Leonardo Alese, Dott. Mag.

Problems on Closed Curves

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Univ.-Prof. Johannes Wallner, Dr. techn.

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AFFIDAVIT

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Introduction

Closed curves are familiar items of everyday human life as we make use of their topological and geometrical properties to wear necklaces, bind objects together, and bound territories. This clearly makes closed curves a very natural mathematical class of objects and possibly the most elementary one to require an analysis able to handle the global character of their defining property. In the simplest scenario, this is the condition for the curve to start and end at the same point. Formally, for real numbers $a < b$, a closed curve γ is a continuous function from the interval $[a, b]$ to $\mathbb{R}^n$ such that $\gamma(a) = \gamma(b)$. Especially in dimension 2, such a simple definition on one hand is so insidiously general that it requires some efforts even when it comes to prove "apparent" properties like the Jordan curve theorem [28], guaranteeing that a simple closed curve splits the plane into two connected components. On the other one it is rich enough to allow surprising conjectures such as the inscribed square problem, asking whether every Jordan curve admits an inscribed square, which has indeed been confirmed only under stronger hypotheses [44]. Often prolific branches of literature have been motivated by extremely practical problems, as the legendary task of queen Dido [20], whose problem of maximizing the area enclosed by a curve of given length is solved by the isoperimetric inequality, which in itself is the starting point of a rich theory [35].

If we require the curve to be C^h, with $\|\gamma'\| \neq 0$ and matching derivatives at the two coinciding ends, then global differential geometry of regular closed curves is just born. The 4-vertex theorem, which bounds the number of local maxima/minima of the curvature function of a simple planar curve, is an iconic result here. It is one of the earliest contributions to this field and has a fascinating story, which starts with its proof at the beginning of the 20th century [33], [25] and culminates in the the establishment of a full converse in 2005 [4], with a truly simple proof obtained only in 1985 [34] (see [9] for an overview). Also in this case, several analogous problems have been studied and generalizations to space curves keep coming with an increasing degree of refinement [16], [15]. One cannot end a paragraph about global differential geometry of curves without mentioning Milnor's results about the total curvature of knots [30] and their predecessor, Fenchel's theorem [11], which in its most basic formulation states that the average curvature of a closed space curve of length L is at least $2\pi/L$ and that the equality holds only for convex planar curves.

A leitmotiv can be identified in the unexhaustive inventory above: all the listed results rely on simple definitions and can be stated in a few lines, but require very different proof techniques. In [12], Fenchel himself argues on the nature of research outcomes involving the differential geometry of closed space curves. Although we deal also with planar curves, the content of the present work is well represented by his comment:

> *The results are often comparatively elementary and seem to be isolated. On the other hand, the intuitive character of the statements and the lack of a general method of approach make the field attractive.*

The first two chapters of this thesis are characterized by results whose nature clashes with intuition, making it sort of hard to believe the statements before reading the proofs, an aspect which is especially surprising given the simplicity of the objects involved. The first chapter is about the possibility of closing a smooth planar curve by splitting and rearranging its arcs, while the second one analyzes the problem of linear interpolation of curvature functions with the constraint of mantaining closedness. The last chapter is set in three-dimensional Euclidean space and looks at curves as a tool to characterize curved-creased origami. Even if we do not discuss only closed constructions, the main motivation of the chapter is the longstanding problem of existence of the so-called folded annulus with multiple creases [5]. Besides, developable surfaces themselves can be regarded as unidimensional objects if one looks at their dual representation, making the whole contribution a good fit for the present dissertation.

Contributions of this thesis

In this work we study several properties related to the differential geometry of closed curves in Euclidean space of dimension 2 and 3. The sequence of chapters is roughly organized in increasing order of technical complexity and, by chance, of smoothness degree of the curves involved (Chapter i deals with curves of regularity C^i).

(1) L. Alese. Closing curves by rearranging arcs. Submitted: May 2020, arXiv:2002.05422.

(2) L. Alese. Affine subspaces of curvature functions from closed planar curves. Submitted: May 2020, arXiv:2006.09678.

(3) L. Alese. Propagation of curved folding: The folded annulus with multiple creases exists. Submitted: August 2020, arXiv:2008.02660.

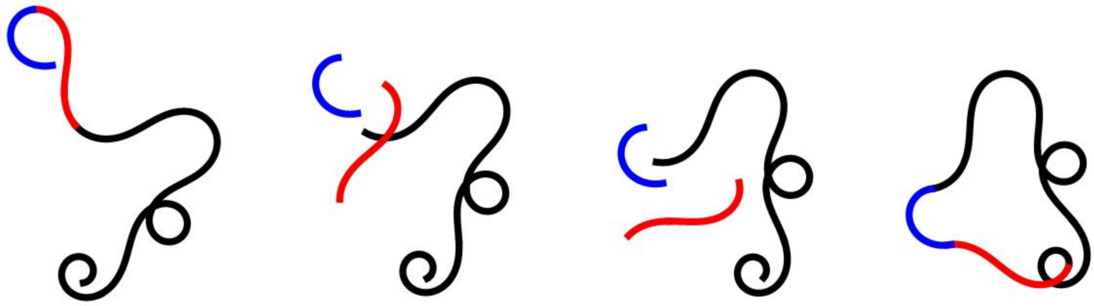

Figure 0.1: A smooth planar curve with total turning number a non-zero integer multiple of 2π can always be split into 3 arcs rearrangeable to a closed smooth curve.

The three chapters which constitute the present dissertation essentially coincide with the corresponding journal articles itemized in the list above. Thus, throughout this work the terms "chapter" and "paper" are to be considered synonyms. To avoid redundancy, the respective bibliographies have been merged into the one at the end of the thesis. We provide a more detailed overview of the content of each chapter in the following subsections.

Chapter 1: Closing curves by rearranging arcs

The core result of the chapter is that a C^1-smooth planar curve, whose tangent fully turns a non-zero number of times, can always be split into 3 arcs rearrangeable to a closed smooth curve (Fig. 0.1). Such a result, which we called the 2-cut theorem is proven with elementary topological methods reminescent of the topological proof of the fundamental theorem of algebra.

Moving from the 2-cut theorem, generalizations are considered, both by weakening the requirements on the total turning angle, and by looking at curves split into several arcs of non-zero length with the additional constraint of using a given permutation to achieve the objective of closing the curves. In the latter case, we provide a full classification, showing that for a smooth planar curve with total turning number $2k\pi$, $k \in \mathbb{Z}$ and $k \neq 0$, there are always cuts allowing such a rearrangement unless the chosen permutation is a cyclic shift. The proof technique is heavily based on the one we use for the simpler 2-cut theorem and relies on its robustness to small perturbations; we mention a few examples from the literature where similar approaches have been succesful, e.g. the converse of the 4-vertex theorem [9].

Finally comments are made about how the tools used in the paper could possibly be applied to address the problem of rearranging arcs of framed curves in higher dimensions.

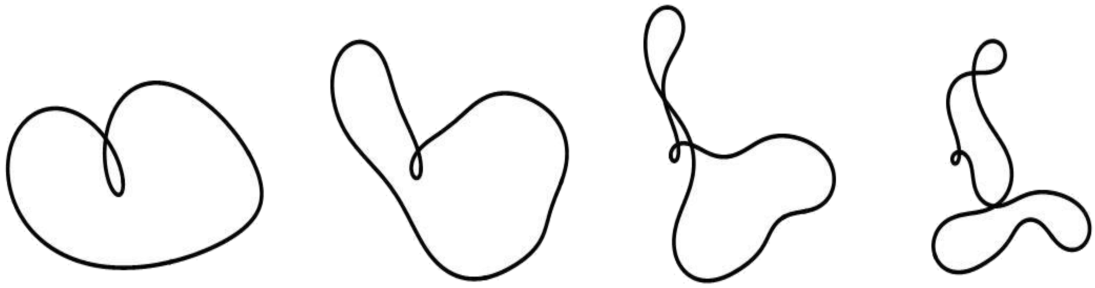

Figure 0.2: A curve is deformed by adding real multiples of a given function to its curvature. If such a function is chosen properly, then all the curves of the family are closed.

Chapter 2: Affine subspaces of curvature functions from closed planar curves

In the second chapter, given a pair of real functions k, f on an interval $[a, b]$ we study the conditions that they must satisfy for $k + \lambda f$ to be the curvature, as a function of arc-length, of a closed planar curve for all real λ (Fig. 0.2). We observe that the closedness constraint involves a function analytic in λ and thus we are are able to prove a series of sufficient and necessary conditions, including $\int_a^b \mathrm{e}^{i\theta} \cdot \phi^n = 0$, $\forall n \in \mathbb{N}$, where θ and ϕ are the turning angle functions associated respectively to k and f. Possibly the most surprising consequence, obtained by way of the Weierstrass approximation theorem, is that the curves associated to the curvature functions on the affine line $k + \lambda f$ are all closed if and only if the curves associated to $k + \lambda f g'(\phi)$ are all closed, where g is any differentiable function with a bounded and integrable derivative. This entails that a single direction suitable for the interpolation implies the existence of an affine space of infinite dimension with the same property. Leveraging this characterization, we also show that the respective behaviour of k and f on the boundary is strongly related, translating into analagous features of the associated family of curves, whose elements turn out to be more regular than originally required.

In the second part of the paper, the same questions we ask for smooth curves are asked again for closed polylines, drawing a strong analogy between the set of conditions obtained in the two cases.

Finally, using the results from previous sections, we provide some insights on the complexity of detecting the closedness of a curve by looking at its curvature. We formally show that it is impossible to develop a sufficient analogue of the 4-vertex theorem, meaning that it is not possible to come up with a procedure that tells whether the curve associated to a curvature function k is closed or not by accessing finitely many evaluations

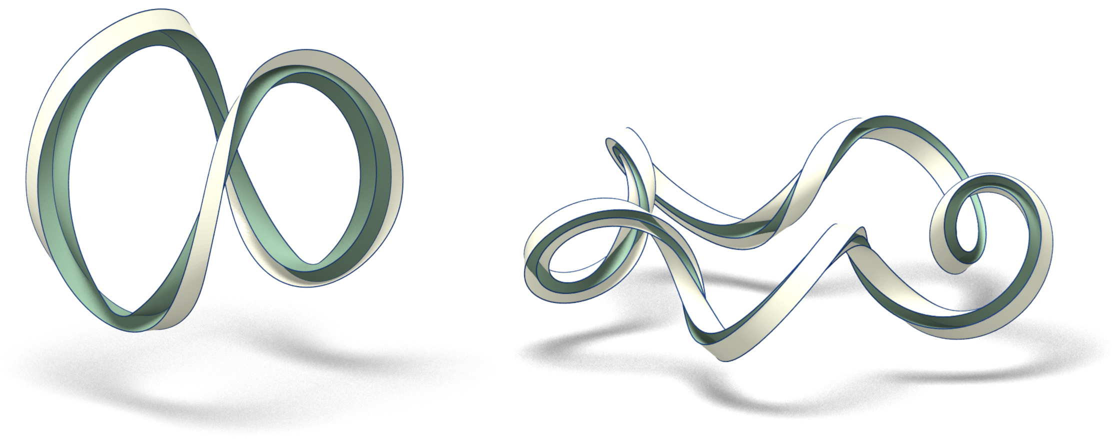

Figure 0.3: Regular developables meeting at curved creases and providing instances of folded annuli with multiple folds.

and/or level sets of k, its derivatives and antiderivatives.

Chapter 3: Propagation of curved folding: The folded annulus with multiple creases exists

In the final chapter we work in three-dimensional Euclidean space and consider curved folds, i.e. continuous surfaces locally isometric to a planar domain, which are piecewise differentiable and exhibit creases along curves. The main goal of the paper is to achieve a better understanding of how creases interact when multiple foldlines are prescribed. We pay special attention to those patterns that present non-trivial combinatorics, such as the annulus folded along concentric circles [5]. Our strategy is to move the attention from the developables involved (the differentiable parts) to the curves bounding them (the ridges) and their relation with the singular points of the surface (the regression curve). This allows a more convenient approach to providing explicit constructions and analyzing their regularity.

With the intent of making the chapter self-contained, we recall how developables admit a ruled parametrization, pointing out how its regularity is the best possible for those surfaces involved in a curved fold. Rephrasing [13], local curved folding is characterized in terms of the geodesic curvature, normal curvature and relative torsion of the ridge with respect to the developables on its two sides, whose dependency can be conveniently expressed by these descriptors with formulae enjoying a nice degree of symmetry.

We describe then how a fold propagates when an additional foldline is prescribed, proving the relation between the normal curvature and relative torsion at the endpoints of a ruling on consecutive ridges. Although such relations are not simple, they describe the phenomenon in its full generality. On the one hand this can be employed to provide explicit parametrizations of annuli folded along multiple concentric circles (Fig. 0.3). On the other hand this can be used to obtain existence and non-existence results of broad applicability, although of somewhat technical nature. More specifically, on the existence side we show that any fold can be propagated to an arbitrary fixed number of additional foldlines, provided that they are close enough to each other, and on the non-existence one that any sequence of folds can be artificially engineered to look armless till a certain ridge and then turn singular in an arbitrarily abrupt manner, pointing out that a possible argument about the foldability of a pattern with infinitely many prescribed foldlines should involve a control mechanism on the derivative of all orders.

1 Closing curves by rearranging arcs

Abstract

In this paper we show how, under surprisingly weak assumptions, one can split a planar curve into three arcs and rearrange them (matching tangent directions) to obtain a closed curve. We also generalize this construction to curves split into k arcs and comment what can be achieved by rearranging arcs for a curve in higher dimensions. Proofs involve only tools from elementary topology, and the paper is mostly self-contained.

1.1 Introduction

In this chapter we study the problem of splitting a given planar curve into arcs and rearrange them (matching tangent directions) in order to make the curve closed. The interest of our result lies in the counterintuitive nature of the statement and in the simplicity of the proof. Using an argument very similar to that involved in the topological proof of the fundamental theorem of algebra, we will show that a one time differentiable curve with total turning angle a nonzero integer multiple of 2π can always be split into three arcs that are rearrangeable to a closed curve.

The operation of joining arcs of curves matching frames at junction points has been considered in various settings, mainly with the goal of constructing closed curves with certain properties. In [2] multiple copies of the same planar curve are joined one after another; if the arc-length integral of the curvature is a rational non-integer multiple of 2π then gluing finitely many copies of the curve will eventually close up the construction to the starting point. In [26] and [31] arcs of helices resp. so-called Salkowski curves are joined to obtain a family of closed space curves of constant curvature which are curvature-continuous (C^2) resp. C^3. In [29] all types of knots and links are realized as C^2 curves of constant curvature by joining arcs of helices.

As for an outline of the contents, in §1.2 we set the notation and recall an elementary topology lemma. In §1.3 we prove that, under very natural assumptions, a C^1 planar curve can be split into 3 arcs that can be rearranged to obtain a closed C^1 curve. In §1.4 this construction is extended to permutations of any number of arcs. In §1.5 generalizations to higher dimensions are discussed and some possible directions for future work are pointed out.

1.2 Notation

We start by settling the language about some natural topological objects. For any $a, b \in \mathbb{R}$ with $a \leq b$, we call *path* a continuous function $w(t)$ over $[a, b]$ to $\mathbb{R}^2$ and *loop* a path l such that $l(a) = l(b)$. A *contraction* of a loop $l : [a, b] \to \mathbb{R}^2$ to a point Q is a continuous family $H(h, t)$ defined for $h, t \in [a, b]$ with $h \leq t$ such that $H(a, t) = l(t)$, $H(h, h) = H(h, b)$ for all $h \in [a, b]$ and $H(b, b) = Q$: in particular, each path $t \mapsto H(h, t)$ is a loop defined over $[h, b]$. A loop l is said to be *contractible* in a subset A of $\mathbb{R}^2$ if there exists a contraction of l whose image is contained in A.

A path $w(t)$ over $[a, b]$ that does not contain $P \in \mathbb{R}^2$ can be expressed as $w(t) = P + \rho(t)(\cos\phi(t), \sin\phi(t))$ where $(\rho(t), \phi(t))$ are its *polar coordinates* with respect to P. Since w is continuous, we will always assume $\rho(t)$ and $\phi(t)$ are continuous as well. We call *winding number* of w with respect to P the value $\phi(b) - \phi(a)$. The next is a result from elementary homotopy theory and the proof given here is the usual one contained in topology textbooks; for more on homotopy theory see for example the first chapter of [19].

Lemma 1.2.1. *Let $l(t)$ be a loop on $[a, b]$ whose image does not contain $P \in \mathbb{R}^2$. Let $(\rho(t), \phi(t))$ be polar coordinates with respect to P and $\phi(b) - \phi(a) = 2k\pi$, $k \in \mathbb{Z}$. If $k \neq 0$, i.e. l has winding number with respect to P different from 0, then l is not contractible in $\mathbb{R}^2 \setminus \{P\}$.*

Proof. Assume for a contradiction that a contraction $H(h, t)$ exists. For each loop $H(h, t)$ we consider polar coordinates $(\rho_h(t), \phi_h(t))$ and the integer k_h that satisfies $\phi_h(b) = \phi_h(a) + 2k_h\pi$. Since a contraction is a continuous function, $\phi_h(t)$ can be chosen to be continuous also in h and such that $\phi_a(t) = \phi(t)$ for all t (we continuously extend the polar coordinates we already have for l). This entails that k_h is also continuous and therefore constant as a function whose image is contained in $\mathbb{Z}$. This implies $k_h = k \neq 0$ for all h. But this is a contradiction since the image of $H(b, t)$ is a single point and therefore $k_b = 0$. $\square$

We formalize now what is meant by *joining arcs of curves matching frames at junction points*. To do that we introduce the concept of *framed* curve $(\gamma, \mathcal{F})(s)$ of $\mathbb{R}^n$, a pair consisting of a C^1 curve $\gamma(s)$ of $\mathbb{R}^n$ with constant speed c, namely a differentiable function from some interval $[a, b]$ to $\mathbb{R}^n$ such that $\|\gamma'\| \equiv c > 0$, and a positive orthonormal basis

$\mathcal{F}(s) = \{f_1(s), f_2(s), ..., f_n(s)\}$ of $\mathbb{R}^n$, continuous in s and such that $f_1(s) = \gamma'(s)/c$. An example of a frame for $n = 3$ and $\gamma \in C^2$ with everywhere nonzero curvature is the Frenet-Serret frame, obtained by taking f_2 as $\gamma''/\|\gamma''\|$ and f_3 as the vector product of f_1 and f_2.

If $(\gamma_1, \mathcal{F}_1)(s_1)$ and $(\gamma_2, \mathcal{F}_2)(s_2)$ are two framed curves parametrized over $[a_1, b_1]$ and $[a_2, b_2]$ with the same constant speed c, denoting with T_{a_2,b_1} the rigid motion of $\mathbb{R}^n$ that shifts the point $\gamma_2(a_2)$ to $\gamma_1(b_1)$ and rotates the frame $\mathcal{F}_2(a_2)$ to $\mathcal{F}_1(b_1)$, we define the *concatenation* of the two curves, for $s \in [0, b_1 - a_1 + b_2 - a_2]$, as

$$\gamma_1 * \gamma_2(s) := \begin{cases} \gamma_1(s + a_1), & s \leq b_1 - a_1 \\ T_{a_2,b_1}\gamma_2(\bar{s}(s)), & s > b_1 - a_1, \end{cases}$$

where $\bar{s}(s) = s - (b_1 - a_1) + a_2$ is the reparametrization achieving $\bar{s}(b_1 - a_1) = a_2$. Operation $*$ rigidly glues γ_2 to the end point of γ_1 by matching frames. It is easy to see that this operation is associative.

1.3 Two-cut theorem

In the following section let $\gamma(s)$ be a C^1 planar curve with constant speed c, parametrized over $[0, 1]$ and framed with $\mathcal{F}(s) = \{f_1(s), f_2(s)\}$ where $f_2(s)$ is obtained by rotating $f_1(s) = \gamma'(s)/c$ counter-clockwise by $\frac{\pi}{2}$. For convenience we assume without loss of generality $\gamma(0) = (0, 0)$ and $f_1(0), f_2(0)$ aligned with the axes of the coordinate system. For any choice of *cuts* c_1, c_2 with $0 \leq c_1 \leq c_2 \leq 1$, we split the curve into three arcs γ_1, γ_2 and γ_3 respectively parametrized over $[0, c_1]$,$[c_1, c_2]$ and $[c_2, 1]$. We define the *rearranged* curve $r_{(c_1,c_2)} : [0, 1] \to \mathbb{R}^2$ as

$$r_{(c_1,c_2)} := \gamma_1 * \gamma_3 * \gamma_2.$$

Figure 1.1 visualizes this construction for different cuts c_1, c_2. Intuitively this is the curve obtained by swapping the middle arc between parameters c_1 and c_2 and the tail of the curve between c_2 and 1. The rearrangement is well defined also if one or more arcs γ_i degenerate to a point since they still inherit from γ the information of a tangent direction.

In Theorem 1.3.1 and Lemma 1.3.2 we will continuously move the two cuts, while tracking the end point of the rearranged curve, defining this way a family of loops whose properties with respect to contractibility imply the existence of cuts such that $\gamma_1 * \gamma_3 * \gamma_2$ is a closed curve. Two main observations will be needed to follow such a construction and it is worth stressing them before moving to the proofs.

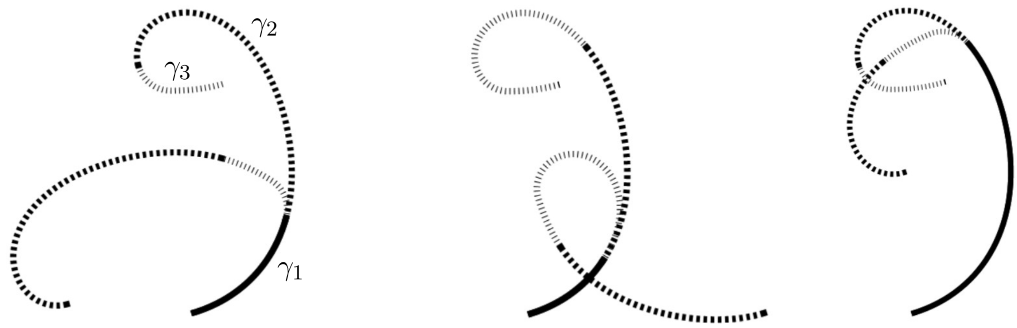

Figure 1.1: Rearrangements of a curve for different cuts. Arcs γ_2 and γ_3 are swapped and tangent directions matched.

- For any $c_1 \in [0,1]$ it holds $r_{(c_1,c_1)}(1) = r_{(c_1,1)}(1) = \gamma(1)$, i.e. $c_2 \mapsto r_{(c_1,c_2)}(1)$ is a loop defined over $[c_1, 1]$.
- If $c_1 = 0$, the rearrangement just swaps the only two arcs in which the curve has been split. Under the hypothesis of Theorem 1.3.1, $\|r_{(0,c_2)}(1) - r_{(0,c_2)}(0)\|$ will not depend on the choice of c_2.

We consider a function $\theta(s)$ such that $\gamma'(s) = c\,(\cos\theta(s), \sin\theta(s))$ and call it *turning angle* function for γ. We also call $\theta(1) - \theta(0)$ the *total turning angle* of γ. Besides, we denote with R_θ the counter-clockwise rotation of angle θ and center in the origin.

Theorem 1.3.1 (Two-cut theorem). *Let $\gamma(s)$ be a C^1 constant speed planar curve over $[0,1]$ and $\theta(s)$ a turning angle function for γ. If $\theta(1) - \theta(0) = m2\pi$ with $0 \neq m \in \mathbb{Z}$ then there exist cuts c_1, c_2 such that the rearranged curve $r_{(c_1,c_2)}$ is a closed C^1 curve.*

Proof. Let us consider the loop $e(t) := r_{(0,t)}(1)$. Explicitly,

$$\begin{aligned} e(t) &= R_{\theta(1)-\theta(t)}(\gamma(t) - \gamma(0)) + R_{\theta(0)-\theta(t)}(\gamma(1) - \gamma(t)) + \gamma(0) \\ &= R_{\theta(1)-\theta(t)}(\gamma(t)) + R_{-\theta(t)}(\gamma(1) - \gamma(t)) \\ &= R_{-\theta(t)}\big(R_{\theta(1)}(\gamma(t)) - \gamma(t) + \gamma(1)\big) = R_{-\theta(t)}\big(\gamma(1)\big), \end{aligned}$$

where we have used the equaliy $R_{\theta(1)} = R_{\theta(0)}$, which follows directly from the hypothesis on the total turning angle, and the assumptions $\gamma(0) = (0,0)$, $\gamma'(0) = (c,0)$. This equation provides polar coordinates $(\|\gamma(1)\|, -\theta(t))$ for $e(t)$ whose image turns out to be a circle of radius $\|\gamma(1)\|$ centered in the origin. By hypothesis we have $\theta(1) - \theta(0) = m2\pi$ with $m \neq 0$ and therefore Lemma 1.2.1 guarantees e is not contractible in $\mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{R}^2 \setminus \{\gamma(0)\}$. Varying $h \in [0,1]$ we obtain a continuous family of loops $r_{(h,t)}(1)$, $t \in [h,1]$, which is in fact a contraction of $e(t)$ to $\gamma(1)$, an operation not possible in $\mathbb{R}^2 \setminus \{(0,0)\}$. Hence it

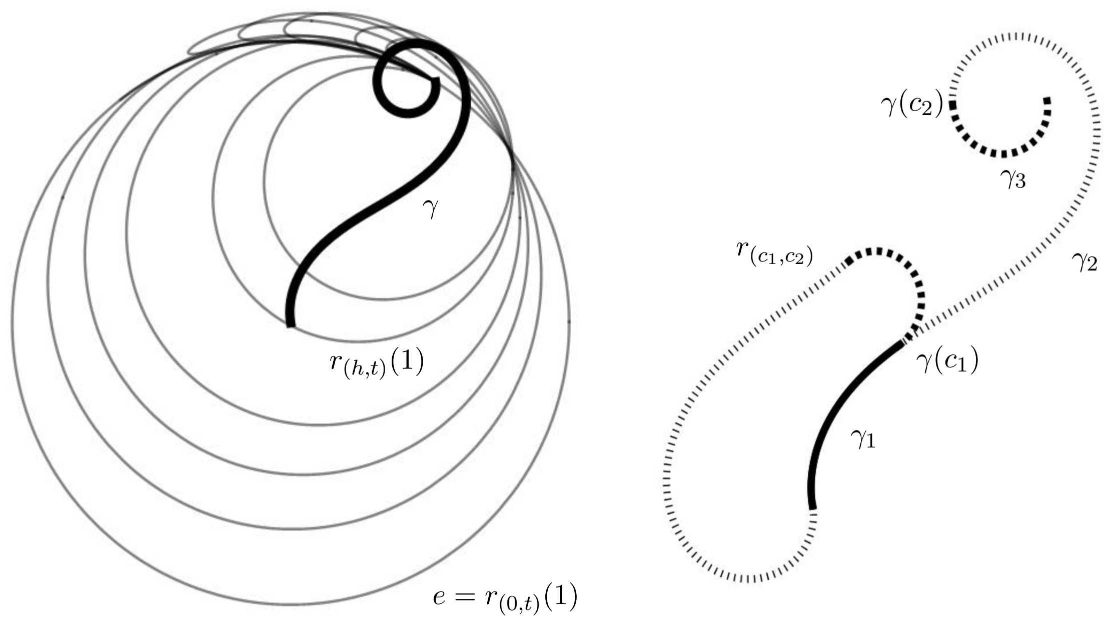

Figure 1.2: For any choice of the first cut h, tracking in t the endpoint of the rearranged curve $r_{(h,t)}$ provides a loop. If the total turning angle of γ is a nonzero multiple of 2π loops in this family start as a circle and contract to $\gamma(1)$, therefore passing through the origin and guaranteeing the rearrangeability to a closed curve.

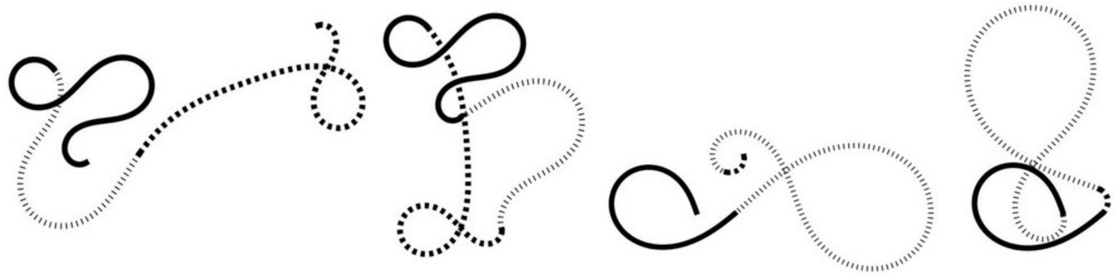

Figure 1.3: More closed rearrangements for curves whose total turning angle is a nonzero integer multiple of 2π.

must be a contraction in $\mathbb{R}^2$ whose image contains $(0, 0)$, which means there exist (h, t) such that $r_{(h,t)}(1) = (0, 0)$, starting point of the curve. Since the total turning angle did not change and it is still an integer multiple of 2π, tangents at the beginning and at the end of the curve match. $\square$

Examples of rearrangements are given in Figure 1.3. As mentioned in the introduction, the proof of Theorem 1.3.1 is similar to the topological proof of the fundamental theorem of algebra. Given a polynomial $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ with $n > 0$, for $\rho \in \mathbb{R}$ large enough the loop $p(\rho e^{it})$, $t \in [0, 2\pi]$ winds around the origin n times. If now ρ continuously decreases to 0 such a loop contracts to a_0, which entails the existence of ρ, t such that ρe^{it} is a root of p. A detailed proof can be found in [19].

We conclude the section with a lemma, which, using the same techinques of the proof of Theorem 1.3.1, generalizes the previous result at the price of a (much) less expressive hypothesis.

Lemma 1.3.2. *Let $\gamma(s)$ be a C^1 constant speed planar curve over $[0,1]$ and $\theta(s)$ a turning angle function for γ. If $|\theta(1) - \theta(0)| \geq 2\pi$ and*

$$\|\gamma(1)\| \geq \sqrt{2\big(1 - \cos\theta(1)\big)} \max_{s\in[0,1]} \|\gamma(s)\|,$$

then there exist cuts c_1, c_2 such that $r_{(c_1,c_2)}$ is closed, i.e., $r_{(c_1,c_2)}(0) = r_{(c_1,c_2)}(1)$ (not necessarily smooth at the end point).

Proof. Again we look at $e(t) = R_{-\theta(t)}\big(R_{\theta(1)}(\gamma(t)) - \gamma(t) + \gamma(1)\big)$. After computing $\|R_{\theta(1)}(\gamma(t)) - \gamma(t)\| = \sqrt{2(1 - \cos\theta(1))}\|\gamma(t)\|$ our hypothesis entails that $R_{\theta(1)}(\gamma(t)) - \gamma(t) + \gamma(1)$ is contained in a ball centered in $\gamma(1)$ which does not contain the origin. If $(\rho(t), \phi(t))$ are polar coordinates for $e(t)$ with respect to the origin this implies $\phi(1) - \phi(0) \in (\theta(0) - \theta(1) - \frac{\pi}{2}, \theta(0) - \theta(1) + \frac{\pi}{2})$. Since e is a loop and $|\theta(1) - \theta(0)| \geq 2\pi$ we get $\phi(1) - \phi(0) = 2m_e\pi$ with $0 \neq m_e \in \mathbb{Z}$ and Lemma 1.2.1 guarantees again that e is not contractible in $\mathbb{R} \setminus \{\gamma(0)\}$. Varying $h \in [0,1]$ we define again a continuous family of loops $r_{(h,t)}(1)$, $t \in [h, 1]$, which contracts to $\gamma(1)$, and we conclude as in the proof of Theorem 1.3.1. □

Note that the argument of Theorem 1.3.1 and Lemma 1.3.2 still works if we want to reach, as the end point of the rearranged curve, any point "inside" the loop $e(t)$, that is any point with respect to which e has winding number different from 0. Moreover, the winding number itself provides a lower bound on the number of possible different rearrangements.

Remark 1.3.3. The conditions we used on the turning angle are sufficient but not necessary for rearrangeability. It is easy to find examples of curves with total turning angle 0, whose associated loop e is not surjective onto the circle in which it is contained, but that, in spite of the failure of the contraction argument, can still be rearranged to closed curves. On the other hand, surjectivity of γ'/c onto S^1 is not sufficient to guarantee that a curve is rearrangeable to a closed one; if arcs of circles and line segments are arranged as in Figure 1.4, the end point $r_{(h,t)}(1)$ does not coincide with the origin for any choice of cuts.

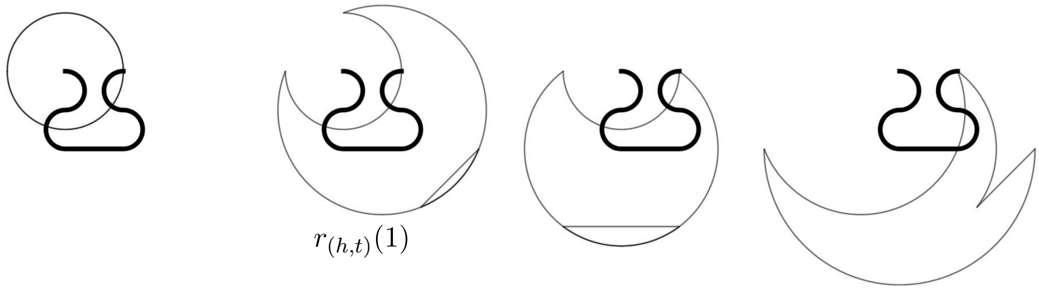

Figure 1.4: In bold a curve whose tangent is surjective onto S^1 but no 3 arcs can be rearranged to obtain a closed curve. The four images show loops in the family $r_{(h,t)}(1)$ for different values of h. Such a family opens up without passing through the origin.

1.4 More permutations

We break now the curve into $k \geq 2$ arcs by chosing cuts in the set

$$D_k := \{(c_1, c_2, ..., c_{k-1}) \in [0,1]^{k-1},\ 0 \leq c_1 \leq c_2 \leq ... \leq c_{k-1} \leq 1\}.$$

In the following, for notation convenience we will refer a few times also to the 0th and k-th component of a string of cuts $C \in D_k$, which we define by setting $c_0 := 0$ and $c_k := 1$. Given $C \in D_k$, we split a framed curve $(\gamma, \mathcal{F})$ over $[0,1]$ into $\gamma_1, \gamma_2, ..., \gamma_k$ respectively defined over $[0, c_1], [c_1, c_2], ..., [c_{k-1}, 1]$. For any element σ of the permutation group S_k of the indices $\{1, 2, ..., k\}$, we define

$$r_{\sigma,C} := \gamma_{\sigma(1)} * \gamma_{\sigma(2)} * ... * \gamma_{\sigma(k)}.$$

In order to compare more easily rearranged curves from different permutations, in the following we will implicitely assume that $r_{\sigma,C}$ is also shifted and rotated such that its starting point and frame coincide with the origin and the axes of the coordinate system. We also define

$$e_\sigma : D_k \to \mathbb{R}^n, \qquad e_\sigma(C) := r_{\sigma,C}(1),$$

which is the (continuous) map from the space of admissible cuts to the end point of the rearranged curve. We call a curve (k,j)*-rearrangeable with respect to a permution* $\sigma \in S_k$ if there exist cuts $C \in D_k$ such that $r_{\sigma,C}$ is a C^j closed curve. Theorem 1.3.1 and Lemma 1.3.2 from §1.3 give conditions that guarantee a curve is $(3,1)$-rearrangeable resp. $(3,0)$-rearrangeable with respect to the permutation (23) of S_3. Note that asking for higher regularity, i.e. $j > 1$, greatly restricts the set of admissible cuts.

We now want to move towards a full characterization of the permutations $\sigma \in S_k$ with respect to which a planar curve can be rearranged. In particular, we will see that this characterization is the same as that of permutations $\sigma \in S_k$ with respect to which a curve can be *properly* rearranged, meaning with this that no arc of the rearranged closed curve $r_{\sigma,C}$ degenerates to a point (in other words in a proper rearrangement we ask for C to be contained in the interior of the set of admissible cuts D_k, denoted with $\mathrm{int}(D_k)$ in the following). A major role will be played by the subgroup of cyclic shifts of S_k,

$$Z_k := \{z_h\}_{h\in\{0,1,\dots,k-1\}} \subseteq S_k \ \text{ with } \ z_h(i) = \begin{cases} i+h, & i \leq k-h \\ i+h-k, & i > k-h. \end{cases}$$

By the end of the section we will have proven the following theorem.

Theorem 1.4.1. *Let $\gamma(s)$ be a C^1 constant speed non-closed planar curve over $[0,1]$, whose turning angle function $\theta(s)$ satisfies $\theta(1)-\theta(0) = m2\pi$ with $0 \neq m \in \mathbb{Z}$. If $\sigma \in S_k$ with $3 \leq k$, then there exist cuts $C \in int(D_k)$ such that the rearranged curve $r_{\sigma,C}$ is closed C^1 if and only if $\sigma \in S_k \setminus Z_k$, i.e. σ is no cyclic shift.*

The next two lemmas explain how Z_k is relevant to our purposes. For the rest of the section, as in the hypothesis of Theorem 1.4.1, we will always assume $\gamma(s)$ is a C^1 constant speed non-closed planar curve defined over $[0,1]$, whose turning angle function $\theta(s)$ satisfies $\theta(1)-\theta(0) = m2\pi$ with $0 \neq m \in \mathbb{Z}$.

Lemma 1.4.2. *Let γ be a curve as described in the previous paragraph. If $z_h \in Z_k$ then there exist no cuts $C \in D_k$ such that $r_{z_h,C}$ is closed.*

Proof. For any cuts, $r_{z_h,C}$ is the same curve as the one obtained by swapping just two arcs partitioning the curve. More precisely, if $C = (c_1, c_2, \dots, c_{k-1})$, it holds $r_{z_h,C} = r_{(12),\bar{C}}$ with $\bar{C} = (c_h)$, and we have already observed in §1.3 that the distance from the origin of the end point of such a curve is independent from the choice of the cut and different from 0. $\square$

Lemma 1.4.3. *Let γ be a curve as described above. If $\sigma \in S_k$ and $z_h \in Z_k$, then γ is $(k,1)$-rearrangeable with respect to σ if and only if γ is $(k,1)$-rearrangeable with respect to the composition $\sigma \cdot z_h$ of these two permutations.*

Proof. As already pointed out, z_h cyclically shifts all elements by the same integer h. Once the curve is rearranged to a closed one, it remains closed if the sequence of arcs is cyclically shifted and the same cuts $C \in D_k$ are used. $\square$

Observing that $(123), (132) \in Z_3$ and that $(13) = (23)\cdot(123)$, $(12) = (23)\cdot(132)$ we conclude the following characterization of $(3,1)$-rearrangeability.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & & \downarrow & & \\ 2 & \cancel{4} & 5 & 1 & 3 \end{array} \overset{F_2}{\longmapsto} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ & \downarrow & & \\ 2 & 4 & 1 & 3 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ & & & \downarrow & & \\ 5 & 1 & 2 & \cancel{4} & 6 & 3 \end{array} \overset{F_4}{\longmapsto} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ & & \downarrow & & \\ 4 & 1 & 2 & 5 & 3 \end{array}$$

Figure 1.5: When the arc in position i of the rearranged curve degenerates, $k-1$ arcs are left and a permutation on S_{k-1} is induced.

Proposition 1.4.4. *Let $\gamma(s)$ be a C^1 constant speed non-closed planar curve over $[0,1]$, whose turning angle function $\theta(s)$ satisfies $\theta(1)-\theta(0)=m2\pi$ with $0 \neq m \in \mathbb{Z}$. Let also σ be a permutation of $\{1,2,3\}$, then γ is $(3,1)$-rearrangeable with respect to σ if and only if σ is a transposition.*

At this point we can make clear how we want to tackle the proof of Theorem 1.4.1. The plan is to first drop the properness constraint and show that some arcs can be collapsed to reduce the problem to a permutation on S_3. After that we will conclude by observing that the topological argument we used for S_3 is robust to perturbations and can be adapted to cuts where the degnerate arcs are inflated a little bit to guarantee properness.

When two cuts in D_k coincide and an arc degenerates we can relabel the indices, inducing a permutation on S_{k-1}. For $i \leq k$, $\sigma \in S_k$, we define $F_i(\sigma) \in S_{k-1}$ by

$$F_i(\sigma)(j) = \begin{cases} \sigma(j), & j < i, \sigma(j) < \sigma(i) \\ \sigma(j)-1, & j < i, \sigma(j) > \sigma(i) \\ \sigma(j+1), & j \geq i, \sigma(j+1) < \sigma(i) \\ \sigma(j+1)-1, & j \geq i, \sigma(j+1) > \sigma(i), \end{cases}$$

where $j \in \{1,2,...,k-1\}$. This definition by cases might be not particularly expressive, while Fig. 1.5 is the better tool to understand the combinatorial meaning of this relabelling. F_i's are not group homomorphisms as shown for example in S_5 by taking $i=2$, $\sigma_1 = (124)(35), \sigma_2 = (134)(25)$ for which $F_2(\sigma_2) \cdot F_2(\sigma_1) \neq F_2(\sigma_2 \cdot \sigma_1)$.

We now need to prove a combinatorial lemma, which will be the core of the inductive construction used in Proposition 1.4.6 to prove the characterization of $(k,1)$-rearrangeability, when cuts are allowed (actually forced) to degenerate.

Lemma 1.4.5. *For $\sigma \in S_k \setminus Z_k$ with $k \geq 4$ and $\sigma(1)=1$, there exists $i \in \{1,2,...,k\}$ such that $F_i(\sigma) \in S_{k-1} \setminus Z_{k-1}$.*

Proof. Let us consider the smallest $r \in \{1,2,...,k\}$ such that $\sigma(r) \neq r$, which exists since σ is not the identical permutation. We set i to the following values, distinguishing 3 cases:

$$i := \begin{cases} 1, & r > 2 \\ \sigma^{-1}(k), & r = 2, \sigma(2) \neq k \\ 3, & r = 2, \sigma(2) = k, \end{cases} \quad \Rightarrow \quad \begin{array}{l} F_i(\sigma)(r-1) = \sigma(r) - 1 \neq r - 1 \\ F_i(\sigma)(2) = \sigma(2) \neq 2 \\ F_i(\sigma)(2) = \sigma(2) - 1 = k - 1 > 2. \end{array}$$

Since for all three cases $F_i(\sigma)(1) = 1$, this concludes the proof. □

Proposition 1.4.6. *Let $\gamma(s)$ be a C^1 constant speed non-closed planar curve over $[0,1]$, whose turning angle function $\theta(s)$ satisfies $\theta(1) - \theta(0) = m2\pi$ with $0 \neq m \in \mathbb{Z}$. If $\sigma \in S_k$ with $4 \leq k$, then there exist cuts $C \in \partial(D_k)$ such that the rearranged curve $r_{\sigma,C}$ is closed C^1 if and only if $\sigma \in S_k \setminus Z_k$.*

Proof. Lemma 1.4.2 rules out permutations in Z_k from the picture. By Lemma 1.4.3 proving rearrangeability with respect to $\sigma \cdot z_h$ for $z_h \in Z_k$ implies also rearrangeability with respect to σ. We can therefore assume, by possibly applying some cyclic shift, that $\sigma(1) = 1$. By Lemma 1.4.5 we can find i such that by taking cuts on ∂D_k with $c_{\sigma(i)-1} = c_{\sigma(i)}$ a permutation in $S_{k-1} \setminus Z_{k-1}$ is induced. The statement follows by induction, once we observe that the base $k = 4$ is guaranteed by Proposition 1.4.4 after one last contraction. □

We can finally prove Theorem 1.4.1 by discussing the robustness of the argument we used in §1.3.

Proof of Theorem 1.4.1. The case $k = 3$ is implied by Proposition 1.4.4 after observing that cuts must be in the interior of D_3 since the curve we want to rearrange is not closed. If $k > 3$, by Proposition 1.4.6 we can find $q_1 < q_2 < q_3 \in \{0, 1, 2, ..., k-1\}$ such that there exist cuts $\bar{C} = (\bar{c}_1, \bar{c}_2, ..., \bar{c}_{k-1})$, satisfying $\bar{c}_j < \bar{c}_{j+1}$ for $j \in \{q_1, q_2, q_3\}$ and $\bar{c}_j = \bar{c}_{j+1}$ otherwise, and such that $r_{\sigma,\bar{C}}$ is closed C^1 (recall that we agreed on $\bar{c}_0 = 0$ and $\bar{c}_k = 1$). We want now to inflate degenerate arcs, making them proper again, without undermining the contraction argument we used for S_3. For $(l_1, l_2) \in D_3$ we let $\delta(l_1, l_2) = \frac{1}{k-2}\min\{l_1, l_2 - l_1, 1 - l_2\}$ and define the *inflated* cuts $I[l_1, l_2] = (c_1, ..., c_{k-1}) \in D_k$ as

$$c_j = \begin{cases} j\delta(l_1, l_2), & j \leq q_1 \\ l_1 + (j - (q_1 + 1))\delta(l_1, l_2), & q_1 < j \leq q_2 \\ l_2 + (j - (q_2 + 1))\delta(l_1, l_2), & q_2 < j \leq q_3 \\ 1 - (k - j)\delta(l_1, l_2), & j > q_3. \end{cases}$$

Figure 1.6 visualizes the inflation operation. Note that if $(l_1, l_2) \in \text{int}(D_3)$ then $I[l_1, l_2] \in \text{int}(D_k)$. Further, no inflation happens if $(l_1, l_2) \in \partial(D_3)$. What we are doing

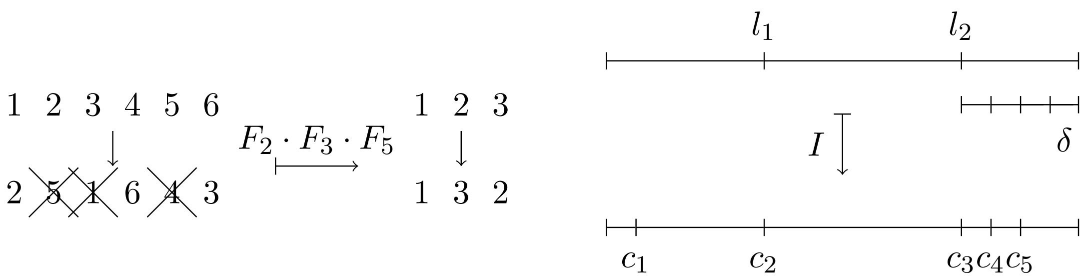

Figure 1.6: Indices are repeatedly contracted from a permution in S_6 to obtain a permutation in S_3. Such indices determine how degenerate cuts are inflated and turned into proper ones.

here is to push the interior of the triangular face $c_j = c_{j+1}$ for $j \notin \{q_1, q_2, q_3\}$, which can be thought as a copy of D_3, towards the interior of D_k. Because of Proposition 1.4.3 we can assume the permutation induced on S_3 is the one swapping arc 2 and 3. We proceed as in §1.3, by considering the loop $e_\sigma(I[0,t])$ defined on $[0,1]$. Since no inflation is happening on ∂D_3, this is exactly the same loop contained in a circle and with winding number $m2\pi$ we described in the proof of Theorem 1.3.1, which we know being contained and not contractible in $\mathbb{R}^2 \setminus \{(0,0)\}$. For $h \in [0,1]$ the family of loops $e_\sigma(I[h,t])$ defined on $[h,1]$ is a contraction to a point different from $(0,0)$, which implies the existence of $(h,t) \in \mathrm{int}(D_3)$ such that $e_\sigma(I[h,t]) = (0,0)$. □

Figure 1.7 shows how a curve split into 6 arcs can be properly rearranged. One can be a bit more precise about the maximum magnitude of the inflation factor δ exploiting uniform continuity of e_σ on D_k. Nevertheless, we preferred the slightly less informative but leaner proof we gave above.

The proof of Theorem 1.4.1, in the way we made sure that the perturbation of our starting cuts would not undermine the features of the loop $e_\sigma(I[0,t])$, presents one further analogy with the topological proof of the fundamental theorem of algebra. If $\bar{p}(z) = z^n$ then it is apparent that the loop $\bar{p}(\rho e^{it})$ for $\rho > 0$ winds around the origin n times, which is a property of little use since it is obvious where the roots of such a polynomial are. If $\bar{p}$ is changed to $p(z) = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ though, the property about the winding number remains unalterd if ρ is chosen large enough to make negligible the contribution of the terms of lower degree and therefore the contraction argument can still be used. The idea of exploiting the robustness of a topological argument, proven to work for a degenerate case, is also reminescent of the proof of the converse of the so called 4-vertex theorem by Gluck [17], generalized on the same line some 30 years later by Dahlberg [4]. A nice survey on this theorem and its proof(s) can be found in [9], where

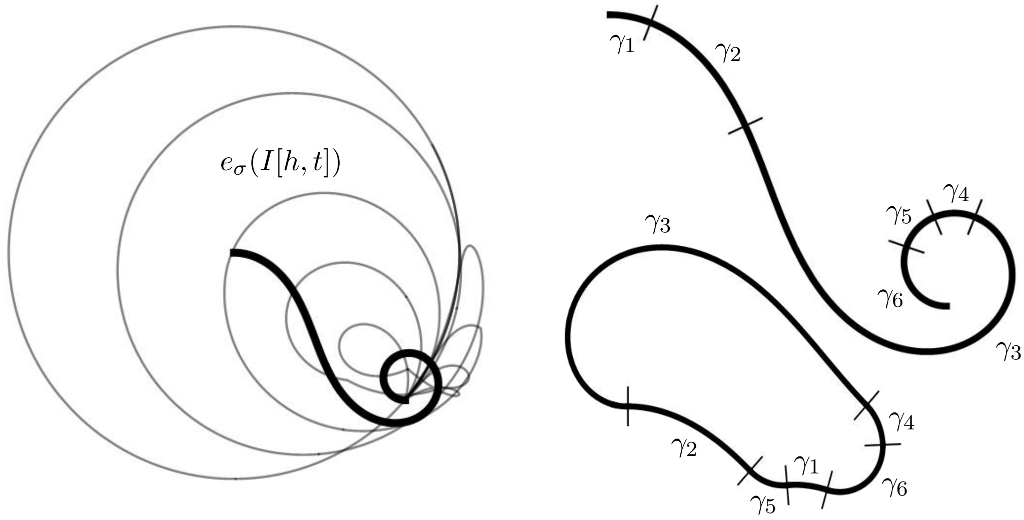

Figure 1.7: Proper rearrangement of a curve split into 6 arcs. The topological argument is the same as in §1.3: $e_\sigma(I[h,t])$ defines a family of loops starting as a circle centered in $(0,0)$ and contracting to $\gamma(1) \neq (0,0)$, which implies the starting point of the curve is contained in one of the intermediate loops of this family.

also the above observation about the fundamental theorem of algebra is pointed out.

1.5 Some comments on higher dimensions and future work

A viable technique for showing $(k,0)$-rearrangeability in $\mathbb{R}^n$ would be to proceed in a manner analogous to the 2D case by assessing the contractibility in $\mathbb{R}^n \setminus \{O\}$ of the image of ∂D_k through e_σ. If $e_\sigma(\partial D_k)$ contains the starting point O then we have already achieved $(k-1,0)$-rearrangeability. Otherwise, up to homeomorphism a map $\partial D_k \cong S^{k-2} \to \mathbb{R}^n \setminus \{O\}$ is induced. If this map is not contractible (it represents a non-trivial element in the $(k-2)$-th homotopy group of $\mathbb{R}^n \setminus \{O\} \simeq S^{n-1}$) then there exist cuts in the interior of D that provides $(k,0)$-rearrangeability. Nevertheless, in this setting the current lack of nice criteria to detect contractibility of the map induced by a certain permutation makes it hard to translate such considerations into explicit statements about rearrangeability.

We conclude with pointing out a few possibilities for future work. A full characterization of planar curves that are $(k,0)$-rearrangeable or $(k,1)$-rearrangeable is an obvious next step. For higher dimensions it would be relevant to better understand the combinatorics of the image of $e_\sigma(\partial D_k)$ and to develop at least some neat sufficient conditions which guarantee that a curve is (k,j)-rearrangeable. C^2 curves in $\mathbb{R}^3$ whose curvature is constant are a promising family to study in this direction; in such class, if the curve is framed with the Frenet-Serret frame, the rearrangement procedure described in §1.4 would provide C^2 regularity also at junction points.

Acknowledgments

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2 Affine subspaces of curvature functions from closed planar curves

Abstract

Given a pair of real functions (k, f), we study the conditions they must satisfy for $k + \lambda f$ to be the curvature in the arc-length of a closed planar curve for all real λ. Several equivalent conditions are pointed out, certain periodic behaviours are shown as essential and a family of such pairs is explicitely constructed. The discrete counterpart of the problem is also studied. Finally, the characterization obtained is used to show that a sufficient analogue of the 4-vertex theorem cannot be developed.

2.1 Introduction

The natural and complete geometric descriptor we associate to a curve is its *curvature*. If $\gamma \in C^2([0, 2\pi], \mathbb{R}^2)$ is an *arc-length parametrized planar curve*, *i.e.*, a twice differentiable function from the interval $I := [0, 2\pi]$ to the real plane such that the norm of its first derivative $\|\gamma'\|$ is constantly equal to 1, we can define a *turning angle* function θ that satisfies $\gamma'(t) = (\cos\theta(t), \sin\theta(t)) = \mathrm{e}^{i\theta(t)}$, where $\mathbb{R}^2$ has been identified with the complex plane $\mathbb{C}$. The *curvature* k of γ is defined as the first derivative θ' of the turning angle function. The other way round, given a continuous curvature $k \in C^0(I, \mathbb{R})$ we can reconstruct by integration, uniquely up to rigid motions, the curve it comes from. In fact, $\theta(t) = \int_0^t k(s)\mathrm{d}s + C$ and $\gamma(t) = \int_0^t \mathrm{e}^{i\theta(s)}\mathrm{d}s + V$. For a more extensive treatment of the subject the reader may refer to [10].

Given another function $f \in C^0(I, \mathbb{R})$, the main question we are interested in this chapter is:

> What are the conditions on k and f for $k + \lambda f$ to be the curvature of a closed curve for all $\lambda \in \mathbb{R}$?

Here and in the following with *closed* we just mean that starting and end point of the curve coincide (we will see though that the nature of the problem entails much stiffer relations also on the derivatives at the extreme points of the curve). Figure 2.1 visualizes the objects we are going to study.

Interpolation of curvature functions is a tool used in computer graphics to gradually transform one curve into another, while mantaining the length of the curve [45]. This method does not perform well when it comes to deform closed curves, since there is no guarantee that the intermediate curves are closed as well; from the point of view of computer graphics this problem can be fixed by approximating the transition curves with closed ones that are not too far away from them [38]. In this paper we approach the problem from the theoretical perspective, exploring the conditions guaranteeing that all the curves are closed over the interpolation of the curvature functions.

In the more general framework of deformations, the evolution of curves under the action of different flows has been studied: in [3] a curvature-based flow is used to transform a shape into another while preserving the length; the closedness of the curve over the process is guaranteed by an extra projection step to the hyperspace of L^2 defined by the constraint $\int_0^{2\pi} k'(s)\gamma(s)\mathrm{d}s = 0$ relating the position of the curve and the first derivative of its curvature, which is interestingly proven as a necessary and sufficient condition for a curve to be closed.

As for an outline of the contents, §2.2 presents the main characterization theorem, proving also that the existence of a single affine line of curvature functions from closed curves is equivalent to the existence of an infinite dimensional affine space of such functions. In §2.3 periodicity properties of k and f are shown. On the existence side, §2.4 deals with the explicit construction of pairs of analytic function (k, f) that satisfy our constraints. In §2.5 we discuss the discrete case. Finally, in §2.6 a *hardness* result on the task of telling whether a curve is closed by looking at its curvature k is obtained. We show that it is not possible to develop a procedure that tells whether the associated curve is closed or not by accessing finitely many evaluations and/or level sets of k, its derivatives and its antiderivatives.

2.2 Equivalent characterizations of closedness

Let γ be a closed C^2 curve defined on the interval $I = [0, 2\pi]$, θ its associated turning angle function and $k = \theta'$ its curvature. For $f \in C^0(I, \mathbb{R})$, we want to answer the question: what are the conditions on f for $k + \lambda f$ to be the curvature of a closed curve for all $\lambda \in \mathbb{R}$? Calling $\phi(t) := \int_0^t f(s)\mathrm{d}s$, this is equivalent to

$$\int_0^{2\pi} \mathrm{e}^{i(\theta(t)+\lambda\phi(t))}\mathrm{d}t = 0, \qquad \forall\, \lambda \in \mathbb{R}.$$

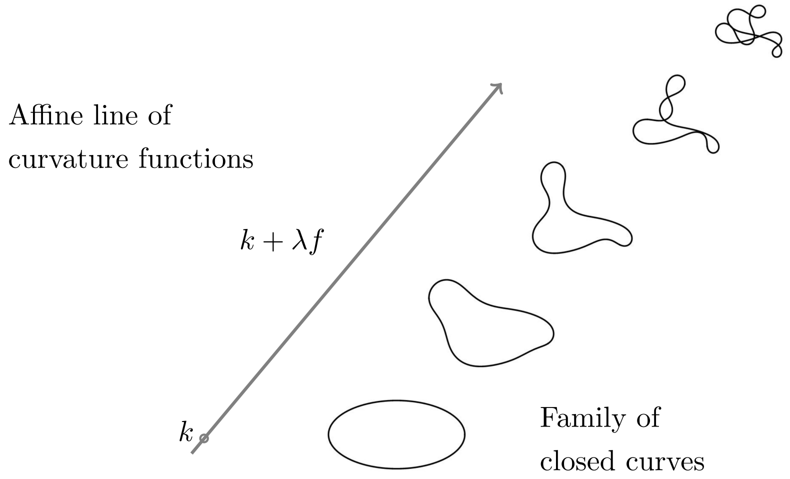

Figure 2.1: An ellipse is deformed by adding multiple of f to its curvature k. If, as in this case, f is chosen properly, then all the curves of the family are closed. We want to study the constraints k and f must satisfy to present such a behaviour.

The function $F(\lambda) := \int_0^{2\pi} \mathrm{e}^{i(\theta(t)+\lambda\phi(t))}\mathrm{d}t$ is analytic in λ. This can be seen for example by giving the following explicit entire expansion for the real part of F (the imaginary part is analogous):

$$F_1(\lambda) = \sum_{c=0}^{\infty} c_n \frac{\lambda^n}{n!}, \quad \text{with } c_n = \begin{cases} (-1)^{\frac{n}{2}} \int_0^{2\pi} \phi(t)^n \cos\theta(t)\mathrm{d}t, & \text{if } n \text{ is even,} \\ (-1)^{\frac{n+1}{2}} \int_0^{2\pi} \phi(t)^n \sin\theta(t)\mathrm{d}t, & \text{if } n \text{ is odd.} \end{cases}$$

This observation alone is enough to conclude the first of our equivalent conditions.

Lemma 2.2.1. *Let $k, f \in C^0(I, \mathbb{R})$. Then the curve with curvature $k + \lambda f$ is closed $\forall\, \lambda \in \mathbb{R} \Leftrightarrow$ we have the equality*

$$\int_0^{2\pi} \mathrm{e}^{i\theta(t)}\phi(t)^n \mathrm{d}t = 0, \quad \forall\, n \in \mathbb{N}_0, \tag{2.1}$$

where $\theta(t) = \int_0^t k(s)\mathrm{d}s$ and $\phi(t) = \int_0^t f(s)\mathrm{d}s$.

Proof. An analytic function is everywhere 0 if and only if all of its derivatives vanish in at least one point. We conclude by computing the n-th derivative of F and evaluating it in $\lambda = 0$, obtaining $0 = F^{(n)}(0) = i^n \int_0^{2\pi} \mathrm{e}^{i\theta(t)}\phi(t)^n\mathrm{d}t$. Note that we could take the derivative within the integral thanks to the Leibniz integral rule. □

We want now to better understand this condition, by discussing some of its implications. Our main tool will be an approximation argument based on the observation that, if ϕ

satisfies the condition above, then for any $N \in \mathbb{N}_0$ and $(c_j)_{j\in\{1,\dots,N\}} \in \mathbb{R}^{N+1}$, also the sum $\sum_{j=0}^{N} c_j \phi^j$ does.

Lemma 2.2.2. *$\theta, \phi \in C^1(I, \mathbb{R})$ satisfy condition (2.1) $\Leftrightarrow$ $\theta, g(\phi)$ (composition of functions) satisfy condition (2.1), for any g bounded and integrable.*

Proof. The 'if' part is trivial. For the 'only if' we use a density property of polynomials in our class of functions to approximate g. More explicitely, for any $n \in \mathbb{N}_0$ and $\varepsilon > 0$, there exists a polynomial $p_{n,\varepsilon}$ of degree $N(n, \varepsilon)$ such that

$$\int_0^{2\pi} \big|g(\phi(t))^n - p_{n,\varepsilon}(\phi(t))\big| \mathrm{d}t < \varepsilon,$$

which implies

$$\begin{aligned}
&|\int_0^{2\pi} \mathrm{e}^{i\theta(t)} g(\phi(t))^n \mathrm{d}t| \\
\leq &|\int_0^{2\pi} \mathrm{e}^{i\theta(t)} \big(g(\phi(t))^n - p_{n,\varepsilon}(\phi(t))\big) \mathrm{d}t| + |\int_0^{2\pi} \mathrm{e}^{i\theta(t)} p_{n,\varepsilon}(\phi(t)) \mathrm{d}t| \\
\leq &\int_0^{2\pi} \big|g(\phi(t))^n - p_{n,\varepsilon}(\phi(t))\big| \mathrm{d}t \leq \varepsilon.
\end{aligned}$$

□

By Lemma 2.2.2, the existence of ϕ satisfying (2.1) implies the existence of an infinite-dimensional affine space through θ whose elements satisfy (2.1) as well. From the perspective of the curvature, what we are saying here is that, choosing g to be C^1, we can pass from f to $fg'(\phi)$ and still have that the curves with curvature functions $k + \lambda f g'(\phi)$ are closed for all λ.

Before moving to the next lemma, which provides a much more local characterization of our constraint, it is convenient to recall that a level set $\phi^{-1}(a) = \{t \mid \phi(t) = a\}$ consists of isolated points, if $\phi'(t) \neq 0$ for all $t \in \phi^{-1}(a)$. For ϕ defined on a compact interval I, level sets of such regular values are therefore finite.

Lemma 2.2.3. *If $\theta, \phi \in C^1(I, \mathbb{R})$ satisfy condition (2.1), then we have the implication*

$$a \neq \phi(0), \phi(2\pi) \text{ is a regular value of } \phi \Rightarrow \sum_{b\in\phi^{-1}(a)} \frac{\mathrm{e}^{i\theta(b)}}{|\phi'(b)|} = 0. \tag{2.2}$$

Proof. In Lemma 2.2.2 we select $g = \chi_{[a,a+\delta]}$, that is the characteristic function of the interval $[a, a+\delta]$, and obtain

$$\int_{\phi^{-1}([a,a+\delta])} \mathrm{e}^{i\theta(t)} \phi(t) \mathrm{d}t = 0, \quad \forall\, \delta \geq 0, a \in \mathbb{R}.$$

We can find $\delta > 0$ such that the restrictions $\{\phi_j\}$ of ϕ to the finitely many components of $\phi^{-1}([a, a+\delta])$ are invertible. Calling $R(\delta) := \int_{\phi^{-1}([a,a+\delta])} \mathrm{e}^{i\theta(t)}\phi(t)\mathrm{d}t$, we compute its derivative with respect to δ

$$R'(\delta) = \sum_j \mathrm{e}^{\theta(\phi_j^{-1}(a+\delta))}\phi(\phi_j^{-1}(a+\delta)) \cdot |(\phi^{-1})'(a+\delta)|,$$

and then, since R is a constant,

$$0 = R'(0) = a \sum_{b \in \phi^{-1}(a)} \frac{\mathrm{e}^{i\theta(b)}}{|\phi'(b)|}. \qquad \square$$

The reason we excluded the level sets $\phi(0)$ and $\phi(2\pi)$ from the constraint on the sum is to avoid to distinguish cases depending on the sign of the derivative at extreme points of the interval: all relevant information is already in the condition for the sum over inner points.

Remark 2.2.4. In order to have the equivalence (2.1) $\Leftrightarrow$ (2.2), in addition we must require $\int_{\phi^{-1}(a)} \mathrm{e}^{i\theta} = 0$, for all $a \in \mathbb{R}$. If ϕ is analytic this requirement is always met.

We collect in one theorem all the conditions we have proven equivalent.

Theorem 2.2.5. *Let $k, f \in C^0(I, \mathbb{R})$ and $\theta(t) = \int_0^t k(s)\mathrm{d}s, \phi(t) = \int_0^t f(s)\mathrm{d}s$. The following conditions are equivalent.*

0. *The curve with curvature $k + \lambda f$ is closed $\forall\, \lambda \in \mathbb{R}$,*
1. *$\int_0^{2\pi} \mathrm{e}^{i\theta(t)}\phi(t)^n \mathrm{d}t = 0, \quad \forall\, n \in \mathbb{N}_0$,*
2. *$\int_0^{2\pi} \mathrm{e}^{i\theta(t)}g(\phi(t))^n \mathrm{d}t = 0, \quad \forall\, n \in \mathbb{N}_0$ and any g bounded and integrable.*

Moreover, they imply

$$a \neq \phi(0), \phi(2\pi) \text{ is a regular value of } \phi \Rightarrow \sum_{b \in \phi^{-1}(a)} \frac{\mathrm{e}^{i\theta(b)}}{|\phi'(b)|} = 0.$$

We also point out the following corollary, which rules out the possibility of vector spaces of curvatures of closed curves.

Corollary 1. *For $f \in C^0(I, \mathbb{R})$, there exists λ such that the curve that has λf as curvature is not closed. More precisely, the set $\Lambda = \{\lambda \in \mathbb{R} : \lambda f$ is the curvature of a closed curve$\}$ does not have accumulation points.*

Proof. Setting $k \equiv 0$, condition (1) of Theorem 2.2.5 becomes $\int_0^{2\pi} \phi(t)^n dt = 0$ for all $n \in \mathbb{N}_0$, which for n even can only be satisfied by $\phi \equiv 0$. On the other hand, the presence of accumulation points in Λ is enough to guarantee $\Lambda = \mathbb{R}$ by the analiticity argument from the beginning of the section, hence entailing the same conclusion. $\square$

2.3 Conditions on the boundary

In this section we discuss some periodicity properties that θ and ϕ must satisfy if the curve with turning angle function $\theta + \lambda\phi$ is closed for all $\lambda \in \mathbb{R}$. We will show that, under the conditions of Theorem 2.2.5, the respective behaviour of θ and ϕ on the boundary is strongly related.

Proposition 2.3.1. *If $\theta, \phi \in C^h(I, \mathbb{R})$ satisfy condition (2.2) and $\phi(0)$ is not a critical value, then $\phi(0) = \phi(2\pi)$ and the derivatives of ϕ, θ obey either*

$$\begin{aligned} \theta(2\pi) - \theta(0) \equiv 0 \mod 2\pi, \ & \theta^{(k)}(0) = \theta^{(k)}(2\pi), & 1 \le k \le h-1, \\ & \phi^{(k)}(0) = \phi^{(k)}(2\pi), & 1 \le k \le h \end{aligned}$$

or

$$\begin{aligned} \theta(2\pi) - \theta(0) \equiv \pi \mod 2\pi, \ & \theta^{(k)}(0) = (-1)^k \theta^{(k)}(2\pi), & 1 \le k \le h-1, \\ & \phi^{(k)}(0) = (-1)^k \phi^{(k)}(2\pi), & 1 \le k \le h. \end{aligned}$$

Proof. Setting $a = \phi(0)$, we consider $\delta > 0$ such that ϕ is invertible on the finitely many components of $\phi^{-1}([a-\delta, a+\delta])$. We then look at the connected components of $\phi^{-1}([a, a+\delta])$ and we use the symbol p_j for the restriction of ϕ to the j-th component, numbered from the left ($j = 1, ..., N_+$, see Figure 2.2). Similarly, functions m_j's are the restriction of ϕ to the connected components of $\phi([a-\delta, a])$. Rewriting condition (2.2) we have

$$\lim_{\varepsilon \to 0^+} \sum_j \frac{\mathrm{e}^{i\theta\left(p_j^{-1}(a+\varepsilon)\right)}}{|\phi'(p_j^{-1}(a+\varepsilon))|} = \lim_{\varepsilon \to 0^-} \sum_j \frac{\mathrm{e}^{i\theta\left(m_j^{-1}(a+\varepsilon)\right)}}{|\phi'(m_j^{-1}(a+\varepsilon))|} = 0.$$

Since limit contributions coming from the restrictions to intervals in the interior of I are equal in the two sums, we have no other choice than $\phi(0) = \phi(2\pi)$, otherwise the contribution from p_1 in the first sum could not be balanced in the limit by any terms of the second sum. Without loss of generality we can assume $\phi'(0) > 0$. We distinguish two cases, depending on the sign of $\phi'(2\pi)$. If $\phi'(2\pi) > 0$, just by rewriting again condition (2.2) while keeping contributions from the two extreme intervals on the left-hand side of the equalities, we have , for $0 < \varepsilon < \delta$

$$\begin{aligned} \frac{\mathrm{e}^{i\theta\left(p_1^{-1}(a+\varepsilon)\right)}}{\phi'(p_1^{-1}(a+\varepsilon))} &= - \sum_{1<j} \frac{\mathrm{e}^{i\theta\left(p_j^{-1}(a+\varepsilon)\right)}}{|\phi'(p_j^{-1}(a+\varepsilon))|}, \\ \frac{\mathrm{e}^{i\theta\left(m_{N_-}^{-1}(a-\varepsilon)\right)}}{\phi'(m_{N_-}^{-1}(a-\varepsilon))} &= - \sum_{j<N_-} \frac{\mathrm{e}^{i\theta\left(m_j^{-1}(a-\varepsilon)\right)}}{|\phi'(m_j^{-1}(a-\varepsilon))|}. \end{aligned}$$

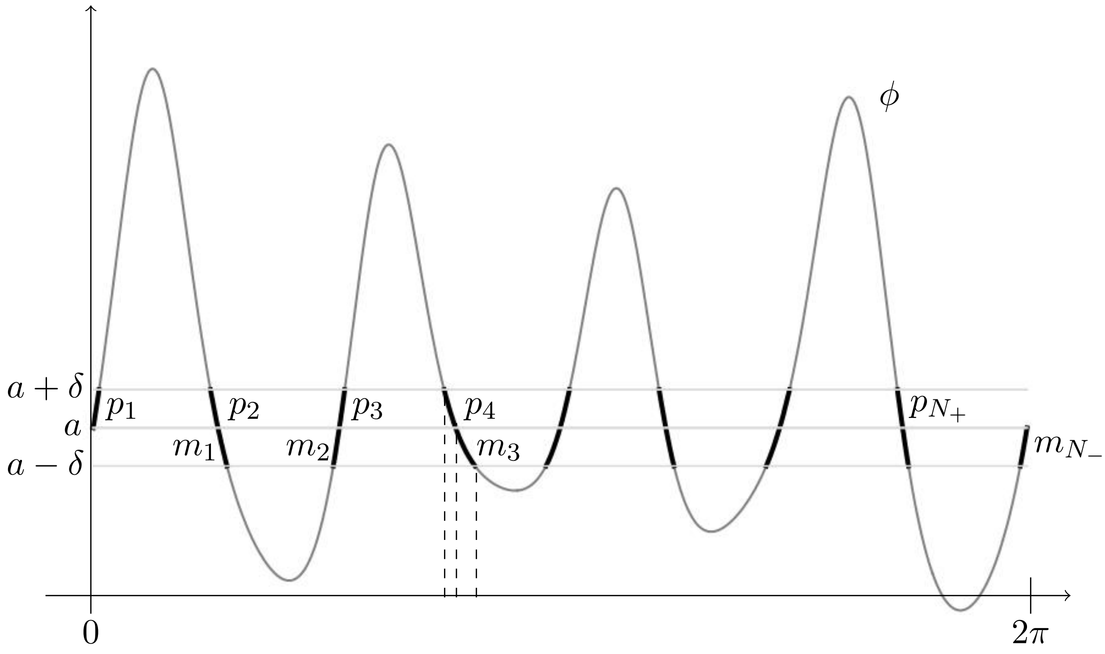

Figure 2.2: The functions $\{p_j\}_{j\in\{1,\dots,N_+\}}$ are the invertible restrictions of ϕ to the finitely many components of $\phi^{-1}([a,a+\delta])$ and analogously $\{m_j\}_{j\in\{1,\dots,N_-\}}$ are the restrictions of ϕ to $\phi^{-1}([a-\delta,a])$.

The sums on the right-hand side of the equations are equal for $\varepsilon = 0$, entailing

$$\frac{\mathrm{e}^{i\theta(0)}}{\phi'(0)} = \frac{\mathrm{e}^{i\theta(2\pi)}}{\phi'(2\pi)},$$

which proves $\theta(2\pi)-\theta(0) \equiv 0 \mod 2\pi$ and $\phi'(0)=\phi'(2\pi)$. Analogously, taking the first derivative of the equations with respect to ε and considering the limit $\varepsilon \to 0$, we conclude

$$\frac{i\mathrm{e}^{i\theta(0)}\theta'(0) - \mathrm{e}^{i\theta(0)}\phi''(0)\frac{1}{\phi'(0)}}{\phi(0)^2} = \frac{i\mathrm{e}^{i\theta(2\pi)}\theta'(2\pi) - \mathrm{e}^{i\theta(2\pi)}\phi''(2\pi)\frac{1}{\phi'(2\pi)}}{\phi(2\pi)^2},$$

which, already knowing the respective relations of θ, ϕ and ϕ' at extreme parameters, and noticing that $\mathrm{e}^{i\theta(0)}$ and $i\mathrm{e}^{i\theta(0)}$ are orthogonal, implies $\theta'(0)=\theta'(2\pi)$ and $\phi''(0)=\phi''(2\pi)$. For the derivatives of higher order, the statement follows analogously by induction.

If $\phi'(2\pi)<0$, we get

$$\frac{\mathrm{e}^{i\theta\left(p_1^{-1}(a+\varepsilon)\right)}}{\phi'(p_1^{-1}(a+\varepsilon))} - \frac{\mathrm{e}^{i\theta\left(p_{N_+}^{-1}(a+\varepsilon)\right)}}{\phi'(p_{N_+}^{-1}(a+\varepsilon))} = -\sum_{1<j<N_+} \frac{\mathrm{e}^{i\theta\left(p_j^{-1}(a+\varepsilon)\right)}}{|\phi'(p_j^{-1}(a+\varepsilon))|},$$

$$0 = -\sum_{j} \frac{\mathrm{e}^{i\theta\left(m_j^{-1}(a-\varepsilon)\right)}}{|\phi'(m_j^{-1}(a-\varepsilon))|}$$

and we conclude again by taking derivatives term by term with respect to ε and using induction. $\square$

Remark 2.3.2. Note that the constraint on the curves associated to $k + \lambda f$ to be closed for all λ just means that starting and end point coincide. Proposition 2.3.1 proves that in this case the function k and f enjoy much stronger periodicity.

Remark 2.3.3. In the hypotheses of Proposition 2.3.1, we obtain an additional constraint on the integral over I of the function f, in fact $\int_0^{2\pi} f(s)\mathrm{d}s = \phi(2\pi) = \phi(0) = 0$. This means that along the affine line $k + \lambda f$ the total turning angle of the associated curve is constant and equal to 0 or π up to multiples of 2π.

2.4 Explicit families of closed curves

In §2.2 and §2.3 we characterized pairs of functions (k, f) such that the curve obtained by integrating the curvature $k+\lambda f$ is closed for all $\lambda \in \mathbb{R}$. In this section we are interested in the existence of such pairs. We show how one can explicitly construct curvature functions with the desired properties.

Lemma 2.4.1. *If $\theta \in C^1(I, \mathbb{R})$, then*

$$\exists \phi \in C^1(I, \mathbb{R}) : \int_0^{2\pi} \mathrm{e}^{i\theta(s)} \phi(s)^n \mathrm{d}s = 0, \ \forall n \in \mathbb{N}$$

$$\Leftrightarrow$$

$$\exists \psi \in C^1(I, \mathbb{R}) : \int_0^{2\pi} \mathrm{e}^{i\theta(s)} \mathrm{e}^{in\psi(s)} \mathrm{d}s = 0, \ \forall n \in \mathbb{N}.$$

Proof. The first existence statement implies the second just by taking $\psi = \phi$ and recalling condition (2) of Theorem 2.2.5, guaranteeing that the composition with a function that is bounded and integrable mantains the desired property. The other way round we pick for example $\phi = \cos(\psi)$ and conclude by observing that $\cos(\psi)^n$ can be rewritten as a linear combination of terms of the form $\cos(h \cdot \psi)$. $\square$

We now consider curves allowing a periodic regular parametrization that can be expressed as a Fourier series with periodic gaps in the coefficients

$$\gamma(t) = \left(\sum_{j=0}^{\infty} a_j \cos(j \cdot t) + b_j \sin(j \cdot t), \sum_{j=0}^{\infty} \bar{a}_j \cos(j \cdot t) + \bar{b}_j \sin(j \cdot t) \right),$$

that is $a_j = b_j = \bar{a}_j = \bar{b}_j = 0$ whenever j is an integer multiple of $M \in \mathbb{N}$. The asymptotics of the coefficients for j going to infinity determines periodicity and differentiability of the function (see for example [23]). From now on we assume that γ is a closed analytic curve, which, in the most trivial case, can simply be obtained by truncating the series and considering a trigonometric polynomial; in this case all the harmonics with index larger than the degree of the polynomial are 0 and therefore there exists always M satisfying the conditions above. By the orthogonality relations between elements of a Fourier basis we have

$$\int_0^{2\pi} \gamma'(t)\cos(n \cdot M \cdot t)dt = (0,0),\ \forall n \in \mathbb{N}.$$

Using complex notation, we write $\gamma'(t)$ as $v(t)\mathrm{e}^{i\theta(t)}$ where $v(t) = \|\gamma'(t)\|$ is the speed of γ and θ is the turning angle associated to the parametrization. After reparametrizing with respect to the arc-length (always possible as long as the curve is regular) we obtain, possibly scaling our curve to a length of 2π,

$$\int_0^{2\pi} \mathrm{e}^{i\theta(t(s))}\cos(n \cdot M \cdot t(s))ds = 0,\ \forall n \in \mathbb{N}.$$

Note that if γ is analytic than also γ', $\|\gamma'\|$, $\int_0^t \|\gamma'\|$ and its inverse are analytic and therefore arc-length parametrization preserves analyticity. By Lemma 2.4.1, $\phi(s) = \cos(l \cdot t(s))$ and $\theta(t(s))$ satisfy condition (1) of Theorem 2.2.5 and therefore the analytic curve obtained by integrating $\mathrm{e}^{i(\theta+\lambda\phi)}$ is closed for all real λ's (note that the functions ϕ and θ constructed this way are in general not periodic of any period smaller than 2π). It is enough to take the derivative with respect to s to get the correspondent curvature functions. Figure 2.1 and Figure 3.8 show families of curves obtained by such a linear modification of the turning angle (or equivalently of the curvature).

2.5 The discrete case

In this section we look at a discretization of the problem we studied in the smooth setting. Consider an arc-length parametrized polyline, that is a finite sequence of *vertices* $(v_j)_{j\in\{1,2,\ldots,N\}} \subset \mathbb{C}$ with $\|v_{j+1} - v_j\| = 1$ for $1 \leq j \leq N-1$. We define the *curvature* k_j at a non-extreme vertex v_j as the counter-clockwise angle between $v_j - v_{j-1}$ and $v_{j+1} - v_j$. The *turning angle* θ_j at an interior vertex v_j is the sum $\sum_{r=2}^{j} k_j$. Also in this setting we can reconstruct, up to rigid motions, a polyline from its curvature, first computing the turning angle (θ_j) and then defining

$$v_1 = 0,\ \ v_2 = 1,\ \ v_j = v_{j-1} + \mathrm{e}^{i\theta_{j-1}} \text{ for } j \geq 3.$$

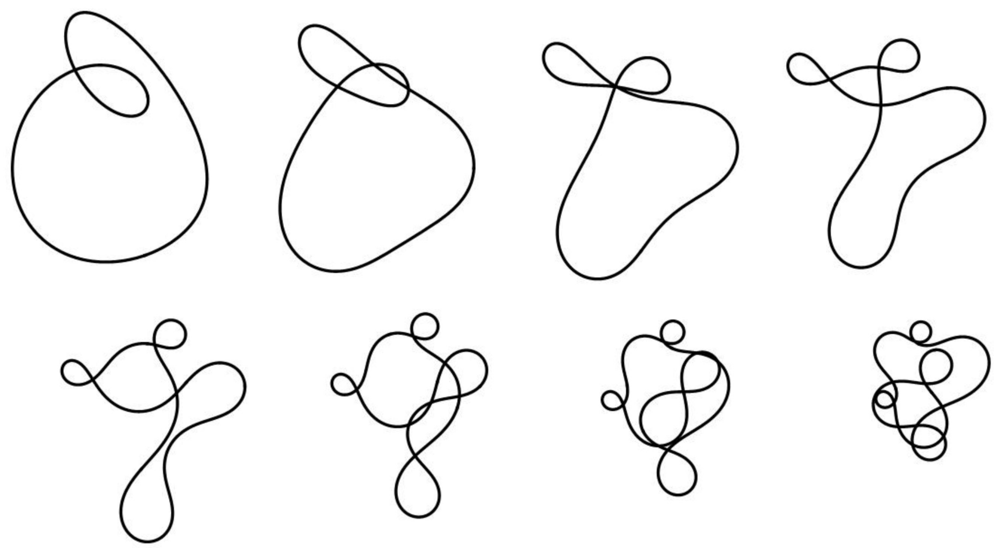

Figure 2.3: The turning angle θ of a trigonometric curve of degree 3 is linearly changed to $\theta + \lambda\phi$ with $\phi(t) = \mathrm{e}^{\cos(4t)} + 2\cos(4t)$ while the curve remains closed. From top left to bottom right λ goes from 0 to 0.7 by 0.1 increments.

We consider now a polyline with N vertices, which is *closed* ($v_1 = v_N$) and whose curvature is (k_j). Given another discrete function $(f_j)_{j\in\{2,\dots,N-1\}} \in \mathbb{R}^{N-2}$, we ask what are the conditions on (f_j) to guarantee that the polyline with curvature $(k_j) + \lambda(f_j)$ is closed for all $\lambda \in \mathbb{R}$. The following theorem answers this question, drawing a strong analogy to Theorem 2.2.5.

Theorem 2.5.1. *Let (k_j) and (f_j) be two discrete functions and $(\theta_j), (\phi_j)$ the turning angles obtained as their respective partial sums. The following conditions are equivalent*

0. The polyline with curvature $(k_j) + \lambda(f_j)$ is closed $\forall\lambda \in \mathbb{R}$,

1. $\sum_{1<j<N} \mathrm{e}^{i\theta_j}\phi_j^n = 0, \ \ \forall n \in \mathbb{N}_0$,

2. $\sum_{j\in\phi^{-1}(a)} \mathrm{e}^{i\theta_j} = 0, \ \ \forall a \in \mathbb{R}$.

Proof. The equivalence (0) $\Leftrightarrow$ (1) is deduced as in §2.2 by taking the n-th derivative with respect to λ of the constant function $1 = -\sum_{1<j<N} \mathrm{e}^{i(\theta_j+\lambda\phi_j)}$. Condition (1) is easily implied by (2), while for the opposite direction we observe that for all $n \in \mathbb{N}, a \in \mathbb{R}\setminus\{0\}$,

$$\sum_{1<j<N} \mathrm{e}^{i\theta_j}\left(\frac{\phi_j}{a}\right)^n = \frac{1}{a^n}\sum_{1<j<N} \mathrm{e}^{i\theta_j}\phi_j^n = 0.$$

If all ϕ_j's are equal to 0, we are done since the polyline associated to the turning angles θ_j is closed. Otherwise, letting $A = \max_j\{|\phi_j|\}$,

$$0 = \lim_{n\to\infty} \sum_{1<j<N} e^{i\theta_j}\left(\frac{\phi_j}{A}\right)^{2n} = \sum_{j\in\phi^{-1}(A)} e^{i\theta_j} + \sum_{j\in\phi^{-1}(-A)} e^{i\theta_j},$$

$$0 = \lim_{n\to\infty} \sum_{1<j<N} e^{i\theta_j}\left(\frac{\phi_j}{A}\right)^{2n+1} = \sum_{j\in\phi^{-1}(A)} e^{i\theta_j} - \sum_{j\in\phi^{-1}(-A)} e^{i\theta_j},$$

which entails $\sum_{j\in\phi^{-1}(A)} e^{i\theta_j} = \sum_{j\in\phi^{-1}(-A)} e^{i\theta_j} = 0$. For $A' = \max_j\{|\phi_j| \mid |\phi_j| < A\}$, it holds analogously

$$\begin{aligned} 0 &= \lim_{n\to\infty} \sum_{1<j<N} e^{i\theta_j}\left(\frac{\phi_j}{A'}\right)^{n} \\ &= \lim_{n\to\infty} \sum_{j\notin\phi^{-1}(\pm A)} e^{i\theta_j}\left(\frac{\phi_j}{A'}\right)^{n} + \left(\frac{\pm A}{A'}\right)^{n} \sum_{j\in\phi^{-1}(\pm A)} e^{i\theta_j} \\ &= \sum_{j\in\phi^{-1}(A')} e^{i\theta_j} \pm \sum_{j\in\phi^{-1}(-A')} e^{i\theta_j} + 0, \end{aligned}$$

and we conclude by iterating the same argument until we exhaust all the finitely many vertices of the polyline. □

In order to find non-trivial pairs such that the polyline associated to $(k_j) + \lambda(f_j)$ is closed for all λ, by Theorem 2.5.1 the polyline associated to (k_j) must possess at least one proper subset $\bar{V} \subset \{2, ..., N-1\}$ of indices that is *balanced*, meaning $\sum_{j\in\bar{V}} e^{i\theta_j} = 0$. A visualization of this behaviour is given in Figure 2.4.

Note that it is easy to construct polylines with no balanced proper subsets of edges. Consider for example n copies of the pair of unit vectors summing up to $(1/n, 0)$ and either the vector $(-1, 0)$ for a polyline with an odd number of edges or the two unit vectors whose sum is $(-1, 0)$ for an even number. Any proper subset of vectors from the "copies" part either consists of a single vector or its elements sum up to a non-unit vector different from 0. In both case it is not possible to counterbalance the sum with the vector(s) on the other side of the y-axis.

2.6 There is no "sufficient" 4-vertex theorem

The 4-vertex theorem provides a necessary condition for a function to be the curvature of a closed planar curve without self-intersections (see [10] or [9] for a comprehensive survey). In this section we use the results from §2.2 to show in a rigorous way that it

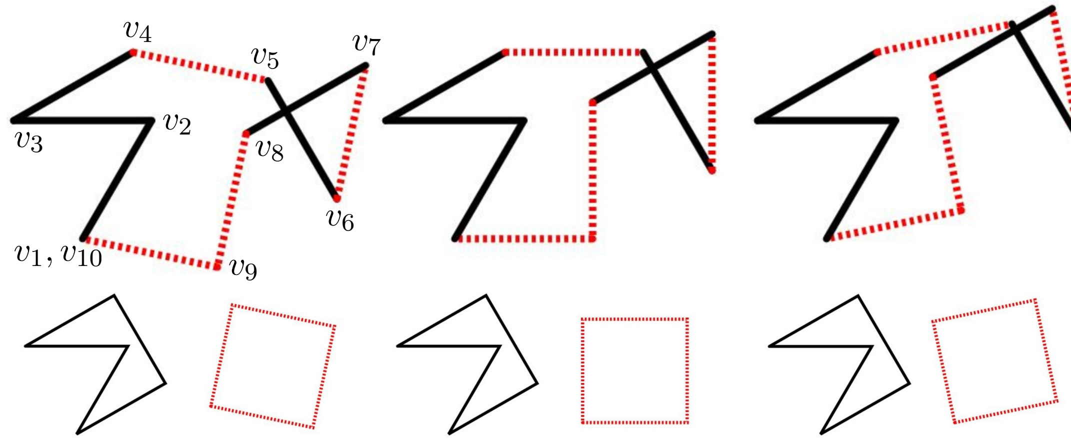

Figure 2.4: The discrete curvature (k_j) of a polyline is modified linearly in λ to $(k_j) + \lambda(f_j)$, with $(f_j) = (0, 0, \phi_1, -\phi_1, \phi_1, -\phi_1, \phi_1, 0)$. Such a curvature vector sums up to the turning angle $(0, 0, \phi_1, 0, \phi_1, 0, \phi_1, \phi_1)$, which rotates, as λ varies, only the dashed edges corresponding to a balanced subset of indices.

is not possible to develop a sufficient condition of the same nature, namely it is not possible to tell whether a curvature function k belongs to an arc-length parametrized closed curve by computing finitely many level sets and evaluations of k, its derivatives and its antiderivatives. The way we want to do this is by first assuming that in the class of analytic functions on I such a procedure exists, and to show afterwards that it is always possible to construct an instance for which such a procedure yields the wrong answer.

We need to formalize what we mean with *procedure*; this is done by first introducing the objects involved one by one, pointing out at the same time their high level meaning. We consider the set of sequences $A = \{\{a_j\}_{j \in \mathbb{N}}\}$, such that $a_j \in \mathbb{R} \cup \Sigma$, with Σ a finite alphabet of symbols; sequences of this type will be used to store the progress of our procedure. Then we fix $L = \{l_j\}$, a countable set of analytic functions on I; this family will generalize the concept of level sets and finite linear combinations of its elements will be considered, selecting coefficients in $C = \{\{c_j\}_{j \in \mathbb{N}_0}\}$, set of real sequences whose terms are 0 for j big enough and $c_0 \in I$ (this is a special term used for evaluating functions at a certain parameter).

Given the real analytic function k that we want to test, our procedure is determined by three functions M, H_k, G.

$$\begin{aligned} M: &\qquad A \to \{E, \cap\} \times \mathbb{Z} \times C, \\ H_k: &\qquad \{E, \cap\} \times \mathbb{Z} \times C \to \{E, \cap\} \times \mathbb{Z} \times A, \\ G: &\qquad A \times \{E, \cap\} \times \mathbb{Z} \times A \to \{\text{YES,NO}\} \cup A, \end{aligned}$$

where M and G can be chosen arbitrarily and represent the functioning of the procedure while H_k depends on the input k and is the only tool we have to extract information about k in the way we are going to specify in the next paragraph.

For $\{a_j\} \in A$, initialized to $a_j = \square$ with $\square \in \Sigma$ for all j, we compute iteratively $G\big(\{a_j\}, H_k\big(M(\{a_j\})\big)\big)$ until an answer YES, the curve with curvature k integrates in the arc-length to a closed curve, or NO, it does not, is output. Concerning the function H_k, for $(\sigma, n, \{c_j\}) \in \{E, \cap\} \times \mathbb{Z} \times C$, we have

$$H_k(\sigma, n, \{c_j\}) = (\sigma, n, \{a_j\}),$$

$$\text{with } \{a_j\} \textit{ storing } \begin{cases} \text{the evaluation } k^{(n)}(c_0), & \text{if } \sigma = E, \\ \text{the solution(s) of } k^{(n)} = \sum_{j \geq 1} c_j l_j, & \text{if } \sigma = \cap, \end{cases}$$

where $k^{(n)}$ denotes the n-th derivative of k for n positive, k itself for $n = 0$ and the n-th antiderivative of k for n negative, meant as the result of taking $-n$ times the operation $\int_0^t$. With *storing* we mean just sequentially writing the result of the evaluation or the solution(s) of the equation if they exist and are finitely many, or using special symbols from Σ if there are no solutions or the two functions coincide (these are the only remaining possibilities in an analytic regime as ours). Unused terms of the sequence are just filled with the blank symbol $\square$.

To summarize, M looks at the current sequence $\{a_j\}$ and determines what is the informations H_k should extract from k or one of its antiderivatives/derivatives. Then, G considers the result output from H_k together with a copy of $\{a_j\}$ and either decides an answer to the problem or rather merges the new information updating the sequence in A. As anticipated, in the following we assume for a contradiction that there exist functions M, G and a set L such that the procedure $\mathfrak{T}$ they define in the sense above is *correct*, meaning that, for any analytic functions k on I, it decides an answer $\mathfrak{T}(k) \in$\{YES,NO\} after finitely many iterations and that $\mathfrak{T}(k)$ =YES if and only if the arc-length parametrized curve with curvature k is closed.

Our strategy is to perturbate the function k of a closed curve to $k + \lambda p$, with p another analytic function on I and λ a real number such that still $\mathfrak{T}(k + \lambda p)$ =YES, but this time the associated curve is not closed anymore, entailing a contradiction.

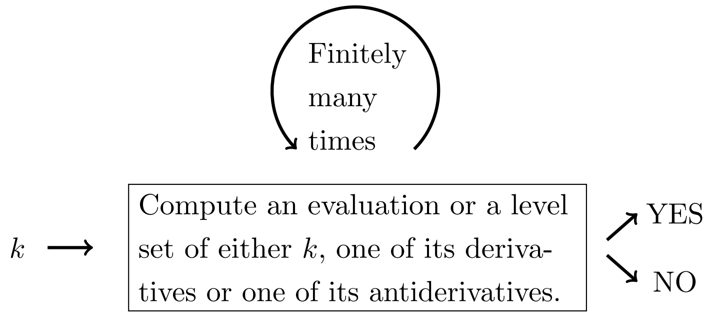

Figure 2.5: High level diagram of the decision procedure. Given a curvature function k, finitely many evaluations and/or level set of k, its derivatives and its antiderivatives are computed to output the answer YES, the associated curve is closed, or NO, it is not.

Lemma 2.6.1. *If $\mathfrak{T}$ is a correct procedure, then there exist k, P non-constant analytic functions on I and $N \in \mathbb{N}$ such that $\mathfrak{T}\big(k + \lambda(\psi \cdot P)^{(N)}\big)$=YES for any ψ analytic on I and for all $\lambda \in \mathbb{R}$.*

Proof. We start by considering the curvature function k of a closed curve such that k, its derivatives and its antiderivatives are independent from the set L, *i.e.* there is no finite combination of elements in L that equals to any of them. Calling $\theta = \int k$, this can be done for example by observing that the condition $\int \mathrm{e}^{i\theta} = 0$ for a curve to be closed allows a family with the cardinality of the continuum of independent functions or, more explicitely, by using the existence result from §2.4 and condition (2) of Theorem 2.2.5 to construct a closed curve with turning angle $\theta + g(\phi)$ with g an appropriate analytic function that guarantees the independency from L.

We construct then the set of triples $S = \{(t_j, d_j, n_j)\} \subset I \times \mathbb{N} \times \mathbb{Z}$, where the t_j's are the single roots of the equations involving $k^{(n_j)}$ that the procedure solved to conclude the answer YES, and the d_j's the respective degrees of such roots. At the same time, we put in S also the triples $(t, 1, n)$ if an evaluation of $k^{(n)}$ has been computed at t over the run. Since k has been chosen independent from L, the set S is the complete record of what information has been extracted from k by H_k. Calling $D := \max_j\{d_j\}$ and $N := \max_j\{|n_j|\}$, we define the polynomial

$$P(t) = t^N \prod_j (t - t_j)^{2N+D}.$$

By construction, k and P as above satisfy, for all ψ and all λ small enough, $\mathfrak{T}\big(k + \lambda(\psi \cdot P)^{(N)}\big)$=YES; in fact, for small perturbations, $\mathfrak{T}$ performs exactly the same sequence of iterations and therefore ouputs the same result. In §2.2 we saw that

$F(\lambda) = \int_0^{2\pi} \mathrm{e}^{\left(i(k+\lambda(\psi\cdot P)^{(N)}\right)}$ is analytic in λ and therefore, if the procedure is correct as we assumed, it actually holds $\mathfrak{T}\big(k + \lambda(\psi \cdot P)^{(N)}\big)$=YES for all $\lambda \in \mathbb{R}$. $\square$

We are ready to prove the theorem promised at the beginning of the section.

Theorem 2.6.2. *There is no sufficient 4-vertex theorem, i.e. there is no correct procedure to determine with finitely many iterations whether the curve associated to the curvature function k is closed by computing finitely many evaluations and/or generalized level sets of k, its derivatives and its antiderivatives.*

Proof. Let k and P be chosen as in Lemma 2.6.1. We construct ψ for which it is apparent that condition (2.2) cannot hold for the pair $\big(k, (\psi \cdot P)^{(N)}\big)$, therefore obtaining a contradiction. We choose $\bar{t}$ where $P(\bar{t}) \neq 0$ and consider the family of *triangle functions* T_δ, attaining the value 0 outside the interval $[\bar{t} - \delta, \bar{t} + \delta]$ and linearly interpolating the value $T_\delta(\bar{t}) = 1/\delta$. By the Stone-Weierstrass theorem, for $\varepsilon > 0$, we can find a polynomial $h_{\varepsilon,\delta}$ such that

$$\sup_I |h_{\varepsilon,\delta} - T_\delta| < \varepsilon \text{ and hence } \sup_I \big|h_{\varepsilon,\delta}^{(-j)} - T_\delta^{(-j)}\big| < \varepsilon(2\pi)^j,\ \forall j \in \mathbb{N},$$

where the superscript $(-j)$ means the j-th antiderivative $\int_0^t$ of a function. For $j \geq 1$, it holds

$$\sup_I h_{\varepsilon,\delta}^{(-j)} < (2\pi)^{j-1} + \varepsilon(2\pi)^j.$$

We consider now

$$\big(h_{\varepsilon,\delta}^{(-N)} \cdot P\big)^{(N)} = h_{\varepsilon,\delta}P + \sum_{1\leq j\leq N} \binom{N}{j} h_{\varepsilon,\delta}^{(-j)} P^{(j)}. \tag{2.3}$$

With $M = \max_{1\leq j\leq N} \sup_I P^{(j)}$, we see that the second term of (2.3) is bounded by $\sum_{1\leq j\leq N} \binom{N}{j}(2\pi)^j(1+\varepsilon)M$, which does not depend on δ. Choosing $\psi = h_{\varepsilon,\delta}$ with ε and δ small enough, the maxima on I of $\big(h_{\varepsilon,\delta}^{(-N)} \cdot P\big)^{(N)}$ are all contained in an arbitrarily small neighborhod of $\bar{t}$. This makes it impossible to satisfy condition (2.2), which is the contradiction we needed to conclude the theorem. $\square$

Remark 2.6.3. Showing the impossibility of a sufficient analogue of the 4-vertex theorem cannot be reduced to a cardinality argument. For example, if we are just interested in constructing a procedure as the one described at the beginning of the section for curvature functions over I that are π-periodic and such that $|k^{(-1)}(s)| \leq \pi$ for $s \leq \pi$, then the associated curve is closed if and only if $k^{(-1)}(\pi) = \pi$ and it is therefore enough to compute such an evaluation to conclude the correct answer. The interplay between closed curves and periodic curvature functions has been characterized in [2].

Future work.

Given the curvature function k of a closed curve, when is it possible to find f such that the curve associated to $k + \lambda f$ is closed for all λ? In §2.4 we identified a class of pairs of functions that satisfies this condition and the next obvious step would be a full characterization in the C^h and analytic setting. Thinking in terms of the turning angle $\theta = \int k$, a possible way of approaching the problem could be by synthesizing a Fourier series for ϕ that would satisfy the family of orthogonality relations $\int \mathrm{e}^{i\theta}\phi^n = 0$ in L^2. The ugliness of the convolution formula for the Fourier coefficients of a product prevented the author from succeeding.

Another nice improvement would be the generalization of the periodicity result from §2.3 to the case $\phi(0)$ being a critical value. This would also make the proof of the non-existence of a "sufficient" 4-vertex theorem in §2.6 more agile.

Acknowledgments

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3 Propagation of curved folding: The folded annulus with multiple creases exists

Abstract

In this paper we consider developable surfaces which are isometric to planar domains and which are piecewise differentiable, exhibiting folds along curves. The paper revolves around the longstanding problem of existence of the so-called folded annulus with multiple creases, which we partially settle by building upon a deeper understanding of how a curved fold propagates to additional prescribed foldlines. After recalling some crucial properties of developables, we describe the local behaviour of curved folding employing normal curvature and relative torsion as parameters and then compute the very general relation between such geometric descriptors at consecutive folds, obtaining novel formulae enjoying a nice degree of symmetry. We make use of these formulae to prove that any proper fold can be propagated to an arbitrary finite number of rescaled copies of the first foldline and to give reasons why problems involving infinitely many foldlines are harder to solve.

3.1 Introduction

In recent years growing attention has been paid to the field of mathematical origami. The process of folding paper with the intent of crafting objects of art dates back to ancient China and Japan; although the earliest hard evidence of such an exercise is from the 16th century, it is possible that paper folding has been already practiced shortly after paper arrived in Japan via Buddhist monks in the 6th century [27]. As objects of combinatorics and kinematics, origami have been studied by many authors over a broad and diverse literature [8], [46].

Moving from the seminal paper [21], the scientific community has also investigated the differential geometry of origami obtained by folding along curves, rather than straight lines. This is due not just to a theoretical interest but also to the role that surfaces obtainable by bending a flat foil (developables) have acquired in the interdependent fields of design, manufacturing and architecture in recent years [7], [24], [36], [40], [41], [42].

Even if the local geometry of folding along a single curve is well understood [13], the case of a nontrivial pattern of foldlines is challenging and may require ad hoc solutions [6] or numerical optimization [22]. The main intention of the present paper is to approach the propagation of a curved fold to the next prescribed foldline from a broad perspective, highlighting the role played by the regression curve of developables and providing formulae that describe the phenomenon in its full generality and complexity but that can still be employed to get new insights on its specificities. Also, we want to address the well known problem concerning the foldability of patterns involving concentric closed convex foldlines and contribute to the issue raised at the very end of [5]:

> ... we conjecture that the circular pleat indeed folds, and that so too does any similar crease pattern consisting of a concentric series of convex smooth curves. Unfortunately a proof remains elusive. Such a proof would be the first proof to our knowledge of the existence of any curved-crease origami model, beyond the local neighborhood of a single crease.

Some existence results were obtained [6] but to the knowledge of the author no progress has been made in constructing examples of folds along multiple concentric curves. We here finally provide explicit instances of such a kind (Fig. 3.1, 3.2), present arguments that guarantee the existence of folds involving any finite number of concentric foldlines and give reasons why the proof still remains *elusive* when it comes to patterns with infinitely many foldlines. We want to stress that in this paper we tackle the curved folding subject from the perspective of isometric maps, without addressing the issue of continuous deformations, which is nevertheless another interesting and relevant topic. In our setting, folds as the one in Fig. 3.2 are legitimate while they would not be possible if one requires the existence of a continuous deformation: in our example the linking number of any two curves bounding a developable strip, which is invariant under isotopy, is different from the linking number of two concentric circles [37].

As for an outline of the content, §3.2 settles the notation about some natural geometric descriptors for curves and surfaces of Euclidean space and recalls how a surface isometric to the plane admits a ruled parametrization. In §3.3 we describe how paper locally folds along a curve by discussing its behaviour in terms of the normal curvature and relative torsion of the ridge; the degree of symmetry of the formulae obtained points out how such parameters are to some extent the natural ones to describe the problem. In §3.4, three methods for folding along a circle and, more in general, along a closed convex curve are described. In §3.5, the formulae describing the relations between two

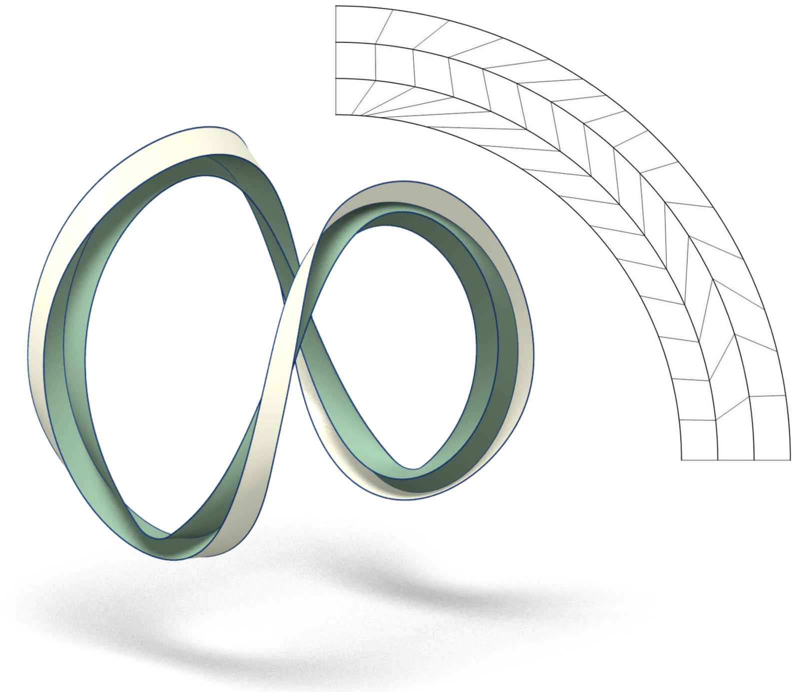

Figure 3.1: Fold along two concentric circles of an annulus with inner radius 0.905 and outer radius 1.19. The two inner developables (green) are obtained by extending the isometry between the unit circle and the rescaled intersection of the unit sphere with the hyperbolic paraboloid of equation $z - 3xy = 0$. The outer developable (white) is induced once the second concentric foldline is prescribed. On the right, we show the ruled structure of one of the circular sectors of the annulus (equivalent up to reflection).

consecutive curved foldlines are presented. In §3.6 we prove that any fold along one foldline can be propagated to any number of rescaled copies of itself, if the scaling factor is small enough. Finally, in §3.7 we discuss how the propagation of a fold can turn singular in an arbitrarily abrupt manner, implying that an existence proof of foldability on any pattern with infinitely many prescribed foldlines must involve a control mechanism on the derivative of all orders. The appendix contains a more thorough discussion of the examples in Fig. 3.1, 3.2, employing the formulae from §3.5 to make apparent the regularity of the developables involved.

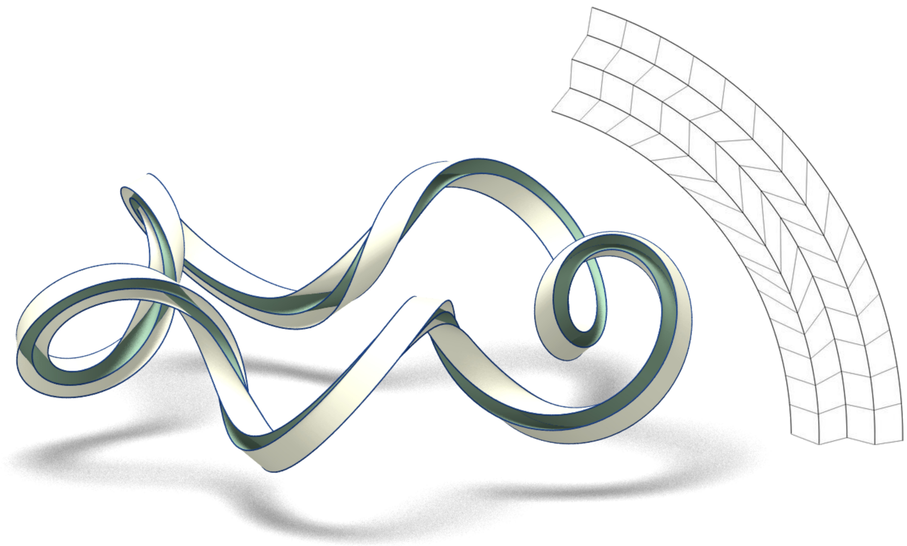

Figure 3.2: Fold along three concentric circles of an annulus with inner radius 0.86 and outer radius 1.14. The two middle developables (green) are obtained by extending the isometry between the unit circle and the toroidal unknot $\omega_{3,(9,2)}$ (see §3.4). The outer and the inner developables (white) are induced once two additional concentric foldlines are prescribed. On the right, we show the ruled structure of one of the circular sectors of the annulus (equivalent up to rotation).

3.2 Space curves and parabolic developables

If γ is an arc-length parametrized C^3 curve, denoting the derivative with respect to the arc-length parameter with a prime, we define the Frenet frame of γ as the triple of orthonormal vectors $\{T, N, B\} := \left\{\gamma', \frac{\gamma''}{\|\gamma''\|}, \gamma' \times \frac{\gamma''}{\|\gamma''\|}\right\}$. At the same time, if the curve is known to be lying on a surface of $\mathbb{R}^3$ whose unit normal at $\gamma(s)$ is $n(s)$, we can define also the Darboux frame as $\{T, u, n\} := \{\gamma', n \times \gamma', n\}$. The coefficients that express the first derivative of such bases with respect to the basis itself have significant geometric meanings,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \qquad \begin{pmatrix} T' \\ u' \\ n' \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_r \\ -k_n & -\tau_r & 0 \end{pmatrix} \begin{pmatrix} T \\ u \\ n \end{pmatrix}.$$

The nonnegative function k is called *curvature* of the curve. The *geodesic curvature* k_g with respect to the given surface is the length of the projection of the *curvature vector* $k \cdot N$ to the *tangent plane* of the surface, spanned by T and u, and signed with respect to u. The *normal curvature* k_n is the signed length of the projection of the same curvature vector to the normal direction n. The function τ is called *torsion* of the curve, while τ_r is the *relative torsion* with respect to the given surface.

For our purposes it will be useful to express the geometric descriptors above as a function of the angle α between the *osculating plane*, spanned by T and N, and the tangent plane of the surface. We measure α anticlockwise by looking at the angle between B and n from the tip of T. Then

$$k_g = \cos(\alpha)k, \qquad k_n = -\sin(\alpha)k, \qquad \tau_r = \tau + \alpha'.$$

For a more extensive treatment and additional insights about the quantities and formulae above the reader may refer to [10, §1-5 and exercise 19 in §3-2].

If a regular surface is of class C^2, we can compute at each point and in each tangent direction v its *normal curvature*, that is the curvature of the section obtained by intersecting the surface with the plane spanned by the tangent v and the normal to the surface n. Varying v, we call *principal curvatures* the maximum k_1 and the minimum k_2 among the normal curvatures. The product $K := k_1 \cdot k_2$ is the *Gaussian curvature* of the point. The Theorema Egregium by Gauss guarantees that its value is preserved under C^2 isometries ([18, pp. 759–760] or [10, §4-3] assuming C^3 regularity). A surface that locally can be obtained as image of a planar domain by a C^h isometry is called a *C^h developable.* If $h \geq 2$, because of the invariance just discussed, its Gaussian curvature must be everywhere 0; we call a point of such a surface *parabolic* if the two principal curvatures satisfy (up to relabelling) $k_1 \neq k_2 = 0$ and *flat* if instead $k_1 = k_2 = 0$.

If parabolic points are dense on the surface, it can be shown that a unique straight line (a *ruling*) passes through any of its points and that the tangent plane of the surface along this line is constant. In the rest of the paper we will be interested in developable surfaces that have only parabolic points; in this case its representation as a family of rulings (*ruled parametrization*) enjoys useful regularity properties. In the following, adapting [18, pp. 769-770], we give a self-contained proof of such a characterization theorem, without making direct use of any concepts other than the geodesic and normal curvature of a curve.

Theorem 3.2.1. *[18] Let S be a C^h developable surface with $h \geq 2$ and no flat points. For any point $p \in S$ there exist a C^h arc-length parametrized curve $\gamma : [-\varepsilon, \varepsilon] \to \mathbb{R}^3$ and a C^{h-1} function $r : [-\varepsilon, \varepsilon] \to \mathbb{R}^3$, with $\|r\| \equiv 1$, such that, in a neighbourhood of p, S can be parametrized as $a(u,t) = \gamma(u) + t \cdot r(u)$. Moreover, fixed $\bar{u}$, the tangent plane of the surface is constant along the ruling $a(\bar{u}, t)$.*

Proof. We consider a C^h isometry $\sigma : D_\varepsilon := [-\varepsilon, \varepsilon]^2 \to S, (u,v) \mapsto \sigma(u,v)$. Being an isometry in this case simply means that

$$\langle \sigma_u, \sigma_u \rangle = \langle \sigma_v, \sigma_v \rangle \equiv 1, \;\; \langle \sigma_u, \sigma_v \rangle \equiv 0,$$

where we denoted with a subscript the derivative with respect to a certain parameter and with $\langle ., . \rangle$ the Euclidean inner product between two vectors. As we did above, also in the following we will often omit the dependency on the parameters. Taking one more derivative with respect to u and v of the relations above, we obtain

$$\begin{aligned} &\langle \sigma_{uu}, \sigma_u \rangle = \langle \sigma_{uu}, \sigma_v \rangle = \langle \sigma_{vv}, \sigma_u \rangle = \\ &\langle \sigma_{vv}, \sigma_v \rangle = \langle \sigma_{uv}, \sigma_u \rangle = \langle \sigma_{uv}, \sigma_v \rangle \equiv 0, \end{aligned}$$

which implies that σ_{uu}, σ_{uv} and σ_{vv} are all equal to the unit normal to the surface $n := \sigma_u \times \sigma_v$ up to a respective signed coefficient L, M and N. We finally observe that

$$0 = \langle \sigma_{uu}, \sigma_v \rangle_v - \langle \sigma_{uv}, \sigma_v \rangle_u = LN - M^2. \tag{3.1}$$

From this condition we see that L and N have the same sign, which we can assume, up to reversing one coordinate, being positive. We now want to find the condition for a curve on the surface to be *asymptotic*, that is to have constant vanishing normal curvature. If $(u(t), v(t))$ is an arc-length parametrized curve in D_ε, then $\gamma := \sigma(u(t), v(t))$ is an arc-length parametrized curve on S and

$$\begin{aligned} \gamma' &= u'\sigma_u + v'\sigma_v, \\ \gamma'' &= \big(L(u')^2 + 2Mu'v' + N(v')^2\big)n + u''\sigma_u + v''\sigma_v. \end{aligned}$$

The conditions of vanishing normal curvature becomes

$$0 = k_n = \langle \gamma'', n \rangle = \big(L(u')^2 + 2Mu'v' + N(v')^2\big) \overset{(3.1)}{=} \big(\sqrt{L}u' + \sqrt{N}v'\big)^2.$$

Since all points of S are parabolic, in D_ε we can assume $\langle \sigma_{uu}, n \rangle = L \neq 0$ (which for ε small enough guarantees also $\langle \sigma_{uu}, n(0,0) \rangle \neq 0$). The Cauchy problem

$$\begin{cases} \sqrt{L}u'(t) + \sqrt{N}v'(t) = 0 \\ (u(0), v(0)) = (s, 0) \end{cases}$$

plus the extra constraint $(u')^2 + (v')^2 = 1$ defines for each s a unique arc-length parametrized asymptotic curve $l_s(t) = \sigma(u(t), v(t))$. Such a curve satisfies $v' \neq 0$.

By observing that

$$\begin{aligned} (\sigma_u)' &= (Lu' + Mv')n \overset{(3.1)}{=} \sqrt{L}(\sqrt{L}u' + \sqrt{N}v')n = 0, \\ (\sigma_v)' &= (Mu' + Nv')n \; = \; \sqrt{N}(\sqrt{L}u' + \sqrt{N}v')n \; = 0, \end{aligned}$$

we also have proven that a curve is asymptotic if and only if the tangent plane to the surface and therefore its normal are constant along it. Hence asymptotic curves are planar, the normal to the plane on which they lay being such that $\langle n_u(s,t), n(0,0)\rangle$ is monotone in s for fixed t.

Locally S must be the envelope of the tangent planes orthogonal to $n(s,0)$, which makes it is easy to see that its asymptotic curves must actually be lines. In fact

$$\begin{aligned} l_{s+h} &\subseteq \{x \in \mathbb{R}^3 \mid \langle x, \sigma_u \times \sigma_v(s+h,0)\rangle = 0\}, \\ l_s &\subseteq \{x \in \mathbb{R}^3 \mid \langle x, \sigma_u \times \sigma_v(s,0)\rangle = 0\}, \end{aligned}$$

and in the limit, by continuity, l_s must lay also on another plane:

$$\begin{aligned} l_s \subseteq & \Big\{x \in \mathbb{R}^3 \mid \langle x, \lim_{h\to 0} \frac{\sigma_u \times \sigma_v(s+h,0) - \sigma_u \times \sigma_v(s,0)}{h}\rangle = 0\Big\} = \\ &= \Big\{x \in \mathbb{R}^3 \mid \langle x, L\sigma_u + M\sigma_v\rangle = 0\Big\}. \end{aligned}$$

The ruled parametrization we were looking for is finally given by setting $\gamma(u) = \sigma(u,0)$ and $r(s)$ as the unit direction of the line l_s. The Cauchy problem we have solved to construct $r(s)$ just implies that r enjoys C^{h-2} regularity. To show that it is actually C^{h-1}, we need to observe that under our assumptions the implicit function theorem guarantees for s and v small enough the existence of a C^{h-1} solution $u(s,v)$ to the equation constraining σ_u to be constant along the parameter v

$$\langle \sigma_u(s,0), n(0,0)\rangle = \langle \sigma_u\big(u(s,v),v\big), n(0,0)\rangle,$$

and that it holds $\sigma(s,0) + \sigma(u(s,v),v) = \gamma(s) + v \cdot r(s)$. □

Remark 3.2.2. For those readers more familiar with the language of fundamental forms, the functions we named as L, N, M are the coefficients of the matrix representing the second fundamental form of the developable surface with respect to the isometric parametrization σ. In our setting the Gaussian curvature can simply be computed as the determinant of such a matrix, which is in fact an equivalent way of obtaining equation (3.1) if one knows that Gaussian curvature is preserved under isometries. Our derivation, which is also the only step of the proof where we use that the regularity of the surface is at least C^3, can be considered an *ad hoc* proof of the Theorema Egregium for developable surfaces. Since the invariance of the Gaussian curvature under isometries can be proven in the more general C^2 case, if we assume this fact, then Theorem 3.2.1 holds for C^2 surfaces as well.

Before moving to the next section we recall the elementary fact at the core of the local geometry of curved folding.

Lemma 3.2.3. *[10, §4-2] The geodesic curvature of a planar curve is preserved under isometries of the planar domain in which it is contained.*

Proof. In analogy with the notation used in the proof of Theorem 3.2.1, let $u(t)$ be the arc-length parametrization of the curve contained in the planar domain D and σ an isometry from D to a developable surface. Calling $\gamma(t) := \sigma(u(t))$, we project its second derivative to the tangent plane spanned by σ_u and σ_v, obtaining

$$k_g = \|u''\sigma_u + v''\sigma_v\| = \sqrt{(u'')^2 + (v'')^2},$$

which is the curvature of the planar curve we started with (for planar curves, curvature and geodesic curvature coincide). □

3.3 Local curved folding

In the following, *folding along* a *foldline*, which is a curve contained in an open domain of $\mathbb{R}^2$, means the isometric mapping of such a planar domain onto two C^1 *good surfaces* (decomposable as a finite complex of C^2 regions joined by vertices and C^2 edges, [5]) that meet with C^0 regularity (and not more) along the image of the foldline. With folding the foldline *onto* a space curve we mean folding along the foldline in such a way that its image under the isometry is the given space curve, which we call the *ridge*. A visualization of a curved fold about a point of a C^2 ridge is given in Fig. 3.3. In [6], where local curved folding onto good surfaces is thoroughly studied, it is shown that, in order to construct an isometry on both sides of the foldine, the regularity of the ridge cannot be C^1 while not being C^2 so, unless the ridge is kinked, its Frenet frame is well-defined. In the rest of the paper we will be mainly interested in folding along foldlines and onto ridges whose regularity is at least C^3.

In order to fix the notation and to explain why Fig. 3.3 is substantially the only way a local curved fold can look like we recall a couple of formulae from [13]. These describe how paper locally folds along a given C^3 foldline with curvature $k_g > 0$ once a C^3 ridge is prescribed. We call n_+ the normal to the developable such that the angle α_+ between the binormal vector B of the Frenet frame of γ and n_+, measured anticlockwise with respect to $T = \gamma'$, has value $0 < \alpha_+ < \frac{\pi}{2}$. Analogously, the normal n_- and the angle α_- are defined for the other developable to satisfy $0 > \alpha_- > -\frac{\pi}{2}$. Since geodesic curvature is preserved by Lemma 3.2.3, denoting with k the curvature of the ridge γ, we have

$$\cos(\alpha_+)k = k_g = \cos(\alpha_-)k,$$

and therefore $\alpha_+ = -\alpha_-$ if, as in our definition of folding, the transition from one side to the other must be just C^0. More precisely, a fold is called *proper* when the above relation

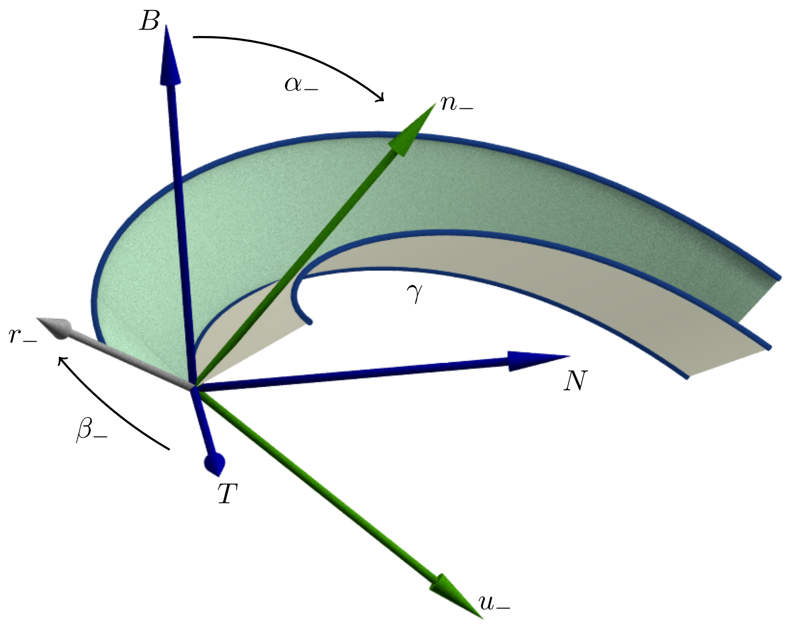

Figure 3.3: $\{T, N, B\}$ is the Frenet frame of the ridge, while $\{T, u_-, n_-\}$ its Darboux frame with respect to the outer green surface oriented by n_-. The tangent plane to the surface is spanned by T and u_- and the the ruling direction r_- lies on it. The Darboux frame of the inner white surface can be obtained by simply rotating $\{T, u_-, n_-\}$ by $-2\alpha_-$ about T.

is well defined and $\alpha_+ \neq 0, \frac{\pi}{2}$ which is the case iff $k > k_g$ and $k_g \neq 0$. For intuition, $\alpha_+ \neq 0$ ensures some folding is actually happening and $\alpha_+ \neq \frac{\pi}{2}$ that the developables on the two sides do not overlap each other.

Lemma 3.3.1. *[13] Given a C^3 foldline $\bar{\gamma}$ and a C^3 ridge of the same length γ with curvatures $0 < k_g < k$, then, on the two sides of the osculating plane of γ, two different proper folds are possible along $\bar{\gamma}$ onto γ and the unit directions of the rulings of the developables are given by*

$$r_S = \frac{\tau_{(r,S)}T - k_{(n,S)}\big(\cos(\alpha_S)N + \sin(\alpha_S)B\big)}{\sqrt{\tau^2_{(r,S)} + k^2_{(n,S)}}}, \quad S \in \{+, -\},$$

where $\{T, N, B\}$ is the Frenet frame of γ. The symbols $\tau_{(r,+)}$, $k_{(n,+)}$ denote the relative torsion resp. the normal curvature of γ with respect to the developable whose normal n_+ forms with B the angle $0 < \alpha_+ < \frac{\pi}{2}$, when measured anticlockwise with respect to T. Analogous notation is used for $\tau_{(r,-)}, k_{(n,-)}$, $-\frac{\pi}{2} < \alpha_- < 0$ and n_- for the second developable.

Proof. Since $k_g < k$ two C^1 continuous families of planes can be defined along γ by rotating its osculating plane about T by the angles α_+ and α_- with $0 < \alpha_+ = -\alpha_-$, such that $k_g = \cos(\alpha_+)k$ for $S \in \{+,-\}$. It is well known that the envelope of such a family of plane locally defines a C^1 developable surface whose rulings can be obtained by intersecting the planes orthogonal to $n_S := -\sin(\alpha_S)N + \cos(\alpha_S)B$ and n'_S, where the derivation is taken with respect to the omitted arc-length parameter (a way of seeing this is by doing reverse engineering on the proof of Theorem 3.2.1). Setting τ as the curvature of γ, we have

$$\begin{aligned} n_S &= -\sin(\alpha_S)N + \cos(\alpha_S)B, \\ n'_S &= k\sin(\alpha_S)T - \big(\tau\cos(\alpha_S) + \alpha'_S\cos(\alpha_S)\big)N - \big(\alpha'_S\sin(\alpha_S) + \tau\sin(\alpha_S)\big)B. \end{aligned}$$

and the two planes orthogonal to such vectors intersect at a line with direction

$$\begin{aligned} n_S \times n'_S &= (\tau + \alpha'_S)T + k\sin(\alpha_S)\cos(\alpha_S)N + k\sin(\alpha_s)^2 B \\ &= \tau_{(r,S)}T - k_{(n,S)}\big(\cos(\alpha_S)N + \sin(\alpha_S)B\big), \end{aligned}$$

which, after normalizing, provides the expression for the unit direction r_S. If we want to extend the isometry on the two sides of $\bar{\gamma}$ to obtain a proper fold we have two possibilities: for v positive, the ruled parametrizations $\gamma + v \cdot r_S$ describes the developables on the same side as B of the osculating plane of γ, while for v negative, the one on the opposite side. $\square$

The next lemma relates the normal curvature and relative torsion of a ridge with respect to one developable to the normal curvature and relative torsion of the same ridge with respect to the developable on the opposite side.

Lemma 3.3.2. *In the notation of Lemma 3.3.1, we have the equalities*

$$\begin{aligned} k_{(n,S)} &= -k_{(n,\bar{S})}, \\ \tau_{(r,S)} &= \tau_{(r,\bar{S})} - 2\alpha'_{\bar{S}} = \tau_{(r,\bar{S})} - 2\frac{k'_g k_{(n,\bar{S})} - k_g k'_{(n,\bar{S})}}{k_g^2 + k_{(n,\bar{S})}^2}, \end{aligned}$$

where $S \in \{+,-\}$ and $\bar{S}$ is the opposite sign of S.

Proof. By direct computation,

$$\begin{aligned} \tau_{(r,S)} &= \tau + \alpha'_S = \tau_{(r,\bar{S})} - \alpha'_{\bar{S}} - \alpha'_{\bar{S}} = \tau_{(r,\bar{S})} - 2\alpha'_{\bar{S}} \\ &= \tau_{(r,\bar{S})} + 2\frac{(\cos(\alpha_{\bar{S}}))'}{\sin(\alpha_{\bar{S}})} = \tau_{(r,\bar{S})} - 2\frac{\Big(k_g/\Big(\sqrt{k_g^2 + k_{(n,\bar{S})}^2}\Big)\Big)'}{k_{(n,\bar{S})}/\Big(\sqrt{k_g^2 + k_{(n,\bar{S})}^2}\Big)} \\ &= \tau_{(r,\bar{S})} - 2\frac{k'_g k_{(n,\bar{S})} - k_g k'_{(n,\bar{S})}}{k_g^2 + k_{(n,\bar{S})}^2}. \end{aligned}$$

$\square$

By calling β_S the functions measuring, anticlockwise with respect to B, the angle between T and r_S, direct computations provide the following lemma.

Lemma 3.3.3. *Given a proper fold along the C^3 foldline $\bar{\gamma}$ onto the C^3 ridge γ, the angles β_S between T and r_S, for $S \in \{+,-\}$, satisfy*

$$\cos(\beta_S) = \frac{\tau_{(r,S)}}{\sqrt{\tau_{(r,S)}^2 + k_{(n,S)}^2}}, \quad \sin(\beta_S) = \frac{-k_{(n,S)}}{\sqrt{\tau_{(r,S)}^2 + k_{(n,S)}^2}}, \quad \cot(\beta_S) = -\frac{\tau_{(r,S)}}{k_{(n,S)}} =: f_S,$$

and

$$\beta_S' = -\frac{(\cos(\beta_S))'}{\sin(\beta_S)} = \frac{\tau_{(r,S)}' k_{(n,S)} - k_{(n,S)}' \tau_{(r,S)}}{\tau_{(r,S)}^2 + k_{(n,S)}^2} = -f_S' \frac{1}{1+f_S^2}.$$

Lemma 3.3.4. *The developable surfaces on the two sides of a proper curved fold have no planar points.*

Proof. By knowing the normal curvature $k_{(n,S)}$ of the ridge with respect to the developable surface and the angle β_S its tangent forms with the ruling direction, we can retrieve the nonzero principal curvature $k_{(p,S)}$ by using Euler's formula [10, §3-2],

$$k_{(p,S)} = \frac{k_{(n,S)}}{\sin(\beta_S)^2}.$$

Since we are considering proper folds, this expression is well defined and nonzero. Finally, it is a classical result that if a ruling contains a parabolic, resp. a flat point, then all of its points must be parabolic, resp. flat [43, Cor. 6, Chap. 5].

□

Although in general the regularity of the ruled parametrization of a developable surface is not greater than C^0 (see [47] for an explicit analysis of this phenomenon), if the developable presents no planar rulings as in the case of a proper fold then the regularity of the surface passes over to the ruled parametrization in the way described in Theorem 3.2.1. In particular, if the foldline and the ridge are of regularity class C^h then the ruled parametrization is C^{h-2}.

The last task we tackle in this section is concerned with locating singular points of the developables of a proper fold, that is identifying the so called *regression curve*, obtained as the envelope of the family of rulings of the developable:

$$R_S = \gamma - \frac{\langle \gamma', r_S' \rangle}{\langle r_S', r_S' \rangle} r_S = \gamma + \frac{\sin(\beta_S)}{\beta_S' + k_g} r_S.$$

This expression is easy to obtain by computing the limit intersection of two rulings approaching each other in the developed state (the formula can for example be found in

[39], where developables of low smoothness are investigated in relation to their regression curve). In the setting of proper folding, if we assume the developable is C^2 then r is C^1 and the regression curve must be projectively continuous, that is it can possibly have points at infinity. If r is just piecewise C^1, the expression above is still well-defined if one allows jump discontinuities to occur.

3.4 Local folding along closed convex curves

In this short section we provide three ways one can construct closed space curves onto which it is locally possible to fold along closed convex foldlines. The fact that the curvature of the ridge must everywhere be strictly greater then the curvature of the foldline (Lemma 3.3.1) implies a necessary condition to proper fold along a closed curve [13]: the total curvature of the ridge must be strictly greater than 2π, preventing it from lying in a plane by Fenchel's theorem [11].

Ridges on a sphere

If a curve on the unit sphere is longer than 2π then, by adequate rescaling, it is possible to fold onto the curve along the unit circle, i.e. it is possible to extend the isometry between the two curves to a local curved fold.

Lemma 3.4.1. *Let ω be a closed C^3 curve of length $L > 2\pi$ on the unit sphere, then it is possible to proper fold along the unit circle onto $\gamma := \frac{2\pi}{L}\omega$.*

Proof. Since ω lies on the unit sphere, its curvature is greater or equal to 1, the normal curvature with respect to the sphere being everywhere 1. Therefore, the curvature k of γ satisfies $k > 1$. This guarantees that the unit circle and the ridge γ satisfy the hypotheses of Lemma 3.3.1.

□

The two inner developables of Fig. 3.1 (in green) are an example of a proper fold along the unit circle obtained by such a construction.

Ridges on a torus

Toroidal curves are another interesting class of space curves suitable for proper folding along any convex closed foldline. For $a \in \mathbb{R}$, $p, q \in \mathbb{N}$ and $\lambda := q/p$, we consider the family of curves on $[0, p2\pi]$ given by

$$\omega_{a,(p,q)}(t) := \big(\big(a + \cos(\lambda t)\big)\cos(t), \big(a + \cos(\lambda t)\big)\sin(t), \sin(\lambda t)\big).$$

For any fixed value of a, the curvature of the curve can be made arbitrarily close to 1 everywhere by picking a large value of q. Since the length of the curve is monotone in a, by rescaling the curve to be of length 1, we can obtain ridges of arbitrarily large minimum curvature. These are therefore suitable to proper fold along any closed convex foldline. By writing down the expression for the torsion one can additionally observe that in this regime its value tends to be 0 everywhere. Since the first derivative with respect to t of the curvature function can be made everywhere arbitrarily small and with that the angle α between the osculating and the tangent planes close to constant, we can even force the rulings emanating from the ridge to be about orthogonal to the tangent direction everywhere along the curve. Self-intersections of the developables obtained may occur.

The idea of employing a toroidal curve as the ridge of a curved fold was already mentioned as an example in [48]. Moreover, in [14] it is shown that in the isotopy class of any C^2 knot of the space there exists a C^∞ knot of constant curvature which is arbitrarily close to the first one both in trajectory and tangent direction. Since the curvature of the approximating knot can be chosen to be any value larger then the maximum curvature of the starting knot, constructions as the one we described for toroidal curves are possible in a much broader setting.

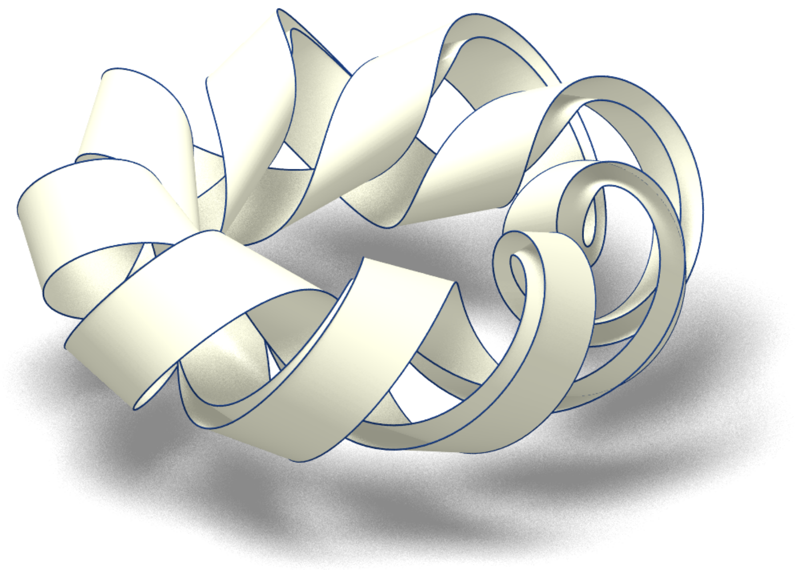

Figure 3.4: Fold along the unit circle of an annulus of width 2/10 onto the rescaled toroidal knot $\omega_{3,(9,2)}$.

Salkowski curves as ridges

As pointed out in [32] it is possible to proper fold along a circle on Salkowski curves; in the following we provide the definition of this family and shortly discuss how the developables obtained are just of regularity C^1, the lowest allowed in a proper fold, and how this impacts their regression curve.

It was originally shown in [31] how to join together arcs of Salkowsky curves and possibly circular helices to get a C^3 closed curve of constant curvature and non-constant continuous torsion. More explicitely, for $r \in \mathbb{R}, q = \frac{r}{\sqrt{1+r^2}}, r \notin \{0, \pm\frac{1}{\sqrt{3}}\}$, we consider the family of curves given by

$$\gamma_r(t) := \frac{q}{r}\left[\begin{pmatrix} -\frac{1-q}{4(1+2q)}\sin((1+2q)t) - \frac{1+q}{4(1-2q)}\sin((1-2q)t) - \frac{1}{2}\sin t \\ \frac{1-q}{4(1+2q)}\cos((1+2q)t) + \frac{1+q}{4(1-2q)}\cos((1-2q)t) + \frac{1}{2}\cos t \\ \frac{1}{4r}\cos(2qt) \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{q^2}{1-4q^2} \\ \frac{1}{4r} \end{pmatrix}\right].$$

Such curves are regular in $(-\frac{\pi}{2q}, \frac{\pi}{2q})$, their curvature and torsion are respectively $k \equiv 1$, $\tau = \tan(rt)$ and the arc-length between 0 and t is $\frac{1}{r}\sin(qt)$. The translation factor on the very right of the expression makes the curve defined in the limit while r approaches a value in $\{0, \pm\frac{1}{\sqrt{3}}\}$.

C^3 closed curves obtained by joining four branches of Salkowski curves are depicted in Figure 3.5. For different choices of r, curves are scaled down to match the length of the unit circle, which is the curve obtained in the limit as r goes to 0. Because of this scaling, their curvature is everywhere greater then 1, which guarantees one can proper fold along the unit circle consistently on the whole closed curve as the local construction of Lemma 3.3.1 is everywhere well defined. The developables obtained are of regularity C^1 allowing a C^0 ruled parametrization and presenting a discontinuous regression curve (Fig. 3.6).

3.5 Propagation to the next foldline

In this section we look at folds involving two foldlines. In particular we first fold along the first foldline by prescribing a ridge as we did in §3.3 and then we induce, if the isometry on one of the two sides extends suitably till the second foldline, a proper fold (consistent with the first one) along such a curve as well.

Let $\bar{\gamma}_1$ and $\bar{\gamma}_2$ be two non-intersecting planar curves of nonzero curvature in an open domain D of $\mathbb{R}^2$. We construct a proper fold along $\bar{\gamma}_1$ onto the ridge γ_1 and assume that the isometries on the two sides of it extend to the whole domain D; this means that exactly one ruling passes through any of the point of D different from those of $\bar{\gamma}_1$ and that the preimage of the regression curve of the developable is not reached within the

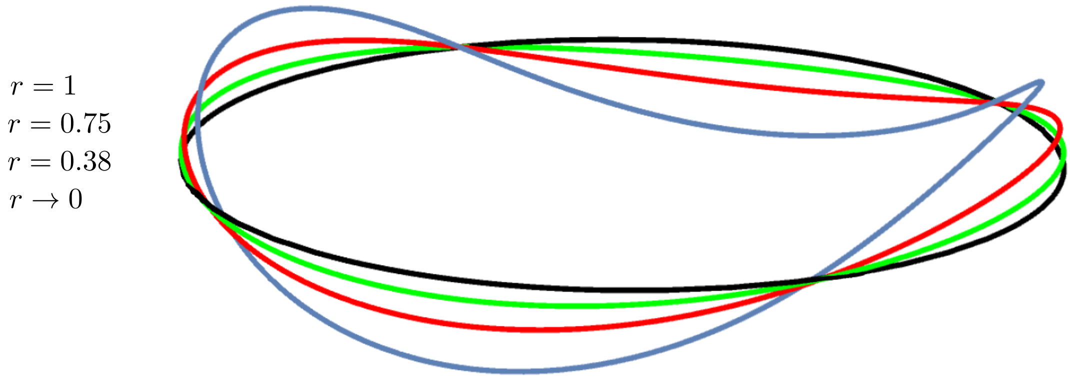

Figure 3.5: C^3 closed curves obtained by joining four branches of Salkowski curves for different choices of r.

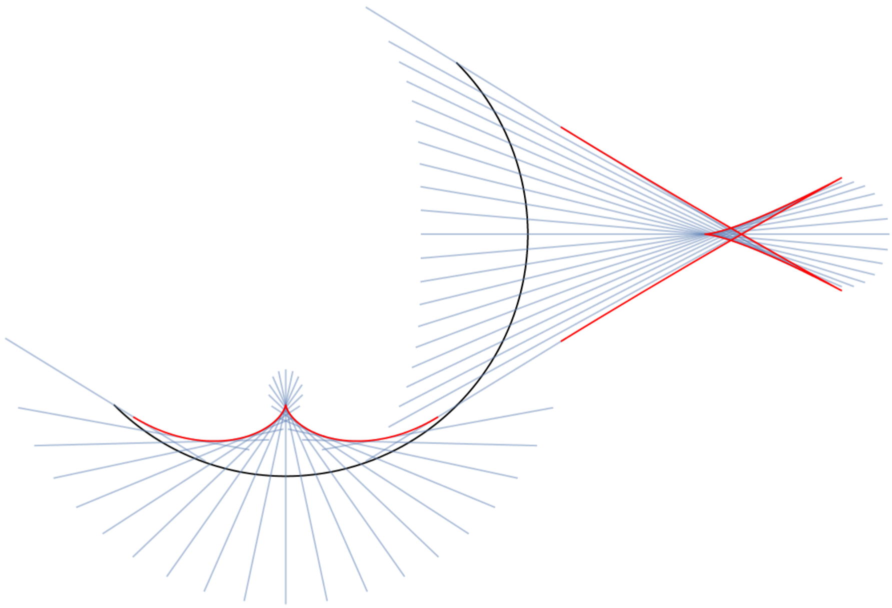

Figure 3.6: Regression curve of one of the developables of a proper fold along a circle on a ridge obtained by joining two Salkowski curves. The regression curve is not even continuous at the junction point.

domain along the ruling. We also assume that no rulings coming out of $\bar{\gamma}_1$ are tangent to $\bar{\gamma}_2$ where they intersect it for the first time. Under these premises, the restriction of the isometry of D to the second foldine induces a ridge γ_2 onto which it is possible to proper fold along $\bar{\gamma}_2$. We guarantee this by arguing that all points of the developables of a proper fold are parabolic by Lemma 3.3.4 and that no rulings are tangent to γ_2;

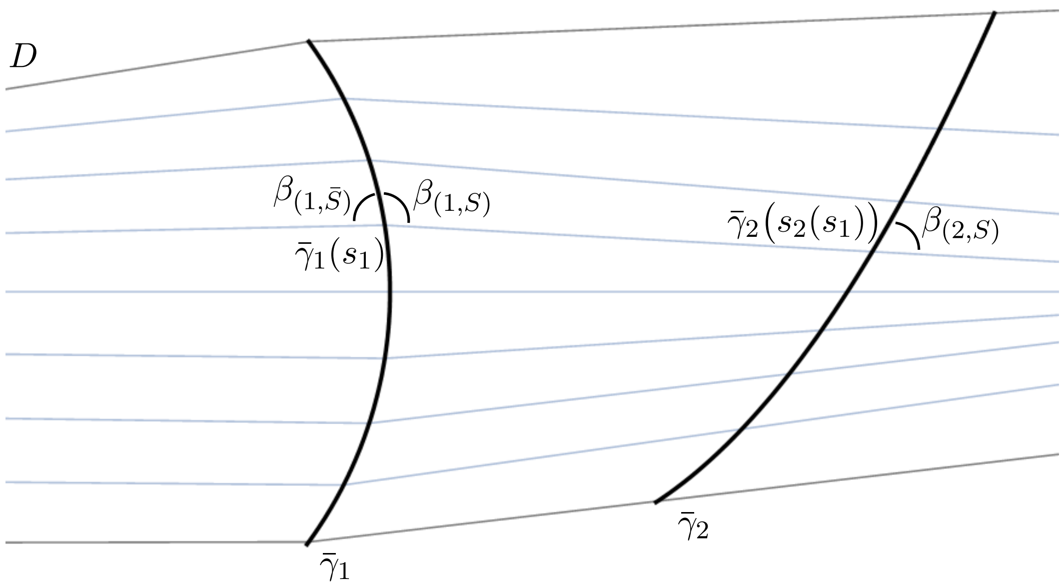

Figure 3.7: One of the two isometries obteined after proper folding along $\bar{\gamma}_1$ extends to the foldline $\bar{\gamma}_2$. The domain has possibly been trimmed to guarantee a bijection provided by the rulings between the foldlines.

therefore, its normal curvature with respect to the developable must be different from zero, ensuring that also $\bar{\gamma}_2$ and γ_2 satisfy the hypotheses of Lemma 3.3.1.

Note that our assumptions do not imply that all the rulings emanating from γ_1 interesect γ_2. Nevertheless, by possibly restricting D we can force a bijection between the two curves mediated by the family of rulings, associating to a point $\gamma_1(s_1)$ the point $\gamma_2\big(s_2(s_1)\big)$, obtained as the first intersection of the ruling emanating from $\gamma_1(s_1)$ with γ_2, s_1 and s_2 being the respective arc-legth parameters (Fig. 3.7).

If we want now to *propagate* the proper fold along $\bar{\gamma}_2$ onto γ_2 and we want it to be consistent with the first one, we have only one choice, since by Lemma 3.3.1 the developables of a proper fold lie on the same side of the osculating plane of the ridge and one of them is already prescribed by the first fold.

In the next lemma we express the normal curvature and the relative torsion of γ_2 with respect to the developable obtained by proper folding onto γ_1, in function of the normal curvature and the relative torsion of γ_1.

Lemma 3.5.1. *Let γ_1 and γ_2 be two non-intersecting curves on a C^2 developable surface whose points are all parabolic. We assume that the two curves have nonzero geodesic curvature, nonzero normal curvature (their tangents are never parallel to the rulings) and* *that a bijection $\gamma_1(s_1) \longleftrightarrow \gamma_2\big(s_2(s_1)\big)$ is induced by considering the first intersection point between γ_2 and the ruling through $\gamma(s_1)$. If δ is the angle between the tangent vectors at correspondent points $\gamma_1'(s_1)$ and $\gamma_2'\big(s_2(s_1)\big)$, measured anticlockwise with respect to the surface normal n, then*

$$\begin{pmatrix} k_{2,n} \\ \tau_{2,r} \end{pmatrix} = \frac{1}{s_2'} \begin{pmatrix} \cos(\delta)k_{1,n} + \sin(\delta)\tau_{1,r} \\ -\sin(\delta)k_{1,n} + \cos(\delta)\tau_{1,r} \end{pmatrix} = \frac{1}{s_2'} R_{-\delta} \begin{pmatrix} k_{1,n} \\ \tau_{1,r} \end{pmatrix} = \frac{k_{2,g}}{\delta' + k_{1,g}} R_{-\delta} \begin{pmatrix} k_{1,n} \\ \tau_{1,r} \end{pmatrix},$$

where $k_{i,g}$,$k_{i,n}$ and $\tau_{i,r}$ respectively are the geodesic curvature, the normal curvature and the relative torsion of γ_i with respect to the developable, for $i \in \{1,2\}$. R_ω denotes the anticlockwise rotation by the angle ω.

Proof. Let n_1, n_2 be the restriction of the surface normal n to the curves γ_1, γ_2 and $\{\gamma_1', u_1, n_1\}$, $\{\gamma_2', u_2, n_2\}$ the respective Darboux frames. Since the surface normal is constant along the ruling we have

$$\begin{aligned} & s_2'(s_1)\Big(-k_{2,n}(s_2(s_1))\gamma_2'\big(s_2(s_1)\big) - \tau_{2,r}(s_2(s_1))u_2\big(s_2(s_1)\big)\Big) \\ &= s_2'(s_1)n_2'\big(s_2(s_1)\big) = \big(n_2\big(s_2(s_1)\big)\big)' = \big(n_1(s_1)\big)' = -k_{1,n}(s_1)\gamma_1'(s_1) - \tau_{1,r}(s_1)u_1(s_1). \end{aligned}$$

The vectors γ_2' and u_2 can be obtained rotating respectively γ_1' and u_1 by δ about the surface normal n and thus, interpreting $k_{1,n}$, $\tau_{1,r}$ and $k_{2,n}$, $\tau_{2,r}$ as coordinates of the same vector in different bases we get

$$\begin{pmatrix} k_{2,n} \\ \tau_{2,r} \end{pmatrix} = \frac{1}{s_2'} R_{-\delta} \begin{pmatrix} k_{1,n} \\ \tau_{1,r} \end{pmatrix}.$$

To express the velocity s_2' we look at the developed state $\bar\gamma_1$, $\bar\gamma_2$ of the two curves and exploit the relation $\bar\gamma_2' = R_\delta \bar\gamma_1'$ at correspondent points. By taking the derivative with respect to s_1 we obtain

$$s_2'\bar\gamma_2'' = \delta' R_\delta R_{\frac{\pi}{2}} \bar\gamma_1' + R_\delta \bar\gamma_1'',$$

which provides $s_2' = (\delta' + k_{1,g})/k_{2,g}$. □

We want now to point out an additional way of computing the normal curvature and the relative torsion of the second ridge once a proper fold is prescribed for the first one. This expression will highlight the role played by the regression curve in the propagation and provide a direct formula for computing the velocity of the parametrization of the second ridge induced by the rulings correspondence.

Lemma 3.5.2. *Let γ_1, γ_2 be two curves on a developable surface as in Lemma 3.5.1. Let also $\beta_{(1,S)}$ be the angle between the tangent $\gamma_1'(s_1)$ and the ruling direction and $\beta_{(2,S)}$*

the one between $\gamma_2'\big(s_2(s_1)\big)$ and the same ruling direction (Fig. 3.7). If $\bar{v}$ is the distance between $\gamma_1(s_1)$ and $\gamma_2\big(s_2(s_1)\big)$, then

$$k_{(2,n,S)} = \left(\frac{\sin(\beta_{(2,S)})}{\sin(\beta_{(1,S)})}\right)^2 \frac{k_{(1,n,S)}}{1 - \bar{v}\frac{\beta'_{(1,S)}+k_{(1,g)}}{\sin(\beta_{(1,S)})}},$$

$$\tau_{(2,r,S)} = -\left(\frac{\cos(\beta_{(2,S)})}{\sin(\beta_{(1,S)})}\right)\left(\frac{\sin(\beta_{(2,S)})}{\sin(\beta_{(1,S)})}\right)\frac{k_{(1,n,S)}}{1 - \bar{v}\frac{\beta'_{(1,S)}+k_{(1,g)}}{\sin(\beta_{(1,S)})}}.$$

Proof. As shown in Lemma 5, Chap. 5 of [43], the nonzero principal curvature along a parabolic ruling $\gamma + v \cdot r$ can be written as

$$k_{(p,S)}(v) = \frac{1}{\big(\sin(\beta_{(1,S)})\big)^2}\frac{k_{(1,n,S)}}{1 - v \cdot \frac{\beta'_{(1,S)}+k_{(1,g)}}{\sin(\beta_{(1,S)})}}.$$

This expression has been constructed by requiring the reciprocal of a linear function to attain the value $k_{(1,n,S)}/\big(\sin(\beta_{(1,S)})\big)^2$ at $v = 0$ and being indeterminate at the parameter v corresponding to the intersection with the regression curve. Evaluating at $\bar{v}$, applying Euler's formula and recalling that $\cot(\beta_{(2,S)}) = -\tau_{(2,r,S)}/k_{(2,n,S)}$, we obtain the desired formulae for $k_{(2,n,S)}$ and $\tau_{(2,r,S)}$. □

Lemma 3.5.3. *Let γ_1, γ_2 be two curves on a developable surface as in Lemma 3.5.1. With $\bar{v}$ as in Lemma 3.5.2, the velocity of the parametrization of the second ridge $\gamma_2\big(s_2(s_1)\big)$ can be expressed as*

$$s_2' = \frac{\sin(\beta_{(1,S)})}{\sin(\beta_{(2,S)})}\left(1 - \bar{v}\frac{\beta'_{(1,S)} + k_{(1,g)}}{\sin(\beta_{(1,S)})}\right).$$

Proof. By Lemma 3.5.1, we have

$$s_2' = \frac{\cos(\delta)k_{(1,n,S)} + \sin(\delta)\tau_{(1,r,S)}}{k_{(2,n,S)}}.$$

We conclude the claim of the lemma by observing that $\beta_{(2,S)} = \beta_{(1,S)} - \delta$ and hence

$$\cos(\delta)k_{(1,n,S)} + \sin(\delta)\tau_{(1,r,S)} = k_{(1,n,S)}(\cos(\delta) - \sin(\delta)\cot(\beta_{(1,S)}) = k_{(1,n,S)}\frac{\sin(\beta_{(2,S)})}{\sin(\beta_{(1,S)})}.$$

□

For what concerns the propagation of a curved fold, we can use Lemma 3.3.2 and Lemma 3.5.1 to compute, as a function of the normal curvature and relative torsion of the ridge γ_1, the normal curvature and relative torsion of the second ridge γ_2 with respect to the developable obtained after proper folding also on the other side of γ_2.

Although the formulae are not simple, such a construction can possibly be iterated to further propagate the fold when several foldlines are prescribed, the propagation being uniquely determined by the chosen foldlines together with the normal curvature and relative torsion of the first ridge. Fig. 3.1 and 3.2 show two examples of a pleated annulus with multiple folds, drawn via their explicit parametrizations, which have been obtained by the propagation process just described. The details on how to guarantee the regularity of such a construction are given in the appendix.

Proposition 3.5.4. *Let γ_1, γ_2 be two curves on a developable surface M as in Lemma 3.5.1, then there is a unique way to propagate the fold onto γ_2. In more detail, there is a unique way to properly fold onto γ_2 (along the foldline with the correspective geodesic curvature) in a consistent way with the pre-existing developable M. The normal curvature and the relative torsion $k_{(2,n,S)}$, $\tau_{(2,r,S)}$ of γ_2 with respect to the new developable can be expressed as*

$$
\begin{aligned}
k_{(2,n,S)} =& - k_{(2,n,\bar{S})}, \\
\tau_{(2,r,S)} =& \tau_{(2,r,\bar{S})} - \frac{2}{(s_2')^2\Big(k^2_{(2,n,\bar{S})} + k^2_{(2,g)}\Big)} \cdot \Bigg(s_2'' k_{(2,n,\bar{S})} k_{(2,g)} \\
& + s_2'\Big(k_{(2,n,\bar{S})} k'_{(2,g)} - \tau_{(2,r,\bar{S})} k_{(2,g)} \big(s_2' k_{(2,g)} - k_{(1,g)}\big)\Big) \\
& - k_{(2,g)} \Big(\cos\big(\beta_{(1,S)} - \beta_{(2,S)}\big) k'_{(1,n,\bar{S})} + \sin\big(\beta_{(1,S)} - \beta_{(2,S)}\big) \tau'_{(1,r,\bar{S})}\Big)\Bigg),
\end{aligned}
$$

where $k_{(1,g)}$, $k_{(1,n,\bar{S})}$, $\tau_{(1,r,\bar{S})}$ and $k_{(2,g)}$, $k_{(2,n,\bar{S})}$, $\tau_{(2,r,\bar{S})}$ respectively are the geodesic curvature, the normal curvature and the relative torsion of γ_1 and γ_2 with respect to M. Finally, $\beta_{(1,S)}$, $\beta_{(2,S)}$ are the angles between the tangents γ_1', γ_2' and the ruling direction r_S at corresponding points $\gamma_1(s_1)$, $\gamma_2\big(s_2(s_1)\big)$.

Proof. Direct computation by Lemma 3.3.2 and Lemma 3.5.1. □

Corollary 2. *We can employ $\delta = \beta_{(1,S)} - \beta_{(2,S)}$ to rewrite the formula for the relative torsion from Proposition 3.5.4 in a slightly more compact way,*

$$
\begin{aligned}
\tau_{(2,r,S)} =& \tau_{(2,r,\bar{S})} - \frac{2}{k^2_{(2,n,\bar{S})} + k^2_{(2,g)}} \left(\frac{k_{(2,g)}}{\delta' + k_{(1,g)}}\right)^2 \cdot \Bigg(k_{(2,n,\bar{S})} \Big(\delta'' + k'_{(1,g)}\Big) \\
& - \tau_{(2,r,\bar{S})} \Big(\delta' + k_{(1,g)}\Big)\delta' - k_{(2,g)} \Big(\cos(\delta) k'_{(1,n,\bar{S})} + \sin(\delta) \tau'_{(1,r,\bar{S})}\Big)\Bigg).
\end{aligned}
$$

Remark 3.5.5. By Lemma 3.5.1 and Lemma 3.5.3, $k_{(2,n,\bar{S})}$, $\tau_{(2,r,\bar{S})}$ and s_2' depend only on the prescribed foldlines and on the values of $k_{(1,n,\bar{S})}$ and $\tau_{(1,r,\bar{S})}$ and their first derivative at the point of interest. By this, s_2'' depends on the derivatives of $k_{(1,n,\bar{S})}$ and $\tau_{(1,r,\bar{S})}$ up to the second order.

Observation 1. *If the two foldlines $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are very close to each other, for example $\bar{\gamma}_2$ being a very gentle offset of $\bar{\gamma}_1$ in the direction of $\bar{\gamma}_1''$, we can approximate $k_{(1,g)} \sim k_{(2,g)}$, $k_{(2,n,\bar{S})} \sim k_{(1,n,\bar{S})}$, $\tau_{(2,r,\bar{S})} \sim \tau_{(1,r,\bar{S})}$ and $\delta \sim 0$ to obtain*

$$\tau_{(2,r,S)} \sim \tau_{(1,r,\bar{S})} - 2\frac{k_{(1,n,\bar{S})}k'_{(1,g)} - k'_{(1,n,\bar{S})}k_{(1,g)}}{\left(k^2_{(1,n,\bar{S})} + k^2_{(1,g)}\right)} \sim \tau_{(1,r,S)}.$$

This, matching the expression for the relative torsion from Lemma 3.3.2, shows that in this extreme setting the developable on the other side of M with respect to γ_2 (the new one we want to define) is approximately a continuation of the developable on the other side of M with respect to γ_1. A more formal discussion of this behaviour will be given in §3.6.

3.6 Propagation to several foldlines

In this section we discuss how for any natural number N, by choosing a family of uniformly rescaled foldlines close enough to each other, it is possible to propagate the proper fold along the first foldline onto an arbitrary ridge to the remaining $N-1$ foldlines in the way we described in §3.5. Fig. 3.8 provides a visualization of what we mean with *uniformly rescaled*; the rigorous definition of such a family is given directly in Theorem 3.6.1.

Theorem 3.6.1. *Let $\bar{\gamma}_1$ be a C^∞ foldine with nonzero curvature along which a proper fold onto the C^∞ ridge γ_1 is locally well defined. Let $p \in \mathbb{R}^2$ be such that no ray $\bar{\gamma}_1 - p$ is parallel to $\bar{\gamma}_1'$ then, for any $N \in \mathbb{N}$, there exists a scaling factor $\bar{c} > 0$ such that for all $0 < c < \bar{c}$ the proper fold along $\bar{\gamma}_1$ propagates to the family of foldlines $\bar{\gamma}_j = (1 + (j-1) \cdot c)(\bar{\gamma}_1 - p) + p$ for $1 < j \leq N$, possibly restricting the definition domain $[a_j, b_j]$ of $\bar{\gamma}_j$ to $[a_j + \rho_j(c), b_j - \rho_j(c)]$ with $\lim_{c \to 0} \rho_j(c) = 0$.*

For the sake of clarity, we proceed by presenting a technical lemma before providing the actual proof of the theorem, which is essentially obtained as a consquence of Observation 1 plus some work to make the argument rigorous. Given a scaling factor c, in the following we will always assume the family $\bar{\gamma}_j$ defined as in the statement of the theorem. If s_1 is the arc-length parameter of $\bar{\gamma}_1$ then the arc-length parameter s_j of $\bar{\gamma}_j$ can be expressed as $l_j(s_1) = s_1/(1 + (j-1) \cdot c)$.

Although the statements of Theorem 3.6.1 and Lemma 3.6.2 are given for a positive value of c, this is just for convenience, and analogous conclusions hold for rescaled copies of $\bar{\gamma}_1$ on the same side of p ($c < 0$).

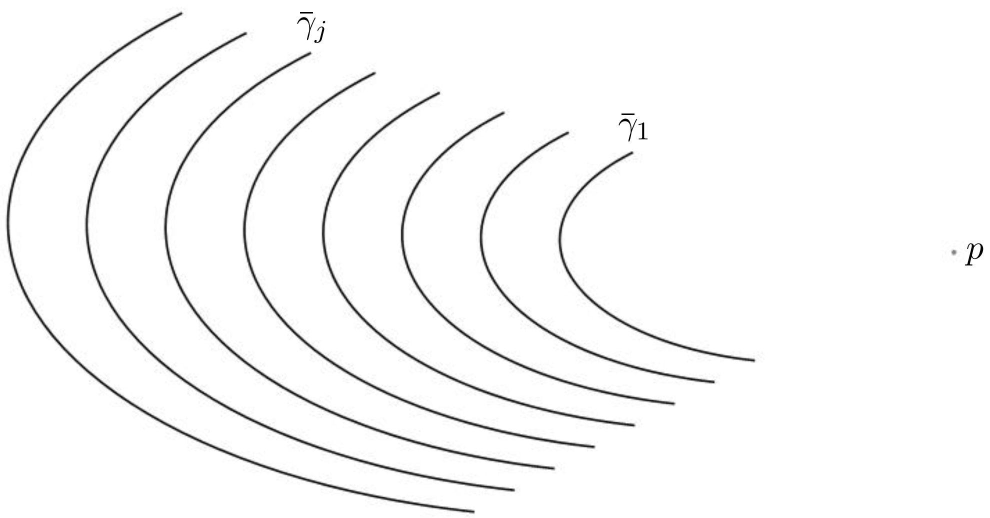

Figure 3.8: Family of foldlines obtained by rescaling $\bar{\gamma}_1$ with respect to the center p.

Lemma 3.6.2. *Let $\bar{\gamma}_1$ be a C^∞ planar curve with nonzero curvature parametrized by arc-length over the interval I. Assume that $p \in \mathbb{R}^2$ is such that no ray $\bar{\gamma}_1 - p$ is parallel to $\bar{\gamma}_1'$. If $\bar{r}$ is a C^∞ family of ruling directions defined on I, such that no direction $\bar{r}$ is parallel to $\bar{\gamma}_1'$, then for any open interval $A \subset I$ and any $j \in \mathbb{N}$ there exists $\bar{c} > 0$ such that for any $0 < c < \bar{c}$ the family of rulings direction $\bar{r}$ identifies a bijection between $\bar{\gamma}_j\big(l_j(A)\big)$ and $\bar{\gamma}_{j+1}\big(s_{j+1}\big(l_j(A)\big)\big)$, constructed by considering the intersection of the line $\bar{\gamma}_j\big(l_j(s_1)\big)+v \cdot \bar{r}(s_1)$ with the curve $\bar{\gamma}_{j+1}$ on the side pointed by $\bar{\gamma}_j - p$.*

Moreover, if β_j and β_{j+1} are the angles between $\bar{r}$ and $\bar{\gamma}_j'$ and $\bar{\gamma}_{j+1}'$ respectively, then

$$\lim_{c\to 0}\Big|\beta_{j+1}^{(h)}\big(s_{j+1}(s_j)\big)-\beta_j^{(h)}(s_j)\Big| = 0, \qquad \forall\, 0 \le h \in \mathbb{N}, \ \textit{and}$$

$$\lim_{c\to 0}\big|s_{j+1}' - 1\big| = \lim_{c\to 0}\Big|s_{j+1}^{(h)}\Big| = 0, \qquad \forall\, 2 \le h \in \mathbb{N},$$

Here all derivatives are taken with respect to s_j and $\bar{r}$ is considered as the function $\bar{r}\big(l_j^{-1}(s_j)\big)$.

Proof. By the definition of $\bar{\gamma}_j$, since by continuity $\lim_{c\to 0} \bar{\gamma}_{j+1}'\big(l_{j+1}(s_1)\big) = \lim_{c\to 0} \bar{\gamma}_j'\big(l_j(s_1)\big) = \bar{\gamma}'(s_1)$ we have

$$\lim_{c\to 0}\big|\beta_{j+1}\big(s_{j+1}(s_j)\big)-\beta_j(s_j)\big| = 0.$$

By the expression for the velocity given in Lemma 3.5.3 we obtain, for $v(c, s_1)$ such that $\bar{\gamma}_j\big(l_j(s_1)\big)+v(c, s_1) \cdot \bar{r}(s_1) = \bar{\gamma}_{j+1}\big(s_{j+1}\big(l_j(s_1)\big)\big)$,

$$\lim_{c\to 0} s_{j+1}' = \lim_{c\to 0} \frac{\sin\beta_j}{\sin\beta_{j+1}}\left(1 - v(c, s_1)\frac{\beta_j' + k_j}{\sin(\beta_j)}\right) = 1,$$

where k_j is the curvature of the j-th foldline. For the derivatives of higher order of s_{j+1}, β_{j+1} and β_j we recall from Lemma 3.5.1 that $s'_{j+1} = \big(k_j + (\beta'_j - \beta'_{j+1})\big)/k_{j+1}$ and therefore

$$\lim_{c\to 0} \beta'_{j+1} = \lim_{c\to 0} \beta'_j - s'_{j+1}k_{j+1} + k_j = \lim_{c\to 0} \beta'_j.$$

The statement follows by alternately taking the derivatives with respect to s_j of the expressions for s'_{j+1} and β'_{j+1} (induction on the derivatives of lower order). $\square$

We are now ready to prove Theorem 3.6.1.

Proof of Theorem 3.6.1. We proceed by induction, assuming the existence of $\bar{c}$ such that the proper fold along $\bar{\gamma}_1$ onto γ_1 propagates to the first N foldlines for all $0 < c < \bar{c}$. More precisely, the inductive hypothesis we want to iterate claims that for $0 < c < \bar{c}$ a bijection between $\bar{\gamma}_j(s_j)$ and $\bar{\gamma}_{j+1}\big(s_{j+1}(s_j)\big)$ (or equivalently between γ_j and γ_{j+1}) is identified by the rulings of the developable between the curves for $j < N$, possibly restricting to suitable open sets. Besides, fixed any $\varepsilon > 0$ and $H \in \mathbb{N}$ we also ask that in the same range of c the normal curvature and the relative torsion of the ridge γ_j with respect to the developable on the opposite side of the scaling center p satisfy

$$\begin{aligned}\left|k^{(h)}_{(j,n,S_j)}\big(s_j(s_{j-1}(...s_2(s_1)))\big) - k^{(h)}_{(1,n,S_j)}(s_1)\right| &< \varepsilon,\\ \left|\tau^{(h)}_{(j,r,S_j)}\big(s_j(s_{j-1}(...s_2(s_1)))\big) - \tau^{(h)}_{(1,r,S_j)}(s_1)\right| &< \varepsilon, \qquad \forall\, j \le N, h \le H,\end{aligned}$$

where S_j has been assumed without loss of generality being $+$ or $-$ if j is respectively odd or even. The derivatives are taken with respect to the arc-length parameter s_j of the ridge of interest. Finally, still in the inductive hypothesis we ask for guarantees that the speed of the reparametrization mediated by the ruling $s_{j+1}(s_j)$ does not deviate too much from the arc-length, requiring also

$$\begin{aligned}\left|s'_{j+1} - 1\right| &< \varepsilon,\\ \left|s^{(h)}_{j+1}\right| &< \varepsilon, \qquad \forall\, j < N, 1 < h \le H.\end{aligned}$$

Assuming the inductive hypothesis for $N \ge 1$, whose basis is simply provided by the knowledge that we can properly fold along $\bar{\gamma}_1$ onto γ_1, we show that it also holds for $N+1$. We carry over the proof for N odd, the even case being analogous. Since the angle between a ruling and the tangent to the ridge is a continuous function of $k_{(N,n,+)}$ and $\tau_{(N,r,+)}$ and their derivatives, for any $\varepsilon > 0$ and $H \in \mathbb{N}$, we can choose $\bar{c}$ small enough to guarantee that

$$\left|\beta^{(h)}_{(j,+)}\big(s_j(s_{j-1}(...s_2(s_1)))\big) - \beta^{(h)}_{(1,+)}(s_1)\right| < \varepsilon, \qquad \forall\, j \le N, h \le H.$$

The fold along $\bar{\gamma}_1$ being proper, the regression curve of the two developables on which γ_1 lies is at nonzero distance from the ridge along the ruling. If ε is small enough, such a property passes over to the regression curves of the developables on the two sides of γ_N. We can require $\bar{c}$ to be small enough to have $\bar{\gamma}_{N+1}$ within the minimum of such distances and hence obtain the desired bijection $\gamma_{N+1}(s_{N+1}(s_N))$ up to restriction of the definition domains.

Next step is to control the normal curvature and the relative torsion of γ_{N+1} with respect to the developable on the negative side. By making use of the inductive hypothesis on the speed of the bijections s_j and possibly further decreasing $\bar{c}$, we have for $l_j(s_1) = s_1/(1+(j-1)\cdot c)$

$$\left|\beta^{(h)}_{(j,+)}\big(l_j(s_1)\big) - \beta^{(h)}_{(1,+)}(s_1)\right| < \varepsilon, \qquad \forall\, j \le N, h \le H.$$

Hence, for $\bar{r}$ chosen to be the development of the rulings direction from γ_1 to γ_2, Lemma 3.6.2 together with the triangular inequality allows us to conclude the desired condition on the speed of the reparametrization $s_{N+1}(s_N)$ and the following bounds on the angle $\beta_{(N+1,+)}$ and its derivatives

$$\left|\beta^{(h)}_{(N+1,+)}\big(s_{N+1}(s_N(\ldots s_2(s_1))\big) - \beta^{(h)}_{(1,+)}(s_1)\right| < \varepsilon.$$

Finally, Proposition 3.5.4 provides expressions for $k_{(N+1,n,-)}$ and $\tau_{(N+1,r,-)}$ which continuously depend on the values of $\beta_{(N,+)}$ and $\beta_{(N+1,+)}$ and their derivatives up to degree 2. For $\bar{c}$ small enough we can therefore guarantee that

$$\left|k_{(N+1,n,-)}\big(s_{N+1}(\ldots s_2(s_1)))\big) - \big(-k_{(1,n,+)}(s_1)\big)\right| < \varepsilon,$$

$$\left|\tau_{(N+1,r,-)}\big(s_{N+1}(\ldots s_2(s_1)))\big) - \left(\tau_{(1,r,+)} - 2\frac{k_{(1,n,+)}k'_{(1,g)} - k'_{(1,n,+)}k_{(1,g)}}{\left(k^2_{(1,n,+)} + k^2_{(1,g)}\right)}\right)(s_1)\right| < \varepsilon,$$

where $k_{(1,g)}$ is the geodesic curvature of $\bar{\gamma}_1$. Again exploiting the condition on the speed of the bijections s_j up to $j = N+1$, the inductive hypothesis on the derivative of $\beta_{(N+1,+)}$ and possibly making $\bar{c}$ smaller enough, for any H we conclude the desired bounds

$$\left|k^{(h)}_{(N+1,n,-)}\big(s_{N+1}(s_{j-1}(\ldots s_2(s_1)))\big) - k^{(h)}_{(1,n,-)}(s_1)\right| < \varepsilon,$$

$$\left|\tau^{(h)}_{(N+1,r,-)}\big(s_{N+1}(s_{j-1}(\ldots s_2(s_1)))\big) - \tau^{(h)}_{(1,r,-)}(s_1)\right| < \varepsilon, \qquad \forall\, h \le H-2.$$

This completes the inductive step and with that the proof of the theorem. □

Observation 2. *If $\bar{\gamma}_1$ is a closed convex curve then the scaling center p must be in its interior and no restriction of the definition domains is ever needed, once a scaling factor c small enough to guarantee the propagation has been found.*

Observation 3. *If we are not interested in mantaining a constant scaling factor c, then it is easy to propagate a proper fold onto an arbitrary ridge to infinitely many additional foldlines. We can in fact just proceed by induction: once the n-th proper fold is determined we prescribe the $(n+1)$-th foldline by scaling the previous one by a factor small enough to make the curve contained in the interior of the domain on which the isometry identifying the next developable is well-defined.*

3.7 Why the propagation to infinitely many prescribed foldlines is hard

In this section we show that the propagation of a proper fold can turn singular with an arbitrarily abrupt behaviour. More precisely, we will show that for any proper fold involving N foldlines in the sense of §3.6, we can construct a proper fold over the first $N-1$ foldlines whose ridges are arbitrarily close to those of the first fold up to the derivative of order 3 but such that a non-singular isometry between the $(N-1)$-th and the N-th foldline cannot be consistently constructed. This will provide evidence that in general inductive strategies taking into account only derivatives up to a finite order cannot be employed to guarantee the propagation of a curved fold to a prescribed infinite family of foldlines.

Proposition 3.7.1. *Let $\bar{\gamma}_j$ be a family of N non-intersecting C^{2N} foldlines such that the proper fold along $\bar{\gamma}_1$ onto the C^{2N} ridge γ_1 propagates sequentially to the foldlines 2 to N identifying bijections between ridges $\gamma_j(s_j) \leftrightarrow \gamma_{j+1}(s_{j+1}(s_j))$ induced by the rulings correspondence. Then, there exists a ridge $\tilde{\gamma}_1$ such that a proper fold along $\bar{\gamma}_1$ onto $\tilde{\gamma}_1$ propagates, possibly up to restriction of the definition domains, to the foldlines 2 to $N-1$, inducing ridges $\tilde{\gamma}_j$, but not to the N-th foldline, some of the rulings emanating from $\bar{\gamma}_{N-1}$ on the side of $\bar{\gamma}_N$ crossing the regression curve before hitting the last foldline. Moreover, for any $\varepsilon > 0$, $\tilde{\gamma}_1$ can be chosen such that*

$$\max\left|k_{(j,n,S_j)} - \tilde{k}_{(j,n,S_j)}\right| < \varepsilon,$$
$$\max\left|\tau_{(j,r,S_j)} - \tilde{\tau}_{(j,r,S_j)}\right| < \varepsilon, \quad \forall\, 1 \leq j \leq N-1.$$

where $k_{(j,n,S_j)}$, $\tau_{(j,r,S_j)}$ and $\tilde{k}_{(j,n,S_j)}$, $\tilde{\tau}_{(j,r,S_j)}$ are the normal curvature and relative torsion respectively of γ_j and $\tilde{\gamma}_j$ with respect to the developable emanating from the ridge j on the side of $j+1$.

Again, we first provide a technical lemma.

Lemma 3.7.2. *In the hypotheses of Theorem 3.7.1, with the notation* $k_1 := k_{(1,n,S_1)}$ *and* $\tau_1 := \tau_{(1,r,S_1)}$*, for* $j > 1$ *we have the equalities*

$$\tau_{(j,r,S_j)} = \tau_1^{(2j-2)} \cdot f_j\left(s_1, k_1, k_1', ..., k_1^{(2j-3)}, \tau_1, \tau_1', ..., \tau_1^{(2j-3)}\right) \\ +g_j\left(s_1, k_1, k_1', ..., k_1^{(2j-2)}, \tau_1, \tau_1', ..., \tau_1^{(2j-3)}\right), \quad and$$

$$k_{(j,n,S_j)} = q_j\left(s_1, k_1, k_1', ..., k_1^{(2j-3)}, \tau_1, \tau_1', ..., \tau_1^{(2j-3)}\right).$$

Here f_j*,* g_j *and* q_j *are* C^∞ *functions depending only on the family of foldlines, and such that* f_j *never attains the value* 0 *if evaluated as above (the argument* s_1 *of* k_1 *and* τ_1*, and* $s_j(s_{j-1}(...s_2(s_1))$ *of* $\tau_{(j,r,S_j)}$*,* $k_{(j,n,S_j)}$ *have been omitted for brevity).*

Proof. We proceed by induction, the basis step being provided by Proposition 3.5.4, where s_2'' is rewritten making use of the expressions for s_2' from Lemma 3.5.3 and for $\beta'_{(1,S_1)}$ from Lemma 3.3.3. We employ a similar strategy to prove the inductive step, applying Proposition 3.5.4 between ridges j and $j+1$. The characterization for $k_{(j+1,n,S_{j+1})} = -k_{(j+1,n,\bar{S}_{j+1})}$ is easily obtained after observing that by Lemma 3.5.1 it depends only on $k_{(j,n,\bar{S}_j)}$ and $\tau_{(j,r,\bar{S}_j)}$ and their first derivative, and concluding by the inductive hypothesis. We look then at the term $s''_{j+1}k_{(j+1,n,\bar{S}_{j+1})}k_{(j+1,g)}$ containing the derivative of highest order of the expression for $\tau_{(j+1,r,\bar{S}_j)}$. Further decomposing s''_{j+1}, by Lemma 3.5.3 and Lemma 3.3.3 we end up looking at

$$-\tau''_{(j,r,S_j)} \frac{\bar{v} k_{(j,n,S_j)}}{\tau^2_{(j,r,S_j)} + k^2_{(j,n,S_j)}} k_{(j+1,n,\bar{S}_{j+1})} k_{(j+1,g)},$$

as the term of highest differential order in $\tau_{(j,r,S_j)}$, where $\bar{v}$ is the distance function between γ_j and γ_{j+1} along the ruling. Again we conclude by induction after observing that the factor multiplying $\tau''_{(j,r,S_j)}$ is nonzero. □

Proof of Proposition 3.7.1. Without loss of generality in the proof argument, we assume $S_1 = +$ and N even with $S_{N-1} = \bar{S}_N = +$.

For any s_1 and $M \in \mathbb{R}$, we can locally perturbate $\tau_1^{(2N-3)}$ to $\tilde{\tau}_1^{(2N-3)}$, for example with a very steep bump function, to have $\tilde{\tau}_1^{(2N-3)}(s_1) = M$, but still for any $\rho > 0$, taking the antiderivatives of $\tilde{\tau}_1^{(2N-3)}$ with suitable boundary conditions

$$\max\left|\tau_1^{(h)} - \tilde{\tau}_1^{(h)}\right| < \rho, \qquad h < 2N - 3,$$

which is possible because we are constraining finitely many antiderivatives defined on compact domains. We define $\tilde{\gamma}_1$ as the ridge having $\tilde{k}_1 := k_1$ and $\tilde{\tau}_1$ as normal curvature and relative torsion and propagate the fold along $\bar{\gamma}_1$ onto such a ridge. By Lemmma

3.7.2 if ρ is small enough, again by continuity and compactness, the normal curvature and the relative torsion of the new ridges $\tilde{\gamma}_1$ to $\tilde{\gamma}_{N-1}$ are arbitrarily close to those of the original ridges γ_1 to γ_{N-1}, which proves the second part from the claim of the theorem.

It remains to show that we can exploit the perturbation freedom we have on $\tilde{\tau}_1^{(2N-3)}$ to (heavily) modify the behaviour of the regression curve of the developable between the ridges $N-1$ and N. We do that by recalling that the distance of such a curve from the ridge along a ruling emanating from $\tilde{\gamma}_{N-1}$ is given by

$$\frac{\sin(\tilde{\beta}_{(N-1,+)})}{\tilde{\beta}'_{(N-1,+)} + k_{(N-1,g)}} \quad \text{and} \quad \tilde{\beta}'_{(N-1,+)} = \frac{\tilde{\tau}'_{(N-1,r,+)}\tilde{k}_{(N-1,n,+)} - \tilde{k}'_{(N-1,n,+)}\tilde{\tau}_{(N-1,r,+)}}{\left(\tilde{\tau}^2_{(N-1,r,+)} + \tilde{k}^2_{(N-1,n,+)}\right)},$$

where $\tilde{\beta}_{(N-1,+)}$ is the angle function between the ruling and the tangent $\tilde{\gamma}'_{N-1}$ and $k_{(N-1,g)}$ is the curvature of the respective foldline. By Lemma 3.7.2, $\tilde{\tau}'_{(N-1,r,+)}$ can be made arbitrarily large/small while keeping all the other functions almost unchanged, and with that, since $\tilde{k}_{(N-1,n,+)} \neq 0$, the same behaviour translates to $\tilde{\beta}'_{(N-1,+)}$, hence forcing the point of the regression curve to be arbitrarily located along the ruling and preventing the isometry to be extended to the final foldline. □

3.8 Future work

The construction of §3.6 is artificial in the measure it forces the existence of finitely many folds by exploiting the local guarantees provided by the properness of the first one. It would be nice to see in future years an existence proof that would work on infinitely many uniformly rescaled foldlines. Proposition 3.7.1 makes it clear that such a proof would depend on the development of an inductive tool allowing a suitable control not only on the local propagation but also on the derivatives of arbitrary order of the curves involved.

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Appendix. Folding the annulus

This appendix is devoted to an explicit application of the formulae obtained in §3.5 to the annulus folded along conentric circles.

Lemma 3.8.1. *Let $\bar{\gamma}$ be a circle of radius R and center in the origin traversed counter-clockwise and r a unit vector forming with the tangent of the circle at $\bar{\gamma}(s)$ the angle $-\pi < \beta < +\pi$ (measured counter-clockwise). The signed distance $\bar{v}$ between $\bar{\gamma}(s)$ and the closest intersection point between the line $\bar{\gamma}(s) + v \cdot r$ and the scaled circle $(1+c)\bar{\gamma}$ with $c \in \mathbb{R}$, whenever well defined, obeys the formula*

$$\bar{v} = R\Big(\sin(\beta) - sgn\big(\sin(\beta)\big)\sqrt{\sin(\beta)^2 + c^2 + 2c}\Big).$$

Besides, the angle δ between $\bar{\gamma}'(s)$ and the tangent with the second circle at the intersection point satisfies

$$\sin(\delta) = \frac{\bar{v} \cdot \cos(\beta)}{R \cdot (1+c)},$$
$$\cos(\delta) = \frac{R - \bar{v} \cdot \sin(\beta)}{R \cdot (1+c)}.$$

Proof. The lemma follows from elementary computations. □

Given a sequence of concentric circles of radius R_j and a ridge suitable to fold along the $\bar{j}$-th one, formulae from §3.3, §3.5 and Lemma 3.8.1 can be iterated to compute explicit parametrizations of the developables involved in the propagated curved fold. In the notation of §3.5, two conditions must be met to guarantee the regularity of such surfaces.

- Setting $c_{(j,\pm)} = R_{j\pm1}/R_j - 1$, the function

  $$\bar{v}_{(j,S_j)} = R_j\Big(\sin(\beta_{(j,S_j)}) - \operatorname{sgn}\big(\sin(\beta_{(j,S_j)})\big)\sqrt{\sin(\beta_{(j,S_j)})^2 + c_{(j,\pm)}^2 + 2c_{(j,\pm)}}\Big)$$

  must be well defined, meaning the ruling intersects the next foldline.

- Let $d_{(j,S_j)} = \sin(\beta_{(j,S_j)})/(\beta'_{(j,S_j)} + k_{(j,g)})$ be the signed distance of the regression curve along the ruling, then either

  $$\operatorname{sgn}\big(d_{(j,S_j)} \cdot \bar{v}_{(j,S_j)}\big) < 0 \text{ or } \left|\bar{v}_{(j,S_j)}\right| \geq \left|d_{(j,S_j)}\right|,$$

  meaning the regression curve is not intersected before the ruling reaches the next foldline.

The regularity of the two folds represented in Fig. 3.1, 3.2 is guaranteed by comparing the values of $\bar{v}_{(j,S_j)}$ and $d_{(j,S_j)}$ computed with the Mathematica code available at [1] (Fig. 3.9, 3.12). For the intersection between the unit sphere and the hyperbolic paraboloid $z - 3xy = 0$ we provide the plots for one of the four arcs equivalent up to reflection, which can be parametrized as

$$\left(t, \sqrt{\frac{1-t^2}{1+9t^2}}, 3t\sqrt{\frac{1-t^2}{1+9t^2}}\right) \text{ on } \left[-\sqrt{(-1+\sqrt{10})/9}, \sqrt{(-1+\sqrt{10})/9}\,\right].$$

For the toroidal curve we use the parametrization from §3.4 and restrict the plot to $[0, 2\pi]$, which corresponds to one of the five arcs equivalent up to rotation. In both cases the parametrizations are not arc-length and rather than scaling down the starting ridge to match the length of the unit circle, equivalently to our purposes we have scaled up the unit circle (and accordingly all the concentric foldlines) to match the length of the ridge. It is worth mentioning that in the hyperbolic paraboloid case, although the ruling structure of the outer strip is rather well-behaved, it is not possible to further propagate the folding to an additional strip of the same width, since some of the rulings would cross the regression curve of the induced developable before they could reach the new outer boundary (Fig. 3.10).

The plots of the normal curvature and those of the relative torsion of the ridges are also provided (Fig. 3.11, 3.13).

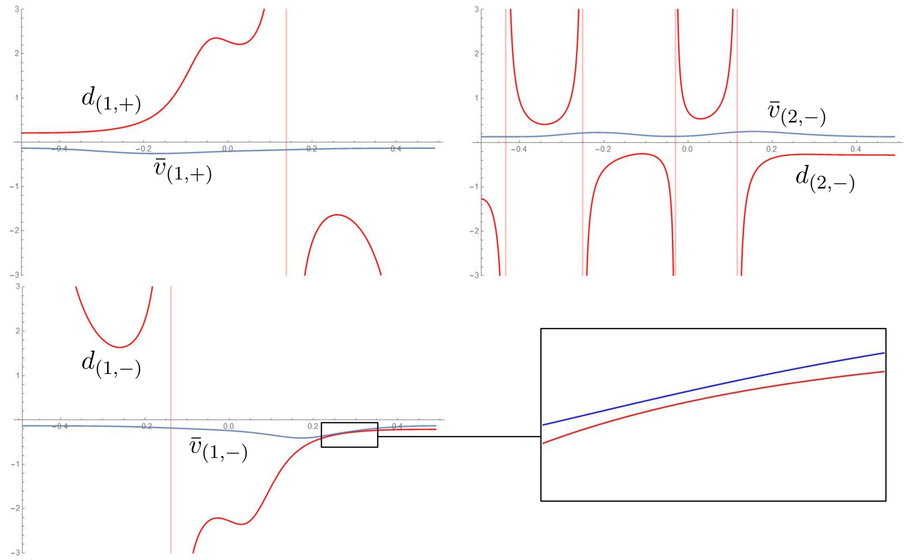

Figure 3.9: For the three developables of Fig. 3.2, comparison of the signed distance along the ruling between concentric circles $d_{(j,S)}$ and between the ridge and the regression curve $\bar{v}_{(j,S)}$. The surfaces are regular since the rulings reach the next foldline without first crossing the regression curve.

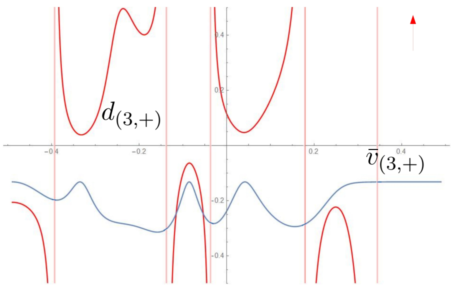

Figure 3.10: The isometry of a possible third outer strip induced by the propagation would turn singular before reaching the boundary.

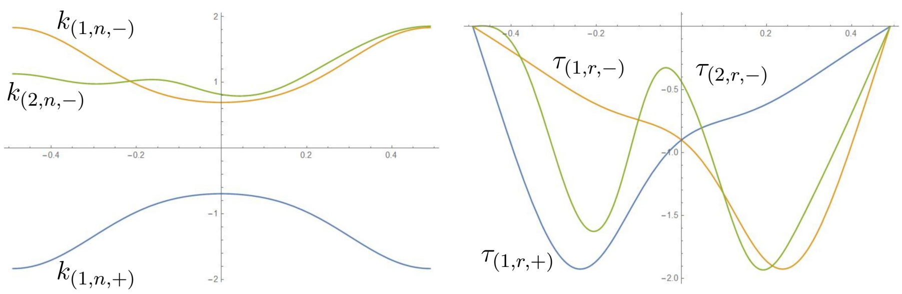

Figure 3.11: In the notation from §3.5, normal curvature and relative torsion of the two ridges in the fold of Fig. 3.1. Note that $k_{(1,n,+)}$, $\tau_{(1,r,+)}$ and $k_{(1,n,-)}$, $\tau_{(1,r,-)}$ are respectively the normal curvature and relative torsion of the first ridge (the one from which the propagation starts) w.r.t. the developables on its two sides.

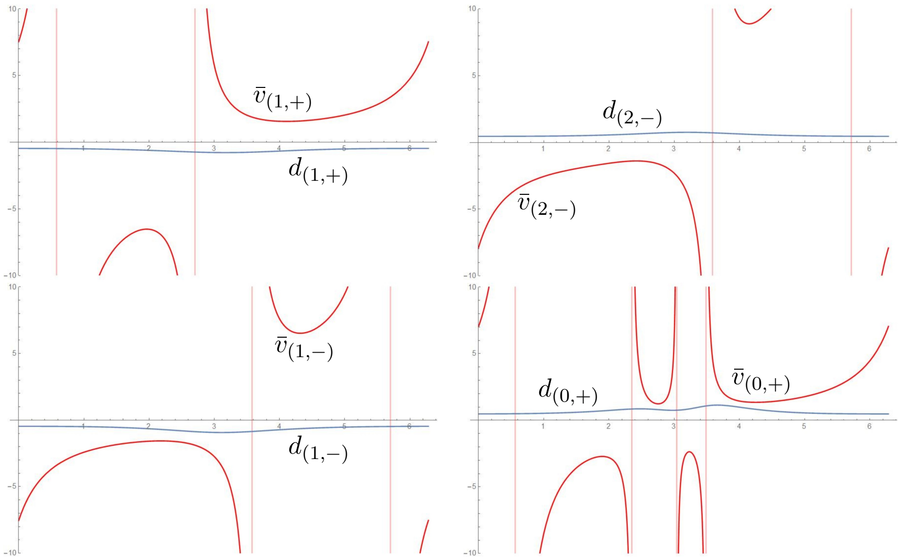

Figure 3.12: For the four developables of Fig. 3.2, comparison of the signed distance along the ruling between concentric circles $d_{(j,S)}$ and between the ridge and the regression curve $\bar{v}_{(j,S)}$. The surfaces are regular since the rulings reach the next foldline without first crossing the regression curve. We refer to the inner ridge as the 0-th one.

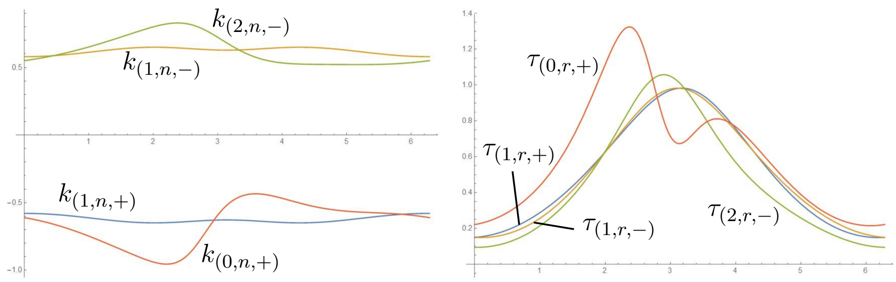

Figure 3.13: In the notation from §3.5, normal curvature and relative torsion of the three ridges in the fold of Fig. 3.2. Note that $k_{(1,n,+)}$, $\tau_{(1,r,+)}$ and $k_{(1,n,-)}$, $\tau_{(1,r,-)}$ are respectively the normal curvature and relative torsion of the first ridge (the one from which the propagation starts) w.r.t. the developables on its two sides. We refer to the inner ridge as the 0-th one.

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