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Geometric Dominating Sets

A Min-Max Version Of The No-Three-In-Line Problem And Related Problems

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Abstract

In this master thesis, we study point sets in the $n \times n$ integer grid that have the property that each point in the grid lies on at least one line defined by two points from the point set. We call sets with this property *geometric dominating sets* and the minimum size of geometric dominating sets in the $n \times n$ grid *geometric domination number*.
Our ultimate goal is to determine the value of the geometric domination number for general n, but first, we provide some historic background and point out similarities to the *Queens-Domination Problem* and the *No-Three-In-Line Problem*.
Then we make several general observations on points and lines in the $n \times n$ grid, which we use subsequently to prove non-trivial lower and upper bounds on the geometric domination number.
Finally, we consider the problem on the discrete torus and use probabilistic methods to obtain lower and upper bounds on the minimum size of geometric dominating sets in this case.

Kurzfassung

In dieser Masterarbeit setzen wir uns mit Punktmengen im $n \times n$ Gitter auseinander, die die Eigenschaft haben, dass jeder Punkt im Gitter auf mindestens einer Linie liegt, die von zwei Punkten in der betrachteten Punktmenge definiert wird. Wir nennen Punktmengen mit dieser Eigenschaft *geometric dominating sets* (geometrische Dominanzmengen) und die minimale Kardinalität solcher Mengen im $n \times n$ Gitter *geometric domination number* (geometrische Dominanzzahl).
Unser Ziel ist es, die geometrische Dominanzzahl für allgemeine n zu bestimmen, aber zuerst liefern wir historischen Kontext und beschreiben Ähnlichkeiten zum Dominanzproblem der Damen und zum No-Three-In-Line Problem.
Des Weiteren machen wir einige allgemeine Beobachtungen zu Punkten und Linien im $n \times n$ Gitter, die wir anschließend dazu verwenden nicht-triviale untere und obere Schranken für die geometrische Dominanzzahl für allgemeine n zu beweisen.
Zu guter Letzt werden wir das Problem auch am diskreten Torus betrachten, wo wir mittels probabilistischer Methoden obere und untere Schranken für die minimale Größe geometrischer Dominanzmengen erhalten.

Acknowledgements

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1 Introduction

In 2017, at the 29th Canadian Conference on Computational Geometry, invited speaker David Eppstein initiated a discussion on the computational complexity of finding the largest subset without three points in a line of a given point set. The question is based on the well-known *No-Three-In-Line Problem* that asks for the largest point set without three points in a line in an $n \times n$ grid, which has intrigued mathematicians such as Paul Erdős, for roughly 100 years now and which seems to be notoriously hard to solve [3].

Further discussions at a workshop in Barbados in 2018 provoked several new ideas, approaches and questions. One of these questions, posed by Oswin Aichholzer, was a min-max version of the No-Three-In-Line Problem. He asked for the smallest number of points in general position in an $n \times n$ grid such that every point in the grid lies on a common line with (at least) two of the points in the set [3].
It turned out that this problem already appeared in 1976 in a mathematical games column by Martin Gardner in the *Scientific American* [12]. However, it seems that no progress has been made, except for the special case where lines are restricted to vertical, horizontal and diagonal lines [6].

In any case, this minimum version might remind one less of the No-Three-In-Line Problem, which itself is based on a mathematical chess puzzle, and more of the *Queens Domination Problem* that asks for a placement of five queens on a chessboard such that every square of the board is attacked by a queen. In a more general setting this problem asks for the domination number of the $n \times n$ queen graph.

Therefore, we will call point sets in the $n \times n$ grid with the property that every point in the grid lies on a common line with (at least) two points from the set, a *(geometric) dominating set* and the smallest size of such a set the *(geometric) domination number* $\mathscr{D}_n$.

After establishing some general results about sets of points and lines in the $n \times n$ grid in Section 2, we will prove non-trivial asymptotic upper and lower bounds and provide computational results for dominating sets in general position in Section 3.
In Section 5, we will discuss the problem on the discrete torus, if n is prime, and prove asymptotic lower and upper bounds with probabilistic methods which we present in Section 4.
But first, we will give some historical context and formalize the question that we have in mind in mathematical terms.

1.1 Some History: Two Mathematical Chess Puzzles

In this subsection, we will give a short introduction to the No-Three-In-Line Problem and the Queen-Domination Problem which are both based on mathematical chess puzzles.

1.1.1 The No-Three-In-Line Problem

In 1917, Henry Ernest Dudeney published a book called *Amusements in Mathematics* [7] that featured more than 400 mathematical puzzles, some of which intrigue researchers to this day. One of them is *A Puzzle With Pawns* that inspired the *No-Three-In-Line Problem.*

A Puzzle With Pawns (H.E. Dudeney [7])
Place two pawns in the middle of the chessboard, one at Q4 and the other at K5. Now, place the remaining fourteen pawns (sixteen in all) so that no three shall be in a straight line in any possible direction.

The generalized version of this puzzle is called the No-Three-In-Line Problem, that asks for the maximum number of points in an $n \times n$ grid such that no three points lie on a common line. This problem is particularly hard to solve and explicit solutions only exist up to $n = 46$ and single solutions for $n = 48, 50, 52$ (See e.g. [9]). According to Achim Flammenkamp [9] the-state-of-the-art computational approach is a sophisticated branch-and-bound algorithm with which he computed all solutions up to $n = 16$.

In the 1960s, Guy and Kelly [13] took a probabilistic approach which we will discuss in Section 4. They conjectured that for large n, there are no point sets in general position of size greater than $(\pi/\sqrt{3})\, n$. However, we will discuss in Section 4.1 that their conjecture is, in the strict sense, based on a wrong assumption.

General constructions of point sets in general position are relatively rare. Paul Erdös was the first to propose a construction of n points in general position, for n prime, to Klaus Friedrich Roth, who first published it in [24].
Roughly twenty years later, Hall, Jackson, Sudbery, and Wild [14] found a construction for a point set in general position of size $(3/2 - \varepsilon)\, n$. To this day, this is the best lower bound on the largest size of a point set in general position in the $n \times n$ grid, for general n.

About a decade ago, Fowler, Groot, Pandya and Snapp considered the No-Three-In-Line Problem on the $n \times n$ torus and proved for n prime that the maximal size of a point set in general position is $n + 1$ by giving an explicit construction and proving its maximality [10] with algebraic methods. Their result sparked some interest in the algebra community and further progress on the maximal size of point sets on the $m \times n$ torus and on higher dimensional tori was made (see e.g. [19], [18]).

Various other versions of the No-Three-In-Line Problem were considered, too. Among them were point sets in general position in higher dimensions and point sets without four coplanar points in higher dimensions (see [23] and [22]).

The minimum variant of the No-Three-In-Line Problem that is the one of the main topics of this thesis, first appeared in the column *Mathematical Games* in the *Scientific American* [12]. Its author, Martin Gardner wrote,

Instead of asking for the maximum number of counters that can be put on an order-n board, no three in line, let us ask for the minimum that can be placed such that adding one more counter on any vacant cell will produce three in line.

According to Gardner, the problem had already been mentioned briefly in a paper by Adena, Holton and Kelly [1]. He mentioned their best results which they obtained by hand for $3 \leq n \leq 10$. These are $4, 4, 6, 6, 8, 8, 12, 12$. Surprisingly, up to $n = 8$, their solutions are indeed optimal solutions as we will see in Section 3.2. However, no further progress was made.

Only in 2012, Cooper, Pikhurko, Schmitt, Warrington [6] picked up the problem and gained access to Gardner's notes and correspondence archived at Stanford University.
They found several letters discussing a variant of the Minimum No-Three-In-Line Problem which they called the *Queens Version*. In this version, lines are restricted to vertical, horizontal and diagonal lines, like the legal movements of a queen. One of the correspondents, John Harris, mentioned in one of his letters that he could prove that n is a lower bound but no further notes on the proof could be found. Cooper et.al. [6] found two different proofs that n is indeed a lower bound in this particular version, except in the case when n is congruent to 3 modulo 4, in which case one less may suffice.
Interestingly, as we will see in Section 3.2, optimal solutions in the general case may be smaller than $n - 1$.

1.1.2 The Queen Domination Problem

Another popular chess puzzle is the *Queen Domination Problem* that Dudeney also published in "Amusements in Mathematics".

The Hat-Peg Puzzle (H.E. Dudeney [7])
Here is a five-queen puzzle that I gave in a fanciful dress in 1897. As the queens were there repre-

sented as hats on sixty-four pegs, I will keep to the title, "The Hat-Peg Puzzle." It will be seen that every square is occupied or attacked [in the way queens are placed in Figure 1]. The puzzle is to remove one queen to a different square so that still every square is occupied or attacked, then move a second queen under a similiar condition, then a third queen and finally a fourth queen. After the fourth move, every square must be attacked or occupied, but no queen must then attack another. Of course, the moves need not be "queen moves"; you can move a queen to any part of the board.

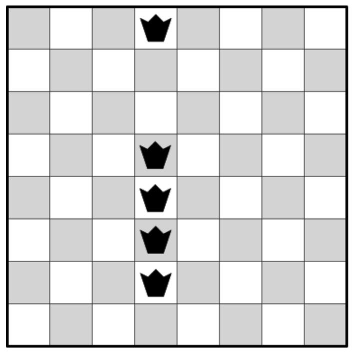

Figure 1: Initial position of the queens in the Hat-Peg Puzzle

In more general terms, the problem asks for the smallest number of queens that need to be placed on an $n \times n$ chessboard such that every square is attacked or occupied by a queen. A variant further asks for the smallest number of queens needed to attack or occupy every square under the condition that no queen attacks another.
These numbers are usually referred to as the queen domination number and the independent queen domination number. The terms derive from the interpretation of the problem as dominating sets of the so called *queen graphs*.

Definition 1 (Queen graphs)**.** The $m \times n$ *queen graph*, denoted by $Q_{m,n}$ is a graph with mn vertices in which each vertex represents a square in an $m \times n$ chessboard and each edge corresponds to a legal move by a queen.

Definition 2 (Dominating sets)**.** For a graph $G = (V, E)$ and a subset $S \subseteq V$, let $N(S)$ be the set of vertices in G which are in S or adjacent to a vertex in S. If $N(S) = V$, then S is called a *dominating set*.
A dominating set of smallest size is called a *minimum dominating set* and its size is known as the *domination number* of the graph.

So, any dominating set of the $n \times n$ queen graph $Q_{n,n}$ corresponds to a placement of queens that attack or occupy every square on an $n \times n$ chessboard. A minimum domination set of $Q_{n,n}$

is a solution to the generalized Queens Domination Problem and if we add the constraint that no queen is to attack another, the minimum dominating set has to be independent too.

Definition 3 (Independent sets)**.** An *independent set* of a graph G is a subset of the vertices such that no two vertices in the subset represent an edge of G.

The first dominating sets and independent dominating sets of $Q_{n,n}$ for small n where published around 1900 by Rouse Ball and Ahrens (see [25], [2]) but general results only appeared around 30 years ago. In 1990, Cockayne, Spencer and Welsh established lower and upper bounds that are summarized in [5]. Spencer's lower bound of $(n-1)/2$ is still the best general lower bound. Weakley [27] later improved this lower bound for $n = 4k + 1$ to $(n + 1)/2$.

In 2001, Östergard and Weakley [20] published the general upper bound $69n/133 + O(1)$ on the domination number and the upper bound $61n/111 + O(1)$ on the independent domination number. Additionally, they computed further optimal solutions, such that for $n \leq 120$, the domination number and independent domination number is either known, or known to have one of two values.

1.2 Geometric Dominating Sets

In the spirit of mathematical chess puzzles, we now ask,

How many pawns do we have to place on a chessboard such that every square lies on a straight line defined by two pawns? How many pawns do we need if no three pawns are allowed to lie on a common line?

As we will see in Section 3.2, the answer is eight and possible solutions are the placements in Figure 2. In fact, there are 228 possibilities to place eight pawns such that every square lies on a line defined by two pawns and no three pawns lie on a line.

But first, let us introduce some notation and definitions to formalize what "every square lies on a line defined by two pawns" really means and describe intuitive formulations like "point p lies on line L" in mathematical terms to prevent any misunderstandings later on.

1.2.1 Definitions

We start by defining lines in the grid. We will define them as finite point sets that consist of all points in the grid that lie on a common continuous line.

Definition 4 (Lines and collinearity)**.** We denote the range from 1 to n by $[n] = \{1, 2, \dots n\}$ and the $n \times n$ grid by $[n] \times [n] = [n]^2$.

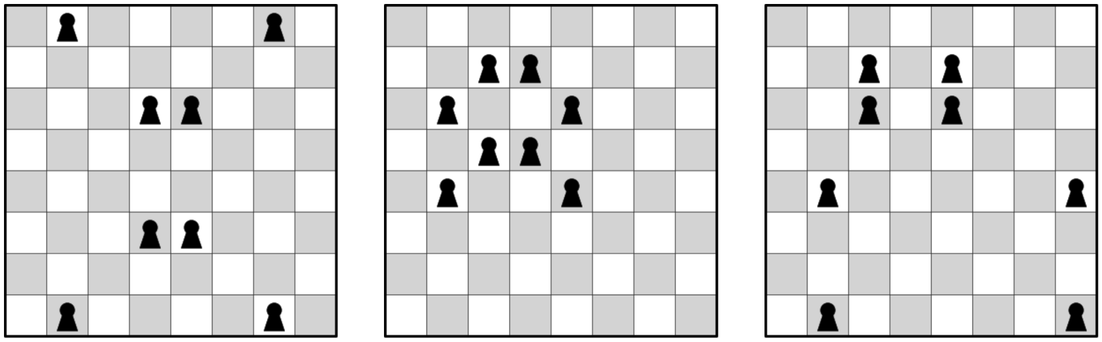

Figure 2: Sample solutions: Every square lies on a line defined by two pawns

A *line* L in $[n]^2$ is a subset $L = \{\mathsf{x} \in [n]^2 \mid \mathsf{x} = \mathsf{z} + t\mathsf{y},\ \mathsf{z}, \mathsf{y} \in \mathbb{R}^2, t \in \mathbb{R}\}$ with $|L| \geq 2$. A point $\mathsf{p} \in [n]^2$ *lies on a line* L, if $\mathsf{p} \in L$. In this case, we also say p and L are *incident*. Two points $\mathsf{x}, \mathsf{y} \in [n]^2$ define the line $\overline{\mathsf{xy}} = \{\mathsf{z} \in [n]^2 \mid \mathsf{z} = \mathsf{x} + t(\mathsf{y} - \mathsf{x}), t \in \mathbb{R}\}$.
Let $\mathsf{x}, \mathsf{y}, \mathsf{z} \in [n]^2$. We call these points *collinear* if they lie on a common line. That is, if $\mathsf{z} \in \overline{\mathsf{xy}}$.
A set of points is called *collinear*, if every three points in the set are collinear.
Conversely, a set $S \subseteq [n]^2$ is called *in general position* if no three points in S are collinear.

Throughout this text, if we talk about lines, we only refer to lines in $[n]^2$ and whenever we mean a line in $\mathbb{R}^2$, that is a set $\{\mathsf{x} \in \mathbb{R}^2 \mid \mathsf{x} = \mathsf{z} + t\mathsf{y},\ \mathsf{y}, \mathsf{z} \in \mathbb{R}^2, t \in \mathbb{R}\}$, we will talk about a continuous line.

Example. Consider the 3×3 grid $[3]^2$. The set $L = \{(1,1), (1,2), (1,3)\}$ is a line and $|L| = 3$ points lie on L. The three points incident to L are collinear.

With respect to the puzzle with pawns, we will identify the $n \times n$ chessboard with $[n]^2$ such that every square on the chessboard naturally corresponds to a point in $[n]^2$ and we will identify the set of pawns as a subset $S \subseteq [n]^2$. Now, we continue to formally describe the property that every square on the chessboard lies on a line defined by two pawns.

Definition 5 (Geometric dominating sets)**.** Let $S \subseteq [n]^2$ and $\mathsf{p} \in [n]^2$. If $\mathsf{p} \in S$ or there exist $\mathsf{x}, \mathsf{y} \in S$ such that $\{\mathsf{x}, \mathsf{y}, \mathsf{p}\}$ are collinear, we call p *dominated (by S)*. Similarly, we will say p is dominated by a line L if $\mathsf{p} \in L$ and if $\mathsf{p} \in S$, we also say that p is dominated by itself. Otherwise, we call p *free*.
If all $\mathsf{p} \in [n]^2$ are dominated by S, we call S a *(geometric) dominating set* or simply *dominating*.
We will call the smallest size of a dominating set of $[n]^2$ the *(geometric) domination number* $\mathscr{D}_n$ and denote the smallest size of a dominating set in general position by $\overline{\mathscr{D}}_n$.

With the Queens Domination Problem in mind, we are tempted to refer to $\overline{\mathscr{D}}_n$ as *independent (geometric) domination number*. However, there is no need to introduce this notion, because our main focus in this thesis are dominating sets that do not need to be in general position.

While the above definition of dominating sets is very natural, note that it is equivalent to the question
What is the smallest subset $S \subseteq [n]^2$ such that for every element $\mathsf{z} \in [n]^2$, $\mathsf{z} \in S$ or there are $a, b \in \mathbb{Z},\ b \neq 0$ and $\mathsf{x} \neq \mathsf{y} \in S$ such that

$$b(\mathsf{z} - \mathsf{x}) = a(\mathsf{y} - \mathsf{x})$$

The above follows, because three distinct points $\mathsf{x}, \mathsf{y}, \mathsf{z}$ are collinear if there is a $t \in \mathbb{R}$ such that $\mathsf{z} = \mathsf{x} + t(\mathsf{y} - \mathsf{x})$. If we look at this equation in the first coordinate, then for $\mathsf{x} = (x_1, x_2)$, $\mathsf{y} = (y_1, y_2)$, $\mathsf{z} = (z_1, z_2)$, this means that $z_1 = x_1 + t(y_1 - x_1)$.
We assume without loss of generality that $x_1 \neq y_1$, otherwise we look at the equation in the second coordinate. It follows that $t = (z_1 - x_1)/(y_1 - x_1) = a/b \in \mathbb{Q}$ and consequently

$$\mathsf{z} = \mathsf{x} + \frac{a}{b}(\mathsf{y} - \mathsf{x}) \quad \Longleftrightarrow \quad b(\mathsf{z} - \mathsf{x}) = a(\mathsf{y} - \mathsf{x})$$

We note that this also means that the absolute values of a and b are at most n.
A set S fulfilling the above conditions is therefore a dominating set and the smallest size of such a set equals $\mathscr{D}_n$.

In the following sections, we will prove upper and lower bounds on the value of $\mathscr{D}_n$ and $\overline{\mathscr{D}}_n$ for a given $n \in \mathbb{N}$ and we will present dominating sets in general position for $n \leq 10$ that we found computationally.
Further, we will consider the problem on the discrete torus. In 2012, Fowler, Groot, Pandya and Snapp solved the No-Three-In-Line Problem on the torus for n prime by using algebraic properties of cyclic subgroups of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. We will use these as well to derive lower and upper bounds on the size of minimum dominating sets.

1.3 Summary Of Results

In Section 2, we will explain a general strategy on how to count lines and dominated points in $[n]^2$. We will compute

- the number of lines incident to a point in the center of $[n]^2$ (Subsection 2.1)
- the maximal number of dominated points by s lines incident to a fixed point (Subsection 2.3, Lemma 5)
- lower and upper bounds on the number of possibilities to dominate a fixed point (Subsection 2.3, Lemma 6).

In Section 3, we will prove that

- $\overline{\mathscr{D}}_n \geq \mathscr{D}_n = \Omega(n^{2/3})$ (Subsection 3.1, Theorem 3)
- $\mathscr{D}_n \leq 2\lceil n/2 \rceil$ (Subsection 3.2, Theorem 4)

and present computational results on dominating sets in general position for $n \leq 10$.
In Section 4, we will introduce probabilistic models and methods which we use to show that on the discrete torus, if n is prime, that the domination number on the torus, denoted by $\mathscr{D}_n^T$, is

- $\mathscr{D}_n^T = \Omega(\sqrt{n})$ (Subsection 5.1, Theorem 8)
- $\mathscr{D}_n^T = O(\sqrt{n \log n})$ (Subsection 5.1, Theorem 9).

2 Points And Lines In $[n]^2$

In this section, we will try to get a basic understanding of point and line sets in grids.
In order to find a lower bound on $\mathscr{D}_n$, it is very useful to know how many points can be dominated by a fixed number of lines incident to a fixed point. Conversely, it is interesting how "easily" a point is dominated. That is, how many possibilities there are to dominate a fixed point. Further we will discuss whether this number is equal for all points in $[n]^2$ and which points provide lower and upper bounds on these numbers.

For simplicity, we will only consider odd $n = 2k+1,\ k \in \mathbb{N}$ in this section.

2.1 Lines

First, we will count the lines incident to the point $\mathsf{c}_n = (k+1, k+1)$ in the center of $[n]^2$. Since $[n]^2$ is the union of two rotated copies and two mirrored and rotated copies of the grey area in Figure 3, we only need to consider all lines that are also incident to a point in

$$A_n = \{(x_1, x_2) \in [n]^2 \mid k+1 \leq x_2 \leq x_1\}.$$

Note however, that we will doublecount the vertical, horizontal and diagonal lines incident to c_n.

Now, the general idea is to count lines by their slope that we will determine by the point in A_n that lies on the line and that is closest to c_n.
For example, if $k = 6$, we can see in Figure 3 that the red points uniquely define the respective lines and the slopes of the lines incident to $(7,7)$ are

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}\right\}$$

We observe that these slopes are exactly the fractions $\frac{j}{i} \leq 1$ with $1 \leq i \leq 6$, $0 \leq j \leq 6$ and greatest common divisor $\gcd(j, i) = 1$.

For general k, let us introduce Euler's phi-function and Farey fractions.

Definition 6 (Euler phi-function)**.** $\varphi(n)$ is the number of non-negative integers k less than n such that $\gcd(k, n) = 1$.

Definition 7 (Farey sequence)**.** The Farey sequence F_n for any $n \in \mathbb{N}$ is the sequence of fractions $\frac{a}{b}$ with $0 \leq a \leq b \leq n$ and $\gcd(a, b) = 1$, arranged in increasing order. These fractions are also called Farey fractions.

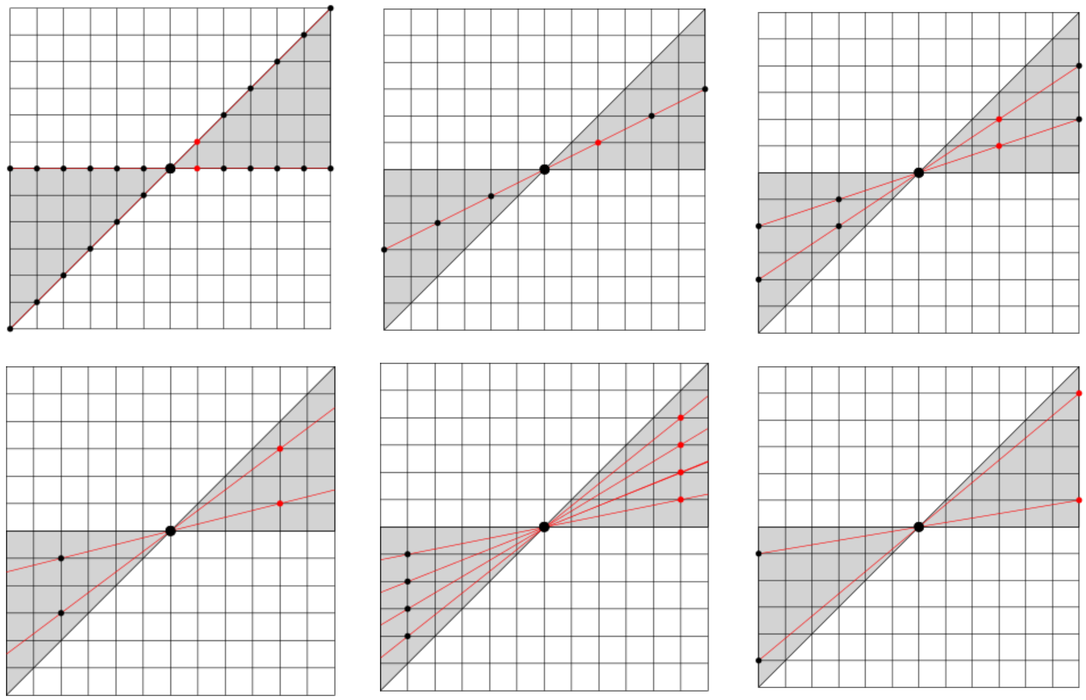

Figure 3: Example $n = 13$: Counting of lines incident to c_n by their slope

Lemma 1. *There is a bijection between the Farey fractions F_k and the lines incident to c_n and a point in $A_n = \{(x_1, x_2) \in [n]^2 \mid k+1 \leq x_2 \leq x_1\}$.*

Proof. Let $\mathcal{L}$ be the set of lines incident to c_n and a point in A_n. We define the function

$$L : F_k \to \mathcal{L}, \quad \frac{j}{i} \mapsto L(i,j) = \{\mathsf{y} \in [n]^2 \mid \mathsf{y} = \mathsf{c}_n + t(i,j),\ t \in \mathbb{R}\,\}$$

As $j \leq i \leq k$, there is at least one $\mathsf{y} \in A_n \setminus \{\mathsf{c}_n\}$ that $L(i,j)$ is incident to. So, the function is well-defined.

For $L(a,b) = L(c,d)$, it follows that $\mathsf{c}_n + (a,b) = \mathsf{c}_n + t(c,d)$ for some $t \in \mathbb{R}$. But a, b, c, d are integers, where $c \neq 0$. So

$$a = tc \quad \Rightarrow \quad t = \frac{a}{c} \in \mathbb{Q}$$

and

$$b = td = \frac{a}{c}d \quad \Rightarrow \quad cb = ad.$$

Since $\gcd(a,b) = \gcd(c,d) = 1$, it has to hold that $a = c$, $b = d$. Hence, the function is injective.

Now, let $L \in \mathcal{L}$ be incident to $\mathsf{x} = (x_1, x_2) \in A_n \setminus \{\mathsf{c}_n\}$. That is

$$L = \{\mathsf{y} \in [n]^2 \mid \mathsf{y} = \mathsf{c}_n + t(\mathsf{x} - \mathsf{c}_n),\ t \in \mathbb{R}\}$$

Further, let $m = \gcd(x_1 - (k+1), x_2 - (k+1))$ and $\mathsf{z} = (z_1, z_2) \in \{1, 2 \ldots, k\} \times \{0, 1, 2 \ldots, k\}$ such that $\mathsf{x} - \mathsf{c}_n = m \cdot \mathsf{z}$. So, $L = \{\mathsf{y} \in [n]^2 \mid \mathsf{y} = \mathsf{c}_n + t\mathsf{z},\ t \in \mathbb{R}\}$ and $\gcd(z_1, z_2) = 1$.
As $\mathsf{x} \in A_n$ and $0 \leq z_2 \leq z_1 \leq k$, the slope of L is $z_2/z_1 \in F_k$. Consequently, the function is surjective. □

The essence of this bijection is the fact that we can identify every line defined by c_n and a point in A_n with its slope. And the slope can be defined directly via the closest point to c_n that lies on the line. In the proof, we identified this point with $\mathsf{x} + \mathsf{z} = (k + 1 + z_1, k + 1 + z_2)$ and the slope with z_2/z_1. In Figure 3, this point is marked red.

Identifying lines with their slope in terms of the completely reduced fraction z_2/z_1 also makes it very obvious that there are $\varphi(k)$ additional lines incident to c_n and a point in A_n if we go from $k-1$ to k. Just think about the Farey sequence F_k, that will contain $\varphi(k)$ more fractions. That is,

$$|F_k| = |F_{k-1}| + \varphi(k) = 1 + \sum_{i=1}^{k} \varphi(i)$$

where $|F_1| = |\{\frac{0}{1}, \frac{1}{1}\}| = 1 + \varphi(1)$.

Luckily, the asymptotic behaviour of $|F_k|$ is a well known result in number theory. The best asymptotic formula was proven by Arnold Walfisz around 1960.

Theorem 1 (Arnold Walfisz [26])**.**

$$\sum_{i=1}^{k} \varphi(i) = \frac{3}{\pi^2} k^2 + O\left(k (\log k)^{\frac{2}{3}} (\log \log k)^{\frac{4}{3}}\right)$$

However, we will restrict our considerations to the use of the simpler expression $\frac{3}{\pi^2} k^2 + O\left(k \log k\right)$.

To get the total number of lines incident to c_n, we multiply by 4 (as $[n]^2$ is the union of four copies of our considered area) and subtract 4 since we doublecount the diagonal, the horizontal and the vertical line incident to c_n. So, the number of lines incident to c_n is

$$4 \cdot |F_k| - 4 = 4 \cdot \sum_{i=1}^{k} \varphi(i) = \frac{12k^2}{\pi^2} + O(k \log k) = \frac{3}{\pi^2} n^2 + O(n \log n)$$

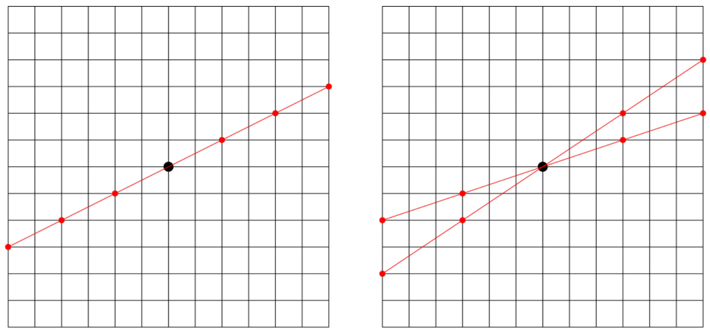

Figure 4: Example $n = 13$: Incident points to different lines in $[13]^2$

2.2 Dominating Points

Now, that we can distinguish the lines incident to c_n by their slope, we can count exactly how many points lie on each line and eventually derive how many points can be dominated at most by lines defined by c_n and one of s additional points.

2.2.1 How do we count dominated points?

Before we compute any numbers, we will discuss how to count dominated points in a structured manner.

Let us consider our example $n = 13$ first. As we can see in Figure 4, the line with slope $1/2$ will be incident to the points

$$\{(7,7) + m(2,1) \mid -3 \leq m \leq 3,\ m \in \mathbb{Z}\}$$

whereas the line with slope $2/3$ will be incident to the points

$$\{(7,7) + m(3,2) \mid -2 \leq m \leq 2,\ m \in \mathbb{Z}\}$$

And we also observe that the line with slope $1/3$ is incident to as many points as the line with slope $2/3$.

Observation 1. A line defined by c_n and a point in A_n with slope $\frac{j}{i}$, where $\gcd(i,j) = 1$, will be incident to exactly $2\lfloor \frac{k}{i} \rfloor + 1$ points in $[n]^2$, because for any point on the line there is a $t \in \mathbb{R}$ such that $(x_1, x_2) = \mathsf{c}_n + t(i,j)$ and

$$1 \leq x_1 = k + 1 + t \cdot i \leq 2k + 1 \quad \Leftrightarrow \quad -k/i \leq t \leq k/i.$$

Since $\gcd(i,j) = 1$ and x_1, x_2 are integers, t has to be an integer too. So, there are $2\lceil \frac{k}{i} \rceil + 1$ choices for t to define a point in $[n]^2$.

The inequality for the second coordinate is in these cases also satisfied since for every point in A_n it holds that $j \leq i$.

Moreover, as we have seen in the previous subsection, there are exactly $4\,\varphi(i)$ distinct lines that are incident to c_n and $2\lfloor \frac{k}{i} \rfloor$ additional points. Since every point in $[n]^2$ lies on exactly one line that is also incident to c_n, we can count the points in $[n]^2$ line by line. As we have to count c_n only once, the sum will be

$$n^2 = (2k+1)^2 = 1 + \sum_{i=1}^{k} 2 \left\lfloor \frac{k}{i} \right\rfloor 4\,\varphi(i)$$

If we divide by 8, we can see that we proved the nice identity,

$$\binom{k+1}{2} = \sum_{i=1}^{k} \left\lfloor \frac{k}{i} \right\rfloor \varphi(i).$$

2.2.2 What is the maximal number of dominated points by lines defined by c_n and one of s other points?

Now, we are actually interested in the maximal number of dominated points by lines defined by c_n and one of s additional points.

For simplicity, let $s = 4\sum_{i=1}^{m} \varphi(i)$. Then it easier to count the number of dominated points. Since we are aiming for an upper bound, we try to place the s points on distinct lines incident to c_n with a lot of points. That is, first we place a point on the diagonal, vertical and horizontal lines, then on all lines with slope $\pm 1/2$ and ± 2, and so on. In short, we will try to order the lines by the amount of points that they dominate and then choose the s lines that dominate the most points.

We think about it in the sense of a greedy algorithm. There will be s steps. In each step, we look for a line incident to c_n where no point has been placed yet. Of all those feasible lines, we choose one of the lines that dominate the most points in $[n]^2$ and place a point on it.
By our observation in the previous subsection, this means, we first choose the $4\,\varphi(1)$ lines incident to $2k$ further points, then the $4\,\varphi(2)$ lines incident to $2\lfloor k/2 \rfloor$ further points and so on.
In the end, after $s = \sum_{i=1}^{m} 4\varphi(i)$ steps, the number of points dominated by these lines (including c_n) is

$$1 + \sum_{i=1}^{m} 2 \left\lfloor \frac{k}{i} \right\rfloor 4\,\varphi(i) \;\leq\; 1 + 8k \sum_{i=1}^{m} \frac{\varphi(i)}{i} = 1 + 4(n-1) \sum_{i=1}^{m} \frac{\varphi(i)}{i} \tag{1}$$

Note, that any placement of s points that contains no point on one of the lines that we have chosen, would yield a lower number of dominated points. So, this is in fact an upper bound on the number of dominated points by s lines incident to c_n.

Since we are interested in an asymptotic formula of this number, we will use another result proven by Walfisz in [26].

Theorem 2 (Arnold Walfisz [26])**.**

$$\sum_{i=1}^{m} \frac{\varphi(i)}{i} = \frac{6}{\pi^2} m + O\left((\log m)^{\frac{2}{3}} (\log \log m)^{\frac{4}{3}} \right)$$

Once again, we will restrict our considerations to the expression $6m/\pi^2 + O(\log m)$. Applying this theorem to Equation 1, we obtain the following result.

Lemma 2. *Let* $\mathsf{c}_n = (k+1, k+1)$*,* $n = 2k+1$ *and* $S \subseteq [n]^2$*, with* $|S| = 4\sum_{i=1}^{m} \varphi(i)$*, where* $1 \leq m \leq k$*. Then the number of dominated points by lines incident to* c_n *and some point in* S *is bounded by*

$$1 + 8 \sum_{i=1}^{m} \left\lfloor \frac{k}{i} \right\rfloor \varphi(i) \;\leq\; \frac{24}{\pi^2} nm + O(n \log m)$$

2.2.3 How "easy" is it to dominate c_n?

This is another very natural question to ask in the search for dominating sets and close to asking "How likely is it to dominate c_n if we choose two random points in $[n]^2$?".
For now, we will count how many ways there are to place two points in $[n]^2$ such that c_n is dominated. The result will be especially interesting when we compare the number to those of other points in $[n]^2$.

Let us first introduce a variable for convenience.

Notation. Let $p \in [n]^2$. We will refer to two points $x, y \neq p$ that dominate p as *dominating pair (of p)* and define d_p to be the number of dominating pairs in $[n]^2$.

Again, we will first only consider lines defined by c_n and a point in A_n and count the possibilities to choose a dominating pair line by line, distinguishing them by their slope. Finally, we will multiply by 4 once again.

We know, there are $\varphi(i)$ lines that are incident to $2\left\lfloor \frac{k}{i} \right\rfloor$ points, excluding c_n. So, there are $\binom{2\lfloor \frac{k}{i} \rfloor}{2}$ choices to dominate c_n by two points one of these lines. Thus, summing over all lines, we get

$$d_{\mathsf{c}_n} = \sum_{i=1}^{k} \binom{2\left\lfloor \frac{k}{i} \right\rfloor}{2} 4\varphi(i) \;\leq\; 8k^2 \sum_{i=1}^{k} \frac{\varphi(i)}{i^2} = 2(n-1)^2 \sum_{i=1}^{k} \frac{\varphi(i)}{i^2} \tag{2}$$

In order to obtain an asymptotic formula for this expression, we could take a detour into analytic number theory (see [4], in particular p.65ff), but we will refrain to go into too many details and compute an asymptotic formula by summation by parts.

Lemma 3.

$$\sum_{i=1}^{m} \frac{\varphi(i)}{i^2} = \frac{6}{\pi^2} \log m + O\left(1\right)$$

Proof. Any tools and facts that we use in this proof can be found in the books by Königsberger [17]. First, we will use the general formula

$$\sum_{i=1}^{m} a_i b_i = A_m b_m + \sum_{i=1}^{m-1} A_i (b_i - b_{i+1}), \quad \text{where } A_i = \sum_{j=1}^{i} a_j$$

So, let $\Phi(m) = \sum_{i=1}^{m} \varphi(i)$. By Theorem 1, $\Phi(m) = 3m^2/\pi^2 + O(m \log m)$ and consequently,

$$\begin{aligned}
\sum_{i=1}^{m} \frac{\varphi(i)}{i^2} &= \frac{1}{m^2}\Phi(m) + \sum_{i=1}^{m-1} \Phi(i) \left(\frac{1}{i^2} - \frac{1}{(i+1)^2} \right) \\
&= \frac{1}{m^2}\Phi(m) + \sum_{i=1}^{m-1} \Phi(i) \left(\frac{2}{i^2(i+1)} - \frac{1}{i^2(i+1)^2} \right) \\
&= \frac{3}{\pi^2} + O\left(\frac{\log m}{m} \right) + \frac{3}{\pi^2} \sum_{i=1}^{m-1} \left(\frac{2}{(i+1)} - \frac{1}{(i+1)^2} \right) + O\left(\sum_{i=1}^{m} \frac{\log i}{i^2} \right)
\end{aligned}$$

We will now need the facts that

$$\lim_{m \to \infty} \left(\sum_{i=1}^{m} \frac{1}{i} - \log m \right) = \gamma$$

where γ denotes the Euler-Mascheroni constant and that

$$\sum_{i=1}^{\infty} \frac{1}{n^\alpha} < \infty \quad \Leftrightarrow \quad 1 < \alpha$$

Further note, that $\log m/\sqrt{m} < 1$ for all $m \geq 1$. Hence,

$$\begin{aligned}
\sum_{i=1}^{m} \frac{\varphi(i)}{i^2} &= \frac{3}{\pi^2} + O\left(\frac{\log m}{m} \right) + \frac{3}{\pi^2} \sum_{i=1}^{m-1} \left(\frac{2}{(i+1)} - \frac{1}{(i+1)^2} \right) + O\left(\sum_{i=1}^{m} \frac{\log i}{i^2} \right) \\
&= \frac{3}{\pi^2} + O\left(1\right) + \frac{3}{\pi^2} \left(2\left(\log m - 1 + \gamma + o(1) \right) - O(1) \right) + O\left(\sum_{i=1}^{m} \frac{1}{i^{3/2}} \right) \\
&= \frac{6}{\pi^2} \log m + O(1)
\end{aligned}$$

$\square$

By applying Lemma 3 to Equation 2, we obtain the upper bound

$$d_{\mathsf{c}_n} = 4\sum_{i=1}^{k} \binom{2\left\lfloor \frac{k}{i} \right\rfloor}{2} \varphi(i) \leq \frac{12}{\pi^2} n^2 \log n + O\left(n^2\right)$$

Conversely, as a lower bound, we would estimate $\binom{2\left\lfloor \frac{k}{i} \right\rfloor}{2} \geq \frac{1}{2}(2(\frac{k}{i}-1)-1)^2 \geq 2\frac{k^2}{i^2} - 6\frac{k}{i}$. But this does not change the main term of our asymptotic result as one can see by using Theorem 2 and Lemma 3.

$$\begin{aligned} 4\sum_{i=1}^{k} \binom{2\left\lfloor \frac{k}{i} \right\rfloor}{2} \varphi(i) \ &\geq \ 8k^2 \sum_{i=1}^{k} \frac{\varphi(i)}{i^2} - 24k \sum_{i=1}^{k} \frac{\varphi(i)}{i} \\ &= 8k^2 \left(\frac{6}{\pi^2} \log k + O(1) \right) - 24k \left(\frac{6}{\pi^2} k + O\left(\log k\right) \right) \\ &= \frac{12}{\pi^2} n^2 \log n + O(n^2) \end{aligned}$$

Hence, we get the following result.

Lemma 4. *Let $\mathsf{c}_n = (k+1, k+1)$, $n = 2k+1$. Then d_{c_n}, the number of dominating pairs of c_n is*

$$d_{\mathsf{c}_n} = 4\sum_{i=1}^{k} \binom{2\left\lfloor \frac{k}{i} \right\rfloor}{2} \varphi(i) = \frac{12}{\pi^2} n^2 \log n + O\left(n^2\right)$$

2.3 General Results

We will now try to understand how the results in the previous subsections change for points that do not lie at the center of $[n]^2$. We will show that in terms of asymptotics, the numbers for c_n provide upper bounds on $\{d_{\mathsf{p}}\}_{\mathsf{p}\in[n]^2}$ and the number of dominated points by s lines incident to an arbitrary point.

Intuitively, this is not very surprising. If we consider a continuous line incident to c_n and translate it, it can only get shorter in the $n \times n$ square. And since the number of two-element subsets of a set grows quadratically with its size, points on the border of $[n]^2$ are very reasonable candidates for a lower bound on $\{d_{\mathsf{p}}\}_{\mathsf{p}\in[n]^2}$. However, in $[n]^2$, lines do not behave as nicely as we would expect them at first.

Figure 5 illustrates for $n = 13$ how the number of incident points of four lines change, if you translate them from $(7, 7)$ to $(9, 8)$. The diagonals dominate less points, the line with slope $1/2$ stays the same, but the number of points incident to the line with slope $1/4$ increases.
In Figure 6, we can see that d_{p} does not show any monotone behaviour either.

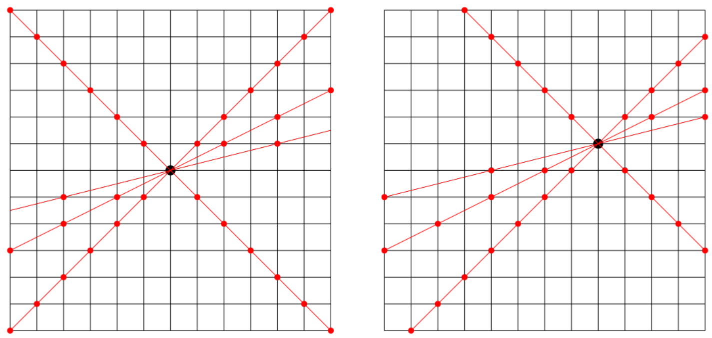

Figure 5: Example $n = 13$: Comparison of number of points on lines incident to (7,7) and (9,8)

						276
					243	248
				270	244	244
			289	267	236	234
		330	286	274	235	231
	331	318	300	269	231	211
396	348	344	303	286	242	236

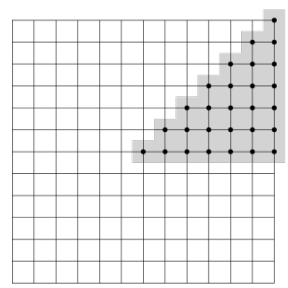

Figure 6: Example $n = 13$: Values of d_{p} for all $\mathsf{p} \in A_{13}$, ordered according to their position

So even though the technical details might seem overbearing at first, we will carefully prove these lower and upper bounds in this subsection.

Lemma 5. *For $\mathsf{p} \in [n]^2$ and $S \subseteq [n]^2$, with $|S| = 4\sum_{i=1}^{m} \varphi(i)$ and $1 \leq m \leq k$, the number of dominated points by lines incident to p and some point in S is bounded by*

$$1 + 8\sum_{i=1}^{m} \frac{k}{i}\varphi(i) \;\; = \;\; \frac{24}{\pi^2}nm + O\left(n \log m\right)$$

Proof. Again, the general idea is to order lines incident to p and lines incident to c_n according to the number of points that they dominate. Then, we compare these numbers line by line and derive that c_n provides an asymptotic upper bound. But first, we observe what happens if we translate a line from p to c_n. Vertical and horizontal lines always dominate n points, but any other line L with slope $\frac{j}{i}$, $\gcd(i,j) = 1$ and $|i|, |j| \leq k$, can be translated to c_n such that the translated line L' is

incident to at least three points. Without loss of generality, we will assume that the line has slope $\frac{j}{i} \in F_k$, where F_k are the Farey fractions.
There are two cases:

1. $|L'| \leq |L|$. So, the number of points that can be dominated by L' is at most the number of points that can be dominated by L and in the next step, we can use the same estimation for this line that we used in the sum for c_n

2. $|L'| > |L|$. Note, that $|L'|$ is bounded from above by $\frac{n-1}{i} + 1$. So, this case can only occur if

$$\left(\frac{2k}{i} + 1\right) - \left(2\left\lfloor\frac{k}{i}\right\rfloor + 1\right) \geq 1 \quad \iff \quad \frac{k}{i} - \left\lfloor\frac{k}{i}\right\rfloor \geq \frac{1}{2}$$

   It is also easy to see that

$$\left(\frac{2k}{i} + 1\right) - \left(2\left\lfloor\frac{k}{i}\right\rfloor + 1\right) = 2\left(\frac{k}{i} - \left\lfloor\frac{k}{i}\right\rfloor\right) < 2$$

   which means a translated line will be incident to at most one more point. But then, by the equation above, $2\frac{k}{i} \geq 2\lfloor\frac{k}{i}\rfloor + 1$. So, we can use the same estimation in the sum over the dominated points as we would in the sum for c_n again.

Any other line incident to p, will only be incident to one other point and could theoretically contribute only one additional dominated point, whereas any line incident to c_n dominates at least two additional points by symmetry.
For a (rough) upper bound on d_{p}, we will again greedily place point after point on one of the lines incident to p that dominate the most points in $[n]^2$ and on which no point was placed yet. Note that these lines do not have to be the same (translated) lines we would choose for c_n. There might be lines incident to c_n that reduce to the single point p if you translate them to p. Conversely, we have seen that every line incident to p will dominate at least as many points minus 1 if you translate it to c_n. Therefore, the line in the i'th step will dominate at most one point more than the line incident to c_n in the i'th step. And even if this is the case, we can use the same estimation as we did in our calculations of the sum for Lemma 2. Consequently, just as in Lemma 2, an upper bound is given by

$$1 + 8\sum_{i=1}^{m} \frac{k}{i}\varphi(i)$$

which, by Theorem 2 is equal to

$$1 + 8\sum_{i=1}^{m} \frac{k}{i}\varphi(i) \;=\; \frac{24}{\pi^2}nm + O\left(n \log m\right)$$

□

Next, we will tackle the number of dominating pairs of a fixed point p.

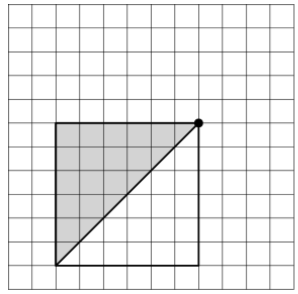

Figure 7: Example $n = 13$: Square incident to a point $\mathsf{p} \in A_n$

Lemma 6. *For all* $\mathsf{p} \in [n]^2$*, it holds that*

$$\frac{3}{2\pi^2} n^2 \log n + O\left(n^2\right) \leq d_{\mathsf{p}} \leq \frac{12}{\pi^2} n^2 \log n + O\left(n^2\right)$$

Proof. Once again for symmetry reasons, we only need to consider points in A_n.

The upper bound follows analogously to the reasoning in the proof of Lemma 5. For $\mathsf{p} \in A_n$, lines with slope $\frac{j}{i}$, $|i|, |j| \leq k$, $\gcd(i,j) = 1$ are incident to at most one point more than the translated line incident to c_n. Nevertheless a constant change of incident points does not affect the main term in our asymptotic expression for the number of dominating points as we have already seen in the proof of Lemma 4. All other lines only consist of two points such that p can only be dominated by itself on these lines and therefore, they do not contribute to d_{p}.

For a simple lower bound, we observe that every point $\mathsf{p} = (p_1, p_2) \in A_n$ lies at the corner of a square $\{(x_1, x_2) \in \mathbb{Z}^2 \mid p_1 - k \leq x_1 \leq p_1,\ p_2 - k \leq x_2 \leq p_2\} \subseteq [n]^2$. We can divide this square by the diagonal incident to p and count dominating pairs in each triangle like in the case of c_n (see Figure 7). This time, we only have two copies of our considered segment which is colored grey in Figure 7. So, there are

$$2 \sum_{i=1}^{k} \varphi(i)$$

lines and a line with slope j/i will only be incident to $\lfloor k/i \rfloor + 1$ points in the square including p. Therefore, similar to Lemma 4,

$$d_{\mathsf{p}} \geq 2 \sum_{i=1}^{k} \binom{\lfloor \frac{k}{i} \rfloor}{2} \varphi(i) = \frac{6}{\pi^2} k^2 \log k + O(k^2) = \frac{3}{2\pi^2} n^2 \log n + O(n^2)$$

$\square$

Remark 1. *The constant in the leading term of the lower bound in Lemma 6 can be improved significantly to* $\frac{9}{2\pi^2}$ *by more detailed considerations. However, even this constant is probably not tight and it would not improve later results.*

Remark 2. *Similarly to Lemma 4, we could prove that there are* $\Theta(n^3)$ *ways to choose three points on a common line that is incident to a fixed point* p.
The n^3 *factor derives from the number of possibilities to choose three points on a line with slope* $j/i \in F_k$*, which is for the lower bound at least*

$$\binom{\lfloor \frac{k}{i} \rfloor}{3} \sim \frac{k^3}{6}\frac{1}{i^3}$$

and for the upper bound at most

$$\binom{2\lfloor \frac{k}{i} \rfloor}{3} \sim \frac{4k^3}{3}\frac{1}{i^3}.$$

At the same time, the factor $\frac{1}{i^3}$ *will give us a sum*

$$1 \le \sum_{i=1}^{k} \frac{\varphi(i)}{i^3} \le \sum_{i=1}^{k} \frac{1}{i^2} < \infty$$

since $\varphi(i) \le i$ *for all* $i \in \mathbb{N}$ *and* $\varphi(1) = 1$.

3 Dominating Sets Of $[n]^2$

We first notice, that we only have a trivial upper bound on the geometric domination number, given by $2n$. In this case, we could place two points in every row (or column) such that all points are dominated. Otherwise, there are no obvious bounds.
In this section, we will present non-trivial upper and lower bounds on the geometric domination number and computational results on optimal solutions on dominating sets in general position.

3.1 Lower Bounds

In this section, we will first prove a simple lower bound on the geometric domination number and then refine the argument by using our results from the previous section.

Let S be a set of s points in $[n]^2$.
First, we fix a point $\mathsf{p} \in S$. There are at most $s-1$ lines defined by p and another point in S, each line dominating at most n points. So, summing over all those lines, at most $(s-1)n$ points will be dominated.
Further, we sum over all points in S and obtain that there are at most $s(s-1)n$ points dominated. Since we doublecount each line defined by two points, we can divide by two and obtain a final upper bound of dominated points by

$$\frac{1}{2}s(s-1)n.$$

Now, if we want S to be a dominating set, it has to hold that

$$\frac{1}{2}s(s-1)n \geq n^2 \quad \Leftrightarrow \quad s^2 - s - 2n \geq 0$$

which is the case if $s \geq \frac{1}{2} + \sqrt{\frac{1}{4} + 2n} \geq \sqrt{2n}$. Hence, $\mathscr{D}_n = \Omega(n^{1/2})$.

The attentive reader might notice that we could have derived the same upper bound by arguing that each pair of points in S defines a line that dominates at most n points, but the initial reasoning can be refined.
As we have seen in Section 2, there are very few lines incident to a point that actually dominate close to n points. In particular, we will use Lemma 5 to improve the lower bound on $\mathscr{D}_n$ significantly.

Theorem 3. *For $n \in \mathbb{N}$, it holds that $\mathscr{D}_n = \Omega(n^{2/3})$.*

Proof. First, let $n = 2k+1,\ k \in \mathbb{N}$ and let S be a set of s points in $[n]^2$, where $\sqrt{2n} \leq s \leq 2n$. (Remember that $2n$ is a trivial upper bound on $\mathscr{D}_n$ and $\sqrt{2n}$ a lower bound.)

Let m be the smallest positive integer such that

$$s - 1 \leq 4 \cdot \sum_{i=1}^{m} \varphi(i) = s'$$

Since we are aiming for an upper bound on dominated points, we can continue to work with s'. Now, it will be easier to apply results from Section 2.
At this point, it is also worth noting that $m = \Omega(n^{1/4})$ since by Theorem 1,

$$s \leq 1 + 4 \cdot \sum_{i=1}^{m} \varphi(i) = \frac{12}{\pi^2} m^2 + O\left(m \log m\right) \quad \text{and} \quad s = \Omega(n^{1/2}).$$

Conversely, since $s \leq 2n$ and

$$s \geq 1 + 4 \cdot \sum_{i=1}^{m-1} \varphi(i) = \frac{12}{\pi^2}(m-1)^2 + O\left(m \log m\right) = \frac{12}{\pi^2} m^2 + O\left(m \log m\right)$$

we know that $m = O(n^{1/2})$.

We will now go along the same lines as in the proof of the simple lower bound at the beginning of the section. Fix a point $\mathsf{p} \in S$. By Lemma 5, the number of dominated points by lines incident to p and one of s' additional points is bounded by

$$1 + 8 \sum_{i=1}^{m} \frac{k}{i} \varphi(i) \;=\; \frac{24}{\pi^2} nm + O\left(n \log m\right) \tag{3}$$

Of course, we want all points in $[n]^2$ dominated, so it has to hold that the overall number of dominated points is at least n^2. As $|S| = s$, we multiply the number in Equation 3 by s and as we doublecount each point on a line, we also divide by 2. Hence, we get the estimate

$$n^2 \leq \frac{1}{2} s \left(1 + 8 \sum_{i=1}^{m} \frac{k}{i} \varphi(i) + O(n)\right) = s \left(\frac{12}{\pi^2} nm + O\left(n \log m\right)\right)$$

Next, we plug in the asymptotic expression for s, such that the inequality above simplifies to

$$\begin{aligned} n^2 &\leq \left(\frac{12}{\pi^2} m^2 + O\left(m \log m\right)\right)\left(\frac{12}{\pi^2} nm + O\left(n \log m\right)\right) \\ &= \frac{144}{\pi^4} nm^3 + O\left(nm^2 \log m\right) \end{aligned}$$

and further,

$$n \leq \frac{144}{\pi^4} m^3 + O\left(m^2 \log m\right) \tag{4}$$

Thus, for all $\varepsilon' > 0$, there is a N such that for all $n \geq N$ it has to hold that

$$m > (1 - \varepsilon') \left(\frac{\pi^4 n}{144}\right)^{1/3}$$

such that Equation 4 is satisfied. Remember that $m = O(n^{1/2})$ such that for every $\varepsilon > 0$ and n large enough, it follows that

$$s = \frac{12}{\pi^2}m^2 + O\left(m \log m\right) = \frac{12}{\pi^2}m^2 + O\left(n^{1/2}\log n\right) > (1-\varepsilon)\left(\frac{\pi^4 n}{144}\right)^{2/3}$$

Consequently, for n odd, $\mathscr{D}_n = \Omega(n^{2/3})$.

Of course, we can easily generalize this result for n even by embedding $[n]^2$ in $[n+1]^2$. We know, that we need $\mathscr{D}_n$ points to dominate the embedded square $[n]^2$ and we only need to place at most two points on each of the lines $\{n+1\} \times [n+1]$ and $[n+1] \times \{n+1\}$ to dominate $[n+1]^2$. That is, in general, $\mathscr{D}_{n+1} - 4 \leq \mathscr{D}_n \leq \mathscr{D}_{n-1} + 4$.
Hence, $\mathscr{D}_n = \Omega(n^{2/3})$. □

Corollary 1. *Since $\overline{\mathscr{D}}_n \geq \mathscr{D}_n$, it holds that $\overline{\mathscr{D}}_n = \Omega(n^{\frac{2}{3}})$.*

3.2 Upper Bounds

A trivial upper bound on $\overline{\mathscr{D}}_n$ is given by $2n$, since any larger point set will contain at least three points in one column and as $\mathscr{D}_n \leq \overline{\mathscr{D}}_n$, both values are trivially bounded by this number.

Finding a non-trivial upper bound on $\overline{\mathscr{D}}_n$ seems complicated as there are hardly any known constructions for point sets in general position in the first place. Paul Erdős [24] proposed the following point set, when n is a prime number

$$\{(i, i^2 \bmod n) \mid 1 \leq i \leq n\}$$

where x mod y means the smallest remainder if x is divided by a multiple of y.
Roughly twenty years later, Hall, Jackson, Sudbery, and Wild [14] improved this lower bound on the largest point set in general position to $(3/2 - \varepsilon)n$.

However, none of these constructions are maximal and therefore dominating sets.

3.2.1 Two simple constructions for an upper bound on $\mathscr{D}_n$

So, let us first have a look at two simple constructions for a non-trivial upper bound on $\mathscr{D}_n$.

Construction 1.
The basic idea is to divide the $n \times n$ grid into nine cells (see Figure 8) and place points on the border of the inner cell.

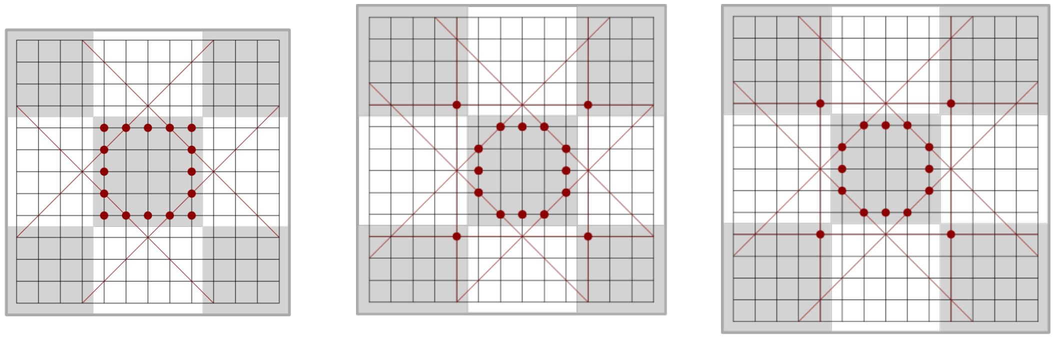

Figure 8: Dominating set constructions for $n = 13, 14, 15$

Then, every point in the cell of a corner of $[n]^2$ lies on a line with slope ± 1, that is, a line parallel to one of the diagonals. All other points lie on horizontal or vertical lines. So let us first consider $n \geq 10$, where $n - 1 \mod 3 = 0$ and let t be the edge length of the inner cell and s be the edge length of the cells in the corners. We want t to be as small as possible, but we need

$$2s + t = n \quad \text{and} \quad 2s - 1 \leq 2(t-1) - 1$$

as there are only $2(t-1) - 1$ lines with slope 1 (and -1 resp.) that could cover the points in the cells in the corners. So, we can choose $t = (n-1)/3 + 1$ and place $4(t-1)$ points.
We also notice that the points on the corner of the inner cell are only needed to define diagonals and can be moved along these diagonals. Thus, we can move the points to the inner corners of the cells in the corners $[n]^2$ (see Figure 8). Hence, all points of the $(n+2) \times (n+2)$ grid are covered as well. The same holds for the $(n+1) \times (n+1)$ grid, if we identify it as a subgrid in the $(n+2) \times (n+2)$ grid (see Figure 8). Consequently,

$$\mathscr{D}_n \leq 4 \left\lfloor \frac{n-1}{3} \right\rfloor \quad \text{for } n \geq 10.$$

Construction 2.
If we take into account lines with slope $\pm i$, $0 \leq i \leq n$ as well, the construction in Figure 9 gives us an even better upper bound. It was found by Aichholzer et.al. [3] by hand and subsequently they verified it computationally.

Clearly, all points in the grey areas are dominated by vertical or horizontal lines.
We also note, that points in the blue areas are dominated by diagonal lines and that we cannot extend the field in the horizontal direction, since the point $(17, 4)$ would not be dominated anymore. Points in the yellow area are dominated by lines with slope ± 2, points in the green area by lines with slope ± 3 and so on.

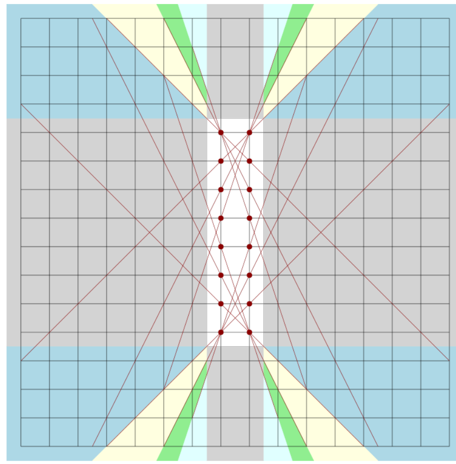

Figure 9: Dominating set construction for $n = 16$

Theorem 4. *For $n \in \mathbb{N}$, it holds that*

$$\mathscr{D}_n \leq 2 \left\lceil \frac{n}{2} \right\rceil .$$

Proof. First, let us consider $n = 2k$. The idea is to choose k points on each of the two vertical lines in the middle as dominating sets (see Fig. 9). That is,

$$S = \{(i, j) \mid k \leq i \leq k+1,\ \lceil k/2 \rceil + 1 \leq j \leq k + \lceil k/2 \rceil\}.$$

Obviously, all $x = (x_1, x_2)$ with $x_1 \in \{k, k+1\}$ are dominated by a vertical line and those with $\lceil k/2 \rceil + 1 \leq x_2 \leq k + \lceil k/2 \rceil$ are dominated by a horizontal line. Further, for symmetry reasons, it is sufficient to prove that points in the lower left rectangle $[k-1] \times [\,\lceil k/2 \rceil\,]$ are indeed dominated.

So let $x = (x_1, x_2) \in [k-1] \times [\,\lceil k/2 \rceil\,]$ and $\Delta = k - x_1$. We will now try to find the line with the smallest slope t that is incident to x and two points in S.
To do so, let t be the smallest positive integer such that $x_2 + t\,\Delta > \lceil k/2 \rceil$. Since $1 \leq \Delta \leq k-1$, this integer t is well defined and in the range from 1 to k.
We claim, that the points $(k, x_2 + t\,\Delta)$ and $(k+1, x_2 + t\,(\Delta + 1))$ are in S. This is the case if $x_2 + t\,(\Delta + 1) \leq k + \lceil k/2 \rceil$. We consider two cases:

1. $\Delta = k - x_1 \geq \lceil k/2 \rceil$. Then $t = 1$ and $x_2 + \Delta + 1 \leq k + \lceil k/2 \rceil$, since $\Delta \leq k - 1$ and $x_2 \leq \lceil k/2 \rceil$. (Note that this case is tight, which is why we cannot extend the grid).

2. $\Delta < \lceil k/2 \rceil$. Since t is the smallest integer such that $t > (\lceil k/2 \rceil - x_2)/\Delta$, we know that

$t \leq (\lceil k/2 \rceil - x_2)/\Delta + 1$ and obtain

$$\begin{aligned} x_2 + t\,(\Delta+1) &\leq x_2 + \left(\frac{\lceil k/2 \rceil - x_2}{\Delta} + 1\right)(\Delta+1) \\ &= \lceil k/2 \rceil + \frac{\lceil k/2 \rceil - x_2}{\Delta} + (\Delta+1) \leq 3\lceil k/2 \rceil - 1 \leq k + \lceil k/2 \rceil \end{aligned}$$

Thus, $(k, x_2 + t\,\Delta)$ and $(k+1, x_2 + t\,(\Delta+1))$ are indeed in S and x is dominated by the line that is defined by the two points.

If $n = 2k-1$, we can embed $[n]^2$ in $[2k]^2$ and obtain the desired upper bound. $\square$

3.2.2 Computational results on $\overline{\mathscr{D}}_n$

For an upper bound, we take a computational approach and naively implement a classic backtracking algorithm.

Backtracking is a recursive procedure that builds up candidates for an optimal solution step by step and as soon as it determines that the candidate cannot be extended to an optimal solution, takes a step back (*backtracks*) and chooses an alternative step.

In our case, it builds up point sets in general position point by point. If a set becomes larger than the currently known smallest dominating set(s) in general position, the algorithm steps back and generates a new point set. If a set dominates the grid and is at most the size of the currently known smallest dominating set(s), it saves the solution and steps back to generate further possible solutions.

Algorithm 1: backtrack(grid, dom_set, p, max, free)

```
Input:  grid, the n × n grid with points marked as free or dominated
        dom_set, a potential candidate for the dominating set
        p, a new point for the dominating set
        max, the size of the smallest known dominating set
        free, a variable that counts the number of free points
begin
    AddPoint(dom_set, p, grid, free);
    if free = 0 then
        if |dom_set|< max then
            max = |dom_set|;
        end
        Save(dom_set);
    else
        if |dom_set|< max then
            p = NextFree(point, grid);
            while p ∈ grid do
                backtrack(grid, dom_set, p, max, free);
                p = NextFree(point, grid);
            end
        end
    end
end
```

The function *AddPoint* adds the point p to the set *dom_set* and updates the grid and the value of the counter $free$. The function *NextFree* searches the grid line by line after the point p and returns the first free point that it finds.
A main program starts the backtracking procedure starting at every point in the $n \times n$ grid with the empty set as initial value for *dom_set*, n^2 for $free$ and $2n$ for max.

Oswin Aichholzer [3] computed all smallest dominating sets in general position with 4-fold symmetries for $n \leq 30$. So, we can choose the initial value of max as the size of these sets to improve the runtime a bit. However, we know that a dominating set has to be at least of size $cn^{2/3}$, which is also a lower bound on the depth of the recursion. The algorithm will therefore at least check every point set in general position of size $cn^{2/3}$ and at most every point set of size $2n$.

Nevertheless, the algorithm produced all optimal solutions up to $n = 10$ and Aichholzer et.al. [3] further computed all optimal solutions for $n = 11$. The results are summarized in Table 1.

n	$\overline{\mathscr{D}}_n$	number of solutions
2	4	1
3	4	5
4	4	2
5	6	152
6	6	8
7	8	4136
8	8	228
9	8	11
10	8	4
11	10	108

Table 1: Values of smallest dominating sets in general position for small n

The value for $n = 10$ was obtained by running the algorithm with starting points $(1, 1)$ to $(1, 5)$, since any dominating set that contains a point from the border can be mirrored or rotated such that it contains one of these points. All other dominating sets would also be dominating sets of the 9×9 grid, but none of the eleven solution for the 9×9 grid do not dominate $[10]^2$.

Aichholzer et.al. [3] further computed that the optimal solutions for $n = 12$ are at most of size 10.

4 Random Point Sets

4.1 Heuristic Arguments From Guy And Kelly [13]

In this section, we will discuss the steps Guy and Kelly took in a probabilistic approach to bound the maximal size of a set in general position in $[n]^2$ [13]. They conjectured that we cannot find a point set of size αn in general position for $\alpha > \left(2\pi^2/3\right)^{1/3} \approx 1.873856$ and n large enough.

First, they calculated the probability that three uniformly at random chosen points in $[n]^2$ lie on a line.
To do so, they counted the number of possibilities to choose three points on a line and then divided this number by the number of possibilities to choose three points from a set of n^2 points.

Theorem 5 (Guy, Kelly [13])**.** *The number t_n of sets of three collinear points that can be chosen from $[n]^2$ is*

$$t_n = \frac{3}{\pi^2} n^4 \log n + O(n^4).$$

Hence, they derived that

$$\mathbb{P}(E) = 1 - \frac{\frac{3}{\pi^2} n^4 \log n + O(n^4)}{\binom{n^2}{3}} = 1 - \frac{18 \log n}{\pi^2 n^2} + O\left(\frac{1}{n^2}\right)$$

where E denotes the event that three randomly chosen points in $[n]^2$ are in general position.

Subsequently, they concluded that *under the assumption that the events that three points of* $2n$ *random points of* $[n]^2$ *are in general position are mutually independent*, the probability that $2n$ random points are in general position is

$$\left(1 - \frac{18 \log n}{\pi^2 n^2} + O\left(\frac{1}{n^2}\right)\right)^{\binom{2n}{3}} = \exp\left(-\frac{24}{\pi^2} n \log n + O(n)\right)$$

where $\exp(x)$ denotes the exponential function.

If this is indeed the probability that a random point set of size $2n$ is in general position, the number of solutions to the No-Three-In-Line Problem has to be

$$\binom{n^2}{2n} \exp\left(-\frac{24}{\pi^2} n \log n + O(n)\right) = O\left(n^{2n - \frac{24}{\pi^2} n} c^n\right)$$

Using Stirling's formula, we can see that this number converges to 0 as $n \to \infty$.

In order to obtain a lower bound for this convergence behavior, they argued that the probability that αn random points lie on a line is

$$\left(1-\frac{18\log n}{\pi^2 n^2}+O\left(\frac{1}{n^2}\right)\right)^{\binom{\alpha n}{3}}=\exp\left(-\frac{3\alpha^3}{\pi^2}n\log n+O(n)\right)$$

and consequently, the number of solutions should be

$$\binom{n^2}{2n}\exp\left(-\frac{3\alpha^3}{\pi^2}n\log n+O(n)\right)=O\left(n^{2-\frac{3\alpha^3}{\pi^2}n}c^n\right) \tag{5}$$

which converges to 0 as $n\to\infty$ if $\alpha>\left(2\pi^2/3\right)^{1/3}\approx 1.873856$.

In 2004, Gabor Ellman pointed out a small error to Richard Guy directly. Apparently, Guy has never published a correction and Ed Pegg only published the correct number for a lower bound on α on his homepage "Math Games" [21].
However, it is easy to see that we have to multiply with the number of subsets of size αn and the correct formula for equation (5) is instead

$$\binom{n^2}{\alpha n}\exp\left(-\frac{3\alpha^3}{\pi^2}n\log n+O(n)\right)=O\left(n^{\alpha-\frac{3\alpha^3}{\pi^2}n}c^n\right)$$

which converges to 0 as $n\to\infty$ if $\alpha>\left(\pi^2/3\right)^{1/2}\approx 1.813799$.

Another thing that stands out about this proof is the assumption that the events that three points of a random subset of $[n]^2$ are in general position are mutually independent. Of course, this is not the case. If the points $\mathsf{p}_1,\mathsf{p}_2,\mathsf{p}_3$ are collinear and $\mathsf{p}_1,\mathsf{p}_2,\mathsf{p}_4$ are collinear the event that $\mathsf{p}_1,\mathsf{p}_3,\mathsf{p}_4$ are collinear is fully determined.

One can easily verify this fact for small n. Consider for example $n=3$:
There are 8 possibilities to choose three collinear points in $[3]^2$, so the probability that three random points are not collinear is

$$1-\frac{8}{\binom{9}{3}}=\frac{19}{21}$$

but given four random points, the probability they are in general position is

$$1-\frac{8\cdot 6}{\binom{9}{4}}=\frac{13}{21}\approx 0,619\neq\left(\frac{19}{21}\right)^4\approx 0,67$$

If we take it further, we know that there are only 2 possibilities to choose six points in general position (see e.g. [9]) in $[3]^2$, but

$$\frac{2}{\binom{9}{6}}\approx 0,02439\neq\left(\frac{19}{21}\right)^{\binom{6}{3}}\approx 0,13511$$

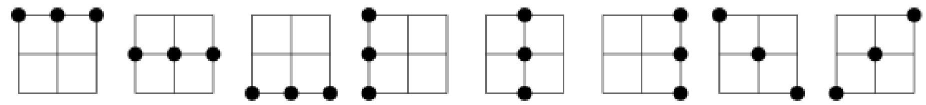

Figure 10: Possible choices for three points in a line for $n = 3$

However, these probabilities could get very close as n grows and the result could asymptotically still hold. According to the number of solutions for small n which can be found on Flammenkamp's homepage [9], the numbers do not suggest that this is true. Nonetheless, Guy and Kelly's computations could still provide an upper bound on the number of solutions as one can see in Table 2.

n	number of solutions	t_n	approx. number of sol.
3	2	8	11,35
4	11	44	131,65
5	32	152	893,63
6	50	372	9664,13
7	132	824	39475,38
8	380	1544	319791,65
9	368	2736	1285895,73

Table 2: Real number of solutions to the No-Three-In-Line Problem compared to the approximated number of solutions by Guy and Kelly's arguments

In the following subsection, we will introduce a number of tools and concepts that are usually used in the study of random graphs that will help us to properly consider the mutual dependence of certain events.
It will turn out, that this approach is not very fruitful with respect to the No-Three-In-Line Problem or the geometric domination number of $[n]^2$, but we will make heavy use of the theory and methods in Section 5, where we prove lower and upper bounds on the geometric domination number on the discrete torus.

4.2 Random Point Sets In $[n]^2$

Similar as in the study of random graphs, we will now introduce two models of random point sets in $[n]^2$ and use them to prove an upper bound for $\mathscr{D}_n$ on the torus.
We will go along the same lines as Frieze and Karonski in their book "Introduction to random

graphs” [11], adapt their definitions to random point sets and use some of the theorems in their book.

First, we define two different random point set models and relate them to each other.

Definition 8 (Random point set models)**.** Let $\mathcal{R}_{n,m}$ be the family of all subsets of size m of $[n]^2$. To every point set $R \in \mathcal{R}_{n,m}$ we assign the probability

$$\mathbb{P}(R) = \binom{n^2}{m}^{-1}$$

Essentially, this means that we start with an empty subset of $[n]^2$ and add m points such that all possible $\binom{n^2}{m}$ choices are equally likely. We denote such a random point set by $R_{n,m}$ and call it a *uniform random point set*.

The other model that we will introduce now is similar. Fix $0 \le p \le 1$. Then for $0 \le m \le n^2$, assign to each point set R with m points a probability

$$\mathbb{P}(R) = p^m (1-p)^{n^2-m}$$

Here, we start with an empty set and perform n^2 Bernoulli experiments inserting points of $[n]^2$ independently with probability p. Therefore, we call such a random point set a *binomial random point set* and denote it by $R_{n,p}$.

Just like the respective random graph models, the random point set models relate to each other such that $R_{n,p}$ conditioned on the event $\{|R_{n,p}| = m\}$ is distributed like $R_{n,m}$.

Lemma 7. $(R_{n,p} \mid |R_{n,p}| = m) \sim R_{n,m}$

Proof. Let $R \subseteq [n]^2$ with $|R| = m$. Then, because $R_{n,p} = R$ only if $|R_{n,p}| = m$, we have

$$\begin{aligned}\mathbb{P}(R_{n,p} = R \mid |R_{n,p}| = m) &= \frac{\mathbb{P}(R_{n,p} = R,\ |R_{n,p}| = m)}{\mathbb{P}(|R_{n,p}| = m)} \\ &= \frac{\mathbb{P}(R_{n,p} = R)}{\mathbb{P}(|R_{n,p}| = m)}\end{aligned}$$

And since $|R_{n,p}|$ is the sum of n^2 independent Bernoulli random variables that take the value 1 if the point is in the set and 0 otherwise, $|R_{n,p}|$ is binomially distributed with success probability p and therefore,

$$\begin{aligned}\mathbb{P}(R_{n,p} = R \mid |R_{n,p}| = m) &= \frac{p^m(1-p)^{n^2-m}}{\binom{n^2}{m}p^m(1-p)^{n^2-m}} \\ &= \binom{n^2}{m}^{-1}\end{aligned}$$

□

Another useful concept that we will use is the idea of monotone properties.

Definition 9 (Monotone properties)**.** A point set property is a subset of all possible point sets in $[n]^2$. We call a point set property P *monotone increasing* if adding a point to a point set does not destroy the property. Conversely, a point set property is *monotone decreasing* if removing a point from a point set does not destroy the property.

By analogous arguments as in [11, p. 5ff], one can formally prove what intuitively seems clear: If P is a monotone increasing property, then the probability that a uniform random point set of size m' is equally or more likely to have the property than a uniform random point set of size $m \leq m'$. The same holds for $R_{n,p}$ and $R_{n,p'}$ if $p \leq p'$.

Lemma 8. *If P is a monotone increasing property, $p \leq p'$ and $m \leq m'$, then*

$$\mathbb{P}(R_{n,p} \in P) \leq \mathbb{P}(R_{n,p'} \in P) \quad \textit{and} \quad \mathbb{P}(R_{n,m} \in P) \leq \mathbb{P}(R_{n,m'} \in P)$$

Of course the converse holds for monotone decreasing properties.

Proof. For the first inequality, we will consider two point sets R_1 and R_2, where R_1 is distributed like $R_{n,p}$ and R_2 is distributed like $R_{n,q}$, where q is such that

$$(1 - p') = (1 - p)(1 - q).$$

Then the probability that a point does not lie in $R_1 \cup R_2$ is $(1 - p')$ and consequently, $R_1 \cup R_2$ is distributed like $R_{n,p'}$. Since P is monotone increasing, it follows that

$$R_1 \in P \quad \Rightarrow \quad R_1 \cup R_2 \in P$$

which means $\mathbb{P}(R_{n,p} \in P) \leq \mathbb{P}(R_{n,p'} \in P)$.

For the second inequality, we build $R_{n,m'}$ from two point sets R_1 and R_2, where R_1 is distributed like $R_{n,m}$ and R_2 is a uniform random point set of size $m' - m$ from the set $[n]^2 \backslash R_{n,m}$.
Again, $R_1 \cup R_2$ is distributed like $R_{n,m'}$ and the inequality follows from the monotonicity of P. $\square$

Clearly, the property of a point set being in general position is a monotone decreasing property while the property of being a dominating set is a monotone increasing property.
Guy and Kelly considered random point sets of fixed size. So, if we want to take a similar approach to our problem, we have to work with uniform random point sets. However, the tools that we want to use can only be applied to binomial random point sets. This is why we need the following lemma that we will prove analogously as a respective lemma for random graphs that can be found in the lecture notes by Joshua Erde [8].

Lemma 9. *Let P be a monotone point set property and let $m, n \in \mathbb{N}$. If we let $p = m/n^2$, then*

$$\mathbb{P}(R_{n,m} \in P) \leq 2\mathbb{P}(R_{n,p} \in P)$$

Again, the proof is absolutely analogous to the proof in the notes. In order to provide accessibility, we will include it anyways.

Proof. We will proof the theorem for monotone decreasing properties, but the case that P is monotone increasing is similar.
By the law of total probability and Lemma 7, it holds that

$$\begin{aligned}
\mathbb{P}(R_{n,p} \in P) &= \sum_{k=0}^{n^2} \mathbb{P}(|R_{n,p}| = k)\mathbb{P}(R_{n,p} \in P \mid |R_{n,p}| = k) \\
&= \sum_{k=0}^{n^2} \mathbb{P}(|R_{n,p}| = k)\mathbb{P}(R_{n,k} \in P) \\
&\geq \sum_{k=0}^{m} \mathbb{P}(|R_{n,p}| = k)\mathbb{P}(R_{n,k} \in P)
\end{aligned}$$

Since P is monotone, we can further estimate

$$\mathbb{P}(R_{n,p} \in P) \geq \mathbb{P}(R_{n,m} \in P) \sum_{k=0}^{m} \mathbb{P}(|R_{n,p}| = k) = \mathbb{P}(R_{n,m} \in P)\mathbb{P}(|R_{n,p}| \leq m)$$

Now, $|R_{n,p}|$ is a binomial random variable with n^2 trials and success probability p and its mean and median coincide. Hence,

$$\mathbb{P}(|R_{n,p}| \leq m) = \frac{1}{2}$$

Rearranging the terms above yields the desired inequality. $\square$

So instead of trying to show like Guy and Kelly that the probability that a uniform random point set R_m is in general position goes to 0, we could try to prove the same for a binomial random point set R_p with $p = m/n^2$ since these models are (in this case) far easier to handle. They offer the possibility of using the following inequalities that were first published in the 1980's and that can be found in Frieze and Karonski's book.

Theorem 6 (Janson's inequality [11])**.** *Let $\mathcal{A}$ be a finite set and R_p, where $p = \{p_i, i \in \mathcal{A}\}$, a random subset in such a way that the elements are chosen independently with $P(i \in R_p) = p_i$ for each $i \in A$.*

Let $\mathcal{B}$ be a family of subsets of $\mathcal{A}$ and, for every $B \in \mathcal{B}$, let I_B be the indicator variable. Then $X = \sum_{B\in\mathcal{B}} I_B$ counts those elements of $\mathcal{B}$ which are entirely contained in R_p. Set

$$\mu = \mathbb{E}(X), \quad \Delta = \frac{1}{2} \sum_{\substack{A\neq B\in\mathcal{B}\\ A\cap B\neq\emptyset}} \mathbb{E}(I_A I_B)$$

Then, it holds that

$$\mathbb{P}(X = 0) \leq \exp\left(-\mu + \Delta\right) \tag{6}$$

If $\Delta > \frac{\mu}{2}$, a stronger bound is given by

$$\mathbb{P}(X = 0) \leq \exp\left(-\frac{\mu^2}{\mu + 2\Delta}\right) \tag{7}$$

In the next subsection, we will use Janson's inequality to bound the probability that a fixed point in $[n]^2$ is free. To do so, we will define $\mathcal{A}$ to be $[n]^2$ and $\mathcal{B}$ to be the set of dominating pairs of this fixed point. Then, Janson's inequality yields an upper bound on the probability that no dominating pair is contained in the random point set.
Subsequently, we will use the union bound to obtain an upper bound on the probability that there is a free point in $[n]^2$.

Theorem 7 (Union bound [16])**.** *Let $\{A_i\}_{i\in\mathbb{N}}$ be a countable family of events. Then*

$$\mathbb{P}\left(\bigcup_{i\in\mathbb{N}} A_i\right) \leq \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

4.3 An Application To Dominating Sets Of $[n]^2$

In this subsection, we will try to prove that $\mathscr{D}_n \leq \alpha n$ with $\alpha < 2$, using random point sets. Unfortunately, we will not be particularly successful. But, as we will see in the next section, the same strategy works very well on the discrete torus.

The general outline of the proof is the following:

1. Fix a point $\mathsf{p} \in [n]^2$ and compute an upper bound on the probability that a random point set of size αn dominates p:

    (a) Use Lemma 9 and bound the probability that a uniform random point set of size αn does not dominate p by the probability that a binomial random point with $p = \alpha/n$ does not dominate p.

(b) Use the fact that p is dominated by $R_{n,\alpha/n}$ if and only if $R_{n,\alpha/n}$ contains a dominating pair of p. Apply Janson's inequality to bound the probability that $R_{n,\alpha/n}$ does not contain a dominating pair of p.

2. Use the union bound to show that the probability that there exists a free point in $R_{\alpha n}$ is smaller than 1. Then, there exists a dominating point set of size αn.

We start with step 1. Let $R_{\alpha n}$ be a uniform random point set and $\mathsf{p} \in [n]^2$. Let P_p be the set of all point sets in $[n]^2$ that do not dominate p. By definition, P_p is a point set property and since any subset of a point set that does not dominate p does not dominate p either, P_p is a decreasing point set property.
By Lemma 9, it holds that

$$\mathbb{P}(R_{\alpha n} \in P_\mathsf{p}) \leq 2\,\mathbb{P}(R_{n,\alpha/n} \in P_\mathsf{p}).$$

Therefore, we will only consider $R_{n,\alpha/n}$ from now on. Note that $R_{n,\alpha/n}$ dominates p if and only if $R_{n,\alpha/n}$ contains a dominating pair of p or $\mathsf{p} \in R_{n,\alpha}$.
So, let $\mathcal{D}_\mathsf{p}$ be the set of dominating pairs of p and for any $D \in \mathcal{D}_\mathsf{p}$, let I_D be the indicator random variable that takes the value 1 if $D \in R_{n,\alpha/n}$ and 0 otherwise. Likewise, let I_p be the indicator random variable that takes the value 1 if $\mathsf{p} \in R_{n,\alpha/n}$ and 0 otherwise. Then, the random variable

$$X = I_\mathsf{p} + \sum_{D \in \mathcal{D}_\mathsf{p}} I_D$$

counts the number of dominating pairs of p in $R_{n,\alpha/n}$ and adds 1 if $\mathsf{p} \in R_{n,\alpha/n}$. That means, $X \geq 1$ if and only if p is dominated by the random point set.
Now, we can apply Janson's inequality, if we identify $\mathcal{A}$ with $[n]^2$, R_p with $R_{n,\alpha/n}$ and $\mathcal{B}$ with $\mathcal{D}_\mathsf{p} \cup \{\mathsf{p}\}$. We just need to calculate μ and Δ.

In order to so, remember that we defined $d_\mathsf{p} = |\mathcal{D}_\mathsf{p}|$ in Section 2.2. Since any dominating pair consists of exactly two points that are independently contained in $R_{n,\alpha/n}$, each with probability α/n, we know that

$$\begin{aligned}
\mu = \mathbb{E}(X) &= \mathbb{E}(I_\mathsf{p}) + \sum_{D \in \mathcal{D}_\mathsf{p}} \mathbb{E}(I_D) \\
&= \mathbb{P}(\mathsf{p} \in R_{n,\alpha/n}) + \sum_{D \in \mathcal{D}_\mathsf{p}} \mathbb{P}(D \in R_{n,\alpha/n}) \\
&= \frac{\alpha}{n} + \sum_{D \in \mathcal{D}_\mathsf{p}} \left(\frac{\alpha}{n}\right)^2 = \frac{\alpha}{n} + d_\mathsf{p} \left(\frac{\alpha}{n}\right)^2.
\end{aligned}$$

For the Δ-term, we will only compute an asymptotic formula. Two distinct dominating pairs C, D of p have a non-empty intersection if and only if their intersection contains exactly one point and since both pairs dominate p the three points in $C \cup D$ are collinear. On the other hand, a

dominating pair of p and $\{\mathsf{p}\}$ will have an empty intersection.
Since we know by Remark 2 that there are $\Theta(n^3)$ possibilities to choose three collinear points on a line incident to p and each of those three points is in $R_{n,\alpha/n}$ independently with probability α/n, we obtain

$$\Delta = \frac{1}{2} \sum_{\substack{C \neq D \in \mathcal{D}_{\mathsf{p}} \cup \{\mathsf{p}\} \\ C \cap D \neq \emptyset}} \mathbb{E}(I_C I_D) = \frac{1}{2} \sum_{\substack{\mathsf{x},\mathsf{y},\mathsf{z} \in [n]^2 \\ \mathsf{x},\mathsf{y},\mathsf{z},\mathsf{p} \text{ collinear}}} \left(\frac{\alpha}{n}\right)^3 = \Theta(1)$$

Using Janson's inequality, we can now derive

$$\begin{aligned} \mathbb{P}(R_{\alpha n} \in P_{\mathsf{p}}) &\leq 2\,\mathbb{P}(R_{n,\alpha/n} \in P_{\mathsf{p}}) = 2\,\mathbb{P}(X = 0) \\ &\leq 2\exp(-\mu + \Delta) \\ &\leq 2\exp\left(-\frac{\alpha}{n} - d_{\mathsf{p}}\left(\frac{\alpha}{n}\right)^2 + \Theta(1)\right) \end{aligned}$$

Finally, as outlined in step 2 at the beginning, we can now use the union bound to bound the probability that $R_{\alpha n}$ is not a dominating set.

$$\begin{aligned} \mathbb{P}(\text{There exists a free point in } T_n) &= \mathbb{P}(R_{\alpha n} \in \bigcup_{\mathsf{p} \in [n]^2} P_{\mathsf{p}}) \\ &\leq \sum_{\mathsf{p} \in [n]^2} \mathbb{P}(R_{\alpha n} \in P_{\mathsf{p}}) \\ &\leq 2 \sum_{\mathsf{p} \in [n]^2} \exp\left(-\frac{\alpha}{n} - d_{\mathsf{p}}\left(\frac{\alpha}{n}\right)^2 + \Theta(1)\right) \end{aligned}$$

By Lemma 6, we know that $d_{\mathsf{p}} \geq \frac{3}{2\pi^2} n^2 \log n + O(n^2)$, so we advance our calculations as follows.

$$\begin{aligned} \mathbb{P}(\text{There exists a free point in } T_n) &\leq 2 \sum_{\mathsf{p} \in [n]^2} \exp\left(-\frac{\alpha}{n} - d_{\mathsf{p}}\left(\frac{\alpha}{n}\right)^2 + \Theta(1)\right) \\ &\leq 2n^2 \exp\left(-\left(\frac{\alpha}{n}\right)^2 \frac{3}{2\pi^2} n^2 \log n + \Theta(1)\right) \\ &= \exp\left(2\log n - \frac{3\alpha^2}{2\pi^2}\log n + \Theta(1)\right) \end{aligned}$$

So, for all n large enough the probability that there exists a free point will be smaller than 1 if we choose α such that

$$2 - \frac{3\alpha^2}{2\pi^2} < 0 \quad \Leftrightarrow \quad \alpha > \frac{2\pi}{\sqrt{3}} > 2$$

This lower bound for α is of course useless because $\alpha = 2$ yields the trivial upper bound $2n$ on $\mathscr{D}_n$. If we could find a better bound for d_{p}, somehow, the lower bound for α could decrease significantly. Let us assume, for example, we could use the upper bound of d_{p} for all $\mathsf{p} \in [n]^2$. Then we would obtain

$$2 - \frac{12\alpha^2}{\pi^2} < 0 \quad \Leftrightarrow \quad \alpha > \frac{\pi}{\sqrt{6}}$$

where we observe that $\pi/\sqrt{6} \approx 1,28255 < 2$.
However, we already showed that $\mathscr{D}_n \leq 2\lceil n/2 \rceil$ in the previous section. So this probabilistic approach does not seem useful overall.

On the other hand, we will see in the next section that on the discrete torus, we can exploit the fact that every point is incident to the same number of lines, that all lines are of equal length and that d_{p} is constant for all $\mathsf{p} \in [n]^2$.

5 Dominating Sets On The Discrete Torus

Roughly ten years ago, Fowler, Groot, Pandya and Snapp considered the No-Three-In-Line Problem on a $n \times n$ torus and proved for n prime that the maximal size of a point set in general position is $n+1$ by giving an explicit construction and proving its maximality [10]. Additionally, they considered $n \times m$ tori, where $m = n^2$ and $m = nk$, $\gcd(n, k) = 1$. In 2016, Misiaka, Stpieńa, Szymaszkiewicza, Szymaszkiewiczb and Zwierzchowski made further progress on $n \times m$ tori and solved the problem for n, m such that $\gcd(n, m)$ is a prime.

In this section, we will not provide any specific constructions for dominating sets. Instead, we will prove lower bounds on the minimum size of dominating sets in a similar manner as in Section 3 and upper bounds with the tools that we introduced in Section 4.

But first, we need to define the discrete torus and state some preliminary facts.

Definition 10. We identify the discrete torus T_n with $\{0, 1, 2, \ldots, n-1\} \times \{0, 1, 2, \ldots, n-1\}$ and, just as in [19], we define lines on the torus to be images of lines in $\mathbb{Z} \times \mathbb{Z}$ under the projection $\pi_n : \mathbb{Z} \times \mathbb{Z} \to [n]^2$ defined as follows

$$\pi_n(x_1, x_2) := (x_1 \bmod n, x_2 \bmod n)$$

where $a \bmod b$ means the smallest non-negative remainder when a is divided by b.
A set of points S on the torus is collinear, if there is a line on the torus that is a superset of S.

In Figure 11, we can see that the line incident to $(0, 0)$ with slope $\frac{1}{2}$ in $[n]^2$ "wraps around the torus" such that it is now incident to the 13 black points. The line incident to $(1, 7)$ with slope $\frac{1}{2}$ now projects to exactly the same line.

By means of group theory, Fowler et.al. proved in their paper that the maximal size of a point set in general position is $n+1$ if n is prime and they obtained several other results. We will just cite their results in order to avoid introducing any more theory.

In the following, let n be prime.
In the proof of Theorem 2.3 in the paper of Fowler et.al [10], the authors showed that every point on the torus is incident to exactly $n+1$ distinct lines and they explicitly stated the *generators* of these lines if we translate them to $(0, 0)$. While they used the term generators in the algebraic sense, we think about these generators in a similar way as we thought about the closest point to c_n on a line in $[n]^2$ in Section 2. These generators uniquely define a line incident to $(0, 0)$ and every point on the line generated by (x_1, x_2) is of the form

$$(tx_1 \bmod n, tx_2 \bmod n), \text{ where } 1 \leq t \leq n.$$

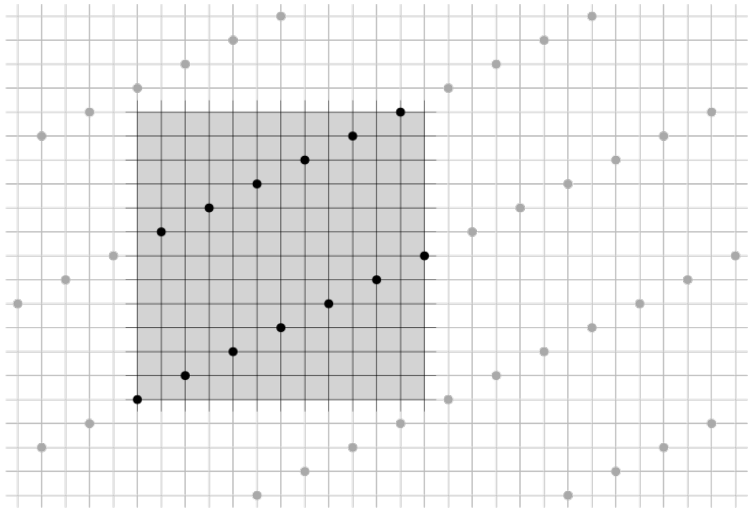

Figure 11: A sample line in black on the 13×13 torus

They identified the generators $G = \{(0,1), (1,0), (1,1), (1,2), \ldots (1, n-1)\}$ of the lines incident to $(0,0)$. Since n is prime, we know that for $1 \leq t_1 < t_2 \leq n$ and $x \in [n-1]$ it holds that

$$(t_1 x \bmod n) \neq (t_2 x \bmod n).$$

It follows for any $(x_1, x_2) \in G$ that

$$(t_1 x_1 \bmod n, t_1 x_2 \bmod n) \neq (t_2 x_1 \bmod n, t_2 x_2 \bmod n).$$

Thus, every line is incident to exactly n points.

Even though it seems obvious, the property that the intersection of two lines contains at most one point is due to the fact that n is prime. We can verify this property in this case easily if we count points by lines. After we count $(0,0)$ once, we are left with $n+1$ lines incident to $(n-1)$ points excluding $(0,0)$. Since every point lies on at least one line incident to $(0,0)$ and

$$1 + (n+1)(n-1) = 1 + n^2 - 1 = n^2$$

the intersection of two distinct lines on T_n, if n is prime, must not consist of more than one point.

This is all that we need for our considerations in this section. So, let us state this observations for future reference.

Observation 2. For n prime, every point on the $n \times n$ torus is incident to exactly $n+1$ distinct lines and every line is incident to exactly n points.

5.1 Lower And Upper Bounds On The Size Of Dominating Sets

The results in this section are achieved by similar considerations as in Section 3.1 and Section 4.3. For convenience, let us denote the minimum size of a dominating set on the $n \times n$ torus by $\mathscr{D}_n^T$.

Theorem 8. *For n prime, $\mathscr{D}_n^T = \Omega(\sqrt{n})$.*

Proof. Let $S \subseteq T_n$. Any pair of points in S defines a line and, by Observation 2, this line will dominate exactly n points. So S dominates at most $\binom{|S|}{2}n$ points.
Hence, if S is a dominating set, it has to hold that

$$\binom{|S|}{2}n \geq n^2 \quad \Leftrightarrow \quad |S|^2 - |S| - 2n \geq 0$$

which is the case if $|S| \geq \frac{1}{4} + \sqrt{\frac{1}{4} + 2n}$. $\square$

Since every line dominates exactly n points, we cannot refine these considerations any further. Surprisingly, the upper bound on the minimum size of dominating sets on the torus is even smaller than the lower bound on the geometric domination number in $[n]^2$.

Theorem 9. *For n prime, $\mathscr{D}_n^T = O(\sqrt{n \log n})$.*

Proof. The proof goes along the exact same lines as our considerations in Section 4.3. We will use random point sets that we defined, strictly speaking, for $[n]^2$. But we can translate $[n]^2$ by $-(1,1)$ such that we obtain the discrete torus $T_n = \{0, 1, 2, \dots n-1\}^2$.
So, let R_m be a uniform random point set of size m on T_n. By our considerations on the lower bound in Theorem 8 and the upper bound on the size of point sets in general position [10], we can assume $\sqrt{2n} \leq m \leq n+1$.
Now, fix $\mathsf{p} \in T_n$ and let P_p be the set of all point sets on T_n that do not dominate p. Again, we note that P_p is a decreasing point set property.
By Lemma 9, it holds that

$$\mathbb{P}(R_m \in P_\mathsf{p}) \leq 2\mathbb{P}(R_{n,m/n^2} \in P_\mathsf{p}).$$

Therefore, we continue by considering $R_{n,m/n^2}$.

$R_{n,m/n^2}$ dominates p if and only if $R_{n,m/n^2}$ contains a dominating pair of p or $\mathsf{p} \in R_{n,m/n^2}$. So, once again, let X be the random variable that counts the number of dominating pairs of p. That is

$$X = I_\mathsf{p} + \sum_{D \in \mathcal{D}_\mathsf{p}} I_D$$

where I_x are indicator random variables and $\mathcal{D}_{\mathsf{p}}$ denotes the set of dominating pairs of p. Note that $X \geq 1$ if and only if p is dominated by the random point set.

Next, we apply Janson's inequality and identify $\mathcal{A}$ with T_n, R_p with $R_{n,m/n^2}$ and $\mathcal{B}$ with $\mathcal{D}_{\mathsf{p}} \cup \{\mathsf{p}\}$. All we have to do is computing μ and Δ.

Since p is incident to $n+1$ lines that consist of n points, there are $(n+1)\binom{n-1}{2}$ distinct dominating pairs of p on T_n. Consequently,

$$
\begin{aligned}
\mu = \mathbb{E}(X) &= \mathbb{P}(\mathsf{p} \in R_{n,m/n^2}) + \sum_{D \in \mathcal{D}_{\mathsf{p}}} \mathbb{P}(D \in R_{n,m/n^2}) \\
&= \frac{m}{n^2} + \sum_{D \in \mathcal{D}_{\mathsf{p}}} \left(\frac{m}{n^2}\right)^2 = \frac{m}{n^2} + \frac{(n+1)(n-1)(n-2)}{2} \left(\frac{m}{n^2}\right)^2 \\
&= \frac{m^2}{2n} + \Theta\left(\left(\frac{m}{n}\right)^2\right)
\end{aligned}
$$

Furthermore, two distinct dominating pairs C, D of p have a non-empty intersection if and only if their intersection contains exactly one point and the three points in $C \cup D$ are collinear. The number of collinear triples that lie on a common line with p is therefore $(n+1)\binom{n-1}{3}$. Hence,

$$
\begin{aligned}
\Delta = \frac{1}{2} \sum_{\substack{C \neq D \in \mathcal{D}_{\mathsf{p}} \cup \{\mathsf{p}\} \\ C \cap D \neq \emptyset}} \mathbb{E}(I_C I_D) &= \frac{1}{2} \sum_{\substack{\mathsf{x},\mathsf{y},\mathsf{z} \in [n]^2 \\ \mathsf{x},\mathsf{y},\mathsf{z},\mathsf{p} \text{ collinear}}} \left(\frac{m}{n^2}\right)^3 \\
&= \frac{(n+1)(n-1)(n-2)(n-3)}{12} \left(\frac{m}{n^2}\right)^3 \\
&= \frac{m^3}{12n^2} + \Theta\left(\left(\frac{m}{n}\right)^3\right)
\end{aligned}
$$

Using Janson's inequality, we derive

$$
\begin{aligned}
\mathbb{P}(R_m \in P_{\mathsf{p}}) \leq 2\,\mathbb{P}(R_{n,m/n^2} \in P_{\mathsf{p}}) &= 2\,\mathbb{P}(X = 0) \leq 2\,\exp(-\mu + \Delta) \\
&= 2\,\exp\left(-\frac{m^2}{2n} + \Theta\left(\left(\frac{m}{n}\right)^2\right) + \frac{m^3}{12n^2} + \Theta\left(\left(\frac{m}{n}\right)^3\right)\right)
\end{aligned}
$$

The sum of the asymptotic terms above is $O(1)$ because we assumed that $m \leq n+1$. Finally, we use the union bound again.

$$\begin{aligned}
\mathbb{P}(\exists \text{ a free point in } T_n) &= \mathbb{P}(R_m \in \bigcup_{\mathsf{p}\in[n]^2} P_{\mathsf{p}}) \\
&\leq \sum_{\mathsf{p}\in[n]^2} \mathbb{P}(R_m \in P_{\mathsf{p}}) \\
&\leq 2n^2 \exp\left(-\frac{m^2}{2n} + \frac{m^3}{12n^2} + O(1)\right) \\
&= \exp\left(2\log n - \frac{m^2}{2n} + \frac{m^3}{12n^2} + O(1)\right)
\end{aligned}$$

If we choose $m = (2+\varepsilon)\sqrt{n\log n}$, with $\varepsilon > 0$, it follows that

$$2\log n - \frac{m^2}{2n} + \frac{m^3}{12n^2} + O(1) \to -\infty \quad \text{as } n \to \infty$$

Consequently, for all n large enough, there is a dominating set of size $O(\sqrt{n\log n})$. $\square$

6 Summary And Future Work

In Sections 2 through 5, we were able to shed some light on dominating sets in $[n]^2$ and on the discrete torus T_n, for n prime. We have shown that

- for all $\varepsilon > 0$ and n large enough, $\overline{\mathscr{D}}_n \geq \mathscr{D}_n > (1-\varepsilon)\left(\frac{\pi^4 n}{144}\right)^{2/3}$ (Subsection 3.1, Theorem 3)
- for all $n \in \mathbb{N}$, $\mathscr{D}_n \leq 2\lceil \frac{n}{2} \rceil$ (Subsection 3.2, Theorem 4)
- the minimum sizes of geometric dominating sets in general positions are (Subsection 3.2.2):
  - for $\overline{\mathscr{D}}_2 = \overline{\mathscr{D}}_3 = \overline{\mathscr{D}}_4 = 4$
  - for $\overline{\mathscr{D}}_5 = \overline{\mathscr{D}}_6 = 6$
  - for $\overline{\mathscr{D}}_7 = \overline{\mathscr{D}}_8 = \overline{\mathscr{D}}_9 = \overline{\mathscr{D}}_{10} = 8$
- for all $n \in \mathbb{N}$, the minimum size of dominating sets on the torus is $\mathscr{D}_n^T \geq \frac{1}{4} + \sqrt{\frac{1}{4} + 2n}$ (Subsection 5.1, Theorem 8)
- and for all $\varepsilon > 0$ and n large enough, $\mathscr{D}_n^T \leq (2+\varepsilon)\sqrt{n \log n}$. (Subsection 5.1, Theorem 9).

However, new questions popped up and the exact minimum size of dominating sets for general n is still unknown.

The search for upper bounds on the geometric domination number seems to be particularly hard, but our results in Section 2 and Subsection 3.2 might be able to fuel computational approaches. In Subsection 2.2, we obtained a general understanding on how to place points with respect to a fixed point in order to dominate the most points. The dominating sets in general position that we found computationally suggest that placing points on horizontal, vertical and diagonal lines with respect to subsequent points might be a very good strategy to maximize dominated points globally. We could therefore try to implement a randomized algorithm that assigns weights to free points, according to their position with respect to subsequently chosen points. The larger the weight of a point, the more likely the algorithm will choose the point next. Conversely, we could assign weights with respect to the likelihood that a point is dominated by the currently chosen points and an additional free point.

From a purely theoretical viewpoint, it would be interesting to compute tight asymptotic expressions of d_{p} for all $\mathsf{p} \in [n]^2$ too.

Aichholzer [3] also asked whether the geometric domination number is monotone and pointed out that, in either case, it would be interesting to look for dominating sets of $[n]^2$ that may include points in $\mathbb{Z}^2$ outside of $[n]^2$. For instance, we know that $\mathscr{D}_2 = 4$, but if we consider dominating sets in $\mathbb{Z}^2$, three points are enough as one can see in Figure 12.

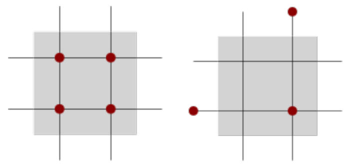

Figure 12: Dominating sets of $[2]^2$, restricted to $[2]^2$ and unrestricted

On the torus, there are several cases that we have not considered yet. In particular, it should be possible to apply our probabilistic strategy if n is not prime, once it is clear how many lines are incident to a fixed point. However, this would probably require to exploit the algebraic properties of the discrete torus which was out of scope of this thesis.

The attentive reader might wonder whether our probabilistic arguments can be applied to the No-Three-In-Line Problem as well. Unfortunately, we were not able to prove Guy and Kelly's conjecture which we discussed in Subsection 4.1. The Δ-term in Janson's inequality, that reflects the variance of the number of collinear triples in a random point set, is too large and, with our strategy, the probability that a random point set of size αn is in general position can only be bounded by e^{-cn} as opposed to Guy and Kelly's bound of $e^{-cn \log n}$.

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