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Diophantine Equations and Linearly Recurrent Sequences

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Affidavit

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Preface

This doctoral thesis contains a collection of papers of the author. Eight of these are already published. All the details can be found in the Publication List following this preface. One paper *On Pillai's problem with k–generalized Fibonacci numbers and powers of* 3, has been accepted for publication in *International Journal of Number Theory*.

The structure of this thesis is as follows. After the introduction, there are four chapters and each corresponds to one publication. There is an appendix which also includes a collection of papers of the author, some of which were published before the actual PhD studies began while others are related papers that were published during the PhD studies. At the beginning of each of the chapters, more information about each publication can be found.

Publications of the author

1. Ddamulira, M. On a problem of Pillai with Fibonacci numbers and powers of 3. *Bol. Soc. Mat. Mex. (3),* 1–15 (2019).
2. Ddamulira, M. On the problem of Pillai with Padovan numbers and powers of 3. *Stud. Sci. Math. Hungar.* **56,** 364–379 (2019).
3. Ddamulira, M. On the problem of Pillai with Tribonacci numbers and powers of 3. *J. Integer Sequences* **22,** Art. 19.5.6, 14 pp (2019).
4. Ddamulira, M. On the x–coordinates of Pell equations that are products of two Lucas numbers. *Fibonacci Quart.* 1–20 (2019).
5. Ddamulira, M. & Luca, F. On the problem of Pillai with k–generalized Fibonacci numbers and powers of 3. *Int. J. Number Theory,* 1–24 (2019).
6. Ddamulira, M. & Luca, F. On the x–coordinates of Pell equations which are k–generalized Fibonacci numbers. *J. Number Theory* **207,** 156–195 (2020).
7. Ddamulira, M., Luca, F. & Rakotomalala, M. Fibonacci numbers which are products of two Pell numbers. *Fibonacci Quart.* **54,** 11–18 (2016).
8. Ddamulira, M., Luca, F. & Rakotomalala, M. On a problem of Pillai with Fibonacci numbers and powers of 2. *Proc. Indian Acad. Sci. Math. Sci.* **127,** 411–421 (2017).
9. Ddamulira, M., Gómez, C. A. & Luca, F. On a problem of Pillai with k–generalized Fibonacci numbers and powers of 2. *Monatsh. Math.* **187,** 635–664 (2018).

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Abstract

The field of transcendence has a variety of subfields including: the transcendence of individual numbers, algebraic independence, transcendence of functions (for example, modular forms, the zeta and j functions, etc.) at particular values, and applications to Diophantine equations. This thesis applies the methods of transcendence to solving Diophantine equations which involve the linearly recurrent sequences (for example, Fibonacci numbers, Lucas numbers, Tribonacci numbers, Pell numbers, Padovan numbers, and the k–generalized Fibonacci numbers).

Contents

1. Introduction

1.1. Linear forms in logarithms and Baker's method

We start by recalling results from the theory of transcendence, and then work towards lower bounds for linear forms in logarithms of algebraic numbers, which are of great importance in effectively solving Diophantine equations. For further details and proofs we refer the reader to the book of Baker [10], and that of Shorey and Tijdeman [58].

First, we state the transcendence result proved in 1934, independently by Gelfond – a Russian mathematician and Schneider – a German mathematician, and thereby solved the famous seventh problem of Hilbert.

A transcendental number is a real number or complex number that is not an algebraic number – that is, a number that is not a root of a nonzero polynomial with integer coefficients. The well-known examples of transcendental numbers are π and e.

Theorem 1.1.1 (Gelfond–Schneider)**.** *Let α, β be algebraic numbers in $\mathbb{C}$, with $\alpha \neq 0,\ 1$ and $\beta \notin \mathbb{Q}$. Then α^{β} is transcendental.*

In Theorem 1.1.1, $\alpha^{\beta} := e^{\beta \log \alpha}$, where $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\log \alpha = \log|\alpha| + i \arg \alpha$. The argument of α is determined only up to a multiple of 2π. Thus, $\log \alpha$ and hence α^{β} are multi-valued. Theorem 1.1.1 holds for any value of $\arg \alpha$. We state some of the immediate corollaries.

Corollary 1.1.2. *Let α be a complex algebraic number with $i\alpha \notin \mathbb{Q}$. Then $e^{\pi\alpha}$ is transcendental.*

Corollary 1.1.3. *Let α, β be algebraic numbers in $\mathbb{C}$, with $\alpha, \beta \neq 0, 1$ such that $\log \alpha$ and $\log \beta$ are linearly independent over $\mathbb{Q}$. Then, for all nonzero algebraic numbers γ and η in $\mathbb{C}$ we have that $\gamma \log \alpha + \eta \log \beta \neq 0$.*

Next, we state a result of Baker in 1966, which is a generalization of Corollary 1.1.3 to linear forms in an arbitrary number of logarithms of algebraic numbers.

Theorem 1.1.4 (Baker)**.** *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers in $\mathbb{C}$ which are different from $0, 1$ and such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$. Then, for every tuple $(\beta_0, \beta_1, \ldots, \beta_n)$ of algebraic numbers in $\mathbb{C}$ different from $(0, 0, \ldots, 0)$ we have*

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n \neq 0.$$

For applications to Diophantine equations, it is important that not only the above linear form is nonzero, but also that we have a strong enough lower bound for the absolute value of this linear form. Below we state a result of Baker in 1975, which is a special case where $\beta_0 = 0$ and $\beta_1, \ldots, \beta_n$ are rational integers.

Theorem 1.1.5 (Baker). *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers in $\mathbb{C}$ different from $0, 1$. Furthermore, let $b_1, \ldots, b_n$ be rational integers such that*

$$b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \neq 0.$$

Then

$$|b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n| \geq (eB)^{-C},$$

where $B := \max\{|b_1|, \ldots, |b_n|\}$ and C is an effectively computable constant depending only on n and on $\alpha_1, \ldots, \alpha_n$.

The statement C is an effectively computable constant means that by going through the proof of Theorem 1.1.5 one can compute an explicit value of C. It is also possible to get rid of the logarithms. Then, Theorem 1.1.5 leads to the following corollary.

Corollary 1.1.6. *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers in $\mathbb{C}$ different from $0, 1$ and let $b_1, \ldots, b_n$ be rational integers such that*

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n} \neq 1.$$

Then

$$\left| \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \right| \geq (eB)^{-C'},$$

where $B := \max\{|b_1|, \ldots, |b_n|\}$ and C' is an effectively computable constant depending only on n and on $\alpha_1, \ldots, \alpha_n$.

For completeness, we give the result of Matveev [50], which is a completely explicit version of Corollary 1.1.6 in the case that $\alpha_1, \ldots, \alpha_n$ are rational numbers. The *height* of a rational number $\alpha = p/q$ with $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$, is defined by $H(\alpha) := \max\{|p|, |q|\}$.

Theorem 1.1.7 (Matveev). *Let $\alpha_1, \ldots, \alpha_n$ be nonzero rational numbers and let $b_1, \ldots, b_n$ be integers such that*

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n} \neq 1.$$

Then

$$\left| \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \right| \geq (eB)^{-C'},$$

where $B := \max\{|b_1|, \ldots, |b_n|\}$ and $C' := \frac{1}{2} e \cdot 30^{n+3} \cdot n^{4.5} \prod_{i=1}^{n} \max\{1, \log H(\alpha_i)\}$.

To prove our main results, we need to use several times a Baker-type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such in the literature like that of Baker and Wüstholz from [12]. We recall the result of Bugeaud, Mignotte, and Siksek ([23], Theorem 9.4, pp. 989), which is a modified version of the result of Matveev [50]. This result is one of our main tools in this thesis.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(x-\eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|,1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p,q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|,q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next chapters of this thesis without reference:

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\le h(\eta_1) + h(\eta_2) + \log 2,\\ h(\eta_1\eta_2^{\pm 1}) &\le h(\eta_1) + h(\eta_2),\\ h(\eta^s) &= |s|h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned} \tag{1.1}$$

For the proofs of (1.1) and further details, we refer the reader to the book of Bombieri and Gubler [14].

With the above notation, Matveev [50] proved the following theorem.

Theorem 1.1.8 (Matveev according to Bugeaud, Mignotte, Siksek)**.** *Let $\eta_1,\ldots,\eta_t$ be positive real numbers in a number field $\mathbb{K}$ of degree D, let $b_1,\ldots,b_t$ be nonzero integers, and assume that*

$$\Lambda := \eta_1^{b_1}\cdots\eta_t^{b_t} - 1, \tag{1.2}$$

is nonzero. Then

$$\log|\Lambda| > -1.4\times 30^{t+3}\times t^{4.5}\times D^2(1+\log D)(1+\log B)A_1\cdots A_t,$$

where

$$B \ge \max\{|b_1|,\ldots,|b_t|\},$$

and

$$A_i \ge \max\{Dh(\eta_i), |\log\eta_i|, 0.16\}, \qquad \textit{for all} \qquad i = 1,\ldots,t.$$

When $t = 2$ and η_1, η_2 are positive and multiplicatively independent, we can use a result of Laurent, Mignotte, and Nesterenko [44]. Namely, let in this case B_1, B_2 be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\eta_i), \frac{|\log \eta_i|}{D}, \frac{1}{D}\right\}, \qquad \text{for} \quad i = 1, 2,$$

and put

$$b' := \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.$$

Put

$$\Gamma := b_1 \log \eta_1 + b_2 \log \eta_2. \tag{1.3}$$

We note that $\Gamma \neq 0$ because η_1 and η_2 are multiplicatively independent. The following result is due to Laurent, Mignotte, and Nesterenko ([44], Corollary 2, pp. 288).

Theorem 1.1.9 (Laurent, Mignotte, Nesterenko)**.** *With the above notations, assuming that η_1, η_2 are positive and multiplicatively independent, then*

$$\log |\Gamma| > -24.34 D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2. \tag{1.4}$$

Note that with Γ given by (1.3), we have $e^\Gamma - 1 = \Lambda$, where Λ is given by (1.2) in case $t = 2$, which explains the connection between Theorem 1.1.8 and Theorem 1.1.9.

1.2. Reduction procedure and continued fractions

During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classical result in the theory of Diophantine approximation.

Lemma 1.2.1 (Legendre)**.** *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction of τ, and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [30], Lemma 5a). The proof is almost identical to the proof of the corresponding result in [30] and the details have been worked out in Lemma 2.9 in [16]. For a real number X, we write $\|X\| := \min\{|X-n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 1.2.2 (Dujella, Pethő)**.** *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let $\varepsilon := \|\mu q\| - M\|\tau q\|$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms in logarithms with bounded integer coefficients in three and four variables. In this case, we use the Lenstra-Lenstra-Lovász lattice basis reduction algorithm (LLL algorithm) that we describe below. Let $\tau_1, \tau_2, \ldots \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \leq X_i. \tag{1.5}$$

We put $X := \max\{X_i\}$, $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \quad \text{for} \quad 1 \leq j \leq t-1 \quad \text{and} \quad \mathbf{b}_t := \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 1.2.3 (LLL algorithm)**.** *Let $X_1, X_2, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ is a fixed sufficiently large constant. With the above notation on the lattice Ω, we consider a reduced base $\{\boldsymbol{b}_i\}$ to Ω and its associated Gram-Schmidt orthogonalization base $\{\boldsymbol{b}_i^*\}$. We set*

$$c_1 := \max_{1 \leq i \leq t} \frac{||\boldsymbol{b}_1||}{||\boldsymbol{b}_i^*||}, \quad \theta := \frac{||\boldsymbol{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2, \quad \textit{and} \quad R := \frac{1}{2}\left(1 + \sum_{i=1}^{t} X_i\right).$$

If the integers x_i are such that $|x_i| \leq X_i$, for $1 \leq i \leq t$ and $\theta^2 \geq Q + R^2$, then we have

$$\left|\sum_{i=1}^{t} x_i\tau_i\right| \geq \frac{\sqrt{\theta^2 - Q} - R}{C}.$$

For the proof and further details, we refer the reader to the book of Cohen. (Proposition 2.3.20 in [27], pp. 58–63).

1.3. The k–generalized Fibonacci sequences

In this section, we recall some of the facts and properties of the k–generalized Fibonacci sequences which will be used later in Chapter 2, Chapter 3, and Chapter 4.

Let $k \geq 2$ be an integer. We consider a generalization of Fibonacci sequence called the k–generalized Fibonacci sequence $\{F_n^{(k)}\}_{n\geq 2-k}$ defined as

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \tag{1.6}$$

with the initial conditions

$$F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0 \quad \text{and} \quad F_1^{(k)} = 1.$$

We call $F_n^{(k)}$ the nth k–generalized Fibonacci number. Note that when $k = 2$, it is the classical Fibonacci number (n-th term, which is denoted by F_n), when $k = 3$ it is the Tribonacci number (n-th term, which is denoted by T_n), and so on.

The first direct observation is that the first $k+1$ nonzero terms in $F_n^{(k)}$ are powers of 2, namely

$$F_1^{(k)} = 1,\ F_2^{(k)} = 1,\ F_3^{(k)} = 2,\ F_4^{(k)} = 4, \ldots, F_{k+1}^{(k)} = 2^{k-1},$$

while the next term in the above sequence is $F_{k+2}^{(k)} = 2^k - 1$. Thus, we have that

$$F_n^{(k)} = 2^{n-2} \qquad \text{holds for all} \quad 2 \leq n \leq k+1. \tag{1.7}$$

We also observe that the recursion (1.6) implies the three–term recursion

$$F_n^{(k)} = 2F_{n-1}^{(k)} - F_{n-k-1}^{(k)} \qquad \text{for all } n \geq 3 \tag{1.8}$$

which shows that the k–Fibonacci sequence grows at a rate less than 2^{n-2}. In fact, the inequality $F_n^{(k)} < 2^{n-2}$ holds for all $n \geq k+2$ (see [17], Lemma 2).

It is known that the characteristic polynomial of the k–generalized Fibonacci numbers $F^{(k)} := \{F_n^{(k)}\}_{n\geq 0}$, namely

$$\Psi_k(x) := x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle. Let $\alpha := \alpha(k)$ denote that single root, which is located between $2\left(1-2^{-k}\right)$ and 2 (see [29]). To simplify notation, in our application we shall omit the dependence on k of α. We shall use $\alpha^{(1)}, \ldots, \alpha^{(k)}$ for all roots of $\Psi_k(x)$ with the convention that $\alpha^{(1)} := \alpha$.

We now consider for an integer $k \geq 2$, the function

$$f_k(z) = \frac{z-1}{2+(k+1)(z-2)} \qquad \text{for} \quad z \in \mathbb{C}. \tag{1.9}$$

With this notation, Dresden and Du presented in [29] the following "Binet–like" formula for the terms of $F^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}) {\alpha^{(i)}}^{n-1}. \tag{1.10}$$

It was proved in [29] that the contribution of the roots which are inside the unit circle to the formula (1.10) is very small, namely that the approximation

$$\left| F_n^{(k)} - f_k(\alpha)\alpha^{n-1} \right| < \frac{1}{2} \quad \text{holds for all } n \geqslant 2-k. \tag{1.11}$$

When $k = 2$, one can easily prove by induction that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{for all} \quad n \geq 1. \tag{1.12}$$

It was proved by Bravo and Luca in [17] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \tag{1.13}$$

holds for all $n \geq 1$ and $k \geq 2$, which shows that (1.12) holds for the k–generalized Fibonacci numbers as well. The observations made from the expressions (1.10) to (1.13) lead us to call α the *dominant root* of $F^{(k)}$.

Before we conclude this section, we present some useful lemmas that will be used in the next chapters of this thesis. The following lemma was proved by Bravo and Luca in [17].

Lemma 1.3.1. *For $k \geq 2$, let α be the dominant root of $F^{(k)}$, and consider the function $f_k(z)$ defined in* (1.9)*. Then:*

(i) Inequalities

$$\frac{1}{2} < f_k(\alpha) < \frac{3}{4} \quad \textit{and} \quad |f_k(\alpha^{(i)})| < 1,\ 2 \leq i \leq k$$

hold. So, the number $f_k(\alpha)$ is not an algebraic integer.

(ii) The logarithmic height of $f_k(\alpha)$ satisfies $h(f_k(\alpha)) < 3\log k$.

Next, we present a useful lemma which is a result due to Cooper and Howard [28].

Lemma 1.3.2. *For $k \geq 2$ and $n \geq k+2$,*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j} 2^{n-(k+1)j-2},$$

where

$$C_{n,j} := (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

In the above, we have denoted by $\lfloor x \rfloor$ the greatest integer less than or equal to x and used the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of a or b is negative.

Before going further, let us see some particular cases of Lemma 1.3.2.

Example 1.3.1. *(i) Assume that $m \in [2, k+1]$. Then $1 < \frac{m+k}{k+1} < 2$, so $\lfloor \frac{m+k}{k+1} \rfloor = 1$. In this case,*

$$F_m^{(k)} = 2^{m-2},$$

a fact which we already knew.

(ii) Assume that $m \in [k+2, 2k+2]$. Then $2 \leq \frac{m+k}{k+1} < 3$, so $\lfloor \frac{m+k}{k+1} \rfloor = 2$. In this case,

$$\begin{aligned} F_m^{(k)} &= 2^{m-2} + C_{m,1} 2^{m-(k+1)-2} \\ &= 2^{m-2} - \left(\binom{m-k}{1} - \binom{m-k-2}{-1} \right) 2^{m-k-3} \\ &= 2^{m-2} - (m-k) \cdot 2^{m-k-3}. \end{aligned}$$

(iii) Assume that $m \in [2k+3, 3k+3]$. Then $3 \leq \frac{m+k}{k+1} < 4$, so $\lfloor \frac{m+k}{k+1} \rfloor = 3$. In this case,

$$\begin{aligned} F_m^{(k)} &= 2^{m-2} + C_{m,1} 2^{m-(k+1)-2} + C_{m,2} 2^{m-2(k+1)-2} \\ &= 2^{m-2} - (m-k) 2^{m-k-3} + \left(\binom{m-2k}{2} - \binom{m-2k-2}{0} \right) 2^{m-2k-4} \\ &= 2^{m-2} - (m-k) 2^{m-k-3} + \left(\frac{(m-2k)(m-2k-1)}{2} - 1 \right) 2^{m-2k-4} \\ &= 2^{m-2} - (m-k) \cdot 2^{m-k-3} + (m-2k+1)(m-2k-2) \cdot 2^{m-2k-5}. \end{aligned}$$

Gómez and Luca in [34] derived from the Cooper and Howard's formula the following asymptotic expansion of $F_m^{(k)}$ valid when $2 \leq m < 2^k$.

Lemma 1.3.3 (Gómez, Luca)**.** *If $m < 2^k$, then the following estimate holds:*

$$F_m^{(k)} = 2^{m-2} \left(1 + \delta_1(m) \frac{k-m}{2^{k+1}} + \delta_2(m) \frac{f(k,m)}{2^{2k+2}} + \zeta(k,m) \right), \tag{1.14}$$

where $f(k,m) := \frac{1}{2}(z-1)(z+2)$; $z = 2k - m$, $\zeta := \zeta(k,m)$ is a real number satisfying

$$|\zeta| < \frac{4m^3}{2^{3k+3}},$$

and $\delta_i(m)$ is the characteristic function of the set $\{m > i(k+1)\}$ for $i = 1, 2$.

1.4. The problem of Pillai

In this section we introduce the problem of Pillai that is studied in Chapter 2 and Chapter 3 for the k–generalized Fibonacci numbers with powers of 2, and with powers 3, respectively and the appendix chapters where it is studied for Fibonacci numbers with powers of 2, Fibonacci numbers with powers of 3, Tribonacci numbers with powers of 3, and Padovan numbers with powers of 3.

A perfect power is a positive integer of the form a^x where $a > 1$ and $x \geq 2$ are integers. Pillai wrote several papers on these numbers. In 1936 and again in 1945 (see [55, 56]), he conjectured that for any given integer $c \geq 1$, the number of positive integer solutions (a,b,x,y), with $x \geq 2$ and $y \geq 2$, to the Diophantine equation

$$a^x - b^y = c, \tag{1.15}$$

is finite. This conjecture, which is still open for all $c \neq 1$, amounts to saying that the distance between two consecutive terms in the sequence of all perfect powers tends to infinity. The case $c = 1$ is Catalan's conjecture which states that the only solution in positive integers to (1.15) for $a,b > 0$, $x,y > 1$ is $x = 2,\ a = 3,\ y = 3,\ b = 2$. This conjecture was proved by Mihăilescu [53]. In 1936 (see [55, 56]), in the special case $(a,b) = (3,2)$ which is a continutation of the work of Herschfeld [38, 39] in 1935, Pillai conjectured that the only integers c admitting at least two representations of the form $2^x - 3^y$ are given by

$$2^3 - 3^2 = 2^1 - 3^1 = -1, \quad 2^5 - 3^3 = 2^3 - 3^1 = 5, \quad 2^8 - 3^5 = 2^4 - 3^1 = 13. \tag{1.16}$$

This was confirmed by Stroeker and Tijdeman [59] in 1982. For small $|c|$ this is not the case. Pillai (see [55, 56]) extended Herschfeld's result to the more general exponential Diophantine equation (1.15) with fixed integers a,b,c with $\gcd(a,b) = 1$ and $a > b \geq 1$. Specifically, Pillai showed that there exists a positive integer $c_0(a,b)$ such that, for $|c| > c_0(a,b)$, equation (1.15) has at most one integer solution (x,y). The general problem of Pillai is difficult to solve and this has motivated the consideration of special cases of this problem. In the past years, several special cases of the problem of Pillai have been studied. Work in this direction began when together with Luca and Rakotomala [8], we studied a variant of (1.15) and replaced a^x with the sequence of Fibonacci numbers F_n and b^y with powers of 2. Futher details on this problem are given in Appendix B. This problem is further studied in Chapter 2 and Chapter 3 for the k–generalized Fibonacci numbers. Other related problems have been studied in [19, 25, 26, 33, 36, 37].

1.5. Pell equations and Dickson polynomials

In this section, we recall some of the facts and properties of Pell equations and Dickson polynomials which will be used later in Chapter 4 and Chapter 5.

Let $d \geq 2$ be a positive integer which is not a perfect square. It is well known that the Pell equation

$$x^2 - dy^2 = \pm 1 \tag{1.17}$$

has infinitely many positive integer solutions (x, y). By putting (x_1, y_1) for the smallest positive solution, all solutions are of the form (x_k, y_k) for some positive integer k, where

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k \qquad \text{for all} \quad k \geq 1. \tag{1.18}$$

Furthermore, the sequence $\{x_k\}_{k\geq 1}$ is binary recurrent. In fact, the following formula

$$x_k = \frac{(x_1 + y_1\sqrt{d})^k + (x_1 - y_1\sqrt{d})^k}{2},$$

holds for all positive integers k.

We put $\delta := x_1 + \sqrt{x_1^2 - \varepsilon}$ for the minimal positive integer x_1 such that

$$x_1^2 - dy_1^2 = \varepsilon, \qquad \varepsilon \in \{\pm 1\}$$

for some positive integer y_1. Then,

$$x_n + \sqrt{d}y_n = \delta^n \qquad \text{and} \qquad x_n - \sqrt{d}y_n = \sigma^n, \qquad \text{where} \qquad \sigma := \varepsilon\delta^{-1}.$$

From the above, we get

$$2x_n = \delta^n + (\varepsilon\delta^{-1})^n \qquad \text{for all} \qquad n \geq 1. \tag{1.19}$$

There is a formula expressing $2x_n$ in terms of $2x_1$ by means of the Dickson polynomial $D_n(2x_1, \varepsilon)$, where

$$D_n(x, \nu) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i}\binom{n-i}{i}(-\nu)^i x^{n-2i}.$$

These polynomials appear naturally in many number theory problems and results, most notably in a result of Bilu and Tichy [13] concerning polynomials $f(X), g(X) \in \mathbb{Z}[X]$ such that the Diophantine equation $f(x) = g(y)$ has infinitely many integer solutions (x, y).

Example 1.5.1. *(i)* $n = 2$. *We have*

$$2x_2 = \sum_{i=0}^{1} \frac{2}{2-i}\binom{2-i}{i}(-\varepsilon)^i(2x_1)^{2-2i} = 4x_1^2 - 2\varepsilon, \quad \textit{so} \quad x_2 = 2x_1^2 - \varepsilon.$$

(ii) $n = 3$. *We have*

$$2x_3 = \sum_{i=0}^{1} \frac{3}{3-i}\binom{3-i}{i}(-\varepsilon)^i(2x_1)^{3-2i} = (2x_1)^3 - 3\varepsilon(2x_1), \quad \textit{so} \quad x_3 = 4x_1^3 - 3\varepsilon x_1.$$

(iii) $n \geq 4$. *We have*

$$\begin{aligned} 2x_n &= \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i}\binom{n-i}{i}(-\varepsilon)^i(2x_1)^{n-2i} \\ &= (2x_1)^n - n\varepsilon(2x_1)^{n-2} + \frac{n(n-3)}{2}(2x_1)^{n-4} + \sum_{i\geq 3}^{\lfloor n/2 \rfloor} \frac{n(-\varepsilon)^i}{n-i}\binom{n-i}{i}(2x_1)^{n-2i}. \end{aligned}$$

Finally, the following lemma is also useful. It is Lemma 7 in [35].

Lemma 1.5.1 (Gúzman Sánchez, Luca)**.** *If* $m \geqslant 1$, $T > (4m^2)^m$, *and* $T > x/(\log x)^m$, *then*

$$x < 2^m T(\log T)^m.$$

2. On a problem of Pillai with k–generalized Fibonacci numbers and powers of 2

The presentation in this chapter is a slightly modified version of the paper [9] with the title *On a problem of Pillai with k–generalized Fibonacci numbers and powers of* 2. This is a joint work with *Carlos Alexis Gómez Ruiz* and *Florian Luca*. The article has been published in *Monatshefte für Mathematik* in December, 2018.

Abstract: In this paper, we find all integers c having at least two representations as a difference between a k–generalized Fibonacci number and a powers of 2 for any fixed $k \geq 4$. This paper extends previous work from [8] for the case $k = 2$ and [19] for the case $k = 3$.

Keywords: Diophantine equations; Pillai's problem; Generalized Fibonacci sequence; Reduction method.

2010 Mathematics Subject Classification: 11D61, 11B39, 11D45, 11J86.

2.1. Introduction

Recently, Ddamulira, Luca, and Rakotomalala [8] considered the Diophantine equation

$$F_n - 2^m = c, \tag{2.1}$$

where c is a fixed integer and $\{F_n\}_{n\geqslant 0}$ is the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 0$. This type of equation can be seen as a variation of Pillai's equation. Ddamulira, et al. proved that the only integers c having at least two representations of the form $F_n - 2^m$ are contained in the set $\mathscr{C} = \{0, -1, 1, -3, 5, -11, -30, 85\}$. Moreover, they computed for each $c \in \mathscr{C}$ all representations of the form (2.1).

Bravo, Luca, and Yazán [19] considered the Diophantine equation

$$T_n - 2^m = c, \tag{2.2}$$

where c is a fixed integer and $\{T_n\}_{n\geqslant 0}$ is the sequence of Tribonacci numbers given by $T_0 = 0$, $T_1 = 1$, $T_2 = 1$, and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geqslant 0$. In their paper, Bravo, et al. proved that the only integers c having at least two representations of the form $T_n - 2^m$ are contained in the set $\mathscr{C} = \{0, -1, -3, 5, -8\}$. In fact, each $c \in \mathscr{C}$ has exactly two representations of the form (2.2).

In the same spirit, Chim, Pink, and Ziegler [25] considered the Diophantine equation

$$F_n - T_m = c, \tag{2.3}$$

where c is a fixed integer. They proved that the only integers c having at least two representations of the form $F_n - T_m$ are contained in the set

$$\mathscr{C} = \{0, 1, -1, -2, -3, 4, -5, 6, 8, -10, 11, -11, -22, -23, -41, -60, -271\}.$$

In particular, they computed for each $c \in \mathscr{C}$ all representations of the form (2.3), showing that each $c \in \mathscr{C}$ has at most four representations.

The purpose of this paper is to generalize the previous results corresponding to (2.1) and (2.2). Let $k \geqslant 2$ be an integer. We consider the k–generalized Fibonacci sequence $\{F_n^{(k)}\}_{n\geqslant 2-k}$ defined in Section 1.3.

In this paper, we find all integers c admitting at least two representations of the form $F_n^{(k)} - 2^m$ for some positive integers k, n and m. This can be interpreted as solving the equation

$$F_n^{(k)} - 2^m = F_{n_1}^{(k)} - 2^{m_1} \quad (= c) \tag{2.4}$$

with $(n, m) \neq (n_1, m_1)$. As we already mentioned, the cases $k = 2$ and $k = 3$ have been solved completely by Ddamulira, et al. [8] and Bravo, et al. [19], respectively. So, we focus on the case $k \geqslant 4$.

We prove the following theorem:

Theorem 2.1.1. *Assume that $k \geq 4$. Then equation (2.4) with $n > n_1 \geq 2$, $m > m_1 \geq 0$ has the following families of solutions (c, n, m, n_1, m_1).*

(i) In the range $2 \leq n_1 < n \leq k+1$, we have the following solution:

$$(0, s, s-2, t, t-2) \qquad \textit{for} \quad 2 \leq t < s \leq k+1.$$

(ii) In the ranges $2 \leq n_1 \leq k+1$ and $k+2 \leq n \leq 2k+2$, we have the following solutions:

(a) when $n_1 = n-1$:

$$\left(2^{k-1} - 1, k+2, k-1, k+1, 0\right).$$

(b) *when* $n_1 < n-1$*:*

$$\left(2^{\gamma}-2^{\rho},k+2^{a}-2^{b},k+2^{a}-2^{b}-2,\gamma+2,\rho\right),$$

with $\gamma = b-3+2^a-2^b$ *and* $\rho = a-3+2^a-2^b$*, where* $a > b \geq 0$*,* $(a,b) \neq (1,0)$ *and* $\gamma+3 \leq k+2$*.*

(iii) *In the range* $k+2 \leq n_1 < n \leq 2k+2$*, we have the following solutions: if the integer a is maximal such that* $2^a \leq k+2$ *satisfies* $a+2^a = k+1+2^b$ *for some positive integer b, then*

$$(-2^{a+2^a-3},k+2^a,k+2^a-2,k+2^b,b+2^b-3).$$

(iv) *If* $n = 2k+3$*, and additionally* $k = 2^t-3$ *for some integer* $t \geq 3$*, then:*

$$(1-2^{t+2^t-3},2^{t+1}-3,2^{t+1}-5,2,t+2^t-3).$$

Equation (2.4) *has no solutions with* $n > 2k+3$*.*

2.2. Parametric families of solutions

Assume that $(n,m) \neq (n_1,m_1)$ are such that

$$F_n^{(k)} - 2^m = F_{n_1}^{(k)} - 2^{m_1}.$$

If $m = m_1$, then $F_n^{(k)} = F_{n_1}^{(k)}$ and since $\min\{n,n_1\} \geq 2$, we get that $n = n_1$. Thus, $(n,m) = (n_1,m_1)$, contradicting our assumption. Hence, $m \neq m_1$, and we may assume without loss of generality that $m > m_1 \geq 0$. Since

$$F_n^{(k)} - F_{n_1}^{(k)} \quad = \quad 2^m - 2^{m_1}, \tag{2.5}$$

and the right–hand side of (2.5) is positive, we get that the left–hand side of (2.5) is also positive and so $n > n_1$. Thus, since $F_1^{(k)} = F_2^{(k)} = 1$, we may assume that $n > n_1 \geq 2$.

We analyze the possible situations.

Case 2.2.1. *Assume that* $2 \leq n_1 < n \leq k+1$.

Then, by (1.7), we have

$$F_{n_1}^{(k)} = 2^{n_1-2} \qquad \text{and} \qquad F_n^{(k)} = 2^{n-2}$$

so, by substituting in (2.5), we get

$$2^m - 2^{m_1} = 2^{n-2} - 2^{n_1-2}.$$

The number on the left–hand side of the above equation is $2^{m-1}+\cdots+2^{m_1}$ and the number on the right–hand side is $2^{n-3}+\cdots+2^{n_1-2}$. So, by the uniqueness of the binary representation we have $m=n-2$ and $m_1=n_1-2$, giving $c=0$. All powers of 2 in the k–generalized Fibonacci sequence are known to be just the numbers $F_s^{(k)}$ with $1\leq s\leq k+1$ (see [17]). This gives (i) from the statement of Theorem 2.1.1.

From now on, we assume that $c\neq 0$.

Case 2.2.2. *Assume that* $2\leq n_1\leq k+1$ *and* $k+2\leq n\leq 2k+2$.

Then, by (1.7) and Example 1.3.1 (ii), we have

$$F_{n_1}^{(k)}=2^{n_1-2}\qquad\text{and}\qquad F_n^{(k)}=2^{n-2}-(n-k)\cdot 2^{n-k-3}.$$

So, by substituting in (2.5) as before, we get

$$2^{n-2}-2^{n_1-2}-(n-k)\cdot 2^{n-k-3}=2^m-2^{m_1}. \tag{2.6}$$

In the left–hand side of the above equation, we have

$$2^{n-2}-2^{n_1-2}-(n-k)\cdot 2^{n-k-3}\geq 2^{n-3}-(n-k)\cdot 2^{n-k-3}>2^{n-4}.$$

Indeed the last inequality is equivalent to $2^{n-4}>(n-k)\cdot 2^{n-k-3}$, or $2^{k-1}>n-k$. Since $n\leq 2k+2$, it suffices that $2^{k-1}>k+2$, which indeed holds for all $k\geq 4$. Furthermore, unless $n_1=n-1$, we have

$$2^{n-2}-2^{n_1-2}-(n-k)\cdot 2^{n-k-3}\geq 2^{n-2}-2^{n-4}-(n-k)\cdot 2^{n-k-3}>2^{n-3},$$

from the preceding argument. Thus, we have either $n_1=n-1$ and then

$$2^{n-3}\geq 2^{n-2}-2^{n_1-2}-(n-k)\cdot 2^{n-k-3}>2^{n-4},$$

which leads to

$$2^{n-3}\geq 2^m-2^{m_1}>2^{n-4},$$

showing that $m=n-3$, or $n_1<n-1$, in which case

$$2^{n-2}>2^{n-2}-2^{n_1-2}-(n-k)\cdot 2^{n-k-3}>2^{n-3},$$

showing that $m=n-2$.

We study the two cases. When $n_1=n-1$, then since $n_1\leq k+1$, it follows that $n\leq k+2$. Since in fact $n\geq k+2$, we get $n=k+2$. Then $m=n-3=k-1$, so from (2.6)

$$2^{k-1}-2^{m_1}=2^m-2^{m_1}=2^k-2^{k-1}-2\cdot 2^{-1}=2^{k-1}-1,$$

showing that $m_1 = 0$. So, we have found the parametric family

$$(n, m, n_1, m_1) = (k+2, k-1, k+1, 0)$$

for which $c = 2^{k-1} - 1$ according to (2.4). This corresponds to situation (ii–a) in the statement of Theorem 2.1.1.

A different possibility is $n_1 < n-1$, in which case $m = n-2$. Now (2.6) leads to

$$2^{n-2} - 2^{n_1-2} - (n-k) \cdot 2^{n-k-3} = 2^{n-2} - 2^{m_1}$$

so

$$(n-k) \cdot 2^{n-k-3} = 2^{m_1} - 2^{n_1-2}.$$

Simplifying the powers of 2, we get

$$n-k = 2^{m_1-(n-k)+3} - 2^{n_1-(n-k)+1}.$$

Thus, $n-k \in [2, k+2]$ is a difference of two powers of 2. Take any number in $[2, k+2]$ which is a difference of two powers of 2. Let it be $2^a - 2^b$. Note that $a > b$ and $(a, b) \neq (1, 0)$. Set

$$n-k = 2^a - 2^b.$$

This gives $n = k + 2^a - 2^b \in [k+2, 2k+2]$. Next we have $n_1 - (n-k) + 1 = b$. Then $n_1 = b + (n-k) - 1$. But $n_1 \leq k+1$. This gives $(b-1) + (n-k) \leq k+1$, so $(b-1) + 2^a - 2^b \leq k+1$. But we started with $2^a - 2^b \in [2, k+2]$. So, in fact we get

$$b + 2^a - 2^b \leq k+2$$

and $2^a - 2^b \geq 2$. If $n-k = 2$, then $(a, b) = (2, 1)$, otherwise $n-k \geq 3$ and $b \geq 0$. Finally, $m_1 + 3 - (n-k) = a$. Thus,

$$m_1 = (a-3) + (n-k) = (a-1) + ((n-k)-2)$$

and this is nonnegative from the preceding discussion. So, the family is

$$(n, m, n_1, m_1) = (k + 2^a - 2^b, k + 2^a - 2^b - 2, b - 1 + 2^a - 2^b, a - 3 + 2^a - 2^b),$$

where (a, b) are such that $a > b \geq 0$, $(a, b) \neq (1, 0)$, and $b + 2^a - 2^b \leq k+2$. Furthermore, by (2.4), we have $c = 2^{b-3+2^a-2^b} - 2^{a-3+2^a-2^b}$. This corresponds to situation (ii–b) in the statement of Theorem 2.1.1.

Case 2.2.3. *Assume that $k+2 \leq n_1 < n \leq 2k+2$.*

Then, by Exapmle 1.3.1 (ii), we have that

$$F_{n_1}^{(k)} = 2^{n_1-2} - (n_1-k)\cdot 2^{n_1-k-3} \qquad \textit{and} \qquad F_n^{(k)} = 2^{n-2} - (n-k)\cdot 2^{n-k-3}.$$

Then by a similar substitution as before, equation (2.5) translates into

$$2^{n-2} - 2^{n_1-2} - \Big((n-k)\cdot 2^{n-k-3} - (n_1-k)\cdot 2^{n_1-k-3}\Big) = 2^m - 2^{m_1}. \tag{2.7}$$

Since $n_1 \le n-1$, the left–hand side is at least

$$\begin{aligned} 2^{n-2} - 2^{n_1-2} &- ((n-k)\cdot 2^{n-k-3} - (n_1-k)\cdot 2^{n_1-k-3}) \\ &\ge 2^{n-2} - 2^{n-3} - ((n-k)\cdot 2^{n-k-3} - (n-k-1)\cdot 2^{n-k-4}) \\ &= 2^{n-3} - (n-k+1)\cdot 2^{n-k-4} > 2^{n-4}, \end{aligned}$$

where the last inequality is equivalent to

$$2^{n-4} > (n-k+1)\cdot 2^{n-k-4},$$

or

$$2^k > n-k+1.$$

Since $n-k \le k+2$, it suffices that $2^k > k+2+1 = k+3$, which holds for $k \ge 4$. Thus, if $n_1 = n-1$, then

$$2^{n-3} > 2^{n-2} - 2^{n-3} - \Big((n-k)\cdot 2^{n-k-3} - (n-k-1)\cdot 2^{n-k-4}\Big) > 2^{n-4},$$

so

$$2^{n-3} > 2^m - 2^{m_1} > 2^{n-4},$$

giving $m = n-3$. In this case, we get from (2.7),

$$2^{n-2} - 2^{n-3} - (n-k+1)\cdot 2^{n-k-4} = 2^{n-3} - 2^{m_1},$$

so

$$(n-k+1)\cdot 2^{n-k-4} = 2^{m_1},$$

giving

$$n-k+1 = 2^{m_1+4-(n-k)}.$$

Thus, $n-k+1 = 2^t$ is a power of two in the interval $[3, k+3]$ (so $t \ge 2$). Further, $n = 2^t + k - 1$, $n_1 = n-1 = 2^t + k - 2$, $m = n-3 = 2^t + k - 4$ and $m_1 = n-k-4+t = 2^t + t - 5$. Since $t \ge 2$, we get that $m_1 > 0$. Hence,

$$(n, m, n_1, m_1) = (k+2^t-1, k+2^t-4, k+2^t-2, t+2^t-5)$$

which corresponds to the parametric family (iii), with $c = 2^{k+2^t-4} + 2^{2^t-4} - 2^{t+2^t-4}$, in the statement of the Theorem 2.1.1.

Next we consider the situation $n_1 < n-1$. We show that there are no solutions in this case. Then,

$$\begin{aligned} 2^{n-2} &> 2^{n-2} - 2^{n_1-2} - ((n-k)\cdot 2^{n-k-3} - (n_1-k)\cdot 2^{n_1-k-3}) \\ &\geq 2^{n-2} - 2^{n-4} - \Big((n-k)\cdot 2^{n-k-3} - (n-k-2)\cdot 2^{n-k-5}\Big) \\ &> 2^{n-3}. \end{aligned}$$

The last inequality is equivalent to

$$2^{n-4} > (n-k)\cdot 2^{n-k-3} - (n-k-2)\cdot 2^{n-k-5},$$

which is implied by

$$2^{n-4} > (n-k)\cdot 2^{n-k-3},$$

or

$$2^{k-1} > n-k.$$

Since $n-k \leq k+2$, it suffices that $2^{k-1} > k+2$ and this holds for all $k \geq 4$. Thus, for $n_1 < n-1$, we have

$$2^{n-2} > F_n^{(k)} - F_{n_1}^{(k)} > 2^{n-3},$$

so

$$2^{n-2} > 2^m - 2^{m_1} > 2^{n-3},$$

showing that $m = n-2$. In this case, we have by (2.7), that

$$2^{n-2} - 2^{n_1-2} - (n-k)\cdot 2^{n-k-3} + (n_1-k)\cdot 2^{n_1-k-3} = 2^{n-2} - 2^{m_1},$$

giving

$$(n-k)\cdot 2^{n-k-3} - (n_1-k)\cdot 2^{n_1-k-3} = 2^{m_1} - 2^{n_1-2}.$$

The left–hand side is positive therefore so is the right–hand side. Thus,

$$2^{n_1-k-3}(2^{n-n_1}(n-k) - (n_1-k)) = 2^{n_1-2}(2^{m_1-n_1+2} - 1). \tag{2.8}$$

To proceed, we write

$$n-k = 2^{\alpha}u \qquad \text{and} \qquad n_1 - k = 2^{\alpha_1}u_1,$$

where α, α_1 are nonnegative and u, u_1 are odd. Since $n-k$, $n_1-k \in [2, k+2]$, it follows $2^{\alpha} \leq k+2$ and $2^{\alpha_1} \leq k+2$. Hence, $\max\{\alpha, \alpha_1\} \leq \log(k+2)/\log 2$. Equation (2.8) becomes

$$2^{n_1-k-3}(2^{\alpha+n-n_1}u - 2^{\alpha_1}u_1) = 2^{n_1-2}(2^{m_1-n_1+2} - 1). \tag{2.9}$$

We distinguish various cases.

Case 2.2.4. $\alpha + n - n_1 = \alpha_1$.

In this case, by (2.9), we have

$$2^{n_1-k-3+\alpha_1}(u-u_1) = 2^{n_1-2}(2^{m_1-n_1+2}-1). \tag{2.10}$$

Note that we cannot have $u = u_1$ (otherwise we get $n = n_1$, a contradiction). Since the exponent of 2 in the right in (2.10) is exactly $n_1 - 2$ and in the left is at least $n_1 - k - 3 + \alpha_1$, we get that $n_1 - 2 \geq n_1 - k - 3 + \alpha_1$, so $k + 1 \geq \alpha_1$, and

$$u - u_1 = 2^{k+1-\alpha_1}(2^{m_1-n_1+2}-1).$$

We deduce that the following inequality holds:

$$2^{k+1-\alpha_1} \mid u - u_1, \quad \text{so} \quad k+1-\alpha_1 \leq \frac{\log(u-u_1)}{\log 2} \leq \frac{\log(k+1)}{\log 2}.$$

Thus,

$$k+1 = (k+1-\alpha_1)+\alpha_1 \leq \frac{\log(k+1)}{\log 2} + \frac{\log(k+2)}{\log 2},$$

which yields

$$2^{k+1} \leq (k+2)(k+1),$$

so $k \leq 3$. So, this case cannot lead to infinitely many solutions.

Case 2.2.5. $\alpha + n - n_1 < \alpha_1$.

In this case, by (2.9), we now have

$$2^{n-k-3+\alpha}(u - 2^{\alpha_1-\alpha-n+n_1}u_1) = 2^{n_1-2}(2^{m_1-n_1+2}-1).$$

Identifying factors which are powers of 2 in both sides, we have

$$n_1 = n + \alpha - k - 1.$$

Since

$$n - n_1 < \alpha_1 - \alpha \leq \alpha_1 \leq \frac{\log(k+2)}{\log 2},$$

we have

$$k+1 = (n-n_1)+\alpha \leq \frac{\log(k+2)}{\log 2} + \frac{\log(k+2)}{\log 2},$$

giving

$$2^{k+1} \leq (k+2)^2,$$

so $k \leq 4$.

Thus, as in the previous case, this situation cannot lead us to infinitely many solutions either.

Case 2.2.6. $\alpha_1 < \alpha + n - n_1$.

In this case, (2.9) becomes

$$2^{n_1-k-3+\alpha_1}(2^{\alpha-\alpha_1+n-n_1}u - u_1) = 2^{n_1-2}(2^{m_1-n_1+2} - 1).$$

Identifying powers of 2 in both sides above, we get

$$k+1 = \alpha_1.$$

Hence,

$$k+1 \leq \frac{\log(k+2)}{\log 2},$$

giving $2^{k+1} \leq k+2$, so $k \leq 1$, a contradiction.

The last parametric family from the statement of Theorem 2.1.1 will be identified in the next section.

2.3. Solutions with $n \geq 2k+3$

From now on, we searched for solutions other than the ones given in Theorem 2.1.1 (i), (ii), and (iii), with the aim is to show that perhaps they are none except for some sporadic ones with $k < k_0$ with some small k_0. Then the problem will be solved by finding individually for every $k \in [4, k_0]$, the values of c such that (2.4) has some solution (n, m, n_1, m_1) with $n > n_1,\ m > m_1$ and determining for each c all such representations. It turns out that this program does not quite work out since along the way we find parametric family (iv) with $n = 2k+3$, but afterwards all does work out and we are able to show that indeed if $n > 2k+3$, then $k \leq 790$.

So, let's get to work. We go back to (2.4) and assume that $n \geq 2k+3$. Suppose first that $m \geq n-1$. We recall equality (2.5):

$$2^m - 2^{m_1} = F_n^{(k)} - F_{n_1}^{(k)}.$$

The left–hand side is positive and

$$2^m - 2^{m_1} \geq 2^{m-1} \geq 2^{n-2} > F_n^{(k)} > F_n^{(k)} - F_{n_1}^{(k)},$$

where we used the fact that $F_n^{(k)} < 2^{n-2}$ for $n \geq k+2$. Thus, $m \leq n-2$. Note that $n \geq 2k+3$, so $n - 2k \geq 3$.

We put $y := n/2^k$, and assume that

$$n^3 < 2^{k-5}, \qquad \text{so} \qquad y < 1/4. \tag{2.11}$$

Thus, by Lemma 1.3.3, we have

$$F_n^{(k)} = 2^{n-2}(1-\zeta), \qquad where \qquad |\zeta| < \frac{1}{2}\left(y+y^2+y^3\right).$$

Similarly,

$$F_{n_1}^{(k)} = 2^{n_1-2}(1-\zeta_1), \qquad where\ \ also \qquad |\zeta_1| < \frac{1}{2}(y+y^2+y^3). \tag{2.12}$$

We get from (2.5)

$$\left|(2^m - 2^{m_1}) - (2^{n-2} - 2^{n_1-2})\right| < (2^{n-2} + 2^{n_1-2})\frac{(y+y^2+y^3)}{2} < 2^{n-2}y. \tag{2.13}$$

If $m \le n-4$, then the left–hand side in (2.13) is at least

$$(2^{n-2} - 2^{n_1-2}) - 2^{n-4} \ge 2^{n-3} - 2^{n-4} \ge 2^{n-4},$$

showing that

$$2^{n-4} \le 2^{n-2}y,$$

giving $y \ge 1/4$, a contradiction to (2.11). Further, assuming that $m = n-3$ but $n_1 < n-1$, the left–hand side in formula (2.13) is at least

$$(2^{n-2} - 2^{n_1-2}) - 2^m \ge 2^{n-2} - 2^{n-4} - 2^{n-3} = 2^{n-4},$$

and we get to the same contradiction to (2.11), namely that $y \ge 1/4$. Thus, we conclude that either $(m, n_1) = (n-3, n-1)$, or $m = n-2$. The first case gives from (2.5)

$$F_n^{(k)} - F_{n-1}^{(k)} = 2^{n-3} - 2^{m_1}. \tag{2.14}$$

Using Lemma 1.3.3, we get

$$F_n^{(k)} = 2^{n-2}\left(1 - \frac{n-k}{2^{k+1}} + \gamma\right) \quad and \quad F_{n-1}^{(k)} = 2^{n-3}\left(1 - \frac{n-k-1}{2^{k+1}} + \gamma_1\right), \tag{2.15}$$

where

$$\max\{|\gamma|, |\gamma_1|\} \le \frac{1}{2}(y^2+y^3) < y^2.$$

Putting these into (2.14), we get

$$\left|-2^{n-k-3}(n-k) + 2^{n-k-4}(n-k-1) + 2^{m_1}\right| < 2^{n-2}|\gamma| + 2^{n-3}|\gamma_1| < 2^{n-1}y^2.$$

In the left–hand side, we have the amount

$$|2^{m_1} - 2^{n-k-4}(n-k+1)|.$$

If $m_1 \leq n-k-4$, then this amount exceeds $2^{n-k-4}(n-k) > 2^{n-k-4}$. If $m_1 > n-k-4$, then the above number can be rewritten as

$$2^{n-k-4}|n-k+1-2^{m_1-(n-k-4)}|.$$

If $n-k+1 \neq 2^{m_1-(n-k-4)}$, then the above amount is $\geq 2^{n-k-4}$. We thus get in all the above instances

$$2^{n-k-4} \leq |2^{m_1} - 2^{n-k-4}(n-k+1)| < 2^{n-1}y^2 < \frac{2^{n-1}n^2}{2^{2k}},$$

giving

$$n^2 > 2^{k-3} \qquad so \qquad n > 2^{(k-3)/2},$$

a contradiction to (2.11). If $n-k+1 = 2^{m_1-(n-k-4)}$, we consider one more term in (2.15):

$$\begin{aligned} F_n^{(k)} &= 2^{n-2} - 2^{n-k-3}(n-k) + 2^{n-2k-5}(n-2k+1)(n-2k-2) + 2^{n-2}\delta, \\ F_{n_1}^{(k)} &= 2^{n-3} - 2^{n-k-4}(n-k-1) + 2^{n-2k-6}(n-2k)(n-2k-3) + 2^{n-3}\delta_1 \end{aligned}$$

where

$$2^{n-2}|\delta| < 2^{n-3}y^3 < 2^{n-3k-3}n^3 \quad and \quad 2^{n-3}|\delta_1| < 2^{n-4}y^3 < 2^{n-3k-4}n^3.$$

Thus, by (2.11),

$$\max\{2^{n-2}|\delta|, 2^{n-3}|\delta_1|\} < 2^{n-2k-8}. \tag{2.16}$$

Putting these into (2.14), we get

$$2^{n-2k-6}|2(n-2k+1)(n-2k-2)-(n-2k)(n-2k-3)| < 2^{n-2}|\delta| + 2^{n-3}|\delta_1| < 2^{n-2k-7}.$$

Taking $w := n-2k$, we have that

$$w^2+w-4 = |2(w+1)(w-2)-w(w-3)| < 1/2,$$

which is a contradiction for all $k \geq 4$, given that $n \geq 2k+3$. So, the situation $(m, n_1) = (n-3, n-1)$ is not possible.

Hence, we continue with the case $m = n-2$. Going back to (2.4), we have

$$F_n^{(k)} - 2^{n-2} = F_{n_1}^{(k)} - 2^{m_1}. \tag{2.17}$$

The number on the left–hand side in (2.17) is negative. We will show that $m_1 \geq n_1 - 2$. Indeed, suppose that $m_1 \leq n_1 - 3$. Since for us $y < 1/4$, we get $|\zeta_1| < 1/2$ (see (2.12)). Further, again by (2.12), we note that $F_{n_1}^{(k)} > 2^{n_1-3} \geq 2^{m_1}$, so the right–hand side in (2.17) is positive, a contradiction. Thus, $m_1 \geq n_1 - 2$. The case $m_1 = n_1 - 2$ leads to

$$F_n^{(k)} - 2^{n-2} = F_{n_1}^{(k)} - 2^{n_1-2}. \tag{2.18}$$

Since $c \neq 0$, it follows that $n_1 \geq k+2$. However, we have the following lemma.

Lemma 2.3.1. *The sequence $\{2^{n-2} - F_n^{(k)}\}_{n \geq k+2}$ is increasing for $n \geq k+3$.*

Proof. We want

$$2^{n-1} - F_{n+1}^{(k)} > 2^{n-2} - F_n^{(k)},$$

which is equivalent to

$$2^{n-2} > F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k+1}^{(k)}.$$

There are $k-1$ terms in the right–hand side. Each of them satisfies $F_{n-1-j}^{(k)} \leq 2^{n-3-j}$ for $j = 0, 1, \ldots, k-2$ because $F_a \leq 2^{a-2}$ holds for all $a \geq 2$. Thus, it suffices that

$$2^{n-2} > 2^{n-3} + 2^{n-4} + \cdots + 2^{n-k-1},$$

which is obvious. □

Thus, (2.18) is impossible. Hence, $m_1 \geq n_1 - 1$. Using the first identity in (2.15), we have that the left–hand side in (2.17) is

$$-2^{n-k-3}(n-k) + 2^{n-2}\gamma, \tag{2.19}$$

where $|\gamma| < (y^2 + y^3)/2 < y^2$. Note that since

$$y^2 = \frac{n^2}{2^{2k}} < \frac{1}{2^{k+3}} \qquad (\text{by } 2.11),$$

it follows that

$$2^{n-2}|\gamma| < 2^{n-k-5}. \tag{2.20}$$

Thus, the left–hand side of (2.17) is in the interval

$$(-2^{n-k-3}(n-k+1/2), -2^{n-k-3}(n-k-1/2)).$$

Now the right–hand side of (2.17) is in the interval $(-2^{m_1}, -2^{m_1-1}]$, where for the right–hand extreme of the interval we used the fact that $F_{n_1}^{(k)} \leq 2^{n_1-2} \leq 2^{m_1-1}$. Comparing them we get

$$-2^{n-k-3}(n-k+1/2) < -2^{m_1-1} \qquad \text{and} \qquad -2^{n-k-3}(n-k-1/2) > -2^{m_1},$$

which gives

$$2^{m_1-1} < 2^{n-k-3}(n-k+1/2) \qquad \text{and} \qquad 2^{n-k-3}(n-k-1/2) < 2^{m_1}.$$

In particular, $m_1 \geq n-k-3$, so

$$2^{m_1-(n-k-3)-1} \leq n-k \leq 2^{m_1-(n-k-3)}.$$

We thus get, from (2.17) and (2.19), that

$$-2^{n-k-3}(n-k-2^{m_1-(n-k-3)}) = F_{n_1}^{(k)} - 2^{n-2}\gamma. \tag{2.21}$$

We distinguish two cases.

Case 2.3.1. *Assume that $n_1 < n-k-1$.*

Then $F_{n_1}^{(k)} < 2^{n_1-2} \leq 2^{n-k-4}$. Using also (2.20), we get

$$2^{n-k-3}\left|(n-k) - 2^{m_1-(n-k-3)}\right| < \max\{F_{n_1}^{(k)}, 2^{n-2}|\gamma|\} < 2^{n-k-4},$$

so $n-k-2^{m_1-(n-k-3)}$ is an integer which is at most $1/2$ in absolute value. Hence, it is zero. Thus, $n-k = 2^{m_1-(n-k-3)}$. We now go one more step and say that

$$\begin{aligned} F_n^{(k)} &= 2^{n-2} - 2^{n-k-3}(n-k) + 2^{n-2k-5}(n-2k+1)(n-2k-2) + 2^{n-2}\delta, \\ F_{n_1}^{(k)} &= 2^{n_1-2}(1-\gamma_1), \end{aligned}$$

where, by (2.16),

$$2^{n-2}|\delta| < 2^{n-2k-8}.$$

Further, by (2.12),

$$2^{n_1-2}|\gamma_1| < 2^{n_1-3}y^2 < 2^{n-3k-4}n^2 < 2^{n-2k-8}.$$

Equation (2.17) now implies that

$$-2^{n-k-3}(n-k) + 2^{n-2k-5}(n-2k+1)(n-2k-2) + 2^{n-2}\delta = 2^{n_1-2} - 2^{n_1-2}\gamma_1 - 2^{m_1},$$

so, given that $n-k = 2^{m_1-(n-k-3)}$,

$$2^{n-2k-5}(n-2k+1)(n-2k-2) - 2^{n_1-2} = -2^{n-2}\delta - 2^{n_1-2}\gamma_1. \tag{2.22}$$

Assume that $n_1 \leq n-2k-4$. Then

$$\begin{aligned} 2^{n-2k-5} \leq 2^{n-2k-5}(n-2k+1)(n-2k-2) &\leq 2^{n_1-2} + 2^{n-2}|\delta| + 2^{n_1-2}|\gamma_1| \\ &< 3 \times 2^{n-2k-7} < 2^{n-2k-5}, \end{aligned}$$

which is a contradiction. Thus, we must have $n_1 \geq n-2k-3$, so

$$\begin{aligned} 2^{n-2k-5}\left|(n-2k+1)(n-2k-2) - 2^{n_1-2-(n-2k-5)}\right| &\leq 2^{n-2}|\delta| + 2^{n_1-2}|\gamma_1| \\ &< 2^{n-2k-7}. \end{aligned}$$

The left–hand side above is an integer divisible by 2^{n-2k-5}. Since it is smaller than 2^{n-2k-7}, it must be the zero integer. Thus, with $w = n-2k$, we have

$$(w+1)(w-2) = 2^{n_1-2-(n-2k-5)}.$$

In the left–hand side above, one of the factors $w-2$ and $w+1$ is odd. Since they are both positive and powers of 2, it follows that the smaller one is 1. Hence, $w = 3$, so

$$w+1 = 2^2 = 2^{n_1-2-(n-2k-5)},$$

giving $n_1 - 2 = n - 2k - 3 = 0$. Thus, $n = 2k+3, n_1 = 2$ and $m = n - 2 = 2k+1$. From equality (2.17), we conclude that

$$2^{2k+1} - 2^k(k+3) = 2^{2k+1} - 2^{m_1}$$

giving $k+3 = 2^t$, for some integer $t \geq 3$ and $m_1 = k+t$. Hence, we obtain the parametric family

$$(n,m,n_1,m_1) = (2^{t+1}-3, 2^{t+1}-5, 2, t+2^t-3)$$

with $c = 1 - 2^{t+2^t-3}$, which corresponds to situation (iv) in the statement of the Theorem 2.1.1.

Case 2.3.2. $n_1 \geq n-k-1$.

The equation that we then get from (2.12) and (2.21) is

$$2^{n-k-3}\left((n-k) - 2^{m_1-(n-k-3)} + 2^{n_1-2-(n-k-3)}\right) = 2^{n_1-2}\gamma_1 + 2^{n-2}\gamma.$$

Given that $m_1 < m$ and that we are in the case $m = n-2$, we have

$$2^{n_1-2}|\gamma_1| \leq 2^{m_1-1}|\gamma_1| \leq 2^{n-k-3}(n-k)y < 2^{n-2k-3}n^2 < 2^{n-k-7}.$$

We thus have

$$2^{n-k-3}|(n-k) - 2^{m_1-(n-k-3)} + 2^{n_1-2-(n-k-3)}| < 2^{n_1-2}|\gamma_1| + 2^{n-2}|\gamma| < 2^{n-k-5},$$

showing the left–hand side is zero. Thus, $a = m_1 - (n-k-3)$, $b = n_1 - 2 - (n-k-3)$ and

$$n - k = 2^a - 2^b.$$

So, $n = k + 2^a - 2^b$. As in previous iterations, we go one step further and write

$$\begin{aligned} F_n^{(k)} &= 2^{n-2} - 2^{n-k-3}(n-k) + 2^{n-2k-5}(n-2k+1)(n-2k-2) + 2^{n-2}\delta, \\ F_{n_1}^{(k)} &= 2^{n_1-2} - 2^{n_1-k-3}(n_1-k) + 2^{n_1-2}\gamma_1. \end{aligned}$$

Inserting these into equation (2.17), we get

$$\begin{aligned} -2^{n-k-3}(n-k) &+ 2^{n-2k-5}(n-2k-2)(n-2k+1) + 2^{n-2}\delta \\ &= 2^{n_1-2} - 2^{n_1-k-3}(n_1-k) - 2^{m_1} + 2^{n_1-2}\gamma_1, \end{aligned}$$

or

$$2^{n-2k-5}(n-2k-2)(n-2k+1) + 2^{n_1-k-3}(n_1-k) = -2^{n-2}\delta + 2^{n_1-2}\gamma_1.$$

We have $n_1 = n-k-1+b$, so $n_1 - k - 3 = n - 2k - 4 + b$ so $n_1 - k - 3 - (n-2k-5) = b+1$ and $n_1 - k = n - 2k - 1 + b$. Thus,

$$2^{n-2k-5}\left((n-2k+1)(n-2k-2) + 2^{b+1}(n-2k-1+b)\right) = -2^{n-2}\delta + 2^{n_1-2}\gamma_1.$$

We already know that $2^{n-2}|\delta| < 2^{n-2k-8}$. Now $n_1 - 2 = n-k-3+b$, so

$$2^{n_1-2} = 2^{n-k-3}2^b.$$

Note that $n-k = 2^a - 2^b \geq 2^{a-1} \geq 2^b$, so $2^b < n$. Thus,

$$2^{n_1-2}|\gamma_1| \leq 2^{n-k-3}ny^2 \leq 2^{n-k-3}n^3/2^{2k} < 2^{n-2k-8},$$

since $n^3 < 2^{k-5}$. Hence, we get

$$2^{n-2k-5}|(n-2k+1)(n-2k-2)+2^{b+1}(n-2k-1+b)| < 2^{n-2k-7},$$

showing that the number in absolute value is zero, which is a contradiction because $n-2k \geq 3$ and $b \geq 0$. In conclusion, there are no solutions with $n > 2k+3$ provided that (2.11) holds. In the next section, we estimate a value of k_0 for which inequality (2.11) is fulfilled for all $k > k_0$.

2.4. Establishing an inequality in terms of n and k and estimating k_0

Since $n > n_1 \geq 2$, we have that $F_{n_1}^{(k)} \leq F_{n-1}^{(k)}$ and therefore

$$F_n^{(k)} = F_{n-1}^{(k)} + \cdots + F_{n-k}^{(k)} \geq F_{n-1}^{(k)} + \cdots + F_{n-k-1}^{(k)} \geq F_{n_1}^{(k)} + \cdots + F_{n-k-1}^{(k)}.$$

So, from the above, (1.13) and (2.5), we have

$$\begin{aligned} \alpha^{n-4} &\leq F_{n-2}^{(k)} \leq F_n^{(k)} - F_{n_1}^{(k)} = 2^m - 2^{m_1} < 2^m, \\ \alpha^{n-1} &\geq F_n^{(k)} > F_n^{(k)} - F_{n_1}^{(k)} = 2^m - 2^{m_1} \geq 2^{m-1}, \end{aligned} \tag{2.23}$$

leading to

$$1 + \left(\frac{\log 2}{\log \alpha}\right)(m-1) < n < \left(\frac{\log 2}{\log \alpha}\right) m + 4. \tag{2.24}$$

It can be noted that the above inequality (2.24) in particular implies that $m < n < 1.2m+4$. Moreover, note that we can assume $n \geq k+2$, since otherwise, this would give us only the solution for $c = 0$, which is family (i) of Theorem 2.1.1.

Assume for technical reasons that $n > 1600$. By (1.11) and (2.5), we get

$$\begin{aligned} \left|f_k(\alpha)\alpha^{n-1} - 2^m\right| &= \left|(f_k(\alpha)\alpha^{n-1} - F_n^{(k)}) + (F_{n_1}^{(k)} - 2^{m_1})\right| \\ &= \left|(f_k(\alpha)\alpha^{n-1} - F_n^{(k)}) + (F_{n_1}^{(k)} - f_k(\alpha)\alpha^{n_1-1}) + (f_k(\alpha)\alpha^{n_1-1} - 2^{m_1})\right| \\ &< \frac{1}{2} + \frac{1}{2} + \alpha^{n_1-1} + 2^{m_1} \\ &< \alpha^{n_1} + 2^{m_1} \\ &< 2\max\{\alpha^{n_1}, 2^{m_1}\}. \end{aligned}$$

In the above, we have also used the fact that $|f_k(\alpha)| < 1$. Dividing through by 2^m we get

$$\left|f_k(\alpha)\alpha^{n-1}2^{-m} - 1\right| < 2\max\left\{\frac{\alpha^{n_1}}{2^m}, 2^{m_1-m}\right\} < \max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\}, \quad (2.25)$$

where for the right–most inequality in (2.25) we used (2.23) and the fact that $\alpha^2 > 2$.

For the left-hand side of (2.25) above, we apply Theorem 1.1.8 with the data

$$t := 3,\ \ \eta_1 := f_k(\alpha),\ \ \eta_2 := \alpha,\ \ \eta_3 := 2,\ \ b_1 := 1,\ \ b_2 := n-1,\ \ b_3 := -m.$$

We begin by noticing that the three numbers η_1, η_2, η_3 are positive real numbers and belong to the field $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. Put

$$\Lambda := f_k(\alpha)\alpha^{n-1}2^{-m} - 1.$$

To see why $\Lambda \neq 0$, note that otherwise, we would then have that $f_k(\alpha) = 2^m\alpha^{-(n-1)}$ and so $f_k(\alpha)$ would be an algebraic integer, which contradicts Lemma 1.3.1 (i).

Since $h(\eta_2) = (\log\alpha)/k < (\log 2)/k$ and $h(\eta_3) = \log 2$, it follows that we can take $A_2 := \log 2$ and $A_3 := k\log 2$. Further, in view of Lemma 1.3.1 (ii), we have that $h(\eta_1) < 3\log k$, so we can take $A_1 := 3k\log k$. Finally, since $\max\{1, n-1, m\} = n-1$, we take $B := n$.

Then, the left–hand side of (2.25) is bounded below, by Theorem 1.1.8, as

$$\log|\Lambda| > -1.4 \times 30^6 \times 3^{4.5} \times k^4(1+\log k)(1+\log n)(3\log k)(\log 2)(\log 2).$$

Comparing with (2.25), we get

$$\min\{(n-n_1-6)\log\alpha, (m-m_1-1)\log 2\} < 4.2 \times 10^{11} k^4 \log^2 k(1+\log n),$$

which gives

$$\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} < 4.25 \times 10^{11} k^4 \log^2 k(1+\log n).$$

Now the argument is split into two cases.

Case 2.4.1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} = (n-n_1)\log\alpha.$

In this case, we rewrite (2.5) as

$$\begin{aligned}\left|f_k(\alpha)\alpha^{n-1} - f_k(\alpha)\alpha^{n_1-1} - 2^m\right| &= \left|(f_k(\alpha)\alpha^{n-1} - F_n^{(k)}) + (F_{n_1}^{(k)} - f_k(\alpha)\alpha^{n_1-1}) - 2^{m_1}\right| \\ &< \frac{1}{2} + \frac{1}{2} + 2^{m_1} \le 2^{m_1+1}.\end{aligned}$$

Dividing through by 2^m gives

$$\left|f_k(\alpha)(\alpha^{n-n_1}-1)\alpha^{n_1-1}2^{-m}-1\right| \quad < \quad 2^{m_1-m+1}. \tag{2.26}$$

Now we put

$$\Lambda_1 := f_k(\alpha)(\alpha^{n-n_1}-1)\alpha^{n_1-1}2^{-m}-1.$$

We apply again Theorem 1.1.8 with the following data

$$t:=3,\ \eta_1 := f_k(\alpha)(\alpha^{n-n_1}-1),\ \eta_2 := \alpha,\ \eta_3 := 2,\ \ b_1 := 1,\ b_2 := n_1-1,\ b_3 := -m.$$

As before, we begin by noticing that the three numbers η_1,η_2,η_3 belong to the field $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K}:\mathbb{Q}] = k$. To see why $\Lambda_1 \neq 0$, note that otherwise, we would get the relation $f_k(\alpha)(\alpha^{n-n_1}-1) = 2^m\alpha^{1-n_1}$. Conjugating this last equation with any automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ such that $\sigma(\alpha) = \alpha^{(i)}$ for some $i \geq 2$, and then taking absolute values, we arrive at the equality $|f_k(\alpha^{(i)})((\alpha^{(i)})^{n-n_1}-1)| = |2^m(\alpha^{(i)})^{1-n_1}|$. But this cannot hold because, $|f_k(\alpha^{(i)})||(\alpha^{(i)})^{n-n_1}-1| < 2$ since $|f_k(\alpha^{(i)})| < 1$ by Lemma 1.3.1 (i), and $|(\alpha^{(i)})^{n-n_1}| < 1$, since $n > n_1$, while $|2^m(\alpha^{(i)})^{1-n_1}| \geq 2$.

Since

$$h(\eta_1) \leq h(f_k(\alpha)) + h(\alpha^{n-n_1}-1) < 3\log k + (n-n_1)\frac{\log\alpha}{k} + \log 2,$$

it follows that

$$kh(\eta_1) < 6k\log k + (n-n_1)\log\alpha < 6k\log k + 2.95\times 10^{11}k^4\log^2 k(1+\log n).$$

So, we can take $A_1 := 3\times 10^{11}k^4\log^2 k(1+\log n)$. Further, as before, we take $A_2 := \log 2$ and $A_3 := k\log 2$. Finally, by recalling that $m < n$, we can take $B := n$.

We then get that

$$\log|\Lambda_1| > -1.4\times 30^6\times 3^{4.5}\times k^3(1+\log k)(1+\log n)(3\times 10^{11}k^4\log^2 k(1+\log n))(\log 2)^2,$$

which yields

$$\log|\Lambda_1| > -4.13\times 10^{22}k^7\log^3 k(1+\log n)^2.$$

Comparing this with (2.26), we get that

$$(m-m_1)\log 2 < 4.2\times 10^{22}k^7\log^3 k(1+\log n)^2.$$

Case 2.4.2. $\min\{(n-n_1)\log\alpha,(m-m_1)\log 2\} = (m-m_1)\log 2.$

In this case, we write (2.5) as

$$\begin{aligned}\left|f_k(\alpha)\alpha^{n-1}-2^m+2^{m_1}\right| \quad &= \quad \left|(f_k(\alpha)\alpha^{n-1}-F_n^{(k)}) + (F_{n_1}^{(k)}-f_k(\alpha)\alpha^{n_1-1}) + f_k(\alpha)\alpha^{n_1-1}\right| \\ &< \quad \frac{1}{2}+\frac{1}{2}+\alpha^{n_1-1} \quad < \quad \alpha^{n_1},\end{aligned}$$

so that

$$\left|f_k(\alpha)(2^{m-m_1}-1)^{-1}\alpha^{n-1}2^{-m_1}-1\right| < \frac{\alpha^{n_1}}{2^m-2^{m_1}} \leq \frac{2\alpha^{n_1}}{2^m} < \alpha^{n_1-n+6}. \tag{2.27}$$

The above inequality (2.27) suggests once again studying a lower bound for the absolute value of

$$\Lambda_2 := f_k(\alpha)(2^{m-m_1}-1)^{-1}\alpha^{n-1}2^{-m_1}-1.$$

We again apply Matveev's theorem with the following data

$$t:=3,\ \eta_1 := f_k(\alpha)(2^{m-m_1}-1)^{-1},\ \eta_2 := \alpha,\ \eta_3 := 2,\ \ b_1 := 1,\ b_2 := n-1, b_3 := -m_1.$$

We can again take $B := n$ and $\mathbb{K} := \mathbb{Q}(\alpha)$, so that $D := k$. We also note that, if $\Lambda_2 = 0$, then $f_k(\alpha) = \alpha^{-(n-n_1)}2^{m_1}(2^{m-m_1}-1)$ implying that $f_k(\alpha)$ is an algebraic integer, which is not the case. Thus, $\Lambda_2 \neq 0$.

Now, we note that

$$h(\eta_1) \leq h(f_k(\alpha)) + h(2^{m-m_1}-1) < 3\log k + (m-m_1+k)\frac{\log 2}{k}.$$

Thus, $kh(\eta_1) < 4k\log k + (m-m_1)\log 2 < 3\times 10^{11}k^4\log^2 k(1+\log n)$, and so we can take $A_1 := 3\times 10^{11}k^4\log^2 k(1+\log n)$. As before, we take $A_2 := \log 2$ and $A_3 := k\log 2$. It then follows from Matveev's theorem, after some calculations, that

$$\log|\Lambda_2| > -4.13\times 10^{22}k^7\log^3 k(1+\log n)^2.$$

From this and (2.27), we obtain that

$$(n-n_1)\log\alpha < 4.2\times 10^{22}k^7\log^3 k(1+\log n)^2.$$

Thus in both Case 2.4.1 and Case 2.4.2, we have

$$\begin{aligned} \min\{(n-n_1)\log\alpha,(m-m_1)\log 2\} &< 4.3\times 10^{11}k^4\log^2 k(1+\log n),\\ \max\{(n-n_1)\log\alpha,(m-m_1)\log 2\} &< 4.2\times 10^{22}k^7\log^3 k(1+\log n)^2. \end{aligned} \tag{2.28}$$

We now finally rewrite equation (2.5) as

$$\left|f_k(\alpha)\alpha^{n-1}-f_k(\alpha)\alpha^{n_1-1}-2^m+2^{m_1}\right| = \left|(f_k(\alpha)\alpha^{n-1}-F_n^{(k)})+(F_{n_1}^{(k)}-f_k(\alpha)\alpha^{n_1-1})\right| < 1.$$

We divide through both sides by $2^m-2^{m_1}$ getting

$$\left|\frac{f_k(\alpha)(\alpha^{n-n_1}-1)}{2^{m-m_1}-1}\alpha^{n_1-1}2^{-m_1}-1\right| < \frac{1}{2^m-2^{m_1}} \leq \frac{2}{2^m} < 2^{5-0.8n}, \tag{2.29}$$

since $n < 1.2m+4$. To find a lower–bound on the left–hand side of (2.29) above, we again apply Theorem 1.1.8 with the data

$$t := 3,\ \eta_1 := \frac{f_k(\alpha)(\alpha^{n-n_1}-1)}{2^{m-m_1}-1},\ \eta_2 := \alpha,\ \eta_3 := 2,\ b_1 := 1,\ b_2 := n_1-1, b_3 := -m_1.$$

We also take $B := n$ and we take $\mathbb{K} := \mathbb{Q}(\alpha)$ with $D := k$. From the properties of the logarithmic height function, we have that

$$\begin{aligned} kh(\eta_1) &\leq k\left(h(f_k(\alpha))+h(\alpha^{n-n_1}-1)+h(2^{m-m_1}-1)\right) \\ &< 3k\log k+(n-n_1)\log\alpha+k(m-m_1)\log 2+2k\log 2 \\ &< 5.3\times 10^{22}k^8\log^3 k(1+\log n)^2, \end{aligned}$$

where in the above chain of inequalities we used the bounds (2.28). So we can take $A_1 := 5.3\times 10^{22}k^8\log^3 k(1+\log n)^2$, and certainly as before we take $A_2 := \log 2$ and $A_3 := k\log 2$. We need to show that if we put

$$\Lambda_3 := \frac{f_k(\alpha)(\alpha^{n-n_1}-1)}{2^{m-m_1}-1}\alpha^{n_1-1}2^{-m_1}-1,$$

then $\Lambda_3 \neq 0$. To see why $\Lambda_3 \neq 0$, note that otherwise, we would get the relation

$$f_k(\alpha)(\alpha^{n-n_1}-1) = 2^{m_1}\alpha^{1-n_1}(2^{m-m_1}-1).$$

Again, as for the case of Λ_1, conjugating the above relation with an automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ such that $\sigma(\alpha)=\alpha^{(i)}$ for some $i\geq 2$, and then taking absolute values, we get that $|f_k(\alpha^{(i)})((\alpha^{(i)})^{n-n_1}-1)| = |2^{m_1}(\alpha^{(i)})^{1-n_1}(2^{m-m_1}-1)|$. This cannot hold true because in the left–hand side we have $|f_k(\alpha^{(i)})||(\alpha^{(i)})^{n-n_1}-1| < 2$, while in the right–hand side we have $|2^{m_1}||(\alpha^{(i)})^{1-n_1}||2^{m-m_1}-1| \geq 2$. Thus, $\Lambda_3 \neq 0$. Then Theorem 1.1.8 gives

$$\log|\Lambda_3| > -1.4\times 30^6\times 3^{4.5}k^{11}(1+\log k)(1+\log n)\left(5.3\times 10^{22}\log^3 k(1+\log n)^2\right)(\log 2)^2,$$

which together with (2.29) gives

$$(0.8n-5)\log 2 < 7.3\times 10^{33}k^{11}\log^4 k(1+\log n)^3.$$

The above inequality leads to

$$n < 5.1\times 10^{34}k^{11}\log^4 k\log^3 n,$$

which can be equivalently written as

$$\frac{n}{(\log n)^3} < 5.1\times 10^{34}k^{11}\log^4 k. \tag{2.30}$$

If $A\geq 10^{30}$, the inequality

$$\frac{x}{(\log x)^3} < A \quad \text{yields} \quad x < 16A\log^3 A.$$

Thus, taking $A := 5.1 \times 10^{34} k^{11} \log^4 k$, inequality (2.30) yields

$$n \quad < \quad 2.8 \times 10^{41} k^{11} \log^7 k. \tag{2.31}$$

We then record what we have proved so far as a lemma.

Lemma 2.4.1. *If (n,m,n_1,m_1,k) is a solution in positive integers to equation* (2.4) *with $(n,m) \neq (n_1,m_1)$, $n > n_1 \geq 2$, $m > m_1 \geq 0$ and $k \geq 4$, we then have that $n < 2.8 \times 10^{41} k^{11} \log^7 k$.*

2.5. Reduction of the bounds on n

2.5.1. The cut-off for k

We have from the above that Baker's method gives

$$n < 2.8 \times 10^{41} k^{11} \log^7 k.$$

Imposing that the above amount is at most $2^{(k-5)/3}$, which would imply inequality (2.11), we get

$$2.8^3 \times 10^{123} k^{33} (\log k)^{21} < 2^k,$$

leading to $k > 790$.

We now reduce the bounds and to do so we make use of Lemma 1.2.2 several times.

2.5.2. The Case of small k

We next treat the cases when $k \in [4, 790]$. We note that for these values of the parameter k, Lemma 2.4.1 gives us absolute upper bounds for n. However, these upper bounds are so large that we wish to reduce them to a range where the solutions can be identified by using a computer. To do this, we return to (2.25) and put

$$\Gamma := (n-1)\log\alpha - m\log 2 + \log\left(f_k(\alpha)\right). \tag{2.32}$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geq 20$. In the case that this condition fails, we consider one of the following inequalities instead:

(i) if $n - n_1 < 20$ but $m - m_1 \geq 20$, we consider (2.26);
(ii) if $n - n_1 \geq 20$ but $m - m_1 < 20$, we consider (2.27);
(iii) if $n - n_1 < 20$ and $m - m_1 < 20$, we consider (2.29).

Let us start by considering (2.25). Note that $\Gamma \neq 0$; thus we distinguish the following cases. If $\Gamma > 0$, then $e^{\Gamma} - 1 > 0$, so from (2.25) we obtain

$$0 < \Gamma < e^{\Gamma} - 1 < \max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\}.$$

Suppose now that $\Gamma < 0$. Since $\Lambda = |e^{\Gamma} - 1| < 1/2$, we get that $e^{|\Gamma|} < 2$. Thus,

$$0 < |\Gamma| \leq e^{|\Gamma|} - 1 = e^{|\Gamma|}|e^{\Gamma} - 1| < 2\max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\}.$$

In any case, we have that the inequality

$$0 < |\Gamma| < 2\max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\} \tag{2.33}$$

always holds. Replacing Γ in the above inequality by its formula and dividing through by $\log 2$, we conclude that

$$0 < \left|(n-1)\left(\frac{\log\alpha}{\log 2}\right) - m + \frac{\log(f_k(\alpha))}{\log 2}\right| < \max\{200\cdot\alpha^{-(n-n_1)},\ 8\cdot 2^{-(m-m_1)}\}.$$

We apply Lemma 1.2.2 with the data

$$k \in [4,790], \qquad \tau_k := \frac{\log\alpha}{\log 2}, \qquad \mu_k := \frac{\log(f_k(\alpha))}{\log 2}, \qquad (A_k, B_k) := (200,\alpha) \quad \text{or} \quad (8,2).$$

We also put $M_k := \lfloor 2.8 \times 10^{41} k^{11} \log^7 k \rfloor$, which is upper bound on n by Lemma 2.4.1. From the fact that α is a unit in $\mathcal{O}_{\mathbb{K}}$, the ring of integers of $\mathbb{K}$, ensures that τ_k is an irrational number. Furthermore, τ_k is transcendantal by Gelfond–Schneider Theorem. A computer search in *Mathematica* showed that the maximum value of $\lfloor \log(200q/\varepsilon)/\log\alpha \rfloor$ is < 1571 and the maximum value of $\lfloor \log(8q/\varepsilon)/\log 2 \rfloor$ is < 1566. Therefore, either

$$n - n_1 < \frac{\log(200q/\varepsilon)}{\log\alpha} < 1571, \ \text{ or } m - m_1 < \frac{\log(8q/\varepsilon)}{\log 2} < 1566.$$

Thus, we have that either $n - n_1 \leq 1571$, or $m - m_1 \leq 1566$.

First, let us assume that $n - n_1 \leq 1571$. In this case we consider the inequality (2.26) and assume that $m - m_1 \geq 20$. We put

$$\Gamma_1 := (n_1 - 1)\log\alpha - m\log 2 + \log(f_k(\alpha)(\alpha^{n-n_1} - 1)).$$

By the same arguments used for proving (2.33), from (2.26) we get

$$0 < |\Gamma_1| < \frac{4}{2^{m-m_1}},$$

and so

$$0 < \left|(n_1-1)\left(\frac{\log\alpha}{\log 2}\right) - m + \frac{\log(f_k(\alpha)(\alpha^{n-n_1}-1))}{\log 2}\right| < 8\cdot 2^{-(m-m_1)}. \tag{2.34}$$

As before, we keep the same τ_k, M_k, $(A_k, B_k) := (8,2)$ and put

$$\mu_{k,l} := \frac{\log(f_k(\alpha)(\alpha^l - 1))}{\log 2}, \qquad k \in [4,790] \quad and \quad l \in [1,1566].$$

We now apply Lemma 1.2.2 to inequality (2.34) for the values of $k \in [4,790]$ and $l \in [1,1571]$. A computer search with *Mathematica* revealed that the maximum value of $\lfloor \log(Aq/\varepsilon)/\log B \rfloor$ over the values of $k \in [4,790]$ and $l \in [1,1571]$ is < 1570. Hence, $m - m_1 \leq 1570$.

Now let us assume that $m - m_1 \leq 1566$. In this case, we consider the inequality (2.27) and assume that $n - n_1 \geq 20$. We put

$$\Gamma_2 := (n-1)\log\alpha - m_1 \log 2 + \log(f_k(\alpha)(2^{m-m_1} - 1)).$$

Then, by the same arguments as before, we get

$$0 < |\Gamma_2| < \frac{2\alpha^6}{\alpha^{n-n_1}}.$$

Replacing Γ_2 in the above inequality by its formula and dividing through by $\log 2$, we finally get that

$$0 < \left| (n-1)\left(\frac{\log\alpha}{\log 2}\right) - m_1 + \frac{\log(f_k(\alpha)(2^{m-m_1} - 1))}{\log 2} \right| < 114 \cdot \alpha^{-(n-n_1)}.$$

We apply Lemma 1.2.2 with the same τ_k, M_k, $(A_k, B_k) := (114, \alpha)$ and put

$$\mu_{k,l} := \frac{\log(f_k(\alpha)(2^l - 1))}{\log 2}, \qquad k \in [4,790] \quad and \quad l \in [1,1566].$$

As before, a computer search with *Mathematica* revealed that the maximum value of

$$\lfloor \log(Aq/\varepsilon)/\log B \rfloor, \qquad for \quad k \in [4,790] \quad and \quad l \in [1,1566]$$

is < 1574. Hence, $n - n_1 \leq 1574$.

To conclude the above computations, we first got that either $n - n_1 \leq 1571$ or $m - m_1 \leq 1566$. If $n - n_1 \leq 1571$, then $m - m_1 \leq 1570$, and if $m - m_1 \leq 1566$, then $n - n_1 \leq 1574$. Thus, in conclusion, we always have that

$$n - n_1 \leq 1574 \qquad and \qquad m - m_1 \leq 1570.$$

Finally, we go to (2.29) and put

$$\Gamma_3 := (n_1 - 1)\log\alpha - m_1 \log 2 + \log\left(\frac{f_k(\alpha)(\alpha^{n-n_1} - 1)}{2^{m-m_1} - 1}\right).$$

Since $n > 1600$, from (2.29) we conclude that

$$0 < |\Gamma_3| < \frac{2^6}{2^{0.8n}}.$$

Hence,

$$0 < \left| (n_1 - 1) \left(\frac{\log \alpha}{\log 2} \right) - m_1 + \frac{\log(f_k(\alpha)(\alpha^l - 1)/(2^j - 1))}{\log 2} \right| < (2^6/\log 2) \cdot 2^{-n},$$

where $(l, j) := (n - n_1,\ m - m_1)$. We apply Lemma 1.2.2 with the same τ_k, M_k, $(A_k, B_k) := (2^6/\log 2, 2)$ and

$$\mu_{k,l,j} := \frac{\log(f_k(\alpha)(\alpha^l - 1)/(2^j - 1))}{\log 2} \quad \text{for} \quad k \in [4, 790], \quad l \in [1, 1574] \quad and \quad j \in [1, 1570].$$

With the help of *Mathematica* we find that the maximum value of

$$\lfloor \log(114q/\varepsilon)/\log 2 \rfloor, \quad for \quad k \in [4, 790], \quad l \in [1, 1574] \quad and \quad j \in [1, 1570]$$

is < 1574. Thus, $n < 1574$, which contradicts the assumption that $n > 1600$ in Section 5.

We finish the resolution of the Diophantine equation (2.4), for this case, with the following procedure. Consider the following equivalent equation to (2.4)

$$F_n^{(k)} - F_{n_1}^{(k)} = 2^m - 2^{m_1}.$$

For $k \in [4, 790]$ and $n \in [k+2, 1600]$, let the set

$$F_{n,k} := \left\{ F_n^{(k)} - F_{n_1}^{(k)} \pmod{10^{20}} : n_1 \in [2, n-1] \right\},$$

and

$$D_{n,k} := \left\{ 2^m - 2^{m_1} \pmod{10^{20}} : m \in [\lfloor c(n-4) \rfloor, \lceil c(n-1) + 1 \rceil],\ m_1 \in [0, m-1] \right\}$$

with $c = \log \alpha / \log 2$. Note that we have used (2.24) to define the range of m in $D_{n,k}$. As in all computations of this paper, with the help of *Mathematica*, we looked for all (n,k) the intersections $F_{n,k} \cap D_{n,k}$. After an extensive search, we obtain that $F_{n,k} \cap D_{n,k}$ contains only the solutions corresponding to the families $(i) - (iv)$ in the statement of Theorem 2.1.1 for the current range of the variables.

This completes the proof in the case of small k.

2.5.3. The Case of large k.

In this case we assume that $k > 790$, we have already shown that the Diophantine equation (2.4) has only the solutions listed in Theorem 2.1.1.

3. On the problem of Pillai with k–generalized Fibonacci numbers and powers of 3

The presentation in this chapter is a slightly modified version of the paper [5] with the title *On the problem of Pillai with k–generalized Fibonacci numbers and powers of* 3. This is a joint work with *Florian Luca*. This paper has been accepted for publication in *International Journal of Number Theory*.

Abstract: In this paper, we find all integers c with at least two representations as a difference between a k-generalized Fibonacci number and a power of 3. This paper continues the previous work of [1] and [3].

Keywords: Generalized Fibonacci numbers; linear forms in logarithms; Baker's method.

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3.1. Introduction

Let $k \geqslant 2$ be an integer. We consider a generalization of Fibonacci sequence called the k–generalized Fibonacci sequence $\{F_n^{(k)}\}_{n\geqslant 2-k}$ defined in Section 1.3.

The generalised Fibonacci analogue of the problem of Pillai under the same conditions as in (1.15), concerns studying for fixed (k,ℓ) all values of the integer c such that the equation

$$F_n^{(k)} - F_m^{(\ell)} = c \tag{3.1}$$

has at least two solutions (n,m). We are not aware of a general treatment of equation (3.1) (namely, considering k and ℓ parameters), although the particular case when $\{k,\ell\} = \{2,3\}$ was treated in [25].

Ddamulira, Gómez, and Luca [9], studied the Diophantine equation

$$F_n^{(k)} - 2^m = c, \tag{3.2}$$

where k is also a parameter, which is a variation of equation (3.1). They determined all integers c such that equation (3.2) has at least two solutions (n,m). These c together with their multiple representations as in (3.2) turned out to be grouped into four parametric families.

In this paper, we study a related problem and we find all integers c admitting at least two representations of the form $F_n^{(k)} - 3^m$ for some positive integers k, n and m. This can be interpreted as solving the equation

$$F_n^{(k)} - 3^m = F_{n_1}^{(k)} - 3^{m_1} \quad (= c) \tag{3.3}$$

with $(n,m) \neq (n_1,m_1)$. The cases $k = 2$ and $k = 3$ have been solved completely by the first author in [1] and [3], respectively. So, we focus on the case $k \geqslant 4$.

Theorem 3.1.1. *For fixed integer $k \geq 4$, the Diophantine equation* (3.3) *with $n > n_1 \geq 2$ and $m > m_1 \geq 1$ has:*

(i) solutions with $c \in \{-1, 5, 13\}$ and $2 \leq n \leq k+1$, which arise from the classical Pillai problem for $(a,b) = (2,3)$, namely:

$$\begin{array}{rcrcl} F_5^{(k)} - 3^2 &=& F_3^{(k)} - 3^1 &=& -1, \quad k \geq 4, \\ F_7^{(k)} - 3^3 &=& F_5^{(k)} - 3^1 &=& 5, \quad k \geq 6, \\ F_{10}^{(k)} - 3^5 &=& F_6^{(k)} - 3^1 &=& 13, \quad k \geq 9; \end{array}$$

(ii) solutions with $c \in \{-25, -7, 5\}$ and $n \geq k+2$ and $k \in \{4,5,6\}$. Futhermore, all the representations of c in this case are given by

$$\begin{array}{rcrcl} F_8^{(4)} - 3^4 &=& F_3^{(4)} - 3^3 &=& -25, \\ F_{10}^{(5)} - 3^5 &=& F_3^{(5)} - 3^2 &=& -7, \\ F_{10}^{(6)} - 3^5 &=& F_6^{(6)} - 3^1 &=& 5. \end{array}$$

for $k = 4, 5$, and 6, respectively.

3.2. The connection with the classical Pillai problem

Assume that $(n,m) \neq (n_1,m_1)$ are such that

$$F_n^{(k)} - 3^m = F_{n_1}^{(k)} - 3^{m_1}.$$

If $m = m_1$, then $F_n^{(k)} = F_{n_1}^{(k)}$ and since $\min\{n, n_1\} \geq 2$, we get that $n = n_1$. Thus, $(n,m) = (n_1, m_1)$, contradicting our assumption. Hence, $m \neq m_1$, and we may assume without loss of generality that $m > m_1 \geq 1$. Since

$$F_n^{(k)} - F_{n_1}^{(k)} = 3^m - 3^{m_1}, \tag{3.4}$$

and the right–hand side of (3.4) is positive, we get that the left–hand side of (3.4) is also positive and so $n > n_1$. Furthermore, since $F_1^{(k)} = F_2^{(k)} = 1$, we may assume that $n > n_1 \geq 2$.

We analyse the possible situations.

Case 3.2.1. *Assume that* $2 \leq n_1 < n \leq k+1$.

Then, by (1.7), we have

$$F_{n_1}^{(k)} = 2^{n_1 - 2} \qquad \textit{and} \qquad F_n^{(k)} = 2^{n-2}$$

so, by substituting them in (3.4), we get

$$2^{n-2} - 3^m = 2^{n_1-2} - 3^{m_1}.$$

By comparing with the classical solutions in (1.16), and by using the fact that $F_n^{(k)}$ is a power of 2 if and only if $n \leq k+1$ (see [20]), we get the solutions

$$\begin{aligned} F_5^{(k)} - 3^2 = F_3^{(k)} - 3^1 &= -1, \quad k \geq 4 \\ F_7^{(k)} - 3^3 = F_5^{(k)} - 3^2 &= 5, \quad k \geq 6, \\ F_{10}^{(k)} - 3^5 = F_6^{(k)} - 3^1 &= 13, \quad k \geq 9. \end{aligned} \tag{3.5}$$

Case 3.2.2. *Assume* $n \geq k+2$.

The following lemma is useful.

Lemma 3.2.1. *For* $n \geq k+2$*, the conditions*

$$F_n^{(k)} - 3^m = F_{n_1}^{(k)} - 3^{m_1} \quad \textit{and} \quad 2^{n-2} - 3^m = 2^{n_1-2} - 3^{m_1}$$

cannot simultaneously hold.

Proof. If they do, then

$$2^{n-2} - F_n^{(k)} = 2^{n_1-2} - F_{n_1}^{(k)}.$$

The sequence $\{2^{n-2}-F_n^{(k)}\}_{n\geq 2}$ is 0 at $n=2,3,\ldots,k+1$ and is 1 at $n=k+2$. We show that from here on it is increasing. That is

$$2^{n-1}-F_{n+1}^{(k)} > 2^{n-2}-F_n^{(k)} \quad \text{holds for} \quad n\geq k+2.$$

This is equivalent to

$$2^{n-2} > F_{n+1}^{(k)}-F_n^{(k)} = F_{n-1}^{(k)}+\cdots+F_{n+1-k}^{(k)},$$

and this last inequality holds true because in the right–hand side we have $F_i \leq 2^{i-2}$ for $i = n+1-k, n+2-k, \ldots, n-1$ and then

$$\sum_{i=n-k+1}^{n-1} F_i \leq \sum_{i=n-k+1}^{n-1} 2^{i-2} < 1+2+\cdots+2^{n-3} < 2^{n-2}.$$

□

3.3. Bounding n in terms of m and k

By the results of the previous section, we assume that $n\geq k+2$. Thus, $2^{n-2}-3^m \neq 2^{n_1-2}-3^{m_1}$. Since $n>n_1\geq 2$, we have that $F_{n_1}^{(k)} \leq F_{n-1}^{(k)}$ and therefore

$$F_n^{(k)} = F_{n-1}^{(k)}+\cdots+F_{n-k}^{(k)} \geq F_{n-1}^{(k)}+\cdots+F_{n-k-1}^{(k)} \geq F_{n_1}^{(k)}+\cdots+F_{n-k-1}^{(k)}.$$

So, from the above, (1.13) and (3.4), we have

$$\begin{aligned} \alpha^{n-4} &\leq F_{n-2}^{(k)} \leq F_n^{(k)}-F_{n_1}^{(k)} = 3^m-3^{m_1} < 3^m, \text{ and} \\ \alpha^{n-1} &\geq F_n^{(k)} > F_n^{(k)}-F_{n_1}^{(k)} = 3^m-3^{m_1} \geq 3^{m-1}, \end{aligned} \tag{3.6}$$

leading to

$$1+\left(\frac{\log 3}{\log\alpha}\right)(m-1) < n < \left(\frac{\log 3}{\log\alpha}\right)m+4. \tag{3.7}$$

We note that the above inequality (3.7) in particular implies that $m<n<1.6m+4$. We assume for technical reasons that $n>600$. By (1.11) and (3.4), we get

$$\begin{aligned} \left|f_k(\alpha)\alpha^{n-1}-3^m\right| &= \left|(f_k(\alpha)\alpha^{n-1}-F_n^{(k)})+(F_{n_1}^{(k)}-3^{m_1})\right| \\ &= \left|(f_k(\alpha)\alpha^{n-1}-F_n^{(k)})+(F_{n_1}^{(k)}-f_k(\alpha)\alpha^{n_1-1})+(f_k(\alpha)\alpha^{n_1-1}-3^{m_1})\right| \\ &< \frac{1}{2}+\frac{1}{2}+\alpha^{n_1-1}+3^{m_1} \\ &< \alpha^{n_1}+3^{m_1} \\ &< 2\max\{\alpha^{n_1},3^{m_1}\}. \end{aligned}$$

In the above, we have also used the fact that $|f_k(\alpha)| < 1$ (see Lemma 1.3.1). Dividing through by 3^m, we get

$$\left|f_k(\alpha)\alpha^{n-1}3^{-m} - 1\right| < 2\max\left\{\frac{\alpha^{n_1}}{3^m}, 3^{m_1-m}\right\} < \max\{\alpha^{n_1-n+6}, 3^{m_1-m+1}\}, \qquad (3.8)$$

where for the right–most inequality in (3.8) we used (3.6) and the fact that $\alpha^2 > 2$.

For the left-hand side of (3.8) above, we apply Theorem 1.1.8 with the data

$$t := 3, \quad \eta_1 := f_k(\alpha), \quad \eta_2 := \alpha, \quad \eta_3 := 3, \quad b_1 := 1, \quad b_2 := n-1, \quad b_3 := -m.$$

We begin by noticing that the three numbers $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers and belong to the field $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. Put

$$\Lambda := f_k(\alpha)\alpha^{n-1}3^{-m} - 1.$$

To see why $\Lambda \neq 0$, note that otherwise, we would then have that $f_k(\alpha) = 3^m\alpha^{-(n-1)}$ and so $f_k(\alpha)$ would be an algebraic integer, which contradicts Lemma 1.3.1 (i).

Since $h(\eta_2) = (\log\alpha)/k < (\log 2)/k$ and $h(\eta_3) = \log 3$, it follows that we can take $A_2 := \log 2$ and $A_3 := k\log 3$. Further, in view of Lemma 1.3.1 (ii), we have that $h(\eta_1) < 3\log k$, so we can take $A_1 := 3k\log k$. Finally, since $\max\{1, n-1, m\} = n-1$, we take $B := n$.

Then, the left–hand side of (3.8) is bounded below, by Theorem 1.1.8, as

$$\log|\Lambda| > -1.4 \times 30^6 \times 3^{4.5} \times k^4(1+\log k)(1+\log n)(3\log k)(\log 2)(\log 3).$$

Comparing with (3.8), we get

$$\min\{(n-n_1-6)\log\alpha, (m-m_1-1)\log 3\} < 6.54 \times 10^{11}k^4\log^2 k(1+\log n),$$

which gives

$$\min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} < 6.60 \times 10^{11}k^4\log^2 k(1+\log n).$$

Now the argument is split into two cases.

Case 3.3.1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} = (n-n_1)\log\alpha.$

In this case, we rewrite (3.4) as

$$\begin{aligned}\left|f_k(\alpha)\alpha^{n-1} - f_k(\alpha)\alpha^{n_1-1} - 3^m\right| &= \left|(f_k(\alpha)\alpha^{n-1} - F_n^{(k)}) + (F_{n_1}^{(k)} - f_k(\alpha)\alpha^{n_1-1}) - 3^{m_1}\right| \\ &< \frac{1}{2} + \frac{1}{2} + 3^{m_1} \le 3^{m_1+1}.\end{aligned}$$

Dividing through by 3^m gives

$$\left|f_k(\alpha)(\alpha^{n-n_1}-1)\alpha^{n_1-1}3^{-m}-1\right| \quad < \quad 3^{m_1-m+1}. \tag{3.9}$$

Now we put

$$\Lambda_1 := f_k(\alpha)(\alpha^{n-n_1}-1)\alpha^{n_1-1}3^{-m}-1.$$

We apply again Theorem 1.1.8 with the following data

$$t:=3, \quad \eta_1 := f_k(\alpha)(\alpha^{n-n_1}-1), \quad \eta_2 := \alpha, \quad \eta_3 := 3, \quad b_1 := 1, \quad b_2 := n_1-1, \quad b_3 := -m.$$

As before, we begin by noticing that the three numbers η_1, η_2, η_3 belong to the field $\mathbb{K} := \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. To see why $\Lambda_1 \neq 0$, note that otherwise, we would get the relation $f_k(\alpha)(\alpha^{n-n_1}-1) = 3^m\alpha^{1-n_1}$. Conjugating this last equation with any automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ such that $\sigma(\alpha) = \alpha^{(i)}$ for some $i \geq 2$, and then taking absolute values, we arrive at the equality $|f_k(\alpha^{(i)})((\alpha^{(i)})^{n-n_1}-1)| = |3^m(\alpha^{(i)})^{1-n_1}|$. But this cannot hold because, $|f_k(\alpha^{(i)})||(\alpha^{(i)})^{n-n_1}-1| < 2$ since $|f_k(\alpha^{(i)})| < 1$ by Lemma 1.3.1 (i), and $|(\alpha^{(i)})^{n-n_1}| < 1$, since $n > n_1$, while $|3^m(\alpha^{(i)})^{1-n_1}| \geq 3$.

Since

$$h(\gamma_1) \leq h(f_k(\alpha)) + h(\alpha^{n-n_1}-1) < 3\log k + (n-n_1)\frac{\log\alpha}{k} + \log 2,$$

it follows that

$$kh(\gamma_1) < 6k\log k + (n-n_1)\log\alpha < 6k\log k + 6.60\times 10^{11}k^4\log^2 k(1+\log n).$$

So, we can take $A_1 := 6.80\times 10^{11}k^4\log^2 k(1+\log n)$. Further, as before, we take $A_2 := \log 2$ and $A_3 := k\log 3$. Finally, by recalling that $m < n$, we can take $B := n$.

We then get that

$$\log|\Lambda_1| > -1.4\times 30^6\times 3^{4.5}\times k^3(1+\log k)(1+\log n)(6.80\times 10^{11}k^4\log^2 k(1+\log n))(\log 2)(\log 3),$$

which yields

$$\log|\Lambda_1| > -7.41\times 10^{22}k^7\log^3 k(1+\log n)^2.$$

Comparing this with (3.9), we get that

$$(m-m_1)\log 3 < 7.50\times 10^{22}k^7\log^3 k(1+\log n)^2.$$

Case 3.3.2. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} = (m-m_1)\log 3$.

In this case, we write (3.4) as

$$\begin{aligned}\left|f_k(\alpha)\alpha^{n-1}-3^m+3^{m_1}\right| &= \left|(f_k(\alpha)\alpha^{n-1}-F_n^{(k)}) + (F_{n_1}^{(k)}-f_k(\alpha)\alpha^{n_1-1}) + f_k(\alpha)\alpha^{n_1-1}\right| \\ &< \frac{1}{2}+\frac{1}{2}+\alpha^{n_1-1} \quad < \quad \alpha^{n_1},\end{aligned}$$

so that

$$\left|f_k(\alpha)(3^{m-m_1}-1)^{-1}\alpha^{n-1}3^{-m_1}-1\right| < \frac{\alpha^{n_1}}{3^m-3^{m_1}} \leq \frac{2\alpha^{n_1}}{3^m} < \alpha^{n_1-n+6}. \tag{3.10}$$

The above inequality (3.10) suggests once again studying a lower bound for the absolute value of

$$\Lambda_2 := f_k(\alpha)(3^{m-m_1}-1)^{-1}\alpha^{n-1}3^{-m_1}-1.$$

We again apply Matveev's theorem with the following data

$$t := 3,\ \eta_1 := f_k(\alpha)(3^{m-m_1}-1)^{-1},\ \eta_2 := \alpha,\ \eta_3 := 3,\ b_1 := 1,\ b_2 := n-1,\ b_3 := -m_1.$$

We can again take $B := n$ and $\mathbb{K} := \mathbb{Q}(\alpha)$, so that $D := k$. We also note that, if $\Lambda_2 = 0$, then $f_k(\alpha) = \alpha^{-(n-n_1)}3^{m_1}(3^{m-m_1}-1)$ implying that $f_k(\alpha)$ is an algebraic integer, which is not the case. Thus, $\Lambda_2 \neq 0$.

Now, we note that

$$h(\eta_1) \leq h(f_k(\alpha)) + h(3^{m-m_1}-1) < 3\log k + (m-m_1+k)\frac{\log 3}{k}.$$

Thus, $kh(\eta_1) < 4k\log k + (m-m_1)\log 3 < 6.80\times 10^{11}k^4\log^2 k(1+\log n)$, and so we can take $A_1 := 6.80\times 10^{11}k^4\log^2 k(1+\log n)$. As before, we take $A_2 := \log 2$ and $A_3 := k\log 3$. It then follows from Matveev's theorem, after some calculations, that

$$\log|\Lambda_2| > -7.41\times 10^{22}k^7\log^3 k(1+\log n)^2.$$

From this and (3.10), we obtain that

$$(n-n_1)\log\alpha < 7.50\times 10^{22}k^7\log^3 k(1+\log n)^2.$$

Thus, in both Case 3.3.1 and Case 3.3.2, we have

$$\begin{aligned} \min\{(n-n_1)\log\alpha,(m-m_1)\log 2\} &< 6.6\times 10^{11}k^4\log^2 k(1+\log n),\\ \max\{(n-n_1)\log\alpha,(m-m_1)\log 2\} &< 7.5\times 10^{22}k^7\log^3 k(1+\log n)^2. \end{aligned} \tag{3.11}$$

We now finally rewrite equation (3.4) as

$$\left|f_k(\alpha)\alpha^{n-1}-f_k(\alpha)\alpha^{n_1-1}-3^m+3^{m_1}\right| = \left|(f_k(\alpha)\alpha^{n-1}-F_n^{(k)})+(F_{n_1}^{(k)}-f_k(\alpha)\alpha^{n_1-1})\right| < 1.$$

We divide through both sides by $3^m-3^{m_1}$ getting

$$\left|\frac{f_k(\alpha)(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\alpha^{n_1-1}3^{-m_1}-1\right| < \frac{1}{3^m-3^{m_1}} \leq \frac{2}{3^m} < 3^{5-0.8n}, \tag{3.12}$$

since $n < 1.6m+4$. To find a lower–bound on the left–hand side of (3.12) above, we again apply Theorem 1.1.8 with the data

$$t := 3,\ \eta_1 := \frac{f_k(\alpha)(\alpha^{n-n_1}-1)}{3^{m-m_1}-1},\ \eta_2 := \alpha,\ \eta_3 := 3,\ b_1 := 1,\ b_2 := n_1-1,\ b_3 := -m_1.$$

We also take $B := n$ and we take $\mathbb{K} := \mathbb{Q}(\alpha)$ with $D := k$. From the properties of the logarithmic height function, we have that

$$\begin{aligned} kh(\eta_1) &\leq k\big(h(f_k(\alpha)) + h(\alpha^{n-n_1}-1) + h(3^{m-m_1}-1)\big) \\ &< 3k\log k + (n-n_1)\log\alpha + k(m-m_1)\log 3 + 2k\log 2 \\ &< 8.3\times 10^{22}k^8\log^3 k(1+\log n)^2, \end{aligned}$$

where in the above chain of inequalities we used the bounds (3.11). So we can take $A_1 := 8.3\times 10^{22}k^8\log^3 k(1+\log n)^2$, and certainly as before we take $A_2 := \log 2$ and $A_3 := k\log 3$. We need to show that if we put

$$\Lambda_3 := \frac{f_k(\alpha)(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\alpha^{n_1-1}3^{-m_1}-1,$$

then $\Lambda_3 \neq 0$. To see why $\Lambda_3 \neq 0$, note that otherwise, we would get the relation

$$f_k(\alpha)(\alpha^{n-n_1}-1) = 3^{m_1}\alpha^{1-n_1}(3^{m-m_1}-1).$$

Again, as for the case of Λ_1, conjugating the above relation with an automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ such that $\sigma(\alpha) = \alpha^{(i)}$ for some $i \geq 2$, and then taking absolute values, we get that $|f_k(\alpha^{(i)})((\alpha^{(i)})^{n-n_1}-1)| = |3^{m_1}(\alpha^{(i)})^{1-n_1}(3^{m-m_1}-1)|$. This cannot hold true because in the left–hand side we have $|f_k(\alpha^{(i)})||(\alpha^{(i)})^{n-n_1}-1| < 2$, while in the right–hand side we have $|3^{m_1}||(\alpha^{(i)})^{1-n_1}||3^{m-m_1}-1| \geq 4$. Thus, $\Lambda_3 \neq 0$. Then Theorem 1.1.8 gives

$$\log|\Lambda_3| > -1.4\times 30^6\times 3^{4.5}k^{11}(1+\log k)(1+\log n)\left(8.3\times 10^{22}\log^3 k(1+\log n)^2\right)(\log 2)(\log 3),$$

which together with (3.12) gives

$$(0.8n-5)\log 3 < 9.05\times 10^{33}k^{11}\log^4 k(1+\log n)^3.$$

The above inequality leads to

$$n < 6.2\times 10^{34}k^{11}\log^4 k\log^3 n,$$

which can be equivalently written as

$$\frac{n}{(\log n)^3} < 6.2\times 10^{34}k^{11}\log^4 k. \tag{3.13}$$

We apply Lemma 1.5.1 with the data $m := 3,\ \ x := n,\ \ T := 6.2\times 10^{34}k^{11}\log^4 k$. Inequality (3.13) yields

$$\begin{aligned} n &< 8\times(6.2\times 10^{34}k^{11}\log^4 k)\log(6.2\times 10^{34}k^{11}\log^4 k)^3 \\ &< 4\times 10^{42}k^{11}(\log k)^7. \end{aligned} \tag{3.14}$$

We then record what we have proved so far as a lemma.

Lemma 3.3.1. *If* (n,m,n_1,m_1,k) *is a solution in positive integers to equation* (3.3) *with* $(n,m) \neq (n_1,m_1)$, $n > \min\{k+2, n_1+1\}$, $n_1 \geq 2$, $m > m_1 \geq 1$ *and* $k \geq 4$, *we then have that* $n < 4 \times 10^{42} k^{11} (\log k)^7$.

3.4. Reduction of the bounds on n

3.4.1. The cutoff k

We have from the above lemma that Baker's method gives

$$n < 4 \times 10^{42} k^{11} (\log k)^7.$$

By imposing that the above amount is at most $2^{k/2}$, we get

$$4 \times 10^{42} k^{11} (\log k)^7 \quad < \quad 2^{k/2}.$$

The inequality above holds for $k > 600$.

We now reduce the bounds and to do so we make use of Lemma 1.2.2 several times.

3.4.2. The Case of small k

We now treat the cases when $k \in [4,600]$. First, we consider equation (3.4) which is equivalent to (3.3). For $k \in [4,600]$ and $n \in [3,600]$, consider the sets

$$F_{n,k} := \left\{ F_n^{(k)} - F_{n_1}^{(k)} (\text{mod } 10^{20}) : \ n \in [3,600], \ n_1 \in [2, n-1] \right\}$$

and

$$D_{n,k} := \left\{ 3^m - 3^{m_1} (\text{mod } 10^{20}) : \ m \in [2,600], \ m_1 \in [1, m-1] \right\}.$$

With the help of *Mathematica*, we intersected these two sets and found the only solutions listed in Theorem 3.1.1.

Next, we note that for these values of k, Lemma 3.3.1 gives us absolute upper bounds for n. However, these upper bounds are so large that we wish to reduce them to a range where the solutions can be easily identified by a computer. To do this, we return to (3.8) and put

$$\Gamma := (n-1)\log\alpha - m\log 3 + \log(f_k(\alpha)). \tag{3.15}$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geq 20$. In the case that this condition fails, we consider one of the following inequalities instead:

(i) if $n-n_1<20$ but $m-m_1\geq 20$, we consider (3.9);
(ii) if $n-n_1\geq 20$ but $m-m_1<20$, we consider (3.10);
(iii) if $n-n_1<20$ but $m-m_1<20$, we consider (3.12).

We start by considering (3.8). Note that $\Gamma\neq 0$; thus we distinguish the following two cases. If $\Gamma>0$, then $e^{\Gamma}-1>0$, then from (3.8) we get

$$0<\Gamma<e^{\Gamma}-1<\max\left\{\alpha^{n_1-n+6},3^{m_1-m+1}\right\}.$$

Next we suppose that $\Gamma<0$. Since $\Lambda=|e^{\Gamma}-1|<\frac{1}{2}$, we get that $e^{|\Gamma|}<2$. Therefore,

$$0<|\Gamma|\leq e^{|\Gamma|}-1=e^{|\Gamma|}|e^{\Gamma}-1|<2\max\left\{\alpha^{n_1-n+6},3^{m_1-m+1}\right\}.$$

Therefeore, in any case, the following inequality holds

$$0<|\Gamma|<2\max\left\{\alpha^{n_1-n+6},3^{m_1-m+1}\right\}. \tag{3.16}$$

By replacing Γ in the above inequality by its formula and dividing through by $\log 3$, we then conclude that

$$0<\left|(n-1)\left(\frac{\log\alpha}{\log 3}\right)-m+\frac{\log(f_k(\alpha))}{\log 3}\right|<\max\left\{(2\alpha^6)\cdot\alpha^{-(n-n_1)},\frac{6}{\log 3}\cdot 3^{-(m-m_1)}\right\}$$

Then, we apply Lemma 1.2.2 with the following data

$$k\in[4,600],\quad \tau_k:=\frac{\log\alpha}{\log 3},\quad \mu_k:=\frac{\log(f_k(\alpha))}{\log 3},\quad (A_k,B_k):=(2\alpha^6,\alpha)\text{ or }\left(\frac{6}{\log 3},3\right).$$

Next, we put $M_k:=\lfloor 4\times 10^{42}k^{11}(\log k)^7\rfloor$, which is the absolute upper bound on n by Lemma 3.3.1. An intensive computer search in *Mathematica*, revealed that the maximum value of $\lfloor\log(2\alpha^6q/\varepsilon)/\log\alpha\rfloor$ is <600 and the maximum value of $\lfloor\log((6/\log 3)q/\varepsilon)/\log 3\rfloor$ is <375. Thus, either

$$n-n_1<\frac{\log(2\alpha^6q/\varepsilon)}{\log\alpha}<600,\quad\text{or } m-m_1<\frac{\log((6/\log 3)q/\varepsilon)}{\log 3}<375.$$

Therefore, we have that either $n-n_1\leq 600$ or $m-m_1\leq 375$.

Now, let us assume that $n-n_1\leq 600$. In this case, we consider the inequality (3.9) and assume that $m-m_1\geq 20$. Then we put

$$\Gamma_1:=(n_1-1)\log\alpha-m\log 3+\log((f_k(\alpha)(\alpha^{n-n_1}-1)).$$

By similar arguments as in the previous step for proving (3.16), from (3.9) we get

$$0<|\Gamma_1|<\frac{6}{3^{m-m_1}},$$

and replacing Γ_1 with its formula and dividing through by $\log 3$ gives

$$0 < \left|(n_1-1)\left(\frac{\log\alpha}{\log 3}\right) - m + \frac{\log(f_k(\alpha)(\alpha^{n-n_1}-1))}{\log 3}\right| < \frac{6}{\log 3}\cdot 3^{-(m-m_1)}. \tag{3.17}$$

As before, we keep the same $\tau_k,\ M_k,\ (A_k,B_k) := ((6/\log 3),3)$ and put

$$\mu_{k,\ell} := \frac{\log(f_k(\alpha)(\alpha^{\ell}-1))}{\log 3},\quad k\in[4,600],\quad \ell := n-n_1\in[1,600].$$

We apply Lemma 1.2.2 to the inequality (3.17) with the above data. A computer search in *Mathematica*, revealed that the maximum value of $\lfloor\log(Aq/\varepsilon)/\log B\rfloor$ over the values of $k\in[4,600]$ and $\ell\in[1,600]$ is <377. Hence, $m-m_1\le 377$.

Next, we assume that $m-m_1\le 375$. Here, we consider the inequality (3.10) and also assume that $n-n_1\ge 20$. We put

$$\Gamma_2 := (n-1)\log\alpha - m_1\log 3 + \log\left(f_k(\alpha)/(3^{m-m_1}-1)\right).$$

Thus, by the same arguments as before, we get

$$0<|\Gamma_2|<\frac{2\alpha^6}{\alpha^{n-n_1}}.$$

By substituting for Γ_2 with its formula and dividing through by $\log 3$ in the above inequality, we get

$$0 < \left|(n-1)\left(\frac{\log\alpha}{\log 3}\right) - m_1 + \frac{\log\left(f_k(\alpha)/(3^{m-m_1}-1)\right)}{\log 3}\right| < \frac{2\alpha^6}{\log 3}\cdot\alpha^{-(n-n_1)}.$$

As before, we apply Lemma 1.2.2 with the same $\tau_k,\ M_k,\ (A_k,B_k) := (2\alpha^6/\log 3,\alpha)$ and put

$$\mu_{k,j} := \frac{\log\left(f_k(\alpha)/(3^{m-m_1}-1)\right)}{\log 3},\quad k\in[4,600],\quad j := m-m_1\in[1,375].$$

A computer search with *Mathematica*, revealed that the maximum value of $\lfloor\log(Aq/\varepsilon)/\log B\rfloor$, for $k\in[4,600]$ and $j\in[1,375]$ is <603. Hence, $n-n_1\le 603$.

To conclude the above computations, first we got that either $n-n_1\le 600$ or $m-m_1\le 375$. If $n-n_1\le 600$, then $m-m_1\le 377$, and if $m-m_1\le 375$, then $n-n_1\le 603$. Therefore, we can conclude that we always have

$$n-n_1\le 603 \quad \text{and } m-m_1\le 377.$$

Finally, we go to (3.12) and put

$$\Gamma_3 := (n_1-1)\log\alpha - m_1\log 3 + \log\left(\frac{f_k(\alpha)(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\right).$$

Since $n > 600$, from (3.12) we can conclude that

$$0 < |\Gamma_3| < \frac{2\cdot 3^5}{3^{0.8n}}.$$

Hence, by substituting for Γ_3 by its formula and dividing through by $\log 3$, we get

$$0 < \left|(n_1-1)\left(\frac{\log\alpha}{\log 3}\right) - m_1 + \frac{\log\left(f_k(\alpha)(\alpha^{n-n_1}-1)/(3^{m-m_1}-1)\right)}{\log 3}\right| < 1328\cdot 3^{-0.8n}.$$

We apply Lemma 1.2.2 with the same τ_k, M_k, $(A_k,B_k) := (1328,3)$, $k\in[4,600]$, and put

$$\mu_{k,l,j} := \frac{\log\left(f_k(\alpha)(\alpha^l-1)/(3^j-1)\right)}{\log 3}, \quad l := n-n_1\in[1,603], \quad j := m-m_1\in[1,377].$$

A computer search in *Mathematica*, revealed that the maximum value of $\lfloor\log(1328q/\varepsilon)/\log 3\rfloor$, for $k\in[4,600]$, $l\in[1,603]$ and $j\in[1,377]$ is < 378. Hence, $n < 473$, which contradicts the assumption that $n > 500$ in the previous section.

3.4.3. The case of large k

We now assume that $k > 600$. Note that for these values of k we have

$$n < 4\times 10^{42}k^{11}(\log k)^7.$$

Since, $n \geq k+2$, we have that $n \geq 602$. The following lemma is useful.

Lemma 3.4.1. *For $1\leq n < 2^{k/2}$ and $k\geq 10$, we have*

$$F_n^{(k)} = 2^{n-2}(1+\zeta) \quad \textit{where} \quad |\zeta| < \frac{5}{2^{k/2}}.$$

Proof. When $n\leq k+1$, we have $F_n^{(k)} = 2^{n-2}$ so we can take $\zeta := 0$. So, assume $k+2\leq n < 2^{k/2}$. It follows from (1.8) in [22] that

$$|f_k(\alpha)\alpha^{n-1} - 2^{n-2}| < \frac{2^n}{2^{k/2}}.$$

By (1.11), we also have $\left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < 1/2$. Thus,

$$\begin{aligned}
|F_n^{(k)} - 2^{n-2}| &\leq |f_k(\alpha)\alpha^{n-1} - 2^{n-2}| + |F_n^{(k)} - f_k(\alpha)\alpha^{n-1}| \\
&< \frac{2^n}{2^{k/2}} + \frac{1}{2} = \frac{2^n}{2^{k/2}}\left(1+\frac{1}{2^{n-k/2+1}}\right) \leq \frac{2^n}{2^{k/2}}\left(1+\frac{1}{2^{k/2+3}}\right) \\
&< \frac{2^n\cdot 1.25}{2^{k/2}} = \left(\frac{5}{2^{k/2}}\right)\cdot 2^{n-2}.
\end{aligned}$$

□

By the above lemma, we can rewrite (3.4) as

$$2^{n-2}(1+\zeta) - 2^{n_1-2}(1+\zeta_1) = 3^m - 3^{m_1}, \qquad \max\{|\zeta|, |\zeta_1|\} < \frac{5}{2^{k/2}}.$$

So,

$$\begin{aligned} |2^{n-2} - 3^m| &= |-\zeta \cdot 2^{n-2} + 2^{n_1-2}(1+\zeta_1) - 3^{m_1}| \\ &\leq 2^{n-2}\left(\frac{5}{2^{k/2}}\right) + 2^{n_1-2}\left(1+\frac{5}{2^{k/2}}\right) + 3^{m_1}. \end{aligned} \tag{3.18}$$

Next, we have

$$2^{n-2} > F_n^{(k)} - F_{n_1}^{(k)} = 3^m - 3^{m_1} \geq 2 \cdot 3^{m-1}, \quad \text{so} \quad 2^{n-2}/3^m > 2/3.$$

Further,

$$\begin{aligned} 3^m > 3^m - 3^{m_1} &= F_n^{(k)} - F_{n_1}^{(k)} \geq F_n^{(k)} - F_{n-1}^{(k)} \\ &\geq F_{n-2}^{(k)} > 2^{n-4}\left(1 - \frac{5}{2^{k/2}}\right) \\ &> 2^{n-4}\left(\frac{27}{32}\right) \quad (k > 10), \end{aligned}$$

so

$$\frac{128}{27} > \frac{2^{n-2}}{3^m} > \frac{2}{3}. \tag{3.19}$$

Going back to (3.18), we have

$$|3^m 2^{-(n-2)} - 1| < \frac{5}{2^{k/2}} + \frac{1.25}{2^{n-n_1}} + \frac{3^{m_1}}{(2/3)3^m} = \frac{5}{2^{k/2}} + 1.5\left(\frac{1}{2^{n-n_1}} + \frac{1}{3^{m-m_1}}\right).$$

Thus,

$$|3^m 2^{-(n-2)} - 1| < 8 \max\left\{\frac{1}{2^{n-n_1}}, \frac{1}{3^{m-m_1}}, \frac{1}{2^{k/2}}\right\}. \tag{3.20}$$

We now apply Theorem 1.1.8 on the left-hand side of (3.20) with the data

$$\Gamma := 3^m 2^{-(n-2)} - 1, \; t := 2, \; \eta_1 := 3, \; \eta_2 := 2, \; b_1 := m, \; b_2 := -(n-2).$$

It is clear that $\Gamma \neq 0$, otherwise we would get $3^m = 2^{n-2}$ which is a contradiction since 3^m is odd while 2^{n-2} is even. We consider the field $\mathbb{K} := \mathbb{Q}$, in this case $D := 1$. Since $h(\eta_1) = h(3) = \log 3$ and $h(\eta_2) = h(2) = \log 2$, we can take $A_1 := \log 3$ and $A_2 := \log 2$. We also take $B := n$. Then, by Theorem 1.1.8, the left-hand side of (3.20) is bounded below as

$$\log|\Gamma| > -5.86 \times 10^8 (1 + \log n). \tag{3.21}$$

By comparing with (3.20), we get

$$\min\{(n-n_1-3)\log 2,\ (m-m_1-2)\log 3,\ (k/2-3)\log 2\} < 5.86\times 10^8(1+\log n),$$

which implies that

$$\min\{(n-n_1)\log 2,\ (m-m_1)\log 3,\ (k/2)\log 2\} < 5.88\times 10^8(1+\log n). \tag{3.22}$$

Now the argument is split into four cases.

Case 3.4.1. $\min\{(n-n_1)\log 2,\ (m-m_1)\log 3,\ (k/2)\log 2\} = (k/2)\log 2.$

In this case, we have

$$(k/2)\log 2 < 5.88\times 10^8(1+\log n),$$

which implies that

$$k < 1.70\times 10^9(1+\log n).$$

Case 3.4.2. $\min\{(n-n_1)\log 2,\ (m-m_1)\log 3,\ (k/2)\log 2\} = (n-n_1)\log 2.$

We rewrite (3.4) as

$$\begin{aligned} |3^m - 2^{n_1-2}(2^{n-n_1}-1)| &= |3^{m_1}+2^{n-2}\zeta-2^{n_1-2}\zeta_1| \\ &< 3^{m_1}+2^{n-2}\left(\frac{10}{2^{k/2}}\right), \end{aligned}$$

which implies that

$$\left|3^m 2^{-n_1}(2^{n-n_1}-1)^{-1}-1\right| < 20\max\left\{\frac{1}{3^{m-m_1}},\frac{1}{2^{k/2}}\right\}. \tag{3.23}$$

We now apply Matveev's theorem, Theorem 1.1.8 on the left-hand side of (3.23) to

$$\Gamma_1 := 3^m 2^{-(n_1-2)}(2^{n-n_1}-1)^{-1}-1,$$

$$t:=3,\ \eta_1:=3,\ \eta_2:=2,\ \eta_3:=2^{n-n_1}-1,\ b_1:=m,\ b_2:=-(n_1-2),\ b_3:=-1.$$

Note that $\Gamma_1\neq 0$. Otherwise, $3^m=2^{n-2}-2^{n_1-2}$, so $n_1=2$, and $2^{n-2}-3^m=1$, so $n\leq 4$ by classical results on Catalan's equation, which is a contradiction because $n\geq k+2>602$. We use the same values, $A_1 := \log 3$, $A_2 := \log 2$, $B := n$ as in the previous step. In order to find A_3, note that

$$h(\eta_3) = h(2^{n-n_1} - 1) \leq (n - n_1 + 1)\log 2 < 5.90 \times 10^8(1 + \log n).$$

So, we take $A_3 := 5.90 \times 10^8(1 + \log n)$. By Theorem 1.1.8, we have

$$\log|\Gamma_1| > -6.43 \times 10^{19}(1 + \log n)^2.$$

By comparing with (3.23), we get

$$\min\{(m - m_1 - 3)\log 3,\ (k/2 - 5)\log 2\} < 6.43 \times 10^{19}(1 + \log n)^2,$$

which implies that

$$\min\{(m - m_1)\log 3, (k/2)\log 2\} < 6.44 \times 10^{19}(1 + \log n)^2.$$

At this step, we have that either

$$(m - m_1)\log 3 < 6.44 \times 10^{19}(1 + \log n)^2$$

or

$$k < 1.86 \times 10^{20}(1 + \log n)^2.$$

Case 3.4.3. $\min\{(n - n_1)\log 2,\ \ (m - m_1)\log 3,\ \ (k/2)\log 2\} = (m - m_1)\log 3.$

We rewrite (3.4) as

$$\begin{aligned} |(3^{m_1}(3^{m-m_1} - 1) - 2^{n-2}| &= |2^{n-2}\zeta - 2^{n_1-2}(1 + \zeta_1)| \\ &< 2^{n-2}\left(\frac{5}{2^{k/2}}\right) + 2^{n_1-2}\left(1 + \frac{5}{2^{k/2}}\right), \end{aligned}$$

which implies that

$$\left|3^{m_1}(3^{m-m_1} - 1)2^{-(n-2)} - 1\right| < 20\max\left\{\frac{1}{2^{n-n_1}}, \frac{1}{2^{k/2}}\right\}. \tag{3.24}$$

We again apply Matveev's theorem, Theorem 1.1.8 on the left-hand side of (3.23) which is

$$\Gamma_2 := 3^{m_1}2^{-(n-2)}(3^{m-m_1} - 1) - 1,$$

$$t := 3,\ \eta_1 := 3,\ \eta_2 := 2,\ \eta_3 := (3^{m-m_1} - 1),\ b_1 := m_1,\ b_2 := -(n-2),\ b_3 := 1.$$

Note that $\Gamma_2 \neq 0$. Otherwise, $3^m - 3^{m_1} = 2^{n-2}$, which is impossible since the left–hand side is a multiple of 3 and the right–hand side isn't. We use the same values, $A_1 := \log 3$, $A_2 := \log 2$, $B := n$ as in the previous steps. In order to determine A_3, note that

$$h(\eta_3) = h(3^{m-m_1} - 1) \leq (m - m_1 + 1)\log 3 < 5.90 \times 10^8 (1 + \log n).$$

So, we take $A_3 := 5.90 \times 10^8 (1 + \log n)$. By Theorem 1.1.8, we have the lower bound

$$\log|\Gamma_2| > -6.43 \times 10^{19} (1 + \log n)^2.$$

By comparing with (3.24), we get

$$\min\{(n - n_1 - 5)\log 3,\ (k/2 - 5)\log 2\} < 6.43 \times 10^{19} (1 + \log n)^2,$$

which implies that

$$\min\{(n - n_1)\log 3, (k/2)\log 2\} < 6.44 \times 10^{19} (1 + \log n)^2.$$

As before, at this step we have that either

$$(n - n_1)\log 3 < 6.44 \times 10^{19} (1 + \log n)^2$$

or

$$k < 1.86 \times 10^{20} (1 + \log n)^2.$$

Therefore, in Case 3.4.1, Case 3.4.2, and Case 3.4.3, we got

$$\begin{aligned} \min\{(n - n_1)\log 2,\ (m - m_1)\log 3,\ (k/2)\log 2\} &< 5.88 \times 10^8 (1 + \log n) \\ \max\{(n - n_1)\log 2,\ (m - m_1)\log 3,\ (k/2)\log 2\} &< 6.44 \times 10^{19} (1 + \log n)^2. \end{aligned} \tag{3.25}$$

Case 3.4.4. $(k/2)\log 2 > 6.44 \times 10^{19} (1 + \log n)^2$.

From the previous analysis, we conclude that one of $(n - n_1)\log 2$ and $(m - m_1)\log 3$ is bounded by $5.88 \times 10^8 (1 + \log n)$ and the other one by $6.44 \times 10^{19} (1 + \log n)^2$. We rewrite (3.4) as

$$\left|3^{m_1}(3^{m-m_1} - 1) - 2^{n_1-2}(2^{n-n_1} - 1)\right| = |\zeta| \cdot 2^{n-2} + |\zeta_1| \cdot 2^{n_1-2} \leq 2^{n-2}\left(\frac{10}{2^{k/2}}\right),$$

which implies that

$$\left|3^{m_1} 2^{-(n_1-2)} \left(\frac{3^{m-m_1} - 1}{2^{n-n_1} - 1}\right) - 1\right| < \frac{20}{2^{k/2}}. \tag{3.26}$$

We apply Matveev's Theorem to

$$\Gamma_3 := 3^{m_1} 2^{-(n_1-2)} \left(\frac{3^{m-m_1}-1}{2^{n-n_1}-1} \right) - 1,$$

with the data

$$t =: 3,\ \eta_1 := 3,\ \eta_2 := 2,\ \eta_3 := \left(\frac{3^{m-m_1}-1}{2^{n-n_1}-1} \right),\ b_1 := m_1,\ b_2 := -(n_1-2),\ b_3 := 1.$$

Note that $\Gamma_3 \neq 0$, otherwise, we get $2^n - 3^m = 2^{n_1} - 3^{m_1}$ which is impossible by Lemma 3.2.1.

As before we take $B := n$, $A_1 := \log 3$, $A_2 := \log 2$. In oder to determine an acceptable value for A_3, note that

$$\begin{array}{rcl} h(\eta_3) & \leq & h(3^{m-m_1}-1) + h(2^{n-n_1}-1) < (m-m_1+1)\log 3 + (n-n_1+1)\log 2 \\ & < & 2 \times 6.46 \times 10^{19}(1+\log n)^2 < 1.30 \times 10^{20}(1+\log n)^2. \end{array}$$

Thus, we take $A_3 := 1.30 \times 10^{20}(1+\log n)^2$. By Theorem 1.1.8, we have

$$\log |\Gamma_3| > -1.86 \times 10^{31}(1+\log n)^3.$$

By comparing with (3.26), we get

$$(k/2-5)\log 2 < 1.86 \times 10^{31}(1+\log n)^3,$$

which implies that

$$k < 5.42 \times 10^{31}(1+\log n)^3. \tag{3.27}$$

Thus, inequality (3.27) holds in all four cases. Since $n < 4 \times 10^{42} k^{11} (\log k)^7$, then

$$k < 5.42 \times 10^{31} \left(1 + \log\left(4 \times 10^{42} k^{11} (\log k)^7\right)\right)^3, \tag{3.28}$$

which gives the absolute upper bounds

$$k < 8.631 \times 10^{40} \ < 10^{41}$$

and

$$m < n < 3.44 \times 10^{506} < 10^{507}.$$

We record what we have proved.

Lemma 3.4.2. *We have*

$$k < 10^{41} \qquad \textit{and} \qquad m < 10^{507}.$$

3.4.4. The final reduction

The previous bounds are too large, so we need to reduce them by applying a Baker-Davenport reduction procedure. First, we go to (3.20) and let

$$z := m\log 3 - (n-2)\log 2.$$

Assume $m - m_1 > 1066$, $n - n_1 > 1690$ and $k > 600$. Then, we note that (3.23) can be rewritten as

$$|e^z - 1| < \max\{2^{n_1-n+3},\ 3^{m_1-m+2},\ 2^{-k/2+3}\}.$$

If $z > 0$, then $e^z - 1 > 0$, so we obtain

$$0 < z < e^z - 1 < \max\{2^{n_1-n+3},\ 3^{m_1-m+2},\ 2^{-k/2+3}\}.$$

Suppose now that $z < 0$. Since $\Gamma = |e^z - 1| < 1/2$, we get that $e^{|z|} < 2$. Thus,

$$0 < |z| \le e^{|z|} - 1 = e^{|z|}|e^z - 1| < 2\max\{2^{n_1-n+3},\ 3^{m_1-m+2},\ 2^{-k/2+3}\}.$$

Therefore, in any case we have that the inequality

$$0 < |z| < 2\max\{2^{n_1-n+3},\ 3^{m_1-m+2},\ 2^{-k/2+3}\} \tag{3.29}$$

always holds. By replacing z in the above inequality by its formula and dividing through by $m\log 2$, we get that

$$0 < \left|\frac{\log 3}{\log 2} - \frac{n}{m}\right| < \max\left\{\frac{24}{2^{n-n_1}m},\ \frac{26}{3^{m-m_1}m},\ \frac{24}{2^{k/2}m}\right\}. \tag{3.30}$$

Then

$$\max\left\{\frac{24}{2^{n-n_1}m},\ \frac{26}{3^{m-m_1}m},\ \frac{24}{2^{k/2}m}\right\} < \frac{1}{2m^2},$$

because $m < 10^{507}$. By the Legendre criterion Lemma 1.2.1, it follows that n/m is a convergent of $\log 3/\log 2$. So n/m is of the form p_ℓ/q_ℓ for some $l = 0,1,2,\ldots,972$. Then $n/m = p_\ell/q_\ell$ implies that $m = dq_\ell$ for some $d \ge 1$. Thus,

$$\frac{1}{(a_{\ell+1}+2)q_k q_{\ell+1}} < \left|\frac{\log 3}{\log 2} - \frac{p_\ell}{q_\ell}\right| < \max\left\{\frac{24}{2^{n-n_1}dq_\ell},\ \frac{26}{3^{m-m_1}dq_\ell},\ \frac{24}{2^{k/2}dq_\ell}\right\}.$$

Since $\max\{a_{l+1} : l = 0,1,2,\ldots,972] = 3308$, we get that

$$\min\{2^{n-n_1}, 3^{m-m_1}, 2^{k/2}\} \le 26\cdot 3310 q_{973}.$$

With the help of *Mathematica*, we have $q_{973} \approx 1.6834 \times 10^{507}$. We then conclude that one of the following inequalities holds:

$$n - n_1 < 1690, \qquad m - m_1 < 1066, \qquad k < 3380.$$

Suppose first that $m - m_1 > 10$ and $k \geq 20$, we go back to (3.23) and let

$$z_1 := m \log 3 - (n_1 - 2) \log 2 - \log(2^{n-n_1} - 1). \tag{3.31}$$

Then we note that (3.23) can be rewritten as

$$|e^{z_1} - 1| < \max\{3^{m_1-m+3}, \ 2^{-k/2+5}\}.$$

This implies that

$$0 < |z_1| < 2 \max\{3^{m_1-m+3}, \ 2^{-k/2+5}\}.$$

This also holds when $m - m_1 < 10$ and $k < 20$. By substituting for z_1 and dividing through by $\log 2$, we get

$$0 < \left| m \left(\frac{\log 3}{\log 2} \right) - (n_1 - 2) + \frac{\log(1/(2^{n-n_1} - 1))}{\log 2} \right| < \max\{98 \cdot 3^{-(m-m_1)}, \ 94 \cdot 2^{-k/2}\}.$$

We put

$$\tau := \frac{\log 3}{\log 2}, \qquad \mu := \frac{\log(1/(2^{n-n_1} - 1))}{\log 2}, \qquad (A, B) := (78, 3) \quad \text{or} \quad (94, 2),$$

where $n - n_1 \in [1, 1690]$. We take $M := 10^{507}$. A computer search in *Mathematica* reveals that $q = q_{977} \approx 5.708 \times 10^{510} > 6M$ and the minimum positive value of $\varepsilon := ||\mu q|| - M||\tau q|| > 0.0186$. Thus, Lemma 1.2.2 tells us that either $m - m_1 \leq 1078$ or $k \leq 3418$.

Next, we suppose that $n - n_1 > 10$, $k > 20$ and go to (3.24) and let

$$z_2 := m_1 \log 3 - (n - 2) \log 2 + \log(3^{m-m_1} - 1). \tag{3.32}$$

Then we also note that (3.24) can be rewritten as

$$|e^{z_2} - 1| < \max\{2^{n_1-n+5}, \ 2^{-k/2+5}\}.$$

This gives

$$0 < |z_2| < 2 \max\{2^{n_1-n+5}, \ 2^{-k/2+5}\}.$$

This also holds for $n - n_1 < 10$ and $k < 20$ as well. By substituting for z_2 and dividing through by $\log 2$, we get

$$0 < \left| m_1 \left(\frac{\log 3}{\log 2} \right) - (n - 2) + \frac{\log(3^{m-m_1} - 1)}{\log 2} \right| < \max\{94 \cdot 2^{-(n-n_1)}, \ 94 \cdot 2^{-k/2}\}.$$

We put

$$\tau := \frac{\log 3}{\log 2}, \qquad \mu := \frac{\log(3^{m-m_1}-1)}{\log 2}, \qquad (A,B) := (94,2),$$

where $m - m_1 \in [1, 1066]$. We keep the same M and q as in the previous step. A computer search in *Mathematica* reveals that the minimum positive value of $\varepsilon := ||\mu q|| - M||\tau q|| > 0.0372$. Thus, Lemma 1.2.2 tells us that either $n - n_1 \leq 1708$ or $k \leq 3416$.

Lastly, we assume that $k > 20$ and go to (3.26) and let

$$z_3 := m_1 \log 3 - (n_1 - 2)\log 2 - \log((3^{m-m_1}-1)/(2^{n-n_1}-1)). \tag{3.33}$$

We note that (3.26) can be rewritten as

$$|e^{z_3} - 1| < 2^{-k/2+5}.$$

This gives

$$0 < |z_3| < 2^{-k/2+6},$$

which also holds when $k < 20$. By substituting for z_3 and dividing through by $\log 2$, we get

$$0 < \left| m_1 \left(\frac{\log 3}{\log 2}\right) - (n_1 - 2) + \frac{\log((3^{m-m_1}-1)/(2^{n-n_1}-1))}{\log 2} \right| < 94 \cdot 2^{-k/2}.$$

We put

$$\tau := \frac{\log 3}{\log 2}, \qquad \mu := \frac{\log((3^{m-m_1}-1)/(2^{n-n_1}-1))}{\log 2}, \qquad (A,B) := (94,2),$$

where $n - n_1 \in [1, 1708]$ and $m - m_1 \in [1, 1074]$. We keep the same M and q as before. A computer search in *Mathematica*, reveals that the minimum positive value of $\varepsilon := ||\mu q|| - M||\tau q|| > 0.00058$. Thus, Lemma 1.2.2 tells us that $k \leq 3428$.

Therefore, in all cases we found out that $k < 3428$ which gives that $n < 7.2741 \times 10^{87} < 10^{88}$. These bounds are still too large. We repeat the above procedure several times by adjusting the values of M with respect to the new bounds of n. We summarise the data for the iterations performed in Table 3.1.

	M	$n - n_1 \leq$	$m - m_1 \leq$	$k \leq$
1	10^{507}	1708	1074	3428
2	10^{88}	319	197	662
3	10^{80}	287	180	590
4	10^{79}	282	180	584
5	10^{79}	282	180	584

Table 3.1.: Computation results

From the data displayed in the above table, it is evident that after four times of the iteration, the upper bound on k stabilizes at 584. Hence, $k < 600$ which contradicts our assumption that $k > 600$. Therefore, we have no further solutions to the Diophantine equation (3.3) with $k > 600$.

4. On the x–coordinates of Pell equations which are k–generalized Fibonacci numbers

The presentation in this chapter is a slightly modified version of the paper [6] with the title *On the x–coordinates of Pell equations which are k–generalized Fibonacci numbers*. This is joint work with *Florian Luca*. The article has been published in *Journal of Number Theory* in February, 2020.

Abstract: In this paper, for an integer $d \geq 2$ which is square free, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 1$, which is a k–generalized Fibonacci number, with a few exceptions that we completely characterize. This paper extends previous work from [47] for the case $k = 2$ and [49] for the case $k = 3$.

Keywords: Pell equation; generalized Fibonacci sequence; linear form in logarithms; reduction method.

2010 Mathematics Subject Classification: 11A25, 11B39, 11J86.

4.1. Introduction

Recently, Luca and Togbé [47] considered the Diophantine equation

$$x_n = F_m, \tag{4.1}$$

where $\{F_m\}_{m\geqslant 0}$ is the sequence of Fibonacci numbers. They proved that equation (4.1) has at most one solution (n,m) in positive integers except for $d = 2$, in which case equation (4.1) has the three solutions $(n,m) = (1,1),(1,2),(2,4)$.

Luca, Montejano, Szalay, and Togbé [49] considered the Diophantine equation

$$x_n = T_m, \tag{4.2}$$

where $\{T_m\}_{m\geqslant 0}$ is the sequence of Tribonacci numbers. They proved that equation (4.2) has at most one solution (n,m) in positive integers for all d except for $d = 2$ when equation (4.2) has the three solutions $(n,m)=(1,1),(1,2),(3,5)$ and when $d=3$ case in which equation (4.2) has the two solutions $(n,m)=(1,3),(2,5)$.

The purpose of this paper is to generalize the previous results. Let $k\geq 2$ be an integer. We consider a generalization of Fibonacci sequence called the k–generalized Fibonacci sequence $\{F_m^{(k)}\}_{m\geq 2-k}$ defined in Section 1.3.

4.2. Main Result

In this paper, we show that there is at most one value of the positive integer x participating in (1.17) which is a k–generalized Fibonacci number, with a couple of parametric exceptions that we completely characterize. This can be interpreted as solving the system of equations

$$x_{n_1}=F_{m_1}^{(k)}, \qquad x_{n_2}=F_{m_2}^{(k)}, \tag{4.3}$$

with $n_2>n_1\geq 1$, $m_2>m_1\geq 2$ and $k\geq 2$. The fact that $F_1^{(k)}=F_2^{(k)}=1$, allows us to assume that $m\geq 2$. That is, if $F_m^{(k)}=1$ for some positive integer m, then we will assume that $m=2$. As we already mentioned, the cases $k=2$ and $k=3$ have been solved completely by Luca and Togbé [47] and Luca, Montejano, Szalay, and Togbé [49], respectively. So, we focus on the case $k\geq 4$.

We put $\varepsilon:=x_1^2-dy_1^2$. Note that $dy_1^2=x_1^2-\varepsilon$, so the pair (x_1,ε) determines d and y_1. Our main result is the following:

Theorem 4.2.1. *Let $k\geq 4$ be a fixed integer. Let $d\geq 2$ be a square-free integer. Assume that*

$$x_{n_1}=F_{m_1}^{(k)}, \qquad \text{and} \quad x_{n_2}=F_{m_2}^{(k)} \tag{4.4}$$

for positive integers $m_2>m_1\geq 2$ and $n_2>n_1\geq 1$, where x_n is the x–coordinate of the nth solution of the Pell equation (1.17)*. Then, either:*

(i) $n_1=1,\ n_2=2,\ m_1=(k+3)/2,\ m_2=k+2$ and $\varepsilon=1$; or
(ii) $n_1=1,\ n_2=3,\ k=3\times 2^{a+1}+3a-5,\ m_1=3\times 2^a+a-1,\ m_2=9\times 2^a+3a-5$ for some positive integer a and $\varepsilon=1$.

4.3. Preliminary Results

The following variation of a result of Luca [46] is useful. Let $P(m)$ denote the largest prime factor of the positive integer m.

Lemma 4.3.1. *If $P(x_n)\leq 5$, then either $n=1$, or $n=2$ and $x_2\in\{3,9,243\}$.*

Proof. In [46] it was shown that if $\varepsilon = 1$ and $P(x_n) \leq 5$, then $n = 1$. We give here a proof for both cases $\varepsilon \in \{\pm 1\}$. Since $x_n = y_{2n}/y_n$, where $y_m = (\delta^m - \sigma^m)/(\delta - \sigma)$, it follows, by Carmichael's Primitive Divisor Theorem [24], that if $n \geq 7$, then x_n has a prime factor which is primitive for y_{2n}. In particular, this prime is $\geq 2n-1 > 5$. Thus, $n \leq 6$. Assume next that $n > 1$. If $n \in \{3,6\}$, then x_n is of the form $x(4x^2 \pm 3)$, where $x = x_\ell$ with $\ell = n/3 \in \{1,2\}$. The factor $4x^2 \pm 3$ is larger than 1 (since $x_n > x_\ell$) odd (hence, coprime to 2), not a multiple of 9, and coprime to 5 since $\left(\frac{\pm 3}{5}\right) = -1$. Thus, the only possibility is $4x^2 \pm 3 = 3$, equation which does not have a positive integer solution x. If $n \in \{2,4\}$, then $x_n = 2x^2 \pm 1$, where $x = x_\ell$ and $\ell = n/2 \in \{1,2\}$. Further, if $\ell = 2$ only the case with the -1 on the right is possible. The expression $2x^2 - 1$ is odd, and coprime to both 3 and 5 since $\left(\frac{2}{3}\right) = \left(\frac{2}{5}\right) = -1$, so the case $x_n = 2x_\ell^2 - 1$ is not possible. Finally, if $x_n = 2x_\ell^2 + 1$, then $n = 2$, $\ell = 1$. Further, $2x^2 + 1$ is coprime to 2 and 5 so we must have $2x^2 + 1 = 3^b$ for some exponent b. Thus, $x^2 = (3^b - 1)/(3 - 1)$, and the only solutions are $b \in \{1,2,5\}$ by a result of Ljunggren [45]. $\square$

Since none of 3, 9, 243 are of the form $F_m^{(k)}$ for any $m \geq 1$, $k \geq 4$, for our practical purpose we will use the implication that if $x_n = F_m^{(k)}$ and $P(x_n) \leq 5$, then $n = 1$.

4.4. A small linear form in logarithms

We assume that (x_1, y_1) is the fundamental solution of the Pell equation (1.17). As in Section 1.3, we set

$$x_1^2 - dy_1^2 =: \varepsilon, \qquad \varepsilon \in \{\pm 1\},$$

and put

$$\delta := x_1 + \sqrt{d}y_1 \qquad \text{and} \qquad \sigma := x_1 - \sqrt{d}y_1 = \varepsilon\delta^{-1}.$$

From (1.18) (or (1.19)), we get

$$x_n = \frac{1}{2}(\delta^n + \sigma^n). \tag{4.5}$$

Since $\delta \geq 1 + \sqrt{2} > 2 > \alpha$, it follows that the estimate

$$\frac{\delta^n}{\alpha^2} \leq x_n < \delta^n \quad \text{holds for all} \quad n \geq 1. \tag{4.6}$$

We now assume, as in the hypothesis of Theorem 4.2.1, that (n_1, m_1) and (n_2, m_2) are pairs of positive integers with $n_1 < n_2$, $2 \leq m_1 < m_2$ and

$$x_{n_1} = F_{m_1}^{(k)} \qquad \text{and} \qquad x_{n_2} = F_{m_2}^{(k)}.$$

By setting $(n,m) = (n_j,m_j)$ for $j \in \{1,2\}$ and using the inequalities (1.13) and (4.6), we get that

$$\alpha^{m-2} \leq F_m^{(k)} = x_n < \delta^n \qquad \text{and} \qquad \frac{\delta^n}{\alpha^2} \leq x_n = F_m^{(k)} \leq \alpha^{m-1}. \tag{4.7}$$

Hence,

$$nc_1 \log\delta \leq m+1 \leq nc_1 \log\delta + 3, \qquad c_1 := 1/\log\alpha. \tag{4.8}$$

Next, by using (1.10) and (4.5), we get

$$\frac{1}{2}(\delta^n + \sigma^n) = f_k(\alpha)\alpha^{m-1} + (F_m^{(k)} - f_k(\alpha)\alpha^{m-1}),$$

so

$$\delta^n (2f_k(\alpha))^{-1}\alpha^{-(m-1)} - 1 = \frac{-\sigma^n}{2f_k(\alpha)\alpha^{m-1}} + \frac{(F_m^{(k)} - f_k(\alpha)\alpha^{m-1})}{f_k(\alpha)\alpha^{m-1}}.$$

Hence, by using (1.11) and Lemma 1.3.1(i), we have

$$|\delta^n (2f_k(\alpha))^{-1}\alpha^{-(m-1)} - 1| \leq \frac{1}{\alpha^{m-1}\delta^n} + \frac{1}{\alpha^{m-1}} < \frac{1.5}{\alpha^{m-1}}. \tag{4.9}$$

In the above, we have used the facts that $1/f_k(\alpha) < 2$, $|F_m^{(k)} - f_k(\alpha)\alpha^{m-1}| < 1/2$, $|\sigma| = \delta^{-1}$, as well as the fact that $\delta > 2$. We let Λ be the expression inside the absolute value of the left–hand side above. We put

$$\Gamma := n\log\delta - \log(2f_k(\alpha)) - (m-1)\log\alpha. \tag{4.10}$$

Note that $e^\Gamma - 1 = \Lambda$. Inequality (4.9) implies that

$$|\Gamma| < \frac{3}{\alpha^{m-1}}. \tag{4.11}$$

Indeed, for $m \geq 3$, we have that $\frac{1.5}{\alpha^{m-1}} < \frac{1}{2}$, and then inequality (4.11) follows from (4.9) via the fact that

$$|e^\Gamma - 1| < x \qquad \text{implies} \qquad |\Gamma| < 2x, \qquad \text{whenever} \qquad x \in (0,1/2), \tag{4.12}$$

with $x := \frac{1.5}{\alpha^{m-1}}$. When $m = 2$, we have $x_n = F_m^{(k)} = 1$, so $n = 1$, $\varepsilon = 1$, $\delta = 1+\sqrt{2}$, and then

$$|\Gamma| = |\log(1+\sqrt{2}) - \log(2f_k(\alpha)\alpha)| < \max\{\log(1+\sqrt{2}), \log(2f_k(\alpha)\alpha)\} < \log 3 < \frac{3}{\alpha},$$

where we used the fact that $1 < 2f_k(\alpha)\alpha < 3$ (see Lemma 1.3.1 (i)). Hence, inequality (4.10) holds for all pairs (n,m) with $x_n = F_m^{(k)}$ with $m \geq 2$.

Let us recall what we have proved, since this will be important later-on.

Lemma 4.4.1. *If (n,m) are positive integers with $m \geq 2$ such that $x_n = F_m^{(k)}$, then with $\delta = x_1 + \sqrt{x_1^2 - \varepsilon}$, we have*

$$|n\log\delta - \log(2f_k(\alpha)) - (m-1)\log\alpha| < \frac{3}{\alpha^{m-1}}. \tag{4.13}$$

4.5. Bounding n in terms of m and k

We next apply Theorem 1.1.8 on the left-hand side of (4.9). First we need to check that

$$\Lambda = \delta^n (2f_k(\alpha))^{-1}\alpha^{-(m-1)} - 1$$

is nonzero. Well, if it were, then $\delta^n = 2f_k(\alpha)\alpha^{m-1}$. So, $2f_k(\alpha) = \delta^n\alpha^{-(m-1)}$ is a unit. To see that this is not so, we perform a norm calculation of the element $2f_k(\alpha)$ in $\mathbb{L} := \mathbb{Q}(\alpha)$. For $i \in \{2,\ldots,k\}$, we have that $|\alpha^{(i)}| < 1$, so that, by the absolute value inequality, we have

$$|2f_k(\alpha^{(i)})| = \frac{2|\alpha^{(i)}-1|}{|2+(k+1)(\alpha^{(i)}-2)|} \le \frac{4}{(k+1)(2-|\alpha^{(i)}|)-2} < \frac{4}{k-1} \le \frac{4}{5} \quad \text{for} \quad k \ge 6.$$

Thus, for $k \ge 6$, using also Lemma 1.3.1 (i), we get

$$|\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(2f_k(\alpha))| < |2f_k(\alpha)|\prod_{i=2}^{k}|2f_k(\alpha^{(i)})| < \frac{3}{2}\left(\frac{4}{5}\right)^{k-1} \le \frac{3}{2}\left(\frac{4}{5}\right)^{5} < 1.$$

This is for $k \ge 6$. For $k = 4,\ 5$ one checks that $|\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(2f_k(\alpha)| < 1$ as well. In fact, the norm of $2f_k(\alpha)$ has been computed (for all $k \ge 2$) in [31], and the formula is

$$|\mathcal{N}_{\mathbb{L}/\mathbb{Q}}(2f_k(\alpha))| = \frac{2^k(k-1)^2}{2^{k+1}k^k-(k+1)^{k+1}}.$$

One can check directly that the above number is always smaller than 1 for all $k \ge 2$ (in particular, for $k = 4,5$). Thus, $\Lambda \ne 0$, and we can apply Theorem 1.1.8. We take

$$t = 3, \quad \eta_1 = \delta, \quad \eta_2 = 2f_k(\alpha), \quad \eta_3 = \alpha, \quad b_1 = n, \quad b_2 = -1, \quad b_3 = -(m-1).$$

We take $\mathbb{K} = \mathbb{Q}(\sqrt{d},\alpha)$ which has degree $D \le 2k$. Since $\delta \ge 1+\sqrt{2} > \alpha$, the second inequality in (4.7) tells us right-away that $n \le m$, so we can take $B := m$. We have $h(\eta_1) = (1/2)\log\delta$ and $h(\eta_3) = (1/k)\log\alpha$. Further,

$$h(\eta_2) = h(2f_k(\alpha)) \le h(2)+h(f_k(\alpha)) < 3\log k+\log 2 < 4\log k \tag{4.14}$$

by Lemma 1.3.1 (ii). So, we can take $A_1 := k\log\delta, A_2 := 8k\log k$ and $A_3 := 2\log 2$. Now Theorem 1.1.8 tells us that

$$\begin{aligned}\log|\Lambda| &> -1.4\times 30^6\times 3^{4.5}\times(2k)^2(1+\log 2k)(1+\log m)(k\log\delta)(8k\log k)(2\log 2),\\ &> -1.6\times 10^{13}k^4(\log k)^2\log(\delta)(1+\log m).\end{aligned}$$

In the above, we used the fact that $k \ge 4$, therefore $2k \le k^{3/2}$, so

$$1+\log(2k) \le 1+1.5\log k < 2.5\log k.$$

By comparing the above inequality with inequality (4.9), we get

$$(m-1)\log\alpha - \log 3 < 1.6\times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m).$$

Thus,

$$(m+1)\log\alpha < 1.7\times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m).$$

Since $\alpha^{m+1} \geq \delta^n$ by the second inequality in (4.7), we get that

$$n < 1.7\times 10^{13}k^4(\log k)^2(1+\log m). \tag{4.15}$$

Furthermore, since $\alpha > 1.927$, we get

$$m < 2.6\times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m). \tag{4.16}$$

We now record what we have proved so far, which are estimates (4.15) and (4.16).

Lemma 4.5.1. *If $x_n = F_m^{(k)}$ and $m \geq 2$, then*

$$n < 1.7\times 10^{13}k^4(\log k)^2(1+\log m) \ \text{ and } \ m < 2.6\times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m).$$

Note that in the above bound, n is bounded only in terms of m and k (but not δ).

4.6. Bounding m_1, n_1, m_2, n_2 in terms of k

Next, we write down inequalities (4.13) for both pairs $(n,m) = (n_j, m_j)$ with $j = 1,2$, multiply the one for $j = 1$ with n_2 and the one with $j = 2$ with n_1, subtract them and apply the triangle inequality to the result to get that

$$\begin{aligned}
& |(n_2-n_1)\log(2f_k(\alpha)) - (n_1m_2 - n_2m_1 + n_2 - n_1)\log\alpha| \\
\leq\ & n_2|n_1\log\delta - \log(2f_k(\alpha)) - (m_1-1)\log\alpha| \\
+\ & n_1|n_2\log\delta - \log(2f_k(\alpha)) - (m_2-1)\log\alpha| \\
\leq\ & \frac{3n_2}{\alpha^{m_1-1}} + \frac{3n_1}{\alpha^{m_2-1}} < \frac{6n_2}{\alpha^{m_1-1}}.
\end{aligned}$$

Therefore, we have

$$|(n_2-n_1)\log(2f_k(\alpha)) - (n_1m_2 - n_2m_1 + n_2 - n_1)\log\alpha| < \frac{6n_2}{\alpha^{m_1-1}}. \tag{4.17}$$

We are now set to apply Theorem 1.1.9 with

$$\eta_1 = 2f_k(\alpha), \quad \eta_2 = \alpha, \quad b_1 = n_2 - n_1, \quad b_2 = -(n_1m_2 - n_2m_1 + n_2 - n_1).$$

The fact that η_1 and η_2 are multiplicatively independent follows because α is a unit and $2f_k(\alpha)$ isn't by a previous argument. Next, we observe that $n_2 - n_1 < n_2$, while by the absolute value of the inequality in (4.17), we have

$$|n_1m_2 - n_2m_1 + n_2 - n_1| \le (n_2 - n_1)\frac{\log(2f_k(\alpha))}{\log\alpha} + \frac{6n_2}{\alpha^{m_1-1}\log\alpha} < 6n_2.$$

In the above, we used that

$$\frac{\log(2f_k(\alpha))}{\log\alpha} < \frac{\log(1.5)}{\log\alpha} < 1 \qquad \text{and} \qquad \frac{6}{\alpha^{m_1-1}\log\alpha} < 5,$$

because $\alpha \ge \alpha_4 > 1.92$ and $m_1 \ge 2$. We take $\mathbb{K} := \mathbb{Q}(\alpha)$ which has degree $D = k$. So, we can take

$$\log B_1 = 4\log k > \max\left\{h(\eta_1), \frac{|\log\eta_1|}{k}, \frac{1}{k}\right\}$$

(see inequality (4.14)), and

$$\log B_2 = \frac{1}{k} = \max\left\{h(\eta_2), \frac{|\log\eta_2|}{k}, \frac{1}{k}\right\}.$$

Thus,

$$b' = \frac{(n_2 - n_1)}{k \times (1/k)} + \frac{|n_1m_2 - n_2m_1 + n_2 - n_1|}{4k\log k} < n_2 + \frac{6n_2}{4k\log k} < 1.3n_2.$$

Now Theorem 1.1.9 tells us that with

$$\Gamma := (n_2 - n_1)\log(2f_k(\alpha)) - (n_1m_2 - n_2m_1 + n_2 - n_1)\log\alpha,$$

we have

$$\log|\Gamma| > -24.34 \times k^4 \left(\max\left\{\log(1.3n_2) + 0.14, \frac{21}{k}, \frac{1}{2}\right\}\right)^2 (4\log k)\left(\frac{1}{k}\right).$$

Thus,

$$\log|\Gamma| > -97.4k^3\log k\left(\max\left\{\log(1.5n_2), \frac{21}{k}, \frac{1}{2}\right\}\right)^2,$$

where we used the fact that $\log(1.3n_2) + 0.14 = \log(1.3 \times e^{0.14}n_2) < \log(1.5n_2)$. By combining the above inequality with (4.17), we get

$$(m_1 - 1)\log\alpha - \log(6n_2) < 97.4k^3\log k\left(\max\left\{\log(1.5n_2), \frac{21}{k}, \frac{1}{2}\right\}\right)^2. \tag{4.18}$$

Since $\log(1.5n_2) \ge \log 3 > 1.098$, the maximum in the right–hand side above cannot be $1/2$. If it is not $\log(1.5n_2)$, we then get

$$1.098 < \log(1.5n_2) \le \frac{21}{k} \le 5.25, \qquad \text{so} \quad k \le 19 \quad \text{and} \quad n_2 \le 127. \tag{4.19}$$

Then, the above inequality (4.18) gives

$$\begin{aligned}(m_1+1)\log\alpha &< 97.4\times 21^2 k\log k+\log(6\times 127)+2\log\alpha\\ &< 4.3\times 10^5 k\log k. \end{aligned}\tag{4.20}$$

Since $\alpha \geq 1.927$, we get that

$$m_1+1 < 6.6\times 10^5 k\log k. \tag{4.21}$$

Further, we have

$$\begin{aligned}(\alpha^{(m_1+1)})^{n_2} &> (3F_{m_1}^{(k)})^{n_2} \geq (2F_{m_1}^{(k)}+1)^{n_2} = (2x_{n_1}+1)^{n_2}\\ &= (\delta^{n_1}+(1+\sigma^{n_1}))^{n_2} > \delta^{n_1 n_2} = (\delta^{n_2})^{n_1}\\ &= (2x_{n_2}-\sigma^{n_2})^{n_1} > 2x_{n_2}-1 > x_{n_2} = F_{m_2}^{(k)} > \alpha^{m_2-2},\end{aligned}$$

so

$$m_2 \leq 1+n_2(m_1+1) < 8.4\times 10^7 k\log k. \tag{4.22}$$

Since $n_1 < n_2$, inequalities (4.19), (4.21) and (4.22) bound m_1, n_1, m_2, n_2 in terms of k when the maximum in the right–hand side of (4.18) is $21/k$.

Assume next that the maximum in the right–hand side of (4.18) is $\log(1.5n_2)$. Then

$$\begin{aligned}(m_1+1)\log\alpha &< 97.4k^3\log k(\log(1.5n_2))^2+2\log\alpha+\log(6n_2)\\ &< 97.4k^3(\log k)(\log 1.5+\log n_2)^2+\log(24n_2)\\ &< 97.5\times 2.56k^3(\log k)(\log n_2)^2+6\log n_2\\ &< 249.6k^3(\log k)(\log n_2)^2+6\log n_2\\ &< 249.6k^3(\log k)(\log n_2)^2\left(1+\frac{6}{249.6k^3(\log k)(\log n_2)}\right)\\ &< 2.5\times 10^2 k^3(\log k)(\log n_2)^2. \end{aligned}\tag{4.23}$$

For the above inequality, we used that $2\log\alpha+\log(6n_2) < \log(24n_2) \leq 6\log n_2$ (since $n_2 \geq 2$ and $\alpha < 2$), the fact that $\log(1.5n_2) < 1.6\log n_2$ holds for $n_2 \geq 2$ and the fact that

$$1+\frac{6}{249.6k^3(\log k)(\log n_2)} < 1.0004 \qquad \text{holds for} \quad k\geq 4 \quad \text{and} \quad n_2 \geq 2.$$

In turn, since $\alpha \geq \alpha_4 \geq 1.927$, (4.23) yields

$$m_1 < 4\times 10^2 k^3(\log k)(\log n_2)^2. \tag{4.24}$$

Since $\alpha^{m_1+1} > \delta^{n_1} \geq \delta$ (see the second relation in (4.9)), we get

$$\log\delta \leq n_1\log\delta < (m_1+1)\log\alpha < 2.5\times 10^2 k^3(\log k)(\log n_2)^2. \tag{4.25}$$

By combining the above inequality with Lemma 4.5.1 for $(n,m) := (n_2,m_2)$ together with the fact that $n_2 < m_2$, we get

$$\begin{aligned} m_2 &< 2.6\times 10^{13}k^4(\log k)^2(\log\delta)(1+\log m_2) \\ &< 2.6\times 10^{13}k^4(\log k)^2(2.5\times 10^2k^3(\log k))(\log m_2)^2(1.92\log m_2) \\ &< 1.25\times 10^{16}k^7(\log k)^3(\log m_2)^3. \end{aligned} \tag{4.26}$$

In the above, we used that $1+\log m_2 \le 1.92\log m_2$ holds for all $m_2 \ge 3$. We now apply Lemma 1.5.1 with $m := 3$ and $T := 1.25\times 10^{16}k^7(\log k)^3$ (which satisfies the hypothesis $T > (4\cdot m^2)^m$), to get

$$\begin{aligned} m_2 &< 8\times 1.25\times 10^{16}k^7(\log k)^3(\log T)^3 \\ &< 10^{17}k^7(\log k)^3(7\log k+3\log\log k+\log(1.25\times 10^{16}))^3 \\ &< 10^{17}\times(4.1\times 10^5)k^7(\log k)^6 \\ &< 4.1\times 10^{22}k^7(\log k)^6. \end{aligned} \tag{4.27}$$

In the above calculation, we used that

$$\left(\frac{7\log k+3\log\log k+\log(10^{16})}{\log k}\right)^3 < 4.1\times 10^5 \qquad \text{for all} \qquad k\ge 4.$$

By substituting the upper bound (4.27) for m_2 in the first inequality of Lemma 4.5.1, we get

$$\begin{aligned} n_2 &< 1.7\times 10^{13}k^4(\log k)^2(1+\log m_2) \\ &< 1.7\times 10^{13}k^4(\log k)^2(1+\log(4.1\times 10^{22})+7\log k+6\log\log k) \\ &< 1.7\times 10^{13}\times 48k^4(\log k)^3 \\ &< 8.2\times 10^{14}k^4(\log k)^3, \end{aligned} \tag{4.28}$$

where we used the fact that

$$\frac{7\log k+6\log\log k+\log(4.1\times 10^{22})+1}{\log k} < 48 \quad \text{for all} \quad k\ge 4.$$

Finally, if we substitute the upper bound (4.28) for n_2 into the inequality (4.23), we get

$$\begin{aligned} (m_1+1)\log\alpha &< 2.5\times 10^2k^3(\log k)(\log n_2)^2 \\ &< 2.5\times 10^2k^3(\log k)(1+\log(4\times 10^{16})+4\log k+3\log\log k)^2 \\ &< 2.5\times 10^2(9.2\times 10^2)k^3(\log k)^3 \\ &< 2.3\times 10^5k^3(\log k)^3. \end{aligned} \tag{4.29}$$

In the above, we used that

$$\left(\frac{4\log k+3\log\log k+\log(3.4\times 10^{16})+1}{\log k}\right)^2 < 9.2\times 10^2 \qquad \text{for all} \qquad k\ge 4.$$

Thus, using $\alpha > 1.927$, we get

$$m_1 < 3.6 \times 10^5 k^3 (\log k)^3. \tag{4.30}$$

Thus, inequalities (4.27), (4.28), (4.30) give upper bounds for m_2, n_2 and m_1, respectively, in the case in which the maximum in the right–hand side of inequality (4.18) is $\log(1.5n_2)$. Comparing inequalities (4.27) with (4.22), (4.28) with (4.19), and (4.29) with (4.21), respectively, we conclude that (4.27), (4.28) and (4.30) always hold. Let us summarise what we have proved again, which are the bounds (4.27), (4.28) and (4.30).

Lemma 4.6.1. *If $x_{n_j} = F_{m_j}^{(k)}$ for $j \in \{1,2\}$ with $2 \le m_1 < m_2$, and $n_1 < n_2$, then*

$$m_1 < 3.6 \times 10^5 k^3 (\log k)^3, \quad m_2 < 4.1 \times 10^{22} k^7 (\log k)^6, \quad n_2 < 8.2 \times 10^{14} k^4 (\log k)^3.$$

Since $n_1 \le m_1$, the above lemma gives bounds for all of m_1, n_1, m_2, n_2 in terms of k only.

4.7. The case $k > 500$

Lemma 4.7.1. *If $k > 500$, then*

$$8m_2^3 < 2^k. \tag{4.31}$$

Proof. In light of the upper bound given by Lemma 4.6.1 on m_2, this is implied by

$$4.1 \times 10^{22} k^7 (\log k)^6 < 2^{k/3-1},$$

which indeed holds for all $k \ge 462$ as confirmed by *Mathematica*. □

From now on, we assume that $k > 500$. Thus, (4.31) holds. The main result of this section is the following.

Lemma 4.7.2. *If $k > 500$, then $m_1 \le k+1$. In particular, $x_{n_1} = F_{m_1}^{(k)} = 2^{m_1-2}$, and $n_1 = 1$.*

For the proof, we go to Lemma 1.3.3 and write for $m := m_j$ with $j = 1,2$ the following approximations

$$F_m^{(k)} = 2^{m-2}(1+\zeta_m) = 2^{m-2}\left(1 + \delta_m \left(\frac{k-m}{2^{k+1}}\right) + \gamma_m\right), \tag{4.32}$$

where $\delta_m \in \{0,1\}$ and

$$\begin{aligned} |\zeta_m| &\le \frac{m}{2^{k+1}} + \frac{m^2}{2^{2k+2}} + \frac{4m^3}{2^{3k+3}} < \frac{1}{2^{2k/3}}\left(\frac{1}{2} + \frac{1}{2^{2+2k/3}} + \frac{1}{2^{4+4k/3}}\right) < \frac{1}{2^{2k/3}}, \\ |\gamma_m| &\le \frac{m^2}{2^{2k+2}} + \frac{4m^3}{2^{3k+3}} < \frac{1}{2^{4k/3}}\left(\frac{1}{2^2} + \frac{1}{2^{2k/3+4}}\right) < \frac{1}{2^{4k/3}}, \end{aligned} \tag{4.33}$$

where we used that $m < 2^{k/3-1}$ (see (4.31)) and $k \geq 4$. We then write

$$|F_m^{(k)} - x_n| = 0,$$

from where we deduce

$$|2^{m-1}(1+\zeta_m) - \delta^n| = \frac{1}{\delta^n}. \tag{4.34}$$

Thus,

$$|2^{m-1} - \delta^n| = \frac{1}{\delta^n} + |\zeta_m| 2^{m-1},$$

so

$$|1 - \delta^n 2^{-(m-1)}| = \frac{1}{2^{m-1}\delta^n} + |\zeta_m| < \frac{1}{2^m} + \frac{1}{2^{2k/3}} \leq \frac{1}{2^{\min\{2k/3-1, m-1\}}}. \tag{4.35}$$

In the above, we used that $\delta^n \geq \delta \geq 1 + \sqrt{2} > 2$. The right–hand side above is $< 1/2$, so we may pass to logarithmic form as in (4.12) to get that

$$|n \log \delta - (m-1)\log 2| < \frac{1}{2^{\min\{2k/3-2, m-2\}}}. \tag{4.36}$$

We write the above inequality for (n_1, m_1) and (n_2, m_2) cross-multiply the one for (n_1, m_1) by n_2 and the one for (n_2, m_2) by n_1 and subtract them to get

$$|(n_1(m_2-1) - n_2(m_1-1))\log 2| < \frac{n_2}{2^{\min\{2k/3-2, m_1-2\}}} + \frac{n_1}{2^{\min\{2k/3-2, m_2-2\}}}.$$

Assume $n_1(m_2-1) \neq n_2(m_1-1)$. Then the left–hand side above is $\geq \log 2 > 1/2$. In particular, either

$$2^{\min\{2k/3-2, m_1-2\}} < 4n_2 \quad \text{or} \quad 2^{\min\{2k/3-2, m_2-2\}} < 4n_1.$$

The first one is weaker than the second one and is implied by the second one, so the first one must hold. If the minimum is $2k/3-2$, we then get

$$2^{2k/3-2} \leq 4n_2 < 2^{k/3+1},$$

because $n_2 \leq m_2 < 2^{k/3-1}$, so $2k/3 - 2 < k/3 + 1$, or $k < 9$, a contradiction. Thus,

$$2^{m_1-2} < 4n_2 < 2^{k/3+1},$$

getting

$$m_1 < k/3 + 3 < k + 2.$$

Thus, by Example 1.3.1 (i), we get that $x_{n_1} = F_{m_1}^{(k)} = 2^{m_1-2}$, which by Lemma 4.3.1, implies that $n_1 = 1$.

So, we got the following partial result.

Lemma 4.7.3. *For $k > 500$, either $n_1 = 1$ and $m_1 < k/3 + 3$, or $n_1/n_2 = (m_1-1)/(m_2-1)$.*

To finish the proof of Lemma 4.7.2, assume for a contradiction that $m_1 \geq k+2$. Lemma 4.7.3 shows that $n_1/n_2 = (m_1-1)/(m_2-1)$. Further, in (4.32), we have $\delta_{m_1} = \delta_{m_2} = 1$. Thus, we can rewrite equation (4.34) using γ_m for both $m \in \{m_1, m_2\}$. We get

$$\left|2^{m-1}\left(1+\frac{k-m}{2^{k+1}}+\gamma_m\right)-\delta^n\right| = \frac{1}{\delta^n},$$

so

$$\left|2^{m-1}\left(1+\frac{k-m}{2^{k+1}}\right)-\delta^n\right| \leq \frac{1}{\delta^n}+2^{m-1}|\gamma_m|,$$

therefore

$$\left|\left(1+\frac{k-m}{2^{k+1}}\right)-\delta^n 2^{-(m-1)}\right| \leq \frac{1}{2^{m-1}\delta^n}+|\gamma_m|.$$

Now $\delta^n \geq \alpha^{m-2}$ by the first inequality in (4.7). Thus,

$$2^{m-1}\delta^n \geq 2^{m-1}\alpha^{m-2} \geq 2^{m-1}2^{0.9(m-2)} > 2^{1.9m-3} > 2^{1.9k} > 2^{4k/3},$$

where we used the fact that $m \geq k+2$ and that $\alpha \geq \alpha_4 = 1.9275\ldots > 2^{0.9}$. Since also $|\gamma_m| \leq \frac{1}{2^{4k/3}}$, we get that

$$\left|\left(1+\frac{k-m}{2^{k+1}}\right)-\delta^n 2^{-(m-1)}\right| < \frac{2}{2^{4k/3}}.$$

The expression $1+(k-m)/2^{k+1}$ is in $[1/2, 2]$. Thus,

$$\left|1-\delta^n 2^{-(m-1)}(1+(k-m)/2^{k+1})^{-1}\right| < \frac{4}{2^{4k/3}}.$$

The right–hand side is $< 1/2$ for all $k \geq 4$. We pass to logarithms via implication (4.12) getting that

$$\left|n\log\delta-(m-1)\log 2-\log\left(1+\frac{k-m}{2^{k+1}}\right)\right| < \frac{8}{2^{4k/3}}.$$

We evaluate the above in $(n,m) := (n_j, m_j)$ for $j = 1, 2$. We multiply the expression for $j = 1$ with n_2, the one with $j = 2$ with n_1, subtract them and use $n_2(m_1-1) = n_1(m_2-1)$, to get

$$\left|n_1\log\left(1+\frac{k-m_2}{2^{k+1}}\right)-n_2\left(1+\frac{k-m_1}{2^{k+1}}\right)\right| < \frac{16n_2}{2^{4k/3}}. \tag{4.37}$$

One checks that in our range we have

$$16n_2 < 2^{k/4}. \tag{4.38}$$

By Lemma 4.6.1, this is fulfilled if

$$16 \times 8.2 \times 10^{14} k^4 (\log k)^3 < 2^{k/4},$$

and *Mathematica* checks that this is so for all $k \geq 346$. Thus, inequality (4.37) implies

$$\left| n_1 \log\left(1+\frac{k-m_2}{2^{k+1}}\right) - n_2\left(1+\frac{k-m_1}{2^{k+1}}\right)\right| < \frac{2^{k/4}}{2^{4k/3}} < \frac{1}{2^{13k/12}}.$$

Using the fact that the inequality

$$|\log(1+x)-x| < 2x^2 \quad \text{holds for} \quad |x| < 1/2,$$

with $x_j := (k-m_j)/2^{k+1}$ for $j = 1,2$, and noting that $2x_j^2 < 2m_2^2/2^{2k+2}$ holds for both $j = 1,2$, we get

$$\left|\frac{n_1(k-m_2)}{2^{k+1}} - \frac{n_2(k-m_1)}{2^{k+1}}\right| < \frac{4n_2m_2^2}{2^{2k+2}} + \frac{1}{2^{13k/12}}.$$

In the right–hand side, we have

$$\frac{4n_2m_2^2}{2^{2k+2}} < \frac{2^{2+(k/4-4)+2(k/3-1)}}{2^{2k+2}} = \frac{1}{2^{13k/12+5}}.$$

Hence,

$$\left|\frac{n_1(k-m_2)}{2^{k+1}} - \frac{n_2(k-m_1)}{2^{k+1}}\right| < \frac{2}{2^{13k/12}}.$$

which implies

$$|n_1(k-m_2) - n_2(k-m_1)| < \frac{4}{2^{k/12}}.$$

Since $k > 500$, the right–hand side is smaller than 1. Since the left–hand side is an integer, it must be the zero integer. Thus,

$$n_1/n_2 = (k-m_1)/(k-m_2).$$

Since also $n_1/n_2 = (m_1-1)/(m_2-1)$, we get that $(m_1-1)/(m_2-1) = (m_1-k)/(m_2-k)$, or $(m_1-1)/(m_1-k) = (m_2-1)/(m_2-k)$. This gives $1+(k-1)/(m_1-k) = 1+(k-1)/(m_2-k)$, so $m_1 = m_2$, a contradiction.

Thus, $m_1 \leq k+1$. By Example 1.3.1 (i), we get that $x_{n_1} = 2^{m_1-2}$, which by Lemma 4.3.1 implies ghat $n_1 = 1$. This finished the proof of Lemma 4.7.2.

4.8. The case $m_1 > 376$

Since $k > 500$, we know, by Lemma 4.7.2, that $m_1 \leq k+1$ and $n_1 = 1$. In this section, we prove that if also $m_1 > 376$, then the only solutions are the ones shown at (i) and (ii) of the Theorem 4.2.1. This finishes the proof of Theorem 4.2.1 in the case $k > 500$ and $m_1 > 376$. The remaining cases are handled computationally in the next section.

4.8.1. A lower bound for m_1 in terms of m_2

The main goal of this subsection is to prove the following result.

Lemma 4.8.1. *Assume that $m_1 > 376$. Then $2^{m_1-6} > \max\{k^4, n_2^2\}$.*

Proof. Assume $m_1 > 376$. We evaluate (4.35) in $(n,m) := (n_2, m_2)$. Further, by Lemma 4.3.1, x_{n_2} is not a power of 2, so $m_2 \geq k+2$, therefore $\min\{2k/3-2, m_2-2\} = 2k/3-2$, getting

$$|n_2 \log\delta - (m_2-1)\log 2| < \frac{1}{2^{2k/3-2}}. \tag{4.39}$$

We write a lower bound for the left–hand side using Theorem 1.1.9. Let

$$\Lambda := n_2 \log\delta - (m_2-1)\log 2. \tag{4.40}$$

We have

$$\eta_1 = \delta, \quad \eta_2 = 2, \quad b_1 = n_2, \quad b_2 = -(m_2-1).$$

We have $\mathbb{K} := \mathbb{Q}(\delta)$ has $D = 2$. Further, $h(\eta_1) = (\log\delta)/2$ and $h(\eta_2) = \log 2$. Thus, we can take $\log B_1 = (\log\delta)/2$, $\log B_2 = \log 2$,

$$b' = \frac{n_2}{2\log 2} + \frac{m_2-1}{\log\delta} < m_2\left(\frac{1}{2\log 2} + \frac{1}{\log(1+\sqrt{2})}\right) < 2m_2.$$

Further, Theorem 1.1.9 is applicable since η_1, η_2 are real positive and multiplicatively independent (this last condition follows because δ is a unit and 2 isn't). Theorem 1.1.9 shows that

$$\log|\Lambda| > -24.34 \cdot 2^4 E^2 (\log\delta/2)\log 2 > -195\log 2(\log\delta)E^2, \;\; E := \max\{\log(3m_2), 10.5\}^2,$$

where we used $\log(3m_2) > 0.14 + \log(2m_2) > 0.14 + \log b'$. Thus,

$$|\Lambda| > 2^{-195(\log\delta)E^2}. \tag{4.41}$$

Comparing (4.39) and (4.41), we get

$$195(\log\delta)E^2 > 2k/3-2. \tag{4.42}$$

Since

$$2^{m_1-1} = 2x_1 = \delta + \frac{\varepsilon}{\delta} > \frac{\delta}{2},$$

we get $\delta < 2^{m_1}$, so $\log\delta < m_1\log 2$. Thus,

$$(m_1\log 2)(195E^2) > 2k/3-2.$$

Now let us assume that in fact the inequality $2^{m_1-6} < \max\{k^4, n_2^2\}$ holds. Assume first that the above maximum is n_2^2. Then $m_1 \log 2 < \log(2^6 n_2^2)$. We thus get that

$$2k/3 - 2 < 195 \log(64 n_2^2) E^2.$$

Since by Lemma 4.6.1, $64 n_2^2 < 64 \times 8.2^2 \times 10^{28} k^8 (\log k)^6$, and $3m_2 < 12.3 \times 10^{22} k^7 (\log k)^5$, we get that

$$2k/3 - 2 < 195 \log(64 \times 8.2^2 \times 10^{28} k^8 (\log k)^6) \max\{10.5, \log(12.3 \times 10^{22} k^7 (\log k)^5)\}^2,$$

which gives $k < 4 \times 10^9$. Thus,

$$n_2 < 8.2 \times 10^{14} k^4 (\log k)^3 < 5 \times 10^{55},$$

and since

$$2^{m_1-6} \le n_2^2 < (5 \times 10^{55})^2,$$

we get $m_1 < 6 + 2(\log 5 \times 10^{55})/(\log 2) < 377$, contradicting the fact that $m_1 > 376$. This was in the case $n_2 \ge k^2$. But if $n_2 < k^2$, then $\max\{n_2^2, k^4\} = k^4$ and the same argument gives us an even smaller bound on k; hence, on m_1. This contradiction finishes the proof of this lemma. □

4.8.2. We have $m_2 - 1 = n_2(m_1 - 1)$

The aim of this subsection is to prove the following result.

Lemma 4.8.2. *If $k > 500$ and $m_1 > 376$, then $n_2(m_1 - 1) = m_2 - 1$.*

For the proof, we write

$$2x_1 = \delta + \frac{\varepsilon}{\delta} = 2F_{m_1}^{(k)} = 2^{m_1-1};$$
$$2x_{n_2} = \delta^{n_2} + \left(\frac{\varepsilon}{\delta}\right)^{n_2} = 2F_{m_2}^{(k)}.$$

Thus,

$$\begin{aligned} 2F_{m_2}^{(k)} &= \sum_{i=0}^{\lfloor n_2/2 \rfloor} \frac{n_2}{n_2 - i} \binom{n_2 - i}{i} (-\varepsilon)^i 2^{(m_1-1)(n_2-2i)} \\ &= 2^{(m_1-1)n_2} \left(1 + \sum_{i=1}^{\lfloor n_2/2 \rfloor} \frac{n_2}{n_2 - i} \binom{n_2 - i}{i} \left(-\frac{\varepsilon}{2^{2(m_1-1)}}\right)^i \right). \end{aligned}$$

Note that

$$\frac{n_2}{n_2 - i} \binom{n_2 - i}{i} < n_2^i.$$

Thus,

$$\left|\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\varepsilon}{2^{2(m_1-1)}}\right)^i\right| < \left(\frac{n_2}{2^{2(m_1-1)}}\right)^i. \qquad (4.43)$$

Since $m_1 > 376$, we have $2^{m_1-6} > n_2^2$ by Lemma 4.8.1. In this case, (4.43) tells us that

$$\left|\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\varepsilon}{2^{2(m_1-1)}}\right)^i\right| < \frac{1}{2^{1.5m_1 i}}\left(\frac{n_2}{2^{0.5m_1-2}}\right)^i < \frac{1}{2^{1.5m_1 i}}\left(\frac{1}{2^i}\right). \qquad (4.44)$$

Combining (4.44) with (4.33),

$$\begin{aligned} 2x_{n_2} &= 2^{(m_1-1)n_2}\left(1+\sum_{i=1}^{\lfloor n_2/2\rfloor}\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\varepsilon}{2^{2(m_1-1)}}\right)^i\right) \\ &:= 2^{(m_1-1)n_2}(1+\zeta'_{n_2}) \\ 2F_{m_2}^{(k)} &= 2^{m_2-1}(1+\zeta_{m_2}), \end{aligned}$$

where

$$\zeta'_{n_2} := \sum_{i=1}^{\lfloor n_2/2\rfloor}\frac{n_2}{n_2-i}\binom{n_2-i}{i}\left(-\frac{\varepsilon}{2^{2(m_1-1)}}\right)^i.$$

Since $2x_{n_2} = 2F_{m_2}^{(k)}$, we then have

$$|2^{(m_1-1)n_2} - 2^{m_2-1}| \le 2^{(m_1-1)n_2}|\zeta'_{n_2}| + 2^{m_2-1}|\zeta_{m_2}|.$$

If $(m_1-1)n_2 \neq m_2-1$, then putting $R := \max\{2^{(m_1-1)n_2}, 2^{m_2-1}\}$, the left–hand side above is $\geq R/2$, while the right-side above is $< R/2$, since

$$|\zeta_{m_2}| < \frac{1}{2^{2k/3}} < \frac{1}{4} \quad \text{and} \quad |\zeta'_{n_2}| < \sum_{i\geq 1}\frac{1}{2^{1.5m_1 i}}\left(\frac{1}{2^i}\right) < \frac{1}{2^{1.5m_1}}\sum_{i\geq 1}\frac{1}{2^i} < \frac{1}{2^{1.5m_1}} < \frac{1}{4}.$$

This contradiction shows that $m_2-1 = n_2(m_1-1)$, which finishes the proof of Lemma 4.8.2.

4.8.3. The case $n_2 = 2$

By Lemma 4.8.2, we get $m_2 = 2m_1 - 1$. Since $m_1 \le k+1$, we get that $m_2 \le 2k+1$. Also, $m_2 \ge k+2$. By Example 1.3.1 (ii), we have

$$F_{m_2}^{(k)} = 2^{m_2-2} - (m_2-k)2^{m_2-k-3} = x_2 = 2x_1^2 - \varepsilon = 2(2^{m_1-2})^2 - \varepsilon.$$

We thus get

$$2^{2m_1-3} - (2m_1-k-1)2^{2m_1-k-4} = 2^{2m_1-3} - \varepsilon.$$

We get that the $\varepsilon = 1$, and further $(2m_1-k-1)2^{2m_1-k-4} = 1$, so $m_1 = (k+3)/2$. This gives the parametric family (i) from Theorem 4.2.1.

4.8.4. The case $n_2 = 3$

By Lemma 4.8.2, we get $m_2 = 3(m_1-1)+1 = 3m_1-2$. Since $m_1 \le k+1$, we get that $m_2 = 3m_1-2 \le 3k+1$. Further, $m_2 \ge k+2$. If $m_2 \in [k+2, 2k+2]$, then, by Example 1.3.1 (ii), we have

$$F_{m_2}^{(k)} = 2^{m_2-2} - (m_2-k)2^{m_2-k-3} = x_3 = 4x_1^3 - 3\varepsilon x_1 = 4(2^{m_1-2})^3 - 3\varepsilon 2^{m_1-2},$$

so $\varepsilon = 1$, and $(3m_1-k-2)2^{3m_1-k-5} = 3 \times 2^{m_1-2}$. This gives

$$(3m_1-k-2)2^{2m_1-k-3} = 3.$$

By unique factorization, we get

$$3m_1-k-2 = 3\times 2^a \qquad \text{and} \qquad 2m_1-k-3 = -a$$

for some integer $a \ge 0$. Solving, we get

$$\begin{aligned} m_1 &= 3\times 2^a + a - 1, \\ k &= 3\times 2^{a+1} + 3a - 5, \end{aligned}$$

and then $m_2 = 3m_1 - 2 = 9\times 2^a + 3a - 5$. The case $a = 0$ gives $k = 1$, which is not convenient so $a \ge 1$. This is the parametric family (ii).

It can also be the case that $m_2 \in [2k+3, 3k+1]$. By Example 1.3.1 (iii), we get

$$4(2^{m_1-2})^3 - 3\varepsilon 2^{m_1-2} = 2^{m_2-2} - (m_2-k)2^{m_2-k-3} - (m_2-2k+1)(m_2-2k-2)2^{m_2-2k-5}.$$

This leads to

$$3\varepsilon 2^{m_1-2} = (3m_1-k-2)2^{3m_1-k-5} - (3m_1-2k-1)(3m_1-2k-4)2^{3m_1-2k-7}.$$

Simplifying 2^{3m_1-2k-7} from both sides of the above equation we get

$$3\varepsilon 2^{2k+5-2m_1} = (3m_1-k-2)2^{k+2} - (3m_1-2k-1)(3m_1-2k-4).$$

Since $m_2 = 3m_1-2 \ge 2k+3$, it follows that $m_2 \ge (2k+5)/3$, so $2k+5-2m_1 \le (2k+5)/3$. It thus follows, by the absolute value inequality, that

$$\begin{aligned} 2^{k+2} &< (3m_1-k-2)2^{k+2} \le 3\cdot 2^{2k+5-2m_1} + (3m_1-2k-1)(3m_1-2k-4) \\ &\le 3\cdot 2^{(2k+5)/3} + (k+2)(k-1), \end{aligned}$$

an inequality which fails for $k \ge 5$. Thus, there are no other solutions in this range for $n_2 = 3$ except for the ones indicated in (ii) of Theorem 4.2.1.

4.8.5. The case $n_2 = 4$

In this case, we have $m_2 = 4(m_1 - 1) + 1 = 4m_1 - 3$. Since $m_1 \le k+1$, we have $m_2 \le 4k+1$. Note that

$$x_4 = 2x_2^2 - 1 = 2(2x_1^2 - \varepsilon)^2 - 1 = 8x_1^4 - 8\varepsilon x_1^2 + 1 = 8(2^{m_1-2})^4 - 8\varepsilon(2^{m_1-2})^2 + 1 \qquad (4.45)$$

is odd. Assume first that $m_2 \in [k+2, 2k+2]$. We then have, by Example 1.3.1,

$$F_{m_2}^{(k)} = 2^{m_2-2} - (m_2 - k)2^{m_2-k-3} = 2^{4m_1-5} - (4m_1 - k - 3)2^{4m_1-k-6}. \qquad (4.46)$$

Comparing (4.46) with (4.45), we get

$$(4m_1 - k - 3)2^{4m_1-k-6} = \varepsilon 2^{2m_1-1} - 1.$$

First, $\varepsilon = 1$. Second, the right–hand side above is odd. This implies that the left–hand side is also odd. Thus, the left–hand side is in $\{1, 3\}$. This is impossible since the right–hand side is at least 2^{753}. Thus, this instance does not give us any solution.

Assume next that $m_2 \in [2k+3, 3k+3]$. Then

$$\begin{aligned} F_{m_2}^{(k)} &= 2^{m_2-2} - (m_2 - k)2^{m_2-k-3} + (m_2 - 2k + 1)(m_2 - 2k - 2)2^{m_2-2k-5} \\ &= 8(2^{m_1-2})^4 - 8\varepsilon(2^{m_1-2})^2 + 1. \end{aligned}$$

Identifying, we get

$$(4m_1 - k - 3)2^{4m_1-k-6} - (4m_1 - 2k - 2)(4m_1 - 2k - 5)2^{4m_1-2k-8} = \varepsilon 2^{2m_1-1} - 1.$$

Note that $4m_1 - 2k - 8$ is even. If $4m_1 - 2k - 8 \ge 0$, then the left–hand side is even and the right–hand side is odd, a contradiction. Thus, we must have $4m_1 - 2k - 8 = -2$. This gives $4m_1 = 2k + 6$, so $m_1 = (k+3)/2$. We thus get

$$(k+3)2^k - 1 = \varepsilon 2^{k+2} - 1.$$

This implies that $\varepsilon = 1$ and $(k+3)2^k = 2^{k+2}$, which leads to $k + 3 = 4$, so $k = 1$, which is impossible. Thus, this instance does not give us a solution either.

Assume finally that $m_2 \in [3k+4, 4k+1]$. Applying the Cooper-Howard formula from Lemma 1.3.2, we get

$$F_{m_2}^{(k)} = 2^{m_2-2} + \sum_{j=1}^{3} C_{m_2,j} 2^{m_2-(k+1)j-2}.$$

Eliminating the main term in the equality $F_{m_2}^{(k)} = x_4$ and changing signs in the remaining equation, we get

$$\sum_{j=1}^{3} -C_{m_2,j} 2^{m_2-(k+1)j-2} = \varepsilon 2^{2m_1-1} - 1. \qquad (4.47)$$

At $j = 3$, the exponent of 2 is $m_2 - 3j - 5$. If this is positive, the left hand side is even and the right–hand side is odd, a contradiction. Thus, $m_2 \in \{3k+4, 3k+5\}$. In this case,

$$-C_{m_2,3}2^{m_2-3k-5} = \left(\binom{m_2-3k}{3} - \binom{m_2-3k-2}{1}\right)2^{m_2-3k-5} \in \{1,7\}.$$

For $j \in \{1,2\}$, $m_2 - j(k+1) - 2 \geq m_2 - 2k - 4 \geq k > 500$. Thus, the left–hand side in (4.47) is congruent to $1,7 \pmod{2^{500}}$, while the right–hand side of (4.47) is congruent to $-1 \pmod{2^{500}}$ because $m_1 > 500$. We thus get $1,7 \equiv -1 \pmod{2^{500}}$, a contradiction. Hence, there are no solutions with $n_2 = 4$.

4.8.6. The case $n_2 \geq 5$

The goal here is to prove the following result.

Lemma 4.8.3. *If $k > 500$ and $m_1 > 376$, then there is no solution with $n_2 \geq 5$.*

We write again the two series for $2x_{n_2} = 2F_{m_2}^{(k)}$:

$$2F_{m_2}^{(k)} = 2^{m_2-1}\left(1 + \frac{k-m_2}{2^{k+1}} + \gamma_{m_2}\right) = 2^{n_2(m_1-1)}\left(1 + \frac{-\varepsilon n_2}{2^{2(m_1-1)}} + \gamma'_{n_2}\right),$$

where

$$|\gamma_{m_2}| < \frac{1}{2^{4k/3}} \qquad \text{and} \qquad |\gamma'_{n_2}| \leq \sum_{i\geq 2} \frac{1}{2^{1.5m_1 i}}\left(\frac{1}{2^i}\right) < \frac{1}{2^{3m_1}}.$$

By Lemma 4.8.2, we have $m_2 - 1 = n_2(m_1 - 1)$ so the leading powers of 2 above cancel, and we get

$$\frac{k-m_2}{2^{k+1}} + \gamma_{m_2} = \frac{-\varepsilon n_2}{2^{2(m_1-1)}} + \gamma'_{n_2}.$$

We would like to derive that this implies that

$$\frac{k-m_2}{2^{k+1}} = \frac{-\varepsilon n_2}{2^{2(m_1-1)}}. \tag{4.48}$$

Well, we distinguish two cases.

Case 4.8.1. *Suppose that $2(m_1 - 1) \geq k + 1$.*

We then write

$$\left|\frac{k-m_2}{2^{k+1}} + \frac{n_2\varepsilon}{2^{2(m_1-1)}}\right| \leq |\gamma_{m_2}| + |\gamma'_{n_2}| \leq \frac{1}{2^{4k/3}} + \frac{1}{2^{3m_1}}. \tag{4.49}$$

Since $2m_1 \geq k+3$, we get $3m_1 > 3k/2 > 4k/3$. Thus,

$$\left| \frac{k-m_2}{2^{k+1}} + \frac{n_2\varepsilon}{2^{2(m_1-1)}} \right| \leq \frac{2}{2^{4k/3}}. \tag{4.50}$$

Suppose further that $m_1 \leq 2k/3$. Multiplying inequality (4.50) across by $2^{2(m_1-1)}$, we get

$$|2^{2(m_1-1)-(k+1)}(k-m_2)+\varepsilon n_2| \leq \frac{2^{2m_1-1}}{2^{4k/3}} \leq \frac{1}{2},$$

and since the left–hand side above is an integer, it must be the zero integer. This proves (4.48) in the current case assuming that $m_1 \leq 2k/3$. If $m_1 > 2k/3$, we deduce from (4.50) that

$$\frac{m_2-k}{2^{k+1}} < \frac{2}{2^{4k/3}} + \frac{n_2}{2^{2(m_1-1)}} < \frac{2+4n_2}{2^{4k/3}} < \frac{5n_2}{2^{4k/3}} < \frac{1}{2^{13k/12}},$$

where in the right–above we used the fact that $8n_2 < 2^{k/4}$ (see (4.38)). We thus get

$$2 \leq m_2 - k < \frac{2^{k+1}}{2^{13k/12}} < \frac{2}{2^{k/12}} < 1,$$

where the right–most inequality holds since $k > 500$. This is a contradiction, so the $m_1 > 2k/3$ cannot occur in this case. This completes the proof of (4.48) in Case 4.8.1.

Case 4.8.2. *Assume that* $2(m_1-1) < k+1$.

We then write

$$\frac{n_2}{2^{2(m_1-1)}} \leq \frac{m_2-k}{2^{k+1}} + |\gamma_{m_2}| + |\gamma'_{n_2}|.$$

Since $|\gamma_{m_2}| < 1/2^{4k/3} < 1/2^{k+1}$ and $|\gamma'_{n_2}| \leq 1/2^{3m_1} < 1/2^{2(m_1-1)}$, we get that

$$\frac{1}{2^{2(m_1-1)}} < \left| \frac{n_2-1}{2^{2(m_1-1)}} \right| \leq \frac{n_2}{2^{2(m_1-1)}} - |\gamma'_{n_2}| \leq \frac{m_2-k}{2^{k+1}} + |\gamma_{m_2}| < \frac{m_2}{2^{k+1}},$$

where we also used that $n_2 > 1$ and $k \geq 2$. Thus,

$$2^{k+1-2(m_1-1)} < m_2.$$

We now go back to (4.49) and write that

$$\left| \frac{k-m_2}{2^{k+1}} + \frac{n_2\varepsilon}{2^{2(m_1-1)}} \right| < \frac{2}{2^{\min\{4k/3,3m_1\}}}.$$

We multiply across by 2^{k+1} getting

$$|(k-m_2) + 2^{k+1-2(m_1-1)}\varepsilon n_2| < \frac{2^{k+2}}{2^{\min\{4k/3,3m_1\}}}.$$

If the minimum on the right above is $4k/3$, then the right–hand side above is smaller than $4/2^{k/3} < 1/2$ since k is large, so the number on the left is zero. If the minimum is $3m_1$, on the right above then

$$|(k-m_2)+2^{k+1-2(m_1-1)}\varepsilon n_2| < \frac{1}{2}\left(\frac{2^{k+1-2(m_1-1)}}{2^{m_1}}\right).$$

Since

$$2^{k+1-2(m_1-1)} < m_2 = n_2(m_1-1) < kn_2 \le \max\{k^2, n_2^2\} < 2^{m_1-6} < 2^{m_1}$$

(here, we used Lemma 4.8.1 for the inequality in the right–hand side above), it follows that

$$|(k-m_2)+2^{k+1-2(m_1-1)}\varepsilon n_2| < \frac{1}{2},$$

so again the left–hand side is 0. Since $m_2 > k$, this implies that $\varepsilon = 1$. We record what we just proved.

Lemma 4.8.4. *If $k > 500$, $m_1 > 376$ and $n_2 \ge 5$, then $m_1 \le k+1$, $n_1 = 1$, $\varepsilon = 1$, $m_2 - 1 = n_2(m_1-1)$ and*

$$\frac{m_2-k}{2^{k+1}} = \frac{n_2}{2^{2(m_1-1)}}.$$

We now get an extra relation. First, from Lemma 4.8.4, we get that

$$n_2 = \left\{\begin{matrix} 2^{2(m_1-1)-(k+1)}(m_2-k) & \text{if} \quad 2(m_1-1) \ge k+1; \\ \frac{m_2-k}{2^{k+1-2(m_1-1)}} & \text{if} \quad 2(m_1-1) < k+1. \end{matrix}\right\}. \tag{4.51}$$

Since $n_2 \ge 5$, we can write more terms.

$$\begin{aligned} 2F_{m_2}^{(k)} &= 2^{m_2-1}\left(1+\frac{k-m_2}{2^{k+1}}+\delta_{m_2}\frac{(m_2-2k+1)(m_2-2k-2)}{2^{2k+2}}+\eta_{m_2}\right) \\ 2x_{n_2} &= 2^{n_2(m_1-1)}\left(1+\frac{-\varepsilon n_2}{2^{2(m_1-1)}}+\frac{n_2(n_2-3)}{2^{4(m_1-1)+1}}+\eta'_{n_2}\right) \end{aligned}$$

In the formula for $F_{m_2}^{(k)}$, we have $\delta_{m_2} = \zeta_{m_2} = 0$ if $m_2 \le 2k+2$. But $m_2 \le 2k+2$ is not possible since then the only terms in the first expansion of $2F_{m_2}^{(k)}$ are the first two which already coincide with the first two terms of the expansion of $2x_{n_2}$, but in the second expansion we have additional terms since $n_2 \ge 5$ while in the first we do not, which is a contradiction. Thus, $m_2 \ge 2k+3$.

Assume that $2(m_1-1) \ge k+1$. In this case, from (4.51), we deduce that

$$n_2 = 2^{2(m_1-1)-(k+1)}(m_2-k) = \frac{m_2-1}{m_1-1}.$$

So, $m_2 - k \mid m_2 - 1$. Thus, $m_2 - k \mid (m_2 - 1) - (m_2 - k) = k - 1$. This shows that $m_2 - k \leq k - 1$, so $m_2 \leq 2k - 1$, a contradiction. Thus, $k + 1 > 2(m_1 - 1)$.

Simplifying again the power of 2 from the two representations of $2x_{n_2} = 2F_{m_2}^{(k)}$ and eliminating the first two terms we get

$$\frac{(m_2 - 2k + 1)(m_2 - 2k - 2)}{2^{2k+3}} + \eta_{m_2} = \frac{n_2(n_2 - 3)}{2^{4(m_1-1)+1}} + \eta'_{n_2}.$$

Here,

$$|\eta_{m_2}| < \frac{4m_2^3}{2^{3k+3}} < \frac{1}{2^{2k+4}} \quad \text{and} \quad |\eta'_{n_2}| \leq \sum_{i\geq 3} \frac{1}{2^{1.5mi}} \left(\frac{1}{2^i}\right) < \frac{1}{2^{4.5m_1+1}},$$

by (4.31) and (4.44). Thus,

$$\left| \frac{(m_2 - 2k + 1)(m_2 - 2k - 2)}{2^{2k+3}} - \frac{n_2(n_2 - 3)}{2^{4(m_1-1)+1}} \right| \leq |\eta_{m_2}| + |\eta'_{n_2}| < \frac{2}{\min\{2^{2k+4}, 2^{4.5m_1+1}\}}. \quad (4.52)$$

Recall that $2(m_1 - 1) < k + 1$. Then, by (4.51), we have $n_2 \mid m_2 - k$. Since also $n_2 \mid m_2 - 1$, it follows that $n_2 \mid (m_2 - 1) - (m_2 - k) = k - 1$. Thus, $n_2 < k$, and since $2^{(k+1)-2(m_1-1)}$ is a divisor of n_2, we conclude that $2^{(k+1)-2(m_1-1)} < k$. We multiply (4.52) across by $2^{2(k+1)}$. We get

$$\left| \frac{(m_2 - 2k + 1)(m_2 - 2k - 2)}{2} - 2^{2(k+1)-4(m_1-1)} \frac{n_2(n_2 - 3)}{2} \right| \leq \frac{2^{2k+3}}{\min\{2^{2k+4}, 2^{4.5m_1+1}\}}.$$

If the minimum above is 2^{2k+4}, then the right–hand side is $< \frac{1}{2} < 1$. The left–hand side is an integer, so it equals 0. If the minimum is $2^{4.5m_1+1}$, then we can rewrite it as

$$\frac{2^{2k+3}}{2^{4.5m_1+1}} = \frac{2^{2(k+1)-4(m_1-1)}}{2^{0.5m_1+4}} < \frac{k^2}{2^{0.5m_1+5}} < 1.$$

The right–most inequality holds because $2^{m_1-6} > k^4$ by Lemma 4.8.1. Hence, the left–hand side above is again 0. We get that

$$(m_2 - 2k + 1)(m_2 - 2k - 2) = 2^{2(k+1)-4(m_1-1)} n_2(n_2 - 3). \quad (4.53)$$

So, let us record the equations we have:

$$\left\{ \begin{array}{ccc} m_2 - 1 & = & n_2(m_1 - 1); \\ b & = & (k+1) - 2(m_1 - 1); \\ n_2 & = & \frac{m_2 - k}{2^b}; \\ (m_2 - 2k + 1)(m_2 - 2k - 2) & = & 2^{2b} n_2(n_2 - 3). \end{array} \right\}. \quad (4.54)$$

with $b > 0$. To finish, we need to prove the following lemma.

Lemma 4.8.5. *There are no integer solutions $(b, k, m_1, m_2, n_1, n_2)$ to system (4.54) with $n_2 \geq 5$ in the range $k > 500$ and $m_1 > 376$.*

Now that we are seeing the light at the end of the tunnel, let's prove Lemma 4.8.5. As we saw, $n_2 \mid (k-1)$. The last equation in system (4.54) is

$$\left(\frac{m_2-1}{n_2}-\frac{2(k-1)}{n_2}\right)(m_2-2k-2)=2^{2b}(n_2-3),$$

or, using the first equation in system (4.54),

$$\left(m_1-1-\frac{2(k-1)}{n_2}\right)(m_2-2k-2)=n_2-3.$$

Now $n_2 < k$ and $m_1 \le k+1$, so from the first equation $m_2 < k^2$. Since $2^b \mid m_2-k$, we get that $2^b < k^2$, so $b < 2(\log k)/(\log 2) < 3\log k$. Since $b=(k+1)-2(m_1-1)$, we get that

$$m_1=\frac{k+3-b}{2}\in\left(\frac{k+3-3\log k}{2},\frac{k+3}{2}\right).$$

In the last equation in the left, at most one of $m_1-1-2(k-1)/n_2$ (divisor of m_2-2k+1) and m_2-2k-2 is even. If the first one is even, then m_2-2k-2 is a divisor of n_2-3. Thus,

$$n_2-3\ge m_2-2k-2=n_2(m_1-1)-2k-1\ge n_2\left(\frac{k+1-3\log k}{2}\right)-2k-1,$$

giving

$$2k-2\ge n_2\left(\frac{k+1-3\log k}{2}-1\right)=n_2\left(\frac{k-1-3\log k}{2}\right).$$

Since $n_2\ge 5$, we get

$$4k-4\ge 5(k-1-3\log k),\qquad \text{or}\qquad k\le 15\log k+1,$$

giving $k\le 63$, a contradiction. Thus, $2^{2b}\mid m_2-2k-2$. Hence,

$$\left(m_1-1-\frac{2(k-1)}{n_2}\right)\left(\frac{m_2-2k-2}{2^{2b}}\right)=n_2-3,$$

and all fractions above are in fact integers. The left–most integer is

$$m_1-1-\frac{2(k-1)}{n_2}\ge\frac{k+1-3\log k}{2}-\frac{2(k-1)}{5}>\frac{k-1}{12}-3$$

since $k>500$. Since this number is a divisor of (so, at most as large as) the number $n_2-3=(k-1)/D-3$ for some integer D, we get that $D\in\{1,2,\ldots,11\}$. Thus, $(k-1)/D\in\{1,\ldots,11\}$, so

$$m_1-1-\frac{2(k-1)}{n_2}\ge\frac{k+1-3\log k}{2}-22=\frac{k-43-3\log k}{2}.$$

Now let us look at the integer $(m_2-2k-2)/2^{2b}$. Assume that it is at least 3. We then get

$$3\left(\frac{k-43-3\log k}{2}\right)\le n_2-3\le k-4,\qquad \text{or}\qquad k\le 121+9\log k,$$

and this is false for $k\ge 500$. Thus, $(m_2-2k-2)/2^{2b}\in\{1,2\}$.

Assume that $(m_2-2k-2)/2^{2b}=1$. Then

$$m_1-1-\frac{2(k-1)}{n_2}=n_2-3.$$

The number in the left hand side is

$$m_1-1-\frac{2(k-1)}{n_2}\ge\frac{k+1-3\log k}{2}-22=\frac{k-43-3\log k}{2}>\frac{k-1}{3}-3$$

(since $k>500$) and also

$$m_1-1-\frac{2(k-1)}{n_2}\le m_1-3\le\frac{k-3}{2}<k-4.$$

Thus, writing again $n_2=(k-1)/D$, we get that

$$n_1-3=\frac{k-1}{D}-3\in\left(\frac{k-1}{3}-3,\frac{k-1}{1}-3\right),$$

showing that $1<D<3$, so $D=2$. Thus, $n_2=(k-1)/2$, and we get that

$$\frac{k-7}{2}=\frac{k-1}{2}-3=n_2-3=m_1-1-\frac{2(k-1)}{n_2}=m_1-1-4=m_1-5,$$

so

$$m_1=\frac{k+3}{2},\qquad \text{so}\qquad b=0,$$

which is impossible.

Assume next that $(m_2-2k-2)/2^b=2$. In this case, we get

$$n_2-3=2\left(m_1-1-\frac{2(k-1)}{n_2}\right).$$

Proceeding as before, we have

$$\begin{aligned}\frac{k-1}{D}-3 &= n_2-3=2\left(m_1-1-\frac{2(k-1)}{n_2}\right)\ge 2\left(\frac{k+1-3\log k}{2}-22\right)\\ &= k-43-3\log k>\frac{k-1}{2}-3,\end{aligned}$$

showing that $D < 2$. Thus, $D = 1$ and so $n_2 = k - 1$. Hence,

$$k - 4 = n_2 - 3 = 2\left(m_1 - 1 - \frac{2(k-1)}{n_2}\right) = 2(m_1 - 1 - 2) = 2(m_1 - 3),$$

so

$$m_1 = \frac{k+2}{2}, \qquad \text{therefore} \qquad b = 1.$$

Thus, $m_2 - 2k - 2 = 2^{2b+1} = 8$. Consequently,

$$8 = (m_2 - 1) - 2k - 1 = n_2(m_1 - 1) - 2k - 1 = \frac{(k-1)k}{2} - 2k - 1 = \frac{k^2 - 5k - 2}{2},$$

giving $k^2 - 5k - 18 = 0$, which is impossible.

So, indeed there are no solutions with $k > 500$ and $m_1 > 376$ other than the ones from (i) and (ii) of Theorem 4.2.1. □

4.9. The computational part $k \le 500$ or $m_1 \le 376$

Throughout this section, we make the following definition.

Definition 4.9.1. Assume that $k \ge 4$, $x_1 \ge 1$, $\varepsilon \in \{\pm 1\}$ are given such that there exist $n_1 \ge 1$ and $m_1 \ge 2$ such that $x_{n_1} = F_{m_1}^{(k)}$. We say that n_1 is minimal if there is are no positive integers $n_0 < n_1$ and $m_0 < m_1$ such that the equality $x_{n_0} = F_{m_0}^{(k)}$ also holds.

The aim of this section is to first show that in the range $k \le 500$ or $m_1 \le 376$, all solutions of $x_{n_1} = F_{m_1}^{(k)}$ with n_1 minimal have $n_1 = 1$. Then we finish the calculations.

4.9.1. The case $k \le 500$

Here, we exploit inequality (4.17), which we consider convenient to remind:

$$|(n_2 - n_1)\log(2f_k(\alpha)) - (n_1 m_2 - n_2 m_1 + n_2 - n_1)\log\alpha| < \frac{6n_2}{\alpha^{m_1-1}}. \tag{4.55}$$

Thus,

$$\left|\chi_k - \frac{N}{n_2 - n_1}\right| < \frac{6n_2}{(n_2 - n_1)\alpha^{m_1-1}\log\alpha}, \qquad \chi_k := \frac{\log(2f_k(\alpha))}{\log\alpha}, \tag{4.56}$$

with $N := n_1 m_2 - n_2 m_1 + n_2 - n_1$. Lemma 4.6.1 shows that

$$n_2 - n_1 < n_2 < 8.2 \times 10^{14} k^4 (\log k)^3 < 10^{29}.$$

The right–hand side of (4.56) can be rewritten as

$$\frac{1}{2(n_2-n_1)^2}\left(\frac{\alpha^{m_1-1}\log\alpha}{12n_2(n_2-n_1)}\right)^{-1}. \tag{4.57}$$

Assume that

$$\frac{\alpha^{m_1-1}}{\log\alpha} > 12(8.2\times10^{14}k^4(\log k)^3)^2. \tag{4.58}$$

Using $\alpha > 1.927$, inequality (4.58) holds with $k \le 500$ for all $m_1 \ge 203$. In this case, inequalities (4.57), (4.56) and Lemma 1.2.1 show that $N/(n_1-n_1) = p_j^{(k)}/q_j^{(k)}$ for some $j \ge 0$, where $p_j^{(k)}/q_j^{(k)}$ is the jth convergent of χ_k. Note that $\chi_k \in (0,1)$ because by Lemma 1.3.1 (i), we have $1 < 2f_k(\alpha) < 1.5 < \alpha$.

We distinguish two cases.

Case 4.9.1. $N \neq 0$.

In this case, $j \ge 1$. Since

$$n_2 - n_1 \le 10^{29} < F_{150} \le q_{150}^{(k)},$$

where F_{150} is the 150th member of the Fibonacci sequence, it follows that if we take

$$a_N := \max\{a_i^{(k)} : 2 \le i \le 150; 4 \le k \le 500\},$$

then Lemma 1.2.1 implies that

$$\frac{1}{(a_N+2)(q_j^{(k)})^2} < \left|\chi_k - \frac{N}{n_2-n_1}\right| < \frac{6n_2}{(n_2-n_1)\alpha^{m_1-1}\log\alpha}.$$

A computer calculation shows that $a_N = 433576$, so $a_N + 2 < 10^6$. Hence,

$$\begin{aligned}\alpha^{m_1-1}\log\alpha &< 6n_2(a+2)(q_j^{(k)})^2(n_2-n_1) < 6\times10^6 n_2^2\\ &< 6\times10^6(8.2\times10^{14}500^4(\log 500)^3)^2,\end{aligned}$$

and using $\alpha \ge 1.927$, we get $m_1 \le 221$.

Case 4.9.2. $N = 0$.

In this case, inequality (4.56) gives

$$\alpha^{m_1-1}\log\alpha < 6n_2\chi_k^{-1} < 6\times(8.2\times10^{14}500^4(\log 500)^3)\chi_k^{-1}.$$

A computation with *Mathematica* reveals that $\chi_k^{-1} < 10^{148}$ for $k \le 500$. Feeding this into the above inequality, we get $m_1 \le 720$. Note that since $N = 0$, we also have $n_1(m_2-1) = n_2(m_1-1)$. In particular, $n_1 = 1$ is not possible in this case.

Let us record what we just proved.

Lemma 4.9.1. *If $k \leq 500$, then the following hold:*

(i) $m_1 \leq 221$*;*
(ii) $m_1 \in [222, 720]$*, but* $n_1 > 1$.

For reasons that will become clear later, we allow $m \leq 1049$ (instead of just $m \leq 720$). To continue, assume first that $x_1 \in \{1,2,3,\ldots,20\}$. We then generate all values of $\delta = x_1 + \sqrt{x_1^2 - \varepsilon}$ for $\varepsilon \in \{\pm 1\}$. We generate $x_{n_1} = (\delta^{n_1} + \sigma^{n_1})/2$, where η is the Galois conjugate of δ in the quadratic field $\mathbb{Q}(\delta)$, for all $1 \leq n \leq m \leq 1049$ and we test for the equation

$$x_n = F_m^{(k)} \qquad 4 \leq k \leq 500, \quad 2 \leq m \leq 1049.$$

The only solutions we find computationally have:

(i) $n = 1$ and $x_1 \in \{1,2,4,8,15,16\}$;
(ii) $n = 2$ and $x_2 \in \{31, 127, 511\}$. These are not minimal because $x_2 = 31 = F_7^{(5)}$ has $\varepsilon = 1$ and for it $x_1 = 4 = F_7^{(4)}$, $x_2 = 127 = F_9^{(7)}$ has $\varepsilon = 1$ and for it $x_1 = 8 = F_9^{(5)}$, while $x_2 = 511 = F_{11}^{(9)}$ has $\varepsilon = 1$ and for it $x_1 = 16 = F_{11}^{(6)}$, as stated in (i) of Theorem 4.2.1 with $k = 7$, 9, and 11, respectively.
(iii) $n = 3$ and $x_3 = 16336 = F_{19}^{(13)}$. This is not minimal since $x_1 = 16 = F_6^{(13)}$, as stated in (ii) of Theorem 4.2.1 with $a = 1$.

Assume now that $x_1 \geq 21$. Then $\delta \geq 21 + \sqrt{440}$. Inequality (A.15) together with the fact that $m_1 \leq 1050$ gives

$$n_1 \leq \frac{(m_1+1)\log\alpha}{\log\delta} \leq \frac{1051\log 2}{\log(21+\sqrt{440})},$$

so $n_1 \leq 194$. Our next goal is to show that in our range $k \leq 500$ and $m \leq 1049$, we must have $n \in \{1,2,3\}$. For this, assume that $n > 3$. Every positive integer > 3 is either divisible by 4, 6, 9 or a prime $p \geq 5$. Thus, we generate the set

$$\mathscr{B} = \{4, 6, 9, p_k : 3 \leq k \leq 44\},$$

a set with 45 elements, where p_k is the kth prime. We use the fact that if $a \mid b$, then x_b is the ath solution of the Pell equation whose first (smallest) x-coordinate is $x_{b/a}$ (that is, δ gets replaced by $\delta^{b/a}$). In particular, x_{n_1} is x_b for some $b \in B$ and some value of x_1. Further, say $y = F_m^{(k)}$ for some $m \in [2, 1049]$ and $k \in [3, 500]$. We then need to solve $x_b = y$. Note that if $z \geq 1$ and $n \geq 2$, then

$$(z^n+1)^{1/n} - z = z\left(\left(1+\frac{1}{z^n}\right)^{1/n} - 1\right) < \frac{1}{nz^{n-1}} \leq \frac{1}{2}. \tag{4.59}$$

Thus,

$$x_b = \left(x_1 + \sqrt{x_1^2-\varepsilon}\right)^b + \left(x_1 - \sqrt{x_1^2-\varepsilon}\right)^b = 2y$$

implies

$$x_1+\sqrt{x_1^2-1}\in((2y-1/2)^{1/b},(2y+1/2)^{1/b}).$$

Further, this leads to

$$2x_1\in((2y-1/2)^{1/n}-1/2,(2y+1/2)^{1/n}+1/2).$$

The length of the interval on the right above is, by (4.59), at most 2, so it contains at most one even integer $2x_1$ and if it contains one, it must be such that

$$x_1=\left\lfloor \frac{1}{2}\left(\left(2y+\frac{1}{2}\right)^{1/b}+\frac{1}{2}\right)\right\rfloor. \tag{4.60}$$

So what we did was for each $y=F_m^{(k)}$ and each $b\in B$, we calculated the last 10-digits of the integer shown at (4.60) (that is, we only calculated it modulo 10^{10}). Then we picked $\varepsilon\in\{\pm 1\}$ and generated $\{x_n\}_{n\geq 0}$ as the sequence given by $x_0:=1$, x_1 given by (4.60) modulo 10^{10} and $x_{n+1}=(2x_1)x_n-\varepsilon x_{n-1} \pmod{10^{10}}$ for all $n\geq 1$. In this way, we never kept more that then last 10 digits of x_n. And we checked whether indeed $x_b\equiv y \pmod{10^{10}}$. Unsurprisingly, no solution was found. We used the same program for $n_1=2,3$. For these we got that all solutions of (i) in our range were candidates for $n_1=2$ and all solutions (ii) in our range were candidates for $n_1=3$. By candidates we meant that we only checked out these equalities modulo 10^{10}. They turn out to be actual solutions for $\varepsilon=1$ (and they are not solutions with $\varepsilon=-1$ just because a number of the form $2^{2j+1}-1$ with $j\geq 2$ cannot be also of the form $2z^2+1$ for some integer z, while a number of the form $4x^3-3x$ for some integer $x>1$ then it cannot be also of the form $4z^3+3z$ for some integer z). Finally, one word about "recognising" y as number of the form $F_m^{(k)}$. It follows from a result of Bravo and Luca [21] that the equation $F_m^{(k)}=F_n^{(\ell)}$ with $m\geq k+2$, $n\geq \ell+2$ and $k>\ell\geq 4$ has no solutions (m,k,n,ℓ). Thus, if we already know a representation of a representation of y as $F_m^{(k)}$ for some m and $k\geq 4$, then it is unique. In particular, for $j\geq 2$, $F_{2j+3}^{(2j+1)}$ is the only representation of $2^{2j+1}-1$ as a $F_m^{(k)}$ for some positive integers m and $k\geq 4$.

4.9.2. The case $m_1\leq 376$

We may assume that $k>500$, otherwise we are in the preceding case. Thus, $k>m_1$, so $n_1=1$. Thus, $\delta=2^{m_1-2}+\sqrt{2^{2m_1-2}-\varepsilon}$ for all $m_1\geq 2$ and $\varepsilon\in\{\pm 1\}$ (except for $m_1=2$, case in which only $\varepsilon=1$ is possible). We now go back to the proof of Lemma 4.8.1 to get that the inequality (4.39), recalled below

$$|n_2\log\delta-(m_2-1)\log 2|<\frac{1}{2^{2k/3-2}} \tag{4.61}$$

implies (4.42), namely

$$2k/3-2<195(\log\delta)\max\{10.5,\log(3m_2)\}^2.$$

For us, $\log\delta \leq m_1 \log 2 \leq 376 \log 2$. Using also the upper bound from Lemma 4.6.1 on m_2, we get

$$2k/3 - 2 < 195 \times 376(\log 2) \max\left\{10.5, \log(3 \times 4.1 \times 10^{22} k^7 (\log k)^6)\right\}^2,$$

leading to $k < 4 \times 10^9$. Thus, by Lemma 4.6.1 again,

$$n_2 < 8.2 \times 10^{14} k^4 (\log k)^3 < 8.2 \times 10^{14} (4 \times 10^9)^4 (\log(4 \times 10^9))^3 < 10^{58}.$$

Now (4.61) gives

$$\left|\frac{\log\delta}{\log 2} - \frac{m_2 - 1}{n_2}\right| < \frac{1}{(\log 2) 2^{2k/3-1} n_2}. \tag{4.62}$$

In our range, the right–hand side above is smaller than $1/(2n_2^2)$. Indeed, this is equivalent to $n_2 < 2^{2k/3-3}(\log 2)$, which holds provided that

$$8.2 \times 10^{14} k^4 (\log k)^3 < 2^{2k/3-3}(\log 2),$$

which indeed holds for all $k > 500$. Thus, $(m_2 - 1)/n_2 = p_j/q_j$ is some convergent of $\log\delta/\log 2$. Since its denominator q_j divides n_2 and

$$q_j \leq n_2 < 10^{58} < F_{299},$$

where F_{299} is the 299th term of the Fibonacci sequence, it follows that $j \leq 298$. We generated the continued fractions of all $\log\delta/\log 2$ for all possibilities for $m_1 \leq 376$, $\varepsilon \in \{\pm 1\}$ and $j \leq 299$ and collected together the obtained values of a_j. The maximum value obtained was 1033566. Hence,

$$\frac{1}{1.1 \times 10^7 n_2^2} < \frac{1}{(a_{j+1}+2) n_2^2} < \left|\frac{\log\delta}{\log 2} - \frac{m_2 - 1}{n_2}\right| < \frac{1}{(\log 2) 2^{2k/3-1} n_2},$$

giving

$$2^{2k/3-2} \log 2 < 1.1 \times 10^7 n_2 < 1.1 \times 10^7 \times (8.2 \times 10^{14} k^4 (\log k)^3),$$

giving $k \leq 166$, a contradiction.

Thus, this case leads to no solution, and we must have $k \leq 500$, $n_1 = 1$ and $m_1 \leq 221$ by Lemma 4.9.1.

4.9.3. The final computations

Now we go to inequality (4.13) for $(n, m) = (n_2, m_2)$:

$$|n_2 \log\delta - \log(2 f_k(\alpha)) - (m_2 - 1) \log\alpha| < \frac{3}{\alpha^{m_2 - 1}}. \tag{4.63}$$

We divide both sides by $\log\alpha$ and get

$$|n_2\tau-(m_2-1)-\mu|<\frac{A}{B^{m_2-1}},\quad (\tau,\mu,A,B):=\left(\frac{\log\delta}{\log\alpha},\frac{\log(2f_k(\alpha)}{\log\alpha},\frac{3}{\log(1.92)},1.92\right).$$

We have

$$n_2\le 8.2\times 10^{14}k^4(\log k)^3\le 8.2\times 10^{14}(500)^4(\log 500)^3<1.3\times 10^{28}:=M.$$

Since $6M<10^{30}<F_{150}$, we try q_λ for some $\lambda\ge 150$. A computer code ran through the range $k\in[4,500]$, $m_1\in[2,221]$ and $\varepsilon\in\{\pm1\}$, generated $\delta=2^{m_1-2}+\sqrt{2^{2(m_1-2)}-\varepsilon}$ (except for $m_1=2$, when only $\varepsilon=1$ is possible), and confirmed the following:

(i) For $4\le k\le 500$ and $\lambda=200$, we have $\varepsilon>0$ in all cases.
(ii) The maximal value of $1+\lfloor\log(Aq_\lambda/\varepsilon)/\log B\rfloor$ in (i) above is 1049.

Applying Lemma 1.2.2, we got that in all cases $m_2\le 1049$ by using either q_{150} or q_{200}. By the calculations from Subsection 4.9.1 where in fact we treated the case $m\le 1049$, we get that (n_2,m_2) is one of the solutions listed in (i) or (ii) of Theorem 4.2.1. This finishes the proof of the theorem. □

5. On the x–coordinates of Pell equations that are products of two Lucas numbers

This chapter contains a presentation of a slightly modified version of the paper [4] with the title *On the x–coordinates of Pell equations that are products of two Lucas numbers*. The article has been published in *The Fibonacci Quarterly* in February, 2020.

Abstract: Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers given by $L_0 = 2,\ L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. In this paper, for an integer $d \geq 2$ that is square-free, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 1$, that is a product of two Lucas numbers, with a few exceptions that we completely characterize.

Keywords: Lucas numbers; Pell equations; Linear forms in logarithms; Baker's method.

2010 Mathematics Subject Classification: 11B39, 11D45, 11D61, 11J86.

5.1. Introduction

Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers given by $L_0 = 2,\ L_1 = 1$, and

$$L_{n+2} = L_{n+1} + L_n$$

for all $n \geq 0$. This is sequence A000032 on the Online Encyclopedia of Integer Sequences (OEIS). The first few terms of this sequence are

$$\{L_n\}_{n\geq 0} \quad = \quad 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, \ldots.$$

Putting $(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ for the roots of the characteristic equation $r^2 - r - 1 = 0$ of the Lucas sequence, the Binet formula for its general terms is given by

$$L_n = \alpha^n + \beta^n, \qquad \text{for all} \quad n \geq 0. \tag{5.1}$$

Furthermore, we can prove by induction that the inequality

$$\alpha^{n-1} \le L_n \le \alpha^{n+2}, \tag{5.2}$$

holds for all $n \ge 0$.

Kafle, et al. [43] considered the Diophantine equation

$$x_n = F_\ell F_m, \tag{5.3}$$

where $\{F_m\}_{m\ge0}$ is the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all $m \ge 0$. They proved that equation (5.3) has at most one solution n in positive integers except for $d = 2,3,5$, for which case equation (5.3) has the solutions $x_1 = 1$ and $x_2 = 3$, $x_1 = 2$ and $x_2 = 26$, $x_1 = 2$ and $x_2 = 9$, respectively.

There are many other researchers who have studied related problems involving the intersection sequence $\{x_n\}_{n\ge1}$ with linear recurrence sequences of interest. For example, see [15, 19, 6, 40–42, 47, 57].

5.2. Main Result

In this paper, we study a similar problem to that of Kafle, et al. [43], but with the Lucas numbers instead of the Fibonacci numbers. That is, we show that there is at most one value of the positive integer x participating in (1.17), that is a product of two Lucas numbers, with a few exceptions that we completely cahracterize. This can be interpreted as solving the Diophantine equation

$$x_k = L_n L_m, \tag{5.4}$$

in nonnegative integers (k,n,m) with $k \ge 1$ and $0 \le m \le n$.

Theorem 5.2.1. *For each square-free integer $d \ge 2$ there is at most one integer k such that the equation* (5.4) *holds, except for $d \in \{2,3,5,15,17,35\}$ for which $x_1 = 1,\ x_2 = 3, x_3 = 7, x_9 = 1393$ (for $d = 2$), $x_1 = 2,\ x_2 = 7$ (for $d = 3$), $x_1 = 2,\ x_2 = 9$ (for $d = 5$), $x_1 = 4,\ x_5 = 15124$ (for $d = 15$), $x_1 = 4,\ x_2 = 33$ (for $d = 17$) and $x_1 = 6,\ x_3 = 846$ (for $d = 35$).*

5.3. Bounding the variables

We assume that (x_1, y_1) is the smallest positive solution of the Pell equation (1.17). As in Section 1.3, we set

$$x_1^2 - dy_1^2 =: \varepsilon, \qquad \varepsilon \in \{\pm1\},$$

and put

$$\delta := x_1 + \sqrt{d}y_1 \qquad \text{and} \qquad \sigma := x_1 - \sqrt{d}y_1 = \varepsilon\delta^{-1}.$$

From (1.18), we get

$$x_k = \frac{1}{2}\left(\delta^k + \sigma^k\right). \tag{5.5}$$

Since $\delta \geq 1+\sqrt{2} > \alpha^{3/2}$, it follows that the estimate

$$\frac{\delta^k}{\alpha^2} \leq x_k < \frac{\delta^k}{\alpha} \quad \text{holds for all} \quad k \geq 1. \tag{5.6}$$

We let $(k,n,m) := (k_i,n_i,m_i)$ for $i = 1,2$ be the solutions of (5.4). By (5.2) and (5.6), we get

$$\alpha^{n+m-2} \leq L_nL_m = x_k < \frac{\delta^k}{\alpha} \quad \text{and} \quad \frac{\delta^k}{\alpha^2} \leq x_k = L_nL_m \leq \alpha^{n+m+4}, \tag{5.7}$$

so

$$kc_1\log\delta - 6 < n+m < kc_1\log\delta + 1 \quad \text{where} \quad c_1 := \frac{1}{\log\alpha}. \tag{5.8}$$

To fix ideas, we assume that

$$n \geq m \qquad \text{and} \qquad k_1 < k_2.$$

We also put

$$m_3 := \min\{m_1,m_2\}, \quad m_4 := \max\{m_1,m_2\}, \quad n_3 := \min\{n_1,n_2\}, \quad n_4 := \max\{n_1,n_2\}.$$

Using the inequality (5.8) together with the fact that $\delta \geq 1+\sqrt{2} = \alpha^{3/2}$ (so, $c_1\log\delta > 3/2$), gives us that

$$\frac{3}{2}k_2 < k_2c_1\log\delta < 2n_2+6 \leq 2n_4+6,$$

so

$$k_1 < k_2 < \frac{4}{3}n_4 + 4. \tag{5.9}$$

Thus, it is enough to find an upper bound on n_4. Substituting (1.10) and (5.5) in (5.4) we get

$$\frac{1}{2}(\delta^k + \sigma^k) = (\alpha^n + \beta^n)(\alpha^m + \beta^m). \tag{5.10}$$

This can be regrouped as

$$\delta^k 2^{-1}\alpha^{-n-m} - 1 = -2^{-1}\sigma^k\alpha^{-n-m} + (\beta\alpha^{-1})^n + (\beta\alpha^{-1})^m + (\beta\alpha^{-1})^{n+m}.$$

Since $\beta = -\alpha^{-1}$, $\sigma = \varepsilon\delta^{-1}$ and using the fact that $\delta^k \geq \alpha^{n+m-1}$ (by (5.7)), we get

$$\begin{aligned}\left|\delta^k 2^{-1}\alpha^{-n-m} - 1\right| &\leq \frac{1}{2\delta^k\alpha^{n+m}} + \frac{1}{\alpha^{2n}} + \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{2(n+m)}}\\ &\leq \frac{\alpha}{2\alpha^{2(n+m)}} + \frac{3}{\alpha^{2m}} \quad < \quad \frac{6}{\alpha^{2m}},\end{aligned}$$

In the above, we have also used the facts that $n \geq m$ and $(1/2)\alpha + 3 < 6$. Hence,

$$\left|\delta^k 2^{-1}\alpha^{-n-m} - 1\right| < \frac{6}{\alpha^{2m}}. \tag{5.11}$$

We let $\Lambda_1 := \delta^k 2^{-1}\alpha^{-n-m} - 1$. We put

$$\Gamma_1 := k\log\delta - \log 2 - (n+m)\log\alpha. \tag{5.12}$$

Note that $e^{\Gamma_1} - 1 = \Lambda_1$. If $m > 100$, then $\frac{6}{\alpha^{2m}} < \frac{1}{2}$. Since $|e^{\Gamma_1} - 1| < 1/2$, it follows that

$$|\Gamma_1| < 2|e^{\Gamma_1} - 1| < \frac{12}{\alpha^{2m}}. \tag{5.13}$$

By recalling that $(k,n,m) = (k_i,n_i,m_i)$ for $i = 1,2$, we get that

$$|k_i\log\delta - \log 2 - (n_i+m_i)\log\alpha| < \frac{12}{\alpha^{2m_i}} \tag{5.14}$$

holds for both $i = 1,2$ provided $m_3 > 100$.

We apply Theorem 1.1.8 on the left-hand side of (5.11). First, we need to check that $\Lambda_1 \neq 0$. Well, if it were, then $\delta^k\alpha^{-n-m} = 2$. However, this is impossible since $\delta^k\alpha^{-n-m}$ is a unit while 2 is not. Thus, $\Lambda_1 \neq 0$, and we can apply Theorem 1.1.8. We take the data

$$t := 3, \quad \eta_1 := \delta, \quad \eta_2 := 2, \quad \eta_3 := \alpha, \quad b_1 := k, \quad b_2 := -1, \quad b_3 := -n-m.$$

We take $\mathbb{K} := \mathbb{Q}(\sqrt{d},\alpha)$ which has degree $D \leq 4$ (it could be that $d = 5$ in which case $D = 2$; otherwise, $D = 4$). Since $\delta \geq 1+\sqrt{2} > \alpha$, the second inequality in (5.8) tells us that $k < n+m$, so we take $B := 2n$. We have $h(\eta_1) = h(\delta) = \frac{1}{2}\log\delta$, $h(\eta_2) = h(2) = \log 2$ and $h(\eta_3) = h(\alpha) = \frac{1}{2}\log\alpha$. Thus, we can take $A_1 := 2\log\delta$, $A_2 := 4\log 2$ and $A_3 := 2\log\alpha$. Now, Theorem 1.1.8 tells us that

$$\begin{aligned}\log|\Lambda_1| &> -1.4\times 30^6\times 3^{4.5}\times 4^2(1+\log 4)(1+\log(2n))(2\log\delta)(4\log 2)(2\log\alpha)\\ &> -2.92\times 10^{13}\log\delta(1+\log(2n)).\end{aligned}$$

By comparing the above inequality with (5.11), we get

$$2m\log\alpha - \log 6 < 2.92\times 10^{13}\log\delta(1+\log(2n)). \tag{5.15}$$

Thus

$$m < 6.06 \times 10^{13} \log \delta (1 + \log(2n)). \tag{5.16}$$

Since, $\delta^k < \alpha^{n+m+6}$, we get that

$$k \log \delta < (n+m+6) \log \alpha \leq (2n+6) \log \alpha, \tag{5.17}$$

which together with the estimate (5.16) gives

$$km < 5.84 \times 10^{13} n (1 + \log(2n)). \tag{5.18}$$

Let us record what we have proved, since this will be important later-on.

Lemma 5.3.1. *If $x_k = L_n L_m$ and $n \geq m$, then*

$$m < 6.06 \times 10^{13} \log \delta (1 + \log(2n)), \quad km < 5.84 \times 10^{13} n (1 + \log(2n)), \quad k \log \delta < 4n \log \alpha.$$

Note that we did not assume that $m_3 > 100$ for Lemma 5.3.1 since we have worked with the inequality (5.11) and not with (5.13). We now again assume that $m_3 > 100$. Then the two inequalities (5.14) hold. We eliminate the term involving $\log \delta$ by multiplying the inequality for $i = 1$ with k_2 and the one for $i = 2$ with k_1, subtract them and apply the triangle inequality as follows

$$\begin{aligned}
& |(k_2 - k_1) \log 2 - (k_2(n_1 + m_1) - k_1(n_2 + m_2)) \log \alpha| \\
& = |k_2(k_1 \log \delta - \log 2 - (n_1 + m_1) \log \alpha) - k_1(k_2 \log \delta - \log 2 - (n_2 + m_2) \log \alpha)| \\
& \leq k_2 |k_1 \log \delta - \log 2 - (n_1 + m_1) \log \alpha| + k_1 |k_2 \log \delta - \log 2 - (n_2 + m_2) \log \alpha| \\
& \leq \frac{12 k_2}{\alpha^{2m_1}} + \frac{12 k_1}{\alpha^{2k_2}} < \frac{24 k_2}{\alpha^{2m_3}}.
\end{aligned}$$

Thus,

$$|\Gamma_2| := |(k_2 - k_1) \log 2 - (k_2(n_1 + m_1) - k_1(n_2 + m_2)) \log \alpha| < \frac{24 k_2}{\alpha^{2m_3}}. \tag{5.19}$$

We are now set to apply Theorem 1.1.9 with the data

$$t := 2, \quad \eta_1 := 2, \quad \eta_2 := \alpha, \quad b_1 := k_2 - k_1, \quad b_2 := k_2(n_1 + m_1) - k_1(n_2 + m_2).$$

The fact that $\eta_1 = 2$ and $\eta_2 = \alpha$ are multiplicatively independent follows because α is a unit while 2 is not. We observe that $k_2 - k_1 < k_2$, whereas by the absolute value of the inequality in (5.19), we have

$$|k_2(n_1 + m_1) - k_1(n_2 + m_2)| \leq (k_2 - k_1) \frac{\log 2}{\log \alpha} + \frac{24 k_2}{\alpha^{2m_3} \log \alpha} < 2k_2,$$

because $m_3 > 10$. We have that $\mathbb{K} := \mathbb{Q}(\alpha)$, which has $D := 2$. So we can take

$$\log B_1 = \max\left\{h(\eta_1), \frac{|\log \eta_1|}{2}, \frac{1}{2}\right\} = \log 2,$$

and

$$\log B_2 = \max\left\{h(\eta_2), \frac{|\log \eta_2|}{2}, \frac{1}{2}\right\} = \frac{1}{2}.$$

Thus,

$$b' = \frac{|k_2 - k_1|}{2\log B_2} + \frac{|k_2(n_1+m_1) - k_1(n_2+m_2)|}{2\log B_1} \le k_2 + \frac{k_2}{\log 2} < 3k_2.$$

Now Theorem 1.1.9 tells us that with

$$\Gamma_2 = (k_2 - k_1)\log 2 - (k_2(n_1+m_1) - k_1(n_2+m_2))\log \alpha,$$

we have

$$\log|\Gamma_2| > -24.34 \times 2^4 \left(\max\{\log(3k_2) + 0.14, 10.5\}\right)^2 \cdot (2\log 2) \cdot (1/2).$$

Thus,

$$\log|\Gamma_2| > -270 \left(\max\{\log(3k_2) + 0.14, 10.5\}\right)^2.$$

By comparing the above inequality with (5.19), we get

$$2m_3 \log \alpha - \log(24k_2) < 270 \left(\max\{\log(3k_2) + 0.14, 10.5\}\right)^2.$$

If $k_2 \le 10523$, then $\log(3k_2) + 0.14 < 10.5$. Thus, the last inequality above gives

$$2m_3 \log \alpha < 270 \times 10.5^2 + \log(24 \times 10523),$$

giving $m_3 < 30942$ in this case. Otherwise, $k_2 > 10523$, and we get

$$2m_3 \log \alpha < 272(1 + \log k_2)^2 + \log(24k_2) < 280(1 + \log k_2)^2,$$

which gives

$$m_3 < 160(1 + \log k_2)^2.$$

We record what we have proved

Lemma 5.3.2. *If* $m_3 > 100$*, then either*

(i) $k_2 \le 10523$ *and* $m_3 < 30942$ *or*
(ii) $k_2 > 10523$*, in which case* $m_3 < 160(1 + \log k_2)^2$.

Now suppose that some m is fixed in (5.4), or at least we have some good upper bounds on it. We rewrite (5.4) using (1.10) and (5.5) as

$$\frac{1}{2}(\delta^k+\sigma^k)=L_m(\alpha^n+\beta^n),$$

so

$$\delta^k\,(2L_m)^{-1}\,\alpha^{-n}-1=-\frac{1}{2L_m}\sigma^k\alpha^{-n}+(\beta\alpha^{-1})^n.$$

Since $m\geq 1$, $\beta=-\alpha^{-1}$, $\sigma=\varepsilon\delta^{-1}$ and $\delta^k>\alpha^{n+m-1}$, we get

$$\begin{aligned}\left|\delta^k\,(2L_m)^{-1}\,\alpha^{-n}-1\right| &\leq \frac{1}{2L_m\delta^k\alpha^n}+\frac{1}{\alpha^{2n}} \leq \frac{\alpha}{\alpha^{2(n+m)}}+\frac{1}{\alpha^{2n}}\\ &\leq \frac{\alpha+1}{\alpha^{2n}} < \frac{6}{\alpha^{2n}},\end{aligned}$$

where we have used the fact that $n\geq m\geq 0$ and $\alpha+1<6$. Hence,

$$|\Lambda_3|:=\left|\delta^k\,(2L_m)^{-1}\,\alpha^{-n}-1\right|<\frac{6}{\alpha^{2n}}. \tag{5.20}$$

We assume that $n_3>100$. In particular, $\frac{6}{\alpha^{2n}}<\frac{1}{2}$ for $n\in\{n_1,n_2\}$, so we get by the previous argument that

$$|\Gamma_3|:=|k\log\delta-\log(2L_m)-n\log\alpha|<\frac{12}{\alpha^{2n}}. \tag{5.21}$$

We are now set to apply Theorem 1.1.8 on the left-hand side of (5.20) with the data

$$t:=3,\quad \eta_1:=\delta,\quad \eta_2:=2L_m,\quad \eta_3:=\alpha,\quad b_1:=k,\quad b_2:=-1,\quad b_3:=-n.$$

First, we need to check that $\Lambda_3:=\delta^k(2L_m)^{-1}\alpha^{-n}-1\neq 0$. If not, then $\delta^k=2L_m\alpha^m$. The left-hand side belongs to the field $\mathbb{Q}(\sqrt{d})$ but not rational while the right-hand side belongs to the field $\mathbb{Q}(\sqrt{5})$. This is not possible unless $d=5$. In this last case, δ is a unit in $\mathbb{Q}(\sqrt{5})$ while $2L_m$ is not a unit in $\mathbb{Q}(\sqrt{5})$ since the norm of this first element is $4L_m^2\neq\pm 1$. So, $\Lambda_3\neq 0$. Thus, we can apply Theorem 1.1.8. We have the field $\mathbb{K}:=\mathbb{Q}(\sqrt{d},\sqrt{5})$ which has degree $D\leq 4$. We also have

$$\begin{aligned}h(\eta_2) &= h(2L_m)=h(2)+h(L_m)\\ &\leq \log 2+(m+1)\log\alpha<2+m\log\alpha\\ &\leq 2.92\times 10^{13}\log\delta(1+\log(2n))\quad \text{by (5.16)}.\end{aligned}$$

So, we take

$$h(\eta_1)=\frac{1}{2}\log\delta,\quad h(\eta_2)=2.92\times 10^{13}\log\delta(1+\log(2n))\quad\text{and}\quad h(\eta_3)=\frac{1}{2}\log\alpha.$$

Then,

$$A_1 := 2\log\delta, \quad A_2 := 1.18\times 10^{14}\log\delta(1+\log(2n)) \quad \text{and} \quad A_3 := 2\log\alpha.$$

Then, by Theorem 1.1.8 we get

$$\begin{aligned}\log|\Lambda_3| &> -1.4\times 30^6\times 3^{4.5}\times 4^2(1+\log 4)(1+\log n)(2\log\delta)\\ &\quad \times(1.18\times 10^{14}\log\delta(1+\log(2n)))(2\log\alpha)\\ &> -8.6\times 10^{26}(1+\log(2n))^2(\log\delta)^2\log\alpha.\end{aligned}$$

Comparing the above inequality with (5.20), we get

$$2n\log\alpha-\log 6 < 8.6\times 10^{26}(1+\log(2n))^2(\log\delta)^2\log\alpha,$$

which implies that

$$n < 4.3\times 10^{26}(1+\log(2n))^2(\log\delta)^2. \tag{5.22}$$

We record what we have proved.

Lemma 5.3.3. *If $x_k = L_nL_m$ with $n\ge m\ge 1$, then we have*

$$n < 4.3\times 10^{26}(1+\log(2n))^2(\log\delta)^2.$$

Note that we did not use the assumption that $m_3 > 100$ of that $n_3 > 100$ for Lemma 5.3.3 since we worked with the inequality (5.20) not with the inequality (5.21). We now assume that $n_3 > 100$ and in particular (5.21) holds for $(k,n,m) = (k_i,n_i,m_i)$ for both $i = 1,2$. By the previous procedure, we also eliminate the term involving $\log\delta$ as follows

$$|k_2\log(2L_{m_1})-k_1\log(2L_{m_2})-(k_2n_1-k_1n_2)\log\alpha| < \frac{12k_2}{\alpha^{2n_1}}+\frac{12k_1}{\alpha^{2n_2}} < \frac{24k_2}{\alpha^{2n_3}}. \tag{5.23}$$

We assume that $\alpha^{2n_3} > 48k_2$. If we put

$$\Gamma_4 := k_2\log(2L_{m_1})-k_1\log(2L_{m_2})-(k_2n_1-k_1n_2)\log\alpha,$$

we have that $|\Gamma_4| < 1/2$. We then get that

$$|\Lambda_4| := |e^{\Gamma_4}-1| < 2|\Gamma_4| < \frac{48k_2}{\alpha^{2n_3}}. \tag{5.24}$$

We apply Theorem 1.1.8 to

$$\Lambda_4 := (2L_{m_1})^{k_2}(2L_{m_2})^{-k_1}\alpha^{-(k_2n_1-k_1n_2)}-1.$$

First, we need to check that $\Lambda_4\neq 0$. Well, if it were, then it would follow that

$$\frac{L_{m_1}^{k_2}}{L_{m_2}^{k_1}} = 2^{k_1-k_2}\alpha^{k_2n_1-k_1n_2}. \tag{5.25}$$

We consider the following Lemma.

Lemma 5.3.4. *The equation* (5.25) *has only many small positive integer solutions* (k_i, n_i, m_i) *for* $i = \{1,2\}$ *with* $k_1 < k_2$ *and* $m_1 \leq m_2 \leq 6$. *Futhermore, none of these solutions lead to a valid solution to the original Diophantine equation* (5.4).

Proof. We suppose that (5.25) holds and assume that $\gcd(k_1, k_2) = 1$. Since $\alpha^{k_2 n_1 - k_1 n_2} \in \mathbb{Q}$, it follows $k_2 n_1 = k_1 n_2$. Thus, if one of the n_1, n_2 is zero, so is the other. Since $n_i \geq m_i$ for $i \in \{1,2\}$, it follows that $n_1 = n_2 = 0$, $m_1 = m_2 = 0$, so $x_{k_1} = x_{k_2}$, therefore $k_1 = k_2$ a contradiction. Thus, n_1 and n_2 are both positive integers. Next $L_{m_1}^{k_2}/L_{m_2}^{k_1} = 2^{k_1 - k_2} < 1$. Thus, $L_{m_1}^{k_2} < L_{m_2}^{k_1} < L_{m_2}^{k_2}$, so $L_{m_1} < L_{m_2}$. This implies that either $(m_1, m_2) = (1,0)$ or $m_1 < m_2$. The case $(m_1, m_2) = (1,0)$ gives $1/2^{k_1} = 2^{k_1 - k_2}$. Thus, $k_2 = 2k_1$ and since $\gcd(k_1, k_2) = 1$, we get $k_1 = 1$, $k_2 = 2$, so $n_2 = 2n_1$. But then $x_2 = x_{k_2} = L_{n_2} L_{m_2} = L_{2n_1} L_0 = 2L_{2n_1}$ is even, a contradiction since $x_2 = 2x_1 \pm 1$ (by Example 1.5.1 (i)) is odd. Thus, $m_1 < m_2$. If $m_2 > 6$, the Carmichael Primitive Divisor Theorem for Lucas numbers shows that L_{m_2} is divisible by a prime $p > 7$ which does not divide L_{m_1}. This is impossible since it contradicts the assumption that (5.25) holds. Thus, $m_2 \leq 6$. Further since $L_{m_1}^{k_2}/L_{m_2}^{k_1} = 1/2^{k_2 - k_1}$ it follows that $L_{m_1}^{k_1} \mid L_{m_1}^{k_2} \mid L_{m_2}^{k_1}$, so $L_{m_1} \mid L_{m_2}$. So, there are three cases that we analyse:

Case 5.4.4.1. $m_1 = 0$, $m_2 \in \{3,6\}$. If $(m_1, m_2) = (0,3)$, then $2^{k_2}/4^{k_1} = 1/2^{2k_1 - k_2} = 1/2^{k_2 - k_1}$. This gives $2k_2 = 3k_1$ and since k_1 and k_2 are coprime, it follows that $k_1 = 2$ and $k_2 = 3$. Then $x_2 = x_{k_1} = L_{n_1} L_{m_1} = L_{n_1} L_0 = 2L_{n_1}$ is even, a contradiction since $x_2 = 2x_1 \pm 1$ is odd. If $(m_1, m_2) = (0,6)$, then $2^{k_2}/18^{k_1} = 1/2^{k_2 - k_1}$, which is impossible since by looking at the exponent of 3 we would get $k_1 = 0$, a contradiction.

Case 5.4.4.2. $m_1 = 2$ and L_{m_2} is a power of 2. The case $m_2 = 0$ has been treated so the only other case left is $m_2 = 3$. In this case, $1/4^{k_1} = 1/2^{k_2 - k_1}$, giving $k_2 = 3k_1$. Thus, since $\gcd(k_1, k_2) = 1$, then $k_1 = 1$ and $k_2 = 3$. Since $k_2 n_1 = k_1 n_2$, we get $n_2 = 3n_1$. Thus, $x_1 = L_{n_1} L_1 = L_{n_1}$ and $x_3 = L_{3n_1} L_3 = 4L_{3n_1}$. Now $x_3 = x_1(4x_1^2 \pm 3)$ (by Example 1.5.1 (ii)) and the second factor is odd, so the power of 2 dividing $4L_{3n_1}$ divides $x_1 = L_{n_1}$. But $4L_{3n_1}$ is a multiple of 8 since L_{3n_1} is even. Thus, $8 \mid L_{n_1}$, which is false.

Case 5.4.4.3. $m_1 = 2$ and $m_2 = 6$. We get $3^{k_2}/(2.3^2)^{k_1} = 1/2^{k_2 - k_1}$. Looking at the exponent of 3, we get $k_2 = 2k_1$ and loking at the exponent of 2 we also get $k_2 = 2k_1$, so $k_1 = 1$ and $k_2 = 2$. Also, $n_2 = 2n_1$. Thus, $x_1 = L_{n_1} L_{m_1} = 3L_{n_1}$ and $x_2 = L_{n_2} L_{m_2} = 18L_{2n_1}$ is even, a contradiction with the fact that $x_2 = 2x_1^2 \pm 1$ is odd. □

So, by Lemma 5.3.4 we have $\Lambda_4 \neq 0$. Thus, we can now apply Theorem 1.1.8 with the data

$$t := 3, \quad \eta_1 := 2L_{m_1}, \quad \eta_2 := 2L_{m_2}, \quad \eta_3 := \alpha, \quad b_1 = k_2,$$
$$b_2 := -k_1, \quad b_3 := -(k_2 n_1 - k_1 n_2).$$

We have $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ which has degree $D := 2$. Also, using (5.9), we can take $B := 4n_4^2$. We can also take $A_1 := 2(2+m_1\log\alpha) \le 4m_1\log\alpha$, $A_2 := 2(2+m_2\log\alpha) \le 4m_2\log\alpha$ and $A_3 := \log\alpha$. Theorem 1.1.8 gives that

$$\begin{aligned}\log|\Lambda_4| &> -1.4\times 30^6\times 3^{4.5}\times 2^2(1+\log 2)(1+\log(4n_4^2))(4m_1\log\alpha)(4m_2\log\alpha)\log\alpha,\\ &> -3.44\times 10^{12}m_1m_2(1+\log(2n_4)).\end{aligned}$$

By comparing this with the inequality (5.24), we get

$$2n_3\log\alpha - \log(48k_2) < 3.44\times 10^{12}m_1m_2(1+\log(2n_4)).$$

Since $k_2 < 4n_4$ and $n_4 > 10$, we get that $\log(48k_2) < 2(1+\log(2n_4))$. Thus,

$$n_3 < 3.58\times 10^{12}m_1m_2(1+\log(2n_4)). \tag{5.26}$$

All this was done under the assumption that $\alpha^{2n_3} > 48k_2$. But if that inequality fails, then

$$n_3 < c_1\log(48k_2) < 12(1+\log(2n_4)),$$

which is much better than (5.26). Thus, (5.26) holds in all cases. Next, we record what we have proved.

Lemma 5.3.5. *Assuming that $n_3 > 100$, then we have*

$$n_3 < 3.58\times 10^{12}m_1m_2(1+\log(2n_4)).$$

We now start finding effective bounds for our variables.

Case 5.4.1. $m_4 \le 100$.

Then $m_1 < 100$ and $m_2 < 100$. By Lemma 5.3.5, we get that

$$n_3 < 3.58\times 10^{16}(1+\log(2n_4)).$$

By Lemma 5.3.1, we get

$$\log\delta < 4n_3\log\alpha < 6.89\times 10^{16}(1+\log(2n_4)).$$

By the inequality (5.8), we have that

$$\begin{aligned} n_4 &\le n_4+m_4-1\\ &< k_2c_1\log\delta\\ &< 1.72\times 10^{27}c_1(1+\log(2n_4))^2(\log\delta)^3 \quad \text{(by (5.9) and Lemma 5.3.3)}\\ &< \frac{1}{\log\alpha}(1.72\times 10^{27}(1+\log(2n_4))^2)(6.89\times 10^{16}(1+\log(2n_4)))^3\\ &< 1.17\times 10^{78}\log(1+\log(2n_4))^5.\end{aligned}$$

With the help of *Mathematica*, we get that $n_4 < 4.6\times 10^{89}$. Thus, using (5.9), we get

$$\max\{k_2,n_4\} < 4.6\times 10^{89}.$$

We record what we have proved.

Lemma 5.3.6. *If* $m_4 := \max\{m_1, m_2\} \le 100$*, then*

$$\max\{k_2, n_4\} < 4.6 \times 10^{89}.$$

Case 5.4.2. $m_4 > 100$.

Note that either $m_3 \le 100$ or $m_3 > 100$ case in which by Lemma 5.3.2 and the inequality (5.9), we have $m_3 \le 160(1+\log(4n_4))^2$ provided that $m_4 > 10000$, which we now assume.

We let $i \in \{1,2\}$ be such that $m_i = m_3$ and j be such that $\{i, j\} = \{1,2\}$. We assume that $n_3 > 100$. We work with (5.21) for i and (5.14) for j and noting the conditions $n_i > 100$ and $m_j = m_4 > 100$ are fullfilled. That is,

$$\begin{aligned} |k_i \log\delta + \log(2L_{m_i}) - n_i \log\alpha| &< \frac{12}{\alpha^{2n_i}}, \\ \left|k_j \log\delta - \log 2 - (n_j + m_j)\log\alpha\right| &< \frac{12}{\alpha^{2m_j}}. \end{aligned}$$

By a similar procedure as before, we eliminate the term involving $\log\delta$. We multiply the first inequality by k_j, the second inequality by k_i, subtract the resulting inequalities and apply the triangle inequalty to get

$$\begin{aligned} \left|k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j))\log\alpha\right| &< \frac{12k_j}{\alpha^{2m_i}} + \frac{12k_i}{\alpha^{2l_j}} \\ &< \frac{24k_2}{\alpha^{2\min\{n_i, m_j\}}}. \end{aligned} \tag{5.27}$$

Assume that $\alpha^{2\min\{n_i, m_j\}} > 48k_2$. We put

$$\Gamma_5 := k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j))\log\alpha.$$

We can write $\Lambda_5 := (2L_{m_i})^{k_j} 2^{-k_i} \alpha^{(k_j n_i - k_i(n_j + m_j))} - 1$. Under the above assumption and using (5.27), we get that

$$|\Lambda_5| = |e^{\Gamma_5} - 1| < 2|\Gamma_5| < \frac{48k_2}{\alpha^{2\min\{n_i, m_j\}}}. \tag{5.28}$$

We are now set to apply Theorem 1.1.8 on Λ_5. First, we need to check that $\Lambda_5 \neq 0$. Well, if it were, then we would get that

$$L_{m_i}^{k_j} = 2^{k_i - k_j} \alpha^{(k_j n_i - k_i(n_j + m_j))}. \tag{5.29}$$

We consider the following lemma.

Lemma 5.3.7. *The equation* (5.29) *has only many small positive integer solutions* $(k_i, k_j, n_i, n_j, m_i, m_j)$ *for* $i, j = \{1,2\}$ *with* $k_1 < k_2$ *and* $m_1 \le m_2 \le 6$*. Futhermore, none of these solutions lead to a valid solution to the original Diophantine equation* (5.4)*.*

Proof. Suppose that (5.29) holds and assume that $\gcd(k_1,k_2)=1$. Since $\alpha^{(k_jn_i-k_i(n_j+m_j))}\in\mathbb{Q}$, then $k_jn_i=k_i(n_j+m_j)$. Next $L_{m_i}^{k_j}=2^{k_i-k_j}$. Thus, $k_i\geq k_j$, so $i=2,\ j=1,\ k_2>k_1$ and $m_2\neq 1$. Since $L_{m_2}>1$ is a power of 2, it follows that $m_2\in\{0,3\}$. Suppose $m_2=0$. Then $L_{m_2}^{k_1}=2^{k_1}=2^{k_2-k_1}$, so $k_2=2k_1$. Hence, $k_1=1$ and $k_2=2$. Further, $n_2=2(n_1+m_1)$. Thus, $x_2=x_{k_2}=L_{n_2}L_{m_2}=2L_{2(n_1+m_1)}$ is even, which false because $x_2=2x_1^2\pm1$ is odd. Suppose next that $m_2=3$. Then $4^{k_1}=2^{k_2-k_1}$. Thus, $k_2=3k_1$, so $k_1=1$ and $k_2=3$. Next, $n_2=3(n_1+m_1)$. Hence, $x_1=x_{k_1}=L_{n_1}L_{m_1}$ and $x_3=x_{k_2}=L_{n_2}L_{m_2}=4L_{3(n_1+m_1)}$. By the previous argument in the proof of Lemma 5.3.4, 8 divides $x_3=x_1(4x_1^2\pm1)$, so $8\mid x_1$. Since $x_1=L_{n_1}L_{m_1}$ and $8\nmid L_n$ for any n, it follows that L_{n_1} and L_{m_1} are both even. Thus, $3\mid n_1$, $3\mid m_1$. Further, one of L_{n_1}, L_{m_1} is a multiple of 4, so one of $n_1,\ m_1$ is odd. Suppose both are odd. Then $4\mid L_{n_1}$, $4\mid L_{m_1}$ so $16\mid x_1\mid x_3\mid 4L_{3(n_1+m_1)}$. This implies that $4\mid L_{3(n_1+m_1)}$, which is false because $3(n_1+m_1)$ is an even multiple of 3, and $2\|L_{6m}$ for any m. Suppose now that one of $n_1,\ m_1$ is an even multiple of 3, and the other is odd. Then $\mathrm{ord}_2(x_1)=3$, where $\mathrm{ord}_2(x)$ is the exponent at which 2 appears in the factorization of x. Hence,

$$3=\mathrm{ord}_2(x_3)=\mathrm{ord}_2(4L_{3(n_1+m_1)})=2+\mathrm{ord}_2(L_{3(n_1+m_1)}),$$

giving $\mathrm{ord}_2(L_{3(n_1+m_1)})=1$, which is again false since $3(n_1+m_1)$ is an odd multiple 3, so a number of the form $3+6m$, and for such numbers we have $4\|L_{3+6m}$. Hence, in all instances we have gotten a contradiction. $\square$

Thus, by Lemma 5.3.7 we have that1 $\Lambda_5\neq0$. So, we can apply Theorem 1.1.8 with the data

$$t:=3,\quad \eta_1:=2L_{m_i},\quad \eta_2:=2\quad \eta_3:=\alpha\quad b_1:=k_j,$$
$$b_2:=-k_i,\quad b_3:=-(k_jn_i-k_i(n_j+m_j)).$$

From the previous calculations, we know that $\mathbb{K}:=\mathbb{Q}(\sqrt{2})$ which has degree $D=2$ and $A_1:=4m_i\log\alpha$, $A_2:=2\log2$ and $A_3:=\log\alpha$. We also take $B:=4n_4^2$. By Theorem 1.1.8, we get that

$$\begin{aligned}\log|\Lambda_5| &> -1.4\times30^6\times3^{4.5}\times2^2(1+\log2)(1+\log(4n_4^2))(4m_i\log\alpha)(2\log2)\log\alpha,\\ &> -5.18\times10^{12}m_i(1+\log(2n_4)).\end{aligned}$$

Comparing the above inequality with (5.28), we get

$$2\min\{n_i,m_j\}\log\alpha-\log(48k_2)<5.12\times10^{12}m_i(1+\log(2n_4)).$$

Since $m_4>100$, we get using (5.9) ($k_2<4n_4$) that,

$$\min\{n_i,n_j\}<5.38\times10^{12}(160(1+\log(4n_4))^2)(1+\log(2n_4))+\frac{c_1}{2}\log(192n_4),$$

which implies that

$$\min\{n_i,m_j\}<1.72\times10^{15}(1+\log(2n_4))^3. \tag{5.30}$$

All this was under the assumptions that $n_4 > 10000$, and that $\alpha^{2\min\{n_i,m_j\}} > 48k_2$. But, still under the condition that $n_4 > 10000$, if $\alpha^{2\min\{n_i,m_j\}} < 48k_2$, then we get an inequality for $\min\{n_i,n_j\}$ which is even much better than (5.30). So, (5.30) holds provided that $n_4 > 10000$. Suppose say that $\min\{n_i,m_j\} = m_j$. Then we get that

$$m_3 < 160(1+\log(4n_4))^2, \quad m_4 < 1.72\times 10^{15}(1+\log(2n_4))^3.$$

By Lemma 5.3.5, since $m_3 > 100$, we get

$$\begin{aligned} n_3 &< (3.58\times 10^{12})(160(1+\log(4n_4))^2)(1+\log(2n_4)) \\ &\quad \times 1.72\times 10^{15}(1+\log(2n_4))^3 \\ &< 1.98\times 10^{30}(1+\log(2n_4))^6. \end{aligned}$$

Together with Lemma 5.3.1, we get

$$\log\delta < 3.80\times 10^{30}(1+\log(2n_4))^6,$$

which together with Lemma 5.3.3 gives

$$n_4 < 4.30\times 10^{26}(1+\log(2n_4))^2(3.80\times 10^{30}(1+\log(2n_4))^6)^2,$$

which implies that

$$n_4 < 6.21\times 10^{87}(1+\log(2n_4))^{14}. \tag{5.31}$$

With the help of *Mathematica*, we get that $n_4 < 1.30\times 10^{122}$. This was proved under the assumption that $n_4 > 10000$, but the situation $n_4 \le 10000$ already provides a better bound than $n_4 < 1.30\times 10^{122}$. Hence,

$$\max\{k_2,n_1,n_2\} < 1.30\times 10^{122}. \tag{5.32}$$

This was when $m_j = \min\{n_i,m_j\}$. Now we assume that $n_i = \min\{n_i,m_j\}$. Then we get

$$n_i < 1.72\times 10^{15}(1+\log(2n_4))^3.$$

By Lemma 5.3.1, we get that

$$\log\delta < 3.31\times 10^{15}(1+\log(2n_4))^3.$$

Now by Lemma 5.3.3 together with Lemma 5.3.1 to bound l_4 give

$$\begin{aligned} n_4 &< 4.30\times 10^{26}(1+\log(2n_4)))^2(3.31\times 10^{15}(1+\log(2n_4))^3)^2 \\ &< 4.72\times 10^{57}(1+\log(2n_4))^{10}. \end{aligned}$$

This gives, $n_4 < 2.44\times 10^{80}$ which is a better bound than 1.30×10^{122}. We record what we have proved.

Lemma 5.3.8. *If* $m_4 := \max\{m_1, m_2\} > 100$ *and* $n_3 := \min\{n_1, n_2\} > 100$*, then*

$$\max\{k_2, n_1, n_2\} < 1.30 \times 10^{122}.$$

It now remains the case when $m_4 > 100$ and $n_3 \leq 100$. But then, by Lemma 5.3.1, we get $\log\delta < 192$ and now Lemma 5.3.1 together with Lemma 5.3.3 give

$$n_4 < 1.56 \times 10^{31}(1 + \log(2n_4))^2,$$

which implies that $n_4 < 10^{36}$ and further $\max\{k_1, n_1, n_2\} < 10^{40}$. We record what we have proved.

Lemma 5.3.9. *If* $m_4 > 100$ *and* $n_3 \leq 100$*, then*

$$\max\{k_1, n_1, n_2\} < 10^{40}.$$

5.4. The final computations

5.4.1. The first reduction

In this subsection we reduce the bounds for k_1, m_1, n_1 and k_2, m_2, n_2 to cases that can be computationally treated. For this we return to the inequalities for Γ_2, Γ_4 and Γ_5.

We return to (5.19) and we set $s := k_2 - k_1$ and $r := k_2(n_1 + m_1) - k_1(n_2 + m_2)$ and divide both sides by $s\log\alpha$ to get

$$\left|\frac{\log 2}{\log\alpha} - \frac{r}{s}\right| < \frac{24k_2}{\alpha^{2m_3} s\log\alpha}. \tag{5.33}$$

We assume that l_3 is so large that the right-hand side of the inequality in (5.33) is smaller than $1/(2s^2)$. This certainly holds if

$$\alpha^{2m_3} > 48k_2^2/\log\alpha. \tag{5.34}$$

Since $k_2 < 1.3 \times 10^{122}$, it follows that the last inequality (5.34) holds provided that $m_3 \geq 589$, which we now assume. In this case r/s is a convergent of the continued fraction of $\tau := \frac{\log 2}{\log\alpha}$ and $s < 1.30 \times 10^{122}$. We are now set to apply Lemma 1.2.1.

We write $\tau := [a_0; a_1, a_2, a_3, \ldots] = [1,2,3,1,2,3,2,4,2,1,2,11,2,1,11,1,1,134,2,2,\ldots]$ for the continued fraction of τ and p_k/q_k for the $k-$th convergent. We get that $r/s = p_j/q_j$ for some $j \leq 237$. Furthermore, putting $a(M) := \max\{a_j : j = 0, 1, \ldots, 237\}$, we get $a(M) := 880$. By Lemma 1.2.1, we get

$$\frac{1}{882s^2} = \frac{1}{(a(M)+2)s^2} \leq \left|\tau - \frac{r}{s}\right| < \frac{24k_2}{\alpha^{2m_3} s\log\alpha},$$

giving

$$\alpha^{2m_3} < \frac{882 \times 24k_2^2}{\log \alpha} < \frac{882 \times 24 \times (1.30 \times 10^{122})^2}{\log \alpha},$$

leading to $m_3 \leq 1190$. We record what we have just proved.

Lemma 5.4.1. *We have* $m_3 := \min\{m_1, m_2\} \leq 1190$.

If $m_1 = m_3$, then we have $i = 1$ and $j = 2$, otherwise $m_2 = m_3$ implying that we have $i = 2$ and $j = 1$. In both cases, the next step is the application of Lemma 1.2.3 (LLL algorithm) for (5.27), where $n_i < 1.30 \times 10^{112}$ and $|k_j n_i - k_i(n_j + m_j)| < 10^{116}$. For each $m_j \in [1, 1190]$ and

$$\Gamma_5 := k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j)) \log \alpha, \tag{5.35}$$

we apply the LLL algorithm on Γ_3 with the data

$$t := 3, \quad \tau_1 := \log(2L_{m_i}), \quad \tau_2 := \log 2, \quad \tau_3 := \log \alpha$$
$$x_1 := k_j, \quad x_2 := -k_i, \quad x_3 := k_j n_i - k_i(n_j + m_j).$$

Further, we set $X := 10^{116}$ as an upper bound to $|x_i|$ for $i = 1, 2, 3$, and $C := (5X)^5$. A computer search in *Mathematica* allows us to conclude, together with the inequality (5.27), that

$$2 \times 10^{-480} < \min_{1 \leq \min\{n_i, m_j\} \leq 1190} |\Gamma_5| < \frac{24k_2}{\alpha^{2\min\{n_i, m_j\}}}. \tag{5.36}$$

Thus, $\min\{n_i, m_j\} \leq 1419$. We assume first that $i = 1$, $j = 2$. Thus, $n_1 \leq 1419$ or $m_j = \min\{n_i, m_j\} \leq 1419$.

Next, we suppose that $m_j = \min\{n_i, m_j\} \leq 1419$. Since $m_1 := m_3 \leq 1190$, we have

$$m_3 := \min\{m_1, m_2\} \leq 1190 \quad \text{and} \quad m_4 := \max\{m_1, m_2\} \leq 1419.$$

Now, returning to the inequality (5.23) which involves

$$\Gamma_4 := k_2 \log(2L_{m_1}) - k_1 \log(2L_{m_2}) - (k_2 n_1 - k_1 n_2) \log \alpha, \tag{5.37}$$

we use again the LLL algorithm to estimate the lower bound for $|\Gamma_4|$ and thus, find a bound for n_1 that is better than the one given in Lemma 5.3.8. We distinguish the cases $m_3 < m_4$ and $m_3 = m_4$.

The case $m_3 < m_4$

We take $m_1 := m_3 \in [1, 1190]$ and $m_2 := m_4 \in [m_3 + 1, 1419]$ and apply Lemma 1.2.3 with the data:

$$t := 3, \quad \tau_1 := 2L_{m_1}, \quad \tau_2 := 2L_{m_2}, \quad \tau_3 := \log\alpha,$$
$$x_1 := k_2, \quad x_2 := -k_1, \quad x_2 := k_1 n_2 - k_2 n_1.$$

We also put $X := 10^{116}$ and $C := (20X)^9$. After a computer search in *Mathematica* together with the inequality (5.23), we can confirm that

$$2 \times 10^{-1120} \leq \min_{\substack{1 \leq m_3 \leq 1190 \\ m_3+1 \leq m_4 \leq 1419}} |\Gamma_4| < 24 k_2 \alpha^{-2n_3}.$$

This leads to the inequality

$$\alpha^{2n_3} < 12 \times 10^{1120} k_2.$$

Sustituting for the bound k_2 given in Lemma 5.3.8, we get that $n_1 := n_3 \leq 2950$.

The case $m_3 = m_4$

In this case $m_1 = m_2 \leq 1419$ and we have

$$\Gamma_4 := (k_2 - k_1)\log(2L_{m_1}) - (k_2 n_1 - k_1 n_2)\log\alpha \neq 0.$$

This is similar to the case we have handled in the previous steps and yields the bound on n_1 which is less than 2950. So in both cases we have $n_1 \leq 2950$. From the fact that

$$\log\delta \leq k_1 \log\delta \leq 4n_1 \log\alpha < 5678,$$

and by considering the inequality given in Lemma 5.3.3, we conclude that

$$n_2 < 1.4 \times 10^{34}(1 + \log(2n_2))^2,$$

which with the help of *Mathematica* yields $n_2 < 1.12 \times 10^{38}$. We summarise the first cycle of our reductions.

$$\max\{k_1, m_1\} \leq n_1 < 2950 \quad \text{and} \quad \max\{k_2, m_2\} \leq n_2 < 1.12 \times 10^{38}. \tag{5.38}$$

From (5.38), we note that the upper bound on n_2 represents a very good reduction of the bound given in Lemma 5.3.8. Hence, we expect that if we restart our reduction cycle with the new bound on n_2, then we get better bounds on n_1 and n_2. Thus, we return to the inequality (5.33) and take $M := 1.12 \times 10^{38}$. A computer seach in *Mathematica* reveals that

$$q_{82} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \leq i \leq 82\} = a_{12} = 134,$$

from which it follows that $m_3 \leq 100$. We now return to (5.35) and we put $X := 1.12 \times 10^{40}$ and $C := (20X)^5$ and then apply the LLL algorithm in Lemma 1.2.3 to $m_3 \in [1,100]$. After a computer search in *Mathematica*, we get

$$1.04 \times 10^{-139} < \min_{1 \leq m_3 \leq 100} |\Gamma_4| < 24k_2\alpha^{-2\min\{n_i,m_j\}},$$

then $\min\{n_i,m_j\} \leq 410$. By continuing under the assumption that $m_j := \min\{n_i,m_j\} \leq 426$, we return to (5.37) and put $X := 1.12 \times 10^{40}$, $C := (20X)^5$ and $M := 1.12 \times 10^{38}$ for the case $m_3 < m_4$ and the case $m_3 = m_4$. After a computer search, we confirm that

$$4.39 \times 10^{-168} < \min_{\substack{1 \leq m_3 \leq 100 \\ m_3+1 \leq m_4 \leq 426}} |\Gamma_4| < 24k_2\alpha^{-2n_3}. \tag{5.39}$$

This gives $n_1 \leq 494$ which holds in both cases. Hence, by a similar procedure given in the first cycle, we get that $n_2 < 3 \times 10^{36}$.

We record what we have proved.

Lemma 5.4.2. *Let* $(k_i,n_i,m_i\}$ *be a solution to the Diophantine equation* $x_{k_i} = L_{n_i}L_{m_i}$*, with* $0 \leq m_i \leq n_i$ *for* $i = 1,2$ *and* $1 \leq k_1 \leq k_2$*, then*

$$\max\{k_1,m_1\} \leq n_1 \leq 494 \quad \textit{and} \quad \max\{k_2,m_2\} \leq n_2 < 3 \times 10^{36}.$$

5.4.2. The final reduction

Returning back to (5.13) and (5.21) and using the fact that (x_1,y_1) is the smallest positive solution to the Pell equation (1.17), we obtain

$$\begin{aligned} x_k &= \frac{1}{2}(\delta^k + \eta^k) = \frac{1}{2}\left(\left(x_1 + y_1\sqrt{d}\right)^k + \left(x_1 - y_1\sqrt{d}\right)^k\right) \\ &= \frac{1}{2}\left(\left(x_1 + \sqrt{x_1^2 \mp 1}\right)^k + \left(x_1 - \sqrt{x_1^2 \mp 1}\right)^k\right) := P_k^{\pm}(x_1). \end{aligned}$$

Thus, we return to the Diophantine equation $x_{k_1} = L_{n_1}L_{m_1}$ and consider the equations

$$P_{k_1}^{+}(x_1) = L_{n_1}L_{m_1} \quad \text{and} \quad P_{k_1}^{-}(x_1) = L_{n_1}L_{m_1}, \tag{5.40}$$

with $k_1 \in [1,500]$, $m_1 \in [0,500]$ and $n_1 \in [m_1+1,500]$.

Besides the trivial case $k_1 = 1$, with the help of a computer search in *Mathematica*, on the above equations in (5.40), we list the only nontrivial solutions in Table 5.1. We also note that

$$7 + 5\sqrt{2} = (1+\sqrt{2})^3,$$

so these solutions come from the same Pell equation with $d = 2$.

$Q^{+}_{k_1}(x_1)$				
k_1	x_1	y_1	d	δ
2	2	1	3	$2+\sqrt{3}$
2	5	2	6	$5+2\sqrt{6}$
2	10	3	11	$10+3\sqrt{11}$
2	4	1	15	$4+\sqrt{15}$
2	6	1	35	$6+\sqrt{35}$

$Q^{-}_{k_1}(x_1)$				
k_1	x_1	y_1	d	δ
2	1	1	2	$1+\sqrt{2}$
2	2	1	5	$2+\sqrt{5}$
2	7	5	2	$7+5\sqrt{2}$
2	4	1	17	$4+\sqrt{17}$
2	26	1	677	$26+\sqrt{677}$
2	179	1	32042	$179+\sqrt{32042}$

Table 5.1.: Solutions to $P^{\pm}_{k_1}(x_1) = L_{n_1}L_{m_1}$

From Table 5.1, we set each $\delta := \delta_t$ for $t = 1,2,\ldots 10$. We then work on the linear forms in logarithms Γ_1 and Γ_2, in order to reduce the bound on n_2 given in Lemma 5.4.2. From the inequality (5.14), for $(k,n,m) := (k_2,n_2,m_2)$, we write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - (n_2+m_2) + \frac{\log 2}{\log(\alpha^{-1})} \right| < \left(\frac{12}{\log \alpha} \right) \alpha^{-2m_2}, \tag{5.41}$$

for $t = 1,2,\ldots 10$.

We put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \qquad \mu_t := \frac{\log 2}{\log(\alpha^{-1})} \qquad \text{and} \quad (A_t, B_t) := \left(\frac{12}{\log \alpha}, \alpha \right).$$

We note that τ_t is transcendental by the Gelfond-Schneider's Theorem and thus, τ_t is irrational. We can rewrite the above inequality, (5.41) as

$$0 < |k_2\tau_t - (n_2+m_2) + \mu_t| < A_t B_t^{-2m_2}, \quad \text{for} \quad t = 1,2,\ldots,10. \tag{5.42}$$

We take $M := 3\times 10^{36}$ which is the upper bound on n_2 according to Lemma 5.4.2 and apply Lemma 1.2.2 to the inequality (5.42). As before, for each τ_t with $t = 1,2,\ldots,10$, we compute its continued fraction $[a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, \ldots]$ and its convergents $p_0^{(t)}/q_0^{(t)}, p_1^{(t)}/q_1^{(t)}, p_2^{(t)}/q_2^{(t)}, \ldots$. For each case, by means of a computer search in *Mathematica*, we find and integer s_t such that

$$q_{s_t}^{(t)} > 18\times 10^{36} = 6M \qquad \text{and} \qquad \varepsilon_t := ||\mu_t q^{(t)}|| - M||\tau_t q^{(t)}| > 0.$$

We finally compute all the values of $b_t := \lfloor \log(A_t q_{s_t}^{(t)}/\varepsilon_t)/\log B_t \rfloor /2$. The values of b_t correspond to the upper bounds on m_2, for each $t = 1,2,\ldots,10$, according to Lemma 1.2.2.

Note that we have a problem at $\delta_7 := 2+\sqrt{5}$. This is because

$$2+\sqrt{5} = 2\left(\frac{1+\sqrt{5}}{2}\right)^2 = 2\alpha^2.$$

So in this case we have $\Gamma_1 := (k_2-1)\log 2 - (n_2+m_2-2k_2)\log\alpha$. Thus,

$$\left|\frac{\log 2}{\log\alpha} - \frac{n_2+m_2-2k_2}{k_2-1}\right| < \frac{12}{(k_2-1)\alpha^{2m_2}\log\alpha}$$

By a similar procedure given in Subsection 5.4.1 with $M := 3\times 10^{36}$, we get that $q_{77} > M$ and $a(M) := \max\{a_i : 0 \le i \le 77\} = 134$. From this we can conclude that $m_2 \le 96$.

The results of the computation for each t are recorded in Table 5.2.

t	δ_t	s_t	q_{s_t}	$\varepsilon_t >$	b_t
1	$2+\sqrt{3}$	68	2.07577×10^{37}	0.319062	94
2	$5+2\sqrt{6}$	91	8.19593×10^{37}	0.087591	97
3	$10+3\sqrt{11}$	67	2.25831×10^{38}	0.316767	96
4	$4+\sqrt{15}$	70	2.78896×10^{37}	0.329388	94
5	$6+\sqrt{35}$	74	1.75745×10^{38}	0.409752	96
6	$1+\sqrt{2}$	76	2.02409×10^{37}	0.263855	94
7	$2+\sqrt{5}$	–	–	–	96
8	$4+\sqrt{17}$	78	4.76137×10^{37}	0.131771	96
9	$26+\sqrt{677}$	65	3.17521×10^{37}	0.356148	94
10	$179+\sqrt{32042}$	77	3.45317×10^{37}	0.384127	94

Table 5.2.: First reduction computation results

By replacing $(k,n,m) := (k_2,n_2,m_2)$ in the inequality (5.21), we can write

$$\left|k_2\frac{\log\delta_t}{\log\alpha} - n_2 + \frac{\log(2L_{m_2})}{\log(\alpha^{-1})}\right| < \left(\frac{12}{\log\alpha}\right)\alpha^{-2n_2}, \tag{5.43}$$

for $t = 1,2,\ldots,10$.

We now put

$$\tau_t := \frac{\log\delta_t}{\log\alpha}, \quad \mu_{t,m_2} := \frac{\log(2L_{m_2})}{\log(\alpha^{-1})} \quad \text{and} \quad (A_t,B_t) := \left(\frac{12}{\log\alpha},\alpha\right).$$

With the above notations, we can rewrite (5.43) as

$$0 < |k_2\tau_t - n_2 + \mu_{t,m_2}| < A_tB_t^{-2n_2}, \quad \text{for} \quad t = 1,2,\ldots 10. \tag{5.44}$$

We again apply Lemma 1.2.2 to the above inequality (5.44), for

$$t = 1,2,\ldots,10, \quad m_2 = 1,2,\ldots,b_t, \quad \text{with} \quad M := 3\times 10^{36}.$$

We take

$$\varepsilon_{t,m_2} := ||\mu_t q^{(t,m_2)}|| - M||\tau_t q^{(t,m_2)}|| > 0,$$

and

$$b_t = b_{t,m_2} := \lfloor \log(A_t q_{s_t}^{(t,m_2)}/\varepsilon_{t,m_2})/\log B_t \rfloor /2.$$

The case $\delta_7 = 2+\sqrt{5}$ is again treated individually by a similar procedure as in the previous step. With the help of Mathematica, we record the results of the computation in Table 5.3.

t	1	2	3	4	5	6	7	8	9	10
$\varepsilon_{t,m_2} >$	0.0145	0.0002	0.0006	0.0034	0.0106	0.0005	–	0.0009	0.0019	0.0010
b_{t,m_2}	97	103	102	99	99	100	102	100	99	100

Table 5.3.: Final reduction computation results

$$\text{Therefore,}\ \max\{b_{t,m_2} : t = 1,2,\ldots,10 \quad \text{and} \quad m_2 = 1,2,\ldots b_t\} \le 103.$$

Thus, by Lemma 1.2.2, we have that $n_2 \le 103$, for all $t = 1,2,\ldots,10$. From the fact that $\delta^k \le \alpha^{n+m+6}$, we can conclude that $k_1 < k_2 \le 198$. Collecting everything together, our problem is reduced to search for the solutions for (5.4) in the following ranges

$$1 \le k_1 < k_2 \le 200, \quad 0 \le m_1 \le n_1 \le 200 \quad \text{and} \quad 0 \le m_2 \le n_2 \le 200.$$

After a computer search on the equation (5.4) on the above ranges, we obtained the following solutions, which are the only solutions for the exceptional d cases we have stated in Theorem 5.2.1:

For the $+1$ case:

$$\begin{aligned}
(d=3) &\qquad x_1 = 2 = L_1L_0, \quad x_2 = 7 = L_4L_1;\\
(d=15) &\qquad x_1 = 4 = L_3L_1 = L_0L_0, \quad x_5 = 15124 = L_{11}L_9;\\
(d=35) &\qquad x_1 = 6 = L_2L_0, \quad x_3 = 846 = L_8L_6.
\end{aligned}$$

For the -1 case:

$$\begin{aligned}
(d=2) &\qquad x_1 = 1 = L_3L_3, \quad x_2 = 3 = L_2L_1, \quad x_3 = 7 = L_4L_1, \quad x_9 = 1393 = L_{11}L_4;\\
(d=5) &\qquad x_1 = 2 = L_1L_0, \quad x_2 = 9 = L_2L_2;\\
(d=17) &\qquad x_1 = 4 = L_3L_1 = L_0L_0, \quad x_2 = 33 = L_5L_2.
\end{aligned}$$

This completes the proof of Theorem 5.2.1. □

Appendix A.

Fibonacci numbers which are products of two Pell numbers

This appendix chapter contains a presentation of a slightly modified version of the paper [7] with the title *Fibonacci numbers which are products of two Pell numbers*. This is a joint work with *Florian Luca* and *Mihaja Rakotomalala*. The article was published in *The Fibonacci Quarterly* in February, 2016.

Abstract: In this paper, we find all Fibonacci numbers which are products of two Pell numbers and all Pell numbers which are products of two Fibonacci numbers.

Keywords: Fibonacci numbers; Pell numbers; Diophantine equations.

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A.1. Introduction

Let $\{F_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ be the sequences of Fibonacci and Pell numbers given by $F_0 = P_0 = 0$, $F_1 = P_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \quad \text{and} \quad P_{n+2} = 2P_{n+1} + P_n \quad \text{for all} \quad n \geq 0,$$

respectively. Their first few terms are
$\{F_n\}_{n\geq 1}$ $1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,4181,\ldots$
$\{P_n\}_{n\geq 1}$ $1,2,5,12,29,70,169,408,985,2378,5741,13860,33461,80782,195025,\ldots.$

Putting $(\alpha,\beta) = ((1+\sqrt{5})/2,(1-\sqrt{5})/2)$ and $(\gamma,\delta) = (1+\sqrt{2},1-\sqrt{2})$ for the pairs of roots of the characteristic equations $x^2-x-1=0$ and $x^2-2x-1=0$ of the Fibonacci and Pell numbers, respectively, then the Binet formulas for their general terms are:

$$F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta} \quad \text{and} \quad P_n = \frac{\gamma^n-\delta^n}{\gamma-\delta} \quad \text{for all} \quad n \geq 0,$$

respectively.

In this note, we study the Diophantine equations

$$F_k = P_m P_n \tag{A.1}$$

and

$$P_k = F_m F_n, \tag{A.2}$$

Our results are:

Theorem A.1.1. *(i) All positive integer solutions (k,m,n) of equation (A.1) have $k = 1,2,5,12$.*
(ii) All positive integer solutions (k,m,n) of equation (A.2) *have $k = 1,2,3,7$.*

It is known that $144 = 12^2$ and $169 = 13^2$ are the largest squares in the Fibonacci and Pell sequences, respectively, and 12 and 13 are Pell and Fibonacci numbers, respectively. So, the above theorem says that there are no larger Fibonacci or Pell numbers which are products of two numbers from the other sequence.

When $m = 1$ in equation (A.1) or $k = 1$ in equation (A.2), the resulting Diophantine equation is of the form

$$U_n = V_m \quad \text{for some} \quad m,n \geq 0, \tag{A.3}$$

where $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ are the Fibonacci and Pell sequences, respectively. More generally, there is a lot of literature on how to solve equations like (A.3) in case $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ are two non degenerate linearly recurrent sequences with dominant roots. See, for example, [51] and [52]. The theory of linear forms in logarithms ła Baker gives that, under reasonable conditions (say, the dominant roots of $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ are multiplicatively independent), equation (A.3) has only finitely many solutions which are effectively computable. In fact, a straightforward linear form in logarithms gives some very large bounds on $\max\{m,n\}$, which then are reduced in practice either by using the LLL algorithm or by using a procedure originally discovered by Baker and Davenport [11] and perfected by Dujella and Pethő [30].

In this paper, we also use a lower bound for linear forms in logarithms due to Matveev [50] and the Dujella-Pethő [30] reduction procedure, to solve equations (A.1) and (A.2).

A.2. Proof of Theorem A.1.1

We ran a computation for $k \leq 400$ and got only the indicated solutions. We now assume that $k > 400$ and that $n > m$. We do not consider the case $n = m$ since they lead to $F_k = \square$ and $P_k = \square$ whose largest solutions are $k = 12$ and $k = 7$, respectively, as we already pointed out in the Introduction. We deal with equation (A.1) first. We use the known inequalities that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{and} \quad \gamma^{n-2} \leq P_n \leq \gamma^{n-1} \quad \text{for all} \quad n \geq 0.$$

Thus,

$$\alpha^{k-2} \le F_k = P_m P_n \le \gamma^{m+n-2} \quad \text{and} \quad \alpha^{k-1} \ge F_k = P_n P_m \ge \gamma^{m+n-4}. \tag{A.4}$$

Hence,

$$1 + c_1(m+n-4) \le k \le 2 + c_1(m+n-2), \quad \text{where} \quad c_1 = \log\gamma/\log\alpha = 1.83157\ldots. \tag{A.5}$$

In particular, $k < 4n$. We get

$$\frac{1}{\sqrt{5}}(\alpha^k - \beta^k) = \frac{1}{8}(\gamma^m - \delta^m)(\gamma^n - \delta^n),$$

which can be regrouped as

$$\left|\frac{\alpha^k}{\sqrt{5}} - \frac{\gamma^{m+n}}{8}\right| = \left|\frac{\beta^k}{\sqrt{5}} - \frac{\gamma^n\delta^m + \gamma^m\delta^n - \delta^{m+n}}{8}\right|.$$

Since $\delta = -\gamma^{-1}$ and $\beta = -\alpha^{-1}$, and the fact that $3/8 < 1/\sqrt{5}$, we get that

$$\left|\frac{\alpha^k}{\sqrt{5}} - \frac{\gamma^{m+n}}{8}\right| < \frac{2}{\sqrt{5}}\max\left\{|\beta|^k, \gamma^{n-m}\right\} = \frac{2\gamma^{n-m}}{\sqrt{5}}. \tag{A.6}$$

Dividing across by $\gamma^{m+n}/8$, we get

$$\left|\frac{8}{\sqrt{5}}\alpha^k\gamma^{-n-m} - 1\right| < \frac{16}{\sqrt{5}\gamma^{2m}}. \tag{A.7}$$

On the left–hand side of (A.7) we apply Theorem 1.1.8 with the data

$$t := 3,\ \eta_1 := 8/\sqrt{5},\ \eta_2 := \alpha,\ \eta_3 := \gamma,\ b_1 := 1,\ b_2 := k,\ b_3 := -m-n.$$

We take $\mathbb{K} := \mathbb{Q}(\sqrt{2}, \sqrt{5})$, for which $D := 4$. Since

$$h(\eta_1) = \log 8,\ h(\eta_2) = (1/2)\log\alpha,\ h(\eta_3) = (1/2)\log\gamma,$$

we take $A_1 := 4\log 8,\ A_2 := 2\log\alpha,\ A_3 := 2\log\gamma$. Finally, we can take $D = 4n$. Note that

$$\Lambda_1 := \frac{8}{\sqrt{5}}\alpha^k\gamma^{-n-m} - 1.$$

The fact that it isn't zero follows from the fact that if it were, we would then get that $\alpha^{-k}\gamma^{m+n} = 8/\sqrt{5}$. However, the left-hand side of the above relation is a unit in $\mathbb{K}$, whereas the right hand side is not as its norm over $\mathbb{K}$ is $2^{12}/5^2$. Thus, $\Lambda_1 \ne 0$. Theorem 1.1.8 gives that

$$\log|\Lambda_1| > -1.4 \times 30^6 \times 3^{4.5} \times 4^2(1+\log 4)(1+\log(4n))(4\log 8)(2\log(\alpha))(2\log\gamma).$$

Comparing the above inequality with (A.7), we get

$$2m\log\gamma - \log(16/\sqrt{5}) < 7.8 \times 10^{13}(1+\log(4n)). \tag{A.8}$$

Hence,

$$m\log\gamma < 4\times 10^{13}(1+\log(4n)). \tag{A.9}$$

Next we return to equation (A.1) and rewrite it as

$$\left|\frac{\alpha^k}{\sqrt{5}P_m}-\frac{\gamma^n}{2\sqrt{2}}\right| = \left|\frac{\beta^k}{\sqrt{5}P_m}-\frac{\delta^n}{2\sqrt{2}}\right| \le \frac{2}{\sqrt{5}}\max\left\{\frac{1}{\alpha^k},\frac{1}{\gamma^n}\right\}.$$

We divide both sides above by $\gamma^n/2\sqrt{2}$ getting

$$\left|\frac{2\sqrt{2}}{\sqrt{5}P_m}\alpha^k\gamma^{-n}-1\right| \le \frac{4\sqrt{2}}{\sqrt{5}}\max\left\{\frac{1}{\alpha^k\gamma^n},\frac{1}{\gamma^{2n}}\right\}.$$

From (A.4), we get that

$$\frac{1}{\alpha^k\gamma^n} = \frac{1/\alpha}{\alpha^{k-1}\gamma^n} \le \frac{(1/\alpha)}{\gamma^{2n+m-4}} = \frac{\gamma^3/\alpha}{\gamma^{2n+m-1}} < \frac{9}{\gamma^{2n}},$$

because $\gamma^3/\alpha < 9$ and $m \ge 1$. Thus,

$$\left|\frac{2\sqrt{2}}{\sqrt{5}P_m}\alpha^k\gamma^{-n}-1\right| \le \frac{4\sqrt{2}\times 9}{\sqrt{5}\gamma^{2n}} = \frac{36\sqrt{2}}{\sqrt{5}\gamma^{2n}}. \tag{A.10}$$

On the left–hand side of (A.10) we apply Theorem 1.1.8 with the data

$$t := 3,\ \eta_1 := \sqrt{5}P_m/2\sqrt{2},\ \eta_2 := \alpha,\ \eta_3 := \gamma,\ b_1 := -1,\ b_2 := k,\ b_3 := -n.$$

We take again $\mathbb{K} := \mathbb{Q}(\sqrt{2},\sqrt{5})$, for which $D := 4$. As before,

$$h(\eta_2) = (1/2)\log\alpha,\ h(\eta_3) = (1/2)\log\gamma,$$

so we can take $A_2 := 2\log\alpha,\ A_3 := 2\log\gamma$. As for $h(\eta_1)$, the polynomial

$$8X^2 - 5P_m^2$$

has η_1 as a root. Thus,

$$\begin{aligned} h(\eta_1) &\le \frac{1}{2}\left(\log 8 + 2\log(\sqrt{5}P_m/2\sqrt{2})\right) \\ &= \log P_m + \log\sqrt{5} \le (m-1)\log\gamma + \log\sqrt{5} \\ &< m\log\gamma. \end{aligned}$$

Using (A.9), we can take

$$A_1 := 16\times 10^{13}(1+\log(4n)) > 4h(\eta_1).$$

Finally, we can take $B := 4n$. Note that

$$\Lambda_2 := \frac{2\sqrt{2}}{\sqrt{5}P_m}\alpha^k\gamma^{-n-m} - 1.$$

Similarly to the argument used to prove that $\Lambda_1 \neq 0$, one justifies that $\Lambda_2 \neq 0$. Theorem 1.1.8 gives that

$$\log|\Lambda_2| > -1.4 \times 30^6 \times 3^{4.5} \times 4^2(1+\log 4)(1+\log(4n))^2 16 \times 10^{13} \times (2\log(\alpha))(2\log\gamma).$$

Comparing this with (A.10), we get

$$2n\log\gamma - \log(36\sqrt{2}/\sqrt{5}) < 1.5 \times 10^{27}(1+\log 4n))^2,$$

giving

$$n < 5 \times 10^{30}. \tag{A.11}$$

The same arguments apply to equation (A.2) (just swap the roles of the pairs (α,β) and (γ,δ) of $1/\sqrt{5}$ and $1/(2\sqrt{2})$. Let us give the details. We assume $m \geq 3$, otherwise $m \in \{1,2\}$, $F_m = 1$ and the solutions of (A.2) are among the solutions to (A.1) with $m = 1$. Inequality (A.5) becomes

$$1+c_2(m+n-4) \leq k \leq 2+c_2(m+n-2), \qquad c_2 = 1/c_1 = \log\alpha/\log\gamma = 0.545979\ldots, \tag{A.12}$$

which implies in particular that $k \leq 3n$. The analog of inequality (A.6) is

$$\begin{aligned}\left|\frac{\gamma^k}{2\sqrt{2}} - \frac{\alpha^{m+n}}{5}\right| &= \left|\frac{\delta^k}{2\sqrt{2}} - \frac{\alpha^n\beta^m + \alpha^m\beta^n - \beta^{m+n}}{5}\right| \\ &\leq \frac{6}{5}\max\left\{|\delta|^k, \alpha^{n-m}\right\} = \frac{6\alpha^{n-m}}{5}.\end{aligned} \tag{A.13}$$

This leads to

$$\left|\frac{5}{2\sqrt{2}}\gamma^k\alpha^{-n-m} - 1\right| < \frac{6}{\alpha^{2m}}, \tag{A.14}$$

which is the analogue of (A.7). We check that the amount Λ_3 in the left–hand side above is non-zero by an argument similar to the one used to prove that Λ_1 and Λ_2 are non-zero, and apply Theorem 1.1.8 to get a lower bound for it, getting

$$\log\Lambda_3 > -1.4 \times 30^6 \times 3^{4.5} \times 4^2(1+\log 4)(1+\log(3n))(4\log 5)(2\log(\alpha))(2\log\gamma).$$

We get that the analog of (A.8) is

$$2m\log\alpha - \log 6 < 5.98 \times 10^{13}(1+\log(3n)),$$

giving

$$m\log\alpha + 1 < 3 \times 10^{13}(1+\log(3n)), \tag{A.15}$$

which is the analog of inequality (A.9). Returning to equation (A.2), we get

$$\left|\frac{\gamma^k}{2\sqrt{2}F_m}-\frac{\alpha^n}{\sqrt{5}}\right|=\left|\frac{\delta^k}{2\sqrt{2}F_m}-\frac{\beta^n}{\sqrt{5}}\right|\leq\frac{2}{\sqrt{5}}\max\left\{\frac{1}{\gamma^k},\frac{1}{\alpha^n}\right\}. \tag{A.16}$$

By (A.12), we get

$$\gamma^k\geq\gamma\alpha^{m+n-4}\geq\gamma\alpha^{-3}\alpha^n,$$

so

$$\frac{1}{\gamma^k}\leq\frac{\alpha^3/\gamma}{\alpha^n}<\frac{2}{\alpha^n}. \tag{A.17}$$

Hence, by (A.16) and (A.17), we get

$$\left|\frac{\sqrt{5}}{2\sqrt{2}F_m}\gamma^k\alpha^{-n}-1\right|<\frac{4}{\alpha^{2n}}. \tag{A.18}$$

This is the analog of (A.10). Writing Λ_4 for the amount under the absolute value in the left–hand side above, we get that it is not 0 by arguments similar to the ones used to prove that $\Lambda_i\neq 0$ for $i=1,2,3$. We apply as we did for Λ_2. Here, $\eta_1:=2\sqrt{2}F_m/\sqrt{5}$ is a root of $5X^2-8F_m^2$. Its height therefore satisfies

$$\begin{aligned} h(\eta_1) &\leq \log F_m+\log 2\sqrt{2}\leq(m-1)\log\alpha+\log 2\sqrt{2}\\ &< m\log\alpha+1<3\times 10^{13}(1+\log(3n)), \end{aligned}$$

by (A.15). We get that

$$\log\Lambda_4>-1.4\times 30^6\times 3^{4.5}\times 4^2(1+\log 4)(1+\log(3n))^2 12\times 10^{13}\times(2\log(\alpha))(2\log\gamma),$$

which together with (A.18) leads to

$$2n\log\alpha-\log 4<1.2\times 10^{27}(1+\log(3n))^2,$$

giving

$$n<7\times 10^{30}.$$

So, comparing the above bound with (A.11), we conclude that both in equation (A.1) and (A.2), we get $n<7\times 10^{30}$. We record what we proved as a lemma.

Lemma A.2.1. *If (k,m,n) are positive integers satisfying one of the equations A.1 or* (A.2) *with $m\leq n$, then $k<4n$ and $n<7\times 10^{30}$.*

Now we need to reduce the bound. To do so, we make use several times of the following result, which is a slight variation of a result due to Dujella and Pethő [30].

We look at (A.7). Assume that $m \geq 20$. Put

$$\Gamma_1 := k \log \alpha - (n+m) \log \gamma + \log(8/\sqrt{5}).$$

Then $|e^{\Gamma_1} - 1| = |\Lambda_1| < 1/4$ by (A.7), which implies that $|\Gamma_1| < 1/2$. Since $|x| < 2|e^x - 1|$ whenever $x \in (-1/2, 1/2)$, we get from $\Lambda_1 = e^{\Gamma_1}$ and (A.7) that

$$|\Gamma_1| < \frac{32}{\sqrt{5}\gamma^{2m}}.$$

If $\Gamma_1 > 0$, then

$$0 < k\left(\frac{\log \alpha}{\log \gamma}\right) - (n+m) + \frac{\log(8/\sqrt{5})}{\log \gamma} < \frac{32}{(\sqrt{5}\log \gamma)\gamma^{2m}} < \frac{17}{\gamma^{2m}}.$$

We apply Lemma 1.2.2 with $M = 3 \times 10^{31}$ (note that $M > 4n > k$),

$$\tau := \frac{\log \alpha}{\log \gamma}, \quad \mu := \frac{\log(8/\sqrt{5})}{\log \delta}, \quad A := 17, \quad B := \gamma^2.$$

Writing $\tau = [a_0, a_1, \ldots]$ as a continued fraction, we get

$$[a_0, \ldots, a_{74}] = \frac{p_{74}}{q_{74}} = \frac{2037068391552562960855777461929676271}{3731035235978315437343082205475618926},$$

and we get $q_{74} > 3 \times 10^{36} > 6M$. We compute $\varepsilon = \|\mu q_{74}\| - M\|\tau q_{74}\| > 0.4$. The reason that we picked the 74th convergent is that both the inequalities $q_{74} > 6M$ and $\varepsilon > 0$ hold. Thus, by Lemma 1.2.2, we get $m \leq 49$. A similar conclusion is reached if we assume that $\Gamma_1 < 0$. This was in the case of inequality (A.7). In the case of inequality (A.14), assuming again that $m \geq 20$, we get that

$$\left|(n+m)\log \alpha - k \log \gamma - \log(5/2\sqrt{2})\right| < \frac{12}{5\alpha^{2m}}.$$

Let Γ_3 be the expression under the absolute value of the left–hand side above. If $\Gamma_3 > 0$, we get

$$0 < (n+m)\left(\frac{\log \alpha}{\log \gamma}\right) - k + \frac{\log(2\sqrt{2}/5)}{\log \gamma} < \frac{12}{(5\log \gamma)\alpha^{2m}} < \frac{3}{\alpha^{2m}}.$$

We keep the same values for $M,\ \tau,\ q$ and only change μ to

$$\mu' := \frac{\log(2\sqrt{2}/5)}{\log \gamma}, \quad A := 3, \quad B := \alpha^2.$$

We get $\varepsilon > 0.2$, and by Lemma 1.2.2, $m \leq 90$. A similar conclusion is reached if $\Gamma_3 < 0$. Thus, $m \leq 90$ in all cases. Now we move on to (A.10). Assume $n > 100$. We then get

$$\left|k \log \alpha - n \log \gamma + \log(2\sqrt{2}/\sqrt{5}P_m)\right| < \frac{72\sqrt{2}}{\sqrt{5}\delta^{2n}}.$$

Let Γ_2 be the expression under the absolute value in the left–hand side above. If $\Gamma_2 > 0$, we then get

$$0 < k\left(\frac{\log\alpha}{\log\gamma}\right) - n + \frac{\log(2\sqrt{2}/(\sqrt{5}P_m))}{\log\gamma} < \frac{72\sqrt{2}}{(\sqrt{5}\log\gamma)\gamma^{2n}} < \frac{52}{\gamma^{2n}}.$$

We keep the same values for $M,\ \tau,\ q$ and only change μ to

$$\mu_m := \frac{\log(2\sqrt{2}/(\sqrt{5}P_m))}{\log\gamma}, \quad A := 52, \quad B := \gamma^2 \quad \text{for all} \quad m = 1,\ldots,90.$$

We get $\varepsilon > 0.019$, so $n \le 53$. A similar conclusion is reached if $\Gamma_2 < 0$. Finally, if instead of (A.10), we have (A.18), then a similar argument leads to

$$\left|n\log\alpha - k\log\gamma + \log(2\sqrt{2}F_m/\sqrt{5})\right| < \frac{4}{\alpha^{2n}}.$$

Putting Γ_4 for the amount under the absolute value in the left–hand side above, we get in case $\Gamma_4 > 0$ that

$$0 < n\left(\frac{\log\alpha}{\log\gamma}\right) - k + \frac{\log(2\sqrt{2}F_m/\sqrt{5})}{\log\gamma} < \frac{4}{\log\gamma\alpha^{2n}} < \frac{5}{\alpha^{2n}}.$$

We keep the same values for $M,\ \tau,\ q$ and only change μ to

$$\mu_m := \frac{\log(2\sqrt{2}F_m/\sqrt{5})}{\log\gamma}, \quad A := 5, \quad B := \alpha^2, \quad \text{for all} \quad m = 1,\ldots,90.$$

We get $\varepsilon > 0.005$, so $n \le 94$. So, in all cases $n \le 94$, so $k < 400$. We generated $\{F_k\}_{1\le k\le 400}$ and $\{P_mP_n\}_{1\le m<n\le 100}$ and intersected them, and also $\{P_k\}_{1\le k\le 400}$ and $\{F_mF_n\}_{1\le m<n\le 100}$ and intersected them and got no other solutions. Hence, Theorem A.1.1 is proved.

A.3. Comments

It is apparent from our proof that the method is more general and shows that every equation of the form

$$U_k = V_mV_n$$

has only finitely effectively computable many positive integer solutions (k,m,n) provided that $\{U_n\}_{n\ge 0}$ and $\{V_n\}_{n\ge 0}$ satisfy a few technical conditions such as:

- (i) they are both non degenerate binary recurrent and have characteristic equations of real roots $\alpha,\ \beta$ and $\delta,\ \gamma$ with $\alpha\beta = \pm 1$ and $\gamma\delta = \pm 1$.
- (ii) $\mathbb{Q}[\alpha]$ and $\mathbb{Q}[\delta]$ are distinct quadratic fields.

In fact, more is true, namely that for fixed k and s, the diophantine equation

$$\prod_{i=1}^{k} F_{n_i} = \prod_{j=1}^{s} P_{m_j}$$

has only finitely many positive integer solutions

$$(n_1, \ldots, n_k, m_1, \ldots, m_k)$$

and all such are effectively computable. Such a statement is not very difficult to prove. A deeper conjecture made in [48] to the effect that the intersection of the multiplicative group generated by $\{F_n\}_{n\geq 1}$ with the multiplicative group generated by Pell numbers $\{P_n\}_{n\geq 1}$ is finitely generated cannot unfortunately be attacked by these methods.

Appendix B.

On a problem of Pillai with Fibonacci numbers and powers of 2

This appendix chapter presents a slightly modified version of the paper [8] with the title *On a problem of Pillai with Fibonacci numbers and powers of* 2. This is a joint work with *Florian Luca* and *Mihaja Rakotomalala*. The article was published in *Proceedings – Mathematical Sciences* in June, 2017.

Abstract: In this paper, we find all all integers c having at least two representations as a difference between a Fibonacci number and a power of 2.

Keywords: Fibonacci numbers; Linear forms in logarithms; Reduction method.

2010 Mathematics Subject Classification: 11B39, 11J86.

B.1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all} \quad n \geq 0.$$

Its first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, \ldots.$$

Here, we study a Pillai-related problem and find all positive integers c admitting two representations of the form $F_n - 2^m$ for some positive integers n and m. We assume that representations with $n \in \{1, 2\}$ (for which $F_1 = F_2$) count as one representation just to avoid trivial "parametric families" such as $1-2^m = F_1 - 2^m = F_2 - 2^m$, and so we always assume that $n \geq 2$. Notice the solutions

$$\begin{aligned}
1 &= 5-4 = 3-2 (= F_5 - 2^2 = F_4 - 2^1),\\
-3 &= 5-8 = 1-4 = 13-16 (= F_5 - 2^3 = F_2 - 2^2 = F_7 - 2^4),\\
5 &= 21-16 = 13-8 (= F_8 - 2^4 = F_7 - 2^3),\\
0 &= 8-8 = 2-2 (= F_6 - 2^3 = F_3 - 2^1),\\
-11 &= 21-32 = 5-16 (= F_8 - 2^5 = F_5 - 2^4),\\
-30 &= 34-64 = 2-32 (= F_9 - 2^6 = F_3 - 2^5)\\
85 &= 4181-4096 = 89-4 (= F_{19} - 2^{12} = F_{11} - 2^2).
\end{aligned} \tag{B.1}$$

We prove the following theorem.

Theorem B.1.1. *The only integers c having at least two representations of the form $F_n - 2^m$ are $c \in \{0, 1, -3, 5, -11, -30, 85\}$. Furthermore, for each c in the above set, all its representations of the form $F_n - 2^m$ with integers $n \geq 2$ and $m \geq 1$ appear in the list* (B.1).

B.2. Proof of Theorem B.1.1

Assume that $(n, m) \neq (n_1, m_1)$ are such that

$$F_n - 2^m = F_{n_1} - 2^{m_1}.$$

If $m = m_1$, then $F_n = F_{n_1}$ and since $\min\{n, n_1\} \geq 2$, we get that $n = n_1 = 2$, so $(n, m) = (n_1, m_1)$, which is not the case. Thus, $m \neq m_1$, and we may assume that $m > m_1$. Since

$$F_n - F_{n_1} = 2^m - 2^{m_1}, \tag{B.2}$$

and the right–hand side is positive, we get that the left–hand side is also positive and so $n > n_1$. Thus, $n \geq 3$ and $n_1 \geq 2$. We use the Binet formula

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \qquad \text{for all} \qquad k \geq 0,$$

where $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ are the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence. It is well-known that

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1} \qquad \text{for all} \qquad k \geq 1.$$

In (B.2) we have

$$\begin{aligned}
\alpha^{n-4} &\leq F_{n-2} \leq F_n - F_{n_1} = 2^m - 2^{m_1} < 2^m,\\
\alpha^{n-1} &\geq F_n > F_n - F_{n_1} = 2^m - 2^{m_1} \geq 2^{m-1},
\end{aligned} \tag{B.3}$$

therefore

$$n-4 < c_1 m \quad \text{and} \quad n-1 > c_1(m-1), \quad \text{where} \quad c_1 = \log 2/\log\alpha = 1.4402\ldots. \tag{B.4}$$

If $n < 400$, then $m < 300$. We ran a computer program for $2 \le n_1 < n \le 400$ and $1 \le m_1 < m < 300$ and found only the solutions from list (B.1). From now, on, $n \ge 400$. By the above inequality (B.4), we get that $n > m$. Thus, we get

$$\begin{aligned}\left|\frac{\alpha^n}{\sqrt{5}} - 2^m\right| &= \left|\frac{\beta^n}{\sqrt{5}} + \frac{\alpha^{n_1}-\beta^{n_1}}{\sqrt{5}} - 2^{m_1}\right| \le \frac{\alpha^{n_1}+2}{\sqrt{5}} + 2^{m_1}\\ &\le \frac{2\alpha^{n_1}}{\sqrt{5}} + 2^{m_1} < 2\max\{\alpha^{n_1}, 2^{m_1}\}.\end{aligned}$$

Dividing by 2^m we get

$$\left|\sqrt{5}^{-1}\alpha^n 2^{-m} - 1\right| < 2\max\left\{\frac{\alpha^{n_1}}{2^m}, 2^{m_1-m}\right\} < \max\{\alpha^{n_1-n+6}, 2^{m_1-m+1}\}, \tag{B.5}$$

where for the last right–most inequality above we used (B.3) and the fact that $2 < \alpha^2$. For the left–hand side above, we use Theorem 1.1.8 with the data

$$t := 3,\ \eta_1 := \sqrt{5},\ \eta_2 := \alpha,\ \eta_3 := 2,\ b_1 := -1,\ b_2 := n,\ b_3 := -m.$$

We take $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ for which $D := 2$. Then we can take $A_1 := 2h(\eta_1) = \log 5$, $A_2 := 2h(\eta_2) = \log\alpha$, $A_3 := 2h(\eta_3) = 2\log 2$. We take $B := n$. We have

$$\Lambda := \sqrt{5}^{-1}\alpha^n 2^{-m} - 1.$$

Clearly, $\Lambda \neq 0$, for if $\Lambda = 0$, then $\alpha^{2n} \in \mathbb{Q}$, which is false. The left–hand side of (B.6) is bounded, by Theorem 1.1.8, as

$$\log|\Lambda| > -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+\log n)(\log 5)(2\log\alpha)(2\log 2).$$

Comparing with (B.5), we get

$$\min\{(n-n_1-6)\log\alpha, (m-m_1-1)\log 2\} < 1.1 \times 10^{12}(1+\log n),$$

which gives

$$\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} < 1.2 \times 10^{12}(1+\log n).$$

Now the argument splits into two cases.

Case B.2.1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 2\} = (n-n_1)\log\alpha.$

In this case, we rewrite (B.2) as

$$\left|\frac{(\alpha^{n-n_1}-1)}{\sqrt{5}}\alpha^{n_1}-2^m\right| = \left|\frac{\beta^n-\beta^{n_1}}{\sqrt{5}}-2^{m_1}\right| < 2^{m_1}+1 \le 2^{m_1+1},$$

so

$$\left|\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)\alpha^{n_1}2^{-m}-1\right| < 2^{m_1-m-1}. \tag{B.6}$$

Case B.2.2. $\min\{(n-n_1)\log\alpha,(m-m_1)\log 2\}=(m-m_1)\log 2.$

In this case, we rewrite (B.2) as

$$\left|\frac{\alpha^n}{\sqrt{5}}-2^{m_1}(2^{m-m_1}-1)\right| = \left|\frac{\beta^n+\alpha^{n_1}-\beta^{n_1}}{\sqrt{5}}\right| < \frac{\alpha^{n_1}+2}{\sqrt{5}} < \alpha^{n_1},$$

so

$$\left|(\sqrt{5}(2^{m-m_1}-1))^{-1}\alpha^n 2^{-m_1}-1\right| < \frac{\alpha^{n_1}}{2^m-2^{m_1}} \le \frac{2\alpha^{n_1}}{2^m} \le 2\alpha^{n_1-n+4} < \alpha^{n_1-n+6}. \tag{B.7}$$

Inequalities (B.6) and (B.7) suggest studying lower bounds for the absolute values of

$$\Lambda_1 := \left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)\alpha^{n_1}2^{-m}-1 \quad \text{and} \quad \Lambda_2 := (\sqrt{5}(2^{m-m_1}-1))^{-1}\alpha^n 2^{-m_1}-1.$$

We apply again Theorem 1.1.8. We take in both cases $t:=3,\ \eta_2:=\alpha,\ \eta_3:=2$. For Λ_1, we have $b_2:=n_1,\ b_3:=-m$, while for Λ_2 we have $b_2:=n,\ b_3:=-m_1$. In both cases we take $B:=n$. We take

$$\eta_1 := \frac{\alpha^{n-n_1}-1}{\sqrt{5}}, \quad \text{or} \quad \eta_1 := \sqrt{5}(2^{m-m_1}-1),$$

according to whether we work with Λ_1 or Λ_2, respectively. For Λ_1 we have $b_1:=1$ and for Λ_2 we have $b_1:=-1$. In both cases $\mathbb{K}:=\mathbb{Q}(\sqrt{5})$ for which $D:=2$. The minimal polynomial of η_1 divides

$$5X^2-5F_{n-n_1}X-((-1)^{n-n_1}+1-L_{n-n_1}) \quad \text{or} \quad X^2-5(2^{m-m_1}-1)^2,$$

respectively, where $\{L_k\}_{k\ge 0}$ is the Lucas companion sequence of the Fibonacci sequence given by $L_0=2,\ L_1=1,\ L_{k+2}=L_{k+1}+L_k$ for $k\ge 0$ for which its Binet formula of the general term is

$$L_k=\alpha^k+\beta^k \qquad \text{for all} \qquad k\ge 0.$$

Thus,

$$h(\eta_1) \le \frac{1}{2}\left(\log 5+\log\left(\frac{\alpha^{n-n_1}+1}{\sqrt{5}}\right)\right) \quad \text{or} \quad \log(\sqrt{5}(2^{m-m_1}-1)), \tag{B.8}$$

respectively. In the first case,

$$h(\eta_1) < \frac{1}{2}\log(2\sqrt{5}\alpha^{n-n_1}) < \frac{1}{2}(n-n_1+4)\log\alpha < 7\times 10^{11}(1+\log n), \tag{B.9}$$

and in the second case

$$h(\eta_1) < \log(8\times 2^{m-m_1}) = (m-m_1+3)\log 2 < 1.3\times 10^{12}(1+\log n).$$

So, in both cases, we can take $A_1 := 2.6\times 10^{12}(1+\log n)$. We have to justify that $\Lambda_i \neq 0$ for $i = 1,2$. But $\Lambda_1 = 0$ means

$$(\alpha^{n-n_1}-1)\alpha^{n_1} = \sqrt{5}\times 2^m.$$

Conjugating this relation in $\mathbb{Q}$, we get that

$$(\alpha^{n-n_1}-1)\alpha^{n_1} = -(\beta^{n-n_1}-1)\beta^{n_1}. \tag{B.10}$$

The absolute value of the left-hand side is at least $\alpha^n - \alpha^{n_1} \geq \alpha^{n-2} \geq \alpha^{398}$, while the absolute value of the right–hand side is at most $(|\beta|^{n-n_1}+1)|\beta|^{n_1} < 2$, which is a contradiction. As for Λ_2, note that $\Lambda_2 = 0$ implies $\alpha^{2n} \in \mathbb{Q}$, which is not possible. We then get that

$$\log|\Lambda_i| > -1.4\times 30^6\times 3^{4.5}\times 2^2(1+\log 2)(1+\log n)(2.6\times 10^{12}(1+\log n))2(\log 2)\log\alpha,$$

for $i = 1,2$. Thus,

$$\log|\Lambda_i| > -1.7\times 10^{24}(1+\log n)^2 \quad \text{for} \quad i = 1,2.$$

Comparing these with (B.6) and (B.7), we get that

$$(m-m_1-1)\log 2 < 1.7\times 10^{24}(1+\log n)^2, \quad (n-n_1-6)\log\alpha < 1.7\times 10^{24}(1+\log n)^2,$$

according to whether we are in Case B.2.1 or in Case B.2.2. Thus, in both Case B.2.1 and Case B.2.2, we have

$$\begin{aligned} \min\{(n-n_1)\log\alpha,(m-m_1)\log 2\} &< 1.2\times 10^{12}(1+\log n)\\ \max\{(n-n_1)\log\alpha,(m-m_1)\log 2\} &< 1.8\times 10^{24}(1+\log n)^2. \end{aligned} \tag{B.11}$$

We now finally rewrite equation (B.2) as

$$\left|\frac{(\alpha^{n-n_1}-1)}{\sqrt{5}}\alpha^{n_1} - 2^{m_1}(2^{m-m_1}-1)\right| = \left|\frac{\beta^n-\beta^{n_1}}{\sqrt{5}}\right| < |\beta|^{n_1} = \frac{1}{\alpha^{n_1}}.$$

We divide both sides above by $2^m - 2^{m_1}$ getting

$$\begin{aligned} \left|\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}(2^{m-m_1}-1)}\right)\alpha^{n_1}2^{-m_1}-1\right| &< \frac{1}{\alpha^{n_1}(2^m-2^{m_1})} \leq \frac{2}{\alpha^{n_1}2^m}\\ &\leq 2\alpha^{4-n-n_1} \leq \alpha^{4-n}, \end{aligned} \tag{B.12}$$

because $\alpha^{n_1} \geq \alpha^2 > 2$. To find a lower-bound on the left–hand side above, we use again Theorem 1.1.8 with the data

$$t := 3,\ \eta_1 := \frac{\alpha^{n-n_1}-1}{\sqrt{5}(2^{m-m_1}-1)},\ \eta_2 := \alpha,\ \eta_3 := 2,\ b_1 := 1,\ b_2 := n_1,\ b_3 := -m_1,\ B := n.$$

We have $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ with $D := 2$. Using the fact that $h(x/y) \leq h(x) + h(y)$ for any two nonzero algebraic numbers x and y, we have

$$\begin{aligned} h(\eta_1) &\leq h\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right) + h(2^{m-m_1}+1) < \log(2\sqrt{5}\alpha^{n-n_1}) + \log(2^{m-m_1}+1) \\ &\leq (n-n_1)\log\alpha + (m-m_1)\log 2 + \log(2\sqrt{5}) + 1 < 2\times 10^{24}(1+\log n)^2, \end{aligned}$$

where in the above chain of inequalities we used the arguments from (B.8) and (B.9) as well as the bound (B.11). So, we can take $A_1 := 4\times 10^{24}(1+\log n)^2$ and certainly $A_2 := \log\alpha$ and $A_3 := 2\log 2$. We need to show that if we put

$$\Lambda_3 := \frac{(\alpha^{n-n_1}-1)}{\sqrt{5}}\alpha^{n_1} - 2^{m_1}(2^{m-m_1}-1),$$

then $\Lambda_3 \neq 0$. But $\Lambda_3 = 0$ leads to

$$(\alpha^{n-n_1}-1)\alpha^{n_1} = \sqrt{5}(2^m - 2^{m_1}),$$

which upon conjugation in $\mathbb{K}$ leads to (B.10), which we have seen that it is impossible. Thus, $\Lambda_3 \neq 0$. Theorem 1.1.8 gives

$$\log|\Lambda_3| > -1.4\times 30^6 \times 3^{4.5}\times 2^2(1+\log 2)(1+\log n)(4\times 10^{24}(1+\log n)^2)2(\log 2)\log\alpha,$$

which together with (B.12) gives

$$(n-3)\log\alpha < 3\times 10^{36}(1+\log n)^3,$$

leading to $n < 7\times 10^{42}$.

Now we need to reduce the bound. To do so, we make use several times of the result due to Dujella and Pethő [30].

We return first to (B.5) and put

$$\Gamma := n\log\alpha - m\log 2 - \log\sqrt{5}.$$

Assume that $\min\{n-n_1, m-m_1\} \geq 20$ and we go to (B.5). This is not a very restrictive assumption since, as we shall see immediately, if this condition fails then we do the following:

(i) if $n-n_1 < 20$ but $m-m_1 \geq 20$, we go to (B.6);
(ii) if $n-n_1 \geq 20$ but $m-m_1 < 20$, we go to (B.7);

(iii) if both $n-n_1<20$ and $m-m_1<20$, we go to (B.12).

In (B.5), since $|e^{\Gamma}-1|=|\Lambda|<1/4$, we get that $|\Gamma|<1/2$. Since $|x|<2|e^x-1|$ holds for all $x\in(-1/2,1/2)$, we get that

$$|\Gamma|<2\max\{\alpha^{n_1-n+6},2^{m-m_1+1}\}\leq\max\{\alpha^{n_1-n+8},2^{m_1-m+2}\}.$$

Assume $\Gamma>0$. Then

$$0<n\left(\frac{\log\alpha}{\log 2}\right)-m+\frac{\log(1/\sqrt{5})}{\log 2}<\max\left\{\frac{\alpha^8}{(\log 2)\alpha^{n-n_1}},\frac{4}{(\log 2)2^{m-m_1}}\right\}.$$

We apply Lemma 1.2.2 with

$$\tau:=\frac{\log\alpha}{\log 2},\quad \mu:=\frac{\log(1/\sqrt{5})}{\log 2},\quad (A,B):=(68,\alpha)\quad\text{or}\quad(6,2).$$

We let $\tau=[a_0,a_1,a_2,\ldots]=[0,1,2,3,1,2,3,2,4,\ldots]$ be the continued fraction of τ. We take $M=7\times10^{42}$. We take

$$\frac{p}{q}=\frac{p_{149}}{q_{149}}=\frac{755830092745232999099612135303693391839418748444717618738467007831418529 20}{1088712850528619465432515952603697382184620103833234826296110844071070900 03}$$

where $q>10^{74}>6M$. We have $\varepsilon>0.09$, therefore either

$$n-n_1\leq\frac{\log(68q/0.09)}{\log\alpha}<369\quad\text{or}\quad m-m_1\leq\frac{\log(6q/0.09)}{\log 2}<253.$$

Thus we have that either $n-n_1\leq368$ or $m-m_1\leq252$. A similar conclusion is obtained when $\Gamma<0$.

In case $n-n_1\leq368$, we go to (B.6). There, we assume that $m-m_1\geq20$. We put

$$\Gamma_1:=n_1\log\alpha-m\log 2+\log\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right).$$

Then (B.6) implies that

$$|\Gamma_1|<\frac{4}{2^{m-m_1}}.$$

Assume $\Gamma_1>0$. Then

$$0<n_1\left(\frac{\log\alpha}{\log 2}\right)-m+\frac{\log((\alpha^{n-n_1}-1)/\sqrt{5})}{\log 2}<\frac{4}{(\log 2)2^{m-m_1}}<\frac{6}{2^{m-m_1}}.$$

We keep the same τ, M, q, $(A,B)=(6,2)$ and put

$$\mu_k:=\frac{\log((\alpha^k-1)/\sqrt{5})}{\log 2},\qquad k=1,2,\ldots,368.$$

We have problems at $k \in \{4,12\}$. We discard these values and we will treat them later. For the remaining values of k, we get $\varepsilon > 0.001$. Hence, by Lemma 1.2.2, we get

$$m - m_1 < \frac{\log(6q/0.001)}{\log 2} < 259.$$

Thus, $n - n_1 \leq 368$ implies $m - m_1 \leq 258$, unless $n - n_1 \in \{4,12\}$. A similar conclusion is reached if $\Gamma_1 < 0$ with the same two exceptions for $n - n_1 \in \{4,12\}$. The reason we have a problem at $k \in \{4,8\}$ is because

$$\frac{\alpha^4 - 1}{\sqrt{5}} = \alpha^2 \quad \text{and} \quad \frac{\alpha^{12} - 1}{\sqrt{5}} = 2^3\alpha^6.$$

So,

$$\Gamma_1 := (n_1+2)\tau - m, \quad \text{or} \quad (n_1+6)\tau - (m-3) \quad \text{when} \quad k = 4,12, \quad \text{respectively.}$$

Thus we get that

$$\left|\tau - \frac{m}{n_1+2}\right| < \frac{4}{2^{m-m_1}(n_1+2)} \quad \text{or} \quad \left|\tau - \frac{m-3}{n_1+6}\right| < \frac{4}{2^{m-m_1}(n_1+6)}.$$

Assume $m - m_1 > 150$. Then $2^{m-m_1} > 8 \times (8 \times 10^{42}) > 8 \times (n_1+6)$, therefore

$$\frac{4}{2^{m-m_1}(n_1+2)} < \frac{1}{2(n_1+2)^2} \quad \text{and} \quad \frac{4}{2^{m-m_1}(n_1+6)} < \frac{1}{2(n_1+6)^2}.$$

By Lemma 1.2.1, it follows that $m/(n_1+2)$ or $(m+3)/(n_1+6)$ are convergents of τ, respectively. So, say one of $m/(n_1+2)$ or $m/(n_1+6)$ is of the form p_k/q_k for some $k = 0,1,2,\ldots,99$. Here we use that $q_{99} > 8 \times 10^{42} > n_1 + 6$. Then

$$\frac{1}{(a_k+2)q_k^2} < \left|\tau - \frac{p_k}{q_k}\right|.$$

Since $\max\{a_k : k = 0,\ldots,99\} = 134$, we get that

$$\frac{1}{136q_k^2} < \frac{4}{2^{m-m_1}q_k} \quad \text{and} \quad q_k \quad \text{divides one of} \quad \{n_1+2,\ n_1+6\}.$$

Thus

$$2^{m-m_1} \leq 4 \times 136(n_1+6) < 4 \times 136 \times 8 \times 10^{42}$$

giving $m - m_1 \leq 151$. Hence, even in the case $n - n_1 \in \{4,12\}$, we still keep the conclusion that $m - m_1 \leq 258$.

Now let us assume that $m - m_1 \leq 252$. Then we go to (B.7). We write

$$\Gamma_2 := n\log\alpha - m_1\log 2 + \log(1/(\sqrt{5}(2^{m-m_1}-1))).$$

We assume that $n - n_1 \geq 20$. Then

$$|\Gamma_2| < \frac{2\alpha^6}{\alpha^{n-n_1}}.$$

Assuming $\Gamma_2 > 0$, we get

$$0 < n\left(\frac{\log\alpha}{\log 2}\right) - m_1 + \frac{\log(1/(\sqrt{5}(2^{m-m_1}-1)))}{\log 2} < \frac{2\alpha^6}{(\log 2)\alpha^{n-n_1}} < \frac{52}{\alpha^{n-n_1}}.$$

We apply again Lemma 1.2.2 with the same τ, q, M, $(A,B) = (52,\alpha)$ and

$$\mu_k := \frac{\log(1/(\sqrt{5}(2^k-1))}{\log 2} \quad \text{for} \quad k = 1,2,\ldots,252.$$

We get $\varepsilon > 0.0005$, therefore

$$n - n_1 < \frac{\log(52q/0.0005)}{\log\alpha} < 379.$$

A similar conclusion is reached when $\Gamma_2 < 0$. To conclude, we first got that either $n - n_1 \leq 368$ or $m - m_1 \leq 252$. If $n - n_1 \leq 368$, then $m - m_1 \leq 258$, and if $m - m_1 \leq 252$, then $n - n_1 \leq 378$. In conclusion, we always have $n - n_1 < 380$ and $m - m_1 < 260$.

Finally we go to (B.12). We put

$$\Gamma_3 := n_1 \log\alpha - m_1 \log 2 + \log\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}(2^{m-m_1}-1)}\right).$$

Since $n \geq 400$, (B.12) tells us that

$$|\Gamma| < \frac{2}{\alpha^{n-3}} = \frac{2\alpha^3}{\alpha^n}.$$

Assume that $\Gamma_3 > 0$. Then

$$0 < n_1\left(\frac{\log\alpha}{\log 2}\right) - m_1 + \frac{\log((\alpha^k-1)/\sqrt{5}(2^\ell-1))}{\log 2} < \frac{2\alpha^3}{(\log 2)\alpha^n} < \frac{13}{\alpha^n}$$

where $(k,\ell) := (n-n_1, m-m_1)$. We apply again Lemma 1.2.2 with the same τ, M, q, $(A,B) = (13,\alpha)$ and

$$\mu_{k,\ell} := \frac{\log((\alpha^k-1)/\sqrt{5}(2^\ell-1))}{\log 2} \quad \text{for} \quad 1 \leq k \leq 379,\ 1 \leq \ell \leq 259.$$

We have a problem at $(k,\ell) = (4,1),\ (12,1)$ (as for the case of (B.6) and additionally for $(k,\ell) = (8,2)$ since

$$\frac{\alpha^8-1}{\sqrt{5}(2^2-1)} = \alpha^4.$$

We discard the cases $(k,\ell) = (4,1),\ (12,1),(8,2)$ for the time being. For the remaining ones, we get $\varepsilon > 7\times 10^{-6}$, so we get

$$n \le \frac{\log(13q/(7\times 10^{-6}))}{\log\alpha} < 385.$$

A similar conclusion is reached when $\Gamma_3 < 0$. Hence $n < 400$. Now we look at the cases $(k,\ell) = (4,1),\ (12,1),\ (8,2)$. The cases $(k,\ell) = (4,1),\ (12,1)$ can be treated as we did before when we showed that $n - n_1 \le 368$ implies $m - m_1 \le 258$. The case when $(k,\ell) = (8,2)$ can be dealt with similarly as well. Namely, it gives

$$|(n_1+4)\tau - m_1| < \frac{13}{\alpha^n}.$$

Hence

$$\left|\tau - \frac{m_1}{n_1+4}\right| < \frac{13}{(n_1+4)\alpha^n}. \tag{B.13}$$

Since $n \ge 400$, then $\alpha^n > 2\times 13\times(8\times 10^{42}) > 2\times 13(n_1+4)$, which shows that the right–hand side of inequality (B.13) is at most $2/(n_1+4)^2$. By Legendre's criterion, $m/(n_1+4) = p_k/q_k$ for some $k = 0,1,\ldots,99$. We then get by an argument similar to a previous one that

$$\alpha^n \le 13\times 136\times(8\times 10^{42})$$

giving $n \le 220$. So, the conclusion is that $n < 400$ holds also in the case of the pair $(k,\ell) = (8,2)$. However, this contradicts our working assumption that $n \ge 400$.

Theorem B.1.1 is therefore proved.

Appendix C.

On a problem of Pillai with Fibonacci numbers and powers of 3

This appendix chapter contains a presentation of a slightly modified version of the paper [9] with the title *On a problem of Pillai with Fibonacci numbers and powers of* 3. The article has appeared online in *Boletín de la Sociedad Matemática Mexicana* in September, 2019.

Abstract: In this paper, we find all integers c having at least two representations as a difference between a Fibonacci number and a power of 3.

Keywords: Fibonacci number; linear form in logarithms; Baker's method.

2010 Mathematics Subject Classification: 11B39, 11J86.

C.1. Introduction

In this paper, we are interested in studying the Diophantine equation

$$F_n - 3^m = c \tag{C.1}$$

for a fixed integer c and variable integers n and m. In particular, we are interested in finding those integers c admitting at least two representations as a difference between a Fibonacci number and a power of 3. This equation is a variant of the Pillai equation (1.15).

C.2. Main Result

The main aim of this paper is to prove the following result. Since $F_1 = F_2 = 1$, we discard the situation when $n = 1$ and just count the solutions for $n = 2$.

Theorem C.2.1. *The only integers c having at least two representations of the form* $F_n - 3^m$ *are* $c \in \{-26, -6, -1, 0, 2, 4, 7, 12\}$. *Furthermore, all the representations of the above integers as* $F_n - 3^m$ *with integers* $n \geq 2$ *and* $m \geq 0$ *are given by*

$$\begin{aligned} -26 &= F_{10} - 3^4 = F_2 - 3^3; \\ -6 &= F_8 - 3^3 = F_4 - 3^2; \\ -1 &= F_6 - 3^2 = F_3 - 3^1 \\ 0 &= F_4 - 3^1 = F_2 - 3^0; \\ 2 &= F_5 - 3^1 = F_4 - 3^0; \\ 4 &= F_7 - 3^2 = F_5 - 3^0; \\ 7 &= F_9 - 3^3 = F_6 - 3^0; \\ 12 &= F_8 - 3^2 = F_7 - 3^0. \end{aligned} \tag{C.2}$$

C.3. Proof of Theorem C.2.1

Assume that there exist nonnegative integers n, m, n_1, m_1 with $\min\{n, n_1\} \geq 2$ and $\min\{m, m_1\} \geq 0$ such that $(n, m) \neq (n_1, m_1)$, and

$$F_n - 3^m = F_{n_1} - 3^{m_1}.$$

Without loss of generality, we can assume that $m \geq m_1$. If $m = m_1$, then $F_n = F_{n_1}$, so $(n, m) = (n_1, m_1)$, which gives a contradiction to our assumption. Thus $m > m_1$. Since

$$F_n - F_{n_1} = 3^m - 3^{m_1}, \tag{C.3}$$

and the right-hand side is positive, we get that the left-hand side is also positive and so $n > n_1$.

Using the Binet formula

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}} \text{ for all } k \geq 0, \tag{C.4}$$

where $(\alpha, \beta) := \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ are the roots of the equation $x^2 - x - 1 = 0$, which is the characteristic equation of the Fibonacci sequence. One can easily prove by induction that

$$\alpha^{k-2} \leq F_n \leq \alpha^{k-1} \text{ for all } k \geq 1. \tag{C.5}$$

Using the equation (C.3), we get

$$\begin{aligned} \alpha^{n-4} &\leq F_{n-2} \leq F_n - F_{n_1} = 3^m - 3^{m_1} < 3^m, \\ \alpha^{n-1} &\geq F_n \geq F_n - F_{n_1} = 3^m - 3^{m_1} \geq 3^{m-1}, \end{aligned} \tag{C.6}$$

from which we get that

$$1+\left(\frac{\log 3}{\log\alpha}\right)(m-1)<n<\left(\frac{\log 3}{\log\alpha}\right)m+4. \tag{C.7}$$

If $n\le 300$, then $m\le 127$. We ran a *Mathematica* program for $2\le n_1<n\le 300$ and $0\le m_1<m\le 127$ and found only the solutions from the list (C.2). From now, we assume that $n>300$ and from (C.7) we have that $m>127$. Therefore, to solve the Diophatine equation (C.1), it suffices to find an upper bound for n.

C.3.1. Bounding n

By substituting the Binet formula (C.4) in the Diophantine equation (C.1), we get

$$\begin{aligned}\left|\frac{\alpha^n}{\sqrt{5}}-3^m\right| &= \left|\frac{\beta^n}{\sqrt{5}}+\frac{\alpha^{n_1}-\beta^{n_1}}{\sqrt{5}}-3^{m_1}\right| \le \frac{\alpha^{n_1}+2}{\sqrt{5}}+3^{m_1}\\ &\le \frac{2\alpha^{n_1}}{\sqrt{5}}+3^{m_1} < 3\max\{\alpha^{n_1},3^{m_1}\}.\end{aligned}$$

Multiplying through by 3^{-m}, using the relation (C.6) and using the fact that $\alpha<3$, we get

$$\left|(\sqrt{5})^{-1}\alpha^n 3^{-m}-1\right|<3\max\left\{\frac{\alpha^{n_1}}{3^m},3^{m_1-m}\right\}<\max\{\alpha^{n_1-n+7},3^{m_1-m+1}\}. \tag{C.8}$$

For the left-hand side, we apply the result of Matveev, Theorem 1.1.8 with the following data

$$t=3,\ \ \gamma_1=\sqrt{5},\ \ \gamma_2=\alpha,\ \ \gamma_3=3,\ \ b_1=-1,\ \ b_2=n,\ \ b_3=-m.$$

Through out we work with the field $\mathbb{K}:=\mathbb{Q}(\sqrt{5})$ with $D=2$. Since $\max\{1,n,m\}\le 2n$, we take $B:=2n$. Furthermore, we take $A_1:=2h(\gamma_1)=\log 5, A_2:=2h(\gamma_2)=\log\alpha, A_3:=2h(\gamma_1)=2\log 3$. We put

$$\Lambda:=(\sqrt{5})^{-1}\alpha^n 3^{-m}-1.$$

First we check that $\Lambda\ne 0$, if it were, then $\alpha^{2n}\in\mathbb{Q}$, a contradiction. Thus, $\Lambda\ne 0$. Then by Matveev's theorem, the left-hand side of (C.8) is bounded as

$$\log|\Lambda|>-1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)(1+\log 2n)(\log 5)(\log\alpha)(2\log 3).$$

By comparing with (C.8), we get

$$\min\{(n-n_1-7)\log\alpha,(m-m_1-1)\log 3\}<1.66\times 10^{12}(1+\log 2n),$$

which gives

$$\min\{(n-n_1)\log\alpha,(m-m_1)\log 3\}<1.67\times 10^{12}(1+\log 2n).$$

Now we split the argument into two cases.

Case C.3.1. $\min\{(n-n_1)\log\alpha,(m-m_1)\log 3\}=(n-n_1)\log\alpha$.

In this case, we rewrite (C.3) as

$$\left|\left(\frac{\alpha^n-\alpha^{n_1}}{\sqrt{5}}\right)-3^m\right|=\left|\left(\frac{\beta^n-\beta^{n_1}}{\sqrt{5}}\right)-3^{m_1}\right| \;<\; 1+3^{m_1} \;\leq\; 3^{m_1+1},$$

which implies

$$\left|\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)\alpha^{n_1}3^{-m}-1\right| \;<\; 3^{m_1-m+1}. \tag{C.9}$$

We put

$$\Lambda_1:=\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)\alpha^{n_1}3^{-m}-1.$$

To see that $\Lambda_1\neq 0$, for if $\Lambda_1=0$, then

$$\alpha^n-\alpha^{n_1}=\sqrt{5}\cdot 3^m.$$

By conjugating the above relation in $\mathbb{K}$, we get that

$$\beta^n-\beta^{n_1}=-\sqrt{5}\cdot 3^m.$$

The absolute value of the left-hand side is at most $|\beta^n-\beta^{n_1}|\leq|\beta|^n+|\beta|^{n_1}<2$, while the absolute value of the right-hand side is at least $|-\sqrt{5}\cdot 3^m|\geq\sqrt{5}>2$ for all $m>127$, which is a contradiction.

We apply Theorem 1.1.8 on the left-hand side of (C.9) with the data

$$t=3,\;\; \gamma_1=\frac{\alpha^{n-n_1}-1}{\sqrt{5}},\;\; \gamma_2=\alpha,\;\; \gamma_3=3,\;\; b_1=1,\;\; b_2=n_1,\;\; b_3=-m.$$

The minimal polynomial of γ_1 divides

$$5X^2-5F_{n-n_1}X-((-1)^{n-n_1}+1-L_{n-n_1}),$$

where $\{L_k\}_{k\geq 0}$ is the Lucas companion sequence of the Fibonacci sequence given by $L_0=2,\;L_1=1,\;L_{k+2}=2L_{k+1}+L_k$ for all $k\geq 0$, for which the Binet formula for its general term is given by

$$L_k=\alpha^k+\beta^k \quad \text{for all } k\geq 0.$$

Thus, we obtain

$$\begin{aligned} h(\gamma_1) \;&\leq\; \frac{1}{2}\left(\log 5+\log\left(\frac{\alpha^{n-n_1}+1}{\sqrt{5}}\right)\right) \;<\; \frac{1}{2}\log(2\sqrt{5}\alpha^{n-n_1})\\ &<\; \frac{1}{2}(n-n_1+2)\log\alpha \;<\; 8.4\times 10^{11}(1+\log 2n). \end{aligned} \tag{C.10}$$

So, we can take $A_1 := 1.67 \times 10^{12}(1+\log 2n)$. Furthermore, as before, we take $A_2 := \log\alpha$ and $A_3 := 2\log 3$. Finally, since $\max\{1, n_1, m\} \le 2n$, we can take $B := 2n$. Then, we get

$$\begin{aligned}\log|\Lambda_1| \; > \; & -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log 2n)(16.8 \times 10^{11}(1+\log 2n)) \\ & \times(\log\alpha)(2\log 3).\end{aligned}$$

Then,

$$\log|\Lambda_1| > -1.72 \times 10^{24}(1+\log 2n)^2.$$

By comparing the above relation with (C.9), we get that

$$(m-m_1)\log 3 < 1.80 \times 10^{24}(1+\log 2n)^2. \tag{C.11}$$

Case C.3.2. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} = (m-m_1)\log 3.$

In this case, we rewrite (C.3) as

$$\left|\frac{\alpha^n}{\sqrt{5}} - (3^{m-m_1}-1)\cdot 3^{m_1}\right| = \left|\frac{\beta^n + \alpha^{n_1} - \beta^{n_1}}{\sqrt{5}}\right| \; < \; \frac{\alpha^{n_1}+2}{\sqrt{5}} \; < \; \alpha^{n_1},$$

which implies that

$$\begin{aligned}|(\sqrt{5}(3^{m-m_1}-1))^{-1}\alpha^n 3^{-m_1} - 1| \; & < \; \frac{\alpha^{n_1}}{3^m - 3^{m_1}} \le \frac{3\alpha^{n_1}}{3^m} \\ & < \; 3\alpha^{n_1-n+4} \; < \; \alpha^{n_1-n+7}.\end{aligned} \tag{C.12}$$

We put

$$\Lambda_2 := (\sqrt{5}(3^{m-m_1}-1))^{-1}\alpha^n 3^{-m_1} - 1.$$

Clearly, $\Lambda_2 \neq 0$, for if $\Lambda_2 = 0$, then $\alpha^{2n} \in \mathbb{Q}$, which is a contradiction. We again apply Theorem 1.1.8 with the following data

$$t=3,\;\; \gamma_1 = \sqrt{5}(3^{m-m_1}-1),\;\; \gamma_2 = \alpha,\;\; \gamma_3 = \alpha,\;\; b_1 = -1,\;\; b_2 = n,\;\; b_3 = -m_1.$$

The minimal polynomial of γ_1 is $X^2 - 5(3^{m-m_1}-1)^2$. Thus,

$$h(\gamma_1) = \log(\sqrt{5}(3^{m-m_1}-1)) < (m-m_1+1)\log 3 < 1.25 \times 10^{12}(1+\log 2n).$$

So, we can take $A_1 := 2.5 \times 10^{12}(1+\log 2n)$. Further, as in the previous applications, we take $A_2 := \log\alpha$ and $A_3 := 2\log 3$. Finally, since $\max\{1, n, m_1\} \le 2n$, we can take $B := 2n$. Then, we get

$$\begin{aligned}\log|\Lambda_2| \; > \; & -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log 2n)(2.5 \times 10^{12} \\ & \times(1+\log 2n))(\log\alpha)(2\log 3).\end{aligned}$$

Thus,

$$\log|A_2| > -2.56\times 10^{24}(1+\log 2n)^2.$$

Now, by comparing with (C.12), we get that

$$(n-n_1)\log\alpha < 2.58\times 10^{24}(1+\log 2n)^2. \tag{C.13}$$

Therefore, in both Case C.3.1 and Case C.3.2, we have

$$\begin{aligned} \min\{(n-n_1)\log\alpha,(m-m_1)\log 3\} &< 1.24\times 10^{12}(1+\log 2n),\\ \max\{(n-n_1)\log\alpha,(m-m_1)\log 3\} &< 2.58\times 10^{24}(1+\log 2n)^2. \end{aligned} \tag{C.14}$$

Finally, we rewrite the equation (C.3) as

$$\left|\frac{(\alpha^{n-n_1}-1)}{\sqrt{5}}\alpha^{n_1}-(3^{m-m_1}-1)\cdot 3^{m_1}\right| = \left|\frac{\beta^n-\beta^{n_1}}{\sqrt{5}}\right| < |\beta|^{n_1} < 1.$$

Dividing through by $3^m-3^{m_1}$, we get

$$\begin{aligned} \left|\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}(3^{m-m_1}-1)}\right)\alpha^{n_1}3^{-m_1}-1\right| &< \frac{1}{(3^m-3^{m_1})}\leq\frac{3}{3^m}\\ &\leq 3\alpha^{-(n-4)} \leq \alpha^{7-n}, \end{aligned} \tag{C.15}$$

since $\alpha<3$ and $\alpha\leq\alpha^{n_1}$. We again apply Theorem 1.1.8 on the left-hand side of (C.15) with the data

$$t=3,\ \gamma_1=\frac{\alpha^{n-n_1}-1}{\sqrt{5}(3^{m-m_1}-1)},\ \gamma_2=\alpha,\ \gamma_3=3,\ b_1=1,\ b_2=n_1,\ b_3=-m_1.$$

By using the algebraic properties of the logarithmic height function, we get

$$\begin{aligned} h(\gamma_1) &= h\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}(3^{m-m_1}-1)}\right) \leq h\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}}\right)+h(3^{m-m_1}-1)\\ &< \frac{1}{2}(n-n_1+4)\log\alpha+(m-m_1)\log 3 < 2.80\times 10^{24}(1+\log 2n)^2, \end{aligned}$$

where in the above inequalities, we used the argument from (C.10) as well as the bounds (C.14). Thus, we can take $A_1 := 5.60\times 10^{24}(1+\log 2n)$, and again as before $A_2 := \log\alpha$ and $A_3 := 2\log 3$. If we put

$$\Lambda_3 := \left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}(3^{m-m_1}-1)}\right)\alpha^{n_1}3^{-m_1}-1,$$

we need to show that $\Lambda_3\neq 0$. If not, $\Lambda_3=0$ leads to

$$\alpha^n-\alpha^{n_1}=\sqrt{5}(3^m-3^{m_1}).$$

A contradiction is reached upon a conjugation in $\mathbb{K}$ and by taking absolute values on both sides. Thus, $\Lambda_3 \neq 0$. Applying Theorem 1.1.8 gives

$$\begin{aligned}\log|\Lambda_3| &> -1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2(1+\log 2)(1+\log 2n)(5.6\times 10^{24}(1+\log 2n)^2)\\ &\quad \times(\log\alpha)(2\log 3),\end{aligned}$$

a comparison with (C.15) gives

$$(n-4) \quad < \quad 3\times 10^{36}(1+\log 2n)^3,$$

or

$$2n \quad < \quad 6.2\times 10^{36}(1+\log 2n)^3. \tag{C.16}$$

Now by applying Lemma 1.5.1 on (C.16) with the data $m := 3$, $T := 6.2\times 10^{36}$, and $x := 2n$, leads to $n < 2\times 10^{40}$.

C.3.2. Reducing the bound for n

We need to reduce the above bound for n and to do so we make use of Lemma 1.2.2 several times. To begin, we return to (C.8) and put

$$\Gamma := n\log\alpha - m\log 3 - \log(\sqrt{5}).$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\}\geq 20$. We go back to the inequalities for Λ, Λ_1 and Λ_2, Since we assume that $\min\{n-n_1,m-m_1\}\geq 20$ we get $|e^{\Gamma}-1| = |\Lambda| < \frac{1}{4}$. Hence, $|\Lambda| < \frac{1}{2}$ and since the inequality $|y| < 2|e^y-1|$ holds for all $y\in\left(-\frac{1}{2},\frac{1}{2}\right)$, we get

$$|\Gamma| < 2\max\{\alpha^{n_1-n+5}, 3^{m_1-m+1}\} \leq \max\{\alpha^{n_1-n+8}, 3^{m_1-m+2}\}.$$

Assume that $\Gamma > 0$. We then have the inequality

$$\begin{aligned}0 < n\left(\frac{\log\alpha}{\log 3}\right) - m + \frac{\log(1/\sqrt{5})}{\log 3} &< \max\left\{\frac{\alpha^8}{(\log 3)\alpha^{n-n_1}}, \frac{6}{(\log 3)3^{m-m_1}}\right\}.\\ &< \max\{45\alpha^{-(n-n_1)}, 8\cdot 3^{-(m-m_1)}\}.\end{aligned}$$

We apply Lemma 1.2.2 with the data

$$\tau := \frac{\log\alpha}{\log 3},\quad \mu := \frac{\log(1/\sqrt{5})}{\log 3},\quad (A,B) := (45,\alpha)\quad \text{or } (8,3).$$

Let $\tau = [a_0; a_1, a_2, \ldots] = [0; 2, 3, 1, 1, 6, 1, 49, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 10, 3, \ldots]$ be the continued fraction of τ. We choose $M := 2\times 10^{40}$ and consider the 91-th convergent

$$\frac{p}{q} = \frac{p_{91}}{q_{91}} = \frac{4876242003851841671302557442327379215121748593365 81}{1113251817385764505972408650620147577750763395186265}.$$

It satisfies $q = q_{91} > 6M$. Furthermore, it yields $\varepsilon > 0.4892$, and therefore either

$$n - n_1 \le \frac{\log(45q/\varepsilon)}{\log\alpha} < 254, \quad \text{or} \quad m - m_1 \le \frac{\log(8q/\varepsilon)}{\log 3} < 110.$$

In the case $\Gamma < 0$, we consider the inequality

$$\begin{aligned} m\left(\frac{\log 3}{\log\alpha}\right) - n + \frac{\log(\sqrt{5})}{\log\alpha} &< \max\left\{\frac{\alpha^8}{\log\alpha}\alpha^{-(n-n_1)}, \frac{8}{\log\alpha}\cdot 3^{-(m-m_1)}\right\} \\ &< \max\{98\alpha^{-(n-n_1)}, 18\cdot 3^{-(m-m_1)}\}. \end{aligned}$$

We then apply Lemma 1.2.2 with the data

$$\tau = \frac{\log 3}{\log\alpha}, \quad \mu = \frac{\log\sqrt{5}}{\log\alpha}, \quad (A,B) = (98,\alpha), \quad \text{or} \quad (18,3).$$

Let $\tau = [a_0; a_1, a_2, \ldots] = [2;3,1,1,6,1,49,1,2,2,1,1,2,1,2,2,1,10,3,12,\ldots]$ be the continued fraction of τ. Again, we choose $M = 2\times 10^{40}$, and in this case we consider the 101-th convergent

$$\frac{p}{q} = \frac{p_{101}}{q_{101}} = \frac{106360048375891410642967692492903700137161881169662}{56228858848524361385900581302251812795713192394033},$$

which satisfies $q = q_{101} > 6M$. Further, this yields $\varepsilon > 0.125$, and therefore either

$$n - n_1 \le \frac{\log(98q/\varepsilon)}{\log\alpha} < 254 \quad , \text{or} \quad m - m_1 \le \frac{\log(18q/\varepsilon)}{\log 3} < 110.$$

These bounds agree with the bounds obtained in the case $\Gamma > 0$. As a conclusion, we have that either $n - n_1 \le 253$ or $m - m_1 \le 109$ whenever $\Gamma \ne 0$.

Now, we distinguish between the cases $n - n_1 \le 253$ and $m - m_1 \le 109$. First, we assume that $n - n_1 \le 253$. In this case we consider the inequality for Λ_1, (C.9) and also assume that $m - m_1 \le 20$. We put

$$\Gamma_1 := n_1\log\alpha - m\log 3 + \log\left(\frac{\alpha^{n-n_1}}{\sqrt{5}}\right).$$

Then inequality (C.9) implies that

$$|\Gamma_1| < \frac{6}{3^{m-m_1}}.$$

If we further assume that $\Gamma_1 > 0$, we then get

$$0 < n_1\left(\frac{\log\alpha}{\log 3}\right) - m + \frac{\log((\alpha^{n-n_1}-1)/\sqrt{5})}{\log 3} < \frac{6}{(\log 3)3^{m-m_1}} < \frac{6}{3^{m-m_1}}.$$

Again we apply Lemma 1.2.2 with the same τ as in the case $\Gamma > 0$. We use the 91-th convergent $p/q = p_{91}/q_{91}$ of τ as before. But in this case we choose $(A,B) := (8,3)$ and use

$$\mu_\ell := \frac{\log((\alpha^\ell - 1)/\sqrt{5})}{\log 3},$$

instead of μ for each possible value of $\ell := n - n_1 \in [1,2,\ldots,253]$. We have problems at $\ell \in \{4,12\}$. We discard these values for now and we will treat them later. For the remaining values of ℓ, we get $\varepsilon > 0.0005$. Hence by Lemma 1.2.2, we get

$$m - m_1 \; < \; \frac{\log(8q/0.0005)}{\log 3} \; < \; 116.$$

Thus, $n - n_1 \leq 253$ implies that $m - m_1 \leq 115$, unless $n - n_1 \in \{4,12\}$. A similar conclusion is reached when $\Gamma_1 < 0$ with the same two exceptions for $n - n_1 \in \{4,12\}$. The reason we have a problem at $\ell \in \{4,12\}$ is because

$$\frac{\alpha^4 - 1}{\sqrt{5}} = \alpha^2, \quad \text{and} \quad \frac{\alpha^{12} - 1}{\sqrt{5}} = 2^3\alpha^6.$$

So, $\Gamma_1 = (n_1+2)\log\alpha - m\log 3$, or $(n_1+6)\log\alpha - (m-3)\log 3$ when $l = 4,12$, respectively. Thus we get that

$$\left|\tau - \frac{m}{n_1+2}\right| < \frac{6}{3^{m-m_1}(n_1+2)}, \quad \text{or} \quad \left|\tau - \frac{m-3}{n_1+6}\right| < \frac{6}{3^{m-m_1}(n_1+6)},$$

respectively. We assume that $m - m_1 > 150$. Then $3^{m-m_1} > 8 \times (4 \times 10^{40}) > 8 \times (n_1+6)$, therefore

$$\frac{6}{3^{m-m_1}(n_1+2)} < \frac{1}{3(n_1+2)^2}, \quad \text{and} \quad \frac{6}{3^{m-m_1}(n_1+6)} < \frac{1}{3(n_1+6)^2}.$$

By Lemma 1.2.1, it follows that $m/(n_1+2)$ or $(m-3)/(n_1+6)$ are convergents of τ, respectively. So, say one of $m/(n_1+2)$ or $(m-3)/(n_1+6)$ is of the form p_k/q_k for some $k = 0,1,2,\ldots,92$. Here, we use that $q_{92} > 4 \times 10^{40} > n+1+6$. Then

$$\frac{1}{(a_k+2)q_k^2} < \left|\tau - \frac{p_k}{q_k}\right|.$$

Since $\max\{a_k : k = 0,1,2,\ldots,92\} = 140$, we get

$$\frac{1}{142q_k^2} < \frac{6}{3^{m-m_1}q_k} \quad \text{and} \quad q_k \quad \text{divides one of} \quad \{n_1+2, n_1+6\}.$$

Thus, we get

$$3^{m-m_1} \leq 6 \times 142(n_1+6) < 6 \times 142 \times 4 \times 10^{40},$$

giving $m-m_1 \leq 92$.

Now let us turn to the case $m-m_1 \leq 109$ and we consider the inequlity for Λ_2, (C.12). We put

$$\Gamma_2 := n\log\alpha - m_1\log 3 + \log(1/(\sqrt{5}(3^{m-m_1}-1))),$$

and we also assume that $n-n_1 \geq 20$. We then have

$$|\Gamma_2| < \frac{2\alpha^8}{\alpha^{n-n_1}}.$$

We assume that Γ_2, then we get

$$0 \; < \; n\left(\frac{\log\alpha}{\log 3}\right) - m_1 + \frac{\log(1/(\sqrt{5}(3^{m-m_1}-1))}{\log\alpha} \; < \; \frac{3\alpha^8}{(\log 3)\alpha^{n-n_1}} \; < \; \frac{130}{\alpha^{n-n_1}}.$$

We apply again Lemma 1.2.2 with the same τ, q, M, $(A,B) := (130,\alpha)$ and

$$\mu_\ell := \frac{\log(1/(\sqrt{5}(3^\ell-1)))}{\log 3} \quad \text{for } \ell = 1,2,\ldots,109.$$

We get $\varepsilon > 0.004$, therefore

$$n-n_1 < \frac{\log(130q/\varepsilon)}{\log\alpha} < 266.$$

A similar conclusion is reached when $\Gamma_2 < 0$. To conclude, we first get that either $n-n_1 \leq 253$ or $m-m_1 \leq 109$. If $n-n_1 \leq 253$, then $m-m_1 \leq 115$, and if $m-m_1 \leq 109$ then $n-n_1 \leq 265$. Thus, we conclude that we always have $n-n_1 \leq 265$ and $m-m_1 \leq 115$.

Finally we go to the inequality of Λ_3, (C.15). We put

$$\Gamma_3 := n_1\log\alpha - m_1\log 3 + \log\left(\frac{\alpha^{n-n_1}-1}{\sqrt{5}(3^{m-m_1}-1)}\right).$$

Since $n \geq 300$, the inequality (C.15) implies that

$$|\Gamma_3| < \frac{3}{\alpha^{n-4}} = \frac{3\alpha^4}{\alpha^n}.$$

Assuming that $\Gamma_3 > 0$, then

$$0 < n_1\left(\frac{\log\alpha}{\log 3}\right) - m_1 + \frac{\log((\alpha^k-1)/(\sqrt{5}(3^\ell-1))}{\log 3} < \frac{3\alpha^4}{(\log 3)\alpha^n} < \frac{20}{\alpha^n},$$

where $(k,\ell) := (n-n_1, m-m_1)$. We again apply Lemma 1.2.2 with the same τ, q, M, $(A,B) := (20,\alpha)$ and

$$\mu_{k,\ell} := \frac{\log((\alpha^k-1)/(\sqrt{5}(3^\ell-1))}{\log 3} \quad \text{for} \quad 1 \leq k \leq 265, \;\; 1 \leq l \leq 115.$$

As before, we have a problem at $(k,\ell) = (4,1),\ (12,1),\ (8,2)$. The cases $(k,\ell) = (4,1),(12,1)$ were treated before in the case of Γ_1. The case $(k,\ell) = (8,2)$ arises because

$$\frac{\alpha^8 - 1}{\sqrt{5}(3^2 - 1)} = \frac{3}{8}\alpha^4,$$

we therefore discard the cases $(k,\ell) := (4,1),\ (12,1),\ (8,2)$ for some time. For the remaining cases, we get $\varepsilon > 0.0015$, so we obtain

$$n \le \frac{\log(20q/\varepsilon)}{\log\alpha} < 264.$$

A similar conclusion is reached when $\Gamma_3 < 0$. Hence, $n < 300$. Now we look at the cases $(k,\ell) = (4,1),\ (12,1),\ (8,2)$. The cases $(k,\ell) = (4,1),\ (12,1)$ can be treated as before when we showed that $n - n_1 \le 263$ implies $m - m_1 \le 115$. The case when $(k,\ell) = (8,2)$ can be delt with in a similar way. Namely, it gives that

$$|(n_1+4)\tau - m_1| < \frac{20}{\alpha^n}.$$

Therefore,

$$\left|\tau - \frac{m_1}{n_1+4}\right| < \frac{20}{(n_1+4)\alpha^n}. \tag{C.17}$$

Since $n \ge 300$, we have $\alpha^n > 2 \times 20 \times (4 \times 10^{40}) > 40(n_1+4)$. This shows that the right hand side of the above inequality, (C.17) is at most $2/(n_1+4)^2$. By Lemma 1.2.1, we get that $m_1/(n_1+4) = p_k/q_k$ for some $k = 1,2,\ldots,92$. We then get by a similar argument as before that

$$\alpha^n < 20 \times 142 \times (4 \times 10^{40}),$$

which gives $n \le 211$. Therefore, the conclusion is that $n < 300$ holds also in the case $(k,\ell) = (8,2)$. However, this contradicts our working assumption that $n > 300$. This completes the proof of Theorem C.2.1.

Appendix D.

On the problem of Pillai with Tribonacci numbers and powers of 3

The presentation in this appendix chapter is a slightly modified version of the paper [3] with the title *On the problem of Pillai with Tribonacci numbers and powers of* 3. This paper has been published in *Journal of Integer Sequences* in August, 2019.

Abstract: Let $(T_n)_{n\geq 0}$ be the sequence of Tribonacci numbers defined by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 0$. In this note, we find all integers c admitting at least two representations as a difference between a tribonacci number and a power of 3.

Keywords: Tribonacci numbers; Linear forms in logarithms; Baker's method.

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D.1. Introduction

We consider the sequence $(T_n)_{n\geq 0}$ of Tribonacci numbers defined by

$$T_0 = 0,\ T_1 = 1,\ T_2 = 1, \text{ and } T_{n+3} = T_{n+2} + T_{n+1} + T_n \text{ for all } n \geq 0.$$

The Tribonacci sequence is sequence A000073 on the Online Encyclopedia of Integer Sequences (OEIS) [54]. The first few terms of the Tribonacci sequence are

$$(T_n)_{n\geq 0} = 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, \ldots.$$

In this paper, we study the Diophantine equation

$$T_n - 3^m = c, \tag{D.1}$$

for a fixed integer c and variable integers n and m. In particular, we are interested in finding those integers c admitting at least two representations as a difference between a Tribonacci number and a power of 3. This equation is a variation of the Pillai equation (1.15).

We discard the situation when $n = 1$ and just count the solutions for $n = 2$ since $T_1 = T_2 = 1$. The reason for the above convention is to avoid trivial parametric families such as $1 - 3^m = T_1 - 3^m = T_2 - 3^m$. Thus, we always assume that $n \geq 2$. The main aim of this paper is to prove the following result.

Theorem D.1.1. *The only integers c having at least two representations of the form $T_n - 3^m$ with $n \geq 2$ and $m \geq 0$, are $c \in \{-2, 0, 1, 4\}$. Furthermore, all the representations of the above integers as $T_n - 3^m$ with integers $n \geq 2$ and $m \geq 0$ are given by*

$$\begin{aligned} -2 &= T_5 - 3^2 = T_2 - 3^1, \\ 0 &= T_9 - 3^4 = T_2 - 3^0, \\ 1 &= T_4 - 3^1 = T_3 - 3^0, \\ 4 &= T_6 - 3^2 = T_5 - 3^1. \end{aligned} \tag{D.2}$$

D.2. Preliminary results

D.2.1. The Tribonacci sequence

The characteristic polynomial of the Tribonacci sequence $(T_n)_{n\geq 0}$ is given by

$$\Psi(x) := x^3 - x^2 - x - 1.$$

$\Psi(x)$ is irreducible in $\mathbb{Q}[x]$, and has a positive real zero

$$\alpha = \frac{1}{3}\left(1 + (19 + 3\sqrt{33})^{1/3} + (19 - 3\sqrt{33})^{1/3}\right),$$

lying strictly outside the unit circle and two complex conjugate zeros β and γ lying strictly inside the unit circle. Furthermore, $|\beta| = |\gamma| = \alpha^{-1/2}$. According to Dresden and Zu [29], a Binet-like formula for the k-generalized Fibonacci sequences is established. For the Tribonacci sequence, it states that

$$T_n = C_\alpha \alpha^{n-1} + C_\beta \beta^{n-1} + C_\gamma \gamma^{n-1} \quad \text{for all } n \geq 1, \tag{D.3}$$

where $C_X = (X-1)/(4X-6)$. Dresden and Zu [29], also showed that the contribution of the complex conjugate zeros β and γ to the right-hand side of (D.3) is very small. More precisely,

$$\left|T_n - C_\alpha \alpha^{n-1}\right| < \frac{1}{2} \quad \text{for all } n \geq 1. \tag{D.4}$$

The minimal polynomial of C_α over the integers is given by

$$44X^3 - 44X^2 + 12X - 1,$$

has zeros $C_\alpha,\ C_\beta,\ C_\gamma$ with $|C_\alpha|,\ |C_\beta|,\ |C_\gamma| < 1$. Numerically,

$$\begin{gathered}1.83 < \alpha < 1.84,\\ 0.73 < |\beta| = |\gamma| = \alpha^{-1/2} < 0.74,\\ 0.61 < |C_\alpha| < 0.62,\\ 0.19 < |C_\beta| = |C_\gamma| < 0.20.\end{gathered}$$

It is also a well known fact (see [19, 8]) that

$$\alpha^{n-2} \le T_n \le \alpha^{n-1} \quad \text{holds for all } n \ge 1. \tag{D.5}$$

Let $\mathbb{K} := \mathbb{Q}(\alpha,\beta)$ be the splitting field of the polynomial Ψ over $\mathbb{Q}$. Then, $[\mathbb{K},\mathbb{Q}] = 6$. Furthermore, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. The Galois group of $\mathbb{K}$ over $\mathbb{Q}$ is given by

$$\mathscr{G} := \mathrm{Gal}(\mathbb{K}/\mathbb{Q}) \cong \{(1),(\alpha\beta),(\alpha\gamma),(\beta\gamma),(\alpha\beta\gamma),(\alpha\gamma\beta)\} \cong S_3.$$

Thus, we identify the automorphisms of $\mathscr{G}$ with the permutations of the zeros of the polynomial Ψ. For example, the permutation $(\alpha\gamma)$ corresponds to the automorphism $\sigma : \alpha \to \gamma,\ \gamma \to \alpha,\ \beta \to \beta$.

D.3. Proof of Theorem D.1.1

Let n, m, n_1, and m_1 be non-negative integers such that $(n,m) \ne (n_1,m_1)$ and

$$T_n - 3^m = T_{n_1} - 3^{m_1}.$$

Without loss of generality, we assume that $m \ge m_1$. If $m = m_1$, then $T_n = T_{n_1}$, so $(n,m) = (n_1,m_1)$, which gives a contradiction to our assumption. Thus, $m > m_1$. Since

$$T_n - T_{n_1} = 3^m - 3^{m_1}, \tag{D.6}$$

and the right-hand side is positive, we get that the left-hand side is also positive and so $n > n_1$. Thus, $n \ge 3$ and $n_1 \ge 2$.

Using the equation (D.6) and the inequality (D.5), we get

$$\begin{aligned}\alpha^{n-4} &\le T_{n-2} \le T_n - T_{n_1} = 3^m - 3^{m_1} < 3^m,\\ \alpha^{n-1} &\ge T_n \ge T_n - T_{n_1} = 3^m - 3^{m_1} \ge 3^{m-1},\end{aligned} \tag{D.7}$$

from which we get that

$$1 + \left(\frac{\log 3}{\log\alpha}\right)(m-1) < n < \left(\frac{\log 3}{\log\alpha}\right)m + 4. \tag{D.8}$$

If $n \le 300$, then $m \le 200$. We ran a Mathematica program for $2 \le n_1 < n \le 300$ and $0 \le m_1 < m \le 200$ and found only the solutions from the list (D.2). From now, we assume that $n > 300$. Note that the inequality (D.8) implies that $m < 0.6n + 0.4$. Therefore, to solve the Diophatine equation (D.1), it suffices to find an upper bound for n.

D.3.1. Bounding n

From (D.3) and (D.4), we get

$$\begin{aligned}\left|C_\alpha\alpha^{n-1}-3^m\right| &= \left|(C_\alpha\alpha^{n-1}-T_n)+(T_{n_1}-3^{m_1})\right|\\ &= \left|(C_\alpha\alpha^{n-1}-T_n)+(T_{n_1}-C_\alpha\alpha^{n_1-1})+(C_\alpha\alpha^{n_1-1}-3^{m_1})\right|\\ &< 1+\frac{7}{10}\alpha^{n_1-1}+3^{m_1}\\ &< \alpha^{n_1}+3^{m_1}\\ &< 2\max\{\alpha^{n_1},3^{m_1}\}.\end{aligned}$$

In the above we have used the fact that $|C_\alpha|<0.62<0.7$. Multiplying through by 3^{-m}, using the relation (D.7) and using the fact that $\alpha<3$, we get

$$\left|C_\alpha\alpha^{n-1}3^{-m}-1\right| \;<\; 2\max\left\{\frac{\alpha^{n_1}}{3^m},3^{m_1-m}\right\} \;<\; \max\{\alpha^{n_1-n+6},3^{m_1-m+1}\}. \tag{D.9}$$

For the left-hand side, we apply the result of Matveev, Theorem 1.1.8 with the following data:

$$t:=3,\;\; \eta_1:=C_\alpha,\;\; \eta_2:=\alpha,\;\; \eta_3:=3,\;\; b_1:=1,\;\; b_2:=n-1,\;\; \text{and}\;\; b_3:=-m.$$

Through out we work with the field $\mathbb{K}:=\mathbb{Q}(\alpha)$ with $D:=3$. Since $\max\{1,n-1,m\}\le n$, we take $B:=n$. The minimal polynomial of C_α over the integers is given by

$$44x^3-44x^2+12x-1.$$

Since $|C_\alpha|,\ |C_\beta|,\ |C_\gamma|<1$, we get that $h(C_\alpha)=\frac{1}{3}\log 44$. So we can take $A_1:=3h(\gamma_1)=\log 44$. We can also take $A_2:=3h(\eta_2)=\log\alpha$ and $A_3:=3h(\eta_3)=3\log 3$. We put

$$\Lambda:=C_\alpha\alpha^{n-1}3^{-m}-1.$$

First we check that $\Lambda\neq 0$, if it were, then $C_\alpha\alpha^{n-1}=3^m\in\mathbb{Z}$. Conjugating this relation by the automorphism $(\alpha\beta)$, we obtain that $C_\beta\beta^{n-1}=3^m$, which is a contradiction because $|C_\beta\beta^{n-1}|<1$ while $3^m\ge 3$ for all $m\ge 1$. Thus, $\Lambda\neq 0$. Hence, by Theorem 1.1.8, the left-hand side of (D.9) is bounded as follows:

$$\log|\Lambda|>-1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2(1+\log 3)(1+\log n)(\log 44)(\log\alpha)(3\log 3).$$

By comparing with (D.9), we get

$$\min\{(n-n_1-5)\log\alpha,(m-m_1-1)\log 3\}<2.06\times 10^{13}(1+\log n),$$

which gives

$$\min\{(n-n_1)\log\alpha,(m-m_1)\log 3\}<2.12\times 10^{13}(1+\log n).$$

Now, we split the argument into two cases.

Case D.3.1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} = (n-n_1)\log\alpha$.

In this case, we rewrite (D.6) as

$$\begin{aligned}\left|C_\alpha\alpha^{n-1} - C_\alpha\alpha^{n_1-1} - 3^m\right| &= \left|C_\alpha\alpha^{n-1} - T_n) + (T_{n_1} - C_\alpha\alpha^{n_1-1}) - 3^{m_1}\right| \\ &< 1 + 3^{m_1} \leq 3^{m_1+1},\end{aligned}$$

which implies

$$\left|C_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1-1}3^{-m} - 1\right| < 3^{m_1-m+1}. \tag{D.10}$$

We put

$$\Lambda_1 := C_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1-1}3^{-m} - 1.$$

As before, we take $\mathbb{K} := \mathbb{Q}(\alpha)$, so we have $D = 3$. We have $\Lambda_1 \neq 0$, for if $\Lambda_1 = 0$, then

$$C_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1-1} = 3^m.$$

By conjugating the above relation by the Galois automorphism $(\alpha\beta)$, we get that

$$C_\beta(\beta^{n-n_1} - 1)\beta^{n_1-1} = 3^m.$$

The absolute value of the left-hand side is at most $|C_\beta(\beta^{n-n_1} - 1)\beta^{n_1-1}| \leq |C_\beta\beta^{n-1}| + |C_\beta\beta^{n_1-1}| < 2$, while the absolute value of the right-hand side is at least $3^m \geq 3$ for all $m \geq 1$, which is a contradiction.

We apply Theorem 1.1.8 on the left-hand side of (D.10) with the following data:

$$t := 3,\ \ \eta_1 := C_\alpha(\alpha^{n-n_1} - 1),\ \ \eta_2 := \alpha,\ \ \eta_3 := 3,\ \ b_1 := 1,\ \ b_2 := n_1 - 1,\ \ \text{and}\ \ b_3 := -m.$$

Since

$$\begin{aligned}h(\eta_1) &\leq h(C_\alpha) + h(\alpha^{n-n_1} - 1) \\ &< \frac{1}{3}\log 44 + \frac{1}{3}(n-n_1)\log\alpha + \log 2 \\ &< \frac{1}{3}(\log 11 + \log 32) + \frac{1}{3} \times 2.12 \times 10^{13}(1+\log n) \\ &< \frac{1}{3} \times 2.50 \times 10^{13}(1+\log n).\end{aligned} \tag{D.11}$$

So, we can take $A_1 := 2.50 \times 10^{13}(1+\log n)$. Furthermore, as before, we take $A_2 := \log\alpha$ and $A_3 := 3\log 3$. Finally, since $\max\{1, n_1 - 1, m\} \leq n$, we can take $B := n$. Then we get

$$\log|\Lambda_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1+\log 3)(1+\log n)(2.50 \times 10^{13}(1+\log n))(\log\alpha)(3\log 3).$$

Then,

$$\log|\Lambda_1| > -1.36\times 10^{25}(1+\log n)^2.$$

By comparing the above relation with (D.10), we get that

$$(m-m_1)\log 3 < 1.40\times 10^{26}(1+\log n)^2. \tag{D.12}$$

Case D.3.2. $\min\{(n-n_1)\log\alpha,(m-m_1)\log 3\} = (m-m_1)\log 3.$

In this case, we rewrite (D.6) as

$$\begin{aligned}\left|C_\alpha\alpha^n-(3^{m-m_1}-1)\cdot 3^{m_1}\right| &= \left|(C_\alpha\alpha^{n-1}-T_n)+(T_{n_1}-C_\alpha\alpha^{n_1-1})+C_\alpha\alpha^{n_1-1}\right|\\ &< 1+\frac{7}{10}\alpha^{n-1} \ < \ \alpha^{n-1} \quad \text{(beacause } n\geq 3),\end{aligned}$$

which implies that

$$\begin{aligned}|C_\alpha(3^{m-m_1}-1)^{-1}\alpha^{n-1}3^{-m_1}-1| &< \frac{\alpha^{n_1}}{3^m-3^{m_1}} \ \leq \ \frac{3\alpha^{n_1}}{3^m}\\ &< 3\alpha^{n_1-n+4} \ < \ \alpha^{n_1-n+6}.\end{aligned} \tag{D.13}$$

We put

$$\Lambda_2 := C_\alpha(3^{m-m_1}-1)^{-1}\alpha^{n-1}3^{-m_1}-1.$$

Clearly, $\Lambda_2\neq 0$, for if $\Lambda_2=0$, then $C_\alpha=(\alpha^{-1})^{n-1}(3^m-3^{m_1})$ implying that C_α is an algebraic integer, a contradiction. We again apply Theorem 1.1.8 with the following data:

$$t:=3,\ \ \eta_1:=C_\alpha(3^{m-m_1}-1)^{-1},\ \ \eta_2:=\alpha,\ \ \eta_3:=\alpha,\ \ b_1:=1,\ \ b_2:=n,\ \ \text{and}\ \ b_3:=-m_1.$$

We note that

$$\begin{aligned}h(\eta_1) &= h(C_\alpha(3^{m-m_1}-1)^{-1}) \ \leq \ h(C_\alpha)+h(3^{m-m_1}-1)\\ &= \frac{1}{3}\log 44+h(3^{m-m_1}-1) \ < \ \log(3^{m-m_1+2})\\ &= (m-m_1+2)\log 3 \ < \ 2.50\times 10^{13}(1+\log n).\end{aligned}$$

So, we can take $A_1 := 7.5\times 10^{13}(1+\log n)$. Further, as in the previous applications, we take $A_2 := \log\alpha$ and $A_3 := 3\log 3$. Finally, since $\max\{1,n-1,m_1\}\leq n$, we can take $B := n$. Then, we get

$$\log|\Lambda_2| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2(1+\log 3)(1+\log n)(7.5\times 10^{13}(1+\log n))(\log\alpha)(3\log 3).$$

Thus,

$$\log|A_2| > -4.08\times 10^{26}(1+\log n)^2.$$

Now, by comparing with (D.13), we get that

$$(n-n_1)\log\alpha < 4.10\times 10^{26}(1+\log n)^2. \tag{D.14}$$

Therefore, in both Case D.3.1 and Case D.3.2, we have

$$\begin{aligned}\min\{(n-n_1)\log\alpha,(m-m_1)\log 3\} &< 2.12\times 10^{13}(1+\log n),\\ \max\{(n-n_1)\log\alpha,(m-m_1)\log 3\} &< 4.10\times 10^{26}(1+\log n)^2.\end{aligned} \tag{D.15}$$

Finally, we rewrite the equation (D.6) as

$$\left|C_\alpha\alpha^{n-1}-C_\alpha\alpha^{n_1-1}-3^m+3^{m_1}\right| = \left|(C_\alpha\alpha^{n-1}-T_n)+(T_{n_1}-C_\alpha\alpha^{n_1-1})\right| \;<\; 1.$$

Dividing through by $3^m-3^{m_1}$, we get

$$\begin{aligned}\left|\frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\alpha^{n_1-1}3^{-m_1}-1\right| &< \frac{1}{3^m-3^{m_1}} \le \frac{3}{3^m}\\ &\le 3\alpha^{-(n-4)} \;\le\; \alpha^{6-n},\end{aligned} \tag{D.16}$$

since $3<\alpha\le\alpha^{n_1}$. We again apply Theorem 1.1.8 on the left-hand side of (D.16) with the following data:

$$t:=3,\;\; \eta_1:=\frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1},\;\; \eta_2:=\alpha,\;\; \eta_3:=3,\;\; b_1:=1,\;\; b_2:=n_1-1,\;\; \text{and}\;\; b_3:=-m_1.$$

By using the algebraic properties of the logarithmic height function, we get

$$\begin{aligned}3h(\eta_1)=3h\left(\frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\right) &\;\le\; h\left(C_\alpha(\alpha^{n-n_1}-1)\right)+h(3^{m-m_1}-1)\\ &< \log 352+(n-n_1)\log\alpha+3(m-m_1)\log 3\\ &< 6.80\times 10^{26}(1+\log n)^2,\end{aligned}$$

where in the above inequalities, we used the argument from (D.11) as well as the bounds (D.15). Thus, we can take $A_1:=6.80\times 10^{26}(1+\log n)$, and again as before $A_2:=\log\alpha$ and $A_3:=3\log 3$. If we put

$$\Lambda_3:=\frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\alpha^{n_1-1}3^{-m_1}-1,$$

we need to show that $\Lambda_3\neq 0$. If not, $\Lambda_3=0$ leads to

$$C_\alpha(\alpha^{n-1}-\alpha^{n_1-1})=3^m-3^{m_1}.$$

A contradiction is reached upon a conjugation by the automorphism $(\alpha\beta)$ in $\mathbb{K}$ and by taking absolute values on both sides. Thus, $\Lambda_3 \neq 0$. Applying Theorem 1.1.8 gives

$$\log|\Lambda_3| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2(1+\log 3)(1+\log n)(6.80\times 10^{26}(1+\log n)^2)(\log\alpha)(3\log 3),$$

a comparison with (D.16) gives

$$(n-6) < 3.70\times 10^{39}(1+\log n)^3,$$

or

$$n < 3.8\times 10^{39}(1+\log n)^3. \tag{D.17}$$

Now, by applying Lemma 1.5.1 on (D.17) with the data $m := 3$, $T := 3.8\times 10^{39}$, and $x := n$, leads to $n < 3\times 10^{46}$.

D.3.2. Reducing the bound for n

We need to reduce the above bound for n and to do so we make use of Lemma 1.2.2 several times. To begin, we return to (D.9) and put

$$\Gamma := (n-1)\log\alpha - m\log 3 + \log C_\alpha.$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geq 20$. We go back to the inequalities for Λ, Λ_1, and Λ_2, Since we assume that $\min\{n-n_1, m-m_1\} \geq 20$ we get $|e^\Gamma - 1| = |\Lambda| < \frac{1}{4}$. Hence, $|\Lambda| < \frac{1}{2}$ and since the inequality $|y| < 2|e^y - 1|$ holds for all $y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get

$$0 < |\Gamma| < 2\max\{\alpha^{n_1-n+6}, 3^{m_1-m+1}\} \leq \max\{\alpha^{n_1-n+8}, 3^{m_1-m+2}\}.$$

By substituting for Γ in the above inequality and dividing through by $\log 3$, we get the inequality

$$0 < \left|(n-1)\left(\frac{\log\alpha}{\log 3}\right) - m + \frac{\log C_\alpha}{\log 3}\right| < \max\left\{\frac{\alpha^8}{(\log 3)\alpha^{n-n_1}}, \frac{9}{(\log 3)3^{m-m_1}}\right\}.$$

We apply Lemma 1.2.2 with the following data

$$\tau := \frac{\log\alpha}{\log 3}, \quad \mu := \frac{\log C_\alpha}{\log 3}, \quad \text{and} \quad (A,B) := \left(\frac{\alpha^8}{\log 3}, \alpha\right) \text{ or } \left(\frac{9}{\log 3}, 3\right).$$

Let $\tau = [a_0; a_1, a_2, \ldots] = [0; 1, 1, 4, 13, 1, 6, 1, 4, 1, 10, 7, 1, 24, 3, 3, 2, 12, 4, 4, \ldots]$ be the continued fraction expansion of τ. We choose $M := 3\times 10^{46}$ which is the upper bound on n. With the help of Mathematica, we find out that the convergent

$$\frac{p}{q} = \frac{p_{88}}{q_{88}} = \frac{383979914200993729068715782793592146551951600940}{692255294546383107303758900444711151890883197059}$$

is such that $q = q_{88} > 6M$. Furthermore, it yields $\varepsilon > 0.0428119$, and therefore either

$$n - n_1 \le \frac{\log\left((\alpha^8/\log 3)q/\varepsilon\right)}{\log\alpha} < 193, \quad \text{or} \quad m - m_1 \le \frac{\log((9/\log 3)q/\varepsilon)}{\log 3} < 105.$$

Thus, we have that either $n - n_1 \le 193$ or $m - m_1 \le 105$.

Now we distinguish between the cases $n - n_1 \le 193$ and $m - m_1 \le 105$. First, we assume that $n - n_1 \le 193$. In this case we consider the inequality for Λ_1, (D.10) and also assume that $m - m_1 \ge 20$. We put

$$\Gamma_1 := (n_1 - 1)\log\alpha - m\log 3 + \log\left(C_\alpha(\alpha^{n-n_1} - 1)\right).$$

Then, inequality (D.10) implies that

$$|\Gamma_1| < \frac{6}{3^{m-m_1}}.$$

If we substitute for Γ_1 in the above inequality and divide through by $\log 3$, we then get

$$0 < \left|(n_1 - 1)\left(\frac{\log\alpha}{\log 3}\right) - m + \frac{\log(C_\alpha(\alpha^{n-n_1} - 1))}{\log 3}\right| < \frac{6}{(\log 3)3^{m-m_1}}.$$

Again we apply Lemma 1.2.2 with the same τ as in the case of Γ. We use the 88-th convergent $p/q = p_{88}/q_{88}$ of τ as before. But in this case we choose $(A,B) := \left(\frac{6}{\log 3}, 3\right)$ and use

$$\mu_\ell := \frac{\log(C_\alpha(\alpha^\ell - 1))}{\log 3},$$

instead of μ for each possible value of $\ell := n - n_1 \in [1, 2, \ldots, 193]$. For all values of ℓ, we get $\varepsilon > 0.0000420218$. Hence, by Lemma 1.2.2, we get

$$m - m_1 < \frac{\log((6/\log 3)q/\varepsilon)}{\log 3} < 110.$$

Thus, $n - n_1 \le 193$ implies that $m - m_1 \le 110$.

Now let us turn to the case $m - m_1 \le 105$ and we consider the inequlity for Λ_2, (D.13). We put

$$\Gamma_2 := (n-1)\log\alpha - m_1\log 3 + \log\left(\frac{C_\alpha}{3^{m-m_1} - 1}\right),$$

and we also assume that $n - n_1 \ge 20$. We then have

$$|\Gamma_2| < \frac{\alpha^8}{\alpha^{n-n_1}}.$$

If we substitute for Γ_2 in the above inequality and divide through by $\log 3$, we then get

$$0 < \left|(n-1)\left(\frac{\log\alpha}{\log 3}\right) - m_1 + \frac{\log(C_\alpha/(3^{m-m_1}-1))}{\log 3}\right| < \frac{\alpha^8}{(\log 3)\alpha^{n-n_1}}.$$

We apply again Lemma 1.2.2 with the same τ, q, M, $(A,B) := \left(\frac{\alpha^8}{\log 3}, \alpha\right)$, and

$$\mu_\ell := \frac{\log(C_\alpha/(3^\ell - 1))}{\log 3} \quad \text{for } \ell = 1,2,\ldots,105.$$

We get $\varepsilon > 0.00218297$, therefore

$$n - n_1 < \frac{\log\left((\alpha^8/\log 3)q/\varepsilon\right)}{\log\alpha} < 198.$$

To conclude, we first get that either $n-n_1 \leq 193$ or $m-m_1 \leq 105$. If $n-n_1 \leq 193$, then $m-m_1 \leq 110$, and if $m-m_1 \leq 105$, then $n-n_1 \leq 198$. Thus, we conclude that we always have $n-n_1 \leq 198$ and $m-m_1 \leq 110$.

Finally, we go to the inequality of Λ_3, (D.16). We put

$$\Gamma_3 := (n_1-1)\log\alpha - m_1\log 3 + \log\left(\frac{C_\alpha(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\right).$$

Since $n > 300$, the inequality (D.16) implies that

$$|\Gamma_3| < \frac{2}{\alpha^{n-6}} = \frac{\alpha^8}{\alpha^n}.$$

Substituting for Γ_3 in the above inequality and dividing through by $\log 3$, we get

$$0 < \left|(n_1-1)\left(\frac{\log\alpha}{\log 3}\right) - m_1 + \frac{\log(C_\alpha(\alpha^k-1)/(3^\ell-1))}{\log 3}\right| < \frac{\alpha^8}{(\log 3)\alpha^n},$$

where $(k,\ell) := (n-n_1, m-m_1)$. We again apply Lemma 1.2.2 with the same τ, q, M, $(A,B) := \left(\frac{\alpha^8}{\log 3}, \alpha\right)$, and

$$\mu_{k,\ell} := \frac{\log(C_\alpha(\alpha^k-1)/(3^\ell-1))}{\log 3} \quad \text{for} \quad 1 \leq k \leq 198 \quad \text{and} \quad 1 \leq \ell \leq 110.$$

For the cases, we get $\varepsilon > 0.0000115272$, so we obtain

$$n \leq \frac{\log\left((\alpha^8/\log 3)q/\varepsilon\right)}{\log\alpha} < 207.$$

Hence, $n \leq 207$. However, this contradicts our working assumption that $n > 300$. This completes the proof of Theorem D.1.1. □

Appendix E.

On the problem of Pillai with Padovan numbers and powers of 3

This appendix chapter contains a presentation of a slightly modified version of the paper [2] with the title *On the problem of Pillai with Padovan numbers and powers of* 3. The article has been published in *Studia Scientiarum Mathematicarum Hungarica* in September, 2019.

Abstract: Let $\{P_n\}_{n\geq 0}$ be the sequence of Padovan numbers defined by $P_0 = 0$, $P_1 = 1,\ P_2 = 1$, and $P_{n+3} = P_{n+1} + P_n$ for all $n \geq 0$. In this paper, we find all integers c admitting at least two representations as a difference between a Padovan number and a power of 3.

Keywords: Padovan numbers; Linear forms in logarithms; Baker's method.

2010 Mathematics Subject Classification: 11B39, 11J86.

E.1. Introduction

We consider the sequence $\{P_n\}_{n\geq 0}$ of Padovan numbers defined by

$$P_0 = 0,\ P_1 = 1,\ P_2 = 1,\ \text{and}\ P_{n+3} = P_{n+1} + P_n\ \text{for all}\ n \geq 0.$$

This is sequence $A000931$ on the On-Line Encyclopedia of Integer Sequences (OEIS) [54]. The first few terms of this sequence are

$$\{P_n\}_{n\geq 0} = 0,1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49,65,86,114,151,\ldots.$$

In this paper, we study the Diophantine equation

$$P_n - 3^m = c \tag{E.1}$$

for a fixed integer c and variable integers n and m. In particular, we are interested in finding those integers c admitting at least two representations as a difference between a Padovan number and a power of 3. This equation is a variation of the Pillai equation (1.15).

E.2. Main Result

We discard the situations when $n = 1$ and $n = 2$ and just count the solutions for $n = 3$ since $P_1 = P_2 = P_3 = 1$. The reason for the above convention is to avoid trivial parametric families such as $1-3^m = P_1-3^m = P_2-3^m = P_3-3^m$. For the same reasons, we discard the situation when $n = 4$ and just count the solutions for $n = 5$ since $P_4 = P_5 = 2$. Thus, we always assume that $n \geq 2$ and $n \neq 4$. The main aim of this paper is to prove the following result.

Theorem E.2.1. *The only integers c having at least two representations of the form $P_n - 3^m$ are $c \in \{-6, 0, 1, 22, 87\}$. Furthermore, all the representations of the above integers as $P_n - 3^m$ with integers $n \geq 3$, $n \neq 4$, and $m \geq 0$ are given by*

$$\begin{aligned} -6 &= P_{13} - 3^3 = P_6 - 3^2; \\ 0 &= P_{10} - 3^2 = P_6 - 3^1 \ (= P_3 - 3^0); \\ 1 &= P_{14} - 3^3 = P_7 - 3^1 \ (= P_5 - 3^0); \\ 22 &= P_{20} - 3^5 = P_{16} - 3^3; \\ 87 &= P_{24} - 3^6 = P_{17} - 3^3. \end{aligned} \tag{E.2}$$

By a recent result of Chim, Pink, and Ziegler [26], there exists an effectively computable bound for c in (E.1). Hence, the main difficulty is to compute the bound for our case and reduce it to a manageable size. To do so, we apply the Baker's theory for linear forms in logarithms of algebraic numbers to establish the bound, and apply a Baker-Davenport reduction procedure to reduce the bound to a manageable size that can be implemented by the computer.

E.3. Preliminary results

E.3.1. The Padovan sequence

Here, we recall some important properties of the Padovan sequence $\{P_n\}_{n\geq 0}$. The characteristic equation

$$\Psi(x) := x^3 - x - 1 = 0$$

has roots $\alpha, \beta, \gamma = \bar{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \qquad \beta = \frac{-(r_1 + r_2) + \sqrt{-3}(r_1 - r_2)}{12}$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Furthermore, the Binet formula is given by

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \qquad \text{for all} \quad n \geq 0, \tag{E.3}$$

where

$$a = \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)}, \quad c = \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \bar{b}. \tag{E.4}$$

Numerically, the following estimates hold:

$$\begin{gathered} 1.32 < \alpha < 1.33 \\ 0.86 < |\beta| = |\gamma| = \alpha^{-\frac{1}{2}} < 0.87 \\ 0.72 < a < 0.73 \\ 0.24 < |b| = |c| < 0.25. \end{gathered} \tag{E.5}$$

By induction, one can easily prove that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 4. \tag{E.6}$$

Let $\mathbb{K} := \mathbb{Q}(\alpha, \beta)$ be the splitting field of the polynomial Ψ over $\mathbb{Q}$. Then $[\mathbb{K}, \mathbb{Q}] = 6$. Furthermore, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. The Galois group of $\mathbb{K}$ over $\mathbb{Q}$ is given by

$$\mathscr{G} := \mathrm{Gal}(\mathbb{K}/\mathbb{Q}) \cong \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \cong S_3.$$

Thus, we identify the automorphisms of $\mathscr{G}$ with the permutations of the roots of the polynomial Ψ. For example, the permutation $(\alpha\beta)$ corresponds to the automorphism $\sigma : \alpha \to \beta, \beta \to \alpha, \gamma \to \gamma$.

E.4. Proof of Theorem E.2.1

Assume that there exist positive integers n, m, n_1, m_1 such that $(n, m) \neq (n_1, m_1)$, and

$$P_n - 3^m = P_{n_1} - 3^{m_1}.$$

In particular, we can assume that $m \geq m_1$. If $m = m_1$, then $P_n = P_{n_1}$, so $(n, m) = (n_1, m_1)$, which gives a contradiction to our assumption. Thus $m > m_1 \geq 0$. Since

$$P_n - P_{n_1} = 3^m - 3^{m_1}, \tag{E.7}$$

and the right-hand side is positive, we get that the left-hand side is also positive and so $n > n_1$. Thus, $n \geq 5$ and $n_1 \geq 3$, because $n \neq 4$.

Using the equation (E.7) and the inequality (E.6), we get

$$\begin{aligned} \alpha^{n-4} &\le P_{n-2} \le P_n - P_{n_1} = 3^m - 3^{m_1} < 3^m, \\ \alpha^{n-1} &\ge P_n \ge P_n - P_{n_1} = 3^m - 3^{m_1} \ge 3^{m-1}, \end{aligned} \tag{E.8}$$

from which we get that

$$1 + \left(\frac{\log 3}{\log \alpha}\right)(m-1) < n < \left(\frac{\log 3}{\log \alpha}\right) m + 4. \tag{E.9}$$

If $n \le 500$, then $m \le 200$. We ran a *Mathematica* program for $2 \le n_1 < n \le 500$ and $0 \le m_1 < m \le 200$ and found only the solutions from the list (E.2). From now, we assume that $n > 500$. Note that the inequality (E.9) implies that $m < \frac{1}{4}n + \frac{3}{4}$. Thus, to solve the Diophatine equation (E.1), it suffices to find an upper bound for n.

E.4.1. Bounding n

By using (E.1) and (E.3) and the estimates (E.5), we get

$$\begin{aligned} a\alpha^n + b\beta^n + c\gamma^n - 3^m &= a\alpha^{n_1} + b\beta^{n_1} + c\gamma^{n_1} - 3^{m_1} \\ |a\alpha^n - 3^m| &= |a\alpha^{n_1} + b(\beta^{n_1} - \beta^n) + c(\gamma^{n_1} - \gamma^n) - 3^{m_1}| \\ &\le a\alpha^{n_1} + |b|(|\beta|^n + |\beta|^{n_1}) + |c|(|\gamma|^n + |\gamma|^{n_1}) + 3^{m_1} \\ &\le a\alpha^{n_1} + 2|b|(|\beta|^n + |\beta|^{n_1}) + 3^{m_1} \\ &\le a\alpha^{n_1} + 4|b||\beta|^{n_1} + 3^{m_1} \\ &< \alpha^{n_1} + 3^{m_1} \\ &< 2\max\{\alpha^{n_1}, 3^{m_1}\}. \end{aligned}$$

Multiplying through by 3^{-m}, using the relation (E.8) and using the fact that $\alpha < 3$, we get

$$|a\alpha^n 3^{-m} - 1| < 2\max\left\{\frac{\alpha^{n_1}}{3^m}, 3^{m_1-m}\right\} < \max\{\alpha^{n_1-n+5}, 3^{m_1-m+1}\}. \tag{E.10}$$

For the left-hand side, we apply the result of Matveev, Theorem 1.1.8 with the following data

$$t = 3,\ \ \gamma_1 = a,\ \ \gamma_2 = \alpha,\ \ \gamma_3 = 3,\ \ b_1 = 1,\ \ b_2 = n,\ \ b_3 = -m.$$

Through out we work with the field $\mathbb{K} := \mathbb{Q}(\alpha)$ with $D = 3$. Since $\max\{1, n, m\} \le n$, we take $B := n$. Further,

$$a = \frac{\alpha(\alpha+1)}{3\alpha^2 - 1},$$

the minimum polynomial of a is $23x^3 - 23x^2 + 6x - 1$, and has roots a, b, c. Also by (E.5), we have $\max\{|a|, |b|, |c|\} < 1$. Thus, $h(\gamma_1) = h(a) = \frac{1}{3}\log 23$. So we can take $A_1 := 3h(\gamma_1) = \log 23$. We can also take $A_2 := 3h(\gamma_2) = \log\alpha$, $A_3 := 3h(\gamma_3) = 3\log 3$. We put

$$\Lambda := a\alpha^n 3^{-m} - 1.$$

First we check that $\Lambda \neq 0$, if it were, then $a\alpha^n = 3^m \in \mathbb{Z}$. Conjugating this relation by the automorphism $(\alpha\beta)$, we obtain that $b\beta^n = 3^m$, which is a contradiction because $|b\beta^n| < 1$ while $3^m \geq 1$ for all $m \geq 0$. Thus, $\Lambda \neq 0$. Then by Matveev's theorem, the left-hand side of (E.10) is bounded as

$$\log|\Lambda| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1+\log 3)(1+\log n)(\log 23)(\log\alpha)(3\log 3).$$

By comparing with (E.10), we get

$$\min\{(n-n_1-5)\log\alpha, (m-m_1-1)\log 3\} < 7.97 \times 10^{12}(1+\log n),$$

which gives

$$\min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} < 7.98 \times 10^{12}(1+\log n). \tag{E.11}$$

Now we split the argument into two cases

Case E.4.1. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} = (n-n_1)\log\alpha$.

In this case, we rewrite (E.7) as

$$\begin{aligned} |a\alpha^n - a\alpha^{n_1} - 3^m| &\leq |b|(|\beta|^n + |\beta|^{n_1}) + |c|(|\gamma|^n + |\gamma|^{n_1}) + 3^{m_1} \\ &\leq 2|b|(|\beta|^n + |\beta|^{n_1}) + 3^{m_1} \\ &\leq 4|b||\beta|^{n_1} + 3^{m_1} \\ &< 1 + 3^{m_1} \leq 3^{m_1+1}, \end{aligned}$$

which implies

$$\left|a(\alpha^{n-n_1}-1)\alpha^{n_1}3^{-m} - 1\right| < 3^{m_1-m+1}. \tag{E.12}$$

We put

$$\Lambda_1 := a(\alpha^{n-n_1}-1)\alpha^{n_1}3^{-m} - 1.$$

Since $\alpha, a, 3 \in \mathbb{Q}(\alpha)$, we take $\mathbb{K} := \mathbb{Q}(\alpha)$, so we have $D = 3$. To see that $\Lambda_1 \neq 0$, for if $\Lambda_1 = 0$, then

$$a(\alpha^{n-n_1}-1)\alpha^{n_1} = 3^m.$$

By conjugating the above relation by the Galois automorphism $(\alpha\beta)$, we get that

$$b(\beta^{n-n_1}-1)\beta^{n_1} = 3^m.$$

The absolute value of the left-hand side is at most $|b(\beta^{n-n_1}-1)\beta^{n_1-1}| \leq |b|(|\beta^n| + |\beta^{n_1}|) < 2|b||\beta|^{n_1} < 1$, while the absolute value of the right-hand side is at least $3^m \geq 1$ for all $m \geq 0$, which is a contradiction. Thus, $\Lambda_1 \neq 0$.

We apply Theorem 1.1.8 on the left-hand side of (E.12) with the data

$$t = 3,\ \gamma_1 = a(\alpha^{n-n_1} - 1),\ \gamma_2 = \alpha,\ \gamma_3 = 3,\ b_1 = 1,\ b_2 = n_1,\ b_3 = -m.$$

Since

$$\begin{aligned} h(\gamma_1) &\leq h(a) + h(\alpha^{n-n_1} - 1) \\ &< \frac{1}{3}\log 23 + \frac{1}{3}(n-n_1)\log\alpha + \log 2 \\ &< \frac{1}{3}(\log 8 + \log 23) + \frac{1}{3} \times 7.98 \times 10^{12}(1+\log n) \quad \text{by (E.11)} \\ &< \frac{1}{3} \times 8.00 \times 10^{12}(1+\log n) \end{aligned} \tag{E.13}$$

So, we can take $A_1 := 8.00 \times 10^{12}(1+\log n)$. Furthermore, as before, we take $A_2 := \log\alpha$ and $A_3 := 3\log 3$. Finally, since $\max\{1, n_1, m\} \leq n$, we can take $B := n$. Then, we get

$$\log|\Lambda_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1+\log 3)(1+\log n)(8.00 \times 10^{12}(1+\log n))(\log\alpha)(3\log 3).$$

Then,

$$\log|\Lambda_1| > -6.38 \times 10^{25}(1+\log n)^2.$$

By comparing the above relation with (E.12), we get that

$$(m-m_1)\log 3 < 6.40 \times 10^{25}(1+\log n)^2. \tag{E.14}$$

Case E.4.2. $\min\{(n-n_1)\log\alpha, (m-m_1)\log 3\} = (m-m_1)\log 3.$

In this case, we rewrite (E.7) as

$$\begin{aligned} \left|a\alpha^n - (3^{m-m_1} - 1)\cdot 3^{m_1}\right| &\leq a\alpha^{n_1} + |b|(|\beta|^n + |\beta|^{n_1}) + |c|(|\gamma|^n + |\gamma|^{n_1}) \\ &\leq a\alpha^{n_1} + 4|b||\beta|^n \\ &< 1 + \frac{3}{4}\alpha^{n_1} \ < \ \alpha^{n_1}, \end{aligned}$$

which implies that

$$\begin{aligned} |a(3^{m-m_1} - 1)^{-1}\alpha^n 3^{-m_1} - 1| &< \frac{\alpha^{n_1}}{3^m - 3^{m_1}} \leq \frac{3\alpha^{n_1}}{3^m} \\ &< 3\alpha^{n_1-n+4} < \alpha^{n_1-n+7}. \end{aligned} \tag{E.15}$$

We put

$$\Lambda_2 := a(3^{m-m_1} - 1)^{-1}\alpha^n 3^{-m_1} - 1.$$

Clearly, $\Lambda_2 \neq 0$, for if $\Lambda_2 = 0$, then $a\alpha^n = 3^m - 3^{m_1}$, by similar arguments of conjugation and taking absolute values on both sides as before we get a contradiction. We again apply Theorem 1.1.8 with the following data

$$t = 3,\ \ \gamma_1 = a(3^{m-m_1} - 1)^{-1},\ \ \gamma_2 = \alpha,\ \ \gamma_3 = \alpha,\ \ b_1 = 1,\ \ b_2 = n,\ \ b_3 = -m_1.$$

We note that

$$\begin{aligned} h(\gamma_1) &= h(a(3^{m-m_1} - 1)^{-1}) \ \leq\ h(a) + h(3^{m-m_1} - 1) \\ &= \frac{1}{3}\log 23 + h(3^{m-m_1} - 1) \ <\ \log(3^{m-m_1+2}) \\ &= (m - m_1 + 2)\log 3 \ <\ 8.00 \times 10^{13}(1 + \log n) \quad \text{by (E.11)}. \end{aligned}$$

So, we can take $A_1 := 2.40 \times 10^{13}(1 + \log n)$. Further, as in the previous applications, we take $A_2 := \log\alpha$ and $A_3 := 3\log 3$. Finally, since $\max\{1, n, m_1\} \leq n$, we can take $B := n$. Then, we get

$$\log|\Lambda_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1 + \log 3)(1 + \log n)(2.40 \times 10^{13}(1 + \log n))(\log\alpha)(3\log 3).$$

Thus,

$$\log|\Gamma_2| > -1.91 \times 10^{26}(1 + \log n)^2.$$

Now, by comparing with (E.15), we get that

$$(n - n_1)\log\alpha < 1.92 \times 10^{26}(1 + \log n)^2. \tag{E.16}$$

Therefore, in both Case E.4.1 and Case E.4.2, we have

$$\begin{aligned} \min\{(n - n_1)\log\alpha, (m - m_1)\log 3\} &< 7.98 \times 10^{12}(1 + \log n), \\ \max\{(n - n_1)\log\alpha, (m - m_1)\log 3\} &< 1.92 \times 10^{26}(1 + \log n)^2. \end{aligned} \tag{E.17}$$

Finally, we rewrite the equation (E.7) as

$$|a\alpha^n - a\alpha^{n_1} - 3^m + 3^{m_1}| = |b\beta^{n_1} + c\gamma^{n_1}| \leq |b||\beta|^{n_1} + |c||\gamma|^{n_1} < 2|b||\beta|^{n_1} \ <\ \frac{1}{2}.$$

Dividing through by $3^m - 3^{m_1}$, we get

$$\begin{aligned} \left|\frac{a(\alpha^{n-n_1} - 1)}{3^{m-m_1} - 1}\alpha^{n_1}3^{-m_1} - 1\right| &< \frac{1}{3^m - 3^{m_1}} \leq \frac{3}{3^m} \\ &\leq 3\alpha^{-(n+n_1-4)} \ \leq\ \alpha^{4-n}, \end{aligned} \tag{E.18}$$

since $1.32 < \alpha < \alpha^{n_1}$. We again apply Theorem 1.1.8 on the left-hand side of (E.18) with the data

$$t = 3,\ \ \gamma_1 = \frac{a(\alpha^{n-n_1} - 1)}{3^{m-m_1} - 1},\ \ \gamma_2 = \alpha,\ \ \gamma_3 = 3,\ \ b_1 = 1,\ \ b_2 = n_1,\ \ b_3 = -m_1.$$

By using the algebraic properties of the logarithmic height function, we get

$$\begin{aligned} 3h(\gamma_1) &= 3h\left(\frac{a(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\right) \leq 3h\left(a(\alpha^{n-n_1}-1)\right)+3h(3^{m-m_1}-1) \\ &< \log 23+3\log 2+3(n-n_1)\log\alpha+3(m-m_1)\log 3 \\ &< 3.86\times 10^{26}(1+\log n)^2, \end{aligned}$$

where in the above inequalities, we used the argument from (E.17). Thus, we can take $A_1 := 3.86\times 10^{26}(1+\log n)$, and again as before $A_2 := \log\alpha$ and $A_3 := 3\log 3$. If we put

$$\Lambda_3 := \frac{a(\alpha^{n-n_1}-1)}{3^{m-m_1}-1}\alpha^{n_1}3^{-m_1}-1,$$

we need to show that $\Lambda_3 \neq 0$. If not, $\Lambda_3 = 0$ leads to

$$a(\alpha^n-\alpha^{n_1}) = 3^m-3^{m_1}.$$

A contradiction is reached upon a conjuagtion by the automorphism $(\alpha\beta)$ in $\mathbb{K}$ and by taking absolute values on both sides. Thus, $\Lambda_3 \neq 0$. Applying Theorem 1.1.8 gives

$$\log|\Lambda_3| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 3^2(1+\log 3)(1+\log n)(3.86\times 10^{26}(1+\log n)^2)(\log\alpha)(3\log 3),$$

a comparison with (E.18) gives

$$(n-4) < 3.08\times 10^{39}(1+\log n)^3,$$

or

$$n < 3.10\times 10^{39}(1+\log n)^3. \tag{E.19}$$

Now by applying Lemma 1.5.1 on (E.19) with the data $m := 3$, $Y := 3.10\times 10^{39}$, and $x := n$, leads to $n < 2\times 10^{46}$.

E.4.2. Reducing the bound for n

We need to reduce the above bound for n and to do so we make use of Lemma 1.2.2 several times. To begin, we return to (E.10) and put

$$\Gamma := n\log\alpha - m\log 3 + \log a.$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geq 20$. We go back to the inequalities for Λ, Λ_1, and Λ_2, Since we assume that $\min\{n-n_1, m-m_1\} \geq 20$ we get $|e^\Gamma - 1| = |\Lambda| < \frac{1}{4}$. Hence, $|\Lambda| < \frac{1}{2}$ and since the inequality $|y| < 2|e^y-1|$ holds for all $y \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get

$$0 < |\Gamma| < 2\max\{\alpha^{n_1-n+5}, 3^{m_1-m+1}\} \leq \max\{\alpha^{n_1-n+6}, 3^{m_1-m+2}\}.$$

Assume that $\Gamma > 0$. We then have the inequality

$$\begin{aligned} n\left(\frac{\log\alpha}{\log 3}\right) - m + \frac{\log a}{\log 3} &< \max\left\{\frac{\alpha^6}{(\log 3)\alpha^{n-n_1}}, \frac{9}{(\log 3)3^{m-m_1}}\right\}. \\ &< \max\{36\cdot\alpha^{-(n-n_1)}, 9\cdot 3^{-(m-m_1)}\}. \end{aligned}$$

We apply Lemma 1.2.2 with the data

$$\tau := \frac{\log\alpha}{\log 3}, \quad \mu := \frac{\log a}{\log 3}, \quad (A,B) := (36,\alpha) \quad \text{or } (9,3).$$

Let $\tau = [a_0; a_1, a_2, \ldots] = [0; 3, 1, 9, 1, 2, 1, 4, 1, 2, 2, 1, 1, 3, 1, 2, 1, 20, 1, 1, 1, 3, 11, 1, \ldots]$ be the continued fraction of τ. We choose $M := 2\times 10^{46}$ which is the upper bound on n. By *Mathematica*, we find out that the convergent

$$\frac{p}{q} = \frac{p_{89}}{q_{89}} = \frac{3123049185137266854491675319812527194766363593581}{12201370578769620000479260876419428374896683408344}$$

is such that $q = q_{89} > 6M$. Furthermore, it yields $\varepsilon > 0.436533$, and therefore either

$$n - n_1 \le \frac{\log(36q/\varepsilon)}{\log\alpha} < 417, \quad \text{or} \quad m - m_1 \le \frac{\log(9q/\varepsilon)}{\log 3} < 105. \tag{E.20}$$

For the case when $\min\{n-n_1, m-m_1\} < 20$, we have that since $\min\{n-n_1, m-m_1\} < 20 < 105$ by (E.20), then (E.20) always holds in both cases.

In the case $\Gamma < 0$, we consider the inequality

$$\begin{aligned} m\left(\frac{\log 3}{\log\alpha}\right) - n + \frac{\log(1/a)}{\log\alpha} &< \max\left\{\frac{\alpha^6}{\log\alpha}\alpha^{-(n-n_1)},\ \frac{9}{\log\alpha}\cdot 3^{-(m-m_1)}\right\} \\ &< \max\{64\alpha^{-(n-n_1)},\ 15\cdot 3^{-(m-m_1)}\}. \end{aligned}$$

We then apply Lemma 1.2.2 with the data

$$\tau := \frac{\log 3}{\log\alpha}, \ \mu := \frac{\log(1/a)}{\log\alpha}, \ (A,B) := (64,\alpha), \quad \text{or} \ (15,3).$$

Let $\tau = [a_0; a_1, a_2, \ldots] = [3; 4, 4, 1, 1, 4, 4, 9, 11, 2, 7, 4, 2, 4, 2, 1, 1, 1, 1, 2, 1, 1, 16, 1, \ldots]$ be the continued fraction of τ. Again, we choose $M = 2\times 10^{46}$, and in this case the convergent $p/q = p_{90}/q_{90}$ is such that $q = q_{90} > 6M$. Further, this yields $\varepsilon > 0.432863$, and therefore either

$$n - n_1 \le \frac{\log(64q/\varepsilon)}{\log\alpha} < 417 \quad \text{or} \quad m - m_1 \le \frac{\log(15q/\varepsilon)}{\log 3} < 105. \tag{E.21}$$

For the case when $\min\{n-n_1, m-m_1\} < 20$, we have that since $\min\{n-n_1, m-m_1\} < 20 < 105$ by (E.21), then (E.21) always holds in both cases.

The bounds in (E.21) agree with the bounds obtained in the case $\Gamma > 0$. As a conclusion, we have that either $n - n_1 \leq 417$ or $m - m_1 \leq 105$ whenever $\Gamma \neq 0$.

Now, we distinguish between the cases $n - n_1 \leq 417$ and $m - m_1 \leq 105$. First, we assume that $n - n_1 \leq 417$. In this case we consider the inequality for Λ_1, (E.12) and also assume that $m - m_1 \geq 20$. We put

$$\Gamma_1 := n_1 \log \alpha - m \log 3 + \log\left(a(\alpha^{n-n_1} - 1)\right).$$

Then inequality (E.12) implies that

$$|\Gamma_1| < \frac{6}{3^{m-m_1}}.$$

If we substitute for Γ_1 in the above inequality, we then get

$$0 < \left| n_1 \left(\frac{\log \alpha}{\log 3} \right) - m + \frac{\log(a(\alpha^{n-n_1} - 1))}{\log 3} \right| < \frac{6}{(\log 3) 3^{m-m_1}} < \frac{6}{3^{m-m_1}}.$$

Again we apply Lemma 1.2.2 with the same τ as in the case $\Gamma > 0$. We use the 89-th convergent $p/q = p_{89}/q_{89}$ of τ as before. But in this case we choose $(A, B) := (9, 3)$ and use

$$\mu_\ell := \frac{\log(a(\alpha^\ell - 1))}{\log 3},$$

instead of μ for each possible value of $\ell := n - n_1 \in [1, 2, \ldots, 417]$. For all values of ℓ, we get $\varepsilon > 0.00287989$. Hence by Lemma 1.2.2, we get

$$m - m_1 \ < \ \frac{\log(9q/\varepsilon)}{\log 3} \ < \ 110. \tag{E.22}$$

For the case $m - m_1 < 20$, we have that since $m - m_1 < 20 < 110$ by (E.22), then (E.22) always holds in both cases. Thus, $n - n_1 \leq 417$ implies that $m - m_1 \leq 110$.

Now, let us turn to the case $m - m_1 \leq 105$ and we consider the inequlity for Λ_2, (E.15). We put

$$\Gamma_2 := n \log \alpha - m_1 \log 3 + \log(a(3^{m-m_1} - 1)),$$

and we also assume that $n - n_1 \geq 20$. We then have

$$|\Gamma_2| < \frac{2\alpha^6}{\alpha^{n-n_1}}.$$

We substitute for Γ_2 in the above inequality and divide through by $\log 3$. Then we get

$$0 \ < \ \left| n \left(\frac{\log \alpha}{\log 3} \right) - m_1 + \frac{\log(a/(3^{m-m_1} - 1))}{\log 3} \right| \ < \ \frac{3\alpha^6}{(\log 3)\alpha^{n-n_1}} \ < \ \frac{106}{\alpha^{n-n_1}}.$$

We apply again Lemma 1.2.2 with the same τ, q, M, $(A,B) := (106,\alpha)$ and

$$\mu_\ell := \frac{\log(a/(3^\ell - 1))}{\log 3} \quad \text{for } \ell := m - m_1 = 1,2,\ldots,105.$$

We get $\varepsilon > 0.00384254$, therefore

$$n - n_1 < \frac{\log(106q/\varepsilon)}{\log\alpha} < 442. \tag{E.23}$$

For the case $n - n_1 < 20$, we have that since $n - n_1 < 20 < 442$ by (E.23), then (E.23) holds in both cases. To conclude, we first get that either $n - n_1 \leq 417$ or $m - m_1 \leq 105$. If $n - n_1 \leq 417$, then $m - m_1 \leq 110$, and if $m - m_1 \leq 105$ then $n - n_1 \leq 442$. Thus, we conclude that we always have $n - n_1 \leq 442$ and $m - m_1 \leq 110$.

Finally we go to the inequality of Λ_3, (E.18). We put

$$\Gamma_3 := n_1 \log\alpha - m_1 \log 3 + \log\left(\frac{a(\alpha^{n-n_1} - 1)}{3^{m-m_1} - 1}\right).$$

Since $n > 500$, the inequality (E.18) implies that

$$|\Gamma_3| < \frac{3}{\alpha^{n-4}} = \frac{3\alpha^6}{\alpha^n}.$$

We substitute for Γ_3 in the above inequality and divide through by $\log 3$, then

$$0 < \left| n_1 \left(\frac{\log\alpha}{\log 3}\right) - m_1 + \frac{\log(a(\alpha^k - 1)/(3^\ell - 1)}{\log 3} \right| < \frac{3\alpha^6}{(\log 3)\alpha^n} < \frac{116}{\alpha^n},$$

where $(k,\ell) := (n - n_1, m - m_1)$. We again apply Lemma 1.2.2 with the same τ, q, M, $(A,B) := (116,\alpha)$ and

$$\mu_{k,\ell} := \frac{\log(a(\alpha^k - 1)/(3^\ell - 1)}{\log 3} \quad \text{for} \quad 1 \leq k \leq 442, \ \ 1 \leq \ell \leq 110.$$

For these cases, we get $\varepsilon > 0.0000160572$, so we obtain

$$n \leq \frac{\log(116q/\varepsilon)}{\log\alpha} < 457.$$

Hence, $n \leq 457$. However, this contradicts our working assumption that $n > 500$. This completes the proof of Theorem E.2.1.

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