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Advanced State Estimation Methods for Complex Dynamical Systems

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Abstract

State estimation is a key concept in modern model-based control and related fields like diagnosis and supervision. The increasing complexity of nowadays systems demands new concepts and tools for state estimation and observer design. Different aspects of complexity are covered in this work, which is divided into two parts.

In the first part, complexity arises from the problem size, i.e., the system order, and from the linear time-varying or nonlinear system dynamics. Hence, this part presents observer design techniques, which allow to reduced the problem size and the computational complexity of the resulting observers. The techniques are based on non-uniform and uniform exponential stability notions for linear time-varying systems. Different detectability concepts, which are the key requirement for a successful observer design, are introduced based on the aforementioned stability notions. Theoretical tools like Lyapunov exponents and the exponential dichotomy spectrum are utilized for the stability analysis of the corresponding observer error dynamics. Moreover, numerical methods for the stability analysis of the underlying system are employed in the observer design. The properties of the resulting observers are investigated by means of simulation examples and experimental data.

The second part of the thesis presents robust estimation concepts in the presence of unknown inputs. Although a large class of disturbances can be modeled as unknown inputs, their presence renders the observer design more complex compared to the undisturbed case. Based on the notions of strong observability and detectability, various linear and nonlinear observer design techniques are proposed for linear time invariant and time varying systems.

Kurzfassung

Die Schätzung unbekannter Zustände mittels Beobachtern ist ein Schlüsselkonzept der modernen modellbasierten Regelungstechnik und verwandten Problemstellungen wie Diagnose und Überwachung technischer Prozesse. Die zunehmende Komplexität heutiger Systeme erfordert neue Konzepte und Werkzeuge für die Zustandsschätzung und den Beobachterentwurf. Dies ist die Motivation für die vorliegende Arbeit, die in zwei Teile gegliedert ist.

Im ersten Teil ergibt sich die Komplexität beim Beobachterentwurf aus der Problemgröße, d.h. der Systemordnung, und aus der linearen zeitvarianten oder nichtlinearen Systemdynamik. Es werden Beobachterentwurfsmethoden vorgestellt, die es erlauben, die Problemgröße und die Komplexität des resultierenden Beobachters zu reduzieren. Die Entwurfsmethoden basieren auf unterschiedlichen Stabilitätskonzepten für lineare zeitvariante Systeme. Im Speziellen werden nicht gleichförmige und gleichförmige exponentielle Stabilität eingehend untersucht. Auf Basis dieser Stabilitätsbegriffe werden verschiedene Detektierbarkeitskonzepte eingeführt, welche eine Grundvoraussetzung für einen erfolgreichen Beobachterentwurf sind. Theoretische Werkzeuge wie Lyapunovexponenten oder das exponentielle Dichotomiespektrum werden für die Stabilitätsanalyse der zugrunde liegenden Schätzfehlerdynamik vorgeschlagen. Darüber hinaus werden numerische Methoden zur Stabilitätsuntersuchung der betrachteten Systeme im Beobachterentwurf verwendet. Die Eigenschaften der resultierenden Beobachter werden mit Hilfe von Simulationsbeispielen und Experimenten untersucht.

Im zweiten Teil der Arbeit werden robuste Schätzmethoden für Systeme mit unbekannten Eingängen vorgestellt. Obwohl eine große Klasse von Störungen als unbekannter Eingang modelliert werden kann, erhöhen solche Störungen die Komplexität beim Beobachterentwurf im Vergleich zum ungestörten Fall. Basierend auf den Konzepten der starken Beobachtbarkeit und Detektierbarkeit werden verschiedene lineare und nichtlineare Beobachterentwurfsmethoden für lineare zeitinvariante und zeitvariante Systeme vorgeschlagen.

Danksagung

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Graz, im Februar 2020 Markus Tranninger

Contents

Chapter 1
Introduction

State estimation is an elementary building block of modern model-based control and related fields. Due to limited sensor information, it is often impracticable or even impossible to obtain the desired information about the system states solely based on measurements. The underlying idea of all state estimation techniques is to combine mathematical models of the considered system with knowledge of input and output data to deduce the true system states. The realization of such concepts is a so-called observer, which allows to obtain reliable information about the system states.

The state estimation problem has a long history in control theory and applications since the ground breaking work of Kalman [KB61] and Luenberger [Lue64] in the 1960s, see also [ORe83] for a comprehensive monograph on this topic. Observers are used in a variety of different applications like model based control, model based diagnosis, parameter estimation, the synchronization of dynamical systems or predictive maintenance and forecasting.

Due to new technologies and emerging communication capabilities between systems, state estimation techniques have a wide range of applications already today and especially in the future. Besides the classical applications in control, potential fields are, e.g., real-time monitoring and forecasting of traffic flow [ABN16; Wit+15] or the detection of malicious attacks in networked control systems [PDB13]. This demands reliable estimation methods despite the increasing complexity of the considered problems.

Complexity can be provoked by different reasons like, e.g., model uncertainties, external disturbances or the problem size. The focus of this thesis is on analysis and design techniques to solve important aspects of state estimation problems for complex dynamical systems. Different aspects of complexity are considered within this work.

The first part of the thesis is devoted to state estimation methods for large scale systems, i.e., the complexity is resulting from the problem size. Linear time varying ordinary differential equations as mathematical models are considered, because they typically arise from the linearization of nonlinear models along a specific trajectory.

One fundamental research question covered in the first part of the work is the existence of an observer and the minimum requirements for this. The existence question is closely related to the so-called detectability of the underlying system. Detectability is not only relevant for the observer design, it is also important for other analysis and design problems such as sensor placement [SL14] or model order reduction [Sch00; Moo81]. The detectability question will be investigated extensively in Chapter 3 and Chapter 4 for the class of (continuous-time) linear time varying systems and a new perspective on the detectability concept will be presented.

As a state of the art approach for the observer design for this system class, a deterministic interpretation of the Kalman-Bucy filter [KB61] is usually the method of choice. This observer relies on the solution of a matrix Riccati differential equation. For systems with a large number of states, however, solving this differential equation becomes computationally challenging or even intractable and thus alternative approaches are presented in this thesis, which allow a reduction of the problem size.

The challenge regarding the detectability analysis lies in the fact that detectability is inherently related to the stability concept of the underlying observer error dynamics. In the time varying setting, stability notions, which all coincide for the time invariant case, differ significantly. If fact, non-uniform and uniform asymptotic or exponential stability notions show different robustness properties with respect to perturbations of the underlying model.

In Chapter 3, so-called exponential stability is considered. The proposed exponential detectability condition and the observer design are based on the so-called Lyapunov exponents. The Lyapunov exponents are characteristic numbers introduced by Lyapunov [Lya92]. They can be seen as an extension of eigenvalues of linear time invariant systems and allow to assess exponential stability properties of the system.

Chapter 4 extends the ideas from Chapter 3 to uniform exponential stability and detectability. For this, a detectability condition based on the so-called exponential dichotomy spectrum is proposed. Based on this detectability condition, an observer design is presented, which guarantees a negative exponential dichotomy spectrum and hence uniform exponential stability of the underlying observer error dynamics. For systems, which possess an exponential dichotomy, the proposed condition is shown to be necessary and sufficient for the existence of an observer with the desired stability properties.

The considered spectral stability concepts utilized in Chapters 3 and 4 are extensively discussed. In both cases, the proposed observer algorithms are investigated in detail and various examples allow to gain insight into the required theoretical concepts.

In Chapter 5, the ideas presented in Chapter 4 are extended to nonlinear systems. This allows to design local observers with guaranteed convergence of the estimation error if the initial estimate lies within some neighborhood of the true system states.

The properties of the proposed observer algorithms are extensively studied with the aid of simulation examples and real measurement data. The presented methods and tools may also foster further research on detectability concepts for linear time varying and nonlinear systems.

In the second part of this thesis, additional disturbances acting on the system are explicitly considered in the observer design. Such disturbances render the observer design more complex in general. However, a large class of disturbances and uncertainties can be modeled as unknown inputs to the considered system. Hence, the second part is devoted to a generalization of observability and detectability concepts for systems with unknown inputs. If the system possesses certain properties, e.g., so-called strong detectability, it is still possible to obtain reliable state information solely based on known input and output information. The essential requirements for the design an unknown input observer for linear time invariant systems are known since [Hau83].

In the first part of Chapter 6, existing unknown input observer design techniques are revisited and a straightforward design procedure is proposed. It is shown that the design of a linear unknown input observer can be reduced to a classical Luenberger observer design. The second part of the chapter deals with the case, where a classical unknown input observer does not exist. By incorporating also time derivatives of the sensor signals, it may still be feasible to estimate the system states in the presence of unknown inputs. The derivative estimation based on higher order sliding mode concepts is known to be immune with respect to certain classes of disturbances [Lev03]. Hence, a class of robust higher order sliding mode based observers is presented in the second part of Chapter 6. Chapter 7 essentially combines results from Chapter 4 and the higher order sliding mode techniques presented in Chapter 6. Extending the latter ideas to the time varying setting allows to tackle the problem of accurate state estimation despite unknown inputs for linear time varying systems.

To sum up, the present work answers relevant and fundamental questions in the field of state estimation and observer design. The theoretical and algorithmic tools presented in this thesis aim towards dependable solutions for the state estimation problem for complex dynamical systems.

Notation

Matrices are printed in bold capital letters, whereas column vectors are bold face lower case letters. The kernel and the image of a matrix are denoted by $\ker(\cdot)$ and $\operatorname{im}(\cdot)$, respectively. The vector $\mathbf{q}_i$ denotes the i-th column of the Matrix $\mathbf{Q}$ whose entries are denoted by q_{ij}. The matrix $\mathbf{I}_n$ is the $n \times n$ identity matrix. Symmetric positive definite (positive semidefinite) matrices $\mathbf{M}^\mathsf{T} = \mathbf{M}$ are denoted by $\mathbf{M} \succ 0$ ($\mathbf{M} \succeq 0$). If for two symmetric matrices $\mathbf{M}_2 - \mathbf{M}_1 \succ 0$ ($\succeq 0$), then $\mathbf{M}_1 \prec \mathbf{M}_2$ ($\mathbf{M}_1 \preceq \mathbf{M}_2$). The unique square root of a positive semidefinite matrix $\mathbf{M}$ is a positive semidefinite matrix $\mathbf{N}$ such that $\mathbf{N}^2 = \mathbf{M}$. This square root is denoted by $\mathbf{M}^{1/2}$. The time derivative $\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}$ is also represented by $\dot{\mathbf{x}}(t)$ and in some cases the time dependency is omitted for the sake of better readability. The 2-norm of a vector or the corresponding induced matrix norm is denoted by $\|\cdot\|$. For two sets A and B, $A \subset B$ ($A \subsetneq B$) denotes that A is a subset (a proper subset) of B.

Chapter 2

Fundamentals of Dynamical System Theory

This chapter recalls basic concepts of stability theory for dynamical systems in Sections 2.1 to 2.3. The second part of the chapter discussed in Sections 2.4 and 2.5 is devoted to observability and detectability notions and concepts, which allow to obtain the system states via the knowledge of the system's inputs and outputs.

2.1 Stability Notions

Stability theory for dynamical systems deals with the stability properties of solutions of differential equations of the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}). \tag{2.1}$$

The nonlinear function $\mathbf{f}(t, \mathbf{x}) : \mathbb{J} \times \mathbb{R}^n \to \mathbb{R}^n$ is piece-wise continuous in the scalar time parameter t and continuous in the state $\mathbf{x}$. The system of differential equations (2.1) is considered for times $t \in \mathbb{J} = [0, \infty)$ and the initial state at the initial time $t_0 \in \mathbb{J}$ is denoted by $\mathbf{x}(t_0) = \mathbf{x}_0$. It is assumed that the solutions of (2.1) are defined and unique for any $\mathbf{x}_0$ and any $t \geq t_0$.

For the stability assessment, one is often interested in the behavior of solutions starting near an equilibrium $\mathbf{0} = \mathbf{f}(t, \mathbf{x})$. It is assumed that $\mathbf{x} = \mathbf{0}$ is such an equilibrium[1] of (2.1) in the following.

Details on different stability concepts can be found, e.g., in [Hah67; HP06; AIW13; Zho16]. The notions used in this work are introduced in the following definition.

Definition 2.1 (stability notions) The zero solution $\mathbf{x} = \mathbf{0}$ of (2.1), is

(i) stable (S) in the sense of Lyapunov if for each $\varepsilon > 0$ and for each $t_0 \in \mathbb{J}$ there

[1] For any other equilibrium $\mathbf{x}_r$, one can introduce the change of coordinates $\mathbf{z} = \mathbf{x} - \mathbf{x}_r$.

exists a scalar $\delta(\varepsilon, t_0) > 0$ such that

$$\|\mathbf{x}(t_0)\| \leq \delta(\varepsilon, t_0) \text{ implies that } \|\mathbf{x}(t)\| \leq \varepsilon \text{ for all } t \geq t_0; \tag{2.2}$$

(ii) uniformly stable (US) if δ in (i) is independent of the initial time t_0;

(iii) attractive (A) if for each $t_0 \in \mathbb{J}$ there exists a scalar $\eta(t_0) > 0$ such that for each $\kappa > 0$ and for all $\mathbf{x}_0$ with $\|\mathbf{x}_0\| \leq \eta$ there exists a $T(\kappa, t_0) > 0$ such that

$$\|\mathbf{x}(t_0)\| \leq \eta(t_0) \text{ implies that } \|\mathbf{x}(t)\| \leq \kappa \text{ for all } t \geq t_0 + T(\kappa, t_0); \tag{2.3}$$

(iv) uniformly attractive (UA) if η and T in (iii) are independent of t_0;

(v) asymptotically stable (AS) if it is stable and attractive;

(vi) uniformly asymptotically stable (UAS) if it is uniformly stable and uniformly attractive;

(vii) exponentially stable (ES) if for some real $\mu > 0$ and every $t_0 \in \mathbb{J}$ there exist scalars $\rho(t_0) > 0$ and $K(t_0) \geq 1$ such that for every $\mathbf{x}_0$ with $\|\mathbf{x}_0\| \leq \rho(t_0)$ one has

$$\|\mathbf{x}(t)\| \leq K(t_0) e^{-\mu(t-t_0)} \|\mathbf{x}(t_0)\| \tag{2.4}$$

for all $t \geq t_0$;

(viii) uniformly exponentially stable (UES) if K and ρ in (vii) are independent of t_0;

(ix) unstable if it is not stable.

It should be remarked that in some standard textbooks, e.g., [Vid02; Hah67], the non-uniform notion of exponential stability is not considered and thus the terminology exponential stability is used for *uniform* exponential stability.

2.2 Stability of Linear Time Varying Systems

In the following, the linear time varying autonomous system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \ \mathbf{x}(t) \in \mathbb{R}^n \tag{2.5}$$

is considered for $t \in \mathbb{J} = [0, \infty)$. It is assumed that $\mathbf{A}(t)$ is continuous and uniformly bounded according to

$$\sup_{\substack{t \in \mathbb{J} \\ \|\mathbf{x}\|=1}} \|\mathbf{A}(t)\mathbf{x}\| = \bar{a} < \infty. \tag{2.6}$$

The following results on the solution of linear systems can be found in classical textbooks, e.g., [CL84; Rug95; HP06]. System (2.5) has the unique solution

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0, \tag{2.7}$$

where $\mathbf{\Phi}(t, t_0)$ is the state transition matrix, $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$ the initial state and $t_0 \in \mathbb{J}$ the considered initial time. This state transition matrix can be obtained by the associated fundamental matrix differential equation

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t), \quad \mathbf{X}(t) \in \mathbb{R}^{n\times n}. \tag{2.8}$$

Using any solution of (2.8) with $\mathbf{X}(0) = \mathbf{X}_0$ as a non-singular matrix, the state transition matrix is given by

$$\mathbf{\Phi}(t, t_0) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0). \tag{2.9}$$

It has the properties

i) $\mathbf{\Phi}(t, t) = \mathbf{I}_n$,

ii) $\mathbf{\Phi}(t, t_1)\mathbf{\Phi}(t_1, t_0) = \mathbf{\Phi}(t, t_0)$ for all $t_0, t_1, t \in \mathbb{J}$ (co-cycle property),

iii) $\mathbf{\Phi}^{-1}(t_1, t_0) = \mathbf{\Phi}(t_0, t_1)$,

iv) $\frac{\partial \mathbf{\Phi}(t,t_0)}{\partial t} = \mathbf{A}(t)\mathbf{\Phi}(t, t_0)$, and

v) $\frac{\partial \mathbf{\Phi}(t_1,t)}{\partial t} = -\mathbf{\Phi}(t_1, t)\mathbf{A}(t)$.

The transition matrix can also be stated in terms of the Peano-Baker series

$$\begin{aligned}\mathbf{\Phi}(t, t_0) = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\tau_1)\,\mathrm{d}\tau_1 + \int_{t_0}^{t} \mathbf{A}(\tau_1) \int_{t_0}^{\tau_1} \mathbf{A}(\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \\ + \int_{t_0}^{t} \mathbf{A}(\tau_1) \int_{t_0}^{\tau_1} \mathbf{A}(\tau_2) \int_{t_0}^{\tau_2} \mathbf{A}(\tau_3)\,\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \cdots,\end{aligned} \tag{2.10}$$

see [Rug95]. If the matrix $\mathbf{A}(t)$ commutes with its integral, i.e., if

$$\mathbf{A}(t)\int_{t_0}^{t} \mathbf{A}(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} \mathbf{A}(\tau)\,\mathrm{d}\tau\,\mathbf{A}(t) \tag{2.11}$$

holds for all t, $t_0 \in \mathbb{J}$, then the state transition matrix is given by

$$\mathbf{\Phi}(t, t_0) = e^{\int_{t_0}^{t} \mathbf{A}(\tau)\,\mathrm{d}\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\int_{t_0}^{t} \mathbf{A}(\tau)\,\mathrm{d}\tau\right]^k. \tag{2.12}$$

In particular, this holds for a constant coefficient matrix $\mathbf{A}(t) = \mathbf{A}$, where the state transition matrix takes the compact form

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)} \tag{2.13}$$

with the matrix exponential

$$e^{\mathbf{A}\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k\tau^k = \mathbf{I} + \mathbf{A}\tau + \frac{1}{2!}\mathbf{A}^2\tau^2 + \frac{1}{3!}\mathbf{A}^3\tau^3 + \cdots . \tag{2.14}$$

For a linear system in the form of (2.5), stability is not a property of the specific equilibrium and is characterized entirely by its state transition matrix $\mathbf{\Phi}(\cdot,\cdot)$. This is summarized in the following.

Lemma 2.2 (stability criteria for linear systems)
System (2.5) *is*

(i) stable, if and only if for each $t_0 \in \mathbb{J}$ *there exists a* $K(t_0) > 0$ *such that*

$$\|\mathbf{\Phi}(t,t_0)\| \le K(t_0) \text{ for all } t \ge t_0; \tag{2.15}$$

(ii) uniformly stable, if and only if K *in (i) is independent of* t_0*;*

(iii) asymptotically stable, if and only if (2.15) *holds and*

$$\lim_{t\to\infty} \|\mathbf{\Phi}(t,0)\| = 0; \tag{2.16}$$

(iv) exponentially stable, if and only if there exists a constant $\mu > 0$ *such that for every* $t_0 \in \mathbb{J}$ *there exists a scalar* $K(t_0) \ge 1$ *such that*

$$\|\mathbf{\Phi}(t,t_0)\| \le K(t_0)e^{-\mu(t-t_0)} \text{ for all } t \ge t_0{}^2; \tag{2.17}$$

(v) uniformly exponentially stable, if and only if K *in (iv) is independent of* t_0*;*

(vi) uniformly asymptotically stable, if and only if it is uniformly exponentially stable.

These relations are well known in the literature. Details can be found for example in [HM80], [Rug95, Chapter 6], [Hah67, Chapter VIII] and [Zho16]. It should be remarked that in the linear case, attractivity is a global property, i.e. if the zero solution is attractive for some $\mathbf{x}_0$ with $\|\mathbf{x}_0\| \le \eta$ and some $\eta > 0$, it is attractive for all $\mathbf{x}_0 \in \mathbb{R}^n$. Hence, (uniform) asymptotic stability is also a global property.

For time invariant systems, the uniform and non-uniform stability notions coincide. Moreover, the eigenvalues of the constant coefficient matrix $\mathbf{A}(t) = \mathbf{A}$ determine the stability characteristics. A negative eigenvalue spectrum $\sigma(\mathbf{A})$ is equivalent to uniform exponential stability and the matrix $\mathbf{A}$ is then called a Hurwitz matrix. The

[2] If this holds for some t_0, it also holds for any other t_0^* because of the co-cycle property of the state transition matrix $\mathbf{\Phi}(t,t_0^*) = \mathbf{\Phi}(t,t_0)\mathbf{\Phi}(t_0,t_0^*)$ and $\|\mathbf{\Phi}(t_0,t_0^*)\|$ as a constant.

(a) linear time varying systems

(b) linear time invariant systems

Fig. 2.1: Relations between stability notions of linear systems.

relations between the different stability notions in the linear time varying and time invariant case are summarized in Fig. 2.1.

A state transformation which preserves the stability properties of the system is called a Lyapunov transformation.

Definition 2.3 (Lyapunov transformation [Adr95]) A smooth and invertible linear change of coordinates $\mathbf{z}(t) = \mathbf{T}(t)\mathbf{x}(t)$ is called a Lyapunov transformation if $\mathbf{T}(t)$, $\mathbf{T}^{-1}(t)$ and $\dot{\mathbf{T}}(t)$ are uniformly bounded for all $t \in \mathbb{J}$.

The transformed system is given by

$$\dot{\mathbf{z}}(t) = \left[\mathbf{T}(t)\mathbf{A}(t)\mathbf{T}^{-1}(t) + \dot{\mathbf{T}}(t)\mathbf{T}^{-1}(t)\right]\mathbf{z}(t). \tag{2.18}$$

Systems (2.5) and (2.18) are also called kinematically similar. Stability of linear time varying systems, which are kinematically similar to time invariant systems, can be assessed via the eigenvalues of the time invariant coefficient matrix obtained via (2.18). A system with a periodic coefficient matrix is always kinematically similar to a system with a constant coefficient matrix [Adr95, Theorem 3.2.2].

In general, the (time-dependent) eigenvalues of a time-varying coefficient matrix do not allow to reason about the stability properties, i.e., negative eigenvalues of $\mathbf{A}(t)$ are neither necessary nor sufficient for stability of the underlying system. This is demonstrated by the following examples.

Example 2.4 (eigenvalues of linear time varying systems, [Poe10])*:* Consider the linear time varying system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 - 2\cos(4t) & 2 + 2\sin(4t) \\ -2 + 2\sin(4t) & -1 + 2\cos(4t) \end{bmatrix} \mathbf{x}(t). \tag{2.19}$$

The point-wise eigenvalues of the coefficient matrix lie at $\sigma(\mathbf{A}(t)) = \{-1\}$ for all $t \in \mathbb{R}$. Nevertheless, this system is unstable because it admits the unbounded solution

$$\mathbf{x}(t) = e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}. \tag{2.20}$$

[3]The example is due to a communication of the author with Prof. J. Reger from TU Ilmenau.

Example 2.5 (eigenvalues of linear time varying systems)*:* Consider the linear time varying system[3]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 + 2\sin(4t) & -2\cos(4t) \\ -2\cos(4t) & -1 - 2\sin(4t) \end{bmatrix} \mathbf{x}(t). \tag{2.21}$$

The point-wise eigenvalues of the coefficient matrix lie at $\sigma(\mathbf{A}(t)) = \{-4, 2\}$ for all $t \in \mathbb{R}$. However, system (2.21) is kinematically similar to a linear time invariant system

$$\dot{\mathbf{z}} = \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix} \mathbf{z} \tag{2.22}$$

via the transformation matrix

$$\mathbf{T}(t) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t). \end{bmatrix} \tag{2.23}$$

The eigenvalues of the constant matrix in (2.22) are $\{-1, -1\}$. Hence, system (2.21) is uniformly exponentially stable.

Different concepts for the generalization of eigenvalues to the time varying setting will be discussed in Chapters 3 and 4.

2.3 Stability Assessment via Lyapunov Functions

The stability definitions require the general solution of the nonlinear system (2.1). This solution is hard to obtain for general nonlinear systems. Lyapunov provided a tool for the stability assessment of equilibrium points (or reference solutions) for general nonlinear systems without explicit knowledge of the solution, hence this concept is also called Lyapunov's direct method. The original result can be found in [Lya92, Section 16, Theorem 1].

Lyapunov's direct method can be applied to equilibria of nonlinear systems in the form of (2.1) and it is again assumed that $\mathbf{x} = \mathbf{0}$ is such an equilibrium. In order to state the required results, the concept of comparison functions is introduced first.

Definition 2.6 (comparison function [HP06]) Let $0 < r_1 \leq \infty$. A function $\alpha : [0, r_1) \to \mathbb{R}_+$ is

(i) a class $\mathcal{K}$ function if $\alpha(\cdot)$ is monotonically increasing, $\alpha(r) > 0$ for $r > 0$ and $\lim_{r\to 0} \alpha(r) = 0$.

(ii) class $\mathcal{K}_\infty$ function if additionally $r_1 = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.

The required results are summarized in the following theorems and are discussed in detail, e.g., in [HP06, Section 3.2] or [Kha02, Chapter 4].

Theorem 2.7 (uniform asymptotic stability [Kha02])
Let $\mathbf{x} = \mathbf{0}$ be an equilibrium of (2.1) and $\mathcal{D} \subset \mathbb{R}^n$ a domain containing $\mathbf{x} = \mathbf{0}$. Let $V(t, \mathbf{x}) : \mathbb{J} \times \mathcal{D} \to \mathbb{R}_+$ be a continuously differentiable function such that

$$\alpha_1(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|), \tag{2.24}$$

with $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ as class $\mathcal{K}$ functions. If

(i) the time derivative of V along the trajectories is negative semidefinite, i.e.,

$$\dot{V}(t, \mathbf{x}) := \frac{\partial V}{\partial t}(t, \mathbf{x}) + \left[\frac{\partial V}{\partial \mathbf{x}}(t, \mathbf{x})\right]^{\mathsf{T}} \mathbf{f}(t, \mathbf{x}) \leq 0 \tag{2.25}$$

for all $t \in \mathbb{J}$ and all $\mathbf{x} \in \mathcal{D}$, then $\mathbf{x} = \mathbf{0}$ is uniformly stable and V is called a Lyapunov function.

(ii) the time derivative is negative definite, i.e.,

$$\dot{V}(t, \mathbf{x}) \leq \alpha_3(\|\mathbf{x}\|) \tag{2.26}$$

for all $t \in \mathbb{J}$ and all $\mathbf{x} \in \mathcal{D}$ with α_3 as a class $\mathcal{K}$ function, then $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable.

(iii) $\dot{V}(t, \mathbf{x})$ is negative definite for all $t \in \mathbb{J}$, $\mathcal{D} = \mathbb{R}^n$ and α_1 and α_2 in (2.24) are class $\mathcal{K}_\infty$ functions, then $\mathbf{x} = \mathbf{0}$ is globally uniformly asymptotically stable.

Theorem 2.8 (uniform exponential stability [Kha02, Theorem 4.10])
Let $\mathbf{x} = \mathbf{0}$ be an equilibrium of (2.1) and $\mathcal{D} \subset \mathbb{R}^n$ a domain containing $\mathbf{x} = \mathbf{0}$. Suppose that the continuously differentiable function $V(t, \mathbf{x}) : \mathbb{J} \times \mathcal{D} \to \mathbb{R}_+$ satisfies

$$\alpha_1 \|\mathbf{x}\|^\gamma \leq V(t, \mathbf{x}) \leq \alpha_2 \|\mathbf{x}\|^\gamma, \ \dot{V}(t, \mathbf{x}) \leq -\alpha_3 \|x\|^\gamma \tag{2.27}$$

for all $t \in \mathbb{J}$ and all $\mathbf{x} \in \mathcal{D}$, where α_1, α_2, α_3 and $\gamma > 0$ are positive constants. Then, the equilibrium $\mathbf{x} = \mathbf{0}$ is uniformly exponentially stable.

If the linear time varying system (2.5) is uniformly exponentially stable, a Lyapunov function exists by a converse theorem presented in [HP06, Theorem 3.3.33 and 3.3.38].

Proposition 2.9 (Lyapunov function for linear time varying systems, [HP06])
Supposed that (2.5) is uniformly exponentially stable, then:

(i) There exists a unique, uniformly bounded solution $\mathbf{P}(t)$ of the matrix differ-

ential equation

$$\dot{\mathbf{P}}(t) + \mathbf{A}^{\mathsf{T}}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t) + \mathbf{Q}(t) = \mathbf{0} \tag{2.28}$$

for some $\mathbf{Q}(t)$ *with* $q_1\mathbf{I}_n \preceq \mathbf{Q}(t) \preceq q_2\mathbf{I}_n$ *and some positive constants* q_1, q_2.

(ii) The only bounded $\mathbf{P}(t)$ *which solves* (2.28) *is given by*

$$\mathbf{P}(t) = \int_t^\infty \mathbf{\Phi}^{\mathsf{T}}(s,t)\mathbf{Q}(t)\mathbf{\Phi}(s,t)\,\mathrm{d}s \text{ with } t \in \mathbb{J}, \tag{2.29}$$

and there exist positive constants p_1, p_2 *such that* $p_1\mathbf{I}_n \preceq \mathbf{P}(t) \preceq p_2\mathbf{I}_n$.

(iii) $V(t,\mathbf{x}) = \mathbf{x}^{\mathsf{T}}\mathbf{P}(t)\mathbf{x}$ *is a Lyapunov function for* (2.5) *with*

$$\dot{V}(t,\mathbf{x}) = -\mathbf{x}^{\mathsf{T}}\mathbf{Q}(t)\mathbf{x}. \tag{2.30}$$

(iv) The constant p_2 *can be bounded according to*

$$p_2 \leq \frac{K^2 q_2}{2\mu}, \tag{2.31}$$

where K *and* μ *are obtained from a bound on the state transition matrix*

$$\|\mathbf{\Phi}(t,t_0)\| \leq Ke^{-\mu(t-t_0)}. \tag{2.32}$$

2.4 Observability and Constructibility

Now, the considered linear time varying system is extended by inputs and outputs according to

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{2.33a}$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{2.33b}$$

with the input $\mathbf{u}(t) \in \mathbb{R}^m$ and the output $\mathbf{y}(t) \in \mathbb{R}^p$ and continuous and uniformly bounded matrices $\mathbf{B}(t)$ and $\mathbf{C}(t)$ of appropriate dimension. Observability is the system property, which allows to determine the initial state by the knowledge of all inputs and outputs on a finite time interval. Details on the following concepts can be found in [Rug95; Che98; Hes09].

Definition 2.10 (observability) System (2.33) is observable on $[t_0, t_1]$ if the initial state $\mathbf{x}_0$ at time t_0 can be uniquely determined by the knowledge of $\mathbf{u}(t)$ and $\mathbf{y}(t)$ on a finite time interval with $t \in [t_0, t_1]$.

The solution of (2.33a) is given by the variation of constants formula

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\,\mathrm{d}s \tag{2.34}$$

with the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$. The output of (2.33) can then be computed according to

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \mathbf{C}(t)\int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\,\mathrm{d}s. \tag{2.35}$$

The initial state can be uniquely determined, if it is distinguishable from all other initial states. Two initial states $\mathbf{x}_0^{(1)}$ and $\mathbf{x}_0^{(2)}$ are indistinguishable for an input $\mathbf{u}(t)$ on $[t_0, t_1]$ if

$$\mathbf{y}^{(1)}(t) = \mathbf{y}^{(2)}(t) \text{ for all } t \in [t_0, t_1]. \tag{2.36}$$

By the two solutions

$$\begin{aligned} \mathbf{y}^{(1)}(t) &= \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0^{(1)} + \mathbf{C}(t)\int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\,\mathrm{d}s \\ \mathbf{y}^{(2)}(t) &= \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0^{(2)} + \mathbf{C}(t)\int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\,\mathrm{d}s \end{aligned} \tag{2.37}$$

it can be seen that $\mathbf{y}^{(1)}(t) - \mathbf{y}^{(2)}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)(\mathbf{x}_0^{(1)} - \mathbf{x}_0^{(2)})$. This shows that indistinguishability of two initial states does not depend on the specific input for linear systems. In fact, without loss of generality, the known input $\mathbf{u}(t)$ can be assumed to be zero and the unobservable subspace is then given by all $\mathbf{x}$ such that the output is zero, i.e.,

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x} = \mathbf{0} \text{ for all } t \in [t_0, t_1]. \tag{2.38}$$

The indistinguishable states can be characterized in terms of the unobservable subspace.

Definition 2.11 (unobservable subspace) The unobservable subspace $\mathcal{UO}_{[t_0,t_1]}$ on a time interval $[t_0, t_1]$, $t_0, t_1 \in \mathbb{J}$ with $t_1 > t_0$ consists of all states $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x} = \mathbf{0} \text{ for all } t \in [t_0, t_1]. \tag{2.39}$$

The quantity

$$\tilde{\mathbf{y}}(t) = \mathbf{y}(t) - \mathbf{C}(t)\int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{B}(s)\mathbf{u}(s)\,\mathrm{d}s \tag{2.40}$$

is completely determined by the input and output and rearranging (2.35) yields

$$\tilde{\mathbf{y}}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0. \tag{2.41}$$

Multiplying both sides of this equation with $\mathbf{\Phi}^\mathsf{T}(t, t_0)\mathbf{C}^\mathsf{T}(t)$ and integrating from t_0 to t_1 gives

$$\int_{t_0}^{t_1} \mathbf{\Phi}^\mathsf{T}(s, t_0)\mathbf{C}^\mathsf{T}(s)\tilde{\mathbf{y}}(s)\,\mathrm{d}s = \int_{t_0}^{t_1} \mathbf{\Phi}^\mathsf{T}(s, t_0)\mathbf{C}^\mathsf{T}(s)\mathbf{C}(s)\mathbf{\Phi}(s, t_0)\,\mathrm{d}s\,\mathbf{x}_0. \tag{2.42}$$

The symmetric positive semidefinite $n \times n$ matrix

$$\mathbf{M}(t_1, t_0) = \int_{t_0}^{t_1} \mathbf{\Phi}^\mathsf{T}(s, t_0)\mathbf{C}^\mathsf{T}(s)\mathbf{C}(s)\mathbf{\Phi}(s, t_0)\,\mathrm{d}s \tag{2.43}$$

is the so-called observability Gramian. The state $\mathbf{x}_0$ can be reconstructed by the algebraic relation (2.42) if the observability Gramian is invertible. Also the converse is true, which is summarized in the following, see [Rug95; Hes09].

Theorem 2.12 (observability)
The following statements are equivalent:

(i) *system* (2.33) *is observable on* $[t_0, t_1]$

(ii) $\det \mathbf{M}(t_1, t_0) \neq 0$

(iii) $\mathcal{UO}_{[t_0,t_1]} = \{\mathbf{0}\}$

Observability uses future measurements and outputs to reconstruct the initial state $\mathbf{x}_0$, see (2.42); the related constructibility concept uses past measurements and outputs and is discussed in the following. The so-called constructibility Gramian

$$\mathbf{N}(t_1, t_0) = \int_{t_0}^{t_1} \mathbf{\Phi}^\mathsf{T}(s, t_1)\mathbf{C}^\mathsf{T}(s)\mathbf{C}(s)\mathbf{\Phi}(s, t_1)\,\mathrm{d}s \tag{2.44}$$

is related to the state $\mathbf{x}(t_1)$ via

$$\int_{t_0}^{t_1} \mathbf{\Phi}(s, t_1)\mathbf{C}^\mathsf{T}(s)\tilde{\mathbf{y}}(s)\,\mathrm{d}s = \mathbf{N}(t_1, t_0)\mathbf{x}(t_1). \tag{2.45}$$

By utilizing the co-cycle property it can be shown that the constructibility Gramian is related to the observability Gramian via

$$\mathbf{N}(t_1, t_0) = \mathbf{\Phi}^\mathsf{T}(t_0, t_1)\mathbf{M}(t_1, t_0)\mathbf{\Phi}(t_0, t_1). \tag{2.46}$$

Due to the invertibility of $\mathbf{\Phi}(\cdot,\cdot)$ for linear continuous-time[4] systems, the system is constructible on a time interval, if and only if it is observable.

The constructibility Gramian satisfies the differential Lyapunov equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{N}(t, t_0) = -\mathbf{A}^\mathsf{T}(t)\mathbf{N}(t, t_0) - \mathbf{N}(t, t_0)\mathbf{A}(t) + \mathbf{C}^\mathsf{T}(t)\mathbf{C}(t) \tag{2.47}$$

with the initial condition $\mathbf{N}(t_0, t_0) = \mathbf{0}$. Moreover, the symmetric matrix $\mathbf{N}$ is positive semidefinite for all $t_1 \geq t_0$, i.e. for all $\boldsymbol{\eta} \in \mathbb{R}^n$ it holds that

$$\boldsymbol{\eta}^\mathsf{T}\mathbf{N}(t_1, t_0)\boldsymbol{\eta} \geq 0 \text{ or equivalently } \mathbf{N}(t_1, t_0) \succeq 0. \tag{2.48}$$

[4]This is not guaranteed in the discrete time case, where the state transition matrix can be singular.

A stronger concept than observability, so-called uniform complete observability, was introduced by Kalman and Bucy [KB61].

Definition 2.13 (Uniform complete observability/constructibility) System (2.33) is uniformly completely observable, if there exist positive constants β_1, β_2 and σ such that

$$\beta_1 \mathbf{I} \preceq \mathbf{M}(t_0+\sigma, t_0) \preceq \beta_2 \mathbf{I} \text{ for all } t_0 \in \mathbb{J}. \tag{2.49}$$

The system is uniformly completely constructible, if there exist positive constants α_1, α_2 and σ such that

$$\alpha_1 \mathbf{I} \preceq \mathbf{N}(t_0+\sigma, t_0) \preceq \alpha_2 \mathbf{I} \text{ for all } t_0 \in \mathbb{J}. \tag{2.50}$$

The upper bound in (2.49) and (2.50) is guaranteed by the uniform boundedness of $\mathbf{A}(t)$ and $\mathbf{C}(t)$. By the Grönwall-Bellman inequality [Adr95, p. 197] one obtains

$$\|\mathbf{x}(t)\| \leq \|\mathbf{x}(t_0)\| e^{\left|\int_{t_0}^{t} \|\mathbf{A}(s)\| \, \mathrm{d}s\right|} \leq \|\mathbf{x}(t_0)\| e^{\bar{a}|t-t_0|} \tag{2.51}$$

for all $t, t_0 \in \mathbb{J}$. Together with the uniform bound on $\mathbf{C}(t)$ this allows to derive a uniform upper bound on $\mathbf{M}(t_0+\sigma, t_0)$ and $\mathbf{N}(t_0+\sigma, t_0)$ for any finite σ. The observability (or constructibility) properties of a system with bounded coefficient matrices are thus determined by the lower bounds on the corresponding Gramians.

Due to relation (2.46), the system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ with output $\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t)$ is uniformly completely observable if and only if it is uniformly completely constructible, because $\mathbf{\Phi}(t_0+\sigma, t_0)$ is uniformly bounded in t_0. This follows directly from (2.51). Hence, the two concepts are equivalent and the suitable concept will be used in this work depending on the situation.

Differential Observability

If the matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are sufficiently often continuously differentiable with uniformly bounded derivatives, it is possible to define a so-called generalized observability matrix [SM67; SM69] according to

$$\mathbf{O}_r(t) = \begin{bmatrix} \mathbf{C}_0(t) \\ \mathbf{C}_1(t) \\ \vdots \\ \mathbf{C}_{r-1}(t) \end{bmatrix} \tag{2.52}$$

for some positive integer r. The components $\mathbf{C}_i(t)$ for $i = 0, \ldots, r-1$ are recursively defined according to

$$\mathbf{C}_{i+1}(t) = \mathbf{C}_i(t)\mathbf{A}(t) + \dot{\mathbf{C}}_i(t), \ \mathbf{C}_0(t) = \mathbf{C}(t). \tag{2.53}$$

Note that in the time invariant case summarized in the following section, this matrix is the classical observability matrix due to Kalman for $r = n$. This generalized observability matrix allows to formulate sufficient conditions for observability and uniform complete observability. The first result is given, e.g., in [Rug95, Theorem 9.10].

Theorem 2.14 (observability condition)
System (2.33) *is observable on a time interval* $[t_0, t_1]$ *if there exists a positive integer* r *such that for some* $t_a \in [t_0, t_1]$

$$\operatorname{rank} \mathbf{O}_r(t_a) = n. \tag{2.54}$$

The generalized observability matrix allows also to state a sufficient condition for uniform complete observability

Theorem 2.15 (uniform complete observability condition)
System (2.33) *is uniformly completely observable on* $\mathbb{J}$ *if there exists a* $\mu > 0$ *and a positive integer* r *such that*

$$\mathbf{O}_r^\mathsf{T}(t)\mathbf{O}_r(t) \succeq \mu \mathbf{I}_n \text{ for all } t \in \mathbb{J}. \tag{2.55}$$

A proof of this result can be found in [BPP10]. Silverman [SM67] introduced the concept of uniform observability on a compact time interval and hence the lower bound in (2.55) is guaranteed if $\operatorname{rank} \mathbf{O}_r = n$ on this compact interval. In the time invariant case, it holds that $r \leq n$. This is not necessarily the case in the time varying setting as demonstrated by the following example.

Example 2.16*:* For the scalar system $\dot{x}(t) = x(t)$ with the output $y(t) = \sin(t)x(t)$, the generalized observability matrix for $r = n = 1$ is given by

$$O_1 = \sin(t) \tag{2.56}$$

and the bound in (2.55) for $O_1^\mathsf{T} O_1 = \sin^2(t)$ cannot hold. However, for $r = 2$ one has

$$\mathbf{O}_2(t) = \begin{bmatrix} \sin(t) \\ \sin(t) + \cos(t) \end{bmatrix} \tag{2.57}$$

and one obtains

$$\mathbf{O}_2^\mathsf{T}(t)\mathbf{O}_2 = 1 + \sin^2(t) + \sin(2t), \tag{2.58}$$

which is positive and uniformly bounded from zero for all t.

Similar to the stability notions, the uniformity (i.e., for all t_0) is always fulfilled for time invariant systems. This fact drastically simplifies the conditions for (uniform complete) observability in the time invariant case, which will be briefly summarized in the following.

2.4.1 Observability of Linear Time Invariant Systems

For constant matrices $\mathbf{A}(t) = \mathbf{A}$ and $\mathbf{C}(t) = \mathbf{C}$, the following statements are equivalent (see e.g. [Che98, Chapter 6]):

(i) The system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},\ \mathbf{y} = \mathbf{C}\mathbf{x}$ is (uniformly completely) observable.

(ii) The observability Gramian

$$\mathbf{M}(t,0) = \int_0^t e^{\mathbf{A}^\mathsf{T} s}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}s}\,\mathrm{d}s. \tag{2.59}$$

is nonsingular for all $t > 0$.

(iii) The $pn \times n$ observability matrix

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \tag{2.60}$$

has full column rank n.

(iv) The $(n+p) \times n$ matrix

$$\begin{bmatrix} \mu\mathbf{I} - \mathbf{A} \\ \mathbf{C} \end{bmatrix} \tag{2.61}$$

has full column rank n for every eigenvalue μ of $\mathbf{A}$.

The observability test via the observability matrix in (iii) is due to Kalman, the criterion in (iv) was presented by Hautus [Hau69]. The latter condition, the so-called eigenvector test in fact states that the system is observable if and only if there is no right eigenvector $\mathbf{p}_i$ of $\mathbf{A}$ such that

$$\mathbf{C}\mathbf{p}_i = \mathbf{0}. \tag{2.62}$$

If this condition does not hold for some eigenvector $\mathbf{p}_i$, the corresponding eigenvalue is called unobservable.

Observability is sufficient for the existence of a Luenberger-type observer [Lue64], i.e., a dynamical system of the form

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}\left[\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)\right]. \tag{2.63}$$

The dynamics of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is given by

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{LC})\mathbf{e}(t). \tag{2.64}$$

Observability guarantees that the eigenvalues of $(\mathbf{A} - \mathbf{LC})$ can be placed arbitrarily by a proper choice of the feedback gain $\mathbf{L}$. Hence, the estimation error system (2.64) can be rendered uniformly exponentially stable.

2.4.2 A Class of Observers for Linear Time Varying Systems

A milestone in state estimation was the Kalman filter in discrete time and the Kalman-Bucy filter in a continuous-time setting [KB61]. In this work, the result of Kalman and Bucy, which was originally developed in a stochastic setting, is employed in a deterministic framework. This allows to formulate an observer for system (2.33) of the form

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{L}(t)\left[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right] \tag{2.65}$$

with the special choice of the feedback gain

$$\mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^{\mathsf{T}}(t). \tag{2.66}$$

The symmetric $n \times n$ matrix $\mathbf{P}(\cdot)$ is the unique solution of the differential Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(t) = \mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^{\mathsf{T}}(t) - \mathbf{P}(t)\mathbf{C}^{\mathsf{T}}(t)\mathbf{C}(t)\mathbf{P}(t) + \mathbf{G}(t), \tag{2.67}$$

with the $n \times n$ matrices $\mathbf{P}(t_0) = \mathbf{P}^{\mathsf{T}}(t_0) \succeq 0$. The positive definite matrix $\mathbf{G}(t) = \mathbf{G}^{\mathsf{T}}(t) \succ 0$ is considered an observer design parameter and will be discussed in detail later. The assumptions, which guarantee that all solutions of (2.67) are uniformly upper and lower bounded, are summarized in the following.

Proposition 2.17 (uniform boundedness of the Riccati equation)
Suppose that the following assumptions hold:

(a) The initial condition $\mathbf{P}(t_0) = \mathbf{P}_0$ is positive definite and there exist positive constants p_{01} and p_{02} such that $p_{01}\mathbf{I} \preceq \mathbf{P}_0 \preceq p_{02}\mathbf{I}$ for all $t_0 \in \mathbb{J}$

(b) The matrix $\mathbf{G}(t)$ is positive definite and there exist positive constants g_1, g_2 such that $g_1\mathbf{I} \preceq \mathbf{G}(t) \preceq g_2\mathbf{I}$ for all $t \in \mathbb{J}$

(c) System (2.33) is uniformly completely constructible, i.e., there exist positive constants α_1, α_2 and σ such that

$$\alpha_1\mathbf{I} \preceq \mathbf{N}(t+\sigma, t) \preceq \alpha_2\mathbf{I} \tag{2.68}$$

holds for all $t \in \mathbb{J}$.

Then, there exist positive constants p_1, p_2 such that all solutions of (2.67) denoted by $\mathbf{\Pi}(t, \mathbf{P}_0, t_0)$ can be bounded according to

$$p_1\mathbf{I}_n \preceq \mathbf{\Pi}(t, \mathbf{P}_0, t_0) \preceq p_2\mathbf{I}_n \tag{2.69}$$

for all $t_0 \in \mathbb{J}$ and all $t \geq t_0$.

The proof is given in Appendix A.1. An alternative argumentation for the existence of a uniform lower bound is given in [Esc18, Proposition 2.5] by providing an upper bound for its inverse.

If $\mathbf{P}_0$ is chosen as a positive definite matrix, $\mathbf{P}(t)$ remains positive definite for all $t \geq t_0$. Hence, $\mathbf{P}(t)$ is invertible and a differential expression for the inverse can be obtained. Differentiating the identity $\mathbf{P}(t)\mathbf{P}^{-1}(t) = \mathbf{I}$ yields

$$\dot{\mathbf{P}}(t)\mathbf{P}^{-1}(t) + \mathbf{P}(t)\dot{\mathbf{P}}^{-1}(t) = \mathbf{0} \tag{2.70}$$

and together with (2.67), a matrix differential equation for the inverse is obtained according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}^{-1}(t) = -\mathbf{P}^{-1}(t)\mathbf{A}(t) - \mathbf{A}^{\mathsf{T}}(t)\mathbf{P}^{-1}(t) - \mathbf{P}^{-1}(t)\mathbf{G}(t)\mathbf{P}^{-1}(t) + \mathbf{C}^{\mathsf{T}}(t)\mathbf{C}(t), \tag{2.71}$$

which is again a differential Riccati equation. This inverse is again uniformly bounded from below and from above, i.e.,

$$\eta_1\mathbf{I} \preceq \mathbf{P}^{-1}(t) \preceq \eta_2\mathbf{I} \tag{2.72}$$

with $\eta_1 = p_2^{-1}$ and $\eta_2 = p_1^{-1}$ for all $t \geq t_0 \geq 0$. This allows to show uniform exponential stability of the resulting observer error dynamics as summarized in the following.

Theorem 2.18 (observer for linear time varying systems)
Assume that system (2.33) *is uniformly completely observable. Let an observer for* (2.33) *be given by* (2.65) *with* $\mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^{\mathsf{T}}(t)$ *and* $\mathbf{P}(t)$ *as the solution of the differential Riccati equation* (2.67) *with* $p_{01}\mathbf{I} \preceq \mathbf{P}_0 \preceq p_{02}\mathbf{I}$ *and* $g_1\mathbf{I} \preceq \mathbf{G}(t) \preceq g_2\mathbf{I}$ *for positive constants* p_{01}, p_{02}, q_1, q_2. *Then, the estimation error dynamics*

$$\dot{\mathbf{e}}(t) = \left[\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)\right]\mathbf{e}(t) \tag{2.73}$$

is uniformly exponentially stable.

Proof. Let $V(t, \mathbf{e}) = \mathbf{e}^{\mathsf{T}}\mathbf{P}^{-1}(t)\mathbf{e}$ be a Lyapunov function candidate. Due to the upper and lower boundedness of $\mathbf{P}^{-1}(t)$ as stated in (2.72), one has

$$\eta_1\|\mathbf{e}\|^2 \leq V(t, \mathbf{e}) \leq \eta_2\|\mathbf{e}\|^2. \tag{2.74}$$

The time derivative of V along the trajectories yields (the time dependency is omitted for the sake of readability)

$$\begin{aligned}
\dot{V} &= \dot{\mathbf{e}}^{\mathsf{T}}\mathbf{P}^{-1}\mathbf{e} + \mathbf{e}^{\mathsf{T}}\dot{\mathbf{P}}^{-1}\mathbf{e} + \mathbf{e}^{\mathsf{T}}\mathbf{P}^{-1}\dot{\mathbf{e}} \\
&= \mathbf{e}^{\mathsf{T}}\left[\mathbf{A}^{\mathsf{T}}\mathbf{P}^{-1} - \mathbf{C}^{\mathsf{T}}\mathbf{C} - \mathbf{P}^{-1}\mathbf{A} - \mathbf{A}^{\mathsf{T}}\mathbf{P}^{-1} - \mathbf{P}^{-1}\mathbf{G}\mathbf{P}^{-1} + \mathbf{C}^{\mathsf{T}}\mathbf{C} + \mathbf{P}^{-1}\mathbf{A} - \mathbf{C}^{\mathsf{T}}\mathbf{C}\right]\mathbf{e} \\
&= -\mathbf{e}^{\mathsf{T}}\left[\mathbf{P}^{-1}\mathbf{G}\mathbf{P}^{-1} + \mathbf{C}^{\mathsf{T}}\mathbf{C}\right]\mathbf{e} \\
&\leq -\mathbf{e}^{\mathsf{T}}\left[\mathbf{P}^{-1}\mathbf{G}\mathbf{P}^{-1}\right]\mathbf{e} \\
&\leq -g_1\eta_1 V
\end{aligned} \tag{2.75}$$

The derivative of V is negative definite. Hence, the estimation error dynamics is uniformly asymptotically or equivalently uniformly exponentially stable [HP06, Theorem 3.2.7]. □

Remark 2.19*:* The Riccati equation (2.71) has a strong similarity to the differential form of the constructibility Gramian (2.47). The additional quadratic term $\mathbf{P}^{-1}(t)\mathbf{Q}(t)\mathbf{P}^{-1}(t)$ acts as a stabilizing term, which guarantees the boundedness of $\mathbf{P}^{-1}(t)$, whereas the constructibility Gramian might grow without bound. A complete derivation of the Kalman-Bucy filter as deterministic observer and the relation to the constructibility Gramian was recently presented in [Esc18].

2.5 Detectability

Uniform complete observability is a strong assumption and may not always be feasible. A natural question is to ask for a minimum requirement which allows to at least asymptotically reconstruct the system states. This concept is called detectability and is very well understood for linear time invariant systems. Hence, it is discussed for this class first. A natural definition of detectability is stated in the following.

Definition 2.20 (detectability for linear time invariant systems) The linear time invariant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \tag{2.76a}$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{2.76b}$$

is detectable if $\mathbf{y}(t) = \mathbf{0}$ for all $t \geq 0$ implies that

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}. \tag{2.77}$$

Together with the definition of the unobservable subspace in Definition 2.11, this basically requires that every trajectory, whose initial condition lies in the unobservable subspace should decay to zero asymptotically.

Several well known detectability criteria for linear time invariant systems exist in literature. The following lemma is known as the Popov-Belevitch-Hautus test for detectability.

Lemma 2.21 (Popov-Belevitch-Hautus test for detectability, [Son13])
The time invariant system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$*,* $\mathbf{y} = \mathbf{C}\mathbf{x}$ *is detectable if and only if for every eigenvalue* μ_i *of* $\mathbf{A}$ *with* $\mathrm{Re}\{\mu_i\} \geq 0$ *it holds that*

$$\operatorname{rank}\begin{bmatrix}\mu_i\mathbf{I} - \mathbf{A}\\ \mathbf{C}\end{bmatrix} = n. \tag{2.78}$$

Similar to the eigenvector test for observability, an equivalent formulation of this Lemma is that there exists no right eigenvector $\mathbf{p}_i$ of $\mathbf{A}$ corresponding to an eigenvalue μ_i with $\mathrm{Re}\{\mu_i\} \geq 0$ such that

$$\mathbf{C}\mathbf{p}_i = \mathbf{0}. \tag{2.79}$$

An alternative interpretation can be derived via the Kalman decomposition of (2.76a), see [Che98, Theorem 6.O6]. If the observability matrix does not have full rank, i.e.,

$$\operatorname{rank} \mathcal{O} = \operatorname{rank} \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} = n_o < n, \tag{2.80}$$

one can construct a nonsingular matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{bmatrix} \tag{2.81}$$

such that the $n_o \times n$ matrix $\mathbf{P}_1$ is formed by n_o linearly independent rows of $\mathcal{O}$. The $n - n_o \times n$ matrix $\mathbf{P}_2$ is chosen such that $\mathbf{P}$ is nonsingular. Then, the state transformation $\mathbf{z} = \mathbf{Px}$ yields

$$\begin{bmatrix} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}, \tag{2.82a}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \end{bmatrix}, \tag{2.82b}$$

where $\mathbf{A}_{11}$ is an $n_o \times n_o$ matrix. The pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ is observable and it can be seen that $\mathbf{z}_2$ does not contribute to the output either directly or via $\mathbf{z}_1$. Hence, it cannot be reconstructed from the measurements in any way and the system is detectable if and only if

$$\dot{\mathbf{z}}_2 = \mathbf{A}_{22}\mathbf{z}_2 \tag{2.83}$$

is asymptotically stable, i.e., if $\mathbf{A}_{22}$ is a Hurwitz matrix. In the time invariant case, detectability is necessary and sufficient for the existence of an observer. This can be seen by designing an observer for a detectable system. Without loss of generality, it can be assumed that the system is already in the form of (2.82). The observer can the be stated as

$$\dot{\hat{\mathbf{z}}}_1 = \mathbf{A}_{11}\hat{\mathbf{z}}_1 + \mathbf{L}(\mathbf{y} - \mathbf{C}_1\mathbf{z}_1) \tag{2.84a}$$

$$\dot{\hat{\mathbf{z}}}_2 = \mathbf{A}_{21}\hat{\mathbf{z}}_1 + \mathbf{A}_{22}\hat{\mathbf{z}}_2, \tag{2.84b}$$

with $\mathbf{z}_2(t) \in \mathbb{R}^{n_o}$, i.e., n_o is the dimension of the unobservable subsystem. The feedback gain $\mathbf{L}$ is chosen such that $(\mathbf{A}_{11} - \mathbf{LC})$ is a Hurwitz matrix. Equation (2.84b) is a trivial observer for $\mathbf{z}_2$. By the detectability assumption, $\mathbf{A}_{22}$ is also a Hurwitz matrix and the dynamics of the estimation error $\mathbf{e} = \mathbf{z} - \hat{\mathbf{z}}$ is given by

$$\begin{bmatrix} \dot{\mathbf{e}}_1 \\ \dot{\mathbf{e}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} - \mathbf{L}\mathbf{C}_1 & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix}. \tag{2.85}$$

This error system is asymptotically stable if and only if system (2.82) is detectable.

For time varying systems, the presented relations and underlying concepts are less studied and detectability is a truly more delicate topic. Although Kalman introduced the aforementioned decomposition originally for time varying systems [Kal63, Theorem 5] point-wise in time, it is hard to construct the time varying subspaces for general time varying systems. Moreover, the dimension of these subspaces might change over time [Kal63]. It is shown in [Tai87], that for the class of so-called constant rank systems, detectability can be characterized in a similar manner as in the time invariant case. Constant rank systems include periodic systems and also systems with real and analytic coefficient matrices, see also [Sil71] for the Kalman decomposition of this system class.

A detectability notion introduced by [AM81] (originally for discrete time systems) states that the pair $(\mathbf{A}(t), \mathbf{C}(t)$ is “uniformly detectable” if there exist constants $s \geq \tau \geq 0$ and constants d, b with $0 \leq d < 1$ and $0 < b < \infty$ such that whenever

$$\|\mathbf{\Phi}(t_0 + \tau, t_0)\boldsymbol{\xi}\| \geq d\|\boldsymbol{\xi}\| \tag{2.86}$$

holds for some $t_0 \in \mathbb{J}$ and some vector $\boldsymbol{\xi}$, then

$$\boldsymbol{\xi}^\mathsf{T}\mathbf{M}(t+s, t)\boldsymbol{\xi} \geq b\boldsymbol{\xi}^\mathsf{T}\boldsymbol{\xi}. \tag{2.87}$$

The matrix $\mathbf{M}$ is the observability Gramian of the pair $(\mathbf{A}(t), \mathbf{C}(t))$. A somehow more natural definition is related to the existence of an observer. This definition is motivated by the fact that in the time invariant case, detectability is necessary and sufficient for the existence of an observer.

In the time varying case, different notions of exponential stability exist. This motivates the introduction of different detectability definitions for time varying systems. The definitions are related to the underlying stability property of the estimation error dynamics. The concepts and corresponding detectability conditions are extensively discussed in the following in Chapters 3 and 4.

Part I

New Perspectives on Detectability and Observer Design

Chapter 3

Detectability and Observer Design for Linear Time Varying Systems

This chapter proposes detectability and state estimation concepts for a class of (large-scale) linear time varying systems. In this context, large-scale refers to a large system order of the underlying linear time varying model of ordinary differential equations (ODEs) with hundreds or even thousands of system states. Such models typically occur as a finite-dimensional approximation of partial differential equations via ODEs and have a variety of applications in control and estimation of distributed parameter systems, computational physics or meteorology, see, e.g., [ABN16].

A very important question in this context is the existence of an observer, i.e., a dynamical system, which allows to reconstruct the system states. The existence of such an observer is closely related to detectability of the underlying system. For linear time invariant systems, detectability is well studied and efficient numerical methods to check this property are available, see e.g. [Ros74]. In the linear time varying setting, the problem of detectability (or the dual concept of stabilizability) is subject of ongoing research, see [FZ18; AIW13; Boc+17], for example, for some recent contributions. The increasing complexity of today's systems and their corresponding models motivates the research on detectability concepts for linear time varying systems. This is of interest because real systems change their behavior over time and linear time varying systems may be obtained by linearizing a nonlinear system along a specific trajectory. Hence, detectability of linear time varying systems is related to the existence of local observers for nonlinear systems, as it is shown in [FZ18].

Different detectability notions are associated with different stability concepts of the resulting observer error dynamics. If the system is uniformly completely observable, there exists a uniformly bounded output feedback gain such that the observer error system is uniformly exponentially stable, see Section 2.4. The gain can be obtained by computing the unique positive definite solution of a differential Riccati equation. For systems with large system order, solving this matrix differential equation might become computationally intractable or the matrices are too large to be even stored in memory [ABN16, Chapter 5]. This also motivates the research on detectability concepts and computationally efficient observer design procedures.

In this chapter, the linear time varying system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),\ \mathbf{x}(t) \in \mathbb{R}^n \tag{3.1a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t),\ \mathbf{y}(t) \in \mathbb{R}^p \tag{3.1b}$$

is considered for $t \in \mathbb{J} = [0, \infty)$ and the matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are assumed to be continuous and uniformly bounded according to

$$\sup_{\substack{t\in\mathbb{J}\\ \|\mathbf{x}\|=1}} \|\mathbf{A}(t)\mathbf{x}\| \leq \bar{a} < \infty, \quad \sup_{\substack{t\in\mathbb{J}\\ \|\mathbf{x}\|=1}} \|\mathbf{C}(t)\mathbf{x}\| \leq \bar{c} < \infty. \tag{3.2}$$

The goal is to asymptotically reconstruct the system states and provide conditions, which guarantee the existence of an observer. Based on these detectability conditions, an output feedback gain $\mathbf{L}(t)$ is designed such that the resulting observer error dynamics

$$\dot{\mathbf{e}} = \left[\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)\right]\mathbf{e}(t) \tag{3.3}$$

are exponentially stable. It should be remarked that observability and detectability of linear systems are system properties and do not depend on known inputs. Hence, without loss of generality, it can be assumed that all inputs are zero, see Section 2.4.

3.1 Related Work and Contribution

A uniform detectability definition for discrete-time linear time varying systems is presented in [AM81, Definition 2.1]. The notion is based on the observability Gramian of the system and states that if a state trajectory is not "fast decaying", it must be observable. This is a generalization of conditions for linear time invariant systems, which basically state that the unstable modes should be observable [Hes09]. The theory presented in this chapter also generalizes this idea, but based on a different approach, which allows a computationally efficient observer design.

In [Tai87], different detectability notions including the one presented in [AM81] are investigated for the class of constant rank systems [SM69]. This class is a special case of linear time varying systems and include systems with time invariant and, more importantly, also analytic coefficient matrices. For constant rank systems, the existence of a Kalman decomposition with a fixed block structure is guaranteed [Tai87], which allows straightforward generalizations of time invariant detectability concepts. It is shown in [Tai87], that different generalizations of detectability (or the dual stabilizability) notions coincide for this system class. Constant rank systems have strong smoothness requirements regarding the system and output matrix, however.

In [AIW13], the concept of non-uniform stabilizability by (a possibly unbounded) state feedback gain is investigated. The authors argue that non-uniform notions of stabilizability and the relations to the solution of the corresponding Riccati equation are less well studied than their corresponding uniform notions. The work is based on an optimal control formulation of the stabilization problem and presents two finite cost conditions for uniform and non-uniform stabilizability. Moreover, their work gives a good overview of the relations between different stability notions for linear time varying systems.

The problem of state estimation in the context of large scale dynamical systems is also known as data assimilation in geosciences and has a variety of practical applications in meteorology or oceanography, see [TP11]. The book [ABN16] gives a comprehensive overview of data assimilation concepts and applications in different fields. State estimation for large scale dynamical systems is the motivation for recently proposed results on the design of numerically efficient observer and filtering algorithms in the discrete [Boc+17; TP11] and continuous-time [FZ18] setting.

In [FZ18], an observer is proposed, which does not rely on the solution of a differential Riccati equation. The feedback gain is designed such that all Lyapunov exponents of the resulting error dynamics are negative, which guarantees (non-uniform) exponential stability. A necessary existence condition for the resulting observer is, however, that the rank of the output matrix is at least equal to the number of non-negative Lyapunov exponents. These characteristic numbers determine the exponential stability properties of a system and will be extensively discussed in the following. The observer presented in [FZ18] might not exist, even if the system is uniformly completely observable [KB61], if the number of outputs is too low. This is shown by a counter example. Hence, the directional detectability notion recently introduced in [FZ18] does not reduce to the well-established detectability notions for linear time invariant systems.

The contribution of this chapter are a detectability condition and an observer design technique, which do not impose the above requirement but still is computationally tractable. The resulting observer can be constructed by solving a differential Riccati equation on a *reduced state-space* related to the non-negative Lyapunov exponents. This reduces computational complexity and guarantees exponential convergence of the estimation error, which is also demonstrated by means of numerical simulation examples. Moreover, the robustness of the proposed observer scheme with respect to bounded measurement noise and model uncertainties is investigated. If the system is not only detectable in the sense that all unstable modes are observable, but has stronger observability properties, a trade-off between the convergence speed of the estimation error and the computational complexity can be achieved.

It is shown that the presented detectability notion and the sufficient condition presented in this chapter reduce to the classical notions and a necessary and sufficient condition, respectively, for the time invariant case.

A part of the content presented in this chapter is adopted from

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The chapter is structured as follows: Section 3.2 summarizes important concepts of Lyapunov's stability theory for linear time varying systems and the numerical approximation of Lyapunov exponents. The main result, which proposes a detectability condition and an observer design procedure, is presented in Section 3.3. Section 3.4 analyzes the robustness of the proposed approach with respect to bounded measurement noise and model uncertainties. A reduced order observer design procedure is presented in Section 3.5 and in Section 3.6 it is shown that the detectability condition is necessary and sufficient in the linear time invariant case. The effectiveness of the proposed approach is demonstrated by means of numerical simulation examples in Section 3.7, and Section 3.8 discusses the obtained results.

3.2 Stability Assessment via Lyapunov Exponents

This section summarizes important concepts of Lyapunov's stability theory and the numerical approximation of Lyapunov exponents for linear time varying systems. In the following, stability properties of the linear time varying autonomous system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathbb{R}^n, \tag{3.4}$$

are considered for $t \in \mathbb{J} = [0, \infty)$.

In many contributions, uniform exponential stability is the desired stability property because it is preserved under sufficiently small nonlinear perturbations [Hah67, Ch. 65]. Compared to non-uniform notions, uniform exponential stability is trickier to verify numerically, however, because it involves the computation of the Bohl exponents [HP06, Def. 3.3.10] or the exponential dichotomy spectrum. Uniform exponential stability of (3.4) will be covered extensively in Chapter 4.

Non-uniform exponential stability is related to the aforementioned Lyapunov exponents and since a few years, efficient numerical methods for the computation of the Lyapunov exponents exist. Moreover, assuming additional regularity conditions of the underlying system, exponential stability is also preserved under "sufficiently small" nonlinear perturbations. This is a motivation to work with exponential stability subsequently.

3.2.1 Lyapunov Exponents and Lyapunov Basis

Characteristic exponents of dynamical systems were introduced by Lyapunov, see [Lya92] for a translation of his monograph *The General Problem of the Stability of Motion*. The Lyapunov exponents provide a measure for the rate of exponential growth or decay of solutions of (3.4) as discussed subsequently. These characteristic exponents are also used as measures for chaos and complexity in (possibly large-scale) nonlinear dynamical systems and are employed in various different fields like, e.g., meteorology and climatology [MMA14; Van17; CLD06], or chaos theory [Dar+12; SM16]. Details and additional background material can be found in [BP02b; DRV97; FZ18] and the references therein.

Exponential stability of (3.4) can be studied using the function

$$\chi^s(\mathbf{x}_0) = \limsup_{t\to\infty} \frac{1}{t} \ln \|\mathbf{\Phi}(t,0)\mathbf{x}_0\|, \tag{3.5}$$

which measures the asymptotic rate of exponential growth or decay of the solution of (3.1a) with initial condition $\mathbf{x}_0$. Taking into account every nonzero $\mathbf{x}_0 \in \mathbb{R}^n$, this function attains at most n distinct values $\lambda_1, \dots, \lambda_s$, $s \leq n$, the so-called (upper) Lyapunov exponents. Each Lyapunov exponent λ_i has a corresponding multiplicity d_i equal to the dimension of the span of all initial conditions satisfying $\chi^s(\mathbf{x}_0) = \lambda_i$ and it holds that $d_1 + \cdots + d_s = n$, see [Wil16]. Without loss of generality it is assumed that the Lyapunov exponents are ordered such that $\lambda_1 > \lambda_2 > \cdots > \lambda_s$.

According to Lyapunov's stability theory, system (3.1a) is exponentially stable if and only if all Lyapunov exponents are negative or equivalently $\lambda_1 < 0$ [HP06, p. 258]; in particular (2.4) holds for any $\mu < -\lambda_1$.

Let the sets $\mathcal{V}_j$ be defined as

$$\mathcal{V}_j = \{\mathbf{x}_0 \in \mathbb{R}^n : \chi^s(\mathbf{x}_0) \leq \lambda_j\}, \ j = 1, \ \dots, \ s. \tag{3.6}$$

It is shown in [BP02b, p. 10] that these $\mathcal{V}_j$ are subspaces of $\mathbb{R}^n$ such that

$$\{0\} =: \mathcal{V}_{s+1} \subsetneq \mathcal{V}_s \subsetneq \dots \subsetneq \mathcal{V}_2 \subsetneq \mathcal{V}_1 = \mathbb{R}^n \tag{3.7}$$

with $n_j = \dim \mathcal{V}_j = \dim \mathcal{V}_{j+1} + d_j$. One obtains $\chi^s(\mathbf{x}_0) = \lambda_j$ if and only if $\mathbf{x}_0 \in \mathcal{V}_j \backslash \mathcal{V}_{j+1}$. A basis $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n\end{bmatrix}$ of $\mathbb{R}^n$ is called an ordered normal Lyapunov basis if

$$\mathcal{V}_i = \operatorname{span}(\mathbf{v}_{n-n_i+1}, \dots, \mathbf{v}_n). \tag{3.8}$$

Note that this choice is not unique. For any ordered Lyapunov basis it holds that

$$\chi^s(\mathbf{v}_j) = \lambda_i \text{ for } n - n_i < j \leq n - n_i + d_i. \tag{3.9}$$

In the case of distinct Lyapunov exponents, every Lyapunov vector $\mathbf{v}_j$ corresponds to a specific Lyapunov exponent such that $\chi^s(\mathbf{v}_j) = \lambda_j$, $j = 1, \ldots, n$.

The following example constructs an ordered normal Lyapunov basis for linear time invariant systems and shows that the Lyapunov exponents correspond to the real part of the eigenvalues of $\mathbf{A}$.

Example 3.1 (Time Invariant System)*:* Let $\mathbf{A}(t) = \mathbf{A}$ be constant and assume that $\mathbf{A}$ is diagonalizable. In this case, the ordered (right) eigenvectors of $\mathbf{A}$ form an ordered normal Lyapunov basis. To see this, let the n eigenvalues be denoted as μ_i, $i = 1, \ldots, n$ and assume that they are ordered according to their real part, i.e., $\mathrm{Re}\{\mu_1\} \geq \mathrm{Re}\{\mu_2\} \geq \ldots \geq \mathrm{Re}\{\mu_n\}$. Let p_i be an eigenvector corresponding to the eigenvalue $\mu_i = \alpha_i + j\beta_i$ such that

$$\mathbf{A}\mathbf{p}_i = \mu_i \mathbf{p}_i \tag{3.10}$$

and moreover

$$\mathbf{\Phi}(t, 0)\mathbf{p}_i = e^{\mu_i t}\mathbf{p}_i. \tag{3.11}$$

Then, by evaluating (3.5) and choosing the (real) initial condition $\mathbf{x}_0 = \mathbf{p}_i + \mathbf{p}_i^*$ it follows that

$$\begin{aligned}\chi^s(\mathbf{p}_i) &= \limsup_{t\to\infty} \frac{1}{t} \ln \|\mathbf{\Phi}(t,0)\mathbf{x}_0\| = \lim_{t\to\infty} \frac{1}{t} \ln \left\| e^{\alpha_i t}\left(e^{j\beta_i t}\mathbf{p}_i + e^{-j\beta_i t}\mathbf{p}_i^*\right)\right\| \\ &= \lim_{t\to\infty} \frac{1}{t}\left[\ln e^{\alpha_i t} + \ln \left\| e^{j\beta_i t}\mathbf{p}_i + e^{-j\beta_i t}\mathbf{p}_i^*\right\|\right] \\ &= \lim_{t\to\infty} \frac{1}{t}\alpha_i t = \alpha_i = \mathrm{Re}\{\mu_i\}.\end{aligned} \tag{3.12}$$

The real parts of the eigenvalues correspond to the Lyapunov exponents of the underlying system. If one assumes (to avoid confusion regarding the multiplicity of eigenvalues and Lyapunov exponents) that all eigenvalues are real-valued and distinct, it follows by (3.8) and (3.9) that

$$\mathcal{V}_i = \mathrm{span}(\mathbf{p}_i, \ldots, \mathbf{p}_n). \tag{3.13}$$

Hence, an ordered normal Lyapunov basis can be constructed by the eigenvectors of $\mathbf{A}$ according to

$$V = \begin{bmatrix}\mathbf{p}_1 & \mathbf{p}_2 & \ldots & \mathbf{p}_n\end{bmatrix}. \tag{3.14}$$

3.2.2 Forward Regularity and Numerical Approximation of the Lyapunov Exponents

Lyapunov exponents are especially important to determine the stability properties of the perturbed equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t, \mathbf{x}) \tag{3.15}$$

with $\mathbf{f}$ as a "sufficiently small" nonlinear perturbation. This means that it is assumed that $\mathbf{f}(t, \mathbf{0}) = \mathbf{0}$ and for every $\mathbf{x}, \mathbf{v} \in \mathbb{R}^n$ it holds that[1]

$$\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{v})\| \leq K\|\mathbf{x} - \mathbf{v}\|(\|\mathbf{x}\| + \|\mathbf{v}\|)^{q-1} \tag{3.16}$$

for some $K > 0$ and $q > 1$. In [BP02b], q is denoted as the order of the perturbation. The concept of *forward regularity* introduced by Lyapunov is crucial for the stability assessment of (3.16). It guarantees the existence of the Lyapunov exponents as points and is discussed subsequently. The *Lyapunov Stability Theorem*, see, e.g. [BP02b, Theorem 1.1.2], states that if all Lyapunov exponents of (3.4) are forward regular and negative, then the zero solution of the perturbed system (3.15) is exponentially stable.

The Lyapunov exponents of (3.4) can be obtained by utilizing the corresponding fundamental matrix differential equation $\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t)$, where the columns of the initial condition $\mathbf{X}(0) = [\mathbf{x}_{1,0} \; \dots \; \mathbf{x}_{n,0}]$ form an ordered normal Lyapunov basis. However, this matrix differential equation is numerically ill conditioned [Ben+80]. Thus, it is common to work with a system in upper triangular form, which can be achieved with the aid of a Lyapunov transformation of system (3.4). Note that Lyapunov transformations preserve the Lyapunov exponents. Based on Perron's lemma [Adr95, Lemma 3.3.1], there exists an orthogonal Lyapunov transformation $\mathbf{R}(t) = \mathbf{Q}^\mathsf{T}(t)\mathbf{X}(t)$ such that $\mathbf{R}(t)$ is upper triangular with a positive diagonal.

This transformation can be obtained by means of the continuous-time QR decomposition. The matrices $\mathbf{Q}$ and $\mathbf{R}$ are the solutions of the differential equations

$$\dot{\mathbf{R}}(t) = \mathbf{B}(t)\mathbf{R}(t), \qquad \mathbf{R}(0) = \mathbf{R}_0, \qquad \mathbf{B} = \mathbf{Q}^\mathsf{T}\mathbf{A}\mathbf{Q} - \mathbf{S}, \tag{3.17}$$

$$\dot{\mathbf{Q}}(t) = \mathbf{Q}(t)\mathbf{S}(t), \qquad \mathbf{Q}(0) = \mathbf{Q}_0, \qquad \mathbf{S} = -\mathbf{S}^\mathsf{T}, \tag{3.18}$$

with skew-symmetric matrix $\mathbf{S}$ according to $s_{ij} = \mathbf{q}_i^\mathsf{T}\mathbf{A}\mathbf{q}_j$, $i > j$, $\mathbf{Q} = [\mathbf{q}_1, \dots, \mathbf{q}_n]$ and a bounded upper triangular matrix $\mathbf{B}$. The initial condition is $\mathbf{X}_0 = \mathbf{Q}_0\mathbf{R}_0$. Note that $\mathbf{Q}$ and $\mathbf{R}$ are uniquely defined if the diagonal of $\mathbf{R}$ is positive, see [DRV97]. Next, forward regularity is characterized in terms of the system in upper triangular form.

[1]Note that the inequality presented in the [BP02b, Theorem 1.1.2] is incorrect. This was corrected in an erratum, see [BP02a].

Definition 3.2 (forward regularity, [FZ18; BP02b])
Let $\mathbf{R}(t) = \begin{bmatrix}\mathbf{r}_1(t) & \cdots & \mathbf{r}_n(t)\end{bmatrix}$ be the unique solution of (3.17) with $\mathbf{X}_0 = \mathbf{Q}_0\mathbf{R}_0$ as ordered normal Lyapunov basis. Then, the Lyapunov exponent $\chi^s(\mathbf{Q}_0\mathbf{r}_{0,i}) = \lambda_i$ is called forward regular, if

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t b_{ii}(s)\,\mathrm{d}s = \liminf_{t\to\infty} \frac{1}{t}\int_0^t b_{ii}(s)\,\mathrm{d}s \tag{3.19}$$

holds with b_{ii} as the corresponding main diagonal element of $\mathbf{B}$ and $\mathbf{r}_{0,i} = \mathbf{r}_i(0)$. System (3.4) is called forward regular if all Lyapunov exponents are forward regular.

It is pointed out in [FZ18; BP02b] that even though this regularity condition seems quite strong, it typically holds when obtaining (3.1a) via the linearization of a nonlinear system along a trajectory. Moreover, systems with periodic or constant coefficient matrices and systems, which are kinematically similar to such systems are forward regular [Adr95, Ch. 3, §5]. It is shown in [BP02b, Theorem 1.3.1] that for a forward regular system with any ordered normal Lyapunov basis $V = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n]$, the Lyapunov exponents can be obtained via

$$\begin{aligned}\chi^s(\mathbf{v}_j) = \lambda_i &= \lim_{t\to\infty}\frac{1}{t}\int_0^t b_{jj}(s)\,\mathrm{d}s\\ &= \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{q}_j^T(s)\mathbf{A}(s)\mathbf{q}_j(s)\,\mathrm{d}s,\end{aligned} \tag{3.20}$$

with $\mathbf{q}_j$ as the j-th column of $\mathbf{Q}$ as in (3.18) and i, j as in (3.8) and (3.9). This important fact allows to approximate the Lyapunov exponents of (3.4) without explicitly computing the possibly unbounded solution of $\mathbf{X}$ or $\mathbf{R}$. Instead, it is possible to solve the differential equation for the orthogonal matrix $\mathbf{Q}$. To approximate the corresponding Lyapunov exponents, the average in (3.20) can be computed for a finite time horizon.

The transformation to an upper triangular system might be time-varying, even if the underlying system is time invariant. This is demonstrated in the next example.

Example 3.3*:* Consider the (linear time invariant) harmonic oscillator

$$\dot{\mathbf{x}} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\mathbf{x} \tag{3.21}$$

with the principal fundamental matrix solution

$$\mathbf{X}(t) = \begin{bmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{bmatrix}. \tag{3.22}$$

This system possesses two eigenvalues $\mu_{12} = \pm j$ and hence one single Lyapunov exponent $\lambda_1 = 0$ with multiplicity $d_1 = 2$. The initial condition $\mathbf{X}(0) = \mathbf{I}$ is already in upper triangular form and one obtains $\mathbf{Q}_0 = \mathbf{I}$ and $\mathbf{R}_0 = \mathbf{I}$. One can verify that

$$\mathbf{R}(t) = \mathbf{I} \text{ and } \mathbf{Q}(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \tag{3.23}$$

solve (3.17) and (3.18) with

$$\mathbf{B}(t) = \mathbf{0} \text{ and } \mathbf{S}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.24}$$

respectively. The Lyapunov exponents can be obtained via the diagonal elements of $\mathbf{B}$, because time invariant systems are always forward regular. This confirms the single Lyapunov exponent $\lambda_1 = 0$ with multiplicity $d_1 = 2$.

If one is only interested in the largest exponents, it is reasonable to only approximate the $k \leq n$ largest Lyapunov exponents. For this it suffices to solve (3.17) on a reduced state space by computing a reduced QR-decomposition, which only takes into account the first k columns of $\mathbf{X}(t)$. Therefore, let the full QR decomposition (3.17), (3.18) be partitioned appropriately as

$$\begin{bmatrix} \dot{\mathbf{R}}_{11} & \dot{\mathbf{R}}_{12} \\ \mathbf{0} & \dot{\mathbf{R}}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{0} & \mathbf{R}_{22} \end{bmatrix}, \tag{3.25}$$

$$\begin{bmatrix} \dot{\bar{\mathbf{Q}}} & \dot{\bar{\mathbf{Q}}}_\perp \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{Q}} & \bar{\mathbf{Q}}_\perp \end{bmatrix} \begin{bmatrix} \mathbf{S}_{11} & -\mathbf{S}_{21}^\mathsf{T} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix}, \tag{3.26}$$

with $\bar{\mathbf{Q}}(t) \in \mathbb{R}^{n\times k}$. A reduced QR decomposition is obtained by only considering the first k columns of $\mathbf{Q}$ and $\mathbf{R}$ according to

$$\dot{\mathbf{R}}_{11}(t) = \mathbf{B}_{11}(t)\mathbf{R}_{11}(t), \quad \mathbf{R}_{11}(0) = \mathbf{R}_{11,0} \in \mathbb{R}^{k\times k}, \tag{3.27}$$

$$\dot{\bar{\mathbf{Q}}}(t) = \bar{\mathbf{Q}}(t)\mathbf{S}_{11}(t) + \bar{\mathbf{Q}}_\perp(t)\mathbf{S}_{21}(t) = \bar{\mathbf{Q}}(t)\mathbf{S}_{11}(t) + \bar{\mathbf{Q}}_\perp(t)\bar{\mathbf{Q}}_\perp^\mathsf{T}(t)\mathbf{A}(t)\bar{\mathbf{Q}}(t), \tag{3.28}$$

with

$$\mathbf{B}_{11}(t) = \bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{A}(t)\bar{\mathbf{Q}}(t) - \mathbf{S}_{11}(t), \tag{3.29}$$

and the entries s_{ij} of $\mathbf{S}_{11}(t) \in \mathbb{R}^{k\times k}$ given by

$$s_{ij}(t) = \begin{cases} -\bar{\mathbf{q}}_i^\mathsf{T}(t)\mathbf{A}(t)\bar{\mathbf{q}}_j(t) & i < j, \\ 0 & i = j, \\ \bar{\mathbf{q}}_i^\mathsf{T}(t)\mathbf{A}(t)\bar{\mathbf{q}}_j(t) & i > j. \end{cases} \tag{3.30}$$

By orthogonality of $\mathbf{Q}$ it follows that $\bar{\mathbf{Q}}_\perp\bar{\mathbf{Q}}_\perp^\mathsf{T} = (\mathbf{I} - \bar{\mathbf{Q}}\bar{\mathbf{Q}}^\mathsf{T})$ and hence

$$\dot{\bar{\mathbf{Q}}}(t) = (\mathbf{I} - \bar{\mathbf{Q}}(t)\bar{\mathbf{Q}}^\mathsf{T}(t))\mathbf{A}(t)\bar{\mathbf{Q}}(t) + \bar{\mathbf{Q}}(t)\mathbf{S}_{11}(t),\ \bar{\mathbf{Q}}(0) = \bar{\mathbf{Q}}_0 \in \mathbb{R}^{n\times k}. \tag{3.31}$$

If only the first k columns of $\mathbf{Q}$ are of interest, one can solve equation (3.27) instead of (3.17) with $\bar{\mathbf{Q}}(t) \in \mathbb{R}^{n\times k}$. This reduces computational complexity if k is small compared to n.

Remark 3.4*:* For numerically solving the differential equation (3.31) to obtain $\bar{\mathbf{Q}}$, a so-called projected integrator [DRV97] should be employed. This is a standard numerical integration scheme like, e.g., the Runge-Kutta algorithm [CK06], combined with the modified Gram-Schmidt algorithm, which is crucial in order to keep $\bar{\mathbf{Q}}$ orthogonal [DRV97]. The utilized modified Gram-Schmidt algorithm is given in Appendix B.1. Since an ordered normal Lyapunov basis is typically not known a priori, $\bar{\mathbf{Q}}_0$ is chosen as a random orthogonal matrix as suggested in [FZ18]. The projected integration causes the span of $\bar{\mathbf{Q}}$ to converge to a span of Lyapunov vectors corresponding to the k leading exponents for any $\bar{\mathbf{Q}}_0$ that has a non-trivial projection onto the first k most dominant directions. Since this is the case generically, i.e. with probability one, $\bar{\mathbf{Q}}_0$ is typically chosen randomly. This choice is theoretically justified by [Ben+80]. For details on the numerical implementation of the continuous QR algorithm, see [DRV97; DV03].

3.3 Detectability and Observer Design

For linear time varying systems, different notions of detectability (or the dual concept of stabilizability) exist in literature, see, e.g. [AIW13; RPK92; Tai87; AM81] and Section 2.5. The link between most available detectability concepts is the question of existence of an observer, i.e., of an output feedback gain $\mathbf{L}(t)$ such that (3.3) is stable with respect to some stability notion. Typically, a uniformly bounded gain $\mathbf{L}$ and uniform exponential stability is required. The condition presented in this chapter is based on the theory of Lyapunov exponents and hence the following definition is introduced.

Definition 3.5 (exponential detectability) System (3.1) is called exponentially detectable if there exists a uniformly bounded output feedback gain $\mathbf{L}(t)$ such that the system

$$\dot{\mathbf{e}} = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{e}, \quad \mathbf{e}(t_0) = \mathbf{e}_0 \in \mathbb{R}^n \tag{3.32}$$

is exponentially stable.

By this definition, exponential detectability is equivalent to the existence of an observer

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{L}(t)\,[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)] \tag{3.33}$$

such that the resulting estimation error dynamics (3.32) with $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is exponentially stable.

The directional detectability definition recently introduced in [FZ18] is sufficient for exponential detectability, because the proof of [FZ18, Proposition 3.7] results in the construction of a feedback gain which satisfies Definition 3.5. The advantage of this approach is that the solution of a differential Riccati equation is not required for the computation of the observer gain. A drawback is, however, that directional detectability does not reduce to classical concepts in the time invariant case. This can be seen by the necessary condition stated in [FZ18, Remark 3.6], which requires the rank of the output matrix $\mathbf{C}(t)$ to be at least equal to the number of non-negative Lyapunov exponents. For the double integrator system

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} \\ \mathbf{y} &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \end{aligned} \tag{3.34}$$

whose Lyapunov exponents are both equal to zero, this requirement clearly cannot be fulfilled, although the system is observable.

It is discussed in Section 2.4 that the problem of state reconstruction can in general be studied with the aid of the observability or constructibility Gramians. The observability Gramian $\mathbf{M}(t, t_0)$ is related to the problem of reconstructing the initial state, whereas the constructibility Gramian $\mathbf{N}(t, t_0)$ is related to reconstruction of the current state $\mathbf{x}(t)$. Uniform complete observability (or constructibility), i.e., the existence of positive constants α_1, α_2 and τ such that

$$\alpha_1 \mathbf{I}_n \preceq \mathbf{M}(t_0 + \tau, t_0) \preceq \alpha_2 \mathbf{I}_n \tag{3.35}$$

for all $t_0 \in \mathbb{J}$ is sufficient for the existence of an observer in the form of (3.33), see Section 2.4.2.

Uniform complete observability is a strong assumption. The basic idea to derive the weaker detectability condition presented in this chapter is to consider the projection of the constructibility Gramian

$$\mathbf{N}(t_1, t_0) = \int_{t_0}^{t_1} \mathbf{\Phi}^\mathsf{T}(s, t_1) \mathbf{C}^\mathsf{T}(s) \mathbf{C}(s) \mathbf{\Phi}(s, t_1) \, \mathrm{d}s \tag{3.36}$$

onto a complement of the stable subspace. This complement is a subspace, which corresponds to the non-negative Lyapunov exponents. A basis for this subspace can be obtained by means of the continuous QR-decomposition. To see this, let the initial condition $\mathbf{X}_0 = [\mathbf{x}_{1,0} \ \mathbf{x}_{2,0} \ \dots \ \mathbf{x}_{n,0}]$ of the fundamental matrix differential equation $\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t)$ be an ordered normal Lyapunov basis such that

$$\chi^s(\mathbf{x}_{1,0}) = \lambda_1, \ \chi^s(\mathbf{x}_{2,0}), \ \dots, \ \chi^s(\mathbf{x}_{n,0}) = \lambda_n. \tag{3.37}$$

If $k^\star$ is the number of non-negative Lyapunov exponents, a complement of the stable subspace is denoted by $\mathcal{U}$ and is given by

$$\mathcal{U} = \operatorname{span} \begin{bmatrix} \mathbf{x}_{0,1} & \cdots & \mathbf{x}_{0,k^\star} \end{bmatrix}. \tag{3.38}$$

Note that unlike the stable subspace spanned by $[\mathbf{x}_{0,k^\star+1},\ \ldots,\ \mathbf{x}_{0,n}]$, this complement is not unique and depends on the chosen Lyapunov basis. Since it is possible to represent $\mathbf{X}_0 = \mathbf{Q}_0\mathbf{R}_0$ via the product of the matrix $\mathbf{Q}_0$ and the upper triangular matrix $\mathbf{R}_0$, one can use the first $k^\star$ columns of $\mathbf{Q}$ instead of $\mathbf{X}$ to obtain a basis for $\mathcal{U}$, because

$$\mathcal{U} = \operatorname{span}\begin{bmatrix}\mathbf{x}_{0,1} & \cdots & \mathbf{x}_{0,k^\star}\end{bmatrix} = \operatorname{span}\begin{bmatrix}\mathbf{q}_{0,1} & \cdots & \mathbf{q}_{0,k^\star}\end{bmatrix} = \operatorname{span}\bar{\mathbf{Q}}_0. \tag{3.39}$$

The Lyapunov exponents are preserved under the flow $\mathbf{\Phi}(\cdot,\cdot)$ and therefore

$$\chi^s(\mathbf{x}_{i,0}) = \chi^s(\mathbf{\Phi}(0,t_1)\mathbf{x}_i(t_1)), \quad i = 1,\ldots,n, \tag{3.40}$$

with $\mathbf{x}_i$ as the i-th column of $\mathbf{X}$. Hence, it is possible to obtain such a space $\mathcal{U}(t)$ for any $t \geq 0$ by only considering the first $k^\star$ columns of $\mathbf{Q}(t)$ which satisfies (3.18). The idea is now that the unstable modes should not be in the nullspace of $\mathbf{N}$, i.e., they should be regularly observed as $t \to \infty$. This leads to the detectability condition and observer design approach presented in the following theorem.

Theorem 3.6 (exponential detectability condition, [Tra+20])
Assume that the Lyapunov exponents $\lambda_1,\ldots,\lambda_n$ of (3.1a) are forward regular with $k^\star$ as the number of non-negative Lyapunov exponents. Let $\mathbf{R}_{11}(t)$ and $\bar{\mathbf{Q}}(t)$ solve (3.27) and (3.31) with some $k \geq k^\star$ and the initial condition $\bar{\mathbf{Q}}(0)\mathbf{R}_{11}(0)$ and as the first columns of some ordered normal Lyapunov basis. Then, system (3.1) is exponentially detectable if there exist positive constants α_1, α_2 and σ such that

$$\begin{aligned}\alpha_1\mathbf{I} \preceq \mathbf{N}_+(t_0+\sigma,t_0) \preceq \alpha_2\mathbf{I} \ \textit{with} \\ \mathbf{N}_+(t,t_0) \coloneqq \bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{N}(t,t_0)\bar{\mathbf{Q}}(t)\end{aligned} \tag{3.41}$$

holds for all t_0. In particular, system (3.32) is exponentially stable with the uniformly bounded feedback gain

$$\mathbf{L}(t) = \bar{\mathbf{Q}}(t)\mathbf{P}(t)\bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{C}^\mathsf{T}(t). \tag{3.42}$$

The positive definite matrix $\mathbf{P}$ is the unique solution of the differential Riccati equation

$$\dot{\mathbf{P}} = \mathbf{B}_{11}\mathbf{P} + \mathbf{P}\mathbf{B}_{11}^\mathsf{T} - \mathbf{P}\bar{\mathbf{Q}}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C}\bar{\mathbf{Q}}\mathbf{P} + \mathbf{G}, \qquad \mathbf{P}(0) \succ 0, \tag{3.43}$$

with $\mathbf{B}_{11}(t)$ as in (3.29) and any continuous positive definite matrix $\mathbf{G}(t)$ satisfying $g_1\mathbf{I} \preceq \mathbf{G}(t) \preceq g_2\mathbf{I}$ for some positive constants $0 < g_1 \leq g_2$.

Proof. The proof uses condition (3.41) to explicitly design the observer (3.33). Differentiating $\mathbf{N}_+(t,t_0)$ with respect to time gives

$$\dot{\mathbf{N}}_+ = \dot{\bar{\mathbf{Q}}}^\mathsf{T}\mathbf{N}\bar{\mathbf{Q}} + \bar{\mathbf{Q}}^\mathsf{T}\dot{\mathbf{N}}\bar{\mathbf{Q}} + \bar{\mathbf{Q}}^\mathsf{T}\mathbf{N}\dot{\bar{\mathbf{Q}}}. \tag{3.44}$$

Inserting (3.31) and the differential relation for the constructibility Gramian

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{N} = -\mathbf{A}^\mathsf{T}\mathbf{N} - \mathbf{N}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{C}, \qquad \mathbf{N}(0) = \mathbf{0}, \tag{3.45}$$

into (3.44) and performing some manipulations, one obtains the Lyapunov differential equation

$$\dot{\mathbf{N}}_+ = -\mathbf{B}_{11}^\mathsf{T}\mathbf{N}_+ - \mathbf{N}_+\mathbf{B}_{11} + \bar{\mathbf{Q}}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C}\bar{\mathbf{Q}} \tag{3.46}$$

with $\mathbf{B}_{11}$ as in (3.29). This is the differential form of the constructibility Gramian of the system

$$\begin{aligned} \dot{\boldsymbol{\xi}}(t) &= \mathbf{B}_{11}(t)\boldsymbol{\xi}(t), \qquad \boldsymbol{\xi}(t) \in \mathbb{R}^k, \\ \mathbf{y}_\xi &= \mathbf{C}(t)\bar{\mathbf{Q}}(t)\boldsymbol{\xi}(t). \end{aligned} \tag{3.47}$$

Hence, condition (3.41) can be interpreted as uniform complete constructibility/observability of system (3.47), see Section 2.4. According to Theorem 2.18, condition (3.41) guarantees the existence of a bounded output feedback gain $\mathbf{L}_+(t)$ such that the system

$$\dot{\boldsymbol{\xi}} = (\mathbf{B}_1 - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}})\boldsymbol{\xi} \tag{3.48}$$

is uniformly exponentially stable. By the choice of $\mathbf{G}$, (3.43) has a unique positive definite solution $\mathbf{P}(t)$ which is uniformly bounded. Moreover, system (3.48) with

$$\mathbf{L}_+(t) = \mathbf{P}(t)\bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{C}^\mathsf{T}(t) \tag{3.49}$$

is uniformly exponentially stable, see Theorem 2.18.

The dynamics of the estimation error (3.32) will be investigated by introducing the change of coordinates $\mathbf{e}_z = \mathbf{Q}^\mathsf{T}\mathbf{e}$. One has

$$\dot{\mathbf{e}}_z = \dot{\mathbf{Q}}^\mathsf{T}\mathbf{e} + \mathbf{Q}^\mathsf{T}\dot{\mathbf{e}}, \tag{3.50}$$

and by skew-symmetry of $\mathbf{S}$ in (3.18) it follows that $\dot{\mathbf{Q}}^\mathsf{T} = -\mathbf{S}\mathbf{Q}^\mathsf{T}$. Hence, the dynamics of $\mathbf{e}_z$ is governed by

$$\begin{aligned} \dot{\mathbf{e}}_z &= -\mathbf{S}\mathbf{Q}^\mathsf{T}\mathbf{e} + \mathbf{Q}^\mathsf{T}\mathbf{A}\mathbf{e} - \mathbf{Q}^\mathsf{T}\mathbf{L}\mathbf{C}\mathbf{e} \\ &= -\mathbf{S}\mathbf{e}_z + \mathbf{Q}^\mathsf{T}\mathbf{A}\mathbf{Q}\mathbf{e}_z - \mathbf{Q}^\mathsf{T}\mathbf{L}\mathbf{C}\mathbf{Q}\mathbf{e}_z \\ &= \mathbf{B}\mathbf{e}_z - \mathbf{Q}^\mathsf{T}\mathbf{L}\mathbf{C}\mathbf{Q}\mathbf{e}_z \end{aligned} \tag{3.51}$$

with $\mathbf{B}(t)$ as in (3.17). Inserting the observer gain (3.42) into $\mathbf{Q}^\mathsf{T}\mathbf{L}\mathbf{C}\mathbf{Q}$, it follows that this matrix is block upper triangular according to

$$\begin{aligned} \mathbf{Q}^\mathsf{T}\mathbf{L}\mathbf{C}\mathbf{Q} &= \begin{bmatrix} \bar{\mathbf{Q}}^\mathsf{T} \\ \bar{\mathbf{Q}}_\perp^\mathsf{T} \end{bmatrix} \bar{\mathbf{Q}}\mathbf{P}\bar{\mathbf{Q}}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{C} \begin{bmatrix} \bar{\mathbf{Q}} & \bar{\mathbf{Q}}_\perp \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}} & \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}_\perp \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned} \tag{3.52}$$

Partitioning $\mathbf{e}_z$ appropriately with $\mathbf{e}_z = [\mathbf{e}_z^{+\mathsf{T}} \;\; \mathbf{e}_z^{-\mathsf{T}}]^\mathsf{T}$ reveals that

$$\dot{\mathbf{e}}_z^+ = (\mathbf{B}_{11} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}})\mathbf{e}_z^+ + (\mathbf{B}_{12} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}_\perp)\mathbf{e}_z^-, \tag{3.53}$$

$$\dot{\mathbf{e}}_z^- = \mathbf{B}_{22}\mathbf{e}_z^-. \tag{3.54}$$

All Lyapunov exponents of (3.54) are negative and thus $\mathbf{e}_z^-$ converges to zero exponentially fast. The dynamics (3.53) of $\mathbf{e}_z^+$ can be interpreted as an unperturbed equation $\dot{\mathbf{e}}_z^+ = (\mathbf{B}_{11} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}})\mathbf{e}_z^+$, which is uniformly exponentially stable, and a bounded perturbation $(\mathbf{B}_{12} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}_\perp)\mathbf{e}_z^-$, which vanishes exponentially fast. Exponential stability of (3.53), (3.54) then follows from [Zho16, Theorem 2].

□

The implementation of the observer requires to solve the differential equations (3.31) and (3.43) simultaneously in order to compute the output feedback gain (3.42). To summarize, the proposed observer is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{L}(t)\left[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right], \tag{3.55a}$$

$$\dot{\mathbf{P}}(t) = \mathbf{B}_{11}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}_{11}^\mathsf{T}(t) - \mathbf{P}(t)\bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{C}^\mathsf{T}(t)\mathbf{C}(t)\bar{\mathbf{Q}}(t)\mathbf{P}(t) + \mathbf{G}(t), \tag{3.55b}$$

$$\dot{\bar{\mathbf{Q}}}(t) = \left[\mathbf{I} - \bar{\mathbf{Q}}(t)\bar{\mathbf{Q}}^\mathsf{T}(t)\right]\mathbf{A}(t)\bar{\mathbf{Q}}(t) + \bar{\mathbf{Q}}(t)\mathbf{S}_{11}(t),\; \bar{\mathbf{Q}}(0) = \bar{\mathbf{Q}}_0 \in \mathbb{R}^{n\times k}, \tag{3.55c}$$

with

$$\mathbf{L}(t) = \bar{\mathbf{Q}}(t)\mathbf{P}(t)\bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{C}^\mathsf{T}(t), \tag{3.55d}$$

$$\mathbf{B}_{11}(t) = \bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{A}(t)\bar{\mathbf{Q}}(t) - \mathbf{S}_{11}(t). \tag{3.55e}$$

The elements s_{ij} of $\mathbf{S}_{11}$ are given by

$$s_{ij}(t) = \begin{cases} -\bar{\mathbf{q}}_i^\mathsf{T}(t)\mathbf{A}(t)\bar{\mathbf{q}}_j(t) & i < j, \\ 0 & i = j, \\ \bar{\mathbf{q}}_i^\mathsf{T}(t)\mathbf{A}(t)\bar{\mathbf{q}}_j(t) & i > j, \end{cases} \tag{3.55f}$$

where $\bar{q}_i(t)$ are the columns of $\bar{\mathbf{Q}}(t)$. The initial conditions are chosen as a positive definite $k \times k$ matrix $\mathbf{P}_0 \succ 0$ and a random orthogonal $n \times k$ matrix $\mathbf{Q}_0$. The matrix $\mathbf{G}(t)$ in (3.55b) is a tuning parameter of the observer. By choosing $\mathbf{G}(t) = g\mathbf{I}$ with $g > 0$, this could be reduced to the tuning of only one parameter. Large values of g tend to increase the solution $\mathbf{P}(t)$ of (3.55b) and the gain $\mathbf{L}_+$. Hence, the convergence speed of $\dot{\mathbf{e}}_z^+ = (\mathbf{B}_1 - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}})\mathbf{e}_z^+$ will be increased. The convergence speed of (3.53) is, however, limited by the autonomous system (3.54).

The following counterexample shows that the proposed condition is not sufficient for exponential detectability if (3.41) just holds for a specific t_0, i.e., there exist systems that are not exponentially detectable but fulfill (3.41) for some values of t_0.

Example 3.7*:* Consider system (3.1) with

$$\begin{aligned} \mathbf{A}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & f(t) \end{bmatrix}, \quad f(t) = \begin{cases} 0 & t \leq T \\ -1 & t > T \end{cases} \\ \mathbf{C}(t) &= \begin{cases} \begin{bmatrix} 1 & -1 \end{bmatrix} & t \leq T \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & t > T \end{cases} \end{aligned} \tag{3.56}$$

and some constant $T > 0$. If $t_0 < T$, the constructibility Gramian is positive semidefinite for all $t > t_0$. Especially, for $t_1 \geq T$ it holds that

$$\begin{aligned} \mathbf{N}(t_1, t_0) &= \int_{t_0}^{T} \mathbf{\Phi}^\mathsf{T}(s,t) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} \mathbf{\Phi}(s,t) ds \\ &= (T - t_0) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned} \tag{3.57}$$

The system is forward regular and

$$\bar{\mathbf{Q}}(t) = \bar{\mathbf{Q}} = \begin{bmatrix} 1 & 0 \end{bmatrix}^\mathsf{T}, \quad \bar{\mathbf{Q}}_\perp = \begin{bmatrix} 0 & 1 \end{bmatrix}^\mathsf{T}, \tag{3.58}$$

where $\mathbf{Q}(t)$ solves (3.18). Thus, condition (3.41) holds, i.e.,

$$\bar{\mathbf{Q}}^\mathsf{T}\mathbf{N}\bar{\mathbf{Q}} = (T - t_0) > 0. \tag{3.59}$$

However, there exists no feedback gain $\mathbf{L}(t)$ such that

$$\dot{\mathbf{x}} = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{x} \tag{3.60}$$

is exponentially stable. To see this, note that for the initial condition $\mathbf{x}_0 = [1\ 1]^\mathsf{T}$, the output $y(t) = \mathbf{C}(t)\mathbf{x}(t) = 0$ for all $t \geq t_0$ and thus (3.60) cannot be exponentially stabilized by any choice of $\mathbf{L}$.

Note that even if the presented counterexample does not fulfill the assumed continuity properties, one can always construct a continuous approximation which shows the same behavior.

3.4 Robustness Analysis

This section discusses the robustness properties of the proposed observer design in the presence of bounded perturbations. The cases of measurement noise and model perturbations are treated separately in the following.

3.4.1 Measurement Noise

Let system (3.1) fulfill the assumptions for the existence of the observer stated in Theorem 3.6. It is furthermore assumed that the corresponding output equation (3.1b) is subject to bounded measurement noise according to

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \boldsymbol{\eta}(t), \ \|\boldsymbol{\eta}(t)\| \leq \bar{\eta} \in \mathbb{R}^+. \tag{3.61}$$

It can be shown that the error dynamics results in

$$\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{e}(t) - \mathbf{L}(t)\boldsymbol{\eta}(t) \tag{3.62}$$

with $\mathbf{L}(t)$ as given in (3.42). Applying again the transformation to upper triangular form $\mathbf{e}_z = \mathbf{Q}^\mathsf{T}\mathbf{e}$ yields

$$\dot{\mathbf{e}}_z(t) = \left[\mathbf{B}(t) - \mathbf{Q}^\mathsf{T}(t)\mathbf{L}(t)\mathbf{C}(t)\mathbf{Q}(t)\right]\mathbf{e}_z(t) - \mathbf{Q}^\mathsf{T}(t)\mathbf{L}(t)\boldsymbol{\eta}(t). \tag{3.63}$$

Partitioning the states according to k and following the same arguments as in the proof of Theorem 3.6, one can derive the perturbed error system according to

$$\dot{\mathbf{e}}_z^+ = (\mathbf{B}_{11} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}})e_z^+ + (\mathbf{B}_{12} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}_\perp)\mathbf{e}_z^- - \mathbf{L}_+\boldsymbol{\eta}, \tag{3.64}$$

$$\dot{\mathbf{e}}_z^- = \mathbf{B}_{22}\mathbf{e}_z^-. \tag{3.65}$$

Because $\mathbf{L}_+(t)$ is uniformly bounded, the measurement noise acts as additional bounded input to the error system. The autonomous system $\dot{\mathbf{e}}_z^+ = (\mathbf{B}_{11} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}})\mathbf{e}_z^+$ is uniformly exponentially stable by the choice of $\mathbf{L}_+$ as in (3.49). This implies that also $\mathbf{e}_z^+(t)$ remains bounded[2], see, e.g. [Rug95, Lemma 12.4]. Hence, the estimation error also remains bounded.

In order to obtain more quantitative results on the bounds of the estimation error, one would need to compute explicit bounds on the solution of the Riccati equation (3.43). The best known bounds, which are still very conservative, are presented in [Buc72b]. These bounds are derived based on the bounds of the corresponding Gramian, see Chapter 2. These values are hard to obtain in practice, however. Increasing the observer gain $\mathbf{L}_+$ via increasing $\mathbf{G}$ tends to make the error dynamics more sensitive to the measurement noise, which may result in a larger bound on the estimation error in the limit.

[2]Note that this is a special case of bounded input bounded output stability with $\mathbf{C}(t) = \mathbf{I}$ as output matrix.

3.4.2 Model Perturbations

Now, assume that there is no measurement noise but a uniformly bounded perturbation of the model according to

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t),\ \sup_{t\in\mathbb{J}} \|\mathbf{f}(t)\| \leq \bar{f}. \tag{3.66}$$

The robustness of the proposed observer is related to bounded input bounded state stability of the resulting error system

$$\dot{\mathbf{e}}(t) = \left[\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)\right]\mathbf{e}(t) + \mathbf{f}(t). \tag{3.67}$$

The transformed error system for $\mathbf{e}_z = \mathbf{Q}^\mathsf{T}\mathbf{e}$ is then obtained analogously to (3.51) with

$$\dot{\mathbf{e}}_z(t) = \left[\mathbf{B}(t) - \mathbf{Q}^\mathsf{T}(t)\mathbf{L}(t)\mathbf{C}(t)\mathbf{Q}(t)\right]\mathbf{e}_z(t) + \mathbf{Q}^\mathsf{T}(t)\mathbf{f}(t) \tag{3.68}$$

and a partitioning of the error system according to k, i.e., the number of columns of $\bar{\mathbf{Q}}$, gives

$$\dot{\mathbf{e}}_z^+ = (\mathbf{B}_1 - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}})\mathbf{e}_z^+ + (\mathbf{B}_2 - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}_\perp)\mathbf{e}_z^- + \bar{\mathbf{Q}}\mathbf{f}, \tag{3.69}$$

$$\dot{\mathbf{e}}_z^- = \mathbf{B}_3\mathbf{e}_z^- + \bar{\mathbf{Q}}_\perp\mathbf{f}. \tag{3.70}$$

Hence, a bounded input $\bar{\mathbf{Q}}_\perp\mathbf{f}$ acts on the exponentially stable system

$$\dot{\mathbf{e}}_z^- = \mathbf{B}_3\mathbf{e}_z^-. \tag{3.71}$$

In order to guarantee a bounded $\mathbf{e}_z^-$ and consequently a bounded estimation error, equation (3.70) is required to be bounded input bounded state stable. This can be guaranteed if (3.71) is not merely exponentially stable but uniformly exponentially stable, which may be the case in various applications. Conditions for uniform exponential stability together with methods for their numerical evaluation are discussed in detail in Chapter 4.

3.5 Reduced Order Observer Design

Based on the detectability condition and the observer presented in Theorem 3.6, it is also possible to design a reduced order observer. For this, the transformation of (3.1a) to the upper triangular form is utilized. The Lyapunov transformation $\mathbf{z}(t) = \mathbf{Q}^\mathsf{T}(t)\mathbf{x}(t)$ with $\mathbf{Q}(t)$ obtained by the full-size QR-decomposition as presented in Section 3.2.2 transforms (3.1a) to the upper triangular form

$$\dot{\mathbf{z}} = \mathbf{B}(t)\mathbf{z}(t). \tag{3.72}$$

Again, $k^\star$ is the number of non-negative Lyapunov exponents and it is assumed that the system is exponentially detectable and that the assumptions stated in Theorem 3.6 hold for some $k \geq k^\star$. Partitioning the new states according to $\mathbf{z}(t) = \begin{bmatrix}\mathbf{z}_+^\mathsf{T}(t) & \mathbf{z}_-^\mathsf{T}(t)\end{bmatrix}^\mathsf{T}$ with $\mathbf{z}_+(t) \in \mathbb{R}^k$ results in

$$\begin{bmatrix}\dot{\mathbf{z}}_+ \\ \dot{\mathbf{z}}_-\end{bmatrix} = \begin{bmatrix}\mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{0} & \mathbf{B}_{22}\end{bmatrix} \begin{bmatrix}\mathbf{z}_+ \\ \mathbf{z}_-\end{bmatrix}. \tag{3.73}$$

Note that $\mathbf{B}_{11}$ coincides with $\mathbf{B}_{11}$ of system (3.47). The dynamics for $\mathbf{z}_+$ is given by

$$\dot{\mathbf{z}}_+ = \mathbf{B}_{11}\mathbf{z}_+ + \mathbf{B}_{12}\mathbf{z}_- \tag{3.74}$$

with $\mathbf{z}_-$ as a bounded input which vanishes exponentially fast. The output of system (3.1) can be written as

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x} = \mathbf{C}\bar{\mathbf{Q}}\mathbf{z}_+ + \mathbf{C}\bar{\mathbf{Q}}_\perp\mathbf{z}_-. \tag{3.75}$$

To obtain an estimate $\hat{\mathbf{z}}$ for $\mathbf{z}_+$, the observer

$$\dot{\hat{\mathbf{z}}}(t) = \mathbf{B}_{11}(t)\hat{\mathbf{z}}(t) + \mathbf{L}_+(t)(\mathbf{y}(t) - \mathbf{C}(t)\bar{\mathbf{Q}}(t)\hat{\mathbf{z}}(t)) \tag{3.76}$$

is proposed with

$$\mathbf{L}_+(t) = \mathbf{P}(t)\bar{\mathbf{Q}}(t)\mathbf{C}^\mathsf{T}(t) \tag{3.77}$$

and $\mathbf{P}(t)$ as the solution of the differential Riccati equation (3.43). The dynamics of the reduced estimation error $\mathbf{e}_{z,+} = \mathbf{z}_+ - \hat{\mathbf{z}}$ is then given by

$$\begin{aligned}\dot{\mathbf{e}}_{z,+} &= \dot{\mathbf{z}}_+ - \dot{\hat{\mathbf{z}}} \\ &= \mathbf{B}_{11}\mathbf{z}_+ + \mathbf{B}_{12}\mathbf{z}_- - \mathbf{B}_{11}\hat{\mathbf{z}} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}\mathbf{z}_+ - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}_\perp\mathbf{z}_- + \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}\hat{\mathbf{z}} \\ &= (\mathbf{B}_{11} - \mathbf{L}_+\mathbf{C}\mathbf{Q})\mathbf{e}_{z,+} + (\mathbf{B}_{12} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}_\perp)\mathbf{z}_-.\end{aligned} \tag{3.78}$$

By the assumption of exponential detectability, the bounded feedback gain $\mathbf{L}_+$ guarantees uniform exponential stability of $\dot{\mathbf{e}}_{z,+} = (\mathbf{B}_{11} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}})\mathbf{e}_{z,+}$. The bounded input $(\mathbf{B}_{12} - \mathbf{L}_+\mathbf{C}\bar{\mathbf{Q}}_\perp)\mathbf{z}_-$ to the error system vanishes exponentially fast and hence the estimation error decays to zero exponentially fast. Because $\mathbf{x} = \bar{\mathbf{Q}}\mathbf{z}_+ + \bar{\mathbf{Q}}_\perp\mathbf{z}_-$, one can reconstruct the estimated states $\hat{\mathbf{z}}$ to the original full-order state space according to

$$\hat{\mathbf{x}}_{\text{red}} = \bar{\mathbf{Q}}\hat{\mathbf{z}}. \tag{3.79}$$

Because $\mathbf{z}_-$ decays to zero exponentially fast, one obtains exponential convergence of the full-order estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}_{\text{red}}(t)$ and

$$\lim_{t\to\infty} \mathbf{x}(t) - \hat{\mathbf{x}}_{\text{red}}(t) = \mathbf{0}. \tag{3.80}$$

3.6 The Time Invariant Case

In the time invariant case, exponential stability implies uniform exponential stability and thus Definition 3.5 is equivalent to other detectability notions available in literature, see e.g. [Hes09, Chapter 16] for a comprehensive overview.

The time invariant version of Theorem 3.6 is in the following shown to be necessary and sufficient for detectability. Some useful results regarding detectability of linear time invariant systems are discussed in Chapter 2. Let system (3.1) be time invariant with $\mathbf{A}(t) = \mathbf{A}$ and $\mathbf{C}(t) = \mathbf{C}$. Let $\mathbf{M}(t, 0)$ be the observability Gramian in the time-invariant case given by

$$\mathbf{M}(t, 0) = \int_0^t e^{\mathbf{A}^\mathsf{T} s}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}s}\,\mathrm{d}s. \tag{3.81}$$

As in Example 3.1, assume for simplicity that the eigenvalues of $\mathbf{A}$ have distinct real parts. The eigenvalues are ordered according to their real value such that $\mathrm{Re}\{\mu_1\} > \mathrm{Re}\{\mu_2\} > \ldots > \mathrm{Re}\{\mu_n\}$, $i = 1, \ldots, n$ and $\mu_{k^\star}$ is the eigenvalue with the smallest non-negative real part, i.e., $\mathrm{Re}\{\mu_{k^\star}\} \geq 0$ and $\mathrm{Re}\{\mu_{k^\star+1}\} < 0$.

Because time invariant systems are a sub-class of time varying systems, the detectability condition (3.41) is sufficient for the necessary and sufficient Popov-Belevitch-Hautus Lemma, see Lemma 2.21. For necessity, it is shown that this condition implies (3.41). Any ordered normal Lyapunov basis $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots \mathbf{v}_n]$ can be written as

$$\mathbf{V} = \mathbf{P}\mathbf{L} \tag{3.82}$$

with $\mathbf{P} = [\mathbf{p}_1\ \mathbf{p}_2\ \ldots\ \mathbf{p}_n]$ as the ordered right eigenvectors of $\mathbf{A}$ and a lower triangular matrix

$$\mathbf{L} = \begin{bmatrix} l_{11} & 0 & \ldots & 0 \\ l_{12} & l_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{1n} & \ldots & \ldots & l_{nn} \end{bmatrix} \tag{3.83}$$

with non-zero diagonal elements l_{ii}. Hence, any $\mathbf{v}_i$ can be written as a linear combination of the linearly independent eigenvectors $\mathbf{p}_j$ with $j \geq i$. Because it is assumed that the system is detectable, it holds that $\mathbf{p}_i \notin \ker\mathbf{C}$ for $i = 1, \ldots, k$ according to Lemma 2.21. Moreover,

$$\mathbf{v}_i^\mathsf{T}\mathbf{M}(t, 0)\mathbf{v}_i > 0,\ i = 1, \ldots, k, \tag{3.84}$$

holds for all $t > 0$ because

$$\begin{aligned} \mathbf{v}_i^\mathsf{T}\mathbf{M}(t, 0)\mathbf{v}_i &= \int_0^t \|\mathbf{C}e^{\mathbf{A}s}\mathbf{v}_i\|^2 ds \\ &= \int_0^t \|\mathbf{C}\mathbf{p}_i e^{\mu_i s} l_{ii} + \ldots + \mathbf{C}\mathbf{p}_n e^{\mu_n s} l_{in}\| ds. \end{aligned} \tag{3.85}$$

Hence, $\mathbf{v}_i^\mathsf{T}\mathbf{M}(t,0)\mathbf{v}_i$ cannot be identically zero, since $\mathbf{C}\mathbf{p}_i \neq \mathbf{0}$. By using similar arguments, one can conclude that

$$\bar{\mathbf{V}}^\mathsf{T}\mathbf{M}(t,0)\bar{\mathbf{V}} \succ 0 \tag{3.86}$$

with $\bar{\mathbf{V}} = [\mathbf{v}_1 \ \dots \ \mathbf{v}_k]$. It holds that

$$\mathbf{h}^T\bar{\mathbf{V}}^\mathsf{T}\mathbf{M}(t,0)\bar{\mathbf{V}}\mathbf{h} > 0 \tag{3.87}$$

for any non-trivial $\mathbf{h} = [\mathbf{h}_1 \ \mathbf{h}_2 \ \dots \ \mathbf{h}_k]^\mathsf{T}$ because $\bar{\mathbf{V}}\mathbf{h}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$ and thus of $\mathbf{p}_1, \dots, \mathbf{p}_k, \dots, \mathbf{p}_n$ with at least one coefficient corresponding to $\mathbf{p}_1, \dots, \mathbf{p}_k$ non-zero.

The observability Gramian can be stated with the constructibility Gramian as

$$\mathbf{M}(t,0) = e^{\mathbf{A}^\mathsf{T}t}\mathbf{N}(t,0)e^{\mathbf{A}t} \tag{3.88}$$

and hence (3.86) is equivalent to

$$\bar{\mathbf{V}}^\mathsf{T}e^{\mathbf{A}^\mathsf{T}t}\mathbf{N}(t,0)e^{\mathbf{A}t}\bar{\mathbf{V}} \succ 0 \quad \text{for all } t > 0. \tag{3.89}$$

Because $\bar{\mathbf{V}}$ is constructed by the first k columns of an ordered normal Lyapunov basis, it is possible to apply the reduced QR-decomposition to $\bar{\mathbf{X}}(t) = e^{\mathbf{A}t}\bar{\mathbf{V}}$ to obtain

$$\mathbf{R}_1^\mathsf{T}(t)\bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{N}(t,0)\bar{\mathbf{Q}}(t)\mathbf{R}_1(t) \succ 0. \tag{3.90}$$

Moreover, $\mathbf{R}_1 \in \mathbb{R}^{k\times k}$ is non-singular and hence

$$\bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{N}(t,0)\bar{\mathbf{Q}}(t) \succ 0 \quad \text{for all } t > 0, \tag{3.91}$$

which is the detectability condition presented in Theorem 3.6 for the time varying case. Since the system is time-invariant, the condition (3.91) does not depend on the initial time and holds uniformly in t_0. To sum up, detectability in the time invariant case implies the detectability condition (3.41) and thus this condition is necessary and sufficient in the linear time invariant case.

3.7 Simulation Studies

3.7.1 Academic Example

Let a linear time invariant system $\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z}$, $\mathbf{y} = \bar{\mathbf{C}}\mathbf{z}$ be given by the constant matrices

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \qquad \bar{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \end{bmatrix}, \tag{3.92}$$

$$\mathbf{A}_{11} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0.5 \end{bmatrix}, \qquad \mathbf{A}_{22} = \begin{bmatrix} -1 & 3 & 1 & 0 \\ 0 & -1.5 & 0 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

$$\mathbf{A}_{21} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{C}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The subsystem $(\mathbf{A}_{11}, \mathbf{C}_1)$ is uniformly completely observable, because

$$\operatorname{rank} \mathcal{O}_{11} = \operatorname{rank} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_1\mathbf{A}_{11} \\ \mathbf{C}_1\mathbf{A}_{11}^2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 3 \\ 4 & 0 & 2.5 \\ 0 & 1 & 4.5 \end{bmatrix} = 3. \tag{3.93}$$

Moreover, since in the linear time invariant case the Lyapunov exponents coincide with the real part of the eigenvalues, the system possesses three positive and four negative Lyapunov exponents given by the diagonal entries of $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$, respectively. Because $\mathbf{A}_{22}$ is a Hurwitz matrix, the system $(\bar{\mathbf{A}}, \bar{\mathbf{C}})$ is (exponentially) detectable [Che98, Theorem 6.O6]. Note that the directional detectability condition of [FZ18] does not hold for this example, because there are more non-negative Lyapunov exponents than linearly independent measurements.

A linear time varying system is now generated by a time varying state transformation $\mathbf{x} = \mathbf{T}(t)\mathbf{z}$ with

$$\mathbf{T}(t) = \mathbf{U}^\mathsf{T} \begin{bmatrix} \cos(t) & -\sin(t) & \mathbf{0} \\ \sin(t) & \cos(t) & \\ \mathbf{0} & & \mathbf{I}_5 \end{bmatrix} \mathbf{U} \tag{3.94}$$

with $\mathbf{U}$ as random orthogonal matrix. This leads to a time varying system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{3.95a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{3.95b}$$

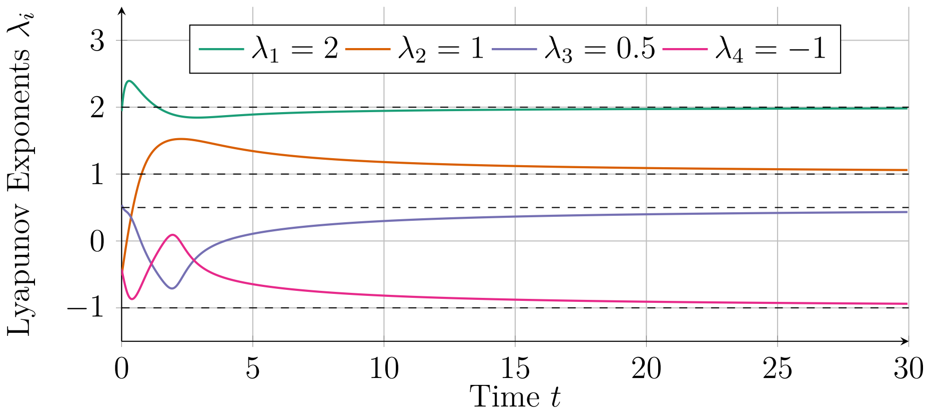


Fig. 3.1: Largest four approximated Lyapunov exponents of (3.95).

with

$$\mathbf{A}(t) = \mathbf{T}(t)\bar{\mathbf{A}}\mathbf{T}^{\mathsf{T}}(t) + \dot{\mathbf{T}}(t)\mathbf{T}^{\mathsf{T}}(t), \text{ and } \mathbf{C}(t) = \bar{\mathbf{C}}\mathbf{T}^{\mathsf{T}}(t). \tag{3.95c}$$

The systems $(\bar{\mathbf{A}}, \bar{\mathbf{C}})$ and $(\mathbf{A}(t), \mathbf{C}(t))$ are kinematically similar and especially have equivalent stability and observability properties, see [Sil71]. The proposed observer is designed for the time varying system[3].

The differential equation (3.1a) for the system and the differential equations (3.55a) and (3.55b) for the observer are solved using fixed step 4th order Runge-Kutta integration with a step-size of $T_s = 0.001$. For solving the differential equation (3.55c) to obtain $\bar{\mathbf{Q}}$, a projected 4th order Runge-Kutta integrator [DRV97] with the same step size is employed. The modified Gram-Schmidt algorithm used in the simulation examples can be found in Appendix B.1. All initial values were chosen randomly with uniform distribution in $(0, 1)$. The initial value $\bar{\mathbf{Q}}(0)$ was orthogonalized after the random number generation.

Figure 3.1 shows the numerical approximation of the four largest Lyapunov exponents and their corresponding asymptotes, which coincide with the four largest eigenvalues of $\bar{\mathbf{A}}$. In Fig. 3.2, the logarithmic estimation errors using the observer gain presented in Theorem 3.6 are depicted. The observer is implemented for $k = 3$, $\mathbf{G} = \mathbf{I}$ and $\mathbf{P}_0 = 10\mathbf{I}$. It can be seen that exponential decay with a rate of e^{-t} is an upper bound for the decay of the estimation errors. This rate corresponds to $\lambda_4 = -1$, i.e. the largest negative Lyapunov exponent which is not modified by the observer gain.

[3]This is a reasonable approach if one assumes $\mathbf{A}(t)$ and $\mathbf{C}(t)$ as known and the transformation matrix $\mathbf{T}(t)$ unknown. Otherwise, it would be possible to design an observer directly for the time invariant system.

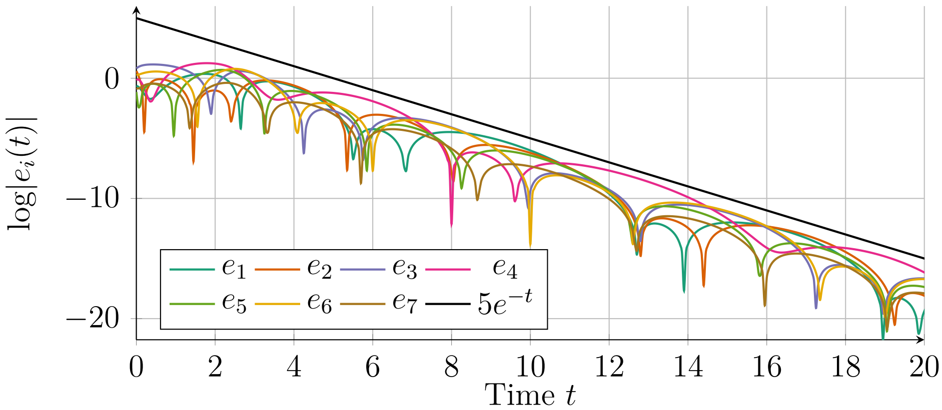


Fig. 3.2: Logarithmic estimation errors $e_i = x_i - \hat{x}_i$ of the observer (3.55) implemented for (3.95).

3.7.2 Lorenz'96 Model

As a more complex simulation example, a linearized version of the Lorenz'96 model, see, e.g., [Boc+17], is investigated in this section. This nonlinear model proposed by E. Lorenz [Lor95] is widely used as a benchmark example in data-assimilation, see, e.g., [FZ18; Boc+17; TP11]. The system is defined by

$$\dot{z}_i = (z_{i+1} - z_{i-2})z_{i-1} - z_i + F, \; i = 1, \dots, n, \tag{3.96}$$

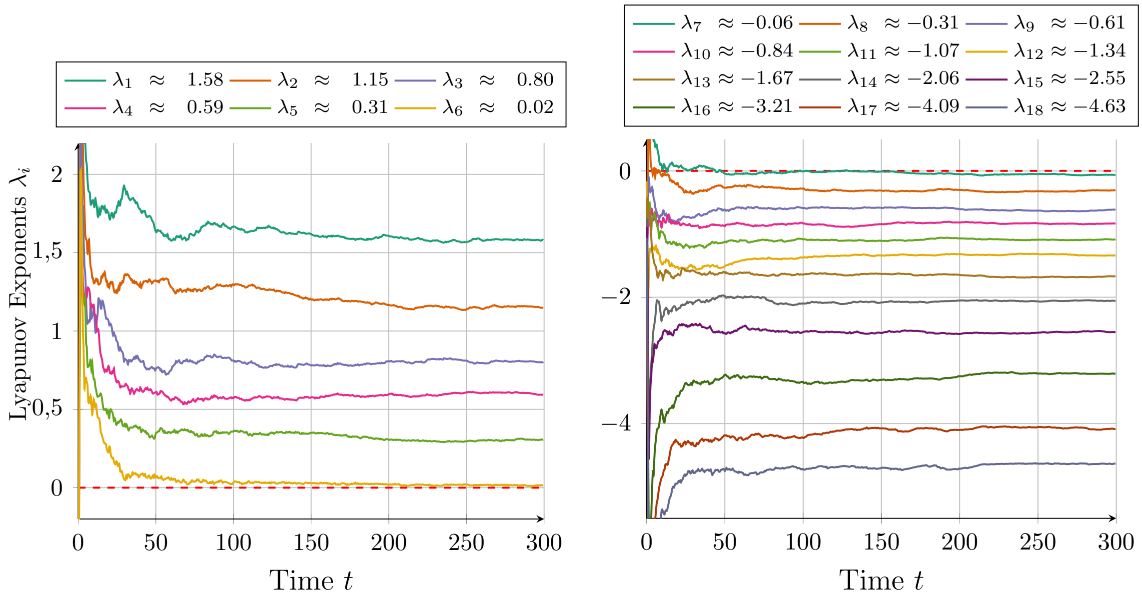
with the notational convention $z_{-1} := z_{n-1}$, $z_0 := z_n$, and $z_{n+1} := z_1$. The state vector is $\mathbf{z} = [z_1 \; \cdots \; z_n]^\mathsf{T} \in \mathbb{R}^n$. The system order and the output are chosen as $n = 18$ and

$$\mathbf{y}(t)^\mathsf{T} = [z_1(t) \; z_5(t) \; z_9(t) \; z_{13}(t) \; z_{17}(t)]. \tag{3.97}$$

The linear time varying system in the form of (3.1) was obtained by linearizing (3.96) and (3.97) along a trajectory. Since no analytical solution for the nonlinear system is available, a numerical solution was computed for the initial condition $z_{i,0} = \sin\left(\frac{i-1}{n} 2\pi\right)$ at time $t_0 = 0$. Together with $F = 8$, the system exhibits a chaotic behavior in this case [Boc+17].

As in the previous example, all differential equations are solved using fixed step 4th order Runge-Kutta integration with a step-size of $T_s = 0.005$. For solving the differential equation (3.55c), a projected integrator [DRV97] is used. The initial condition $\bar{\mathbf{Q}}(0)$ is chosen as a random orthogonal matrix.

In Fig. 3.3a, the approximated values of the six largest Lyapunov exponents for the obtained linear time varying system are shown. The remaining (negative) exponents are depicted in Fig 3.3b. The values are approximated by evaluating (3.20) on a finite horizon. It can be seen that six exponents are non-negative and $\lambda_7 \approx -0.06$ is negative, but very close to zero.



(a) Positive Lyapunov exponents.

(b) Negative Lyapunov Exponents.

Fig. 3.3: Approximated Lyapunov exponents of the linearized Lorenz'96 model.

The initial estimation error was chosen to be $\mathbf{e}_0 = \boldsymbol{\eta}$ where the components of $\boldsymbol{\eta}$ are normally distributed with $\eta_i \sim \mathcal{N}(0,1),\ i = 1,\ldots,n$. The norm of the estimation error for $k = 6$ and ten simulation runs with different initial conditions is shown in Fig. 3.4. The convergence is very slow, since it is governed by the largest negative Lyapunov exponent $\lambda_7 \approx -0.06$. To increase the convergence speed, the observer was implemented for $k = 8$, which means that also negative Lyapunov exponents are modified by the observer gain. The result for ten simulation runs is depicted in Fig. 3.5. The exponential convergence is faster compared to $k = 6$ and the convergence speed is determined by $\lambda_9 \approx -0.61$.

3.8 Discussion

This chapter presents a detectability condition and an observer design technique for forward regular linear time varying systems. The condition guarantees exponential decay of the estimation error. The existence of the observer is related to the unique positive definite solution of a reduced order differential Riccati equation obtained by the projection onto a subspace of lower dimension.

An advantage of the presented observer is that solving a Riccati equation is only required on a reduced state space of dimension k with $k^\star \leq k \leq n$, where $k^\star$ is the number of non-negative Lyapunov exponents. This drastically reduces computational

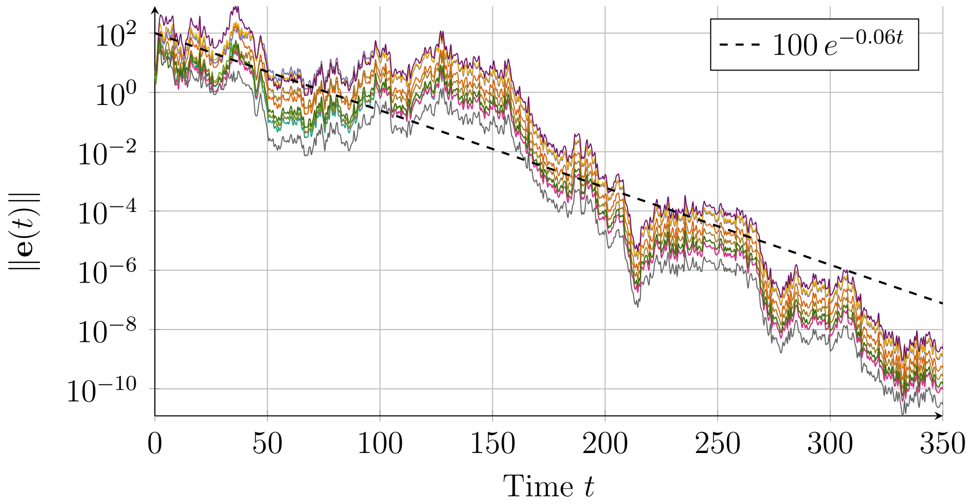


Fig. 3.4: Norm of estimation error of the observer (3.55) implemented for the linearized Lorenz'96 model for $k = 6$ and ten simulation runs.

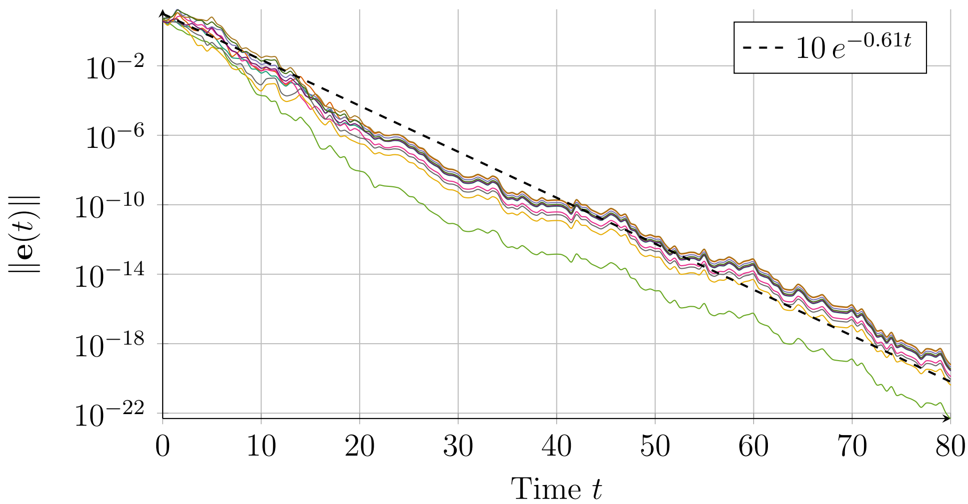


Fig. 3.5: Norm of estimation error of the observer (3.55) implemented for the linearized Lorenz'96 model for $k = 8$ and ten simulation runs.

complexity if only a few Lyapunov exponents of the original system are non-negative. Compared to the $(\frac{n^2-n}{2}+n)+n=\frac{1}{2}n^2+\frac{3}{2}n$ state variables of a full order Riccati equation and observer, the number of state variables in the proposed scheme is reduced to $n+nk+\left(k+\frac{k^2-k}{2}\right)=n+k\left(n+1+\frac{k-1}{2}\right)$ for (3.43), (3.31), and (3.33).

The convergence speed of the proposed (minimal order) observer is limited by the largest negative Lyapunov exponent $\lambda_{k^\star+1}$ for $k=k^\star$. It may happen that the convergence is prohibitively slow, i.e., that $\lambda_{k^\star+1}$ is not small enough. In this case, the number of columns in $\bar{\mathbf{Q}}$ can be chosen larger than $k^\star$ in order to increase convergence speed of the resulting observer. Especially if the system is uniformly completely observable, condition (3.41) trivially holds for any number of columns in $\bar{\mathbf{Q}}$. This allows a trade-off between computational complexity and convergence speed of the estimation error.

Forward regularity allows to numerically approximate the Lyapunov exponents of the considered system and hence it is possible to estimate the number of non-negative exponents in advance. This assumption is hard to verify for specific applications, however. An open question is, if forward regularity is preserved under output feedback. In particular, if the original system $\dot{\mathbf{x}}(t)=\mathbf{A}(t)\mathbf{x}(t)$ is forward regular, it is not clear at this point if the error system $\dot{\mathbf{e}}(t)=[\mathbf{A}(t)-\mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{e}(t)$ remains forward regular for the proposed choice of $\mathbf{L}$. However, this would be desirable in order to preserve robustness of the error system with respect to nonlinear perturbations as discussed in Section 3.2.2.

It is stated in [Lui07, p. 15] that within the context of ergodic theory, regularity is "typical under fairly general assumptions", when the time varying matrix $\mathbf{A}(t)$ arises from the variational equation of a nonlinear system. This is theoretically justified by Oseledets' multiplicative ergodic theorem, see, e.g. [Lui07, Theorem 1.6]. Regularity for systems with inputs is, however, not guaranteed. It is argued in [CK91] that the multiplicative ergodic theorem cannot be applied to systems with inputs and hence different concepts have to be employed. The known system inputs do not occur in the observer error dynamics and hence this is no limitation for linear time varying systems. On the other hand, if the coefficient matrix arises form the variational equation of a nonlinear system with inputs, the regularity assumption might not be fulfilled.

In order to circumvent the forward regularity assumption, a different spectral concept is investigated in the following chapter.

Chapter 4

Uniform Detectability and Subspace Observer Design

The ideas presented in the previous chapter are stated under the assumption of forward regularity, which guarantees the existence of the Lyapunov exponents as points. This property is hard to verify for specific problems. Hence, an alternative concept, the so-called exponential dichotomy spectrum, to assess the stability properties of the underlying system is discussed here.

Different spectral concepts for linear time varying systems are presented in literature. Exponential stability resulting from a negative Lyapunov spectrum in the general non-regular case is not robust with respect to small nonlinear perturbations of the system. Especially for the observer design, one may demand strong robustness properties of the underlying estimation error dynamics and hence uniform exponential stability of the estimation error dynamics is desirable. This chapter presents methods, which allow to achieve this as an extension of the concepts presented in Chapter 3.

As in Chapter 3, the linear time varying system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),\ \mathbf{x}(t) \in \mathbb{R}^n \tag{4.1a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t),\ \mathbf{y}(t) \in \mathbb{R}^p \tag{4.1b}$$

is considered for $t \in \mathbb{J} = [0, \infty)$ and the matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are assumed to be continuous and uniformly bounded.

4.1 Related Work and Contribution

A good overview of spectral concepts for linear time varying systems, their relations and their numerical approximation is given in [DV07]. The theoretical and algorithmic foundations for the numerical approximations of various spectra were laid by L. Dieci and E. S. Van Vleck in [DV95; DRV97; DV02; DV03; DV08; DJV10]. Uniform exponential stability is related to the so-called exponential dichotomy spectrum Σ_{ED} of the system. This spectrum was introduced by R. Sacker and G. Sell in [SS78] and is hence also known as Sacker-Sell spectrum.

If the system possesses an exponential dichotomy, there exists a splitting of the state space into (time-varying) stable and unstable subspaces. This is the fundamental idea of the detectability notion for systems with exponential dichotomy presented in [IM07] from a theoretical point of view. It basically states that the unstable subspace should be uniformly completely observable. The approach presented in this chapter is different and does not require the system to possess an exponential dichotomy in general. The presented concept relies on the exponential dichotomy spectrum, which is a generalization of the eigenvalue spectrum of time invariant systems. In contrast to the Lyapunov spectrum, this spectral concept is uniform with respect to the initial time. A new condition for *uniform exponential detectability* for a class of linear time varying systems allows to design a full or reduced order observer based on a *reduced order* Riccati differential equation. The resulting estimation error dynamics is uniformly exponentially stable. It is shown that for a class of systems with exponential dichotomy, the provided condition is *necessary and sufficient* for uniform exponential detectability and hence it is the minimum requirement for the observer design if one aims to obtain a uniformly exponentially stable error system. To the best of the author's knowledge, an observer design based on the exponential dichotomy spectrum has not been considered in the literature before.

The concept of an exponential dichotomy is recalled in Section 4.2. Important properties and results for systems in upper triangular form, which possess an exponential dichotomy are presented therein. Conditions for an exponential dichotomy of scalar systems, see Section 4.2.1, are the basis for the spectral concepts introduced in Section 4.3. The focus in this section is on different spectral concepts for systems in upper triangular form obtained via the continuous QR-decomposition. Methods for the numerical approximation of the exponential dichotomy spectrum, which will be exploited for the observer design, are introduced in Section 4.4. Section 4.5 combines results from the previous sections and presents the main contribution of this chapter, namely a detectability condition and an observer design technique based on the exponential dichotomy spectrum. Section 4.6 discusses the benefits and limitations of the proposed approach and points out possible future research directions. An overview of the contributions presented in this chapter is depicted in Fig. 4.1.

4.2 Exponential Dichotomy

The concept of an exponential dichotomy plays an important role in the subsequent derivations. An exponential dichotomy basically allows to split the state space into

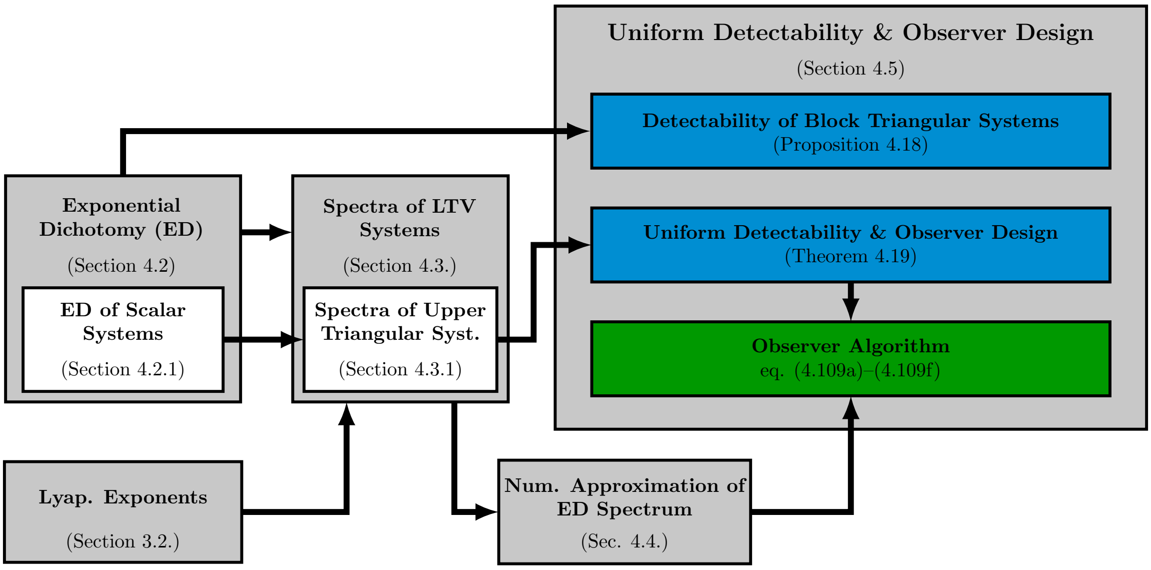


Fig. 4.1: Structure of Chapter 4.

two invariant subspaces, where all solutions in one subspace decay exponentially and uniformly with respect to the initial time, whereas the solutions in the other subspace grow uniformly with an exponential rate. This is stated formally in the following definition.

Definition 4.1 (exponential dichotomy, [DV07; DEV11; Cop78]) System (4.1a) admits an exponential dichotomy on $\mathbb{J}$ if there exists a projection matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$, i.e., a matrix such that $\mathbf{P}^2 = \mathbf{P}$, and constants $K \geq 1$ and $\alpha > 0$ such that

$$\|\mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(t_0)\| \leq Ke^{-\alpha(t-t_0)} \qquad \text{for } t \geq t_0 \geq 0; \tag{4.2a}$$

$$\|\mathbf{X}(t)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(t_0)\| \leq Ke^{\alpha(t-t_0)} \qquad \text{for } 0 \leq t \leq t_0, \tag{4.2b}$$

for some fundamental matrix solution $\mathbf{X}(t)$.

It should be remarked that any projection matrix of rank $k \leq n$ is similar to the projection matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{0}_{(n-k)\times(n-k)} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_k \end{bmatrix}, \tag{4.3}$$

see [Cop67]. One can immediately see that system (4.1a) is uniformly exponentially stable if and only if it admits an exponential dichotomy with $\mathbf{P} = \mathbf{I}_n$, because then (4.2a) reduces to a bound on the state transition matrix $\mathbf{\Phi}(t, t_0) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)$, which coincides with the definition of uniform exponential stability, see Definition 2.1 and Lemma 2.2.

The exponential dichotomy can be characterized in the following equivalent form

Lemma 4.2 (exponential dichotomy and subspaces)
System (4.1a) *has an exponential dichotomy with projection* $\mathbf{P}$ *and a corresponding fundamental matrix solution* $\mathbf{X}(t)$*, if and only if there exist constants* L, M, $N \geq 1$ *and* $\alpha > 0$ *such that*

$$\|\mathbf{X}(t)\mathbf{P}\boldsymbol{\xi}\| \leq Le^{-\alpha(t-t_0)}\|\mathbf{X}(t_0)\mathbf{P}\boldsymbol{\xi}\|, \tag{4.4a}$$

$$e^{\alpha(t-t_0)}\|\mathbf{X}(t_0)(\mathbf{I}-\mathbf{P})\boldsymbol{\xi}\| \leq M\|\mathbf{X}(t)(\mathbf{I}-\mathbf{P})\boldsymbol{\xi}\|, \tag{4.4b}$$

$$\|\mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(t)\| \leq N \tag{4.4c}$$

holds for any non-trivial $\boldsymbol{\xi} \in \mathbb{R}^n$ *and all* $0 \leq t_0 \leq t$.

Proof. Let system (4.1a) possess an exponential dichotomy and assume that $0 \leq t_0 \leq t$. By (4.2a) one has

$$\|\mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(t_0)\mathbf{v}(t_0)\| \leq Ke^{-\alpha(t-t_0)}\|\mathbf{v}(t_0)\| \tag{4.5}$$

for any $\mathbf{v}(t) \in \mathbb{R}^n$. Now, let $\mathbf{v}(t) = \mathbf{X}(t)\mathbf{P}\boldsymbol{\xi}$ for any constant vector $\boldsymbol{\xi} \in \mathbb{R}^n$. Hence, by utilizing $\mathbf{P}^2 = \mathbf{P}$, the relation

$$\|\mathbf{X}(t)\mathbf{P}\boldsymbol{\xi}\| \leq Ke^{-\alpha(t-t_0)}\|\mathbf{X}(t_0)\mathbf{P}\boldsymbol{\xi}\| \tag{4.6}$$

holds, which is relation (4.4a) for $L = K$. From (4.2b) and a swap of the variables t_0 and t, one has

$$\|\mathbf{X}(t_0)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(t)\mathbf{v}'(t)\| \leq Ke^{-\alpha(t-t_0)}\|\mathbf{v}'(t)\| \tag{4.7}$$

for any $\mathbf{v}'(t) \in \mathbb{R}^n$. By setting $\mathbf{v}'(t) = \mathbf{X}(t)(\mathbf{I}-\mathbf{P})\boldsymbol{\xi}$ with any constant vector $\boldsymbol{\xi} \in \mathbb{R}^n$, the inequality (4.4b) is obtained according to

$$e^{\alpha(t-t_0)}\|\mathbf{X}(t_0)(\mathbf{I}-\mathbf{P})\boldsymbol{\xi}\| \leq K\|\mathbf{X}(t)(\mathbf{I}-\mathbf{P})\boldsymbol{\xi}\| \tag{4.8}$$

with $M = K$. The third relation (4.4c) is directly obtained with $N = K$ by setting $t = t_0$ in (4.2a). To show that (4.4) implies (4.2), let $\boldsymbol{\xi}(t) = \mathbf{X}^{-1}(t)\mathbf{w}$ for any non-trivial vector $\mathbf{w} \in \mathbb{R}^n$. Then,

$$\|\mathbf{X}(t)\mathbf{P}(t)\boldsymbol{\xi}(t_0)\| \leq Le^{-\alpha(t-t_0)}\|\mathbf{X}(t_0)\mathbf{P}\boldsymbol{\xi}(t_0)\| \tag{4.9}$$

or equivalently

$$\|\mathbf{X}(t)\mathbf{P}(t)\mathbf{X}^{-1}(t_0)\mathbf{w}\| \leq Le^{-\alpha(t-t_0)}\|\mathbf{X}(t_0)\mathbf{P}\mathbf{X}^{-1}(t_0)\mathbf{w}\|, \tag{4.10}$$

which implies that

$$\|\mathbf{X}(t)\mathbf{P}(t)\mathbf{X}^{-1}(t_0)\| \leq LNe^{-\alpha(t-t_0)}. \tag{4.11}$$

Using the same $\boldsymbol{\xi}(t)$ in (4.4b), one gets

$$\|\mathbf{X}(t_0)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(t)\mathbf{w}\| \leq e^{-\alpha(t-t_0)}M\|\mathbf{X}(t)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(t)\mathbf{w}\|, \tag{4.12}$$

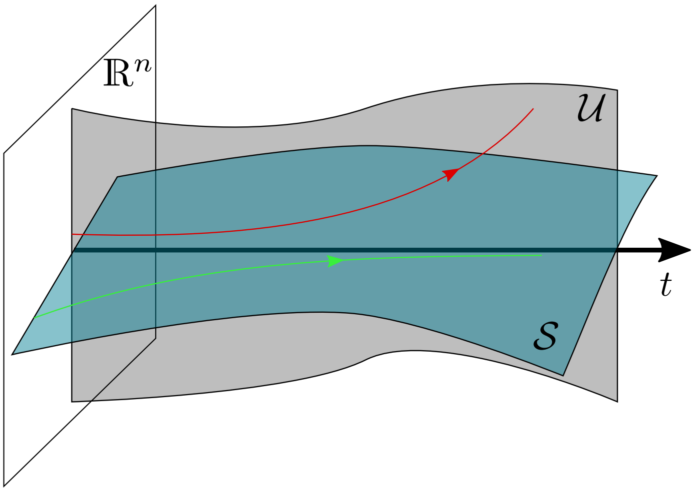


Fig. 4.2: Idea of exponential dichotomy.

where $\|\mathbf{X}(t)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(t)\| \leq N'$ holds for some $N' \geq 1$. Thus, one obtains

$$\|\mathbf{X}(t_0)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(t)\| \leq MN'e^{-\alpha(t-t_0)}, \tag{4.13}$$

for all $0 \leq t_0 \leq t$. Now, by setting $K = \max(LN, MN')$ and swapping back the time variables in (4.13), one obtains (4.2a) and (4.2b) from (4.9) and (4.13), respectively. □

The idea of an exponential dichotomy is sketched out in Figure 4.2. With the characterization from Lemma 4.2, the idea of an exponential dichotomy is pointed out in [Cop78] as follows. Under the assumption that the projection matrix $\mathbf{P}$ has rank k, condition (4.4a) states that there is a k-dimensional subspace $\mathcal{S}(t) = \operatorname{im}\mathbf{X}(t)\mathbf{P}$ of $\mathbb{R}^n$, where the solutions tend to zero uniformly and exponentially for $t \to \infty$. Condition (4.4b) states that there is a complementary $(n-k)$-dimensional subspace $\mathcal{U}(t)$ such that the solutions tend towards infinity uniformly and exponentially for $t \to \infty$.

An interesting result from [Cop78] shows implications of an exponential dichotomy in terms of the existence of bounded solutions of a perturbed system.

Lemma 4.3 (exponential dichotomy and bounded solutions, [Cop78])
Let the perturbed system corresponding to (4.1a) *be given by*

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t). \tag{4.14}$$

Assume that (4.1a) *possesses an exponential dichotomy with a projection matrix* $\mathbf{P}$ *and a corresponding fundamental matrix solution* $\mathbf{X}(t)$*. Then, for any bounded continuous function* $\mathbf{f}(t)$*, there exists at least one bounded solution of the perturbed equation* (4.14)*.*

The proof is given in Appendix A.2. Also the converse is true as shown in [Cop78, Proposition 2]. Another important property of systems with an exponential dichotomy is the so-called reducibility to block diagonal form [Cop67]. This property will be used in the following sections to obtain some insight regarding the proposed detectability concepts.

Definition 4.4 (reducibility to block diagonal form) System (4.1a) is reducible to block diagonal form with dimension k, if there exists a Lyapunov transformation $\mathbf{x}(t) = \mathbf{S}(t)\mathbf{z}(t)$, which transforms (4.1a) to the kinematically similar system

$$\dot{\mathbf{z}}(t) = \mathbf{D}(t)\mathbf{z}(t) \text{ with a block diagonal matrix } \mathbf{D}(t) = \begin{bmatrix} \mathbf{D}_1(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2(t) \end{bmatrix}, \tag{4.15}$$

where $\mathbf{D}_2(t)$ is a $k \times k$ matrix.

It can be assumed without loss of generality that the projection matrix $\mathbf{P}$ of a system with exponential dichotomy is already in the form of (4.3). A result presented in [Cop67, Lemma 2 and Lemma 3] is summarized in the next Lemma.

Lemma 4.5 (reducibility for systems with exponential dichotomy)
Let system (4.1a) *have an exponential dichotomy with* $\mathbf{P}$ *in the form of* (4.3) *and a corresponding fundamental matrix solution* $\mathbf{X}(t)$*. Then, there exists a Lyapunov transformation*

$$\mathbf{S}(t) = \mathbf{X}(t)\mathbf{T}^{-1}(t) \tag{4.16}$$

with a symmetric positive definite $\mathbf{T}(t)$ *such that*

$$\mathbf{T}^2(t) = \mathbf{P}\mathbf{X}^\mathsf{T}(t)\mathbf{X}(t)\mathbf{P} + (\mathbf{I} - \mathbf{P})\mathbf{X}^\mathsf{T}(t)\mathbf{X}(t)(\mathbf{I} - \mathbf{P}). \tag{4.17}$$

This transformation reduces (4.1a) *to the block diagonal system* (4.15) *with* $\mathbf{D}_2(t)$ *as a matrix of dimension* $k \times k$*. Moreover,* $\dot{\mathbf{z}}(t) = \mathbf{D}(t)\mathbf{z}(t)$ *has an exponential dichotomy with the same projection matrix* $\mathbf{P}$ *and the transformed coefficient matrix is given by*

$$\mathbf{D}(t) = \mathbf{S}^{-1}(t)\mathbf{A}(t)\mathbf{S}(t) - \mathbf{S}^{-1}(t)\dot{\mathbf{S}}(t). \tag{4.18}$$

For the two special cases $\mathbf{P} = \mathbf{0}$ and $\mathbf{P} = \mathbf{I}$ one gets the straightforward relation $\mathbf{D}(t) = \mathbf{A}(t)$ and hence these two trivial cases are neglected in the following. The system in block diagonal form is decoupled and the two independent systems

$$\dot{\mathbf{z}}_1(t) = \mathbf{D}_1(t)\mathbf{z}_1(t) \tag{4.19a}$$

and

$$\dot{\mathbf{z}}_2(t) = \mathbf{D}_2(t)\mathbf{z}_2(t) \tag{4.19b}$$

are of order $n - k$ and k, respectively. System (4.19a) is a so-called anti-stable system because it has an exponential dichotomy with projection $\mathbf{P}_1 = \mathbf{0}$ and the corresponding state transition matrix is bounded according to

$$\|\mathbf{\Phi}_1(t,t_0)\| \leq K e^{\alpha(t-t_0)} \quad \text{for } 0 \leq t \leq t_0, \tag{4.20}$$

which follows directly from Definition 4.1. Then, by again changing the variables t and t_0 and assuming that $0 \leq t_0 \leq t$, one gets

$$\|\mathbf{\Phi}_1(t_0,t)\| \leq K e^{-\alpha(t-t_0)} \quad \text{for } 0 \leq t_0 \leq t. \tag{4.21}$$

With $\mathbf{\Phi}_1(t_0,t) = \mathbf{\Phi}^{-1}(t,t_0)$ and the inequality

$$\frac{1}{\|\mathbf{\Phi}(t,t_0)\|} \leq \|\mathbf{\Phi}(t,t_0)\|, \tag{4.22}$$

a lower bound on the state transition matrix can be obtained according to

$$e^{\alpha(t-t_0)} \leq K\|\mathbf{\Phi}(t,t_0)\| \quad \text{for } 0 \leq t_0 \leq t \tag{4.23}$$

with a positive constant $\alpha > 0$.

The second system (4.19b) has an exponential dichotomy with $\mathbf{P}_2 = \mathbf{I}_k$ and hence it is uniformly exponentially stable. Next, it will be shown that this structure can also be exploited for systems in block triangular form.

Lemma 4.6 (exponential dichotomy of block triangular systems)
Let a system $\dot{\mathbf{x}}(t) = \mathbf{B}(t)\mathbf{x}(t)$ with a bounded $\mathbf{B}(t) \in \mathbb{R}^{n\times n}$ have a block triangular structure partitioned according to

$$\begin{bmatrix}\dot{\mathbf{x}}_1(t)\\ \dot{\mathbf{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\mathbf{B}_{11}(t) & \mathbf{B}_{12}(t)\\ \mathbf{0} & \mathbf{B}_{22}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}. \tag{4.24}$$

The block matrices $\mathbf{B}_{11}(t)$, $\mathbf{B}_{12}(t)$ and $\mathbf{B}_{22}(t)$ are of dimension $(n-k)\times(n-k)$, $(n-k)\times k$ and $k \times k$, respectively, with $1 \leq k \leq n-1$. It is assumed that $\dot{\mathbf{x}}_1(t) = \mathbf{B}_{11}(t)\mathbf{x}(t)$ has an exponential dichotomy with $\mathbf{P}_1 = \mathbf{0}$ and $\dot{\mathbf{x}}_2(t) = \mathbf{B}_{22}\mathbf{x}_2(t)$ has an exponential dichotomy with $\mathbf{P}_2 = \mathbf{I}_k$. Then, system (4.24) has an exponential dichotomy with the projection

$$\mathbf{P} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_k\end{bmatrix}. \tag{4.25}$$

Moreover, it is reducible to the block diagonal form (4.15) with $\mathbf{D}_1(t) = \mathbf{B}_{11}(t)$.

The proof can be found in Appendix A.3.

4.2.1 Exponential Dichotomy of Scalar Systems

For a scalar system

$$\dot{x}(t) = a(t)x(t), \qquad t \in \mathbb{J}, \tag{4.26}$$

the state transition matrix can be given explicitly as

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau}, \qquad t,\, t_0 \in \mathbb{J}, \tag{4.27}$$

see Section 2.1. In this case, system (4.26) possesses an exponential dichotomy if and only if there exists an $\alpha > 0$ and $K \geq 1$ such that (4.2a) holds for $P = 1$ or (4.2b) holds for $P = 0$. Now, let (4.2a) hold for the scalar system (4.26). Hence, the system is uniformly exponentially stable and the state transition matrix can be bounded according to

$$e^{\int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau} \leq K e^{-\alpha(t-t_0)}. \tag{4.28}$$

for all $t \geq t_0 \geq 0$. Taking the logarithm on both sides of the inequality and substituting t by $t_0 + T$ with some $T \geq 0$ one obtains

$$\int_{t_0}^{t_0+T} a(\tau)\,\mathrm{d}\tau \leq -\alpha T + \ln K. \tag{4.29}$$

For a positive interval $T > 0$, the relation can be rewritten as

$$\frac{1}{T}\int_{t_0}^{t_0+T} a(\tau)\,\mathrm{d}\tau \leq -\alpha + \frac{\ln K}{T} \text{ for all } t_0 \in \mathbb{J} \text{ and all } T > 0. \tag{4.30}$$

Hence, taking the supremum over all t_0 and assuming that $T \to \infty$ the inequality still holds and

$$\beta_1 = \limsup_{T\to\infty} \sup_{t_0 \in \mathbb{J}} \frac{1}{T}\int_{t_0}^{t_0+T} a(\tau)\,\mathrm{d}\tau \leq -\alpha < 0, \tag{4.31}$$

with β_1 as the so-called upper Bohl exponent of the scalar system (4.26), see also [DKS02, Chapter 3].

On the other hand, let (4.26) possess an exponential dichotomy with $P = 0$ and hence (4.2b) holds and

$$\Phi(t_0, t) = e^{\int_{t}^{t_0} a(\tau)\,\mathrm{d}\tau} \leq K e^{-\alpha(t-t_0)} \text{ for all } t \geq t_0 \geq 0. \tag{4.32}$$

Taking again the logarithm on both sides of the latter equation, substituting $t = t_0 + T$ and swapping the integration limits gives

$$-\int_{t_0}^{t_0+T} a(\tau)\,\mathrm{d}\tau \leq -\alpha T + \ln K \tag{4.33}$$

or equivalently

$$\int_{t_0}^{t_0+T} a(\tau)\,\mathrm{d}\tau \geq \alpha T - \ln K \text{ for all } t_0 \in \mathbb{J} \text{ and all } T \geq 0. \tag{4.34}$$

As before, dividing by $T > 0$, taking the infimum over all t_0 and assuming that $T \to \infty$, one obtains

$$\alpha_1 = \liminf_{T\to\infty} \inf_{t_0\in\mathbb{J}} \int_{t_0}^{t_0+T} a(\tau)\,\mathrm{d}\tau \geq \alpha > 0. \tag{4.35}$$

The quantity α_1 is the so-called lower Bohl exponent of (4.26), see [DKS02, Chapter 3] and [BK01].

In fact, the upper and lower Bohl exponents of the scalar equation entirely characterize the exponential dichotomy as summarized in the following, see also [DKS02; BP15]

Lemma 4.7 (exponential dichotomy of a scalar system)
Let the lower and upper Bohl exponent of system (4.26) *be given by*

$$\alpha_1 = \liminf_{T\to\infty} \inf_{t_0\in\mathbb{J}} \int_{t_0}^{t_0+T} a(\tau)\,\mathrm{d}\tau \text{ and } \beta_1 = \limsup_{T\to\infty} \sup_{t_0\in\mathbb{J}} \frac{1}{T} \int_{t_0}^{t_0+T} a(\tau)\,\mathrm{d}\tau, \tag{4.36}$$

respectively. The system possesses an exponential dichotomy if and only if

$$0 < \alpha_1 \leq \beta_1 \quad \text{or} \quad \alpha_1 \leq \beta_1 < 0. \tag{4.37}$$

This property is useful for obtaining the so-called exponential dichotomy spectrum presented in the following section.

4.3 Spectra of Linear Time Varying Systems

In order to show the relation between different spectral concepts, the so-called Lyapunov spectrum for the case of non-regular systems as introduced in [DV03] is discussed first. Let λ_i be the (sorted) upper Lyapunov exponents of (4.1a) as defined in Section 3.2.1. Now, consider the adjoint system

$$\dot{\mathbf{y}}(t) = -\mathbf{A}^{\mathsf{T}}(t)\mathbf{y}(t) \tag{4.38}$$

of (4.1a), see [Adr95, Ch. 1]. Let $-\mu_i$ be the (sorted) upper Lyapunov exponents of (4.38). The μ_i are also called lower Lyapunov exponents of (4.1a) and are ordered according to $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ with $\lambda_i \geq \mu_i$. For forward regular systems, it holds that $\mu_i = \lambda_i$ for $i = 1, \ldots, n$, i.e., the upper and lower Lyapunov exponents coincide and the Lyapunov spectrum reduces to points [DV03]. This is not the case in general, and hence a continuous spectrum, the Lyapunov spectrum is introduced in [DV03]. It is defined as

$$\Sigma_L \coloneqq \bigcup_{j=1}^{n} [\mu_j, \lambda_j]. \tag{4.39}$$

It is shown in [DV07, Theorem 6.2] that the Lyapunov spectrum can be obtained from the diagonal elements of the system transformed to upper triangular form if the Lyapunov exponents are stable, i.e., the exponents of the perturbed equation

$$\dot{\boldsymbol{\xi}} = [\mathbf{A}(t) + \mathbf{F}(t)]\,\boldsymbol{\xi} \tag{4.40}$$

depend continuously on the perturbation $\mathbf{F}(t)$.

Definition 4.8 (stability of the Lyapunov exponents, [Adr95]) The upper (lower) Lyapunov exponents of system (4.1a) are stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{\substack{t \in \mathbb{J} \\ \|\mathbf{x}\|=1}} \|\mathbf{F}(t)\mathbf{x}\| < \delta \text{ implies that } |\tilde{\lambda}_i - \tilde{\lambda}'_i| < \varepsilon \tag{4.41}$$

for $i = 1, \ldots, n$ with $\tilde{\lambda}_i$ and $\tilde{\lambda}'_i$ as the upper (lower) Lyapunov exponents of the original system (4.1a) and the perturbed system (4.40), respectively.

Note that the exponents of scalar linear systems are always stable and stable exponents do not change under vanishing perturbations, i.e., $\|\mathbf{F}(t)\| \to 0$ implies $\lambda_i = \lambda'_i$, see [Adr95, Theorem 5.2.1].

A concept, which guarantees stability of the Lyapunov exponents is the so-called integral separateness.

Definition 4.9 (integral separation, [Adr95]) The bounded continuous functions $f(t)$ and $g(t)$ are integrally separated if there exist constants $a > 0$ and d such that

$$\int_s^t [f(\tau) - g(\tau)]\,\mathrm{d}\tau \geq a(t-s) - d,\ t \geq s. \tag{4.42}$$

for all $t \geq s \geq 0$ and $i = 1, \ldots, n-1$.

It should be remarked that in this definition, the ordering is important. Integral separation is also an important property for fundamental matrix solutions.

Definition 4.10 (integrally separated fundamental solution, [Adr95]) The fundamental matrix solution $\mathbf{X}(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \ldots & \mathbf{x}_n(t) \end{bmatrix}$ is integrally separated if there exist constants $a > 0$ and $d > 0$ such that

$$\frac{\|\mathbf{x}_i(t)\|}{\|\mathbf{x}_i(s)\|} \frac{\|\mathbf{x}_{i+1}(s)\|}{\|\mathbf{x}_{i+1}(t)\|} \geq d e^{a(t-s)} \tag{4.43}$$

holds for $i = 1, \dots, n-1$ and all $t, s \in \mathbb{J}$ such that $t \geq s$.

If the system possesses distinct Lyapunov exponents, they are stable if and only if there exists an integrally separated fundamental matrix solution [Adr95, Theorem 5.4.8]. Moreover, integral separateness of all diagonal elements $b_{ii}(t)$ of a system in upper triangular form guarantees stable Lyapunov exponents and is a standard assumption in works on the numerical approximation of the Lyapunov spectrum [DV07].

It should be remarked that integral separation is a generic property of linear systems with bounded and continuous coefficient matrices as argued in [Pal79; DE06; DV08]. This motivates the use of integral separateness as a standard assumption if one seeks to approximate the Lyapunov spectrum via the diagonal elements of $\mathbf{B}(t)$, although it might by hard to verify, especially if some Lyapunov exponents are close to each other. This issue is discussed in detail by means of numerical examples in [DV03].

As an alternative to the Lyapunov spectrum, a spectrum based on an exponential dichotomy of the shifted system was first introduced by Sacker and Sell in [SS78].

Definition 4.11 (exponential dichotomy spectrum) The exponential dichotomy spectrum Σ_{ED} of the system $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ is the set of all values $\mu \in \mathbb{R}$ for which the shifted system $\dot{\mathbf{y}} = [\mathbf{A}(t) - \mu\mathbf{I}]\mathbf{y}(t)$ does not admit an exponential dichotomy.

Following this definition, the system itself possesses an exponential dichotomy if 0 is not contained in Σ_{ED}. The exponential dichotomy spectrum is given by a collection of $k \leq n$ disjoint closed intervals

$$\Sigma_{ED} \coloneqq [\bar{a}_1, \bar{b}_1] \cup \dots \cup [\bar{a}_k, \bar{b}_k] \tag{4.44}$$

with $\bar{b}_1 \geq \bar{a}_1 > \bar{b}_2 \geq \bar{a}_2 > \dots > \bar{b}_k \geq \bar{a}_k$. The Lyapunov spectrum is contained in Σ_{ED}, i.e., $\Sigma_L \subseteq \Sigma_{ED}$, see [SS78, Theorem 4] or [DV03, Theorem 6.2]. If the spectral intervals reduce to points, the system possesses a so-called point spectrum. This also implies that $\Sigma_L = \Sigma_{ED}$. Important classes of systems with point spectra are periodic and linear time invariant systems.

Every system in the form of (4.1a) can be transformed to an upper triangular system by means of the continuous QR-decomposition as already discussed in Chapter 3. For upper triangular systems, the exponential dichotomy spectrum can be obtained by only considering the diagonal elements. Hence, upper triangular systems play a crucial role for the numerical approximation of spectral intervals. This will be discussed in detail in the following.

4.3.1 Spectra of Upper Triangular Systems

The relation between the Lyapunov spectrum and the exponential dichotomy spectrum is now discussed for a system in upper triangular form

$$\dot{\mathbf{z}}(t) = \mathbf{B}(t)\mathbf{z}(t) \tag{4.45}$$

with

$$\mathbf{B}(t) = \begin{bmatrix} b_{11}(t) & b_{12}(t) & \cdots & b_{1n}(t) \\ 0 & b_{22}(t) & \cdots & b_{2n}(t) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & b_{nn}(t) \end{bmatrix}. \tag{4.46}$$

It is assumed that this upper triangular system is obtained from (4.1a) via a Lyapunov transformation $\mathbf{z}(t) = \mathbf{Q}^\mathsf{T}\mathbf{x}(t)$. The matrix $\mathbf{Q}(t)$ is the solution of the continuous QR-decomposition, where the initial condition $\mathbf{X}(0) = \mathbf{Q}(0)\mathbf{R}(0)$ forms an ordered normal Lyapunov basis, see Section 3.2.2.

For forward regular systems, the Lyapunov spectrum reduces to points, which can be obtained from the diagonal of $\mathbf{B}(t)$, see Section 3.2.2. This is not possible in general. Hence, the computed Lyapunov spectrum

$$\Sigma_{CL} = \bigcup_{j=1}^{n} [\lambda_j^{\mathrm{i}}, \lambda_j^{\mathrm{s}}] \tag{4.47}$$

with

$$\lambda_j^{\mathrm{i}} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t b_{jj}(\tau)\,\mathrm{d}\tau \quad \text{and} \quad \lambda_j^{\mathrm{s}} = \limsup_{t\to\infty} \frac{1}{t}\int_0^t b_{jj}(\tau)\,\mathrm{d}\tau \tag{4.48}$$

is introduced in [DV07]. It is argued in the proof of [DV07, Theorem 6.3] that in general, it only holds that $\Sigma_{CL} \subseteq \Sigma_L$ because Σ_L is a function of the entire fundamental matrix, while Σ_{CL} is a function of the diagonal of an upper triangular fundamental matrix solution. However, if the Lyapunov exponents are stable, the spectrum can be obtained via the diagonal elements of $\mathbf{B}(t)$ and $\Sigma_{CL} = \Sigma_L$, see [DV07, Theorem 6.2]. Stable exponents are guaranteed by integral separation of the diagonal elements $b_{ii}(t)$ as introduced in Definition 4.9.

According to [DPR16, Proposition 5], the exponential dichotomy spectrum can always be obtained from the diagonal of $\mathbf{B}(t)$ without requiring forward regularity or stability of the Lyapunov exponents. The exponential dichotomy spectrum is entirely characterized[1] by the lower and upper Bohl exponents of the scalar systems

$$\dot{\xi}(t) = b_{jj}(t)\xi(t) \tag{4.49}$$

[1]Note that this is not true on the whole real time axis $\mathbb{J} = \mathbb{R}$, see [BP15].

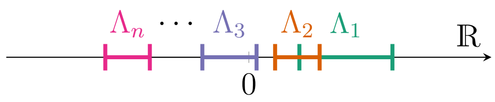


Fig. 4.3: The spectral Intervals Λ_j of Σ_{ED} might be overlapping.

defined by

$$\alpha_j = \liminf_{t\to\infty} \left[\inf_{t_0\in\mathbb{J}} \frac{1}{t} \int_{t_0}^{t_0+t} b_{jj}(\tau)\,\mathrm{d}\tau \right] \tag{4.50}$$

and

$$\beta_j = \limsup_{t\to\infty} \left[\sup_{t_0\in\mathbb{J}} \frac{1}{t} \int_{t_0}^{t_0+t} b_{jj}(\tau)\,\mathrm{d}\tau \right]. \tag{4.51}$$

for $j = 1, \ldots, n$, see Section 4.2.1.

The exponential dichotomy spectrum Σ_{ED} is then given by

$$\Sigma_{ED} = \bigcup_{j=1}^{n} \Lambda_j \text{ with } \Lambda_j = [\alpha_j, \beta_j], \tag{4.52}$$

which is a direct consequence of Lemma 4.7 and [DPR16, Proposition 5]. The intervals Λ_j might overlap as sketched out in Figure 4.3. The union of the intervals Λ_j gives the collection of at most n compact disjoint intervals as introduced in (4.44).

A different yet equivalent characterization of the exponential dichotomy spectrum is given in [DV02; DV07] in terms of integral separation of the auxiliary systems

$$\dot{\boldsymbol{\eta}}_j = \begin{bmatrix} \lambda & 0 \\ 0 & b_{jj}(t) \end{bmatrix} \boldsymbol{\eta}_j, \quad j = 1, \ldots, n; \tag{4.53}$$

and

$$\dot{\boldsymbol{\zeta}}_j = \begin{bmatrix} b_{jj}(t) & 0 \\ 0 & \lambda \end{bmatrix} \boldsymbol{\zeta}_j, \quad j = 1, \ldots, n \tag{4.54}$$

with some $\lambda \in \mathbb{R}$.

It is shown in [DV02, Theorem 2.29], that the systems (4.53) and (4.54) are both not integrally separated if and only if $\lambda \in \Lambda_j$ and hence

$$\Lambda_j = \{\lambda \in \mathbb{R} : \text{(4.53) and (4.54) both are not integrally separated}\}. \tag{4.55}$$

In particular, (4.53) is integrally separated if there exist constants $a > 0$ and d such that

$$\int_{t_0}^{t} [\lambda - b_{jj}(\tau)]\,\mathrm{d}\tau \geq a(t - t_0) - d \tag{4.56}$$

Table 4.1: Endpoints of spectral intervals.

	lower endpoints	upper endpoints
Σ_{CL}	$\lambda_j^{\mathrm{i}} = \liminf_{t\to\infty} \frac{1}{t}\int_0^t b_{jj}(\tau)\,\mathrm{d}\tau$	$\lambda_j^{\mathrm{s}} = \limsup_{t\to\infty} \frac{1}{t}\int_0^t b_{jj}(\tau)\,\mathrm{d}\tau$
Σ_{ED}	$\alpha_j = \liminf_{t\to\infty} \left[\inf_{t_0\in\mathbb{J}} \frac{1}{t}\int_{t_0}^{t_0+t} b_{jj}(\tau)\,\mathrm{d}\tau\right]$	$\beta_j = \limsup_{t\to\infty} \left[\sup_{t_0\in\mathbb{J}} \frac{1}{t}\int_{t_0}^{t_0+t} b_{jj}(\tau)\,\mathrm{d}\tau\right]$

for all $t \geq t_0 \geq 0$ whereas (4.54) is integrally separated if

$$\int_{t_0}^{t} [b_{jj}(\tau) - \lambda]\,\mathrm{d}\tau \geq \bar{a}(t - t_0) - \bar{d} \tag{4.57}$$

holds for some constants $\bar{a} > 0$ and $\bar{d}$ and for all $t \geq t_0 \geq 0$. This characterization is useful for the numerical approximation of the exponential dichotomy spectrum as discussed in the following section.

For general systems in the form of (4.1a), one has the relation

$$\Sigma_{CL} \subseteq \Sigma_L \subseteq \Sigma_{ED}, \tag{4.58}$$

see [DV07]. If the exponential dichotomy spectrum of (4.1a) is negative, or equivalently, $\max \Sigma_{ED} < 0$, then the system possesses an exponential dichotomy with $\mathbf{P} = \mathbf{I}$ and hence it is uniformly exponentially stable.

Ordering of Spectral Intervals If the upper triangular form (4.45) is obtained via the continuous QR decomposition initialized with an ordered normal Lyapunov basis, then a certain ordering of the spectral intervals can be expected, even though the specific spectral intervals might not be disjoint. The endpoints of intervals for the discussed spectral concepts are summarized in Table 4.1. From these endpoints it follows that

$$\alpha_j \leq \lambda_j^{\mathrm{i}} \leq \lambda_j^{\mathrm{s}} \leq \beta_j. \tag{4.59}$$

For systems with point spectrum, one obtains

$$\alpha_j = \lambda_j^{\mathrm{i}} = \lambda_j^{\mathrm{s}} = \beta_j. \tag{4.60}$$

If the Lyapunov exponents of (4.1a) are stable, the lower and upper Lyapunov exponents correspond to λ_j^{i} and λ_j^{s}, respectively. Then, the endpoints of the spectral intervals of $\Sigma_L = \Sigma_{CL}$ are ordered according to

$$\lambda_1^{\mathrm{s}} \geq \lambda_2^{\mathrm{s}} \geq \cdots \geq \lambda_n^{\mathrm{s}} \text{ and } \lambda_1^{\mathrm{i}} \geq \lambda_2^{\mathrm{i}} \geq \cdots \geq \lambda_n^{\mathrm{i}} \tag{4.61}$$

in addition to $\lambda_j^{\mathrm{i}} \leq \lambda_j^{\mathrm{s}}$. Such an ordering cannot be guaranteed for the exponential dichotomy spectrum in general. However, the upper and lower Lyapunov exponents are contained within the exponential dichotomy spectral intervals. This suggests that especially for systems with an exponential dichotomy, a certain ordering of the spectral intervals may be expected. If the system has an exponential dichotomy, zero cannot be included in the spectral intervals Λ_j. However, the upper and lower Lyapunov exponents λ_j^{s} and λ_j^{i} are contained within the Λ_js. Hence, the number of positive upper Bohl exponents has to be equal to the number of positive upper Lyapunov exponents.

4

4.4 Numerical Approximation of Σ_{ED}

This section provides tools for the numerical approximation of the spectral intervals Λ_j via the diagonal elements of $\mathbf{B}(t)$. Let the quantities

$$\alpha_j^H = \inf_t \frac{1}{H} \int_t^{t+H} b_{jj}(\tau)\mathrm{d}\tau \text{ and } \beta_j^H = \sup_t \frac{1}{H} \int_t^{t+H} b_{jj}(\tau)\mathrm{d}\tau, \tag{4.62}$$

be defined for some $H > 0$. These quantities form the so-called integral separation spectrum

$$\Sigma_{IS}^H = \bigcup_{j=1}^{n} [\alpha_i^H, \beta_i^H]. \tag{4.63}$$

It is stated in [DV03, Theorem 8.4] and [DV07, Theorem 2.8] that for any $H > 0$, $\Lambda_j = [\alpha_j, \beta_j] \subseteq [\alpha_j^H, \beta_j^H]$. It is furthermore claimed that for $H > 0$ sufficiently large it holds that $[\alpha_j^H, \beta_j^H] \subseteq \Lambda_j$ and hence $[\alpha_j^H, \beta_j^H] = \Lambda_j$. The latter statement is not correct. The following counterexample shows that the existence of a (finite) $H_0 > 0$ such that $[\alpha_j^H, \beta_j^H] \subseteq \Lambda_j$ for $H \geq H_0$ cannot be guaranteed.

Example 4.12*:* Consider the scalar system

$$\dot{x} = \frac{1}{1+t} x \tag{4.64}$$

which admits the fundamental solution $X(t) = (1 + t)$. The exponential dichotomy spectrum is $\Sigma_{ED} = \{0\}$ and so $\alpha_1 = \beta_1 = 0$. The computation of β_1^H gives

$$\begin{aligned} \beta_1^H &= \sup_t \frac{1}{H} \int_t^{t+H} \frac{1}{1+\tau} \mathrm{d}\tau \\ &= \sup_t \frac{1}{H} \ln(1+\tau)\Big|_t^{t+H} \\ &= \sup_t \frac{1}{H} \ln(1 + \frac{H}{1+t}) \\ &= \frac{1}{H} \ln(1+H) \end{aligned} \tag{4.65}$$

with the supremum at $t = 0$. Hence, for any finite $H > 0$ it holds that $\beta_1^H > 0 = \beta_1$.

It is still theoretically possible to approximate Σ_{ED} up to any desired accuracy by making $H > 0$ sufficiently large. This is summarized in the following result.

Proposition 4.13 (approximation of the spectral intervals)
Let α_j^H and β_j^H be defined as in (4.62). Then, for every $H > 0$ it holds that

$$\alpha_j^H \leq \alpha_j \leq \beta_j \leq \beta_j^H. \tag{4.66}$$

Moreover, for every $\varepsilon > 0$ there exists an $H_0 > 0$ such that for all $H \geq H_0$ it holds that

$$\begin{aligned} \alpha_j - \varepsilon \leq \alpha_j^H &\leq \alpha_j \\ \beta_j \leq \beta_j^H &\leq \beta_j + \varepsilon. \end{aligned} \tag{4.67}$$

Proof. The relation (4.66) for any $H > 0$ follows from the first part of the proof of [DV03, Theorem 8.4]. The second part of the proof follows ideas of the second part of the proof in [DV03, Theorem 8.4], but it shows that for a finite H, Λ_j can be approximated merely up to an ε distance. To see this, select $\lambda = \beta_j + \varepsilon \notin \Lambda_j$. With this choice, (4.53) is integrally separated and there exist constants $a > 0$ and d such that

$$\int_s^t [(\beta_j + \varepsilon) - b_{jj}(\tau)] \,\mathrm{d}\tau \geq a(t-s) - d, \; t \geq s. \tag{4.68}$$

One can select an H_0 sufficiently large such that $a - d/H_0 > a/2$ and it then holds that

$$\frac{1}{H}\int_t^{t+H} (\beta_j + \varepsilon) - b_{jj}(\tau)\mathrm{d}\tau \geq a - \frac{d}{H} > \frac{a}{2} \tag{4.69}$$

for any $H \geq H_0$. Rewriting (4.69) according to

$$\frac{1}{H}\int_t^{t+H} (\beta_j + \varepsilon)\,\mathrm{d}\tau - \frac{1}{H}\int_t^{t+H} b_{jj}(\tau)\,\mathrm{d}\tau = \beta_j + \varepsilon - \beta_j^H > \frac{a}{2} \tag{4.70}$$

reveals that $\beta_j + \varepsilon > \beta_j^H$ for any $H \geq H_0$.

For any $\lambda > \beta_j + \varepsilon$, system (4.53) is still integrally separated with

$$\int_s^t [\lambda - b_{jj}(\tau)]\,\mathrm{d}\tau \geq (a + \lambda - \beta_j - \varepsilon)(t-s) - d, \; t \geq s \tag{4.71}$$

and moreover

$$\frac{1}{H_0}\int_t^{t+H_0} [\lambda - b_{jj}(\tau)]\,\mathrm{d}\tau \geq (a + \lambda - \beta_j - \varepsilon) - \frac{d}{H_0} \geq a - \frac{d}{H_0} > \frac{a}{2}. \tag{4.72}$$

Therefore, $\lambda > \beta_j^H$ for $H \geq H_0$ and $\lambda \geq \beta_j + \varepsilon$. Analogous arguments hold for the lower bound $a_j - \varepsilon$. □

In order to compute the continuous-time moving average[2] required in (4.62), the

[2]This moving average is called Steklov average in [DV03]

quantities $v_j(t)$ governed by the differential equations

$$\dot{v}_j(t) = b_{jj}(t) - b_{jj}(t-H),\ v_j(0) = 0 \tag{4.73}$$

with $b_{jj}(t) = 0$ for $t < 0$ are defined and one has the relation

$$\alpha_j^H = \inf_{t\geq 0} \frac{1}{H} v_j(t+H) \text{ and } \beta_j^H = \sup_{t\geq 0} \frac{1}{H} v_j(t+H). \tag{4.74}$$

These quantities can be evaluated on-line using a suitable numerical integration scheme. To numerically approximate (4.74), the infimum and supremum are approximated by the minimum and maximum up to a finite time instant T_f for a given $H > 0$. If one is merely interested to assess uniform exponential stability rather than to obtain the whole spectral information, it suffices to approximate the β_i^Hs and check if they are negative for a sufficiently large H. Of course this is only possible up to a sufficiently large, but still finite time $t \leq T_f$.

Remark 4.14*:* In [DV07], another spectrum is introduced which is essentially obtained by α_j^H and β_j^H for $H \to 0$. This results in

$$\alpha_j^0 = \inf_t b_{jj}(t) \text{ and } \beta_j^0 = \sup_t b_{jj}(t). \tag{4.75}$$

These quantities can be approximated by the minimum and maximum of the diagonal elements of $\mathbf{B}(t)$ and it holds that $\alpha_j^0 \leq \alpha_j^H \leq \beta_j^H \leq \beta_j^0$ for any $H > 0$.

4.5 Uniform Detectability and Observer Design

In this section, a uniform notion of detectability is considered. It is shown that the presented conditions are related to observability on a subspace. In contrast to Chapter 3, the goal is now to obtain a uniformly exponentially stable observer error system. Let the considered detectability notion be introduced in the next definition.

> **Definition 4.15** (uniform exponential detectability) System (4.1) is called uniformly exponentially detectable if there exists a uniformly bounded output feedback gain $\mathbf{L}(t)$ such that the system
>
> $$\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{e}(t) \tag{4.76}$$
>
> is uniformly exponentially stable.

Before the main result of this chapter is presented, some important and insightful aspects for systems, which possess an exponential dichotomy are discussed. It will be shown that for this system class, uniform complete observability of the unstable subspace is necessary and sufficient for uniform exponential detectability. For this, the result presented in [ZZ15, Lemma 1] is needed, which states that uniform complete observability is preserved under output feedback.

Lemma 4.16 (uniform complete observability under output feedback, [ZZ15])
The pair $(\mathbf{A}(t), \mathbf{C}(t))$ *is uniformly completely observable if and only if for any bounded and integrable matrix* $\mathbf{L}(t)$*, the pair* $(\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t), \mathbf{C}(t))$ *is uniformly completely observable.*

Uniform complete observability is equivalent to the existence of constants β_1, β_2 and σ such that the observability Gramian

$$\mathbf{M}(t_0 + \sigma, t_0) = \int_{t_0}^{t_0+\sigma} \mathbf{\Phi}^\mathsf{T}(s, t_0)\mathbf{C}^\mathsf{T}(s)\mathbf{C}(s)\mathbf{\Phi}(s, t_0)\,\mathrm{d}s \tag{4.77}$$

satisfies

$$\beta_1\mathbf{I}_n \preceq \mathbf{M}(t_0 + \sigma, t_0) \preceq \beta_2\mathbf{I}_n. \tag{4.78}$$

for all $t_0 \in \mathbb{J}$, see also Section 2.4. According to Lemma 4.16 and [ZZ15, Lemma 2], this also guarantees the existences of constants $\bar{\beta}_1$ and $\bar{\beta}_2$ such that

$$\bar{\beta}_1\mathbf{I}_n \preceq \int_{t_0}^{t_0+\sigma} \mathbf{\Phi}_e^\mathsf{T}(s, t_0)\mathbf{C}^\mathsf{T}(s)\mathbf{C}(s)\mathbf{\Phi}_e(s, t_0)\,\mathrm{d}s \preceq \bar{\beta}_2\mathbf{I}_n \tag{4.79}$$

holds for the same σ and for all $t_0 \in \mathbb{J}$, where the matrix $\mathbf{\Phi}_e(\cdot, \cdot)$ is the state transition matrix of (4.76).

The next result deals with anti-stable systems, i.e., systems in the form of (4.1a), which possess an exponential dichotomy with $\mathbf{P} = \mathbf{0}$.

Proposition 4.17 (uniform exponential detectability of anti-stable systems)
Let (4.1a) *possess an exponential dichotomy with* $\mathbf{P} = \mathbf{0}$*. Then, system* (4.1) *is uniformly exponentially detectable if and only if it is uniformly completely observable.*

Proof. Sufficiency follows directly from the observer design presented in Section 2.4.2. For necessity, let the system be uniformly exponentially detectable but not uniformly completely observable. With the aid of the variational equation, the solution of (4.76) can be stated as

$$\mathbf{e}(t) = \mathbf{\Phi}(t, t_0)\mathbf{e}(t_0) - \int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{L}(s)\mathbf{C}(s)\mathbf{\Phi}_e(s, t_0)\mathbf{e}(t_0)\,\mathrm{d}s. \tag{4.80}$$

Hence, the relation

$$\mathbf{\Phi}_e(t_2, t_1) = \mathbf{\Phi}(t_2, t_1) - \int_{t_1}^{t_2} \mathbf{\Phi}(t_2, s)\mathbf{L}(s)\mathbf{C}(s)\mathbf{\Phi}_e(s, t_1)\,\mathrm{d}s \tag{4.81}$$

holds for all t_1, t_2. Multiplication of (4.81) with $\mathbf{\Phi}(t_1, t_2)$ gives

$$\mathbf{\Phi}(t_1, t_2)\mathbf{\Phi}_e(t_2, t_1) = \mathbf{I} - \int_{t_1}^{t_2} \mathbf{\Phi}(t_1, s)\mathbf{L}(s)\mathbf{C}(s)\mathbf{\Phi}_e(s, t_1)\,\mathrm{d}s. \tag{4.82}$$

For some $s \geq t_1$, one can bound $\mathbf{\Phi}(t_1, t_2)$ according to

$$\|\mathbf{\Phi}(t_1, s)\| \leq Ke^{-\alpha(s-t_1)} \leq K \tag{4.83}$$

for some $K,\ \alpha > 0$, see (4.2b). By the uniform exponential detectability assumption, system (4.76) is uniformly exponentially stable for a bounded $\|\mathbf{L}(t)\| \leq K_L$ and

$$\|\mathbf{\Phi}_e(t_2, t_1)\| \leq K_e e^{-\mu(t_2-t_1)} \tag{4.84}$$

holds for all $t_2 \geq t_1$ and some $\mu > 0$ and $K_e \geq 1$. For any vector $\boldsymbol{\xi}$ of appropriate dimension, the inequality

$$KK_e\|\boldsymbol{\xi}\|e^{-\mu(t_2-t_1)} \geq \|\boldsymbol{\xi}\| - KK_L \int_{t_1}^{t_2} \|\mathbf{C}(s)\mathbf{\Phi}_e(s, t_1)\boldsymbol{\xi}\|\,\mathrm{d}s \tag{4.85}$$

is fulfilled. The assumption that the pair $(\mathbf{A}(t), \mathbf{C}(t))$ is not uniformly completely observable implies that $(\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t), \mathbf{C}(t))$ is also not uniformly completely observable. Hence, for any $\sigma > 0$ and any $\rho > 0$ there exists a non-trivial vector $\boldsymbol{\eta}$ such that for some t_0 it holds that

$$\boldsymbol{\eta}^\mathsf{T} \int_{t_0}^{t_0+\sigma} \mathbf{\Phi}_e^\mathsf{T}(s, t_0)\mathbf{C}^\mathsf{T}(s)\mathbf{C}(s)\mathbf{\Phi}_e(s, t_0)\,\mathrm{d}s\boldsymbol{\eta} < \rho\|\boldsymbol{\eta}\|^2 \tag{4.86}$$

or equivalently

$$\int_{t_0}^{t_0+\sigma} \|\mathbf{C}(s)\mathbf{\Phi}_e(s, t_0)\boldsymbol{\eta}\|^2\,\mathrm{d}s < \rho\|\boldsymbol{\eta}\|^2. \tag{4.87}$$

By Schwarz's inequality, it holds for some vector function $\mathbf{f}(t)$ that

$$\int_{t_0}^{t_0+\sigma} \|\mathbf{f}(s)\|\,\mathrm{d}s \leq \sqrt{\sigma}\sqrt{\int_{t_0}^{t_0+\sigma} \|\mathbf{f}(s)\|^2\,\mathrm{d}s}. \tag{4.88}$$

Hence, applying this inequality to (4.87), one obtains

$$\int_{t_0}^{t_0+\sigma} \|\mathbf{C}(s)\mathbf{\Phi}_e(s, t_0)\boldsymbol{\eta}\|\,\mathrm{d}s < \sqrt{\rho\sigma}\|\boldsymbol{\eta}\|. \tag{4.89}$$

Now, let $t_1 = t_0$ and $t_2 = t_0 + \sigma$ in (4.85) and choose $\boldsymbol{\xi} = \boldsymbol{\eta}$ and t_0 such that (4.86) is fulfilled. By selecting $\sigma = \frac{\ln(3KK_e)}{\mu}$ and $\rho = (9K^2K_L^2\sigma)^{-1}$ and combining (4.89) with (4.85) one obtains

$$\frac{1}{3}\|\boldsymbol{\eta}\| = KK_e\|\boldsymbol{\eta}\|e^{-\mu\sigma} \geq \|\boldsymbol{\eta}\| - KK_L\sqrt{\rho\sigma}\|\boldsymbol{\eta}\| = \frac{2}{3}\|\boldsymbol{\eta}\|. \tag{4.90}$$

This is a contradiction and hence uniform exponential detectability implies uniform complete observability for anti-stable systems. □

The proof was inspired by the proof of Theorem 3 in [IMK72]. There, the goal was to show that for systems with bounded coefficient matrices, the dual concept uniform complete controllability is equivalent to complete stabilizability with arbitrary decay rate. A key difference to the present proof is that the decay rate has to be chosen in an appropriate way. This is avoided in the proof of Proposition 4.17 by utilizing the fact that the system is anti-stable.

To sum up, Proposition 4.17 states that uniform complete observability is the minimum requirement in order to obtain a uniformly exponentially stable error system by a bounded feedback gain for anti-stable systems. This allows to formulate necessary and sufficient conditions for uniform detectability of block triangular systems which possess an exponential dichotomy.

Proposition 4.18 (detectability of block triangular systems)
Let a system $\dot{\mathbf{x}}(t) = \mathbf{B}(t)\mathbf{x}(t)$ with a bounded $\mathbf{B}(t) \in \mathbb{R}^{n\times n}$ have a block triangular structure with $\mathbf{x}_1(t) \in \mathbb{R}^{n-k}$ and $\mathbf{x}_2(t) \in \mathbb{R}^{k}$ partitioned according to

$$\begin{bmatrix}\dot{\mathbf{x}}_1(t)\\ \dot{\mathbf{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\mathbf{B}_{11}(t) & \mathbf{B}_{12}(t)\\ \mathbf{0} & \mathbf{B}_{22}(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}. \tag{4.91a}$$

The block matrices $\mathbf{B}_{11}(t)$, $\mathbf{B}_{12}(t)$ and $\mathbf{B}_{22}(t)$ are of appropriate dimension. The system output is given by

$$\mathbf{y}(t) = \begin{bmatrix}\mathbf{C}_1(t) & \mathbf{C}_2(t)\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(t)\\ \mathbf{x}_2(t)\end{bmatrix}, \tag{4.91b}$$

where $\mathbf{C}_1(t)$ and $\mathbf{C}_2(t)$ are uniformly bounded matrices of appropriate dimensions. It is assumed, that (4.91a) *possesses an exponential dichotomy with*

$$\mathbf{P} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_k\end{bmatrix}. \tag{4.92}$$

Then, system (4.91) *is uniformly exponentially detectable if and only if the pair* $(\mathbf{B}_{11}(t), \mathbf{C}_1(t))$ *is uniformly completely observable.*

Proof. It follows form Lemma 4.6 that system (4.91a) is reducible to block diagonal form $\dot{\mathbf{z}}(t) = \mathbf{D}(t)\mathbf{z}(t)$ with $\mathbf{D}_1(t) = \mathbf{B}_{11}(t)$. The block diagonal system has an exponential dichotomy with the same projection $\mathbf{P}$. It is shown in the proof of Lemma 4.6 that the transformation matrix is given by

$$\mathbf{S}(t) = \begin{bmatrix}\mathbf{I}_{n-k} & \mathbf{S}_{12}(t)\\ \mathbf{0} & \mathbf{S}_{22}(t)\end{bmatrix}. \tag{4.93}$$

The matrices $\mathbf{S}_{12}(t)$ and $\mathbf{S}_{22}(t)$ are stated in detail Appendix A.3. The important point in this proof is that

$$\begin{aligned} \mathbf{y}(t) = \mathbf{C}(t)\mathbf{S}(t)\mathbf{z}(t) &= \begin{bmatrix} \mathbf{C}_1(t) & \mathbf{C}_2(t) \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{S}_{12}(t) \\ \mathbf{0} & \mathbf{S}_{22}(t) \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{C}_1(t) & \tilde{\mathbf{C}}_2(t) \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix}, \end{aligned} \tag{4.94}$$

with $\tilde{\mathbf{C}}_2(t) = \mathbf{C}_1(t)\mathbf{S}_{12}(t) + \mathbf{C}_2(t)\mathbf{S}_{22}(t)$. The uniformly exponentially stable system $\dot{\mathbf{z}}_2(t) = \mathbf{D}_2(t)\mathbf{z}_2(t)$ is always uniformly exponentially detectable. Hence, it suffices to show that the anti-stable system

$$\dot{\mathbf{z}}_1(t) = \mathbf{D}_1(t)\mathbf{z}_1(t), \tag{4.95a}$$

$$\mathbf{y}_1(t) = \mathbf{C}_1(t)\mathbf{z}_1(t) \tag{4.95b}$$

is uniformly exponentially detectable if and only if the assumptions stated in the proposition hold. For this, let

$$\dot{\hat{\mathbf{z}}}_1(t) = \mathbf{D}_1(t)\hat{\mathbf{z}}_1(t) + \mathbf{L}(t)\left[\mathbf{y}(t) - \mathbf{C}_1(t)\hat{\mathbf{z}}_1(t)\right] \tag{4.96}$$

be an observer for (4.95a) with some uniformly bounded feedback matrix $\mathbf{L}(t)$. The dynamics of the estimation error $\mathbf{e}_1(t) = \mathbf{z}_1(t) - \hat{\mathbf{z}}_1(t)$ is given by

$$\dot{\mathbf{e}}_1(t) = \left[\mathbf{D}_1(t) - \mathbf{L}(t)\mathbf{C}_1(t)\right]\mathbf{e}_1(t) + \mathbf{L}(t)\tilde{\mathbf{C}}_2(t)\mathbf{z}_2(t). \tag{4.97}$$

The input to this error system vanishes uniformly and exponentially. According to Proposition 4.17, there exists a uniformly bounded feedback gain $\mathbf{L}(t)$ such that $\dot{\mathbf{e}}_1(t) = \left[\mathbf{D}_1(t) - \mathbf{L}(t)\mathbf{C}_1(t)\right]\mathbf{e}_1(t)$ is uniformly exponentially stable if and only if the pair $(\mathbf{D}_1(t), \mathbf{C}_1(t))$ with $\mathbf{D}_1(t) = \mathbf{B}_{11}(t)$ is uniformly completely observable. □

If the system does not admit an exponential dichotomy, it is still possible state sufficient conditions for uniform exponential detectability. Let $\mathbf{X}(t) = \mathbf{Q}(t)\mathbf{R}(t)$ be the unique continuous QR-decomposition of a fundamental matrix solution $\mathbf{X}(t)$ of system (4.1) with $\mathbf{X}_0$ as an ordered normal Lyapunov basis. Then, the spectral intervals $\Lambda_j = [\alpha_j, \beta_j]$ can be obtained from the diagonal elements of the coefficient matrix in upper triangular form according to

$$b_{jj}(t) = \mathbf{q}_j(t)^\mathsf{T}\mathbf{A}(t)\mathbf{q}_j(t). \tag{4.98}$$

The endpoints α_j and β_j, i.e., the upper and lower Bohl exponents, are given in Table 4.1. For stability assessment, one is merely interested in the upper Bohl exponents. The ultimate goal is to design an observer feedback gain such that all upper Bohl exponents of the observer error system (4.76) are negative. This is the idea for the detectability condition proposed in the following.

Theorem 4.19 (uniform detectability condition and observer design)
It is assumed that the system

$$\dot{\mathbf{z}} = \mathbf{B}(t)\mathbf{z}(t) \tag{4.99a}$$
$$\mathbf{y}(t) = \mathbf{C}_z(t)\mathbf{z}(t) \tag{4.99b}$$

is obtained from (4.1) *via* $\mathbf{z}(t) = \mathbf{Q}^\mathsf{T}(t)\mathbf{x}(t)$ *and* $\mathbf{Q}(t)$ *from the continuous QR decomposition with* $\mathbf{Q}(0)$ *as ordered normal Lyapunov basis. Hence,* $\mathbf{B}(t)$ *is an upper triangular matrix with diagonal elements* $b_{jj}(t)$, $j = 1, \ldots, n$. *Let* β_j *be the upper Bohl exponent obtained from* $b_{jj}(t)$ *according to* (4.51). *Let* $0 \leq j^\star \leq n$ *be the smallest integer such that* $\beta_j < 0$ *for all* $j > j^\star$. *Then,* (4.99) *can be partitioned according to*

$$\begin{bmatrix} \dot{\mathbf{z}}_+(t) \\ \dot{\mathbf{z}}_-(t) \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11}(t) & \mathbf{B}_{12}(t) \\ \mathbf{0} & \mathbf{B}_{22}(t) \end{bmatrix} \begin{bmatrix} \mathbf{z}_+(t) \\ \mathbf{z}_-(t) \end{bmatrix} \tag{4.100a}$$

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{C}_{z+}(t) & \mathbf{C}_{z-}(t) \end{bmatrix} \begin{bmatrix} \mathbf{z}_+(t) \\ \mathbf{z}_-(t) \end{bmatrix} \tag{4.100b}$$

with $\mathbf{B}_{11}(t)$ *as a* $j^\star \times j^\star$ *matrix.*
System (4.99) *is uniformly exponentially detectable if the pair* $(\mathbf{B}_{11}(t), \mathbf{C}_{z+}(t))$ *is uniformly completely observable. In particular,*

$$\dot{\mathbf{e}}_z(t) = [\mathbf{B}(t) - \mathbf{L}_z(t)\mathbf{C}_z(t)]\,\mathbf{e}_z(t) \tag{4.101}$$

is uniformly exponentially stable with the feedback gain

$$\mathbf{L}_z(t) = \begin{bmatrix} \mathbf{P}_{z+}(t)\mathbf{C}_{z+}^\mathsf{T}(t) \\ \mathbf{0}_{(n-j^\star \times p)} \end{bmatrix} \tag{4.102}$$

where the positive definite $j^\star \times j^\star$ *matrix* $\mathbf{P}_{z+}$ *is the unique uniformly bounded positive definite solution of the differential Riccati equation*

$$\dot{\mathbf{P}}_{z+}(t) = \mathbf{B}_{11}\mathbf{P}_{z+}(t) + \mathbf{P}_{z+}(t)\mathbf{B}_{11}^\mathsf{T}(t) - \mathbf{P}_{z+}(t)\mathbf{C}_{z+}^\mathsf{T}(t)\mathbf{C}_{z+}(t)\mathbf{P}_{z+}(t) + \mathbf{G}(t), \tag{4.103}$$

with $\mathbf{P}_{z+}(t_0) \succ 0$ *and any continuous positive definite matrix* $\mathbf{G}(t)$ *satisfying* $g_1\mathbf{I} \preceq \mathbf{G}(t) \preceq g_2\mathbf{I}$ *for some positive constants* $g_1 \leq g_2$.

The proof follows analogously to the proof of Theorem 3.6. Because the free system $\mathbf{z}_-(t) = \mathbf{B}_{22}(t)\mathbf{z}_-(t)$ is now uniformly exponentially stable by virtue of having only negative Bohl exponents, the error system remains uniformly exponentially stable as well.

The following example shows that in general, this condition is not necessary.

Example 4.20*:* Consider the second order system

$$\dot{x}_1(t) = -\frac{1}{5}x_1(t) + x_2(t) \tag{4.104a}$$
$$\dot{x}_2(t) = \left[-\frac{7}{5} + \sin(\ln(t)) + \cos(\ln(t))\right] x_2(t) \tag{4.104b}$$

with the output

$$y(t) = x_2(t). \tag{4.104c}$$

The system is already in upper triangular form and it will first be shown that the Lyapunov exponents are ordered. Because (4.104a) is time invariant, $\alpha_1 = \lambda_1^i = \lambda_1^s = \beta_1 = \frac{1}{5}$. For the second spectral interval, consider the differential equation

$$\dot{\zeta}(t) = [\sin(\ln(t)) + \cos(\ln(t))]\,\zeta(t). \tag{4.105}$$

This scalar system was proposed by O. Perron as an example for a non-regular system where the Lyapunov and Bohl exponents do not coincide. It is shown in [DKS02, p. 124] that this system has the exponents $\alpha_{1,\zeta} = -\sqrt{2}$, $\lambda_{1,\zeta}^i = -1$, $\lambda_{1,\zeta}^s = 1$ and $\beta_{1,\zeta} = \sqrt{2}$. Equation (4.104b) has the same spectrum as (4.105) shifted by $-\frac{7}{5}$ and hence the upper Lyapunov exponent is negative with $\lambda_2^s = -\frac{2}{5}$ and less than λ_1^s. However, the upper Bohl exponent $\beta_2 = \beta_{1,\zeta} - \frac{7}{5}$ is positive and hence $j^\star = 2$. System (4.104) cannot be uniformly completely observable because the state x_1 does not contribute to the output, neither directly nor indirectly via x_2. The system is uniformly exponentially detectable, however, which can be seen by the observer

$$\begin{bmatrix}\dot{\hat{x}}_1(t)\\ \dot{\hat{x}}_2(t)\end{bmatrix} = \begin{bmatrix}-\frac{1}{5} & 1\\ 0 & -\frac{7}{5} + \sin(\ln(t)) + \cos(\ln(t))\end{bmatrix}\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}[x_2(t) - \hat{x}_2(t)] \tag{4.106}$$

The error dynamics is given by

$$\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\end{bmatrix} = \begin{bmatrix}-\frac{1}{5} & 1\\ 0 & -\frac{12}{5} + \sin(\ln(t)) + \cos(\ln(t))\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}. \tag{4.107}$$

The upper Bohl exponents of (4.107) are negative and thus the error system is uniformly exponentially stable and the system (4.104) is uniformly exponentially detectable.

It should be again remarked that if (4.99a) possesses an exponential dichotomy with

$$\mathbf{P} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{(n-j^\star)}\end{bmatrix}, \tag{4.108}$$

then, the presented detectability condition is necessary and sufficient. The exponential dichotomy spectrum is also ordered for systems with point spectra, e.g., time invariant or periodic systems. The numerical examples presented in Chapter 5 show that a certain ordering of the spectral intervals can be expected based on the ordering of the Lyapunov exponents for physically motivated systems.

The observer in original coordinates is structurally equivalent to the algorithm stated in (3.55a)-(3.55f). In order to obtain the feedback gain in the original coordinates, $\mathbf{L}_z(t)$ from (4.102) can be reconstructed to the original coordinates via $\mathbf{L}(t) = \bar{\mathbf{Q}}(t)\mathbf{L}_z(t)$. For the sake of self-containment, the observer design is summarized as

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{L}(t)\left[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right], \tag{4.109a}$$

$$\dot{\mathbf{P}}(t) = \mathbf{B}_{11}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}_{11}^{\mathsf{T}}(t) - \mathbf{P}(t)\bar{\mathbf{Q}}^{\mathsf{T}}(t)\mathbf{C}^{\mathsf{T}}(t)\mathbf{C}(t)\bar{\mathbf{Q}}(t)\mathbf{P}(t) + \mathbf{G}(t), \tag{4.109b}$$

$$\dot{\bar{\mathbf{Q}}}(t) = \left[\mathbf{I} - \bar{\mathbf{Q}}(t)\bar{\mathbf{Q}}^{\mathsf{T}}(t)\right]\mathbf{A}(t)\bar{\mathbf{Q}}(t) + \bar{\mathbf{Q}}(t)\mathbf{S}_{11}(t),\ \bar{\mathbf{Q}}(0) = \bar{\mathbf{Q}}_0 \in \mathbb{R}^{n\times k}, \tag{4.109c}$$

with

$$\mathbf{L}(t) = \bar{\mathbf{Q}}(t)\mathbf{P}(t)\bar{\mathbf{Q}}^{\mathsf{T}}(t)\mathbf{C}^{\mathsf{T}}(t), \tag{4.109d}$$

$$\mathbf{B}_{11}(t) = \bar{\mathbf{Q}}^{\mathsf{T}}(t)\mathbf{A}(t)\bar{\mathbf{Q}}(t) - \mathbf{S}_{11}(t), \tag{4.109e}$$

and the elements s_{ij} of $\mathbf{S}_{11}$ given by

$$s_{ij}(t) = \begin{cases} -\bar{\mathbf{q}}_i^{\mathsf{T}}(t)\mathbf{A}(t)\bar{\mathbf{q}}_j(t) & i < j, \\ 0 & i = j, \\ \bar{\mathbf{q}}_i^{\mathsf{T}}(t)\mathbf{A}(t)\bar{\mathbf{q}}_j(t) & i > j. \end{cases} \tag{4.109f}$$

The columns of $\bar{\mathbf{Q}}(t)$ are denoted by $\bar{\mathbf{q}}_i(t)$. The initial conditions are chosen as a positive definite $k \times k$ matrix $\mathbf{P}_0 \succ 0$ and a random orthogonal $n \times k$ matrix $\mathbf{Q}_0$. The matrix $\mathbf{G}(t)$ in (4.109b) is again a tuning parameter of the observer.

The matrix $\bar{\mathbf{Q}}(t)$ is the solution of the matrix differential equation (4.109c). The integer k in the observer algorithm is now chosen as $k \geq j^\star$. If the original system is uniformly completely observable, it can be regarded as a tuning parameter in the observer design to achieve a trade-off between the convergence speed and the computational complexity resulting from the dimension of the considered Riccati differential equation. In particular, it determines how many spectral intervals of the original system are modified in the observer error dynamics. The reduced order observer design proposed in Section 3.5 can be adopted to the present concept in a straightforward manner. Moreover, uniform exponential stability implies exponential stability and hence all the properties of the observer concept discussed in Chapter 3 are valid for the observer resulting from the feedback gain (4.102). Uniform exponential stability shows stronger robustness properties than merely exponential stability, which will be discussed in the following.

4.5.1 Robustness Considerations

In the following, the perturbed system (4.1) with a uniformly bounded perturbation $\sup_{t\in\mathbb{J}} \|\mathbf{f}(t)\| \leq \bar{f} < \infty$ of the state equation according to

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{4.110}$$

is considered. Under the assumptions stated in Theorem 4.19, the perturbation free error system $\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{e}(t)$ is uniformly exponentially stable. The solution of the perturbed error system

$$\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{e}(t) + \mathbf{f}(t) \tag{4.111}$$

is given by

$$\mathbf{e}(t) = \mathbf{\Phi}_e(t, t_0)\mathbf{e}_0 + \int_{t_0}^{t} \mathbf{\Phi}_e(t, s)\mathbf{f}(s)\,\mathrm{d}s. \tag{4.112}$$

The transition matrix can be bounded by

$$\|\mathbf{\Phi}_e(t, t_0)\| \leq K_e e^{-\gamma(t-t_0)} \tag{4.113}$$

for some $\gamma > 0$ and hence

$$\|\mathbf{e}(t)\| \leq K_e\|\mathbf{e}_0\| + \frac{K_e}{\gamma}\bar{f}, \tag{4.114}$$

see, e.g., Appendix A.2. This so-called bounded input bounded state stability is guaranteed by uniform exponential stability of the perturbation free system.

If $\lim_{t\to\infty} \|\mathbf{f}(t)\| = 0$, then also $\lim_{t\to\infty} \|\mathbf{e}(t)\| = 0$, see [Hah67, Theorem 59.1]. The arguments provided in the proof of this Theorem are the following. The part of the solution arising from $\mathbf{\Phi}(t, t_0)\mathbf{e}_0$ vanishes exponentially fast due to (4.113). Provided that $\lim_{t\to\infty} \|\mathbf{f}(t)\| = 0$, for each $\varepsilon > 0$ one can select a t_1 such that $\|\mathbf{f}(t)\| \leq \varepsilon$ for all $t \geq t_1$. For $t_0 \leq t \leq t_1$ one has $\|\mathbf{f}(t)\| \leq \bar{f}$ and thus

$$\begin{aligned}
\left\|\int_{t_0}^{t} \mathbf{\Phi}_e(t, s)\mathbf{f}(s)\,\mathrm{d}s\right\| &\leq \int_{t_0}^{t_1} \|\mathbf{\Phi}_e(t, s)\mathbf{f}(s)\|\,\mathrm{d}s + \int_{t_1}^{t} \|\mathbf{\Phi}_e(t, s)\mathbf{f}(s)\|\,\mathrm{d}s \\
&\leq \bar{f}K_e \int_{t_0}^{t_1} e^{-\gamma(t-s)}\,\mathrm{d}s + \varepsilon K_e \int_{t_1}^{t} e^{-\gamma(t-s)}\,\mathrm{d}s \\
&= \frac{\bar{f}K_e}{\gamma} e^{-\gamma t}\left(e^{\gamma t_1} - e^{\gamma t_0}\right) + \varepsilon\frac{K_e}{\gamma}\left(1 - e^{\gamma(t_1-t)}\right).
\end{aligned} \tag{4.115}$$

This norm becomes arbitrarily small as t increases and ε decreases.

An alternative way to show the above properties is to utilize the Lyapunov function provided in [HP06, Theorem 3.3.33 and 3.3.38], see Section 2.2. The autonomous error system $\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\mathbf{e}(t) = \mathbf{A}_e(t)\mathbf{e}(t)$ is uniformly exponentially stable and hence

$$V(t, \mathbf{e}) = \mathbf{e}^\mathsf{T}\mathbf{P}_L(t)\mathbf{e} \tag{4.116}$$

is a Lyapunov function for the autonomous error system with a symmetric positive matrix $p_1\mathbf{I}_n \preceq \mathbf{P}_L(t) \preceq p_2\mathbf{I}_n$ and some positive constants $p_1 \leq p_2$. The matrix $\mathbf{P}_L(t)$ fulfills the differential equation

$$\dot{\mathbf{P}}_L(t) + \mathbf{A}_e^\mathsf{T}(t)\mathbf{P}_L(t) + \mathbf{P}_L(t)\mathbf{A}_e(t) + \mathbf{Q}(t) = \mathbf{0} \tag{4.117}$$

for some $\mathbf{Q}(t)$ with $q_1\mathbf{I}_n \preceq \mathbf{Q}(t) \preceq q_2\mathbf{I}_n$ and q_1, q_2 some positive constants.

The Lyapunov function candidate (4.116) is now used for the perturbed error system (4.111) and the time derivative along the trajectory yields

$$\begin{aligned} \dot{V}(t, \mathbf{e}) &= -\mathbf{e}^\mathsf{T}\mathbf{Q}(t)\mathbf{e} + 2\mathbf{f}^\mathsf{T}(t)\mathbf{P}(t)\mathbf{e} \\ &\leq -q_1\|\mathbf{e}\|^2 + 2p_2\|\mathbf{f}(t)\|\|\mathbf{e}\|. \end{aligned} \tag{4.118}$$

For a bounded $\mathbf{f}(t)$, the derivative of V is negative definite if $\|\mathbf{e}\| > \frac{2p_2\bar{f}}{q_1}$ and hence the estimation error converges in a bounded neighborhood of the origin. If $\lim_{t\to\infty} \|\mathbf{f}(t)\| = 0$, also the estimation tends to zero asymptotically.

This Lyapunov function will become useful in the following chapter, when the observer design is extended to nonlinear systems. Extensive simulation studies for the proposed concept applied to relevant test examples are also presented in the following chapter.

4.6 Discussion

This chapter proposes a condition for uniform exponential detectability and a corresponding observer design technique such that the resulting estimation error dynamics are uniformly exponentially stable. As an extension to the concepts presented in Chapter 3, the observer gain modifies specific spectral intervals of the exponential dichotomy spectrum. For uniformly completely observable systems, this allows to design computationally efficient observers in the sense that only “unstable or slowly converging modes” are modified by the observer gain and hence solving a differential Riccati equation is required on a state space of reduced dimension only.

For systems, which admit an exponential dichotomy, it is shown, that uniform complete observability on the unstable subspace is necessary and sufficient for uniform exponential detectability. As a result, the condition presented in Theorem 4.19 is necessary and sufficient for uniform exponential detectability for a class of linear time varying systems.

The observer design relies on the transformation of the original system to triangular form via the continuous QR-decomposition as discussed already in Chapter 3. A key assumption here is that this transformation is initialized with an ordered normal Lyapunov basis. Such a basis is typically not known a priori. However, a random choice worked reliable in all numerical simulations, which are presented in the following in Chapter 5. This topic is also discussed in Remark 3.4 of Chapter 3. In [DE06; DV07] the continuous singular value decomposition is proposed as an alternative to the QR-decomposition for the computation of spectra of dynamical systems. This algorithm does not require to be initialized with an ordered normal basis and any fundamental matrix solution can be used, but at the price of an increased computational complexity. Moreover, the same authors propose methods to numerically approximate the projection matrix $\mathbf{P}$ for systems with exponential dichotomy in [DEV10] based on the QR and the singular value decomposition. A possible future research question is, if this information can be exploited in the observer design or for numerical detectability checks.

The presented observer is extended for a class of nonlinear systems in Chapter 5. The concepts are evaluated in various numerical simulation examples and compared with an extended Kalman-Bucy filter for a specific application with real measurement data. This gives interesting insights into the properties of the observer and the assumptions stated in Theorem 4.19.

Chapter 5

Extended Subspace Observer Design

In this chapter, the approach presented in Chapter 4 is extended to a class of non-linear systems. It is shown that the estimation error dynamics of the resulting observer is locally uniformly exponentially stable.

Because the Riccati differential equation is solved on a reduced state-space only, the ideas for the convergence analysis of the extended Kalman-Bucy filter cannot be applied in the present setting. However, the local stability result utilizes a converse Lyapunov theorem and hence under some assumptions stated subsequently, local uniform asymptotic stability of the observer error dynamics can be guaranteed.

Sections 5.3 to 5.5 present representative nonlinear examples, where the proposed observer is applied. First, the full nonlinear Lorenz model already mentioned in Section 3.7.2 is investigated in Section 5.3. As a second example, a discretized version of a specific partial differential equation, the so-called Burgers equation, is considered in Section 5.4. In Section 5.5, the algorithm is applied to the problem of wafer surface temperature estimation. This last example demonstrates the effectiveness of the presented estimation concept compared with the extended Kalman-Bucy filter using experimental data.

In the following, the nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)),\ t \in \mathbb{J} \tag{5.1a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{5.1b}$$

is considered with the state $\mathbf{x}(t) \in \mathbb{R}^n$, the output $\mathbf{y}(t) \in \mathbb{R}^p$ and the input $\mathbf{u}(t) \in \mathbb{R}^m$. Again it is assumed that the output matrix $\mathbf{C}(t)$ is uniformly bounded and moreover it is assumed that $\mathbf{f}(\cdot, \cdot)$ is continuously differentiable.

It should be remarked that the case of measurements linear in the states is treated here to demonstrate the key idea. An extension to a nonlinear output map is possible by a straightforward application of the concepts presented in [RU96; RSU98].

5.1 Extended Kalman-Bucy Filter as a Deterministic Observer

The idea of the deterministic interpretation of extended Kalman-Bucy filter (EKBF) is, to obtain the time varying matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$ by linearization along the estimated trajectory [RU96]. The algorithm considered in this work is summarized in the following:

$$\hat{\mathbf{x}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t)) + \mathbf{P}(t)\mathbf{C}^\mathsf{T}(t)\left[\mathbf{y}(t) - \mathbf{h}(\hat{x}(t))\right], \ \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0, \tag{5.2a}$$

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A}^\mathsf{T}(t) + \mathbf{A}(t)\mathbf{P}(t) - \mathbf{P}(t)\mathbf{C}^\mathsf{T}(t)\mathbf{C}(t)\mathbf{P}(t) + \mathbf{G}(t), \mathbf{P}(t) \in \mathbb{R}^{n\times n} \tag{5.2b}$$

$$\mathbf{A}(t) = \frac{\partial}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))\bigg|_{(\hat{\mathbf{x}}(t),\mathbf{u}(t))}. \tag{5.2c}$$

The initial condition $\mathbf{P}_0$ is chosen as a positive definite matrix and the matrix $\mathbf{G}(t)$ is a positive definite tuning parameter, see Section 2.4.2.

Local stability results for the dynamics of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ are presented in [RU96; RSU98; Kre03; BS15]. A key assumption of these stability proofs is the existence of positive constants p_1, p_2 such that $p_1\mathbf{I}_n \preceq \mathbf{P}(t) \preceq p_2\mathbf{I}_n$ holds for (5.2b). This assumption can be guaranteed by uniform complete observability of the observer trajectory, which might not be a trajectory of the original system in general. However, this assumption cannot be dropped easily as extensively discussed in [BS15; Kar+18]. The key idea of the convergence result presented in [RSU98] is to use $\mathbf{P}^{-1}(t)$ in a quadratic Lyapunov function for the (nonlinear) estimation error dynamics, i.e. $\mathbf{V}(\mathbf{e}, t) = \mathbf{e}^\mathsf{T}\mathbf{P}^{-1}\mathbf{e}$. It can then be shown that $\mathbf{e}(t) = \mathbf{0}$ is a locally uniformly exponentially stable equilibrium, i.e., the estimation error converges to zero if $\mathbf{e}(0) = \mathbf{x}(0) - \hat{\mathbf{x}}(0)$ is sufficiently small.

5.2 Extended Subspace Observer

Based on the ideas proposed in Chapter 4, an observer for a class of nonlinear systems is presented here. The proposed algorithm is summarized as

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t), \mathbf{u}(t)) + \mathbf{L}(t)\left[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right], \tag{5.3a}$$

$$\dot{\mathbf{P}}(t) = \mathbf{B}_{11}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{B}_{11}^{\mathsf{T}}(t) - \mathbf{P}(t)\bar{\mathbf{C}}^{\mathsf{T}}(t)\bar{\mathbf{C}}(t)\mathbf{P}(t) + \mathbf{G}(t), \mathbf{P}(t) \in \mathbb{R}^{k\times k} \tag{5.3b}$$

$$\dot{\bar{\mathbf{Q}}}(t) = \left[\mathbf{I} - \bar{\mathbf{Q}}(t)\bar{\mathbf{Q}}^{\mathsf{T}}(t)\right]\mathbf{A}(t)\bar{\mathbf{Q}}(t) + \bar{\mathbf{Q}}(t)\mathbf{S}_{11}(t),\ \bar{\mathbf{Q}}(t) \in \mathbb{R}^{n\times k}, \tag{5.3c}$$

with

$$\mathbf{L}(t) = \bar{\mathbf{Q}}(t)\mathbf{P}(t)\bar{\mathbf{C}}^{\mathsf{T}}(t), \tag{5.3d}$$

$$\bar{\mathbf{C}}(t) = \mathbf{C}(t)\mathbf{Q}(t), \tag{5.3e}$$

$$\mathbf{A}(t) = \frac{\partial}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))\bigg|_{(\hat{\mathbf{x}}(t), \mathbf{u}(t))}, \tag{5.3f}$$

$$\mathbf{B}_{11}(t) = \bar{\mathbf{Q}}^{\mathsf{T}}(t)\mathbf{A}(t)\bar{\mathbf{Q}}(t) - \mathbf{S}_{11}(t), \tag{5.3g}$$

and the elements s_{ij} of $\mathbf{S}_{11}$ given by

$$s_{ij}(t) = \begin{cases} -\bar{\mathbf{q}}_i^{\mathsf{T}}(t)\mathbf{A}(t)\bar{\mathbf{q}}_j(t) & i < j, \\ 0 & i = j, \\ \bar{\mathbf{q}}_i^{\mathsf{T}}(t)\mathbf{A}(t)\bar{\mathbf{q}}_j(t) & i > j. \end{cases} \tag{5.3h}$$

The number of columns in $\bar{\mathbf{Q}}(t)$ has to be chosen such that $j^\star \leq k \leq n$ with $j^\star$ as given in Theorem 4.19. In the following, a convergence result for this observer is provided.

5.2.1 Convergence Analysis

The dynamics of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is given by

$$\dot{\mathbf{e}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}) - \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}) - \mathbf{L}(t)\left[\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right]. \tag{5.4}$$

By substituting $\mathbf{x}$ with $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{e}$ in (5.4) and performing a Taylor series expansion of $\mathbf{f}(\cdot, \cdot)$ around $\mathbf{e} = \mathbf{0}$ one obtains

$$\mathbf{f}(\hat{\mathbf{x}} + \mathbf{e}, \mathbf{u}) = \mathbf{f}(\hat{\mathbf{x}}, \mathbf{u}) + \mathbf{A}(t)\mathbf{e} + \boldsymbol{\eta}(\mathbf{e}, \hat{\mathbf{x}}, \mathbf{u}), \tag{5.5}$$

where

$$\mathbf{A}(t) = \frac{\partial}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{u})\bigg|_{(\hat{\mathbf{x}}(t), \mathbf{u}(t))} \tag{5.6}$$

and $\boldsymbol{\eta}$ is the remainder of the Taylor series truncated after the linear term. Hence, the estimation error dynamics can be stated as

$$\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{e}(t) + \boldsymbol{\eta}(\mathbf{e}, \hat{\mathbf{x}}, \mathbf{u}). \tag{5.7}$$

The following additional assumptions are standard assumptions in the convergence analysis of the extended Kalman-Bucy filter, see, e.g. [RU96; RSU98].

(a1) The matrix $\mathbf{A}(t)$ is uniformly bounded

(a2) There exist positive constants p_1 and p_2 such that $p_1\mathbf{I}_k \preceq \mathbf{P}(t) \preceq p_2\mathbf{I}_k$ holds for $\mathbf{P}(t)$ in (5.3b).

(a3) There exist positive constants ε and κ such that

$$\|\boldsymbol{\eta}(\mathbf{e}, \mathbf{x}, \mathbf{u})\| \leq \kappa\|\mathbf{e}(t)\|^2 \tag{5.8}$$

holds for all $\mathbf{x}, \mathbf{e} \in \mathbf{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$ with $\|\mathbf{e}(t)\| \leq \varepsilon$.

Remark 5.1*:* Assumption (a3) is fulfilled, e.g., if $\mathbf{f}(\cdot,\cdot)$ is at least two times continuously differentiable and the corresponding Hessian matrix of each component of $\mathbf{f}(\cdot,\cdot)$ is bounded, see [RU96; FZ18] and [Heu04, Section 168]. Let therefore the components of $\mathbf{f}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} f_1(\mathbf{x}, \mathbf{u}) & \cdots & f_n(\mathbf{x}, \mathbf{u}) \end{bmatrix}^\mathsf{T}$ be denoted by f_i, $i = 1, \ldots, n$.
Then, κ is given by

$$\kappa = \frac{1}{2} \max_{i=1}^{n} \sup_{\substack{\mathbf{x}\in\mathbb{R}^n \\ \mathbf{u}\in\mathbb{R}^m}} \|\mathbf{H}_{f_i}(\mathbf{x}, \mathbf{u})\| \tag{5.9}$$

with the Hessian matrix of f_i given by

$$\mathbf{H}_{f_i}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \frac{\partial^2 f_i}{\partial x_1 \partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial^2 f_i}{\partial x_1 \partial x_2}(\mathbf{x}, \mathbf{u}) & \cdots & \frac{\partial^2 f_i}{\partial x_1 \partial x_n}(\mathbf{x}, \mathbf{u}) \\ \frac{\partial^2 f_i}{\partial x_2 \partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial^2 f_i}{\partial x_2 \partial x_2}(\mathbf{x}, \mathbf{u}) & \cdots & \frac{\partial^2 f_i}{\partial x_2 \partial x_n}(\mathbf{x}, \mathbf{u}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_i}{\partial x_n \partial x_1}(\mathbf{x}, \mathbf{u}) & \frac{\partial^2 f_i}{\partial x_n \partial x_2}(\mathbf{x}, \mathbf{u}) & \cdots & \frac{\partial^2 f_i}{\partial x_n \partial x_n}(\mathbf{x}, \mathbf{u}) \end{bmatrix} \tag{5.10}$$

and $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^\mathsf{T}$.

To show local stability of the estimation error dynamics, the standard approach used in the stability analysis for the extended Kalman-Bucy filter discussed in Section 5.1 and [RU96; RSU98] cannot be applied, because the Riccati equation is solved only on a reduced-order subspace. However, a local convergence result is obtained by utilizing the Lyapunov function for linear time varying systems presented in Section 2.3, Proposition 2.9.

Theorem 5.2 (extended subspace observer)
Let a nonlinear system be given by (5.1) *and the observer for this system by equations* (5.3)*. Moreover, let the assumptions (a1) to (a3) hold. Then, the estimation error dynamics* (5.7) *resulting from this observer is locally uniformly exponentially stable.*

Proof. If the assumptions (a1) to (a3) hold, the linear time varying system

$$\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)]\,\mathbf{e}(t) \tag{5.11}$$

is uniformly exponentially stable with the feedback gain $\mathbf{L}(t)$ as in (5.3d) due to Theorem 4.19. The state transition matrix of this (unperturbed) system can then be bounded by

$$\|\mathbf{\Phi}(t_1, t_0)\| \leq K e^{-\gamma(t_1 - t_0)} \tag{5.12}$$

for some positive constants $K \geq 1$ and $\gamma > 0$. Moreover, as shown in [HP06] and already discussed in Section 2.3, there exists a Lyapunov function $V(t, \mathbf{e}) = \mathbf{e}^\mathsf{T}\mathbf{P}_L(t)\mathbf{e}$ with positive constants $\bar{p}_1$ and $\bar{p}_2$ and a positive definite $n \times n$ matrix $\mathbf{P}_L(t)$ such that $\bar{p}_1\mathbf{I}_n \preceq \mathbf{P}_L(t) \preceq \bar{p}_2\mathbf{I}_n$. The matrix $\mathbf{P}_L(t)$ is the unique positive definite solution of

$$\dot{\mathbf{P}}_L(t) + \mathbf{A}_e^\mathsf{T}(t)\mathbf{P}_L(t) + \mathbf{P}_L\mathbf{A}_e(t) + \mathbf{Q}_L(t) = \mathbf{0} \tag{5.13}$$

with $\mathbf{A}_e(t) = \mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)$ and $\mathbf{Q}_L(t)$ as any positive definite matrix bounded by positive constants q_1 and q_2 such that $q_1\mathbf{I}_n \preceq \mathbf{Q}_L(t) \preceq q_2\mathbf{I}_n$.

The function $V(t, \mathbf{e}) = \mathbf{e}^\mathsf{T}\mathbf{P}_L(t)\mathbf{e}(t)$ is now used as a Lyapunov function candidate for the perturbed error system (5.7) and it is assumed that $\|\mathbf{e}(0)\| \leq \varepsilon$. The time derivative along the trajectory can then be obtained according to

$$\begin{aligned} \dot{V}(t, \mathbf{e}) &= \dot{\mathbf{e}}^\mathsf{T}\mathbf{P}_L\mathbf{e} + \mathbf{e}^\mathsf{T}\dot{\mathbf{P}}_L\mathbf{e} + \mathbf{e}^\mathsf{T}\mathbf{P}_L\dot{\mathbf{e}} \\ &= -\mathbf{e}^\mathsf{T}\mathbf{Q}_L\mathbf{e} + 2\boldsymbol{\eta}(\mathbf{e}, \hat{\mathbf{x}}, \mathbf{u})^\mathsf{T}\mathbf{P}_L\mathbf{e} \\ &\leq -q_1\|\mathbf{e}\|^2 + 2\bar{p}_2\|\boldsymbol{\eta}\|\|\mathbf{e}(t)\| \\ &\leq -q_1\|\mathbf{e}\|^2 + 2\bar{p}_2\kappa\|\mathbf{e}\|^3 \\ &\leq (-q_1 + 2\bar{p}_2\kappa\|\mathbf{e}\|)\|\mathbf{e}\|^2 \end{aligned} \tag{5.14}$$

For $\|\mathbf{e}\| \leq \min\left(\frac{q_1}{4\bar{p}_2\kappa}, \varepsilon\right)$, it holds that

$$\dot{V}(\mathbf{e}, t) \leq -\frac{1}{2}q_1\|\mathbf{e}\|^2. \tag{5.15}$$

According to Theorem 2.8, the error dynamics is locally uniformly exponentially stable because of (5.15) and $\bar{p}_1\|\mathbf{e}\|^2 \leq V(t, \mathbf{e}) \leq \bar{p}_2\|\mathbf{e}\|^2$. □

To further investigate the bound on the initial error, it is now assumed for simplicity that $\varepsilon \gg \frac{q_1}{4\bar{p}_2\kappa}$ and that $\mathbf{Q}(t) = q\mathbf{I}_n$, i.e., $q_1 = q_2 = q$. Then, the relation

$$\bar{p}_2 \leq \frac{K^2 q}{2\gamma} \tag{5.16}$$

with K and γ as in (5.12) can be used to simplify the bound on the initial error, see Section 4.5.1 and [HP06]. Hence, convergence is guaranteed for

$$\|\mathbf{e}(t_0)\| \leq \frac{q_1}{4\bar{p}_2\kappa} \leq \frac{\gamma}{2K^2\kappa} \tag{5.17}$$

with γ as the rate of exponential decay. For the proposed observer, γ is bounded by the k-th upper Bohl exponent β_k such that $\gamma < \beta_k$, because the $(k+1)$-th exponent is not modified by the observer gain. This suggests that the region of convergence can be increased by taking more exponents into account, i.e., by increasing the number of columns in $\bar{\mathbf{Q}}$, if the system is not merely uniformly exponentially detectable but has stronger observability properties. This effect was also recognized for the numerical simulation examples presented in the following.

5.3 Lorenz'96 Model

The nonlinear Lorenz'96 model [Lor95] was already briefly introduced in Chapter 3. It is a system of nonlinear differential equations recursively defined by

$$\dot{z}_i = (z_{i+1} - z_{i-2})z_{i-1} - z_i + F, \qquad i = 1, \dots, n, \tag{5.18}$$

with the notational convention $z_{-1} := z_{n-1}$, $z_0 := z_n$, and $z_{n+1} := z_1$. The state vector is $\mathbf{z} = [z_1 \ \cdots \ z_n]^\mathsf{T} \in \mathbb{R}^n$. For $F = 8$, this model exhibits a chaotic behavior. The characteristic behavior of chaotic systems is that a small perturbation of the initial state yields entirely different trajectories. This qualifies such systems as benchmark examples for state estimation [FZ18; PCT13; BC17]. The evolution of the system states over time for a model of order $n = 40$ is shown in Fig. 5.1.

The following simulations are carried out for a model of order $n = 18$. The output of the model is chosen as $\mathbf{y}(t) = \mathbf{C}_p\mathbf{z}(t)$ with a constant $p \times n$ matrix $\mathbf{C}_p$. The rows of $\mathbf{C}_p$ are chosen such that one state is exclusively measured in one row and the "sensors" are distributed with equal distance over all state variables. The matrix is given by

$$\mathbf{C}_p = \begin{bmatrix} \mathbf{e}_1^\mathsf{T} \\ \mathbf{e}_{d+1}^\mathsf{T} \\ \cdots \\ \mathbf{e}_{(p-1)d+1}^\mathsf{T} \end{bmatrix}, \tag{5.19}$$

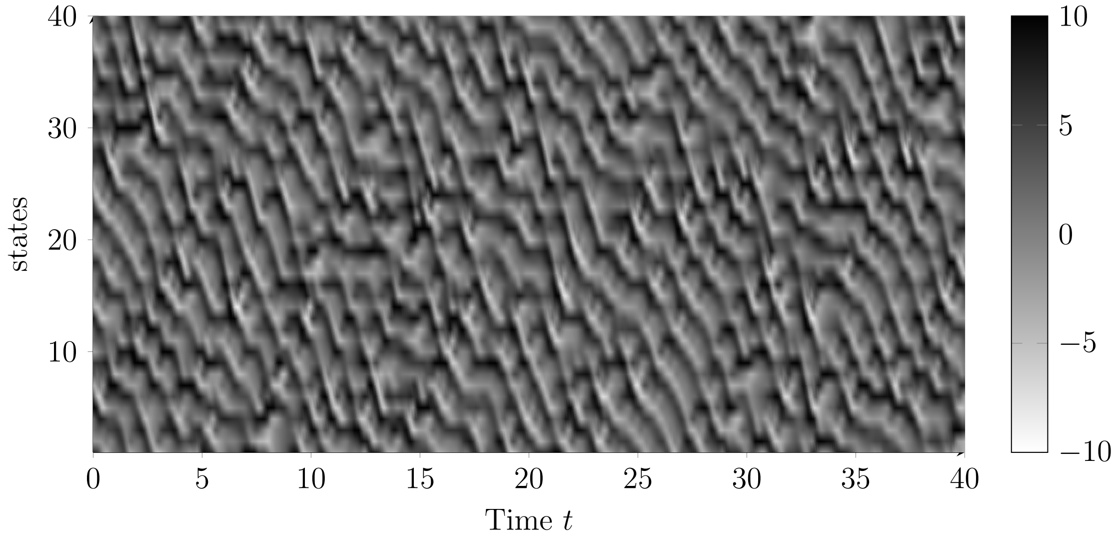


Fig. 5.1: Evolution of the system states of the Lorenz'96 model over time. The values between the states are interpolated linearly.

where $\mathbf{e}_i$ is the i-th unit vector and $d = \left\lfloor \frac{n}{p} \right\rfloor$, where $\lfloor \cdot \rfloor$ denotes rounding to the nearest integer towards zero.

The initial condition is chosen as $z_{i,0} = \sin\left(\frac{i-1}{n} 2\pi\right)$ at initial time $t_0 = 0$. For evaluation purposes, also different initial conditions were chosen. The obtained results were quantitatively similar and in particular the approximated Lyapunov exponents and the upper and lower Bohl exponents were approximately the same.

First, the spectral intervals of the system are investigated. All differential equations are solved using fixed step 4th order Runge-Kutta integration with a step-size of $T_s = 0.005$. For solving the differential equation (3.31) to obtain $\bar{\mathbf{Q}}$, a projected integrator [DRV94] is implemented, where the orthonormalization is carried out after each simulation time step.

The obtained results for all $n = 18$ spectral intervals are given in Table 5.1. The final time of this simulation is chosen as $T_f = 1500$ in order to be able to choose the averaging window length H for the approximation of the exponential dichotomy spectral intervals sufficiently large. For a window length of $H = 300$, the 7^{th} approximated upper Bohl exponent is positive whereas for an increased window length $H = 800$, this value is negative. This critical part of the spectrum is depicted in Fig. 5.2 and Fig. 5.3 for $H = 300$ and $H = 800$, respectively. The spectral intervals Λ_7 to Λ_{18} are thus negative and moreover all intervals are disjoint. All spectral intervals are depicted in Fig. 5.20 and Fig. 5.21 at the end of this chapter. The results show that the number of non-negative Lyapunov exponents $k^\star$ is equal to the number of non-negative upper Bohl exponents $j^\star$. The infimum and supremum of the diagonal elements in order to

Table 5.1: Approximated spectral intervals of the Lorenz'96 model.

	Σ_{CL}	$\Sigma_{ED} \approx \Sigma_{IS}^{H=300}$		$\Sigma_{ED} \approx \Sigma_{IS}^{H=800}$		Σ_{IS}^{0}	
interval	λ_i	$\alpha_i^{H=300}$	$\beta_i^{H=300}$	$\alpha_i^{H=800}$	$\beta_i^{H=800}$	α_i^0	β_i^0
1	1.545	1.430	1.658	1.4870	1.5920	-5.488	6.820
2	1.211	1.139	1.274	1.1920	1.2410	-4.846	6.322
3	0.878	0.790	0.959	0.8630	0.9190	-5.319	6.704
4	0.570	0.494	0.643	0.5360	0.5930	-5.191	6.839
5	0.283	0.231	0.333	0.2600	0.2900	-6.822	5.650
6	0.003	-0.011	0.021	-0.0050	0.0080	-6.015	5.434
7	-0.045	-0.096	0.015	-0.0760	-0.0320	-6.138	5.044
8	-0.296	-0.365	-0.228	-0.3310	-0.2760	-5.514	4.691
9	-0.579	-0.639	-0.526	-0.5880	-0.5560	-5.759	4.153
10	-0.829	-0.898	-0.778	-0.8510	-0.8200	-6.273	4.521
11	-1.103	-1.159	-1.051	-1.1300	-1.0910	-6.682	3.963
12	-1.349	-1.395	-1.283	-1.3690	-1.3300	-6.897	3.464
13	-1.635	-1.705	-1.573	-1.6580	-1.6120	-8.166	3.930
14	-2.022	-2.088	-1.955	-2.0390	-1.9970	-9.605	2.851
15	-2.524	-2.595	-2.434	-2.5470	-2.5040	-9.176	2.719
16	-3.240	-3.390	-3.124	-3.2830	-3.1840	-9.755	2.150
17	-4.148	-4.279	-4.015	-4.1670	-4.0980	-12.623	1.024
18	-4.720	-4.835	-4.596	-4.7560	-4.6780	-11.459	1.320

obtain Σ_{IS}^0 are approximated by the minimum and maximum of the b_{ii} on the time interval $[300, 1500]$. This suggests that for this specific example one cannot draw any conclusion about the spectral intervals from Σ_{IS}^0, because $\beta_i^0 > 0$ and $\alpha_i^0 < 0$ for all $i = 1, \ldots, n$.

Now, the design of the extended subspace observer is carried out for $p = 5$ as the dimension of the output vector $\mathbf{y}(t)$. The initial estimate is chosen as $\hat{\mathbf{x}}_0 = \mathbf{x}_0 + \mathbf{e}_0$ with $\mathbf{e}_0$ as a random perturbation. The components $e_{0,i}$ of $\mathbf{e}_0$ are chosen to be uniformly distributed on an interval $(-\delta, \delta)$ with $\delta > 0$ as a simulation parameter. For $k \leq 5 < j^\star$, no convergence of the estimation error could be achieved independently of the size of the initial perturbation. This coincides with the observations from the approximated spectral intervals. Hence, in a first step, $k = k^\star = j^\star = 6$ is chosen in the observer design and an ensemble simulation with $N = 50$ simulation runs is carried out. The matrix $\mathbf{G}(t)$ in the differential Riccati equation (5.3b) is chosen as a constant matrix $\mathbf{G} = 10\mathbf{I}_k$ in the following simulations.

The (point-wise) minimum and maximum estimation errors together with the median

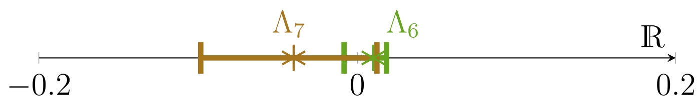


Fig. 5.2: Critical part of the approximated Lorenz'96 exponential dichotomy spectrum, $H = 300$.

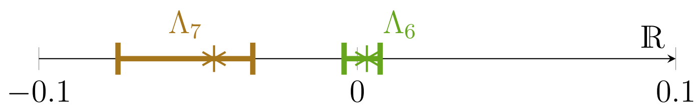


Fig. 5.3: Critical part of the approximated Lorenz'96 exponential dichotomy spectrum, $H = 800$.

and the 80%-quantile are depicted in Fig. 5.4. The latter quantity is a (point-wise) upper bound for 80% of the estimation errors. The expected minimum convergence rate of the observer error, i.e., the approximated 7-th upper Bohl exponent is also depicted. This indicates that the estimation error converges exponentially with the rate $\beta_7^{800} \approx -0.0320$. This convergence rate is very small and, as predicted by theory, the stability of the nonlinear error system is very sensitive to the size of the initial estimation error $\mathbf{e}_0$. This behavior can be seen in the simulations and hence the interval bound for the uniform distribution of the initial error is chosen as $\delta = 10^{-4}$.

In order to increase the convergence speed and to decrease sensitivity with respect to the magnitude of the initial perturbation, the number of columns in $\bar{\mathbf{Q}}$ was increased to $k = 7$. The results of this ensemble simulation is depicted in Fig. 5.5. The resulting exponential convergence is now achieved at an approximate rate of $\beta_8^{800} \approx -0.2760$, because this is the first Bohl exponent which is not modified by the observer feedback gain. The convergence is faster compared to $k = 6$ and moreover the robustness with respect to the initial perturbation is improved. Hence, the size of the initial perturbation could be increased to $\delta = 10^{-3}$.

Taking into account one more spectral interval to be modified via the observer gain, the convergence speed can again be increased as depicted in Fig 5.6 for $k = 8$. The size of the initial perturbation was again increased to $\delta = 10^{-2}$ and a convergence of the estimation error was achieved for all simulation runs.

In order to analyze the detectability properties of the Lorenz'96 model, the observer was implemented for $k = j^\star = 6$ with the matrix $\mathbf{G}(t) = \mathbf{0}$. Then, the obtained differential Riccati equation is a differential equation for the inverse of the constructibility Gramian on the considered subspace as already discussed in Section 2.4.2. This inverse, i.e., $\mathbf{P}(t_0)$, cannot be initialized correctly, because the constructibility Gramian is zero at the initial time. However, $\mathbf{P}(t)$ converges to the true inverse $\mathbf{N}^{-1}(t, 0)$ exponentially fast, see [Esc18]. This suggests that if $\mathbf{P}(t)$ has a uniform upper bound, the constructibility Gramian is uniformly bounded from below and the linearization along the estimated trajectory is uniformly completely constructible on the considered

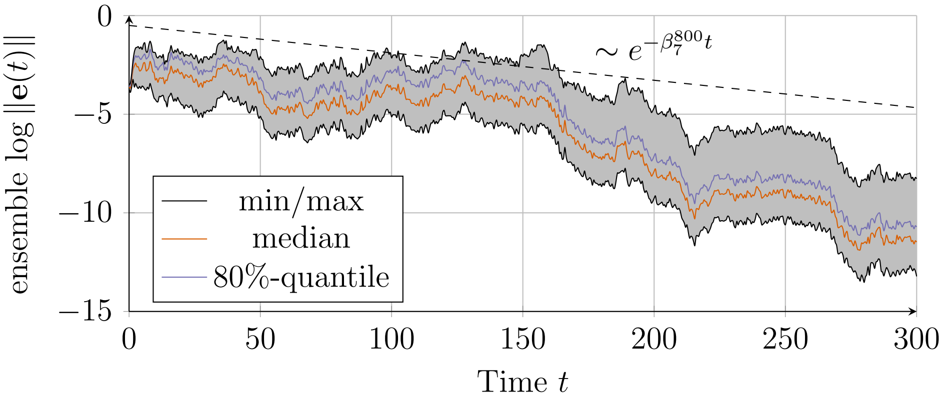


Fig. 5.4: Ensemble estimation error for the Lorenz'96 model with $k = 6$ and $N = 50$ simulation runs.

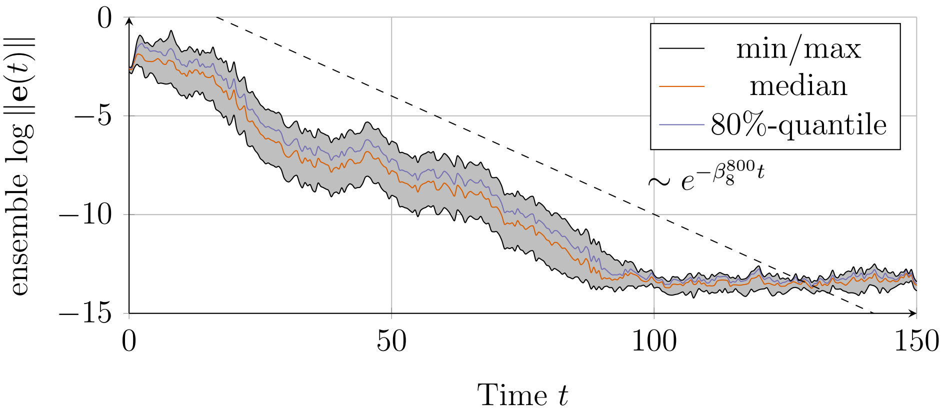


Fig. 5.5: Ensemble estimation error for the Lorenz'96 model with $k = 7$ and $N = 50$ simulation runs.

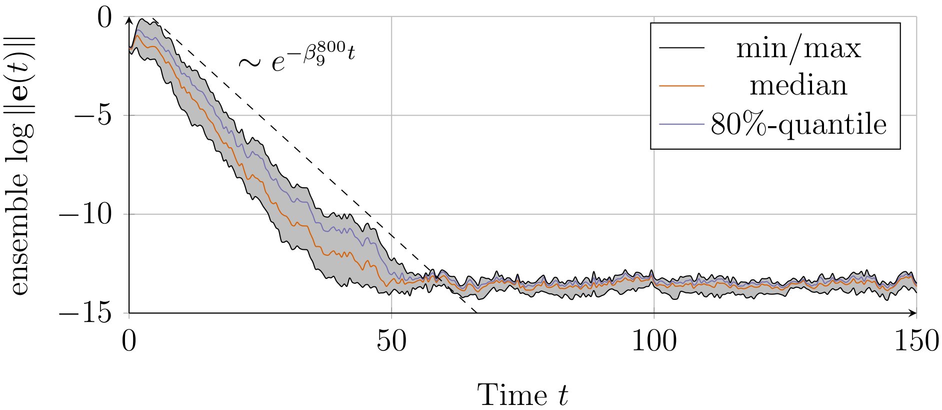


Fig. 5.6: Ensemble estimation error for the Lorenz'96 model with $k = 8$ and $N = 50$ simulation runs.

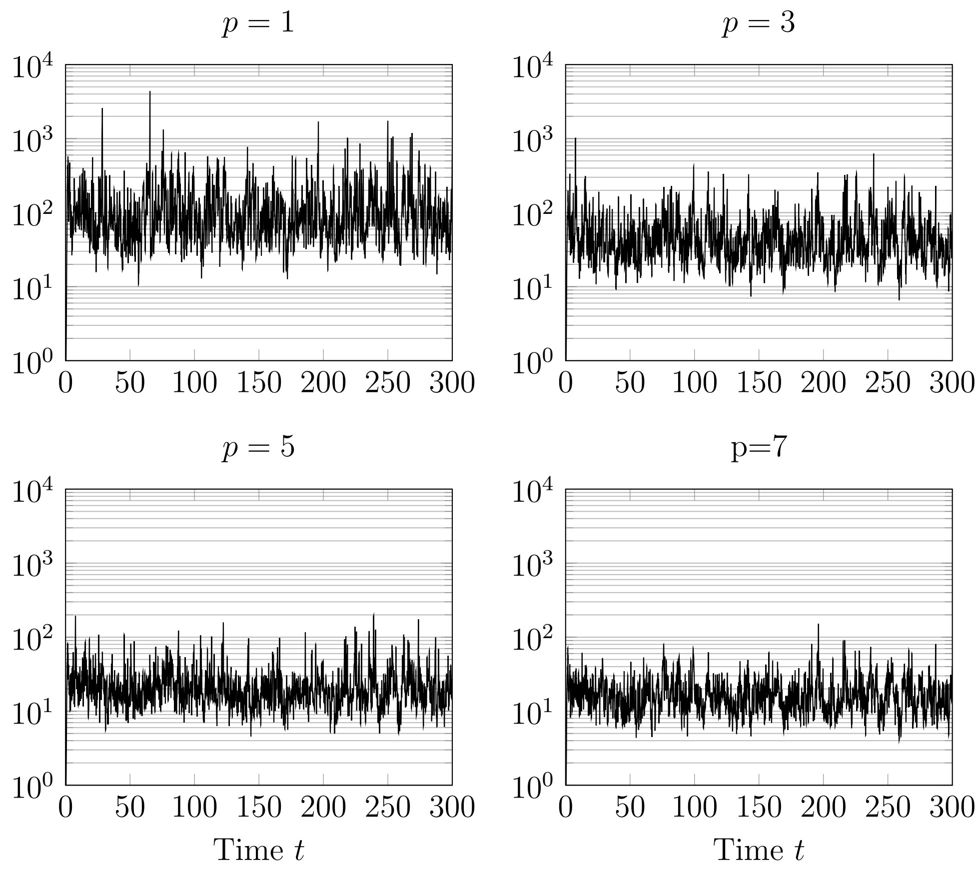


Fig. 5.7: Maximum eigenvalue of $\mathbf{P}(t)$ for the Lorenz'96 model with different measurement configurations.

subspace. The largest eigenvalue of $\mathbf{P}(t)$ for different measurement configurations is depicted in Fig. 5.7. This result indicates that the considered trajectory is uniformly completely constructible on the unstable subspace for any number of measurements.

An additional insight can be obtained by the investigation of the smallest eigenvalue of $\mathbf{P}(t)$. The number of measurements was now again chosen to be $p = 5$. If one considers only the unstable modes in the observer gain, i.e. $k = 6$, the smallest eigenvalue is uniformly bounded from below. However, if one takes into account an additional mode in the observer gain by choosing $k = 7$, the smallest eigenvalue tends to zero, see Fig. 5.8. This indicates that the observer gain is "loosing strength" in the already uniformly asymptotically stable directions. This is avoided by choosing $\mathbf{G}(t)$ as a positive definite matrix as proposed in the present observer design. The effect is also discussed for a deterministic interpretation of the Kalman-Bucy filter in [Esc18, p. 20] and for the discrete time setting in [Boc+17].

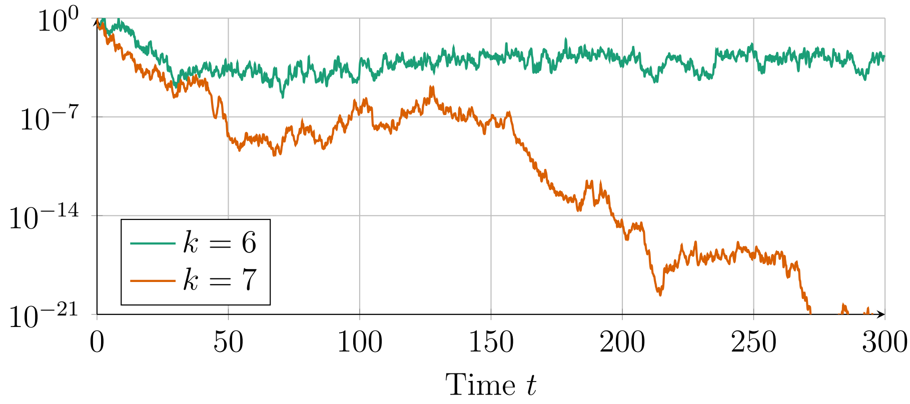


Fig. 5.8: Minimum eigenvalue of $\mathbf{P}(t)$ for the Lorenz'96 model with different values for k and $g = 0$ (minimum energy estimate).

5.4 Burgers Equation

Burgers equation is a nonlinear scalar partial differential equation which occurs in various problems like acoustics, fluid dynamics and macroscopic vehicular traffic models [MH78; Nag02; BLB10]. This makes it interesting as a benchmark example for state estimation [FZ18]. In this section, a semi-discretization of the so-called inviscid Burgers equation

$$\frac{\partial z}{\partial t} + z\frac{\partial z}{\partial x} = 0 \tag{5.20}$$

is considered.

The semi-discretization taken from [FZ18] results in a set of nonlinear ordinary differential equations and is given by

$$\dot{z}_i = -\frac{1}{6\Delta x}[z_i(z_{i+1} - z_{i-1})] + (z_{i+1}^2 - z_{i-1}^2),\ i = 1, \dots, n,\ \Delta x = \frac{2\pi}{n} \tag{5.21}$$

with the notational convention $z_{-1} \coloneqq z_{n-1}$, $z_0 \coloneqq z_n$, $z_{n+1} \coloneqq z_1$ and n as the system order dimension. This discretization, discussed in detail in [Jam07], preserves the quadratic energy $\sum_i z_i^2$ and hence $\|\mathbf{z}(t)\|$ with the state vector

$$\mathbf{z}(t) = \begin{bmatrix} z_1(t) & z_2(t) & \cdots & z_n(t) \end{bmatrix}^\mathsf{T}. \tag{5.22}$$

is constant.

The system order was chosen to be $n = 18$ and the system output was chosen as $\mathbf{y}(t) = \mathbf{C}_p\mathbf{z}(t)$ with $\mathbf{C}_p$ as in (5.19). The dimension of the measurement vector was chosen to be $p = 6$. The initial condition $\mathbf{z}(0)$ was chosen randomly from the interval $(0, 1)$ assuming a uniform distribution. The evolution of the system states is shown in Fig. 5.9.

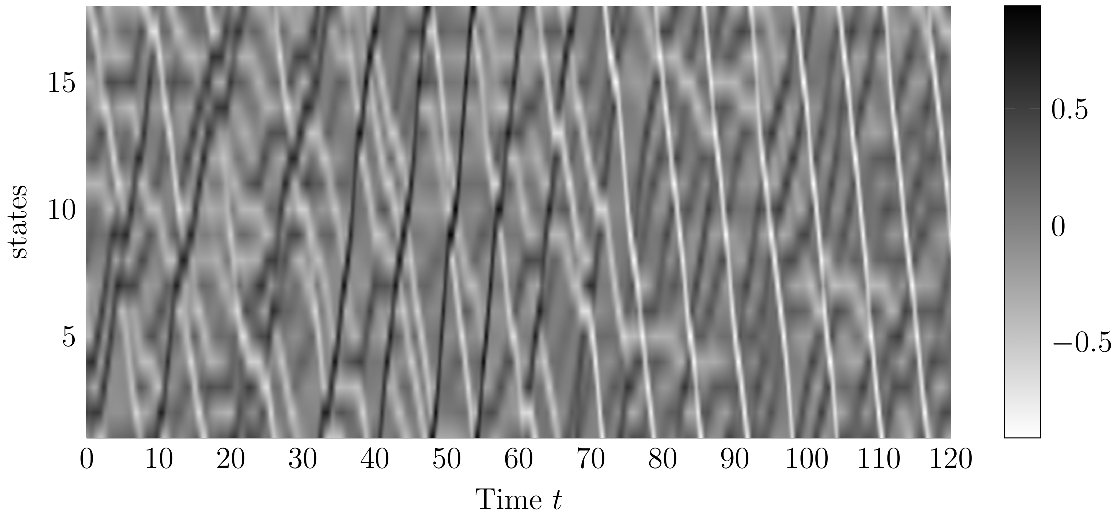


Fig. 5.9: Evolution of the system states of the semi-discretized inviscid Burgers' equation.

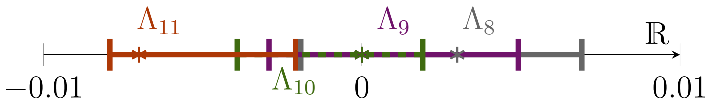


Fig. 5.10: Critical part of the exponential dichotomy spectrum for $H = 1100$.

Similar to the Lorenz'96 model, the spectral intervals of the linearized Burgers equation were approximated via the concepts presented in Chapter 4. The simulation step size and the final simulation time were chosen to be $T_s = 0.005$ and $T_f = 2000$. The critical part of the spectral intervals together with the approximated Lyapunov exponents are depicted in Fig. 5.10. The obtained numerical values (rounded to the third decimal) are given in Tab. 5.2 and a graphical representation is shown in Fig. 5.22.

It can be seen that especially around zero, the spectral intervals overlap. However, the number of non-negative upper Bohl exponents coincides with the number of non-negative Lyapunov exponents and $j^\star = 10$. This also coincides with the numerical results obtained in [FZ18]. Analogous to the Lorenz'96 model, a simulation of the proposed observer was carried out with $k = j^\star = 10$ and $N = 50$ simulation runs with small random initial errors. It should be remarked that for $k < 10$, no convergence of the observer error could be achieved. The results of the ensemble simulation for $k = 10$ are depicted in Fig. 5.11. The first negative upper Bohl exponent $\beta_{11} \approx -0.002$ governs the convergence speed of the estimation error and hence the convergence is very slow. By increasing the number of considered spectral intervals k up to $k = 13$, the convergence speed and the robustness with respect to the size of the initial error can again be increased. The ensemble simulation results for $k = 11$, $k = 12$ and $k = 13$ are depicted in Fig. 5.12, Fig. 5.13 and Fig. 5.14, respectively. These simulation results

Table 5.2: Approximated spectral intervals of Burgers model.

	Σ_{CL}	$\Sigma_{ED} \approx \Sigma_{IS}^{H=500}$		$\Sigma_{ED} \approx \Sigma_{IS}^{H=1100}$		Σ_{IS}^{0}	
interval	λ_i	$\alpha_i^{H=500}$	$\beta_i^{H=500}$	$\alpha_i^{H=1100}$	$\beta_i^{H=1100}$	α_i^0	β_i^0
1	0.483	0.428	0.549	0.449	0.511	-1.838	2.394
2	0.372	0.343	0.399	0.363	0.379	-1.786	2.001
3	0.292	0.251	0.322	0.284	0.309	-1.496	1.934
4	0.214	0.193	0.234	0.200	0.221	-1.484	1.814
5	0.162	0.119	0.192	0.142	0.170	-1.774	1.792
6	0.104	0.087	0.121	0.097	0.111	-1.328	1.660
7	0.049	0.040	0.064	0.047	0.055	-1.761	1.540
8	0.003	-0.009	0.014	-0.002	0.007	-1.652	1.509
9	0.000	-0.006	0.010	-0.003	0.005	-1.626	1.462
10	0.000	-0.010	0.007	-0.004	0.002	-1.030	1.228
11	-0.007	-0.018	0.004	-0.008	-0.002	-2.038	1.585
12	-0.059	-0.071	-0.043	-0.063	-0.053	-1.911	1.349
13	-0.104	-0.120	-0.081	-0.110	-0.095	-1.735	1.079
14	-0.144	-0.162	-0.122	-0.151	-0.130	-1.923	1.179
15	-0.220	-0.248	-0.190	-0.236	-0.207	-2.049	1.487
16	-0.293	-0.320	-0.267	-0.301	-0.286	-2.121	1.651
17	-0.364	-0.395	-0.315	-0.384	-0.343	-2.736	1.721
18	-0.489	-0.550	-0.441	-0.515	-0.456	-3.147	1.052

suggest that the rate of convergence is governed by the first upper Bohl exponent, which is not modified by the observer gain.

5.5 Wafer Temperature Profile Estimation

The problem investigated in this section is the temperature profile estimation of silicon wafers heated by light emitting diodes (LEDs). The ultimate control goal of this application is to ensure a uniform temperature profile of the wafer's surface following a desired temperature. This is required, e.g., for activating dopant or repairing damage of the wafer after ion implantations. Another application could be the treatment of the wafer surface with reactive gases to, e.g., remove hard-baked photoresist with the aid of ozone gas.

The surface temperature is usually measured only at specific points using a pyrometer. This makes it necessary to estimate the temperature profile on the whole wafer surface

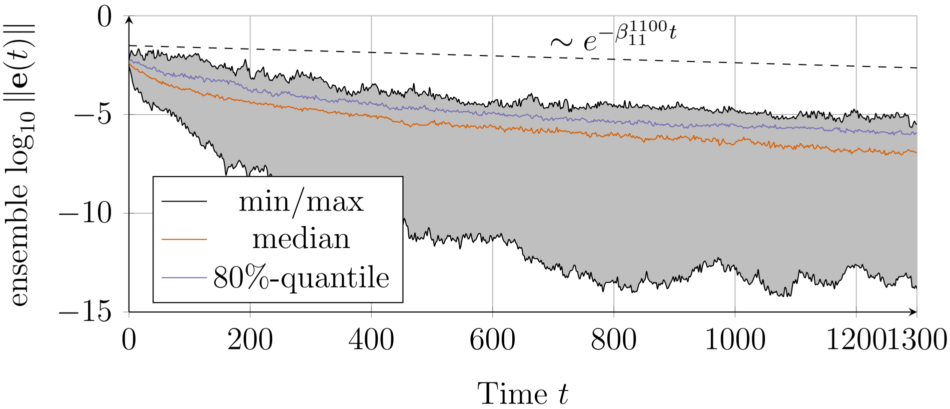


Fig. 5.11: Ensemble estimation error for the Burgers model with $k = 10$ and $N = 50$ simulation runs.

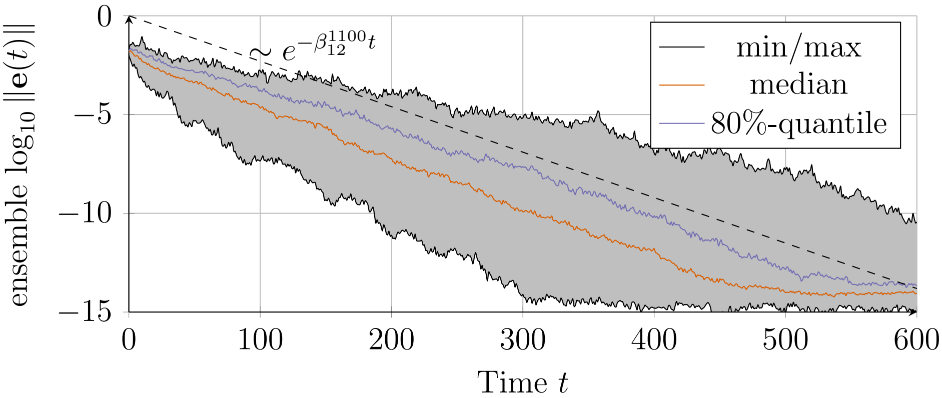


Fig. 5.12: Ensemble estimation error for the Burgers model with $k = 11$ and $N = 50$ simulation runs.

in order to apply model-based control techniques, as proposed, e.g., in [Kle+18; Kle+19].

In the considered setup, the silicon wafer is mounted on a chuck and rotates in a magnetic field in the process chamber. A static heating plate equipped with high power LEDs is mounted below the rotating chuck. The LEDs emit light with a wavelength of approximately 450 nm, which is absorbed by the wafer. The LEDs are grouped into four heating zones. All LEDs of one group are controlled simultaneously with a desired electrical power. Pyrometers are used in the present setup to measure the temperature at specific points. Four pyrometers are available to measure the temperatures in the four heating zones, see Fig. 5.15. One of these is used for the state estimation, the others for validation.

[Kle+18] presents a model for the heat transport in a rotating silicon wafer. The wafer

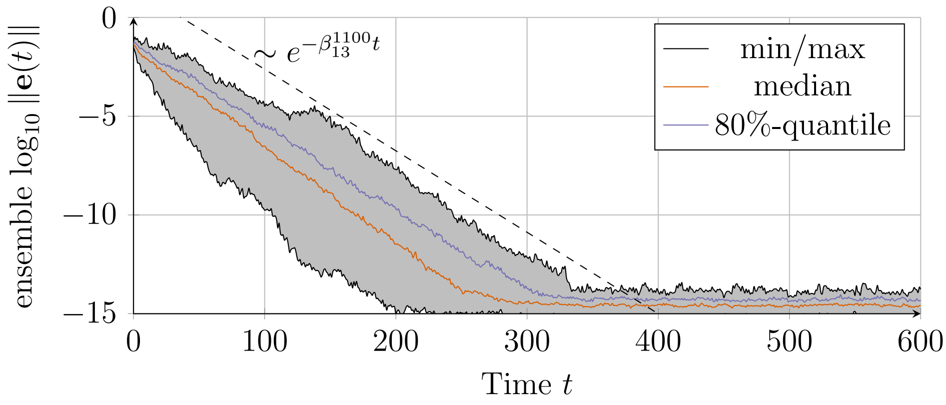


Fig. 5.13: Ensemble estimation error for the Burgers model with $k = 12$ and $N = 50$ simulation runs.

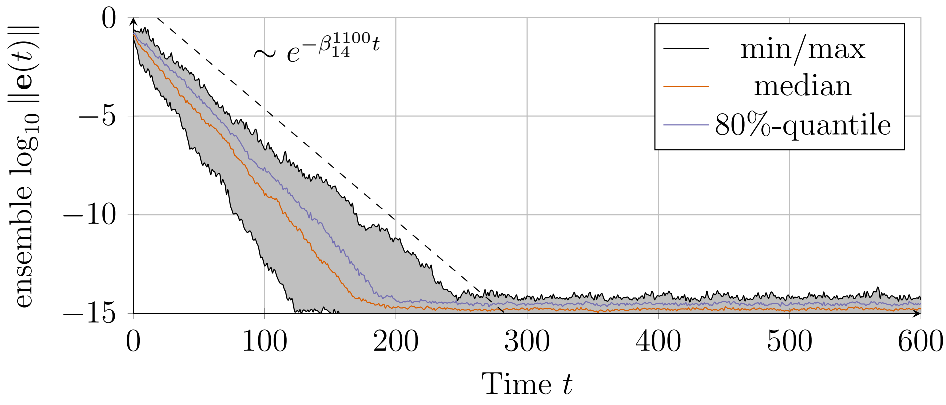


Fig. 5.14: Ensemble estimation error for the Burgers model with $k = 13$ and $N = 50$ simulation runs.

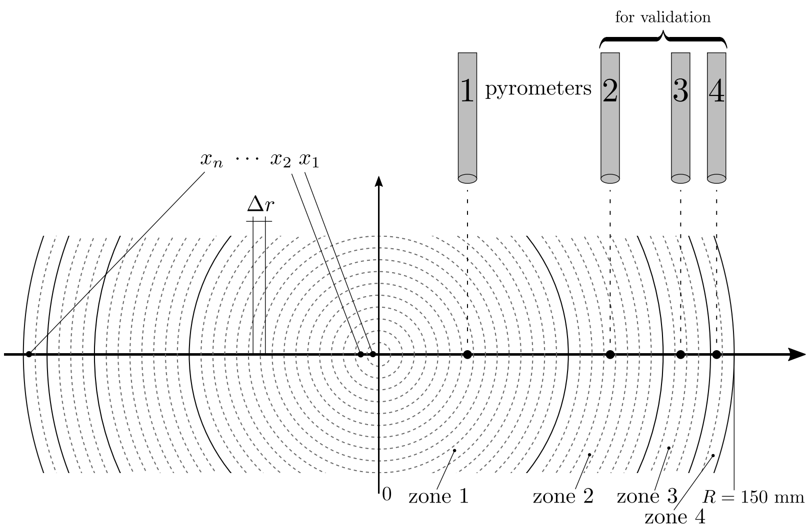


Fig. 5.15: Radial symmetric spatial discretization of the wafer together with the four available heating zones.

is heated via the aforementioned heating plate equipped with high power LEDs. The model is based on the heat equation and takes into account significant effects like the light absorption of the wafer and cooling effects caused by radiation. The distributed parameter model is discretized in space in order to obtain a set of (nonlinear) ordinary differential equations.

The basis of the mathematical model proposed in [Kle+18] is the heat transfer equation for rotation symmetric problems. This one dimensional partial differential equation is then discretized spatially with $n = 30$ grid points into concentric rings, see Fig. 5.15. The final nonlinear model is given by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, \mathbf{u}) = \underbrace{\mathbf{A}_1(\mathbf{x})\mathbf{x}}_{①} - \underbrace{\varepsilon \mathbf{A}_2(\mathbf{x}) \begin{bmatrix} x_1^4 \\ x_2^4 \\ \vdots \\ x_n^4 \end{bmatrix}}_{②} + \underbrace{\mathbf{B}(\mathbf{x})\mathbf{u}}_{③} . \tag{5.23}$$

The state x_i represents the temperature in the center of the i-th ring. The first part ① of the model is related to the lossless heat equation. The heat losses through radiation are described by ②, where ε is the total emissivity, which depends on the particular wafer type. The heat input through light absorption is described by ③. The input $\mathbf{u} \in \mathbb{R}^4$ is the electrical power supplied to the LED groups in the four heating zones. Details on the derivation on the model and the specific structure and properties of $\mathbf{A}_1(\mathbf{x})$, $\mathbf{A}_2(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ can be found in [Kle+18].

Only pyrometer 1, which measures the temperature in zone 1, is used for the observer design and hence the model possesses a scalar output

$$y = x_p = \mathbf{C}\mathbf{x}. \tag{5.24}$$

The constant output matrix is given by $\mathbf{C} = \mathbf{e}_p^\mathsf{T}$ and $\mathbf{e}_p$ as the unit vector corresponding to the position of pyrometer 1. The other pyrometers present in the setup are used for validation purposes in the experiments presented later on.

The emissivity parameter ε is a characteristic quantity for a specific wafer type and in particular depends on its dopant level and coating. The parameter varies from approximately 0.2 for a "bare silicon" wafer to 0.9 for a "highly doped" wafer. In some cases, the type of the processed wafer is not known and hence it is desired to estimate the total emissivity ε together with the temperature profile. In order to account for this, the state vector is extended according to $\tilde{\mathbf{x}}^\mathsf{T} = \begin{bmatrix}\mathbf{x}^\mathsf{T} & \varepsilon\end{bmatrix}$ and a new model

$$\begin{aligned} \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}}, \mathbf{u}) = \begin{bmatrix} \mathbf{A}_1(\mathbf{x})\mathbf{x} - \varepsilon \mathbf{A}_2(\mathbf{x}) \begin{bmatrix} x_1^4 \\ x_2^4 \\ \vdots \\ x_n^4 \end{bmatrix} + \mathbf{B}(\mathbf{x})\mathbf{u} \\ 0 \end{bmatrix} \\ y &= \begin{bmatrix}\mathbf{C} & 0\end{bmatrix} \tilde{\mathbf{x}} \end{aligned} \tag{5.25}$$

is obtained where ε is modeled as an unknown constant.

Simulation studies showed that system (5.23) possesses a negative exponential dichotomy spectrum in the presence of pseudo-random inputs. This allows to implement a trivial observer (TO) without measurement injection as it is also discussed in [Kle+19]. However, the performance of this observer is not satisfying because the convergence behavior cannot be influenced. Moreover, it does not yield an estimate for the total emissivity. The extended Kalman-Bucy filter and the extended subspace observer presented in Sections 5.1 and 5.2 are thus implemented for the nonlinear model (5.25). Due to the structure, this model possesses a Bohl exponent at zero, which has to be modified in the observer error dynamics. In the following, the extended subspace observer (ESO) in various configurations is compared with the extended Kalman-Bucy filter (EKBF). All algorithms are implemented with the fixed step-size solver `ode2 (Heun)` and a step-size of $T_s = 0.1$ s. The ESO is implemented for $k = 6$, $k = 3$ and $k = 1$. The tuning parameters are parametrized via scalars according to $\mathbf{G} = g\mathbf{I}$ and $\mathbf{P}(t_0) = p_0\mathbf{I}$ and identity matrices of appropriate dimensions. For the EKBF and the ESO with $k = 3$ and $k = 6$, the parameters are chosen as $g = 10^{-3}$ and $p_0 = 1$; for the ESO with $k = 1$, $p_0 = 1$ and $g = 1$ was chosen to obtain a similar convergence speed as for the other observers.

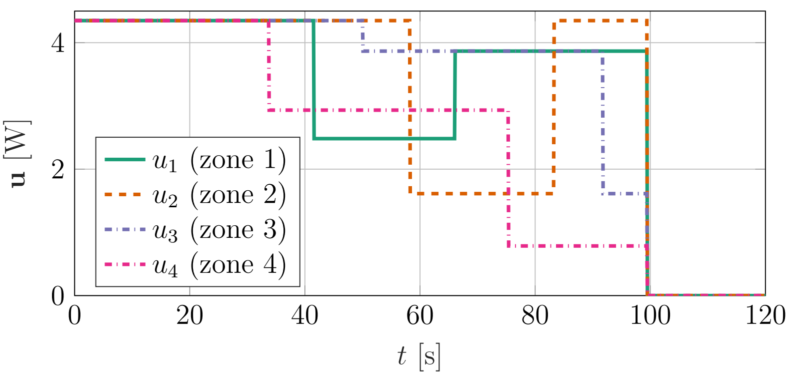


Fig. 5.16: Electrical power (input) for the LED heating plate in the four heating zones.

The experiments are conducted with a pseudo-random electrical heat power input as depicted in Fig. 5.16. The obtained measurements are fed off-line into the SIMULINK model with the implemented observer algorithms. In Fig. 5.17 the temperature estimates in the four heating zones for the different observers are compared with the measurements of the pyrometers. Note that below 300 °C, the pyrometers do not provide reliable measurements due to the measurement principle. One can see that the temperature estimates are qualitatively comparable for all implemented observers. The emissivity estimates are depicted in Fig. 5.18. The estimates are reasonable because the experiment was carried out with a highly doped wafer with an emissivity of approximately 0.85.

Remarkably, the ESO for $k = 1$ requires to solve a scalar Riccati differential equation only but still allows to get reasonable temperature and emissivity estimates. In contrast to the extended Kalman-Bucy filter, which requires to solve a set of differential equations of order $n + (\frac{n^2-n}{2} + n) = \frac{1}{2}n^2 + \frac{3}{2}n$, the proposed ESO requires to solve a differential equation of order $n + nk + (k + \frac{k^2-k}{2}) = n + k(n + 1 + \frac{k-1}{2})$. For the ESO with $k = 1$, this reduces to a differential equation of order 61 compared to 495 for the EKBF. The whole temperature profile estimate for the ESO with $k = 1$ is depicted in Fig. 5.19 together with all pyrometer measurements. This also shows the satisfying performance of the proposed observer design technique.

To sum up, the extended subspace observer presented in this chapter allows a straightforward observer design resulting in a computationally efficient observer. It turns out that for the present application, it suffices to solve a scalar Riccati differential equation to achieve an estimation performance similar to the full order extended Kalman-Bucy filter.

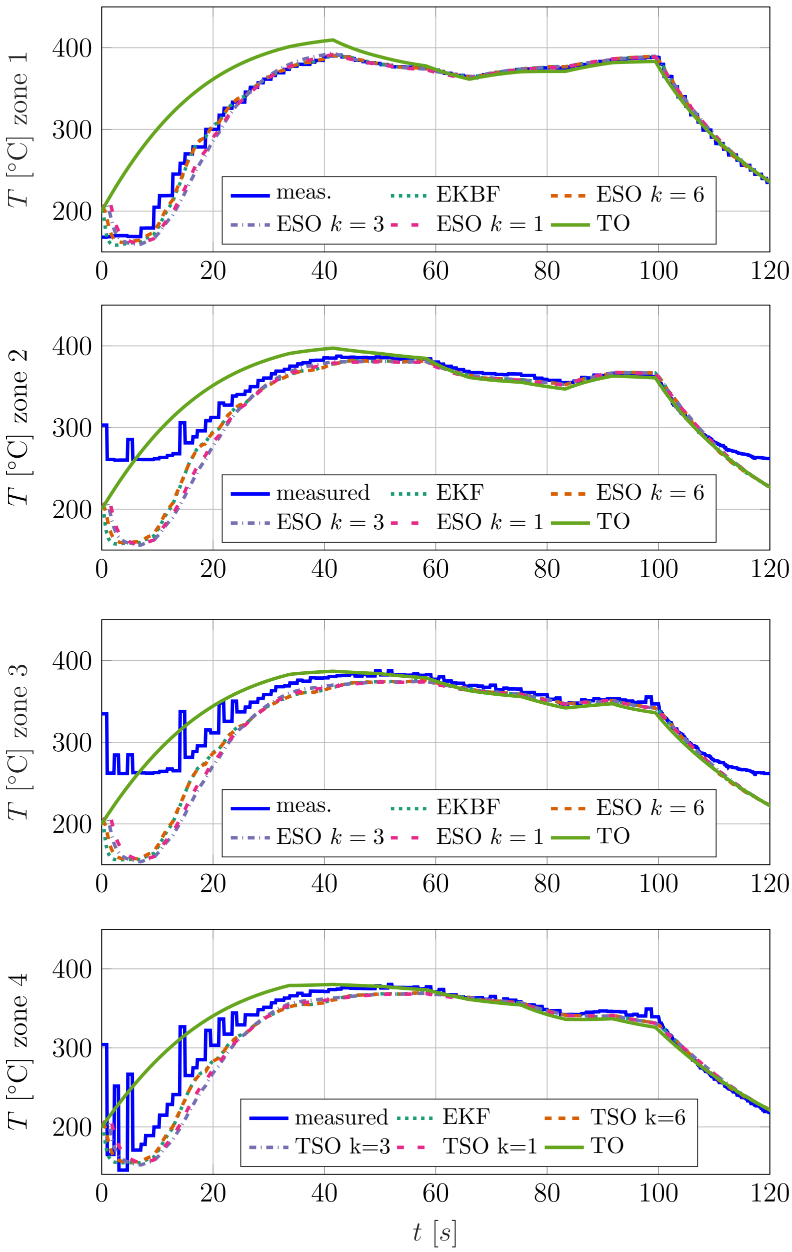


Fig. 5.17: Comparison of estimates and measurements of the wafer temperature in the four zones.

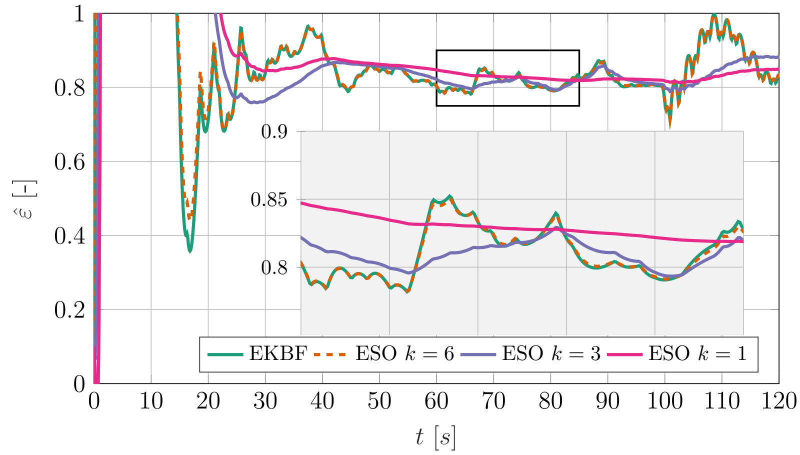


Fig. 5.18: Comparison of the total emissivity estimation for different observers.

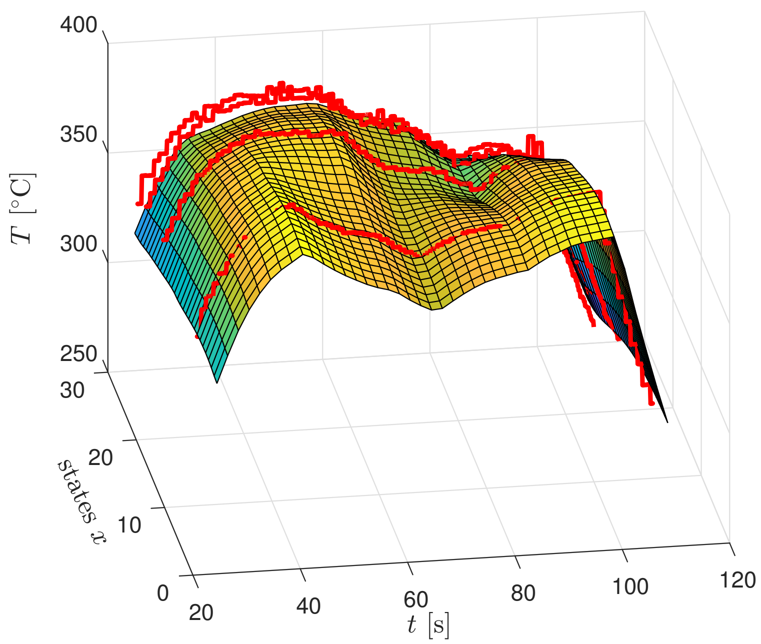


Fig. 5.19: ESO estimation results together with the wafer temperature measurements (red) for $k = 1$.

5.6 Discussion

This chapter proposes an observer design technique for nonlinear systems. The approach is based on the observer design for linear time varying systems presented in Chapter 4. Using a similar idea as the extended Kalman-Bucy filter, the nonlinear system is linearized along the estimated trajectory. Together with the ideas from Chapter 4, the solution of the (reduced order) Riccati differential equation used in the observer gain is then computed on-line from this linearization. Local uniform exponential stability of the estimation error dynamics is shown by utilizing a Lyapunov function. In fact, the obtained results suggest that the convergence speed and the region of convergence depend on the largest upper Bohl exponents not modified by the observer gain. The properties of the proposed observer were extensively studied by using numerical benchmark examples and real measurement data.

A key assumption in the stability proof is, that the (reduced order) Riccati equation has a uniformly bounded solution. This is also a standard assumption for the extended Kalman filter, see, e.g., [RSU98] for the extended Kalman-Bucy filter and [BRD97; RU99; BOE08] for the discrete time extended Kalman filter. The solution of the Riccati equation depends, however, on the specific estimate and hence uniform complete observability of the observer trajectory instead of the system's trajectory guarantees the existence of these bounds [Kar+18]. It would be interesting to relax this assumption and proof the convergence properties of the extended subspace observer solely based on the observability properties of the underlying system.

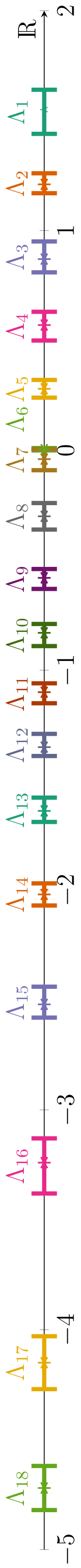


Fig. 5.20: Approximated exponential dichotomy spectrum of the Lorenz 96 model for $H = 300$.

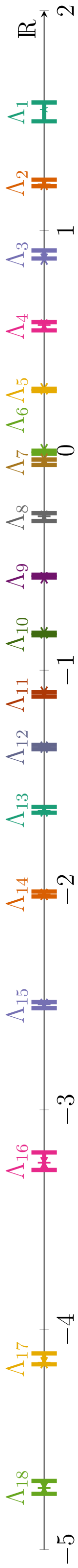


Fig. 5.21: Approximated exponential dichotomy spectrum of the Lorenz 96 model for $H = 800$.

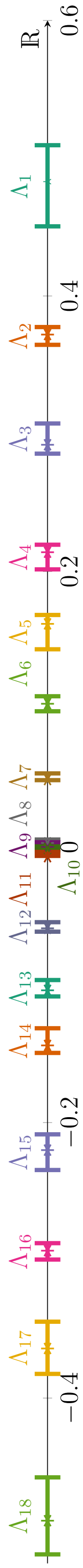


Fig. 5.22: Approximated exponential dichotomy spectrum of the Burgers equation for $H = 1100$.

Part II

State Estimation in the Presence of Unknown Inputs

Chapter 6

Observers for Linear Time Invariant Systems with Unknown Inputs

Real-world systems are often influenced by uncertain or unknown external disturbances. Accurate knowledge of the system states despite these uncertainties is often essential, for example, when considering control or fault detection problems [Pat97]. Since a large class of uncertainties and faults can be modeled as unknown inputs [PF89], dependable methods for state estimation in the presence of unknown inputs are desirable. A natural question deals with the existence of observers, i.e. dynamical systems, which allow to obtain state estimates without the knowledge of the unknown inputs but just by the knowledge of the system model, known inputs and outputs. The overall problem setup is sketched in Fig. 6.1.

Unknown input observers are proposed for a variety of different applications, e.g., in distributed and decentralized control. There, such observers can be used as local observers [PUP11]. The best known application for unknown input observers is fault detection and isolation (FDI), see [CP98] for a comprehensive overview. To solve the fault detection problem, sensor, actuator or system faults are modeled as unknown

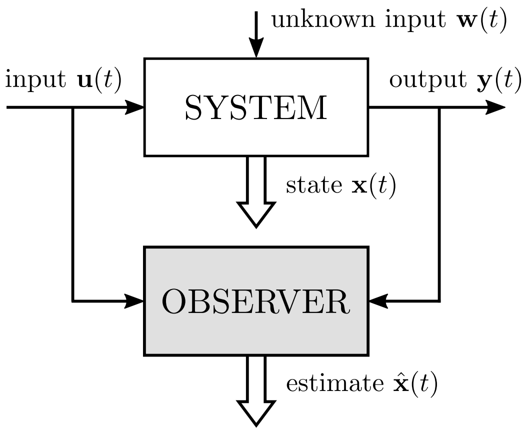


Fig. 6.1: General problem setup: The goal is to estimate the system states despite unknown inputs.

inputs. The basic idea for fault isolation is to use a bank of unknown input observers, where each observer is insensitive to one (or more) faults but sensitive to all others. The evaluation of all observer residuals, i.e., the output errors, then allows to determine the faulty component [CP98].

Another topic of increasing interest, especially for large scale networked control systems, is security and the detection of malicious attacks. In this setting, compared to pure networked computer systems, the interaction between physical systems and networked computer systems makes the attack detection more complex. Hence, it is argued in [PDB13] that information security concepts should be complemented by control theoretic methods for attack detection in cyber physical systems. The detectability of cyber attacks and the design of attack detection monitors is strongly related to the existence of unknown input observers [PDB13; CC17]. The variety of possible applications for unknown input observers motivates the research presented in this chapter. Different observer design techniques are presented throughout the chapter and the requirements for the existence of the observers are thoroughly investigated.

6.1 Related Work and Contribution

The problem of state estimation for the class of linear time invariant systems in the presence of unknown inputs has received considerable attention in the last decades, see, e.g., [Hau83; Kur83; HM92; Kra95; CYH97; ES98; Val99; SH07; IM20]. Different strategies for state estimation in the presence of unknown inputs are present in literature, such as classical linear approaches [Hau83] and first-order sliding mode based unknown input observers [ES98]. In addition to the strong detectability property, which is a prerequisite to solve the problem in general, these observers require the system to fulfill the so-called rank condition. In the single-input single-output case, this condition corresponds to the system's relative degree being zero or one.

For strongly observable systems, [Kra95] proposes a strategy to relax this restriction assuming availability of the output's time derivatives. It does not consider the design of the required differentiator, however.

Sliding mode techniques, such as the robust exact differentiator proposed in [Lev03] can be used to obtain exact derivatives for arbitrary signals with bounded derivatives. Based on this differentiator, several observer designs that do *not* impose the rank condition are proposed in literature. Some of these contributions require differentiability of the unknown input [FLD07] or additional structural system properties [FB06]. Other approaches, such as [FDL11], for example, are based on a large number of state transformations and thus are quite complex to design from a practical point of view. In [BF10; FBF11], a design based on invariant subspace methods, closely related to the work in [Kra95], is presented. In the case of non-differentiable unknown inputs, these techniques involve the open-loop integration of measured outputs, however, which in practice may lead to numerical problems.

In the first part of this chapter in Section 6.2, the basic concepts related to the observer design for systems with unknown inputs are recalled. An insightful interpretation of the existence conditions for linear unknown input observers is presented in Section 6.3. Using this interpretation, the resulting observer design is straightforward and reduces to a classical Luenberger observer design for observable or detectable systems.

The rest of the chapter is devoted to the case where a linear unknown input observer without using derivatives of the output does not exist. The relations between various concepts for strong detectability and observability are discussed and a novel observer design technique is proposed in Section 6.4. To solve the state estimation problem if the rank condition is not fulfilled, derivatives of the output signals are required in general [Hau83]. It is proposed in Section 6.5 to employ sliding mode concepts to obtain the required derivatives. For unstable systems, the two-stage estimation procedure presented in [BPF07; BF10] is adopted to avoid issues with unbounded derivatives.

For strongly observable systems, the presented observer design is straightforward without any state transformations; for strongly detectable systems, it is generalized by means of invariant subspace methods. Ultimately, a numerically stable observer for systems with bounded non-differentiable unknown inputs is obtained. The presented approach is based on the ideas of [Kra95; BFP09] and avoids open loop integration. The step-by-step observer design is summarized in Section 6.6 and the implementation and performance of the proposed observers is demonstrated by means of numerical examples in Section 6.7. In Section 6.8, the obtained results are discussed and possible future research directions are pointed out.

A part of the content presented in this chapter is adopted from

M. Tranninger et al. “Exact State Reconstruction for LTI-Systems with Non-Differentiable Unknown Inputs.” In: *2019 18th European Control Conference (ECC)*. June 2019, pp. 3096–3102. DOI: 10.23919/ECC.2019.8796142, © IEEE 2019.

6.2 Strong Observability and Detectability

This section discusses important aspects related to state estimation in the presence of unknown inputs. The required system properties are recalled and examples give better insight into the discussed theory.

Let a linear time invariant system be given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{D}\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad t \in \mathbb{J} = [0, \infty), \tag{6.1a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}(t) + \mathbf{F}\mathbf{w}(t) \tag{6.1b}$$

with $\mathbf{x}(t) \in \mathbb{R}^n$ as the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ as the known input, $\mathbf{w}(t) \in \mathbb{R}^q$ as the unknown input, $\mathbf{y}(t) \in \mathbb{R}^p$ as the output, and constant matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{F}$ of appropriate dimension. Without loss of generality, it can always be assumed that all known inputs are zero in the observability analysis, see Section 2.4. Hence, it suffices to investigate the (strong) observability properties of the system

$$\Sigma: \begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{D}\mathbf{w} & (6.2a) \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{F}\mathbf{w}, & (6.2b) \end{cases}$$

which is also referred to as the quadruple $(\mathbf{A}, \mathbf{D}, \mathbf{C}, \mathbf{F})$. Another standard assumption is that

$$\operatorname{rank} \begin{bmatrix} \mathbf{D} \\ \mathbf{F} \end{bmatrix} = q. \tag{6.3}$$

If this is not the case and $\operatorname{rank} [\mathbf{D}^\mathsf{T}\ \mathbf{F}^\mathsf{T}]^\mathsf{T} = r < q$, one can always construct a new input $\bar{\mathbf{w}} \in \mathbb{R}^r$ and a decomposition such that

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{F} \end{bmatrix} \mathbf{w} = \begin{bmatrix} \bar{\mathbf{D}} \\ \bar{\mathbf{F}} \end{bmatrix} \mathbf{T}\mathbf{w} = \begin{bmatrix} \bar{\mathbf{D}} \\ \bar{\mathbf{F}} \end{bmatrix} \bar{\mathbf{w}}, \tag{6.4}$$

where $\bar{\mathbf{w}} = \mathbf{T}\mathbf{w}$ and the matrix $[\bar{\mathbf{D}}^\mathsf{T}\ \bar{\mathbf{F}}^\mathsf{T}]^\mathsf{T}$ has full rank r, see [Val99].

The following notions were introduced by M. Hautus in [Hau83].

Definition 6.1 (strong observability) System Σ is called strongly observable if for all initial states $\mathbf{x}_0$ and any input $\mathbf{w}(t)$, $\mathbf{y}(t) = \mathbf{0}$ for all $t \in \mathbb{J}$ implies that $\mathbf{x}(t) = \mathbf{0}$ for all $t \in \mathbb{J}$.

Definition 6.2 (strong detectability) System Σ is called strongly detectable if for all initial states $\mathbf{x}_0$ and any input $\mathbf{w}(t)$, $\mathbf{y}(t) = \mathbf{0}$ for all $t \in \mathbb{J}$ implies $\mathbf{x}(t) \to \mathbf{0}$ for $t \to \infty$.

Definition 6.3 (strong* detectability) System Σ is called strong* detectable if for all initial states $\mathbf{x}_0$ and any input $\mathbf{w}(t)$, $\mathbf{y}(t) \to \mathbf{0}$ for $t \to \infty$ implies $\mathbf{x}(t) \to \mathbf{0}$ for $t \to \infty$.

It should be remarked that if $\mathbf{w}$ is identically zero for all $t \in \mathbb{J}$, the notions of strong observability and strong detectability reduce to the classical concepts of observability and detectability, respectively, see Section 2.4.

Conditions for strong detectability and observability are often stated in terms of invariant zeros of the system Σ. These zeros are characterized by the so-called Rosenbrock matrix, see, e.g., [TSH12, Chapter 7]. The important concepts are introduced in the following.

Definition 6.4 (polynomial matrix and normal rank) An $m \times n$ polynomial matrix $\mathbf{P}(s)$ is a matrix with polynomial entries in the form

$$\mathbf{P}(s) = \begin{bmatrix} p_{11}(s) & \cdots & p_{1n}(s) \\ \vdots & \ddots & \vdots \\ p_{m1}(s) & \cdots & p_{mn}(s) \end{bmatrix}. \tag{6.5}$$

The normal rank of $\mathbf{P}(s)$ is defined as

$$\operatorname{normrank} \mathbf{P} = \max\{\operatorname{rank} \mathbf{P}(\lambda) \mid \lambda \in \mathbb{C}\}. \tag{6.6}$$

Hence, for almost all $s \in \mathbb{C}$, one has $\operatorname{rank} \mathbf{P}(s) = \operatorname{normrank} \mathbf{P}$

An important polynomial matrix is the so-called Rosenbrock matrix proposed in [Ros67]. It is a useful representation of the dynamical system Σ and can be obtained from the Laplace transform of (6.2). In the following, this matrix is used to characterize the so-called invariant zeros of Σ.

Definition 6.5 (invariant zeros) The invariant zeros of Σ are the values $s = \lambda \in \mathbb{C}$ such that the Rosenbrock matrix

$$\mathbf{P}(s) = \begin{bmatrix} s\mathbf{I}_n - \mathbf{A} & -\mathbf{D} \\ \mathbf{C} & \mathbf{F} \end{bmatrix} \tag{6.7}$$

exhibits a rank loss, i.e.,

$$\operatorname{rank} \mathbf{P}(\lambda) < \operatorname{normrank} \mathbf{P}. \tag{6.8}$$

The following relations are stated in [Hau83].

Proposition 6.6 (strong observability and detectability conditions)
System Σ is

(i) strongly observable, if and only if $\operatorname{rank} \mathbf{P}(\lambda) = n + q$ *for all* $\lambda \in \mathbb{C}$.

(ii) strongly detectable, if and only if $\operatorname{rank} \mathbf{P}(\lambda) = n + q$ *for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\{\lambda\} \geq 0$.

(iii) strong detectable, if and only if it is strongly detectable and additionally*

$$\operatorname{rank} \begin{bmatrix} \mathbf{CD} & \mathbf{F} \\ \mathbf{F} & \mathbf{0} \end{bmatrix} = \operatorname{rank} \mathbf{F} + \operatorname{rank} \begin{bmatrix} \mathbf{D} \\ \mathbf{F} \end{bmatrix} = \operatorname{rank} \mathbf{F} + q. \tag{6.9}$$

Remark 6.7*:* For the characterization of strong observability and detectability, [Hau83] introduces a definition of zeros. However, this definition is slightly different from the notion of invariant zeros introduced in Definition 6.5, which can also be found, e.g., in [TSH12]. Both definitions coincide under the assumption that normrank $\mathbf{P}(s) = n + q$. In this case, condition *(i)* essentially requires that the system has no invariant zeros. Condition *(ii)* then states that the system is strongly detectable if and only if it possesses only invariant zeros λ with $\mathrm{Re}\{\lambda\} < 0$. It is stated in [TSH12, Lemma 8.9] that normrank $\mathbf{P}(s) = n +$ normrank $\mathbf{G}(s)$ with the transfer matrix

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{D} + \mathbf{F}. \tag{6.10}$$

Hence, normrank $\mathbf{P}(s) = n + q$ if and only if normrank $\mathbf{G}(s) = q$. A system for which the latter relation holds is also called left-invertible. It is assumed in the following that the system is left-invertible. A necessary condition for left-invertability of $\mathbf{G}(s)$ is that $p \geq q$, i.e., that there are more linearly independent measurements than unknown inputs. Consequently, this condition is also necessary for strong detectability.

Condition (6.9) is the so-called rank-condition and is a basic requirement for the design of a linear unknown input observer without using derivatives of the output signal.

One can see from the above conditions that strong observability implies strong detectability. Moreover, strong* detectability implies strong detectability. The converse is not true as shown by the following example borrowed from [Hau83].

Example 6.8*:* The double integrator system

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{aligned} \tag{6.11}$$

with the unknown input w is observable and hence also detectable. Its transfer function is given by

$$G(s) = \frac{1}{s^2} \tag{6.12}$$

and it has normrank $G(s) = 1 = q$. The system has no zeros and so it is strongly observable and strongly detectable. Let the system's output be given by

$$y(t) = \frac{(\sin t)^2}{t}. \tag{6.13}$$

Differentiating the output allows to fully determine the states and the input according to

$$x_1(t) = y(t) = \frac{(\sin t)^2}{t} \tag{6.14}$$

$$x_2(t) = \dot{y}(t) = 2(\cos t)^2 - \frac{(\sin t)^2}{t^2} \tag{6.15}$$

$$w(t) = \ddot{y}(t) = -4t(\sin t)^2 + \frac{2(\sin t)^2 - 2t^2(\cos t)^2}{t^3}. \tag{6.16}$$

Although it holds that

$$\lim_{t\to\infty} y(t) = 0, \tag{6.17}$$

the state x_2 cannot tend to zero because

$$\lim_{t\to\infty}\left[x_2(t) - 2(\cos t)^2\right] = -\lim_{t\to\infty}\frac{(\sin t)^2}{t} = 0 \tag{6.18}$$

and so x_2 tends to $2(\cos t)^2$.

The relations between the different observability and detectability concepts are summarized in Fig. 6.2.

An alternative characterization of strong observability can be given in terms of invariant subspaces, see e.g. [TSH12; Won12].

Definition 6.9 (Controlled Invariant Subspace [TSH12])
A subspace $\mathcal{W} \subset \mathbb{R}^n$ of system Σ is called controlled invariant if for any $\mathbf{x}_0 \in \mathcal{W}$ there exists an input $\mathbf{w}(t)$ such that $\mathbf{x}(t) \in \mathcal{W}$ for all $t \geq 0$.

Definition 6.10 (Weakly Unobservable Subspace [TSH12])
A point $\mathbf{x}_0 \in \mathbb{R}^n$ is called weakly unobservable if there exists an input $\mathbf{w}(t)$ such that the corresponding output $\mathbf{y}(t) = \mathbf{0}$ for $t \geq 0$. The set of all weakly unobservable points is denoted by $\mathcal{V}(\Sigma)$ and is called the weakly unobservable subspace.

The weakly unobservable subspace is a so-called output nulling invariant subspace as introduced in [And75]. This allows to characterize the weakly unobservable subspace in a slightly different but equivalent form. For an arbitrary $\mathbf{v} \in \mathcal{V}(\Sigma)$ there exists

observability	$\Rightarrow$	detectability		
$\Uparrow$		$\Uparrow$		
strong observability	$\Rightarrow$	strong detectability	$\Leftarrow$	strong* detectability

Fig. 6.2: Relation between observability and detectability concepts

some $\mathbf{w} \in \mathbb{R}^q$ and $v_1 \in \mathcal{V}(\Sigma)$ with

$$\mathbf{v}_1 = \mathbf{A}\mathbf{v} + \mathbf{D}\mathbf{w} \tag{6.19a}$$
$$\mathbf{0} = \mathbf{C}\mathbf{v} + \mathbf{D}\mathbf{w}. \tag{6.19b}$$

The following result is stated in [And75].

Lemma 6.11 (output nulling invariant subspace)
A subspace $\mathcal{W}$ of $\mathbb{R}^n$ is an output nulling invariant subspace, if and only if for some state feedback matrix $\mathbf{K}$ of appropriate dimension it holds that

$$(\mathbf{A} + \mathbf{D}\mathbf{K})\mathcal{W} \subset \mathcal{W} \tag{6.20a}$$
$$(\mathbf{C} + \mathbf{F}\mathbf{K})\mathcal{W} = \mathbf{0}. \tag{6.20b}$$

Relation (6.20a) states that for all vectors $\mathbf{w} \in \mathcal{W}$, it follows that $(\mathbf{A} + \mathbf{D}\mathbf{K})\mathbf{w} \in \mathcal{W}$. The second relation (6.20b) is equivalent to $(\mathbf{C} + \mathbf{F}\mathbf{K})\mathbf{w} = \mathbf{0}$ for all $\mathbf{w} \in \mathcal{W}$.

This result will be useful for the decomposition of strongly detectable systems in Section 6.4.2.

It is shown in [TSH12, Theorem 7.16] that the following statements are equivalent:

i) Σ is strongly observable;

ii) $\mathcal{V}(\Sigma) = 0$;

iii) The pair $(\mathbf{A} + \mathbf{D}\mathbf{K}, \mathbf{C} + \mathbf{F}\mathbf{K})$ is observable for all state feedback matrices $\mathbf{K}$ of appropriate dimension.

For systems without direct feed-through, $\mathcal{V}(\Sigma)$ is the largest controlled invariant subspace contained in $\ker \mathbf{C}$, see [TSH12]. Recursive algorithms to compute a basis for the weakly unobservable subspace are stated, e.g., in [Mol76; TSH12; Won12].

The following result is needed throughout this chapter.

Lemma 6.12 (strong detectability under state feedback and output injection)
System Σ is strongly observable (strongly detectable), if and only if the quadruple $\Sigma_K = (\mathbf{A} + \mathbf{D}\mathbf{K}, \mathbf{D}, \mathbf{C} + \mathbf{F}\mathbf{K}, \mathbf{F})$ is strongly observable (strongly detectable) for any state feedback matrix $\mathbf{K}$ of appropriate dimension.
Moreover, system Σ is strongly observable (strongly detectable), if and only if the quadruple $\Sigma_L = (\mathbf{A} + \mathbf{L}\mathbf{C}, \mathbf{D} + \mathbf{L}\mathbf{F}, \mathbf{C}, \mathbf{F})$ is strongly observable (strongly detectable) for any output injection matrix $\mathbf{L}$ of appropriate dimension.

Proof. Assume that Σ is strongly observable (strongly detectable). Consequently,

$$\operatorname{rank} \mathbf{P}(s) = \operatorname{rank} \begin{bmatrix} s\mathbf{I}_n - \mathbf{A} & -\mathbf{D} \\ \mathbf{C} & \mathbf{F} \end{bmatrix} = n + q \;\; \forall s \in \mathbb{C} \;\; (\forall s \in \mathbb{C} \text{ with } \operatorname{Re}\{s\} \geq 0)\,. \tag{6.21}$$

For the system under state feedback $\mathbf{w} = \mathbf{K}\mathbf{x} + \tilde{\mathbf{w}}$ with $\tilde{\mathbf{w}}$ as a new input and $\mathbf{K}$ as a feedback matrix of appropriate dimension, the Rosenbrock matrix $\mathbf{P}_K$ of Σ_K is given by

$$\mathbf{P}_K(s) = \begin{bmatrix} s\mathbf{I}_n - (\mathbf{A} + \mathbf{D}\mathbf{K}) & -\mathbf{D} \\ (\mathbf{C} + \mathbf{F}\mathbf{K}) & \mathbf{F} \end{bmatrix}. \tag{6.22}$$

It directly follows that

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} s\mathbf{I}_n - \mathbf{A} & -\mathbf{D} \\ \mathbf{C} & \mathbf{F} \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} s\mathbf{I}_n - \mathbf{A} & -\mathbf{D} \\ \mathbf{C} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{K} & \mathbf{I}_q \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} s\mathbf{I}_n - (\mathbf{A} + \mathbf{D}\mathbf{K}) & -\mathbf{D} \\ (\mathbf{C} + \mathbf{F}\mathbf{K}) & \mathbf{F} \end{bmatrix}. \end{aligned} \tag{6.23}$$

For the output injection case, let $\mathbf{L}$ be any injection gain of appropriate dimension. Analogously to (6.23) it follows that

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} s\mathbf{I}_n - \mathbf{A} & -\mathbf{D} \\ \mathbf{C} & \mathbf{F} \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} \mathbf{I}_n & -\mathbf{L} \\ \mathbf{0} & \mathbf{I}_p \end{bmatrix} \begin{bmatrix} s\mathbf{I}_n - \mathbf{A} & -\mathbf{D} \\ \mathbf{C} & \mathbf{F} \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} s\mathbf{I}_n - (\mathbf{A} + \mathbf{L}\mathbf{C}) & -(\mathbf{D} + \mathbf{L}\mathbf{F}) \\ \mathbf{C} & \mathbf{F} \end{bmatrix} = \operatorname{rank} \mathbf{P}_L(s) \end{aligned} \tag{6.24}$$

with $\mathbf{P}_L(s)$ as the Rosenbrock matrix of system Σ_L. □

The result states that the rank of the Rosenbrock matrix is invariant with respect to state feedback and output injection. This determines the name *invariant zeros* defined via the rank of the Rosenbrock matrix [CLS04, Sec. 3.6].

6.3 Linear Unknown Input Observer Design

Linear unknown input observer design techniques were proposed by various authors in the continuous-time or discrete time setting, see [Hau83; Kur83; HM94] and [CYH97; Val99] for the continuous and discrete time case, respectively. The proposed approaches are based either on frequency domain ideas as in [Hau83] or state space concepts based on algebraic considerations [Val99; HM94].

Different yet equivalent necessary and sufficient existence conditions are presented in the various contributions. Their equivalence to strong* detectability as introduced by Hautus is shown for the continuous and discrete time setting in [HM94] and [Val99], respectively.

The following derivation of an unknown input observer gives insight into this existence condition and results in a straightforward observer design. First, the case without direct feed-through is investigated. The general case is discussed in detail together with a design example in Section 6.3.2.

6.3.1 No Direct Feed-Through

It is assumed that there is no direct feed-through from the unknown inputs to the outputs, i.e., $\mathbf{F} = \mathbf{0}$, and the considered system (6.1) is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{D}\mathbf{w} \tag{6.25a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}. \tag{6.25b}$$

In the present case, the rank condition (6.9) simplifies to

$$\operatorname{rank}\mathbf{C}\mathbf{D} = \operatorname{rank}\mathbf{D} = q \tag{6.26}$$

and it is assumed that this condition holds.

Differentiating the output gives

$$\dot{\mathbf{y}} = \mathbf{C}\dot{\mathbf{x}} = \mathbf{C}\mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{D}\mathbf{w}. \tag{6.27}$$

Because the rank condition holds, $\mathbf{CD}$ has full column rank q and there exists a left inverse $(\mathbf{CD})^\dagger$ such that $(\mathbf{CD})^\dagger\mathbf{CD} = \mathbf{I}_q$.

This can be utilized to express the unknown input according to

$$\mathbf{w} = (\mathbf{CD})^\dagger \left[\dot{\mathbf{y}} - \mathbf{C}\mathbf{A}\mathbf{x}\right]. \tag{6.28}$$

By using this input in system (6.25), one obtains

$$\dot{\mathbf{x}} = \underbrace{\left[\mathbf{A} - \mathbf{D}(\mathbf{CD})^\dagger\mathbf{CA}\right]}_{\tilde{\mathbf{A}}}\mathbf{x} + \mathbf{D}(\mathbf{CD})^\dagger\dot{\mathbf{y}} \tag{6.29a}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{6.29b}$$

If the original system (6.25) is strongly observable (strongly detectable), the pair $(\tilde{\mathbf{A}}, \mathbf{C})$ is observable (detectable). This is an immediate consequence of Lemma 6.12 and the relation between the observability (detectability) and strong observability (strong detectability) notions, see Fig. 6.2.

Under the assumption that the output's derivative $\dot{\mathbf{y}}$ is known exactly, one can design a Luenberger observer for system (6.29) according to

$$\dot{\hat{\mathbf{x}}} = \tilde{\mathbf{A}}\hat{\mathbf{x}} + \mathbf{D}(\mathbf{CD})^\dagger\dot{\mathbf{y}} + \mathbf{L}\left[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}\right]. \tag{6.30}$$

The observer gain $\mathbf{L}$ has to be chosen such that $\tilde{\mathbf{A}} - \mathbf{LC}$ is a Hurwitz matrix. In the strongly observable case, the observer dynamics can be assigned arbitrarily. In the strongly detectable case, the invariant zeros of (6.25) are the unobservable modes of the pair $(\tilde{\mathbf{A}}, \mathbf{C})$. To see this, let λ be an invariant zero of the left-invertible system (6.25). In this case, $\operatorname{rank} \mathbf{P}(\lambda) < n+q$ and there exists a nonzero vector $[\mathbf{x}_0^\mathsf{T} \ \mathbf{w}_0^\mathsf{T}]^\mathsf{T}$ such that

$$\mathbf{P}(\lambda)\begin{bmatrix}\mathbf{x}_0\\ \mathbf{w}_0\end{bmatrix} = \begin{bmatrix}\lambda\mathbf{I}_n - \mathbf{A} & -\mathbf{D}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}_0\\ \mathbf{w}_0\end{bmatrix} = \mathbf{0}. \tag{6.31}$$

Now, let the input be $\mathbf{w}(t) = e^{\lambda t}\mathbf{w}_0$. The state $\mathbf{x}(s)$ considered in the Laplace domain can then be obtained according to

$$\mathbf{x}(s) = (s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{x}_0 + (s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{D}\mathbf{w}(s) = (s\mathbf{I}_n - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{D}(s\mathbf{I}_n - \lambda\mathbf{I}_n)^{-1}\mathbf{w}_0\right]. \tag{6.32}$$

With relation (6.31), one has $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{x}_0 = \mathbf{D}\mathbf{w}_0$ and hence it follows that

$$\mathbf{x}(s) = (s\mathbf{I}_n - \mathbf{A})^{-1}\left[\mathbf{x}_0 + (s\mathbf{I}_n - \lambda\mathbf{I}_n)^{-1}(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{x}_0\right], \tag{6.33}$$

because $(s\mathbf{I}_n - \lambda\mathbf{I}_n)^{-1}$ is a scalar matrix. By using the identity

$$(s\mathbf{I}_n - \lambda\mathbf{I}_n)^{-1}(s\mathbf{I}_n - \lambda\mathbf{I}_n) = \mathbf{I}_n \tag{6.34}$$

one can summarize the term $[\mathbf{x}_0 + (s\mathbf{I}_n - \lambda\mathbf{I})^{-1}(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{x}_0]$ according to

$$(s\mathbf{I}_n - \lambda\mathbf{I}_n)^{-1}(s\mathbf{I}_n - \lambda\mathbf{I}_n)\mathbf{x}_0 + (s\mathbf{I}_n - \lambda\mathbf{I}_n)^{-1}(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{x}_0 = (s\mathbf{I}_n - \lambda\mathbf{I}_n)^{-1}(s\mathbf{I}_n - \mathbf{A})\mathbf{x}_0 \tag{6.35}$$

and by combining (6.33) and (6.35) one gets

$$\mathbf{x}(s) = (s\mathbf{I} - \lambda\mathbf{I})^{-1}\mathbf{x}_0. \tag{6.36}$$

Hence, $\mathbf{x}(t) = e^{-\lambda t}\mathbf{x}_0$ is the solution of (6.25a) for the chosen input $\mathbf{w}(t)$. The corresponding output $\mathbf{y}(t)$ is equal to zero for all t, because $\mathbf{C}\mathbf{x}_0 = \mathbf{0}$ follows from (6.31).

Because (6.25) and (6.29) are equivalent systems, $\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0$ is also a solution of (6.29a). Moreover, $\mathbf{y}(t) = \mathbf{0}$ for all t implies that $\dot{\mathbf{y}}(t) = \mathbf{0}$ and hence $\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0$ is an unobservable mode of the free system $\dot{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{x}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$. This shows that the invariant zeros λ of (6.25) are unobservable eigenvalues of the pair $(\tilde{\mathbf{A}}, \mathbf{C})$.

The error dynamics for the estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ is obtained by combining (6.29a) and (6.30) according to

$$\dot{\mathbf{e}} = \left[\tilde{\mathbf{A}} - \mathbf{L}\mathbf{C}\right]\mathbf{e}. \tag{6.37}$$

Remarkably, these error dynamics are independent of the unknown input. In order to avoid the differentiation of the output signal, the observer (6.30) is implemented in a slightly different form. An auxiliary variable

$$\mathbf{z} = \hat{\mathbf{x}} - \mathbf{D}(\mathbf{C}\mathbf{D})^{\dagger}\mathbf{y} \tag{6.38}$$

is introduced, which allows to state the observer according to

$$\dot{\mathbf{z}}(t) = \tilde{\mathbf{A}}\hat{\mathbf{x}}(t) + \mathbf{L}\left[\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)\right], \qquad \mathbf{z}(0) = \mathbf{z}_0, \tag{6.39a}$$
$$\hat{\mathbf{x}}(t) = \mathbf{z}(t) + \mathbf{D}(\mathbf{CD})^\dagger \mathbf{y}(t). \tag{6.39b}$$

Differentiating (6.39b) yields the observer proposed in (6.30).

Remark 6.13*:* The unknown input observers presented, e.g., in [Val99; HM94; CYH97] are very similar to the observer (6.39), although their derivation follows a different idea. The ansatz is a generalized Luenberger observer of the form

$$\dot{\mathbf{z}} = \mathbf{N}\mathbf{z} + \mathbf{M}\mathbf{y} \tag{6.40a}$$
$$\hat{\mathbf{x}} = \mathbf{z} + \mathbf{E}\mathbf{y} \tag{6.40b}$$

and conditions for the matrices $\mathbf{N}$, $\mathbf{M}$ and $\mathbf{E}$ are derived such that the estimation error dynamics are independent of the unknown input and is asymptotically stable. The observer (6.39) is equivalent to a generalized Luenberger observer in the form of (6.40). This can be seen by replacing $\hat{\mathbf{x}}$ in (6.39a) with (6.39b), which yields

$$\dot{\mathbf{z}} = \underbrace{\left(\tilde{\mathbf{A}} - \mathbf{LC}\right)}_{\mathbf{N}}\mathbf{z} + \underbrace{\left[\left(\tilde{\mathbf{A}} - \mathbf{LC}\right)\mathbf{D}(\mathbf{CD})^\dagger + \mathbf{L}\right]}_{\mathbf{M}}\mathbf{y} \tag{6.41a}$$
$$\hat{\mathbf{x}} = \mathbf{z} + \underbrace{\mathbf{D}(\mathbf{CD})^\dagger}_{\mathbf{E}}\mathbf{y}. \tag{6.41b}$$

Estimation of the Unknown Input

Equation (6.28) allows to design an estimator for the unknown input. This requires the derivative of output to be available. This derivative may be obtained by utilizing higher order sliding mode techniques as discussed in Section 6.4 or other derivative estimation techniques like algebraic derivative estimation or causal approximations of linear differentiators [ZRH07; RJ09]. It is assumed now that this derivative is known exactly in the following. The estimate

$$\hat{\mathbf{w}}(t) = (\mathbf{CD})^\dagger\left[\dot{\mathbf{y}}(t) - \mathbf{CA}\hat{\mathbf{x}}(t)\right] + \mathbf{L}_w\left[\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)\right]. \tag{6.42}$$

is proposed in [CYH97] for the discrete time setting and is adapted here accordingly. The $q \times p$ matrix $\mathbf{L}_w$ is a design parameter. The unknown input estimation error $\mathbf{e}_w = \mathbf{w} - \hat{\mathbf{w}}$ can be obtained as

$$\mathbf{e}_w = -\left[(\mathbf{CD})^\dagger\mathbf{CA} - \mathbf{L}_w\mathbf{C}\right]\mathbf{e} \tag{6.43}$$

and hence depends on on the state estimation error. A bound on the estimation error for the unknown input is given by

$$\|\mathbf{e}_w\| \leq \| \underbrace{(\mathbf{CD})^\dagger \mathbf{CA} + \mathbf{L}_w \mathbf{C}}_{\mathbf{G}_w} \| \|\mathbf{e}\|. \tag{6.44}$$

The norm of $\mathbf{G}_w$ can be minimized in the least squares sense by the choice

$$\mathbf{L}_w = -(\mathbf{CD})^\dagger \mathbf{CAC}^\mathsf{T}(\mathbf{CC}^\mathsf{T})^{-1}. \tag{6.45}$$

6.3.2 The Direct Feed-Trough Case

Under the assumption that $\Sigma := (\mathbf{A}, \mathbf{D}, \mathbf{C}, \mathbf{F})$ is strong* detectable, this section discusses the design of a linear unknown input observer for system (6.2) with direct feed-through. Strong* detectability implies that that all invariant zeros of the left invertible system $(\mathbf{A}, \mathbf{D}, \mathbf{C}, \mathbf{F})$ are in $\mathbb{C}^-$ and that the rank condition

$$\operatorname{rank} \begin{bmatrix} \mathbf{CD} & \mathbf{F} \\ \mathbf{F} & \mathbf{0} \end{bmatrix} = \operatorname{rank} \mathbf{F} + \operatorname{rank} \begin{bmatrix} \mathbf{D} \\ \mathbf{F} \end{bmatrix} = k + q \tag{6.46}$$

holds, where $\operatorname{rank} \mathbf{F} = k \leq q$.

The output is directly influenced by the unknown input and the ideas presented in Section 6.3 are not directly applicable. Hence an output transformation $\tilde{\mathbf{y}} = \mathbf{Ty}$ is applied first, where $\mathbf{T}$ is chosen as a full row rank $(p - k) \times p$ Matrix such that $\mathbf{TF} = \mathbf{0}$. The new output is given by

$$\tilde{\mathbf{y}} = \mathbf{Ty} = \mathbf{TCx} + \underbrace{\mathbf{TF}}_{=\mathbf{0}} \mathbf{w}. \tag{6.47}$$

The derivative of this new output, contrary to $\dot{\mathbf{y}}$, exists also in the case of non-differentiable unknown inputs and is given by

$$\dot{\tilde{\mathbf{y}}} = \mathbf{T}\dot{\mathbf{y}} = \mathbf{TC}\dot{\mathbf{x}} = \mathbf{TCAx} + \mathbf{TCDw}. \tag{6.48}$$

Combining (6.48) with the original output equation $\mathbf{y} = \mathbf{Cx} + \mathbf{Fw}$ yields

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{T}\dot{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{TCA} \end{bmatrix} \mathbf{x} + \underbrace{\begin{bmatrix} \mathbf{F} \\ \mathbf{TCD} \end{bmatrix}}_{\mathbf{\Gamma}} \mathbf{w} \tag{6.49}$$

If the $(2p - k) \times q$ matrix $\mathbf{\Gamma}$ has full column rank q, the unknown input can be obtained by using a left inverse $\mathbf{\Gamma}^\dagger$ of $\mathbf{\Gamma}$ according to

$$\mathbf{w} = \mathbf{\Gamma}^\dagger \begin{bmatrix} \mathbf{y} \\ \mathbf{T}\dot{\mathbf{y}} \end{bmatrix} - \mathbf{\Gamma}^\dagger \begin{bmatrix} \mathbf{C} \\ \mathbf{TCA} \end{bmatrix} \mathbf{x}. \tag{6.50}$$

This can be achieved if and only if the rank condition holds. It is shown in [Val99] that

$$\operatorname{rank}\begin{bmatrix}\mathbf{CD} & \mathbf{F}\\ \mathbf{F} & \mathbf{0}\end{bmatrix} = \operatorname{rank}\mathbf{F} + \operatorname{rank}\begin{bmatrix}\mathbf{F}\\ \mathbf{TCD}\end{bmatrix} = k + \operatorname{rank}\mathbf{\Gamma}. \tag{6.51}$$

To see this, let a rank factorization of $\mathbf{F}$ be given by $\mathbf{F} = \bar{\mathbf{F}}\bar{\mathbf{L}}$ where $\bar{\mathbf{F}}$ is a full column rank $p \times k$ matrix and $\bar{\mathbf{L}}$ is a full row rank $k \times q$ matrix. There exists a non-singular $p \times p$ matrix $[\mathbf{H}^\mathsf{T}\ \mathbf{T}^\mathsf{T}]^\mathsf{T}$ with a matrix $\mathbf{H}$ of appropriate dimension[1] such that

$$\begin{bmatrix}\mathbf{H}\\ \mathbf{T}\end{bmatrix}\bar{\mathbf{F}} = \begin{bmatrix}\mathbf{I}_k\\ \mathbf{0}\end{bmatrix}. \tag{6.52}$$

Then, by reordering the block matrix

$$\operatorname{rank}\begin{bmatrix}\mathbf{CD} & \mathbf{F}\\ \mathbf{F} & \mathbf{0}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{F} & \mathbf{CD}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}, \tag{6.53}$$

and multiplying the latter matrix with the full rank $2p \times 2p$ matrix

$$\begin{bmatrix}\mathbf{0} & \mathbf{I}_p\\ \mathbf{T} & \mathbf{0}\\ \mathbf{H} & \mathbf{0}\end{bmatrix} \tag{6.54}$$

it follows that

$$\operatorname{rank}\begin{bmatrix}\mathbf{CD} & \mathbf{F}\\ \mathbf{F} & \mathbf{0}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{0} & \mathbf{I}_p\\ \mathbf{T} & \mathbf{0}\\ \mathbf{H} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{F} & \mathbf{CD}\\ \mathbf{0} & \mathbf{F}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{0} & \mathbf{F}\\ \mathbf{0} & \mathbf{TCD}\\ \bar{\mathbf{L}} & \mathbf{HCD}\end{bmatrix}. \tag{6.55}$$

Because the matrix $\bar{\mathbf{L}}$ has rank k, the matrix

$$\mathbf{\Gamma} = \begin{bmatrix}\mathbf{F}\\ \mathbf{TCD}\end{bmatrix} \tag{6.56}$$

has full column rank q if and only if the rank condition (6.46) holds. It should be remarked that if $\mathbf{F} = \mathbf{0}$, the matrix $\mathbf{T} = \mathbf{I}_p$ can be chosen and the design reduces to the one presented in Section 6.3.1.

The computed $\mathbf{w}$ is now used in the original system description (6.2) to obtain an unknown input free representation. For this, let a state feedback gain $\mathbf{K}$ be defined according to

$$\mathbf{K} = \mathbf{\Gamma}^\dagger\begin{bmatrix}\mathbf{C}\\ \mathbf{TCA}\end{bmatrix}. \tag{6.57}$$

[1]This transformation can be obtained, e.g., by computing the reduced row echelon form of $\bar{\mathbf{F}}$.

By combining (6.50), (6.57) and (6.2) one obtains the equivalent system representation

$$\dot{\mathbf{x}} = \underbrace{(\mathbf{A} - \mathbf{DK})}_{=\tilde{\mathbf{A}}}\mathbf{x} + \mathbf{D}\boldsymbol{\Gamma}^{\dagger}\begin{bmatrix}\mathbf{y}\\ \mathbf{T}\dot{\mathbf{y}}\end{bmatrix} \tag{6.58a}$$

$$\mathbf{y} = \underbrace{(\mathbf{C} - \mathbf{FK})}_{=\tilde{\mathbf{C}}}\mathbf{x} + \mathbf{F}\boldsymbol{\Gamma}^{\dagger}\begin{bmatrix}\mathbf{y}\\ \mathbf{T}\dot{\mathbf{y}}\end{bmatrix}. \tag{6.58b}$$

It is shown in Lemma 6.12 that the pair $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$ is observable (detectable) if the original system is strongly observable (detectable). If this is the case, a feedback gain $\mathbf{L}$ can be designed such that $(\tilde{\mathbf{A}} - \mathbf{L}\tilde{\mathbf{C}})$ is a Hurwitz matrix. As in Section 6.3.1, it is assumed in the first step that $\dot{\tilde{\mathbf{y}}} = \mathbf{T}\dot{\mathbf{y}}$ is known exactly. The proposed observer can then be stated as

$$\dot{\hat{\mathbf{x}}} = \tilde{\mathbf{A}}\hat{\mathbf{x}} + \mathbf{D}\boldsymbol{\Gamma}^{\dagger}\begin{bmatrix}\mathbf{y}\\ \mathbf{T}\dot{\mathbf{y}}\end{bmatrix} + \mathbf{L}\left(\mathbf{y} - \tilde{\mathbf{C}}\hat{\mathbf{x}} - \mathbf{F}\boldsymbol{\Gamma}^{\dagger}\begin{bmatrix}\mathbf{y}\\ \mathbf{T}\dot{\mathbf{y}}\end{bmatrix}\right). \tag{6.59}$$

The dynamics of the estimation error $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ are given by

$$\dot{\mathbf{e}} = (\tilde{\mathbf{A}} - \mathbf{L}\tilde{\mathbf{C}})\mathbf{e}. \tag{6.60}$$

Again, the unknown input does not appear in the estimation error dynamics and the error system is asymptotically stable for a suitable choice of $\mathbf{L}$.

In order to get rid of the derivative of $\tilde{\mathbf{y}}$, one can use a similar change of variables as already introduced in Section 6.3.1. Let the auxiliary variable $\mathbf{z}$ be given by

$$\mathbf{z} = \hat{\mathbf{x}} - (\mathbf{D} - \mathbf{LF})\,\boldsymbol{\Gamma}^{\dagger}\begin{bmatrix}\mathbf{0}\\ \mathbf{T}\mathbf{y}\end{bmatrix}. \tag{6.61}$$

Then, the observer (6.59) is equivalent to the implementation

$$\dot{\mathbf{z}} = \tilde{\mathbf{A}}\hat{\mathbf{x}} + \mathbf{L}(\mathbf{y} - \tilde{\mathbf{C}}\hat{\mathbf{x}}) + (\mathbf{D} - \mathbf{LF})\boldsymbol{\Gamma}^{\dagger}\begin{bmatrix}\mathbf{y}\\ \mathbf{0}\end{bmatrix}, \tag{6.62a}$$

$$\hat{\mathbf{x}} = \mathbf{z} + (\mathbf{D} - \mathbf{LF})\boldsymbol{\Gamma}^{\dagger}\begin{bmatrix}\mathbf{0}\\ \mathbf{T}\mathbf{y}\end{bmatrix}. \tag{6.62b}$$

Hence, the explicit differentiation can be avoided by a suitable implementation of the observer. The following example demonstrates the straightforward design procedure for the proposed observer.

Example 6.14*:* Let the system (6.2) be given with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The Rosenbrock matrix

$$\mathbf{P}(s) = \left[\begin{array}{ccc|c} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right] \tag{6.63}$$

has full column rank $n+q$ for all values of s and the system is strongly observable. Moreover, the rank condition holds and the system is strong* detectable.
A decoupling matrix is given by $\mathbf{T} = [0\ 1]$ and consequently $\mathbf{\Gamma} = [1\ 0\ 1]^\mathsf{T}$ by using (6.56). A left inverse $\mathbf{\Gamma}^\dagger$ is then given by

$$\mathbf{\Gamma}^\dagger = (\mathbf{\Gamma}^\mathsf{T}\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^\mathsf{T} = \begin{bmatrix} 1/2 & 0 & 1/2 \end{bmatrix}. \tag{6.64}$$

The state feedback $\mathbf{K}$ according to (6.57) to obtain the new system representation (6.58) can be computed according to

$$\mathbf{K} = \mathbf{\Gamma}^\dagger \begin{bmatrix} \mathbf{C} \\ \mathbf{TCA} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \end{bmatrix}. \tag{6.65}$$

The original system is given in controllable canonical form and also $\mathbf{FK}$ takes a simple form. This allows to directly state the auxiliary system (6.58) with

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1/2 & -1/2 & 0 \end{bmatrix} \text{ and } \tilde{\mathbf{C}} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{6.66}$$

It follows directly that $\tilde{\mathbf{A}}$ is no Hurwitz matrix and so a feedback matrix $\mathbf{L}$ has to be designed to obtain an asymptotically stable error system. The gain

$$\mathbf{L} = \begin{bmatrix} 4 & -2 & -5 \\ 0 & 1 & 2 \end{bmatrix}^\mathsf{T} \tag{6.67}$$

places the eigenvalues of $(\tilde{\mathbf{A}} - \mathbf{L}\tilde{\mathbf{C}})$ at $\{-1, -2, -2\}$. The unknown input is chosen as $w(t) = \sin(t)$ and the observer (6.62) is simulated with $\mathbf{x}_0 = [1\ 1\ 1]^\mathsf{T}$ and $\mathbf{z}_0 = [0\ 0\ 0]^\mathsf{T}$. The true and estimated states are depicted in Figure 6.3. Moreover, it can be seen in Figure 6.4 that the estimation error converges to zero and is not influenced by the unknown input.

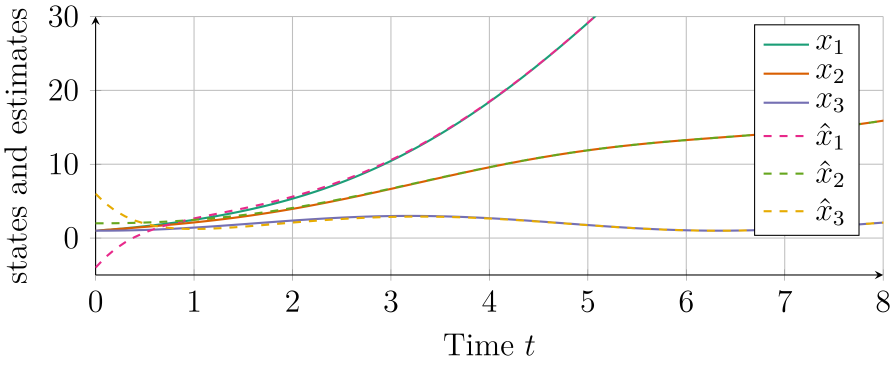


Fig. 6.3: Simulated states and corresponding estimates in the presence of the unknown input.

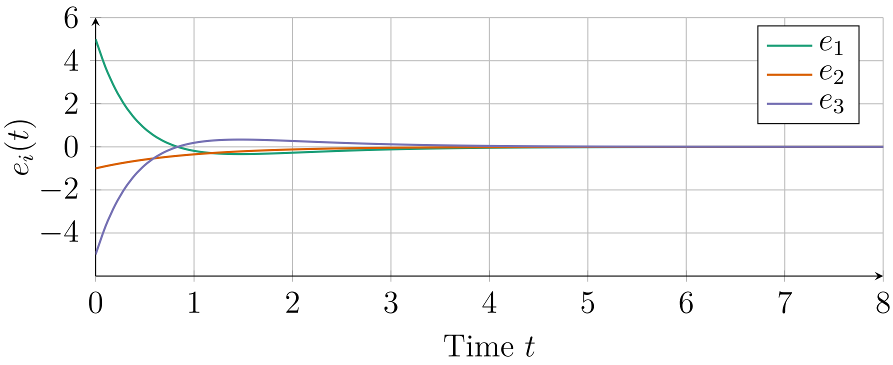


Fig. 6.4: The estimation errors are not influenced by the unknown input.

6.4 Derivative-based Unknown Input Observer

If the rank condition is not fulfilled, derivatives of the output have to be taken into account in order to design unknown input observers. The methods presented in this section are based on the ideas of [Kra95]. A link to invariant subspace methods is established and a computationally stable reconstruction scheme is presented for the case of non-differentiable unknown inputs. First, the basic idea is shown for differentiable unknown inputs. Then, a technique which removes this requirement is proposed. This part is also presented in [Tra+19].

6.4.1 State Reconstruction for Strongly Observable Systems

Again, system (6.2) is considered. Under the assumption that the unknown input is sufficiently often differentiable, the $(r-1)$ derivatives of the system's output (6.1b) are given by

$$
\begin{aligned}
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{F}\mathbf{w} \\
\dot{\mathbf{y}} &= \mathbf{C}\mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{D}\mathbf{w} + \mathbf{F}\dot{\mathbf{w}} \\
\ddot{\mathbf{y}} &= \mathbf{C}\mathbf{A}^2\mathbf{x} + \mathbf{C}\mathbf{A}\mathbf{D}\mathbf{w} + \mathbf{C}\mathbf{D}\dot{\mathbf{w}} + \mathbf{F}\ddot{\mathbf{w}} \\
&\vdots \\
\mathbf{y}^{(r-1)} &= \mathbf{C}\mathbf{A}^{r-1}\mathbf{x} + \mathbf{C}\mathbf{A}^{r-2}\mathbf{D}\mathbf{w} + \dots + \mathbf{C}\mathbf{D}\mathbf{w}^{(r-2)} + \mathbf{F}\mathbf{w}^{(r-1)}
\end{aligned}
\tag{6.68}
$$

with $r \in \mathbb{Z}^+$. The abbreviations

$$
\begin{aligned}
\tilde{\mathbf{y}}_r &= \begin{bmatrix} \mathbf{y}^\mathsf{T} & \dot{\mathbf{y}}^\mathsf{T} & \cdots & \mathbf{y}^{(r-1)\mathsf{T}} \end{bmatrix}^\mathsf{T}, \\
\tilde{\mathbf{w}}_r &= \begin{bmatrix} \mathbf{w}^\mathsf{T} & \dot{\mathbf{w}}^\mathsf{T} & \cdots & \mathbf{w}^{(r-1)\mathsf{T}} \end{bmatrix}^\mathsf{T}, \\
\mathbf{O}_r &= \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{r-1} \end{bmatrix}, \ \mathbf{J}_r = \begin{bmatrix} \mathbf{F} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\mathbf{D} & \mathbf{F} & \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\mathbf{A}\mathbf{D} & \mathbf{C}\mathbf{D} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{F} & \mathbf{0} \\ \mathbf{C}\mathbf{A}^{r-2}\mathbf{D} & \cdots & \cdots & \mathbf{C}\mathbf{D} & \mathbf{F} \end{bmatrix}
\end{aligned}
\tag{6.69}
$$

are introduced such that one has

$$
\tilde{\mathbf{y}}_r = \mathbf{O}_r\mathbf{x} + \mathbf{J}_r\tilde{\mathbf{w}}_r \tag{6.70}
$$

with $\mathbf{O}_r$ as an $rp \times n$ matrix and $\mathbf{J}_r$ an $rp \times rq$ matrix. Note that for $r = n$, $\mathbf{O}_n$ is the classical Kalman observability matrix. Let $\mathbf{M}_r$ be an $rp \times rp$ matrix such that

$$
\ker \mathbf{M}_r = \text{im } \mathbf{J}_r, \tag{6.71}
$$

and thus $\mathbf{M}_r\mathbf{J}_r = \mathbf{0}$ holds. This annihilating matrix can be constructed for example by using the singular value decomposition as discussed in Appendix B.2. Multiplying (6.70) with $\mathbf{M}_r$ from the left yields

$$
\mathbf{M}_r\tilde{\mathbf{y}}_r = \mathbf{M}_r\mathbf{O}_r\mathbf{x} + \mathbf{M}_r\mathbf{J}_r\mathbf{w}_r = \mathbf{M}_r\mathbf{O}_r\mathbf{x}. \tag{6.72}
$$

If $\mathbf{M}_r\mathbf{O}_r$ is left invertible, the states can be reconstructed by using the output and its $(r-1)$ derivatives according to

$$
\mathbf{x}(t) = (\mathbf{M}_r\mathbf{O}_r)^\dagger\mathbf{M}_r\tilde{\mathbf{y}}_r(t) \tag{6.73}
$$

with $(\mathbf{M}_r\mathbf{O}_r)^\dagger$ as a left inverse of $\mathbf{M}_r\mathbf{O}_r$.

Note that if $\mathbf{w}$ is not differentiable, the derivatives of $\mathbf{y}_r$ in general do not exist either. Because the system is linear and time invariant, one possible remedy is to exchange the order of differentiation and decoupling as presented in [Kra95; FBF11]. However, as this procedure involves open loop integration of the output $\mathbf{y}$, the resulting estimation scheme is not computationally stable. As an alternative, stable pre-filtering of $\mathbf{y}(t)$ is proposed here instead. To relax the differentiability assumption on the unknown input, consider (6.73) in the Laplace domain

$$\mathbf{x}(s) = (\mathbf{M}_r\mathbf{O}_r)^{\dagger}\mathbf{M}_r \begin{bmatrix} \mathbf{y}(s) \\ s\mathbf{y}(s) \\ \vdots \\ s^{r-1}\mathbf{y}(s) \end{bmatrix} \tag{6.74}$$

with zero initial conditions. Dividing this equation by a Hurwitz polynomial

$$\mu(s) = s^{r-1} + a_{r-2}s^{r-2} + \ldots + a_1 s + a_0 \tag{6.75}$$

of degree $r-1$ yields

$$\frac{1}{\mu(s)}\mathbf{x}(s) = (\mathbf{M}_r\mathbf{O}_r)^{\dagger}\mathbf{M}_r \begin{bmatrix} \frac{1}{\mu(s)}\mathbf{y}(s) \\ \frac{s}{\mu(s)}\mathbf{y}(s) \\ \vdots \\ \frac{s^{r-1}}{\mu(s)}\mathbf{y}(s) \end{bmatrix}. \tag{6.76}$$

Introducing

$$\boldsymbol{\psi}(s) = (\mathbf{M}_r\mathbf{O}_r)^{\dagger}\mathbf{M}_r \begin{bmatrix} \frac{1}{\mu(s)}\mathbf{y}(s) \\ \frac{s}{\mu(s)}\mathbf{y}(s) \\ \vdots \\ \frac{s^{r-1}}{\mu(s)}\mathbf{y}(s) \end{bmatrix}, \tag{6.77}$$

it can be concluded that

$$\mathbf{x}(s) = \mu(s)\boldsymbol{\psi}(s). \tag{6.78}$$

Note that now only filtered versions of the output appear in $\boldsymbol{\psi}$ and no time derivatives of $\mathbf{w}$ occur. Thus, $\mathbf{x}(t)$ can be determined as a linear combination of $\boldsymbol{\psi}$ and its derivatives according to

$$\mathbf{x}(t) = \boldsymbol{\psi}^{(r-1)}(t) + a_{r-2}\boldsymbol{\psi}^{(r-2)}(t) + \ldots + a_1\dot{\boldsymbol{\psi}}(t) + a_0\boldsymbol{\psi}(t). \tag{6.79}$$

In contrast to [Kra95; FBF11], this allows a computationally stable state reconstruction in the case of non-differentiable unknown inputs.

These derivations show that if the generalized inverse $(\mathbf{M}_r\mathbf{O}_r)^\dagger$ exists, the states can be reconstructed by using the output and its derivatives without knowledge of the unknown input. The question is, under which condition this inverse exists. Conditions for invertability of $\mathbf{M}_r\mathbf{O}_r$ are given in terms of state space conditions for strong observability in [Kra95]. This work is linked to the concept of weakly unobservable subspaces by the following lemma.

Lemma 6.15 (weakly unobservable subspace)
The weakly unobservable subspace $\mathcal{V}(\Sigma)$ of system Σ is given by

$$\mathcal{V}(\Sigma) = \ker(\mathbf{M}_n\mathbf{O}_n). \tag{6.80}$$

The proof can be found in Appendix A.4. The system is strongly observable, i.e. $\mathcal{V}(\Sigma) = \mathbf{0}$, if and only if $(\mathbf{M}_n\mathbf{O}_n)$ is left invertible and the states can be reconstructed by using $\boldsymbol{\psi}$ and its derivatives. For practical applications, the number of needed derivatives should be kept at a minimum. According to [Kra95, Proposition 1],

$$\operatorname{rank}\begin{bmatrix}\mathbf{O}_r & \mathbf{J}_r\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{I}_n & \mathbf{0}\\ \mathbf{O}_r & \mathbf{J}_r\end{bmatrix} \tag{6.81}$$

is a necessary and sufficient condition for invertability of $\mathbf{M}_r\mathbf{O}_r$. Based on this condition, it is possible to select the smallest integer r such that (6.81) holds; only derivatives up to $r-1$ are needed in this case. Note that $\nu \leq r \leq n$ holds with ν as the observability index [Che98], because rank $\mathbf{O}_r < n$ for $r < \nu$.

6.4.2 System Decomposition for Strongly Detectable Systems

If Σ is not strongly observable but strongly detectable, the system can be decomposed into a strongly observable and a weakly unobservable part as shown, e.g., in [BFP09; And75]. This decomposition is shown in the following. The system decomposition is based on the computation of a basis for the weakly unobservable subspace. This basis can be determined by using Lemma 6.15 as an alternative to the invariant subspace algorithm presented in [TSH12; Won12]. Let

$$\mathbf{M}_n\mathbf{O}_n = \mathbf{USV} = \begin{bmatrix}\mathbf{U}_1 & \mathbf{U}_2\end{bmatrix}\begin{bmatrix}\mathbf{S}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1^T\\ \mathbf{V}_2^T\end{bmatrix} \tag{6.82}$$

be the singular value decomposition of $\mathbf{M}_n\mathbf{O}_n$. Then, the columns of the orthogonal matrix $\mathbf{V}_2$, corresponding to the zero singular values, form a basis of $\mathcal{V}(\Sigma)$. As shown in [And75], the algebraic equation

$$\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix}\mathbf{V}_2 = \begin{bmatrix}\mathbf{V}_2 & \mathbf{D}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix} \tag{6.83}$$

for some matrices $\mathbf{X}$ and $\mathbf{Y}$ of appropriate dimensions has a solution

$$\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix} = \begin{bmatrix}\mathbf{V}_2 & \mathbf{D}\\ \mathbf{0} & \mathbf{F}\end{bmatrix}^{\dagger} \begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} \mathbf{V}_2. \tag{6.84}$$

With the abbreviation $\mathbf{K} = -\mathbf{Y}\mathbf{V}_2^{\mathsf{T}}$, the relations

$$\begin{aligned}(\mathbf{A}+\mathbf{D}\mathbf{K})\mathbf{V}_2 &= \mathbf{V}_2\mathbf{X},\\ (\mathbf{C}+\mathbf{F}\mathbf{K})\mathbf{V}_2 &= \mathbf{0},\end{aligned} \tag{6.85}$$

hold. Applying an input transformation

$$\bar{\mathbf{w}} = \mathbf{w} - \mathbf{K}\mathbf{x} \tag{6.86}$$

one obtains

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{A}+\mathbf{D}\mathbf{K})\mathbf{x} + \mathbf{D}\bar{\mathbf{w}},\\ \mathbf{y} &= (\mathbf{C}+\mathbf{F}\mathbf{K})\mathbf{x} + \mathbf{F}\bar{\mathbf{w}}.\end{aligned} \tag{6.87}$$

As it can be seen from (6.85) and Lemma 6.11, the columns of $\mathbf{V}_2$ span an output nulling subspace of this system.

Now, a state transformation $\bar{\mathbf{x}} = \mathbf{V}\mathbf{x}$ with $\mathbf{V}$ given in (6.82) is applied to system (6.87). Due to the orthogonality of $\mathbf{V}_1$ and $\mathbf{V}_2$ and the relations (6.83) and (6.85) one gets

$$\mathbf{V}_1^{\mathsf{T}}(\mathbf{A}+\mathbf{D}\mathbf{K})\mathbf{V}_1 = \mathbf{V}_1^{\mathsf{T}}\mathbf{A}\mathbf{V}_1; \qquad \mathbf{V}_1^{\mathsf{T}}(\mathbf{A}+\mathbf{D}\mathbf{K})\mathbf{V}_2 = \mathbf{V}_1^{\mathsf{T}}\mathbf{V}_2\mathbf{X} = \mathbf{0}; \tag{6.88a}$$

$$\mathbf{V}_2^{\mathsf{T}}(\mathbf{A}+\mathbf{D}\mathbf{K})\mathbf{V}_1 = \mathbf{V}_2^{\mathsf{T}}\mathbf{A}\mathbf{V}_2; \qquad \mathbf{V}_2^{\mathsf{T}}(\mathbf{A}+\mathbf{D}\mathbf{K})\mathbf{V}_2 = \mathbf{X}. \tag{6.88b}$$

Introducing the new system states $\bar{\mathbf{x}}_1 = \mathbf{V}_1^{\mathsf{T}}\mathbf{x}$ and $\bar{\mathbf{x}}_2 = \mathbf{V}_2^{\mathsf{T}}\mathbf{x}$ gives the final structure of the decomposed system according to

$$\begin{bmatrix}\dot{\bar{\mathbf{x}}}_1\\ \dot{\bar{\mathbf{x}}}_2\end{bmatrix} = \begin{bmatrix}\mathbf{A}_1 & \mathbf{0}\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix}\begin{bmatrix}\bar{\mathbf{x}}_1\\ \bar{\mathbf{x}}_2\end{bmatrix} + \begin{bmatrix}\mathbf{D}_1\\ \mathbf{D}_2\end{bmatrix}\bar{\mathbf{w}} \tag{6.89a}$$

$$\mathbf{y} = \mathbf{C}_1\bar{\mathbf{x}}_1 + \mathbf{F}\bar{\mathbf{w}} \tag{6.89b}$$

with $\mathbf{A}_1 = \mathbf{V}_1^{\mathsf{T}}\mathbf{A}\mathbf{V}_1$, $\mathbf{A}_3 = \mathbf{V}_2^{\mathsf{T}}\mathbf{A}\mathbf{V}_1$, and $\mathbf{A}_4 = \mathbf{X}$, see (6.88a). The new input matrix is given by $\mathbf{D}_1 = \mathbf{V}_1^{\mathsf{T}}\mathbf{D}$ and $\mathbf{D}_2 = \mathbf{V}_2^{\mathsf{T}}\mathbf{D}$. Because $(\mathbf{C}+\mathbf{F}\mathbf{K})\mathbf{V}_2 = \mathbf{0}$, the state $\bar{\mathbf{x}}_2$ does not contribute to the output and hence the output is given by (6.89b).

It is shown in [BFP09] that using this transformation, the subsystem specified by $(\mathbf{A}_1, \mathbf{D}_1, \mathbf{C}_1, \mathbf{F})$ is strongly observable and that the eigenvalues of $\mathbf{A}_4$ coincide with the invariant zeros of Σ.

Due to strong observability of $(\mathbf{A}_1, \mathbf{D}_1, \mathbf{C}_1, \mathbf{F})$, the states $\bar{\mathbf{x}}_1$ can be reconstructed by using the methods presented in Section 6.4.1. Moreover, the matrix $[\mathbf{D}_1^\mathsf{T}\ \mathbf{F}^\mathsf{T}]$ has full column rank q. Thus $\bar{\mathbf{w}}$ can be determined according to

$$\bar{\mathbf{w}} = \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{F} \end{bmatrix}^\dagger \begin{bmatrix} \dot{\bar{\mathbf{x}}}_1 - \mathbf{A}_1 \bar{\mathbf{x}}_1 \\ \mathbf{y} - \mathbf{C}_1 \bar{\mathbf{x}}_1 \end{bmatrix}. \tag{6.90}$$

By applying similar ideas as in Section 6.3 in order to avoid using the derivative $\dot{\bar{\mathbf{x}}}_1$ explicitly, the following estimate $\boldsymbol{\xi}_2$ for $\bar{\mathbf{x}}_2$ is proposed in [BFP09]:

$$\dot{\boldsymbol{\eta}} = \mathbf{A}_3 \bar{\mathbf{x}}_1 + \mathbf{A}_4 \boldsymbol{\xi}_2 - \mathbf{D}_2 \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{F} \end{bmatrix}^\dagger \begin{bmatrix} \mathbf{A}_1 \bar{\mathbf{x}}_1 \\ \mathbf{C}_1 \bar{\mathbf{x}}_1 - \mathbf{y} \end{bmatrix}, \tag{6.91a}$$

$$\boldsymbol{\xi}_2 = \boldsymbol{\eta} + \mathbf{D}_2 \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{F} \end{bmatrix}^\dagger \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \mathbf{0} \end{bmatrix}. \tag{6.91b}$$

Because $\bar{\mathbf{x}}_1$ is assumed to be known exactly, the estimation error $\mathbf{e}_2 = \bar{\mathbf{x}}_2 - \boldsymbol{\xi}_2$ is governed by the dynamics

$$\dot{\mathbf{e}}_2 = \mathbf{A}_4 \mathbf{e}_2. \tag{6.92}$$

These estimation error dynamics are asymptotically stable if and only if all eigenvalues of $\mathbf{A}_4$ (which correspond to the invariant zeros of Σ) are in $\mathbb{C}^-$, i.e., if and only if the system Σ is strongly detectable, see also [BFP09]. Even if the derivatives of the output are known exactly, the states can only be reconstructed asymptotically if the system is not strongly observable but merely strongly detectable.

6.5 Higher Order Sliding Mode Observer

Now, the robust observer design for system Σ with known and unknown inputs as in (6.1) is discussed. It is assumed that

(a1) system $(\mathbf{A}, \mathbf{D}, \mathbf{C}, \mathbf{F})$ is strongly observable.

(a2) the unknown input $\mathbf{w}(t)$ is bounded according to $\sup_{t \in \mathbf{J}} \|\mathbf{w}(t)\|_\infty \leq w^+ < \infty$.

If the system is not strongly observable but strongly detectable, the decomposition presented in Section 6.4.2 has to be carried out before the observer design and the strategy presented in this chapter can be applied to the strongly observable part. This estimate has then to be combined with the observer (6.91). If the unknown input is not bounded but has bounded derivatives, it is still possible to design an unknown input observer using the concepts presented in [BF10].

Because higher order sliding mode differentiators as presented in [Lev03] are exact in the absence of noise and show good robustness properties if noise is present, this is the method of choice here in order to obtain the time derivatives. This class of differentiators is shortly recapitulated in the following.

6.5.1 Higher Order Sliding Mode Differentiator

Consider a signal $f_0(t)$ where the r-th derivative $f_0^{(r)}(t)$ exists and has a known Lipschitz constant L_d, i.e., $|f_0^{(r+1)}(t)| < L_d$. The differentiator proposed in [Lev03] is defined in the recursive form

$$\begin{aligned}
\dot{z}_0 &= v_0 = -\lambda_r L_d^{1/(r-1)} |z_0 - f_0(t)|^{r/(r+1)} \operatorname{sign}(z_0 - f(t)) + z_1, \\
\dot{z}_1 &= v_1 = -\lambda_{r-1} L_d^{1/r} |z_1 - v_0|^{(r-1)/r} \operatorname{sign}(z_1 - v_0) + z_2, \\
&\vdots \\
\dot{z}_r &= -\lambda_0 L_d \operatorname{sign}(z_r - v_{r-1}),
\end{aligned} \tag{6.93}$$

with sufficiently large parameters λ_i. Because the right hand side of (6.93) is discontinuous, all solutions are understood in the sense of Filippov, see [Fil88] or [Sht+14, Chapter 2]. An established choice of parameters for a differentiator up to order 5 is $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$, as proposed in [Lev03]. A constructive design paradigm for the parameters of arbitrary order differentiators are presented in [RS18]. It is shown in [Lev03] that in the absence of measurement noise, $|z_i - f_0^{(i)}(t)| = 0$ holds after a finite transient time for $i = 0, \ldots, r$. Hence, this differentiator can be used to exactly reconstruct the r derivatives of $f_0(t)$ in finite time.

Following the notation from [GFD16], $\mathcal{D}_r^i \lfloor f_0(t) \rceil$ represents the signal z_i, $i = 1, \ldots, r$ in (6.93), which is obtained from the application of an r-th order differentiator to the signal $f_0(t)$.

For the numerical implementation of this differentiator a discretization procedure, which eliminates discretization chattering, is presented in [KR18].

6.5.2 Observer Design

To apply the higher order sliding mode differentiator to obtain the $(r-1)$ derivatives (6.79), the highest derivative is required to be Lipschitz continuous [Lev03] with a known Lipschitz constant.

As argued in [FLD07], the higher order sliding mode differentiator might not be directly applicable for unstable systems in the presence of known inputs. Thus, a cascaded estimation scheme as proposed in [FLD07; BFP09] is utilized. The structure of the cascaded observer is depicted in Fig. 6.5. It consists of a classical Luenberger observer to achieve a bounded estimation error and eliminate the influence of the known input. The estimation error is then reconstructed as shown in Section 6.4.1, where the required derivatives are obtained with the aid of higher order sliding mode differentiators.

For the Luenberger observer

$$\dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}_s(\mathbf{y} - \mathbf{C}\tilde{\mathbf{x}}), \tag{6.94}$$

a gain $\mathbf{L}_s$ is designed such that the resulting error dynamics

$$\begin{aligned} \dot{\mathbf{e}} &= (\mathbf{A} - \mathbf{L}_s\mathbf{C})\mathbf{e} + (\mathbf{D} - \mathbf{L}_s\mathbf{F})\mathbf{w} \\ \mathbf{e}_y &= \mathbf{C}\mathbf{e} + \mathbf{F}\mathbf{w} \end{aligned} \tag{6.95}$$

with $\mathbf{e} = \mathbf{x} - \tilde{\mathbf{x}}$ are asymptotically stable for $\mathbf{w} = \mathbf{0}$. This is always possible if the quadruple $(\mathbf{A}, \mathbf{D}, \mathbf{C}, \mathbf{F})$ is strongly observable.

Since $\mathbf{w}$ is bounded by $\|\mathbf{w}(t)\|_\infty \leq \mathbf{w}^+ < \infty$ according to assumption (a2), the estimation error then converges into a bounded neighborhood around the origin, i.e., there exists a finite time T and a constant $\mathbf{e}^+$ such that $\|\mathbf{e}(t)\|_\infty \leq \mathbf{e}^+$ holds for all $t \geq T$, see, e.g., [Son91].

The methods presented in Section 6.4.1 are now applied to (6.95) to reconstruct the estimation error. This is always possible, because the error system is strongly observable with respect to the unknown input if and only if system (6.2) is strongly observable, see Lemma 6.12.

To reconstruct the estimation error, equation (6.78) is adapted according to

$$\mathbf{e}(s) = \mu(s)(\mathbf{M}_r\mathbf{O}_r)^\dagger\mathbf{M}_r \begin{bmatrix} \frac{1}{\mu(s)}\mathbf{e}_y(s) \\ \frac{s}{\mu(s)}\mathbf{e}_y(s) \\ \vdots \\ \frac{s^{r-1}}{\mu(s)}\mathbf{e}_y(s) \end{bmatrix} = \mu(s)\boldsymbol{\psi}(s). \tag{6.96}$$

with $\mathbf{M}_r$ and $\mathbf{O}_r$ computed for the error system (6.95), i.e., with $\mathbf{A}-\mathbf{L}_s\mathbf{C}$ and $\mathbf{D}-\mathbf{L}_s\mathbf{F}$ instead of $\mathbf{A}$ and $\mathbf{D}$, respectively. By using the derivatives of $\boldsymbol{\psi}$, the estimation error can be reconstructed according to

$$\mathbf{e}(t) = \boldsymbol{\psi}^{(r-1)} + a_{r-2}\boldsymbol{\psi}^{(r-2)} + \ldots + a_1\dot{\boldsymbol{\psi}} + a_0\boldsymbol{\psi}. \tag{6.97}$$

Therein, a_0, …, a_{r-2} are the coefficients of a Hurwitz polynomial $\mu(s)$.

The derivatives of $\boldsymbol{\psi}$ are obtained by component-wise application of the robust exact differentiator proposed in [Lev03], see Section 6.5.1.

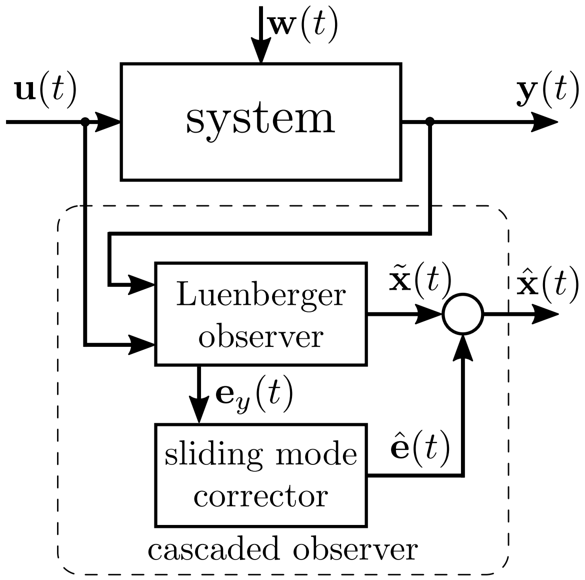


Fig. 6.5: Cascaded observer structure.

In order to apply this higher order sliding mode differentiator, a bound on the components of the r-th derivative of $\boldsymbol{\psi}$ needs to be known. Here, r is the smallest integer such that $\mathbf{M}_r\mathbf{O}_r$ is left invertible, see Section 6.4. Defining

$$\boldsymbol{\gamma} = \dot{\mathbf{e}} = (\mathbf{A} - \mathbf{L}_s\mathbf{C})\mathbf{e} + (\mathbf{D} - \mathbf{L}_s\mathbf{F})\mathbf{w} \tag{6.98}$$

and $\mathbf{z} = \boldsymbol{\psi}^{(r)}$. By linearity and (6.96), one obtains

$$s\mathbf{e}(s) = \boldsymbol{\gamma}(s) = s\mu(s)\boldsymbol{\psi}(s). \tag{6.99}$$

Multiplying the latter equation with s^{r-1} and rearranging the terms gives

$$\mathbf{z}(s) = s^r\boldsymbol{\psi}(s) = \frac{s^{r-1}}{\mu(s)}\boldsymbol{\gamma}(s) \tag{6.100}$$

or equivalently

$$\mathbf{z}(s) = \mathbf{G}(s)\boldsymbol{\gamma}(s) \text{ with } \mathbf{G}(s) = \mathbf{I}_n\frac{s^{r-1}}{\mu(s)}. \tag{6.101}$$

Let

$$\begin{aligned} \dot{\boldsymbol{\omega}} &= \boldsymbol{\Lambda}\boldsymbol{\omega} + \boldsymbol{\Pi}\boldsymbol{\gamma}, \\ \mathbf{z} &= \boldsymbol{\Gamma}\boldsymbol{\omega} + \boldsymbol{\Omega}\boldsymbol{\gamma}, \end{aligned} \tag{6.102}$$

be a minimal realization of the filtering transfer matrix $\mathbf{G}(s)$. Then, one can combine (6.95) and (6.102) to a state space description with input $\mathbf{w}$ and output $\mathbf{z}$ as

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{e}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} &= \begin{bmatrix} (\mathbf{A} - \mathbf{L}_s\mathbf{C}) & \mathbf{0} \\ \boldsymbol{\Pi}(\mathbf{A} - \mathbf{L}_s\mathbf{C}) & \boldsymbol{\Lambda} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \boldsymbol{\omega} \end{bmatrix} + \begin{bmatrix} \mathbf{D} - \mathbf{L}_s\mathbf{F} \\ \boldsymbol{\Pi}(\mathbf{D} - \mathbf{L}_s\mathbf{F}) \end{bmatrix} \mathbf{w}, \\ \mathbf{z} &= \begin{bmatrix} \boldsymbol{\Omega}(\mathbf{A} - \mathbf{L}_s\mathbf{C}) & \boldsymbol{\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \boldsymbol{\omega} \end{bmatrix} + \boldsymbol{\Omega}(\mathbf{D} - \mathbf{L}_s\mathbf{F})\mathbf{w}. \end{aligned} \tag{6.103}$$

Denoting the impulse response matrix of (6.103) by $\mathcal{G}_{wz}(t)$, the corresponding $\mathcal{L}_\infty$ gain can be computed according to

$$g_{wz} = \int_0^\infty \|\mathcal{G}_{wz}(t)\|_\infty \mathrm{d}t \tag{6.104}$$

with $\|.\|_\infty$ as the induced infinity norm, i.e. the row sum norm. Because $\mathbf{z}$ corresponds to $\boldsymbol{\psi}^{(r)}$ as defined in (6.100), a bound on $\boldsymbol{\psi}^{(r)}$ is given by $\|\boldsymbol{\psi}^{(r)}\| \leq g_{wz}w^+$ for a zero initial estimation error $\mathbf{e}_0 = \mathbf{0}$. For non-zero initial conditions, it holds that for every $\varepsilon > 0$ there exists a finite time T such that

$$\|\boldsymbol{\psi}^{(r)}(t)\| \leq g_{wz}w^+ + \varepsilon \text{ for all } t \geq T. \tag{6.105}$$

Hence, the Lipschitz constant L_d in the higher order sliding mode differentiator design can be chosen as

$$L_d = g_{wz}w^+ + \varepsilon \tag{6.106}$$

with some arbitrarily small $\varepsilon > 0$. For this choice, the $r-1$ derivatives of $\boldsymbol{\psi}(t)$ are obtained in finite time if the higher order sliding mode differentiator is applied (component-wise) to $\boldsymbol{\psi}(t)$.

6.6 Summary of the Higher Order Sliding Mode Observer Design

Based on the previous results, the observer design for system (6.1) under the assumptions (a1) and (a2) is summarized as

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}_s(\mathbf{y}(t) - \mathbf{C}\tilde{\mathbf{x}}(t)) \tag{6.107a}$$
$$\dot{\mathbf{z}}(t) = \mathbf{S}\mathbf{z}(t) + \mathbf{Q}(\mathbf{y}(t) - \mathbf{C}\tilde{\mathbf{x}}(t)) \tag{6.107b}$$
$$\boldsymbol{\psi}(t) = \mathbf{H}\mathbf{z}(t) + \mathbf{N}(\mathbf{y}(t) - \mathbf{C}\tilde{\mathbf{x}}(t)) \tag{6.107c}$$
$$\hat{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t) + \mathbf{L}_c\mathbf{h}(\boldsymbol{\psi}) \tag{6.107d}$$

The cascaded observer scheme, see Fig. 6.5, consists of the Luenberger observer (6.107a) and the higher order sliding mode corrector (6.107b)-(6.107d). All parts of the observer design are discussed in detail in the following.

The feedback gain $\mathbf{L}_s$ of the Luenberger observer (6.107a) has to be chosen such that $\tilde{\mathbf{A}} = (\mathbf{A} - \mathbf{L}_s\mathbf{C})$ is a Hurwitz matrix. This is always possible because of assumption (a1).

Now, let $\mathbf{O}_r$ and $\mathbf{J}_r$ be defined for the error system as

$$\mathbf{O}_r = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\tilde{\mathbf{A}} \\ \mathbf{C}\tilde{\mathbf{A}}^2 \\ \vdots \\ \mathbf{C}\tilde{\mathbf{A}}^{r-1} \end{bmatrix}, \ \mathbf{J}_r = \begin{bmatrix} \mathbf{F} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\mathbf{D} & \mathbf{F} & \vdots & \mathbf{0} & \mathbf{0} \\ \mathbf{C}\tilde{\mathbf{A}}\mathbf{D} & \mathbf{C}\mathbf{D} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{F} & \mathbf{0} \\ \mathbf{C}\tilde{\mathbf{A}}^{r-2}\mathbf{D} & \cdots & \cdots & \mathbf{C}\mathbf{D} & \mathbf{F} \end{bmatrix}, \tag{6.108}$$

with $r \leq n$ as the smallest integer such that

$$\operatorname{rank} \begin{bmatrix} \mathbf{O}_r & \mathbf{J}_r \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{O}_r & \mathbf{J}_r \end{bmatrix} \tag{6.109}$$

holds. Then, design a decoupling matrix $\mathbf{M}_r$ such that

$$\ker \mathbf{M}_r = \operatorname{im} \mathbf{J}_r. \tag{6.110}$$

The system (6.107b), (6.107c) is a minimal realization of the $n \times p$ transfer matrix $\mathbf{G}_\psi(s)$ with

$$\boldsymbol{\psi}(s) = \mathbf{G}_\psi(s)\mathbf{e}_y(s), \tag{6.111}$$

$\mathbf{G}_\psi(s) = \mathbf{H}(s\mathbf{I}_n - \mathbf{S})^{-1}\mathbf{Q} + \mathbf{N}$ and $\mathbf{e}_y(s) = \mathbf{y}(s) - \mathbf{C}\tilde{\mathbf{x}}(s)$, see [Che98, Ch. 7]. The transfer matrix is given by

$$\mathbf{G}_\psi(s) = (\mathbf{M}_r\mathbf{O}_r)^\dagger \mathbf{M}_r \begin{bmatrix} \frac{1}{\mu(s)}\mathbf{I}_p \\ \frac{s}{\mu(s)}\mathbf{I}_p \\ \vdots \\ \frac{s^{r-1}}{\mu(s)}\mathbf{I}_p \end{bmatrix} \tag{6.112}$$

with a Hurwitz polynomial $\mu(s) = s^{r-1} + a_{r-2}s^{r-2} + \cdots + a_1 s + a_0$ as a design parameter.

The nonlinear functional $\mathbf{h}(\cdot)$ is the realization of the higher order sliding mode differentiator according to

$$\mathbf{h}(\boldsymbol{\psi}) = \begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \\ \hline \mathcal{D}^1_{r-1}\lfloor\psi_1(t)\rceil \\ \vdots \\ \mathcal{D}^1_{r-1}\lfloor\psi_n(t)\rceil \\ \hline \vdots \\ \hline \mathcal{D}^{r-1}_{r-1}\lfloor\psi_1(t)\rceil \\ \vdots \\ \mathcal{D}^{\nu-1}_{\nu-1}\lfloor\psi_n(t)\rceil \end{bmatrix}. \tag{6.113}$$

The notation $\mathcal{D}_r^i\lfloor f(t)\rceil$ represents the estimated i-th derivative of the signal $f(t)$ obtained from the application of an r-th order differentiator, see Section 6.5.1.

The correction gain $\mathbf{L}_c$ is then defined by the coefficients of $\mu(s)$ according to

$$\mathbf{L}_c = \begin{bmatrix} a_0\mathbf{I}_n & a_1\mathbf{I}_n & \cdots & a_{r-1}\mathbf{I}_n \end{bmatrix}. \tag{6.114}$$

For a sufficiently large differentiator gain L_d, i.e., $L_d > g_{wz}w^+$ with g_{wz} given by (6.104), it holds that

$$\mathbf{h}(\boldsymbol{\psi}(t)) - \begin{bmatrix} \boldsymbol{\psi}(t) \\ \dot{\boldsymbol{\psi}}(t) \\ \vdots \\ \boldsymbol{\psi}^{(r-1)} \end{bmatrix} = \mathbf{0} \tag{6.115}$$

for all $t \geq t_f$ and t_f as some finite time instant. This implies that the estimation error of the Luenberger observer $\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t)$ can be reconstructed in finite time. The estimate $\hat{\mathbf{e}}$ for $\mathbf{e}$ is given by

$$\hat{\mathbf{e}}(t) = \mathbf{L}_c\mathbf{h}(\boldsymbol{\psi}(t)). \tag{6.116}$$

With $\hat{\mathbf{x}} = \tilde{\mathbf{x}} + \hat{\mathbf{e}}$ one achieves that

$$\mathbf{x}(t) - \hat{\mathbf{x}}(t) = \mathbf{0} \tag{6.117}$$

for all $t \geq t_f$.

6.7 Numerical Example

This section exemplarily shows the design of the higher order sliding mode observer presented in Sections 6.5 and 6.6. Let system Σ be given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -3 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 0 \end{bmatrix}, \ \mathbf{F} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.118}$$

It is assumed that the unknown input w is bounded according to $|w(t)| \leq 1$. The Rosenbrock matrix is given by

$$\mathbf{P}(s) = \left[\begin{array}{ccc|c} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ -1 & 2 & s+3 & -1 \\ \hline 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right] \tag{6.119}$$

and rank $\mathbf{P}(s) = 5$ for all values $s \in \mathbb{C}$; thus the system is strongly observable. The matrix $\mathbf{A}$ is in companion form and its characteristic polynomial can be directly found as $p(s) = s^3 + 3s^2 + 2s - 1$. This is no Hurwitz polynomial and the system is unstable. A Luenberger observer gain is designed according to

$$\mathbf{L}_s = \begin{bmatrix} 3 & 0 & 0 \\ -2 & -1 & 1 \end{bmatrix}^T \tag{6.120}$$

such that the set of eigenvalues of $\tilde{\mathbf{A}} = (\mathbf{A} - \mathbf{L}_s\mathbf{C})$ is given by $\{-1, -2, -3\}$. The observability index of the pair $(\mathbf{A}, \mathbf{C})$ and also of $(\tilde{\mathbf{A}}, \mathbf{C})$ is $\nu = 2$.

However, the invertability condition stated in (6.109) is not satisfied for $r = \nu = 2$ but for $r = 3$ and thus derivatives up to order 2 are required. The output error of the Luenberger observer is filtered according to (6.111) with $\mu(s) = (s + 5)^2$.

For the implementation of the component-wise higher order sliding mode differentiator, the discrete time implementation proposed in [Koc+19] and provided by the robust exact differentiator toolbox [Rei+18] was utilized.

The unknown input $w(t)$ was chosen as a periodic square wave signal with period 2, a pulse width of 1, and an amplitude of 1. The initial system states were chosen as $x_0 = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T$. The simulations were carried out in MATLAB/SIMULINK with the fixed step solver `ode1` and a step size of $T_s = 10^{-4}$.

In Fig. 6.6, the true and reconstructed system states using the proposed cascaded observer scheme are shown. The maximum estimation errors of the Luenberger observer, and the proposed observer are shown in Fig. 6.7. The error of the proposed observer converges into a small vicinity of zero due to discretization effects of the robust exact differentiator. Discretization chattering is avoided successfully due to the chosen discretization scheme. In order to increase the asymptotic accuracy, the step size can be reduced. The norm of the estimation error for different choices of the step size is depicted in Fig. 6.8.

6.8 Discussion

This chapter presented various observer design techniques for linear time invariant systems in the presence of unknown inputs. For strongly observable and strongly detectable systems, which additionally fulfill the so-called rank condition, linear unknown input observers can be designed. The observer design presented in Section 6.3 is straightforward and basically reduces to a classical Luenberger observer design.

If the rank condition is not fulfilled, a derivative based unknown input observer design is discussed in Section 6.4. In order to obtain the required derivatives, a higher

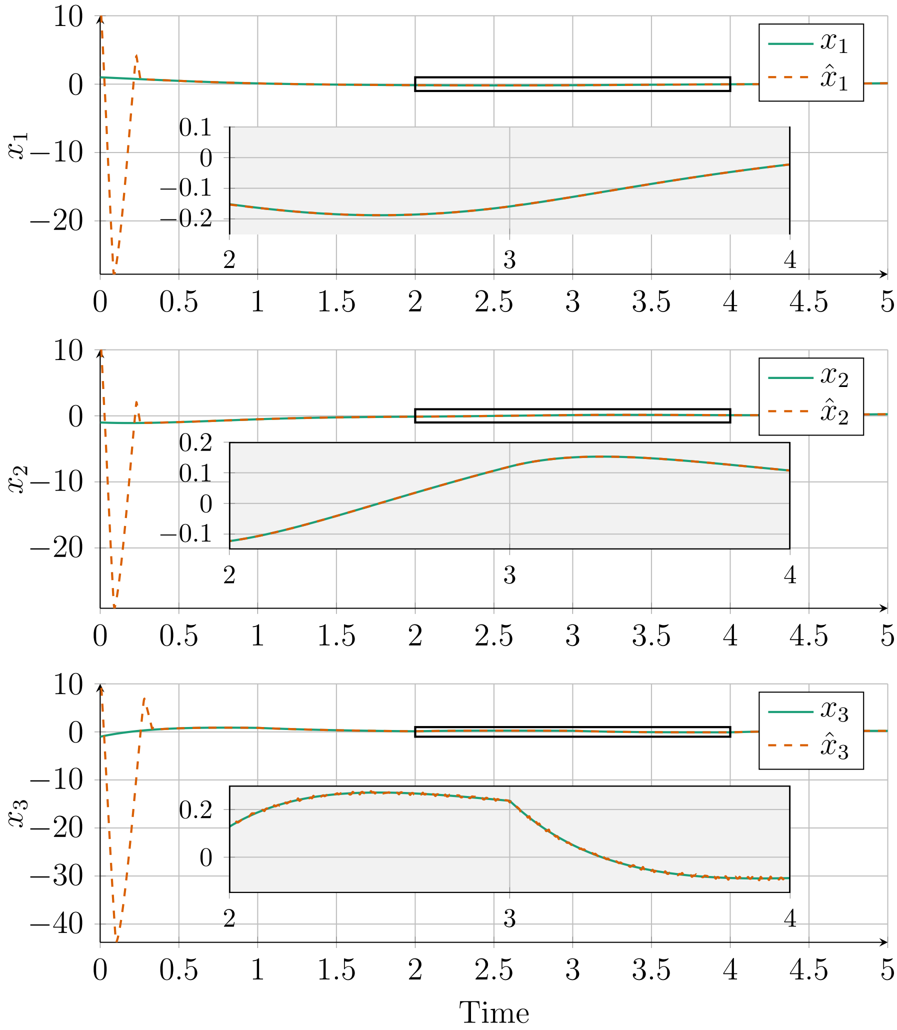


Fig. 6.6: True and estimated system states of system (6.118).

order sliding mode observer design is presented in Section 6.5. This observer design guarantees finite time exact state reconstruction for strongly observable systems.

A drawback of the proposed approach might be the computational complexity arising from the number of state variables of the observer due to the Luenberger observer and the additional pre-filtering in case of non-differentiable unknown inputs. A major step in simplifying the observer design and the computational complexity of the resulting observer would be the possibility to avoid the Luenberger observer as a "stabilizer". This is possible without any modifications for asymptotically stable systems. For unstable systems, a first approach to avoid this stabilization in the single input single output case is the approach recently presented in [NKR19]. Future research is devoted to extending this concept to the multivariable case. This may drastically simplify

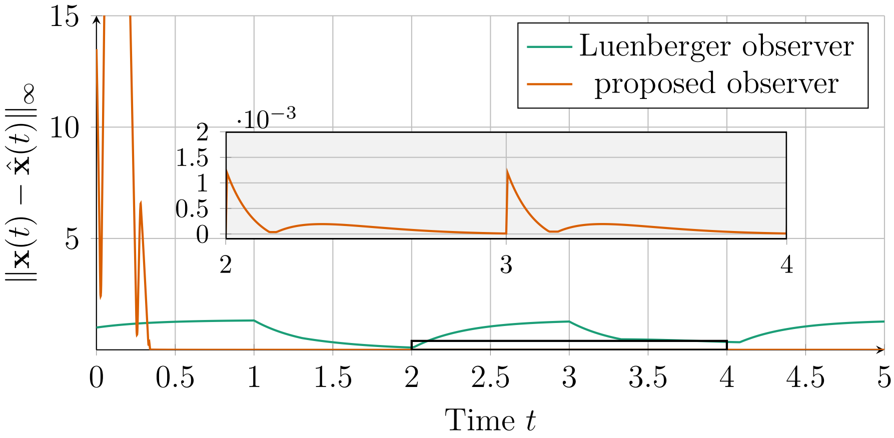


Fig. 6.7: Norm of the estimation errors of a Luenberger observer and the proposed higher order sliding mode scheme for system (6.118).

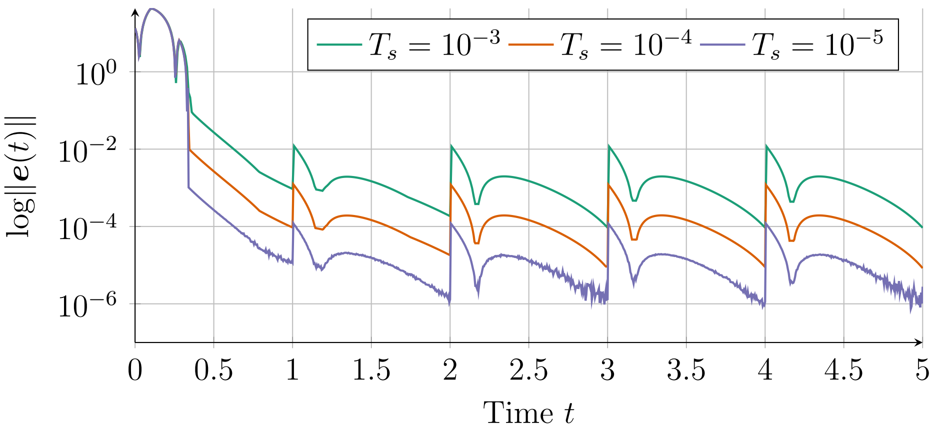


Fig. 6.8: Norm of the estimation error of the higher order sliding mode observer for different simulation step sizes.

existing robust observer design techniques for systems which do not fulfill the rank condition.

Chapter 7

Observers for Linear Time Varying Systems with Unknown Inputs

This chapter extends the ideas for the design of unknown input observers presented in the previous chapter to the time varying case. In particular, the higher order sliding mode observer design discussed in Section 6.5 is considered for a more general class of linear time varying systems.

7.1 Related Work and Contribution

Strong observability for linear time varying systems is barely studied in literature. Kratz and Liebscher proposed a characterization of strong observability in [KL98] as a generalization of the time invariant case presented in [Kra95]. This work is the basis for results presented in this chapter.

The observer proposed in [GFD16] was a first step towards higher order sliding mode observer design for linear time varying systems in the presence of unknown inputs. The concept is based on a deterministic interpretation of the Kalman filter combined with a higher order sliding mode differentiator in a cascaded observer structure. However, the existence of such an observer is not guaranteed a priori by the considered condition for strong observability because the employed filter in the cascaded scheme relies on a uniform complete observability assumption. The condition for strong observability assumed to hold in [GFD16] does not necessarily imply uniform complete observability. This is shown by an example.

Moreover, the so-called "stabilizer" (the first observer in the cascaded observer structure) is employed to yield bounded derivatives. This allows a successful application of the higher order sliding mode differentiation algorithm. Hence, it is reasonable to employ a "stabilizer" of minimum order and in particular the observer presented in Chapter 4 turns out to be well suited for this purpose. In fact, a notion of uniform strong observability is presented in this chapter which implies uniform complete observability and guarantees the existence of the observer presented in Section 4.5. It is moreover shown that also in the time varying case strong observability is preserved under output feedback. This is essential in order to extend the idea of the cascaded observer structure presented in Section 6.5 to the time varying setting.

The basic ideas discussed in this chapter were presented in

M. Tranninger et al. "Sliding Mode Tangent Space Observer for LTV Systems with Unknown Inputs." In: *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, Dec. 2018. DOI: 10.1109/cdc.2018.8619848, © IEEE 2018.

The present chapter significantly improves the initially proposed concepts. The observer used as a stabilizer in the cascaded observer structure in the above work is based on the method presented in [FZ18]. As already discussed, this observer requires additional existence conditions, whereas the concept of uniform strong observability guarantees the existence of an observer as introduced in Theorem 4.19. The resulting estimation error remains (uniformly) bounded in the presence of bounded unknown inputs and a sliding mode differentiator can be applied to the output error signals.

Section 7.2 introduces strong observability for linear time varying systems. The cascaded observer is presented in Section 7.3. The proposed observer is applied to a numerical simulation example in Section 7.4. A detailed discussion of the proposed concept together with future research directions can be found in Section 7.5.

7.2 Strong observability of linear time varying systems

In this section, the time varying system from Chapter 4 is extended by additional (known and unknown) inputs. It is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{D}(t)\mathbf{w}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0,\ t \in \mathbb{J} = [0, \infty), \tag{7.1a}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \tag{7.1b}$$

with $\mathbf{u}(t) \in \mathbb{R}^m$ as the known input and $\mathbf{w}(t) \in \mathbb{R}^q$ as the unknown input. Moreover, it is assumed that the matrices $\mathbf{A}(t)$, $\mathbf{D}(t)$, $\mathbf{C}(t)$ and the unknown input $\mathbf{w}(t)$ are uniformly bounded and sufficiently often differentiable with uniformly bounded derivatives.

Because the known input can always be canceled out in the observer error dynamics only the triple $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ of system (7.1) is considered subsequently. First, the definition of strong observability as an extension of the time invariant case is introduced.

Definition 7.1 (strong observability [KL98; Hau83]) The triple $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ is strongly observable on some non-degenerate time interval $\mathbb{I} \subseteq \mathbb{J}$ if for

$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{w}(t)$, a zero output $\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) = \mathbf{0}$ for all $t \in \mathbb{I}$ implies that $\mathbf{x}(t) = \mathbf{0}$ for any input $\mathbf{w}(t)$ and all $t \in \mathbb{I}$.

Similar to the time invariant case covered in Section 6.4, the abbreviations

$$\begin{aligned} \tilde{\mathbf{y}}_r(t) &= \begin{bmatrix} \mathbf{y}^\mathsf{T}(t) & \dot{\mathbf{y}}^\mathsf{T}(t) & \cdots & \mathbf{y}^{(r-1)\mathsf{T}}(t) \end{bmatrix}^\mathsf{T}, \\ \tilde{\mathbf{w}}_r &= \begin{bmatrix} \mathbf{w}^\mathsf{T}(t) & \dot{\mathbf{w}}^\mathsf{T}(t) & \cdots & \mathbf{w}^{(r-2)\mathsf{T}}(t) \end{bmatrix}^\mathsf{T} \end{aligned} \tag{7.2}$$

for the output, the unknown input and their corresponding derivatives with r as some positive integer are introduced. It is then possible to state the algebraic relation

$$\tilde{\mathbf{y}}_r(t) = \mathbf{O}_r(t)\mathbf{x}(t) + \mathbf{J}_r(t)\tilde{\mathbf{w}}_r(t) \tag{7.3}$$

with $\mathbf{O}_r(t)$ as the generalized observability matrix

$$\mathbf{O}_r(t) = \begin{bmatrix} \mathbf{C}_0(t) \\ \mathbf{C}_1(t) \\ \vdots \\ \mathbf{C}_{r-1}(t) \end{bmatrix} \tag{7.4}$$

with the $\mathbf{C}_i(t)$ for $i = 0, \ldots, r-1$ recursively defined according to

$$\mathbf{C}_{i+1}(t) = \mathbf{C}_i(t)\mathbf{A}(t) + \dot{\mathbf{C}}_i(t), \; \mathbf{C}_0(t) = \mathbf{C}(t), \tag{7.5}$$

see Section 2.4. The $pr \times q(r-1)$ matrix $\mathbf{J}_r$ is given by

$$\mathbf{J}_r(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{D}_{1,0}(t) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{D}_{2,0}(t) & \mathbf{D}_{2,1}(t) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{D}_{r-1,0}(t) & \mathbf{D}_{r-1,1}(t) & \cdots & \mathbf{D}_{r-1,r-2}(t) \end{bmatrix}. \tag{7.6}$$

The matrices $\mathbf{D}_{i,j}(t)$ of dimension $p \times q$ can be state recursively

$$\begin{aligned} \mathbf{D}_{\alpha+1,\alpha}(t) &= \mathbf{C}_0(t)\mathbf{D}(t) && \text{for } 0 \leq \alpha \leq r-1, \\ \mathbf{D}_{\alpha+1,0}(t) &= \mathbf{C}_\alpha(t)\mathbf{D}(t) + \dot{\mathbf{D}}_{\alpha,0}(t) && \text{for } 1 \leq \alpha \leq r-1, \\ \mathbf{D}_{\alpha+1,\beta}(t) &= \mathbf{D}_{\alpha,\beta-1}(t) + \dot{\mathbf{D}}_{\alpha,\beta}(t) && \text{for } 1 \leq \beta < \alpha \leq r-1. \end{aligned} \tag{7.7}$$

The zero blocks in $\mathbf{J}_r(t)$ are also of dimension $p \times q$.

Based on the matrices $\mathbf{O}_r(t)$ and $\mathbf{J}_r(t)$, a necessary and sufficient condition for strong observability is given in [KL98], which is presented in the following theorem.

Theorem 7.2 (strong observability, [KL98])
System (7.1) *is strongly observable on* $\mathbb{I} \subseteq \mathbb{J}$, *if and only if*

$$\operatorname{rank} \mathbf{S}_n(t) = \operatorname{rank} \mathbf{S}_n^*(t) \tag{7.8}$$

with

$$\mathbf{S}_n(t) = \begin{bmatrix} \mathbf{O}_n(t) & \mathbf{J}_n(t) \end{bmatrix}, \quad \mathbf{S}_n^*(t) = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{O}_n(t) & \mathbf{J}_n(t) \end{bmatrix}. \tag{7.9}$$

holds for all $t \in \mathbb{I}$ *except on a nowhere dense subset*

Strong observability on the whole time interval $\mathbb{J}$ allows the reconstruction of the system states based on the output and its derivatives with the aid of (7.3). The approach is a straightforward extension of the time invariant concepts presented in Chapter 6.

In a first step, a decoupling matrix $\mathbf{M}_r(t)$ of dimension $pr \times pr$ is designed such that

$$\ker \mathbf{M}_r(t) = \operatorname{im} \mathbf{J}_r(t) \text{ for all } t \in \mathbb{J}. \tag{7.10}$$

Multiplying (7.3) with $\mathbf{M}_r$ yields

$$\mathbf{M}_r(t)\tilde{\mathbf{y}}(t) = \mathbf{M}_r(t)\mathbf{O}_r(t)\mathbf{x}(t). \tag{7.11}$$

It is shown in [KL98] that if (7.9) holds for all $t \in \mathbb{J}$, then a left inverse $[\mathbf{M}_n(t)\mathbf{O}_n(t)]^\dagger$ exists for all $t \in \mathbb{J}$.

The goal is to utilize this left inverse in the observer design and hence a stronger property is desired. This leads to the concept of uniform strong observability introduced in the following.

Definition 7.3 (uniform strong observability) The triple $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ is uniformly strongly observable on $\mathbb{J}$ if there exists a $\mu > 0$ and some integer r such that

$$[\mathbf{M}_r(t)\mathbf{O}_r(t)]^\mathsf{T} [\mathbf{M}_r(t)\mathbf{O}_r(t)] \succeq \mu \mathbf{I}_n \tag{7.12}$$

with $\mathbf{O}_r$ given in (7.4) and $\mathbf{M}_r$ given in (7.10) holds for all $t \in \mathbb{J}$. The strong observability index ν is the smallest r for which (7.12) holds.

Without loss of generality, it can be assumed that $\|\mathbf{M}_r(t)\| = 1$ for all t. It follows from the proof of Theorem 2 and Corollary 2 in [KL98] that uniform strong observability implies strong observability. Moreover, uniform strong observability of $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ on $\mathbb{J}$ implies uniform complete observability of the pair $(\mathbf{A}(t), \mathbf{C}(t))$. This is a direct consequence of the sufficient condition for uniform complete observability introduced in Theorem 2.15, see Section 2.4.

Strong observability of $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ does not imply uniform complete observability of the pair $(\mathbf{A}(t), \mathbf{C}(t))$. This is shown by the following example.

Example 7.4*:* Consider the system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & e^{-t} \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t), \tag{7.13a}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \tag{7.13b}$$

One has

$$\mathbf{O}_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix} \text{ and } \mathbf{J}_2 = \mathbf{0}_{2\times 2}. \tag{7.14}$$

Hence, $\operatorname{rank} \mathbf{S}_2(t) = \operatorname{rank} \mathbf{S}_2^*(t) = 2 = n$ and the system is strongly observable on $\mathbb{J}$. The state transition matrix is given by

$$\mathbf{\Phi}(t, t_0) = \begin{bmatrix} 1 & -\left(e^{-t} - e^{-t_0}\right) \\ 0 & 1 \end{bmatrix}. \tag{7.15}$$

The observability Gramian $\mathbf{M}(t+\sigma, t)$ for some $t \in \mathbb{J}$ and a fixed $\sigma > 0$ is given by

$$\mathbf{M}(t+\sigma, t) = \begin{bmatrix} \sigma & m_{12}(\sigma)e^{-t} \\ m_{12}(\sigma)e^{-t} & m_{22}(\sigma)e^{-2t} \end{bmatrix} \tag{7.16}$$

with

$$m_{12}(\sigma) = \left[\sigma - 1 + e^{-\sigma}\right],$$
$$m_{22}(\sigma) = \left[\sigma + \frac{1}{2}\left(1 - e^{-2\sigma}\right) - 2\left(1 - e^{-\sigma}\right)\right].$$

The eigenvalues of $\mathbf{M}(t+\sigma, t)$ are the roots of the characteristic polynomial

$$\begin{aligned} \Delta(s) = \det[s\mathbf{I}_2 - \mathbf{M}] &= \det \begin{bmatrix} s - \sigma & -m_{12}e^{-t} \\ -m_{12}e^{-t} & s - m_{22}e^{-2t} \end{bmatrix} \\ &= s^2 - s(\sigma + m_{22}e^{-2t} + m_{12}^2 e^{-2t}) \end{aligned} \tag{7.17}$$

with

$$s_{12} = \frac{\sigma + m_{22}e^{-2t}}{2} \pm \sqrt{\frac{(\sigma + m_{22}e^{-2t})^2}{2} - m_{12}e^{-2t}}. \tag{7.18}$$

For $t \to \infty$, this yields

$$\lim_{t\to\infty} s_1 = \sigma, \quad \lim_{t\to\infty} s_2 = 0. \tag{7.19}$$

Hence, the pair $(\mathbf{A}(t), \mathbf{C}(t))$ cannot be uniformly completely observable.

The condition stated in Equation (7.12) guarantees the existence of a left inverse $(\mathbf{M}_r(t)\mathbf{O}_r(t))^\dagger$ and can be directly used to reconstruct the states by using the system output and its derivatives. This is used to design a higher order sliding mode observer in the following.

7.3 Sliding Mode Subspace Observer Design

The previously presented results are now combined in order to design the observer using a step by step procedure. The following statements are assumed to hold for system (7.1):

(A1) The triple $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ is uniformly strongly observable on $\mathbb{J}$ with the strong observability index ν.

(A2) The matrices $\mathbf{A}(t)$, $\mathbf{D}(t)$ and $\mathbf{C}(t)$ are at least $\nu-1$, $\nu-1$ and ν times continuously differentiable with uniformly bounded derivatives.

(A3) The unknown input $\mathbf{w}(t)$ is uniformly bounded and $\nu - 1$ times continuously differentiable with uniformly bounded derivatives according to

$$\|\mathbf{w}^{(i)}\| \leq w_+^i \text{ for } i = 0, \ldots \nu - 1. \tag{7.20}$$

The observer

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{L}_s(t)(\mathbf{y}(t) - \mathbf{C}(t)\tilde{\mathbf{x}}(t)) \tag{7.21a}$$

$$\hat{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t) + \mathbf{L}_c(t)\mathbf{h}(\mathbf{e}_y) \tag{7.21b}$$

is proposed to reconstruct the states of system (7.1) in finite time. The observer gain $\mathbf{L}_s(t)$, the reconstruction gain $\mathbf{L}_c(t)$ and the nonlinear vector function $\mathbf{h}(\mathbf{e}_y(t))$ of dimension $p(\nu - 1)$ are design parameters of the observer and discussed in detail in the following. The structure of the observer is depicted in Fig. 7.1.

Observer Gain The observer gain $\mathbf{L}_s(t)$ is obtained via the subspace observer design presented in Theorem 4.19, Chapter 4. Uniform strong observability of $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ implies uniform complete observability of the pair $(\mathbf{A}(t), \mathbf{C}(t))$ and hence the uniform detectability condition is fulfilled for every $k \leq n$. Hence, $j^\star \leq k \leq n$ can always be chosen, where $j^\star$ is defined Theorem 4.19 and k is a design parameter. The gain is then given by

$$\mathbf{L}_s(t) = \bar{\mathbf{Q}}(t)\mathbf{P}(t)\bar{\mathbf{Q}}^\mathsf{T}(t)\mathbf{C}^\mathsf{T}(t), \tag{7.22}$$

where the matrices $\mathbf{P}(t)$ and $\mathbf{C}(t)$ are the solutions to the differential equations (4.109b) and (4.109c), respectively. The algorithm is given in (4.109).

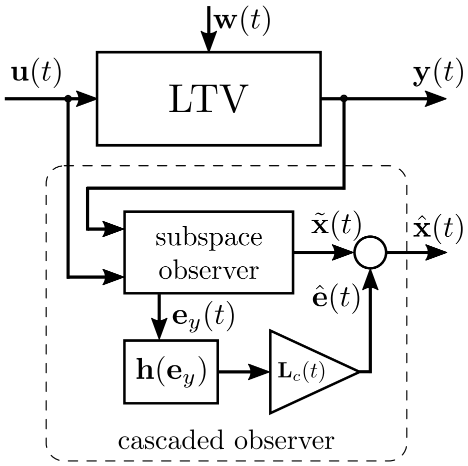


Fig. 7.1: Cascaded observer structure.

Remark 7.5*:* As a computationally more efficient alternative, one can replace (7.21a) by the reduced observer

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{B}_{11}(t)\mathbf{z}(t) + \mathbf{L}_{+}(t)\left[\mathbf{y}(t) - \mathbf{C}(t)\bar{\mathbf{Q}}(t)\mathbf{z}(t)\right], \ \mathbf{z}_0 \in \mathbb{R}^k \\ \hat{\mathbf{x}}(t) &= \bar{\mathbf{Q}}(t)\mathbf{z}(t), \ \mathbf{L}_{+}(t) = \mathbf{P}(t)\bar{\mathbf{Q}}(t)\mathbf{C}^{\mathsf{T}}(t). \end{aligned} \tag{7.23}$$

if the known input is zero. This reduced observer design is discussed in Section 3.5. One merely has to replace $k \geq k^{\star}$ by $k \geq j^{\star}$. Then, the resulting (full order) estimation error dynamics are uniformly exponentially stable for $\mathbf{w}(t) = 0$ and the following discussion are also valid for this choice.

Reconstruction Gain The state reconstruction discussed in Section 7.2 is now carried out for the observer error system with $\tilde{\mathbf{A}}(t) = \mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)$ and the output matrix $\mathbf{C}(t)$. The observability matrix of the error system with the strong observability index ν is denoted by $\mathbf{O}_{\nu}(t)$ and $\mathbf{M}_{\nu}(t)$ is given in (7.10) for the pair $(\tilde{\mathbf{A}}(t), \mathbf{C}(t))$. Then, the reconstruction gain is given by

$$\mathbf{L}_c(t) = \left[\mathbf{M}_{\nu}(t)\mathbf{O}_{\nu}(t)\right]^{\dagger}\mathbf{M}_{\nu}(t). \tag{7.24}$$

The left inverse $[\mathbf{M}_{\nu}(t)\mathbf{O}_{\nu}(t)]^{\dagger}$ exists for all $t \in \mathbb{J}$ except possibly on a nowhere dense subset, because strong observability is preserved under output feedback as shown in the following

Lemma 7.6 (strong observability under output feedback)
The error system

$$\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}_s(t)\mathbf{C}(t)] + \mathbf{D}(t)\mathbf{w}(t) \tag{7.25a}$$
$$\mathbf{e}_y(t) = \mathbf{C}(t)\mathbf{e}(t) \tag{7.25b}$$

with a uniformly bounded feedback gain $\mathbf{L}_s(t)$ *is strongly observable if and only if the triple* $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ *is strongly observable.*

Proof. The assumption $\mathbf{e}_y(t) = \mathbf{0}$ yields $\dot{\mathbf{e}}(t) = \mathbf{A}(t)\mathbf{e}(t) + \mathbf{D}(t)\mathbf{w}(t)$ for (7.25a). This implies that $\mathbf{e}(t) \equiv \mathbf{0}$ if the triple $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ is strongly observable which proofs sufficiency. For necessity, assume that $(\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t), \mathbf{D}(t), \mathbf{C}(t))$ is not strongly observable. This means that one can find an input $\mathbf{w}(t)$ such that $\mathbf{e}_y(t) = \mathbf{0}$ for all t and $\mathbf{e}(t) \neq \mathbf{0}$. Thus, it would be also possible to find $\mathbf{w}(t)$ such that for system (3.1) $\mathbf{y}(t) = \mathbf{0}$ and $\mathbf{x}(t) \neq \mathbf{0}$ holds, which is a contradiction. □

To guarantee the existence of the left inverse $[\mathbf{M}_\nu(t)\mathbf{O}_\nu(t)]^\dagger$ for all $t \in \mathbb{J}$, uniform strong observability of the error system has to be assumed. It can be expected that this property is preserved under output injection. However, this remains to be shown rigorously.

Higher Order Sliding Mode Differentiator The higher order sliding mode differentiation algorithm discussed in Section 6.5.1 is now applied component-wise to the output error. It is assumed that the Lipschitz constant L_d required in the differentiator algorithm is chosen sufficiently large. The existence of this constant is shown in Section 7.3.1.

The output error of the previous observer is denoted by $\mathbf{e}_y(t) = \mathbf{y}(t) - \mathbf{C}(t)\tilde{\mathbf{x}}(t)$ with the components

$$\mathbf{e}_y(t) = \begin{bmatrix} e_{y,1}(t) & e_{y,2}(t) & \cdots & e_{y,p}(t) \end{bmatrix}^\mathsf{T}. \tag{7.26}$$

Then, the functional $\mathbf{h}_\nu(\mathbf{e}_y)$ represents the component-wise application of the higher order sliding mode differentiator according to

$$\mathbf{h}_\nu(\mathbf{e}_y) = \begin{bmatrix} e_{y,1}(t) \\ \vdots \\ e_{y,p}(t) \\ \hdashline \mathcal{D}^1_{\nu-1}\lfloor e_{y,1}(t) \rceil \\ \vdots \\ \mathcal{D}^1_{\nu-1}\lfloor e_{y,p}(t) \rceil \\ \hdashline \vdots \\ \hdashline \mathcal{D}^{\nu-1}_{\nu-1}\lfloor e_{y,1}(t) \rceil \\ \vdots \\ \mathcal{D}^{\nu-1}_{\nu-1}\lfloor e_{y,p}(t) \rceil \end{bmatrix}. \tag{7.27}$$

7.3.1 Convergence Analysis

For the specific choice of the observer gain $\mathbf{L}_s(t)$, the dynamics of the observer error $\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t)$ given by

$$\dot{\mathbf{e}}(t) = [\mathbf{A}(t) - \mathbf{L}_s(t)\mathbf{C}(t)]\,\mathbf{e}(t) + \mathbf{D}(t)\mathbf{w}(t) \tag{7.28}$$

is bounded input bounded state stable, i.e., for any bounded input $\mathbf{w}(t)$, the estimation error converges into a bounded neighborhood of the origin. Because of uniform exponential stability of the error system for $\mathbf{w}(t) = \mathbf{0}$, one has the estimate

$$\|\mathbf{e}(t)\| \leq K_e\|\mathbf{e}_0\| + \frac{K_e}{\gamma}\bar{d}w_+^0 \tag{7.29}$$

for all $t \in \mathbb{J}$ with $\|\mathbf{D}(t)\| \leq \bar{d}$. The constants $K_e \geq 1$ and $\gamma > 0$ are bounds on the state transition matrix

$$\|\mathbf{\Phi}(t, t_0)\| \leq K_e e^{-\gamma(t-t_0)}, \tag{7.30}$$

see Section 4.5.1. Hence, the estimation error is uniformly bounded by

$$e_+ = K_e\|\mathbf{e}_0\| + \frac{K_e}{\gamma}\bar{d}w_+^0. \tag{7.31}$$

The generalized observability matrix of the error system is given by

$$\mathbf{O}_\nu(t) = \begin{bmatrix} \mathbf{C}_0(t) \\ \vdots \\ \mathbf{C}_{\nu-1}(t) \end{bmatrix} \tag{7.32}$$

with the $p \times n$ block matrix $\mathbf{C}_i(t)$ recursively defined for the pair $(\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t), \mathbf{C}(t))$ as in (7.5). Then, the ν-th derivative of the output error $\mathbf{e}_y(t)$ can be bounded according to

$$\|\mathbf{e}_y^{(\nu)}(t)\| \leq \left\|\mathbf{C}_{\nu-1}(t)\,[\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)] + \dot{\mathbf{C}}_{n-1}(t)\right\|\,\|\mathbf{e}(t)\| + \sum_{i=0}^{\nu-1}\|\mathbf{D}_{\nu,i}(t)\mathbf{w}^{(i)}(t)\| \tag{7.33}$$

with $\mathbf{D}_{i,j}(t)$ as defined in (7.7). Due to the smoothness and boundedness assumptions in (A2) and (A3), the existence of positive constants $\bar{d}_{\nu,i}$, $i = 0, \ldots, \nu_1$ and $\bar{c}_\nu$ is guaranteed such that

$$\|\mathbf{D}_{\nu,i}(t)\| \leq \bar{d}_{\nu,i} \text{ and } \left\|\tilde{\mathbf{C}}_{\nu-1}(t)\,[\mathbf{A}(t) - \mathbf{L}(t)\mathbf{C}(t)] + \dot{\mathbf{C}}_{n-1}(t)\right\| \leq \bar{c}_\nu \tag{7.34}$$

for all $t \in \mathbb{J}$. This allows to state a uniform bound on the ν-th derivative of the estimation error according to

$$\|\mathbf{e}_y^{(\nu)}(t)\| \leq \bar{c}_\nu e_+ + \sum_{i=0}^{\nu-1}\bar{d}_{\nu,i}w_+^i. \tag{7.35}$$

Hence, for a sufficiently large L_d in the differentiation algorithm,

$$\tilde{\mathbf{e}}_y(t) - \mathbf{h}_\nu(\mathbf{e}_y(t)) = \mathbf{0} \tag{7.36}$$

with $\tilde{\mathbf{e}}_y(t) = \begin{bmatrix}\mathbf{e}_y^{\mathsf{T}}(t) & \dot{\mathbf{e}}_y^{\mathsf{T}}(t) & \cdots & \mathbf{e}_y^{(\nu-1)\mathsf{T}}(t)\end{bmatrix}^{\mathsf{T}}$ holds after some finite time $t_f \geq 0$. This guarantees that the state can be reconstructed in finite time and

$$\mathbf{x}(t) = \hat{\mathbf{x}}(t) \tag{7.37}$$

holds for all $t \geq t_f$ if the error system is uniformly strongly observable.

The bound for the differentiator gain in (7.35) might be very conservative and hence L_d should be considered as a tuning parameter in the implementation.

7.4 Numerical Example

Consider a system in the form of (3.1) with the time varying dynamic matrix as

$$A(t) = \begin{bmatrix} -2 & a_{12} & 0 & 0 & a_{15} & -0.12 & 0.42 & 0.92 \\ -a_{12} & -3.2 & a_{23} & 0.51 & a_{25} & -0.23 & 0 & -0.31 \\ 0 & -a_{23} & -4.4 & 0.48 & -0.80 & 0.53 & 0 & 0.17 \\ 0 & -0.51 & -0.48 & 3.35 & 0.64 & 0.59 & 0 & a_{48} \\ -a_{15} & -a_{25} & 0.80 & -0.64 & 1.80 & -0.62 & 0.31 & 0.50 \\ 0.12 & 0.23 & -0.53 & -0.59 & 0.62 & -2.45 & -0.67 & -0.48 \\ -0.42 & 0 & 0 & 0 & -0.31 & 0.67 & -3.47 & 0 \\ -0.92 & 0.31 & -0.17 & -a_{48} & -0.50 & 0.48 & 0 & -4.71 \end{bmatrix} \tag{7.38}$$

with

$$\begin{aligned} a_{12} &= 0.23\sin(0.5t), & a_{15} &= -0.25\sin(0.5t), \\ a_{23} &= 0.083\sin(0.3t), & a_{25} &= 0.09\sin(0.3t), \\ a_{48} &= -0.055\sin(0.3t). \end{aligned}$$

Moreover,

$$\begin{aligned} B^{\mathsf{T}} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.67 & 0.8 \\ 0.98 & 0.63 & 0 & 0.54 & 0.54 & 0 & 0 & 0 \end{bmatrix}, \\ D^{\mathsf{T}} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{7.39}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{7.40}$$

The unknown input is specified as

$$w(t) = 0.3 + 10\sin(2\pi 0.1t) + 3\sin(2\pi 0.4t). \tag{7.41}$$

Because the system is unstable, the simulations are carried out for the stabilized system. The input $\mathbf{u}(t)$ is known and the particular controller design does not influence the observer error dynamics. The simulation is carried out using a forward Euler integration scheme with a step size of $T_s = 10^{-3}$. The sliding mode differentiator is implemented in discrete time using the discretization scheme recently presented in [Koc+19] to avoid discretization chattering. It can be verified that $(\mathbf{A}(t), \mathbf{D}(t), \mathbf{C}(t))$ is strongly observable with index $\nu = 2$.

The system is periodic and hence it possesses a point spectrum and it suffices to numerically approximate the (forward regular) Lyapunov exponents and the observer methods presented Chapters 3 and 4 coincide. The approximated Lyapunov exponents after a final time $T_f = 500$ are $\lambda_1 \approx 2.9$, $\lambda_2 \approx 1.86$, $\lambda_3 \approx -2.30$, $\lambda_4 \approx \lambda_5 \approx -3.00$, $\lambda_6 \approx -3.16$, and $\lambda_7 \approx \lambda_8 \approx -4.19$. Hence, the dimension of the unstable subspace is $k = j^\star = 2$ and the autonomous system possesses an exponential dichotomy. The system states are depicted in Fig. 7.2 and the norm of the reconstruction error $\|\mathbf{x}(t) - \hat{\mathbf{x}}(t)\|$ is shown in Fig. 7.3.

The error of the observer without correction is bounded despite the unknown input. The state variables x_1 to x_4 are measured directly and hence the reconstruction error is zero. The estimation error for the remaining states x_5 to x_8 converge to a small vicinity of the origin due to numerical errors as shown in Fig. 7.4. A discussion of the achievable numerical accuracy can be found in [GFD16]. Because of the chosen discretization scheme for the higher order sliding mode differentiator, discretization chattering is avoided successfully.

7.5 Discussion

The observer design presented in this chapter allows to exactly reconstruct the system states in finite time in the presence of unknown inputs. However, the implementation of the cascaded observer may be computationally complex for specific problems. Especially, determining the decoupling matrix $\mathbf{M}_\nu(t)$ might be costly if the structure of $\mathbf{J}_\nu(t)$ changes over time. In this case, a basis for $\operatorname{im}\mathbf{J}_\nu(t)$ has to be computed point wise via, e.g., QR or SVD methods. It would be interesting to use continuous methods for the computation of this basis via smooth matrix decomposition concepts as discussed in [DE99]. However, the matrix $\mathbf{J}(t)$ is usually rank deficient, which does not allow to apply these algorithms directly. For a large strong observability index ν, the (time-varying) observability matrix depends on the observer gain and its derivatives.

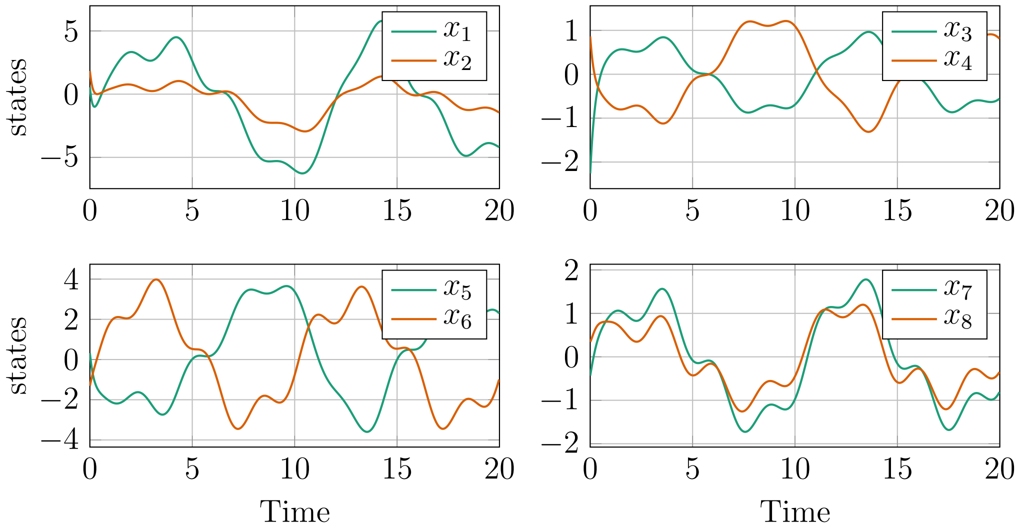


Fig. 7.2: Evolution of the system states in the presence of the unknown input.

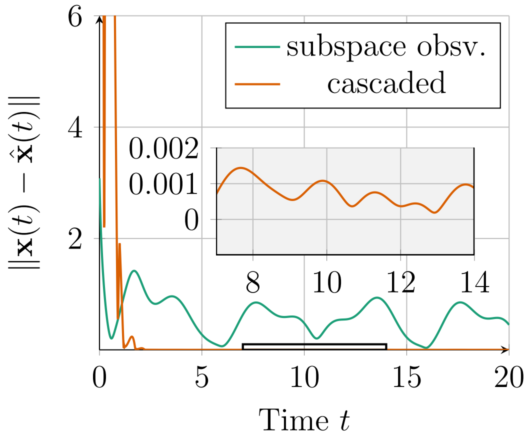


Fig. 7.3: Norm of the estimation error for SO and cascaded observer.

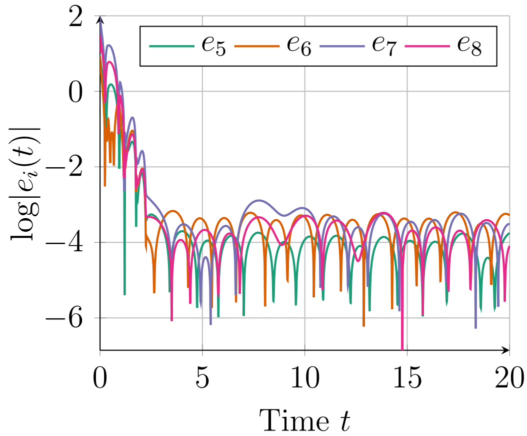


Fig. 7.4: Logarithmic estimation errors of cascaded observer.

If $\mathbf{w}(t)$ is not differentiable, the derivatives of $\mathbf{e}_y(t)$ might not exist either. For the time invariant case, this problem was solved by introducing additional filtering of the output in Section 6.4. This technique essentially uses time invariance and is thus not applicable here directly. However, the problem could be circumvented by using stepwise differentiation of the output as a generalization of the invariant subspace algorithm presented in [BPF07; Mol76] for the time invariant setting. This strategy does not allow to design the decoupling matrix in one step and moreover does not allow to utilize higher order sliding mode differentiators. Hence, this concept is beyond the scope of this chapter.

Although it might be expected that uniform strong observability is preserved under output feedback, this remains to be shown rigorously. By an extension of the rank condition to linear time varying systems, it may be reasonable to extend design techniques for linear observers presented in Section 6.3 to the time varying setting. Together with the detectability condition presented in Chapter 4, this may allow to generalize the notion of strong* detectability to the linear time varying case and results in a computationally efficient observer design.

Chapter 8 Summary and Outlook

This doctoral thesis presents new theoretical concepts and algorithms for state estimation of complex dynamical systems. Different observer concepts are proposed for a variety of possible scenarios.

In the first part of the thesis, new detectability conditions and observer design techniques for large scale linear and nonlinear systems are presented. Detectability is a key requirement for any successful observer design and the presented detectability concepts and conditions allow to understand this requirement more deeply. The proposed observer design is based on a thorough investigation of the considered detectability concepts. Insights are given via many examples and real experimental data. This thesis only considers continuous-time models. The continuous-time point of view allows to extend the present concepts to resource-constraint environments, where event triggered measurements are the method of choice. The investigation of discretization effects and discrete time implementations is subject to future research. To tackle the scalability problem for large scale systems in a possibly distributed environment, a distributed version of the proposed algorithms would be desirable. This is an open question and subject to future research but may be inspired by recent progress in distributed Kalman filtering [FC18]. In all scenarios, the influence of communication network effects like time delays or intermittent data has to be investigated thoroughly.

The second part of the thesis was devoted to state estimation in the presence of unknown inputs. A simple yet effective observer design for the class of linear time invariant systems which fulfill a so-called rank condition was discussed. If this condition does not hold but the system is at least strongly detectable, an asymptotic state estimate can be obtained by means of a higher order sliding mode observer. If the system is strongly observable, the state can even be reconstructed exactly in finite time.

For the class of uniformly strongly observable linear time varying systems, the higher order sliding mode observer design was combined with the observer concepts presented in the first part of the thesis. This combination allows to reconstruct the states of linear time varying systems in finite time. A combination of these concepts with the detectability conditions presented in the first part of the thesis could be a basis for future research on strong detectability concepts for linear time varying and nonlinear systems.

Appendix

Appendix A

Proofs

A.1 Proof of Proposition 2.17

The lower bound presented in [Buc72b] holds after some time $t_0 + \sigma > t_0$ because the initial condition could be positive semidefinite, i.e., $\mathbf{P}(t_0) \succeq 0$. Here, the lower bound is guaranteed for all $t \geq t_0$ by the choice of $\mathbf{P}(t_0)$ and $\mathbf{G}(t)$ as shown afterwards. First, the upper bound p_2 is derived by combining results from [Buc67; Buc72b]. The bounds developed by R. Bucy are based on the Gramians $\mathbf{N}(t, t_0)$ and

$$\mathbf{C}_G(t, t_0) = \int_{t_0}^{t} \mathbf{\Phi}(t, s)\mathbf{G}(s)\mathbf{\Phi}^{\mathsf{T}}(t, s)\,\mathrm{d}s. \tag{A.1}$$

The matrix $\mathbf{C}_G$ is the so-called controllability Gramian of the pair $(\mathbf{A}(t), \mathbf{G}^{1/2}(t))$. Moreover, the pair $(\mathbf{A}(t), \mathbf{G}^{1/2}(t))$ is uniformly completely controllable if there exist positive constants γ_1, γ_2 and σ such that

$$\gamma_1 \mathbf{I} \preceq \mathbf{C}_G(t + \sigma, t) \preceq \gamma_2 \mathbf{I} \tag{A.2}$$

holds for all $t \in \mathbb{J}$ [KB61; Buc72b]. Without loss of generality it can be assumed that σ is the same in (2.68) and (A.2). Especially if $\mathbf{G}(t)$ is chosen as in (b), the uniform complete controllability condition is always fulfilled for every $\sigma > 0$. The matrix $\mathbf{G}(t)$ is considered as a design parameter in this work and hence this can always be achieved. The first bound required is given in [Buc67, Lemma 1] according to

$$\mathbf{\Pi}(t, \mathbf{P}(t_0), t_0) \preceq \mathbf{\Phi}(t, t_0)\mathbf{P}(t_0)\mathbf{\Phi}^{\mathsf{T}}(t, t_0) + \mathbf{C}_G(t, t_0) \text{ for all } t \geq t_0. \tag{A.3}$$

For the present choice of $\mathbf{P}(t_0)$ and $\mathbf{Q}(t)$ this bound can be simplified to

$$\begin{aligned} \Pi(t, \mathbf{P}(t_0), t_0) &\preceq p_{02}\|\mathbf{\Phi}(t, t_0)\|^2\mathbf{I} + g_2 \int_{t_0}^{t} \|\mathbf{\Phi}(t, t_0)\|^2 \,\mathrm{d}s\mathbf{I} \\ &\preceq p_{02}e^{2\bar{a}(t-t_0)}\mathbf{I} + \frac{g_2}{2\bar{a}}[e^{2\bar{a}(t-t_0)} - 1]\mathbf{I} \end{aligned} \tag{A.4}$$

A second bound needed to derive a uniform p_2 is given[1] in [Buc72b, Theorem 2.1] according to

$$\mathbf{\Pi}(t, \mathbf{P}(t_0), t_0) \preceq \mathbf{N}^{-1}(t, t-\sigma) + \frac{\gamma_2 \alpha_2 n^2}{\gamma_1 \alpha_1} \mathbf{C}_G(t, t-\sigma) \text{ for all } t > t_0 + \sigma \tag{A.5}$$

and hence

$$\mathbf{\Pi}(t, \mathbf{P}(t_0), t_0) \preceq \frac{\gamma_1 + \gamma_2^2 \alpha_2 n^2}{\gamma_1 \alpha_1} \mathbf{I} \text{ for all } t > t_0 + \sigma. \tag{A.6}$$

By setting $t = t_0 + \sigma$ in (A.4) and noting that the bound is non decreasing with respect to t, one obtains a uniform upper bound for all solutions to (2.67) according to

$$p_2 = \max\left(p_{02} e^{2\bar{a}\sigma} + \frac{g_2}{2\bar{a}}[e^{2\bar{a}\sigma} - 1], \frac{\gamma_1 + \gamma_2^2 \alpha_2 n^2}{\gamma_1 \alpha_1}\right). \tag{A.7}$$

To see that the solutions to (2.67) are also uniformly bounded from below by some constant p_1, if they are initialized with a positive definite matrix, assume that $\mathbf{P}(t_0) \succ p_{01}\mathbf{I}$. If $\mathbf{P}(t_1)$ gets positive semidefinite for some $t_1 > t_0$, there exists a non-trivial vector $\mathbf{v}$ such that $\mathbf{v}^\mathsf{T}\mathbf{P}(t_1)\mathbf{v} = 0$. Multiplying (2.67) with $\mathbf{v}^\mathsf{T}$ from the left and $\mathbf{v}$ from the right and evaluating the derivative at $t = t_1$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{v}^\mathsf{T}\mathbf{P}(t)\mathbf{v}\Big|_{t=t_1} = \mathbf{v}^\mathsf{T}\mathbf{G}(t_1)\mathbf{v} > 0. \tag{A.8}$$

This derivative is positive because of (b) in Proposition 2.17 and hence due to continuity and since $\mathbf{v}^\mathsf{T}\mathbf{P}(t_0)\mathbf{v} > 0$, the solution of (2.67) must remain positive definite and uniformly bounded from below with some positive p_1.

A.2 Proof of Lemma 4.3

First, it will be shown that

$$\mathbf{x}(t) = \int_0^t \mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s - \int_t^\infty \mathbf{X}(t)(\mathbf{I} - \mathbf{P})\mathbf{X}^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s \tag{A.9}$$

is a solution of (4.14). Differentiation of both integrals in (A.9) according to Leibnitz's rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s = \mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(t)\mathbf{f}(t) + \mathbf{A}(t) \int_0^t \mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s \tag{A.10}$$

[1] A similar bound was given initially in [Buc67, Lemma 4]. This bound is incorrect, see, e.g., [HFA72; Buc72a]. The correct bound is given in [Buc72b]

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s = -\,\mathbf{X}(t)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(t)\mathbf{f}(t) + \mathbf{A}(t)\int_t^\infty \mathbf{X}(t)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s. \tag{A.11}$$

Hence,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{A.12}$$

and (A.9) is a solution of (4.14). In order to show that this solution is bounded, both integrals are shown to be bounded utilizing Definition 4.1 and $\sup_{t\in\mathbb{J}}\|\mathbf{f}(t)\| = \bar{f} < \infty$. The first integral in (A.9) can be bounded according to

$$\begin{aligned}\left\|\int_0^t \mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s\right\| &\le \bar{f}\int_0^t \|\mathbf{X}(t)\mathbf{P}\mathbf{X}^{-1}(s)\|\,\mathrm{d}s \le \bar{f}K\int_0^t e^{-\alpha(t-s)}\,\mathrm{d}s \\ &= \frac{K}{\alpha}\bar{f}\left(1-e^{-\alpha t}\right) \le \frac{K}{\alpha}\bar{f}.\end{aligned} \tag{A.13}$$

The second integral is bounded according to

$$\begin{aligned}\left\|\int_t^\infty \mathbf{X}(t)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(s)\mathbf{f}(s)\,\mathrm{d}s\right\| &\le \bar{f}\int_t^\infty \|\mathbf{X}(t)(\mathbf{I}-\mathbf{P})\mathbf{X}^{-1}(s)\|\,\mathrm{d}s \\ &\le \bar{f}K\int_t^\infty e^{\alpha(t-s)}\,\mathrm{d}s = \frac{K}{\alpha}\bar{f}.\end{aligned} \tag{A.14}$$

This shows that

$$\|\mathbf{x}(t)\| \le \frac{2K}{\alpha}\bar{f} \tag{A.15}$$

for all $t \in \mathbb{J}$. □

A.3 Proof of Lemma 4.6

It is stated in [DEV10, Theorem 24] that system (4.24) has an exponential dichotomy with projection

$$\mathbf{P} = \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_k\end{bmatrix} \tag{A.16}$$

and the fundamental matrix solution

$$\mathbf{X}(t) = \begin{bmatrix}\mathbf{X}_{11}(t) & \mathbf{X}_{12}(t)\\ \mathbf{0} & \mathbf{X}_{22}(t)\end{bmatrix}. \tag{A.17}$$

The matrices $\mathbf{X}_{11}(t)$ and $\mathbf{X}_{22}(t)$ are any fundamental matrix solutions of the systems $\mathbf{x}_1(t) = \mathbf{B}_{11}(t)\mathbf{x}_1(t)$ and $\mathbf{x}_2(t) = \mathbf{B}_{22}(t)\mathbf{x}_2(t)$, respectively. The matrix $\mathbf{X}_{12}(t)$ is given by

$$\mathbf{X}_{12}(t) = -\mathbf{X}_{11}(t)\int_t^\infty \mathbf{X}_{11}^{-1}(\tau)\mathbf{B}_{12}(\tau)\mathbf{X}_{22}(\tau)\,\mathrm{d}\tau. \tag{A.18}$$

Hence, it remains to be shown that the transformation to the block diagonal form (4.15) does not change the upper block, i.e., $\mathbf{D}_1(t) = \mathbf{B}_{11}(t)$. The transformation matrix is given by (4.16) with fundamental matrix solution (A.17). It follows from (4.17) with $\mathbf{P}$ as in (A.16) that

$$\mathbf{T}^2(t) = \begin{bmatrix} \mathbf{X}_{11}^\mathsf{T}(t)\mathbf{X}_{11}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^\mathsf{T}(t)\mathbf{M}(t) \end{bmatrix} \tag{A.19}$$

with $\mathbf{M}^\mathsf{T}(t)\mathbf{M}(t) = \mathbf{X}_{12}^\mathsf{T}(t)\mathbf{X}_{12}(t) + \mathbf{X}_{22}^\mathsf{T}(t)\mathbf{X}_{22}(t)$. Hence, $\mathbf{T}(t)$ is block diagonal with

$$\mathbf{T}(t) = \begin{bmatrix} \mathbf{X}_{11}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{M}(t) \end{bmatrix} \tag{A.20}$$

and the transformation matrix is given by

$$\mathbf{S}(t) = \mathbf{X}(t)\mathbf{T}^{-1}(t) = \begin{bmatrix} \mathbf{X}_{11}(t) & \mathbf{X}_{12}(t) \\ \mathbf{0} & \mathbf{X}_{22}(t) \end{bmatrix} \begin{bmatrix} \mathbf{X}_{11}^{-1}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{X}_{12}(t)\mathbf{M}^{-1}(t) \\ \mathbf{0} & \mathbf{X}_{22}(t)\mathbf{M}^{-1}(t) \end{bmatrix}. \tag{A.21}$$

Its inverse is given by

$$\mathbf{S}^{-1}(t) = \begin{bmatrix} \mathbf{I} & -\mathbf{X}_{12}(t)\mathbf{X}_{22}^{-1}(t) \\ \mathbf{0} & \mathbf{M}\mathbf{X}_{22}^{-1}(t) \end{bmatrix} \tag{A.22}$$

and the time derivative of $\mathbf{S}(t)$ can be stated as

$$\dot{\mathbf{S}}(t) = \begin{bmatrix} \mathbf{0} & \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{X}_{12}(t)\mathbf{M}^{-1}(t)\right] \\ \mathbf{0} & \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{X}_{22}(t)\mathbf{M}^{-1}(t)\right] \end{bmatrix} \tag{A.23}$$

By a straightforward computation using (A.21), (A.22) and (A.23) in

$$\mathbf{D}(t) = \mathbf{S}^{-1}(t)\mathbf{B}(t)\mathbf{S}(t) - \mathbf{S}^{-1}(t)\dot{\mathbf{S}}(t), \tag{A.24}$$

one can see that the upper left block remains unchanged for an upper block triangular coefficient matrix and hence $\mathbf{D}_1(t) = \mathbf{B}_{11}(t)$. □

A.4 Proof of Lemma 6.15

First, consider the case without direct feed-through, i.e. $\mathbf{F} = \mathbf{0}$. Under the assumption that $\mathbf{x}_0 \notin \ker \mathbf{M}_n\mathbf{O}_n$ it follows from equation (6.72) that

$$\mathbf{M}_n\tilde{\mathbf{y}}_{n,0} = \mathbf{M}_n\mathbf{O}_n\mathbf{x}_0 \neq \mathbf{0}. \tag{A.25}$$

Thus $\tilde{\mathbf{y}}_{n,0} \neq \mathbf{0}$, i.e. $\mathbf{y}$ or at least one time derivative is non-zero, and it follows directly that $\mathbf{x}_0 \notin \mathcal{V}(\Sigma_d)$.

Now, assume that there exists some non-trivial $\mathbf{x}_0 \in \ker \mathbf{M}_n \mathbf{O}_n$. Since

$$\mathbf{O}_n \mathbf{x}_0 \in \ker \mathbf{M}_n = \mathrm{im}\ \mathbf{J}_n, \tag{A.26}$$

there exists a vector $\tilde{\mathbf{w}}_{n,0} \in \mathbb{R}^{nq}$ such that

$$\mathbf{0} = \mathbf{O}_n \mathbf{x}_0 + \mathbf{J}_n \tilde{\mathbf{w}}_{n,0} \tag{A.27}$$

holds.

The goal is now to construct an input $\mathbf{w}(t)$ such that $\mathbf{y}(t) = \mathbf{0}$ for all t. This is achieved by specifying the derivatives of $\mathbf{w}$ evaluated at $t = 0$. Therefore, let

$$\mathbf{w}_0^{(k)} = \left. \frac{\mathrm{d}^k \mathbf{w}(t)}{\mathrm{d}t^k} \right|_{t=0} \tag{A.28}$$

denote the k-th derivative of $\mathbf{w}$ at $t = 0$.

The time derivatives of the states at $t = 0$ can be determined recursively by differentiating (6.2a), i.e.,

$$\mathbf{x}_0^{(\beta)} = \mathbf{A}\mathbf{x}_0^{(\beta-1)} + \mathbf{D}\mathbf{w}_0^{(\beta-1)} \tag{A.29}$$

for $\beta = 1, \dots, n-1$ and $\mathbf{x}_0^{(\beta)}$ as $\mathbf{x}^{(\beta)}|_{t=0}$. The vectors $\mathbf{x}_0^{(\beta)}$ thus are defined using the components $\mathbf{w}_0^{(\beta-1)}$ of $\tilde{\mathbf{w}}_{n,0}$, see (6.69). To fully determine $\mathbf{w}(t)$, its higher time derivatives have to be defined. Differentiating the output at $t = 0$ yields

$$\begin{aligned}
\mathbf{0} &= \mathbf{y}_0 = \mathbf{C}\mathbf{x}_0, \\
\mathbf{0} &= \dot{\mathbf{y}}_0 = \mathbf{C}\dot{\mathbf{x}}_0 = \mathbf{C}(\mathbf{A}\mathbf{x}_0 + \mathbf{D}\mathbf{w}_0), \\
\mathbf{0} &= \ddot{\mathbf{y}}_0 = \mathbf{C}\ddot{\mathbf{x}}_0 = \mathbf{C}\left(\mathbf{A}^2\mathbf{x}_0 + \mathbf{A}\mathbf{D}\mathbf{w}_0 + \mathbf{D}\dot{\mathbf{w}}_0\right), \\
&\vdots \\
\mathbf{0} &= \mathbf{y}_0^{(n-1)} = \mathbf{C}\mathbf{x}_0^{(n-1)} = \mathbf{C}\left(\mathbf{A}^{n-1}\mathbf{x}_0 + \sum_{\mu=0}^{n-1} \mathbf{A}^{n-1-\mu}\mathbf{D}\mathbf{w}_0^{(\mu)}\right).
\end{aligned} \tag{A.30}$$

due to (A.27) and (A.29). The initial state $\mathbf{x}_0$ and its time derivatives are linearly dependent since $\ker \mathbf{C} \subset \mathbb{R}^n$ and $\mathbf{x}_0$, $\dot{\mathbf{x}}_0$, $\dots$, $\mathbf{x}_0^{(n-1)} \in \ker \mathbf{C}$. Hence, $\mathbf{x}_0^{(n-1)}$ can be written as linear combination

$$\mathbf{x}_0^{(n-1)} = \sum_{\mu=0}^{n-2} \alpha_\mu \mathbf{x}_0^{(\mu)} \tag{A.31}$$

with real-valued constants α_0, $\dots$, α_{n-2}. The goal is now to choose all higher order derivatives $\mathbf{w}_0^{(n)}$, $\mathbf{w}_0^{(n+1)}, \dots$ such that $\mathbf{y}_0$ and all its derivatives are identically zero. For the n-th derivative combining equations (A.29) and (A.31) yields

$$\mathbf{x}_0^{(n)} = \mathbf{A}\mathbf{x}_0^{(n-1)} + \mathbf{D}\mathbf{w}_0^{(n-1)}. \tag{A.32}$$

Noting that the component $\mathbf{w}_0^{(n-1)}$ of $\tilde{\mathbf{w}}_{n,0}$ can be chosen arbitrarily in (A.27) it is defined as

$$\mathbf{w}_0^{(n)} = \sum_{\mu=0}^{n-1} \alpha_\mu \mathbf{w}_0^{(\mu)}, \tag{A.33}$$

as suggested in the proof of Proposition 2 in [Kra95]. By combining (A.29) and (A.33) it follows that

$$\mathbf{x}_0^{(n)} = \sum_{\mu=0}^{n-2} \alpha_\mu \mathbf{x}_0^{(\mu+1)} \tag{A.34}$$

and thus $\mathbf{x}_0^{(n)} \in \ker \mathbf{C}$. This can be generalized to define all higher order derivatives of $\mathbf{x}_0$ and $\mathbf{w}_0$ according to the recursion

$$\mathbf{x}_0^{(\nu)} = \sum_{\mu=0}^{n-2} \alpha_\mu \mathbf{x}_0^{(\nu-n+\mu+1)}, \; \nu > n-1 \tag{A.35}$$

and

$$\mathbf{w}_0^{(\nu)} = \sum_{\mu=0}^{n-2} \alpha_\mu \mathbf{w}_0^{(\nu-n+\mu+1)}, \; \nu \geq n-1. \tag{A.36}$$

It follows by induction that $\mathbf{x}_0^{(\nu)} \in \ker \mathbf{C}$ for all $\nu \geq 0$. With $\mathbf{w}(t)$ as

$$\mathbf{w}(t) = \sum_{\nu=0}^{\infty} \frac{\mathbf{w}_0^{(\nu)}}{\nu!} t^\nu, \tag{A.37}$$

the output is thus equal to zero for all times, i.e.,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) = \mathbf{C} \sum_{\nu=0}^{\infty} \frac{\mathbf{x}_0^{(\nu)}}{\nu!} t^\nu = \mathbf{0} \text{ for all } t \geq 0. \tag{A.38}$$

Hence, $\mathbf{x}_0 \in \mathcal{V}(\Sigma_d)$ which completes the proof.

For the case with direct feed-through, i.e., $\mathbf{F} \neq \mathbf{0}$, the first part of the proof still holds. It states that if $\mathbf{x}_0 \notin \ker \mathbf{M}_n \mathbf{O}_n$, it cannot be contained in the weakly unobservable subspace, because the output $\mathbf{y}(t) \neq \mathbf{0}$ for all t.

For the second part, consider the auxiliary system

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \tilde{\mathbf{A}}\boldsymbol{\eta} + \tilde{\mathbf{D}}\mathbf{w} \\ \mathbf{z} &= \tilde{\mathbf{C}}\boldsymbol{\eta} \end{aligned} \tag{A.39}$$

with

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{x}^\mathsf{T} & \boldsymbol{\xi}^\mathsf{T} \end{bmatrix}^\mathsf{T}, \qquad \tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}, \tag{A.40}$$

$$\tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{D} \\ \mathbf{F} \end{bmatrix}. \tag{A.41}$$

The output $\mathbf{z} = \boldsymbol{\xi}$ of this auxiliary system is the integral of the original output, i.e., $\dot{\boldsymbol{\xi}} = \mathbf{y}$. Differentiating the auxiliary output n times yields

$$\begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \\ \vdots \\ \mathbf{z}^{(n)} \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{\mathbf{C}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}} \\ \vdots \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}}^{n} \end{bmatrix}}_{\tilde{\mathbf{O}}} \boldsymbol{\eta} + \underbrace{\begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \\ \tilde{\mathbf{C}}\tilde{\mathbf{D}} & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \tilde{\mathbf{C}}\tilde{\mathbf{A}}^{n-1}\tilde{\mathbf{D}} & \cdots & \tilde{\mathbf{C}}\tilde{\mathbf{D}} \end{bmatrix}}_{\tilde{\mathbf{J}}} \begin{bmatrix} \mathbf{w} \\ \dot{\mathbf{w}} \\ \vdots \\ \mathbf{w}^{n-1} \end{bmatrix}. \tag{A.42}$$

Due to the structure of $\tilde{\mathbf{A}}$, $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{C}}$, it follows that

$$\tilde{\mathbf{O}} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_p \\ \mathbf{O}_n & \mathbf{0} \end{bmatrix} \text{ and } \tilde{\mathbf{J}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{J}_n \end{bmatrix} \tag{A.43}$$

One can choose a matrix

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{I}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_n \end{bmatrix} \tag{A.44}$$

with $\mathbf{M}_n$ as in (6.71) such that $\ker \tilde{\mathbf{M}} = \operatorname{im} \tilde{\mathbf{J}}$ holds. If $\mathbf{x}_0 \in \ker(\mathbf{M}_n\mathbf{O}_n)$, then

$$\boldsymbol{\eta}_0 = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix} \in \ker(\tilde{\mathbf{M}}\tilde{\mathbf{O}}), \tag{A.45}$$

because

$$\tilde{\mathbf{M}}\tilde{\mathbf{O}} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_p \\ \mathbf{M}_n\mathbf{O}_n & \mathbf{0} \end{bmatrix}. \tag{A.46}$$

With the same argumentation as in the case without direct feed through, there exists a $\mathbf{w}(t)$ such that $\mathbf{z}(t) = \mathbf{0}$ for all $t \geq 0$. Hence, $\mathbf{y}(t) = \mathbf{0}$ for all $t \geq 0$ which completes the proof. □

B

Appendix B

Additional Topics

B.1 Modified Gram-Schmidt Algorithm

The following code is a MATLAB implementation of the modified Gram-Schmidt orthogonalization procedure:

```
function [Q,R] = mgs(A)

p = size(A,2);
Q = A;
R = zeros(p,p);

for m=1:p
for j=1:m-1
R(j,m) = Q(:,j)'*Q(:,m);
Q(:,m) = Q(:,m) - R(j,m)*Q(:,j);
end
R(m,m) = norm(Q(:,m));
Q(:,m) = Q(:,m)/R(m,m);
end
```

B.2 Construction of Annihilation Matrix $\ker \mathbf{M} = \operatorname{im} \mathbf{J}$

Let $\mathbf{J} \in \mathbb{R}^{n\times m}, m \leq n$ and assume that rank $\mathbf{J} = r \leq m$ holds. The goal is to construct an annihilation matrix $\mathbf{M}$ such that $\ker \mathbf{M} = \operatorname{im} \mathbf{J}$.

The matrix $\mathbf{J}$ can be decomposed by using the singular value decomposition according to

$$\mathbf{J} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \mathbf{\Delta}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^\mathsf{T} \\ \mathbf{V}_2^\mathsf{T} \end{bmatrix} = \mathbf{U}\mathbf{\Delta}\mathbf{V}, \tag{B.1}$$

see, e.g., [PSI11] for details. The matrices $\mathbf{U} \in \mathbb{R}^{n\times n}$ and $\mathbf{V} \in \mathbb{R}^{m\times m}$ are orthogonal. Furthermore, $\boldsymbol{\Delta} \in \mathbb{R}^{n\times n}$ with $\boldsymbol{\Delta}_1 \in \mathbb{R}^{r\times r}$ and zero matrices of appropriate dimensions. The matrix $\boldsymbol{\Delta}_1$ is a diagonal matrix according to

$$\boldsymbol{\Delta}_1 = \mathrm{diag}(\delta_1,\, \delta_2,\, \dots,\, \delta_r),\; \delta_1 \geq \delta_2 \geq \dots \geq \delta_r, \tag{B.2}$$

with $\delta_1, \dots, \delta_r$ as the sorted non-zero singular values of $\mathbf{J}$. The following properties of the singular value decomposition are utilized:

i) $\ker \mathbf{J} = \text{colspan } \mathbf{V}_2^\mathsf{T}$;

ii) $\text{im } \mathbf{J} = \text{colspan } \mathbf{U}_1$,

see [PSI11]. The goal is to design a Matrix $\mathbf{M}$ such that

$$\ker \mathbf{M} = \mathrm{im}\, \mathbf{J}, \tag{B.3}$$

and thus $\mathbf{MJ} = \mathbf{0}$. One possibility is to design the matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ according to

$$\mathbf{M} = \begin{bmatrix} \mathbf{I}_{(n-r)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}_2^\mathsf{T} \\ \mathbf{U}_1^\mathsf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{U}_2^\mathsf{T} \\ \mathbf{0} \end{bmatrix} \tag{B.4}$$

with $\mathbf{U}_2^\mathsf{T} \in \mathbb{R}^{(n-r)\times n}$ and a zero matrix of appropriate dimension. Then, according to i) above,

$$\ker \mathbf{M} = \text{colspan } \mathbf{U}_1 = \mathrm{im}\, \mathbf{J} \tag{B.5}$$

and moreover

$$\mathbf{MJ} = \begin{bmatrix} \mathbf{U}_2^\mathsf{T} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Delta}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^\mathsf{T} \\ \mathbf{V}_2^\mathsf{T} \end{bmatrix} = \mathbf{0}. \tag{B.6}$$

This follows from the orthogonality of $\mathbf{U}$ and hence $\mathbf{U}_2^\mathsf{T}\mathbf{U}_1 = \mathbf{0}$.

List of Figures

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