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A space–time finite element method for the wave equation using a modified Hilbert transformation

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Preface

As a student in the Masters degree with emphasis on Technomathematics at the Graz University of Technology, my aim is to apply mathematics in engineering. This thesis gave me the opportunity to take part in ongoing research on developing a tool, which can be widely used. As a model problem for a hyperbolic partial differential equation, which are found in various applications, we considered the wave equation and proposed a method using the so–called modified Hilbert transformation. This operator was first introduced by Prof. Dr. Olaf Steinbach and gives raise to an unconditionally stable method for the heat equation, as results by Marco Zank show. Also for the wave equation the results in this work are promising. Unfortunately, the theoretical breakthrough is still outstanding and further considerations need to be made. This thesis shall therefore be seen as a first cobblestone on a longer road to be paved.

Contents

Abstract

In this thesis a space–time finite element method for the wave equation, using a modified Hilbert transformation is introduced in subspaces of the Sobolev space $H^1(Q)$. An ordinary partial differential equation is derived, which corresponds to the wave equation and is of particular interest for a theoretical analysis of solvability and stability. A numerical analysis for this ODE is performed, where unconditional stability can be observed. Further, using discrete tensor spaces, a conforming Galerkin–Bubnov finite element discretisation for the one dimensional wave equation is investigated for the proposed method. The results indicate optimal convergence rates and stability. In order to classify the findings, a comparison to a related variational formulation is carried out, for which theory and numerical results already exist ([9]).

Diese Arbeit präsentiert eine Raum-Zeit finite Elemente Methode für die Wellen–gleichung mit einer modifizierten Hilbert-Transformation in Teilräumen des Sobolev–raumes $H^1(Q)$. Eine gewöhnliche Differentialgleichung, die für die theoretische Behandlung von Lösbarkeit und Stabilität von Interesse ist, wird aus der Variationsformulierung abgeleitet und numerisch analysiert. Aus den Ergebnissen lässt sich unbedingte Stabilität erkennen. Des Weiteren wird eine konforme Galerkin-Bubnov finite Elemente Diskretisierung für die eindimensionale Wellengleichung in diskreten Tensorprodukt-Räumen herangezogen, um die vorgeschlagene Methode zu testen. Auch hier zeigen sich Stabilität und optimale Konvergenzraten. Um die Resultate besser einordnen zu können, wird mit einer ähnlichen Formulierung, für die es bereits theoretische und numerische Analysis gibt (siehe [9]), verglichen.

1 Preliminaries

In this chapter the basis for the theoretical discussion in this thesis is briefly outlined. Firstly, Sobolev spaces on intervals and space–time cylinders are revised. Further, the definition of the modified Hilbert transformation and some of its properties are stated. In the end, the discrete trial spaces used for a numerical discussion are introduced. Most of the chapter follows [12, Chapter 2]. For further reading on Sobolev spaces see also [1, 7, 11]. For the modified Hilbert transformation see [9, 10, 12].

1.1 Sobolev Spaces on intervals

Let $0 < T < \infty$ be fixed and let $s \geq 0$. The Hilbert space $H^s(0,T)$ is the Sobolev space of real–valued functions endowed with the Sobolev–Slobodeckij inner product $\langle \cdot, \cdot \rangle_{H^s(0,T)}$ and induced norm $\|\cdot\|_{H^s(0,T)}$. Note, that for $s > \frac{1}{2}$ it holds that $H^s(0,T) \subset C([0,T])$. Hence for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$ the Hilbert spaces

$$H^s_{0,}(0,T) := \left\{ v \in H^s(0,T) \middle| v(0) = 0 \right\},$$
$$H^s_{,0}(0,T) := \left\{ v \in H^s(0,T) \middle| v(T) = 0 \right\}.$$

are well–defined as closed subspaces of $H^s(0,T)$.

In particular, for $s = 1$ the Hilbert spaces $H^1_{0,}(0,T)$ and $H^1_{,0}(0,T)$ are endowed with the inner product

$$\langle u, v \rangle_{H^1_{0,}(0,T)} = \langle u, v \rangle_{H^1_{,0}(0,T)} := \int_0^T \partial_t u(t) \partial_t v(t)\, dt = \langle \partial_t u, \partial_t v \rangle_{L^2(0,T)}$$

and the induced norm

$$|u|_{H^1(0,T)} := \sqrt{\int_0^T |\partial_t u(t)|^2\, dt} = \|\partial_t u\|_{L^2(0,T)}.$$

For $s = \frac{1}{2}$ one defines via function space interpolation the Sobolev spaces [7, Remarque 11.4, p.75]

$$H^{1/2}_{0,}(0,T) = [L^2(0,T), H^1_{0,}(0,T)]_{1/2}$$

with Hilbertian norm

$$\|v\|_{H^{1/2}_{0,}(0,T)} = \sqrt{\|v\|^2_{H^{1/2}(0,T)} + \int_0^T \frac{|v(t)|^2}{t}\, dt}$$

and

$$H^{1/2}_{,0}(0,T) = [L^2(0,T), H^1_{,0}(0,T)]_{1/2}$$

with Hilbertian norm

$$\|v\|_{H^{1/2}_{,0}(0,T)} = \sqrt{\|v\|^2_{H^{1/2}(0,T)} + \int_0^T \frac{|v(t)|^2}{T-t}\, dt}.$$

The representations

$$H^{1/2}_{0,}(0,T) = \left\{ u_{|(0,T)} \middle| u \in H^{1/2}(-\infty,T) \text{ with } u(t) = 0 \text{ for } t < 0 \right\}$$

and

$$H^{1/2}_{,0}(0,T) = \left\{ u_{|(0,T)} \middle| u \in H^{1/2}(0,\infty) \text{ with } u(t) = 0 \text{ for } t > T \right\}$$

hold true. This means especially that $H^{1/2}_{0,}(0,T)$ consists of all functions $u \in H^{1/2}(0,T)$ which can be extended by zero to the left such that the extension

$$\tilde{u}(t) = \begin{cases} u(t), & t \in (0,T), \\ 0, & t < 0, \end{cases}$$

is a function in $H^{1/2}(-\infty,T)$. Analogously, $u \in H^{1/2}_{,0}(0,T)$ admits a zero extension to the right which fulfils to be in $H^{1/2}(0,\infty)$. Note that in particular the constant function

$$\mathbf{1}(t) := 1,\, t \in (0,T),$$

has the property that $\mathbf{1} \in H^{1/2}(0,T)$, but $\mathbf{1} \notin H^{1/2}_{0,}(0,T)$ and $\mathbf{1} \notin H^{1/2}_{,0}(0,T)$ [11, cf. p.159]. Further it holds true that $C_0^\infty(0,T)$ is dense in $H^{1/2}(0,T)$ and even in $H^{1/2}_{0,}(0,T)$ and $H^{1/2}_{,0}(0,T)$ [12, Theorem 2.2.2, p.18].

For $s = 1$ and $s = \frac{1}{2}$ the dual spaces $[H^s_{0,}(0,T)]'$ and $[H^s_{,0}(0,T)]'$ are characterized as completion of $L^2(0,T)$ with respect to the norms

$$\|g\|_{[H^s_{0,}(0,T)]'} := \sup_{0 \neq v \in H^s_{0,}(0,T)} \frac{|\langle g, v\rangle|}{|v|_{H^1(0,T)}}$$

and

$$\|f\|_{[H^s_{,0}(0,T)]'} := \sup_{0 \neq w \in H^s_{,0}(0,T)} \frac{|\langle f, w\rangle|}{|w|_{H^1(0,T)}}.$$

1.2 Sobolev spaces on space-time cylinders Q

Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain, where $d = 1, 2, 3$ and $0 < T < \infty$. The space-time cylinder is defined as $Q := \Omega \times (0, T)$. In order to define Sobolev spaces on Q Bochner spaces are introduced. For further references see [12, Chapter 2.4].

Let H be a seperable Hilbert space. Then the Bochner space $L^2((0, T); H)$ is the space of classes of measurable functions $u : (0, T) \to H$ such that

$$u(t) \in H, \text{ for almost all } t \in (0, T),$$

and

$$\|u\|_{L^2((0,T);H)} = \sqrt{\int_0^T \|u(t)\|_H^2 \, dt} < \infty.$$

Endowed with the inner product

$$\langle u, v\rangle_{L^2((0,T);H)} := \int_0^T \langle u(t), v(t)\rangle_H \, dt$$

$L^2((0, T); H)$ is a Hilbert space.

For $m \in \mathbb{N}_0$ one defines the Bochner Sobolev spaces

$$H^m((0, T); H) := \left\{ u \in L^2((0, T); H) \middle| \, \partial_t^{(i)} u \in L^2((0, T), H) \text{ for all } i = 1, ..., m \right\}.$$

Endowed with the inner product

$$\langle u, v\rangle_{H^m((0,T);H)} := \int_0^T \langle u(t), v(t)\rangle \, dt + \sum_{i=1}^m \int_0^T \langle \partial_t^{(i)} u(t), \partial_t^{(i)} v(t)\rangle \, dt,$$

for $u, v \in H^m((0, T); H)$, the Bochner Sobolev space $H^m((0, T), H)$ is a Hilbert space. In particular for $m = 1$ by the Sobolev embedding theorem it holds true that

$$H^1((0, T); H) \subseteq C([0, T]; H)$$

and thus the closed subspaces

$$H^1_{0,}((0, T); H) := \left\{ u \in H^1((0, T); H) \middle| \, u(0) = 0 \text{ in } H \right\}$$

and

$$H^1_{,0}((0, T); H) := \left\{ u \in H^1((0, T); H) \middle| \, u(T) = 0 \text{ in } H \right\}$$

are well–defined.

Now, for $0 \leq r, s \in \mathbb{R}$ one defines the anisotropic Sobolev spaces on $Q = (0,T) \times \Omega$

$$H^{r,s}(Q) := L^2((0,T); H^r(\Omega)) \cap H^s((0,T); L^2(\Omega)),$$

which are Hilbert spaces with respect to the inner product

$$\begin{aligned}\langle u, v\rangle_{H^{r,s}(Q)} &:= \int_0^T \langle u(\cdot,t), v(\cdot,t)\rangle_{H^r(\Omega)}\,dt + \int_\Omega \langle u(x,\cdot), v(x,\cdot)\rangle_{H^s((0,T))}\,dx \\ &= \langle u, v\rangle_{L^2((0,T);H^r(\Omega))} + \langle u, v\rangle_{H^s((0,T);L^2(\Omega))},\end{aligned}$$

for $u, v \in H^{r,s}(Q)$. Note that for $r = s = 1$ it holds that

$$H^1(Q) := H^{1,1}(Q) \subseteq C([0,T]; L^2(\Omega)).$$

The Sobolev space with homogenuous boundary conditions on the domain Ω is defined as

$$H^{1,1}_{0;}(Q) := H^1((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega)).$$

Endowed with the inner product

$$\begin{aligned}\langle u, v\rangle_{H^{1,1}_{0;}(Q)} &:= \int_\Omega \int_0^T \Big(\partial_t u(x,t)\partial_t v(x,t) + \nabla_x u(x,t)\cdot\nabla_x v(x,t)\Big)\,dtdx \\ &= \langle \partial_t u, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)},\end{aligned}$$

for $u, v \in H^{1,1}_{0;}(Q)$, it is a Hilbert space, with induced norm

$$|u|_{H^1(Q)} = \sqrt{\|\partial_t u\|^2_{L^2(Q)} + \|\nabla_x u\|^2_{L^2(Q)}}.$$

Note that due to the Poincaré inequality the semi–norm $|\cdot|_{H^1(Q)}$ defines a to $\|\cdot\|_{H^1(Q)}$ equivalent norm in $H^{1,1}_{0;}(Q)$.

Further, subspaces of $H^{1,1}_{0;}(Q)$ with initial or final conditions in time are defined

$$H^{1,1}_{0;0,}(Q) := H^1_{0,}((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega))$$

and

$$H^{1,1}_{0;,0}(Q) := H^1_{,0}((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega)),$$

admitting the inner product $\langle\cdot,\cdot\rangle_{H^{1,1}_{0;}}(Q)$ and the norm $|\cdot|_{H^1(Q)}$.

The characterisation of the dual spaces $[H^{1,1}_{0;0,}(Q)]'$ and $[H^{1,1}_{0;,0}(Q)]'$ is by completion of $L^2(Q)$ with respect to the Hilbertian norms

$$\|g\|_{[H^{1,1}_{0;0,}(Q)]'} := \sup_{0\neq v\in H^{1,1}_{0;0,}(Q)} \frac{|\langle g, v\rangle_Q|}{|v|_{H^1(Q)}}$$

and

$$\|f\|_{[H^{1,1}_{0;,0}(Q)]'} := \sup_{0\neq w\in H^{1,1}_{0;,0}(Q)} \frac{|\langle f, w\rangle_Q|}{|w|_{H^1(Q)}}.$$

1.3 The modified Hilbert transformation

Following the ideas of [9, Section 2.4] and [12, Section 3.4.2] in this section a brief overview of the modified Hilbert transformation is given. Define the functions

$$V_k(t) := \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right),\ k \in \mathbb{N}_0. \tag{1.1}$$

They are eigenfunctions of the second time derivative, i.e.

$$-\partial_{tt} V_k(t) = \frac{\lambda_k}{T^2} V_k(t), \quad \lambda_k = \left(\frac{\pi}{2} + k\pi\right)^2$$

and obey the properties [12, Section 3.4.1] for $i, j \in \mathbb{N}_0$

$$V_i \in H^1_{0,}(0,T), \quad \langle V_i, V_j\rangle_{L^2(0,T)} = \frac{T}{2}\delta_{i,j}, \quad \langle V_i, V_j\rangle_{H^1_{0,}(0,T)} = \frac{\lambda_i}{2T}\delta_{i,j}.$$

Thus $\{V_k\}_{k\in\mathbb{N}_0}$ build an orthogonal basis in $L^2(0,T)$ and in $H^1_{0,}(0,T)$. By interpolation arguments they also build an orthogonal basis in $H^{1/2}_{0,}(0,T)$. For $v \in L^2(0,T)$, the representation

$$v(t) = \sum_{k=0}^{\infty} v_k V_k(t), \quad v_k := \frac{2}{T}\int_0^T v(t) V_k(t)\, dt \tag{1.2}$$

holds true and the norm is given by

$$\|v\|^2_{L^2(0,T)} = \frac{T}{2}\sum_{k=0}^{\infty} |v_k|^2.$$

Further, the expansion also converges for $v \in H^1_{0,}(0,T)$. Hence

$$\partial_t v(t) = \frac{1}{T}\sum_{k=0}^{\infty}\left(\frac{\pi}{2} + k\pi\right) v_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) \tag{1.3}$$

converges in $L^2(0,T)$ with norm

$$\|\partial_t v\|^2_{L^2(0,T)} = \frac{1}{2T}\sum_{k=0}^{\infty}\left(\frac{\pi}{2} + k\pi\right)^2 |v_k|^2.$$

By interpolation one defines

$$\|v\|^2_{H^s_{0,}(0,T)} := \frac{T}{2}\sum_{k=0}^{\infty}\left(\frac{\left(\frac{\pi}{2} + k\pi\right)}{T}\right)^{2s} |v_k|^2.$$

In particular, for $s = \frac{1}{2}$ defining the norm

$$\|v\|_{H^{1/2}_{0,}(0,T),F} := \sqrt{\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)|v_k|^2}$$

gives an equivalent norm in $H^{1/2}_{0,}(0,T)$ [12, Theorem 3.4.2, p.74]. This yields the representation

$$H^{1/2}_{0,}(0,T) = \left\{v \in L^2(0,T)\middle| v(t) = \sum_{k=0}^{\infty} v_k V_k(t),\, \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)|v_k|^2 < \infty\right\}.$$

Further, define the functions

$$W_k(t) := \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right),\ k \in \mathbb{N}_0. \tag{1.4}$$

They also fulfil

$$-\partial_{tt}W_k(t) = \frac{\lambda_k}{T^2}W_k(t), \quad \lambda_k = \left(\frac{\pi}{2}+k\pi\right)^2$$

and have the properties for $i, j \in \mathbb{N}_0$

$$W_i \in H^1_{,0}(0,T), \quad \langle W_i, W_j\rangle_{L^2(0,T)} = \frac{T}{2}\delta_{i,j}, \quad \langle W_i, W_j\rangle_{H^1_{,0}(0,T)} = \frac{\lambda_i}{2T}\delta_{i,j}.$$

By the same arguments as above, for $w \in H^{1/2}_{,0}(0,T)$, defining

$$w_k := \frac{2}{T}\int_0^T w(t)W_k(t)\,dt \tag{1.5}$$

yields the representation

$$H^{1/2}_{,0}(0,T) = \left\{w \in L^2(0,T)\middle| w(t) = \sum_{k=0}^{\infty} w_k W_k(t),\, \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)|w_k|^2 < \infty\right\}$$

and an equivalent norm

$$\|w\|_{H^{1/2}_{,0},F} := \sqrt{\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)|w_k|^2}.$$

Consider for $v \in H^{1/2}_{0,}(0,T)$ the distributional derivative with expansion as in (1.3), i.e. for $w \in H^{1/2}_{,0}(0,T)$ one has

$$\begin{aligned}\langle \partial_t v, w\rangle_{(0,T)} &= \int_0^T \frac{1}{T}\sum_{k=0}^{\infty} v_k\left(\frac{\pi}{2}+k\pi\right)\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) w(t)\,dt \\ &= \frac{1}{T}\sum_{k=0}^{\infty} v_k\left(\frac{\pi}{2}+k\pi\right)\langle W_k, w\rangle_{L^2(0,T)}.\end{aligned}$$

Thus, choosing

$$w(t) := \sum_{\ell=0}^{\infty} v_\ell W_\ell(t),$$

one finds by L^2-orthogonality of W_k the ellipticity estimate

$$\begin{aligned}\langle \partial_t v, w\rangle_{(0,T)} &= \frac{1}{T}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} v_k v_\ell\left(\frac{\pi}{2}+k\pi\right)\langle W_k, W_\ell\rangle_{L^2(0,T)} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)|v_k|^2 = \|v\|^2_{H^{1/2}_{0,}(0,T)}.\end{aligned}$$

This motivates the definition of the *modified Hilbert transformation*, for $v \in L^2(0,T)$ as

$$\mathcal{H}_T v(t) = \mathcal{H}_T\left(\sum_{k=0}^{\infty} v_k V_k(t)\right) := \sum_{k=0}^{\infty} v_k W_k(t), \tag{1.6}$$

with coefficients as in (1.2). In particular applying the operator $\mathcal{H}_T$ to a function $v \in H^1_{0,}(0,T)$ changes the boundary condition from an inital condition $v(0) = 0$ to a final condition $\mathcal{H}_T v(T) = 0$. One can show that the modified Hilbert transformation is bijectiive and norm preserving as a map $\mathcal{H}_T : L^2(0,T) \to L^2(0,T)$ but also as map $\mathcal{H}_T : H^1_{0,}(0,T) \to H^1_{,0}(0,T)$ and $\mathcal{H}_T : H^{1/2}_{0,}(0,T) \to H^{1/2}_{,0}(0,T)$ [12, c.f. p.82].

The inverse operator for $w \in L^2(0,T)$ is given as

$$\mathcal{H}_T^{-1} w(t) = \mathcal{H}_T\left(\sum_{k=0}^{\infty} w_k W_k(t)\right) := \sum_{k=0}^{\infty} w_k V_k(t), \tag{1.7}$$

with coefficients as in (1.5).

The following lemma shows the relation between the modified Hilbert transformation and its inverse.

Lemma 1.1 ([12, Lemma 3.4.7, p.83]). *For $u, w \in L^2(0,T)$ the equality*

$$\langle u, \mathcal{H}_T w\rangle_{L^2(0,T)} = \langle \mathcal{H}_T^{-1} u, w\rangle_{L^2(0,T)}$$

is valid.

The above relation also gives insight into the relation between the modified Hilbert transformation and the time reversal operator.

Lemma 1.2. *Let $\iota_T : H^1_{0,}(0,T) \to H^1_{,0}(0,T)$ be the time reversal operator, i.e.*

$$\iota_T u(t) := u(T-t).$$

For $u \in H^1_{0,}(0,T)$ the application of the inverse Hilbert transformation behaves like

$$\mathcal{H}_T^{-1} u(t) = (\mathcal{H}_T u(T-\cdot))(T-t) = \iota_T \mathcal{H}_T \iota_T u(t),\ \forall t \in (0,T). \tag{1.8}$$

Further representation (1.8) holds true for $u \in L^2(0,T)$ pointwise almost everywhere.

Proof. First note that it holds

$$(\iota_T V_i)(t) = V_i(T-t) = (-1)^i \cos\left(\left(\frac{\pi}{2} + i\pi\right)\frac{t}{T}\right) = (-1)^i W_i(t).$$

Since $\{V_i\}_{i\in\mathbb{N}_0}$ build a basis, the identity (1.8) holds if

$$\langle \mathcal{H}_T^{-1} u - \iota_T \mathcal{H}_T \iota_T u, V_i\rangle_{L^2(0,T)} = 0,\ \forall i \in \mathbb{N}_0.$$

Further, since $u \in L^2(0,T)$, the representation

$$u(t) = \sum_{k=0}^{\infty} u_k V_k(t)$$

holds true. Therefore, by linearity of $\mathcal{H}_T$ and ι_T it is sufficient to show for all $k \in \mathbb{N}_0$

$$\langle \mathcal{H}_T^{-1} V_k - \iota_T \mathcal{H}_T \iota_T V_k, V_i\rangle_{L^2(0,T)} = 0,\ \forall i \in \mathbb{N}_0.$$

Using Lemma 1.1 and $\langle u, \iota_T v\rangle_{L^2(0,T)} = \langle \iota_T u, v\rangle_{L^2(0,T)}$ one finds

$$\begin{aligned}
\langle \mathcal{H}_T^{-1} V_k, V_i\rangle_{L^2(0,T)} &= \langle V_k, \mathcal{H}_T V_i\rangle_{L^2(0,T)} = \langle V_k, W_i\rangle_{L^2(0,T)} = (-1)^i \langle V_k, \iota_T V_i\rangle_{L^2(0,T)} \\
&= (-1)^i \langle \iota_T V_k, V_i\rangle_{L^2(0,T)} = (-1)^i \langle \iota_T V_k, \mathcal{H}_T^{-1} W_i\rangle_{L^2(0,T)} \\
&= (-1)^i \langle \mathcal{H}_T \iota_T V_k, W_i\rangle_{L^2(0,T)} = \langle \mathcal{H}_T \iota_T V_k, \iota_T V_i\rangle_{L^2(0,T)} \\
&= \langle \iota_T \mathcal{H}_T \iota_T V_k, V_i\rangle_{L^2(0,T)}.
\end{aligned}$$

□

Moreover, the application of the time derivative on the modified Hilbert transformation is studied.

Lemma 1.3 ([9, Lemma 2.4, p.162]). *For $u \in H^{1/2}_{0,}(0,T)$ one has*

$$\langle \partial_t \mathcal{H}_T u, v \rangle_{(0,T)} = -\langle \mathcal{H}_T^{-1} \partial_t u, v \rangle_{(0,T)}, \text{ for all } v \in H^{1/2}_{0,}(0,T).$$

The next Lemma shows the positive semidefinitness of the modified Hilbert transformation.

Lemma 1.4 ([12, Lemma 3.4.8, p.83]). *For all $v \in L^2(0,T)$ the inequality*

$$\langle v, \mathcal{H}_T v \rangle_{L^2(0,T)} \geq 0$$

is valid.

Despite the theoretical point of view the representation of the modified Hilbert transformation as a series expansion is not handy for implementation. An alternative representation can be given as a Cauchy principal value integral. For $v \in L^2(0,T)$ it holds that (see [10, Lemma 2.1])

$$\mathcal{H}_T v(t) = \int_0^T K(s,t) v(s)\, ds,\ t \in (0,T), \tag{1.9}$$

with the kernel function

$$K(s,t) = \frac{1}{2T} \left[\frac{1}{\sin\left(\frac{\pi(s+t)}{2T}\right)} + \frac{1}{\sin\left(\frac{\pi(s-t)}{2T}\right)} \right].$$

If $v \in H^1_{0,}(0,T)$, applying integration by parts yields [10, Corollary 2.2]

$$\mathcal{H}_T v(t) = -\frac{1}{\pi} \int_0^T \partial_t v(s) \ln \left[\tan\left(\frac{\pi(s+t)}{4T} \right) \tan\left(\frac{\pi(|t-s|)}{4T} \right) \right] ds,\ t \in (0,T). \tag{1.10}$$

Using this expression an efficient implementation can be achived see [10].

1.4 Discretisation and trial spaces

In this section the discrete trial spaces used for the numerical analysis are described. First, the trial spaces on intervals are discussed. Then tensor product trial spaces are outlined.

For an interval $(0,b)$, $b \in \mathbb{R}_{>0}$ a decompositions consisiting of finite elements $\tau_\ell = (s_{\ell-1}, s_\ell)$ such that

$$\overline{(0,b)} = \bigcup_{\ell=1}^{N} \overline{\tau_\ell}$$

is considered. The local mesh size $h_\ell = |\tau_\ell| = s_\ell - s_{\ell-1}$ and the global mesh size $h = \max_{\ell=1,\dots,N} h_\ell$ are defined. A particular choice is uniform refinement where $h = h_\ell = \frac{b}{N}$ for all $\ell = 1, \dots, N$. The finite element space of piecewise linear, globally continous functions on $(0,b)$ is introduced as

$$S_h^1(0,b) = \mathrm{span}\{\varphi_\ell^1 | \ell = 0, \dots, N\}$$

with basis functions (see Fig. 1.1)

$$\varphi_\ell^1(s) = \begin{cases} 1, & s = s_\ell, \\ 0, & s = s_k \neq s_\ell, \\ \text{linear}, & \text{else.} \end{cases}$$

In addition the subspaces with homogenuous initial and boundary conditions $S_{h;0,}^1(0,b) \subseteq S_h^1(0,b)$ and $S_{h;0}^1(0,b) \subseteq S_h^1(0,b)$ are defined as

$$\begin{aligned} S_{h;0,}^1(0,b) &:= S_h^1(0,b) \cap H_{0,}^1(0,b) = \mathrm{span}\{\varphi_\ell^1 | \ell = 1, \dots, N\}, \\ S_{h;0}^1(0,b) &:= S_h^1(0,b) \cap H_0^1(0,b) = \mathrm{span}\{\varphi_\ell^1 | \ell = 1, \dots, N-1\} \end{aligned}$$

The approximation properties are stated in the next theorem.

Theorem 1.1 ([8, Theorem 9.10, p.220])**.** *Let $u \in H^s(0,b)$ with $s \in [\sigma, 2]$ and $\sigma = 0, 1$. Then there holds the approximation property*

$$\inf_{v_h \in S_h^1(0,b)} \|u - v_h\|_{H^\sigma(0,b)} \leq c h^{s-\sigma} |u|_{H^s(0,b)}.$$

Furthermore, the space of splines of second order is introduced as

$$S_h^2(0,b) := \mathrm{span}\{b_\ell | \ell = -1, \dots, N\},$$

with quadratic basis functions (see Fig. 1.1) which can be defined recursively see [3, B-spline property, p.90]. Additionally the subspaces with boundary conditions

$$\begin{aligned} S_{h;0,}^2(0,b) &:= S_h^2(0,b) \cap H_{0,}^2(0,b) = \mathrm{span}\{b_\ell | \ell = 1, \dots, N\}, \\ S_{h;0}^2(0,b) &:= S_h^2(0,b) \cap H_0^1(0,b) = \mathrm{span}\{b_\ell | \ell = 0, \dots, N-1\} \end{aligned}$$

are defined. Note that for $v_h \in S_{h;0,}^2(0,b)$ it holds that $v_h(0) = \partial_s v_h(0) = 0$. The following approximation properties hold true.

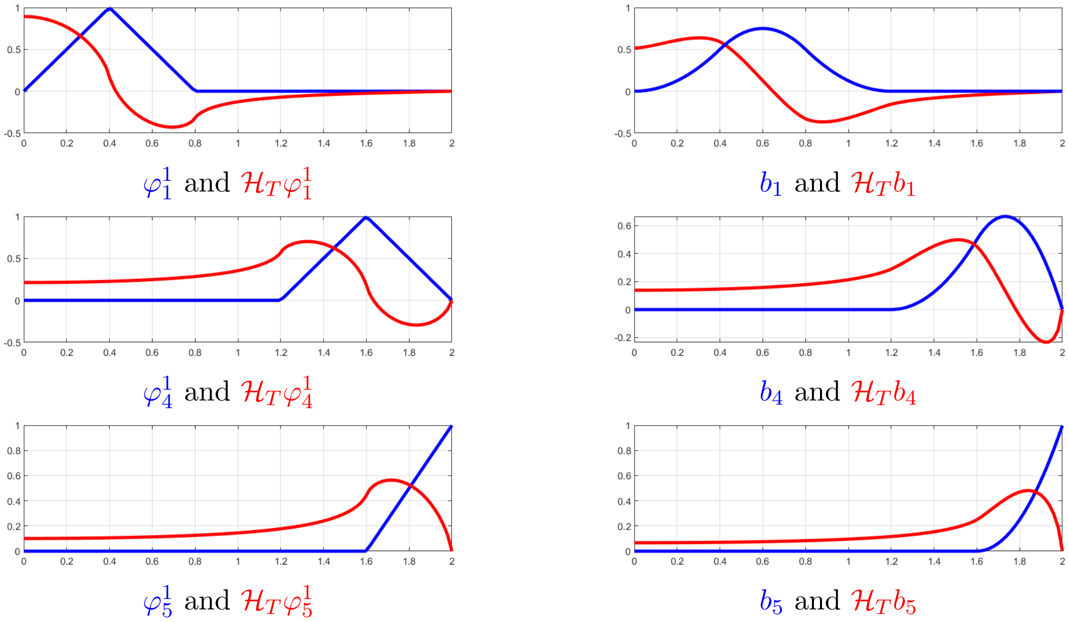

Figure 1.1: Basis functions and their modified Hilbert transformation on $(0, 2)$, $N = 5$

Theorem 1.2 ([5, c.f. Theorem 18, p.51]). *Let $u \in H^s(0, b)$ with $s \in [\sigma, 3]$ and $\sigma = 0, 1$. Then there holds*

$$\inf_{v_h \in S_h^2(0,b)} \|u - v_h\|_{H^\sigma(0,b)} \leq c h^{s-\sigma} |u|_{H^s(0,b)}.$$

For the one–dimensional wave equation tensor product trial spaces are considered. Let $T > 0$ be a terminal time and $L > 0$ be a parameter and $\Omega = (0, L)$ be an interval. Let further N_t be the number of elements for the discretisation in time $(0, T)$ and N_x be the number of elements for Ω. Define $h = \max\{h_t, h_x\}$ where h_t and h_x are the global mesh sizes on $(0, T)$ and Ω respectively. Then define the tensor product finite element spaces on the space–time cylinder $Q = (0, L) \times (0, T)$

$$\begin{aligned} Q_h^1(Q) &:= S_{h_x}^1(0, L) \otimes S_{h_t}^1(0, T), \\ Q_h^2(Q) &:= S_{h_x}^2(0, L) \otimes S_{h_t}^2(0, T). \end{aligned}$$

The representations for $u_h \in Q_h^1(Q)$ and $v_h \in Q_h^2(Q)$ hold

$$u_h(x, t) = \sum_{i=0}^{N_x} \sum_{j=0}^{N_t} u_{i,j} \varphi_i^1(x) \varphi_j^1(t), \quad v_h(x, t) = \sum_{i=-1}^{N_x} \sum_{j=-1}^{N_t} v_{i,j} b_i(x) b_j(t).$$

Moreover, define the subspaces with boundary conditions

$$\begin{aligned} Q_{h;0;0,}^1(Q) &:= Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q) = S_{h;0}^1(0, L) \otimes S_{h;0,}^1(0, L), \\ Q_{h;0;0,}^2(Q) &:= Q_h^2(Q) \cap H_{0;0,}^{1,2}(Q) = S_{h;0}^2(0, L) \otimes S_{h;0,}^2(0, T). \end{aligned}$$

2 Wave equation and Theory

In what follows the wave equation, as model problem for a hyperbolic PDE,

$$\begin{aligned} \partial_{tt}u(x,t) - \Delta_x u(x,t) &= f(x,t), && \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &= 0, && \text{for } (x,t) \in \Sigma = \partial\Omega \times [0,T], \\ u(x,0) = \partial_t u(x,0) &= 0, && \text{for } x \in \Omega, \end{aligned} \tag{2.1}$$

where $\Omega \subseteq \mathbb{R}^d$ is a Lipschitz domain with $d = 1, 2, 3$ and $T > 0$ is a terminal time and f is a given right–hand side, is considered. There are many different approaches for the numerical approximation of the solution of equation (2.1), e.g reformulations as a first order system in spatial and/or time variable, discontinuous Galerkin methods and semi–group theory [6, Chapter 3, Chapter 4],[2, Section 9.3]. These are not in the scope of this work. This thesis focuses on a variational formulation in subspaces of $H^1(Q)$, where the modified Hilbert transformation (see section 1.3) is used acting on the time variable. Numerical examples suggest an unconditionally stable method, i.e. no CFL–condition is required.

In this chapter the space–time variational formulation for the wave equation using the modified Hilbert transformation is stated. A corresponding ordinary differential equation is derived, which is further investigated to gain a better understanding of the unconditional stability. In order to classify the method, a comparison using a different transformation operator $\overline{\mathcal{H}}_T$ is done, for which a theoretical and numerical analysis exist (see [9]). In Chapter 3 numerical anaylsis for the ordinary differential equation is carried out. Using discrete tensor product spaces numerical examples for the one–dimensional wave equation are given in Chapter 4.

2.1 Variational formulation for the wave equation

For a given right hand side $f \in [H^{1,1}_{0;,0}]'$ the variational formulation for equation (2.1) to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)} = \langle f, w \rangle_Q, \text{ for all } w \in H^{1,1}_{0;,0}(Q), \tag{2.2}$$

is investigated. Note that ansatz and test space are not equal and that the boundary condition $u(0) = 0$ is considered in a strong sense, but $\partial_t u(0) = 0$ is incorporated in a weak sense. Consider the functions V_k as in 1.1, which build an orthogonal basis in $H^1_{0,}(0,T)$, and additionally the eigenfunctions of the spatial Laplacian

$$-\Delta_x \phi_i = \mu_i \phi_i \text{ in } \Omega, \quad \phi_i = 0 \text{ on } \Gamma, \quad \|\phi_i\|_{L^2(\Omega)} = 1, \tag{2.3}$$

and recall that ϕ_i are an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H^1_0(\Omega)$. For the eigenvalues μ_i it holds

$$0 < \mu_1 \le \mu_2 \le \mu_3 \le ... \text{ and } \lim_{i\to\infty} \mu_i = \infty.$$

Having these properties, for $u \in H^{1,1}_{0;0,}(Q)$ one finds the representation

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} u_{i,k} V_k(t)\phi_i(x) = \sum_{i=1}^{\infty} U_i(t)\phi_i(x), \quad U_i(t) = \sum_{k=0}^{\infty} u_{i,k} V_k(t) \tag{2.4}$$

with coefficients

$$u_{i,k} = \frac{2}{T}\int_0^T \int_\Omega u(x,t) V_k(t)\phi_i(x)\, dx\, dt.$$

Analogously, using the functions W_k as in (1.4), which are an orthogonal basis in $H^1_{,0}(0,T)$, for $w \in H^{1,1}_{0;,0}(Q)$ the representation

$$w(x,t) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} w_{i,k} W_k(t)\phi_i(x) = \sum_{i=1}^{\infty} W_i(t)\phi_i(x), \quad W_i(t) = \sum_{k=0}^{\infty} w_{i,k} W_k(t) \tag{2.5}$$

with coefficients

$$w_{i,k} = \frac{2}{T}\int_0^T \int_\Omega w(x,t) W_k(t)\phi_i(x)\, dx\, dt$$

holds true. Now define the function $v \in H^{1,1}_{0;0,}(Q)$ as

$$v(x,t) := \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} w_{i,k} V_k(t)\phi_i(x).$$

Applying the modified Hilbert transformation only on the time dependent part it follows

$$\mathcal{H}_T v(x,t) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} w_{i,k} (\mathcal{H}_T V_k)(t)\phi_i(x) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} w_{i,k} W_k(t)\phi_i(x) = w(x,t).$$

Thus, by bijectivity of the modified Hilbert transformation the variational formulation (2.2) is equivalent to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u, \partial_t \mathcal{H}_T v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \mathcal{H}_T v\rangle_{L^2(Q)} = \langle f, \mathcal{H}_T v\rangle_{(0,T)}, \text{ for all } v \in H^{1,1}_{0;0,}(Q).$$

In particular the ansatz and test space are equal. Furthermore, by Lemma 1.1 and Lemma 1.3 this is equivalent to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$\langle \mathcal{H}_T \partial_t u, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \mathcal{H}_T v\rangle_{L^2(Q)} = \langle f, \mathcal{H}_T v\rangle_Q, \text{ for all } v \in H^{1,1}_{0;0,}(Q). \tag{2.6}$$

Of course, one might also think of different ways to obtain equal ansatz and test spaces, e.g. the time reversal map ι_T (see Lemma 1.2). Another possibility, as presented in [9], is to transform the space $H^{1,1}_{0;0,}(Q)$ into $H^{1,1}_{0;,0}(Q)$ using the operator

$$\begin{aligned} \overline{\mathcal{H}}_T &: H^1_{0,}(0,T) \to H^1_{,0}(0,T), \\ (\overline{\mathcal{H}}_T v)(t) &:= v(T) - v(t) \end{aligned} \tag{2.7}$$

on the time dependent part. This operator is norm preserving and bijective between $H^1_{0,}(0,T)$ and $H^1_{,0}(0,T)$. Thus, formulation (2.2) is also equivalent to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u, \partial_t \overline{\mathcal{H}}_T v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} = \langle f, \overline{\mathcal{H}}_T v\rangle_Q, \text{ for all } v \in H^{1,1}_{0;0,}(Q). \tag{2.8}$$

2.2 The ordinary differential equation $\partial_{tt}u + \mu u = f$

Consider the formulation (2.2) and the ansatz (2.4) for $u \in H^{1,1}_{0;0,}(Q)$. Testing with a function $w(x,t) = W(t)\phi_j(x) \in H^{1,1}_{0;,0}(Q)$, $j \in \mathbb{N}$, where $W \in H^1_{,0}(0,T)$, results in

$$-\langle \partial_t U_j, \partial_t W\rangle_{L^2(0,T)} + \mu_j \langle U_j, W\rangle_{L^2(0,T)} = \langle f, W\phi_j\rangle_Q.$$

Observe that by the Poincaré inequality it holds that

$$\begin{aligned} |\langle f, W\phi_j\rangle| &\le \|f\|_{[H^{1,1}_{0;,0}(Q)]'} \|W\phi_j\|_{H^{1,1}_{0;,0}(Q)} \\ &= \|f\|_{[H^{1,1}_{0;,0}(Q)]'} \sqrt{\|W\|^2_{H^1_{,0}(0,T)} + \mu_j \|W\|^2_{L^2(0,T)}} \\ &= \|f\|_{[H^{1,1}_{0;,0}(Q)]'} \sqrt{1 + C_p \mu_j} \|W\|_{H^1_{,0}(0,T)}. \end{aligned}$$

Hence, $\langle F_j, W\rangle_{(0,T)} := \langle f, W\phi_j\rangle_Q$ fulfils $F_j \in [H^1_{,0}(0,T)]'$ and the variational formulation for the wave equation (2.2) is equivalent to find for each $j \in \mathbb{N}$ the coefficient function $U_j \in H^1_{0,}(0,T)$ such that

$$-\langle \partial_t U_j, \partial_t W\rangle_{L^2(0,T)} + \mu_j \langle U_j, W\rangle_{L^2(0,T)} = \langle F_j, W\rangle_{(0,T)}, \text{ for all } W \in H^1_{,0}(0,T). \tag{2.9}$$

Remark 2.1. *Formulation (2.9) is of particular interest in theory, since if a uniform stability estimate with respect to $j \in \mathbb{N}$ holds, the coefficient functions U_j can be uniquely determined and a unique solution $u(x,t) := \sum_{j=1}^\infty U_j(t)\phi_j(x)$ and a stability estimate for formulation (2.2) for the wave equation can be derived.*

Remark 2.2. *As for the wave equation one might test with* $W = \mathcal{H}_T V$ *or* $W = \overline{\mathcal{H}}_T V$ *for* $V \in H^1_{0,}(0,T)$ *in (2.9), to get equal ansatz and test spaces.*

If the functions U_j *obey higher regularity, throwing over a derivative one obtains the formulation to find* $U_j \in H^s_{0,}(0,T)$, $s \geq \frac{3}{2}$ *such that*

$$\langle \partial_{tt} U_j, W \rangle_{(0,T)} + \mu_j \langle U_j, W \rangle_{L^2(0,T)} = \langle F_j, W \rangle_{(0,T)}, \text{ for all } W \in H^{2-s}_{,0}(0,T), \tag{2.10}$$

which is a variational formulation of the ordinary differential equation

$$\begin{aligned} &\partial_{tt} U_j(t) + \mu_j U_j(t) = F_j(t), \text{ for } t \in (0,T), \\ &U_j(0) = \partial_t U_j(0) = 0. \end{aligned}$$

3 Numerical analysis for $\partial_{tt}u + \mu u = f$

Motivated by section 2.2 for $\mu > 0$ the model problem

$$\begin{aligned} \partial_{tt}u(t) + \mu u(t) &= f(t), \text{ for } t \in (0,T), \\ u(0) = \partial_t u(0) &= 0, \end{aligned}$$

and the variational formulation to find $u \in H^1_{0,}(0,T)$ such that

$$-\langle \partial_t u, \partial_t w \rangle_{L^2(0,T)} + \mu \langle u, w \rangle_{L^2(0,T)} = \langle f, w \rangle_{(0,T)}, \text{ for all } w \in H^1_{,0}(0,T), \tag{3.1}$$

are considered. First, an equivalent formulation using the operator $\overline{\mathcal{H}}_T$, as defined in (2.7), is derived and theoretical results for this formulation are repeated (see [9, Chapter 4]). Further, an equivalent formulation applying the modified Hilbert transformation is stated. Both variational formulations are then discretised in a conforming Galerkin-Bubnov setting and the numerical results are compared.

Define the bilinear form

$$\begin{aligned} &a(\cdot,\cdot) : H^1_{0,}(0,T) \times H^1_{,0}(0,T) \to \mathbb{R}, \\ &a(u,v) := -\langle \partial_t u, \partial_t w \rangle_{L^2(0,T)} + \mu \langle u, w \rangle_{L^2(0,T)}. \end{aligned}$$

In order to obtain equal ansatz and test spaces in formulation (3.1), the bijective transformation operator $\overline{\mathcal{H}}_T v(t) = v(T) - v(t) \in H^1_{,0}(0,T)$ is applied to test functions $v \in H^1_{0,}(0,T)$. This gives the equivalent formulation to find $u \in H^1_{0,}(0,T)$ such that

$$a(u, \overline{\mathcal{H}}_T v) = \langle f, \overline{\mathcal{H}}_T v \rangle_{(0,T)}, \text{ for all } v \in H^1_{0,}(0,T). \tag{3.2}$$

The following theorem states the unique solvability of (3.2) and gives a μ-dependent stability estimate. The proof of the unique solvability is repeated in order to outline the argumentation.

Theorem 3.1 ([12, Theorem 4.2.5, p.133])**.** *Let $f \in [H^1_{,0}(0,T)]'$ be given. There exists a unique solution $u \in H^1_{0,}(0,T)$ of the variational formulation (3.2). Furthermore, the solution operator*

$$\mathcal{L} : [H^1_{,0}(0,T)]' \to H^1_{0,}(0,T),\ \mathcal{L}f := u$$

is an isomorphism, satisfying

$$|u|_{H^1(0,T)} = |\mathcal{L}f|_{H^1(0,T)} \le \frac{2 + \sqrt{\mu}T}{2} \|f\|_{[H^1_{,0}(0,T)]'}. \tag{3.3}$$

Moreover, the inequality (3.3) is optimal with respect to μ and T.

Proof. By the Riesz representation theorem, (3.2) is equivalent to the operator equation

$$\mathcal{A}u + \mu\mathcal{C}u = \hat{f},$$

where $\mathcal{A} : H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'$ defined as

$$\langle \mathcal{A}u, v\rangle := -\langle \partial_t u, \partial_t \overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} = \langle \partial_t u, \partial_t v\rangle_{L^2(0,T)}, \text{ for } u, v \in H^1_{0,}(0,T)$$

is elliptic, and thus invertible, and $\mathcal{C} : H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'$ defined as

$$\langle \mathcal{C}u, v\rangle := \langle u, \overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} = \langle u, v(T) - v\rangle_{L^2(0,T)}, \text{ for } u, v \in H^1_{0,}(0,T),$$

is compact. The right–hand side $\hat{f} : H^1_{0,}(0,T) \to \mathbb{R}$ given as

$$\langle \hat{f}, v\rangle := \langle f, \overline{\mathcal{H}}_T v\rangle_{(0,T)}, \text{ for } v \in H^1_{0,}(0,T),$$

satisfies

$$|\langle \hat{f}, v\rangle| \le \|f\|_{[H^1_{,0}(0,T)]'} \|\overline{\mathcal{H}}_T v\|_{H^1_{,0}(0,T)} = \|f\|_{[H^1_{,0}(0,T)]'} \|v\|_{H^1_{0,}(0,T)}$$

for all $v \in H^1_{0,}(0,T)$ and thus $\hat{f} \in [H^1_{0,}(0,T)]'$. Hence, applying the Fredholm alternative it is sufficient to show that the homogenuous problem admits only the trivial solution. Let $u \in H^1_{0,}(0,T)$ be a solution of $(\mathcal{A} + \mu\mathcal{C})u = 0$, i.e.

$$\langle \partial_t u, \partial_t w\rangle_{L^2(0,T)} = \mu\langle u, w\rangle_{L^2(0,T)}, \text{ for all } w \in H^1_{,0}(0,T).$$

This is a weak formulation of the eigenvalue problem

$$\begin{aligned} &\partial_{tt}u(t) = \mu u(t), \qquad \text{for } t \in (0,T), \\ &u(0) = \partial_t u(0) = 0, \end{aligned}$$

which only has the trivial solution $u \equiv 0$.

For the proof of the stability estimate see [12, Lemma 4.2.3, p.132]. □

Remark 3.1. *For a more regular right–hand side $f \in L^2(0,T)$ the stability estimate independent of μ*

$$|u|^2_{H^1(0,T)} + \mu\|u\|^2_{L^2(0,T)} \le \frac{T^2}{2}\|f\|^2_{L^2(0,T)},$$

holds for the unique solution $u \in H^1_{0,}(0,T)$ of (3.2) [12, Lemma 4.2.7, p.135]. Building up on this result a stabilized method is derived in [12].

A conforming Galerkin-Bubnov finite element discretisation of (3.2) with linear ansatz and test functions is to find $u_{h_t} \in S^1_{h_t;0,}(0,T)$ such that

$$\langle \partial_t u_{h_t}, \partial_t v_{h_t} \rangle_{L^2(0,T)} + \mu \langle u_{h_t}, v_{h_t}(T) - v_{h_t} \rangle_{L^2(0,T)} = \langle f, \overline{\mathcal{H}}_T v_{h_t} \rangle, \tag{3.4}$$

for all $v_{h_t} \in S^1_{h_t;0,}(0,T)$.

In Theorem 3.3 a compact pertubation argument is used. Thus, for formulation (3.4) applying the theory for numerical solutions of elliptic operator equations with compact pertubations, unique solvability and discrete stability follow demanding a sufficiently small mesh size

$$h_t \leq \frac{\sqrt{3}\pi}{\sqrt{2}(2 + \sqrt{\mu}T)\mu T} \tag{3.5}$$

and further the error estimate

$$|u - u_{h_t}|_{H^1(0,T)} \leq \hat{c}(T,\mu) h_t \|u\|_{H^2(0,T)} \tag{3.6}$$

holds for a constant $\hat{c}(T,\mu) > 0$ when assuming $u \in H^2(0,T) \cap H^1_{0,}(0,T)$.

A different approach to find equal ansatz and test spaces in the variational formulation (3.1) is to transform test functions $v \in H^1_{0,}(0,T)$ using the modified Hilbert transformation $\mathcal{H}_T v \in H^1_{,0}(0,T)$. An equivalent formulation is therefore to find $u \in H^1_{0,}(0,T)$ such that

$$a(u, \mathcal{H}_T v) = \langle f, \mathcal{H}_T v \rangle_{(0,T)}, \text{ for all } v \in H^1_{0,}(0,T). \tag{3.7}$$

Since the formulations (3.7) and (3.2) are equivalent, unique solvability follows as in Theorem 3.3. But, numerical examples suggest that formulation (3.7) is unconditionally stable, i.e. a stability estimate without dependency on μ holds. Although, a proof of this is behavior is still outstanding, some obervations are stated. First note that with Lemma 1.1 and Lemma 1.3 for $u, v \in H^1_{0,}(0,T)$ it holds

$$\begin{aligned} a(u, \mathcal{H}_T v) &= -\langle \partial_t u, \partial_t \mathcal{H}_T v \rangle_{L^2(0,T)} + \mu \langle u, \mathcal{H}_T v \rangle_{L^2(0,T)} \\ &= \langle \mathcal{H}_T \partial_t u, \partial_t v \rangle_{L^2(0,T)} + \mu \langle u, \mathcal{H}_T v \rangle_{L^2(0,T)}. \end{aligned}$$

In particular, using Lemma 1.4, it follows for $v = u \in H^1_{0,}(0,T)$ that

$$a(u, \mathcal{H}_T u) \geq 0.$$

By Lemma 1.2, another reformulation keeps the modified Hilbert tranformation on the test functions by using the time reversal map

$$a(u, \mathcal{H}_T v) = \langle \iota_T \partial_t u, \mathcal{H}_T \iota_T \partial_t v \rangle_{L^2(0,T)} + \mu \langle u, \mathcal{H}_T v \rangle_{L^2(0,T)}.$$

A Galerkin-Bubnov finite element discretisation of (3.7) in a conforming trial space of linear functions is to find $u_{h_t} \in S^1_{h_t;0,}(0,T)$ such that

$$\langle \mathcal{H}_T \partial_t u_{h_t}, \partial_t v_{h_t} \rangle_{L^2(0,T)} + \mu \langle u_{h_t}, \mathcal{H}_T v_{h_t} \rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v_{h_t} \rangle_{(0,T)}, \tag{3.8}$$

for all $v_{h_t} \in S^1_{h_t;0,}(0,T)$.

3.1 Discrete inf–sup constants

The μ–independence of the stability estimate for the ordinary differential equation $\partial_{tt}u + \mu u = f$ plays a decisive role for the extension of a method to the wave equation (2.1) (see Remark 2.1). The constant c of the stability estimate

$$|u|_{H^1(0,T)} \leq c\|f\|_{[H^1_{,0}(0,T)]'}$$

is related to the inf–sup constant

$$c_s = \inf_{0\neq u\in H^1_{0,}(0,T)} \sup_{0\neq w\in H^1_{,0}(0,T)} \frac{a(u,w)}{|u|_{H^1(0,T)}|w|_{H^1(0,T)}}$$

by the equality $c_s = c^{-1}$, since for the solution $u \in H^1_{0,}(0,T)$ it holds

$$c_s|u|_{H^1_{0,}(0,T)} \leq \sup_{0\neq w\in H^1_{,0}(0,T)} \frac{a(u,w)}{|w|_{H^1_{,0}(0,T)}} = \sup_{0\neq w\in H^1_{,0}(0,T)} \frac{\langle f,w\rangle_{(0,T)}}{|w|_{H^1_{,0}(0,T)}} = \|f\|_{[H^1_{,0}(0,T)]'}.$$

First, the inf–sup constant for formulation (3.2) is considered. Estimate (3.3) predicts the behavior $c_s \propto \mu^{-\frac{1}{2}}$. In order to estimate c_s, the discrete inf–sup constant for linear ansatz and test functions

$$\tilde{c}_s := \inf_{0\neq u_{h_t}\in S^1_{h_t;0,}(0,T)} \sup_{0\neq v_{h_t}\in S^1_{h_t;0,}(0,T)} \frac{a(u_{h_t}, \overline{\mathcal{H}}_T v_{h_t})}{|u_{h_t}|_{H^1(0,T)}|v_{h_t}|_{H^1(0,T)}} \tag{3.9}$$

is computed. This is achieved by solving a generalized eigenvalue problem (see [4, Remark 3.159, p.126]) as follows. The discrete variational formulation (3.4) is equivalent to the system of linear equations

$$K^{\overline{\mathcal{H}}_T}_{h_t}\underline{u}_{h_t} := \left(B^{\overline{\mathcal{H}}_T}_{h_t} + \mu C^{\overline{\mathcal{H}}_T}_{h_t}\right)\underline{u}_{h_t} = \underline{f}^{\overline{\mathcal{H}}_T}_{h_t}, \tag{3.10}$$

with stiffness matrix $B^{\overline{\mathcal{H}}_T}_{h_t} \in \mathbb{R}^{N_t\times N_t}$

$$B^{\overline{\mathcal{H}}_T}_{h_t}[i,j] := \langle \partial_t\varphi^1_j, \partial_t\varphi^1_i\rangle_{L^2(0,T)} \tag{3.11}$$

and mass matrix $C^{\overline{\mathcal{H}}_T}_{h_t} \in \mathbb{R}^{N_t\times N_t}$

$$C^{\overline{\mathcal{H}}_T}_{h_t}[i,j] := \langle \varphi^1_j, \varphi^1_i(T) - \varphi^1_i\rangle_{L^2(0,T)} \tag{3.12}$$

and with right–hand side $\underline{f}^{\overline{\mathcal{H}}_T}_{h_t} \in \mathbb{R}^{N_t}$

$$\underline{f}^{\overline{\mathcal{H}}_T}_{h_t}[i] := \langle f, \varphi^1_i(T) - \varphi^1_i\rangle_{(0,T)},$$

where $\varphi_i^1 \in S^1_{h_t;0,}(0,T)$ are the basis functions. Defining the usual stiffness matrix $A_{h_t} \in \mathbb{R}^{N_t \times N_t}$ for the space $S^1_{h_t;0,}(0,T)$ by

$$A_{h_t}[i,j] := \langle \partial_t \varphi_j^1, \partial_t \varphi_i^1 \rangle_{L^2(0,T)}$$

the generalized eigenvalue problem to be solved is

$$\left(K_{h_t}^{\overline{\mathcal{H}}_T}\right)^\top A_{h_t}^{-1} K_{h_t}^{\overline{\mathcal{H}}_T} \underline{u} = \lambda A_{h_t} \underline{u}. \tag{3.13}$$

The discrete inf–sup constant is then given as the square root of the minimal eigenvalue of (3.13)

$$\tilde{c}_s = \sqrt{\lambda_{min}}.$$

Table 3.1 presents the discrete inf–sup constant for different levels of refinement and different parameters μ. The results confirm the theoretical dependency on μ, i.e. $\tilde{c}_s = \tilde{c}_s(\mu) \propto \mu^{-\frac{1}{2}}$.

h_t	$\tilde{c}_s(h_t;1)$	$\tilde{c}_s(h_t;62.5)$	$\tilde{c}_s(h_t;125)$	$\tilde{c}_s(h_t;250)$	$\tilde{c}_s(h_t;500)$	$\tilde{c}_s(h_t;1000)$
0.500	0.744	0.143	0.131	0.170	0.272	0.484
0.250	0.735	0.254	0.157	0.013	0.003	0.002
0.125	0.732	0.202	0.162	0.126	0.087	0.000
0.063	0.731	0.188	0.140	0.107	0.083	0.065
0.031	0.731	0.185	0.135	0.098	0.072	0.055
0.016	0.731	0.184	0.134	0.097	0.070	0.050
0.008	0.731	0.184	0.134	0.096	0.069	0.049

Table 3.1: inf–sup constants $\tilde{c}_s(h;\mu)$ for formulation (3.2) for linear ansatz and test functions, $T = 2$

Next, the discrete inf–sup constant for the variational formulation (3.7) using the modified Hilbert transformation for linear ansatz and test functions is computed. Analogously, the generalized eigenvalue problem (3.13) has to be solved, with stiffness and mass matrices $B_{h_t}^{\mathcal{H}_T}, C_{h_t}^{\mathcal{H}_T} \in \mathbb{R}^{N_t \times N_t}$

$$B_{h_t}^{\mathcal{H}_T}[i,j] := \langle \mathcal{H}_T \partial_t \varphi_j^1, \partial_t \varphi_i^1 \rangle_{L^2(0,T)}, \tag{3.14}$$

$$C_{h_t}^{\mathcal{H}_T}[i,j] := \langle \varphi_j^1, \mathcal{H}_T \varphi_i^1 \rangle_{L^2(0,T)}. \tag{3.15}$$

The results in table 3.2 indicate μ–robustness of $\tilde{c}_s$, but also show a linear dependency of the constant on the mesh size, i.e. $\tilde{c}_s = \tilde{c}_s(h_t) \propto h_t$. Table 3.3 shows the same behavior for ansatz and test functions of second order.

Remark 3.2. *The modified Hilbert transformation of a basis function does not have local support anymore (see Fig. 1.1). Thus, an efficient evaluation of the matrix entries for the stiffness and mass matrices $B_{h_t}^{\mathcal{H}_T}$ and $C_{h_t}^{\mathcal{H}_T}$ is needed. This can be achieved using the integral representation (1.10) of the modified Hilbert transformation. For details see [10].*

h_t	$\tilde{c}_s(h_t;1)$	$\tilde{c}_s(h_t;62.5)$	$\tilde{c}_s(h_t;125)$	$\tilde{c}_s(h_t;250)$	$\tilde{c}_s(h_t;500)$	$\tilde{c}_s(h_t;1000)$
0.500	0.197	0.559	1.217	2.530	5.153	10.398
0.250	0.105	0.012	0.076	0.243	0.567	1.212
0.125	0.053	0.045	0.031	0.013	0.032	0.114
0.063	0.027	0.026	0.024	0.016	0.016	0.003
0.031	0.013	0.013	0.013	0.012	0.011	0.009
0.016	0.007	0.007	0.007	0.006	0.006	0.006
0.008	0.003	0.003	0.003	0.003	0.003	0.003

Table 3.2: inf–sup constants $\tilde{c}_s(h;\mu)$ for formulation (3.7) for linear ansatz and test functions, $T = 2$

h_t	$\tilde{c}_s(h_t;1)$	$\tilde{c}_s(h_t;62.5)$	$\tilde{c}_s(h_t;125)$	$\tilde{c}_s(h_t;250)$	$\tilde{c}_s(h_t;500)$	$\tilde{c}_s(h_t;1000)$
0.500	0.327	0.548	1.055	2.071	4.104	8.171
0.250	0.199	0.073	0.096	0.320	0.683	1.384
0.125	0.111	0.072	0.056	0.028	0.032	0.166
0.063	0.059	0.049	0.043	0.036	0.029	0.020
0.031	0.030	0.028	0.027	0.026	0.024	0.021

Table 3.3: inf–sup constants $\tilde{c}_s(h;\mu)$ for formulation (3.7) for ansatz and test space $S^2_{h;0,}(0,T)$, $T = 2$

3.2 Test examples

To see how the errors of the numerical approximation and the convergence orders behave for different μ the following test example is considered

$$u(t) = \sin\left(\frac{5\pi}{4}t\right)^2.$$

First, the variational formulation (3.4) using the operator $\overline{\mathcal{H}}_T$ is considered for linear ansatz and test functions, which is equivalent to the system of linear equations (3.10). The theory predicts stability only for sufficiently small mesh size (see (3.5)). Tables 3.4 and 3.5 present the results for different μ and confirm this behavior. The optimal convergence order is obtained for $\mu = 1$, but for $\mu = 1000$ in the first steps the error gets worse.

Next, the variational formulation (3.8) using the modified Hilbert transformation is investigated. This formulation is equivalent to the system of equations

$$\left(B^{\mathcal{H}_T}_{h_t} + \mu C^{\mathcal{H}_T}_{h_t}\right)\underline{u}_{h_t} = \underline{f}^{\mathcal{H}_T}_{h_t},$$

N	h_t	$\|u_{h_t} - u\|_{L^2(0,T)}$	eoc	$\|u_{h_t} - u\|_{H^1(0,T)}$	eoc
4	0.50	$4.99 \cdot 10^{-1}$		$3.47 \cdot 10^{0}$	
8	0.25	$1.63 \cdot 10^{-1}$	1.62	$2.09 \cdot 10^{0}$	0.73
16	0.13	$4.36 \cdot 10^{-2}$	1.90	$1.10 \cdot 10^{0}$	0.93
32	0.06	$1.11 \cdot 10^{-2}$	1.98	$5.54 \cdot 10^{-1}$	0.98
64	0.03	$2.78 \cdot 10^{-3}$	1.99	$2.78 \cdot 10^{-1}$	1.00
128	0.02	$6.97 \cdot 10^{-4}$	2.00	$1.39 \cdot 10^{-1}$	1.00

Table 3.4: Formulation (3.4): Errors and eoc for u and $\mu = 1$, $T = 2$

N	h_t	$\|u_{h_t} - u\|_{L^2(0,T)}$	eoc	$\|u_{h_t} - u\|_{H^1(0,T)}$	eoc
4	0.50	$7.85 \cdot 10^{0}$		$3.97 \cdot 10^{1}$	
8	0.25	$2.40 \cdot 10^{2}$	-4.93	$2.47 \cdot 10^{3}$	-5.96
16	0.13	$4.43 \cdot 10^{1}$	2.44	$1.11 \cdot 10^{3}$	1.16
32	0.06	$9.59 \cdot 10^{-3}$	12.17	$6.21 \cdot 10^{-1}$	10.80
64	0.03	$2.74 \cdot 10^{-3}$	1.81	$2.90 \cdot 10^{-1}$	1.10
128	0.02	$7.14 \cdot 10^{-4}$	1.94	$1.41 \cdot 10^{-1}$	1.04

Table 3.5: Formulation (3.4): Errors and eoc for u and $\mu = 1000$, $T = 2$

with stiffness and mass matrices as in (3.14) and (3.15) and right–hand side $\underline{f}_{h_t}^{\mathcal{H}_T}$ with entries

$$\underline{f}_{h_t}[i] = \langle f, \mathcal{H}_T \varphi_i^1 \rangle_{L^2(0,T)},$$

for $\varphi_i^1 \in S^1_{h_t;0,}(0,T)$ as basis functions. Tables 3.6 and 3.7 list the results for $\mu = 1$ and $\mu = 1000$ and show unconditional stability and optimal orders of convergence for linear ansatz and test functions. This supports the μ–independent stability observed in the inf–sup constants in section 3.1.

N	h_t	$\|u_{h_t} - u\|_{L^2(0,T)}$	eoc	$\|u_{h_t} - u\|_{H^1(0,T)}$	eoc
4	0.50	$2.57 \cdot 10^{0}$		$9.40 \cdot 10^{0}$	
8	0.25	$7.08 \cdot 10^{-1}$	1.86	$5.19 \cdot 10^{0}$	0.86
16	0.13	$1.20 \cdot 10^{-1}$	2.56	$1.89 \cdot 10^{0}$	1.46
32	0.06	$2.74 \cdot 10^{-2}$	2.13	$8.94 \cdot 10^{-1}$	1.08
64	0.03	$6.67 \cdot 10^{-3}$	2.04	$4.42 \cdot 10^{-1}$	1.01
128	0.02	$1.66 \cdot 10^{-3}$	2.01	$2.22 \cdot 10^{-1}$	0.99

Table 3.6: Formulation (3.8): Errors and eoc for u and $\mu = 1$, $T = 2$

N	h_t	$\|u_{h_t} - u\|_{L^2(0,T)}$	eoc	$|u_{h_t} - u|_{H^1(0,T)}$	eoc
4	0.50	$9.57 \cdot 10^{-1}$		$6.69 \cdot 10^{0}$	
8	0.25	$2.06 \cdot 10^{-1}$	2.21	$3.47 \cdot 10^{0}$	0.95
16	0.13	$3.87 \cdot 10^{-2}$	2.42	$1.45 \cdot 10^{0}$	1.26
32	0.06	$3.39 \cdot 10^{-2}$	0.19	$1.71 \cdot 10^{0}$	-0.24
64	0.03	$7.66 \cdot 10^{-3}$	2.15	$5.48 \cdot 10^{-1}$	1.65
128	0.02	$1.85 \cdot 10^{-3}$	2.05	$2.38 \cdot 10^{-1}$	1.20

Table 3.7: Formulation (3.8): Errors and eoc for u and $\mu = 1000$, $T = 2$

An interesting behavior is seen when looking at the condition of the mass matrix $C_{h_t}^{\mathcal{H}_T}$ in table 3.8. For same order ansatz and test functions the modified Hilbert transformation behaves like a first order derivative operator, i.e. $\kappa(C_{h_t}^{\mathcal{H}_T}) \propto h_t^{-1}$. Whereas the stiffness matrix $B_{h_t}^{\mathcal{H}_T}$ behaves like a ususal second order derivative operator, althogh the modified Hilbert transformation is used.

N	$\sigma_{max}(B_{h_t})$	$\sigma_{min}(B_{h_t})$	$\kappa(B_{h_t})$	$\sigma_{max}(C_{h_t})$	$\sigma_{min}(C_{h_t})$	$\kappa(C_{h_t})$
4	$2.72 \cdot 10^{0}$	$3.15 \cdot 10^{-1}$	$8.63 \cdot 10^{0}$	$4.14 \cdot 10^{-1}$	$5.75 \cdot 10^{-2}$	$7.21 \cdot 10^{0}$
8	$6.11 \cdot 10^{0}$	$1.87 \cdot 10^{-1}$	$3.27 \cdot 10^{1}$	$2.37 \cdot 10^{-1}$	$1.69 \cdot 10^{-2}$	$1.40 \cdot 10^{1}$
16	$1.28 \cdot 10^{1}$	$1.01 \cdot 10^{-1}$	$1.26 \cdot 10^{2}$	$1.23 \cdot 10^{-1}$	$4.64 \cdot 10^{-3}$	$2.65 \cdot 10^{1}$
32	$2.60 \cdot 10^{1}$	$5.28 \cdot 10^{-2}$	$4.92 \cdot 10^{2}$	$6.23 \cdot 10^{-2}$	$1.22 \cdot 10^{-3}$	$5.09 \cdot 10^{1}$
64	$5.21 \cdot 10^{1}$	$2.69 \cdot 10^{-2}$	$1.94 \cdot 10^{3}$	$3.12 \cdot 10^{-2}$	$3.15 \cdot 10^{-4}$	$9.92 \cdot 10^{1}$
128	$1.04 \cdot 10^{2}$	$1.36 \cdot 10^{-2}$	$7.62 \cdot 10^{3}$	$1.56 \cdot 10^{-2}$	$7.61 \cdot 10^{-5}$	$2.05 \cdot 10^{2}$

Table 3.8: Table of singular values and condition numbers of $B_{h_t}^{\mathcal{H}_T}$ und $C_{h_t}^{\mathcal{H}_T}$ for linear ansatz and test functions, $T = 2$

4 Numerical examples for the wave equation

For a terminal time $T > 0$, an spatial parameter $L > 0$ and a given right–hand side $f \in [H^{1,1}_{0;,0}(Q)]'$ the one–dimensional wave equation

$$\begin{aligned} \partial_{tt}u(x,t) - \partial_{xx}u(x,t) &= f(x,t), && \text{for } (x,t) \in Q = (0,L) \times (0,T), \\ u(x,t) &= 0, && \text{for } (x,t) \in \Sigma = \{0,L\} \times [0,T], \\ u(x,0) = \partial_t u(x,0) &= 0, && \text{for } x \in (0,L), \end{aligned} \tag{4.1}$$

and the variational formulation to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$a(u,w) = \langle f, w \rangle_Q, \text{ for all } w \in H^{1,1}_{0;,0}(Q), \tag{4.2}$$

with the bilinear form

$$\begin{aligned} &a(\cdot,\cdot) : H^{1,1}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}, \\ &a(u,w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \partial_x u, \partial_x w \rangle_{L^2(Q)}, \end{aligned}$$

are considered. The previous observations suggest that the equivalent variational formulation to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$a(u, \mathcal{H}_T v) = \langle f, \mathcal{H}_T v \rangle_Q, \text{ for all } v \in H^{1,1}_{0;0,}(Q), \tag{4.3}$$

is unconditionally stable. This section is devoted to investigate formulation (4.3) in a discrete setting using tensor product spaces. For a classification the findings are compared to the results in [9] using the operator $\overline{\mathcal{H}}_T$. As a test example the function

$$u(x,t) = \sin(\pi x) t^2 (x-t)^2, \; (x,t) \in Q,$$

is considered.

At the end, a numerical approximation for a test example with singularity in the right–hand side is computed. For formulation (4.3) it is observed that no special treatment for such class of functions is required.

4.1 A conforming Galerking–Bubnov method using $\overline{\mathcal{H}}_T$

As a reference method the variational formulation to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$a(u, \overline{\mathcal{H}}_T v) = \langle f, \overline{\mathcal{H}}_T v\rangle_Q, \text{ for all } v \in H^{1,1}_{0;0,}(Q), \tag{4.4}$$

letting the operator $\overline{\mathcal{H}}_T$ act on the time dependent part is considered, where theory and numerical analysis are outlined in [9, Section 5]. The conforming Galerkin–Bubnov variational formulation of (4.4), with notation of section 1.4, is to find $u_h \in Q^1_{h;0;0,}(Q)$ such that

$$-\langle \partial_t u_h, \partial_t \overline{\mathcal{H}}_T v_h\rangle_{L^2(Q)} + \langle \partial_x u_h, \partial_x \overline{\mathcal{H}}_T v_h\rangle_{L^2(0,T)} = \langle f, \overline{\mathcal{H}}_T v_h\rangle_Q, \text{ for all } v_h \in Q^1_{h;0;0,}(Q). \tag{4.5}$$

Due to the tensor product structure, after reordering the degrees of freedom appropriately, (4.5) is equivalent to the system of linear equations

$$K_h \underline{u}_h = \underline{F}_h, \tag{4.6}$$

with system matrix

$$K_h \underline{u}_h := B^{\overline{\mathcal{H}}_T}_{h_t} \otimes M_{h_x} + C^{\overline{\mathcal{H}}_T}_{h_t} \otimes A_{h_x} \in \mathbb{R}^{N_t(Nx-1)\times N_t(N_x-1)},$$

where stiffness and mass matrices in time $B^{\overline{\mathcal{H}}_T}_{h_t}, C^{\overline{\mathcal{H}}_T}_{h_t} \in \mathbb{R}^{N_t\times N_t}$ are as in (3.11) and (3.12) and the spatial stiffness and mass matrices $A_{h_x}, M_{h_x} \in \mathbb{R}^{(N_x-1)\times(N_x-1)}$ are

$$A_{h_x}[i,j] := \langle \partial_x \varphi^1_j, \partial_x \varphi^1_i\rangle_{L^2(0,L)}, \tag{4.7}$$

$$M_{h_x}[i,j] := \langle \varphi^1_j, \varphi^1_i\rangle_{L^2(0,L)}, \tag{4.8}$$

for basis functions $\varphi^1_i \in S^1_{h;0}(0,L)$. The right–hand side $\underline{F}_h \in \mathbb{R}^{N_t(N_x-1)}$ is given as

$$\underline{F}_h[j + (i-1)N_t] = \int_0^T \int_\Omega f(x,t)\varphi^1_i(x)\overline{\mathcal{H}}_T\varphi^1_j(t)\, dx\, dt,$$

for $i = 1, ..., (N_x - 1)$, $j = 1, ..., N_t$. Stabilty and error estimates for this formulation follow for sufficiently small mesh size h_t, which is mentioned in (3.5), but also when a root condition is satified (see [9, Remark 4.4]). This results in the CFL-condition

$$h_t \le h_x. \tag{4.9}$$

For the test example u errors and convergence rates are stated in tables 4.1 and 4.2. Optimal orders of convergnece and stability can be observed if the CFL-condition (4.9) is fulfiled. Though, the condition seems to be sharp, as violation leads to non–convergence.

N	h_x	h_t	$\|u_h - u\|_{L^2(Q)}$	eoc	$\|u_h - u\|_{H^1(Q)}$	eoc
32	0.2500	0.2500	$2.21 \cdot 10^{-1}$		$3.46 \cdot 10^{0}$	
128	0.1250	0.1250	$5.45 \cdot 10^{-2}$	2.02	$1.71 \cdot 10^{0}$	1.02
512	0.0625	0.0625	$1.36 \cdot 10^{-2}$	2.01	$8.53 \cdot 10^{-1}$	1
2,048	0.0313	0.0313	$3.39 \cdot 10^{-3}$	2	$4.26 \cdot 10^{-1}$	1
8,192	0.0156	0.0156	$8.48 \cdot 10^{-4}$	2	$2.13 \cdot 10^{-1}$	1

Table 4.1: $L^2(Q)$ and $H^1(Q)$ errors and eocs for formulation (4.5) on $Q = (0,1) \times (0,2)$ satisfying (4.9)

N	h_x	h_t	$\|u_h - u\|_{L^2(Q)}$	eoc	$\|u_h - u\|_{H^1(Q)}$	eoc
16	0.2500	0.5000	$2.70 \cdot 10^{-1}$		$4.01 \cdot 10^{0}$	
64	0.1250	0.2500	$6.59 \cdot 10^{-2}$	2.04	$1.97 \cdot 10^{0}$	1.02
256	0.0625	0.1250	$5.42 \cdot 10^{-1}$	−3.04	$2.97 \cdot 10^{1}$	−3.91
1,024	0.0313	0.0625	$4.83 \cdot 10^{5}$	−19.77	$5.50 \cdot 10^{7}$	−20.82
4,096	0.0156	0.0313	$6.09 \cdot 10^{7}$	−6.98	$1.28 \cdot 10^{10}$	−7.86

Table 4.2: $L^2(Q)$ and $H^1(Q)$ errors and eocs for formulation (4.5) on $Q = (0,1) \times (0,2)$ violating (4.9)

4.2 A conforming Galerkin–Bubnov method using $\mathcal{H}_T$

As observed in section 3.1 the inf–sup constant for the variational formulation using $\overline{\mathcal{H}}_T$ is μ–dependent. Thus, no unconditional stability was expected. On the other hand, the inf–sup constant using the modified Hilbert transformation seems to be μ–robust. Hence, an unconditionally stable extension to the wave equation is imaginable. This shall further be investigated.

The discrete Galerkin–Bubnov variational formulation using the modified Hilbert transformation (c.f. (2.6)) is to find $u_h \in Q^1_{h;0;0,}(Q)$ such that

$$\langle \mathcal{H}_T \partial_t u_h, \partial_t v_h \rangle_{L^2(Q)} + \langle \partial_x u_h, \partial_x \mathcal{H}_T v_h \rangle_{L^2(Q)} = \langle f, \mathcal{H}_T v_h \rangle_Q, \text{ for all } v_h \in Q^1_{h;0;0,}(Q). \tag{4.10}$$

Analogously to formulation (4.5), (4.10) is equivalent to the system of linear equations (4.6) with system matrix

$$K_h \underline{u}_h := B^{\mathcal{H}_T}_{h_t} \otimes M_{h_x} + C^{\mathcal{H}_T}_{h_t} \otimes A_{h_x} \in \mathbb{R}^{N_t(Nx-1) \times N_t(N_x-1)},$$

where temporal stiffness and mass matrices $B^{\mathcal{H}_T}_{h_t}, C^{\mathcal{H}_T}_{h_t} \in \mathbb{R}^{N_t \times N_t}$ are as in (3.14) and (3.15), spatial stiffness and mass matrices as in (4.7) and (4.8) and the corresponding right–hand side is given as

$$\underline{F}_h[j + (i-1)N_t] = \int_0^T \int_\Omega f(x,t)\varphi_i^1(x)\mathcal{H}_T\varphi_j^1(t)\,dx\,dt,$$

for $i = 1, ..., (N_x - 1)$, $j = 1, ..., N_t$.

Again, the test example u is considered. The results in tables 4.3 and 4.4 show optimal orders of convergence and stability, even when the CFL-condition (4.9) is violated.

Also for ansatz and test space $Q^2_{h;0;0,}(Q)$ of functions of second order, optimal convergence rates and stability are observed in table 4.5. Note that Q is chosen differently, since the temporal matrix entries are computed using singular integrals, which are too inexact if the level of refinement in time is high.

N	h_x	h_t	$\|u_h - u\|_{L^2(Q)}$	eoc	$\|u_h - u\|_{H^1(Q)}$	eoc
32	0.2500	0.2500	$3.83 \cdot 10^{-1}$		$4.38 \cdot 10^{0}$	
128	0.1250	0.1250	$9.27 \cdot 10^{-2}$	2.04	$2.09 \cdot 10^{0}$	1.07
512	0.0625	0.0625	$2.27 \cdot 10^{-2}$	2.03	$1.03 \cdot 10^{0}$	1.02
2,048	0.0313	0.0313	$5.63 \cdot 10^{-3}$	2.01	$5.12 \cdot 10^{-1}$	1.01
8,192	0.0156	0.0156	$1.38 \cdot 10^{-3}$	2.02	$2.54 \cdot 10^{-1}$	1.01

Table 4.3: $L^2(Q)$ and $H^1(Q)$ errors and eocs for formulation (4.10) on $Q = (0,1)\times(0,2)$ satisfying (4.9)

N	h_x	h_t	$\|u_h - u\|_{L^2(Q)}$	eoc	$\|u_h - u\|_{H^1(Q)}$	eoc
16	0.2500	0.5000	$1.74 \cdot 10^{0}$		$1.15 \cdot 10^{1}$	
64	0.1250	0.2500	$3.06 \cdot 10^{-1}$	2.51	$3.46 \cdot 10^{0}$	1.74
256	0.0625	0.1250	$7.14 \cdot 10^{-2}$	2.1	$1.56 \cdot 10^{0}$	1.15
1,024	0.0313	0.0625	$1.73 \cdot 10^{-2}$	2.05	$7.55 \cdot 10^{-1}$	1.05
4,096	0.0156	0.0313	$4.28 \cdot 10^{-3}$	2.01	$3.75 \cdot 10^{-1}$	1.01

Table 4.4: $L^2(Q)$ and $H^1(Q)$ errors and eocs for formulation (4.10) on $Q = (0,1)\times(0,2)$ violating (4.9)

4.2.1 Singularities in the right–hand side

In order to see how the variational formulation (4.3) handles singularities in the right–hand side the test example

$$v(x,t) = \sin(\pi x)t^2(T-t)^{\frac{3}{4}},\ (x,t) \in Q,$$

N	h_x	h_t	$\|u_h - u\|_{L^2(Q)}$	eoc	$\|u_h - u\|_{H^1(Q)}$	eoc
16	0.7500	0.5000	$8.94 \cdot 10^{-1}$		$4.85 \cdot 10^{0}$	
64	0.3750	0.2500	$9.14 \cdot 10^{-2}$	3.29	$1.07 \cdot 10^{0}$	2.18
256	0.1875	0.1250	$8.79 \cdot 10^{-3}$	3.38	$2.34 \cdot 10^{-1}$	2.2
1,024	0.0938	0.0625	$1.10 \cdot 10^{-3}$	3	$5.58 \cdot 10^{-2}$	2.07

Table 4.5: $L^2(Q)$ and $H^1(Q)$ errors and eocs for formulation (4.10) on $Q = (0,3)\times(0,2)$ with $Q^2_{h;0;0,}(Q)$ as ansatz and test space

is considered. Note that $v \in H^{\frac{5}{4}-\varepsilon}(Q)$, $\varepsilon > 0$, and the singularity occurs for $t = T$. Table 4.6 shows stability of the variational formulation using the modified Hilbert transformation. Decreased L^2-convergence rate can be seen, as expected. Although, no H^1–convergence appears for linear ansatz and test functions table 4.7 shows that using ansatz and test space $Q^2_{h;0;0,}(Q)$ H^1-convergnce can be achieved with rates compatible to the stabilized method using $\overline{\mathcal{H}}_T$ as computed in [12, Table 4.13, p.172].

N	h_x	h_t	$\|u_h - u\|_{L^2(Q)}$	eoc	$\|u_h - u\|_{H^1(Q)}$	eoc
160	0.2500	0.2500	$2.02 \cdot 10^{1}$		$2.12 \cdot 10^{2}$	
640	0.1250	0.1250	$9.79 \cdot 10^{0}$	1.05	$1.81 \cdot 10^{2}$	0.23
2,560	0.0625	0.0625	$5.26 \cdot 10^{0}$	0.9	$1.86 \cdot 10^{2}$	$-4.32 \cdot 10^{-2}$
10,240	0.0313	0.0313	$2.78 \cdot 10^{0}$	0.92	$1.95 \cdot 10^{2}$	$-7.03 \cdot 10^{-2}$

Table 4.6: $L^2(Q)$ and $H^1(Q)$ errors and eocs for formulation (4.10) and test example v on $Q = (0,1) \times (0,10)$

N	h_x	h_t	$\|u_h - u\|_{L^2(Q)}$	eoc	$\|u_h - u\|_{H^1(Q)}$	eoc
160	0.2500	0.2500	$1.30 \cdot 10^{1}$		$8.72 \cdot 10^{1}$	
640	0.1250	0.1250	$8.11 \cdot 10^{0}$	0.68	$7.45 \cdot 10^{1}$	0.23
2,560	0.0625	0.0625	$4.31 \cdot 10^{0}$	0.91	$6.10 \cdot 10^{1}$	0.29
10,240	0.0313	0.0313	$2.03 \cdot 10^{0}$	1.08	$4.52 \cdot 10^{1}$	0.43

Table 4.7: $L^2(Q)$ and $H^1(Q)$ errors and eocs for formulation (4.10) with ansatz and test space $Q^2_{h;0;0,}(Q)$ and test example v on $Q = (0,1) \times (0,10)$

5 Conclusions

Although a precise theoretical treatment of the proposed variational formulation to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u, \partial_t \mathcal{H}_T v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \mathcal{H}_T v\rangle_{L^2(Q)} = \langle f, \mathcal{H}_T v\rangle_Q, \text{ for all } v \in H^{1,1}_{0;0,}(Q),$$

is still outstanding, the method seems suitable for the wave equation. This is indicated by Chapter 3, which shows that the variational formulation of the corresponding ordinary differential equation to find $u \in H^1_{0,}(0,T)$ such that

$$-\langle \partial_t u, \partial_t \mathcal{H}_T v\rangle_{L^2(0,T)} + \mu\langle u, \mathcal{H}_T v\rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v\rangle_{(0,T)}, \text{ for all } v \in H^1_{0,}(0,T),$$

is uniformely stable in $\mu > 0$, for a right–hand side $f \in [H^1_{,0}(0,T)]'$. Further, examples for the one–dimensional wave equation show stablity and optimal orders of convergence, where no CFL–condition is demanded.

A theoretical discussion is still crucial. Ongoing work is to show the observed error estimates for the approximation of the wave equation. Further, the proof of the uniform stability estimate in μ for the variational formulation of the ODE is needed. This goes hand in hand with proving the discrete inf–sup stability.

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