contribution form the gauge-fixing action that will show up in the frame-extended second order action and thus also in the Hessian, is:

$$\begin{aligned} S^{(2)}_{GF}\left[g^c_{\mu\nu},\gamma_{\mu\nu},\zeta\right] &= \frac{1}{4\kappa^2}\frac{1}{2\zeta}\int d^4x d^{(0)}\hat{g}^{\mu\nu}_c H^{(1)}_\mu H^{(1)}_\nu \\ &\overset{(C.25)}{=} \frac{1}{4\kappa^2}\frac{1}{2\zeta}\int d^4x\sqrt{|g^c|}\hat{g}^{\mu\nu}_c\Big(\partial^\alpha\gamma_{\alpha\mu}\partial^\beta\gamma_{\beta\nu} - 2\gamma_{\rho\sigma}\hat{g}^{\beta\sigma}_c\left(\partial^\rho g^c_{\beta\mu}\right)\partial^\lambda\gamma_{\lambda\nu} \\ &\quad +\hat{g}^{\alpha\rho}_c\hat{g}^{\beta\lambda}_c\left(\partial^\sigma g^c_{\alpha\mu}\right)\left(\partial^\epsilon g^c_{\beta\nu}\right)\gamma_{\rho\sigma}\gamma_{\lambda\epsilon}\Big) \end{aligned} \tag{3.88}$$

It will be again convenient to rewrite all partial derivatives in terms of covariant derivatives w.r.t. $g^c_{\mu\nu}$. The rewriting of (3.88) with covariant derivatives is discussed in detail in App. C.4. The result is presented below:

$$\begin{aligned} S^{(2)}_{GF}\left[g^c_{\mu\nu},\gamma_{\mu\nu},\zeta\right] &\overset{(C.29)}{\simeq} \frac{1}{4\kappa^2}\frac{1}{2\zeta}\int d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\Big(-\mathcal{A}^{\alpha\beta\mu\nu}_2 + 2\left(\hat{g}^{\beta\nu}_c{\Gamma_c}^{\alpha\rho}{}_\rho - \hat{g}^{\rho\alpha}_c{\Gamma_c}^{\nu\beta}{}_\rho\right)\nabla^\mu_c \\ &\quad +\mathcal{I}^{\alpha\beta\mu\nu}_1\Big)\gamma_{\mu\nu} \end{aligned} \tag{3.89}$$

Where the auxiliary object $\mathcal{I}^{\alpha\beta\mu\nu}_1$ is defined as follows:

$$\mathcal{I}^{\alpha\beta\mu\nu}_1 \equiv \frac{\Lambda}{3}\left(\hat{g}^{\alpha\beta}_c\hat{g}^{\mu\nu}_c - 4\mathcal{A}^{\alpha\beta\mu\nu}_1\right) + \hat{g}^{\beta\nu}_c{\Gamma_c}^{\alpha\rho}{}_\rho{\Gamma_c}^{\mu\sigma}{}_\sigma + g^c_{\rho\sigma}{\Gamma_c}^{\rho\alpha\beta}{\Gamma_c}^{\sigma\mu\nu} - 2\hat{g}^{\rho\alpha}_c{\Gamma_c}^{\nu\beta}{}_\rho{\Gamma_c}^{\mu\sigma}{}_\sigma \tag{3.90}$$

It is evident that the chart specific expression of the gauge-fixing action will be very simple in the cartesian Minkowski chart as all Christoffel Symbols Γ_c and thus also $\mathcal{I}_1$ vanish. The frame-extended Hessian can now be written as:

$$\begin{aligned}\mathcal{D}^{\alpha\beta\mu\nu}_{\text{f.e.}} &= \frac{1}{2}\left(\mathcal{A}_1^{\alpha\beta\mu\nu} - \hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Box_c - \left(1 - \frac{1}{2\zeta}\right)\mathcal{A}_2^{\alpha\beta\mu\nu} + \mathcal{A}_3^{\alpha\beta\mu\nu} \\ &\quad - \frac{1}{\zeta}\left(\hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho - \hat{g}_c^{\rho\alpha}\Gamma_c{}^{\nu\beta}{}_\rho\right)\nabla_c^\mu + \left(\mathcal{A}_1^{\alpha\beta\mu\nu} - \frac{1}{2}\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Lambda - \frac{1}{2\zeta}\mathcal{I}_1^{\alpha\beta\mu\nu} \qquad (3.91)\end{aligned}$$

Graviton propagator

Now that we have discussed the FMS expansion of the EH action, we can use the results to calculate the graviton propagator $\langle\gamma_{\alpha\beta}\gamma_{\mu'\nu'}\rangle$ that occurs in the FMS expansion of the geon propagator. In particular, we will evaluate the frame-fixed correlation function at tree-level. We will calculate the propagator via the corresponding vertex function at tree-level, i.e. the two-point vertex function. We can make use of the fact that the generating functional of all tree-level vertex functions, also referred to as vertices, is the action itself [35]. Therefore, we can obtain the n-point vertex function $\Gamma_{\text{tl}}^{(n)\mu_1\nu_1\ldots\mu_n\nu_n}(\mathscr{P}_1,\ldots,\mathscr{P}_n)$ via the functional derivative of the action w.r.t. the dynamical d.o.f. which are set to zero in the end [35]:

$$i\Gamma_{\text{tl}}^{(n)\mu_1\nu_1\ldots\mu_n\nu_n}(\mathscr{P}_1,\ldots,\mathscr{P}_n) = \left.\frac{i\delta^2 S[g_{\mu\nu}]}{\delta g_{\mu_1\nu_1}(\mathscr{P}_1)\ldots\delta g_{\mu_n\nu_n}(\mathscr{P}_n)}\right|_{g=0} \qquad (3.92)$$

This provides us with the following identity for the two-point vertex function at tree-level:

$$i\Gamma_{\text{tl}}^{(2)\alpha\beta\mu\nu}(\mathscr{P},\mathscr{Q}) = \left.\frac{i\delta^2 S[g_{\mu\nu}]}{\delta g_{\alpha\beta}(\mathscr{P})\,\delta g_{\mu\nu}(\mathscr{Q})}\right|_{g=0} \qquad (3.93)$$

We can then obtain the propagator from the vertex function by using the fact that the two-point vertex function represents the negative inverse propagator (in particular, the such obtained correlation function will be connected):

$$i\Gamma_{\text{tl}}^{(2)\alpha\beta\mu\nu}(\mathscr{P},\mathscr{Q}) = -\left[\langle g_{\alpha\beta}(\mathscr{P})\,g_{\mu\nu}(\mathscr{Q})\rangle\right]^{-1}, \qquad (3.94)$$

where the superscript -1 is supposed to denote the formal inverse.

These are the general relations, however, as argued before, unless the PI is treated in a frame-fixed setting, these vertex functions will all vanish. In the frame-fixed setting we consider $\gamma_{\mu\nu}$ as the dynamical field and rewrite the action with the FMS expansion, therefore we get the following relation for the two-point vertex function:

$$i\Gamma_{\text{tl}}^{(2)\alpha\beta\mu\nu}(\mathscr{P},\mathscr{Q}) = \left.\frac{i\delta^2 S\left[g^c_{\mu\nu},\gamma_{\mu\nu}\right]}{\delta\gamma_{\alpha\beta}(\mathscr{P})\,\delta\gamma_{\mu\nu}(\mathscr{Q})}\right|_{\gamma=0} \qquad (3.95)$$

From this expression it is immediately evident that only the second order part of the action in the FMS expansion can contribute to the two-point vertex function, since the lower contributions vanish upon second differentiation, whereas all higher order contributions will still contain γ-fields and after setting $\gamma_{\mu\nu}$ to zero, these will vanish as well. Therefore we can easily infer from (3.63) that the tree-level two-point vertex function is proportional to the (symmetric) Hessian (3.68):

$$i\Gamma^{(2)\alpha\beta\mu\nu}_{\text{tl}}\left(\mathscr{P},\mathscr{Q}\right) = \left.\frac{i\delta^2 S^{(2)}_{\text{EH}}\left[g^c_{\mu\nu},\gamma_{\mu\nu}\right]}{\delta\gamma_{\alpha\beta}\left(\mathscr{P}\right)\delta\gamma_{\mu\nu}\left(\mathscr{Q}\right)}\right|_{\gamma=0} \sim \mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}} \tag{3.96}$$

In order to obtain the exact expression for (3.96), we need to evaluate the functional differentiation. For this we assume that we have a coordinatization with $x_{\mathscr{P}}$ and $x_{\mathscr{Q}}$ representing the coordinates of the two events $\mathscr{P}$ and $\mathscr{Q}$. We can invoke the general definitions of functional differentiation as provided in Ref. [35], with the slight change that we need to use the generalization of the Dirac δ-function to curved spacetime, when differentiating w.r.t. to fields [36, 37]:

$$\frac{\delta\gamma_{\mu\nu}\left(x_{\mathscr{P}}\right)}{\delta\gamma_{\alpha\beta}\left(x_{\mathscr{Q}}\right)} \equiv \delta_\mu{}^\alpha\delta_\nu{}^\beta\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right) \text{ , with} \tag{3.97}$$

$$\begin{aligned}\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right) &\equiv \frac{\delta^{(4)}\left(x_{\mathscr{P}}-x_{\mathscr{Q}}\right)}{\sqrt{|g^c\left(x_{\mathscr{P}}\right)|}} = \frac{\delta^{(4)}\left(x_{\mathscr{P}}-x_{\mathscr{Q}}\right)}{\sqrt{|g^c\left(x_{\mathscr{Q}}\right)|}}\\ &= \left(g^c\left(x_{\mathscr{P}}\right)g^c\left(x_{\mathscr{Q}}\right)\right)^{-1/4}\delta^{(4)}\left(x_{\mathscr{P}}-x_{\mathscr{Q}}\right) \text{ and}\end{aligned} \tag{3.98}$$

$$\delta^{(4)}\left(x_{\mathscr{P}}-x_{\mathscr{Q}}\right) \equiv \delta\left(x^0_{\mathscr{P}}-x^0_{\mathscr{Q}}\right)\delta\left(x^1_{\mathscr{P}}-x^1_{\mathscr{Q}}\right)\delta\left(x^2_{\mathscr{P}}-x^2_{\mathscr{Q}}\right)\delta\left(x^3_{\mathscr{P}}-x^3_{\mathscr{Q}}\right) \tag{3.99}$$

Starting with the l.h.s. of (3.96), we can insert (3.63) (modulo the prefactor) and using (3.97), we obtain:

$$\begin{aligned}&\left.\frac{\delta^2}{\delta\gamma_{\alpha\beta}\left(x_{\mathscr{P}}\right)\delta\gamma_{\mu'\nu'}\left(x_{\mathscr{Q}}\right)}\int_{\mathcal{M}} d^4z\sqrt{|g^c(z)|}\gamma_{\alpha\beta}(z)\mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}}(z)\gamma_{\mu\nu}(z)\right|_{\gamma=0} =\\ &= \int_{\mathcal{M}} d^4z\sqrt{|g^c(z)|}\Big(\delta_4\left(z,x_{\mathscr{Q}}\right)\mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}}(z)\delta_4\left(z,x_{\mathscr{P}}\right)+\delta_4\left(z,x_{\mathscr{P}}\right)\mathcal{D}^{\mu\nu\alpha\beta}_{\text{symm.}}(z)\delta_4\left(z,x_{\mathscr{Q}}\right)\Big)\\ &= \Big(\mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}}\left(x_{\mathscr{P}}\right)+\mathcal{D}^{\mu\nu\alpha\beta}_{\text{symm.}}\left(x_{\mathscr{P}}\right)\Big)\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right)\\ &\overset{(3.66)}{=} 2\mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}}\left(x_{\mathscr{P}}\right)\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right)\end{aligned} \tag{3.100}$$

Thus we arrive at the following result for (3.96):

$$i\Gamma^{(2)\alpha\beta\mu\nu}_{\text{tl}}\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right) = \frac{i}{2\kappa^2}\mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}}\left(x_{\mathscr{P}}\right)\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right) \overset{(3.94)}{=} -\left[\langle\gamma_{\alpha\beta}\left(x_{\mathscr{P}}\right)\gamma_{\mu\nu}\left(x_{\mathscr{Q}}\right)\rangle_{\text{tl}}\right]^{-1}. \tag{3.101}$$

The tree-level propagator of the γ-field is then defined via the inversion of the vertex function:

$$\begin{aligned}
&\int_{\mathcal{M}} d^4z\sqrt{|g^c(z)|}\left[\langle\gamma_{\alpha\beta}\left(x_{\mathscr{P}}\right)\gamma_{\mu\nu}\left(z\right)\rangle_{\text{tl}}\right]^{-1}\langle\gamma_{\mu\nu}(z)\gamma_{\lambda\epsilon}\left(x_{\mathscr{Q}}\right)\rangle_{\text{tl}} = \\
&\overset{(3.101)}{=} \int_{\mathcal{M}} d^4z\sqrt{|g^c(z)|}\left(-\frac{i}{2\kappa^2}\mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}}\left(x_{\mathscr{P}}\right)\delta_4\left(x_{\mathscr{P}},z\right)\right)\langle\gamma_{\mu\nu}(z)\gamma_{\lambda\epsilon}\left(x_{\mathscr{Q}}\right)\rangle_{\text{tl}} \\
&= -\frac{i}{2\kappa^2}\mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}}\left(x_{\mathscr{P}}\right)\langle\gamma_{\mu\nu}\left(x_{\mathscr{P}}\right)\gamma_{\lambda\epsilon}\left(x_{\mathscr{Q}}\right)\rangle_{\text{tl}} \overset{!}{=} \mathbb{I}^{\alpha\beta}{}_{\lambda\epsilon}\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right)\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right)
\end{aligned} \tag{3.102}$$

Here $\mathbb{I}^{\alpha\beta}{}_{\lambda\epsilon}\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right)$ denotes the generalization of the unit tensor to curved spacetimes. While for Minkowski space the unit tensor structure is simple and can be expressed purely in Kronecker deltas

$$\mathbb{I}^{\alpha\beta}{}_{\lambda\epsilon} = \frac{1}{2}\left(\delta^{\alpha}{}_{\lambda}\delta^{\beta}{}_{\epsilon} + \delta^{\alpha}{}_{\epsilon}\delta^{\beta}{}_{\lambda}\right), \tag{3.103}$$

this becomes more involved in curved spacetimes. The reason for this is that the co- and contravariant indices refer to two different events, i.e. $\mathscr{P}$ and $\mathscr{Q}$, and in a curved spacetime this involves the parallel propagator for consistency reasons [36]. This is discussed in more detail in Sec. 4.1.1.

Now we reintroduce the convention of Sec. 3.2.3 that primed indices (or just an additional prime for scalars) denote objects w.r.t. event $\mathscr{Q}$, while unprimed objects are related to event $\mathscr{P}$. With the ansatz [11]

$$\langle\gamma_{\mu\nu}\gamma_{\lambda'\epsilon'}\rangle_{\text{tl}} = iG_{\mu\nu;\lambda'\epsilon'} \tag{3.104}$$

the inversion equation (3.102) can be written as the Green's function equation (GFE)

$$\frac{1}{2\kappa^2}\mathcal{D}^{\alpha\beta\mu\nu}_{\text{symm.}}G_{\mu\nu;\lambda'\epsilon'} \overset{!}{=} \mathbb{I}^{\alpha\beta}{}_{\lambda'\epsilon'}\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right). \tag{3.105}$$

Alternatively, we can also use contractions with g^c to lower the first two indices on $\mathcal{D}$symm., thus we obtain

$$\frac{1}{2\kappa^2}\mathcal{D}^{\text{symm.}\,\mu\nu}_{\alpha\beta}G_{\mu\nu;\lambda'\epsilon'} \overset{!}{=} \mathbb{I}_{\alpha\beta;\lambda'\epsilon'}\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right). \tag{3.106}$$

which is a version of the GFE that will be very useful later on. The object $G_{\mu\nu;\lambda'\epsilon'}$ which we introduced in the ansatz (3.104) will be referred to as graviton Green's function from now on.

The solution of this Green's function equation will provide us with the tree-level expression of the propagator $\langle\gamma_{\alpha\beta}\gamma_{\mu'\nu'}\rangle$, this can then be used to calculate the geon propagator (3.48) in the tree-level approximation to second order in the FMS expansion. Thus we have now everything that we need to consider the geon propagator in different cases for the VEV, which will be done in the following two chapters.

[11] Since $G_{\mu\nu;\lambda'\epsilon'}$ is defined via γ, it shares the same symmetry properties, i.e. $G_{\mu\nu;\lambda'\epsilon'} = G_{(\mu\nu);(\lambda'\epsilon')} = G_{(\lambda'\epsilon');(\mu\nu)}$.

4. Minkowski graviton and geon

In the previous chapters we have discussed the setup of the classical and the quantum theory as well as the FMS mechanism as a tool to extract information about the geon from the graviton propagator $\langle \gamma_{\mu\nu}\gamma_{\lambda'\epsilon'} \rangle$ in a frame-fixed setting, which can be evaluated with perturbative methods. We have also discussed how the determination of the propagator $\langle \gamma_{\mu\nu}\gamma_{\lambda'\epsilon'} \rangle$ reduces to a Green's function problem at tree-level. In this chapter we will be concerned with the solution of the GFE (3.105) and the evaluation of the geon in the FMS expansion to second order, under the assumption that the VEV $g^c_{\mu\nu}$ represents a flat spacetime. Such a Minkowski spacetime has vanishing cosmological constant and is Ricci-flat, i.e. $\Lambda = 0$ and all "classical" curvature quantities vanish, see App. A.2. As discussed in Sec. 3.2.1, we will be using the cartesian coordinatization

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{4.1}$$

for Minkowski space, which is in accordance with the Haywood condition that is used for the frame-fixing procedure. This implies that the VEV $g^c_{\mu\nu}$ reduces to the standard Minkowski metric in cartesian coordinates

$$g^c_{\mu\nu} \equiv \eta_{\mu\nu}, \text{ with} \tag{4.2}$$

$$[\eta_{\mu\nu}] \equiv \mathrm{diag}\,(-1, 1, 1, 1) \text{ and} \tag{4.3}$$

$$\sqrt{|g^c|} \equiv \sqrt{|\eta|} = 1. \tag{4.4}$$

The fact that the Minkowski spacetime is Ricci-flat allows for the metric $\eta_{\mu\nu}$ to be globally constant, which greatly simplifies the structure of the related geometric quantities, since the Christoffel symbols vanish and the corresponding covariant derivative reduces to the partial derivative,

$$\Gamma^{\rho}_{c\,\mu\nu} \equiv 0 \text{ and} \tag{4.5}$$

$$\nabla^{\mu}_{c} \equiv \partial^{\mu}. \tag{4.6}$$

4.1. Minkowski space graviton propagator

We can now study the effect of the selected VEV and the corresponding cartesian frame on the (symmetric) Hessian (3.68), which occurs in the Green's function equation for the graviton two-point function $\langle \gamma_{\mu\nu}\gamma_{\lambda'\epsilon'} \rangle$. Since Λ is zero for Minkowski space, all terms in the Hessian (and in the related auxiliary objects) that are proportional to the cosmological constant drop out. We choose to denote quantities that are specific to flat space with a subscript **M**:

$$\mathcal{D}_{\mathrm{M}}^{\alpha\beta\mu\nu} \equiv \frac{1}{2}\left(\mathcal{A}_{1,\mathrm{M}}^{\alpha\beta\mu\nu} - \eta^{\alpha\beta}\eta^{\mu\nu}\right)\Box_{\mathrm{M}} - \mathcal{A}_{2,\mathrm{M}}^{\alpha\beta\mu\nu} + \mathcal{A}_{3,\mathrm{M}}^{\alpha\beta\mu\nu}, \text{ with} \tag{4.7}$$

$$\mathcal{A}_{1,\mathrm{M}}^{\alpha\beta\mu\nu} \equiv \frac{1}{2}\left(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu}\right) \tag{4.8}$$

$$\mathcal{A}_{2,\mathrm{M}}^{\alpha\beta\mu\nu} \equiv \frac{1}{4}\left(\eta^{\beta\mu}\partial^{\alpha}\partial^{\nu} + \eta^{\beta\nu}\partial^{\alpha}\partial^{\mu} + \eta^{\alpha\mu}\partial^{\beta}\partial^{\nu} + \eta^{\alpha\nu}\partial^{\beta}\partial^{\mu}\right) \tag{4.9}$$

$$\mathcal{A}_{3,\mathrm{M}}^{\alpha\beta\mu\nu} \equiv \frac{1}{2}\left(\eta^{\alpha\beta}\partial^{\mu}\partial^{\nu} + \eta^{\mu\nu}\partial^{\alpha}\partial^{\beta}\right) \tag{4.10}$$

The anticommutators in the auxiliary objects $\mathcal{A}_{2,\mathrm{M}}^{\alpha\beta\mu\nu}$ and $\mathcal{A}_{3,\mathrm{M}}^{\alpha\beta\mu\nu}$ are not required anymore, as the partial derivatives commute. Furthermore, on the r.h.s. of the GFE (3.105) the invariant δ-function $\delta_4\left(x_{\mathscr{P}}, x_{\mathscr{Q}}\right)$ reduces to the standard Dirac δ-function $\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right)$, due to (4.4). From (3.106) it follows that the GFE that we need to solve is

$$\frac{1}{2\kappa^2}\mathcal{D}^{\mathrm{M}}_{\alpha\beta}{}^{\mu\nu} G_{\mu\nu;\lambda'\epsilon'} \stackrel{!}{=} \mathbb{I}_{\alpha\beta;\lambda'\epsilon'}\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right). \tag{4.11}$$

Based on which approach we choose, either the frame-extended or the frame-adapted one, there will be further alterations to $\mathcal{D}_{\mathrm{M}}$.

In the frame-adapted scheme all auxiliary objects $\mathcal{J}_j^{\alpha\beta\mu\nu}$ (see (3.78) - (3.81)) as well as $\mathcal{T}^{\rho\sigma}{}_{\nu}$ (see (3.77)) vanish, since both the Christoffel symbols $\Gamma^{\rho}_{c\mu\nu}$ and the cosmological constant Λ are zero in Minkowski space. Consequently $\mathcal{A}_{2,\,\mathrm{f.a.}}^{\alpha\beta\mu\nu}$ and $\mathcal{A}_{3,\,\mathrm{f.a.}}^{\alpha\beta\mu\nu}$ are also identical to zero and the frame-adapted Hessian (3.84) reduces to

$$\begin{aligned}\mathcal{D}^{\mathrm{M,\,f.a.}}_{\alpha\beta}{}^{\mu\nu} &\equiv \frac{1}{2}\left(\mathcal{A}^{1,\mathrm{M}}_{\alpha\beta}{}^{\mu\nu} - \eta_{\alpha\beta}\eta^{\mu\nu}\right)\Box_{\mathrm{M}} \\ &\stackrel{(4.8)}{=} \frac{1}{2}\left(\frac{1}{2}\left(\delta_{\alpha}{}^{\mu}\delta_{\beta}{}^{\nu} + \delta_{\alpha}{}^{\nu}\delta_{\beta}{}^{\mu}\right) - \eta_{\alpha\beta}\eta^{\mu\nu}\right)\Box_{\mathrm{M}},\end{aligned} \tag{4.12}$$

where we have already used $\eta_{\rho\sigma}$ to lower the first two indices of the Hessian, so as to obtain the version that occurs in (4.11).

For the frame-extended Hessian, $\mathcal{I}_1^{\alpha\beta\mu\nu}$ (see (3.90)) and the first-derivative term vanish and the $\mathcal{A}_j^{\alpha\beta\mu\nu}$ reduce to the $\mathcal{A}_{j,\mathrm{M}}^{\alpha\beta\mu\nu}$, thus we obtain

$$\mathcal{D}^{\mathrm{M,\,f.e.}}_{\alpha\beta}{}^{\mu\nu} \equiv \frac{1}{2}\left(\mathcal{A}^{1,\mathrm{M}}_{\alpha\beta}{}^{\mu\nu} - \eta_{\alpha\beta}\eta^{\mu\nu}\right)\Box_{\mathrm{M}} - \left(1 - \frac{1}{2\zeta}\right)\mathcal{A}^{2,\mathrm{M}}_{\alpha\beta}{}^{\mu\nu} + \mathcal{A}^{3,\mathrm{M}}_{\alpha\beta}{}^{\mu\nu}. \tag{4.13}$$

These two alternative expressions for the Hessian (4.7) can now be used in the GFE (4.11), depending on which scheme (frame-extended or frame-adapted) we want to use. What is usually done for quantum fluctuations on a Minkowski background is that under exploitation of Poincaré symmetry one performs the Fourier transformation to momentum space, which allows for the GFE to be written as an algebraic rather than a differential equation [25, 32, 38]. We can do this for our approach too, as long as we stay at the lowest order of the FMS expansion, however, in the case of a VEV corresponding to a curved manifold this is in general not possible anymore as Poincaré symmetry is lost. Therefore, to not restrict ourselves to a very specific case, we want to evaluate the graviton propagator as well as the geon propagator in position space.

4.1.1. Bitensor structure in the Green's function equation

To solve the GFE we may use the fact that the expectation value $\langle\gamma_{\mu\nu}\gamma_{\lambda'\epsilon'}\rangle$ will live on the same manifold which corresponds to $g^c_{\mu\nu}$. Since we have chosen to work with maximally symmetric spacetimes for the VEV, the bitensor $\langle\gamma_{\mu\nu}\gamma_{\lambda'\epsilon'}\rangle$ will be a maximally symmetric bitensor [33]. This piece of information allows us to simplify the GFE significantly, since for maximally symmetric bitensors we can use the theorem discussed in Ref. [33]: "*Any maximally symmetric bitensor can be expressed as a sum of products of the objects $g^c_{\mu\nu}$, $g^c_{\lambda'\epsilon'}$, r^c, n_μ, $n_{\lambda'}$ and $g_{\mu\lambda'}$ with any consistent combination of indices on these. The coefficients of the summands are functions only of the geodesic distance $r^c(\mathscr{P},\mathscr{Q})$.*" The objects used in this decomposition are the metric tensors at both spacetime events $g^c_{\mu\nu}$ and $g^c_{\lambda'\epsilon'}$, the tangent vectors of the geodesic connecting the two events, n^μ and $n^{\lambda'}$, at the endpoints[1], and the parallel propagator $g_{\mu\lambda'}$. These basis objects and their properties are discussed in further detail in App. D.2 and they are referred to as Allen-Jacobson basis in the following. Based on these the graviton propagator $\langle\gamma_{\mu\nu}\gamma_{\lambda'\epsilon'}\rangle$ and therefore also $G_{\mu\nu;\lambda'\epsilon'}$ can be decomposed into five contributions [39, 34]:

$$\begin{aligned}
G_{\mu\nu;\lambda'\epsilon'}(r^c) &= g^c_{\mu\nu}g^c_{\lambda'\epsilon'}a(r^c) + \left(g_{\mu\lambda'}g_{\nu\epsilon'} + g_{\mu\epsilon'}g_{\nu\lambda'}\right)b(r^c) \\
&\quad + \left(n_\mu n_{\lambda'}g_{\nu\epsilon'} + n_\mu n_{\epsilon'}g_{\nu\lambda'} + n_\nu n_{\lambda'}g_{\mu\epsilon'} + n_\nu n_{\epsilon'}g_{\mu\lambda'}\right)c(r^c) \\
&\quad + \left(n_\mu n_\nu g^c_{\lambda'\epsilon'} + g^c_{\mu\nu}n_{\lambda'}n_{\epsilon'}\right)d(r^c) + n_\mu n_\nu n_{\lambda'}n_{\epsilon'}e(r^c) \qquad (4.14)\\
&= \sum_{j=1}^{5} O^{(j)}_{\mu\nu;\lambda'\epsilon'}f_j(r^c) \qquad (4.15)
\end{aligned}$$

In particular, we can now also express the bitensor structure on the r.h.s. of (4.11) within this basis

$$\mathbb{I}_{\mu\nu;\lambda'\epsilon'} = \frac{1}{2}\left(g_{\mu\lambda'}g_{\nu\epsilon'} + g_{\mu\epsilon'}g_{\nu\lambda'}\right) = \frac{1}{2}O^{(2)}_{\mu\nu;\lambda'\epsilon'}. \qquad (4.16)$$

This has the proper action of a unity tensor and respects the fact that the primed and unprimed indices live at different events in spacetime. The great advantage of this basis decomposition of the Green's function $G_{\mu\nu;\lambda'\epsilon'}(r^c)$ is that we can reduce the second order partial differential equation (4.11) in the coordinates to a system of coupled ordinary differential of second order in the geodesic distance r^c. This is possible due to the theorem stated above, which allows us to write any tensor in a decomposition of the type (4.14) and in particular also the covariant derivative of any tensor, e.g.

$$\nabla^c_\mu f_j(r^c) = \frac{df_j(r^c)}{dr^c}\nabla^c_\mu r^c = f'_j(r^c)n_\mu. \qquad (4.17)$$

To make this reduction to ordinary differential equations we now need to know the action of second derivatives on the components $O^{(j)}_{\mu\nu;\lambda'\epsilon'}f_j(r^c)$ [2], of which there are two types

[1] Actually in the decomposition we use the dual vectors $n_\mu = g^c_{\mu\nu}n^\nu$ and $n_{\lambda'} = g^c_{\lambda'\epsilon'}n^{\epsilon'}$ of the tangent vectors.

[2] Here the index j is not subject to the Einstein sum convention as it is neither a spacetime index, nor a spatial index.

occurring in the Hessian. For one we have the d'Alembertian, for which we can use the identity

$$\Box_c \left(O^{(j)}_{\mu\nu;\lambda'\epsilon'} f_j(r^c) \right) = \left(\Box_c f_j(r^c) \right) O^{(j)}_{\mu\nu;\lambda'\epsilon'} + f_j(r^c) \left(\Box_c O^{(j)}_{\mu\nu;\lambda'\epsilon'} \right), \tag{4.18}$$

in combination with

$$\Box_c f_j(r^c) = f_j''(r^c) + 3A(r^c) f_j'(r^c), \tag{4.19}$$

and the results for $\Box_c O^{(j)}_{\mu\nu;\lambda'\epsilon'}$ stated in Table D.2 [39]. The d'Alembertian of the $O^{(j)}_{\mu\nu;\lambda'\epsilon'}$ can again be expressed in terms of the $O^{(j)}_{\mu\nu;\lambda'\epsilon'}$, since we do not introduce any additional indices. On the other hand, the second type of derivatives that occur are the non-minimal derivatives

$$\begin{aligned}
\nabla^\rho_c \nabla^\sigma_c \left(O^{(j)}_{\mu\nu;\lambda'\epsilon'} f_j(r^c) \right) \overset{(4.17)}{=} & \; f_j(r^c) \nabla^\rho_c \nabla^\sigma_c O^{(j)}_{\mu\nu;\lambda'\epsilon'} + n^\rho n^\sigma O^{(j)}_{\mu\nu;\lambda'\epsilon'} f_j''(r^c) \\
& + f_j'(r^c) \Big(n^\rho \nabla^\sigma_c O^{(j)}_{\mu\nu;\lambda'\epsilon'} + n^\sigma \nabla^\rho_c O^{(j)}_{\mu\nu;\lambda'\epsilon'} \\
& + A(r^c) \left(\hat{g}^{\rho\sigma}_c - n^\rho n^\sigma \right) O^{(j)}_{\mu\nu;\lambda'\epsilon'} \Big).
\end{aligned} \tag{4.20}$$

Therefore, if we want to discuss the action of the frame-extended Hessian (4.13), we also need to know the explicit expressions for $\nabla^\rho_c \nabla^\sigma_c O^{(j)}_{\mu\nu;\lambda'\epsilon'}$ and $\nabla^\rho_c O^{(j)}_{\mu\nu;\lambda'\epsilon'}$. These are provided in Table D.2 for a Minkowski spacetime.

So far the discussion of the decomposition in the Allen-Jacobson basis works for any maximally symmetric spacetime, however for the Minkowski spacetime under consideration here, we can again make the substitutions discussed in the beginning of Sec. 4 in combination with setting $A(r^c)$ to $\frac{1}{r^c}$, see Table D.1 and Ref. [33].

With this information we can now compute the action of the Hessian (4.12) as well as (4.13) on the basis decomposition (4.14) and reduce everything to a set of coupled ordinary differential equations. We will discuss this approach first in the context of the frame-extended scheme.

4.1.2. Frame-extended scheme

We shall now discuss the action of the frame-extended Hessian (4.13) on the graviton propagator with the ansatz (4.14), using the properties of the Allen-Jacobson basis discussed above and in App. D.2. The action of the minimal-derivative contribution, which is proportional to the Minkowski d'Alembertian $\Box_{\mathrm{M}}$, is discussed separately of the non-minimal-derivative part, as we can reuse the results also for the frame-adapted approach.

For the minimal-derivative contribution we can use the fact that the tensor structure in front of the d'Alembertian just depends on the Minkowski metric, see (4.8) and (4.13), and therefore we can contract it with $G_{\mu\nu;\lambda'\epsilon'}$, before applying the d'Alembertian. Using the results in Table D.2 we thus obtain

$$\begin{aligned}\frac{1}{2}\left(\mathcal{A}_{\alpha\beta}^{1,\mathrm{M}\mu\nu}-\eta_{\alpha\beta}\eta^{\mu\nu}\right)G_{\mu\nu;\lambda'\epsilon'} \overset{(4.14)}{=} \frac{1}{2}\Big(&\left(-3a-2b-d\right)O^{(1)}+bO^{(2)}+cO^{(3)}+dO_{\mathrm{L}}^{(4)}\\ &+\left(4c-3d-e\right)O_{\mathrm{R}}^{(4)}+eO^{(5)}\Big)\,,\end{aligned} \tag{4.21}$$

where the indices of the basis objects $O^{(j)}$ are suppressed on the r.h.s., since they are always of the form $\{\alpha\beta;\lambda'\epsilon'\}$. Furthermore, also the argument r^c of the scalar coefficient functions is suppressed for brevity. Now we can apply $\Box_{\mathrm{M}}$ to (4.21) and using (4.18) as well as the expressions for $\Box_{\mathrm{M}}O^{(j)}$, listed in Table D.2, this yields

$$\begin{aligned}\Box_{\mathrm{M}}\frac{1}{2}\left(\mathcal{A}_{\alpha\beta}^{1,\mathrm{M}\mu\nu}-\eta_{\alpha\beta}\eta^{\mu\nu}\right)G_{\mu\nu;\lambda'\epsilon'} = \frac{1}{2}\Big(&O^{(1)}\left(-3\Box_{\mathrm{M}}a-2\Box_{\mathrm{M}}b-\left(\Box_{\mathrm{M}}+4r_c^{-2}\right)d\right.\\ &\left.+8r_c^{-2}c-2r_c^{-2}e\right)+O^{(2)}\left(\Box_{\mathrm{M}}b-4r_c^{-2}c\right)\\ &+O^{(3)}\left(\left(\Box_{\mathrm{M}}-8r_c^{-2}\right)c-2r_c^{-2}e\right)\\ &+O_{\mathrm{L}}^{(4)}\left(\left(\Box_{\mathrm{M}}-8r_c^{-2}\right)d+2r_c^{-2}e\right)\\ &+O_{\mathrm{R}}^{(4)}\left(4\left(\Box_{\mathrm{M}}-8r_c^{-2}\right)c-3\left(\Box_{\mathrm{M}}-8r_c^{-2}\right)d\right.\\ &\left.-\left(\Box_{\mathrm{M}}-10(r^c)^{-2}\right)e\right)+O^{(5)}\left(\Box_{\mathrm{M}}-24r_c^{-2}\right)e\Big)\,.\end{aligned} \tag{4.22}$$

The non-minimal contribution of the Hessian (4.13) stems from the auxiliary objects $\mathcal{A}_{\alpha\beta}^{2,\mathrm{M}\mu\nu}$ and $\mathcal{A}_{\alpha\beta}^{3,\mathrm{M}\mu\nu}$. Starting from (4.9) and (4.10), one can show that these expressions can be written proportional to the non-minimal second (covariant) derivative $\partial_\rho\partial_\sigma$:

$$\mathcal{A}_{2,\mathrm{M}}^{\alpha\beta\mu\nu} = \eta^{\rho(\alpha}\eta^{\beta)(\mu}\eta^{\nu)\sigma}\partial_\rho\partial_\sigma \tag{4.23}$$

$$\mathcal{A}_{3,\mathrm{M}}^{\alpha\beta\mu\nu} = \frac{1}{2}\left(\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\nu}\eta^{\alpha\rho}\eta^{\beta\sigma}\right)\partial_\rho\partial_\sigma. \tag{4.24}$$

Therefore, these two contributions can be easily evaluated as soon as we know the second non-minimal derivative of the ansatz, $\partial_\rho\partial_\sigma G_{\mu\nu;\lambda'\epsilon'}$. This can again be evaluated with the relations in Table D.2:

$$
\begin{aligned}
\partial_\rho\partial_\sigma G_{\mu\nu;\lambda'\epsilon'} = &\left(r_c^{-1}a'\eta_{\rho\sigma} + \left(a'' - r_c^{-1}a'\right) n_\rho n_\sigma\right) O^{(1)}_{\mu\nu;\lambda'\epsilon'} + 2r_c^{-2}d\ \eta_{\mu(\rho}O^{(1)}_{\sigma)\nu;\lambda'\epsilon'} \\
&+ \left(r_c^{-1}b'\eta_{\rho\sigma} + \left(b'' - r_c^{-1}b'\right) n_\rho n_\sigma\right) O^{(2)}_{\mu\nu;\lambda'\epsilon'} + r_c^{-2}\left(d\ \eta_{\mu\nu} + e\ n_\mu n_\nu\right) O^{(2)}_{\rho\sigma;\lambda'\epsilon'} \\
&+ r_c^{-1}\left(-4c' + 4r_c^{-1}c\right) n_{(\rho|}n_{(\mu}O^{(2)}_{\nu)|\sigma);\lambda'\epsilon'} \\
&+ \left(r_c^{-1}\left(c' - 2r_c^{-1}c\right)\eta_{\rho\sigma} + \left(c'' - 5r_c^{-1}c' + 8r_c^{-2}c\right) n_\rho n_\sigma\right) O^{(3)}_{\mu\nu;\lambda'\epsilon'} \\
&+ r_c^{-1}\left(\left(-d' + 2r_c^{-1}d\right)\eta_{\mu\nu} + \left(-e' + 4r_c^{-1}e\right) n_\mu n_\nu\right) O^{(3)}_{\rho\sigma;\lambda'\epsilon'} \\
&+ r_c^{-1}\left(d' - 2r_c^{-1}d\right)\left(\eta_{\rho\sigma}O^{(4)}_{\rho\sigma;\lambda'\epsilon'} + 4\eta_{(\mu|(\rho}O^{(4)}_{\mathrm{L}\ \rho)|\sigma);\lambda'\epsilon'}\right) \\
&+ \left(d'' - 5r_c^{-1}d' + 8r_c^{-2}d\right) n_\rho n_\sigma O^{(4)}_{\mathrm{R}\ \mu\nu;\lambda'\epsilon'} + 4r_c^{-2}e\ \eta_{(\mu|(\rho}O^{(4)}_{\mathrm{R}\ \rho)|\sigma);\lambda'\epsilon'} \\
&+ \left(\left(d'' - 5r_c^{-1}d' + 8r_c^{-2}d + e'' - 9r_c^{-1}e' + 24r_c^{-2}e\right) n_\rho n_\sigma\right. \\
&\left.+ r_c^{-1}\left(e' - 4r_c^{-1}e\right)\eta_{\rho\sigma}\right) O^{(5)}_{\mu\nu;\lambda'\epsilon'} + 4r_c^{-1}\left(e' - 4r_c^{-1}e\right)\eta_{(\mu|(\rho}O^{(5)}_{\rho)|\sigma);\lambda'\epsilon'} \\
&+ 8r_c^{-1}\left(c' - 2r_c^{-1}c\right) n_{(\rho}\eta_{\sigma)(\mu}g_{\nu)(\lambda'}n_{\epsilon')} - 8r_c^{-2}e\ n_{(\mu}\eta_{\nu)(\rho}g_{\sigma)(\lambda'}n_{\epsilon')}
\end{aligned}
\tag{4.25}
$$

We can now use the expression for $\partial_\rho\partial_\sigma G_{\mu\nu;\lambda'\epsilon'}$ to evaluate (4.23) and (4.24) and with this the non-minimal contribution to the Hessian (4.13) becomes

$$
\begin{aligned}
&\eta_{\alpha(\rho}\eta_{\sigma)\beta}\left(\left(\frac{1}{2\zeta} - 1\right)\eta^{\omega(\rho}\eta^{\sigma)(\mu}\eta^{\nu)\tau} + \frac{1}{2}\left(\eta^{\rho\sigma}\eta^{\mu\omega}\eta^{\nu\tau} + \eta^{\mu\nu}\eta^{\rho\omega}\eta^{\sigma\tau}\right)\right)\partial_\omega\partial_\tau G_{\mu\nu;\lambda'\epsilon'} = \\
&= \frac{1}{2}\Big(O^{(1)}\left(a'' + 5r_c^{-1}a' + 4r_c^{-1}b' - 4r_c^{-1}c' - 8r_c^{-2}c + d'' + 5r_c^{-1}d' + 2r_c^{-2}d\right. \\
&\quad \left.+\zeta^{-1}\left(2r_c^{-1}a' + 2r_c^{-1}d' - 4r_c^{-2}c - 6r_c^{-2}d\right)\right) \\
&\quad +O^{(2)}\left(-2r_c^{-1}b' + 2r_c^{-1}c' + 4r_c^{-2}c + 2r_c^{-2}d + r_c^{-2}e\right. \\
&\quad \left.+\zeta^{-1}\left(2r_c^{-1}b' - 2r_c^{-1}c' - 8r_c^{-2}c + 2r_c^{-2}d\right)\right) \\
&\quad +O^{(3)}\left(b'' - r_c^{-1}b' - c'' - r_c^{-1}c' + 4r_c^{-2}c - 2r_c^{-1}d' + 4r_c^{-2}d + 5r_c^{-2}e\right. \\
&\quad \left.+\zeta^{-1}\left(-b'' + r_c^{-1}b' + c'' + 5r_c^{-1}c' - 12r_c^{-2}c - 2r_c^{-1}d' + 4r_c^{-2}d - r_c^{-1}e' - 3r_c^{-2}e\right)\right) \\
&\quad +O^{(4)}\left(2b'' - 2r_c^{-1}b' + \zeta^{-1}\left(-4r_c^{-1}c' + 8r_c^{-2}c + 2r_c^{-1}d' - 4r_c^{-2}d + 2r_c^{-1}e' + 6r_c^{-2}e\right)\right) \\
&\quad +O^{(4)}_{\mathrm{L}}\left(2a'' - 2r_c^{-1}a' + 4r_c^{-1}c' - 8r_c^{-2}c - d'' - 5r_c^{-1}d' + 12r_c^{-2}d\right) \\
&\quad +O^{(4)}_{\mathrm{R}}\left(-4c'' - 8r_c^{-1}c' + 24r_c^{-2}c + d'' + 5r_c^{-1}d' - 12r_c^{-2}d + e'' + 5r_c^{-1}e' - 2r_c^{-2}e\right) \\
&\quad +O^{(5)}\left(2d'' - 10r_c^{-1}d' + 16r_c^{-2}d - e'' - 5r_c^{-1}e' + 32r_c^{-2}e\right. \\
&\quad \left.+\zeta^{-1}\left(-4c'' + 20r_c^{-1}c' - 32r_c^{-2}c + 2d'' - 10r_c^{-1}d' + 16r_c^{-2}d + 2e'' - 24r_c^{-2}e\right)\right)\Big)\,,
\end{aligned}
\tag{4.26}
$$

where we have again suppressed the indices on the basis tensors $O^{(j)}$.

The intermediate expressions (4.22) and (4.26), together with (4.11) and (4.16), then combine to the final version for the Green's function equation in the frame-extended approach. As can be seen from (4.22) and (4.26), we can separate the contributions to each tensor structure in the Allen-Jacobson basis and the prefactors to each tensor structure provides us with the l.h.s. of a scalar equation. Since the r.h.s. of (4.11) only has a contribution proportional to $O^{(2)}$, we can set the prefactor to every $O^{(j)}$ to zero, except for the prefactor of $O^{(2)}$, which is set equal to

$$\frac{1}{2}\delta^{(4)}\left(x_{\mathscr{P}}-x_{\mathscr{Q}}\right).$$

Therefore, we end up with six coupled ordinary differential equations in the scalar expansion coefficients $\{a(r^c), b(r^c), c(r^c), d(r^c), e(r^c)\}$. If we collect the latter in the vector $\vec{f} = \{a(r^c), b(r^c), c(r^c), d(r^c), e(r^c)\}$ and define $\vec{f}'$ and $\vec{f}''$ to contain the first and second derivatives of the scalar coefficients w.r.t. r^c, we can write the system of equations in matrix-vector form:

$$\frac{1}{4\kappa^2}\left(M_2\vec{f}'' + r_c^{-1}M_1\vec{f}' + r_c^{-2}M_0\vec{f}\right) = \vec{r}, \tag{4.27}$$

$$\text{with } \vec{r} = \{0, \frac{1}{2}\delta^{(4)}\left(x_{\mathscr{P}}-x_{\mathscr{Q}}\right), 0, 0, 0\}. \tag{4.28}$$

The coefficient matrices are of the form

$$M_2 = \begin{bmatrix} -2 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1-\zeta^{-1} & \zeta^{-1} & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & -4\zeta^{-1} & 2(1+\zeta^{-1}) & 2\zeta^{-1} \end{bmatrix}, \tag{4.29}$$

$$M_1 = \begin{bmatrix} 2(-2+\zeta^{-1}) & -2 & -4 & 2(1+\zeta^{-1}) & 0 \\ 0 & 1+2\zeta^{-1} & 2(1-\zeta^{-1}) & 0 & 0 \\ 0 & -(1-\zeta^{-1}) & 2+5\zeta^{-1} & -2(1+\zeta^{-1}) & 0 \\ -2 & -2 & 4(1-\zeta^{-1}) & 2(-1+\zeta^{-1}) & 2\zeta^{-1} \\ 0 & -2 & 4(1-\zeta^{-1}) & 2(-2+\zeta^{-1}) & 2(1+\zeta^{-1}) \\ 0 & 0 & 20\zeta^{-1} & -10(1+\zeta^{-1}) & -2 \end{bmatrix} \text{ and} \tag{4.30}$$

$$M_0 = \begin{bmatrix} 0 & 0 & -4\zeta^{-1} & -(2+6\zeta^{-1}) & -2 \\ 0 & 0 & -8\zeta^{-1} & 2(1+\zeta^{-1}) & 1 \\ 0 & 0 & -4(1+3\zeta^{-1}) & 4(1+\zeta^{-1}) & 3(1-\zeta^{-1}) \\ 0 & 0 & 8(-1+\zeta^{-1}) & 4(1-\zeta^{-1}) & 2(1+3\zeta^{-1}) \\ 0 & 0 & 8(-1+\zeta^{-1}) & 4(3-\zeta^{-1}) & 2(4+3\zeta^{-1}) \\ 0 & 0 & -32\zeta^{-1} & 16(1+\zeta^{-1}) & 8(1-3\zeta^{-1}) \end{bmatrix}. \tag{4.31}$$

Unfortunately this system of equations cannot be solved, unless additional auxiliary conditions can be found that would allow to decouple the equations. Such a system of equations is studied in the simpler case of the photon propagator in Ref. [33] and an auxiliary condition is found through studying the two-point correlation function of the electromagnetic field-strength tensor. In principle we could do something similar here as well with the Riemann-tensor, however, the resulting correlation function turns out to be very complicated and no useful condition has been obtained so far. Furthermore, a solution for the GFE was obtained in the frame-adapted scheme, thus making it unnecessary to further consider the case of the frame-extended Hessian. The former will be discussed in the section below.

4.1.3. Frame-adapted scheme

The methods employed in this section are motivated by the problems discussed in Ref. [40, 41, 42].

In Sec. 3.2.3 we discussed, how the Hessian can be simplified in the frame-adapted scheme, and the Minkowski-specific form of the frame-adapted Hessian was shown to be (4.12). This term is the same as the minimal-derivative term (4.22) discussed in the previous section. Thus we already know the l.h.s. of the Green's function equation (4.11) in the frame-adapted approach. However, if we now just adopt the r.h.s. of the GFE (4.11) in this approach, we will run into inconsistencies. The reason for this is that in the frame-fixed picture, with the δ-function kept explicit in the PI, the fluctuation field $\gamma_{\mu\nu}$ has the property (3.24), which naturally leads to a propagator satisfying the corresponding condition

$$\hat{g}_c^{\mu\nu}\partial_\mu\langle\gamma_{\nu\alpha}\gamma_{\lambda'\epsilon'}\rangle = \left(\hat{g}_c^{\mu\rho}\hat{g}_c^{\nu\sigma}\partial_\mu g^c_{\nu\alpha}\right)\langle\gamma_{\rho\sigma}\gamma_{\lambda'\epsilon'}\rangle + O\left(\gamma^3\right), \tag{4.32}$$

on each index [43, 41]. With the ansatz (3.104) this is also valid for $G_{\mu\nu;\lambda'\epsilon'}$ and in the case of $g^c_{\mu\nu}$ corresponding to the constant Minkowski metric $\eta_{\mu\nu}$, this translates into the transversality condition

$$\partial^\mu G_{\mu\nu;\lambda'\epsilon'} = \partial^\nu G_{\mu\nu;\lambda'\epsilon'} = \partial^{\lambda'} G_{\mu\nu;\lambda'\epsilon'} = \partial^{\epsilon'} G_{\mu\nu;\lambda'\epsilon'} = 0 + O\left(\gamma^3\right), \tag{4.33}$$

where $\eta_{\mu\nu}$ is again defined to be the index shifter. If we now invoke this property with one of the open indices of $G_{\mu\nu;\lambda'\epsilon'}$ on the l.h.s. of the GFE, this yields

$$\frac{1}{2\kappa^2}\partial^{\lambda'}\mathcal{D}^{\mathrm{M,\ f.a.}\,\mu\nu}_{\alpha\beta}G_{\mu\nu;\lambda'\epsilon'} = \frac{1}{2\kappa^2}\mathcal{D}^{\mathrm{M,\ f.a.}\,\mu\nu}_{\alpha\beta}\partial^{\lambda'}G_{\mu\nu;\lambda'\epsilon'} = 0 + O\left(\gamma^3\right), \tag{4.34}$$

where we have used, that the partial derivative $\partial^{\lambda'}$ commutes with the Hessian. Whereas, applied to the r.h.s. of the unmodified Green's function equation (4.11) the partial derivative results in

$$\frac{1}{2}O^{(2)}_{\alpha\beta;\lambda'\epsilon'}\partial^{\lambda'}\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) \neq 0. \tag{4.35}$$

So as to resolve this inconsistency, the r.h.s. of the Green's function equation (4.11) has to be modified such that it complies with the transversality property (4.33) of the propagator. Based on Ref. [42] and [41] which discuss a similar problem for the transverse photon propagator and the graviton propagator under use of the de Donder condition, the following ansatz for the modified r.h.s. of the GFE is proposed

$$\begin{aligned}\text{r.h.s.} = {} & O^{(2)}_{\alpha\beta;\lambda'\epsilon'}\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) + \left[A_1 g_{(\alpha|\epsilon'}\partial_{|\beta)}\partial_{\lambda'} + A_2 g_{(\alpha|\lambda'}\partial_{|\beta)}\partial_{\epsilon'} + A_3\eta_{\lambda'\epsilon'}\partial_\alpha\partial_\beta\right]\phi\left(r^c\right) \\ & + A_4\partial_\alpha\partial_\beta\partial_{\lambda'}\partial_{\epsilon'}\xi\left(r^c\right),\end{aligned} \tag{4.36}$$

where the A_j are as of now undetermined constant coefficients, and $\phi\left(r^c\right)$ and $\xi\left(r^c\right)$ are arbitrary functions of the geodesic distance. Formally this corresponds to the basis tensor of the Allen-Jacobson basis, where the geodesic tangents are replaced by covariant derivatives. Acting the derivative $\partial^{\lambda'}$ on this ansatz and using (4.17) results in[3]

$$\begin{aligned}\partial^{\lambda'}\left(\text{r.h.s.}\right) = \left(-n_\alpha g_{\beta\epsilon'} - n_\beta g_{\alpha\epsilon'}\right)\frac{d\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right)}{dr^c} + A_1 g_{(\alpha|\epsilon'}\partial_{|\beta)}\Box_{\mathrm{M}}\phi\left(r^c\right)\\ - A_2\partial_{(\alpha}\partial_{\beta)}\partial_{\epsilon'}\phi\left(r^c\right) + A_3\partial_{\epsilon'}\partial_{(\alpha}\partial_{\beta)}\phi\left(r^c\right) + A_4\partial_{\epsilon'}\partial_{(\alpha}\partial_{\beta)}\Box_{\mathrm{M}}\xi\left(r^c\right). \qquad (4.37)\end{aligned}$$

If we now choose the functions $\phi\left(r^c\right)$ and $\xi\left(r^c\right)$ such that the scalar equations

$$\Box_{\mathrm{M}}\phi\left(r^c\right) = \delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) \text{ and} \qquad (4.38)$$

$$\Box_{\mathrm{M}}\xi\left(r^c\right) = \phi\left(r^c\right) \qquad (4.39)$$

are satisfied, then the derivative of the r.h.s. reduces to

$$\begin{aligned}\partial^{\lambda'}\left(r.h.s.\right) &= \left(\frac{A_1}{2} - 1\right)\left(n_\alpha g_{\beta\epsilon'} + n_\beta g_{\alpha\epsilon'}\right)\frac{d\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right)}{dr^c}\\ &\quad + \partial_\alpha\partial_\beta\partial_{\epsilon'}\left(-A_2\phi(r^c) + A_3\phi(r^c) + A_4\Box_{\mathrm{M}}\xi(r^c)\right)\\ &= \left(\frac{A_1}{2} - 1\right)\left(n_\alpha g_{\beta\epsilon'} + n_\beta g_{\alpha\epsilon'}\right)\frac{d\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right)}{dr^c}\\ &\quad + \left(-A_2 + A_3 + A_4\right)\partial_\alpha\partial_\beta\partial_{\epsilon'}\phi(r^c). \qquad (4.40)\end{aligned}$$

For this expression to be equal to zero, the unknown coefficients A_j need to fulfill the conditions

$$\frac{A_1}{2} - 1 = 0 \text{ and } -A_2 + A_3 + A_4 = 0. \qquad (4.41)$$

If we now perform the same calculation with $\partial^{\epsilon'}$ instead of $\partial^{\lambda'}$, the two additional conditions

$$\frac{A_2}{2} - 1 = 0 \text{ and } -A_1 + A_3 + A_4 = 0, \qquad (4.42)$$

need to hold for the resulting expression to vanish. Thus we can determine the four unknown coefficients:

$$A_1 = A_2 = 2 \text{ and } A_3 = A_4 = \frac{A_1}{2}. \qquad (4.43)$$

With this we have now a consistent expression for the r.h.s. of the Green's function equation and the full equation becomes

$$\begin{aligned}\frac{1}{\kappa^2}\mathcal{D}^{\mathrm{M,\,f.a.}\,\mu\nu}_{\alpha\beta}G_{\mu\nu;\lambda'\epsilon'} = O^{(2)}_{\alpha\beta;\lambda'\epsilon'}\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right)\\ + \left[2g_{(\alpha|\epsilon'}\partial_{|\beta)}\partial_{\lambda'} + 2g_{(\alpha|\lambda'}\partial_{|\beta)}\partial_{\epsilon'} + \eta_{\lambda'\epsilon'}\partial_\alpha\partial_\beta\right]\phi\left(r^c\right)\\ + \partial_\alpha\partial_\beta\partial_{\lambda'}\partial_{\epsilon'}\xi\left(r^c\right). \qquad (4.44)\end{aligned}$$

[3]The derivative of the δ-function w.r.t. to the geodesic distance is discussed in [36], but it is actually not necessary to have an explicit expression for it in this discussion.

The ansatz (4.36) is actually missing a factor $\frac{1}{2}$ as compared to the original r.h.s. of (4.11) and to correct for this, we multiplied the l.h.s. of the equation by two.

The auxiliary functions $\phi(r^c)$ and $\xi(r^c)$ can be calculated from the equations (4.38) and (4.39), in fact $\phi(r^c)$ is nothing but the propagator of a scalar massless field, which is given by

$$\phi(r^c) = \frac{f_1}{r_c^2}, \tag{4.45}$$

see Ref. [44]. With this result we can then also solve the equation for $\xi(r^c)$

$$\xi(r^c) = \frac{x_1}{r_c^2} + \frac{f_1}{2}\log(r^c). \tag{4.46}$$

The constants f_1 and x_1 are just numerical factors [44, 38]. We can now again use the identity (4.17), in combination with the properties of the Allen-Jacobson basis elements, to rewrite the r.h.s. of the Green's function equation in terms of the basis bitensors $O^{(j)}$ with scalar prefactors:

$$\begin{aligned} r.h.s. = {} & O^{(1)}(\psi+\chi) + O^{(2)}\left(\delta^{(4)}(x_{\mathscr{P}} - x_{\mathscr{Q}}) + \psi - 2\chi\right) \\ & + O^{(3)}(-\sigma+\omega) + O^{(4)}_{\mathrm{L}}(\sigma+\omega) + O^{(4)}_{\mathrm{R}}\sigma + O^{(5)}\tau, \end{aligned} \tag{4.47}$$

where we have suppressed the indices $\{\alpha\beta; \lambda'\epsilon'\}$ on each basis bitensor. And we have introduced the following short-hand notation

$$\begin{aligned} &\omega \equiv \phi'' - r_c^{-1}\phi', && \psi \equiv r_c^{-2}\xi'' - r_c^{-3}\xi', \\ &\chi \equiv r_c^{-1}\phi', && \sigma \equiv r_c^{-1}\xi''' - 3r_c^{-2}\xi'' + 3r_c^{-3}\xi', \\ &\tau \equiv \xi'''' - r_c^{-1}\xi''' - 5\sigma. \end{aligned} \tag{4.48}$$

Now we can combine (4.22) and (4.47) in the modified Green's function equation (4.44) and we can again separate the contribution to each basis object $O^{(j)}$ into six scalar differential equations of the form

$$\begin{aligned} &O^{(1)} : 3\Box_{\mathrm{M}}\tilde{a} + 2\Box_{\mathrm{M}}\tilde{b}\left(\Box_{\mathrm{M}} + 4(r^c)^{-2}\right)\tilde{d} - 8(r^c)^{-2}\tilde{c} + 2(r^c)^{-2}\tilde{e} = \psi + \chi \\ &O^{(2)} : -\Box_{\mathrm{M}}\tilde{b} + 4(r^c)^{-2}\tilde{c} = \delta^{(4)}(x_{\mathscr{P}} - x_{\mathscr{Q}}) + \psi - 2\chi \\ &O^{(3)} : -\left(\Box_{\mathrm{M}} - 8(r^c)^{-2}\right)\tilde{c} + 2(r^c)^{-2}\tilde{e} = -\sigma + \omega \\ &O^{(4)}_{\mathrm{L}} : -\left(\Box_{\mathrm{M}} - 8(r^c)^{-2}\right)\tilde{d} - 2(r^c)^{-2}\tilde{e} = \sigma + \omega \\ &O^{(4)}_{\mathrm{R}} : -4\left(\Box_{\mathrm{M}} - 8(r^c)^{-2}\right)\tilde{c} + 3\left(\Box_{\mathrm{M}} - 8(r^c)^{-2}\right)\tilde{d} + \left(\Box_{\mathrm{M}} - 10(r^c)^{-2}\right)\tilde{e} = \sigma \\ &O^{(5)} : -\left(\Box_{\mathrm{M}} - 24(r^c)^{-2}\right)\tilde{e} = \tau, \end{aligned} \tag{4.49}$$

where the scalar functions with a tilde are defined via the original functions as follows

$$\tilde{f}_j = -\frac{1}{2\kappa^2} f_j. \tag{4.50}$$

This system of ordinary differential equations is in fact soluble and the solutions to the five scalar coefficient functions $\{a(r^c), b(r^c), c(r^c), d(r^c), e(r^c)\}$ are provided in App. D.3. However, we will see in the next section that we do not actually need the solutions, but the system of equations (4.49) and the knowledge that it has a solution is enough to determine the geon propagator in the FMS expansion.

4.2. Minkowski space geon propagator - frame-adapted scheme

We can now express the general result for $D^{(2)}\left(r^c\left(\mathscr{P},\mathscr{Q}\right)\right)$, see (3.48), in the Minkowski specific case with partial derivatives and $\Lambda = 0$. Due to the fact that we are evaluating the graviton propagator in the frame-adapted scheme, (4.32) and (4.33) are satisfied and the corresponding terms in the geon propagator vanish, see below

$$\begin{aligned}
D^{(2)}_{\text{tl}}\left(r^c\left(\mathscr{P},\mathscr{Q}\right)\right) &= \partial^\mu\partial^\nu\partial^{\rho'}\partial^{\sigma'}\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle_{\text{tl}} \\
&\quad - \left(\eta^{\rho'\sigma'}\Box'_M\partial^\mu\partial^\nu + \eta^{\mu\nu}\Box_M\partial^{\rho'}\partial^{\sigma'}\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle_{\text{tl}} \\
&\quad + \Box_M\Box'_M\eta^{\mu\nu}\eta^{\lambda'\epsilon'}\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle_{\text{tl}} \\
&\overset{(4.33)}{=} \Box_M\Box'_M\eta^{\mu\nu}\eta^{\lambda'\epsilon'}\langle\gamma_{\mu\nu}\gamma_{\lambda'\epsilon'}\rangle_{\text{tl}} \\
&\overset{(3.104)}{=} i\Box'_M\Box_M G^\alpha{}_{\alpha;}{}^{\lambda'}{}_{\lambda'} \\
&\overset{(4.14)}{=} i\Box'_M\Box_M\left(16a + 8b - 4c + 8d + 4e\right) \\
&\overset{(4.50)}{=} -2i\kappa^2\Box'_M\Box_M\left(16\tilde{a} + 8\tilde{b} - 4\tilde{c} + 8\tilde{d} + 4\tilde{e}\right),
\end{aligned} \tag{4.51}$$

where we used the trace properties of the Allen-Jacobson basis, which are listed in Table D.2. Rearranging the six equations from the system of ODE's (4.49) to

$$\Box_M\tilde{a} = \frac{1}{3}\left(-2\Box_M\tilde{b} - \Box_M\tilde{d} + 8r_c^{-2}\tilde{c} - 4r_c^{-2}\tilde{d} - 2r_c^{-2}\tilde{e} + \psi + \chi\right) \tag{4.52}$$

$$\Box_M\tilde{b} = 4r_c^{-2}\tilde{c} - \delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) - \psi + 2\chi \tag{4.53}$$

$$\Box_M\tilde{c} = 8r_c^{-2}\tilde{c} + 2r_c^{-2}\tilde{e} + \sigma - \omega \tag{4.54}$$

$$\Box_M\tilde{d} = 8r_c^{-2}\tilde{d} - 2r_c^{-2}\tilde{e} - \sigma - \omega \tag{4.55}$$

$$\Box_M\tilde{e} = 24r_c^{-2}\tilde{e} - \tau, \tag{4.56}$$

and inserting (4.53) and (4.55) into (4.52) allows us to rewrite the action of the first d'Alembertian on $G^\alpha{}_{\alpha;}{}^{\lambda'}{}_{\lambda'}$, in (4.51), as

$$\Box_M G^\alpha{}_{\alpha;}{}^{\lambda'}{}_{\lambda'} = -\frac{2\kappa^2}{3}\left(8\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) + 4\omega - 20\sigma - 3\tau + 24\psi\right). \tag{4.57}$$

As can be seen, all scalar coefficient functions of $G_{\mu\nu;\lambda'\epsilon'}$ cancel and only the auxiliary functions (4.48) remain. Thus we do not actually need to know the explicit solution for scalar coefficient functions as was already mentioned in the previous section. Furthermore, the remaining expression in terms of $\omega, \sigma, \tau, \psi$ satisfies the equation

$$4\omega - 20\sigma - 3\tau + 24\psi = \delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right), \tag{4.58}$$

as is shown in App. D.4. Consequently, the only term that remains in the expression of the geon propagator is

$$\begin{aligned} D^{(2)}_{\mathrm{tl}}\left(r^c\left(\mathscr{P},\mathscr{Q}\right)\right) &= -2i\kappa^2\Box'_M\Box_M G^{\alpha}{}_{\alpha;}{}^{\lambda'}{}_{\lambda'} \\ &= -\Box'_M\frac{2}{3}i\kappa^2\left(8\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) + 4\omega - 20\sigma - 3\tau + 24\psi\right) \\ &\overset{(4.58)}{=} -\Box'_M\frac{2}{3}i\kappa^2\left(9\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right)\right) \\ &= -i6\kappa^2\Box'_M\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right). \end{aligned} \tag{4.59}$$

With this result we can finally provide an expression for the geon propagator in Minkowski space, to second order in the FMS expansion and with the tree-level approximation for the graviton propagator[4],

$$\left\langle \mathcal{R}\left(\mathscr{P}\right)\mathcal{R}\left(\mathscr{Q}\right)\right\rangle_{\text{connected,tl}} \overset{(3.10)}{=} -i6\kappa^2\Box'_M\delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) + O\left(\gamma^3\right). \tag{4.60}$$

The leading order result for the geon propagator is merely proportional to a derivative factor of the Dirac δ-function which vanishes away form coincidence. Hence, for our purposes the leading order contribution to the geon propagator can be assumed to vanish [45] and therefore the geon as a composite particle does not exist in the approximations that we considered.

As was pointed out in Sec.3.2.1, for the cartesian Minkowski chart we could also use the de Donder condition instead of the Haywood condition for the frame-fixing procedure. This was not done in position space here, but for a momentum space calculation of the quantities discussed above, see Ref. [38] and [46]. Similar position space calculations for maximally symmetric spaces were also conducted in Ref. [31] and [40], again with the de Donder condition, and the vanishing of the correlation function (4.60) to second order is also seen. However, it must be noted that the calculations in Ref. [38, 46, 31, 40] are based on a very different approach to what is pursued in this work, i.e. the combination of a particular frame-fixing with the metric split and the FMS mechanism. It is only the simplicity of Minkowski space that causes the expressions that occur in all cases to be very similar and thus allows for a comparison of the results. This will not work in the case of de Sitter space for instance.

[4]The subscript **tl** on the geon propagator is merely there to indicate that the graviton propagator, which served as an auxiliary quantity in our calculation, was only evaluated in the tree-level approximation.

5. Calculations with de Sitter space

We have now discussed the simplest case for the choice of the VEV, i.e. Minkowski spacetime, and the geon resulting from the respective calculation turned out to be non-existent in the considered approximations. Now we want to discuss the second, more complicated choice for the VEV that was proposed in Sec. 3, namely de Sitter space. As it will turn out the de Sitter space calculation comes with several complications which will prevent us in the end from providing a final result either for the graviton or the geon in this work.

For the frame-fixing procedure with the Haywood condition the cartesian Robertson-Walker coordinate chart (3.17) was selected as a suitable choice and this is what we are going to use for $g^c_{\mu\nu}$ in this chapter,

$$\left[g^c_{\mu\nu}\right] = \mathrm{diag}\left(-1, e^{2Ht}, e^{2Ht}, e^{2Ht}\right). \tag{5.1}$$

The difference to Minkowski space now arises in the fact that the manifold associated with this metric is not Ricci-flat anymore, since the cosmological constant is chosen as $\Lambda \geq 0$. Therefore, the metric components depend on the coordinates (in this case the spatial components of the metric depend on the coordinate time) and the Christoffel symbols $\Gamma^{\rho}_{c\,\mu\nu}$ do not vanish. Consequently, also the covariant derivatives do not reduce to partial derivatives anymore and thus do not commute. With this in mind we can now look at the Green's function equation for the graviton propagator and discuss how to solve it. While we have considered both, the frame-extended and frame-adapted scheme for the Minkowski space case, we will only consider the latter scheme here, since we have not yet been able to solve the frame-extended GFE in the much simpler framework of Minkowski space and it will become much more involved in the case of de Sitter space.

5.1. de Sitter space graviton propagator

Neither the Christoffel symbols nor the cosmological constant are vanishing in the de Sitter space case and therefore we can directly adopt the Green's function equation (3.106) without any formal changes to the involved expressions. Now for the frame-adapted scheme we can consider the respective changes to the Hessian and again the auxiliary objects $\mathcal{J}_j^{\alpha\beta\mu\nu}$ (see (3.78) - (3.81)) as well as $\mathcal{T}^{\rho\sigma}{}_{\nu}$ (see (3.77)) do not vanish in the de Sitter case and we can use the general expressions. The only simplification that we can make, is that we can represent all the $\mathcal{J}_j^{(\alpha\beta)(\mu\nu)}$ and $\hat{g}^{\mu\nu}_c \nabla^{\rho}_c \Gamma^{c\,(\alpha\beta)}_{\ \rho}$ in terms of the metric with the decomposition

$$\frac{1}{2}\left(\mathcal{J}_j^{(\alpha\beta)(\mu\nu)}+\mathcal{J}_j^{(\mu\nu)(\alpha\beta)}\right)=A_j\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}+B_j\mathcal{A}_1^{\alpha\beta\mu\nu}\text{and} \tag{5.2}$$

$$\frac{1}{2}\left(\hat{g}_c^{\mu\nu}\nabla_c^\rho\Gamma_\rho^{c(\alpha\beta)}+\hat{g}_c^{\alpha\beta}\nabla_c^\rho\Gamma_\rho^{c(\mu\nu)}\right)=A_5\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}+B_5\mathcal{A}_1^{\alpha\beta\mu\nu}. \tag{5.3}$$

Therefore (3.82) and (3.83) can be written in a much more compact way, as

$$\mathcal{A}_{2,\text{f.a.,dS}}^{\alpha\beta\mu\nu}=\left(A_1-\frac{9}{48}\Lambda\right)\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}+\left(\frac{3}{4}\Lambda+B_1\right)\mathcal{A}_1^{\alpha\beta\mu\nu} \tag{5.4}$$

$$\mathcal{A}_{3,\text{f.a.,dS}}^{\alpha\beta\mu\nu}=\left(A_3-\frac{A_5}{2}\right)\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}+\left(B_3-\frac{B_5}{2}\right)\mathcal{A}_1^{\alpha\beta\mu\nu}+\frac{1}{2}\left(\hat{g}_c^{\alpha\beta}\Gamma_\rho^{c(\mu\nu)}\nabla_c^\rho-\hat{g}_c^{\mu\nu}\Gamma_\rho^{c(\alpha\beta)}\nabla_c^\rho\right). \tag{5.5}$$

Since we are not actually going to use them in this work, we do not provide the numerical values of the coefficients A_j and B_j. We can use this to provide the frame-adapted Hessian (3.84) in its simplified form

$$\begin{aligned}\mathcal{D}_{\text{f.a.,dS}}^{\alpha\beta\mu\nu}\equiv{}&\frac{1}{2}\left(\mathcal{A}_1^{\alpha\beta\mu\nu}-\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Box_c+\frac{1}{2}\left(\hat{g}_c^{\alpha\beta}\Gamma_\rho^{c(\mu\nu)}\nabla_c^\rho-\hat{g}_c^{\mu\nu}\Gamma_\rho^{c(\alpha\beta)}\nabla_c^\rho\right)\\&+\left(A_3-A_1-\frac{A_5}{2}-\frac{15}{48}\Lambda\right)\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}+\left(B_3-B_1-\frac{B_5}{2}+\frac{\Lambda}{2}\right)\mathcal{A}_1^{\alpha\beta\mu\nu}.\end{aligned} \tag{5.6}$$

At this point we run into a complication, namely in the derivation of the Green's function equation (3.106) we used the symmetrized Hessian, which satisfies the symmetry $\mathcal{D}^{\alpha\beta\mu\nu}=\mathcal{D}^{\mu\nu\alpha\beta}$, however the first-derivative term in the frame-adapted Hessian above does not satisfy this symmetry property. Therefore we have to replace the result of (3.100) with

$$\left(\mathcal{D}_{\text{f.a.,dS}}^{\alpha\beta\mu\nu}\left(x_{\mathscr{P}}\right)+\mathcal{D}_{\text{f.a.,dS}}^{\mu\nu\alpha\beta}\left(x_{\mathscr{P}}\right)\right)\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right) \tag{5.7}$$

So as to be able to use the formulas derived form (3.100), we define a new frame-adapted Hessian (it is denoted just with a subscript **dS** and we are skipping the subscript **f.a.**) based on the sum of the two

$$\begin{aligned}\mathcal{D}_{\text{dS}}^{\alpha\beta\mu\nu}\equiv{}&\frac{1}{2}\left(\mathcal{A}_1^{\alpha\beta\mu\nu}-\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Box_c\\&+\frac{1}{4}\left(\hat{g}_c^{\alpha\beta}\Gamma_\rho^{c(\mu\nu)}+\hat{g}_c^{\mu\nu}\Gamma_\rho^{c(\alpha\beta)}-\hat{g}_c^{\mu\nu}\Gamma_\rho^{c(\alpha\beta)}-\hat{g}_c^{\alpha\beta}\Gamma_\rho^{c(\mu\nu)}\right)\nabla_c^\rho\\&+\left(A_3-A_1-\frac{A_5}{2}-\frac{15}{48}\Lambda\right)\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}+\left(B_3-B_1-\frac{B_5}{2}+\frac{\Lambda}{2}\right)\mathcal{A}_1^{\alpha\beta\mu\nu},\text{ with}\end{aligned} \tag{5.8}$$

$$\mathcal{D}_{\text{f.a.,dS}}^{\alpha\beta\mu\nu}\left(x_{\mathscr{P}}\right)+\mathcal{D}_{\text{f.a.,dS}}^{\mu\nu\alpha\beta}\left(x_{\mathscr{P}}\right)=2\mathcal{D}_{\text{dS}}^{\alpha\beta\mu\nu}. \tag{5.9}$$

With this correction we can reuse all equations derived from (3.100) by replacing $\mathcal{D}_{\text{symm.}}^{\alpha\beta\mu\nu}$ with $\mathcal{D}_{\text{dS}}^{\alpha\beta\mu\nu}$. Consequently, the Green's function that we need to solve to obtain the de

Sitter graviton is

$$\frac{1}{2\kappa^2}\mathcal{D}^{\mathrm{dS}}_{\alpha\beta}{}^{\mu\nu}G_{\mu\nu;\lambda'\epsilon'} \stackrel{!}{=} \mathbb{I}_{\alpha\beta;\lambda'\epsilon'}\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right) + (\text{other terms}), \tag{5.10}$$

see (3.106). The "other terms" on the r.h.s. of the Green's function equation are there, because we have the same issue that we encountered in the frame-adapted scheme for the Minkowski space graviton also for the de Sitter case. Namely, the propagator needs to satisfy the condition (4.32) or the proper covariant version

$$\nabla^\alpha_c\langle\gamma_{\alpha\beta}\gamma_{\lambda'\epsilon'}\rangle \stackrel{(C.18)}{=} \mathcal{T}^{\rho\sigma}{}_\beta\langle\gamma_{\rho\sigma}\gamma_{\lambda'\epsilon'}\rangle + O\left(\gamma^3\right) \Leftrightarrow \nabla^\alpha_c G_{\alpha\beta;\lambda'\epsilon'} \quad = \mathcal{T}^{\rho\sigma}{}_\beta G_{\rho\sigma;\lambda'\epsilon'} + O\left(\gamma^3\right), \tag{5.11}$$

on all indices. Only in this case the $\mathcal{T}^{\rho\sigma}{}_\nu$ does not vanish, therefore this condition does not have the simple interpretation of transversality that it had in the Minkowski case. Due to this the l.h.s. of the GFE satisfies the property

$$\begin{aligned}\nabla^{\lambda'}_c\frac{1}{2\kappa^2}\mathcal{D}^{\mathrm{dS}}_{\alpha\beta}{}^{\mu\nu}G_{\mu\nu;\lambda'\epsilon'} &= \frac{1}{2\kappa^2}\mathcal{D}^{\mathrm{dS}}_{\alpha\beta}{}^{\mu\nu}\nabla^{\lambda'}_c G_{\mu\nu;\lambda'\epsilon'}\\ &= \frac{1}{2\kappa^2}\mathcal{D}^{\mathrm{dS}}_{\alpha\beta}{}^{\mu\nu}\mathcal{T}^{\rho'\sigma'}{}_{\epsilon'}G_{\mu\nu;\rho'\sigma'},\end{aligned} \tag{5.12}$$

and we again need to modify the r.h.s. such that

$$\nabla^{\lambda'}_c\left(\mathbb{I}_{\alpha\beta;\lambda'\epsilon'}\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right) + (\text{other terms})\right) = \mathcal{T}^{\rho'\sigma'}{}_{\epsilon'}\left(\mathbb{I}_{\alpha\beta;\rho'\sigma'}\delta_4\left(x_{\mathscr{P}},x_{\mathscr{Q}}\right) + (\text{other terms})\right). \tag{5.13}$$

To discuss the modification of the r.h.s. of the equation as well as the restriction that (5.11) puts on $G_{\mu\nu;\lambda'\epsilon'}$, we need to use a basis for the bitensor structure. At first we again considered the Allen-Jacobson basis which was used for the Minkowski space graviton (see Sec. 4.1.1) and whose application to de Sitter space is extensively discussed in Ref. [33, 39, 31, 40, 34]. However, the basis is defined in an abstract way that does not provide the components of the objects $g^c_{\mu\nu}$, $g^c_{\lambda'\epsilon'}$, r^c, n_μ, $n_{\lambda'}$ and $g_{\mu\lambda'}$ with respect to a given coordinate chart. This poses a problem for the de Sitter space calculation, because we would need to know the action or rather the decomposition of the tensor $\mathcal{T}^{\rho\sigma}{}_\nu$ in terms of the basis elements

$$\mathcal{T}^{\rho\sigma}{}_\nu = h(r^c)\hat{g}^{\rho\sigma}_c n_\nu + k(r^c)2n^{(\rho}\delta^{\sigma)}{}_\nu + l(r^c)n^\rho n^\sigma n_\nu, \tag{5.14}$$

to describe its action on the decomposition of the propagator (4.14). But there is no obvious way to compute the unknown functions $h(r^c), k(r^c)$ and $l(r^c)$, without resorting to coordinates. Due to this we need to identify a different basis whose coordinate components are known, given a chosen coordinate system. The great advantage of the Allen-Jacobson basis is that we can reduce the system of PDE's in the four coordinates to a system of ODE's in the geodesic distance, thus it is desirable to find another basis, that allows for a similar simplification in the system and also for an expansion of $\mathcal{T}^{\rho\sigma}{}_\nu$ in the basis elements. One such basis is described in [47, 42, 48, 49] for the conformally flat chart of de Sitter space. It is defined via an invariant length function that can be related to the geodesic distance in the end and by applying derivatives to the length function one can generate basis tensors that are analogous to the Allen-Jacobson basis, but feature an explicit coordinate representation. We can define the same invariant length also for the cartesian Robertson-Walker coordinate system and use it to define just such a basis tailored to our coordinate chart. The derivation of this invariant length function between two events with coordinates $x_{\mathscr{P}}$ and $x_{\mathscr{Q}}$ is discussed in Ref. [50, 51, 52] and it is given by the following expression

$$p\left(\mathscr{P},\mathscr{Q}\right) = H^{-2}\cosh\left[H\Delta t^{-}\right] - \frac{1}{2}e^{H\Delta t^{+}}\left(\Delta\vec{x}\right)^{2}, \text{ with} \qquad (5.15)$$
$$\Delta t^{-} = t_{\mathscr{P}} - t_{\mathscr{Q}},\ \Delta t^{+} = t_{\mathscr{P}} + t_{\mathscr{Q}} \text{ and } \left(\Delta\vec{x}\right)^{2} = \left(\vec{x}_{\mathscr{P}} - \vec{x}_{\mathscr{Q}}\right)\left(\vec{x}_{\mathscr{P}} - \vec{x}_{\mathscr{Q}}\right).$$

The arrows denote 3-vectors. By applying covariant derivatives to this length function (which reduce to partial derivatives as p is a biscalar), we can derive three objects that can be used to construct the basis. The invariant length p and the geodesic distance r^c are connected through the equation

$$r^{c}\left(\mathscr{P},\mathscr{Q}\right) = H^{-1}\arccos\left[p\left(\mathscr{P},\mathscr{Q}\right)\right]. \qquad (5.16)$$

The fundamental basis elements are then $\left\{g^{c}_{\mu\nu}, g^{c}_{\lambda'\epsilon'}, p_{\mu\lambda'}, p_{\mu}, p_{\lambda'}\right\}$, with

$$p_{\mu} \equiv \nabla^{c}_{\mu}p = \partial_{\mu}p = \begin{bmatrix} -H^{-1}e^{-H\Delta t^{-}} + Hp \\ -e^{H\Delta t^{+}}\left(\Delta\vec{x}\right) \end{bmatrix}, \qquad (5.17)$$

$$p_{\sigma'} \equiv \nabla^{c}_{\sigma'}p = \partial_{\sigma'}p = \begin{bmatrix} -H^{-1}e^{H\Delta t^{-}} + Hp \\ e^{H\Delta t^{+}}\left(\Delta\vec{x}\right) \end{bmatrix} \text{ and} \qquad (5.18)$$

$$p_{\mu\sigma'} \equiv \nabla^{c}_{\mu}\nabla^{c}_{\sigma'}p = \partial_{\mu}\partial_{\sigma'}p = \begin{bmatrix} -2\cosh\left[H\Delta t^{-}\right] + H^{2}p & He^{H\Delta t^{+}}\left(\Delta\vec{x}\right)^{T} \\ -He^{H\Delta t^{+}}\left(\Delta\vec{x}\right) & e^{H\Delta t^{+}}\mathbb{I}_{3\times 3} \end{bmatrix}. \qquad (5.19)$$

For the coordinate representation we use a $3+1$ decomposition, where the vectors (5.17) and (5.18) are represented by 2×1 matrices, with the time component displayed separately and the spatial components collected in 3-vectors, denoted by an arrow above the symbol. For the rank-2 tensor (5.19) the temporal component is in the (00)-element, the components that mix space and time are collected in the (01)- and (10)-element, which are a row- and a column-vector respectively, and the spatial components sit in the (11)-element in the form of a 3×3 matrix. Useful identities of the basis elements, regarding their normalization and behavior under covariant differentiation, are collected in Table 5.1.

Table 5.1.: Normalization of the basis elements and useful derivative identities.

Normalization of the basis vectors:

$$\hat{g}_c^{\mu\nu} p_\mu p_\nu = H^2 \left(-p^2 + H^{-4}\right)$$
$$\hat{g}_c^{\lambda'\epsilon'} p_{\lambda'} p_{\epsilon'} = H^2 \left(-p^2 + H^{-4}\right)$$

Analog of the parallel propagator:

$$\hat{g}_c^{\mu\nu} p_\nu p_{\mu\sigma'} = -H^2 p p_{\sigma'}$$
$$\hat{g}_c^{\rho'\sigma'} p_{\sigma'} p_{\mu\rho'} = -H^2 p p_\mu$$
$$\hat{g}_c^{\mu\nu} p_{\mu\rho'} p_{\nu\sigma'} = g^c_{\rho'\sigma'} - H^2 p_{\rho'} p_{\sigma'}$$
$$\hat{g}_c^{\rho'\sigma'} p_{\mu\rho'} p_{\nu\sigma'} = g^c_{\mu\nu} - H^2 p_\mu p_\nu$$

Covariant differentiation of the basis elements, first order:

$$\nabla^c_\mu p_\nu = -H^2 p g^c_{\mu\nu}$$
$$\nabla^c_{\rho'} p_{\sigma'} = -H^2 p g^c_{\rho'\sigma'}$$
$$\nabla^c_\nu p_{\mu\rho'} = -H^2 g^c_{\mu\nu} p_{\rho'}$$
$$\nabla^c_{\rho'} p_{\mu\sigma'} = -H^2 g^c_{\rho'\sigma'} p_\mu$$

Action of d'Alembertian on basis elements:

$$\Box_c p_\mu = -H^2 p_\mu$$
$$\Box'_c p_\mu = -4H^2 p_\mu$$
$$\Box_c p_{\mu\rho'} = -H^2 p_{\mu\rho'} = \Box'_c p_{\mu\rho'}$$

With these basis elements we can again define the decomposition of the graviton propagator into basis tensors with respective coefficient functions

$$\begin{aligned} G_{\mu\nu;\lambda'\epsilon'}(p) &= g^c_{\mu\nu} g^c_{\lambda'\epsilon'} a(p) + \left(p_{\mu\lambda'} p_{\nu\epsilon'} + p_{\mu\epsilon'} p_{\nu\lambda'}\right) b(p) \\ &\quad + H^2 \left(p_\mu p_{\lambda'} p_{\nu\epsilon'} + p_\mu p_{\epsilon'} p_{\nu\lambda'} + p_\nu p_{\lambda'} p_{\mu\epsilon'} + p_\nu p_{\epsilon'} p_{\mu\lambda'}\right) c(p) \\ &\quad + H^2 \left(p_\mu p_\nu g^c_{\lambda'\epsilon'} + g^c_{\mu\nu} p_{\lambda'} p_{\epsilon'}\right) d(p) + H^4 p_\mu p_\nu p_{\lambda'} p_{\epsilon'} e(p) \qquad (5.20) \\ &= \sum_{j=1}^{5} S^{(j)}_{\mu\nu;\lambda'\epsilon'} f_j(p). \qquad (5.21) \end{aligned}$$

The powers of H that are used as prefactors for the basis tensors are there to make all of them dimensionless, since also the correlation function $\langle \gamma_{\mu\nu} \gamma_{\lambda'\epsilon'} \rangle$ should be dimensionless.[1] The basis decomposition would again allow for a rewriting of the GFE in terms of a system of ODE's w.r.t. p, via the identity

$$\nabla^c_\mu f_j(p) = f'_j(p) p_\mu, \qquad (5.22)$$

where the prime now denotes a derivative w.r.t. p. But we will now show that this basis, although it provides a representation of $\mathcal{T}^{\rho\sigma}{}_\nu$, is actually also not suitable for treating the de Sitter GFE.

[1] The constant H has mass dimension 1, while the vectors have mass dimension -1 and the analog of the parallel propagator $p_{\mu\lambda'}$ has mass dimension 0. In fact, as defined in (5.16) the invariant length has not the dimension of length but rather mass dimension -2. This could easily be remedied by an additional factor of H in the definition (5.16), but it amounts to the same, if we just correct the basis tensors accordingly.

As discussed above, the important part for the frame-adapted scheme is that we can modify the r.h.s. of the GFE such that the condition (5.13) is satisfied. Just as for the Allen-Jacobson basis we can assume that the r.h.s. can also be expressed in terms of a decomposition analogous to (5.21). Thus to study the condition (5.13) we need to know the expressions for the first covariant derivatives of the basis tensors as well as the contractions with $\mathcal{T}^{\rho\sigma}{}_{\nu}$. The former are easy to work out with the identities in Table 5.1 and the explicit results are given in App. E.1. It are the contractions of the basis tensors $S^{(j)}_{\mu\nu;\lambda'\epsilon'}$ with $\mathcal{T}^{\rho\sigma}{}_{\nu}$, however, which are problematic. This can be seen on the explicit example of $S^{(2)}_{\mu\nu;\lambda'\epsilon'}$. This contraction should have the decomposition

$$\mathcal{T}^{\rho\sigma}{}_{\nu}S^{(2)}_{\mu\nu;\lambda'\epsilon'} = s_{(2,1)}(p)H^3 p_\nu p_{\lambda'} p_{\epsilon'} + s_{(2,2)}(p)Hp_\nu g^c_{\lambda'\epsilon'} + s_{(2,3)}(p)2Hp_{\nu(\lambda'}p_{\epsilon')}, \tag{5.23}$$

if we assume that this is still a maximally symmetric bitensor and that the basis elements $\left\{g^c_{\mu\nu}, g^c_{\lambda'\epsilon'}, p_{\mu\lambda'}, p_\mu, p_{\lambda'}\right\}$ fulfill the same completeness theorem as the basis elements of the Allen-Jacobson basis, see Ref. [42]. We can now compute the resulting coefficient functions $s_{(2,k)}(p)$ (for $k = 1, 2, 3$). First we project the contraction $\mathcal{T}^{\rho\sigma}{}_{\nu}S^{(2)}_{\mu\nu;\lambda'\epsilon'}$ (which needs to be evaluated with the coordinate representations of $\mathcal{T}^{\rho\sigma}{}_{\nu}$ and $S^{(2)}_{\mu\nu;\lambda'\epsilon'}$) onto each of the three basis tensors in the decomposition (5.23):

$$H^3 p^\nu p^{\lambda'} p^{\epsilon'}\left(\mathcal{T}^{\rho\sigma}{}_{\nu}S^{(2)}_{\mu\nu;\lambda'\epsilon'}\right) = H^8 p^2 \Xi \tag{5.24}$$

$$Hp^\nu \hat{g}_c^{\lambda'\epsilon'}\left(\mathcal{T}^{\rho\sigma}{}_{\nu}S^{(2)}_{\mu\nu;\lambda'\epsilon'}\right) = -H^4 \Xi \tag{5.25}$$

$$2Hp^{\nu(\lambda'}p^{\epsilon')}\left(\mathcal{T}^{\rho\sigma}{}_{\nu}S^{(2)}_{\mu\nu;\lambda'\epsilon'}\right) = 42H^4 pp_0 + 2H^6 p\Xi \text{ , with} \tag{5.26}$$

$$\Xi = H^{-1}p^\alpha p_\mu p_\nu \bar{\mathcal{T}}^{\mu\nu}{}_{\alpha} = p_0\left(8H^2\left(H^{-4}+p^2\right) - 2\left(p_0\right)^2\right) \tag{5.27}$$

Then we also compute the same contractions with the decomposition (5.23), using the identities in Table 5.1. This yields the following matrix-vector equation for the coefficient functions $s_{(2,k)}(p)$:

$$\begin{bmatrix} H^1 2\Delta^3 & H^8\Delta^2 & -2H^{10}p\Delta^2 \\ H^8\Delta^2 & 4H^4\Delta & -2H^6 p\Delta \\ -2H^{10}p\Delta^2 & -2H^6 p\Delta & 2\left(H^8p^2\Delta + 4H^4\Delta - H^8\Delta^2\right)\end{bmatrix}\begin{bmatrix} s_{(2,1)}(p) \\ s_{(2,2)}(p) \\ s_{(2,3)}(p)\end{bmatrix} = \begin{bmatrix} H^8p^2\Xi \\ -H^4\Xi \\ 42H^4pp_0 + 2H^6p\Xi\end{bmatrix} \tag{5.28}$$

The inversion of this system provides the solution for the coefficient functions $s_{(2,k)}(p)$:

$$\begin{bmatrix} s_{(2,1)}(p) \\ s_{(2,2)}(p) \\ s_{(2,3)}(p)\end{bmatrix} = \begin{bmatrix} \frac{H^4\Xi\left(\Delta^2-2p^4-\Delta p^2\right)-126\Delta H^2p^2p_0-4\Xi\left(\Delta+4p^2\right)}{3H^4\Delta^3\left(-4+H^4\left(p^2+\Delta\right)\right)} \\ -\frac{\Xi\left(\Delta+p^2\right)}{3\Delta^2} \\ -\frac{p\left(H^2\Xi\left(\Delta+p^2\right)+21\Delta p_0\right)}{\Delta^2\left(H^4\left(\Delta+p^2\right)-4\right)}\end{bmatrix} \tag{5.29}$$

This result demonstrates how the basis leads to inconsistent results. Due to the completeness of the chosen basis [42] it should be possible to express the contraction $\mathcal{T}^{\rho\sigma}{}_{\nu}S^{(2)}_{\mu\nu;\lambda'\epsilon'}$

with the decomposition (5.23), where the coefficient functions $s_{(2,k)}(p)$ only depend on the invariant length p. However, the coefficient functions that we obtain through explicit calculation contain not only p but also p_0, which is the zeroth component of p_μ, and therefore the assumption that the $s_{(2,k)}(p)$ merely depend on the invariant length is violated. Similar relations will also hold for the contractions of the other basis tensors, as well as for the decomposition of the tensorial prefactor of the first-order covariant derivative in the Hessian (5.8).

Therefore it is not possible to re-express either $\mathcal{T}^{\rho'\sigma'}{}_{\epsilon'}(\text{r.h.s})_{\alpha\beta;\rho'\sigma'}$ in (5.13) or the action of the Hessian on the ansatz (5.21) consistently in the chosen basis and this approach fails. An alternative approach would be to treat the Green's function equation explicitly as a system of coupled PDE's, however, since the GFE has in total four open indices, this means that we would have to treat in total 4^4 equations. In reality it will be less then half of this number, due to the symmetry of the tensor structure, but it would still represent a formidable system of coupled partial differential equations. Thus this explicit approach is not further pursued. Up to this point it was not possible to identify a suitable basis that can be used to solve the Green's function equation for the graviton propagator. Thus no final result for the de Sitter graviton propagator could be obtained which could be used to evaluate the de Sitter geon.

6. Conclusions

We set out to study the quantized version of the concept of a gravitational geon, to find out if it can be described as a composite d.o.f. in the theory of gravitation. In particular we were interested in the mass of such a hypothetical particle which, if non-vanishing, would allow for an interpretation of the geon as a purely gravitational dark matter candidate. For our investigation we studied the quantized version of Einstein-Hilbert gravity under the assumption that it is well-defined non-perturbatively. As discussed in Sec. 2 there are indications that the Einstein-Hilbert action might need modification by higher powers of the curvature scalar to achieve this, but in this work we only treat the naive model without modifications to the Einstein-Hilbert action. Based on this starting point we identified the two-point correlation function of the Ricci scalar as a suitable choice for the propagator of the geon and we motivated that the event-dependence of the correlation function can be described in terms of the expectation values of the geodesic distance between the events [1]

$$\langle \mathcal{R}(\mathscr{P})\mathcal{R}(\mathscr{Q})\rangle = D\left(r\left(\mathscr{P},\mathscr{Q}\right)\right). \tag{6.1}$$

This manifestly diffeomorphism invariant and therefore physical characterization of the geon formed the central object of our investigation. In the following the similarities between the Brout-Englert-Higgs effect in electroweak physics and the observed, gravitational properties of the universe were discussed in Sec. 3. Based on these the gravitational analog of the Fröhlich-Morchio-Strocchi mechanism was introduced to simplify the calculation, as was proposed in [1]. Indeed, the special combination of frame-fixing and splitting the metric into a contribution from the VEV $\langle g_{\mu\nu}\rangle = g^c_{\mu\nu}$ and from the fluctuation field $\gamma_{\mu\nu}$, allowed us to expand the geon propagator into an expression proportional to the two-point correlation functions in $\gamma_{\mu\nu}$, accurate to second order in $\gamma_{\mu\nu}$ (FMS expansion),

$$\begin{aligned}\langle \mathcal{R}(\mathscr{P})\mathcal{R}(\mathscr{Q})\rangle = {} & \nabla^\mu_c\nabla^\nu_c\nabla^{\rho'}_c\nabla^{\sigma'}_c\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle \\ & - \left(\nabla^\mu_c\nabla^\nu_c\hat{g}^{\rho'\sigma'}_c\Box'_c + \hat{g}^{\mu\nu}_c\Box_c\nabla^{\rho'}_c\nabla^{\sigma'}_c\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle \\ & - \Lambda\left(\nabla^\mu_c\nabla^\nu_c\hat{g}^{\rho'\sigma'}_c + \hat{g}^{\mu\nu}_c\nabla^{\rho'}_c\nabla^{\sigma'}_c\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle \\ & + \hat{g}^{\mu\nu}_c\Box_c\hat{g}^{\rho'\sigma'}_c\Box'_c\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle \\ & + \Lambda\left(\hat{g}^{\mu\nu}_c\Box_c\hat{g}^{\rho'\sigma'}_c + \hat{g}^{\mu\nu}_c\hat{g}^{\rho'\sigma'}_c\Box'_c\right)\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle \\ & + \Lambda^2\hat{g}^{\mu\nu}_c\hat{g}^{\rho'\sigma'}_c\langle\gamma_{\mu\nu}\gamma_{\rho'\sigma'}\rangle. \end{aligned} \tag{6.2}$$

The latter which is referred to as graviton propagator, was then studied with perturbative methods, under the additional approximation that the fluctuation field vanishes at infinity, which allowed us to ignore total derivatives in the Einstein-Hilbert action. For the case that the VEV can be chosen to be the flat Minkowski spacetime $g^c_{\mu\nu} = \eta_{\mu\nu}$, see Sec. 3 and 4, the graviton propagator was evaluated at tree-level and the corresponding expression for the geon was computed to be

$$\langle \mathcal{R}\left(\mathscr{P}\right) \mathcal{R}\left(\mathscr{Q}\right)\rangle_{\text{connected,tl}} = -i6\kappa^2 \Box'_M \delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) + O\left(\gamma^3\right). \tag{6.3}$$

This result for Minkowski space tells us that under the considered approximations the geon does not exist as a particle d.o.f. of the quantized theory. Thus, in a universe that is flat on macroscopic scales the geon at lowest order in the FMS expansion does not represent a suitable dark matter candidate.

The more complicated case of a universe that is curved on macroscopic scales was studied in Sec. 5, by using a de Sitter space metric for the VEV. It was attempted to apply the same methods that were successful for the Minkowski space calculation also to the case of de Sitter space. However, due to the special condition

$$\nabla^\mu_c \gamma_{\mu\nu} = \mathcal{T}^{\rho\sigma}{}_\nu \gamma_{\rho\sigma} + O\left(\gamma^2\right), \tag{6.4}$$

that the frame-fixing condition imposes on the fluctuation field $\gamma_{\mu\nu}$, it was so far not possible to identify a suitable bitensor basis that can be used to calculate the graviton propagator in a way that is consistent with this condition. Therefore, we are not able to provide a leading-order result for the de Sitter space geon in this work and we cannot draw any conclusions regarding the qualification of the de Sitter space geon as a dark matter candidate.

The condition (6.4) is analogous to the interpolating gauges in QED, which mix Landau-type gauges with axial gauges, with the generalization that we do not use a vector for the axial gauge, but the tensor $\mathcal{T}^{\rho\sigma}{}_\nu$. One hint regarding the reason for the failure of the bases that we discussed so far might come from the general treatment of axial gauges. If an axial gauge is employed, the basis used for the decomposition of the tensor structure usually also contains the vector that is used in the gauge condition [53]. In our case this would amount to including $\mathcal{T}^{\rho\sigma}{}_\nu$ as an additional basis element in whichever basis we construct. This is indeed a case that was not considered in this work and it might be possible to obtain a solution for the graviton in this way.

Albeit not having a final expression for the geon, we can make one interesting observation already at the level of the Hessian (5.8), for de Sitter space. Namely, as compared to the Minkowski space Hessian (4.12) we do not only have derivative terms, but also terms proportional to the cosmological constant and the metric $g^c_{\mu\nu}$. These terms reduce to mass terms in the equations for the scalar coefficient functions of the tensor decomposition, or they do so at least for the bases that were considered so far. Although these bases were in the end not suitable for the solution of the problem, should it be possible to identify a basis that allows for a consistent treatment, it is expected that something similar will occur. This would result in an effectively massive structure of the graviton propagator and consequently the geon propagator. However, a similar cancellation as in Minkowski space could possibly spoil this result. Though, ultimately it is necessary to obtain a solution for the graviton propagator in this framework to be sure.

Appendices

A. Conventions

A.1. Notational conventions

We follow the standard conventions in defining the symmetrization and antisymmetrization operations [15]:

$$T^{(\alpha\beta)} = \frac{1}{2}\left(T^{\alpha\beta} + T^{\beta\alpha}\right) \tag{A.1}$$

$$T^{[\alpha\beta]} = \frac{1}{2}\left(T^{\alpha\beta} - T^{\beta\alpha}\right) \tag{A.2}$$

We also use the following specific convention for symmetrization:

$$T^{(\alpha|\rho|\beta)} = \frac{1}{2}\left(T^{\alpha\rho\beta} + T^{\beta\rho\alpha}\right) \tag{A.3}$$

A.2. Curvature tensors for maximally symmetric spaces

Maximally symmetric spaces play an important role in the calculations of this work, since they allow for a relatively simple structure of the graviton propagator, discussed in Sec. 3. In particular, only symmetric spaces allow for a function of two events to be parametrized purely by the minimal geodesic distance between the two in a straightforward manner. Should the spacetime be less symmetric, such as the FLRW spacetimes, one already requires the additional information such as the spatial distance. Maximally symmetric spaces also allow for very simple expressions of the curvature tensors in terms of the Ricci scalar, which is an identity that will be employed at several steps of the calculations. Also, the Ricci scalar is constant for maximally symmetric spaces and specified purely by the cosmological constant Λ [39]:

$$R_{\mu\nu\rho\sigma} = \frac{\mathcal{R}}{d(d-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{A.4}$$

$$R_{\mu\nu} = \frac{\mathcal{R}}{d}g_{\mu\nu} \tag{A.5}$$

$$\mathcal{R} = \frac{2d}{d-2}\Lambda \tag{A.6}$$

Where d denotes the dimension of the considerd spacetime, in our case $3+1$.

B. Explicit results in Einstein-Hilbert gravity

In the following we present the most explicit versions of the Ricci scalar and the Einstein-Hilbert Lagrangian in Haywood gauge. The full Ricci scalar with no simplifications looks as follows:

$$\mathcal{R} = -\tfrac{1}{4} g^{\alpha\beta} g^{\zeta\eta} \Big(-4\partial_\eta \partial_\beta g_{\alpha\zeta} + 4\partial_\eta \partial_\zeta g_{\alpha\beta} + g^{\kappa\lambda}\big(2\partial_\eta g_{\beta\lambda}\partial_\kappa g_{\alpha\zeta} - 3\partial_\kappa g_{\alpha\zeta}\partial_\lambda g_{\beta\eta} + \partial_\zeta g_{\alpha\beta}(\partial_\eta g_{\kappa\lambda} - 4\partial_\lambda g_{\eta\kappa}) + 4\partial_\beta g_{\alpha\zeta}\partial_\lambda g_{\eta\kappa}\big)\Big) \tag{B.1}$$

Now we can identify the terms that are proportional to the Haywood coordinate condition, (3.14)(3.15):

$$\mathcal{R}^H = -\tfrac{1}{4} g^{\alpha\beta} \bigg(4H^\zeta (\partial_\beta g_{\alpha\zeta} - \partial_\zeta g_{\alpha\beta}) + g^{\zeta\eta}\Big(-4\partial_\eta \partial_\beta g_{\alpha\zeta} + 4\partial_\eta \partial_\zeta g_{\alpha\beta} + g^{\kappa\lambda}\big(\partial_\zeta g_{\alpha\beta}\partial_\eta g_{\kappa\lambda} + \partial_\kappa g_{\alpha\zeta}(2\partial_\eta g_{\beta\lambda} - 3\partial_\lambda g_{\beta\eta})\big)\Big)\bigg) \tag{B.2}$$

Setting H^μ to zero provides us with the Ricci scalar in Haywood gauge.

$$\mathcal{R}^{H=0} = -\tfrac{1}{4} g^{\alpha\beta} g^{\zeta\eta} \Big(-4\partial_\eta \partial_\beta g_{\alpha\zeta} + 4\partial_\eta \partial_\zeta g_{\alpha\beta} + g^{\kappa\lambda}\big(\partial_\zeta g_{\alpha\beta}\partial_\eta g_{\kappa\lambda} + \partial_\kappa g_{\alpha\zeta}(2\partial_\eta g_{\beta\lambda} - 3\partial_\lambda g_{\beta\eta})\big)\Big) \tag{B.3}$$

Einstein-Hilbert Lagrangian in the FMS expansion

Now we can also look at the Einstein-Hilbert Lagrangian in Haywood gauge, which is given by $\sqrt{|g|}\mathcal{R}^H$. In particular we are now also interested in the form of this Lagrangian, when we perform the FMS expansion (3.5). To zeroth order we just get the Einstein-Hilbert Lagrangian related to the classical metric $g^c_{\mu\nu}$, i.e. $\sqrt{|g^c|}\,(\mathcal{R}^c)^H$. For the higher orders, we need to expand the determinant as well as the Ricci scalar in terms of $\gamma_{\mu\nu}$ and the respective fluctuation tensor occurring in the split of the inverse metric $g^{\mu\nu} = \hat{g}_c^{\mu\nu} + \hat{\gamma}^{\mu\nu}$ [1], we do not consider the Woodbury matrix identity yet. This provides us

[1] With $\hat{g}_c^{\mu\nu}$ chosen such that $\hat{g}_c^{\mu\nu} g^c_{\nu\rho} = \delta^\mu{}_\nu$.

with the following first and second order contributions:

$$
\begin{aligned}
\left(\sqrt{|g|}\mathcal{R}^{H=0}\right)^{(1)} = -\tfrac{1}{8}\sqrt{|g_c|}\hat{g}_c{}^{\alpha\beta}\Big(& 8\hat{\gamma}^{\zeta\eta}(\partial_\beta\partial_\alpha g^c{}_{\zeta\eta} - 2\partial_\eta\partial_\beta g^c{}_{\alpha\zeta} + \partial_\eta\partial_\zeta g^c{}_{\alpha\beta}) \\
&+\hat{g}_c{}^{\zeta\eta}\big(-8\partial_\eta\partial_\beta\gamma_{\alpha\zeta} + 8\partial_\eta\partial_\zeta\gamma_{\alpha\beta} + 2\hat{\gamma}^{\kappa\lambda}(2\partial_\beta g^c{}_{\eta\lambda}\partial_\zeta g^c{}_{\alpha\kappa} \\
&-6\partial_\zeta g^c{}_{\alpha\kappa}\partial_\eta g^c{}_{\beta\lambda} + 2\partial_\zeta g^c{}_{\alpha\beta}\partial_\eta g^c{}_{\kappa\lambda} + 4\partial_\eta g^c{}_{\beta\lambda}\partial_\kappa g^c{}_{\alpha\zeta} \\
&-3\partial_\kappa g^c{}_{\alpha\zeta}\partial_\lambda g^c{}_{\beta\eta} + \partial_\kappa g^c{}_{\alpha\beta}\partial_\lambda g^c{}_{\zeta\eta}) \\
&+\hat{g}_c{}^{\kappa\lambda}(4\partial_\zeta g^c{}_{\alpha\beta}\partial_\eta\gamma_{\kappa\lambda} + 8\partial_\eta\gamma_{\beta\lambda}\partial_\kappa g^c{}_{\alpha\zeta} + \hat{g}_c{}^{\mu\nu}\gamma_{\alpha\beta}\partial_\kappa g^c{}_{\zeta\eta}\partial_\lambda g^c{}_{\mu\nu} \\
&-12\partial_\kappa g^c{}_{\alpha\zeta}\partial_\lambda\gamma_{\beta\eta} - 4\gamma_{\alpha\beta}\partial_\lambda\partial_\eta g^c{}_{\zeta\kappa} + 4\gamma_{\alpha\beta}\partial_\lambda\partial_\kappa g^c{}_{\zeta\eta} \\
&+2\hat{g}_c{}^{\mu\nu}\gamma_{\alpha\beta}\partial_\lambda g^c{}_{\eta\nu}\partial_\mu g^c{}_{\zeta\kappa} - 3\hat{g}_c{}^{\mu\nu}\gamma_{\alpha\beta}\partial_\mu g^c{}_{\zeta\kappa}\partial_\nu g^c{}_{\eta\lambda})\big)\Big)
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
\left(\sqrt{|g|}\mathcal{R}^{H=0}\right)^{(2)} &= \sqrt{|g_c|}\hat{\gamma}^{\alpha\beta}\hat{\gamma}^{\zeta\eta}(\partial_\eta\partial_\beta g^c{}_{\alpha\zeta} - \partial_\eta\partial_\zeta g^c{}_{\alpha\beta}) \\
&+ \tfrac{1}{32}\sqrt{|g_c|}\hat{g_c}^{\alpha\beta}\Big(8\hat{\gamma}^{\zeta\eta}\big(-4(\partial_\beta\partial_\alpha\gamma_{\zeta\eta} - 2\partial_\eta\partial_\beta\gamma_{\alpha\zeta} + \partial_\eta\partial_\zeta\gamma_{\alpha\beta}) \\
&+\hat{\gamma}^{\kappa\lambda}(3\partial_\alpha g^c{}_{\zeta\kappa}\partial_\beta g^c{}_{\eta\lambda} - \partial_\alpha g^c{}_{\zeta\eta}\partial_\beta g^c{}_{\kappa\lambda} - 2\partial_\zeta g^c{}_{\alpha\beta}\partial_\eta g^c{}_{\kappa\lambda} \\
&-4\partial_\beta g^c{}_{\eta\lambda}\partial_\kappa g^c{}_{\alpha\zeta} - 2\partial_\eta g^c{}_{\beta\lambda}\partial_\kappa g^c{}_{\alpha\zeta} + 6\partial_\kappa g^c{}_{\alpha\zeta}\partial_\lambda g^c{}_{\beta\eta})\big) \\
&-\hat{g_c}^{\zeta\eta}\Big(16\hat{\gamma}^{\kappa\lambda}(\partial_\alpha g^c{}_{\kappa\lambda}\partial_\beta\gamma_{\zeta\eta} + 2\partial_\beta\gamma_{\eta\lambda}\partial_\zeta g^c{}_{\alpha\kappa} - 6\partial_\zeta g^c{}_{\alpha\kappa}\partial_\eta\gamma_{\beta\lambda} \\
&+\partial_\zeta g^c{}_{\alpha\beta}\partial_\eta\gamma_{\kappa\lambda} + \gamma_{\alpha\beta}\partial_\eta\partial_\zeta g^c{}_{\kappa\lambda} + 2\partial_\eta\gamma_{\beta\lambda}\partial_\kappa g^c{}_{\alpha\zeta} \\
&+2\partial_\zeta g^c{}_{\alpha\kappa}\partial_\lambda\gamma_{\beta\eta} - 3\partial_\kappa g^c{}_{\alpha\zeta}\partial_\lambda\gamma_{\beta\eta} + \partial_\kappa g^c{}_{\alpha\beta}\partial_\lambda\gamma_{\zeta\eta} \\
&-2\gamma_{\alpha\beta}\partial_\lambda\partial_\eta g^c{}_{\zeta\kappa} + \gamma_{\alpha\beta}\partial_\lambda\partial_\kappa g^c{}_{\zeta\eta}) + \hat{g_c}^{\kappa\lambda}\big(8\partial_\zeta\gamma_{\alpha\beta}\partial_\eta\gamma_{\kappa\lambda} \\
&+16\partial_\eta\gamma_{\beta\lambda}\partial_\kappa\gamma_{\alpha\zeta} - 24\partial_\kappa\gamma_{\alpha\zeta}\partial_\lambda\gamma_{\beta\eta} + 8\hat{g_c}^{\mu\nu}\gamma_{\alpha\beta}\partial_\kappa g^c{}_{\zeta\eta}\partial_\lambda\gamma_{\mu\nu} \\
&-16\gamma_{\alpha\beta}\partial_\lambda\partial_\eta\gamma_{\zeta\kappa} + 16\gamma_{\alpha\beta}\partial_\lambda\partial_\kappa\gamma_{\zeta\eta} + 16\hat{g_c}^{\mu\nu}\gamma_{\alpha\beta}\partial_\lambda\gamma_{\eta\nu}\partial_\mu g^c{}_{\zeta\kappa} \\
&+4\hat{\gamma}^{\mu\nu}\gamma_{\alpha\beta}(2\partial_\eta g^c{}_{\lambda\nu}\partial_\kappa g^c{}_{\zeta\mu} - 6\partial_\kappa g^c{}_{\zeta\mu}\partial_\lambda g^c{}_{\eta\nu} + 2\partial_\kappa g^c{}_{\zeta\eta}\partial_\lambda g^c{}_{\mu\nu} \\
&+4\partial_\lambda g^c{}_{\eta\nu}\partial_\mu g^c{}_{\zeta\kappa} - 3\partial_\mu g^c{}_{\zeta\kappa}\partial_\nu g^c{}_{\eta\lambda} + \partial_\mu g^c{}_{\zeta\eta}\partial_\nu g^c{}_{\kappa\lambda}) \\
&-2\hat{g_c}^{\mu\nu}\hat{g_c}^{\xi\rho}\gamma_{\alpha\zeta}\gamma_{\beta\eta}\partial_\mu g^c{}_{\kappa\lambda}\partial_\nu g^c{}_{\xi\rho} + \hat{g_c}^{\mu\nu}\hat{g_c}^{\xi\rho}\gamma_{\alpha\beta}\gamma_{\zeta\eta}\partial_\mu g^c{}_{\kappa\lambda}\partial_\nu g^c{}_{\xi\rho} \\
&-24\hat{g_c}^{\mu\nu}\gamma_{\alpha\beta}\partial_\mu g^c{}_{\zeta\kappa}\partial_\nu\gamma_{\eta\lambda} + 8\hat{g_c}^{\mu\nu}\gamma_{\alpha\zeta}\gamma_{\beta\eta}\partial_\nu\partial_\lambda g^c{}_{\kappa\mu} \\
&-4\hat{g_c}^{\mu\nu}\gamma_{\alpha\beta}\gamma_{\zeta\eta}\partial_\nu\partial_\lambda g^c{}_{\kappa\mu} - 8\hat{g_c}^{\mu\nu}\gamma_{\alpha\zeta}\gamma_{\beta\eta}\partial_\nu\partial_\mu g^c{}_{\kappa\lambda} \\
&+4\hat{g_c}^{\mu\nu}\gamma_{\alpha\beta}\gamma_{\zeta\eta}\partial_\nu\partial_\mu g^c{}_{\kappa\lambda} - 4\hat{g_c}^{\mu\nu}\hat{g_c}^{\xi\rho}\gamma_{\alpha\zeta}\gamma_{\beta\eta}\partial_\nu g^c{}_{\lambda\rho}\partial_\xi g^c{}_{\kappa\mu} \\
&+2\hat{g_c}^{\mu\nu}\hat{g_c}^{\xi\rho}\gamma_{\alpha\beta}\gamma_{\zeta\eta}\partial_\nu g^c{}_{\lambda\rho}\partial_\xi g^c{}_{\kappa\mu} + 6\hat{g_c}^{\mu\nu}\hat{g_c}^{\xi\rho}\gamma_{\alpha\zeta}\gamma_{\beta\eta}\partial_\xi g^c{}_{\kappa\mu}\partial_\rho g^c{}_{\lambda\nu} \\
&-3\hat{g_c}^{\mu\nu}\hat{g_c}^{\xi\rho}\gamma_{\alpha\beta}\gamma_{\zeta\eta}\partial_\xi g^c{}_{\kappa\mu}\partial_\rho g^c{}_{\lambda\nu})\big)\Big)\Big)
\end{aligned}
\tag{B.5}
$$

Now we may also employ the Woodbury matrix identity to identify $\hat{\gamma}^{\mu\nu}$ with a series in $\gamma_{\mu\nu}$, allowing us to consider all expressions to second order in terms of the dynamical field $\gamma_{\mu\nu}$ alone. The zeroth order expression remains unchanged.

$$\begin{aligned}\left(\sqrt{|g|}R^H\right)^{(1)} &= -\tfrac{1}{8}\sqrt{|g_c|}\hat{g}_c{}^{\alpha\beta}\hat{g}_c{}^{\zeta\eta}\big(-8\partial_\eta\partial_\beta\gamma_{\alpha\zeta}+8\partial_\eta\partial_\zeta\gamma_{\alpha\beta}+\hat{g}_c{}^{\kappa\lambda}(4\partial_\zeta g^c{}_{\alpha\beta}\partial_\eta\gamma_{\kappa\lambda}\\ &\quad+8\partial_\eta\gamma_{\beta\lambda}\partial_\kappa g^c{}_{\alpha\zeta}+\hat{g}_c{}^{\mu\nu}\gamma_{\alpha\beta}\partial_\kappa g^c{}_{\zeta\eta}\partial_\lambda g^c{}_{\mu\nu}-12\partial_\kappa g^c{}_{\alpha\zeta}\partial_\lambda\gamma_{\beta\eta}\\ &\quad-4\gamma_{\alpha\beta}\partial_\lambda\partial_\eta g^c{}_{\zeta\kappa}+4\gamma_{\alpha\beta}\partial_\lambda\partial_\kappa g^c{}_{\zeta\eta}+2\hat{g}_c{}^{\mu\nu}\gamma_{\alpha\beta}\partial_\lambda g^c{}_{\eta\nu}\partial_\mu g^c{}_{\zeta\kappa}\\ &\quad-3\hat{g}_c{}^{\mu\nu}\gamma_{\alpha\beta}\partial_\mu g^c{}_{\zeta\kappa}\partial_\nu g^c{}_{\eta\lambda})\big)\end{aligned} \tag{B.6}$$

$$\begin{aligned}\left(\sqrt{|g|}R^H\right)^{(2)} &= \tfrac{1}{32}\sqrt{|g_c|}\hat{g}_c{}^{\alpha\beta}\hat{g}_c{}^{\zeta\eta}\hat{g}_c{}^{\kappa\lambda}\Big(-8\big(\partial_\zeta\gamma_{\alpha\beta}\partial_\eta\gamma_{\kappa\lambda}+2\partial_\eta\gamma_{\beta\lambda}\partial_\kappa\gamma_{\alpha\zeta}-3\partial_\kappa\gamma_{\alpha\zeta}\partial_\lambda\gamma_{\beta\eta}\\ &\quad-2\gamma_{\alpha\beta}\partial_\lambda\partial_\eta\gamma_{\zeta\kappa}-4\gamma_{\alpha\zeta}(\partial_\eta\partial_\beta g^c{}_{\kappa\lambda}-2\partial_\lambda\partial_\eta g^c{}_{\beta\kappa}+\partial_\lambda\partial_\kappa g^c{}_{\beta\eta})+2\gamma_{\alpha\beta}\partial_\lambda\partial_\kappa\gamma_{\zeta\eta}\big)\\ &\quad+\hat{g}_c{}^{\mu\nu}\big(2\gamma_{\alpha\zeta}(-12\partial_\beta g^c{}_{\kappa\mu}\partial_\eta g^c{}_{\lambda\nu}+4\partial_\beta g^c{}_{\kappa\lambda}\partial_\eta g^c{}_{\mu\nu}+8\partial_\kappa g^c{}_{\beta\eta}\partial_\lambda g^c{}_{\mu\nu}\\ &\quad+16\partial_\eta g^c{}_{\lambda\nu}\partial_\mu g^c{}_{\beta\kappa}+8\partial_\lambda g^c{}_{\eta\nu}\partial_\mu g^c{}_{\beta\kappa}-24\partial_\mu g^c{}_{\beta\kappa}\partial_\nu g^c{}_{\eta\lambda}+\hat{g}_c{}^{\xi\rho}\gamma_{\beta\eta}\partial_\mu g^c{}_{\kappa\lambda}\partial_\nu g^c{}_{\xi\rho}\\ &\quad-4\gamma_{\beta\eta}\partial_\nu\partial_\lambda g^c{}_{\kappa\mu}+4\gamma_{\beta\eta}\partial_\nu\partial_\mu g^c{}_{\kappa\lambda}+2\hat{g}_c{}^{\xi\rho}\gamma_{\beta\eta}\partial_\nu g^c{}_{\lambda\rho}\partial_\xi g^c{}_{\kappa\mu}-3\hat{g}_c{}^{\xi\rho}\gamma_{\beta\eta}\partial_\xi g^c{}_{\kappa\mu}\partial_\rho g^c{}_{\lambda\nu})\\ &\quad-\gamma_{\alpha\beta}(8\partial_\kappa g^c{}_{\zeta\eta}\partial_\lambda\gamma_{\mu\nu}+16\partial_\lambda\gamma_{\eta\nu}\partial_\mu g^c{}_{\zeta\kappa}+\hat{g}_c{}^{\xi\rho}\gamma_{\zeta\eta}\partial_\mu g^c{}_{\kappa\lambda}\partial_\nu g^c{}_{\xi\rho}-24\partial_\mu g^c{}_{\zeta\kappa}\partial_\nu\gamma_{\eta\lambda}\\ &\quad-4\gamma_{\zeta\eta}\partial_\nu\partial_\lambda g^c{}_{\kappa\mu}+4\gamma_{\zeta\eta}\partial_\nu\partial_\mu g^c{}_{\kappa\lambda}+2\hat{g}_c{}^{\xi\rho}\gamma_{\zeta\eta}\partial_\nu g^c{}_{\lambda\rho}\partial_\xi g^c{}_{\kappa\mu}-3\hat{g}_c{}^{\xi\rho}\gamma_{\zeta\eta}\partial_\xi g^c{}_{\kappa\mu}\partial_\rho g^c{}_{\lambda\nu}))\Big)\end{aligned} \tag{B.7}$$

Ricci scalar in the FMS expansion

We can consider the same FMS expansion also for the curvature scalar alone, without the metric determinant, which is then the object used as the geon bound-state operator. For simplicity the superscript $H = 0$ is skipped, but all quantities below are still evaluated in the Haywood frame. First we again consider the expansion with the object $\hat{\gamma}^{\mu\nu}$ still occurring as an "independent" quantity. We again provide first and second order separately and skip the zeroth order term, which is just the classical Ricci scalar $\mathcal{R}^c$:

$$\begin{aligned}\mathcal{R}^{(1)} &= -\tfrac{1}{4}\hat{g}_c{}^{\alpha\beta}\Big(4\hat{\gamma}^{\mu\nu}(\partial_\beta\partial_\alpha g^c{}_{\mu\nu}-2\partial_\nu\partial_\beta g^c{}_{\alpha\mu}+\partial_\nu\partial_\mu g^c{}_{\alpha\beta})\\ &\quad+\hat{g}_c{}^{\mu\nu}\Big(-4\partial_\nu\partial_\beta\gamma_{\alpha\mu}+4\partial_\nu\partial_\mu\gamma_{\alpha\beta}+\hat{\gamma}^{\rho\sigma}(2\partial_\beta g^c{}_{\nu\sigma}\partial_\mu g^c{}_{\alpha\rho}-6\partial_\mu g^c{}_{\alpha\rho}\partial_\nu g^c{}_{\beta\sigma}\\ &\quad+2\partial_\mu g^c{}_{\alpha\beta}\partial_\nu g^c{}_{\rho\sigma}+4\partial_\nu g^c{}_{\beta\sigma}\partial_\rho g^c{}_{\alpha\mu}-3\partial_\rho g^c{}_{\alpha\mu}\partial_\sigma g^c{}_{\beta\nu}+\partial_\rho g^c{}_{\alpha\beta}\partial_\sigma g^c{}_{\mu\nu})\\ &\quad+2\hat{g}_c{}^{\rho\sigma}\big(\partial_\mu g^c{}_{\alpha\beta}\partial_\nu\gamma_{\rho\sigma}+\partial_\rho g^c{}_{\alpha\mu}(2\partial_\nu\gamma_{\beta\sigma}-3\partial_\sigma\gamma_{\beta\nu})\big)\Big)\Big)\end{aligned} \tag{B.8}$$

$$
\begin{aligned}
\mathcal{R}^{(2)} = &\hat{\gamma}^{\alpha\beta}\hat{\gamma}^{\mu\nu}(\partial_\nu\partial_\beta {g^c}_{\alpha\mu} - \partial_\nu\partial_\mu {g^c}_{\alpha\beta}) \\
&- \tfrac{1}{4}{\hat{g}_c}^{\alpha\beta}\Bigg(\hat{\gamma}^{\mu\nu}\big(4(\partial_\beta\partial_\alpha\gamma_{\mu\nu} - 2\partial_\nu\partial_\beta\gamma_{\alpha\mu} + \partial_\nu\partial_\mu\gamma_{\alpha\beta}) + \hat{\gamma}^{\rho\sigma}(-3\partial_\alpha {g^c}_{\mu\rho}\partial_\beta {g^c}_{\nu\sigma} \\
&+\partial_\alpha {g^c}_{\mu\nu}\partial_\beta {g^c}_{\rho\sigma} + 2\partial_\mu {g^c}_{\alpha\beta}\partial_\nu {g^c}_{\rho\sigma} + 4\partial_\beta {g^c}_{\nu\sigma}\partial_\rho {g^c}_{\alpha\mu} + 2\partial_\nu {g^c}_{\beta\sigma}\partial_\rho {g^c}_{\alpha\mu} \\
&-6\partial_\rho {g^c}_{\alpha\mu}\partial_\sigma {g^c}_{\beta\nu})\big) + {\hat{g}_c}^{\mu\nu}\Big({\hat{g}_c}^{\rho\sigma}\big(\partial_\mu\gamma_{\alpha\beta}\partial_\nu\gamma_{\rho\sigma} + \partial_\rho\gamma_{\alpha\mu}(2\partial_\nu\gamma_{\beta\sigma} - 3\partial_\sigma\gamma_{\beta\nu})\big) \\
&+2\hat{\gamma}^{\rho\sigma}(\partial_\alpha {g^c}_{\rho\sigma}\partial_\beta\gamma_{\mu\nu} + 2\partial_\beta\gamma_{\nu\sigma}\partial_\mu {g^c}_{\alpha\rho} - 6\partial_\mu {g^c}_{\alpha\rho}\partial_\nu\gamma_{\beta\sigma} + \partial_\mu {g^c}_{\alpha\beta}\partial_\nu\gamma_{\rho\sigma} \\
&+2\partial_\nu\gamma_{\beta\sigma}\partial_\rho {g^c}_{\alpha\mu} + 2\partial_\mu {g^c}_{\alpha\rho}\partial_\sigma\gamma_{\beta\nu} - 3\partial_\rho {g^c}_{\alpha\mu}\partial_\sigma\gamma_{\beta\nu} + \partial_\rho {g^c}_{\alpha\beta}\partial_\sigma\gamma_{\mu\nu})\Big)\Bigg)
\end{aligned}
\tag{B.9}
$$

Now we can invoke the Woodbury matrix identity and express everything in terms of g^c and γ:

$$
\begin{aligned}
\mathcal{R}^{(1)} = &-\tfrac{1}{2}{\hat{g}_c}^{\alpha\beta}{\hat{g}_c}^{\mu\nu}\Big(-2\partial_\nu\partial_\beta\gamma_{\alpha\mu} + 2\,\partial_\nu\partial_\mu\gamma_{\alpha\beta} + {\hat{g}_c}^{\rho\sigma}\big(\partial_\mu {g^c}_{\alpha\beta}\partial_\nu\gamma_{\rho\sigma} \\
&+\partial_\rho\, {g^c}_{\alpha\mu}(2\partial_\nu\gamma_{\beta\sigma} - 3\,\partial_\sigma\gamma_{\beta\nu})\big)\Big)
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
\mathcal{R}^{(2)} =&\tfrac{1}{4}{\hat{g}_c}^{\alpha\beta}{\hat{g}_c}^{\mu\nu}{\hat{g}_c}^{\rho\sigma}\big(-\partial_\mu\gamma_{\alpha\beta}\partial_\nu\gamma_{\rho\sigma} + 4\gamma_{\alpha\mu}\partial_\nu\partial_\beta {g^c}_{\rho\sigma} - 2\partial_\nu\gamma_{\beta\sigma}\partial_\rho\gamma_{\alpha\mu} + 3\partial_\rho\gamma_{\alpha\mu}\partial_\sigma\gamma_{\beta\nu} \\
&-8\gamma_{\alpha\mu}\partial_\sigma\partial_\nu {g^c}_{\beta\rho} + 4\gamma_{\alpha\mu}\partial_\sigma\partial_\rho {g^c}_{\beta\nu} + {\hat{g}_c}^{\tau\lambda}\gamma_{\alpha\mu}(-3\partial_\beta {g^c}_{\rho\tau}\partial_\nu {g^c}_{\sigma\lambda} + \partial_\beta {g^c}_{\rho\sigma}\partial_\nu {g^c}_{\tau\lambda} \\
&+2\partial_\rho {g^c}_{\beta\nu}\partial_\sigma {g^c}_{\tau\lambda} - 6\partial_\lambda {g^c}_{\nu\sigma}\partial_\tau {g^c}_{\beta\rho} + 4\partial_\nu {g^c}_{\sigma\lambda}\partial_\tau {g^c}_{\beta\rho} + 2\partial_\sigma {g^c}_{\nu\lambda}\partial_\tau {g^c}_{\beta\rho})\big)
\end{aligned}
\tag{B.11}
$$

C. Useful identities for the FMS expansion

C.1. Woodbury matrix identity

In order to treat the inverse of the quantum fluctuations $\gamma_{\alpha\beta}$ of the classical metric, we can use the **Woodbury matrix identity**. To be clear, by the inverse of the fluctuation field we do not mean it's proper inverse, i.e. with the property $\hat{\gamma}^{\alpha\beta}\gamma_{\beta\epsilon} = \delta^{\alpha}{}_{\epsilon}$, but rather we mean the tensor occurring in the split of the full inverse metric, with $g^{\alpha\beta} = \hat{g}_c^{\alpha\beta} + \hat{\gamma}^{\alpha\beta}$ and $g^{\alpha\beta}g_{\beta\epsilon} = \delta^{\alpha}{}_{\epsilon}$. So the point is that we know $g_{\alpha\beta} = g^c_{\alpha\beta} + \gamma_{\alpha\beta}$ and we want to have the inverse to $g_{\alpha\beta}$ in terms of the inverse of $g^c_{\alpha\beta}$ and the standard quantum fluctuation field $\gamma_{\alpha\beta}$. Now the point is that for the inverse of the sum of two matrices we can apply the Woodbury matrix identity:

$$(A - B)^{-1} = A^{-1} + A^{-1}B\,(A - B)^{-1} \tag{C.1}$$

In general this would not be of much help, but in a perturbative treatment of the theory, where we assume $\gamma_{\alpha\beta}$ and its inverse to be small, we can use the identity as a recursion formula to get to an expression of the inverse of the sum in terms of the inverse of A and just B, and not its inverse, to arbitrary high orders in B. By substituting $A \to g_c$, $A^{-1} \to {g_c}^{-1}$ and $B \to -\gamma$, we arrive at the following expression:

$$\begin{aligned}(g_c + \gamma)^{-1} &= {g_c}^{-1} - {g_c}^{-1}\gamma(g_c + \gamma)^{-1} \\ &= \sum_{k=0}^{\infty}\left(-{g_c}^{-1}\gamma\right)^k {g_c}^{-1} \\ &= {g_c}^{-1} - {g_c}^{-1}\gamma {g_c}^{-1} + {g_c}^{-1}\gamma {g_c}^{-1}\gamma {g_c}^{-1} + O(\gamma^3)\end{aligned} \tag{C.2}$$

Now using the physics convention that $\hat{g}_c^{\alpha\beta}$ denotes ${g_c}^{-1}$ and that $g^{\alpha\beta}$ denotes g^{-1}, we get the following result in the index notation of tensor calculus:

$$\begin{aligned}g^{\mu\nu} &= \left(g^c_{\mu\nu} + \gamma_{\mu\nu}\right)^{-1} \\ &= \hat{g}_c^{\mu\nu} - \hat{g}_c^{\mu\alpha}\gamma_{\alpha\beta}{g_c}^{\beta\nu} + \hat{g}_c^{\mu\alpha}\gamma_{\alpha\beta}\hat{g}_c^{\beta\lambda}\gamma_{\lambda\epsilon}\hat{g}_c^{\epsilon\nu} + O\left(\gamma^3\right)\end{aligned} \tag{C.3}$$

And now we can require that we want to write the inverse of the full metric tensor in terms of the inverse of the classical metric tensor and the additional quantum fluctuations $g^{\alpha\beta} = \hat{g}_c^{\alpha\beta} + \hat{\gamma}^{\alpha\beta}$, as discussed above, and from this we get the shape of the tensor $\hat{\gamma}^{\alpha\beta}$ in terms of the quantum fluctuation field $\gamma_{\alpha\beta}$, up to arbitrary order in γ:

$$\hat{\gamma}^{\mu\nu}\left[\gamma_{\alpha\beta}\right] = -\hat{g}_c^{\mu\alpha}\hat{g}_c^{\nu\beta}\gamma_{\alpha\beta} + \hat{g}_c^{\mu\alpha}\hat{g}_c^{\beta\lambda}\hat{g}_c^{\nu\epsilon}\gamma_{\alpha\beta}\gamma_{\lambda\epsilon} + O\left(\gamma^3\right) \tag{C.4}$$

The above definition of $\gamma^{\alpha\beta}$, see (C.4), is also compatible with the general requirement of the full inverse metric, up to the order in perturbation theory that we use, i.e. $g^{\alpha\beta}g_{\beta\epsilon} = \delta^{\alpha}{}_{\epsilon} + O\left(\gamma^3\right)$, just as it should be.

C.2. Covariant Gauss-Stokes theorem

The covariant Gauss-Stokes theorem is discussed in detail in Sec. 5.5 and 16.3 of Ref. [18] and it states

$$\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\nabla^c_\mu v^\mu = 0, \tag{C.5}$$

if the arbitrary tangent vector v^μ vanishes sufficiently rapidly at infinity.

We can use this in combination with the Leibnitz rule to perform manipulations under the integral sign which shift the (covariant) derivatives from one object to another, if we assume that any suitable contraction of $\gamma_{\mu\nu}$ with $\hat{g}_c^{\mu nu}$ to a vector vanishes sufficiently rapidly. The two manipulations that were used most often, are

$$\begin{aligned}
\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu\gamma_{\mu\nu} &= \cancel{\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\nabla_c^\mu\left(\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\nu\gamma_{\mu\nu}\right)} \\
&\quad - \int_{\mathcal{M}} d^4x\sqrt{|g^c|}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\gamma_{\alpha\beta}\nabla_c^\nu\gamma_{\mu\nu} \\
&\simeq -\cancel{\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\nabla_c^\nu\left(\hat{g}_c^{\alpha\beta}\gamma_{\mu\nu}\nabla_c^\mu\gamma_{\alpha\beta}\right)} \\
&\quad + \int_{\mathcal{M}} d^4x\sqrt{|g^c|}\hat{g}_c^{\alpha\beta}\gamma_{\mu\nu}\nabla_c^\nu\nabla_c^\mu\gamma_{\alpha\beta} \\
&\simeq \int_{\mathcal{M}} d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\hat{g}_c^{\mu\nu}\nabla_c^\alpha\nabla_c^\beta\gamma_{\mu\nu} \text{ and}
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu\gamma_{\mu\nu} &= \cancel{\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\nabla_c^\mu\left(\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\nu\gamma_{\mu\nu}\right)} \\
&\quad - \int_{\mathcal{M}} d^4x\sqrt{|g^c|}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\gamma_{\alpha\beta}\nabla_c^\nu\gamma_{\mu\nu} \\
&\simeq -\cancel{\int_{\mathcal{M}} d^4x\sqrt{|g^c|}\nabla_c^\nu\left(\hat{g}_c^{\alpha\beta}\gamma_{\mu\nu}\nabla_c^\mu\gamma_{\alpha\beta}\right)} \\
&\quad + \int_{\mathcal{M}} d^4x\sqrt{|g^c|}\hat{g}_c^{\alpha\beta}\gamma_{\mu\nu}\nabla_c^\nu\nabla_c^\mu\gamma_{\alpha\beta} \\
&\simeq \int_{\mathcal{M}} d^4x\sqrt{|g^c|}\gamma_{\alpha\beta}\hat{g}_c^{\mu\nu}\nabla_c^\alpha\nabla_c^\beta\gamma_{\mu\nu}
\end{aligned} \tag{C.7}$$

Where we have simply renamed the indices and exploited the symmetry of $\gamma_{\mu\nu}$ in the last step.

C.3. Symmetrized Hessian

In this appendix we want to provide a detailed discussion of the symmetrization of the Hessian (3.64), introduced to simplify the second order contribution to the EH action in the FMS expansion in Sec. 3.2.3. We want to achieve that the obtained Hessian features the symmetry relation (3.66) manifestly. The terms that need to be considered are $\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu}$ and the non-minimal derivative terms $-\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha$ and $\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu$. We can modify the contractions $\gamma_{\alpha\beta}\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu}\gamma_{\mu\nu}$, $\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu}$ and $\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu\gamma_{\mu\nu}$ in the action, via the standard procedure of writing the term twice with a factor one-half up front and then shifting the indices on the second term by renaming contracted indices and using that $\gamma_{\mu\nu} = \gamma_{\nu\mu}$. We only need to perform this once for $\gamma_{\alpha\beta}\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu}\gamma_{\mu\nu}$ to obtain the following identity, for which the r.h.s. already possesses the symmetries (3.66):

$$\gamma_{\alpha\beta}\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu}\gamma_{\mu\nu} = \frac{1}{2}\gamma_{\alpha\beta}\left(\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu} + \hat{g}_c^{\alpha\nu}\hat{g}_c^{\beta\mu}\right)\gamma_{\mu\nu} \equiv \gamma_{\alpha\beta}\mathcal{A}_1^{\alpha\beta\mu\nu}\gamma_{\mu\nu} \tag{C.8}$$

The same procedure works also for $\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu}$, only that we have to perform it several times, as the object is less symmetric:

$$\begin{aligned}\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu} &= \frac{1}{2}\gamma_{\alpha\beta}\left(\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha + \hat{g}_c^{\beta\nu}\nabla_c^\mu\nabla_c^\alpha\right)\gamma_{\mu\nu}\\ &= \frac{1}{4}\gamma_{\alpha\beta}\left(\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha + \hat{g}_c^{\beta\nu}\nabla_c^\mu\nabla_c^\alpha + \hat{g}_c^{\alpha\mu}\nabla_c^\nu\nabla_c^\beta + \hat{g}_c^{\alpha\nu}\nabla_c^\mu\nabla_c^\beta\right)\gamma_{\mu\nu}\end{aligned} \tag{C.9}$$

Now we apply the procedure once more for each term in (C.9) and switch the covariant derivatives on the additional terms, using the following commutator relation for a maximally symmetric VEV (see A.2):

$$\begin{aligned}\nabla_c^\alpha\nabla_c^\nu\gamma_{\mu\nu} &= \nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu} + [\nabla_c^\alpha, \nabla_c^\nu]\gamma_{\mu\nu}\\ &= \nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu} + R_c^{\alpha\nu}{}_\mu{}^\tau\gamma_{\tau\nu} + R_c^{\alpha\nu}{}_\nu{}^\tau\gamma_{\mu\tau}\\ &= \nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu} + \frac{\Lambda}{3}\left(\delta_\mu{}^\alpha\hat{g}_c^{\tau\nu} - \delta_\mu{}^\nu\hat{g}_c^{\tau\alpha}\right)\gamma_{\tau\nu} - \Lambda\hat{g}_c^{\tau\alpha}\gamma_{\mu\tau}\end{aligned} \tag{C.10}$$

Thus we obtain the equality

$$\begin{aligned}\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu} &= \frac{1}{8}\gamma_{\alpha\beta}\Big(\hat{g}_c^{\beta\mu}\left(\nabla_c^\nu\nabla_c^\alpha + \nabla_c^\alpha\nabla_c^\nu + [\nabla_c^\nu, \nabla_c^\alpha]\right)\\ &\quad+\hat{g}_c^{\beta\nu}\left(\nabla_c^\mu\nabla_c^\alpha + \nabla_c^\alpha\nabla_c^\mu + [\nabla_c^\mu, \nabla_c^\alpha]\right)\\ &\quad+\hat{g}_c^{\alpha\mu}\left(\nabla_c^\nu\nabla_c^\beta + \nabla_c^\beta\nabla_c^\nu + \left[\nabla_c^\nu, \nabla_c^\beta\right]\right)\\ &\quad+\hat{g}_c^{\alpha\nu}\left(\nabla_c^\mu\nabla_c^\beta + \nabla_c^\beta\nabla_c^\mu + \left[\nabla_c^\mu, \nabla_c^\beta\right]\right)\Big)\gamma_{\mu\nu}\\ &= \frac{1}{8}\gamma_{\alpha\beta}\Big(\hat{g}_c^{\beta\mu}\{\nabla_c^\nu, \nabla_c^\alpha\} + \hat{g}_c^{\beta\nu}\{\nabla_c^\mu, \nabla_c^\alpha\} + \hat{g}_c^{\alpha\mu}\{\nabla_c^\nu, \nabla_c^\beta\} + \hat{g}_c^{\alpha\nu}\{\nabla_c^\mu, \nabla_c^\beta\}\\ &\quad+\frac{\Lambda}{6}\left(4\mathcal{A}_1^{\alpha\beta\mu\nu} - \hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Big)\gamma_{\mu\nu}\end{aligned} \tag{C.11}$$

$$\equiv \gamma_{\alpha\beta}\mathcal{A}_2^{\alpha\beta\mu\nu}\gamma_{\mu\nu} \quad \text{, with} \tag{C.12}$$

$$\{\nabla_c^\mu, \nabla_c^\nu\} = \nabla_c^\mu\nabla_c^\nu + \nabla_c^\nu\nabla_c^\mu \quad . \tag{C.13}$$

The very same steps can now also be performed for $\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu\gamma_{\mu\nu}$ and we also need to employ (C.6) to obtain:

$$\begin{aligned}\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu\gamma_{\mu\nu} &= \frac{1}{2}\gamma_{\alpha\beta}\left(\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu + \hat{g}_c^{\mu\nu}\nabla_c^\alpha\nabla_c^\beta\right)\gamma_{\mu\nu} \\ &= \frac{1}{4}\gamma_{\alpha\beta}\left(\hat{g}_c^{\alpha\beta}\{\nabla_c^\mu,\nabla_c^\nu\} + \hat{g}_c^{\mu\nu}\{\nabla_c^\alpha,\nabla_c^\beta\} + \hat{g}_c^{\alpha\beta}\left[\nabla_c^\mu,\nabla_c^\nu\right] + \hat{g}_c^{\mu\nu}\left[\nabla_c^\alpha,\nabla_c^\beta\right]\right)\gamma_{\mu\nu} \\ &= \frac{1}{4}\gamma_{\alpha\beta}\left(\hat{g}_c^{\alpha\beta}\{\nabla_c^\mu,\nabla_c^\nu\} + \hat{g}_c^{\mu\nu}\{\nabla_c^\alpha,\nabla_c^\beta\}\right)\gamma_{\mu\nu} \\ &\equiv \gamma_{\alpha\beta}\mathcal{A}_3^{\alpha\beta\mu\nu}\gamma_{\mu\nu} \end{aligned} \tag{C.14}$$

Where we have used the two identities $\gamma_{\alpha\beta}\left[\nabla_c^\alpha,\nabla_c^\beta\right]\gamma_{\mu\nu} = 0$ and $\left[\nabla_c^\mu,\nabla_c^\nu\right]\gamma_{\mu\nu} = 0$, which follow from the fact that we contract as symmetric object (γ) with an antisymmetric one (the commutator).

The three equalities (C.8), (C.13) and (C.14) now allow us to replace the contractions $\gamma_{\alpha\beta}\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu}\gamma_{\mu\nu}$, $\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu}$ and $\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu\gamma_{\mu\nu}$ in the actions by contractions with the auxiliary objects $\mathcal{A}_j^{\alpha\beta\mu\nu}$, with $j = 1, 2, 3$. Consequently, the corresponding Hessian $\mathcal{D}_{\text{symm}}^{\alpha\beta\mu\nu}$ will now contain the $\mathcal{A}_j^{\alpha\beta\mu\nu}$ and since these satisfy the property

$$\mathcal{A}_j^{\alpha\beta\mu\nu} = \mathcal{A}_j^{(\alpha\beta)(\mu\nu)} = \mathcal{A}_j^{(\mu\nu)(\alpha\beta)} \quad \text{for} \quad j \;=\; 1, 2, s3, \tag{C.15}$$

the Hessian will have the same symmetry, which is what we set out for, i.e. $\mathcal{D}_{\text{symm}}^{\alpha\beta\mu\nu}$ fulfills (3.66). The explicit form of $\mathcal{D}_{\text{symm}}^{\alpha\beta\mu\nu}$ is the given by:

$$\mathcal{D}_{\text{symm.}}^{\alpha\beta\mu\nu} \equiv \frac{1}{2}\left(\mathcal{A}_1^{\alpha\beta\mu\nu} - \hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Box_c - \mathcal{A}_2^{\alpha\beta\mu\nu} + \mathcal{A}_3^{\alpha\beta\mu\nu} + \left(\mathcal{A}_1^{\alpha\beta\mu\nu} - \frac{1}{2}\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu}\right)\Lambda \tag{C.16}$$

C.4. Covariant frame identities and frame-fixing functional

Covariant frame identities

In this section we want to derive the covariant counterparts to the frame identities (3.24) and (3.26), which were derived in Sec. 3.2.2. We begin with the identity (3.24). This identity contains a partial derivative of the fluctuation field γ. This can be exchanged with a covariant derivative by making use of substitution rule (C.17) which follows from the definition (2.14):

$$\begin{aligned}\nabla_\nu^c\gamma_{\mu\alpha} &= \partial_\nu\gamma_{\mu\alpha} - \Gamma_{c\,\nu\mu}^{\rho}\gamma_{\rho\alpha} - \Gamma_{c\,\nu\alpha}^{\rho}\gamma_{\mu\rho} \\ \Longleftrightarrow \partial_\nu\gamma_{\mu\alpha} &= \nabla_\nu^c\gamma_{\mu\alpha} + \Gamma_{c\,\nu\mu}^{\rho}\gamma_{\rho\alpha} + \Gamma_{c\,\nu\alpha}^{\rho}\gamma_{\mu\rho}\end{aligned} \tag{C.17}$$

Thus we can rewrite the identity (3.24) accordingly:

$$\begin{aligned}
\hat{g}_c^{\mu\alpha}\partial_\alpha\gamma_{\mu\nu} = \partial^\mu\gamma_{\mu\nu} &= \nabla_c^\mu\gamma_{\mu\nu} + \left(\Gamma^{\rho\mu}_{c\;\;\mu}\delta_\nu{}^\sigma + \Gamma^{\rho\sigma}_{c\;\;\nu}\right)\gamma_{\rho\sigma} = \left(\hat{g}_c^{\mu\sigma}\partial^\rho g^c_{\mu\nu}\right)\gamma_{\rho\sigma} + O\left(\gamma^2\right) \\
\Longleftrightarrow \nabla_c^\mu\gamma_{\mu\nu} &= -\Gamma^{\rho\mu}_{c\;\;\mu}\gamma_{\rho\nu} + \left(-\Gamma^{\rho\sigma}_{c\;\;\nu} + \hat{g}_c^{\mu\sigma}\partial^\rho g^c_{\mu\nu}\right)\gamma_{\rho\sigma} + O\left(\gamma^2\right) \\
&= -\Gamma^{\rho\mu}_{c\;\;\mu}\gamma_{\rho\nu} + \hat{g}_c^{\rho\alpha}\hat{g}_c^{\sigma\beta}\Gamma^c_{\nu\alpha\beta}\gamma_{\rho\sigma} + O\left(\gamma^2\right) \\
&= \left(\Gamma^{c\;\rho\sigma}_{\nu} - \Gamma^{\rho\mu}_{c\;\;\mu}\delta_\nu{}^\sigma\right)\gamma_{\rho\sigma} + O\left(\gamma^2\right) \\
&\equiv \mathcal{T}^{\rho\sigma}{}_\nu\gamma_{\rho\sigma} + O\left(\gamma^2\right)
\end{aligned} \tag{C.18}$$

Based on (C.18) we can introduce several new identities:

$$\begin{aligned}
\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\alpha\nabla_c^\nu\gamma_{\mu\nu} &= \nabla_c^\alpha\cancel{\left(\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\nu\gamma_{\mu\nu}\right)} - \hat{g}_c^{\beta\mu}\nabla_c^\alpha\gamma_{\alpha\beta}\nabla_c^\nu\gamma_{\mu\nu} \\
&\overset{(C.18)}{\simeq} -\hat{g}_c^{\beta\mu}\mathcal{T}^{\rho\sigma}{}_\beta\gamma_{\rho\sigma}\mathcal{T}^{\lambda\epsilon}{}_\mu\gamma_{\lambda\epsilon} + O\left(\gamma^3\right) \\
&= -\gamma_{\alpha\beta}\hat{g}_c^{\rho\sigma}\mathcal{T}^{\alpha\beta}{}_\rho\mathcal{T}^{\mu\nu}{}_\sigma\gamma_{\mu\nu} + O\left(\gamma^3\right) \\
&\equiv \gamma_{\alpha\beta}\mathcal{J}_1^{\alpha\beta\mu\nu}\gamma_{\mu\nu} + O\left(\gamma^3\right)
\end{aligned} \tag{C.19}$$

The auxiliary object $\mathcal{J}_1^{\alpha\beta\mu\nu}$ can also be expressed explicitly in terms of the Christoffel symbols:

$$\begin{aligned}
\mathcal{J}_1^{\alpha\beta\mu\nu} &= \hat{g}_c^{\rho\sigma}\mathcal{T}^{\alpha\beta}{}_\rho\mathcal{T}^{\mu\nu}{}_\sigma \\
&\overset{(C.18)}{=} \Gamma^{\nu\alpha\beta}_c\Gamma^{\mu\lambda}_{c\;\;\lambda} + \Gamma^{\beta\mu\nu}_c\Gamma^{\alpha\lambda}_{c\;\;\lambda} - \Gamma^{\rho\alpha\beta}_c\Gamma^{c\mu\nu}_\rho - \hat{g}_c^{\beta\nu}\Gamma^{\alpha\lambda}_{c\;\;\lambda}\Gamma^{\mu\tau}_{c\;\;\tau}
\end{aligned} \tag{C.20}$$

If we take a look at the terms which occur in the symmetrized Hessian (3.68), (C.16), we see that it will be useful to also have a version of the identity (C.19) with the covariant derivatives in the reverse order. To switch the order of the covariant derivatives we can use the previously introduced commutator relation (C.10):

$$\begin{aligned}
\gamma_{\alpha\beta}\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha\gamma_{\mu\nu} &\overset{(C.10)}{=} \gamma_{\alpha\beta}\left[\hat{g}_c^{\beta\mu}\nabla_c^\alpha\nabla_c^\nu\gamma_{\mu\nu} - \frac{\Lambda}{3}\left(\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu} - \hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu}\right) + \Lambda\hat{g}_c^{\alpha\nu}\hat{g}_c^{\beta\mu}\right]\gamma_{\mu\nu} \\
&\overset{(C.19)}{=} \gamma_{\alpha\beta}\left[\mathcal{J}_1^{\alpha\beta\mu\nu} - \frac{\Lambda}{3}\left(\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu} - \hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu}\right) + \Lambda\hat{g}_c^{\alpha\nu}\hat{g}_c^{\beta\mu}\right]\gamma_{\mu\nu} + O\left(\gamma^3\right) \\
&\equiv \gamma_{\alpha\beta}\mathcal{J}_2^{\alpha\beta\mu\nu}\gamma_{\mu\nu} + O\left(\gamma^3\right)
\end{aligned} \tag{C.21}$$

Two additional identities can be obtained with the help of (C.18), namely

$$\begin{aligned}
\gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\nabla_c^\nu\gamma_{\mu\nu} &\overset{(C.18)}{=} \gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\nabla_c^\mu\left(\Gamma^{c\;\rho\sigma}_{\mu} - \Gamma^{\rho\nu}_{c\;\;\nu}\delta_\mu{}^\sigma\right)\gamma_{\rho\sigma} \\
&= \gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\left(\left(\nabla_c^\rho\Gamma^{c\mu\nu}_\rho - \nabla_c^\nu\Gamma^{\mu\rho}_{c\;\;\rho} + \Gamma^{c\mu\nu}_\rho\nabla_c^\rho\right)\gamma_{\mu\nu} - \Gamma^{\rho\nu}_{c\;\;\nu}\nabla_c^\mu\gamma_{\mu\rho}\right) \\
&\overset{(C.18)}{=} \gamma_{\alpha\beta}\hat{g}_c^{\alpha\beta}\left(\left(\nabla_c^\rho\Gamma^{c\mu\nu}_\rho - \nabla_c^\nu\Gamma^{\mu\rho}_{c\;\;\rho} - \Gamma^{\rho\sigma}_{c\;\;\sigma}\Gamma^{c\mu\nu}_\rho + \Gamma^{\mu\rho}_{c\;\;\rho}\Gamma^{\nu\sigma}_{c\;\;\sigma}\right) + \Gamma^{c\mu\nu}_\rho\nabla_c^\rho\right)\gamma_{\mu\nu} \\
&\equiv \gamma_{\alpha\beta}\left(\mathcal{J}_3^{\alpha\beta\mu\nu} + \hat{g}_c^{\alpha\beta}\Gamma^{c\mu\nu}_\rho\nabla_c^\rho\right)\gamma_{\mu\nu}
\end{aligned} \tag{C.22}$$

and

$$
\begin{aligned}
\gamma_{\alpha\beta}\hat{g}_c^{\mu\nu}\nabla_c^{\alpha}\nabla_c^{\beta}\gamma_{\mu\nu} &\overset{(C.6)}{\simeq} \gamma_{\mu\nu}\hat{g}_c^{\mu\nu}\nabla_c^{\alpha}\nabla_c^{\beta}\gamma_{\alpha\beta} \\
&\overset{(C.22)}{=} \gamma_{\mu\nu}\left(\mathcal{J}_3^{\mu\nu\alpha\beta} + \hat{g}_c^{\mu\nu}\Gamma_{\rho}^{c\,\alpha\beta}\nabla_c^{\rho}\right)\gamma_{\alpha\beta} \\
&\overset{(C.7)}{\simeq} \gamma_{\mu\nu}\mathcal{J}_3^{\mu\nu\alpha\beta}\gamma_{\alpha\beta} - \gamma_{\alpha\beta}\hat{g}_c^{\mu\nu}\nabla_c^{\rho}\left(\Gamma_{\rho}^{c\,\alpha\beta}\gamma_{\mu\nu}\right) \\
&= \gamma_{\alpha\beta}\left(\left(\mathcal{J}_3^{\mu\nu\alpha\beta} - \hat{g}_c^{\mu\nu}\nabla_c^{\rho}\Gamma_{\rho}^{c\,\alpha\beta}\right) - \hat{g}_c^{\mu\nu}\Gamma_{\rho}^{c\,\alpha\beta}\nabla_c^{\rho}\right)\gamma_{\mu\nu} \\
&\equiv \gamma_{\alpha\beta}\left(\mathcal{J}_4^{\mu\nu\alpha\beta} - \hat{g}_c^{\mu\nu}\Gamma_{\rho}^{c\,\alpha\beta}\nabla_c^{\rho}\right)\gamma_{\mu\nu} \qquad \text{(C.23)}
\end{aligned}
$$

These identities can now be used to simplify the Hessian (3.68) in the frame-adapted approach, see Sec. 3.2.3.

Covariant frame-fixing functional

If we choose to use the approach via the frame-fixing action, i.e. the frame-expanded approach, see Sec. 3.2.3, then we also need to have a version of the frame-fixing functional with covariant derivatives. This will be discussed in the following. As discussed the only relevant contribution to the frame-fixing functional is the term (3.88):

$$\begin{aligned} H^{(1)}_\mu &= -\hat{g}_c^{\alpha\rho}\hat{g}_c^{\beta\sigma}\gamma_{\rho\sigma}\partial_\alpha g^c_{\beta\mu} + \hat{g}_c^{\alpha\beta}\partial_\alpha\gamma_{\beta\mu} && \text{(C.24)} \\ \hat{g}_c^{\mu\nu}H^{(1)}_\mu H^{(1)}_\nu &= \hat{g}_c^{\mu\nu}\Big(\partial^\alpha\gamma_{\alpha\mu}\partial^\beta\gamma_{\beta\nu} - 2\gamma_{\rho\sigma}\hat{g}_c^{\beta\sigma}\left(\partial^\rho g^c_{\beta\mu}\right)\partial^\lambda\gamma_{\lambda\nu} \\ &\quad + \hat{g}_c^{\alpha\rho}\hat{g}_c^{\beta\lambda}\left(\partial^\sigma g^c_{\alpha\mu}\right)\left(\partial^\epsilon g^c_{\beta\nu}\right)\gamma_{\rho\sigma}\gamma_{\lambda\epsilon}\Big) && \text{(C.25)} \end{aligned}$$

We shall discuss the part of (C.25) which contains two derivatives first:

$$\begin{aligned} \hat{g}_c^{\rho\sigma}\left(\partial^\alpha\gamma_{\alpha\rho}\right)\left(\partial^\mu\gamma_{\mu\sigma}\right) &\overset{(C.17)}{=} \hat{g}_c^{\rho\sigma}\Big(\nabla_c^\alpha\gamma_{\alpha\rho} + \Gamma_c{}^{\lambda\alpha}{}_\alpha\gamma_{\lambda\rho} + \Gamma_c{}^{\lambda\alpha}{}_\rho\gamma_{\alpha\lambda}\Big)\times \\ &\quad\times\left(\nabla_c^\mu\gamma_{\mu\sigma} + \Gamma_c{}^{\tau\mu}{}_\mu\gamma_{\tau\sigma} + \Gamma_c{}^{\tau\mu}{}_\sigma\gamma_{\mu\tau}\right) \\ &= \hat{g}_c^{\rho\sigma}\nabla_c^\alpha\gamma_{\alpha\rho}\nabla_c^\mu\gamma_{\mu\sigma} + \gamma_{\alpha\beta}2\Big(\hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho + \hat{g}_c^{\rho\nu}\Gamma_c{}^{\alpha\beta}{}_\rho\Big)\nabla_c^\mu\gamma_{\mu\nu} \\ &\quad + \gamma_{\alpha\beta}\Big(\hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho\Gamma_c{}^{\mu\sigma}{}_\sigma + \hat{g}_c^{\beta\sigma}\Gamma_c{}^{\alpha\rho}{}_\rho\Gamma_c{}^{\nu\mu}{}_\sigma \\ &\quad + \hat{g}_c^{\rho\nu}\Gamma_c{}^{\alpha\beta}{}_\rho\Gamma_c{}^{\mu\sigma}{}_\sigma + \hat{g}_c^{\rho\sigma}\Gamma_c{}^{\alpha\beta}{}_\rho\Gamma_c{}^{\mu\nu}{}_\sigma\Big)\gamma_{\mu\nu} \\ &\overset{(C.6)}{\simeq} \gamma_{\alpha\beta}\Big(-\hat{g}_c^{\beta\mu}\nabla_c^\alpha\nabla_c^\nu + 2\Big(\hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho + \hat{g}_c^{\rho\nu}\Gamma_c{}^{\alpha\beta}{}_\rho\Big)\nabla_c^\mu \\ &\quad + \hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho\Gamma_c{}^{\mu\sigma}{}_\sigma + \hat{g}_c^{\beta\sigma}\Gamma_c{}^{\alpha\rho}{}_\rho\Gamma_c{}^{\nu\mu}{}_\sigma \\ &\quad + \hat{g}_c^{\rho\nu}\Gamma_c{}^{\alpha\beta}{}_\rho\Gamma_c{}^{\mu\sigma}{}_\sigma + \hat{g}_c^{\rho\sigma}\Gamma_c{}^{\alpha\beta}{}_\rho\Gamma_c{}^{\mu\nu}{}_\sigma\Big)\gamma_{\mu\nu} \\ &\overset{(C.10)}{=} \gamma_{\alpha\beta}\Big(-\hat{g}_c^{\beta\mu}\nabla_c^\nu\nabla_c^\alpha + \frac{\Lambda}{3}\Big(\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu} - 4\hat{g}_c^{\alpha\mu}\hat{g}_c^{\beta\nu}\Big) \\ &\quad + 2\Big(\hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho + \hat{g}_c^{\rho\nu}\Gamma_c{}^{\alpha\beta}{}_\rho\Big)\nabla_c^\mu \\ &\quad + \hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho\Gamma_c{}^{\mu\sigma}{}_\sigma + \hat{g}_c^{\beta\sigma}\Gamma_c{}^{\alpha\rho}{}_\rho\Gamma_c{}^{\nu\mu}{}_\sigma \\ &\quad + \hat{g}_c^{\rho\nu}\Gamma_c{}^{\alpha\beta}{}_\rho\Gamma_c{}^{\mu\sigma}{}_\sigma + \hat{g}_c^{\rho\sigma}\Gamma_c{}^{\alpha\beta}{}_\rho\Gamma_c{}^{\mu\nu}{}_\sigma\Big)\gamma_{\mu\nu} \\ &\overset{(C.13)}{=} \gamma_{\alpha\beta}\Big(-\mathcal{A}_2^{\alpha\beta\mu\nu} + \frac{\Lambda}{3}\Big(\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu} - 4\mathcal{A}_1^{\alpha\beta\mu\nu}\Big) \\ &\quad + 2\Big(\hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho + \hat{g}_c^{\rho\nu}\Gamma_c{}^{\alpha\beta}{}_\rho\Big)\nabla_c^\mu \\ &\quad + \hat{g}_c^{\beta\nu}\Gamma_c{}^{\alpha\rho}{}_\rho\Gamma_c{}^{\mu\sigma}{}_\sigma + \hat{g}_c^{\beta\sigma}\Gamma_c{}^{\alpha\rho}{}_\rho\Gamma_c{}^{\nu\mu}{}_\sigma \\ &\quad + \hat{g}_c^{\rho\nu}\Gamma_c{}^{\alpha\beta}{}_\rho\Gamma_c{}^{\mu\sigma}{}_\sigma + \hat{g}_c^{\rho\sigma}\Gamma_c{}^{\alpha\beta}{}_\rho\Gamma_c{}^{\mu\nu}{}_\sigma\Big)\gamma_{\mu\nu} && \text{(C.26)} \end{aligned}$$

The expression is already rearranged such that we can write it as a quadratic form. Next we consider the first-derivative term in (C.25):

$$\begin{aligned}
-2\gamma_{\rho\sigma}\hat{g}_c^{\beta\sigma}\left(\partial^\rho g^c_{\beta\mu}\right)\partial^\lambda\gamma_{\lambda\nu} \overset{(C.17)}{=} & -2\gamma_{\alpha\beta}\Big(\hat{g}_c^{\rho\beta}\left(\partial^\alpha g^c_{\rho\lambda}\right)\hat{g}_c^{\lambda\nu}\nabla_c^\mu + \hat{g}_c^{\rho\beta}\left(\partial^\alpha g^c_{\rho\lambda}\right)\hat{g}_c^{\lambda\nu}\Gamma_c^{\mu\sigma}{}_\sigma \\
& +\hat{g}_c^{\rho\beta}\left(\partial^\alpha g^c_{\rho\lambda}\right)\hat{g}_c^{\lambda\sigma}\Gamma_c^{\mu\nu}{}_\sigma\Big)\gamma_{\mu\nu} \\
= & -2\gamma_{\alpha\beta}\Big(\Big(\hat{g}_c^{\rho\alpha}\Gamma_c^{\nu\beta}{}_\rho + \hat{g}_c^{\rho\nu}\Gamma_c^{\alpha\beta}{}_\rho\Big)\nabla_c^\mu + \hat{g}_c^{\rho\alpha}\Gamma_c^{\nu\beta}{}_\rho\Gamma_c^{\mu\sigma}{}_\sigma \\
& +\hat{g}_c^{\rho\nu}\Gamma_c^{\alpha\beta}{}_\rho\Gamma_c^{\mu\sigma}{}_\sigma + \hat{g}_c^{\rho\alpha}\Gamma_c^{\sigma\beta}{}_\rho\Gamma_c^{\mu\nu}{}_\sigma + \hat{g}_c^{\rho\sigma}\Gamma_c^{\alpha\beta}{}_\rho\Gamma_c^{\mu\nu}{}_\sigma\Big)\gamma_{\mu\nu}
\end{aligned} \tag{C.27}$$

In the last step we have used (C.17) with γ replaced by g^c, combined with the metric postulate (see Sec. 2.1.1), to rewrite partial derivatives of the VEV g^c with the "classical" Christoffel symbols. We can do the same also for the last term in (C.25):

$$\begin{aligned}
\hat{g}_c^{\alpha\rho}\hat{g}_c^{\beta\lambda}\left(\partial^\sigma g^c_{\alpha\mu}\right)\left(\partial^\epsilon g^c_{\beta\nu}\right)\gamma_{\rho\sigma}\gamma_{\lambda\epsilon} &= \gamma_{\alpha\beta}\hat{g}_c^{\rho\sigma}\hat{g}_c^{\beta\lambda}\hat{g}_c^{\mu\tau}\left(\partial^\alpha g^c_{\lambda\rho}\right)\left(\partial^\nu g^c_{\tau\sigma}\right)\gamma_{\mu\nu} \\
&= \gamma_{\alpha\beta}\Big(\hat{g}_c^{\mu\sigma}\Gamma_c^{\alpha\beta}{}_\rho\Gamma_c^{\rho\nu}{}_\sigma + \hat{g}_c^{\rho\sigma}\Gamma_c^{\alpha\beta}{}_\rho\Gamma_c^{\mu\nu}{}_\sigma \\
&\quad +\hat{g}_c^{\beta\rho}\Gamma_c^{\sigma\alpha}{}_\rho\Gamma_c^{\mu\nu}{}_\sigma + g^c_{\rho\sigma}\Gamma_c^{\rho\alpha\beta}\Gamma_c^{\sigma\mu\nu}\Big)\gamma_{\mu\nu}
\end{aligned} \tag{C.28}$$

Now we can combine the partial results (C.26), (C.27) and (C.28) to obtain the complete version of (C.25) with covariant derivatives:

$$\begin{aligned}
\hat{g}_c^{\mu\nu}H_\mu^{(1)}H_\nu^{(1)} \simeq \gamma_{\alpha\beta}\Big(&-\mathcal{A}_2^{\alpha\beta\mu\nu} + \frac{\Lambda}{3}\left(\hat{g}_c^{\alpha\beta}\hat{g}_c^{\mu\nu} - 4\mathcal{A}_1^{\alpha\beta\mu\nu}\right) + 2\left(\hat{g}_c^{\beta\nu}\Gamma_c^{\alpha\rho}{}_\rho - \hat{g}_c^{\rho\alpha}\Gamma_c^{\nu\beta}{}_\rho\right)\nabla_c^\mu \\
&+\hat{g}_c^{\beta\nu}\Gamma_c^{\alpha\rho}{}_\rho\Gamma_c^{\mu\sigma}{}_\sigma + g^c_{\rho\sigma}\Gamma_c^{\rho\alpha\beta}\Gamma_c^{\sigma\mu\nu} - 2\hat{g}_c^{\rho\alpha}\Gamma_c^{\nu\beta}{}_\rho\Gamma_c^{\mu\sigma}{}_\sigma\Big)\gamma_{\mu\nu} \\
\equiv \gamma_{\alpha\beta}\Big(&-\mathcal{A}_2^{\alpha\beta\mu\nu} + 2\left(\hat{g}_c^{\beta\nu}\Gamma_c^{\alpha\rho}{}_\rho - \hat{g}_c^{\rho\alpha}\Gamma_c^{\nu\beta}{}_\rho\right)\nabla_c^\mu + \mathcal{I}_1^{\alpha\beta\mu\nu}\Big)\gamma_{\mu\nu}
\end{aligned} \tag{C.29}$$

Many terms in the contributions that do not contain any derivatives can be made to cancel by renaming indices and exploiting the symmetry of γ and g^c, thus the final expression for $\mathcal{I}_1^{\alpha\beta\mu\nu}$ can be brought to a very compact form.

D. Minkowski space calculation

D.1. Bitensors in maximally symmetric spacetimes

We are considering maximally symmetric spacetimes for the VEV in the frame-fixing procedure, therefore all two-point functions that are computed fall into the category of bitensors, whose definition and properties are discussed in detail in Ref. [34]. A particularly important property is their behavior under covariant differentiation, since this is very crucial for the calculations in Sec. 4 and 5, we briefly discuss it here. In this chapter we again adopt the convention that primed and unprimed indices refer to different events in spacetime. Therefore, a bitensor of type

$$T_{\mu_1 \ldots \mu_k; \nu'_1 \ldots \nu'_{k'}} \in T_{\mathscr{P}}\mathcal{M} \otimes \cdots \otimes T_{\mathscr{P}}\mathcal{M} \otimes T_{\mathscr{Q}}\mathcal{M} \otimes \cdots \otimes T_{\mathscr{Q}}\mathcal{M},$$

has k indices which refer to the tangent space $T_{\mathscr{P}}\mathcal{M}$ at event $\mathscr{P}$ and k' indices which refer to the tangent space $T_{\mathscr{Q}}\mathcal{M}$ at event $\mathscr{Q}$. The covariant derivative of a bitensor w.r.t. event $\mathscr{P}$ is then denoted $\nabla_\rho T_{\mu_1 \ldots \mu_k; \nu'_1 \ldots \nu'_{k'}}$ and can be obtained by (i) fixing the value of the coordinate of $\mathscr{Q}$ and (ii) taking the covariant derivative of the resulting object as if it were a tensor of type $T_{\mu_1 \ldots \mu_k}$ [34]. Especially, for the covariant derivative w.r.t. event $\mathscr{P}$ of a tensor purely defined at event $\mathscr{Q}$ (which is just a special type of bitensor) this implies that $\nabla_\rho T_{\nu'_1 \ldots \nu'_{k'}}$ vanishes, contrary to what we would get if we would just treat this as a constant tensor.

If we now want to compute two-point functions such as $\langle \gamma_{\mu\nu} \gamma_{\lambda'\epsilon'} \rangle$, the resulting expectation value is w.r.t. the VEV that we have selected with the frame-fixing procedure (see Sec. 3). Since we are using maximally symmetric spaces for the VEV, this implies that the resulting bitensor structure is also maximally symmetric, i.e. $\langle \gamma_{\mu\nu} \gamma_{\lambda'\epsilon'} \rangle$ represents a maximally symmetric bitensor [34, 33, 39]. Since in maximally symmetric spaces any function of two events can only depend on the geodesic distance between them, we can express every maximally symmetric bitensor in terms of the basis tensors and corresponding coefficients which depend only on the geodesic distance [33]. It will therefore be very useful to have a tensor basis for maximally symmetric spacetimes, as this will allow us to reduce the tensorial equations for $\langle \gamma_{\mu\nu} \gamma_{\lambda'\epsilon'} \rangle$ to scalar equations in the basis coefficients.

D.2. Properties of the Allen-Jacobson basis

B. Allen and T. Jacobson showed how to construct an abstract tensor basis on any maximally symmetric spacetime in Ref. [33]. Hence, this basis will be referred to as Allen-Jacobson basis and it will be the basis that is employed for the position space calculation of the Minkowski graviton and geon, see Sec.4. In principle this basis could also be used for the de Sitter space calculation, however due to its abstract nature it will turn out to be unsuitable for our approach (see Sec.5). The following presentation of the Allen-Jacobson basis is based on [33, 39, 34].

Starting from the shortest geodesic distance $r^c(\mathscr{P},\mathscr{Q})$ connecting the two points $\mathscr{P}$ and $\mathscr{Q}$, one can define the unit tangent vectors

$$n^{\mu}(\mathscr{P},\mathscr{Q}) \equiv \nabla^{\mu}_{c} r^{c}(\mathscr{P},\mathscr{Q}) \,\text{and} \tag{D.1}$$

$$n^{\mu'}(\mathscr{P},\mathscr{Q}) \equiv \nabla^{\mu'}_{c} r^{c}(\mathscr{P},\mathscr{Q}) \,, \tag{D.2}$$

at the end-points of the geodesic as well as the parallel propagator $g^{\mu}{}_{\nu'}(\mathscr{P},\mathscr{Q})$ along the geodesic. We are going to suppress the arguments from now on as the (un-)primed indices suffice to tell at which event the objects are considered. The parallel propagator takes a vector at event $\mathscr{P}$ and parallel-transports it to event $\mathscr{Q}$ along the shortest geodesic connecting them. In the context of tensors or bitensors it can be generalized to merely parallel-transporting single indices. The most important properties of the parallel propagator are discussed in Ref. [36, 33] and listed in Table D.1. For possible differences in the basis objects, due to space- or timelike separation of the endpoint, see Ref. [34].

We are going to employ the Allen-Jacobson basis only for the flat-space case, and therefore we provide the following properties of the basis elements under covariant differentiation only for the special case of a Minkowski spacetime.

As is discussed in Sec. 4.1.1, the graviton propagator can be expressed in terms of bitensors build out of products of the Allen-Jacobson basis elements, in combination with prefactors that depend only on the geodesic distance r^c. The bitensors that appear in the decomposition, as well as results regarding the covariant derivatives and trace of these objects, are listed below. These properties are obtained from the relations listed in Table D.1 and are taken in part from Ref. [39].

D.3. Solution of the graviton propagator in the frame-adapted scheme

The general solutions for the 5 coefficient functions $\{\tilde{a}(r^c), \tilde{b}(r^c), \tilde{c}(r^c), \tilde{d}(r^c), \tilde{e}(r^c)\}$ can easily be obtained by solving the equation for $O^{(5)}$ first, followed by $O^{(3)}$ and $O^{(4)}_{\mathrm{L}}$. Then we can use $O^{(4)}_{\mathrm{L}}$ to fix one of the coefficients in the general solutions for $\{\tilde{c}(r^c), \tilde{d}(r^c), \tilde{e}(r^c)\}$. After that we can solve the equations for $O^{(2)}$ and then $O^{(1)}$ can be treated at last. The equations were treated away from coincidence, just as in Ref. [39, 31, 34], thus additional ultra-local terms, which are proportional to the δ-distribution might be missing. In this

Table D.1.: Fundamental properties of the Allen-Jacobson basis elements, take from Ref. [33] and [36]

Properties of the parallel propagator:

$$
\begin{aligned}
v^{\alpha'} &= g^{\alpha'}{}_{\beta} v^{\beta} \\
w_{\alpha'} &= g^{\beta}{}_{\alpha'} w_{\beta} \\
g^{c}_{\alpha\beta} &= g^{\alpha'}{}_{\alpha} g^{\beta'}{}_{\beta} g^{c}_{\alpha'\beta'} \\
g^{\alpha'}{}_{\beta} g^{\beta}{}_{\mu'} &= \delta^{\alpha'}{}_{\mu'} \\
g^{\alpha}{}_{\beta'} g^{\beta'}{}_{\mu} &= \delta^{\alpha}{}_{\mu} \\
g_{\alpha\mu'} &= g^{c}_{\alpha\beta} g^{\beta}{}_{\mu'}
\end{aligned}
$$

Action of the parallel propagator on the geodesic tangents:

$$
\begin{aligned}
n_{\alpha'} &= -g^{\beta}{}_{\alpha'} n_{\beta} \\
n_{\alpha} &= -g^{\beta'}{}_{\alpha} n_{\beta'}
\end{aligned}
$$

Normalization of geodesic tangents:

$$
\begin{aligned}
n^{\mu} n_{\mu} &= 1 \\
n^{\mu'} n_{\mu'} &= 1
\end{aligned}
$$

Covariant derivatives of basis elements in Minkowski space:

$$
\begin{aligned}
\partial_{\mu} n_{\nu} &= (r^{c})^{-1} \left(g^{c}_{\mu\nu} - n_{\mu} n_{\nu} \right) \\
\partial_{\mu} n_{\lambda'} &= -(r^{c})^{-1} \left(g_{\mu\lambda'} - n_{\mu} n_{\lambda'} \right) \\
\partial_{\mu} g_{\nu\lambda'} &= 0 \\
\partial^{\mu} n_{\mu} &= (d-1)(r^{c})^{-1} \\
\partial^{\mu} \left(n_{\mu} n_{\lambda'} \right) &= (d-1) n_{\lambda'}
\end{aligned}
$$

Where d=4 for a 4-dimensional spacetime.

case the general solutions for $\{\tilde{a}(r^c), \tilde{b}(r^c), \tilde{c}(r^c), \tilde{d}(r^c), \tilde{e}(r^c)\}$ are:

$$\tilde{a} = a_2 + \frac{e_2}{48\mu^6} - \frac{d_2}{2\mu^4} - \frac{x_1}{2\mu^4} - \frac{a_1}{2\mu^2} - \frac{d_1}{2}\mu^2 + \frac{e_1}{48}\mu^4 \tag{D.3}$$

$$\tilde{b} = b_2 + \frac{e_2}{48\mu^6} + \frac{c_2}{2\mu^4} - \frac{x_1}{2\mu^4} - \frac{b_1}{2\mu^2} + \frac{c_1}{2}\mu^2 + \frac{e_1}{48}\mu^4 \tag{D.4}$$

$$\tilde{c} = c_1\mu^2 + \frac{c_2}{\mu^4} + \frac{e_2 + 3f_1\mu^4 + e_1\mu^10}{8\mu^6} \tag{D.5}$$

$$\tilde{d} = d_1\mu^2 + \frac{d_2}{\mu^4} + \frac{-e_2 + 5f_1\mu^4 - e_1\mu^10}{8\mu^6} \tag{D.6}$$

$$\tilde{e} = e_1\mu^4 + \frac{e_2}{\mu^6} + \frac{24x_1 - f_1\mu^2}{2\mu^4} \tag{D.7}$$

For the frame-adapted approach we also required that the transversality condition (4.33) is fulfilled. Acting the derivative on any index of $G_{\mu\nu;\lambda'\epsilon'}$ and using the ansatz (4.14) with the properties of the Allen-Jacobson basis, this provides three conditions [39]

$$a' + d' + 3r_c^{-1}d - 2r_c^{-1}c = 0, \tag{D.8}$$

$$e' + 3r_c^{-1}e + d' - 2r_c^{-1}d\,(\mu) - 2c' + 4r_c^{-1}c = 0 \text{ and} \tag{D.9}$$

$$-b' - r_c^{-1}d - c' + 4r_c^{-1}c = 0. \tag{D.10}$$

And the three transversality conditions translate to the following relations:

$$
\begin{aligned}
e_1 &= 0 && \text{(D.11)} \\
48c_2 - 24d_2 - 48x_1 &= 0 && \text{(D.12)} \\
2c_2 - d_2 - 2x_1 &= 0 && \text{(D.13)} \\
\frac{f_1}{8} - b_1 &= 0 && \text{(D.14)} \\
5c_1 - d_1 &= 0 && \text{(D.15)}
\end{aligned}
$$

These can be applied to the solutions without any problem and therefore the system is solved.

D.4. Identity for auxiliary functions in frame-adapted scheme

Based on the definitions

$$
\begin{aligned}
\Box_{\mathrm{M}}\phi\left(r^c\right) &= \delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right) \text{ and} && \text{(D.16)} \\
\Box_{\mathrm{M}}\xi\left(r^c\right) &= \phi\left(r^c\right), && \text{(D.17)}
\end{aligned}
$$

we can provide an expression for ξ'' in terms of ϕ and ξ', by using the identity (4.19). It has the form

$$
\xi'' = \phi - 3r_c^{-1}\xi'. \tag{D.18}
$$

By deriving both sides w.r.t. to the geodesic distance we can then also provide expressions of ξ''' and ξ'''' in terms of derivatives of ϕ up to second order and ξ'

$$
\begin{aligned}
\xi''' &= \phi' - 3r_c^{-1}\phi + 12r_c^{-2}\xi' \text{ and} \\
\xi'''' &= \phi'' - 3r_c^{-1}\phi' + 15r_c^{-2}\phi - 60r_c^{-3}\xi'. && \text{(D.19)}
\end{aligned}
$$

Now we can insert these two identities into the definition of the auxiliary functions (4.48) and the expression (4.58) becomes

$$
\begin{aligned}
4\omega - 20\sigma - 3\tau + 24\psi &\overset{(D.19)}{=} \phi'' + 3r_c^{-1}\phi' + 117r_c^{-3}\xi' - 39r_c^{-2}\left(\phi - \xi''\right) \\
&\overset{(4.19)}{=} \Box_{\mathrm{M}}\phi - 39r_c^{-2}\phi + 39r_c^{-2}39\Box_{\mathrm{M}}\xi \\
&\overset{(D.17)}{=} \Box_{\mathrm{M}}\phi \\
&\overset{(D.16)}{=} \delta^{(4)}\left(x_{\mathscr{P}} - x_{\mathscr{Q}}\right). && \text{(D.20)}
\end{aligned}
$$

Table D.2.: The most important properties of the parallel propagator are discussed in Ref. [36, 33] and listed in Table D.1.

Bitensors in the graviton propagator:

$$O^{(1)}_{\mu\nu;\lambda'\epsilon'} = \eta_{\mu\nu}\eta_{\lambda'\epsilon'}$$
$$O^{(2)}_{\mu\nu;\lambda'\epsilon'} = 2g_{\mu(\lambda'|}g_{\nu|\epsilon')}$$
$$O^{(3)}_{\mu\nu;\lambda'\epsilon'} = 4n_{(\mu}g_{\nu)(\lambda'}n_{\epsilon')}$$
$$O^{(4)}_{L\ \mu\nu;\lambda'\epsilon'} = n_\mu n_\nu \eta_{\lambda'\epsilon'}$$
$$O^{(4)}_{R\ \mu\nu;\lambda'\epsilon'} = \eta_{\mu\nu} n_{\lambda'} n_{\epsilon'}$$
$$O^{(4)}_{\mu\nu;\lambda'\epsilon'} = O^{(4)}_{L\ \mu\nu;\lambda'\epsilon'} + O^{(4)}_{R\ \mu\nu;\lambda'\epsilon'}$$
$$O^{(5)}_{\mu\nu;\lambda'\epsilon'} = n_\mu n_\nu n_{\lambda'} n_{\epsilon'}$$

Partial trace of the basis tensors:

$$\eta^{\mu\nu}O^{(1)}_{\mu\nu;\lambda'\epsilon'} = 4\eta_{\lambda'\epsilon'}$$
$$\eta^{\mu\nu}O^{(2)}_{\mu\nu;\lambda'\epsilon'} = 2\eta_{\lambda'\epsilon'}$$
$$\eta^{\mu\nu}O^{(3)}_{\mu\nu;\lambda'\epsilon'} = -4n_{\lambda'}n_{\epsilon'}$$
$$\eta^{\mu\nu}O^{(4)}_{L\ \mu\nu;\lambda'\epsilon'} = \eta_{\lambda'\epsilon'}$$
$$\eta^{\mu\nu}O^{(4)}_{R\ \mu\nu;\lambda'\epsilon'} = 4n_{\lambda'}n_{\epsilon'}$$
$$\eta^{\mu\nu}O^{(5)}_{\mu\nu;\lambda'\epsilon'} = n_{\lambda'}n_{\epsilon'}$$

Contraction with $\mathcal{A}^{1,\mathrm{M}}$:

$$\mathcal{A}^{1,\mathrm{M}\mu\nu}_{\alpha\beta}O^{(j)}_{\mu\nu;\lambda'\epsilon'} = O^{(j)}_{\alpha\beta;\lambda'\epsilon'}$$

Non-minimal, covariant derivative of the basis tensors, first order:

$$\partial_\alpha O^{(1)}_{\mu\nu;\lambda'\epsilon'} = 0 \text{ and } \partial_\alpha O^{(2)}_{\mu\nu;\lambda'\epsilon'} = 0$$
$$\begin{aligned}\partial_\alpha O^{(3)}_{\mu\nu;\lambda'\epsilon'} = r_c^{-1}\big(&4\eta_{\alpha(\mu}g_{\nu)(\lambda'}n_{\epsilon')}\\ &-2n_\alpha O^{(3)}_{\mu\nu;\lambda'\epsilon'} - 2n_{(\mu}O^{(2)}_{\nu)\alpha;\lambda'\epsilon'}\big)\end{aligned}$$
$$\begin{aligned}\partial_\alpha O^{(4)}_{\mu\nu;\lambda'\epsilon'} = r_c^{-1}\big(&2\eta_{\alpha(\mu}n_{\nu)}\eta_{\lambda'\epsilon'}\\ &-2n_\alpha O^{(4)}_{\mu\nu;\lambda'\epsilon'} - 2\eta_{\mu\nu}g_{\alpha(\lambda'}n_{\epsilon')}\big)\end{aligned}$$
$$\begin{aligned}\partial_\alpha O^{(5)}_{\mu\nu;\lambda'\epsilon'} = r_c^{-1}\big(&2\eta_{\alpha(\mu}n_{\nu)}n_{\lambda'}n_{\epsilon'}\\ &-4n_\alpha O^{(5)}_{\mu\nu;\lambda'\epsilon'} - 2n_\mu n_\nu g_{\alpha(\lambda'}n_{\epsilon')}\big)\end{aligned}$$

Second order:

$$\partial_\alpha\partial_\beta O^{(1)}_{\mu\nu;\lambda'\epsilon'} = 0 \text{ and } \partial_\alpha\partial_\beta O^{(2)}_{\mu\nu;\lambda'\epsilon'} = 0$$
$$\begin{aligned}\partial_\alpha\partial_\beta O^{(3)}_{\mu\nu;\lambda'\epsilon'} = r_c^{-2}\big(&-16n_{(\alpha}\eta_{\beta)(\mu}g_{\nu)(\lambda'}n_{\epsilon')}\\ &+2\left(4n_\alpha n_\beta - \eta_{\alpha\beta}\right)O^{(3)}_{\mu\nu;\lambda'\epsilon'}\\ &+4\big(2n_{(\alpha|}n_{(\mu} - \eta_{(\alpha|(\mu}\big)\times\\ &\times O^{(2)}_{\mu)|\beta);\lambda'\epsilon'}\big)\end{aligned}$$
$$\begin{aligned}\partial_\alpha\partial_\beta O^{(4)}_{\mu\nu;\lambda'\epsilon'} = r_c^{-2}\big(&2\big(\eta_{\alpha(\mu}\eta_{\nu)\beta}\\ &-4n_{(\alpha}\eta_{\beta)(\mu}n_{\nu)}\big)\eta_{\lambda'\epsilon'}\\ &+2\left(4n_\alpha n_\beta - \eta_{\alpha\beta}\right)O^{(4)}_{\mu\nu;\lambda'\epsilon'}\\ &+\eta_{\mu\nu}\left(O^{(2)}_{\alpha\beta;\lambda'\epsilon'} + 2O^{(3)}_{\alpha\beta;\lambda'\epsilon'}\right)\big)\end{aligned}$$
$$\begin{aligned}\partial_\alpha\partial_\beta O^{(5)}_{\mu\nu;\lambda'\epsilon'} = r_c^{-2}\big(&4\left(6n_\alpha n_\beta - \eta_{\alpha\beta}\right)\times\\ &\times O^{(5)}_{\mu\nu;\lambda'\epsilon'} + 2\big(\eta_{\alpha(\mu}\eta_{\nu)\beta}\\ &-8n_{(\alpha}\eta_{\beta)(\mu}n_{\nu)}\big)n_{\lambda'}n_{\epsilon'}\\ &+n_\mu n_\nu\left(O^{(2)}_{\alpha\beta;\lambda'\epsilon'} + 4O^{(3)}_{\alpha\beta;\lambda'\epsilon'}\right)\\ &-8\eta_{(\alpha|(\mu}n_{\nu)}g_{|\beta)(\lambda'}n_{\epsilon')}\big)\end{aligned}$$

d'Alembertian of these basis tensors (indices are suppressed):

$$\Box_{\mathrm{M}}O^{(1)} = 0$$
$$\Box_{\mathrm{M}}O^{(2)} = 0$$
$$\Box_{\mathrm{M}}O^{(3)} = -4r_c^{-2}\left(2O^{(3)} + O^{(2)}\right)$$
$$\Box_{\mathrm{M}}O^{(4)}_L = 2r_c^{-2}\left(O^{(1)} - 4O^{(4)}_L\right)$$
$$\Box_{\mathrm{M}}O^{(4)}_R = 2r_c^{-2}\left(O^{(1)} - 4O^{(4)}_R\right)$$
$$\Box_{\mathrm{M}}O^{(5)} = 2r_c^{-2}\left(-12O^{(5)} - O^{(3)} + O^{(4)}\right)$$

E. de Sitter space calculation

E.1. Additional properties of the employed basis

Table E.1.: Fundamental properties of the de Sitter space basis elements.

Bitensors in the graviton propagator:

$$S^{(1)}_{\mu\nu;\lambda'\epsilon'} = g^c_{\mu\nu} g^c_{\lambda'\epsilon'}$$
$$S^{(2)}_{\mu\nu;\lambda'\epsilon'} = 2p_{\mu(\lambda'|}p_{\nu|\epsilon')}$$
$$S^{(3)}_{\mu\nu;\lambda'\epsilon'} = 4H^2 p_{(\mu}p_{\nu)(\lambda'}p_{\epsilon')}$$
$$S^{(4)}_{\mathrm{L}\ \mu\nu;\lambda'\epsilon'} = H^2 p_\mu p_\nu g^c_{\lambda'\epsilon'}$$
$$S^{(4)}_{\mathrm{R}\ \mu\nu;\lambda'\epsilon'} = H^2 g^c_{\mu\nu} p_{\lambda'} p_{\epsilon'}$$
$$S^{(4)}_{\mu\nu;\lambda'\epsilon'} = S^{(4)}_{\mathrm{L}\ \mu\nu;\lambda'\epsilon'} + S^{(4)}_{\mathrm{R}\ \mu\nu;\lambda'\epsilon'}$$
$$S^{(5)}_{\mu\nu;\lambda'\epsilon'} = H^4 p_\mu p_\nu p_{\lambda'} p_{\epsilon'}$$
$$T^{(1)}_{\nu;\lambda'\epsilon'} = H^3 p_\nu p_{\lambda'} p_{\epsilon'}$$
$$T^{(2)}_{\nu;\lambda'\epsilon'} = H p_\nu g^c_{\lambda'\epsilon'}$$
$$T^{(3)}_{\nu;\lambda'\epsilon'} = H\left(p_{\nu\lambda'} p_{\epsilon'} + p_{\nu\epsilon'} p_{\lambda'}\right)$$

Covariant derivatives of basis elements, first order:

$$\nabla^\mu_c \left[{}_{\mu\nu} S_{1\lambda'\epsilon'}\right] = 0$$
$$\nabla^\mu_c \left[{}_{\mu\nu} S_{2\lambda'\epsilon'}\right] = -5HT^{(3)}_{\nu;\lambda'\epsilon'}$$
$$\nabla^\mu_c \left[{}_{\mu\nu} S_{3\lambda'\epsilon'}\right] = H\left(-20T^{(1)}_{\nu;\lambda'\epsilon'} + 4T^{(2)}_{\nu;\lambda'\epsilon'} -2H^2 p T^{(3)}_{\nu;\lambda'\epsilon'}\right)$$
$$\nabla^\mu_c \left[{}_{\mu\nu} S_{4\lambda'\epsilon'}\right] = H\left(-5H^2 p T^{(2)}_{\nu;\lambda'\epsilon'} + T^{(3)}_{\nu;\lambda'\epsilon'}\right)$$
$$\nabla^\mu_c \left[{}_{\mu\nu} S_{5\lambda'\epsilon'}\right] = -7H^3 p T^{(1)}_{\nu;\lambda'\epsilon'}$$

With the knowledge about the behavior of the basis tensors under covariant differentiation, see Table above, we can compute the derivative part of the condition (5.11), using the ansatz (5.21) and write it in the following decomposition

$$\nabla^\mu_c G_{\mu\nu;\lambda'\epsilon'} = t_1(p)T^{(1)}_{\nu;\lambda'\epsilon'} + t_2(p)T^{(2)}_{\nu;\lambda'\epsilon'} + t_3(p)T^{(3)}_{\nu;\lambda'\epsilon'}, \text{ with} \tag{E.1}$$
$$t_1(p) = -7H^3 pe(p) - 20Hc(p) + H^{-1}\Delta e'(p) - 2H^{-1}pc'(p) + H^{-3}d'(p), \tag{E.2}$$
$$t_2(p) = 4Hc(p) - 5H^3 pd(p) + H^{-1}a'(p) + H\Delta d'(p) \text{ and} \tag{E.3}$$
$$t_3(p) = -5Hb(p) - 2H^3 pc(p) + Hd(p) - Hpb'(p) + H\Delta c'(p). \tag{E.4}$$
$$\Delta = -p^2 + H^{-4} \tag{E.5}$$

As can be seen from this, the l.h.s. of the condition (5.11) is easily treated in our constructed basis, it is only the r.h.s. that creates problems, as is discussed in Sec. 5.1.

Bibliography

[1] A. Maas, "The Fröhlich-Morchio-Strocchi mechanism and quantum gravity," *SciPost Phys.*, vol. 8, p. 51, 2020.

[2] J. L. Feng, "Dark Matter Candidates from Particle Physics and Methods of Detection," *Ann. Rev. Astron. Astrophys.*, vol. 48, pp. 495–545, 2010.

[3] N. Bernal, X. Chu, S. Kulkarni, and J. Pradler, "Self-interacting dark matter without prejudice," *Phys. Rev. D*, vol. 101, no. 5, p. 055044, 2020.

[4] A. Einstein, "Does the inertia of a body depend upon its energy-content," *Ann Phys*, vol. 18, pp. 639–641, 1905.

[5] A. Deur, "Relativistic corrections to the rotation curves of disk galaxies," 4 2020.

[6] J. A. Wheeler, "Geons," *Physical Review*, vol. 97, no. 2, p. 511, 1955.

[7] D. R. Brill and J. B. Hartle, "Method of the self-consistent field in general relativity and its application to the gravitational geon," *Physical Review*, vol. 135, no. 1B, p. B271, 1964.

[8] P. R. Anderson and D. R. Brill, "Gravitational geons revisited," *Physical Review D*, vol. 56, no. 8, p. 4824, 1997.

[9] G. P. Perry and F. I. Cooperstock, "Stability of gravitational and electromagnetic geons," *Classical and Quantum Gravity*, vol. 16, no. 6, p. 1889, 1999.

[10] F. Cooperstock, V. Faraoni, and G. Perry, "Probing the gravitational geon," *International Journal of Modern Physics D*, vol. 5, no. 04, pp. 375–406, 1996.

[11] B. Guiot, A. Borquez, A. Deur, and K. Werner, "Graviballs and dark matter," *arXiv preprint arXiv:2006.02534*, 2020.

[12] S. Coleman and L. Smarr, "Are there geon analogues in sourceless gauge-field theories?," *Communications in Mathematical Physics*, vol. 56, no. 1, pp. 1–9, 1977.

[13] V. Mathieu, N. Kochelev, and V. Vento, "The physics of glueballs," *International Journal of Modern Physics E*, vol. 18, no. 01, pp. 1–49, 2009.

[14] A. Maas, "Brout-Englert-Higgs physics: From foundations to phenomenology," *Prog. Part. Nucl. Phys.*, vol. 106, pp. 132–209, 2019.

[15] R. M. Wald, *General Relativity.* Chicago, USA: Chicago Univ. Pr., 1984.

[16] C. W. Misner, K. Thorne, and J. Wheeler, *Gravitation.* San Francisco: W. H. Freeman, 1973.

[17] W. Cordeiro dos Santos, “Non-coordinates basis in General Relativity and Cartan’s structure equations,” 11 2017.

[18] M. Blau, *Lecture notes on general relativity.* Albert Einstein Center for Fundamental Physics, 2011.

[19] H.-S. Roh, “Fundamental constants and conservation laws,” 2 2001.

[20] R. Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety*, vol. 3 of *100 Years of General Relativity.* World Scientific, 2017.

[21] J. M. Pawlowski and M. Reichert, “Quantum gravity: a fluctuating point of view,” 7 2020.

[22] J. B. Jiménez, L. Heisenberg, and T. S. Koivisto, “The Geometrical Trinity of Gravity,” *Universe*, vol. 5, no. 7, p. 173, 2019.

[23] F. W. Hehl, *Gauge Theory of Gravity and Spacetime*, vol. 13, pp. 145–169. 2017.

[24] R. Sondenheimer, “Analytical relations for the bound state spectrum of gauge theories with a Brout-Englert-Higgs mechanism,” *Phys. Rev. D*, vol. 101, no. 5, p. 056006, 2020.

[25] S. Weinberg and O. Greenberg, “The quantum theory of fields, vol. 1: Foundations,” *PhT*, vol. 48, no. 11, p. 78, 1995.

[26] S. M. Carroll, “The Cosmological constant,” *Living Rev. Rel.*, vol. 4, p. 1, 2001.

[27] M. E. Peskin and D. V. Schroeder, *An Introduction to quantum field theory.* Reading, USA: Addison-Wesley, 1995.

[28] T. Muller and F. Grave, “Catalogue of Spacetimes,” 4 2009.

[29] J. Haywood, “The equations of motion and coordinate condition in general relativity,” *Proceedings of the Physical Society. Section A*, vol. 65, no. 3, p. 170, 1952.

[30] P. Mora, N. Tsamis, and R. Woodard, “Graviton Propagator in a General Invariant Gauge on de Sitter,” *J. Math. Phys.*, vol. 53, p. 122502, 2012.

[31] M. Turyn, “The graviton propagator in maximally symmetric spaces,” *Journal of mathematical physics*, vol. 31, no. 3, pp. 669–679, 1990.

[32] H. Nishino and Y. Fujii, “On the gauge conditions in quantum gravity,” *Progress of Theoretical Physics*, vol. 58, no. 1, pp. 381–383, 1977.

[33] B. Allen and T. Jacobson, “Vector two-point functions in maximally symmetric spaces,” *Communications in Mathematical Physics*, vol. 103, no. 4, pp. 669–692, 1986.

[34] G. Pérez-Nadal, A. Roura, and E. Verdaguer, "Stress tensor fluctuations in de sitter spacetime," *Journal of Cosmology and Astroparticle Physics*, vol. 2010, no. 05, p. 036, 2010.

[35] M. Bohm, A. Denner, and H. Joos, *Gauge theories of the strong and electroweak interaction.* 2001.

[36] E. Poisson, A. Pound, and I. Vega, "The motion of point particles in curved spacetime," *Living Reviews in Relativity*, vol. 14, no. 1, p. 7, 2011.

[37] B. S. DeWitt and G. Esposito, "An Introduction to quantum gravity," *Int. J. Geom. Meth. Mod. Phys.*, vol. 5, pp. 101–156, 2008.

[38] L. Shao, D. Shao, C. Shao, and H. Noda, "Calculation of curvature vacuum correlations in R-gravity," *Gen. Rel. Grav.*, vol. 33, pp. 1519–1534, 2001.

[39] B. Allen and M. Turyn, "An evaluation of the graviton propagator in de sitter space," *Nuclear Physics B*, vol. 292, pp. 813–852, 1987.

[40] M. Turyn, "Curvature fluctuations in maximally symmetric spaces," *Physical Review D*, vol. 39, no. 10, p. 2951, 1989.

[41] S.-P. Miao, N. Tsamis, and R. Woodard, "The graviton propagator in de donder gauge on de sitter background," *Journal of mathematical physics*, vol. 52, no. 12, p. 122301, 2011.

[42] N. Tsamis and R. Woodard, "Maximally symmetric vector propagator," *Journal of mathematical physics*, vol. 48, no. 5, p. 052306, 2007.

[43] S. Weinberg, *The quantum theory of fields. Vol. 2: Modern applications.* Cambridge University Press, 8 2013.

[44] H.-H. Zhang, K.-X. Feng, S.-W. Qiu, A. Zhao, and X.-S. Li, "On analytic formulas of Feynman propagators in position space," *Chin. Phys. C*, vol. 34, pp. 1576–1582, 2010.

[45] G. Modanese, "Vacuum correlations at geodesic distance in quantum gravity," *Riv. Nuovo Cim.*, vol. 17N8, pp. 1–62, 1994.

[46] L. Shao, D. Shao, and C. Shao, "Vacuum correlations in quantized gravity," *Gen. Rel. Grav.*, vol. 35, pp. 1161–1173, 2003.

[47] E. Kahya and R. Woodard, "Charged scalar self-mass during inflation," *Physical Review D*, vol. 72, no. 10, p. 104001, 2005.

[48] R. Woodard, "de sitter breaking in field theory," in *Deserfest*, pp. 339–351, World Scientific, 2006.

[49] V. K. Onemli and R. P. Woodard, "Super-acceleration from massless, minimally coupled $\phi 4$," *Classical and Quantum Gravity*, vol. 19, no. 17, p. 4607, 2002.

[50] N. Tsamis and R. Woodard, "The structure of perturbative quantum gravity on a de sitter background," *Communications in mathematical physics*, vol. 162, no. 2, pp. 217–248, 1994.

[51] M. Spradlin, A. Strominger, and A. Volovich, "Les houches lectures on de sitter space,(2001)," *arXiv preprint hep-th/0110007*.

[52] M. Takook, "Quantum field theory in the bulk de sitter space-time," tech. rep., 2014.

[53] D. Capper and G. Leibbrandt, "On Ward Identities in a General Axial Gauge. 2. Quantum Gravity," *Phys. Rev. D*, vol. 25, p. 1009, 1982.