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# On Fresen's Proof of the CLT for Convex Bodies

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# AFFIDAVIT

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# Abstract

In 2019 Daniel Fresen worked out a proof of the central limit theorem (CLT) for convex bodies which he claimed to be significantly simpler than the original proof published by Bo'az Klartag in ["A Central Limit Theorem for Convex Sets". In: *Invent. Math.* 168.1 (Jan. 2007), pp. 91–131]. In the present thesis we discuss Fresen's proof in detail and assess the claim of simplicity.

In the first chapter firstly we formulate the problem, that is, how a CLT for convex bodies is to be understood: a CLT for convex bodies means that the projections onto onedimensional subspaces of the uniform distribution on an isotropic convex body in $\mathbb{R}^n$ are close to an appropriately scaled normal distribution, with both the measure of the exceptional subspaces and the deviation from normality converging to zero as $n \to \infty$. Secondly we trace the milestones in the history of that problem, citing the important precursors and subsequent results.

The second chapter treats the mathematical background needed to establish Fresen's arguments, some of the results being of general use, others in specific need of proving the CLT for convex bodies.

The third chapter is entirely devoted to elaborate on the details of Fresen's proof. The three main steps of it are as follows: first, show that the projections onto most low-dimensional subspaces of the original distribution convolved with a normal distribution of small variance are approximately radially symmetric; the main tools employed are Fourier-transform and concentration of meausre on the sphere. Second, establish a "thin-shell-property" for these low-dimensional projections. And third, from that property derive that most onedimensional projections of the original distribution are close to normal.

In the fourth and final chapter we summarize our conclusions drawn from working on the proof.

# Kurzfassung

2019 arbeitete Daniel Fresen eine Beweis des zentralen Grenzwertsatzes (ZGS) für konvexe Körper aus, der seiner Behauptung nach wesentlich einfacher sei als der ursprüngliche Beweis, den Bo'az Klartag in [“A Central Limit Theorem for Convex Sets”. In: *Invent. Math.* 168.1 (Jän. 2007), SS. 91–131] veröffentlicht hatte. In der vorliegenden Arbeit diskutieren wir Fresens Beweis im Detail und bewerten den Anspruch der Einfachheit.

Im ersten Kapitel formulieren wir erstens das Problem, das heißt, wie ein ZGS für konvexe Körper zu verstehen sei: Ein ZGS für konvexe Körper heißt, dass die Projektionen auf eindimensionale Unterräume der Gleichverteilung auf einem isotropen konvexen Körper im $\mathbb{R}^n$ nahe einer entsprechend skalierten Normalverteilung sind, wobei sowohl das Maß der Ausnahmeunterräume als auch die Abweichung von der Normalverteilung für $n \to \infty$ gegen null konvergieren. Zweitens spüren wir den Meilensteinen in der Geschichte dieses Problems nach, wobei wir wichtige Vorgänger- und Nachfolgeergebnisse anführen.

Das zweite Kapitel behandelt den mathematischen Hintergrund, der zum Aufstellen von Fresens Argumenten benötigt wird; einige Ergebnisse sind von allgemeinem Nutzen, andere werden speziell für den Beweis des ZGS für konvexe Körper gebraucht.

Das dritte Kapitel ist ganz der Ausarbeitung der Details von Fresens Beweis gewidmet. Die drei Hauptschritte desselben sind wie folgt: Als Erstes zeigt man, dass die Projektionen auf die meisten niederdimensionalen Unterräume der Ausgangsverteilung, gefaltet mit einer Normalverteilung kleiner Varianz, annähernd radialsymmetrisch sind; die Hauptwerkzeuge dabei sind Fouriertransformation und Maßkonzentration auf der Sphäre. Als Zweites wird eine „Dünne-Schale-Eigenschaft“ (engl. „thin-shell-property“) dieser niederdimensionalen Projektionen nachgewiesen. Und als Drittes wird aus jener Eigenschaft hergeleitet, dass die meisten eindimensionalen Projektionen der Ausgangsverteilung nahe einer Normalverteilung sind.

Im vierten und letzten Kapitel fassen wir unsere Schlüsse zusammen, die wir aus der Arbeit am Beweis gezogen haben.

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Mein inniger Dank gilt zuvorderst meinen Eltern, die mich all die Jahre unterstützt haben, und meiner Familie.

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# Contents

# List of Tables

# Basic notation

$\mathbb{N}$ denotes the set of natural numbers starting with one, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; interval-notation is used whenever confusion with intervals of reals is unlikely. Throughout let $n \in \mathbb{N}$ be the dimension of the space, arbitrary unless explicitly stated otherwise. $o \in \mathbb{R}^n$ is the zero-vector. We set $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{R}_{>0} := (0, \infty)$.

For sets $A, B \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}$, the *Minkowski-sum* is defined by $A + B := \{x + y | x \in A \wedge y \in B\}$, with the special case of translation $a + B := \{a\} + B$; furthermore $\Lambda A := \{\lambda a | \lambda \in \Lambda \wedge a \in A\}$, with the special cases of scaling $\lambda A := \{\lambda\} A$ and $\Lambda a := \Lambda\{a\}$, in particular $\mathbb{R}a$ for $a \neq o$ is the onedimensional subspace with direction $a$.

$\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, that is $\langle x, y \rangle := x^\mathsf{T} y$ for $x, y \in \mathbb{R}^n$ (considered to be column-vectors, $^\mathsf{T}$ denotes matrix-transpose), and $\|\cdot\|$ is the standard Euclidean norm, that is $\|x\| := \langle x, x \rangle^{1/2}$; $B^n := \{x \in \mathbb{R}^n | \|x\| \leq 1\}$ is the closed unit-ball and $S^{n-1} := \{x \in \mathbb{R}^n | \|x\| = 1\}$ the unit-sphere.

All random-variables are defined on a common probability-space $(\Omega, \mathcal{F}, \mathbb{P})$, expectation and variance w.r.t. $\mathbb{P}$ are denoted by $\mathbb{E}$ and Var respectively. For a topological space $X$ let $\mathcal{B}(X)$ be its Borel-$\sigma$-algebra. $v_n$ is the Lebesgue-measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, additionally for $n = 0$ we set $\mathbb{R}^0 := \{0\}$ and $v_0 := \delta_0$ (the Dirac-measure at 0). $\sigma_{n-1}$ is the surface-measure on $S^{n-1}$ (viewed as an embedded manifold) scaled to $\sigma_{n-1}(S^{n-1}) = 1$, it is the unique orthogonally invariant probability-measure on $(S^{n-1}, \mathcal{B}(S^{n-1}))$. Any nonnegative, measurable function whose integral is 1 is called a *density*; integrals of vector- or matrix-valued functions are defined componentwise. For $A \subset \mathbb{R}^n$ define the indicator-function $1_A \colon \mathbb{R}^n \to \mathbb{R}$ by $1_A(x) := 1$, if $x \in A$, and $1_A(x) := 0$ otherwise. log is the natural logarithm.

The harpoon $\upharpoonright$ is used for various senses of restrictions: for a map $f \colon A \to B$ and $C \subset A$, the map $f{\upharpoonright}C \colon C \to B$ is usually defined by $(f{\upharpoonright}C)(x) := f(x)$ for all $x \in C$, but for a $\sigma$-algebra $\mathcal{A}$ and a measure $\mu \colon \mathcal{A} \to [0, \infty]$ and $C \in \mathcal{A}$ we define $\mathcal{A}{\upharpoonright}C := \{A \cap C | A \in \mathcal{A}\}$ and the measure $\mu{\upharpoonright}C \colon \mathcal{A}{\upharpoonright}C \to [0, \infty]$ by $(\mu{\upharpoonright}C)(A \cap C) := \mu(A \cap C)$ for all $A \in \mathcal{A}$.

Other notation shall be introduced where appropriate. As far as possible the notation of cited authors is adapted to ours.

# 1 Introduction

## 1.1 Problem-formulation

A *central limit-theorem (CLT)* is the statement that a sequence of random-variables $(X_n)_{n\geq 1}$ converges in distribution to a normal, or *Gaußian*, distribution. There are various setups: the $X_n$'s may be scalar or multivariate, and accordingly the limit distribution is one- or multi-dimensional; often some standardization is demanded such that the limit distribution has zero expectation and unit (co-)variance; the $X_n$'s may be compound, that is, of the form $X_n = \sum_{k=1}^{K_n} X_{n,k}$ (with $(K_n)_{n\geq 1} \subset \mathbb{N}$, often strictly increasing); other variations are possible. The task usually is to formulate sufficient conditions (and also necessary ones if possible) on $(X_n)_{n\geq 1}$ in order for a CLT to hold.

A classical CLT – if not considered to be downright *the* CLT – is that of Lindeberg; throughout $\varphi$ denotes the probability-density-function (PDF) of the standard normal distribution, $\varphi(z) = (2\pi)^{-\frac{n}{2}}\,\mathsf{e}^{-\frac{1}{2}\|z\|^2}$ with $z \in \mathbb{R}^n$ (dimension as the situation demands), and in the onedimensional case $\Phi$ its cumulative distribution-function (CDF), $\Phi(z) = \int_{-\infty}^{z} \varphi(t)\,dt$ with $z \in \mathbb{R}$.

**Theorem 1.1** (CLT, Lindeberg 1922)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of independet random-variables which are centred, i.e.* $\mathbb{E}[X_n] = 0$*, and* $\mathrm{Var}[X_n] \in \mathbb{R}_{>0}$*; denote* $s_n^2 := \sum_{i=1}^{n} \mathrm{Var}[X_i]$*. If the* Lindeberg-condition

$$\forall \varepsilon \in \mathbb{R}_{>0}\colon \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E}\big[X_i^2 1_{[|X_i|>\varepsilon s_n]}\big] = 0, \tag{1.1}$$

*holds, then* $\big(\frac{1}{s_n}\sum_{i=1}^{n} X_i\big)_{n\geq 1}$ *converges in distribution to a standard normal distribution, that is*

$$\forall t \in \mathbb{R}\colon \lim_{n\to\infty} \Big|\mathbb{P}\Big[\frac{1}{s_n}\sum_{i=1}^{n} X_i \leq t\Big] - \Phi(t)\Big| = 0.$$

*If in particular the random-variables $X_n$ are independent and identically distributed ("i.i.d.") with* $\mathbb{E}[X_1] = 0$ *and* $\mathrm{Var}[X_1] = \sigma^2 \in \mathbb{R}_{>0}$*, then condition* (1.1) *is satisfied and hence* $\big(\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{n} X_i\big)_{n\geq 1}$ *converges in distribution to a standard normal distribution.*
(Multivariate versions do exist.)

While this CLT often is understood as a statement about the asymptotic distribution of the mean $\frac{1}{n}\sum_{i=1}^{n} X_i$, and this view is widespread and indeed useful especially in statistics, it also allows for another interpretation, as follows. For simplicity assume that all $X_n$ are scalar, centred and have variance one, then $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} X_i = \langle \theta^{(n)}, X^{(n)}\rangle$ if we put $\theta^{(n)} := \frac{1}{\sqrt{n}}(1,\dots,1)^\mathsf{T}$ and $X^{(n)} := (X_1,\dots,X_n)^\mathsf{T}$. Note that $\theta^{(n)} \in S^{n-1}$, hence $\langle\theta^{(n)}, X^{(n)}\rangle$ can be considered as the projection (always orthogonal here) of $X^{(n)}$ onto the onedimensional subspace $\mathbb{R}\theta^{(n)}$, and $\mathbb{E}[X^{(n)}] = o$, $\mathrm{Var}[X^{(n)}] = I_n$ (the $(n \times n)$-unit-matrix). Therefore we can state that there exists a onedimensional projection of $X^{(n)}$ which approximately follows a standard normal distribution.

This reformulation opens the door to further generalization: do not consider the sequence $(X_n)_{n\geq 1}$ of random-variables, but rather the joint distribution $\mu_n$ of $(X_1,\dots,X_n)^\mathsf{T}$ on $\mathbb{R}^n$ and drop the independence of the coordinates; also take projections onto $k$-dimensional subspaces, for $k \in [1, n-1]$. The image-measure under the projection onto a $k$-dimensional subspace shall be referred to as a *$k$-dimensional marginal* of the original measure. *The task now is to identify those high-dimensional distributions for which there exist approximately normal $k$-dimensional marginals, or more strongly, for which most marginals are approximately normal.*

"Approximately normal" usually means, close to a normal distribution at least with respect to the weak topology of (finite) measures; stronger modes of convergence are welcome. The meaning of "most marginals" can be motivated in the following way: as already noted, onedimensional projections are induced by unit-vectors, that is, elements of $S^{n-1}$; the sphere has finite surface-measure, which therefore can be scaled to have total mass 1; "most directions $\theta$" then means, for all $\theta$ in a subset of $S^{n-1}$ with measure close to one. Preferably the measure of the set of exceptional directions should converge to zero for $n \to \infty$.

As the CLT of Lindeberg shows, independence of coordinates imposes sufficient structure on high-dimensional distributions in order to yield approximate normality of low-dimensional marginals. Another possible structure is convexity, which issues in a CLT for convex sets, the history of which is rolled up briefly in Section 1.3 (see also Klartag [24, p. 402]). But before that we take a closer look at two benchmark examples.

## 1.2 Two examples: the cube and the ball

**The cube** We consider the cube $C^n := [-\sqrt{3}, \sqrt{3}]^n \subset \mathbb{R}^n$; let $X$ be uniformly distributed on the cube. By the *uniform distribution* on a set $A \in \mathcal{B}(\mathbb{R}^n)$ we mean the probability-measure on $\mathcal{B}(\mathbb{R}^n)$ given by $B \mapsto \frac{v_n(B\cap A)}{v_n(A)}$, provided $v_n(A) \in \mathbb{R}_{>0}$. Because of symmetry $\mathbb{E}[X] = o$ and $\mathbb{E}[X_iX_j] = 0$ for $i, j \in [1, n]$ with $i \neq j$ follow immediately. The density of $X$ is $\frac{1}{v_n(C^n)}\,1_{C^n}(x) = \prod_{i=1}^n(\frac{1}{v_1(C^1)}\,1_{C^1}(x_i))$ for $x \in \mathbb{R}^n$, from the product-representation follow independence and identical distribution on $C^1$ of the coordinates $X_i$ $(i \in [1, n])$, and their common variance is

$$\mathbb{E}[X_1^2] = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{x^2}{2\sqrt{3}}\,dx = \frac{\sqrt{3}^3 - (-\sqrt{3})^3}{6\sqrt{3}} = 1.$$

As already stated, Lindeberg's CLT implies approximate normal distribution of $\sum_{i=1}^n \frac{1}{\sqrt{n}} X_i = \langle\frac{1}{\sqrt{n}}(1,\dots,1)^\mathsf{T}, X\rangle$, but we would like to try general $\theta \in S^{n-1}$ and assert approximate normality of $\langle\theta, X\rangle$. To that end let $(X_n)_{n\geq 1}$ be i.i.d. random-variables with $X_1$ distributed uniformly on $C^1$, then $X^n := (X_1, \ldots, X_n)$ is distributed uniformly on $C^n$, and let $(\tau_n)_{n\geq 1} \subset \mathbb{R}$ not constantly zero. Define $s_n := \big(\sum_{i=1}^n \tau_i^2\big)^{1/2}$ and set $\theta^n := \frac{1}{s_n}(\tau_1, \ldots, \tau_n)$ as soon as $s_n > 0$, say for $n \geq N \in \mathbb{N}$, then $\theta^n \in S^{n-1}$. Note $\langle \theta^n, X^n \rangle = \frac{1}{s_n}\sum_{i=1}^n \tau_i X_i$, and $(\tau_n X_n)_{n\geq 1}$ still consists of independent, centred random-variables; therefore the task is to assert that this sequence fulfills Lindeberg's condition (1.1).

First we remark that Lindeberg's condition in general implies *Feller's condition:*

$$\lim_{n\to\infty} \max\Big\{ \frac{\mathrm{Var}[\tau_i X_i]}{s_n^2} \Big| i \in [1,n] \Big\} = 0,$$

which in our case is equivalent to $\lim_{n\to\infty} \max\{|\theta_i^n| | i \in [1,n]\} = 0$, and we are going to show that this implies Lindeberg's condition (this converse is not generally true). So let $\lim_{n\to\infty} \max\{|\theta_i^n| | i \in [1,n]\} = 0$ and let $\varepsilon \in \mathbb{R}_{>0}$, then there exists $n_0 \geq N$ such that $\max\{|\theta_i^n| | i \in [1,n]\} < \frac{\varepsilon}{\sqrt{3}}$ for all $n \geq n_0$; this implies $\frac{|\tau_i|}{s_n} < \frac{\varepsilon}{\sqrt{3}}$ for all $i \in [1,n]$, for any $n \geq n_0$. Let $n \geq n_0$ and $i \in [1,n]$, then because of $|X_i| \leq \sqrt{3}$ we have $|\tau_i X_i| < \varepsilon s_n$, hence $1_{[|\tau_i X_i| \geq \varepsilon s_n]} = 0$ and thus $\frac{1}{s_n}\sum_{i=1}^n \mathbb{E}[|\tau_i X_i| 1_{[|\tau_i X_i| \geq \varepsilon s_n]}] = 0$ for all $n \geq n_0$, consequently Lindeberg's condition is satisfied and the CLT holds, that is, $\langle \theta^n, X^n \rangle$ approximately follows a standard normal distribution.

We want to estimate how many directions $\theta^n \in S^{n-1}$ (in the sense of $\sigma_{n-1}$) fulfill the condition $\lim_{n\to\infty} \max\{|\theta_i^n| | i \in [1,n]\} = 0$. To do that we fix $n \in \mathbb{N}$ and $\varepsilon \in (0,1]$ and find an upper bound for $\sigma_{n-1}\big\{\theta \in S^{n-1} \big| \max\{|\theta_i| | i \in [1,n]\} \geq \varepsilon\big\}$; certainly this is at most $2n\,\sigma_{n-1}\{\theta \in S^{n-1} | \theta_1 \geq \varepsilon\}$ (by union-bound and orthogonal invariance of the sphere). By choosing a suitable parametrization of the sphere, for example, and also anticipating Lemma 2.3, p. 30, we have

$$\begin{aligned}
\sigma_{n-1}\{\theta \in S^{n-1} | \theta_1 \geq \varepsilon\} &= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})} \int_{\arcsin(\varepsilon)}^{\frac{\pi}{2}} \cos(\phi)^{n-1}\, d\phi \\
&= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})} \int_0^{\sqrt{1-\varepsilon^2}} x^{n-2}\,(1-x^2)^{-\frac{1}{2}}\, dx \leq \\
&\leq \frac{(\frac{n-1}{2})^{\frac{1}{2}}}{\sqrt{\pi}}\,\frac{1}{\varepsilon} \int_0^{\sqrt{1-\varepsilon^2}} x^{n-2}\, dx = \\
&= \frac{(1-\varepsilon^2)^{\frac{n-1}{2}}}{\varepsilon\sqrt{2\pi(n-1)}}.
\end{aligned}$$

We observe that for any $\varepsilon$ bounded away from zero, $\lim_{n\to\infty} \sigma_{n-1}\big\{\theta \in S^{n-1} \big| \max\{|\theta_i| | i \in [1,n]\} \geq \varepsilon\big\} = 0$, that is, the set of directions for which the CLT may fail is negligible for high dimensions, roughly speaking; or conversely, if the direction is chosen at random uniformly from the unit-sphere, with high probability the respective marginal will be approximately normally distributed. Obvious elements of the failure-set are the canonical unit-vectors, because $\langle e_i^n, X^n \rangle = X_i$ which is uniformly distributed by assumption, independent of dimension.

**The ball** Now take the ball $\sqrt{n+2}\,B^n$, and again let $X$ be uniformly distributed on it. The result for this case is also referred to as *Maxwell's principle* or *Maxwell–Borel-lemma*, and can be found, e.g., in Rassoul-Agha and Seppäläinen [36, p. 81], or Johnston and Prochno [19]. As in the case of the cube, $\mathbb{E}[X] = o$ and $\mathbb{E}[X_i X_j] = 0$ $(i \neq j)$ follow from symmetry, as does $\mathrm{Var}[X_1] = \cdots = \mathrm{Var}[X_n]$, and by Cavalieri's principle there holds (recall $v_n(B^n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$)

$$\begin{aligned}
\mathrm{Var}[X_1] &= \frac{1}{v_n(\sqrt{n+2}\,B^n)} \int_{-\sqrt{n+2}}^{\sqrt{n+2}} x_1^2 v_{n-1}\big(\sqrt{n+2-x_1^2}\,B^{n-1}\big)\,dx_1 \\
&= \frac{v_{n-1}(B^{n-1})}{v_n(B^n)(n+2)^{n/2}} \int_{-\sqrt{n+2}}^{\sqrt{n+2}} x^2\,(n+2-x^2)^{\frac{n-1}{2}}\,dx \\
&= \frac{\Gamma(\frac{n+2}{2})}{\sqrt{\pi}\,\Gamma(\frac{n+1}{2})(n+2)^{n/2}}\;\frac{\Gamma(\frac{3}{2})\Gamma(\frac{n+1}{2})(n+2)^{\frac{n+2}{2}}}{\Gamma(\frac{n+4}{2})} = 1.
\end{aligned}$$

In particular, the components are uncorrelated, but since the density cannot be written as a product of onedimensional densities they are not independent, therefore application of Lindeberg's CLT is out of question.

Nevertheless we try and even consider $k$-dimensional marginals, where $k$ may depend on $n$ such that $\frac{k}{\sqrt{n}} \to 0$ as $n \to \infty$ (so the case of constant $k$ is covered; we do not claim that this is the best possible dependence of $k$ on $n$). Because of the orthogonal invariance of the ball it suffices to consider the subspace $E = \mathbb{R}^k \times \{0\}^{n-k}$ spanned by the first $k$ canonical unit-vectors, and we identify $E = \mathbb{R}^k$; we write $p_E$ for the orthogonal projection onto $E$. The density of $X$ is $\frac{1}{v_n(\sqrt{n+2}\,B^n)}\,\mathbb{1}_{\sqrt{n+2}\,B^n}$; let $A \in \mathcal{B}(\mathbb{R}^k)$, then the distribution of $p_E \circ X$ is given by

$$\begin{aligned}
\mathbb{P}[p_E \circ X \in A] &= \mathbb{P}[(X_1,\dots,X_k)^\mathsf{T} \in A \wedge (X_{k+1},\dots,X_n)^\mathsf{T} \in \mathbb{R}^{n-k}] \\
&= \int_A \int_{\mathbb{R}^{n-k}} \frac{\mathbb{1}_{\sqrt{n+2}\,B^n}(x_1,\dots,x_n)}{v_n(\sqrt{n+2}\,B^n)}\,d(x_{k+1},\dots,x_n)\,d(x_1,\dots,x_k),
\end{aligned}$$

and this shows that $p_E \circ X$ has the following density $(x \in \mathbb{R}^k)$,

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}^{n-k}} \frac{1}{v_n(\sqrt{n+2}\,B^n)}\,\mathbb{1}_{\sqrt{n+2}\,B^n}(x,y)\,dy \\
&= \frac{1}{v_n(B^n)(n+2)^{\frac{n}{2}}} \int_{\mathbb{R}^{n-k}} \mathbb{1}_{\sqrt{n+2-\|x\|^2}\,B^{n-k}}(y)\,dy \\
&= \frac{v_{n-k}\big(\sqrt{n+2-\|x\|^2}\,B^{n-k}\big)}{v_n(B^n)(n+2)^{\frac{n}{2}}} = \frac{v_{n-k}(B^{n-k})(n+2-\|x\|^2)^{\frac{n-k}{2}}}{v_n(B^n)(n+2)^{\frac{n}{2}}} \\
&= \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-k}{2}+1)(1+\frac{n}{2})^{\frac{k}{2}}}\,(2\pi)^{-\frac{k}{2}}\left(1 - \frac{\|x\|^2}{n+2}\right)^{\frac{n-k}{2}}.
\end{aligned}$$

Now we have $\big(1 - \frac{\|x\|^2}{n+2}\big)^{\frac{n-k}{2}} \to \exp\big(-\frac{\|x\|^2}{2}\big)$ as $n \to \infty$, therefore the latter part of $f$ converges precisely to $(2\pi)^{-\frac{k}{2}}\,\mathrm{e}^{-\frac{1}{2}\|x\|^2}$, the density of the $k$-dimensional standard normal distribution. It remains to prove that the former part converges to one; to that end we write first

$$\Gamma\Big(\frac{n}{2}+1\Big) = \Gamma\Big(\frac{n}{2}+1-\Big\lfloor\frac{k}{2}\Big\rfloor\Big)\prod_{i=1}^{\lfloor k/2\rfloor}\Big(\frac{n}{2}+1-i\Big),$$

for an estimate from above we again employ Lemma 2.3, p. 30, and get

$$\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-k}{2}+1)(\frac{n}{2}+1)^{\frac{k}{2}}} \le \frac{(\frac{n-k}{2}+1)^{\frac{k}{2}-\lfloor\frac{k}{2}\rfloor}}{(\frac{n}{2}+1)^{\frac{k}{2}}}\prod_{i=1}^{\lfloor k/2\rfloor}\Big(\frac{n}{2}+1-i\Big) = \\ = \Big(\frac{n+2-k}{n+2}\Big)^{\frac{k}{2}-\lfloor\frac{k}{2}\rfloor}\prod_{i=1}^{\lfloor k/2\rfloor}\frac{n+2-2i}{n+2} \le 1,$$

and from below, using $1+x \le \mathsf{e}^x$, we have

$$\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-k}{2}+1)(\frac{n}{2}+1)^{\frac{k}{2}}} \ge \frac{(\frac{n-k}{2}+1)(\frac{n}{2}+1-\lfloor\frac{k}{2}\rfloor)^{\frac{k}{2}-\lfloor\frac{k}{2}\rfloor-1}}{(\frac{n}{2}+1)^{\frac{k}{2}}}\prod_{i=1}^{\lfloor k/2\rfloor}\Big(\frac{n}{2}+1-i\Big) = \\ = \frac{n+2-k}{n+2}\Big(\frac{n+2-2\lfloor\frac{k}{2}\rfloor}{n+2}\Big)^{\frac{k}{2}-\lfloor\frac{k}{2}\rfloor-1}\prod_{i=1}^{\lfloor k/2\rfloor}\frac{n+2-2i}{n+2} \ge \\ \ge \frac{n+2-k}{n+2}\Big(\frac{n+2-2\lfloor\frac{k}{2}\rfloor}{n+2}\Big)^{\frac{k}{2}-\lfloor\frac{k}{2}\rfloor-1}\Big(\frac{n+2-2\lfloor\frac{k}{2}\rfloor}{n+2}\Big)^{\lfloor\frac{k}{2}\rfloor} \ge \\ \ge \frac{n+2-k}{n+2}\Big(\frac{n+2-2\lfloor\frac{k}{2}\rfloor}{n+2}\Big)^{\frac{k}{2}-\lfloor\frac{k}{2}\rfloor-1}\exp\Big(-\frac{2\lfloor\frac{k}{2}\rfloor^2}{n+2-2\lfloor\frac{k}{2}\rfloor}\Big),$$

and this lower bound converges to one as $n \to \infty$ (here we have used $\lim_{n\to\infty}\frac{k}{\sqrt{n}} = 0$). Thus the densities converge pointwise and hence convergence in distribution is established.

## 1.3 Overview of the history of the CLT for convex bodies

The history of the CLT for convex bodies reaches back about 150 years and has its actual roots, as do various other mathematical results, in physics, namely in the kinetic theory of gases. As recounted in Diaconis and Freedman [9, pp. 418f.] (see there for further references), it was E. Borel, while actually working on a theorem by J. C. Maxwell, who stated that the first few $k$ components of a point $(x_1,\dots,x_n)^\mathsf{T}$ distributed uniformly on $S^{n-1}$ asymptotically follow a normal distribution and are independent. The connection with physics is the following: consider an insulated system of $\frac{n}{3}$ free particles, the $i$-th particle having velocity $(x_{3i-2},x_{3i-1},x_{3i})^\mathsf{T}$ and each having the same mass $m$, then because of conservation of energy the total kinetic energy $\frac{m}{2}\sum_{i=1}^n x_i^2 = E$ is constant, hence $(x_1,\dots,x_n)^\mathsf{T} \in \sqrt{\frac{2E}{m}}\,S^{n-1}$; if now uniform distribution on the sphere is assumed we have precisely the situation studied by Borel (assuming random behaviour in space, velocity or other suitable quantities is standard in statistical mechanics).

Another precursor presented in [9, p. 419f.] is P. Lévy who studied means of operators $U$ on $L^2([0,1])$. Because there is no orthogonally invariant probability-measure on the unit-sphere of that space, Lévy used a discretization where he approximated a function $f$ from $L^2([0,1])$ by those $f_n$ which are constant on subintervals of equal length $\frac{1}{n}$, the constants being the averages of the original function over the respective subintervals; then the $L^2$-norm of $f$ equals that of $f_n$ and that equals the Euclidean norm of the vector of constants divided by $\sqrt{n}$, in particular the $L^2$-unit-sphere is approximated by $\sqrt{n}\,S^{n-1}$. We thus get an approximate mean $M_n(U)$ of $U$ by averaging $U(f_n)$ over $\sqrt{n}\,S^{n-1}$, and the limit $M(U) = \lim_{n\to\infty} M_n(U)$ is declared the mean of $U$. If $U$ depends on the evaluation of $f$ on few points, then $M(U)$ happens to be a mean with respect to the standard normal distribution.

As the title of [9] suggests, the authors did not strive for a CLT but rather for a representation-theorem; they proved that an orthogonally invariant distribution is close in total-variation-distance to a scale mixture of independent standard normal distributions. But in order to do this, they showed that the distribution of the first $k$ components of a point uniformly distributed on a sphere is close to a properly scaled normal distribution, and then they used that the uniform distributions on spheres are extreme points of the convex set of orthogonally invariant distributions.

The next major step was Brehm and Voigt [6]. At this point we should explain the notion of *convex body:*

**Definition 1.2.** A *convex body* is a compact, convex set $K \subset \mathbb{R}^n$ with nonempty interior (then by convexity $K$ is the closure of its interior). It is called *isotropic* iff $v_n(K) = 1$ (not for all authors this is part of the defintion), $\int_K x\,dx = o$ and there is a constant $L_K \in \mathbb{R}_{>0}$ such that $\int_K \langle\theta, x\rangle^2\,dx = L_K^2$ for all $\theta \in S^{n-1}$ (pictorially speaking, the body has unit-volume, barycentre at the origin and its ellipsoid of inertia is a sphere).

Isotropy also allows for a stochastic interpretation: let $K \subset \mathbb{R}^n$ be an isotropic convex body and let $X$ be a random-variable having uniform distribution on $K$. Then the density of $X$ is $1_K$ and therefore $\int_K x\,dx = \mathbb{E}[X] = o$ and $\int_K \langle\theta, x\rangle^2\,dx = \mathbb{E}[\langle\theta, X\rangle^2] = \theta^\mathsf{T}\,\mathbb{E}[XX^\mathsf{T}]\theta = \theta^\mathsf{T}\,\mathrm{Var}[X]\theta = L_K^2$, from which $\mathrm{Var}[X] = L_K^2 I_n$ follows, that is the coordinates of $X$ are uncorrelated and have equal variance.

Brehm and Voigt investigated the cross-sections of convex bodies, that is the functions $\varphi_{K,\theta}(t) := v_{n-1}\{x \in K | \langle\theta, x\rangle = t\}$ (note that $\{x \in \mathbb{R}^n | \langle\theta, x\rangle = t\}$ is the $(n-1)$-dimensional hyperplane with normal vector $\theta \in S^{n-1}$ passing through $t\theta$; more important for us, $\varphi_{K,\theta}$ is the density of the marginal on $\mathbb{R}\theta$ of the uniform distribution on $K$), and show convergence of those in $L^1$ and uniformly to a normal density for $K$ being the hypercube $[-\frac{1}{2}, \frac{1}{2}]^n$, the ball $B^n$ (also for cross-sections of codimension $k$),[1] the cross-polytope, i.e. the $l^1$-ball, with $\theta = \frac{1}{\sqrt{n}}(1, \dots, 1)^\mathsf{T}$, and the standard-simplex $\Delta^n$ for a discrete set of

[1] Our analysis of the cube and the ball given in Section 1.2 is somewhat similar to that of Brehm and Voigt, and in some points based on it, though significantly coarser.

directions. They also noted that, naturally, convergence to a normal distribution cannot be expected for every $\theta \in S^{n-1}$: take the hypercube and, say, the first coordinate-axis, then $\varphi_{K,\theta} = 1_{[-1/2,1/2]}$ for any $n \in \mathbb{N}$ and hence normality will never be achieved. They already remark:

> "It appears to be a known conjecture among specialists that this is a general phenomenon: For large dimensions the function $\varphi_{K,u}$ should be close to a Gaussian density for all isotropic normed convex bodies $K$ and for 'most' directions $u \in S^{n-1}$. More precisely, the density corresponding to $K$ should be the Gaussian with variance $L_K^2$." [6, p. 438]

Around the time of [6] also Anttila, Ball and Perissinaki [1] busied themselves with the conjecture (of which Brehm and Voigt became well aware, as they admit). They worked with symmetric isotropic convex bodies, so they additionally assumed $K = -K := \{-x | x \in K\}$, and that assumtion they needed because they were going to define a norm based on $K$. As an aside, they used $\rho$ for what was $L_K$ in [6]. As far as we can tell, the authors of [1] were the first to make the following hypothesis explicit in the context of convex bodies [1, p. 4723]:

**Concentration-Hypothesis.** *For a given $\varepsilon < \frac{1}{2}$ we say that $K$ satisfies the $\varepsilon$-concentration hypothesis if*

$$\mathbb{P}\Big[\Big|\frac{\|X\|}{\sqrt{n}} - \rho\Big| \geq \varepsilon\rho\Big] \leq \varepsilon. \tag{1.2}$$

Here $X$ is a random-variable distributed uniformly on $K$. In the further course of the studies on the CLT-problem for convex bodies this concentration-hypothesis has also become known as "thin-shell-hypotehsis" or "thin-shell-property", because the pictorial interpretation is that at least the fraction $1-\varepsilon$ of the mass of $K$ lies within a thin spherical shell with thickness $2\varepsilon$ times radius. The authors of [1] referred back to earlier ideas of Diaconis and Freedman [8], of Sudakov [37] and of von Weizsäcker [38]. They stated the conjecture that *all* isotropic convex bodies satisfy (1.2) with $\varepsilon \leq C\,\frac{\log(n)}{\sqrt{n}}$, and proved that for $l^p$-balls $\varepsilon \leq Cn^{-\frac{1}{3}}$ (recall that, for $p \in \mathbb{R}_{\geq 1}$, the $l^p$-ball is $\{x \in \mathbb{R}^n | \sum_{i=1}^n |x_i|^p \leq 1\}$ and the $l^\infty$-ball is $\{x \in \mathbb{R}^n | \forall i \in [1,n] \colon |x_i| \leq 1\}$), and that concentration also holds for uniformly convex symmetric bodies contained in a small Euclidean ball (i.e. radius about $\sqrt{n}$). Their CLT takes the subsequent form [1, Theorem 4].

**Theorem 1.3.** *Under the concentration-hypothesis* (1.2)*, for $\delta > 0$ we have*

$$\sigma_{n-1}\Big\{\theta \in S^{n-1} \Big| \forall t \in \mathbb{R} \colon \Big|\int_{-t}^{t} g_\theta(s)\,ds - \int_{-t}^{t} g(s)\,ds\Big| \leq \delta + 4\varepsilon + \frac{c}{\sqrt{n}}\Big\} \geq 1 - 4\sqrt{n}\log(n)\mathrm{e}^{-\frac{n\delta^2}{50}}.$$

In their notation $g_\theta$ is the cross-section-function in direction $\theta$, that is, the same as $\varphi_{K,\theta}$ in [6], and $g$ is the normal density with variance $\rho^2$. Note that because of symmetricity of the involved distributions the "symmetric CDF" $t \mapsto \int_{-t}^{t} g(s)\,ds$ carries the same information as the usual "left tail" CDF $t \mapsto \int_{-\infty}^{t} g(s)\,ds$.

We also stress the two major points of their proof: on the one hand use (1.2) to prove closeness of the normal distribution and an average distribution, that means, take the CDF $F_\theta$ of the projected measure for each $\theta \in S^{n-1}$ and then compute the (pointwise) average over the sphere $F = \int_{S^{n-1}} F_\theta \, d\sigma_{n-1}(\theta)$; and on the other hand use Lipschitz-continuity and concentration on the sphere in order to show that for most $\theta$, $F_\theta$ is close to $F$ (the relevant result of Lévy is reproduced as Theorem 2.27, p. 61, in the present work).

Perhaps a minor, but not unimportant, contribution of Brehm, Hinow, Vogt and Voigt [5] was that for logarithmically concave densities several modes of convergence towards a normal distribution are equivalent; in particular uniform convergence of the CDFs, of the PDFs and $L^1$-convergence of the PDFs are equivalent. Here, a function $f\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ is called logarithmically concave if $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^\lambda$ for all $x, y \in \mathbb{R}$ and $\lambda \in (0,1)$; the cross-section-functions discussed so far have that property.

Where the mathematicians hitherto worked with bodies, Bobkov [2] took the next step and generalized the task to distributions. This begs the question, which might be the right generalization, but to the specialists of the asymptotic theory of convex bodies the answer was obvious: consider *logarithmically concave* distributions. For short, a measure $\mu$ is called logarithmically concave if for any compact sets $A, B \in \mathbb{R}^n$ and $\lambda \in (0,1)$ there holds $\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda} \mu(B)^\lambda$; the uniform distribution on any convex body is such (the similarity to 'logarithmically concave function' is no coincidence).

Indeed, Bobkov demonstrated more: he investigated $\mathbb{R}^n$-valued random-variables $X$ ($n \geq 2$) with general distribution, but the property $\operatorname{Var}[X] = I_n$; as before let $F_\theta$ for $\theta \in S^{n-1}$ denote the CDF of $\langle \theta, X \rangle$, and $F$ the average of the $F_\theta$'s over $S^{n-1}$. The *Lévy-distance* between two CDFs $G, G'$ is defined as $L(G, G') := \inf\{\varepsilon \in \mathbb{R}_{>0} | \forall t \in \mathbb{R}\colon G'(t-\varepsilon) - \varepsilon \leq G(t) \leq G'(t+\varepsilon) + \varepsilon\}$, it induces the weak convergence. $\|\cdot\|_{L^\infty(\mathbb{R})}$ is the (essential) supremum-norm on $\mathbb{R}$. Then the following holds [2, Theorem 1.1, Corollary 2.5].

**Theorem 1.4.** *If* $\operatorname{Var}[X] = I_n$ *holds true, for all* $\delta \in \mathbb{R}_{>0}$,

$$\sigma_{n-1}\{\theta \in S^{n-1} | L(F_\theta, F) \geq \delta\} \leq 4n^{\frac{3}{8}} \, \mathrm{e}^{-\frac{1}{4}n\delta^4}.$$

*If in addition* (1.2) *holds (with* $\rho = 1$*), then for all* $\delta \in \mathbb{R}_{>0}$,

$$\sigma_{n-1}\big\{\theta \in S^{n-1} \big| \|F_\theta - \Phi\|_{L^\infty(\mathbb{R})} \geq 4\varepsilon + \delta\big\} \leq 4n^{\frac{3}{8}} \, \mathrm{e}^{-cn\delta^4},$$

*where* $c \in \mathbb{R}_{>0}$ *is a universal constant.*

Note the generality of the result: $X$ even need not be centred, the only requirements are unit covariance-matrix and the thin-shell-property. In the case of logarithmically concave distributions the result was sharpened to the following [2, Theorem 1.2], where a random-vector $X$ or its distribution is termed *isotropic* iff it is centred and $\operatorname{Var}[X] = I_n$.

**Theorem 1.5.** *Assume a random-vector $X$ has an isotropic logarithmically concave distribution. Then, for all* $\delta \in \mathbb{R}_{>0}$,

$$\sigma_{n-1}\big\{\theta \in S^{n-1} \big| \exists t \in \mathbb{R}\colon \mathrm{e}^{c|t|} |F_\theta(t) - F(t)| \geq \delta\big\} \leq C\sqrt{n}\log(n)\mathrm{e}^{-cn\delta^2},$$

*where* $C, c \in \mathbb{R}_{>0}$ *are universal constants.*

We should mention that in the study of asymptotic convex geometry "universal constant" or "absolute constant" means, not depending on any parameter or quantity, not even and especially not on the dimension $n$; also to be mentioned is that the values of constants may change, and usually do change, from one appearance to the next. Bobkov did not write down an analogue to his Corollary 2.5 for the logarithmically concave case, that is, comparing $F_\theta$ and $\Phi$.

An important observation by Bobkov is that, applying Čebyšëv's inequality, $\mathrm{Var}[\|X\|^2] \leq Cn$ is sufficient for $X$ to satisfy the thin-shell-property, in particular $\varepsilon \leq Cn^{-\frac{1}{3}}$ then. The uniform distributions on $l^p$-balls and on the uniformly convex bodies studied in [1] fulfill this "small-variance-property".

Now the course seemed clear: whosoever affirmed the thin-shell-property for logarithmically concave distributions, would prove the CLT for convex bodies at the same time. This was accomplished by Klartag [20] in what is, we deem, rightly called a breakthrough and seminal work. He proved the following [20, Theorem 1.4].

**Theorem 1.6.** *Let* $X$ *be a random-vector with an isotropic, logarithmically concave distribution in* $\mathbb{R}^n$*. Then for all* $\varepsilon \in [0, 1]$*,*

$$\mathbb{P}\Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| \geq \varepsilon\Big] \leq Cn^{-c\varepsilon^2},$$

*where* $C, c \in \mathbb{R}_{>0}$ *are universal constants.*

This established the thin-shell-property (1.2) with $\varepsilon = C\sqrt{\frac{\log(\log(n))}{\log(n)}}$; granted, that result did not yet fit the expectation of [1], but nevertheless it was a great achievement of Klartag. His version of the CLT for convex bodies then reads as follows [20, Theorem 1.1]. (Note that the name "CLT for convex bodies" has stuck although ever since [2] the more general setting of logarithmically concave distributions has been studied.)

**Theorem 1.7.** *There exist null-sequences* $(\varepsilon_n)_{n\geq 1}$ *and* $(\delta_n)_{n\geq 1}$ *with* $\varepsilon_n \leq C\sqrt{\frac{\log(\log(n))}{\log(n)}}$ *and* $\delta_n \leq \mathsf{e}^{-cn^{0.99}}$*, where* $C, c \in \mathbb{R}_{>0}$ *are universal constants, for which the following holds: let* $n \geq 1$ *and let* $X$ *be a random-vector in* $\mathbb{R}^n$ *with an isotropic, logarithmically concave distribution, then*

$$\sigma_{n-1}\{\theta \in S^{n-1} | d_{\mathrm{TV}}(\langle\theta, X\rangle, Z) > \varepsilon_n\} < \delta_n,$$

*where* $Z$ *is a scalar random-variable with standard normal distribution.*

Klartag employed the *total-variation-distance* $d_{\mathrm{TV}}$ which essentially is the $L^1$-distance of the respective densities. He also proved a result for projections onto $k$-dimensional subspaces for $k \leq c\,\frac{\log(n)}{\log(\log(n))}$, and additionally a somewhat stronger result for so-called unconditional distributions (those which are invariant under all coordinate-reflexions $x_i \mapsto -x_i$ for $i \in [1, n]$), in this latter case $\theta = \frac{1}{\sqrt{n}}(1, \ldots, 1)^{\mathsf{T}}$ is always admissible.

Klartag concedes, "The quantitative estimate we provide for $\varepsilon_n$ is rather poor. (...) [W]e are still lacking the precise Berry-Esseen type bound. A plausible guess might be that the logarithmic dependence should be replaced by a power-type decay, in the bound for $\varepsilon_n$."

In some sense the hunt for a CLT for convex bodies has found its end with [20], the subsequent contributions have dealt more with improving the bounds of the thin-shell-property. Others tried different proof-techniques, for example Fleury, Guédon and Paouris [11]: they again worked with bodies, not distributions, and produced the following [11, Theorem 1].

**Theorem 1.8.** *There exists $c \in \mathbb{R}_{>0}$ such that for every isotropic convex body $K \subset \mathbb{R}^n$ and every $\varepsilon \in (0,1)$,*

$$v_n\Big\{x \in K \Big| \Big|\frac{\|x\|}{\sqrt{n}\,L_K} - 1\Big| \geq \varepsilon\Big\} \leq 2\mathrm{e}^{-c\sqrt{\varepsilon}\log(n)^{1/12}}.$$

Hence the thin-shell-property is satisfied with $\varepsilon = C\,\frac{\log(\log(n))^2}{\log(n)^{1/6}}$. A markedly better result was obtained by Klartag again [21]; he proved $\mathbb{P}\big[\big|\frac{\|X\|}{\sqrt{n}} - 1\big| \geq \varepsilon\big] \leq C_\alpha \exp(-c_\alpha \varepsilon^{\frac{10}{3}-\alpha} n^{\frac{1}{3}-\alpha})$ for any small $\alpha \in \mathbb{R}_{>0}$ (smaller values worsen the constants) in [21, Theorem 4.4], which leads to a thin-shell-estimate with $\varepsilon \approx n^{-\frac{1}{14}}$ and corresponding power-type-bounds for the CLT itself. Although the value of the exponents was not yet satisfying, this was a clear improvement over the logarithmic bounds found until then; this was also valid for the dependence of $k$ on $n$ for $k$-dimensional marginals. The measure of the set of exceptional directions (or subspaces) remained essentially unchanged.

It was also Klartag who proved sharp Berry-Esseen-type bounds for unconditional, isotropic, logarithmically concave distributions in [23, Theorem 1].

**Theorem 1.9.** *Under said assumtions,*

$$\sup\Big\{\Big|\mathbb{P}\Big[\alpha \leq \frac{1}{\sqrt{n}}\sum_{i=1}^n X_i \leq \beta\Big] - \int_\alpha^\beta \varphi(t)\,dt\Big|\Big|\alpha, \beta \in \mathbb{R}, \alpha \leq \beta\Big\} \leq \frac{C}{n},$$

*where $C \in \mathbb{R}_{>0}$ is a universal constant. Moreover, for any $\theta = (\theta_1, \ldots, \theta_n) \in S^{n-1}$,*

$$\sup\Big\{\Big|\mathbb{P}[\alpha \leq \langle\theta, X\rangle \leq \beta] - \int_\alpha^\beta \varphi(t)\,dt\Big|\Big|\alpha, \beta \in \mathbb{R}, \alpha \leq \beta\Big\} \leq C\sum_{i=1}^n \theta_i^4.$$

As ingredient to the proof Klartag established $\mathrm{Var}[\|X\|^2] \leq Cn$ under the aforementioned assumtions, which bound had already been mentioned by Bobkov [2] (see above). The conjecture is that this holds for *all* isotropic logarithmically concave distributions. An equivalent statement is $\mathbb{E}[(\|X\| - \sqrt{n})^2] \leq C$ with a universal constant $C \in \mathbb{R}_{>0}$, this is what currently is called the "thin-shell-conjecture" in the asymptotic geometry of convex bodies.

Further steps were the works by Fleury [10], where he showed $\mathbb{P}\big[\big|\frac{\|X\|}{\sqrt{n}} - 1\big| \geq \varepsilon\big] \leq C\mathsf{e}^{-cn^{1/8}\varepsilon}$ and $\mathbb{P}\big[\frac{\|X\|}{\sqrt{n}} - 1 \geq \varepsilon\big] \leq C\mathsf{e}^{-cn^{1/4}\varepsilon^2}$, either for all $\varepsilon \in [0,1]$, and Guédon and Milman [18] with $\mathbb{P}\big[\big|\frac{\|X\|}{\sqrt{n}} - 1\big| \geq \varepsilon\big] \leq C\mathsf{e}^{-c\sqrt{n}\min\{\varepsilon,\varepsilon^3\}}$ for all $\varepsilon \in \mathbb{R}_{\geq 0}$; this also implies $\mathbb{E}[(\|X\| - \sqrt{n})^2] \leq Cn^{\frac{2}{3}}$. As far as we are aware, the best recent result concerning the thin-shell-conjecture is Lee and Vempala [27, Corollary 9], extended by the same authors in [29, Corollary 13], where they obtained $\mathbb{E}[(\|X\| - \sqrt{n})^2] \leq Cn^{\frac{1}{2}}$; but they provided no estimates of the deviation-probability $\mathbb{P}\big[\big|\frac{\|X\|}{\sqrt{n}} - 1\big| \geq \varepsilon\big]$, since they exploited the connexions with various other conjectures for convex bodies, central among them the *Kannan-Lovász-Simonovits-conjecture.* We refer to the cited articles and especially to [28] by the same authors.

The relevant contribution to the CLT for convex bodies for the present thesis is Fresen [14]. As the title of his work suggests, he did not aim for improved bounds on either the deviation-probability $\mathbb{P}\big[\big|\frac{\|X\|}{\sqrt{n}} - 1\big| \geq \varepsilon\big]$ or on the expectation $\mathbb{E}[(\|X\| - \sqrt{n})^2]$, but he had revisited the CLT itself and wanted to "present a simple proof that is self-contained (except for very classical results such as the Prékopa-Leindler inequality) and is accessible to anyone" [14, p. 1]. Our declared task is to thoroughly discuss Fresen's proof and to possibly assess whether he has achieved the aimed simplicity. Throughout the present thesis, whenever we make statements related to Fresen's work we refer to the just cited article [14] and omit the citation-mark; other works by Fresen are properly cited.

Also in what follows, *Remarks* are distinguished from *Comments:* the former refer to purely mathematical contents; whereas the latter deal with Fresen's article and need not always be mathematical in substance, and they are concluded with the sign $\Diamond$.

# 2 Mathematical preparations

## 2.1 Various technical results

This section contains various technical results that share no other feature than being of use in what follows thereafter. The first two lemmata treat the interplay of surface-measures on the sphere and the Grassmannian and the Haar-measure on the orthogonal group. Here the orthogonal group is denoted by $\mathrm{O}_n(\mathbb{R})$ and its normed Haar-measure by $\mu_{\mathrm{O}}$ (i.e. $\mu_{\mathrm{O}}(\mathrm{O}_n(\mathbb{R})) = 1$); recall that $\sigma_{n-1}$ is the unique orthogonally invariant probability-measure on $S^{n-1}$. The *Grassmannian manifold* $G_{n,k}$ is the compact manifold of all $k$-dimensional subspaces of $\mathbb{R}^n$, for $k \in [0, n]$, and $\sigma_{n,k}$ is the measure induced on it by $\mu_{\mathrm{O}}$ via $\sigma_{n,k}(A) := \mu_{\mathrm{O}}\{U \in \mathrm{O}_n(\mathbb{R}) | UF \in A\}$ for $A \in \mathcal{B}(G_{n,k})$, where $F \in G_{n,k}$ is a fixed subspace (whose choice does not matter); this measure too is orthogonally invariant.

**Lemma 2.1.** *For any $\theta \in S^{n-1}$ and $A \in \mathcal{B}(S^{n-1})$ there holds*

$$\mu_{\mathrm{O}}\{U \in \mathrm{O}_n(\mathbb{R}) | U\theta \in A\} = \sigma_{n-1}(A).$$

*Proof.* Let $\theta \in S^{n-1}$. We are using the fact that $\sigma_{n-1}$ is the unique orthogonally invariant measure on $S^{n-1}$ with total mass 1. Call the measure on the left-hand-side $\mu$, then

$$\begin{aligned}\mu(S^{n-1}) &= \mu_{\mathrm{O}}\{U \in \mathrm{O}_n(\mathbb{R}) | U\theta \in S^{n-1}\} \\ &= \mu_{\mathrm{O}}(\mathrm{O}_n(\mathbb{R})) = 1,\end{aligned}$$

because $U\theta \in S^{n-1}$ for all $U \in \mathrm{O}_n(\mathbb{R})$. For orthogonal invariance let $A \in \mathcal{B}(S^{n-1})$ and $V \in \mathrm{O}_n(\mathbb{R})$, then

$$\{U \in \mathrm{O}_n(\mathbb{R}) | V^{-1}U\theta \in A\} = \{U \in \mathrm{O}_n(\mathbb{R}) | U\theta \in A\},$$

because if $U \in \mathrm{O}_n(\mathbb{R})$ with $V^{-1}U\theta \in A$, then $V^{-1}U \in \mathrm{O}_n(\mathbb{R})$ is in the set on the right-hand-side; and conversely, if $U$ satisfies $U\theta \in A$, then also $V^{-1}(VU)\theta \in A$ and hence $VU \in \mathrm{O}_n(\mathbb{R})$ is in the set on the left-hand-side. From this follows

$$\begin{aligned}\mu(VA) &= \mu_{\mathrm{O}}\{U \in \mathrm{O}_n(\mathbb{R}) | U\theta \in VA\} \\ &= \mu_{\mathrm{O}}\{U \in \mathrm{O}_n(\mathbb{R}) | V^{-1}U\theta \in A\} \\ &= \mu_{\mathrm{O}}\{U \in \mathrm{O}_n(\mathbb{R}) | U\theta \in A\} = \mu(A).\end{aligned}$$

□

**Lemma 2.2.** *Let $k \in [1, n]$ and for each $E \in G_{n,k}$ let $\sigma_E$ denote the normed surface-measure on $S_E := E \cap S^{n-1}$ (note that $S_E$ is an orthogonal image of $S^{k-1} \times \{0\}^{n-k}$).*

*Then, for any $A \in \mathcal{B}(S^{n-1})$,*

$$\sigma_{n-1}(A) = \int_{G_{n,k}} \sigma_E(E \cap A)\, d\sigma_{n,k}(E).$$

*Proof.* Call the measure defined by the integral $\mu$; as with Lemma 2.1 we show orthogonal invariance of $\mu$ and $\mu(S^{n-1}) = 1$. The latter is straightforward:

$$\begin{aligned}\mu(S^{n-1}) &= \int_{G_{n,k}} \sigma_E(E \cap S^{n-1})\, d\sigma_{n,k}(E) \\ &= \int_{G_{n,k}} \sigma_E(S_E)\, d\sigma_{n,k}(E) = \int_{G_{n,k}} 1\, d\sigma_{n,k}(E) \\ &= \sigma_{n,k}(G_{n,k}) = 1.\end{aligned}$$

For the former let $A \in \mathcal{B}(S^{n-1})$ and $U \in \mathrm{O}_n(\mathbb{R})$. We are going to use the transformation-formula (see, e.g., Çınlar [7, Chapter 1, Theorem 5.2] or Klenke [25, Theorem 4.10]) and orthogonal invariance of $\sigma_{n,k}$ and obtain

$$\begin{aligned}\mu(UA) &= \int_{G_{n,k}} \sigma_E(E \cap UA)\, d\sigma_{n,k}(E) \\ &= \int_{G_{n,k}} \sigma_{UE}(UE \cap UA)\, d(\sigma_{n,k} \circ U^{-1})(E) \\ &= \int_{G_{n,k}} \sigma_E(E \cap A)\, d\sigma_{n,k}(E) = \mu(A).\end{aligned}$$ □

*Remarks.* One stochastic interpretation of the lemma above is that the uniform distribution on $S^{n-1}$ can be simulated by first choosing uniformly a $k$-dimensional subspace of $\mathbb{R}^n$ and then choosing uniformly a point on the unit-sphere of that subspace.

*Comment.* The lemma is the essence of Fresen's casual remark, "[n]ote that $P_{\theta'} P_E = P_\theta$ where $\theta$ is uniformly distributed in $S^{n-1}$" (p. 8, lines 4 and 5). His $E$ is a $G_{n,k}$-valued $\sigma_{n,k}$-distributed random-variable and $\theta'$ is a $S_E$-valued $\sigma_E$-distributed random-variable; he calls them just "uniformly distributed". ♢

The next lemma is concerned with an estimate of the Eulerian gamma-function $\Gamma$ which in some sense generalizes the well-known relation $\Gamma(x+1) = x\Gamma(x)$. There also exist other bounds of various sharpness (e.g., by Gautschi [15]). We present the result given by Wendel in [39].

**Lemma 2.3.** *For any $x \in \mathbb{R}_{>0}$ and $s \in [0,1]$,*

$$x(x+s)^{s-1}\,\Gamma(x) \le \Gamma(x+s) \le x^s\,\Gamma(x).$$

*Proof.* Let $x \in \mathbb{R}_{>0}$. For $s \in \{0,1\}$ there is nothing to prove, hence let $s \in (0,1)$. Recall Hölder's inequality

$$\int_A f(t)g(t)\, dt \le \left(\int_A f(t)^p\, dt\right)^{\frac{1}{p}} \left(\int_A g(t)^q\, dt\right)^{\frac{1}{q}},$$

where $A \in \mathcal{B}(\mathbb{R})$, $f, g\colon A \to [0,\infty]$ are measurable, and $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We set $A = \mathbb{R}_{>0}$, $f(t) := (t^{x-1}\,\mathsf{e}^{-t})^{1-s}$, $g(t) := (t^x\,\mathsf{e}^{-x})^s$, and $p := \frac{1}{1-s}$, $q := \frac{1}{s}$. This implies $f(t)g(t) = t^{x+s-1}\,\mathsf{e}^{-t}$, and inserting into Hölder's inequality we get

$$\begin{aligned}\Gamma(x+s) &= \int_0^\infty t^{x+s-1}\,\mathsf{e}^{-t}\,dt \leq \\ &\leq \Big(\int_0^\infty t^{x-1}\,\mathsf{e}^{-t}\,dt\Big)^{1-s}\Big(\int_0^\infty t^{x}\,\mathsf{e}^{-t}\,dt\Big)^{s} = \\ &= \Gamma(x)^{1-s}\,\Gamma(x+1)^s \\ &= \Gamma(x)^{1-s}\,x^s\,\Gamma(x)^s = x^s\,\Gamma(x).\end{aligned}$$

This also implies the lower bound because

$$x\Gamma(x) = \Gamma(x+1) = \Gamma\big((x+s)+(1-s)\big) \leq (x+s)^{1-s}\,\Gamma(x+s). \qquad \square$$

*Remarks.* The statement can actually be strengthened to: for $x \in \mathbb{R}_{>0}$ and $s \in (0,1)$, $\Gamma(x+s) < x^s\,\Gamma(x)$ holds. Since $t \mapsto f(t)^{1/(1-s)} = t^{x-1}\,\mathsf{e}^{-t}$ and $t \mapsto g(t)^{1/s} = t^x\,\mathsf{e}^{-t}$ are linearly independent in $L^1(\mathbb{R}_{>0})$ for $s \in (0,1)$, Hölder's inequality has strict inequality in that case.

The following two lemmata deal with the CDF of the standard normal distribution, $\Phi$. The first of these is from [2, Lemma 2.6] (verbatim also in Brazitikos et al. [4, Equation (12.1.5)]); the second is used without comment by Fresen.

**Lemma 2.4.** *Let $t \in \mathbb{R}_{\geq 0}$. For all $\alpha \in [0,1]$ there holds $\Phi(t) - \Phi(\alpha t) \leq \frac{1}{2}(1-\alpha)$, and for all $\alpha \in [1,\infty)$ there holds $\Phi(\alpha t) - \Phi(t) \leq \frac{1}{\sqrt{2\pi\mathsf{e}}}(\alpha - 1)$.*

*Remarks.* The two inequalities can be fused to $|\Phi(\alpha t) - \Phi(t)| \leq \frac{1}{2}|\alpha - 1|$ for all $t, \alpha \in \mathbb{R}_{\geq 0}$.

*Proof.* Note that, because $\Phi' = \varphi$ is increasing on $\mathbb{R}_{\leq 0}$ and decreasing on $\mathbb{R}_{\geq 0}$, $\Phi$ is convex on $\mathbb{R}_{\leq 0}$ and concave on $\mathbb{R}_{\geq 0}$.

Validity is obvious for $t = 0$, so let $t > 0$. Let $\alpha \in [0,1]$, then by concavity and $\frac{1}{2} = \Phi(0) \leq \Phi(t) \leq 1$,

$$\begin{aligned}\Phi(\alpha t) &\geq \Phi(0) + \frac{\Phi(t)-\Phi(0)}{t-0}(\alpha t - 0) = \Phi(t) - (\Phi(t)-\Phi(0)) + \alpha(\Phi(t)-\Phi(0)) = \\ &= \Phi(t) - (\Phi(t)-\Phi(0))(1-\alpha) \geq \Phi(t) - \frac{1}{2}(1-\alpha).\end{aligned}$$

If $\alpha \in [1,\infty)$, then again by concavity and by $x\varphi(x) \leq \frac{1}{\sqrt{2\pi\mathsf{e}}}$ for $x \in \mathbb{R}_{\geq 0}$ (maximum attained at 1 as revealed by discussion),

$$\Phi(\alpha t) \leq \Phi(t) + \varphi(t)(\alpha t - t) = \Phi(t) + t\varphi(t)(\alpha - 1) \leq \Phi(t) + \frac{1}{\sqrt{2\pi\mathsf{e}}}(\alpha - 1). \qquad \square$$

**Lemma 2.5.** *Let $t \in \mathbb{R}$ and $\nu, \sigma \in [0,1]$, then*

$$\Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) \geq \Phi(t) - \frac{\nu+\sigma}{\sqrt{2\pi}} \quad \text{and} \quad \Phi\Big(\frac{t+\nu}{\sqrt{1+\sigma^2}}\Big) \leq \Phi(t) + \frac{\nu+\sigma}{\sqrt{2\pi}}.$$

*Proof.* First we consider $t \in \mathbb{R}_{\geq 0}$. We prove the upper bound first. Since $\frac{t+\nu}{\sqrt{1+\sigma^2}} \geq 0$, $\varphi(t) \leq \varphi(0) = \frac{1}{\sqrt{2\pi}}$, and $\frac{1}{\sqrt{1+\sigma^2}} \leq 1$, we get by concavity

$$\Phi\Big(\frac{t+\nu}{\sqrt{1+\sigma^2}}\Big) \leq \Phi(t) + \varphi(t)\Big(\frac{t+\nu}{\sqrt{1+\sigma^2}} - t\Big) \leq$$
$$\leq \Phi(t) + \frac{1}{\sqrt{2\pi}}\,\nu \leq \Phi(t) + \frac{\nu+\sigma}{\sqrt{2\pi}}.$$

For the lower bound consider three cases: first let $t-\nu \leq 0$, then $t-\nu \leq \frac{t-\nu}{\sqrt{1+\sigma^2}} \leq 0 \leq t$, hence $t - \frac{t-\nu}{\sqrt{1+\sigma^2}} \leq t-(t-\nu) = \nu \leq \nu+\sigma$. By convexity on $\mathbb{R}_{\leq 0}$,

$$\Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) \geq \Phi(0) + \varphi(0)\,\frac{t-\nu}{\sqrt{1+\sigma^2}},$$

by concavity on $\mathbb{R}_{\geq 0}$,

$$\Phi(t) \leq \Phi(0) + \varphi(0)t,$$

and thus

$$\Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) \geq \Phi(t) + \varphi(0)\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}} - t\Big) \geq \Phi(t) - \frac{\nu+\sigma}{\sqrt{2\pi}}.$$

Now let $t-\nu \in [0, 2+\sqrt{2}]$, then $t - \frac{t-\nu}{\sqrt{1+\sigma^2}} \leq \nu+\sigma$ holds (this can be proved by rearranging the inequality to $t-\nu \leq \frac{\sigma\sqrt{1+\sigma^2}}{\sqrt{1+\sigma^2}-1}$; the right-hand-side is decreasing in $\sigma$ and so takes its minumum value $2+\sqrt{2}$ at $\sigma = 1$), thus again by concavity,

$$\Phi(t) \leq \Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) + \varphi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big)\Big(t - \frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) \leq \Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) + \frac{\nu+\sigma}{\sqrt{2\pi}}.$$

Finally let $t-\nu \geq 2+\sqrt{2}$. By Lemma 2.4,

$$\Phi(t) = \Phi\Big(\frac{t\sqrt{1+\sigma^2}}{t-\nu}\,\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) \leq \Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) + \frac{1}{\sqrt{2\pi\mathsf{e}}}\Big(\frac{t\sqrt{1+\sigma^2}}{t-\nu} - 1\Big);$$

the function $x \mapsto \sqrt{1+x^2}$ is convex on $\mathbb{R}_{\geq 0}$, hence $\sqrt{1+x^2} \leq 1+(\sqrt{2}-1)x$ for $x \in [0,1]$; using this fact together with $\sigma \leq 1$ leads us to

$$\frac{t\sqrt{1+\sigma^2}}{t-\nu} - 1 = \sqrt{1+\sigma^2} + \frac{\nu\sqrt{1+\sigma^2}}{t-\nu} - 1 \leq$$
$$\leq (\sqrt{2}-1)\sigma + \frac{\sqrt{2}}{2+\sqrt{2}}\,\nu = (\sqrt{2}-1)(\nu+\sigma);$$

In total this yields

$$\Phi(t) \leq \Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) + \frac{\sqrt{2}-1}{\sqrt{2\pi\mathsf{e}}}\,(\nu+\sigma),$$

from which the statement follows because of $\frac{\sqrt{2}-1}{\sqrt{\mathsf{e}}} \leq 1$.

Now let $t \in \mathbb{R}_{\leq 0}$. Using $\Phi(z) = 1 - \Phi(-z)$ for $z \in \mathbb{R}$ we see

$$\Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) = 1 - \Phi\Big(\frac{-t+\nu}{\sqrt{1+\sigma^2}}\Big) \geq 1 - \Phi(-t) - \frac{\nu+\sigma}{\sqrt{2\pi}} = \Phi(t) - \frac{\nu+\sigma}{\sqrt{2\pi}}$$

on the one hand, and on the other

$$\Phi\Big(\frac{t+\nu}{\sqrt{1+\sigma^2}}\Big) = 1 - \Phi\Big(\frac{-t-\nu}{\sqrt{1+\sigma^2}}\Big) \leq 1 - \Phi(-t) + \frac{\nu+\sigma}{\sqrt{2\pi}} = \Phi(t) + \frac{\nu+\sigma}{\sqrt{2\pi}}. \qquad \square$$

The last result in this section is about approximation of points on the sphere by a finite subset, typically referred to as a net. It will be important that the set does not have too many points. The proof of the latter is based on a standard volume-comparison-argument.

**Lemma 2.6.** *For any $\varepsilon \in (0,1]$ there exists a set $\mathcal{N} \subset S^{n-1}$ with $|\mathcal{N}| \leq (\frac{3}{\varepsilon})^n$ and*

$$\forall \theta \in S^{n-1} \exists \theta' \in \mathcal{N} \colon \|\theta - \theta'\| \leq \varepsilon.$$

*($\mathcal{N}$ is an* $\varepsilon$-net *for, or* $\varepsilon$-dense *in, $S^{n-1}$.)*

*Proof.* Let $\varepsilon \in (0,1]$. Let $N \in \mathbb{N}$ denote the minimal cardinality of an $\varepsilon$-net for $S^{n-1}$ and $M \in \mathbb{N}$ the maximal cardinality of an *$\varepsilon$-separated set* $\mathcal{M} \subset S^{n-1}$, that is, if $x, y \in \mathcal{M}$ with $x \neq y$, then $\|x - y\| > \varepsilon$. First we show $N \leq M$.

Let $\mathcal{M} \subset S^{n-1}$ be $\varepsilon$-separated with $|\mathcal{M}| = M$. We prove that $\mathcal{M}$ is an $\varepsilon$-net; then $N \leq M$ follows by definiton. Let $\theta \in S^{n-1}$, w.l.o.g. $\theta \notin \mathcal{M}$, then $|\mathcal{M} \cup \{\theta\}| > M$, hence it is not $\varepsilon$-separated, therefore there are $\theta', \theta'' \in \mathcal{M} \cup \{\theta\}$ with $\theta' \neq \theta''$ and $\|\theta' - \theta''\| \leq \varepsilon$; as $\mathcal{M}$ is $\varepsilon$-separated, $\theta \in \{\theta', \theta''\}$, say $\theta = \theta''$, thus $\|\theta - \theta'\| \leq \varepsilon$.

Now we show $M \leq (\frac{3}{\varepsilon})^n$. Let $\mathcal{M} \subset S^{n-1}$ be as before, then the balls $\theta + \frac{\varepsilon}{2} B^n$, for $\theta \in \mathcal{M}$, are pairwise disjoint, and $\bigcup_{\theta \in \mathcal{M}} (\theta + \frac{\varepsilon}{2} B^n) \subset (1 + \frac{\varepsilon}{2}) B^n$. Passing to the volumes yields

$$|\mathcal{M}| \Big(\frac{\varepsilon}{2}\Big)^n v_n(B^n) = v_n\Big[\bigcup_{\theta \in \mathcal{M}} \Big(\theta + \frac{\varepsilon}{2} B^n\Big)\Big] \leq \Big(1 + \frac{\varepsilon}{2}\Big)^n v_n(B^n),$$

and because of $\varepsilon \leq 1$ this gives us

$$M = |\mathcal{M}| \leq \Big(\frac{2}{\varepsilon} + 1\Big)^n \leq \Big(\frac{3}{\varepsilon}\Big)^n. \qquad \square$$

## 2.2 Log-concave functions and measures

In this section we give a few very basic definitons and results about logarithmically concave functions, measures and random-variables, plus some surrounding facts. As stated by Fresen, the contribution of his proof is simplicity, and indeed little background from the asymptotic theory of convex sets is needed. The first result is the classical inequality of Prékopa and Leindler. Prékopa in [33, Theorem 1] first proved the special case for $n = 1$ and $\lambda = \frac{1}{2}$, Leindler in [30] then extended the result to arbitrary $\lambda \in [0,1]$, and lastly Prékopa again in [34, Theorem 3] generalized it to $n \in \mathbb{N}$. The proof is adapted from [4, Theorem 1.2.3] (they only treat the integrable case).

**Theorem 2.7** (Prékopa-Leindler)**.** *Let $f, g, h\colon \mathbb{R}^n \to [0,\infty]$ be measurable and let $\lambda \in (0,1)$. If for every $x, y \in \mathbb{R}^n$,*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda}\, g(y)^{\lambda}, \tag{2.1}$$

*then,*

$$\int_{\mathbb{R}^n} h\, dv_n \geq \Big(\int_{\mathbb{R}^n} f\, dv_n\Big)^{1-\lambda} \Big(\int_{\mathbb{R}^n} g\, dv_n\Big)^{\lambda}.$$

*Proof. Case 1,* $f$ and $g$ are integrable. Induction on $n$. $n = 1$: For $\int_{\mathbb{R}} f\, dv_1 = 0$ or $\int_{\mathbb{R}} g\, dv_1 = 0$ there is nothing to prove, so let either integral be positive. Then the map $x \mapsto \big(\int_{\mathbb{R}} f\, dv_1\big)^{-1} \int_{-\infty}^{x} f\, dv_1$ is continuous and increasing, with limits 0 for $x \to -\infty$ and 1 for $x \to \infty$; define $X\colon (0,1) \to \mathbb{R}$ by $X(t) := \inf\big\{x \in \mathbb{R} \big| \big(\int_{\mathbb{R}} f\, dv_1\big)^{-1} \int_{-\infty}^{x} f\, dv_1 > t\big\}$ (in essence the right-continuous quantile-function), then $\int_{-\infty}^{X(t)} f\, dv_1 = t \int_{\mathbb{R}} f\, dv_1$ for all $t \in (0,1)$ by continuity, and $X$ is strictly increasing (because the former relation implies, if $X(s) = X(t)$, then $s = t$), therefore $X$ is differentiable almost everywhere and there holds $f(X(t))X'(t) = \int_{\mathbb{R}} f\, dv_1$. Analogously define $Y\colon (0,1) \to \mathbb{R}$ with $g$ instead of $f$.

Now we set $Z := (1-\lambda)X + \lambda Y$, then $Z$ is strictly increasing and differentiable almost everywhere. With the substitution $z = Z(t)$ and the arithmetic-geometric-means-inequality applied to $Z'$ we may conclude,

$$\begin{aligned}
\int_{\mathbb{R}} h(z)\, dz &= \int_0^1 h(Z(t))Z'(t)\, dt \\
&= \int_0^1 h((1-\lambda)X(t) + \lambda Y(t))((1-\lambda)X'(t) + \lambda Y'(t))\, dt \\
&\geq \int_0^1 f(X(t))^{1-\lambda}\, g(Y(t))^{\lambda}\, X'(t)^{1-\lambda}\, Y'(t)^{\lambda}\, dt = \\
&= \int_0^1 \big(f(X(t))X'(t)\big)^{1-\lambda} \big(g(Y(t))Y'(t)\big)^{\lambda}\, dt \\
&= \Big(\int_{\mathbb{R}} f\, dv_1\Big)^{1-\lambda} \Big(\int_{\mathbb{R}} g\, dv_1\Big)^{\lambda}.
\end{aligned}$$

$n-1 \to n$: Let $n \in \mathbb{N}$, $n \geq 2$, such that the statement holds for functions on $\mathbb{R}^{n-1}$. Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfy (2.1); for each $s \in \mathbb{R}$ define $f_s, g_s, h_s\colon \mathbb{R}^{n-1} \to \mathbb{R}_{\geq 0}$ by $f_s(x) := f(x,s)$, analogously $g_s, h_s$. Let $x, y \in \mathbb{R}^{n-1}$ and $s, t \in \mathbb{R}$, then

$$\begin{aligned}
h_{(1-\lambda)s+\lambda t}((1-\lambda)x + \lambda y) &= h((1-\lambda)(x,s) + \lambda(y,t)) \\
&\geq f(x,s)^{1-\lambda}\, g(y,t)^{\lambda} = f_s(x)^{1-\lambda}\, g_t(y)^{\lambda},
\end{aligned}$$

so by the induction-hypothesis

$$\int_{\mathbb{R}^{n-1}} h_{(1-\lambda)s+\lambda t}\, dv_{n-1} \geq \Big(\int_{\mathbb{R}^{n-1}} f_s\, dv_{n-1}\Big)^{1-\lambda} \Big(\int_{\mathbb{R}^{n-1}} g_t\, dv_{n-1}\Big)^{\lambda}.$$

Setting $F, G, H\colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ to be $F(s) := \int_{\mathbb{R}^{n-1}} f_s \, dv_{n-1}$, analogously $G, H$, we see that they satisfy (2.1) for $n = 1$, hence by the induction-basis we conclude

$$\int_{\mathbb{R}^n} h \, dv_n = \int_{\mathbb{R}} H \, dv_1 \geq \Big( \int_{\mathbb{R}} F \, dv_1 \Big)^{1-\lambda} \Big( \int_{\mathbb{R}} G \, dv_1 \Big)^{\lambda} = \\ = \Big( \int_{\mathbb{R}^n} f \, dv_n \Big)^{1-\lambda} \Big( \int_{\mathbb{R}^n} g \, dv_n \Big)^{\lambda}.$$

*Case 2, $f$ or $g$ is not integrable.* Then we approximate by $f_m := \min\{f, m\} 1_{mB^n}$ and $g_m := \min\{g, m\} 1_{mB^n}$, for $m \in \mathbb{N}$; from this follows that $f_m$ and $g_m$ are integrable, $(f_m)_{m \geq 1}$ and $(g_m)_{m \geq 1}$ are monotonically increasing and $(f_m)_{m \geq 1} \to f$, $(g_m)_{m \geq 1} \to g$ pointwise, hence by monotone convergence also the respective integrals converge. By assumption

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} \, g(y)^{\lambda} \geq f_m(x)^{1-\lambda} \, g_m(y)^{\lambda}$$

for all $x, y \in \mathbb{R}$ and $m \in \mathbb{N}$, therefore by case 1

$$\int_{\mathbb{R}^n} h \, dv_n \geq \Big( \int_{\mathbb{R}^n} f_m \, dv_n \Big)^{1-\lambda} \Big( \int_{\mathbb{R}^n} g_m \, dv_n \Big)^{\lambda}$$

for every $m \in \mathbb{N}$, and by taking the limit as $m \to \infty$ the claim is proved. □

*Remarks.* 1. The theorem can be generalized to an arbitrary number of factors, though for the present work this is not relevant.

2. The statement can be reformulated so as to be a reverse Hölder's inequality: set $p := \frac{1}{1-\lambda}$ and $q := \frac{1}{\lambda}$, then $p, q \in \mathbb{R}_{>1}$ and $\frac{1}{p} + \frac{1}{q} = 1$; furthermore use $f^p$ and $g^q$ instead of $f$ and $g$, resp., then there holds: if $h(z) \geq \sup\{f(x)g(y) \mid \frac{x}{p} + \frac{y}{q} = z\}$, then $\int_{\mathbb{R}^n} h \, dv_n \geq \big( \int_{\mathbb{R}^n} f^p \, dv_n \big)^{1/p} \big( \int_{\mathbb{R}^n} g^q \, dv_n \big)^{1/q}$, or more concisely in terms of the $L^p$-norms, $\|h\|_{L^1(\mathbb{R}^n)} \geq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}$.

Since Fresen's proof makes use of projections of random-variables on subspaces (eventually onedimensional for the final result) and the involved distributions have $v_n$-densities, it makes sense to study the densities of the projections. Here we need to explain how integration over a subspace is to be understood. Let $E \in G_{n,k}$ with $k \in [0, n]$; if $k = 0$, i.e. $E = \{o\}$, then we choose the Dirac-measure on $E$ and hence $\int_E f(x) \, dx = f(o)$ for every function $f\colon E \to [-\infty, \infty]$. If $k \geq 1$, then $E$ is an embedded differentiable manifold, hence the integral $\int_E f(x) \, dx$ exists for suitable measurable functions $f\colon E \to [-\infty, \infty]$ (see, e.g., Forster [12, Chapter 14] concerning integration over embedded manifolds). In particular we may choose an orthonormal basis $u_1, \dots, u_k \in E$ and with this the map $\kappa\colon \mathbb{R}^k \to \mathbb{R}^n$ defined by $\kappa(t) := \sum_{i=1}^k t_i u_i$ is a linear isometric parametrization of $E$ with Gramian determinant one, hence $\int_E f(x) \, dx = \int_{\mathbb{R}^k} (f \circ \kappa) \, dv_k$ whenever the integral is defined; in particular $f \in L^1(E)$ iff $f \circ \kappa \in L^1(\mathbb{R}^k)$.

Furthermore, $(E, \langle \cdot, \cdot \rangle)$ is a Euclidean vector-space in itself and $\tilde{\kappa}\colon \mathbb{R}^k \to E$, defined by $\tilde{\kappa}(t) := \kappa(t)$, is an isometric isomorphism, and for the adjoint operators there hold $\tilde{\kappa}^* = \kappa^* \restriction E$ and therefore $(\kappa^* \kappa)(t) = t$ for all $t \in \mathbb{R}^k$ and $(\kappa \kappa^*)(x) = x$ for all $x \in E$.

[*Proof.* We are going to use indices in order to make explicit on which space the respective inner products are defined. Obviously $\kappa^*\restriction E$ is a linear map from $E$ to $\mathbb{R}^k$. Moreover, for any $\xi \in \mathbb{R}^k$ and $x \in E$ we have

$$\langle \tilde{\kappa}^*(x), \xi\rangle_{\mathbb{R}^k} = \langle x, \tilde{\kappa}(\xi)\rangle_E = \langle x, \kappa(\xi)\rangle_{\mathbb{R}^n} = \langle \kappa^*(x), \xi\rangle_{\mathbb{R}^k},$$

and from that the desired identity follows. □]

For a subset, especially a subspace, $A \subset \mathbb{R}^n$ let $A^\perp := \{y \in \mathbb{R}^n | \forall x \in A\colon \langle x, y\rangle = 0\}$ denote the orthogonal complement; for $x \in \mathbb{R}^n$ we simplify $x^\perp := \{x\}^\perp$.

**Definition 2.8.** Let $f\colon \mathbb{R}^n \to [0,\infty]$ be measurable and let $E \subset \mathbb{R}^n$ be a subspace, then the *marginal* of $f$ on $E$ is defined as the map $\pi_E f\colon E \to [0,\infty]$ given by

$$\pi_E f(x) := \int_{E^\perp} f(x+y)\,dy.$$

*Remarks.* If $f$ is integrable, then $\int_E \pi_E f(x)\,dx = \int_E \int_{E^\perp} f(x+y)\,dy\,dx = \int_{\mathbb{R}^n} f(x)\,dx < \infty$, hence $\pi_E f$ is integrable too, and so takes the value $\infty$ on a null-set.

Recall that, given a subspace $E \in G_{n,k}$, $p_E\colon \mathbb{R}^n \to E \hookrightarrow \mathbb{R}^n$ denotes the orthogonal projection onto $E$.

**Lemma 2.9.** *Let $f, g\colon \mathbb{R}^n \to [0,\infty]$ be measurable and let $E \subset \mathbb{R}^n$ be a subspace.*

1. *If we define the measure $\mu$ by $d\mu = f\,dv_n$, then $\pi_E f$ is the density of $\mu \circ p_E^{-1}$ with respect to integration on $E$.*
2. *$\pi_E(f * g) = \pi_E f * \pi_E g$, where on the right-hand-side we have convolution on $E$, that is $(F * G)(x) = \int_E F(x-y)G(y)\,dy$ for $F, G\colon E \to [0,\infty]$ measurable.*

*Proof.* 1. Let $A \in \mathcal{B}(E)$, then $p_E^{-1}(A) = A + E^\perp$; for any $x \in E$ and $y \in E^\perp$ there holds $1_{A+E^\perp}(x+y) = 1_A(x)$, therefore

$$\begin{aligned}
\mu(p_E^{-1}(A)) = \mu(A + E^\perp) &= \int_{\mathbb{R}^n} 1_{A+E^\perp}(x) f(x)\,dx \\
&= \int_E \int_{E^\perp} 1_{A+E^\perp}(x+y) f(x+y)\,dy\,dx = \int_E 1_A(x) \int_{E^\perp} f(x+y)\,dy\,dx \\
&= \int_E 1_A(x)\pi_E f(x)\,dx.
\end{aligned}$$

2. Let $x \in E$, then

$$\begin{aligned}
\pi_E(f * g)(x) &= \int_{E^\perp} (f * g)(x+y)\,dy \\
&= \int_{E^\perp} \int_{\mathbb{R}^n} f(x+y-z)g(z)\,dz\,dy \\
&= \int_E \int_{E^\perp} g(z'+z'') \int_{E^\perp} f(x+y-z'-z'')\,dy\,dz''\,dz'
\end{aligned}$$

$$= \int_E \pi_E f(x - z') \int_{E^\perp} g(z' + z'')\, dz''\, dz'$$
$$= \int_E \pi_E f(x - z') \pi_E g(z')\, dz'$$
$$= (\pi_E f * \pi_E g)(x),$$

where from the second to the third line we have swapped the integrals and split up $\mathbb{R}^n = E \times E^\perp$, writing $z = z' + z''$, and from the third to the fourth line we have substituted $y' := y - z''$. □

*Remarks.* The important stochastic interpretation of the above lemma is as follows: let $X$ be a random-vector with distribution with $v_n$-density $f$, then the distribution of $p_E \circ X$ has density $\pi_E f$ with respect to integration over $E$.

Analogously to the marginals on subspaces we also need the Fourier-transform on subspaces.

**Definition 2.10** (Fourier-transform on subspaces)**.** Let $E \subset \mathbb{R}^n$ be a nonzero subspace, then the *Fourier-transform* on $E$ is the map $\mathcal{F}_E \colon L^1(E) \to C_0(E)$ defined by

$$\mathcal{F}_E f(\xi) = \int_E \mathsf{e}^{-2\pi \mathsf{i}\langle \xi, x\rangle} f(x)\, dx.$$

Instead of $\mathcal{F}_{\mathbb{R}^n}$ we may simply write $\mathcal{F}$.

*Remarks.* 1. If $\kappa \colon \mathbb{R}^k \to \mathbb{R}^n$ is a parametrization of $E \in G_{n,k}$ as described on p. 35, then we can express

$$\mathcal{F}_E f(\xi) = \int_E \mathsf{e}^{-2\pi \mathsf{i}\langle \xi, x\rangle} f(x)\, dx = \int_{\mathbb{R}^k} \mathsf{e}^{-2\pi \mathsf{i}\langle \xi, \kappa(t)\rangle} f(\kappa(t))\, dt$$
$$= \int_{\mathbb{R}^k} \mathsf{e}^{-2\pi \mathsf{i}\langle \kappa^*(\xi), t\rangle} f(\kappa(t))\, dt = \mathcal{F}_{\mathbb{R}^k}(f \circ \kappa)(\kappa^*(\xi)),$$

where $f \in L^1(E)$ and $\xi \in E$. In particular, for $E = \mathbb{R}^n$ we may use the identical parametrization and then Definition 2.10 is consistent with the usual definition of the Fourier-transform. This representation also shows that $\mathcal{F}_E$ maps to $C_0(E)$ indeed.

2. If $f$ is the distribution-density of an $\mathbb{R}^n$-valued random-variable $X$, then we can express $\mathcal{F}f(\xi) = \mathbb{E}[\mathsf{e}^{-2\pi \mathsf{i}\langle \xi, X\rangle}]$, with $\xi \in \mathbb{R}^n$.

The following lemma collects some basic facts concerning the Fourier-transform on subspaces; in particular, it shows that Fourier-transforms and marginals on subspaces go together well. For the properties of the Fourier-transform on $\mathbb{R}^n$ used in the proof we refer the reader to, e.g., Grafakos [16, Chapter 2.2].

**Lemma 2.11.** *Let $E \in G_{n,k}$ with $k \in [1, n]$.*

1. *Let $f, g \in L^1(E)$, then $\mathcal{F}_E(f * g) = \mathcal{F}_E f \cdot \mathcal{F}_E g$.*
2. *Let $f \in L^1(E)$ such that $\mathcal{F}_E f \in L^1(E)$, then the* inversion-formula *holds:*

$$f(x) = \int_E \mathsf{e}^{2\pi \mathsf{i}\langle \xi, x\rangle} \mathcal{F}_E f(\xi)\, d\xi.$$

*3. Let $f\colon \mathbb{R}^n \to [0,\infty]$ be integrable, then*

$$(\mathcal{F}f)\restriction E = \mathcal{F}_E(\pi_E f).$$

*In particular, if $(\mathcal{F}f)\restriction E \in L^1(E)$, the inversion-formula reads ($x \in E$)*

$$\pi_E f(x) = \int_E \mathsf{e}^{2\pi \mathsf{i}\langle \xi, x\rangle}\, \mathcal{F}f(\xi)\, d\xi.$$

*Proof.* Throughout this proof we fix a parametrization $\kappa\colon \mathbb{R}^k \to \mathbb{R}^n$ of $E$ as described on p. 35.

1. First note with regard to the convolution that, for any $t \in \mathbb{R}^k$,

$$\begin{aligned}(f * g)(\kappa(t)) &= \int_E f(\kappa(t) - x)g(x)\, dx = \int_{\mathbb{R}^k} f(\kappa(t) - \kappa(s))g(\kappa(s))\, ds \\ &= \int_{\mathbb{R}^k} (f\circ\kappa)(t-s)(g\circ\kappa)(s)\, ds = ((f\circ\kappa) * (g\circ\kappa))(t).\end{aligned}$$

This implies, for any $\xi \in E$,

$$\begin{aligned}\mathcal{F}_E(f*g)(\xi) &= \mathcal{F}_{\mathbb{R}^k}((f*g)\circ\kappa)(\kappa^*(\xi)) = \mathcal{F}_{\mathbb{R}^k}((f\circ\kappa)*(g\circ\kappa))(\kappa^*(\xi)) \\ &= \big(\mathcal{F}_{\mathbb{R}^k}(f\circ\kappa)\cdot\mathcal{F}_{\mathbb{R}^k}(g\circ\kappa)\big)(\kappa^*(\xi)) \\ &= \mathcal{F}_{\mathbb{R}^k}(f\circ\kappa)(\kappa^*(\xi))\mathcal{F}_{\mathbb{R}^k}(g\circ\kappa)(\kappa^*(\xi)) \\ &= \mathcal{F}_E f(\xi)\mathcal{F}_E g(\xi) = (\mathcal{F}_E f\cdot\mathcal{F}_E g)(\xi).\end{aligned}$$

2. We have $\mathcal{F}_E f = \mathcal{F}_{\mathbb{R}^k}(f\circ\kappa)\circ\kappa^* \in L^1(E)$ iff $\mathcal{F}_{\mathbb{R}^k}(f\circ\kappa)\circ\kappa^*\circ\kappa = \mathcal{F}_{\mathbb{R}^k}(f\circ\kappa) \in L^1(\mathbb{R}^k)$, hence inversion is applicable to $\mathcal{F}_{\mathbb{R}^k}(f\circ\kappa)$ and this yields, for any $x \in E$,

$$\begin{aligned}\int_E \mathsf{e}^{2\pi\mathsf{i}\langle\xi,x\rangle}\,\mathcal{F}_E f(\xi)\, d\xi &= \int_E \mathsf{e}^{2\pi\mathsf{i}\langle\xi,x\rangle}\,\mathcal{F}_{\mathbb{R}^k}(f\circ\kappa)(\kappa^*(\xi))\, d\xi \\ &= \int_{\mathbb{R}^k} \mathsf{e}^{2\pi\mathsf{i}\langle\kappa(t),x\rangle}\,\mathcal{F}_{\mathbb{R}^k}(f\circ\kappa)(t)\, dt \\ &= \int_{\mathbb{R}^k} \mathsf{e}^{2\pi\mathsf{i}\langle t,\kappa^*(x)\rangle}\,\mathcal{F}_{\mathbb{R}^k}(f\circ\kappa)(t)\, dt \\ &= (f\circ\kappa)(\kappa^*(x)) = f(x).\end{aligned}$$

3. Let $\xi \in E$, then,

$$\begin{aligned}\mathcal{F}_E(\pi_E f)(\xi) &= \int_E \mathsf{e}^{-2\pi\mathsf{i}\langle\xi,x\rangle}\,\pi_E f(x)\, dx \\ &= \int_E \mathsf{e}^{-2\pi\mathsf{i}\langle\xi,x\rangle}\int_{E^\perp} f(x+y)\, dy\, dx \\ &= \int_E\int_{E^\perp} \mathsf{e}^{-2\pi\mathsf{i}\langle\xi,x+y\rangle}\, f(x+y)\, dy\, dx \\ &= \int_{\mathbb{R}^n} \mathsf{e}^{-2\pi\mathsf{i}\langle\xi,x\rangle}\, f(x)\, dx \\ &= \mathcal{F}f(\xi),\end{aligned}$$

where in the third line we have used $\langle\xi,y\rangle = 0$ for $\xi \in E$, $y \in E^\perp$. □

We now formally define logarithmically concave functions, measures and random-variables. In the sequel we shorten the term to 'log-concave'.

**Definition 2.12** (log-concavity). 1. A function $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called *log-concave* iff for any $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$,

$$f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{2.2}$$

2. A finite measure $\mu$ on $\mathbb{R}^n$ is called *log-concave* iff for any compact sets[1] $A, B \subset \mathbb{R}^n$ and $\lambda \in (0,1)$,

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda} \mu(B)^{\lambda}. \tag{2.3}$$

3. A $\mathbb{R}^n$-valued random-variable is called *log-concave* iff its distribution is a log-concave measure.

*Remarks.* 1. A function $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is log-concave iff $-\log \circ f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex (here $\log(0) := -\infty$). Hence its support is a convex subset of $\mathbb{R}^n$ and it is continuous on the relative interior of its support.

[*Proof.* For any $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$ there holds

$$f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$$

if and only if

$$-\log\bigl(f((1-\lambda)x + \lambda y)\bigr) \leq -(1-\lambda)\log(f(x)) - \lambda \log(f(y)). \qquad \square]$$

2. The characteristic function $1_K$ of a convex subset $K \subset \mathbb{R}^n$ is log-convex.

[*Proof.* Let $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$. If $x, y \in K$, then also $(1-\lambda)x + \lambda y \in K$, hence both sides evaluate to 1; if $x \notin K$ or $y \notin K$, then the right-hand-side is 0; in either case the inequality (2.2) is satisfied. $\square$]

3. The function $x \mapsto \mathsf{e}^{-\|x\|^2}$ is log-concave. By applying Lemma 2.14 below it follows that any Gaußian density is log-concave and Theorem 2.15 implies that the normal distribution is log-concave.

4. The definition of log-concave measures can be extended to Radon-measures (i.e., locally finite, inner regular measures); any finite measure on $\mathcal{B}(\mathbb{R}^n)$ is a Radon-measure (see, e.g., [25, Theorem 13.6]). We will have no need of such a generalization.

As per the first remark, the support of a log-concave function is convex; the same is true for log-concave measures with the appropriate definition of 'support' inserted: the *support of the measure* $\mu$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is the (closed) set

$$\{x \in \mathbb{R}^n | \forall \varepsilon \in \mathbb{R}_{>0}\colon \mu(x + \varepsilon B^n) > 0\}.$$

(We provide no symbol because it would be rarely used.)

[1] The use of compact sets avoids the difficulty that in general $(1-\lambda)A + \lambda B$ need not be measurable even if $A, B$ are so. If $A, B$ are compact, then so is $A \times B$, the map $(x, y) \mapsto (1-\lambda)x + \lambda y$ is continuous and $(1-\lambda)A + \lambda B$ is the image of $A \times B$ under that map and hence it is compact, therefore measurable.

**Lemma 2.13.** *The support of any log-concave measure $\mu$ is convex.*

*Proof.* Let $x, y \in \mathbb{R}^n$ be in the support of $\mu$ and let $\lambda \in (0,1)$; we have to show $\mu((1-\lambda)x + \lambda y + \varepsilon B^n) > 0$ for any $\varepsilon \in \mathbb{R}_{>0}$, so let $\varepsilon \in \mathbb{R}_{>0}$. We note

$$(1-\lambda)x + \lambda y + \varepsilon B^n = (1-\lambda)(x + \varepsilon B^n) + \lambda(y + \varepsilon B^n)$$

(inclusion from left to right is valid for any set instead of $\varepsilon B^n$, the converse only holds for convex sets). Because $x$ and $y$ are in the support of $\mu$, we know $\mu(x + \varepsilon B^n) > 0$ and $\mu(y + \varepsilon B^n)$, and log-concavity finally implies

$$\mu\big((1-\lambda)x + \lambda y + \varepsilon B^n\big) = \mu\big((1-\lambda)(x + \varepsilon B^n) + \lambda(y + \varepsilon B^n)\big) \geq$$
$$\geq \mu(x + \varepsilon B^n)^{1-\lambda}\, \mu(y + \varepsilon B^n)^{\lambda} > 0. \qquad \square$$

The classes of log-concave functions and measures enjoy stability under certain important operations:

**Lemma 2.14.** *Let $\phi\colon \mathbb{R}^m \to \mathbb{R}^n$ be affine ($m \in \mathbb{N} \cup \{0\}$), let $\alpha \in \mathbb{R}_{\geq 0}$ and let $E \subset \mathbb{R}^n$ be a $k$-dimensional subspace ($k \in [1, n]$).*

*A. Let $f, g\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be log-concave functions.*
*1. $\alpha f$ is log-concave.*
*2. $f \circ \phi$ is log-concave.*
*3. $fg$ is log-concave.*
*4. If $f$ is integrable, then $\pi_E f$ is log-concave.*
*5. If $f$ and $g$ are integrable, then $f * g$ is log-concave.*

*B. Let $\mu\colon \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ and $\nu\colon \mathcal{B}(\mathbb{R}^m) \to \mathbb{R}_{\geq 0}$ be log-concave measures.*
*1. $\alpha\mu$ is log-concave.*
*2. $\nu \circ \phi^{-1}$ is log-concave.*
*3. If $K \subset \mathbb{R}^n$ is closed and convex, then $\mu\lceil K$ is log-concave.*
*4. $\mu \otimes \nu$ is log-concave.*
*5. If $m = n$, then $\mu * \nu$ is log-concave.*

*Proof.* A. Let $x, y \in \mathbb{R}^n$ (except for 4.) and $\lambda \in (0,1)$.
1. This follows from multiplying (2.2) with $\alpha$ and using $\alpha = \alpha^{1-\lambda}\,\alpha^{\lambda}$ on the right-hand-side.
2. $-\log \circ f$ is convex, therefore also $(-\log \circ f) \circ \phi = -\log \circ (f \circ \phi)$ and hence $f \circ \phi$ is log-concave.
3. Writing out (2.2) for either $f$ and $g$ and multiplying the inequalities, while using that all terms involved are nonnegative, yields the result.
4. Let $x, y \in E$, let $U := (u_1, \ldots, u_{n-k})$ be an orthonormal basis of $E^{\perp}$ and let $\xi, \eta \in \mathbb{R}^{n-k}$, then

$$f\big((1-\lambda)x + \lambda y + U((1-\lambda)\xi + \lambda\eta)\big) = f\big((1-\lambda)(x + U\xi) + \lambda(y + U\eta)\big) \geq$$
$$\geq f(x + U\xi)^{1-\lambda}\, f(y + U\eta)^{\lambda},$$

so the Prékopa-Leindler-inequality (with $h(\xi) := f((1-\lambda)x+\lambda y+U\xi)$, $F(\xi) := f(x+U\xi)$ and $g(\xi) := f(y+U\xi)$) implies

$$\begin{aligned}\pi_E f((1-\lambda)x+\lambda y) &= \int_{\mathbb{R}^{n-k}} f((1-\lambda)x+\lambda y+U\xi)\,d\xi \geq \\ &\geq \left(\int_{\mathbb{R}^{n-k}} f(x+U\xi)\,d\xi\right)^{1-\lambda}\left(\int_{\mathbb{R}^{n-k}} f(y+U\xi)\,d\xi\right)^{\lambda} = \\ &= \pi_E f(x)^{1-\lambda}\pi_E f(y)^{\lambda}.\end{aligned}$$

5. The maps $(x,y)\mapsto y$ and $(x,y)\mapsto x-y$, defined on $\mathbb{R}^n\times\mathbb{R}^n$, are affine, hence by 2. $(x,y)\mapsto f(x-y)$ and $(x,y)\mapsto g(y)$ are log-concave, then by 3. $h(x,y) := f(x-y)g(y)$ is log-concave; now by 4. $\pi_{\mathbb{R}^n\times\{o\}}h$ is log-concave, and finally, since $x\in\mathbb{R}^n\mapsto(x,o)\in\mathbb{R}^n\times\mathbb{R}^n$ is again affine, $f*g$ is log-concave, because $(f*g)(x) = \pi_{\mathbb{R}^n\times\{o\}}h(x,o) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$.

B. Let $A,B\subset\mathbb{R}^n$ be compact and let $\lambda\in(0,1)$.

1. Finiteness is clear; furthermore multiply both sides of (2.3) with $\alpha=\alpha^{1-\lambda}\,\alpha^{\lambda}$.

2. Clearly $(\nu\circ\phi^{-1})(\mathbb{R}^n)=\nu(\mathbb{R}^m)<\infty$. For the rest we have to anticipate Theorem 2.16:[2] For $\nu$ there exist $k\in[0,m]$, an injective affine map $\psi\colon\mathbb{R}^k\to\mathbb{R}^m$ and a log-concave density $f\colon\mathbb{R}^k\to\mathbb{R}_{\geq 0}$ such that, for all $A\in\mathcal{B}(\mathbb{R}^m)$, $\nu(A)=\int_{\psi^{-1}(A)} f\,dv_k=\int_{\mathbb{R}^k}(1_A\circ\psi)f\,dv_k$; this is equivalent to $\int_{\mathbb{R}^m} h\,d\nu=\int_{\mathbb{R}^k}(h\circ\psi)f\,dv_k$ for all measurable $h\colon\mathbb{R}^m\to\mathbb{R}_{\geq 0}$.
We know $(1-\lambda)(\phi\circ\psi)^{-1}(A)+\lambda(\phi\circ\psi)^{-1}(B)\subset(\phi\circ\psi)^{-1}((1-\lambda)A+\lambda B)$, this implies

$$1_{(1-\lambda)A+\lambda B}\big((\phi\circ\psi)((1-\lambda)x+\lambda y)\big)\geq 1_A((\phi\circ\psi)(x))^{1-\lambda}1_B((\phi\circ\psi)(y))^{\lambda},$$

together with $f((1-\lambda)x+\lambda y)\geq f(x)^{1-\lambda}\,f(y)^{\lambda}$ and Prékopa–Leindler (using $F(\xi) := 1_A((\phi\circ\psi)(\xi))f(\xi)$, $g(\xi) := 1_B((\phi\circ\psi)(\xi))f(\xi)$ and $h(\xi) := 1_{(1-\lambda)A+\lambda B}((\phi\circ\psi)(\xi))f(\xi)$) we conclude

$$\begin{aligned}\nu\big(\phi^{-1}((1-\lambda)A+\lambda B)\big) &= \int_{\mathbb{R}^m} 1_{(1-\lambda)A+\lambda B}(\phi(x))\,d\nu(x) \\ &= \int_{\mathbb{R}^k} 1_{(1-\lambda)A+\lambda B}(\phi(\psi(\xi)))f(\xi)\,d\xi \\ &\geq \left(\int_{\mathbb{R}^k} 1_A((\phi\circ\psi)(\xi))f(\xi)\,d\xi\right)^{1-\lambda}\left(\int_{\mathbb{R}^k} 1_B((\phi\circ\psi)(\xi))f(\xi)\,d\xi\right)^{\lambda} = \\ &= \left(\int_{\mathbb{R}^m} 1_A(\phi(x))\,d\nu(x)\right)^{1-\lambda}\left(\int_{\mathbb{R}^m} 1_B(\phi(x))\,d\nu(x)\right)^{\lambda} \\ &= \nu(\phi^{-1}(A))^{1-\lambda}\,\nu(\phi^{-1}(B))^{\lambda}.\end{aligned}$$

3. Recall $(\mu\restriction K)(A)=\mu(A\cap K)$. Finiteness follows from $\mu(K)\leq\mu(\mathbb{R}^n)<\infty$. Both $A\cap K$ and $B\cap K$ are compact by closedness of $K$, and $(1-\lambda)(A\cap K)+\lambda(B\cap K)\subset((1-$

[2] The problem is that $\phi^{-1}(A)$ need not be compact any more; this is always the case if $\phi$ is not injective and $\phi^{-1}(A)\neq\emptyset$.

$\lambda)A+\lambda B)\cap K$ (because: let $x\in A\cap K$ and $y\in B\cap K$, then $(1-\lambda)x+\lambda y\in(1-\lambda)A+\lambda B$ and $(1-\lambda)x+\lambda y\in K$ by convexity of $K$), hence

$$\begin{aligned}(\mu\restriction K)((1-\lambda)A+\lambda B)&=\mu\big(((1-\lambda)A+\lambda B)\cap K\big)\geq\mu\big((1-\lambda)(A\cap K)+\lambda(B\cap K)\big)\\&\geq\mu(A\cap K)^{1-\lambda}\,\mu(B\cap K)^{\lambda}=(\mu\restriction K)(A)^{1-\lambda}\,(\mu\restriction K)(B)^{\lambda}.\end{aligned}$$

4. First we have $(\mu\otimes\nu)(\mathbb{R}^n\times\mathbb{R}^m)=\mu(\mathbb{R}^n)\nu(\mathbb{R}^m)<\infty$. Again we invest Theorem 2.16, let $k$, $\psi$, $f$ be as before. Moreover, for $A\subset\mathbb{R}^n\times\mathbb{R}^m$ and $y\in\mathbb{R}^m$ define $A_y:=\{x\in\mathbb{R}^n|(x,y)\in A\}$; if $A$ is measurable (compact), then also $A_y$.
Now let $A,B\subset\mathbb{R}^n\times\mathbb{R}^m$ be compact, then, for any $y_1,y_2\in\mathbb{R}^m$, $(1-\lambda)A_{y_1}+\lambda B_{y_2}\subset((1-\lambda)A+\lambda B)_{(1-\lambda)y_1+\lambda y_2}$ and so

$$\mu\big(((1-\lambda)A+\lambda B)_{(1-\lambda)y_1+\lambda y_2}\big)\geq\mu((1-\lambda)A_{y_1}+\lambda B_{y_2})\geq\mu(A_{y_1})^{1-\lambda}\,\nu(B_{y_2})^{\lambda};$$

also $f((1-\lambda)\xi_1+\lambda\xi_2)\geq f(\xi_1)^{1-\lambda}\,f(\xi_2)^{\lambda}$ for all $\xi_1,\xi_2\in\mathbb{R}^k$. Thus we may apply Prékopa–Leindler to $F(\xi):=\mu(A_{\psi(\xi)})f(\xi)$, $g(\xi):=\mu(B_{\psi(\xi)})f(\xi)$ and $h(\xi):=\mu\big(((1-\lambda)A+\lambda B)_{\psi(\xi)}\big)f(\xi)$ and therewith obtain

$$\begin{aligned}(\mu\otimes\nu)((1-\lambda)A+\lambda B)&=\int_{\mathbb{R}^m}\mu\big(((1-\lambda)A+\lambda B)_y\big)\,d\nu(y)\\&=\int_{\mathbb{R}^k}\mu\big(((1-\lambda)A+\lambda B)_{\psi(\xi)}\big)f(\xi)\,d\xi\\&\geq\Big(\int_{\mathbb{R}^k}\mu(A_{\psi(\xi)})f(\xi)\,d\xi\Big)^{1-\lambda}\Big(\int_{\mathbb{R}^k}\mu(B_{\psi(\xi)})f(\xi)\,d\xi\Big)^{\lambda}=\\&=\Big(\int_{\mathbb{R}^m}\mu(A_y)\,d\nu(y)\Big)^{1-\lambda}\Big(\int_{\mathbb{R}^m}\mu(B_y)\,d\nu(y)\Big)^{\lambda}\\&=(\mu\otimes\nu)(A)^{1-\lambda}\,(\mu\otimes\nu)(B)^{\lambda}.\end{aligned}$$

5. This follows from 2. and 4., because $\mu*\nu=(\mu\otimes\nu)\circ\phi^{-1}$ with $\phi(x,y):=x+y$ by definition. □

Log-concave functions and measures are closely related to each other, as the following two theorems show.

**Theorem 2.15.** *Let $f\colon\mathbb{R}^n\to\mathbb{R}_{\geq 0}$ be log-concave, let $a+E\subset\mathbb{R}^n$ be the affine hull of its support, with some $a\in\mathbb{R}^n$ and $E\in G_{n,k}$, $k\in[0,n]$, suppose $\int_E f(a+x)\,dx<\infty$, and define $\mu\colon\mathcal{B}(\mathbb{R}^n)\to[0,\infty]$ by $\mu(A):=\int_E 1_A(a+x)f(a+x)\,dx$ for $A\in\mathcal{B}(\mathbb{R}^n)$, then $\mu$ is log-concave.*

*Proof.* $\mu(\mathbb{R}^n)<\infty$ is immediate.
Suppose $E=\mathbb{R}^n$ first. Let $A,B\subset\mathbb{R}^n$ be compact and let $\lambda\in(0,1)$. We observe

$$1_{(1-\lambda)A+\lambda B}((1-\lambda)x+\lambda y)\geq 1_A(x)^{1-\lambda}\,1_B(y)^{\lambda}$$

for any $x, y \in \mathbb{R}^n$, for if $x \in A$ and $y \in B$, then both sides equal 1, or if $x \notin A$ or $y \notin B$, then the right-hand-side equals 0. Setting $F := 1_A f$, $g := 1_B f$ and $h := 1_{(1-\lambda)A+\lambda B} f$, the premises for applying the Prékopa-Leindler-inequality are met and we get

$$\begin{aligned}\mu((1-\lambda)A+\lambda B) &= \int_{\mathbb{R}^n} 1_{(1-\lambda)A+\lambda B} f \, dv_n \geq \\ &\geq \left(\int_{\mathbb{R}^n} 1_A f \, dv_n\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} 1_B f \, dv_n\right)^{\lambda} = \\ &= \mu(A)^{1-\lambda} \, \mu(B)^{\lambda}.\end{aligned}$$

Now let $k := \dim(E) < n$. Let $\kappa \colon \mathbb{R}^k \to \mathbb{R}^n$ be a parametrization of $E$ as described on p. 35 and set $\phi := a + \kappa$, then $\phi$ is affine and thus $f \circ \phi$ is log-concave and defines a log-concave measure $\nu$ on $\mathbb{R}^k$. Then, for any $A \in \mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned}\mu(A) &= \int_E 1_A(a+x) f(a+x) \, dx = \int_{\mathbb{R}^k} 1_A(\phi(\xi)) f(\phi(\xi)) \, d\xi \\ &= \int_{\mathbb{R}^k} 1_{\phi^{-1}(A)}(\xi) f(\phi(\xi)) \, d\xi = \nu(\phi^{-1}(A));\end{aligned}$$

so if $A, B \subset \mathbb{R}^n$ are compact and $\lambda \in (0,1)$, then $\phi^{-1}((1-\lambda)A+\lambda B) \supset (1-\lambda)\phi^{-1}(A) + \lambda\phi^{-1}(B)$, $\phi^{-1}(A)$ and $\phi^{-1}(B)$ are compact, and log-concavity of $\mu$ follows from that of $\nu$. □

*Remarks.* As a special instance the measure induced by $\alpha 1_K$ with a convex set $K \subset \mathbb{R}^n$, where $v_n(K) < \infty$, and $\alpha \in \mathbb{R}_{>0}$, is log-concave, therefore so is the uniform distribution on a convex body (take $K$ a convex body and $\alpha = v_n(K)^{-1}$). If we followed the more general definition using Radon-measures also Lebesgue-measure itself would be log-concave.

The converse result to Theorem 2.15 goes back to Borell [3, Theorem 3.2]; actually his theorem covers both the necessity- and sufficiency-parts and is valid for Radon-measures.

**Theorem 2.16.** *Let $\mu \colon \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$ be a log-concave measure, then there exist a $k \in [0, n]$, a finite measure $\nu$ on $\mathbb{R}^k$ with a log-concave $v_k$-density and an affine map $\phi \colon \mathbb{R}^k \to \mathbb{R}^n$ such that $\mu = \nu \circ \phi^{-1}$.*
*(W/o proof.)*

A log-concave function need not be integrable, take for instance $f(x_1, \ldots, x_n) := \mathsf{e}^{x_1}$; but if it is, it vanishes at infinity with exponential rate, and therefore moments of arbitrary order exist (proof from Klartag [22, Lemma 2.1], also in [4, Lemma 2.2.1]):

**Lemma 2.17.** *Let $f \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be log-concave with $\int_{\mathbb{R}^n} f \, dv_n \in \mathbb{R}_{>0}$, then there exist constants $A, B \in \mathbb{R}_{>0}$ such that, for all $x \in \mathbb{R}^n$, $f(x) \leq A\mathsf{e}^{-B\|x\|}$.*

*Proof.* Because of $\int_{\mathbb{R}^n} f \, dv_n > 0$, there is a $c \in (0,1)$ with $v_n\big(f^{-1}[(c, \infty)]\big) > 0$, the latter set is convex by log-concavity of $f$, and since it has nonzero volume its interior is nonempty; let $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$ with $x_0 + rB^n \subset f^{-1}[(c,\infty)]$. It suffices to consider $x_0 = o$; else define $f_0(x) := f(x - x_0)$, then $f(x) = f_0(x + x_0) \le A\mathsf{e}^{-B\|x+x_0\|} \le A\mathsf{e}^{-B(\|x\|-\|x_0\|)} = (A\mathsf{e}^{B\|x_0\|})\mathsf{e}^{-B\|x\|}$.

The set $f^{-1}[(\frac{c}{\mathsf{e}},\infty)]$ is convex, contains $f^{-1}[(c,\infty)]$ and by integrability of $f$, $v_n\big(f^{-1}[(\frac{c}{\mathsf{e}},\infty)]\big) \in \mathbb{R}_{>0}$, thence it also is bounded, so let $R \in \mathbb{R}_{>0}$ with $f^{-1}[(\frac{c}{\mathsf{e}},\infty)] \subset \frac{R}{2}B^n$. (An unbounded convex set containing a ball also contains a half-cylinder with infinite height and therefore has infinite volume.) Then, for $x \in \mathbb{R}^n$ with $\|x\| \ge R$, $R\frac{x}{\|x\|} \notin f^{-1}[(\frac{c}{\mathsf{e}},\infty)]$ and thus $f(R\frac{x}{\|x\|}) \le \frac{c}{\mathsf{e}}$, and also $r\frac{x}{\|x\|} \in f^{-1}[(c,\infty)]$, thus $f(r\frac{x}{\|x\|}) \ge c$. Now write $R\frac{x}{\|x\|} = \frac{\|x\|-R}{\|x\|-r}\, r\frac{x}{\|x\|} + \frac{R-r}{\|x\|-r}\, x$, then by log-concavity

$$\frac{c}{\mathsf{e}} \ge f\Big(R\frac{x}{\|x\|}\Big) \ge f\Big(r\frac{x}{\|x\|}\Big)^{\frac{\|x\|-R}{\|x\|-r}} f(x)^{\frac{R-r}{\|x\|-r}} \ge c^{\frac{\|x\|-R}{\|x\|-r}} f(x)^{\frac{R-r}{\|x\|-r}}$$

and so,

$$f(x) \le c^{-\frac{\|x\|-R}{R-r}}\, c^{\frac{\|x\|-r}{R-r}}\, \mathsf{e}^{-\frac{\|x\|-r}{R-r}} \le \big(c\mathsf{e}^{\frac{r}{R-r}}\big)\mathsf{e}^{-\frac{\|x\|}{R}}.$$

If $\|x\| \le R$, then let $y \in \frac{x}{2} + \frac{r}{2}B^n$, this implies $\|2y - x\| = 2\|y - \frac{x}{2}\| \le 2\,\frac{r}{2} = r$, and from this and from $y = \frac{1}{2}\,(2y - x) + \frac{1}{2}\,x$ and log-concavity follows

$$f(y) \ge f(2x - y)^{\frac{1}{2}}\, f(x)^{\frac{1}{2}} \ge c^{\frac{1}{2}}\, f(x)^{\frac{1}{2}}$$

Thence,

$$\infty > \int_{\mathbb{R}^n} f(y)\,dy \ge \int_{\frac{x}{2}+\frac{r}{2}B^n} f(y)\,dy \ge c^{\frac{1}{2}}\, v_n\Big(\frac{r}{2}B^n\Big) f(x)^{\frac{1}{2}},$$

or,

$$f(x) \le c^{-1}\, v_n\Big(\frac{r}{2}B^n\Big)^{-2}\Big(\int_{\mathbb{R}^n} f\,dv_n\Big)^2 =: M \le (\mathsf{e}M)\mathsf{e}^{-\frac{\|x\|}{R}},$$

and thus the claim follows with $A := \max\big\{c\mathsf{e}^{\frac{r}{R-r}}, \mathsf{e}M\big\}$ and $B := \frac{1}{R}$. □

The following concept of isotropy in the context of convex bodies means that a body has the same moment of inertia for every axis of rotation (along with its barycentre sitting at the origin; see also Definition 1.2); in the context of probability measures it is more or less another term for ‘standardized’, that is, having zero expectation and unit covariance, but also used for functions and measures.

**Definition 2.18** (isotropy). 1. A density-function $f\colon \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is said to be *isotropic* iff

- $\int_{\mathbb{R}^n} x f(x)\,dx = o$ ($f$ is *centred*) and
- $\int_{\mathbb{R}^n} xx^{\mathsf{T}} f(x)\,dx = I_n$,

provided that the integrals exist.

2. A probability-measure $\mu\colon \mathcal{B}(\mathbb{R}^n) \to [0,\infty]$ is *isotropic* iff

- $\int_{\mathbb{R}^n} x \, d\mu(x) = o$ ($\mu$ is *centred*) and
- $\int_{\mathbb{R}^n} xx^\mathsf{T} \, d\mu(x) = I_n$,

provided that the integrals exist.

3. A $\mathbb{R}^n$-valued random-variable $X$ is *isotropic* iff its distribution is an isotropic measure (equivalently: $\mathbb{E}[X] = o$ and $\mathrm{Var}[X] = \mathbb{E}[XX^\mathsf{T}] = I_n$, provided that the moments exist).

The following lemma shows that isotropy can be achieved without great difficulty.

**Lemma 2.19.** *1. Let $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable with $\int_{\mathbb{R}^n} f \, dv_n \in \mathbb{R}_{>0}$, such that $\int_{\mathbb{R}^n} xf(x)\, dx$ and $\int_{\mathbb{R}^n} xx^\mathsf{T} f(x)\, dx$ are defined and the latter is nonsingular, then there exist a constant $\alpha \in \mathbb{R}_{>0}$ and a bijective affine map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\alpha f \circ \phi$ is isotropic.*

*2. Let $\mu\colon \mathcal{B}(\mathbb{R}^n) \to [0,\infty]$ be a finite, non-trivial measure such that $\int_{\mathbb{R}^n} x\, d\mu(x)$ and $\int_{\mathbb{R}^n} xx^\mathsf{T}\, d\mu(x)$ are defined and the latter is nonsingular, then there exist a constant $\alpha \in \mathbb{R}_{>0}$ and a bijective affine map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\alpha\mu \circ \phi$ is isotropic.*

*Proof.* 1. Define

$$b := \frac{\int_{\mathbb{R}^n} xf(x)\, dx}{\int_{\mathbb{R}^n} f(x) dx} \in \mathbb{R}^n,$$
$$A := \left[ \frac{\int_{\mathbb{R}^n} (x-b)(x-b)^\mathsf{T} f(x)\, dx}{\int_{\mathbb{R}^n} f(x)\, dx} \right]^{1/2} \in \mathrm{M}_n(\mathbb{R})$$

and

$$\alpha := \frac{\det(A)}{\int_{\mathbb{R}^n} f(x)\, dx} \in \mathbb{R}_{\geq 0},$$

then $A$ is symmetric and positive-definite, and hence $\alpha > 0$. The function defined by $f'(x) = \alpha f(Ax+b)$ is isotropic because

$$\begin{aligned} \int_{\mathbb{R}^n} f'(x)\, dx &= \alpha \int_{\mathbb{R}^n} f(Ax+b)\, dx = \frac{\alpha}{\det(A)} \int_{\mathbb{R}^n} f(y)\, dy = 1, \\ \int_{\mathbb{R}^n} xf'(x)\, dx &= \alpha \int_{\mathbb{R}^n} xf(Ax+b)\, dx = \frac{\alpha}{\det(A)} \int_{\mathbb{R}^n} A^{-1}(y-b) f(y)\, dy \\ &= \frac{A^{-1}}{\int_{\mathbb{R}^n} f\, dv_n} \left( \int_{\mathbb{R}^n} yf(y)\, dy - b \int_{\mathbb{R}^n} f(y)\, dy \right) = o, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} xx^\mathsf{T} f'(x)\, dx &= \alpha \int_{\mathbb{R}^n} xx^\mathsf{T} f(Ax+b)\, dx = \frac{\alpha}{\det(A)} \int_{\mathbb{R}^n} A^{-1}(y-b)(y-b)^\mathsf{T} A^{-1} f(y)\, dy \\ &= \frac{A^{-1}}{\int_{\mathbb{R}^n} f\, dv_n} \int_{\mathbb{R}^n} (y-b)(y-b)^\mathsf{T} f(y)\, dy\, A^{-1} = I_n. \end{aligned}$$

2. Analogous to above define

$$\alpha := \mu(\mathbb{R}^n)^{-1},$$
$$b := \alpha \int_{\mathbb{R}^n} x \, d\mu(x), \quad \text{and}$$
$$A := \left( \alpha \int_{\mathbb{R}^n} (x-b)(x-b)^\mathsf{T} \, d\mu(x) \right)^{\frac{1}{2}},$$

the affine map $\phi(x) := Ax + b$ and therewith $\mu' := \alpha\mu \circ \phi$. Applying the transformation-formula $\int_{\mathbb{R}^n} h(x)\, d\mu'(x) = \alpha \int_{\mathbb{R}^n} h(A^{-1}(x-b))\, d\mu(x)$ proves the statement. □

*Remarks.* If the function or measure is log-concave to begin with, then by Lemma 2.19 also the transformed function and measure, resp., will be log-concave.

The following simple lemma connects the expectation of the squared norm with the covariance-matrix.

**Lemma 2.20.** *Let* $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be a density with* $A = \int_{\mathbb{R}^n} xx^\mathsf{T} f(x)\, dx$*, then* $\int_{\mathbb{R}^n} \|x\|^2 f(x)\, dx = \operatorname{tr}(A)$.

*Proof.* This is a consequence of cyclic invariance of the trace, that is, $\|x\|^2 = x^\mathsf{T}x = \operatorname{tr}(x^\mathsf{T}x) = \operatorname{tr}(xx^\mathsf{T})$; as a linear operator the trace commutes with the integral. □

The next result is Grünbaum's lemma, going back to Grünbaum [17, Theorem 2 and Remark (iii)] and quoted as given in [4, Lemma 2.2.6].

**Lemma 2.21** (Grünbaum)**.** *Let* $\mu$ *be a centred log-concave probability-measure with full-dimensional support, then, for every* $\theta \in S^{n-1}$,

$$\mu\{x \in \mathbb{R}^n | \langle \theta, x \rangle \leq 0\} \geq \frac{1}{\mathsf{e}}.$$

*Remarks.* If the support of $\mu$ is not full-dimensional, it is contained in a hyperplane, and then for any $\theta \in S^{n-1}$ orthogonal to that hyperplane the stated measure is 1.

*Proof.* Let $\theta \in S^{n-1}$. Note that, since $\mu$ has full-dimensional support, it has a log-concave $v_n$-density $f$ by Theorem 2.16, hence it is absolutely continuous.

*Special case.* There exists $M \in \mathbb{R}_{>0}$ such that $\mu\{x \in \mathbb{R}^n || \langle \theta, x \rangle| > M\} = 0$. Define $F(t) := \mu\{x \in \mathbb{R}^n | \langle \theta, x \rangle \leq t\}$ for $t \in \mathbb{R}$, then $F$ is the CDF of the marginal of $\mu$ on $\mathbb{R}\theta$, hence absolutely continuous and increasing, with $F(t) = 0$ for $t \leq -M$ and $F(t) = 1$ for $t \geq M$; $F$ also is log-concave for the following reason: we set $A := \{(x, s) \in \mathbb{R}^n \times \mathbb{R} | \langle \theta, x \rangle \leq s\}$, then $A$ is convex, therefore $1_A$ is log-concave, thus $(x,t) \mapsto 1_A(x,t)f(x)$ is log-concave and so is $t \mapsto \int_{\mathbb{R}^n} 1_A(x,t) f(x)\, dx = F(t)$ (Lemma 2.14). We have to show $F(0) \geq \frac{1}{\mathsf{e}}$.

Because $\mu$ is centred, so are all its onedimensional marginals, and combined with partial integration this yields

$$0 = \int_{-M}^{M} tF'(t)\, dt = [tF(t)]_{-M}^{M} - \int_{-M}^{M} F(t)\, dt = M - \int_{-M}^{M} F(t)\, dt,$$

equivalently,

$$\int_{-M}^{M} F(t)\,dt = M.$$

$F$ is log-concave, hence $\log \circ F$ is concave, therefore $\log(F(t)) \leq \log(F(0)) + \frac{F'(0)}{F(0)}\,t$ for all $t \in \mathbb{R}$, that is $F(t) \leq F(0)\mathsf{e}^{\alpha t}$ with $\alpha := \frac{F'(0)}{F(0)} \in \mathbb{R}_{>0}$; w.l.o.g. let $\frac{1}{\alpha} < M$. Additionally we know $F \leq 1$, and therewith we can estimate

$$\begin{aligned} M = \int_{-M}^{M} F(t)\,dt &\leq \int_{-\infty}^{1/\alpha} F(0)\mathsf{e}^{\alpha t}\,dt + \int_{1/\alpha}^{M} 1\,dt \\ &= \frac{F(0)}{\alpha}\left[\mathsf{e}^{\alpha t}\right]_{-\infty}^{1/\alpha} + M - \frac{1}{\alpha} = \frac{\mathsf{e}F(0)}{\alpha} + M - \frac{1}{\alpha}, \end{aligned}$$

and $F(0) \geq \frac{1}{\mathsf{e}}$ follows by rearrangement.

*General case.* For any $M \in \mathbb{N}$ we set $H_M := \{x \in \mathbb{R}^n \,|\, |\langle \theta, x\rangle| \leq M\}$, then $H_M$ is closed, convex and $\mu(H_M) > 0$ (otherwise the convex support of $\mu$ would be contained in one of the halfspaces $\{x \in \mathbb{R}^n | \langle \theta, x\rangle < -M\}$ or $\{x \in \mathbb{R}^n | \langle \theta, x\rangle > M\}$, but then also the barycentre would lie in one of those and hence could not be zero) and $\lim_{M\to\infty} \mu(H_M) = 1$ by monotone convergence; furthermore we define $b_M := \frac{1}{\mu(H_M)} \int_{\mathbb{R}^n} 1_{H_M}(x) x\,d\mu(x)$ (the barycentre of $\mu$ restricted to $H_M$), then by dominated convergence $\lim_{M\to\infty} b_M = \int_{\mathbb{R}^n} x\,d\mu(x) = o$ since $\mu$ is centred, therefore there exists $B \in \mathbb{R}_{>0}$ such that $\|b_M\| \leq B$ for all $M \in \mathbb{N}$. With this we define the measure $\mu_M$ ($M \in \mathbb{N}$) by

$$\mu_M(A) := \frac{\mu\big((b_M + A) \cap H_M\big)}{\mu(H_M)} = \frac{1}{\mu(H_M)} \int_{\mathbb{R}^n} 1_{H_M}(x) 1_A(x - b_M)\,d\mu(x)$$

for any $A \in \mathcal{B}(\mathbb{R}^n)$. Obviously $\mu_M$ is a probability-measure with $\mu_M\{x \in \mathbb{R}^n \,|\, |\langle \theta, x\rangle| > M + B\} = 0$, moreover $\mu_M$ is centred. Also, $\mu_M$ is log-concave because it is the restriction to a closed, convex set and translation of the log-convex measure $\mu$. Furthermore, $\mu_M$ cannot be degenerate, otherwise its mass would be concentrated in the set $E \cap H_M$ with a proper subspace $E$, and then $\mu$ would be concentrated on $b_M + E$, which it is not. The special case now implies $\mu_M\{x \in \mathbb{R}^n | \langle \theta, x\rangle \geq 0\} \geq \frac{1}{\mathsf{e}}$, valid for all $M \in \mathbb{N}$.

We have $(1_{H_M}(x))_{M\geq 1} \to 1$ for every $x \in \mathbb{R}^n$. Let $A \in \mathcal{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n \setminus \partial A$, then there is $\varepsilon \in \mathbb{R}_{>0}$ with $x + \varepsilon B^n \subset \mathbb{R}^n \setminus \partial A$, and by $(b_M)_{M\geq 1} \to o$ there exists $M_0 \in \mathbb{N}$ such that $b_M \in \varepsilon B^n$ and therefore $x - b_M \in \mathbb{R}^n \setminus \partial A$ for any $M \geq M_0$, which implies $(1_A(x - b_M))_{M\geq 1} \to 1_A(x)$; hence if $\mu(\partial A) = 0$, then $(1_{H_M}(x) 1_A(x - b_M))_{M\geq 1} \to 1_A(x)$ $\mu$-almost everywhere, and from dominated convergence follows $(\mu_M(A))_{M\geq 1} \to \mu(A)$. As remarked at the beginning of the proof, the boundary of any halfspace has $\mu$-measure zero, also $\mu_M\{x \in \mathbb{R}^n | \langle \theta, x\rangle \geq 0\} \geq \frac{1}{\mathsf{e}}$ for all $M \geq 1$, and hence the claim follows. □

*Comment.* It is unclear to us what theoretical background Fresen expects from his readers, as on the one hand he claims that his proof is accessible to anyone, but on the other hand he calls the proving of Grünbaum's lemma an interesting exercise (page 8); though the proof cited above does not seem to us trivial at all. ◊

The following is an examplification of the rigidity of log-concave functions, it can be considered a strengthening of Fradelizi's theorem (as in [4, Theorem 2.2.2]) in the isotropic case.

**Lemma 2.22.** *Let* $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *be an isotropic log-concave density, then* $f(o) \geq 2^{-7n}$.

*Proof.* The set $f^{-1}[(f(o),\infty)] \subset \mathbb{R}^n$ is convex by log-concavity, with $o \notin f^{-1}[(f(o),\infty)]$, hence there exists $\theta \in S^{n-1}$ with $\langle\theta,x\rangle \leq \langle\theta,o\rangle = 0$ for all $x \in f^{-1}[(f(o),\infty)]$, or, equivalently, $\langle\theta,x\rangle > 0$ implies $f(x) \leq f(o)$, for all $x \in \mathbb{R}^n$. (Note that if $f^{-1}[(f(o),\infty)] = \emptyset$, any $\theta \in S^{n-1}$ works.)

Let $\mu$ denote the measure with $v_n$-density $f$, then, applying Lemma 2.21 (because of absolute continuity $>$ instead of $\geq$ does not hurt), for any $r \in \mathbb{R}_{>0}$,

$$
\begin{aligned}
\frac{1}{\mathsf{e}} &\leq \mu\{x \in \mathbb{R}^n | \langle\theta,x\rangle > 0\} = \\
&= \mu\{x \in \mathbb{R}^n | \langle\theta,x\rangle > 0 \wedge \|x\| < r\} + \mu\{x \in \mathbb{R}^n | \langle\theta,x\rangle > 0 \wedge \|x\| \geq r\} \leq \\
&\leq \int_{\{x\in\mathbb{R}^n | \langle\theta,x\rangle>0\wedge\|x\|<r\}} f(x)\,dx + \mu\{x \in \mathbb{R}^n | \|x\| \geq r\} \leq \\
&\leq f(o) v_n\big(\{x \in \mathbb{R}^n | \langle\theta,x\rangle > 0\} \cap rB^n\big) + \mu\{x \in \mathbb{R}^n | \|x\| \geq r\} = \\
&= \frac{1}{2} f(o) v_n(rB^n) + \mu\{x \in \mathbb{R}^n | \|x\| \geq r\},
\end{aligned}
$$

and therefore

$$
\mu\{x \in \mathbb{R}^n | \|x\| \geq r\} \geq \frac{1}{\mathsf{e}} - \frac{r^n}{2} f(o) v_n(B^n).
$$

Now substitute $r = Av_n(B^n)^{-1/n}$ with $A \in \mathbb{R}_{>0}$; by Lemma 2.20 and using Markov's inequality we obtain

$$
\begin{aligned}
n &= \int_{\mathbb{R}^n} \|x\|^2 f(x)\,dx \geq A^2 v_n(B^n)^{-2/n} \mu\{x \in \mathbb{R}^n | \|x\| \geq Av_n(B^n)^{-1/n}\} \geq \\
&\geq A^2 v_n(B^n)^{-2/n}\Big(\frac{1}{\mathsf{e}} - \frac{A^n v_n(B^n)^{-1}}{2} f(o) v_n(B^n)\Big) = \\
&= v_n(B^n)^{-2/n}\Big(\frac{A^2}{\mathsf{e}} - \frac{A^{n+2}}{2} f(o)\Big);
\end{aligned}
$$

the last expression is maximized at $A = \big(\frac{4}{\mathsf{e}(n+2)f(o)}\big)^{1/n}$, and plugging in yields

$$
n \geq v_n(B^n)^{-2/n}\Big(\frac{4}{\mathsf{e}(n+2)f(o)}\Big)^{2/n} \frac{n}{\mathsf{e}(n+2)}
$$

and thence

$$
f(o) \geq 4\big(\mathsf{e}(n+2)\big)^{-1-\frac{n}{2}} v_n(B^n)^{-1};
$$

we know $v_n(B^n) = \pi^{n/2}\Gamma(\frac{n+2}{2})^{-1} = \pi^{n/2}\frac{n+2}{2\Gamma(\frac{n+2}{2}+1)}$, so investing Stirling's approximation $\Gamma(x+1) \geq \sqrt{2\pi x}(\frac{x}{\mathsf{e}})^x$ we get

$$f(o) \geq 4\pi^{-\frac{n}{2}}\big(\mathsf{e}(n+2)\big)^{-\frac{n+2}{2}}\frac{2}{n+2}\sqrt{\pi(n+2)}\,(n+2)^{\frac{n+2}{2}}(2\mathsf{e})^{-\frac{n+2}{2}} = \\ = \frac{4\sqrt{\pi}}{\mathsf{e}^2}\frac{1}{\sqrt{n+2}}(\mathsf{e}\sqrt{2\pi})^{-n}.$$

By discussing $x \mapsto \sqrt{x+2}\,(\mathsf{e}\sqrt{2\pi})^x\,2^{-7x}$ for $x \geq 1$ we find out that its global maximum is smaller than $\frac{4\sqrt{\pi}}{\mathsf{e}^2}$, and hence the claim is established. □

*Comment.* We do not understand how Fresen obtains the worse estimate $Cn^{-\frac{3}{2}}(\mathsf{e}\sqrt{2\pi})^{-n}$ instead of our sharper one, $Cn^{-\frac{1}{2}}(\mathsf{e}\sqrt{2\pi})^{-n}$, as he does not spell out more details. His introduction of the sequence $(\alpha_n)_{n\geq 1}$ and statement of $\lim_{n\to\infty}\alpha_n = 1$ seem to us an unnecessary complication. A bit confusing are his ommissions of quantifications in lines 4 (generalization over $x \in \mathbb{R}^n$) and 8 (generalization over $A \in \mathbb{R}_{>0}$). Besides, in the fifth line from below there should stand $v_n\big(\alpha_n\sqrt{\frac{n}{2\pi\mathsf{e}}}B^n\big)$. ♢

*Remarks.* If $f$ is a centred but not necessarily isotropic, log-concave density, we know by Lemma 2.19 that $f' := \alpha f \circ A$ with $A := \big(\int_{\mathbb{R}^n} xx^{\mathsf{T}} f(x)\,dx\big)^{1/2}$ (abusing notation by writing $A$ both for the matrix and the induced linear map) and $\alpha := \det(A)$ is isotropic, thus $f'(o) = \alpha f(Ao) = \alpha f(o) \geq 2^{-7n}$, therefore $f(o) \geq \alpha^{-1}2^{-7n} = \det(A)^{-1}2^{-7n}$.

Lemma 2.17 can be strengthened under the assumtion of isotropy (the constants are no longer depending on the function); the case $n = 1$ is proved in [2, Lemma 3.2]:

**Lemma 2.23.** *There exist constants $A_n, B_n \in \mathbb{R}_{>0}$ such that for any isotropic, log-concave density $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ there holds, for all $x \in \mathbb{R}^n$, $f(x) \leq A_n\mathsf{e}^{-B_n\|x\|}$.*

*Proof.* We do not present a complete proof here, but restrict ourselves to plugging together the main ingredients. One is found in Lovász and Vempala [31, Theorem 5.14, (d)]: for any isotropic, log-concave density $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ there holds $f(o) \leq \frac{18^n}{v_n(B^n)} \leq 2\sqrt{n}(20n)^{\frac{n}{2}}$. (The proof is somewhat similar to that of our Lemma 2.22, which actually is part of the cited Theorem 5.14, (d), and also uses Grünbaum's lemma. The proof of our Lemma 2.22 expands the sketch given by Fresen, and does not follow [31].)

The other is [21, Corollary 2.4], stating that for any isotropic, log-concave function $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ we have $f(x) \leq f(o)\mathsf{e}^{Cn-c\|x\|}$ for all $x \in \mathbb{R}^n$, with universal constants $C, c \in \mathbb{R}^n$. [21] gives no explicit values for the constants, but retracing the steps of the proof (which also involves an excursus to [22, Lemma 2.7]) we may reconstruct $C = 2$ and $c = \frac{1}{48\log(80\mathsf{e}^4)} \geq \frac{1}{402.34}$.

The claim follows with $A_n := \frac{(18\mathsf{e}^2)^n}{v_n(B^n)}$ and $B_n := \frac{1}{48\log(80\mathsf{e}^4)}$ (note that the latter does not depend on dimension). □

The last lemma in this section already is in dedicate preparation for the proof of the main Theorem 3.1, it treats the radial part of a log-concave density when transforming integrals to polar coordinates and essentially tells that this radial part is strongly peaked, and if the locations of peaks for different angles do not differ too much, they are actually close to a multiple of $\sqrt{n}$.

**Lemma 2.24.** *Let $n \geq 2$, let $f\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a differentiable log-concave function with $\int_0^\infty f(tx)\,dt \in \mathbb{R}_{>0}$ for every $x \in S^{n-1}$. Let $x \in S^{n-1}$.*

*1. Define the function $F_x\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $F_x(t) := t^{n-1} f(tx)$, then $F_x$ is a log-concave function (considered as a function on $\mathbb{R}$ by extending with value zero).*

*2. $F_x$ has a unique maximizer $t_x \in \mathbb{R}_{>0}$, and there holds*

$$-t_x \frac{d}{dt} \log(f(tx))\big|_{t=t_x} = n-1.$$

*3. For all $u \in [0,1]$,*

$$\int_{(1-u)t_x}^{(1+u)t_x} F_x(t)\,dt \geq \big(1 - 3\mathsf{e}^{-\frac{\log(\mathsf{e}/2)}{2} nu^2}\big) \int_0^\infty F_x(t)\,dt.$$

*4. Let $f$ additionally be a centred density with $\int_{\mathbb{R}^n} xx^\mathsf{T} f(x)\,dx = \sigma^2 I_n$ for some $\sigma \in \mathbb{R}_{>0}$ and let there exist a $\gamma \in (0, \frac{1}{2}]$ such that $1 - \gamma \leq \frac{t_x}{t_y} \leq \frac{1}{1-\gamma}$ for all $x, y \in S^{n-1}$. For any $x \in S^{n-1}$ pick a random-variable $X_x$ with density $(\int_0^\infty F_x\,dv_1)^{-1} F_x$. Let there exist $C_1, C_2 \in \mathbb{R}_{>0}$ such that, for any $x \in S^{n-1}$, $|\mathbb{E}[X_x] - t_x| \leq C_1 \frac{t_x}{\sqrt{n}}$ and $\operatorname{Var}[X_x] \leq C_2 \frac{t_x^2}{n}$. Then there exists $C \in \mathbb{R}_{>0}$ such that, if $\gamma + \frac{C_1}{\sqrt{n}} \leq \frac{1}{4}$ holds, then, for any $x \in S^{n-1}$,*

$$\Big|\frac{t_x}{\sigma\sqrt{n}} - 1\Big| \leq 3\gamma + \frac{C}{\sqrt{n}}.$$

*Proof.* 1. It suffices to show that $t \mapsto t^{n-1}$ is log-concave; then, because $t \mapsto tx$ is affine, $t \mapsto f(tx)$ is log-conacave and thus $F_x$ as the product of log-concave functions is so itself.

Let $s, t \in \mathbb{R}_{\geq 0}$, $s \neq t$, and $\lambda \in (0,1)$. By the inequality of arithmetic and geometric means,

$$s^{1-\lambda} t^\lambda < (1-\lambda)s + \lambda t,$$

and thence

$$((1-\lambda)s + \lambda t)^{n-1} > (s^{1-\lambda} t^\lambda)^{n-1} = (s^{n-1})^{1-\lambda}\,(t^{n-1})^\lambda.$$

So actually $F_x$ is strictly log-concave on its support.

2. By Lemma 2.17, $F_x(t)$ decays at least exponentially to zero for $t \to \infty$. Furthermore $F_x(0) = 0$, and by the assumptions on $f$, values of $t$ with $F_x(t) > 0$ exist and $F_x$ is continuous, hence $F_x$ attains a global maximum; by strict log-convexity the maximizer is unique. The characterization of $t_x$ is a rearrangement of $F_x'(t_x) = 0$.

3. For brevity we call $c := \log(\frac{e}{2})$. The function $t \mapsto \log(f(tx))$ is concave and hence lies below its tangent at $t_x$; together with 2., this means, for all $t \in \mathbb{R}_{\geq 0}$,

$$\log(f(tx)) \leq \log(f(t_x x)) + \frac{d}{dt}\log(f(tx))\big|_{t=t_x}(t-t_x) = \\ = \log(f(t_x x)) - (n-1)\Big(\frac{t}{t_x}-1\Big).$$

Also $\log(1+z) \leq z - cz^2$ for all $z \in [-1,1]$, and therefore

$$\log\Big(\frac{t}{t_x}\Big) - \Big(\frac{t}{t_x}-1\Big) \leq -c\Big(\frac{t}{t_x}-1\Big)^2$$

for all $t \in \mathbb{R}_{\geq 0}$ with $\big|\frac{t}{t_x}-1\big| \leq 1$. These facts now imply

$$\begin{aligned} F_x(t) &= \exp\big((n-1)\log(t) + \log(f(tx))\big) \leq \\ &\leq \exp\Big((n-1)\log\Big(\frac{t}{t_x}\Big) + (n-1)\log(t_x) + \log(f(t_x x)) - (n-1)\Big(\frac{t}{t_x}-1\Big)\Big) = \\ &= F_x(t_x)\exp\Big((n-1)\log\Big(\frac{t}{t_x}\Big) - (n-1)\Big(\frac{t}{t_x}-1\Big)\Big) \leq \\ &\leq F_x(t_x)\exp\Big(-c(n-1)\Big(\frac{t}{t_x}-1\Big)^2\Big), \end{aligned}$$

still for all $t \in \mathbb{R}_{\geq 0}$ with $\big|\frac{t}{t_x}-1\big| \leq 1$.

Let $t \in (t_x, 2t_x]$. If $s \in [t_x, t]$, then $s = \frac{t-s}{t-t_x}\,t_x + \frac{s-t_x}{t-t_x}\,t$, and by log-concavity,

$$\begin{aligned} F_x(s) &\geq F_x(t_x)^{\frac{t-s}{t-t_x}}\,F_x(t)^{\frac{s-t_x}{t-t_x}} = F_x(t)\Big(\frac{F_x(t_x)}{F_x(t)}\Big)^{\frac{t-s}{t-t_x}} \geq \\ &\geq F_x(t)\exp\Big(c(n-1)\,\frac{(t-t_x)(t-s)}{t_x^2}\Big); \end{aligned}$$

integrating yields

$$\begin{aligned} \int_0^\infty F_x(s)\,ds &\geq \int_{t_x}^t F_x(s)\,ds \geq \\ &\geq F_x(t)\int_{t_x}^t \exp\Big(c(n-1)\,\frac{(t-t_x)(t-s)}{t_x^2}\Big)\,ds = \\ &= -\frac{t_x^2 F_x(t)}{c(n-1)(t-t_x)}\Big[\exp\Big(c(n-1)\,\frac{(t-t_x)(t-s)}{t_x^2}\Big)\Big]_{t_x}^t \\ &= \frac{t_x^2 F_x(t)}{c(n-1)(t-t_x)}\Big\{\exp\Big(c(n-1)\Big(\frac{t}{t_x}-1\Big)^2\Big)-1\Big\}, \end{aligned}$$

thus,

$$\frac{t_x^2 F_x(t)}{c(n-1)(t-t_x)} \leq \Big\{\exp\Big(c(n-1)\Big(\frac{t}{t_x}-1\Big)^2\Big)-1\Big\}^{-1}\int_0^\infty F_x(s)\,ds.$$

If $s \in [t, \infty)$, then $t = \frac{s-t}{s-t_x}\, t_x + \frac{t-t_x}{s-t_x}\, s$, so again by log-concavity,

$$F_x(t) \geq F_x(t_x)^{\frac{s-t}{s-t_x}}\, F_x(s)^{\frac{t-t_x}{s-t_x}},$$

or,

$$\begin{aligned} F_x(s) &\leq F_x(t_x)^{-\frac{s-t}{t-t_x}}\, F_x(t)^{\frac{s-t_x}{t-t_x}} = F_x(t)\Big(\frac{F_x(t)}{F_x(t_x)}\Big)^{\frac{s-t}{t-t_x}} \leq \\ &\leq F_x(t)\exp\Big(-c(n-1)\,\frac{(t-t_x)(s-t)}{t_x^2}\Big); \end{aligned}$$

again integrate,

$$\begin{aligned} \int_t^\infty F_x(s)\,ds \leq F_x(t)\int_t^\infty &\exp\Big(-c(n-1)\,\frac{(t-t_x)(s-t)}{t_x^2}\Big)\,ds = \\ &= -\frac{t_x^2 F_x(t)}{c(n-1)(t-t_x)}\,\Big[\exp\Big(-c(n-1)\,\frac{(t-t_x)(s-t)}{t_x^2}\Big)\Big]_t^\infty \\ &= \frac{t_x^2 F_x(t)}{c(n-1)(t-t_x)} \leq \\ &\leq \Big\{\exp\Big(c(n-1)\Big(\frac{t}{t_x}-1\Big)^2\Big)-1\Big\}^{-1}\int_0^\infty F_x(s)\,ds, \end{aligned}$$

where we have used our former result. Rewriting with $u := \frac{t}{t_x} - 1 \in (0,1]$ produces

$$\int_{(1+u)t_x}^\infty F_x(t)\,dt \leq \Big\{\exp\Big(c(n-1)u^2\Big)-1\Big\}^{-1}\int_0^\infty F_x(t)\,dt.$$

Now consider $t \in [0, t_x)$. As before, first let $s \in [t, t_x]$, then $s = \frac{t_x-s}{t_x-t}\, t + \frac{s-t}{t_x-t}\, t_x$, hence

$$\begin{aligned} F_x(s) &\geq F_x(t)^{\frac{t_x-s}{t_x-t}}\, F_x(t_x)^{\frac{s-t}{t_x-t}} = F_x(t)\Big(\frac{Fx(t_x)}{F_x(t)}\Big)^{\frac{t-s}{t-t_x}} \geq \\ &\geq F_x(t)\exp\Big(c(n-1)\,\frac{(t-t_x)(t-s)}{t_x^2}\Big), \end{aligned}$$

from this follows

$$\begin{aligned} \int_0^\infty F_x(s)\,ds \geq \int_t^{t_x} &F_x(s)\,ds \geq \\ &\geq F_x(t)\int_t^{t_x}\exp\Big(c(n-1)\,\frac{(t-t_x)(t-s)}{t_x^2}\Big)\,ds = \\ &= -\frac{t_x^2 F_x(t)}{c(n-1)(t-t_x)}\,\Big[\exp\Big(c(n-1)\,\frac{(t-t_x)(t-s)}{t_x^2}\Big)\Big]_t^{t_x} \\ &= \frac{t_x^2 F_x(t)}{c(n-1)(t_x-t)}\,\Big\{\exp\Big(c(n-1)\,\Big(\frac{t}{t_x}-1\Big)^2\Big)-1\Big\}, \end{aligned}$$

so,

$$\frac{t_x^2 F_x(t)}{c(n-1)(t_x-t)} \le \Big\{\exp\Big(c(n-1)\left(\frac{t}{t_x}-1\right)^2\Big)-1\Big\}^{-1}\int_0^\infty F_x(s)\,ds.$$

For $s\in[0,t]$ we have $t=\frac{t_x-t}{t_x-s}\,s+\frac{t-s}{t_x-s}\,t_x$, consequently,

$$F_x(t)\ge F_x(s)^{\frac{t_x-t}{t_x-s}}\,F_x(t_x)^{\frac{t-s}{t_x-s}},$$

thence,

$$\begin{aligned} F_x(s) &\le F_x(t)^{\frac{t_x-s}{t_x-t}}\,F_x(t_x)^{-\frac{t-s}{t_x-t}} = F_x(t)\Big(\frac{F_x(t)}{F_x(t_x)}\Big)^{\frac{s-t}{t-t_x}} \le \\ &\le F_x(t)\exp\Big(-c(n-1)\,\frac{(t-t_x)(s-t)}{t_x^2}\Big); \end{aligned}$$

from that by integrating,

$$\begin{aligned} \int_0^t F_x(s)\,ds &\le F_x(t)\int_0^t \exp\Big(-c(n-1)\,\frac{(t-t_x)(s-t)}{t_x^2}\Big)\,ds = \\ &= -\frac{t_x^2F_x(t)}{c(n-1)(t-t_x)}\,\Big[\exp\Big(-c(n-1)\,\frac{(t-t_x)(s-t)}{t_x^2}\Big)\Big]_0^t \\ &= \frac{t_x^2F_x(t)}{c(n-1)(t_x-t)}\,\Big\{1-\exp\Big(c(n-1)\,\frac{t(t-t_x)}{t_x^2}\Big)\Big\} \le \\ &\le \Big\{\exp\Big(c(n-1)\left(\frac{t}{t_x}-1\right)^2\Big)-1\Big\}^{-1}\int_0^\infty F_x(s)\,ds; \end{aligned}$$

by substituting $u:=1-\frac{t}{t_x}\in(0,1]$ this reads

$$\int_0^{(1-u)t_x} F_x(t)\,dt \le \Big\{\exp\Big(c(n-1)u^2\Big)-1\Big\}^{-1}\int_0^\infty F_x(t)\,dt.$$

Putting things together, we obtain

$$\int_{(1-u)t_x}^{(1+u)t_x} F_x(t)\,dt \ge \Big\{1-2\Big[\exp\Big(c(n-1)u^2\Big)-1\Big]^{-1}\Big\}\int_0^\infty F_x(t)\,dt,$$

and it remains to bound the term within the braces. First note

$$1-2\Big[\exp\Big(c(n-1)u^2\Big)-1\Big]^{-1}\le 0 \Longleftrightarrow |u|\le\Big(\frac{\log(3)}{c(n-1)}\Big)^{\frac12}.$$

Hence, it is sufficient to consider the range $|u|\in\big[\big(\frac{\log(3)}{c(n-1)}\big)^{1/2},1\big]$. We have

$$\Big[\exp\Big(c(n-1)u^2\Big)-1\Big]^{-1} = \exp\Big(-c(n-1)u^2\Big)\Big[1-\exp\Big(-c(n-1)u^2\Big)\Big]^{-1},$$

the inverted term is decreasing in $|u|$ and thus

$$\Big[1-\exp\Big(-c(n-1)u^2\Big)\Big]^{-1} \le \Big[1-\exp\Big(-c(n-1)\,\frac{\log(3)}{c(n-1)}\Big)\Big]^{-1} = \frac{3}{2},$$

therefore, for $|u| \ge \big(\frac{\log(3)}{c(n-1)}\big)^{1/2}$,

$$1-2\Big[\exp\Big(c(n-1)u^2\Big)-1\Big]^{-1} \ge 1-3\exp\Big(-c(n-1)u^2\Big),$$

and for smaller $|u|$ the right-hand-side is again negative. Lastly, $n-1 \ge \frac{n}{2}$ for $n \ge 2$, and so

$$1-3\exp\Big(-c(n-1)u^2\Big) \ge 1-3\exp\Big(-\frac{c}{2}\,nu^2\Big).$$

4. Here we set $c := \frac{\log(\mathrm{e}/2)}{2}$. Let $x \in S^{n-1}$. Then for any $y \in S^{n-1}$ we get the upper bound

$$\begin{aligned}\mathbb{E}[X_y^2] &= \mathbb{E}[X_y]^2 + \mathrm{Var}[X_y] \le \\ &\le \Big(t_y + C_1\,\frac{t_y}{\sqrt{n}}\Big)^2 + C_2\,\frac{t_y^2}{n} = \\ &= t_y^2\Big(1+\frac{2C_1}{\sqrt{n}}+\frac{C_1^2+C_2}{n}\Big),\end{aligned}$$

now we have $\frac{1}{n} \le \frac{1}{\sqrt{2n}}$ for $n \ge 2$ and $t_y^2 \le \frac{t_x^2}{(1-\gamma)^2} \le (1+6\gamma)t_x^2$ by exploiting convexity of $\gamma \mapsto \frac{1}{(1-\gamma)^2}$ and $\gamma \in [0,\frac{1}{2}]$, so we may continue,

$$\begin{aligned}\mathbb{E}[X_y^2] &\le t_x^2(1+6\gamma)\Big(1+\Big(2C_1+\frac{C_1^2+C_2}{\sqrt{2}}\Big)\frac{1}{\sqrt{n}}\Big) = \\ &= t_x^2\Big(1+6\gamma+(1+6\gamma)\Big(2C_1+\frac{C_1^2+C_2}{\sqrt{2}}\Big)\frac{1}{\sqrt{n}}\Big) \le \\ &\le t_x^2\Big(1+6\gamma+\frac{2C}{\sqrt{n}}\Big),\end{aligned}$$

where we have defined $C := 4C_1+\sqrt{2}(C_1^2+C_2)$. Using Lemma 2.20 and polar coordinates, we obtain

$$\begin{aligned}n\sigma^2 &= \int_{\mathbb{R}^n} \|\xi\|^2 f(\xi)\,d\xi \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_{S^{n-1}}\int_0^\infty r^2 F_y(r)\,dr\,d\sigma_{n-1}(y) \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_{S^{n-1}} \mathbb{E}[X_y^2]\int_0^\infty F_y(r)\,dr\,d\sigma_{n-1}(y) \le \\ &\le t_x^2\Big(1+6\gamma+\frac{2C}{\sqrt{n}}\Big)\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_{S^{n-1}}\int_0^\infty F_y(r)\,dr\,d\sigma_{n-1}(y) =\end{aligned}$$

$$= t_x^2\Big(1 + 6\gamma + \frac{2C}{\sqrt{n}}\Big).$$

From this follows

$$\frac{t_x}{\sigma\sqrt{n}} \geq \Big(1 + 6\gamma + \frac{2C}{\sqrt{n}}\Big)^{-\frac{1}{2}}.$$

Convexity of $\xi \mapsto (1+\xi)^{-1/2}$ implies $(1+\xi)^{-1/2} \geq 1 - \frac{\xi}{2}$ for all $\xi \in (-1, \infty)$ and therefore

$$\frac{t_x}{\sigma\sqrt{n}} \geq 1 - 3\gamma - \frac{C}{\sqrt{n}}.$$

Similarly we bound the expectation from below (note that by our assumtions $\frac{C_1}{\sqrt{n}} \leq \frac{1}{4}$),

$$\begin{aligned}
\mathbb{E}[X_y^2] \geq \mathbb{E}[X_y]^2 \geq \Big(t_y - C_1\frac{t_y}{\sqrt{n}}\Big)^2 &= \\
&= t_y^2\Big(1 - \frac{2C_1}{\sqrt{n}} + \frac{C_1^2}{n}\Big) \\
&\geq t_x^2(1-\gamma)^2\Big(1 - \frac{2C_1}{\sqrt{n}}\Big) \\
&\geq t_x^2(1-2\gamma)\Big(1 - \frac{2C_1}{\sqrt{n}}\Big) = \\
&= t_x^2\Big(1 - 2\gamma - \frac{2C_1}{\sqrt{n}} + \frac{4C_1\gamma}{\sqrt{n}}\Big) \geq \\
&\geq t_x^2\Big(1 - 2\gamma - \frac{2C_1}{\sqrt{n}}\Big),
\end{aligned}$$

where we have used $(1-\gamma)^2 \geq 1 - 2\gamma$ (by convexity). This we use in order to estimate in its turn

$$\begin{aligned}
n\sigma^2 &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_{S^{n-1}} \mathbb{E}[X_y^2]\int_0^\infty F_y(r)\,dr\,d\sigma_{n-1}(y) \geq \\
&\geq t_x^2\Big(1 - 2\gamma - \frac{2C_1}{\sqrt{n}}\Big).
\end{aligned}$$

Hence,

$$\frac{t_x}{\sigma\sqrt{n}} \leq \Big(1 - 2\gamma - \frac{2C_1}{\sqrt{n}}\Big)^{-\frac{1}{2}}.$$

Again by convexity we have $(1+x)^{-1/2} \leq 1 - 2(\sqrt{2}-1)x$ for $x \in [-\frac{1}{2}, 0]$, and by our assumptions we may use this to get

$$\frac{t_x}{\sigma\sqrt{n}} \leq 1 + 4(\sqrt{2}-1)\gamma + \frac{4(\sqrt{2}-1)C_1}{\sqrt{n}}.$$

From this the statement follows since $4(\sqrt{2}-1) \leq 3$ and $4(\sqrt{2}-1)C_1 \leq C$. $\square$

*Comment.* Though Fresen does not state it, point 3. essentially is [20, Lemma 4.5] (apart from the range of the relative deviation: $[0, \frac{1}{2}]$ for the former, $[0, 1]$ for the latter; we have adapted the constants so to cover the larger range). In our view, Fresen's proof has two advantages: it rquires $F_x$ to be differentiable only once (versus $C^2$ in [20]), and the techniques themselves are mostly elementary. Point 4. provides an analogue to [20, Lemma 4.6, (ii)]. ◊

## 2.3 Thin-shell-property and central limit-theorem

In this section we provide important stochastic tools, some of which bear an interest of their own. The first result regards the spherical properties, so to say, of the $n$-dimensional standard normal distribution.

**Lemma 2.25.** *Let $n \geq 2$ and let $Z$ be an $\mathbb{R}^n$-valued random-variable with standard normal distribution.*

1. *$\frac{Z}{\|Z\|}$ has distribution $\sigma_{n-1}$ and is independent of $\|Z\|$.*
2. *If $X$ is an $\mathbb{R}^n$-valued random-variable with distribution $\mu$ and independent of $Z$, then $Y := \langle \frac{X}{\|X\|}, \frac{Z}{\|Z\|} \rangle$ has the same distribution as $\frac{Z_1}{\|Z\|}$ and is independent of $\|X\|$.*
3. *For any $\varepsilon \in \mathbb{R}_{>0}$ there holds*

$$\mathbb{P}\Big[\Big|\frac{\|Z\|}{\sqrt{n}} - 1\Big| \geq \varepsilon\Big] \leq \frac{\mathsf{e} + 2}{2\sqrt{2}}\, \mathsf{e}^{-\frac{1}{2}n\varepsilon^2}.$$

*Proof.* 1. This actually holds for any radially symmetric distribution which puts no mass on the origin. See, e.g., Prochno, Thäle and Turchi [35, Proposition 3.3] for an even more general result. Write $\varphi(z) = \psi(\|z\|)$ with $\psi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, then by transforming to polar coordinates,

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} \varphi(z)\, dz = \int_{\mathbb{R}^n} \psi(\|z\|)\, dz \\ &= \int_0^\infty \int_{S^{n-1}} r^{n-1}\, \psi(\|r\theta\|)\, \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\, d\sigma_{n-1}(\theta)\, dr = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^\infty r^{n-1}\, \psi(r)\, dr. \end{aligned}$$

Now let $A \in \mathcal{B}(S^{n-1})$, then $\frac{Z}{\|Z\|} \in A$ iff $Z \in \mathbb{R}_{>0}A$ because the former implies $\|Z\| > 0$ and $Z \in \|Z\|A \subset \mathbb{R}_{>0}A$, and the latter means $Z = \lambda a$ for some $\lambda \in \mathbb{R}_{>0}$, $a \in A$, therefore $\|Z\| = \lambda$ and hence $\frac{Z}{\|Z\|} = a \in A$. With this we get

$$\begin{aligned} \mathbb{P}\Big[\frac{Z}{\|Z\|} \in A\Big] &= \mathbb{P}[Z \in \mathbb{R}_{>0}A] = \int_{\mathbb{R}_{>0}A} \varphi(z)\, dz \\ &= \int_0^\infty \int_A r^{n-1}\, \psi(r)\, \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\, d\sigma_{n-1}(\theta)\, dr \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^\infty r^{n-1}\, \psi(r)\, dr \int_A d\sigma_{n-1}(\theta) = \sigma_{n-1}(A). \end{aligned}$$

For the independence-part let $A \in \mathcal{B}(S^{n-1})$ and $B \in \mathcal{B}(\mathbb{R}_{\geq 0})$, then via a similar reasoning as above, $\frac{Z}{\|Z\|} \in A \wedge \|Z\| \in B$ iff $Z \in BA$; moreover $1_{BA}(r\theta) = 1_B(r)1_A(\theta)$ for $r \in \mathbb{R}_{\geq 0}$ and $\theta \in S^{n-1}$. Thus,

$$\begin{aligned}
\mathbb{P}\Big[\frac{Z}{\|Z\|} \in A \wedge \|Z\| \in B\Big] &= \mathbb{P}[Z \in BA] = \int_{\mathbb{R}^n} 1_{BA}(z)\varphi(z)\,dz \\
&= \int_{\mathbb{R}_{\geq 0}} \int_{S^{n-1}} r^{n-1}\, 1_{BA}(r\theta)\psi(r)\,\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,d\sigma_{n-1}(\theta)\,dr \\
&= \int_B r^{n-1}\,\psi(r)\,\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,dr \int_A d\sigma_{n-1}(\theta) \\
&= \sigma_{n-1}(A) \int_B \int_{S^{n-1}} r^{n-1}\,\psi(r)\,\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\,d\sigma_{n-1}(\theta)\,dr \\
&= \sigma_{n-1}(A) \int_{\{z \in \mathbb{R}^n | \|z\| \in B\}} \varphi(z)\,dz \\
&= \mathbb{P}\Big[\frac{Z}{\|Z\|} \in A\Big]\,\mathbb{P}[\|Z\| \in B].
\end{aligned}$$

2. For the distribution of $Y$ let $t \in \mathbb{R}$ and note that because of 1., $\mathbb{P}\big[\frac{Z_1}{\|Z\|} \leq t\big] = \sigma_{n-1}\{\theta \in S^{n-1} | \theta_1 \leq t\}$. Thus,

$$\begin{aligned}
\mathbb{P}[Y \leq t] &= (\mu \otimes \sigma_{n-1})\Big\{(x,\theta) \in \mathbb{R}^n \times S^{n-1} \Big| \Big\langle \frac{x}{\|x\|}, \theta \Big\rangle \leq t\Big\} \\
&= \int_{\mathbb{R}^n} \int_{S^{n-1}} 1_{[\langle \frac{x}{\|x\|},\theta\rangle \leq t]}(x,\theta)\,d\sigma_{n-1}(\theta)\,d\mu(x) \\
&= \int_{\mathbb{R}^n} \sigma_{n-1}\Big\{\theta \in S^{n-1} \Big| \Big\langle \frac{x}{\|x\|}, \theta \Big\rangle \leq t\Big\}\,d\mu(x) \\
&= \int_{\mathbb{R}^n} \sigma_{n-1}\{\theta \in S^{n-1} | \theta_1 \leq t\}\,d\mu(x) \\
&= \sigma_{n-1}\{\theta \in S^{n-1} | \theta_1 \leq t\},
\end{aligned}$$

where we have used rotational invariance of $\sigma_{n-1}$ (rotate $\frac{x}{\|x\|}$ to $e_1$). For the independence-part let $s, t \in \mathbb{R}$ and $s \geq 0$, yielding

$$\begin{aligned}
\mathbb{P}\Big[\|X\| \leq s \wedge \Big\langle \frac{X}{\|X\|}, \frac{Z}{\|Z\|} \Big\rangle \leq t\Big] &= \int_{\mathbb{R}^n} \int_{S^{n-1}} 1_{[\|x\| \leq s] \cap [\langle \frac{x}{\|x\|},\theta\rangle \leq t]}(x,\theta)\,d\sigma_{n-1}(\theta)\,d\mu(x) \\
&= \int_{[\|x\| \leq s]} \sigma_{n-1}\Big\{\theta \in S^{n-1} \Big| \Big\langle \frac{x}{\|x\|}, \theta \Big\rangle \leq t\Big\}\,d\mu(x) \\
&= \mu\{x \in \mathbb{R}^n | \|x\| \leq s\}\,\sigma_{n-1}\{\theta \in S^{n-1} | \theta_1 \leq t\} \\
&= \mathbb{P}[\|X\| \leq s]\,\mathbb{P}[Y \leq t],
\end{aligned}$$

where again we have used rotational invariance of $\sigma_{n-1}$, as before.

3. Let $\varepsilon \in \mathbb{R}_{>0}$; we will estimate the upper and lower tails separately, first the former. There holds

$$\begin{aligned}\mathbb{P}\Big[\frac{\|Z\|}{\sqrt{n}} - 1 \ge \varepsilon\Big] &= \mathbb{P}[\|Z\| \ge (1+\varepsilon)\sqrt{n}] = \mathbb{P}\big[Z \in [(1+\varepsilon)\sqrt{n}, \infty)S^{n-1}\big] \\ &= \int_{(1+\varepsilon)\sqrt{n}}^{\infty} \int_{S^{n-1}} r^{n-1} (2\pi)^{-\frac{n}{2}} \, \mathsf{e}^{-\frac{1}{2}r^2} \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \, d\sigma_{n-1}(\theta) \, dr \\ &= \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_{(1+\varepsilon)\sqrt{n}}^{\infty} r^{n-1} \, \mathsf{e}^{-\frac{1}{2}r^2} \, dr;\end{aligned}$$

transforming $r = x + \sqrt{n}$, we get

$$r^{n-1} = (x+\sqrt{n})^{n-1} = n^{\frac{n-1}{2}} \Big(1 + \frac{x}{\sqrt{n}}\Big)^{n-1} \le n^{\frac{n-1}{2}} \, \mathsf{e}^{x\sqrt{n}},$$

so the integral can be estimated by

$$\begin{aligned}\int_{(1+\varepsilon)\sqrt{n}}^{\infty} r^{n-1} \, \mathsf{e}^{-\frac{1}{2}r^2} \, dr &\le n^{\frac{n-1}{2}} \int_{\varepsilon\sqrt{n}}^{\infty} \mathsf{e}^{x\sqrt{n}} \, \mathsf{e}^{-\frac{1}{2}(x+\sqrt{n})^2} \, dx = \\ &= n^{\frac{n-1}{2}} \, \mathsf{e}^{-\frac{n}{2}} \int_{\varepsilon\sqrt{n}}^{\infty} \mathsf{e}^{-\frac{1}{2}x^2} \, dx \le \\ &\le \sqrt{\frac{\pi}{2}} \, n^{\frac{n-1}{2}} \, \mathsf{e}^{-\frac{n}{2}} \, \mathsf{e}^{-\frac{1}{2}n\varepsilon^2},\end{aligned}$$

where we have used the estimate $\int_{\delta}^{\infty} \varphi(t) \, dt \le \frac{1}{2} \varphi(\delta)$. By Stirling's formula we also have $\Gamma(\frac{n}{2}) = \frac{2}{n} \Gamma(\frac{n}{2}+1) \ge \frac{2}{n} \sqrt{\pi n} \, n^{\frac{n}{2}} \, 2^{-\frac{n}{2}} \, \mathsf{e}^{-\frac{n}{2}}$; thus we obtain

$$\begin{aligned}\mathbb{P}\Big[\frac{\|Z\|}{\sqrt{n}} - 1 \ge \varepsilon\Big] &\le \frac{2^{1-\frac{n}{2}}}{\frac{2}{n} \sqrt{\pi n} \, n^{\frac{n}{2}} \, 2^{-\frac{n}{2}} \, \mathsf{e}^{-\frac{n}{2}}} \sqrt{\frac{\pi}{2}} \, n^{\frac{n-1}{2}} \, \mathsf{e}^{-\frac{n}{2}} \, \mathsf{e}^{-\frac{1}{2}n\varepsilon^2} = \\ &= \frac{1}{\sqrt{2}} \, \mathsf{e}^{-\frac{1}{2}n\varepsilon^2}.\end{aligned}$$

Concerning the lower tail we have

$$\begin{aligned}\mathbb{P}\Big[\frac{\|Z\|}{\sqrt{n}} - 1 \le -\varepsilon\Big] &= \mathbb{P}[\|Z\| \le (1-\varepsilon)\sqrt{n}] \\ &= \frac{2^{1-\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_{0}^{(1-\varepsilon)\sqrt{n}} r^{n-1} \, \mathsf{e}^{-\frac{1}{2}r^2} \, dr;\end{aligned}$$

with the substitution $r = x + \sqrt{n}$, the domain of integration becomes $[-\sqrt{n}, -\varepsilon\sqrt{n}]$, and as before $(x+\sqrt{n})^{n-1} = n^{\frac{n-1}{2}} (1 + \frac{x}{\sqrt{n}})^{n-1}$; estimating that from above is more involving this time: consider $f(x) := (1 + \frac{x}{\sqrt{n}})^{n-1} \, \mathsf{e}^{-x\sqrt{n}}$ for $x \in [-\sqrt{n}, 0]$, the critical points are $-\sqrt{n}$ (only for $n \ge 3$) and $-\frac{1}{\sqrt{n}}$, the respective function-values are $f(-\sqrt{n}) = 0$ and

$f(-\frac{1}{\sqrt{n}}) = \mathsf{e}(1-\frac{1}{n})^{n-1}$; because of $f(-\sqrt{n}) = 0 \leq f(x)$, $f(0) = 1$, and $f(-\frac{1}{\sqrt{n}}) \in [1, \frac{\mathsf{e}}{2}]$ we know $f(x) \leq \frac{\mathsf{e}}{2}$ and thus $(1+\frac{x}{\sqrt{n}})^{n-1} \leq \frac{\mathsf{e}}{2}\,\mathsf{e}^{x\sqrt{n}}$; so the integral is bounded from above as follows,

$$\begin{aligned}\int_{-\sqrt{n}}^{-\varepsilon\sqrt{n}} \Big(1+\frac{x}{\sqrt{n}}\Big)^{n-1} \mathsf{e}^{-\frac{1}{2}(x+\sqrt{n})^2}\,dx &\leq \frac{\mathsf{e}}{2}\int_{-\sqrt{n}}^{-\varepsilon\sqrt{n}} \mathsf{e}^{x\sqrt{n}}\,\mathsf{e}^{-\frac{1}{2}(x+\sqrt{n})^2}\,dx \leq \\ &\leq \frac{\mathsf{e}}{2}\,\mathsf{e}^{-\frac{n}{2}} \int_{-\infty}^{-\varepsilon\sqrt{n}} \mathsf{e}^{-\frac{1}{2}x^2}\,dx \leq \\ &\leq \sqrt{\frac{\pi}{2}}\,\frac{\mathsf{e}}{2}\,\mathsf{e}^{-\frac{n}{2}}\,\mathsf{e}^{-\frac{1}{2}n\varepsilon^2};\end{aligned}$$

using the same estimate for $\Gamma(\frac{n}{2})$ as above, we arrive at

$$\begin{aligned}\mathbb{P}\Big[\frac{\|Z\|}{\sqrt{n}} - 1 \leq -\varepsilon\Big] &\leq \frac{2^{1-\frac{n}{2}}}{\frac{2}{n}\sqrt{\pi n}\, n^{\frac{n}{2}}\, 2^{-\frac{n}{2}}\,\mathsf{e}^{-\frac{n}{2}}}\, n^{\frac{n-1}{2}} \sqrt{\frac{\pi}{2}}\,\frac{\mathsf{e}}{2}\,\mathsf{e}^{-\frac{n}{2}}\,\mathsf{e}^{-\frac{1}{2}n\varepsilon^2} = \\ &= \frac{\mathsf{e}}{2\sqrt{2}}\,\mathsf{e}^{-\frac{1}{2}n\varepsilon^2}.\end{aligned}$$

Adding the inequalities for upper and lower tail finishes the proof. □

The next lemma is intimately related to the concentration of measure on the sphere, saying that the volume of a spherical cap approximately follows a normal distribution with standard deviation $\frac{1}{\sqrt{n}}$; this has the effect that in high dimensions an only slightly "fattened" equator already carries most of the mass (of course because of orthogonal invariance this holds for every great circle), see also [24, p. 403]. The interested reader is referred to the textbook of Ledoux [26] for the basics of concentration of measure.

**Lemma 2.26.** *Let $n \geq 2$. Define $\Phi_n\colon \mathbb{R} \to [0,1]$ by $\Phi_n(t) := \sigma_{n-1}\{\theta \in S^{n-1} | \theta_1 \leq \frac{t}{\sqrt{n}}\}$, then*

$$\|\Phi_n - \Phi\|_{L^\infty(\mathbb{R})} \leq \frac{2}{\sqrt{n}}.$$

*Remarks.* The bound actually is poor; graphical experiments suggest something like at most $\frac{0.15}{\sqrt{n}}$, and perhaps even $\frac{0.2}{n}$, but the proof does not yield as much.

*Proof.* Let $Z$ be an $\mathbb{R}^n$-valued random-variable with standard normal distribution, then Lemma 2.25, 1., implies $\Phi_n(t) = \mathbb{P}\big[\frac{Z_1}{\|Z\|} \leq \frac{t}{\sqrt{n}}\big]$. Note that because of the symmetry of $S^{n-1}$ there holds $\Phi_n(-t) = 1 - \Phi_n(t)$, the same goes for $\Phi$ and hence it suffices to consider $|\Phi_n(t) - \Phi(t)|$ for $t \in \mathbb{R}_{\geq 0}$. Also the statement is trivially valid for $n \leq 4$, so w.l.o.g. assume $n \geq 5$.

First let $t \geq \sqrt{n}$, then $\Phi_n(t) = 1$, hence

$$|\Phi_n(t) - \Phi(t)| = 1 - \Phi(t) \leq \frac{1}{t}\,\varphi(t) \leq \frac{1}{\sqrt{2\pi}\sqrt{n}}.$$

Now consider $t \in [0, \sqrt{n})$. Denote $W := (Z_2, \dots, Z_n)^\mathsf{T}$, then $Z_1$ and $W$ are independent Gaußian random-variables, and $\|Z\|^2 = Z_1^2 + \|W\|^2$. Thus,

$$\begin{aligned}
|\Phi_n(t) - \Phi(t)| &= \Big|\mathbb{P}\Big[Z_1 \leq \frac{t}{\sqrt{n}}\,\|Z\|\Big] - \Phi(t)\Big| = \Big|\mathbb{P}\Big[Z_1 \leq 0 \vee Z_1^2 \leq \frac{t^2}{n}\,\|Z\|^2\Big] - \Phi(t)\Big| \\
&= \Big|\mathbb{P}\Big[Z_1 \leq 0 \vee \frac{n-t^2}{n}\,Z_1^2 \leq \frac{t^2}{n}\,\|W\|^2\Big] - \Phi(t)\Big| \\
&= \Big|\mathbb{P}\Big[Z_1 \leq \frac{t}{\sqrt{n-t^2}}\,\|W\|\Big] - \Phi(t)\Big| \\
&= \Big|\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\frac{t}{\sqrt{n-t^2}}\|w\|} (2\pi)^{-\frac{1}{2}} \mathsf{e}^{-\frac{1}{2}z_1^2}\,dz_1\,(2\pi)^{-\frac{n-1}{2}} \mathsf{e}^{-\frac{1}{2}\|w\|^2}\,dw - \Phi(t)\Big| \\
&= \Big|\int_{\mathbb{R}^{n-1}} \Phi\Big(\frac{t\|w\|}{\sqrt{n-t^2}}\Big)(2\pi)^{-\frac{n-1}{2}} \mathsf{e}^{-\frac{1}{2}\|w\|^2}\,dw - \Phi(t)\Big| \\
&= \Big|\mathbb{E}\Big[\Phi \circ \frac{t\|W\|}{\sqrt{n-t^2}}\Big] - \Phi(t)\Big| \leq \\
&\leq \mathbb{E}\Big[\Big|\Phi \circ \frac{t\|W\|}{\sqrt{n-t^2}} - \Phi(t)\Big|\Big].
\end{aligned}$$

For $t^2 \geq \frac{\sqrt{n}}{2}$ we use $1 - \Phi(s) \leq \frac{1}{2}\mathsf{e}^{-\frac{s^2}{2}}$ and $\mathbb{E}[\mathsf{e}^{-s\|Z\|^2}] = \mathbb{E}[\mathsf{e}^{-sZ_1^2}]^n = (1+2s)^{-\frac{n}{2}}$, either for $s \in \mathbb{R}_{\geq 0}$, also $(1+\frac{x}{n})^n \leq \mathsf{e}^x$ (for $x \in [-n, \infty)$), and $\mathsf{e}^{-x} \leq \frac{1}{\mathsf{e}x}$ (for $x \in \mathbb{R}_{\geq 0}$); this yields

$$\begin{aligned}
\mathbb{E}\Big[\Big|\Phi \circ \frac{t\|W\|}{\sqrt{n-t^2}} - \Phi(t)\Big|\Big] &\leq \frac{1}{2}\,\mathbb{E}\big[\mathsf{e}^{-\frac{t^2}{2(n-t^2)}\|W\|^2}\big] + \frac{1}{2}\,\mathsf{e}^{-\frac{1}{2}t^2} = \\
&= \frac{1}{2}\Big(1 + 2\,\frac{t^2}{2(n-t^2)}\Big)^{-\frac{n-1}{2}} + \frac{1}{2}\,\mathsf{e}^{-\frac{1}{2}t^2} \\
&= \frac{1}{2}\Big(1 - \frac{t^2}{n}\Big)^{\frac{n-1}{2}} + \frac{1}{2}\,\mathsf{e}^{-\frac{1}{2}t^2} \leq \\
&\leq \frac{1}{2}\,\mathsf{e}^{-\frac{n-1}{2n}t^2} + \frac{1}{2}\,\mathsf{e}^{-\frac{1}{2}t^2} \\
&\leq \frac{1}{2}\,\frac{2}{\mathsf{e}t^2}\Big(\frac{n}{n-1} + 1\Big) \\
&\leq \frac{31}{15\mathsf{e}}\,\frac{1}{t^2} \leq \frac{1}{t^2} \leq \frac{2}{\sqrt{n}}.
\end{aligned}$$

In the case $t^2 \leq \frac{\sqrt{n}}{2} \leq \frac{n}{4}$ we use Lemmata 2.4 and 2.25, 3., and $|x-1| = \frac{|x^2-1|}{|x+1|} \leq |x^2-1|$ (for $x \in \mathbb{R}_{\geq 0}$) to obtain

$$\begin{aligned}
\mathbb{E}\Big[\Big|\Phi \circ \frac{t\|W\|}{\sqrt{n-t^2}} - \Phi(t)\Big|\Big] &\leq \frac{1}{2}\,\mathbb{E}\Big[\Big|\frac{\|W\|}{\sqrt{n-t^2}} - 1\Big|\Big] \\
&\leq \frac{\sqrt{n-1}}{2\sqrt{n-t^2}}\,\mathbb{E}\Big[\Big|\frac{\|W\|}{\sqrt{n-1}} - 1\Big|\Big] + \frac{1}{2}\Big|\frac{\sqrt{n-1}}{\sqrt{n-t^2}} - 1\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\sqrt{n-1}}{2\sqrt{n-t^2}} \int_0^\infty \mathbb{P}\Big[\Big|\frac{\|W\|}{\sqrt{n-1}} - 1\Big| \ge \varepsilon\Big]\, d\varepsilon + \frac{1}{2}\Big|\frac{n-1}{n-t^2} - 1\Big| \\
&\le \frac{\sqrt{n-1}}{2\sqrt{n-t^2}} \frac{\mathsf{e}+2}{2\sqrt{2}} \int_0^\infty \mathsf{e}^{-\frac{1}{2}(n-1)\varepsilon^2}\, d\varepsilon + \frac{|t^2-1|}{2(n-t^2)} = \\
&= \frac{\sqrt{n-1}}{\sqrt{n-t^2}} \frac{\mathsf{e}+2}{4\sqrt{2}} \frac{1}{2}\sqrt{\frac{2\pi}{n-1}} + \frac{|t^2-1|}{2(n-t^2)} \\
&= \frac{(\mathsf{e}+2)\sqrt{\pi}}{8} \frac{1}{\sqrt{n-t^2}} + \frac{|t^2-1|}{2(n-t^2)} \le \\
&\le \frac{(\mathsf{e}+2)\sqrt{\pi}}{4\sqrt{3}} \frac{1}{\sqrt{n}} + \frac{\sqrt{n}/2}{2\cdot 3n/4} \le \frac{2}{\sqrt{n}}. \qquad \square
\end{aligned}
$$

*Comment.* Unfortunately Fresen's hints towards a proof (page 10) are too scant for us in order to reconstruct it; he only writes down (essentially) $\Phi_n(t) - \Phi_n(-t) = \mathbb{P}\big[|Z_1| \le \frac{t}{\sqrt{n-t^2}}\big(\sum_{i=2}^n Z_i^2\big)^{1/2}\big]$ (he inadvertently misses the square-root over the sum) and suggests using $\Phi((1+\delta)t) - \Phi(t) \le C\delta$ and Gaußian concentration about $\sqrt{n}$; as an alternative he speaks of considering the density $\Phi_n'$. He gives no further information on what he has in mind about how to string the ingredients together to a full-fledged proof. This is why here we have presented the proof of [2, Lemma 2.7], given also in [4, p. 396]. ◊

The theorem of Lévy given below, taken from [4, Theorem 1.7.9], builds upon the concentration of measure on the sphere; its usual interpretation is that a Lipschitz-continuous function on the sphere is "almost constant" on "most of its domain". If the mean used below is replaced by the *Lévy-mean* (also called *median*), then an analogous statement holds for other probability-spaces where concentration of measure can be observed.

Recall that for metric spaces $(X, d_X)$, $(Y, d_Y)$ a function $f\colon X \to Y$ is called Lipschitz-continuous with constant $L \in \mathbb{R}_{\ge 0}$ iff $d_Y(f(x_1), f(x_2)) \le L d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. The standard metric on $S^{n-1}$ is the arclength or *geodesic distance* given by $d(\theta, \theta') := \arccos\langle\theta, \theta'\rangle$; the Euclidean metric on $\mathbb{R}^n$ induces the *chordal metric* on $S^{n-1}$.

**Theorem 2.27** (Lévy). *Let $f\colon S^{n-1} \to \mathbb{R}$ be Lipschitz-continuous with constant $L \in \mathbb{R}_{>0}$ and set*

$$M := \int_{S^{n-1}} f(\theta)\, d\sigma_{n-1}(\theta).$$

*Then, for every $\varepsilon \in \mathbb{R}_{>0}$,*

$$\sigma_{n-1}\big\{\theta \in S^{n-1} \big| |f(\theta) - M| \ge \varepsilon\big\} \le 2\exp\Big(-\frac{(n-1)\varepsilon^2}{2L^2}\Big).$$

*(W/o proof.)*

*Remarks.* Because of $\|\theta - \theta'\| \le d(\theta, \theta')$, the Lipschitz-constant computed with respect to the arclength is bounded from above by the Lipschitz-constant with respect to Euclidean distance, hence also the latter may be used.

One of the key-statements concerning almost normal marginals of high-dimensional random-vectors is the following theorem (formulated and proved here for log-concave distributions; see Theorem 1.4 on p. 24 for the more general version given in [2]).

**Theorem 2.28** (thin shell implies CLT)**.** *There exist constants $C, c \in \mathbb{R}_{>0}$ such that the following holds: let $\varepsilon \in (0, \frac{1}{2})$ and let $X$ be an $\mathbb{R}^n$-valued isotropic log-concave random-variable. If $X$ has the* thin-shell-property

$$\mathbb{P}\Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| \geq \varepsilon\Big] \leq \varepsilon,$$

*then, for any $\delta \in \mathbb{R}_{>0}$, there holds*

$$\sigma_{n-1}\Big\{\theta \in S^{n-1}\Big|\sup\big\{|\mathbb{P}[\langle\theta, X\rangle \leq t] - \Phi(t)|\big|t \in \mathbb{R}\big\} \geq \delta + \frac{3\varepsilon}{2}\Big\} \leq C\sqrt{n}\,\mathsf{e}^{-cn\delta^2}.$$

*Comment.* Fresen omits the role of $\delta$ by seemingly equalling it with $\varepsilon$; while not harmful in itself, it bereaves the theorem of some of its generality and, more importantly, it hinders its application for the proof of the main Theorem 3.1 where we indeed make use of choosing $\delta$ independently from $\varepsilon$ (albeit the former is going to be a multiple of the latter; see page 80).

A minor slip is his writing $\varepsilon > 0$, where the literature usually restricts to $\varepsilon \in (0, \frac{1}{2})$, and at least our proof explicitly needs an upper bound smaller than 1. A matter of taste is his longwinding formulation of the premises instead of something shorter like, "let $\mu$ be an isotropic, log-concave probability measure on $\mathbb{R}^k$ with density $f$". The measure in line 5 of the theorem should be $\sigma_{k-1}$. ♢

*Proof.* Call $f$ the density of $X$. Define the map $F\colon S^{n-1}\times\mathbb{R} \to \mathbb{R}$ by $F(\theta, t) := \mathbb{P}[\langle\theta, X\rangle \leq t]$; we are going to show that $F(\cdot, t)$ is Lipschitz-continuous for any $t \in \mathbb{R}$. First let $t \in \mathbb{R}$ and $\theta_1, \theta_2 \in S^{n-1}$ with $\|\theta_1 - \theta_2\| \leq \frac{1}{10}$, w.l.o.g. $\theta_1 \neq \theta_2$; denote $H_i := \{x \in \mathbb{R}^n|\langle\theta_i, x\rangle \leq t\}$, $i \in \{1, 2\}$, then

$$\begin{aligned}|F(\theta, t) - F(\theta_2, t)| &= |\mathbb{P}[X \in H_1] - \mathbb{P}[X \in H_2]|\\ &= |\mathbb{P}[X \in H_1 \setminus H_2] + \mathbb{P}[X \in H_1 \cap H_2]\\ &\qquad - \mathbb{P}[X \in H_2 \setminus H_1] - \mathbb{P}[X \in H_2 \cap H_1]| \leq\\ &\leq \mathbb{P}[X \in H_1 \setminus H_2] + \mathbb{P}[X \in H_2 \setminus H_1] = \mathbb{P}[X \in H_1 \triangle H_2];\end{aligned}$$

Let $E \in G_{n,2}$ be the linear hull of $\{\theta_1, \theta_2\}$, $\alpha := \arccos\langle\theta_1, \theta_2\rangle \in (0, \frac{\pi}{2})$, then by the Gram–Schmidt-process $\big\{\theta_1, \frac{\theta_2 - \cos(\alpha)\theta_1}{\sin(\alpha)}\big\}$ is an orthonormal basis of $E$ and every $x \in \mathbb{R}^n$ can be expressed as $x = x_1\theta_1 + x_2\frac{\theta_2 - \cos(\alpha)\theta_1}{\sin(\alpha)} + y$ with unique $x_1, x_2 \in \mathbb{R}$ and $y \in E^\perp$, this means

$$\begin{aligned}x \in H_1 \triangle H_2 &\iff x \in H_1 \setminus H_2 \,\dot\vee\, x \in H_2 \setminus H_1\\ &\iff (\langle\theta_1, x\rangle \leq t \wedge \langle\theta_2, x\rangle > t) \,\dot\vee\, (\langle\theta_1, x\rangle > t \wedge \langle\theta_2, x\rangle \leq t)\\ &\iff (x_1 \leq t \wedge \cos(\alpha)x_1 + \sin(\alpha)x_2 > t)\\ &\qquad \dot\vee\, (x_1 > t \wedge \cos(\alpha)x_1 + \sin(\alpha)x_2 \leq t).\end{aligned}$$

Furthermore, $1_{H_1 \triangle H_2}(x+y) = 1_{H_1 \triangle H_2}(x)$ for all $x \in E$ and $y \in E^\perp$, and therefore (we write $f_E(x_1, x_2) := \pi_E f(x_1\theta_1 + x_2 \frac{1-\cos(\alpha)\theta_1}{\sin(\alpha)}))$

$$\begin{aligned}|F(\theta_1,t) - F(\theta_2,t)| \leq &\int_E \int_{E^\perp} 1_{H_1\triangle H_2}(x+y) f(x+y)\, dy\, dx = \\ = &\int_{-\infty}^t \int_{\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}}^\infty f_E(x_1,x_2)\, dx_2\, dx_1 \\ &+ \int_t^\infty \int_{-\infty}^{\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}} f_E(x_1,x_2)\, dx_2\, dx_1.\end{aligned}$$

Because $X$ is log-concave and isotropic, $f$ is a log-concave, isotropic density, hence $\pi_E f$ is so and also $f_E$, thus by Lemma 2.23 there exist constants $A, B \in \mathbb{R}_{>0}$ independent of $f_E$ such that $f_E(x_1,x_2) \leq A\mathrm{e}^{-B\|(x_1,x_2)^\mathsf{T}\|} \leq A\mathrm{e}^{-B(|x_1|+|x_2|)}$ for all $(x_1,x_2)^\mathsf{T} \in \mathbb{R}^2$. First consider the case $t \geq 0$, then the first integral is

$$\begin{aligned}\int_{-\infty}^t \int_{\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}}^\infty f_E(x_1,x_2)\, dx_2\, dx_1 \leq A &\int_{-\infty}^t \mathrm{e}^{-Bx_1} \int_{\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}}^\infty \mathrm{e}^{-Bx_2}\, dx_2\, dx_1 = \\ &= \frac{A}{B} \int_{-\infty}^t \mathrm{e}^{-Bx_1} \mathrm{e}^{-B\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}}\, dx_1 \\ &= \frac{A}{B}\, \mathrm{e}^{-\frac{Bt}{\sin(\alpha)}} \int_{-\infty}^t \mathrm{e}^{B\frac{\cos(\alpha)-\sin(\alpha)}{\sin(\alpha)}x_1}\, dx_1 \\ &= \frac{A\sin(\alpha)}{B^2(\cos(\alpha)-\sin(\alpha))}\, \mathrm{e}^{-\frac{Bt}{\sin(\alpha)}(1-\cos(\alpha)+\sin(\alpha))};\end{aligned}$$

the exponent there is $1 + \frac{1-\cos(\alpha)}{\sin(\alpha)} = 1 + \frac{2\sin(\alpha/2)^2}{2\sin(\alpha/2)\cos(\alpha/2)} = 1 + \tan(\frac{\alpha}{2}) \geq 1$, furthermore, if we set $s := \|\theta_1 - \theta_2\|$, then $s = 2\sin(\frac{\alpha}{2})$ and thence $\frac{\sin(\alpha)}{\cos(\alpha)-\sin(\alpha)} = \frac{2\sin(\frac{\alpha}{2})}{\frac{2\cos(\frac{\alpha}{2})^2-1}{\cos(\frac{\alpha}{2})} - 2\sin(\frac{\alpha}{2})} = \frac{s}{\frac{2-s^2}{\sqrt{4-s^2}} - s} \leq \frac{200}{179}\, s$ by using $s \leq \frac{1}{10}$. This gives

$$\int_{-\infty}^t \int_{\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}}^\infty f_E(x_1,x_2)\, dx_2\, dx_1 \leq \frac{200A}{179B^2}\, \mathrm{e}^{-Bt}\, \|\theta_1 - \theta_2\|.$$

The second integral can be estimated as

$$\begin{aligned}\int_t^\infty \int_{-\infty}^{\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}} f_E(x_1,x_2)\, dx_2\, dx_1 \leq A &\int_t^\infty \mathrm{e}^{-Bx_1} \int_{-\infty}^{\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}} \mathrm{e}^{Bx_2}\, dx_2\, dx_1 = \\ &= \frac{A}{B}\, \mathrm{e}^{\frac{Bt}{\sin(\alpha)}} \int_t^\infty \mathrm{e}^{-B\frac{\sin(\alpha)+\cos(\alpha)}{\sin(\alpha)}x_1}\, dx_1 \\ &= \frac{A\sin(\alpha)}{B^2(\sin(\alpha)+\cos(\alpha))}\, \mathrm{e}^{-\frac{Bt}{\sin(\alpha)}(\sin(\alpha)+\cos(\alpha)-1)};\end{aligned}$$

now $1 - \frac{1-\cos(\alpha)}{\sin(\alpha)} = 1 - \frac{\sin(\frac{\alpha}{2})}{\cos(\frac{\alpha}{2})} = 1 - \frac{s}{\sqrt{4-s^2}} \geq 1 - \frac{1}{\sqrt{399}}$ and $\frac{\sin(\alpha)}{\cos(\alpha)+\sin(\alpha)} \leq \frac{\sin(\alpha)}{\cos(\alpha)} = \frac{2\sin(\frac{\alpha}{2})}{\frac{2\cos(\frac{\alpha}{2})^2-1}{\cos(\frac{\alpha}{2})}} = \frac{s}{\frac{2-s^2}{\sqrt{4-s^2}}} \leq \frac{200}{199}\, s$, which yields

$$\int_t^\infty \int_{-\infty}^{\frac{t-\cos(\alpha)x_1}{\sin(\alpha)}} f_E(x_1,x_2)\,dx_2\,dx_1 \leq \frac{200A}{199B^2}\,\mathrm{e}^{-B(1-\frac{1}{\sqrt{399}})t}\,\|\theta_1-\theta_2\|,$$

and taking everything together we have shown

$$|F(\theta_1,t)-F(\theta_2,t)| \leq \frac{400A}{179B^2}\,\mathrm{e}^{-B(1-\frac{1}{\sqrt{399}})t}\,\|\theta_1-\theta_2\|.$$

For the case $t \leq 0$ we observe

$$\begin{aligned} F(\theta,t) &= \mathbb{P}[\langle\theta,X\rangle \leq t] = 1-\mathbb{P}[\langle\theta,X\rangle \geq t] \\ &= 1-\mathbb{P}[\langle-\theta,X\rangle \leq -t] = 1-F(-\theta,-t) = 1-F(-\theta,|t|) \end{aligned}$$

and therefore

$$|F(\theta_1,t)-F(\theta_2,t)| = |F(-\theta_1,|t|)-F(-\theta_2,|t|)| \leq \frac{400A}{179B^2}\,\mathrm{e}^{-B(1-\frac{1}{\sqrt{399}})|t|}\,\|\theta_1-\theta_2\|.$$

Now let $\theta_1,\theta_2$ have arbitrary distance, then along the shorter arc of the great circle through $\theta_1$ and $\theta_2$ choose points $\theta^{(0)}=\theta_1,\ldots,\theta^{(N)}=\theta_2$ such that $\|\theta^{(i)}-\theta^{(i-1)}\| \leq \frac{1}{10}$ and the arcs spanned by successive points intersect at most at the endpoints, for a suitable $N \in \mathbb{N}$.[3] Then

$$\sum_{i=1}^N \|\theta^{(i)}-\theta^{(i-1)}\| \leq \sum_{i=1}^N d(\theta^{(i-1)},\theta^{(i)}) = d(\theta_1,\theta_2) \leq \frac{\pi}{2}\,\|\theta_1-\theta_2\|$$

and therefore

$$\begin{aligned} |F(\theta_1,t)-F(\theta_2,t)| &\leq \sum_{i=1}^N |F(\theta^{(i-1)},t)-F(\theta^{(i)},t)| \\ &\leq \sum_{i=1}^N \frac{400A}{179B^2}\,\mathrm{e}^{-B(1-\frac{1}{\sqrt{300}})|t|}\,\|\theta^{(i-1)}-\theta^{(i)}\| \\ &\leq \frac{200\pi A}{179B^2}\,\mathrm{e}^{-B(1-\frac{1}{\sqrt{399}})|t|}\,\|\theta_1-\theta_2\|, \end{aligned}$$

that is, $F(\cdot,t)$ is Lipschitz-continuous with constant at most $L(t) := \frac{200\pi A}{179B^2}\,\mathrm{e}^{-B(1-\frac{1}{\sqrt{399}})|t|}$. We aim for an application of Theorem 2.27, so we define the average distribution-function $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(t) := \int_{S^{n-1}} F(\theta,t)\,d\sigma_{n-1}(\theta) \quad \big[= (\mathbb{P}_X \otimes \sigma_{n-1})\{(x,\theta) \in \mathbb{R}^n \times S^{n-1} | \langle\theta,x\rangle \leq t\}\big].$$

[3] Effectively this can be done in the following way: orthonormalize $\{\theta_1,\theta_2\}$ as before, set $\alpha := \arccos\langle\theta_1,\theta_2\rangle \in (0,\pi]$, $\alpha_0 := 2\arcsin(\frac{1}{20})$, $N := \lceil\frac{\alpha}{\alpha_0}\rceil$ and take the points $\theta^{(i)} := \cos(\frac{i\alpha}{N})\theta_1 + \sin(\frac{i\alpha}{N})\frac{\theta_2-\cos(\alpha)\theta_1}{\sin(\alpha)}$ for $i \in [0,N]$; if $\theta_2 = -\theta_1$, then choose any $\theta_2' \notin \{\pm\theta_1\}$ for the orthonormalization.

Let $\delta \in \mathbb{R}_{>0}$, w.l.o.g. $\delta \leq 1$ (since distribution-functions cannot be more than 1 apart), set $m := \lfloor \frac{1}{\delta} \rfloor \in \mathbb{N}$, and for $i \in [1, m]$ define $t_i := F^{-1}(\frac{i}{m+1}) \in \mathbb{R}$ (in the sense of quantile-function), then

$$\begin{aligned}\sigma_{n-1}\{\theta \in S^{n-1} \big| \exists i \in [1,m] \colon |F(\theta, t_i) - F(t_i)| \geq \delta\} \leq \\ &\leq \sum_{i=1}^{m} \sigma_{n-1}\{\theta \in S^{n-1} \big| |F(\theta, t_i) - F(t_i)| \geq \delta\} \\ &\leq \sum_{i=1}^{m} 2\mathrm{e}^{-\frac{(n-1)}{2L(t_i)^2}\delta^2} \\ &\leq 2m\mathrm{e}^{-\frac{1}{4L(0)^2}n\delta^2} \leq \frac{2}{\delta}\,\mathrm{e}^{-\frac{1}{4L(0)^2}n\delta^2}\end{aligned}$$

(keep in mind that $L(0) = \frac{200\pi A}{179B^2}$ is an absolute constant); now if $n\delta^2 \geq 4L(0)^2$, then $\frac{2}{\delta} \leq \frac{\sqrt{n}}{L(0)}$, and if $n\delta^2 \leq 4L(0)^2$, then $\frac{\sqrt{n}}{L(0)}\,\mathrm{e}^{-\frac{1}{4L(0)^2}n\delta^2} \geq \frac{1}{\mathrm{e}L(0)}$. Therefore, define $C := \max\{\mathrm{e}, \frac{1}{L(0)}\}$, then $\frac{2}{\delta} \leq L(0)C\,\frac{\sqrt{n}}{L(0)} = C\sqrt{n}$ for large $\delta$, and $C\sqrt{n}\mathrm{e}^{-\frac{1}{4L(0)^2}n\delta^2} \geq 1$ for small $\delta$; as $\sigma_{n-1} \leq 1$ we have, for all $\delta \in (0, 1]$,

$$\sigma_{n-1}\{\theta \in S^{n-1} \big| \exists i \in [1,m] \colon |F(\theta, t_i) - F(t_i)| \geq \delta\} \leq C\sqrt{n}\,\mathrm{e}^{-\frac{1}{4L(0)^2}n\delta^2}.$$

This shall be extended to all $t \in \mathbb{R}$: let $\theta \in S^{n-1}$ such that $|F(\theta, t_i) - F(t_i)| < \delta$ for all $i \in [1, m]$. First case, $t < t_1$: then $0 \leq F(\theta, t) \leq F(\theta, t_1)$ and $0 \leq F(t) \leq F(t_1) = \frac{1}{m+1} < \delta$, thus

$$\begin{aligned}F(\theta, t) - F(t) &\geq -F(t_1) > -\delta > -2\delta, \\ F(\theta, t) - F(t) &\leq F(\theta, t_1) = F(\theta, t_1) - F(t_1) + F(t_1) < 2\delta.\end{aligned}$$

Second case, $t_i \leq t \leq t_{i+1}$ for some $i \in [1, m-1]$: then $F(\theta, t_i) \leq F(\theta, t) \leq F(\theta, t_{i+1})$, $F(t_i) \leq F(t) \leq F(t_{i+1})$ and $F(t_{i+1}) - F(t_i) = \frac{1}{m+1} < \delta$, hence

$$\begin{aligned}F(\theta, t) - F(t) &\geq F(\theta, t_i) - F(t_{i+1}) = F(\theta, t_i) - F(t_i) - (F(t_{i+1}) - F(t_i)) > -2\delta, \\ F(\theta, t) - F(t) &\leq F(\theta, t_{i+1}) - F(t_i) = F(\theta, t_{i+1}) - F(t_{i+1}) + F(t_{i+1}) - F(t_i) < 2\delta.\end{aligned}$$

Third case, $t \geq t_m$: then $F(\theta, t_m) \leq F(\theta, t) \leq 1$ and $1 - \delta < 1 - \frac{1}{m+1} = F(t_m) \leq F(t) \leq 1$, therefore

$$\begin{aligned}F(\theta, t) - F(t) &\geq F(\theta, t_m) - 1 = F(\theta, t_m) - F(t_m) - (1 - F(t_m)) > -2\delta, \\ F(\theta, t) - F(t) &\leq 1 - F(t_m) < \delta < 2\delta.\end{aligned}$$

These lines have shown

$$\sigma_{n-1}\{\theta \in S^{n-1} \big| \exists t \in \mathbb{R} \colon |F(\theta, t) - F(t)| \geq 2\delta\} \leq C\sqrt{n}\,\mathrm{e}^{-\frac{1}{4L(0)^2}n\delta^2}.$$

In order to compare $F$ to $\Phi$ let $Z$ be a $S^{n-1}$-valued, $\sigma_{n-1}$-distributed random-variable independent of $X$, then by Lemma 2.25, 2., $Y := \left\langle \frac{X}{\|X\|}, Z \right\rangle$ is identically distributed to

$Z_1$ and is independent of $\|X\|$; this also means $\mathbb{P}[Y \le t] = \Phi_n(t\sqrt{n})$ with $\Phi_n$ as defined in Lemma 2.26. Let $t \in \mathbb{R}_{\ge 0}$, then

$$\begin{aligned}
F(t) &= (\mathbb{P}_X \otimes \sigma_{n-1})\{(x,\theta) \in \mathbb{R}^n \times S^{n-1} | \langle \theta, x\rangle \le t\} \\
&= \mathbb{P}[\langle Z, X\rangle \le t] = \mathbb{P}\Big[Y \le \frac{t}{\|X\|}\Big] \\
&= \mathbb{P}\Big(\Big[Y \le \frac{t}{\|X\|}\Big] \Big| \Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| \le \varepsilon\Big]\Big) \mathbb{P}\Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| \le \varepsilon\Big] \\
&\quad + \mathbb{P}\Big(\Big[Y \le \frac{t}{\|X\|}\Big] \Big| \Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| > \varepsilon\Big]\Big) \mathbb{P}\Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| > \varepsilon\Big].
\end{aligned}$$

For an upper bound we estimate the second and third probabilities by 1 and use the thin-shell-property for the forth one, for the first one we exploit the condition $\big|\frac{\|X\|}{n} - 1\big| \le \varepsilon$ equivalent to $\frac{1}{(1+\varepsilon)\sqrt{n}} \le \frac{1}{\|X\|} \le \frac{1}{(1-\varepsilon)\sqrt{n}}$ and therewith get

$$\begin{aligned}
F(t) &\le \mathbb{P}\Big(\Big[Y \le \frac{t}{(1-\varepsilon)\sqrt{n}}\Big] \Big| \Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| \le \varepsilon\Big]\Big) \cdot 1 + 1 \cdot \varepsilon = \\
&= \Phi_n\Big(\frac{t}{1-\varepsilon}\Big) + \varepsilon \le \Phi\Big(\frac{t}{1-\varepsilon}\Big) + \frac{2}{\sqrt{n}} + \varepsilon \\
&\le \Phi(t) + \frac{2}{\sqrt{n}} + \Big(1 + \sqrt{\frac{2}{\mathsf{e}\pi}}\Big)\varepsilon \le \Phi(t) + \frac{2}{\sqrt{n}} + \frac{3\varepsilon}{2},
\end{aligned}$$

where furthermore we have used independence of $Y$ and $\|X\|$, Lemma 2.26 to compare $\Phi_n$ and $\Phi$, and Lemma 2.4 to get $\Phi\big(\frac{t}{1-\varepsilon}\big) \le \Phi(t) + \frac{1}{\sqrt{2\pi\mathsf{e}}}\big(\frac{1}{1-\varepsilon} - 1\big) = \Phi(t) + \frac{1}{\sqrt{2\pi\mathsf{e}}}\frac{\varepsilon}{1-\varepsilon} \le \Phi(t) + \varepsilon\sqrt{\frac{2}{\mathsf{e}\pi}}$.

For a lower bound we estimate the third and fourth probabilites by 0 and take the complements in the first and second ones, that is,

$$\begin{aligned}
F(t) &\ge \Big\{1 - \mathbb{P}\Big(\Big[Y > \frac{t}{\|X\|}\Big] \Big| \Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| \le \varepsilon\Big]\Big)\Big\}\Big\{1 - \mathbb{P}\Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| > \varepsilon\Big]\Big\} \\
&\ge \Big\{1 - \mathbb{P}\Big(\Big[Y > \frac{t}{(1+\varepsilon)\sqrt{n}}\Big] \Big| \Big[\Big|\frac{\|X\|}{\sqrt{n}} - 1\Big| \le \varepsilon\Big]\Big)\Big\}(1-\varepsilon) = \\
&= \Phi_n\Big(\frac{t}{1+\varepsilon}\Big)(1-\varepsilon) \ge \Phi_n\Big(\frac{t}{1+\varepsilon}\Big) - \varepsilon \ge \Phi\Big(\frac{t}{1+\varepsilon}\Big) - \frac{2}{\sqrt{n}} - \varepsilon \ge \\
&\ge \Phi(t) - \frac{2}{\sqrt{n}} - \Big(1 + \frac{1}{\sqrt{2\mathsf{e}\pi}}\Big)\varepsilon \ge \Phi(t) - \frac{2}{\sqrt{n}} - \frac{3\varepsilon}{2},
\end{aligned}$$

where $\Phi(t) \le \Phi\big(\frac{t}{1+\varepsilon}\big) + \frac{1}{\sqrt{2\pi\mathsf{e}}}(1 + \varepsilon - 1) = \Phi\big(\frac{t}{1+\varepsilon}\big) + \frac{\varepsilon}{\sqrt{2\pi\mathsf{e}}}$ again by Lemma 2.4.

For $t \le 0$ we remark that $-Z$ has the same distribution as $Z$ and thence

$$\begin{aligned}
F(t) &= \mathbb{P}[\langle Z, X\rangle \le t] = 1 - \mathbb{P}[\langle Z, X\rangle \ge t] \\
&= 1 - \mathbb{P}[\langle -Z, X\rangle \le -t] = 1 - \mathbb{P}[\langle Z, X\rangle \le -t] = 1 - F(-t),
\end{aligned}$$

and because the same symmetry-relation holds *mutatis mutandis* for $\Phi$, the estimates derived above transfer to the case $t \le 0$. Finally let $\theta \in S^{n-1}$ such that $|F(\theta,t) - F(t)| < 2\delta$ for all $t \in \mathbb{R}$, then, for all $t \in \mathbb{R}$,

$$|F(\theta,t) - \Phi(t)| \le |F(\theta,t) - F(t)| + |F(t) - \Phi(t)| < 2\delta + \frac{2}{\sqrt{n}} + \frac{3\varepsilon}{2}.$$

We already know $C\sqrt{n}\,\mathsf{e}^{-\frac{1}{4L(0)^2}n\delta^2} \ge 1$ for $n\delta^2 \le 4L(0)^2$, equivalently $\frac{2}{\sqrt{n}} \ge \frac{\delta}{L(0)}$; by estimating $2\delta + \frac{2}{\sqrt{n}} + \frac{3\varepsilon}{2} \ge (2 + \frac{1}{L(0)})\delta + \frac{3\varepsilon}{2}$ the failure-set becomes larger, but since $C\sqrt{n}\,\mathsf{e}^{-\frac{1}{4L(0)^2}n\delta^2} \ge 1$ for those $n$ and $\delta$ anyway, it does not matter; on the other hand, for $n\delta^2 \ge 4L(0)^2$ the good set becomes larger by estimating $2\delta + \frac{2}{\sqrt{n}} + \frac{3\varepsilon}{2} \le (2 + \frac{1}{L(0)})\delta + \frac{3\varepsilon}{2}$, so the lower bound $1 - C\sqrt{n}\,\mathsf{e}^{-\frac{1}{4L(0)^2}n\delta^2}$ is valid still. This means that in any case we may absorb $\frac{2}{\sqrt{n}}$ into the $\delta$-term and get

$$\sigma_{n-1}\Big\{\theta \in S^{n-1}\Big|\exists t \in \mathbb{R}\colon |F(\theta,t) - \Phi(t)| \ge \Big(2 + \frac{1}{L(0)}\Big)\delta + \frac{3\varepsilon}{2}\Big\} \le C\sqrt{n}\,\mathsf{e}^{-\frac{1}{4L(0)^2}n\delta^2}.$$

Replacing $\delta$ by $\frac{L(0)}{2L(0)+1}\,\delta$ yields the desired result. □

*Comment.* Fresen's proof has the merit of the simpler demonstration of the Lipschitz-continuity of $F(\cdot,t)$.

Unfortunately it contains several typos: line 3 should read $[\dots] \le \mu(\{\langle x,\theta_1\rangle \le t\} \triangle \{\langle x,\theta_2\rangle \le t\})$ because $M$ already denotes the measure of a halfspace, hence taking the symmetric difference and the measure do not make sense; also equality will usually be violated as is evident from spelling out the details. The same wrong equality-sign stands in line 6, there also the integration-boundaries have incorrect 1 instead of $t$, and $q$ is a function of both $x$ and $y$. In line 10 the argument of the exponential function should read $-c|x| - c|y|$ (absolute values missing and mismatching names of variables); the absolute values again are missing in line 12, and for an unknown reason the constant has been changed to $c'$; also the upper boundary of the inner integral should be $t - y\tan(\beta)$. In lines 14 and 15 $n$ ought to be replaced by $k$. On page 10, line 1, $Y$ should rather be called “scalar” since as an inner product it is onedimensional. Lastly, $Y$ is identical in distribution to $\sqrt{k}\,\theta_1$. (The missing square-root in line 7 from below has alrady been hinted at.) ◊

# 3 The main theorem and its proof

This whole chapter is devoted to the proof of the following theorem.

**Theorem 3.1** (CLT for convex sets)**.** *There exist null-sequences* $(\varepsilon_n)_{n\geq 1} \subset (0,1]$ *and* $(\delta_n)_{n\geq 1} \subset (0,1]$*, such that for any* $n \geq 1$ *and any* $\mathbb{R}^n$*-valued isotropic, log-concave random-variable* $X$*,*

$$\sigma_{n-1}\big\{\theta \in S^{n-1}\big|\sup\big\{|\mathbb{P}[\langle\theta, X\rangle \leq t] - \Phi(t)|\big|t \in \mathbb{R}\big\} \geq \varepsilon_n\big\} \leq \delta_n.$$

*Comment.* Fresen writes $\omega_n$ instead of $\delta_n$ (used by Klartag), but this is just notation; much more unusual is Fresen's speaking of the "Haar-measure" on the sphere: this seems to be an idiosyncrasy of Fresen, since he also uses that wording in the appendix and, e.g., [13, p. 3]. Nevertheless we deem this improper because a Haar-measure always is defined on a group, and neither the sphere $S^{n-1}$ nor the Grassmannian $G_{n,k}$ is a group. It is true that the orthogonal group $\mathrm{O}_n(\mathbb{R})$ acts transitively on either manifold and hence its Haar-measure (which it is indeed) can be transferred to orthogonally invariant measures on said manifolds; in the case of the sphere this induced measure equals, up to scaling, the usual surface-measure. In view of this we do not see why Fresen does not simply speak of the normalized surface measure. ♢

Let $f$ denote the density of $X$. In a nutshell the main-steps of Fresen's proof are as follows: show that a slightly modified version of $f$ is almost radially symmetric on any subspace of some suitably small dimension, then that along any ray it is highly concentrated around its maximizer, and finally derive a thin-shell-estimate from that; Theorem 2.28 yields the result.

W.l.o.g. we may assume $n \geq 3$; in any case the statement is trivially fulfilled if $\delta_n = \varepsilon_n = 1$ for small $n$ up to an absolute bound.

Let $Z$ be an $\mathbb{R}^n$-valued standard Gaußian variable independent of $X$ and let $\sigma \in (0,1]$; set $Y := X + \sigma Z$, then $Y$ has density $h := f * \varphi_\sigma$, with $\varphi_\sigma$ being the density of $\sigma Z$. By Lemma 2.14, 5., $Y$ is log-concave with $\mathbb{E}[Y] = o$ and $\mathrm{Var}[Y] = (1+\sigma^2)I_n$, and $h$ is infinitely often differentiable. The Fourier-transform is $\mathcal{F}h = \mathcal{F}f \cdot \mathcal{F}\varphi_\sigma$; because $\mathcal{F}f$ is bounded and $\mathcal{F}\varphi_\sigma$ is integrable, $\mathcal{F}h$ is integrable too and we may use the inversion-formula.

## 3.1 The first step.

Let $\xi_1, \xi_2 \in \mathbb{R}^n$, then, using $|\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{\mathrm{i}y}| \leq |x-y|$ for $x, y \in \mathbb{R}$ and isotropy of $X$,

$$|\mathcal{F}f(\xi_1) - \mathcal{F}f(\xi_2)| = \big|\mathbb{E}[\mathrm{e}^{-2\pi\mathrm{i}\langle\xi_1,X\rangle}] - \mathbb{E}[\mathrm{e}^{-2\pi\mathrm{i}\langle\xi_2,X\rangle}]\big| \leq$$

$$\leq \mathbb{E}[|\mathsf{e}^{-2\pi\mathsf{i}\langle\xi_1,X\rangle} - \mathsf{e}^{-2\pi\mathsf{i}\langle\xi_2,X\rangle}|] \leq$$
$$\leq 2\pi\,\mathbb{E}[|\langle\xi_1-\xi_2,X\rangle|] \leq 2\pi\,\mathbb{E}[\langle\xi_1-\xi_2,X\rangle^2]^{1/2} =$$
$$= 2\pi\big((\xi_1-\xi_2)^{\mathsf{T}}\,\mathbb{E}[XX^{\mathsf{T}}](\xi_1-\xi_2)\big)^{1/2} = 2\pi\|\xi_1-\xi_2\|,$$

so $\mathcal{F}f$ is at most $2\pi$-lipschitz.

*Comment.* Fresen's calculation at this point is a bit awkward: on the one hand, in order to pull out $\|\xi_1-\xi_2\|$ he must assume that this term is nonzero, but nowhere does he say so, and on the other hand the gain of doing so is not immediately apparent because he presupposes $\operatorname{Var}[X]=I_n$, but here he uses the property $\mathbb{E}[\langle\theta,X\rangle^2]=1$ for all $\theta\in S^{n-1}$ of isotropy.

Without need he also introduces $E=UF$ already at this point. ◊

Now let $\varepsilon\in(0,\frac{1}{2}]$, $c_1,\delta\in\mathbb{R}_{>0}$, and let $k\in\mathbb{N}$ with $k\geq 2$, $k\leq n-1$ and $k\leq\frac{c_1}{-\log(\varepsilon)}\,\delta^2 n$.[1] Let $F\subset\mathbb{R}^n$ be a $k$-dimensional subspace (considered fixed, e.g. $F=\mathbb{R}^k\times\{0\}^{n-k}$), then $S_F=S^{n-1}\cap F$ is a $(k-1)$-dimensional sphere, hence by Lemma 2.6 there exists an $\varepsilon$-net $\mathcal{N}\subset S_F$ with $|\mathcal{N}|\leq(\frac{3}{\varepsilon})^k$. We denote $M(t):=\int_{S^{n-1}}\mathcal{F}f(t\theta)\,d\sigma_{n-1}(\theta)$ for $t\in\mathbb{R}_{\geq 0}$. The map $\xi\mapsto\mathcal{F}f(t\xi)$ has Lipschitz-constant at most $2\pi t$. Let $t_m:=(1+\varepsilon)^m\frac{\sqrt{k}}{\sigma}$ for $m\in\mathbb{Z}$, then applying Theorem 2.27 and Lemma 2.1 together with subadditivity we obtain, with $C_1\in\mathbb{R}_{>0}$,

$$\mu_{\mathrm{O}}\Big\{U\in\mathrm{O}_n(\mathbb{R})\Big|\forall m\in\mathbb{N}_0\forall\theta\in\mathcal{N}\colon|\mathcal{F}f(Ut_m\theta)-M(t_m)|<C_1t_m\Big(\delta+\sqrt{\frac{\log(1+m)}{n}}\Big)\Big\}=$$
$$=1-\mu_{\mathrm{O}}\Big\{U\in\mathrm{O}_n(\mathbb{R})\Big|\exists m\in\mathbb{N}_0\exists\theta\in\mathcal{N}\colon|\mathcal{F}f(Ut_m\theta)-M(t_m)|\geq$$
$$\geq C_1t_m\Big(\delta+\sqrt{\frac{\log(1+m)}{n}}\Big)\Big\}\geq$$
$$\geq 1-\sum_{m=0}^{\infty}\sum_{\theta\in\mathcal{N}}\mu_{\mathrm{O}}\Big\{U\in\mathrm{O}_n(\mathbb{R})\Big||\mathcal{F}f(Ut_m\theta)-M(t_m)|\geq C_1t_m\Big(\delta+\sqrt{\frac{\log(1+m)}{n}}\Big)\Big\}\geq$$
$$\geq 1-\sum_{m=0}^{\infty}\sum_{\theta\in\mathcal{N}}2\exp\Big(-\frac{(n-1)}{2(2\pi)^2t_m^2}\,C_1^2t_m^2\Big(\delta+\sqrt{\frac{\log(1+m)}{n}}\Big)^2\Big)\geq$$
$$\geq 1-2\Big(\frac{3}{\varepsilon}\Big)^k\sum_{m=0}^{\infty}\exp\Big(-\frac{C_1^2n}{(4\pi)^2}\Big(\delta^2+\frac{\log(1+m)}{n}\Big)\Big);$$

from $\varepsilon\leq\frac{1}{2}$ and $k\leq\frac{c_1}{-\log(\varepsilon)}n\delta^2$ follows $(\frac{3}{\varepsilon})^k\leq\exp\big(\frac{c_1\log(6)}{\log(3)}n\delta^2\big)$, and the sum yields

$$\sum_{m=0}^{\infty}\exp\Big(-\frac{C_1^2n}{(4\pi)^2}\Big(\delta^2+\frac{\log(1+m)}{n}\Big)\Big)=\exp\Big(-\frac{C_1^2}{(4\pi)^2}\,n\delta^2\Big)\cdot\zeta\Big(\frac{C_1^2}{(4\pi)^2}\Big),$$

[1] Fresen inadvertently misses the square of $\delta$.

provided $\frac{C_1}{4\pi} > 1$, equivalently $C_1 > 4\pi$, and $\zeta$ is the Riemann-zeta-function; so we can state

$$\mu_{\mathrm{O}}\Big\{U \in \mathrm{O}_n(\mathbb{R}) \Big| \forall m \in \mathbb{N}_0 \forall \theta \in \mathcal{N} \colon |\mathcal{F}f(Ut_m\theta) - M(t_m)| < C_1 t_m\Big(\delta + \sqrt{\frac{\log(1+m)}{n}}\Big)\Big\} \geq$$
$$\geq 1 - 2\zeta\Big(\frac{C_1^2}{(4\pi)^2}\Big)\exp\Big[\Big(\frac{c_1\log(6)}{\log(2)} - \frac{C_1^2}{(4\pi)^2}\Big)n\delta^2\Big] =: 1 - \frac{C_2}{2}\,\mathrm{e}^{-c_2 n\delta^2},$$

where $c_2 > 0$ iff $c_1 < \frac{\log(2)}{\log(6)(4\pi)^2}C_1^2$. Since it is cancelled during the calculations, we may replace $t_m$ by $t_{-m}$ in the result above. In particular, if $n$ is large enough and $\delta$ is not too small, then the set in question has positive measure, tending to 1 at an exponential rate as $n \to \infty$.

This result shows that, roughly speaking, for most orthogonal transformations, $\mathcal{F}f$ is close to its spherical mean on the transform of the grid $\{t_m\theta | m \in \mathbb{Z} \wedge \theta \in \mathcal{N}\}$.

*Comment.* One of the more confusing habits of Fresen in this paper is his use of the same symbols $C$ and $c$ for different constants throughout the work (even taking into account the common practice referred to on page 25 that the values may change). At some points this can be justified by arguing with "adapting the constants", but at other points we are dealing with independent entities the graphical identification of which can be confusing or even detrimental to the validity of the argument. At this point he even does not introduce the constants, they appear on the scene without further comment.

In order ro counteract any ambiguity we have opted to distinguish the constants carefully by indexing them, and we also explicitly state their relations to previously used ones; an overview of all constants is given in Table 3.1 on page 87.

Another comment necessary at this point is Fresen's writing of $\log(m)$ which is $-\infty$ for $m = 0$; one might think that he had $m \geq 1$ in mind, but he expressly uses $m = 0$ too. In the further course of the proof this causes $\log\big(\log\big(\frac{\sigma\|\xi\|}{\sqrt{k}} \vee \frac{\sqrt{k}}{\sigma\|\xi\|}\big)\big) < 0$ for $\frac{\sigma\|\xi\|}{\sqrt{k}} \in (\frac{1}{\mathrm{e}}, \mathrm{e})$, so the square-root is not real there and the estimate becomes useless for considerably many $\xi$. The situation is easy to mend, and we have done so by using $\log(m+1)$. It is unclear how that error slipped his attention. ♢

We wish to extend this result from the grid $t_m\theta$, with $m \in \mathbb{Z}$ and $\theta \in \mathcal{N}$, to all of $F$. Because $M(0) = \int_{S^{n-1}} \mathcal{F}f(o)\,d\sigma_{n-1}(\theta) = \mathcal{F}f(o) = \mathcal{F}f(Uo)$ for all $U \in \mathrm{O}_n(\mathbb{R})$, it suffices to consider $\xi \in F \setminus \{o\}$. Take such a $\xi$, then $\frac{\xi}{\|\xi\|} \in S_F$, hence there is $\theta \in \mathcal{N}$ with $\big|\frac{\xi}{\|\xi\|} - \theta\big| \leq \varepsilon$. We treat two cases, $\|\xi\| \geq \frac{\sqrt{k}}{\sigma}$ and $\|\xi\| \leq \frac{\sqrt{k}}{\sigma}$. In the first case we set $m := \big\lfloor \log(1+\varepsilon)^{-1}\log\big(\frac{\sigma\|\xi\|}{\sqrt{k}}\big)\big\rfloor \in \mathbb{N}_0$ and $\xi' := t_m\theta$, then $(U \in \mathrm{O}_n(\mathbb{R}))$

$$|\mathcal{F}f(U\xi) - M(\|\xi\|)| \leq |\mathcal{F}f(U\xi) - \mathcal{F}f(U\xi')| + |\mathcal{F}f(U\xi') - M(\|\xi'\|)| + |M(\|\xi'\|) - M(\|\xi\|)|$$
$$\leq 2\pi\|\xi - \xi'\| + |\mathcal{F}f(U\xi') - M(\|\xi'\|)| + |M(\|\xi'\|) - M(\|\xi\|)|.$$

the last term satisfies

$$|M(\|\xi'\|) - M(\|\xi\|)| = \left|\int_{S^{n-1}} \mathcal{F}f(\|\xi'\|\theta)\,d\sigma_{n-1}(\theta) - \int_{S^{n-1}} \mathcal{F}f(\|\xi\|\theta)\,d\sigma_{n-1}(\theta)\right| \leq$$

$$\begin{aligned}
&\le \int_{S^{n-1}} |\mathcal{F}f(\|\xi'\|\theta) - \mathcal{F}f(\|\xi\|\theta)|\, d\sigma_{n-1}(\theta) \le \\
&\le 2\pi \int_{S^{n-1}} \big\| \|\xi'\|\theta - \|\xi\|\theta \big\|\, d\sigma_{n-1}(\theta) \le \\
&\le 2\pi\|\xi - \xi'\|;
\end{aligned}$$

for the middle term we may take such $U \in \mathrm{O}_n(\mathbb{R})$ that $|\mathcal{F}f(Ut_m\theta) - M(t_m)| \le C_1 t_m\big(\delta + \sqrt{\frac{\log(1+m)}{n}}\big)$, or since $t_m = \|\xi'\| = (1+\varepsilon)^m \frac{\sqrt{k}}{\sigma}$ and thus $m = \log(1+\varepsilon)^{-1}\log\big(\frac{\sigma\|\xi'\|}{\sqrt{k}}\big)$,

$$|\mathcal{F}f(U\xi') - M(\|\xi'\|)| \le C_1\left[\delta + \sqrt{\frac{1}{n}\log\Big(1 + \log(1+\varepsilon)^{-1}\log\Big(\frac{\sigma\|\xi'\|}{\sqrt{k}}\Big)\Big)}\right]\|\xi'\|;$$

by using $\log(1+\varepsilon) \le 1$ and subadditivity of the square-root we have

$$\sqrt{\frac{1}{n}\log\Big(1 + \log(1+\varepsilon)^{-1}\log\Big(\frac{\sigma\|\xi'\|}{\sqrt{k}}\Big)\Big)} \le \sqrt{-\frac{\log(\log(1+\varepsilon))}{n}} + \sqrt{\frac{1}{n}\log\Big(1 + \log\Big(\frac{\sigma\|\xi'\|}{\sqrt{k}}\Big)\Big)}.$$

If we put $\alpha \ge \frac{\log(\log(3/2))}{\log(1/2)} \approx 1.31$, then $\varepsilon^\alpha \le 2\log(\frac{3}{2})\varepsilon \le \log(1+\varepsilon)$ for all $\varepsilon \in [0, \frac{1}{2}]$ (the second inequality stems from concavity of logarithm), this implies $-\log(\log(1+\varepsilon)) \le -\alpha\log(\varepsilon)$ which we use for the first square-root-term. By our choices of $m$ and $\theta$, $\|\xi'\| \le \|\xi\| \le (1+\varepsilon)\|\xi'\|$ and $\big\|\frac{\xi}{\|\xi\|} - \frac{\xi'}{\|\xi'\|}\big\| \le \varepsilon$, these imply $\|\xi - \xi'\| \le 2\varepsilon\|\xi\|$, therefore,

$$\begin{aligned}
&|\mathcal{F}f(U\xi) - M(\|\xi\|)| \le \\
&\quad \le 4\pi\|\xi - \xi'\| + C_1\left[\delta + \sqrt{\alpha}\sqrt{-\frac{\log(\varepsilon)}{n}} + \sqrt{\frac{1}{n}\log\Big(1 + \log\Big(\frac{\sigma\|\xi'\|}{\sqrt{k}}\Big)\Big)}\right]\|\xi'\| \le \\
&\quad \le \left[8\pi\varepsilon + C_1\delta + C_1\sqrt{\alpha}\sqrt{-\frac{\log(\varepsilon)}{n}} + C_1\sqrt{\frac{1}{n}\log\Big(1 + \log\Big(\frac{\sigma\|\xi\|}{\sqrt{k}}\Big)\Big)}\right]\|\xi\|.
\end{aligned}$$

In the case $\|\xi\| \le \frac{\sqrt{k}}{\sigma}$ choose $\theta \in \mathcal{N}$ with $\big\|\frac{\xi}{\|\xi\|} - \theta\big\| \le \varepsilon$ as before, and $m := \big\lfloor \log(1+\varepsilon)^{-1}\log\big(\frac{\sqrt{k}}{\sigma\|\xi\|}\big)\big\rfloor \in \mathbb{N}_0$, and set $\xi' := t_{-m}\theta$. With this choice $\frac{\|\xi'\|}{1+\varepsilon} \le \|\xi\| \le \|\xi'\|$ holds, hence $\|\xi'\| \le \frac{3}{2}\|\xi\|$ and $\frac{\sqrt{k}}{\sigma\|\xi'\|} \le \frac{\sqrt{k}}{\sigma\|\xi\|}$, furthermore $\|\xi - \xi'\| \le 2\varepsilon\|\xi\|$. The rest of the calculations is virtually the same as above, apart from $m = \log(1+\varepsilon)^{-1}\log\big(\frac{\sqrt{k}}{\sigma\|\xi'\|}\big)$ and an additional factor of $\frac{3}{2}$ for the middle term, so we have

$$\begin{aligned}
&|\mathcal{F}f(U\xi) - M(\|\xi\|)| \le \\
&\quad \le \left[8\pi\varepsilon + \frac{3C_1}{2}\delta + \frac{3C_1}{2}\sqrt{\alpha}\sqrt{-\frac{\log(\varepsilon)}{n}} + \frac{3C_1}{2}\sqrt{\frac{1}{n}\log\Big(1 + \log\Big(\frac{\sqrt{k}}{\sigma\|\xi\|}\Big)\Big)}\right]\|\xi\|.
\end{aligned}$$

To simplify matters, we enlarge the right-hand-sides by taking $C_3 := \max\big\{8\pi, \frac{3C_1\sqrt{\alpha}}{2}\big\}$ and also write the maximum $\frac{\sigma\|\xi\|}{\sqrt{k}} \vee \frac{\sqrt{k}}{\sigma\|\xi\|}$ in the logarithm; the set of $U$'s for which the inequalities holds is not diminished by these actions, its measure still is at least $1-\frac{C_2}{2}\,\mathsf{e}^{-c_2 n\delta^2}$ for either one. Thus with the intersection-bound $\mu_{\mathrm{O}}(A\cap B)\geq \mu_{\mathrm{O}}(A)+\mu_{\mathrm{O}}(B)-1$ we have found

$$\mu_{\mathrm{O}}\Big\{U\in \mathrm{O}_n(\mathbb{R})\Big|\forall\xi\in F\colon |\mathcal{F}f(U\xi)-M(\|\xi\|)|\leq \\ \leq C_3\Big[\delta+\varepsilon+\sqrt{-\frac{\log(\varepsilon)}{n}}+\sqrt{\frac{1}{n}\log\Big(1+\log\Big(\frac{\sigma\|\xi\|}{\sqrt{k}}\vee\frac{\sqrt{k}}{\sigma\|\xi\|}\Big)\Big)}\Big]\|\xi\|\Big\}\geq \\ \geq 1-C_2\mathsf{e}^{-c_2 n\delta^2}.$$

We use our freedom with $\varepsilon$ to choose it such that $g_n(x):=x+\sqrt{-\frac{\log(x)}{n}}$ is (nearly) minimized. We have $\lim_{x\to 0+}g_n(x)=\infty$, $g_n(1)=1$, $g_n$ is strictly convex on $(0,\frac{1}{\sqrt{\mathsf{e}}}]$ and strictly concave on $[\frac{1}{\sqrt{\mathsf{e}}},1]$, and $g_n'(\frac{1}{\sqrt{\mathsf{e}}})=1-\sqrt{\frac{\mathsf{e}}{2n}}>0$ iff $n>\frac{\mathsf{e}}{2}$, that is $n\geq 2$, which in our case is fulfilled; hence $g_n$ has a unique minimizer on $(0,\frac{1}{\sqrt{\mathsf{e}}}]$. Looking at the first derivative, the minimizer is the solution of $x\sqrt{-\log(x)}=\frac{1}{2\sqrt{n}}$ in that interval. This is equivalent to solving $-x^2\log(x^2)=\frac{1}{2n}$; now $G(x):=-x\log(x)$ is strictly concave on $[0,1]$ with $G(0)=G(1)=0$ and global maximum $\frac{1}{\mathsf{e}}$ attained at $\frac{1}{\mathsf{e}}$, this shows $G(x)\geq x$ for all $x\in[0,\frac{1}{\mathsf{e}}]$; in particular from $G(x^2)=\frac{1}{2n}$ follows $x^2\leq\frac{1}{2n}$ and hence $x\leq\frac{1}{\sqrt{2n}}$. Because $g_n(\frac{1}{\sqrt{2n}})<1$ and even $\lim_{n\to\infty}g_n(\frac{1}{\sqrt{2n}})=0$ monotonically, we choose $\varepsilon=\frac{1}{\sqrt{2n}}$ as the approximate minimizer (as $n\geq 3$, also $\varepsilon\leq\frac{1}{2}$ holds). Furhermore $g_n(\frac{1}{\sqrt{2n}})=\frac{1}{\sqrt{2n}}+\sqrt{\frac{\log(2n)}{2n}}=\sqrt{\frac{\log(n)}{n}}\big(\frac{1}{\sqrt{2\log(n)}}+\sqrt{\frac{\log(2)+\log(n)}{2\log(n)}}\big)\leq\sqrt{\frac{\log(n)}{n}}\big(\frac{1}{\sqrt{2\log(3)}}+\sqrt{\frac{\log(2)}{2\log(3)}+\frac{1}{2}}\big)\leq 1.58\sqrt{\frac{\log(n)}{n}}$ for $n\geq 3$, and we update $C_4:=C_3\big(\frac{1}{\sqrt{2\log(3)}}+\sqrt{\frac{\log(2)}{2\log(3)}+\frac{1}{2}}\big)=C_3\,\frac{1+\sqrt{\log(6)}}{\sqrt{\log(9)}}$.

*Comment.* Fresen's $\varepsilon=\sqrt{\frac{\log(n)}{n}}$ seems to be a slip of the pen, writing the desired result in the upper bound for the actual value of $\varepsilon$; as is evident from his later $k\leq c\delta^2\log(n)^{-1}n$ (page 8, lines 3, 4) he had $\varepsilon=\frac{C}{n^\alpha}$ in mind, perhaps $\alpha\in\{\frac{1}{2},1\}$, though the details are unrecoverable. ♢

As $UF\in G_{n,k}$ and $\|U\xi\|=\|\xi\|$ for any $U\in\mathrm{O}_n(\mathbb{R})$ and $\xi\in\mathbb{R}^n$, we may rewrite,

$$\sigma_{n,k}\Big\{E\in G_{n,k}\Big|\forall\xi\in E\colon |\mathcal{F}f(\xi)-M(\|\xi\|)|\leq \\ \leq C_4\Big[\delta+\sqrt{\frac{\log(n)}{n}}+\sqrt{\frac{1}{n}\log\Big(1+\log\Big(\frac{\sigma\|\xi\|}{\sqrt{k}}\vee\frac{\sqrt{k}}{\sigma\|\xi\|}\Big)\Big)}\Big]\|\xi\|\Big\}\geq 1-C_2\mathsf{e}^{-c_2 n\delta^2}.$$

Let $E\in G_{n,k}$ from the set above. As noted near the beginning of the proof, $\mathcal{F}h$ is integrable and hence the inversion-formula may be applied, and by Lemma 2.11 $\mathcal{F}h\restriction E=$

$\mathcal{F}_E \pi_E h$. Therefore, for all $x \in E$,

$$\pi_E h(x) = \int_E \mathsf{e}^{2\pi\mathsf{i}\langle \xi, x\rangle}\, \mathcal{F}h(\xi)\, d\xi.$$

We have $\mathcal{F}h = \mathcal{F}f \cdot \mathcal{F}\varphi_\sigma$, where $\mathcal{F}\varphi_\sigma(\xi) = \mathsf{e}^{-2\pi^2\sigma^2\|\xi\|^2}$ $(\xi \in \mathbb{R}^n)$; let $x \in E$ and $U \in \mathrm{O}_n(\mathbb{R})$ with $UE = E$, then

$$\begin{aligned}
|\pi_E h(Ux) - \pi_E h(x)| &= \\
&= \left| \int_E \mathsf{e}^{2\pi\mathsf{i}\langle \xi, Ux\rangle}\, \mathcal{F}h(\xi)\, d\xi - \int_E \mathsf{e}^{2\pi\mathsf{i}\langle \xi, x\rangle}\, \mathcal{F}h(\xi)\, d\xi \right| \\
&= \left| \int_E \mathsf{e}^{2\pi\mathsf{i}\langle \xi, x\rangle}\, \mathcal{F}h(U\xi)\, d\xi - \int_E \mathsf{e}^{2\pi\mathsf{i}\langle \xi, x\rangle}\, \mathcal{F}h(\xi)\, d\xi \right| \le \\
&\le \int_E |\mathcal{F}h(U\xi) - \mathcal{F}h(\xi)|\, d\xi = \\
&= \int_E |\mathcal{F}f(U\xi) - \mathcal{F}f(\xi)| \mathcal{F}\varphi_\sigma(\xi)\, d\xi \le \\
&\le \int_E \big(|\mathcal{F}f(U\xi) - M(\|\xi\|)| + |\mathcal{F}f(\xi) - M(\|\xi\|)|\big) \mathcal{F}\varphi_\sigma(\xi)\, d\xi \le \\
&\le 2C_4 \int_E \left[\delta + \sqrt{\frac{\log(n)}{n}} + \sqrt{\frac{1}{n}\log\Big(1 + \log\Big(\frac{\sigma\|\xi\|}{\sqrt{k}} \vee \frac{\sqrt{k}}{\sigma\|\xi\|}\Big)\Big)}\right] \|\xi\| \mathcal{F}\varphi_\sigma(\xi)\, d\xi;
\end{aligned}$$

the integrand is radially symmetric, so we transform to $k$-dimensional polar coordinates and immediately get a factor $\frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}$ from the angular part. The $\big(\delta + \sqrt{\frac{\log(n)}{n}}\big)$-part becomes a simple gamma-integral after a substitution and using Lemma 2.3:

$$\begin{aligned}
\int_0^\infty \mathsf{e}^{-2\pi^2\sigma^2 r^2}\, r^k\, dr &= \frac{(2\pi^2\sigma^2)^{-\frac{k+1}{2}}}{2} \int_0^\infty s^{\frac{k-1}{2}}\, \mathsf{e}^{-s}\, ds \\
&= \frac{(2\pi^2\sigma^2)^{-\frac{k+1}{2}}}{2}\, \Gamma\Big(\frac{k+1}{2}\Big) \le \frac{(2\pi^2\sigma^2)^{-\frac{k+1}{2}}}{2\sqrt{2}}\, \sqrt{k}\, \Gamma\Big(\frac{k}{2}\Big).
\end{aligned}$$

For the second part we first estimate $\log(1+x) \le x$ $(x \in \mathbb{R}_{\ge 0})$ to get rid of the double-logarithm, then use the inequality $\log(x) \le \frac{2}{\mathsf{e}}\, x^{1/2}$ $(x \in [1, \infty))$ to eliminate the remaining logarithm, split the integration-domain at $r = \frac{\sqrt{k}}{\sigma}$ to take care of the maximum, perform the same substitution as before, and again apply Lemma 2.3; this yields

$$\int_0^\infty \sqrt{\log\Big(1 + \log\Big(\frac{\sigma r}{\sqrt{k}} \vee \frac{\sqrt{k}}{\sigma r}\Big)\Big)}\, r^k\, \mathsf{e}^{-2\pi^2\sigma^2 r^2}\, dr \le$$

$$\leq \frac{(2\pi^2\sigma^2)^{-\frac{k+1}{2}}}{2}\sqrt{\frac{2}{\mathsf{e}}}\biggl((2k)^{\frac{1}{8}}\,\pi^{\frac{1}{4}}\int_0^{2\pi^2 k} s^{\frac{k}{2}-\frac{5}{8}}\,\mathsf{e}^{-s}\,ds$$
$$+\,(2k)^{-\frac{1}{8}}\,\pi^{-\frac{1}{4}}\int_{2\pi^2 k}^{\infty} s^{\frac{k}{2}-\frac{3}{8}}\,\mathsf{e}^{-s}\,ds\biggr) \leq$$
$$\leq \frac{(2\pi^2\sigma^2)^{-\frac{k+1}{2}}}{\sqrt{2\mathsf{e}}}\Bigl((2k)^{\frac{1}{8}}\,\pi^{\frac{1}{4}}\,\Gamma\Bigl(\frac{k}{2}+\frac{3}{8}\Bigr)+(2k)^{-\frac{1}{8}}\,\pi^{-\frac{1}{4}}\,\Gamma\Bigl(\frac{k}{2}+\frac{5}{8}\Bigr)\Bigr) \leq$$
$$\leq \frac{(2\pi^2\sigma^2)^{-\frac{k+1}{2}}}{\sqrt{2\mathsf{e}}}\,(2^{-\frac{1}{4}}\,\pi^{\frac{1}{4}}+2^{-\frac{3}{4}}\,\pi^{-\frac{1}{4}})\sqrt{k}\,\Gamma\Bigl(\frac{k}{2}\Bigr);$$

since $n \geq 3$, we also have $\sqrt{\frac{1}{n}} \leq \delta + \sqrt{\frac{\log(n)}{n}}$, and this leads to

$$|\pi_E h(Ux) - \pi_E h(x)| \leq 2C_4\Biggl(\delta + \sqrt{\frac{\log(n)}{n}}\Biggr)\frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}\,\frac{(2\pi^2\sigma^2)^{-\frac{k+1}{2}}}{2}$$
$$\cdot\Bigl(\frac{1}{\sqrt{2}}+\sqrt{\frac{2}{\mathsf{e}}}(2^{-\frac{1}{4}}\,\pi^{\frac{1}{4}}+2^{-\frac{3}{4}}\,\pi^{-\frac{1}{4}})\Bigr)\sqrt{k}\,\Gamma\Bigl(\frac{k}{2}\Bigr) =$$
$$= C_4\,\frac{\sqrt{2\mathsf{e}}+2^{\frac{5}{4}}\,\pi^{\frac{1}{4}}+2^{\frac{3}{4}}\,\pi^{-\frac{1}{4}}}{\sqrt{\pi\mathsf{e}}}\,(2\pi\sigma^2)^{-\frac{k+1}{2}}\Biggl(\delta+\sqrt{\frac{\log(n)}{n}}\Biggr)\sqrt{k}; \tag{3.1}$$

for future reference we define $C_5 := C_4\,\frac{\sqrt{2\mathsf{e}}+2^{\frac{5}{4}}\,\pi^{\frac{1}{4}}+2^{\frac{3}{4}}\,\pi^{-\frac{1}{4}}}{\sqrt{\pi\mathsf{e}}} \approx 2.314C_4$. This inequality now says that $\pi_E h$ is almost radially symmetric; to be precise, we have shown the following,

$$\sigma_{n,k}\Biggl\{E \in G_{n,k}\,\Bigg|\,\exists x, y \in E\colon\,\|x\| = \|y\| \wedge |\pi_E h(x) - \pi_E h(y)| >$$
$$> C_5(2\pi\sigma^2)^{-\frac{k+1}{2}}\Biggl(\delta+\sqrt{\frac{\log(n)}{n}}\Biggr)\sqrt{k}\Biggr\} \leq C_2\mathsf{e}^{-c_2 n\delta^2}. \tag{3.2}$$

## 3.2 The second step.

As a reminder, we want to show that on any ray the radial density is concentrated around its maximizer. First we show that for any two directions the maximizers are relatively near. Following the notation of Lemma 2.24, for any $x \in S_E$ let $h_x(t) := -\log(\pi_E h(tx))$ be defined on $\mathbb{R}_{>0}$; because $h$ is log-concave, $h_x$ is convex, and because $\pi_E h = \pi_E f * \pi_E \varphi_\sigma$ and $\pi_E\varphi_\sigma$ is smooth, so is $\pi_E h$ and hence $h_x$.

Let $x, y \in S_E$; by Lemma 2.24, 1. and 2., the function $t \mapsto H_x(t) := t^{k-1}\pi_E h(tx) = t^{k-1}\,\mathsf{e}^{-h_x(t)}$ has a unique global maximizer $t_x \in \mathbb{R}_{>0}$ charcterized by $t_x h_x'(t_x) = k-1$; the same holds for $y$. W.l.o.g. let $t_x < t_y$ (for $t_x = t_y$ there is nothing to prove). By convexity,

$$h_x(t_y) - h_x(t_x) \geq h_x'(t_x)(t_y - t_x) = (k-1)\,\frac{t_y - t_x}{t_x},$$

$$h_y(t_y) - h_x(t_x) \le -h'_y(t_y)(t_x - t_y) = (k-1)\,\frac{t_y - t_x}{t_y},$$

and hence

$$\begin{aligned}\max\{|h_x(t) - h_y(t)||t \in \{t_x, t_y\}\} &\ge \frac{|h_x(t_x) - h_y(t_x)| + |h_x(t_y) - h_y(t_y)|}{2} \\ &\ge \frac{-(h_x(t_x) - h_y(t_x)) + (h_x(t_y) - h_y(t_y))}{2} = \\ &= \frac{(h_x(t_y) - h_x(t_x)) - (h_y(t_y) - h_y(t_x))}{2} \ge \\ &\ge \frac{k-1}{2}\Big(\frac{t_y - t_x}{t_x} - \frac{t_y - t_x}{t_y}\Big) = \\ &= \frac{k-1}{2}\,\frac{(t_y - t_x)^2}{t_x t_y}.\end{aligned}$$

We are going to prove $h_x(t) - h_y(t) < 1$ for either $t \in \{t_x, t_y\}$ by contradiction. Assume that there exists $t \in \{t_x, t_y\}$ with $h_x(t) - h_y(t) \ge 1$. Since $h$ is the density of $Y$ with $\mathrm{Var}[Y] = (1+\sigma^2)I_n$, $\pi_E h$ is the density of $p_E \circ Y$ with $\mathrm{Var}[p_E \circ Y] = (1+\sigma^2)\,p_E = (1+\sigma^2)I_E$; applying the remark after Lemma 2.22 with $A = \big(\int_E x x^{\mathsf T} \pi_E h(x)\,dx\big)^{1/2} = \mathrm{Var}[p_E \circ Y]^{1/2}$ we obtain $\pi_E h(o) \ge \det((1+\sigma^2)I_E)^{-1/2} 2^{-7k} = (1+\sigma^2)^{-k/2} 2^{-7k}$. Thus by convexity on the one hand $h_y(0) \ge h_y(t) + h'_y(t)(-t)$ and on the other hand $h'_y(t_x) \le h'_y(t_y)$, and by our assumption,

$$\begin{aligned}|\mathsf{e}^{-h_x(t)} - \mathsf{e}^{-h_y(t)}| &= \mathsf{e}^{-h_y(t)}|\mathsf{e}^{-(h_x(t)-h_y(t))} - 1| \ge \\ &\ge \mathsf{e}^{-h_y(0)-h'_y(t)t}\,(1 - \mathsf{e}^{-1}) \\ &\ge (1-\mathsf{e}^{-1})\pi_E h(o)\mathsf{e}^{-h'_y(t_y)t_y} \\ &\ge (1-\mathsf{e}^{-1})(1+\sigma^2)^{-\frac{k}{2}} 2^{-7k} \mathsf{e}^{-k+1} = (\mathsf{e}-1)(1+\sigma^2)^{-\frac{k}{2}} 2^{-7k} \mathsf{e}^{-k}. \quad (3.3)\end{aligned}$$

But from (3.1) of the first step,

$$|\mathsf{e}^{-h_x(t)} - \mathsf{e}^{-h_y(t)}| = |\pi_E h(tx) - \pi_E h(ty)| \le C_5 (2\pi\sigma^2)^{-\frac{k+1}{2}} \left(\delta + \sqrt{\frac{\log(n)}{n}}\right)\sqrt{k}, \quad (3.4)$$

and by the right choice of parameters (for which see (3.13), third step) this upper bound is smaller than the lower bound and we have arrived at a contradiction. This proves $h_x(t) - h_y(t) < 1$ for either $t \in \{t_x, t_y\}$; let $t \in \{t_x, t_y\}$ be the maximizer of $|h_x - h_y|$. The function $z \mapsto |\mathsf{e}^z - 1|$ for $z \in \mathbb{R}$ is concave on $\mathbb{R}_{\le 0}$ and convex on $\mathbb{R}_{\ge 0}$, therefore $|\mathsf{e}^z - 1| \ge -(1-\mathsf{e}^{-1})z = (1-\mathsf{e}^{-1})|z|$ for $z \in [-1, 0]$ and $|\mathsf{e}^z - 1| \ge z \ge (1-\mathsf{e}^{-1})|z|$ for $z \ge 0$. This leads to

$$\begin{aligned}|\mathsf{e}^{-h_x(t)} - \mathsf{e}^{-h_y(t)}| &= \mathsf{e}^{-h_y(t)}|\mathsf{e}^{h_y(t)-h_x(t)} - 1| \\ &\ge \mathsf{e}^{-h_y(0)-h'_y(t)t}(1-\mathsf{e}^{-1})|h_y(t) - h_x(t)|\end{aligned}$$

$$\geq (1+\sigma^2)^{-\frac{k}{2}} 2^{-7k} \mathsf{e}^{-k+1}(1-\mathsf{e}^{-1}) \, \frac{k-1}{2} \, \frac{(t_y - t_x)^2}{t_x t_y}$$
$$\geq \frac{(\mathsf{e}-1)(k-1)}{2} \, (1+\sigma^2)^{-\frac{k}{2}} 2^{-7k} \mathsf{e}^{-k} \, \frac{(t_y - t_x)^2}{t_y^2},$$

combined with (3.4) we get

$$\begin{aligned} \frac{t_y - t_x}{t_y} &\leq \Big( \frac{2}{(\mathsf{e}-1)(k-1)} \Big)^{\frac{1}{2}} (1+\sigma^2)^{\frac{k}{4}} 2^{\frac{7k}{2}} \mathsf{e}^{\frac{k}{2}} \sqrt{C_5} (2\pi\sigma^2)^{-\frac{k+1}{4}} \Bigg( \delta + \sqrt{\frac{\log(n)}{n}} \Bigg)^{\frac{1}{2}} k^{\frac{1}{4}} \\ &\leq \Big( \frac{2C_5}{(\mathsf{e}-1)\sqrt{2\pi}} \Big)^{\frac{1}{2}} \Big( \frac{\sqrt{k}}{k-1} \Big)^{\frac{1}{2}} \mathsf{e}^{(\frac{7}{2}\log(2) + \frac{1}{2} - \frac{1}{4}\log(2\pi))k} \\ &\quad \cdot (1+\sigma^2)^{\frac{k}{4}} \sigma^{-\frac{k+1}{2}} \Bigg( \delta^{\frac{1}{2}} + \Big( \frac{\log(n)}{n} \Big)^{\frac{1}{4}} \Bigg) \leq \\ &\leq \gamma := C_6 \mathsf{e}^{c_3 k} (1+\sigma^2)^{\frac{k}{4}} \sigma^{-\frac{k+1}{2}} \Bigg( \delta^{\frac{1}{2}} + \Big( \frac{\log(n)}{n} \Big)^{\frac{1}{4}} \Bigg), \end{aligned} \tag{3.5}$$

where we have used $\frac{\sqrt{k}}{k-1} \leq \sqrt{2}$ for $k \geq 2$ and have defined $C_6 := \big( \frac{2C_5\sqrt{2}}{(\mathsf{e}-1)\sqrt{\pi}} \big)^{1/2}$ and $c_3 := \frac{1}{2} + \frac{7}{2}\log(2) - \frac{1}{4}\log(2\pi) \approx 2.467$. For the sequel we assume that the parameters are such that $\gamma \in (0, \frac{1}{2}]$ (this shall be justified below).

From $\mathbb{E}[p_E \circ Y] = o$ we also can infer $\int_0^\infty \pi_E h(tx)\, dt \in \mathbb{R}_{>0}$ for any $x \in S_E$; otherwise $\int_0^\infty \pi_E h(tx)\, dt = 0$ for some $x \in S_E$ implies $\pi_E h(tx) = 0$ for $t \in \mathbb{R}_{>0}$ and therefore the support of $\pi_E h$ is contained in a halfspace with $o$ at the boundary which does not contain $\mathbb{R}_{>0}x$, but then $\mathbb{E}[p_E \circ Y] = \int_E x \pi_E h(x)\, dx$ lies in the open halfspace and thus cannot be $o$. For this reason we may apply Lemma 2.24 to get, for any $u \in [0,1]$ and with $c_4 := \frac{\log(\mathsf{e}/2)}{2}$,

$$\int_{(1-u)t_x}^{(1+u)t_x} t^{k-1} \pi_E h(tx)\, dt \geq (1 - 3\mathsf{e}^{-c_4 k u^2}) \int_0^\infty t^{k-1} \pi_E h(tx)\, dt. \tag{3.6}$$

*Comment.* There is little need for comment on the second step of Fresen's proof, yet his demonstration of $\frac{t_y}{t_x} - 1 \leq \gamma$, though perhaps not the most elegant one, still is a great feat.

Why he only allows for $u \in [0, \frac{1}{2}]$ is unclear, given that a larger range can be achieved without difficulty, as shown in our proof of the corresponding Lemma 2.24, 3. The Taylor-expansion used on page 6, line 4, may be an overkill as only a quadratic estimate is aimed at. $\diamondsuit$

## 3.3 The third step.

Recall that we want to establish the thin-shell-property for $p_E \circ Y$ in order to get a CLT for it and then transfer the latter result to the original $X$ on the whole of $\mathbb{R}^n$. Let $x \in S_E$

and $u \in [0,1]$,[2] then for any $y \in S_E$ we know $1-\gamma \le \frac{t_x}{t_y} \le \frac{1}{1-\gamma}$ and therefore

$$(1-2(u+\gamma))t_x \le ((1-2\gamma)-2u)\frac{1}{1-\gamma}\,t_y = \Big(1-\frac{\gamma}{1-\gamma}-\frac{2}{1-\gamma}\,u\Big)t_y \le (1-u)t_y$$

and

$$(1+2(u+\gamma))t_x \ge \Big(1+\frac{u+\gamma}{1-\gamma}\Big)(1-\gamma)t_y = (1+u)t_y;$$

this leads to (transform to polar coordinates)

$$\begin{aligned}
\mathbb{P}\Big[\Big|\frac{\|p_E \circ Y\|}{t_x}-1\Big| \le 2(u+\gamma)\Big] =& \\
&= \mathbb{P}\big[(1-2(u+\gamma))t_x \le \|p_E \circ Y\| \le (1+2(u+\gamma))t_x\big] \\
&= \int_{\{y\in E|(1-2(u+\gamma))t_x \le \|y\| (1+2(u+\gamma))t_x\}} \pi_E h(y)\,dy \\
&= \frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}\int_{S_E}\int_{(1-2(u+\gamma))t_x}^{(1+2(u+\gamma))t_x} t^{k-1}\pi_E h(ty)\,dt\,d\sigma_{k-1}(y) \ge \\
&\ge \frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}\int_{S_E}\int_{(1-u)t_y}^{(1+u)t_y} t^{k-1}\pi_E h(ty)\,dt\,d\sigma_{k-1}(y) \\
&\ge (1-3\mathsf{e}^{-c_4 k u^2})\frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}\int_{S_E}\int_0^\infty t^{k-1}\pi_E h(ty)\,dt\,d\sigma_{k-1}(y) = \\
&= 1-3\mathsf{e}^{-c_4 k u^2},
\end{aligned}$$

because the last double integral is just $\int_E \pi_E h(y)\,dy = 1$.

*Comment.* At this point we would like to use Lemma 2.24, 4., in order to replace $t_x$ by $\sqrt{1+\sigma^2}\sqrt{k}$, but there are complications. Fresen elaborates in his note "Radius of the thin shell" on page 8 how this could be achieved: the density $\pi_E h$ meets the requirements of Lemma 2.24, hence 3. implies, for any $y \in S_E$ and $u \in [0,1]$,

$$\int_{(1-u)t_y}^{(1+u)t_y} t^{k-1}\,\pi_E h(t)\,dt \ge (1-3\mathsf{e}^{-cku^2})\int_0^\infty t^{k-1}\pi_E h(t)\,dt,$$

or, equivalently, if we define a random-variable $X_y$ as in Lemma 2.24,

$$\mathbb{P}\Big[\Big|\frac{X_y}{t_y}-1\Big| \ge u\Big] \le 3\mathsf{e}^{-cku^2}, \tag{3.7}$$

still valid for all $u \in [0,1]$. This should yield upper bounds for $|\mathbb{E}[W_y]-t_y|$ and $\mathrm{Var}[W_y]$ via

$$\big|\mathbb{E}[X_y]-t_y\big| \le \mathbb{E}[|X_y-t_y|] = \int_0^\infty \mathbb{P}[|X_y-t_y| \ge t]\,dt \overset{!}{\le} C_1'\,\frac{t_x}{\sqrt{k}}$$

[2] $B_2^n$ instead of $S^{n-1}$ on page 6, line 3 from below, clearly is an oversight.

and

$$\mathrm{Var}[X_y] = \mathrm{Var}[X_y - t_y] \leq \mathbb{E}[(X_y - t_y)^2] = \int_0^\infty \mathbb{P}[(X_y - t_y)^2 \geq t]\, dt \overset{!}{\leq} C_2' \frac{t_x^2}{k},$$

for some $C_1', C_2' \in \mathbb{R}_{>0}$. The problem is that we need to bound the tail-probabilities $\mathbb{P}[|W_y - t_y| \geq u]$ for all $u \in \mathbb{R}_{\geq 0}$, but with (3.7) we only have estimates for $u \in [0, 1]$. We do not know if log-convexity could be exploited in the right manner in order to get useful bounds for the present situation. Once the estimates are accepted, the remaining part of Fresen's calculations at this point does not pose serious problems; the only point to mention is that $\mathbb{E}[\|\pi_E Y\|^2]$ equals $(1+\sigma^2)k$, not just $k$. We have spelled out the details in our proof of Lemma 2.24, 4.

Klartag circumvents the problem in [20, Lemma 4.6] by working on the whole space instead of one-dimensional subspaces, and by using his Lemma 2.1 which gives a uniform exponential bound on tail-probabilities.

For the remainder of the proof we assume the existence of $C_7 \in \mathbb{R}_{>0}$ such that $\big|\frac{t_x}{\sqrt{1+\sigma^2}\sqrt{k}} - 1\big| \leq 3\gamma + \frac{C_7}{\sqrt{k}}$, as stated by the conclusion of Lemma 2.24, 4. ◊

Now $\big|\frac{\|p_E \circ Y\|}{t_x} - 1\big| \leq 2(u+\gamma)$ implies

$$\begin{aligned}
\Big|\frac{\|p_E \circ Y\|}{\sqrt{1+\sigma^2}\sqrt{k}} - 1\Big| &\leq \Big|\frac{\|p_E \circ Y\|}{\sqrt{1+\sigma^2}\sqrt{k}} - \frac{t_x}{\sqrt{1+\sigma^2}\sqrt{k}}\Big| + \Big|\frac{t_x}{\sqrt{1+\sigma^2}\sqrt{k}} - 1\Big| \leq \\
&\leq \frac{t_x}{\sqrt{1+\sigma^2}\sqrt{k}}\Big|\frac{\|p_E \circ Y\|}{t_x} - 1\Big| + 3\gamma + \frac{C_7}{\sqrt{k}} \\
&\leq \Big(1 + 3\gamma + \frac{C_7}{\sqrt{k}}\Big) 2(u+\gamma) + 3\gamma + \frac{C_7}{\sqrt{k}} \\
&\leq \Big(2 + 6\gamma + \frac{2C_7}{\sqrt{k}} + \frac{C_7}{u\sqrt{k}}\Big) u + \Big(5 + 6\gamma + \frac{C_7}{\sqrt{k}}\Big)\gamma \\
&\leq (5 + (\sqrt{2} + \alpha')C_7)u + (8 + \sqrt{2}^{-1} C_7)\gamma \\
&\leq \alpha''(u+\gamma),
\end{aligned}$$

where we have bounded $\frac{1}{u\sqrt{k}} \leq \alpha' \in \mathbb{R}_{>0}$, which will be justified by our choice below, and put $\alpha'' := \max\{5 + (\sqrt{2}+\alpha')C_7, 8 + \frac{C_7}{\sqrt{2}}\}$. Thus

$$\mathbb{P}\Big[\Big|\frac{\|p_E \circ Y\|}{\sqrt{1+\sigma^2}\sqrt{k}} - 1\Big| \geq \alpha''(u+\gamma)\Big] \leq \mathbb{P}\Big[\Big|\frac{\|p_E \circ Y\|}{t_x} - 1\Big| \geq 2(u+\gamma)\Big] \leq 3\mathsf{e}^{-c_4 k u^2}.$$

Since $\alpha'' \geq 3$, we can symmetrize this to

$$\begin{aligned}
\mathbb{P}\Big[\Big|\frac{\|p_E \circ Y\|}{\sqrt{1+\sigma^2}\sqrt{k}} - 1\Big| \geq \alpha''\big(u + \gamma + \mathsf{e}^{-c_4 k u^2}\big)\Big] &\leq \mathbb{P}\Big[\Big|\frac{\|p_E \circ Y\|}{\sqrt{1+\sigma^2}\sqrt{k}} - 1\Big| \geq \alpha''(u+\gamma)\Big] \\
&\leq 3\mathsf{e}^{-c_4 k u^2} \\
&\leq \alpha''\big(u + \gamma + \mathsf{e}^{-c_4 k u^2}\big),
\end{aligned}$$

so we have the thin-shell-property with $\varepsilon = \alpha''\big(u + \gamma + \mathsf{e}^{-c_4 k u^2}\big)$ (again, $\varepsilon \leq \frac{1}{2}$ shall be ensured; note that this $\varepsilon$ is different from that used in Step 1 whose value has been set to $\frac{1}{\sqrt{2n}}$). Therefore Theorem 2.28 asserts the existence of constants $C_8, c_5 \in \mathbb{R}_{>0}$ such that, by plugging in $\delta = u + \gamma + \mathsf{e}^{-c_4 k u^2}$ in said theorem,

$$\sigma_E\Big\{\theta \in S_E \Big| \exists t \in \mathbb{R}\colon \Big|\mathbb{P}\Big[\Big\langle \theta, \frac{p_E \circ Y}{\sqrt{1+\sigma^2}}\Big\rangle \leq t\Big] - \Phi(t)\Big| \geq \Big(1 + \frac{3\alpha''}{2}\Big)\big(u + \gamma + \mathsf{e}^{-c_4 k u^2}\big)\Big\}$$
$$\leq C_8 \sqrt{k} \exp\big[-c_5 k \big(u + \gamma + \mathsf{e}^{-c_4 k u^2}\big)^2\big]; \quad (3.8)$$

note that $\mathbb{P}\big[\langle \theta, \frac{p_E \circ Y}{\sqrt{1+\sigma^2}}\rangle \leq t\big] - \Phi(t) = \mathbb{P}[\langle \theta, p_E \circ Y\rangle \leq s] - \Phi\big(\frac{s}{\sqrt{1+\sigma^2}}\big)$ via the substitution $s = t\sqrt{1+\sigma^2}$.

*Comment.* We think Fresen's application of Theorem 2.28 is a bit unhappy, because the shell in question has radius $t_x$, but in the version of the theorem given in his appendix and also in the literature cited by him the shell is supposed to have radius equal to the square-root of the dimension, and only in the aforementioned note "Radius of the thin shell" – which he does not mention at the relevant point in the proof – does he address that issue; so for an unprepared reader not familiar with Theorem 2.28 its application on page 7 may seem dubious. It is for this reason that we have rearranged the arguments so that a thin-shell-estimate with the "correct" radius is established for which Theorem 2.28 can be used without doubt. ♢

Let $\theta \in S_E$ from the set above. Now $\langle \theta, p_E \circ Y\rangle = \langle \theta, p_E \circ X\rangle + \sigma\langle \theta, p_E \circ Z\rangle$, where $\langle \theta, p_E \circ Z\rangle$ has standard normal distribution.[3] Let $t \in \mathbb{R}$ and $\nu \in (0, 1]$, then

$$\langle \theta, p_E \circ Y\rangle \leq t - \nu \Longrightarrow \langle \theta, p_E \circ X\rangle \leq t \vee \langle \theta, p_E \circ Z\rangle \leq -\frac{\nu}{\sigma};$$

by subadditivity, (3.8) and Lemma 2.5 this leads to a lower bound for $\mathbb{P}[\langle \theta, p_E \circ X\rangle \leq t]$, to wit,

$$\begin{aligned}
\mathbb{P}[\langle \theta, p_E \circ X\rangle \leq t] &\geq \mathbb{P}[\langle \theta, p_E \circ Y\rangle \leq t - \nu] - \mathbb{P}\Big[\langle \theta, p_E \circ Z\rangle \leq -\frac{\nu}{\sigma}\Big] \\
&\geq \Phi\Big(\frac{t-\nu}{\sqrt{1+\sigma^2}}\Big) - \Big(1 + \frac{3\alpha''}{2}\Big)\big(u + \gamma + \mathsf{e}^{-c_4 k u^2}\big) - 1 + \Phi\Big(\frac{\nu}{\sigma}\Big) \\
&\geq \Phi(t) - \frac{\nu + \sigma}{\sqrt{2\pi}} - \Big(1 + \frac{3\alpha''}{2}\Big)\big(u + \gamma + \mathsf{e}^{-c_4 k u^2}\big) - \frac{1}{2}\,\mathsf{e}^{-\frac{1}{2}\frac{\nu^2}{\sigma^2}} \\
&\geq \Phi(t) - C_9\big(\nu + \sigma + u + \gamma + \mathsf{e}^{-c_4 k u^2} + \mathsf{e}^{-\frac{\nu^2}{2\sigma^2}}\big),
\end{aligned}$$

where of course $C_9 := \max\big\{\frac{1}{\sqrt{2\pi}}, 1 + \frac{3\alpha''}{2}, \frac{1}{2}\big\} = 1 + \frac{3\alpha''}{2}$. Similarly we have

$$\langle \theta, p_E \circ X\rangle \leq t \Longrightarrow \langle \theta, p_E \circ Y\rangle \leq t + \nu \vee \langle \theta, p_E \circ Z\rangle \geq \frac{\nu}{\sigma}$$

[3] Because a linearly transformed Gaußian variable again is Gaußian, $\langle \theta, p_E \circ Z\rangle = \langle p_E^*(\theta), Z\rangle = \langle \theta, Z\rangle$, $\mathbb{E}[\langle \theta, Z\rangle] = \langle \theta, \mathbb{E}[Z]\rangle = \langle \theta, o\rangle = 0$ and $\mathrm{Var}[\langle \theta, Z\rangle] = \langle \theta, \mathrm{Var}[Z]\theta\rangle = \langle \theta, I_n\theta\rangle = \|\theta\|^2 = 1$.

and by the same means as before we get an upper bound,

$$\begin{aligned}\mathbb{P}[\langle\theta,p_E\circ X\rangle\le t] &\le \mathbb{P}[\langle\theta,p_E\circ Y\rangle\le t+\nu]+\mathbb{P}\Big[\langle\theta,p_E\circ Z\rangle\le\frac{\nu}{\sigma}\Big]\\ &\le \Phi\Big(\frac{t+\nu}{\sqrt{1+\sigma^2}}\Big)+\Big(1+\frac{3\alpha''}{2}\Big)\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big)+1-\Phi\Big(\frac{\nu}{\sigma}\Big)\\ &\le \Phi(t)+\frac{\nu+\sigma}{\sqrt{2\pi}}+\Big(1+\frac{3\alpha''}{2}\Big)\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big)+\frac{1}{2}\mathsf{e}^{-\frac{1}{2}\frac{\nu^2}{\sigma^2}}\\ &\le \Phi(t)+C_9\big(\nu+\sigma+u+\gamma+\mathsf{e}^{-c_4ku^2}+\mathsf{e}^{-\frac{\nu^2}{2\sigma^2}}\big).\end{aligned}$$

We introduce the error-bound

$$\varepsilon_n := C_9\big(\nu+\sigma+u+\gamma+\mathsf{e}^{-c_4ku^2}+\mathsf{e}^{-\frac{\nu^2}{2\sigma^2}}\big). \tag{3.9}$$

(This is not quite the $\varepsilon_n$ as stated in the theorem, for which we must take the maximum with 1.) Thus, so far we have shown

$$\begin{aligned}\sigma_E\big\{\theta\in S_E\big|\exists t\in\mathbb{R}\colon |\mathbb{P}[\langle\theta,p_E\circ X\rangle\le t]-\Phi(t)|\ge\varepsilon_n\big\}&\\ \le C_8\sqrt{k}\exp\big[-c_5k\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big)^2\big].&\end{aligned} \tag{3.10}$$

Until now we have been working with $\theta$ from subspaces $E$ for which the "almost radially symmetric"-property (3.1) holds; we want to know for which $\theta\in S^{n-1}$ (in the sense of measure) the "almost normally distributed"-property of $\langle\theta,X\rangle$ holds. First note $\langle\theta,p_E\circ X\rangle=\langle p_E^*(\theta),X\rangle=\langle\theta,X\rangle$ for any $\theta\in S_E$, so we can drop $p_E$ in (3.10). We consider the "failure-set", i.e. the set of those $\theta\in S^{n-1}$ for which the normal approximation possibly does not hold. To that end call $\mathcal{A}\subset G_{n,k}$ the set on the left-hand-side of (3.2), then $\sigma_{n,k}(\mathcal{A})\le C_2\mathsf{e}^{-c_2n\delta^2}$; and for each[4] $E\in\mathcal{A}^\mathsf{c}$ call $\mathcal{B}_E$ the set on the left-hand-side of (3.10), then $\sigma_E(\mathcal{B}_E)\le C_8\sqrt{k}\exp\big[-c_5k\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big)^2\big]$. Call the failure-set $\Theta$, that is

$$\Theta := \big\{\theta\in S^{n-1}\big|\exists t\in\mathbb{R}\colon |\mathbb{P}[\langle\theta,X\rangle\le t]-\Phi(t)|\ge\varepsilon_n\big\}. \tag{3.11}$$

Then $\Theta\cap E=\mathcal{B}_E$ for each $E\in\mathcal{A}^\mathsf{c}$. Using Lemma 2.2, we finally obtain

$$\begin{aligned}\sigma_{n-1}(\Theta)=1-\sigma_{n-1}(\Theta^\mathsf{c})=1-\int_{G_{n,k}}\sigma_E(\Theta^\mathsf{c}\cap E)\,d\sigma_{n,k}(E)\le&\\ \le 1-\int_{\mathcal{A}^\mathsf{c}}\sigma_E(\Theta^\mathsf{c}\cap E)\,d\sigma_{n,k}(E)=1-\int_{\mathcal{A}^\mathsf{c}}\sigma_E(\mathcal{B}_E^\mathsf{c})\,d\sigma_{n,k}(E)\le&\\ \le 1-\int_{\mathcal{A}^\mathsf{c}}\big(1-C_8\sqrt{k}\exp\big[-c_5k\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big)^2\big]\big)\,d\sigma_{n,k}(E)=&\\ =1-\big(1-C_8\sqrt{k}\exp\big[-c_5k\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big)^2\big]\big)\,\sigma_{n,k}(\mathcal{A}^\mathsf{c})\le&\\ \le 1-\big(1-C_8\sqrt{k}\exp\big[-c_5k\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big)^2\big]\big)\big(1-C_2\mathsf{e}^{-c_2n\delta^2}\big)\le&\end{aligned}$$

[4] $A^\mathsf{c}$ denotes the complement of the set $A$ when the basic set is clear.

$$\leq C_8\sqrt{k}\exp\bigl[-c_5k\bigl(u+\gamma+\mathsf{e}^{-c_4ku^2}\bigr)^2\bigr]+C_2\mathsf{e}^{-c_2n\delta^2} =: \delta_n. \tag{3.12}$$

(Here again, for the $\delta_n$ of the theorem the maximum with 1 must be taken. Additionally, in order for the computation above to work we must impose that the bounds on $\sigma_E(\mathcal{B}_E)$ and $\sigma_{n,k}(\mathcal{A})$ are at most 1. This is garanteed by proving $\delta_n \leq 1$ which shall be done below.)

*Comment.* Again the order of Fresen's arguments is clouding what is going on: at the end of page 7 he is still working in $E$, thus it is difficult to see why the failure-probability should contain the summand $C\exp(-c\delta n^2)$, and he also gives no argument for its inclusion; only on the next page does he make the jump from $S_E$ to $S^{n-1}$. ◊

Finally we choose the parameters:[5]

$$\begin{aligned} \sigma &= \frac{1}{\log(n+1)}, & k &= \frac{c_6\log(n+1)}{\log(\log(n+2))}, \quad \delta = \frac{\log(n+1)}{\sqrt{n}}, \\ u &= \frac{C_{10}\log(\log(n+2))}{\sqrt{\log(n+1)}}, & \nu &= \frac{C_{10}}{\sqrt{\log(n+1)}}, \end{aligned} \tag{3.13}$$

with $C_{10}, c_6 \in \mathbb{R}_{>0}$ for the fine tuning, and show that all requirements needed along the way are satisfied. Obviously all parameters are positive for any $n \in \mathbb{N}$. We note the basic asymptotic behaviour: $k \ll n$ and $\lim_{n\to\infty} k = \infty$, so $k \geq 2$ is eventually fulfilled (actually in order to make sense $k$ has to be rounded to an integer, but for the asymptotics this is immaterial), and $\lim_{n\to\infty}\delta = \lim_{n\to\infty}\sigma = \lim_{n\to\infty}u = \lim_{n\to\infty}\nu = 0$; furthermore $\sigma \leq 1$ for all $n \geq 2$.

The first is $k \leq \frac{c_1}{-\log(\varepsilon)}n\delta^2$ on page 70; on page 73 we have set $\varepsilon = \frac{1}{\sqrt{2n}}$, plugging in yields that $k \leq \frac{2c_1}{\log(2n)}n\delta^2$ must be met; we have

$$\begin{aligned} \frac{c_6\log(n+1)}{\log(\log(n+2))} &\leq \frac{2c_1 n}{\log(2n)}\,\frac{\log(n+1)^2}{n} = \frac{2c_1\log(n+1)^2}{\log(n)\bigl(1+\frac{\log(2)}{\log(n)}\bigr)} \\ \iff c_6 &\leq \frac{2c_1}{1+\frac{\log(2)}{\log(n)}}\,\frac{\log(n+1)}{\log(n)}\log(\log(n+2)) \geq \\ &\geq \frac{2c_1}{1+\frac{\log(2)}{\log(3)}}\cdot 1\cdot\log(\log(5)) = c_1\,\frac{\log(9)\log(\log(5))}{\log(6)}, \end{aligned}$$

so by imposing $c_6 \leq c_1\frac{\log(9)\log(\log(5))}{\log(6)}$ that relation is satisfied.

In the second step we have derived lower and upper bounds for $|\mathsf{e}^{-h_x(t)} - \mathsf{e}^{-h_y(t)}|$ and promised that the lower will be greater than the upper. The lower bound (3.3) is

$$(\mathsf{e}-1)(1+\sigma^2)^{-\frac{k}{2}}\,2^{-7k}\,\mathsf{e}^{-k} = (\mathsf{e}-1)\mathsf{e}^{-\log(128\mathsf{e})\frac{c_6\log(n+1)}{\log(\log(n+2))}}\Bigl(1+\frac{1}{\log(n+1)^2}\Bigr)^{-\frac{c_6\log(n+1)}{2\log(\log(n+1))}}$$

[5] Curiously, Fresen writes $C_2$ without ever having used $C_1$ before.

$$
\begin{aligned}
&= (\mathsf{e}-1)(n+1)^{-\frac{c_6 \log(128\mathsf{e})}{\log(\log(n+1))}} \\
&\quad \cdot (n+1)^{-\frac{c_6 \log(1+\log(n+1)^2)}{2\log(\log(n+2))}} (n+1)^{\frac{c_6 \log(\log(n+1))}{\log(\log(n+2))}} \\
&= (\mathsf{e}-1)\Big(\frac{n+1}{n}\Big)^{c_6\left[-\frac{\log(128\mathsf{e})}{\log(\log(n+2))}-\frac{\log(1+\log(n+1)^2)}{2\log(\log(n+2))}+\frac{\log(\log(n+1))}{\log(\log(n+2))}\right]} \\
&\quad \cdot n^{c_6\left[-\frac{\log(128\mathsf{e})}{\log(\log(n+2))}-\frac{\log(1+\log(n+1)^2)}{2\log(\log(n+2))}+\frac{\log(\log(n+1))}{\log(\log(n+2))}\right]},
\end{aligned}
$$

and the term in brackets in the exponents can be bounded from below as follows ($n \geq 3$ always),

$$
\begin{aligned}
-\frac{\log(128\mathsf{e})}{\log(\log(n+2))} &- \frac{\log(1+\log(n+1)^2)}{2\log(\log(n+2))} + \frac{\log(\log(n+1))}{\log(\log(n+2))} \geq \\
&\geq -\frac{\log(128\mathsf{e})}{\log(\log(5))} - \frac{\log\big(1+\frac{1}{\log(n+1)^2}\big)}{2\log(\log(n+2))} \\
&\geq -\frac{\log(128\mathsf{e})}{\log(\log(5))} - \frac{\log\big(1+\frac{1}{\log(4)^2}\big)}{2\log(\log(5))} \geq -12.74,
\end{aligned}
$$

in addition $\frac{n+1}{n} \leq \frac{4}{3}$, this yields

$$
(\mathsf{e}-1)(1+\sigma^2)^{-\frac{k}{2}}\, 2^{-7k}\, \mathsf{e}^{-k} \geq (\mathsf{e}-1)\Big(\frac{4}{3}\Big)^{-12.74c_6} n^{-12.74c_6} \geq 1.669 n^{-\frac{1}{10}},
$$

if we demand $12.74c_6 \leq \frac{1}{10}$, that is $c_6 \leq \frac{1}{127.4}$.

For the upper bound (3.4) we obtain

$$
\begin{aligned}
C_5(2\pi\sigma^2)^{-\frac{k+1}{2}}\Bigg(\delta+\sqrt{\frac{\log(n)}{n}}\Bigg)\sqrt{k} &= \frac{C_5}{\sqrt{2\pi}}\sqrt{2\pi}^{-\frac{c_6\log(n+1)}{2\log(\log(n+2))}}\Big(\frac{1}{\log(n+1)}\Big)^{-\frac{c_6\log(n+1)}{\log(\log(n+2))}-1} \\
&\quad \cdot \Bigg(\frac{\log(n+1)}{\sqrt{n}}+\sqrt{\frac{\log(n)}{n}}\Bigg)\sqrt{\frac{c_6\log(n+1)}{\log(\log(n+2))}} \\
&= \frac{C_5\sqrt{c_6}}{\sqrt{2\pi}\sqrt{n}}(n+1)^{-\frac{c_6\log(\sqrt{2\pi})}{\log(\log(n+2))}}(n+1)^{\frac{c_6\log(\log(n+1))}{\log(\log(n+2))}} \\
&\quad \cdot \Big(1+\frac{\sqrt{\log(n)}}{\log(n+1)}\Big)\frac{\log(n+1)^{\frac{5}{2}}}{\sqrt{\log(\log(n+2))}} \\
&= \frac{C_5\sqrt{c_6}}{\sqrt{2\pi}\sqrt{n}}\, n^{c_6\frac{\log(\log(n+1))-\log(\sqrt{2\pi})}{\log(\log(n+2))}} \\
&\quad \cdot \Big(\frac{n+1}{n}\Big)^{c_6\frac{\log(\log(n+1))-\log(\sqrt{2\pi})}{\log(\log(n+2))}} \\
&\quad \cdot \Big(1+\frac{\sqrt{\log(n)}}{\log(n+1)}\Big)\frac{\log(n+1)^{\frac{5}{2}}}{\sqrt{\log(\log(n+2))}};
\end{aligned}
$$

in the exponents we simply estimate $\frac{\log(\log(n+1))-\log(\sqrt{2\pi})}{\log(\log(n+2))} \leq \frac{\log(\log(n+1))}{\log(\log(n+2))} \leq 1$; furthermore $\frac{n+1}{n} \leq \frac{4}{3}$ and $\frac{\sqrt{\log(n)}}{\log(n+1)} \leq \frac{\sqrt{\log(n+1)}}{\log(n+1)} = \frac{1}{\sqrt{\log(n+1)}} \leq \frac{1}{\sqrt{\log(4)}}$ and $\frac{1}{\sqrt{\log(\log(n+2))}} \leq \frac{1}{\sqrt{\log(\log(5))}}$, also $\log(n+1)^{\frac{5}{2}} = \left(\frac{\log(n+1)}{\log(n)}\right)^{\frac{5}{2}} \log(n)^{\frac{5}{2}} \leq \left(\frac{\log(4)}{\log(3)}\right)^{\frac{5}{2}} \log(n)^{\frac{5}{2}}$; moreover a standard-calculation shows $n^{c_6-\frac{1}{10}} \log(n)^{\frac{5}{2}} \leq \left(\frac{25}{\mathsf{e}(1-10c_6)}\right)^{\frac{5}{2}}$, equivalently $n^{c_6} \log(n)^{\frac{5}{2}} \leq \left(\frac{25}{\mathsf{e}(1-10c_6)}\right)^{\frac{5}{2}} n^{\frac{1}{10}}$ (note that from before we already know $c_6 \leq \frac{1}{127.4}$, hence $1-10c_6 > 0$). Taken together we have proved

$$C_5(2\pi\sigma^2)^{-\frac{k+1}{2}}\left(\delta + \sqrt{\frac{\log(n)}{n}}\right)\sqrt{k} \leq \frac{C_5\sqrt{c_6}}{\sqrt{2\pi\log(\log(5))}}\left(\frac{4}{3}\right)^{c_6}\left(1+\frac{1}{\sqrt{\log(4)}}\right) \cdot \left(\frac{25\log(4)}{\mathsf{e}\log(3)(1-10c_6)}\right)^{\frac{5}{2}} n^{-\frac{1}{2}+\frac{1}{10}} \leq 53.5 C_5 n^{-\frac{1}{2}+\frac{1}{10}}.$$

In any case the upper bound transgresses the lower bound for $n > \left(\frac{53.5C_5}{1.669}\right)^{\frac{10}{3}}$ at the latest.

Next we want to verify $\gamma \in (0, \frac{1}{2}]$, recall its definition in (3.5):

$$\begin{aligned} C_6\mathsf{e}^{c_3k}(1+\sigma^2)^{\frac{k}{4}}\sigma^{-\frac{k+1}{2}}\left(\delta^{\frac{1}{2}} + \left(\frac{\log(n)}{n}\right)^{\frac{1}{4}}\right) &= C_6\mathsf{e}^{\frac{c_3c_6\log(n+1)}{\log(\log(n+2))}}\left(1+\frac{1}{\log(n+1)^2}\right)^{\frac{c_6\log(n+1)}{4\log(\log(n+2))}} \\ &\quad \cdot \log(n+1)^{\frac{c_6\log(n+1)}{2\log(\log(n+2))}+\frac{1}{2}} \\ &\quad \cdot \left(\frac{\sqrt{\log(n+1)}}{n^{\frac{1}{4}}} + \frac{\log(n)^{\frac{1}{4}}}{n^{\frac{1}{4}}}\right) \\ &= \frac{C_6}{n^{\frac{1}{4}}}(n+1)^{c_6\left[\frac{4c_3+\log(1+\log(n+1)^{-2})}{4\log(\log(n+2))}+\frac{\log(\log(n+1))}{2\log(\log(n+2))}\right]} \\ &\quad \cdot \log(n+1)\left(1+\frac{\log(n)^{\frac{1}{4}}}{\log(n+1)^{\frac{1}{2}}}\right); \end{aligned}$$

we are applying the usual estimation-techniques: concerning the brackets-term we have $\frac{4c_3+\log(1+\log(n+1)^{-2})}{4\log(\log(n+2))} \leq \frac{4c_3+\log(1+\log(4)^{-2})}{4\log(\log(5))} \leq 5.41$, then $\frac{\log(\log(n+1))}{\log(\log(n+2))} \leq 1$, next $(n+1)^{\rho} = \left(1+\frac{1}{n}\right)^{\rho} n^{\rho} \leq \left(\frac{4}{3}\right)^{\rho} n^{\rho}$, furthermore $\log(n+1) = \frac{\log(n+1)}{\log(n)}\log(n) \leq \frac{\log(4)}{\log(3)}\log(n)$, also $\frac{\log(n)^{1/4}}{\log(n+1)^{1/2}} \leq \frac{\log(n+1)^{1/4}}{\log(n+1)^{1/2}} = \log(n+1)^{-\frac{1}{4}} \leq \log(4)^{-\frac{1}{4}}$, and finally $n^{5.91c_6-\frac{1}{4}}\log(n) \leq \frac{20}{\mathsf{e}(1-118.2c_6)} n^{-\frac{1}{5}}$, which is valid because of $c_6 \leq \frac{1}{127.4}$; this sums up to

$$\gamma \leq C_6\left(\frac{4}{3}\right)^{5.91c_6}\frac{\log(4)}{\log(3)}\left(1+\log(4)^{-\frac{1}{4}}\right)\frac{20}{\mathsf{e}(1-118.2c_6)} n^{-\frac{1}{5}} \leq 250.4 C_6 n^{-\frac{1}{5}},$$

which implies $\lim_{n\to\infty}\gamma = 0$ and hence $\gamma \leq \frac{1}{2}$ eventually (e.g. $n \geq (500.8C_6)^5$).

The application of Lemma 2.24, 3., for (3.6) requires $u \in [0,1]$, which is guaranteed for $n$ sufficiently large (in this case, very; roughly $n \geq \mathsf{e}^{6.32C_{10}^4} - 2$) by $\lim_{n\to\infty} u = 0$, as already noted above. The proximity-condition of $t_x$ and $\sqrt{1+\sigma^2}\sqrt{k}$ stated on page 79, if it were deducible from Lemma 2.24, 4., would need $\gamma + \frac{C_1'}{\sqrt{k}} \leq \frac{1}{4}$; again this is fulfilled by $\gamma$ converging to zero and $k$ converging to infinity.

The next case is the bound $\frac{1}{u\sqrt{k}} \leq \alpha'$ on page 79:

$$\begin{aligned}\frac{1}{u\sqrt{k}} &= \frac{\sqrt{\log(n+1)}}{C_{10}\log(\log(n+2))}\frac{\sqrt{\log(\log(n+2))}}{\sqrt{c_6}\sqrt{\log(n+1)}}\\ &= \frac{1}{C_{10}\sqrt{c_6}}\frac{1}{\sqrt{\log(\log(n+2))}}\\ &\leq \frac{1}{C_{10}\sqrt{c_6\log(\log(5))}},\end{aligned}$$

so we may effectively take $\alpha' := \frac{1}{C_{10}\sqrt{c_6\log(\log(5))}}$. This also reveals $\lim_{n\to\infty}\frac{1}{u\sqrt{k}} = 0$, equivalently $\lim_{n\to\infty}(ku^2) = \infty$, and hence $\lim_{n\to\infty}\varepsilon = \lim_{n\to\infty}\alpha''\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big) = 0$, therefore $\varepsilon \leq \frac{1}{2}$ as claimed on page 80 is (eventually) achieved. The satisfying of $\nu \leq 1$ on page 80 again is a simple consequence of $\lim_{n\to\infty}\nu = 0$, more precisely is holds for $n \geq \mathsf{e}^{C_{10}^2} - 1$.

*Comment.* Fresen does not address this one as he has not overtly used it; we do not know how, if at all, he managed to get the thin-shell-estimate with the "correct" radius without running into this term, since like elsewhere he gives no details. ♢

We are now closing in on the sequences $(\delta_n)_{n\geq 1}$ and $(\varepsilon_n)_{n\geq 1}$ as stated in the theorem. $\delta_n$ is the upper bound for the failure-probability in (3.12), thus

$$\begin{aligned}\delta_n &= C_2\mathsf{e}^{-c_2n\delta^2} + C_9\sqrt{k}\exp\big[-c_5k\big(u+\gamma+\mathsf{e}^{-c_4ku^2}\big)^2\big] \leq C_2\mathsf{e}^{-c_2n\delta^2} + C_9\sqrt{k}\,\mathsf{e}^{-c_5ku^2} =\\ &= C_2\mathsf{e}^{-c_2n\frac{\log(n+1)^2}{n}} + C_9\frac{\sqrt{c_6\log(n+1)}}{\sqrt{\log(\log(n+2))}}\,\mathsf{e}^{-\frac{c_5c_6\log(n+1)}{\log(\log(n+2))}\frac{C_{10}^2\log(\log(n+2))^2}{\log(n+1)}} \leq\\ &\leq C_2\mathsf{e}^{-c_2\log(n+1)^2} + \frac{C_9\sqrt{c_6\log(n+1)}}{\sqrt{\log(\log(5))}}\,\mathsf{e}^{-c_5c_6C_{10}^2\log(\log(n+2))} =\\ &= C_2(n+1)^{-c_2\log(n+1)} + \frac{C_9\sqrt{c_6\log(n+1)}}{\sqrt{\log(\log(5))}}\log(n+2)^{-c_5c_6C_{10}^2} \leq\\ &\leq C_2n^{-c_2\log(4)} + \frac{C_9\sqrt{c_6}}{\sqrt{\log(\log(5))}}\log(n+2)^{\frac{1}{2}-c_5c_6C_{10}^2} =\\ &= \Big(C_2n^{-c_2\log(4)}\log(n)^{c_5c_6C_{10}^2-\frac{1}{2}} + \frac{C_9\sqrt{c_6}}{\sqrt{\log(\log(5))}}\Big)\log(n)^{\frac{1}{2}-c_5c_6C_{10}^2} \leq\\ &\leq \Big[C_2\Big(\frac{c_5c_6C_{10}^2-\frac{1}{2}}{\mathsf{e}c_2\log(4)}\Big)^{c_5c_6C_{10}^2-\frac{1}{2}} + \frac{C_9\sqrt{c_6}}{\sqrt{\log(\log(5))}}\Big]\log(n)^{\frac{1}{2}-c_5c_6C_{10}^2},\end{aligned}$$

where we require $\frac{1}{2} - c_5c_6C_{10}^2 < 0$, that is $C_{10} > \frac{1}{\sqrt{2c_5c_6}}$, then $\lim_{n\to\infty}\delta_n = 0$ and the convergence can be accelerated by taking larger values of $C_{10}$.

The error-bound $\varepsilon_n$ for the difference of the distribution-functions in (3.9) is determined as follows,

$$
\begin{aligned}
\frac{\varepsilon_n}{C_9} &= \nu + \sigma + u + \gamma + \mathsf{e}^{-c_4 k u^2} + \mathsf{e}^{-\frac{\nu^2}{2\sigma^2}} \\
&\le \frac{C_{10}}{\sqrt{\log(n+1)}} + \frac{1}{\log(n+1)} + \frac{C_{10}\log(\log(n+2))}{\sqrt{\log(n+1)}} \\
&\quad + 250.4 C_6 n^{-\frac{1}{5}} + \mathsf{e}^{-c_4 c_6 C_{10}^2 \log(\log(n+2))} + \mathsf{e}^{-\frac{C_{10}^2}{2\log(n+1)}\log(n+1)^2} = \\
&= \frac{C_{10}}{\sqrt{\log(n+1)}} + \frac{1}{\log(n+1)} + \frac{C_{10}\log(\log(n+2))}{\sqrt{\log(n+1)}} \\
&\quad + 250.4 C_6 n^{-\frac{1}{5}} + \log(n+2)^{-c_4 c_6 C_{10}^2} + (n+1)^{-\frac{1}{2}C_{10}^2} \\
&= \frac{\log(\log(n+2))}{\sqrt{\log(n+1)}} \Big( \frac{C_{10}}{\log(\log(n+2))} + \frac{1}{\log(\log(n+2))\sqrt{\log(n+1)}} + C_{10} \\
&\quad + \frac{250.4 C_6 n^{-\frac{1}{5}} \sqrt{\log(n+1)}}{\log(\log(n+2))} + \frac{\log(n+2)^{-c_4 c_6 C_{10}^2}\sqrt{\log(n+1)}}{\log(\log(n+2))} \\
&\quad + \frac{(n+1)^{-\frac{1}{2}C_{10}^2}\sqrt{\log(n+1)}}{\log(\log(n+2))} \Big);
\end{aligned}
$$

as usually, $\frac{1}{\log(\log(n+2))} \le \frac{1}{\log(\log(5))}$, $\frac{1}{\sqrt{\log(n+1)}} \le \frac{1}{\sqrt{\log(4)}}$, $n^{-\frac{1}{5}}\sqrt{\log(n+1)} \le \sqrt{\frac{\log(4)}{\log(3)}} \cdot (n^{-\frac{2}{5}}\log(n))^{\frac{1}{2}} \le \sqrt{\frac{\log(4)}{\log(3)}}\sqrt{\frac{5}{2\mathsf{e}}}$, $(n+1)^{-\frac{1}{2}C_{10}^2}\sqrt{\log(n+1)} = ((n+1)^{-C_{10}^2}\log(n+1))^{\frac{1}{2}} \le \frac{1}{C_{10}\sqrt{\mathsf{e}}}$, and for the second-to-last term we additionally demand $\frac{1}{2} - c_4 c_6 C_{10}^2 < 0$, so $C_{10} > \frac{1}{\sqrt{2c_4c_6}}$, thence $\log(n+2)^{-c_4c_6C_{10}^2}\sqrt{\log(n+1)} \le \log(n+2)^{\frac{1}{2}-c_4c_6C_{10}^2} \le \log(5)^{\frac{1}{2}-c_4c_6C_{10}^2}$; put together this reads

$$
\begin{aligned}
\varepsilon_n \le \frac{C_9}{\log(\log(5))} \Big( & C_{10} + \frac{1}{\sqrt{\log(4)}} + C_{10}\log(\log(5)) \\
& + 250.4 C_6 \sqrt{\frac{5\log(4)}{2\mathsf{e}\log(3)}} + \log(5)^{\frac{1}{2}-c_4c_6C_{10}^2} + \frac{1}{C_{10}\sqrt{\mathsf{e}}} \Big) \frac{\log(\log(n+2))}{\sqrt{\log(n+1)}}
\end{aligned}
$$

and from this $\lim_{n\to\infty} \varepsilon_n = 0$ readily follows, and the proof is complete. □

For the convenience of the reader we provide an overview over the constants used in this proof in Table 3.1; note that $A, B$ themselves come from Lemma 2.23 for $n = 2$, therefore $A = \frac{324\mathsf{e}^4}{\pi} \approx 5631$ and $B = \frac{1}{48\log(80\mathsf{e}^4)} \approx \frac{1}{402.34}$.

We also give a numerical example, as far as possible: we start with $C_1 = 4\pi\sqrt{2} \approx 17.8$, then $C_2 = 4\zeta(2) = \frac{2\pi^2}{3} \approx 6.58$ and $c_1 < \frac{2\log(2)}{\log(6)}$, for instance $c_1 = \frac{\log(2)}{\log(6)} \approx 0.387$, herewith $c_2 = 1$; $c_3 \approx 2.47$ and $c_4 \approx 0.1534$ are fixed in any case; furthermore $c_6 \le \min\{0.2257, 0.007\,85\} = 0.007\,85$. $\alpha \approx 1.3024$ is given, from that follow $C_3 \approx 30.422$, $C_4 \approx 48.00$, $C_5 \approx 111.05$ and $C_6 \approx 10.155$. From $A$ and $B$ result $c_5 \approx 6.11 \cdot 10^{-21}$

| constant | remarks |
|---|---|
| $\alpha \geq \frac{\log(\log(\frac{3}{2}))}{\log(\frac{1}{2})}$ | introduced on p. 72, used for $C_3$ |
| $\alpha' = \frac{1}{C_{10}\sqrt{c_6 \log(\log(5))}}$ | introduced on p. 79, formula on p. 85 |
| $\alpha'' = \max\{5 + (\alpha' + \sqrt{2})C_7, 8 + \frac{C_7}{\sqrt{2}}\}$ | introduced on p. 79, used for $C_9$ |
| $c_1 < \frac{\log(2)}{\log(6)(4\pi)^2} C_1^2$ | introduced on p. 70, condition on p. 71 |
| $c_2 = \frac{C_1^2}{(4\pi)^2} - \frac{c_1 \log(6)}{\log(2)}$ | introduced on p. 71 |
| $c_3 = \frac{1}{2} + \frac{7}{2}\log(2) - \frac{1}{4}\log(2\pi)$ | introduced on p. 77 |
| $c_4 = \frac{\log(\frac{\mathsf{e}}{2})}{2}$ | introduced on p. 77, from Lemma 2.24 |
| $c_5 = \frac{1}{4}\left(\frac{400\pi A}{179B^2} + 1\right)^{-2}$ | introduced on p. 80, from Theorem 2.28 |
| $c_6 \leq \min\{c_1 \frac{\log(9)\log(\log(5))}{\log(6)}, \frac{1}{127.4}\}$ | introduced in (3.13), conditions on pp. 82, 83 |
| $C_1 > 4\pi$ | introduced on p. 70, condition on p. 71 |
| $C_2 = 4\zeta\left(\frac{C_1^2}{(4\pi)^2}\right)$ | introduced on p. 71 |
| $C_3 = \max\{8\pi, \frac{3C_1\sqrt{\alpha}}{2}\}$ | introduced on p. 72 |
| $C_4 = C_3 \frac{1+\sqrt{\log(6)}}{\sqrt{\log(9)}}$ | introduced on p. 73 |
| $C_5 = C_4 \frac{\sqrt{2\mathsf{e}}+2^{5/4}\pi^{1/4}+2^{3/4}\pi^{-1/4}}{\sqrt{\mathsf{e}\pi}}$ | introduced on p. 75 |
| $C_6 = \sqrt{\frac{2C_5\sqrt{2}}{(\mathsf{e}-1)\sqrt{\pi}}}$ | introduced on p. 77 |
| $C_7 \in \mathbb{R}_{>0}$ | introduced on p. 79, from Lemma 2.24, 4. |
| $C_8 = \max\{\mathsf{e}, \frac{179B^2}{200\pi A}\}$ | introduced on p. 80, from Theorem 2.28 |
| $C_9 = 1 + \frac{3\alpha''}{2}$ | introduced on p. 80 |
| $C_{10} > \frac{1}{\sqrt{2\min\{c_4,c_5\}c_6}}$ | introduced in (3.13), conditions on pp. 85, 86 |

Table 3.1: Overview of constants in proof of Theorem 3.1.

and $C_8 = \max\{\mathsf{e}, 3.13 \cdot 10^{-10}\} = \mathsf{e} \approx 2.72$; we continue with $C_{10} > 1.0214 \cdot 10^{11}$, e.g. $C_{10} = 1.1 \cdot 10^{11}$, this produces $\alpha' = 1.49 \cdot 10^{-10}$. Clearly the constants' values span several orders of magnitude; as an extreme case we note the bound $n \geq 10^{4.02 \cdot 10^{44}}$ for $u \leq 1$ (this is still a good deal smaller than Skewes's number).

## Note on the mode of convergence

Theorem 3.1 states that most onedimensional marginals are close to the standard normal distribution in the sense of the uniform distance between the distribution-functions (especially among statisticians also called *Kolmogorov-(Smirnov-)distance*); usually this is stronger than weak convergence of measures, because the latter only demands pointwise convergence on all continuity-points of the limit-CDF (that is all of $\mathbb{R}$ in the case of the normal distribution). As Fresen points out, [20, Theorem 1.1] is formulated in terms of the total-variation-distance which in the case of log-concave distributions is immaterial.

The total-variation-distance $d_{\mathrm{TV}}$ of the probability-measures $\mu$ and $\nu$ on $\mathbb{R}^n$ is defined as

$$d_{\mathrm{TV}}(\mu,\nu) := 2\sup\bigl\{|\mu(A)-\nu(A)|\big|A\in\mathcal{B}(\mathbb{R}^n)\bigr\} = \Bigl\|\frac{d\mu}{d\lambda}-\frac{d\nu}{d\lambda}\Bigr\|_{L^1(\mathbb{R}^n,\lambda)},$$

where $\lambda$ is any $\sigma$-finite measure with $\mu \ll \lambda$ and $\nu \ll \lambda$ (a *dominating measure*, for instance $\lambda = \frac{1}{2}(\mu+\nu)$)

The relevant result is the following, from [5, Theorem 3.3, Remark 3.4]:

**Theorem 3.2.** *There exists an increasing function $\beta_1\colon (0,\infty)\to(0,2]$ with*

$$\beta_1(t) = O\bigl(t(-\log(t))^{\frac{1}{2}}\bigr) \qquad \bigl(t\in\bigl(0,(2\pi)^{-\frac{1}{2}}\bigr]\bigr)$$

*with the following property: for any log-concave density $f\colon\mathbb{R}\to\mathbb{R}_{\geq 0}$ with CDF $F$ and for any $\sigma\in\mathbb{R}_{>0}$ such that $\|f\|_{L^\infty(\mathbb{R})}\leq\frac{1}{\sigma}$ there holds*

$$\|F-\Phi_\sigma\|_{L^\infty(\mathbb{R})} \leq \|f-\varphi_\sigma\|_{L^1(\mathbb{R})} \leq \beta_1(\sigma\|f-\varphi_\sigma\|_{L^\infty(\mathbb{R})}) \leq \beta_1\bigl(\sqrt{5}\|F-\Phi_\sigma\|^{\frac{1}{2}}_{L^\infty(\mathbb{R})}\bigr),$$

*where $\varphi_\sigma$ and $\Phi_\sigma$ are the PDF and CDF respectively of the centred normal distribution with variance $\sigma^2$.*

*If $f$ is centred and $\sigma^2=\int_{\mathbb{R}} t^2 f(t)\,dt$, then $\|f\|_{L^\infty(\mathbb{R})}\leq\frac{1}{\sigma}$ is always satisfied.*
*(W/o proof.)*

This is applicable to Theorem 3.1 because $X$ is isotropic and log-concave, therefore $\langle\theta,X\rangle$ is so for any $\theta\in S^{n-1}$, hence it has an isotropic log-concave density $f_\theta$, so $\sigma=1$; and with dominating measure $\lambda=v_1$ we have $d_{\mathrm{TV}}(\mathbb{P}_{\langle\theta,X\rangle},\gamma_1)=\|f_\theta-\varphi\|_{L^1(\mathbb{R})}$.

For the sake of completeness we mention the more general result from Meckes and Meckes [32, Corollary 8]; here, as already used above, $\gamma_n$ denotes the standard normal distribution on $\mathbb{R}^n$.

**Theorem 3.3.** *A sequence of log-concave measures on $\mathbb{R}^n$ converges weakly to $\gamma_n$ if and only if it converges in total variation to $\gamma_n$.*
*(W/o proof.)*

# 4 Concluding remarks

Here we recapitulate our observations regarding Fresen's proof of the CLT for convex bodies. In particular, we want to assess whether his claim, that the proof be accessible to anyone, can be justified.

For short, our answer is that it depends on how much of Fresen's article one takes into account – either only the proof as contained within Section 2 without the two notes at the end of that section, or including those notes, or even Section 3 – and how much mathematical background can be taken for granted.

In the strictest sense of 'accessible to anyone', we think that Fresen's claim cannot be upheld: the theorem of Lévy for Lipschitz-functions on the sphere (Theorem 2.27) might not be known to every mathematician, and not everybody might be comfortable handling the Haar-measure on $\mathrm{O}_n(\mathbb{R})$ and the uniform measures on $S^{n-1}$ and on $G_{n,k}$ and their interrelations. Perhaps these measure-theoretic results are the main hindrances for a general audience.

If we assume familiarity with the results just named and if we exclude the aforementioned two notes, then we deem Fresen's proof indeed elementary. His style in this article is extremely compressed, and in one or two cases the order of the arguments may be unsatisfying, but most of the gaps can be filled in with more or less effort and relativly simple instruments. We do not claim that the right technique for a particular intermediary result is immediately clear, and writing out the details may be tedious at times. As an example for such elementary, but lengthy, calculations we refer to the sub-proof that all parameters have the right asymptotic behaviour on pp. 82ff. For the sake of a clearer arrangement of the main proof, we have removed a few of the intermediary results and reformulated them as independent lemmas or theorems, if possible. This especially concerns Step 2, most of which we have relegated to Lemma 2.24.

Now we include the two notes at the end of Fresen's Section 2 into our considerations. The second one, "Lower bound on $P_E f(0)$", is less problematic. It makes use of Grünbaum's lemma (Lemma 2.21), the proof of which we deem nontrivial but itself requiring no further deep results. The first note, "Radius of the thin shell", has one crucial fault, which we already have pointed out on p. 78f.: Fresen invests estimates of expectation and variance which to us seem unattainable.

Fresen's Section 3 is declared as an appendix and thus not part of the main proof; nevertheless, it is part of his article and therefore we look into it too. As already noted on p. 67, its greatest merit is the simpler (in terms of applied techniques) proof of Lipschitz-continuity. However, for us this is tainted by downplaying, as we might call it, several important ingredients: one is his "elementary fact" of the exponential bound on an isotropic log-concave function which, as far as we have found out, is not at all elementary, at least as concerns its proof. Another is the use of important properties of the normal distribution and the uniform distribution on the sphere, which we have summarized in Lemmas 2.25 and 2.26. We do not say that using these results is improper, but rather their seeming belittlement.

In total, we agree that over long stretches Fresen's proof is elementary in its techniques, but it needs more special results to completely retrace it than Fresen explicitly states. In spite of its defects we regard his article as a decent contribution to the theory.

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