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Random walks, frogs, and the cost of asymmetry

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AFFIDAVIT

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Chapter 1

Introduction

Since March 2020 life in Europe changed drastically. Countries decided one after the other to close their borders, to ask for social distancing and finally to impose a lockdown. This change of life was all due to the global spread of a virus called COVID-19. The virus was newly discovered in Wuhan (China) in December 2019. Quickly, it spread over whole Europe and North America and by now causes thousands of deaths all over the world. While writing this chapter the number of infected people is sharply decreasing in Austria. Fortunately, it looks like the political actions taken by the Austrian government were effective, but in other parts of the world, the virus is still spreading rapidly. All these political decisions would not have been possible without the use of mathematical models. Mainly, the effective reproduction rate r_e was used to measure the effictiveness of the actions taken, and much discussed in popular media formats. The reproduction rate is defined as the expected number of individuals infected by one infected individual. It may vary during the development of the pandemic due to interventions, mutations of the virus, the immunization of the population, seasonality and many other factors. The aim set by politicians was always to achieve a reproduction rate below one. One stochastic process, which is heavily used in this thesis bears a likeness to a basic version of simulating the spread of a virus, while being too simplistic as a proper model for a real-world virus. It is called the Galton–Watson process. Here, the starting point is one individual; this individual gives birth to a random number of further individuals with a certain probability. These individuals again give birth to a random number of new individuals independently of each other and the generation before. Moreover, every individual bears a certain random number of further indivudals with the same probability as the first ever existing indivual did. One realization of the Galton–Watson process simulates one potential development of a population. The process continues from generation to generation until there is either no new individual in the next generation (i.e. the process dies out), or it repeats itself infinitely often. The Galton–Watson process was established by the British scientists Francis Galton and Henry William Watson in 1873 studying the extinction of Aristocratic surnames. They invented a model about how many male children an Aristocrat has. Together they published a paper "On the probability of the extinction of families" see [66]. There, they prove the basic result about the extinction of the process described above: if one individual always gives birth to at least one other individual, the process will survive in any case. Therefore we assume, that with a certain probability, one individual bears zero other individuals. Then, the process will die out almost surely, i.e. there are no new individuals in the next generation if the expected number of newly born individuals is at most one. On the other side, if this expected number is greater than one, the process will die out only with a probability strictly less than one.

The Galton–Watson process itself is the basic model for so-called branching processes. It can be modified, for example, by allowing immigration in some generations or creating dependence of some kind in between individuals. But we will continue with the classical model, which will be an essential part of the thesis. A realization of the Galton–Watson process may be visualized by its family tree, see Figure 1.1. The tree has one root o, which

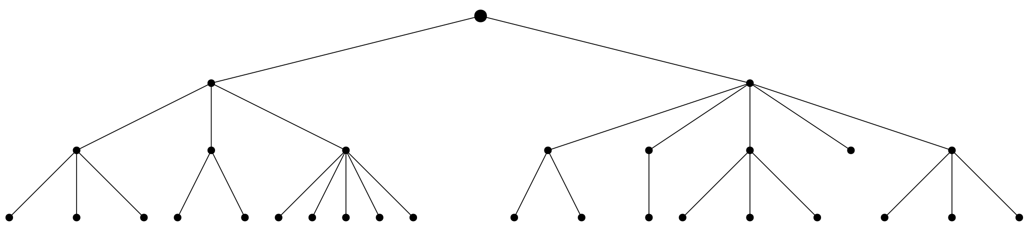

Figure 1.1: The first four generations of a realization of a Galton–Watson process, where one individual gives birth to between 0 and 5 other individuals.

represents the first existing individual. This individual gives birth to k other individuals, visualized by k vertices connected to the root. These individuals again bear children resulting in vertices, connected to their corresponding vertex. We speak of mother vertices – the vertices giving birth in the generation before – and their children. These trees might be finite or infinite, according to the distribution of how many new individuals will appear. Although they are randomly generated, they have a sort of "symmetry": looking from one vertex into the subtree starting at this vertex, the subtree looks in distribution the same as the whole tree. It is a consequence of the fact that each individual gives birth independently of the others to a number of individuals, which is always distributed the same. We call the family trees of a Galton–Watson process Galton–Watson trees. Let us think about these trees now as a graph. We have mother vertices connected to their children. The number of children of each vertex is equally distributed according to a given distribution. Let v be a vertex other than the root with k children. Then, the vertex itself has degree $k + 1$, as it is also connected to its mother vertex. The root has no mother vertex. Therefore, it is only connected to its children. Such a Galton–Watson tree is one example for the class of graphs, called trees. These are graphs consisting of vertices and edges, which are connected and do not contain cycles. A cycle would imply that we can reach a vertex from our starting point over two different paths. Not having cycles yields the important property, that we have unique paths connecting two vertices. Another example of trees are homogeneous trees. These are infinite trees, where every vertex has the same degree, see Figure 1.2. In contrast to Galton–Watson trees, the tree in Figure 1.2 does not have a dedicated root, which has no mother vertex. Nevertheless, we can choose a root here as well, which would have one child more than all other vertices.

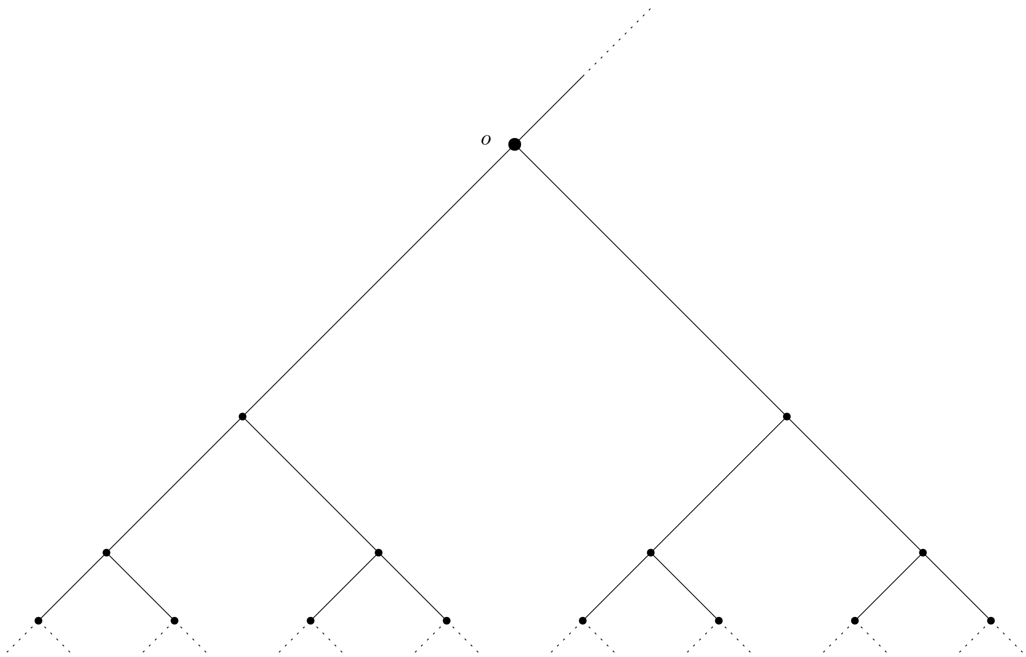

Figure 1.2: The homogeneous tree with degree 3.

On these trees, we might imagine a particle wandering around. The vertices are junctions and the edges are roads connecting the junctions. At every junction, the particle chooses randomly one of the roads with the same probability. If a vertex has degree k, the particle wanders to one of its neighboring vertices with probability $1/k$. Observing this particle, we can track its path, the areas it is visiting or the speed, with which it is walking away from its starting point. Such a particle randomly wandering around on the tree is another random process, called "Simple Random Walk". The statistician Karl Pearson first mentioned the name random walk in 1905, see [51]. Such random walks can be observed on many different graphs. The most common ones are the infinite ray or the higher dimensional integer lattices. The starting point of the intense study of random walks was set by the Hungarian mathematician George Pólya in 1919, see [55], studying whether a random walk will return to its starting point again almost surely. This property is called recurrence. If the random walk returns to its starting point with probability smaller than one, then we call the random walk transient. Another equivalent definition of recurrence is that the random walk visits its starting point infinitely often almost surely. Equivalently, it is called transient, if it visits its starting point only finitely many times, almost surely. In his famous theorem Pólya proved that a simple random walk is recurrent if and only if the dimension of the lattice is at most 2. This means that a random walk will reach its starting point on the infinite ray and the two-dimensional grid again almost surely. How is a random walk behaving on trees concerning recurrence and transience?

A simple random walk is always transient on any homogenous tree with degree greater than three, see [68]. On finite trees, the simple random walk is always recurrent. Conditioning a Galton–Watson tree to be infinite and assuming that a vertex does not have only one child with probability one, the simple random walk is always transient as well. There are many modifications to the simple random walk for achieving recurrence on trees. For example, one can put a certain drift towards the root of a tree. Then, choosing the drift properly according to the degree of the tree, we can assure that a random walk starting at the root will visit the root almost surely, again.

Another option to achieve that the starting point of a random walk will be revisited again is putting more particles than one onto a tree. From now on, we assume that the first particle always starts at the root. We imagine that once a random walk particle started walking to a neighboring vertex, the particle will split into a random number of new particles. These particles will choose their next step according to a simple random walk independently of each other. Reaching the next vertex, they will split independently of each other into a random number of particles while each of these random numbers is equally distributed. This continues in each step. The number of particles walking around on the tree is increasing with each time step. Therefore, the probability of the root being reached is increasing, as well. Such a process is called recurrent if the root is visited infinitely often by some particles almost surely. The branching behaviour is exactly the same as that of a Galton–Watson process. The process just described is one example of a branching Markov chain. The underlying graph, to which the process is adapted is a tree, the branching evolves according to a Galton–Watson process, and the particles move according to a simple random walk. For this specific branching Markov chain, it is well known that the process is transient if and only if the expected reproduction number is less than or equal to the reciprocal of the spectral radius, see [18]. The spectral radius, one might say, is a measure of how branched a tree is. The branching Markov chain can vary in all three aspects of its definition: the underlying graph, how the particles move along the graph and the branching behaviour.

We could also increase the number of particles walking on the tree differently than by creating a branching Markov chain. Let us begin by putting on each vertex of the tree a random and equally distributed number of sleeping particles. Then, the random walk starts at the root and chooses randomly one of the neighboring vertices. As soon as the random walk steps on that chosen vertex it will wake up the sleeping particles and these particles will start moving around independently of each other according to a simple random walk and wake up particles by itself. Telcs and Wormald firstly studied this process in [63] under the name "Egg model". Later Rick Durrett established the name "frog model" and refers to the particles as frogs. We notice the difference to the branching Markov chain: if a vertex is visited for the first time, we can choose the distribution of the frogs and the branching of the branching Markov chain in a way, such that both branch into the same number of particles. But if a vertex is revisited, then the branching Markov chain will continue to produce new particles, while the frog model will not wake up new frogs. This ends up in more particles moving around in the branching Markov chain than frogs in the frog model. Moreover, we notice a certain dependence between the frogs, as it is important to know whether a vertex was already visited or not. The recurrence and transience of the frog model on trees have been studied only very recently. For the homogeneous tree of degree three, it was shown that one sleeping frog per vertex is enough to make the model recurrent. For higher degrees, we can always choose a distribution of the number of sleeping frogs, such that the process is recurrent or transient. For the frog model on Galton–Watson trees, Michelen and Rosenberg have shown very recently that there are a recurrence and transience phase on Galton–Watson trees where every vertex has at least two successors.

In the second chapter of this thesis, we fill this gap: we prove that there is a transient phase for the frog model on Galton–Watson trees, also allowing vertices with one or no child. The main tool for proving this is the comparison of the frog model to the branching Markov chain. As mentioned before, in a proper branching Markov chain, more particles are walking around than frogs in the frog model. Hence, using a proper coupling of the two processes, the transience of the frog model follows from the transience of the coupled branching Markov chain. In all the processes of this chapter, the reproduction rate plays an important role for our results; on one side the reproduction rate of the underlying Galton–Watson tree, on the other side the reproduction rate of the particles in the branching Markov chain. We observe in which cases we can guarantee a threshold for the reproduction rate of the particles in the branching Markov chain, such that the process is transient and can be coupled with the frog model, implying a transient phase of the frog model. This chapter is based on joint work with Sebastian Müller.

For the third chapter of the thesis we go from the frog model back to a single simple random walk without any branching, but we modify it in a different way. Here, we let the random walk face a random risk of dying at every vertex of the tree. Only after surviving this risk, the random walk chooses one of its neighbors with equal probability. Facing the risk of dying at each vertex, we might ask, if the random walk will ever reach vertices very far away. Therefore, we observe the so-called Lyapunov exponents which describe the decay of the probability of reaching vertices far away, while prolonging the distance. There are two different ways of treating the random risk. On one side, we can freeze the potential risk of dying at each vertex. This is the so-called quenched setting. On the other side, we can average the risk, which is called annealed. The chapter states a variational formula giving the annealed Lyapunov exponent as a minimization problem involving quenched exponents and entropy. The entropy can be understood more or less as the distance between two probability measures. The first part of this chapter about the variational formula on the integers is the discrete analogue of a result by Alain-Sol Sznitman. The second part of this chapter gives its generalization to trees. This chapter is an extension to my Master thesis under the supervision of Martin Zerner.

In the last part of the thesis we drop having random walks on graphs and look at the graphs themselves. More precisely, we consider only regular graphs of degree three and with the property that all vertices look the same in a special way. These graphs are vertex-transitive and cubic. We have seen already one example for such graphs in Figure 1.2. Looking the same can be imagined like this: we can grab any two vertices of a graph, a and c in Figure 1.3. Then we exchange them including all connected edges and vertices, but the look of the graph does not change. The obtained graph is still a tetrahedron, see Figure 1.3. Now, we would like to modify the graph such that we cannot find any two

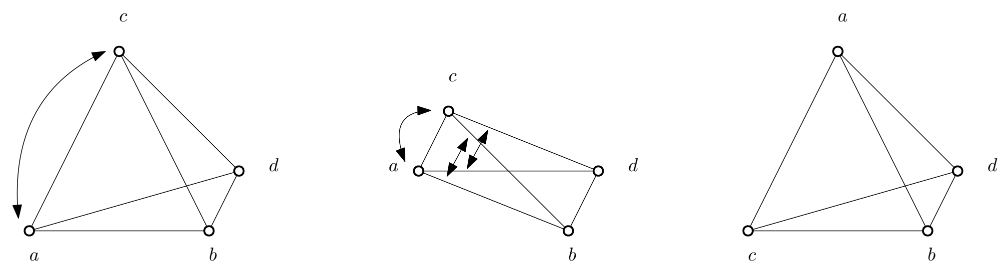

Figure 1.3: Exchanging vertices a and c in a tetrahedron.

vertices that can be exchanged without changing the look of the graph. We will do this by coloring the vertices in different colors and postulate that only vertices with the same color can be exchanged. We will need two or more colors, see Figure 1.4: having one black vertex, we can still exchange the other three vertices; also another one or two black vertices would not fix all vertices; we need at least one more color. Now, we could still exchange the two white vertices. Adding another red vertex, we could exchange the two red vertices. Therefore, we need four colors. Using this coloring, we cannot exchange any two vertices

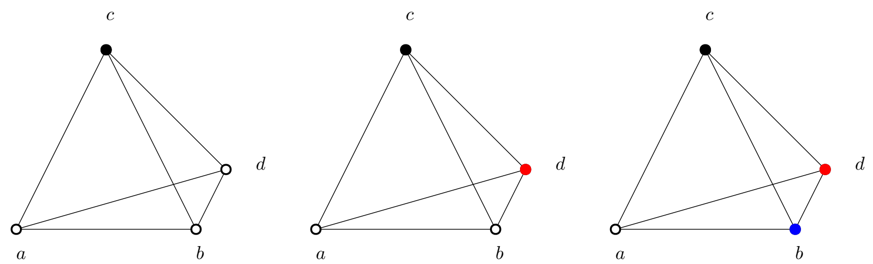

Figure 1.4: The coloring of a tetraeder, such that every vertex is distinguishable.

without changing the coloring of the graph. All vertices are distinguishable. We call such a graph 4-distinguishable, as four colors suffice to make all vertices distinguishable. We are mainly interested in 2-distinguishable graphs. For example, coloring a hexagon as shown in Figure 1.5, we notice that we cannot exchange any two vertices: the black vertices have either one or no black neighbor or a different number of white neighbors; the same holds for the white vertices with respect to black vertices. Looking at the graphs, where we only need two colors to make all vertices distinguishable, we call the set of black vertices an asymmetrizing set. Such a set can be infinite or finite. If the number of black vertices is finite, we refer to it as asymmetrizing cost. If it is infinite, we look at the ratio of the size of the asymmetrizing set to the size of the graph and take its infimum. This is called asymmetrizing density. We will show for vertex transitive cubic graphs with an

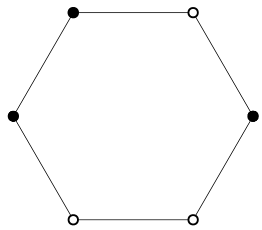

Figure 1.5: The hexagon is a 2-distinguishable graph.

asymmetrizing set, that the cost is less than five or the density is either zero or strictly positive between zero and one, unless the graph is especially highly symmetrical and finite. Moreover, we will present the first known class of graphs with strictly positive density, namely the SPX(2,n) graphs, see Figure 1.6. This part is based on joint work with Wilfried Imrich and Thomas Lachmann. The project started in a course for doctoral students by Wilfried Imrich.

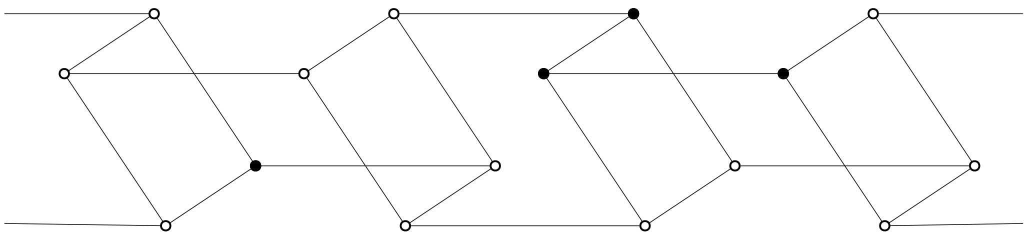

Figure 1.6: The SPX(2,1) with an asymmetrizing coloring. In any other quadrangle, we need to color one vertex black.

The purpose of this introduction has been to give an overview of the contents of the thesis for a large audience. For this reason and better readability, many possible citations were skipped but can be found to a wide extent in the introductory part of each chapter. The mathematical definitions of the introduced concepts can be found in the preliminaries and the corresponding chapters.

We continue with basic concepts before closing the introductory part with an overview of the upcoming chapters.

1.1 Preliminaries

We start by defining a Galton–Watson process. The definitions and results are taken from [37]. Let X be a nonnegative, integer valued random variable and let $p_k := \mathbb{P}[X = k]$.

Moreover, let $(X_i^{(n)})_{i,n\in\mathbb{N}}$ be independently and identically distributed random variables with the same distribution as X.

Definition 1.1.1 (Galton–Watson process) The sequence of random variables $(Z_n)_{n\geq 0}$ defined by $Z_0 = 1$ and

$$Z_n := \sum_{i=1}^{Z_{n-1}} X_i^{(n)}$$

for $n \in \mathbb{N}$ is called a Galton–Watson process with offspring distribution $(p_k)_{k\in\mathbb{N}}$.

Recalling the description of the process from the introduction, Z_n represents the number of individuals in generation n and $X_i^{(n)}$ is the number of children of the ith individual of generation n. The mean offspring of each individual is

$$m := \sum_{k\geq 0} k\, p_k\,.$$

We define by $q := \lim_{n\to\infty} \mathbb{P}[Z_n = 0]$ the probability that the Galton–Watson process will die out. Then, we have the following important result:

Theorem 1.1.2 *We assume that $p_1 \neq 1$. Then, the Galton–Watson process $(Z_n)_{n\geq 0}$ will die out almost surely, i.e. $q = 1$, if and only if $m \leq 1$.*

Galton–Watson trees are the family trees of a Galton–Watson process. The individuals are represented by vertices. Each vertex is connected by an edge to its ancestor. According to Theorem 1.1.2, the tree is finite almost surely if and only if $m \leq 1$. Trees are special types of graphs.

Definition 1.1.3 (Graph) A graph is a pair $G := (V, E)$, where V is the set of vertices and E is the set of edges. An edge is any set $\{v, w\}$ of two vertices $v, w \in V$.

We will only consider undirected, finite or countable graphs. Most of the time, we denote the edges by e. The degree of a vertex $deg(v)$, $v \in V$, is defined as the number of neighbors of v. The neighbors of v are the vertices $w \in V$, where $\{v, w\}$ is an edge in E. A graph is called regular, if every vertex has the same degree.

Definition 1.1.4 (Path) A path of length n in a graph is a sequence of edges $e_1, \ldots, e_n$, with a corresponding sequence of vertices $v_1, \ldots, v_{n+1}$, such that $e_i = \{v_i, v_{i+1}\}$ for all $i = 1, \ldots, n$, and all vertices v_i are distinct.

We want to mention that we will speak of paths later in the context of Markov chains, as well. There, following the standard terminology in probability theory, a path is defined as above but skipping the assumption of distinct vertices. In the context of random walks the path is the trajectory of a realization of the simple random walk. As we are only dealing with Markov chains on trees in this thesis, the path defined as above will be called shortest path. This is the unique shortest path at the same time. For now, we stick to the definition as in Definition 1.1.4.

The distance $d(v, w)$ between two vertices $v, w \in V$ is then defined as the length of the shortest path between the two vertices. With the help of paths, we give the precise definition of trees.

Definition 1.1.5 (Tree) A tree T is a graph, where every two vertices are connected by a unique path.

We notice that the unique path is also the unique shortest path. This is equivalent to the fact that a tree does not contain any cycles. We can also choose one vertex of a tree and label it as the root of the tree. This defines in relation to the root for every vertex one neighbor closer to the root and none, one or multiple neighbors further away from the root. We call these vertices the predecessor or ancestor and the successors. An infinite tree T is called homogeneous of degree $k+1$ if all vertices $v \in T$ have degree $deg(v) = k+1$. We denote it by T_{k+1}. If there is a dedicated root in the tree, then we can determine the successors of each vertex. In T_{k+1} every vertex apart from the root has k successors.

The property of having a unique path between any pair of distinct vertices is essential in studying random walks and the frog model on trees and will be heavily used in the corresponding chapters.

A random walk is a stochastic process describing the movement of a particle in a defined state space. These can be a graph, the integers or also the real line. In this thesis we will only consider random walks in discrete time and with countable state space. Random walks are a special case of time homogeneous denumerable Markov chains. Especially, the so-called Markov property will be important in Chapter 2 and 3. We will now give a short introduction to Markov chains and some of their properties in the form we will need later on. All definitions and properties with their proofs can be found in [69].

Definition 1.1.6 (Time homogeneous denumerable Markov chain) Let X be a denumerable set, the state space, and $Z := (Z_n)_{n\geq 0}$ a sequence of X-valued random variables. We call Z a denumerable time homogeneous Markov chain if it has the following two properties:

1. Markov property: For all $i_0, \ldots, i_{n+1} \in X$ which satisfy $\mathbb{P}[Z_0 = i_0, \ldots, Z_n = i_n] > 0$ it holds
$$\mathbb{P}[Z_{n+1} = i_{n+1} | Z_n = i_n, \ldots, Z_0 = i_0] = \mathbb{P}[Z_{n+1} = i_{n+1} | Z_n = i_n] \,.$$

2. Time homogeneity: For all $n, m \in \mathbb{N}$ and $i, j \in X$ it holds
$$\mathbb{P}[Z_{n+1} = j | Z_n = i] = \mathbb{P}[Z_{m+1} = j | Z_m = i] \,.$$

In other words, only the current state is important for predicting the future step of a Markov chain. Using the time homogeneity we can define $p(i, j) := \mathbb{P}[Z_{n+1} = j | Z_n = i]$

independent of n, assuming that $\mathbb{P}[Z_n = i] > 0$ and we denote by $P = (p(i,j))_{i,j \in X}$ the transition matrix of the Markov chain. A denumerable time homogeneous Markov chain is also very often denoted by (X, P). The initial distribution of the Markov chain is given by the probability measure ν defined by $\nu(i) = \mathbb{P}[Z_0 = i]$. Given an initial distribution, we write $\mathbb{P}_i[Z_n = j]$ for $\mathbb{P}[Z_n = j | Z_0 = i]$. This is the probability of reaching j in n steps after starting in i, and we shorten the notation further to $p^{(n)}(i,j) := \mathbb{P}_i[Z_n = j]$.

As an example of a Markov chain we present the simple random walk on the integers and the simple random walk on trees. Actually, $\mathbb{Z}$ is isomorphic to the homogeneous tree T_2 and we give only the definition of simple random walk on trees, following [68].

Definition 1.1.7 (Simple random walk) Let T be a tree with root. The simple (symmetric) random walk on T is the Markov chain with state space T and transition probabilities

$$p(x,y) = \begin{cases} \frac{1}{deg(x)} & \text{if } x \sim y \\ 0 & \text{else} \end{cases}$$

where $x \sim y$ denotes that x and y are neighbors.

This is an example for a Markov chain adapted to a graph. The graph takes the role of the state space and the transition probabilities follow the neighbor-relation of the graph. One realization of the movement of a random walk is a trajectory $[v_0, v_1 \ldots, v_n, \ldots]$ with $v_i \in T$. Hence, we call the distribution according to the random walk moves also the path measure P, which is given by

$$\mathsf{P}_{v_0}[v_1, v_2, \ldots, v_n] = p(v_0, v_1)p(v_1, v_2) \cdot \ldots p(v_{n-1}, v_n)$$

for all finite paths. One main point of interest in studying Markov chains is the property of recurrence and transience.

Definition 1.1.8 (Recurrence, transience) Let $x \in X$ be a state of the Markov chain Z. We call x recurrent if

$$\mathbb{P}_x[\,\exists n > 0 \,:\, Z_n = x\,] = 1\,.$$

If this is not the case x is called transient.

There are many equivalent definitions of recurrence, for example, a state x is recurrent if

$$\mathbb{P}_x[\,Z_n \,\text{returns to}\, x \,\text{infinitely many times}\,] = 1\,.$$

Moreover, if every state can be reached from any state, either all states are recurrent, or all states are transient. Such Markov chains are called irreducible. All our Markov chains will be irreducible. An important tool for studying recurrence and transience is the Green function, a special generating function.

Definition 1.1.9 (Green function) Let (X, P) be a Markov chain. Then, we define its Green function by

$$G(x, y|z) := \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n$$

for $x, y \in X$ and $z \in \mathbb{C}$.

We observe that

$$G(x, y|1) = \sum_{n=0}^{\infty} p^{(n)}(x, y) = \mathbb{E}_x[\,\text{number of visits to}\, y]$$

which is the expected number of visits of the Markov chain starting in x to state y and $\mathbb{E}_x$ denotes the expected value conditioned on the Markov chain starting in x. Using the Green function we obtain another equivalent definition of recurrence. A state x is recurrent if $G(x, x|1) = \infty$, see Theorem 3.4 in [69]. The radius of convergence for the Green function is given by

$$\mathsf{r}(x, y) := \left(\limsup_n \left(p^{(n)}(x, y) \right)^{\frac{1}{n}} \right)^{-1} \geq 1\,.$$

On the other hand, $\mathsf{r}(x, y)^{-1}$ describes the exponential decay of $(p^{(n)}(x, y))_{n \geq 0}$ for n tending to infinity. For an irreducible Markov chain, this decay is independent of x and y and we can define:

Definition 1.1.10 (Spectral radius) Let (X, P) be an irreducible Markov chain. Then we call

$$\rho(P) := \limsup_n \left(p^{(n)}(x, y) \right)^{\frac{1}{n}}\ ,\ x, y \in X$$

the spectral radius of P.

In the case of the simple random walk, where the transition matrix is directly associated with the underlying tree T, we might also write $\rho(T)$. Having a spectral radius $\rho(T) < 1$ yields that $G(x, x|1)$ converges, and therefore the simple random walk is transient. The homogeneous tree T_{d+1} for $d \geq 1$ has spectral radius

$$\rho(T_{d+1}) = \frac{2\sqrt{d}}{d+1}\,.$$

Hence, the simple random walk is recurrent for $d = 1$ (then X is isomorphic to the integers) and transient for $d \geq 2$. More results about the spectral radius of different trees are presented in Section 2.6.

Combining the two first main stochastic processes – a Galton–Watson process and a Markov chain – we obtain a branching Markov chain. We follow the definitions in [18] and state their result about the transience of branching Markov chains. Let (X, P) be a

Markov chain as above. Additionally, let μ_k be the probability that the Galton– Watson process has k children satisfying

$$\sum_{k\geq 1} \mu_k = 1 \text{ and } m := \sum_{k\geq 1} k\mu_k < \infty\,,$$

and let $\mu := (\mu_k)_{k\geq 1}$.

Definition 1.1.11 (Branching Markov chain) A branching Markov chain with constant mean offspring is the triple $\mathrm{BMC}(X, P, \mu)$ defined as the particle system evolving as follows: at time 0 we start with one particle in the initial state $x \in X$. Then, the particle moves according to P to another state of X, generates k particles with probability μ_k and dies. These k particles move, independently of each other and the history of the process, according to P to a next state of X, and branch independently of each other according to μ and die again. In this way the process continues. In each time step the particles move and branch independently of each other and independently of the history of the process.

Since we assume that the number of offsprings is bigger than 1, the number of particles is increasing over time. Constant mean offspring means that in each state the branching of the particles follows the same rule. This could be modified, such that the branching of the particles is different in each state of X, but we will only need constant mean offspring. If the Markov chain (X, P) is recurrent by itself, we know that also infinitely many particles return to each state in the BMC. If the Markov chain is transient, then the probability that a single particle returns to its origin is smaller than one. But having more than one particle performing the Markov chain, we might ask, in which case we can guarantee that infinitely many particles reach the starting point of the initial particle. Therefore, let $M(n)$ be the total number of particles at time n and let $x_1(n), \ldots, x_{M(n)}(n)$ denote the position of the particles at time n. Then, we call $\mathrm{BMC}(X, P, \mu)$ transient if

$$\mathbb{P}\left[\sum_{n=1}^{\infty}\sum_{i=1}^{M(n)} \mathbf{1}_{\{x_i(n)=x\}} = \infty\right] = 0$$

for all $x \in X$. Otherwise, we call the branching Markov chain recurrent. The criterion from [18], which states in which case a branching Markov chain will be transient, will be one of our key tools.

Theorem 1.1.12 *Let* $\mathrm{BMC}(X, P, \mu)$ *be a branching Markov chain with an irreducible Markov chain* (X, P) *and constant mean offspring* $m > 1$*. Then, the* $\mathrm{BMC}(X, P, \mu)$ *is transient if and only if*

$$m \leq \frac{1}{\rho(P)}\,.$$

Let us consider now the simple random walk on a homogeneous tree T_{d+1} as the Markov chain in the branching Markov chain. We want to determine how we have to choose the distribution μ of the branching mechanism such that $\mathrm{BMC}(X, P, \mu)$ is recurrent. We recall that the spectral radius of a homogeneous tree is $\rho(T_{d+1}) = (2\sqrt{d})\,(d+1)^{-1}$ and that the random walk is transient if and only if $d \geq 2$. Then the BMC is recurrent if and only if the mean offspring satisfies

$$m > \frac{d+1}{2\sqrt{d}}\,.$$

In the branching Markov chain we have the useful property that all particles branch and move independently of each other. Now, we modify the model in the following way. At every state of the Markov chain only once a particle generates new particles according to a given distribution and the particle itself survives. Future visits of particles to an already visited state do not generate new particles. This model is called frog model.

Definition 1.1.13 (Frog model) Let X be a graph with a dedicated origin o, P the path measure of the particles moving on X and $\eta := (\eta_x)_{x\in X}$ a sequence of identically distributed, non-negative and integer-valued random variables. Then, we call the random interacting particle system evolving in the following way the frog model $\mathrm{FM}(X, \mathsf{P}, \eta)$: there is one awake particle at time 0 at the origin o. On all the other vertices, there are η_x sleeping particles placed. Now, the awake particle moves to another vertex according to P and wakes up the sleeping particles at this vertex. These particles start moving independently of each other according to P. As soon as sleeping particles are encountered, they wake up and start moving independently according to P.

In a zoomorphic way, it is common to refer to the particles in this model as frogs. They jump on the graph and wake up sleeping frogs. The frog model is called transient if, almost surely with respect to the frog distribution and the path measure, only a finite number of particles return to the origin o.

Stepping away from probability theory but sticking to observing graphs, we will give some basic definitions related to graphs, which will be helpful for the last chapter of the thesis. We already know the definition of a graph G and of its degree, see Definition 1.1.3. If every vertex has the same degree, we call the graph regular. If the graph is 3-regular, we call it cubic. A graph is connected, if we find for any two vertices $v, w \in G$ a path connecting v, w. Now, we define when two graphs are isomorphic and give the definition of graph automorphisms. Then, we introduce vertex-transitivity and the coloring of graphs.

Definition 1.1.14 (Isomorphic, automorphism) Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. The two graphs are isomorphic if there exists a bijection $\gamma : V_1 \mapsto V_2$ such that for any two vertices $v_1, v_2 \in V_1$ it holds

$$v_1 \sim v_2 \iff \gamma(v_1) \sim \gamma(v_2)\,.$$

The bijection γ is called an isomorphism. An automorphism is an isomorphism from a graph G to itself.

The set of all automorphisms of a graph G forms a group with respect to concatenation of mappings. Indeed, the identity $v \to v$ for all $v \in V$ represents the neutral element. The concatenation of two automorphisms is again an automorphism and we can find for each automorphism an inverse automorphism, such that their concatenation is the identity. We call the set of all automorphism $Aut(G)$.

Definition 1.1.15 (Vertex-transitivity) A graph G is called vertex-transitive if for any two vertices $v, w \in G$ there is an automorphism $\gamma : V \mapsto V$ such that

$$\gamma(v) = w \,.$$

We call a graph edge transitive if for any two edges $\{v, w\}, \{v', w'\} \in E$ there exists an automorphism $\gamma : V \mapsto V$, such that

$$\{\gamma(v), \gamma(w)\} = \{v', w'\} \,.$$

We are interested in the question on how many vertices we have to mark as "different" in a suitable sense, such that the only automorphism mapping "different" vertices only onto "different" vertices is the identity. We call this breaking all automorphisms. This will be done by assigning colors to the vertices of the graph. In a second step, we consider only graphs, where two colors are enough to achieve the breaking of all automorphisms, and determine how many vertices we have to color in the second color.

Definition 1.1.16 (coloring, distinguishing number) A d-coloring is a map $c : V \mapsto \{1, 2, \ldots, d\}$. Each number refers to a color. Then, we define the distinguishing number $D(G)$ as the smallest d to which there exists a d-coloring such that the only automorphism mapping vertices onto vertices of the same color is the identity. In this case we say that the identity is the only color-preserving automorphism.

We call the distinguishing number also the asymmetrizing coloring number. If the distinguishing number of a graph G is 2, then there exists a 2-coloring such that the only color-preserving automorphism is the identity. We refer to the colors as black and white. The two sets of vertices colored in different colors are called asymmetrizing sets. The smallest possible size of such a set is the asymmetrizing cost of G

1.2 Overview

The second chapter of the thesis concerns the transience of the frog model on Galton–Watson trees. It is based on the joint work *On transience of frogs on Galton-Watson trees* with Sebastian Müller.

The third chapter contains the study of quenched and annealed Lyapunov exponents on trees. This is based on my paper *The relation between quenched and annealed Lyapunov exponents in random potential on trees*, which is published in *Stochastic Processes and their Applications*, see [67].

The last part deals with the cost of coloring vertex-transitive cubic graphs. It is based on the joint work *The cost of asymmetrizing vertex-transitive cubic graphs* with Wilfried Imrich and Thomas Lachmann.

Chapter 2

On transience of frogs on Galton-Watson trees

We consider a random interacting particle system, known as the *frog model*, on infinite Galton–Watson trees allowing offspring 0 and 1. The system starts with one awake particle (frog) at the root of the tree and a random number of sleeping particles at the other vertices. Awake frogs move according to simple random walk on the tree and as soon as they encounter sleeping frogs, those will wake up and move independently according to simple random walk. The frog model is called transient if there are almost surely only finitely many particles returning to the root. In this chapter we prove a 0–1-law for transience of the frog model and show the existence of a transient phase for certain classes of Galton–Watson trees.

2.1 The frog model

The frog model $\mathrm{FM}(X, \eta, \mathsf{P})$ is a random interacting particle system, consisting of three parts: a graph X with a dedicated root, a (random) configuration $(\eta(x))_{x\in X}$ of sleeping frogs on each vertex described by a sequence of independent and identically distributed nonnegative random variables with common measure η and the path measure P describing the movement of the particles – also called frogs. We denote the common mean of each $\eta(x), x \in X$, by $\bar{\eta}$ and assume throughout the entire chapter that

$$\bar{\eta} < \infty\,.$$

The model starts by definition with one awake frog at the root o of the graph X and sleeping particles according to η at the other vertices. The awake frogs move independently on the graph with respect to P. When a vertex with sleeping frogs is visited for the first time, the sleeping frogs at this vertex wake up and start to move according to P independently of the other frogs. The different frog models can vary in the underlying graph, the initial distribution of the sleeping frogs (deterministic or random) and the path measure of the awake frogs. Unless it is not specified otherwise, we assume that the frogs move according to simple random walk, write from now on SRW instead of P, and shorten the notation from $\mathrm{FM}(X, \eta, \mathsf{SRW})$ to $\mathrm{FM}(X, \eta)$. More precisely, for $v, w \in X$ we consider the transition probabilities $p(v, w) = \frac{1}{deg(v)}$ if v is a neighbour of w, and 0, otherwise.

In 1999 the frog model was originally introduced as "egg model" in [63] and later on Rick Durrett established the name "frog model". One main point of interest since its introduction was studying the recurrence and transience of the frog model. Let $\mathsf{FM} := \mathsf{FM}_X := \mathsf{SRW} \times \eta$ denote the probability measure on paths of all frogs (following the dynamics of a SRW) given by choosing an independent and identically distributed initial frog configuration according to η on the graph X. Moreover, we define the random variable

$$\nu := \#\,\text{of frogs returning to the root}\,,$$

which is the number of visits to the root in the frog model. Then, we define recurrence and transience in the following way:

Definition 2.1.1 Let X be a graph with a dedicated root o. The frog model $\mathrm{FM}(X, \eta)$ is called transient if

$$\mathsf{FM}[\nu < \infty] = 1\,,$$

that is, there are FM-almost surely only finitely many frogs returning to the root. Otherwise the frog model is called recurrent.

Studying transience and recurrence of the frog model is only interesting when the single random walk is transient. The first result concerning the question about recurrence was in the aforementioned article [63], where Telcs and Wormald showed that $\mathrm{FM}(\mathbb{Z}^d, \delta_1, \mathsf{SRW})$ is recurrent for all $d \in \mathbb{N}$. Later Gantert and Schmidt showed conditions for recurrence for the frog model with drift on the integers in [19]. This was generalized to higher dimensions and a drift in the direction of one axis by Döbler and Pfeifroth [15] and Döbler *et al.* [14].

In 2002, Alves, Machado and Popov [2] studied the frog model on trees with the modification, that the frogs can die with a certain probability p in each step. Let p_c denote the smallest p such that the frog model survives with positive probability. In [2] they are proving in which cases there exists a phase transition, that is $0 < p_c < 1$, on homogeneous trees and integer lattices. Moreover, they have proven phase transitions between transience and recurrence with respect to the survival probability. In 2005 there was the first improvement of the upper bound of p_c by Lebensztayn, Machado and Popov [40]. Recently, Lebensztayn and Utria improved the result again in [42] and proved an upper bound for p_c on biregular trees in [41]. Another modification of the frog model was considered by Deijfen, Hirscher and Lopes in [10] and by Deijfen and Rosengren in [11]. These two papers work on a two-type frog model performing lazy random walk. They show that two populations of frogs on $\mathbb{Z}_d$ can coexist under certain conditions on the path measure of the frogs. Moreover, the coexistence of the frog model does not depend on the shape of the initially activated sets and their frog configuration.

The question if $\mathrm{FM}(T_{d+1}, \delta_1, \mathsf{SRW})$ on the homogeneous tree T_{d+1} is recurrent or transient remained open for quite some time. In 2017 Hoffmann, Johnson and Junge could show in [25], that $\mathrm{FM}(T_{d+1}, \delta_1, \mathsf{SRW})$ is recurrent for $d = 2$ and transient for $d \geq 5$. This result was extended by Rosenberg [59] showing that the alternating tree $T_{3,2}$ with offspring 3 and 2 is recurrent. Studying the frog model on trees was continued by modifying the frog configuration $(\eta(x))_{x \in T_{d+1}}$ to $pois(\mu)$-distributed frogs. Hoffmann, Johnson and Junge proved in [24] the existence of a critical parameter μ_c, bounded by $Cd < \mu_c(d) < C'd \log d$ with C, C' constants, such that $\mathrm{FM}(T_{d+1}, pois(\mu), \mathsf{SRW})$ is recurrent for $\mu > \mu_c$, and transient for $\mu < \mu_c$. Johnson and Junge improved the bounds to $0.24d \leq \mu_c(d) \leq 2.28d$ for sufficiently large d in [34].

The subtlety of the question of recurrence and transience is also reflected in the result by Johnson and Rolla [35]. In fact, transience and recurrence are sensitive not just to the expectation of the frogs but to the entire distribution of the frogs. This is in contrast to closely related models like branching random walk and activated random walk.

Very recently, Michelen and Rosenberg proved in [47] the existence of a phase transition between transience and recurrence on Galton–Watson trees. This was done for trees of at least offspring two. In this chapter we want to answer an open question which appeared in [47], and extend their result. We will prove the existence of a transient phase for supercritical Galton–Watson trees with bounded offspring but also allowing offspring 0 and 1. As in the references above we assume that the initial distribution is random according to a distribution η with finite first moment. We start with showing a 0–1-law for transience.

Theorem 2.1.2 *Let* GW *be the measure of a Galton–Watson tree and* T *a realization. Then it holds*

$$\mathsf{GW}[\ \mathrm{FM}(\mathsf{T}, \eta) \textit{ is transient} \,|\, \mathsf{T} \textit{ is infinite}] \in \{0, 1\}.$$

Michelen and Rosenberg recently proved a stronger 0–1-law for recurrence and transience in [47]. We learned about their proof after writing our first version. While both proofs rely on the stationarity of the augmented Galton-Watson measure, our proof differs in the connection between the ordinary Galton–Watson measure and the augmented Galton–Watson measure. In [39] Kosygina and Zerner proved a 0–1-law for transience and recurrence of the frog model on quasi-transitive graphs.

The main result of the chapter is the existence of a transient phase while allowing offspring 0 and 1:

Theorem 2.1.3 *Let* GW *be a Galton–Watson measure defined by* $(p_i)_{i \geq 0}$*. We assume that* $d_{max} = \max\{i : p_i > 0\} < \infty$ *and set* $d_{min} := \min\{i \geq 2 : p_i > 0\}$*. Then, for any choice of* p_0 *and* p_1 *there exist some constants* $c_d = c_d(p_0, p_1)$ *and* $c_\eta = (p_0, p_1, d_{max})$ *such that for* $d_{min} \geq c_d$ *the frog model* $\mathrm{FM}(\mathsf{T}, \eta, \mathsf{SRW})$ *is transient* GW*-almost surely (conditioned on* T *to be infinite) if* $\bar{\eta} < c_\eta$*.*

We recall that $\bar{\eta}$ is the expected value of the number of sleeping frogs at each vertex. The assumption of finite maximum offspring is needed to control the possible number of attached bushes in a Galton–Watson tree allowing offspring 0. The proof of Theorem 2.1.3 gives bounds on the constants. These bounds can certainly be improved in refining the

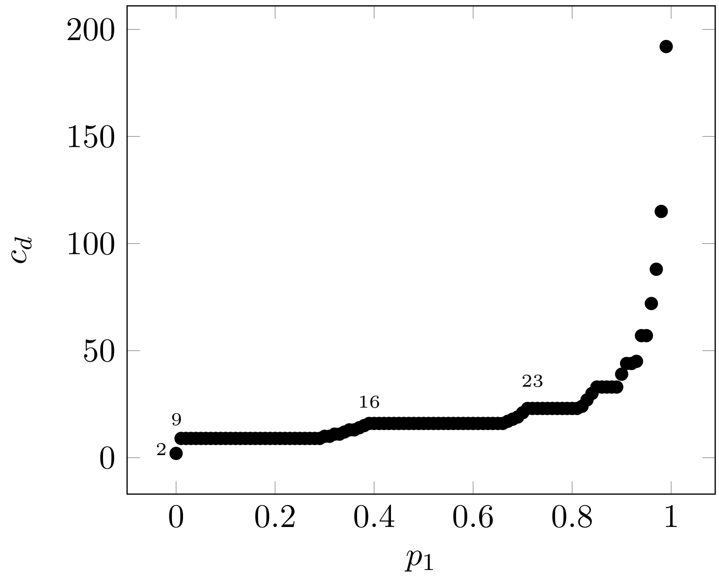

Figure 2.1: The minimal $d_{\min} = c_d$ for each p_1, such that there exists a transient phase of the frog model; with $p_0 = 0$. The mesh size for p_1 is 0.01.

involved estimates, see Figure 2.1 for some explicit values. We believe that a different approach or a new perspective is needed to prove the following conjecture.

Conjecture 2.1.4 *For every supercritical Galton–Watson measure there exists a transient regime.*

For proving Thorem 2.1.3 we compare the frog model with a branching Markov chain (BMC). In contrast to the frog model, the particles in the BMC branch at every vertex, regardless if they visited the vertex already or not. Therefore, there are more particles in the BMC than in the frog model and we can couple the two models. In this way, transience of the BMC implies transience of the frog model. The same kind of approach was already used for example in the proofs of transience in [24] and [34].

While on homogeneous trees the existence of a transient branching Markov chain is guaranteed, this is no longer true in general for Galton–Watson trees. Namely, allowing the particle to have 0 and 1 offspring creates stretches and finite bushes in the family tree. Such trees have a spectral radius equal to 1 and therefore the branching Markov chain is always recurrent on such trees, see [18]. To tackle this problem, we first modify the Galton–Watson trees and then adapt the branching Markov chain to get a dominating process. Firstly, we start with dealing with arbitrary long stretches. This turns out to be more difficult than expected, since a direct coupling of the frog model and the branching Markov chain is not possible. For this reason we compare the expected number of returns to the root of the frog model with the expected number of returns of annother, appropriate branching Markov chain. Next, we treat the case of appearing bushes and possible stretches. This part is essentially a rather straightforward generalization of the first part. The main idea is to control the bushes and the "backbone" (the tree without bushes) separately. The backbone is essentially a Galton–Watson tree with stretches and the bushes just increase the number of frogs per vertex.

The chapter is structured in the following way. In Section 2.2 we give an introduction to Galton–Watson trees and state some useful structural results. Then, we recall the definition of a branching Markov chain together with the above stated transience criterion in Section 2.3. The 0–1-law is proved in Section 2.5. The proof of Theorem 2.1.3 will be split in three parts. In Subsection 2.5.1 we treat the case of no bushes and no stretches ($p_1 = 0, p_0 = 0$), in Subsection 2.5.2 the case when there are no bushes, but stretches ($p_1 > 0, p_0 = 0$), and in Subsection 2.5.3 the case when we have bushes and possibly also stretches ($p_0 > 0$).

2.2 Galton–Watson trees

The Galton–Watson tree (GW-tree) is the family tree of a Galton–Watson process. This latter process starts with one particle at time 0 and at each discrete time step every particle generates new particles independently of the previous history and the other particles of the same generation. More formally, let Y be a non-negative integer valued random variable with $p_k := \mathbb{P}[Y = k]$ for each $k \in \mathbb{N}$ and let $m := \sum_{k\geq 0} k\, p_k$ be the mean of Y. Moreover, let $Y_i^{(n)}$, $i, n \in \mathbb{N}$, be independent and identically distributed random variables with the same distribution as Y. Then, the Galton–Watson process is defined by $Z_0 := 1$ and

$$Z_n := \sum_{i=1}^{Z_{n-1}} Y_i^{(n)}$$

for $n \geq 1$. The random variable Z_n represents the number of particles in the n-th generation. A GW-process with $p_0 > 0$ will survive with positive probability, that is $\mathbb{P}[Z_n > 0 \text{ for all } n > 0] > 0$, if and only if $m > 1$. We introduce $\mathbb{T}$ as the random variable for the family tree of the GW-process and its corresponding measure by GW. Moreover, we denote by $\mathsf{T} := \mathbb{T}(\omega)$ a fixed realization of $\mathbb{T}$. In the remaining chapter we only consider GW-trees with bounded number of offspring: There exists a $d_{max} \in \mathbb{N}$, such that $\sum_{k=0}^{d_{max}} p_k = 1$. For a more detailed introduction to GW-processes and trees we refer to Chapter 5 in [45].

In the case where $p_0 > 0$ the GW-tree contains a.s. finite bushes. We will distinguish between two types of vertices.

Definition 2.2.1 We call a vertex $v \in \mathsf{T}$ of type g if it lies on an infinite geodesic starting from the root. Otherwise we call vertex v of type b.

If a vertex of type b is the descendant of a type g vertex we call it of type $\mathsf{b_r}$ and speak of it as the root of the finite bush that consists of its descendants.

We set

$$f(r) := \mathbb{E}\left[r^Z\right] = \sum_{k\geq 0} r^k p_k$$

the generating function of the GW-process and q the smallest solution of $f(r) = r$.

Let us consider the case where $p_0 > 0$. Conditioned on nonextinction the tree $\mathbb{T}$ is distributed as a tree $\bar{\mathbb{T}}$ generated as follows, e.g. see Proposition 5.28 in [45]: We start with a tree $\mathbb{T}^*$ generated according to the generating function

$$f^*(s) := \frac{f(q + (1-q)s) - q}{1-q}\,.$$

This tree will serve as the backbone of $\bar{\mathbb{T}}$ and looks like a supercritical GW-tree without leaves. All vertices in this tree are of type g. To each of the vertices of $\mathbb{T}^*$ we attach a random number of independent copies of a sub-critical GW-tree generated according to

$$\tilde{f}(s) := \frac{f(qs)}{q}\,.$$

These are finite bushes consisting of vertices of type b. The resulting tree $\bar{\mathbb{T}}$ has the same law as $\mathbb{T}$, conditioned on nonextiction and is a multitype GW-tree with vertices of type b and g. We denote the measure generating $\bar{\mathbb{T}}$ by $\mathsf{GW}^{\mathsf{mult}}$.

Let $(Z_n^{sub})_{n\geq 0}$ denote the subcritical Galton–Watson process with probability generating function $\tilde{f}$ and $\mathbb{T}^{sub}$ its family tree. We know that $\mathbb{E}[Z_1^{sub}] < 1$ and moreover it holds, e.g., Theorem 2.6.1 in [33] that

$$\lim_{n\to\infty} \frac{\mathbb{P}[Z_n^{sub} > 0]}{\mathbb{E}[Z_1^{sub}]^n} = c \quad \text{and} \quad \mathbb{E}\left[|\mathbb{T}^{sub}|\right] < \infty\,. \tag{2.1}$$

Now, if we assume that $p_1 > 0$ the resulting GW-tree may contain arbitrary long stretches. We want to show that this tree generated by GW is equivalent to a tree generated in three steps where firstly the tree without stretches is generated, secondly the location of the stretch is determined and thirdly the stretches are inserted. Therefore we define a new GW-measure using the modified offspring distribution

$$\hat{p}_k := \frac{p_k}{1-p_1}$$

for $k = 0, 2, \ldots, N$ and let $\mathsf{GW}_{\mathsf{bg}}$ be the measure generating a tree with this distribution. Let us denote a tree generated by $\mathsf{GW}_{\mathsf{bg}}$ with $\mathbb{T}_{\mathsf{bg}}$. In the next step every vertex will be independently labeled with bs with probability p_1, which denotes the starting point of a stretch. If such a vertex has no offspring we attach one vertex, otherwise insert a vertex with offspring one in between the vertex and its descendants, see Figure 2.2. We write for such a tree $\mathbb{T}_{\mathsf{p}\times\mathsf{bg}}$. In the next step, the length of the stretch attached to a vertex with label bs will be distributed according to L were L is geometrically distributed $geo(p_1)$ and we obtain a tree $\mathbb{T}_{\mathsf{s}\times\mathsf{p}\times\mathsf{bg}}$. This yields $1 + geo(p_1)$ distributed vertices with offspring one in a row. The length of the stretches will be determined for each stretch starting point independently and identically distributed. We will call this measure of selecting a stretch point PER and the one of choosing the length of the stretch by ST. We denote by $\mathbb{T}_{\mathsf{s}\times\mathsf{p}\times\mathsf{bg}}$ the tree constructed in the three steps according to $\mathsf{ST} \times \mathsf{PER} \times \mathsf{GW}_{\mathsf{bg}}$. The resulting tree has the same distribution as the tree constructed as follows: We start with a root and proceed inductively. Every new vertex

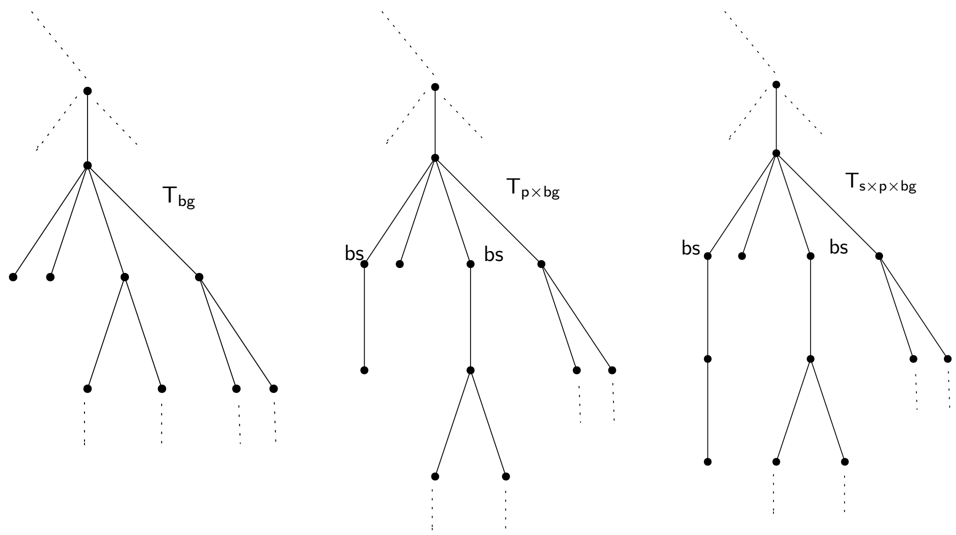

Figure 2.2: Realizations of $\mathbb{T}_{\mathsf{s}\times\mathsf{p}\times\mathsf{bg}}$ by $\mathsf{ST}\times\mathsf{PER}\times\mathsf{GW}_{\mathsf{bg}}$ step by step.

- has 0 descendants (new vertices) with probability $\hat{p}_0(1-p_1)=p_0$,
- has $k\geq 2$ descendants with probability $\hat{p}_k(1-p_1)=p_k$,
- is the starting point of a stretch with length $\ell+1$ and the end of the stretch has $k=0,2,\dots,N$ descendants with probability $p_1p_1^{\ell}(1-p_1)\hat{p}_k=p_1p_1^{l}p_k$.

Moreover, it holds for any finite tree T, that $\mathsf{GW}(\mathsf{T}')=\mathsf{ST}\times\mathsf{PER}\times\mathsf{GW}_{\mathsf{bg}}(\mathsf{T}'')$ where T' and T'' are GW-trees starting with T. Therefore the two measures GW and $\mathsf{ST}\times\mathsf{PER}\times\mathsf{GW}_{\mathsf{bg}}$ are equivalent on the space of all rooted locally finite trees.

2.3 Branching Markov chain

One method for proving transience of the frog model relies on the comparison of the frog model to a branching Markov chain (BMC). This is a labelled Galton–Watson process or tree-indexed Markov chain, [4], where the labels correspond to the position of the particles. In our setting the particles will move on a tree T according to the transition operator $P=(p(v,w))_{v,w\in T}$ of a simple random walk (SRW). We note $p^{(n)}(v,w)$ for the n-step probabilities. If T is connected, the SRW is irreducible and the spectral radius

$$\rho(P):=\limsup_{n\to\infty}\left(p^{(n)}(v,w)\right)^{\frac{1}{n}},v,w\in T$$

is well-defined and takes values in $(0,1]$.

We add the branching mechanism that in every vertex $v\in T$ a particle arriving at v branches according to a branching distribution $\mu(v)$; i.e. each $\mu(v)$ is a measure on $\mathbb{N}$.

We denote by μ the whole sequence $(\mu(v))_{v\in T}$. The expected value of each branching distribution is

$$\bar{\mu}(v) = \sum_{k\in\mathbb{N}} k\mu_k(v)$$

for all $v \in T$ where $\mu_k(v)$ is the probability that a particle jumping in v branches into $k \in \mathbb{N}$ particles. We set $\mathrm{BMC}(T, P, \mu)$ for this branching Markov chain.

Similarly to the frog model, the BMC is called transient if the root will be visited almost surely only by finitely many particles. Otherwise it is called recurrent. A particular case of the transience criterion for BMC given in [18] is the following.

Theorem 2.3.1 *Let T be a locally finite tree and P the transition of the SRW on T. We assume that all branching distributions $\mu(v)$, $v \in T$, have the same mean $\bar{\mu} > 1$. Then the* $\mathrm{BMC}(T, P, \mu)$ *is transient if and only if*

$$\bar{\mu} \leq \frac{1}{\rho(P)}\,.$$

2.4 0–1-law for transience

Before proving the existence of a transient phase for the frog model we want to show that the existence of a transient phase does not depend on the specific realization of the GW-tree. In other words, we show that the frog model is either transient for GW-almost all infinite trees or recurrent for GW-almost all infinite trees.

The proof of this 0–1-law, Theorem 2.1.2, relies on the concept of the environment viewed by the particle. We prove that the events of transience and recurrence are invariant under re-rooting and hence the 0–1-law follows from the ergodicity of the augmented GW-measure.

The augmented Galton–Watson measure, denoted by AGW, is a stationary version of the usual Galton–Watson measure. This measure is defined just like GW except that the number of children of the root has the law of $Y + 1$; i.e. the root has $k + 1$ children with probability p_k. The measure AGW can also be constructed as follows: choose two independent copies T_1 and T_2 with roots o_1 and o_2 according to GW and connect the two roots by one edge to obtain the tree T with the root o_1. We write $\mathsf{T} = \mathsf{T}_1 \bullet\!\!-\!\!\bullet\, \mathsf{T}_2$.

We consider the Markov chain on the state space of rooted trees. If we change the root of a tree T to a vertex $v \in T$, we denote the new rooted tree by $\mathsf{MoveRoot}(T, v)$. We define a Markov chain on the space of rooted trees as:

$$p_{\mathsf{SRW}}((T, v), (T', w)) = \begin{cases} \frac{1}{deg(v)}, & \text{if } v \sim w \text{ and } (T', w) = \mathsf{MoveRoot}(T, w), \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 3.1 and Theorem 8.1 in [44] it holds that this Markov chain with transition probabilities p_{SRW} and the initial distribution AGW is stationary and ergodic conditioned on non-extinction of the Galton–Watson tree.

Lemma 2.4.1 *The events of transience and recurrence of the frog model are invariant under changing the root of the underlying rooted tree $T = (T, o)$, i.e.* $\mathrm{FM}(T, \eta)$ *is transient if and only if* $\mathrm{FM}(\mathsf{MoveRoot}(T, v), \eta)$ *is transient for some (all) $v \in T$.*

Proof As the case of finite trees is trivial we consider an infinite rooted tree (T, o) and let $v \sim o$. We proof that transience of (T, o) implies transience of (T, v) by assuming the opposite. If $\mathrm{FM}(\mathsf{MoveRoot}((T, v), \eta)$ is recurrent, then there exists some $k \in \mathbb{N}$ such that with positive probability infinitely many frogs visit v conditioned on $\eta(o) = k$. In the frog model $\mathrm{FM}((T, o), \eta)$ conditioned on $\eta(v) = k$, the starting frog in o jumps to v with positive probability. Again with positive probability at the second step all frogs awaken in v jump back to o while the frog that came from o is assumed to stay in v for one time step. Note that this has no influence on transience or recurrence of the process. This recreates the same initial configuration of $\mathrm{FM}((T, v), \eta)$ conditioned on $\eta(o) = k$ with the difference that more frogs are already woken up. By assumption in this process infinitely many particles visit v with positive probability, and hence, by the Borel–Cantelli Lemma, also o is visited infinitely many times with positive probability. A contradiction. The claim for arbitrary v now follows by induction and connectedness of the tree. $\square$

Proof (Theorem 2.1.2) By the ergodicity of the Markov chain with transition probabilities p_{SRW} and Lemma 2.4.1, it holds that

$$\mathsf{AGW}[\mathrm{FM}(\mathsf{T}, \eta) \;\text{ transient} \mid \mathsf{T}\,\text{infinite}] \in \{0, 1\}\,.$$

We prove first that

$$\mathsf{GW}[\mathrm{FM}(\mathsf{T}, \eta)\,\text{is transient}] > 0$$

implies

$$\mathsf{AGW}[\mathrm{FM}(\mathsf{T}, \eta) \;\text{ transient}] > 0\,.$$

Let T_1 be a realization on which the frog model is transient. Then, there exists some ball $\mathcal{B}$ around the root o_1 such that no frog awaken outside this ball $\mathcal{B}$ will visit the origin o_1. Let T_2 be an independent realization according to GW and let $\mathsf{T} := \mathsf{T}_1 \bullet\!\!-\!\!\bullet \mathsf{T}_2$.

In the frog model on (T, o_1) the starting frog jumps into T_1 at time $n = 1$ with positive probability. Now, since every frog is transient, with positive probability all frogs in the set $\mathcal{B}$ that are woken up will never cross the additional edge (o_1, o_2) and we obtain that $\mathsf{AGW}[\mathrm{FM}(\mathsf{T}, \eta)\,\text{transient}] > 0$. We write $\mathsf{AGW}_\infty[\cdot] := \mathsf{AGW}[\cdot \mid \mathsf{T}\,\text{infinite}]$ and define GW_∞ similarly. The 0–1-law gives that $\mathsf{AGW}_\infty[\mathrm{FM}(\mathsf{T}, \eta)\,\text{transient}] = 0$ implies $\mathsf{GW}_\infty[\mathrm{FM}(\mathsf{T}, \eta)\,\text{transient}] = 0$.

It remains to show that

$$\mathsf{GW}_\infty[\mathrm{FM}(\mathsf{T}, \eta)\,\text{recurrent}] > 0$$

implies that

$$\mathsf{AGW}_\infty[\mathrm{FM}(\mathsf{T}, \eta)\,\text{recurrent}] > 0\,.$$

Let T_1 and T_2 be two recurrent realizations of GW_∞ and let $\mathsf{T} := \mathsf{T}_1 \bullet\!\!-\!\!\bullet \mathsf{T}_2$. Each copy $\mathsf{T}_i, i \in \{1, 2\}$, is recurrent with positive probability. Hence, we have to verify that the possibility that frogs can change from one T_i to the other does not change this property. Let us say that every frog originally in T_1 wears a red T-shirt and every frog in T_2 wears a blue T-shirt. Now, every frog that jumps from o_1 to o_2 leaves its red T-shirt in a stack in o_1. In the same way every frog leaving o_1 to o_2 leaves its blue T-shirt in a stack in o_2. A frog arriving from o_1 to o_2 takes a blue T-shirt from the stack. If the stack is empty, the frog "creates" a new blue shirt. We proceed similarly for the frogs that arrive in o_1 coming from o_2. The frog model $\mathrm{FM}(\mathsf{T},\eta)$ starts with one awoken frog in a red T-shirt in o_1. Once a frog visits o_2, the blue frog model $\mathrm{FM}(\mathsf{T}_2,\eta)$ is started and a red shirt is left in o_1. Conditioned on the event that $\mathrm{FM}(\mathsf{T}_2,\eta)$ is recurrent a blue frog will eventually jump from o_2 to o_1 and put on the red shirt. In this way, every red shirt is finally put on and the distribution of the red frogs in $\mathrm{FM}(\mathsf{T},\eta)$ equals the distribution of the frogs in $\mathrm{FM}(\mathsf{T}_1,\eta)$ with possible additional frogs. In other words, $\mathrm{FM}(\mathsf{T},\eta)$ is recurrent with positive probability.

Finally, we can conclude

$$\mathsf{GW}[\mathrm{FM}(\mathsf{T},\eta) \text{ is transient} \,|\, \mathsf{T} \text{ is infinite}] \in \{0,1\}\,. \qquad \square$$

2.5 Transience of the frog model

2.5.1 No bushes, no stretches

We start with considering GW-trees $\mathbb{T}$ with $p_0+p_1=0$. By Lemma 2.6.5 we know that $\rho(\mathbb{T})<1$ and hence Theorem 2.3.1 guarantees a transient phase for BMC on such GW-trees $\mathbb{T}$. Coupling the frog model with an appropriate branching Markov chain implies a transient phase for the frog model.

Lemma 2.5.1 *Consider a Galton–Watson measure* GW *with* $p_0+p_1=0$ *and* $m>1$*. Then, for* GW*-almost all trees* T *the frog model with* η *distributed number of frogs per vertex is transient if mean* $\bar\eta \le \frac{d+1}{2\sqrt{d}}-1$ *where* $d := min\{k : p_k>0\}$*.*

Proof The proof relies on the fact that the $\mathrm{BMC}(\mathsf{T},P,\mu)$, where $\mu(v)$ fulfills $\mu_k(v) = \mathbb{P}[\eta(v)+1=k]$ for each $k\ge 1$ and $v\in\mathsf{T}$, stochastically dominates the frog model. We use a coupling of the frog model with a BMC such that at most as many frogs (in the frog model) as particles (in the BMC) visit the root. More precisely, in both models we start with one frog, respectively particle, at the root and couple them. A particle of the BMC that is coupled to a frog x in the frog model is denoted by x'. The "additional" particles in the BMC, in the meaning that they have no counterpart in the frog model, will move and branch without having any influence on the coupling. Let $(\mathsf{f}_v)_{v\in\mathsf{T}}$ be a realization of the sleeping frogs. If a first coupled particle arrives at v it branches into f_v+1 particles. The awakened frogs and newly created particles are coupled. If more than one coupled particle arrives at v for the first time at the same moment, we choose (randomly) one of these, let it have f_v+1 offspring and couple the resulting particles with the frogs as above. The offspring of the other particles (those that are coupled to the remaining frogs arriving at v) are chosen i.i.d. according to $\mu(v)$ and one of them (randomly chosen) is coupled to each corresponding frog. Similarly, if a vertex $v \in \mathsf{T}$ will be visited a second time by a frog, no new frogs will wake up but the particle will branch again into a random $\mu(v)$ distributed number of particles and we couple the frog arriving at v with one (randomly chosen) of the particles. In this way every awake frog is coupled with a particle of the BMC. Hence if the BMC is transient, then also the frog model is transient. The mean offspring $\bar{\mu}$ of BMC(T, P, μ) is constant

$$\bar{\mu} := \bar{\mu}(v) = \mathbb{E}[\eta(v)] + 1 = \bar{\eta} + 1$$

for any $v \in \mathsf{T}$ as $\eta(v)$ are independent and identically distributed. Using Theorem 2.3.1 it follows that the BMC is transient if and only if

$$\bar{\eta} + 1 \leq \frac{1}{\rho(\mathsf{T})}\,.$$

By Lemma 2.6.5 it holds that $\rho(\mathsf{T}) = \rho(T_{d+1}) = \frac{2\sqrt{d}}{d+1}$, where $d := min\{k : p_k > 0\}$ and T_{d+1} is the homogeneous degree with offspring d. Hence, FM(T, η) is transient if we choose η such that it holds

$$\bar{\eta} \leq \frac{1}{\rho(\mathsf{T})} - 1\,.$$

□

Throughout this section, we shall make frequent use of several known results which we have assembled in Section 2.6 below in the form of an appendix.

2.5.2 No bushes, but stretches

In the case $p_0 + p_1 > 0$ a direct coupling as in the proof of Lemma 2.5.1 does not allow to prove transience since every non-trivial BMC is recurrent. This is due to the existence of bushes or stretches in the Galton–Watson tree and the fact that the spectral radius of such trees is a.s. equal to 1, see Lemma 2.6.5. We will start with dealing with stretches and then continue with treating bushes and stretches at the same time. The case of stretches uses a different method than in Lemma 2.5.1. We modify the model, such that we wake up all frogs in a stretch, if the beginning of a stretch is visited for the first time. The awoken frogs are placed according to the first exit measures (of a SRW) at the ends of this stretch. Moreover we send every frog entering a stretch immediately to one of the ends of the stretch; again according to the exit measures. This makes it possible to consider the stretch as one vertex. However, the original length of the stretch is important for the path measure. We treat the path measure by introducing two step probabilities. Since a direct coupling between the frog model and a BMC is no longer possible, we compare the expected number of returns to the root of the frog model with those in a suitable, different BMC.

Proposition 2.5.2 *Consider a Galton–Watson measure* GW *with* $0 < p_1 < 1$, $p_0 = 0$ *and mean* $m > 1$. *We assume that* $d_{max} = \max\{i : p_i > 0\} < \infty$ *and set* $d_{min} := \min\{i \geq 2 :$

$p_i > 0\}$. *Then, for any choice of* p_1 *there exist constants* $c_d = c_d(p_1)$ *and* $c_\eta = (c_d, d_{max})$ *such that for* $d_{min} > c_d$ *the frog model* $\mathrm{FM}(\mathsf{T}, \eta, \mathsf{SRW})$ *is transient* GW*-almost surely (conditioned on* T *to be infinite) if* $\bar{\eta} < c_\eta$.

Proof Let T be an infinite realization of GW. As $0 < p_1 < 1$ we can consider T constructed according to $\mathsf{ST} \times \mathsf{PER} \times \mathsf{GW}_{\mathsf{bg}}$, see Section 2.2. Using this construction we label its vertices in the following way, see also Figure 2.3:

- **label** bs: a vertex of degree 2 with a mother vertex of degree strictly larger than 2;
- **label** es: a vertex of degree 2 with a child of degree strictly larger than 2;
- **label** s: a vertex of degree 2 with all two neighbours of degree 2;
- **label** n: a vertex of degree higher than 2.

These labels help us to identify the stretches and their starting and end points. More precisely, a stretch is a path $[v_1, \dots, v_n]$ where v_1 has label bs and v_n has label es and all vertices $v_i, i \in \{2, \dots, n-1\}$, are labeled with s. As mentioned above a BMC on a GW-tree with $0 < p_1 < 1$ would a.s. be recurrent. To find a dominating BMC, which has a transient phase, we consider two modified state spaces T' and T'_N.

Construction of a dominating frog model FM' on T and T'

We modify the frog model in the following way. Frogs in the new FM' behave as in FM on vertices that are not in stretches. Once a frog enters a stretch we add more particles in the following way. Let $[v_1, \dots, v_\ell]$ be a stretch of length $\ell = \ell_{v_1}$ and u the mother vertex of v_1 and w the child of v_ℓ, see Figure 2.3. Here, v_1 is the first vertex in a stretch, i.e. a vertex with label bs. Now, if a first frog jumps on v_1, all frogs from the stretch are activated and placed on u and v, respectively, according to their exit measures. For any later visit any frog entering the stretch is immediately placed on u or v according to the exit measure of the stretch. The exit measures are solutions of a ruin problem. Similar to the proof of Lemma 2.5.1, we can couple FM and FM' such that

$$\nu \preccurlyeq \nu'$$

where ν' is the number of visits to the root in FM' and conclude that transience of FM' implies transience of FM.

Concerning the stretches, in the definition of FM' only their "exit measures" play a role. The model FM' can therefore live on the tree T' but has to incorporate the length of each stretch. The modified frog model then evolves on a new state space T', constructed as follows: Let $[v_1, \dots, v_\ell] \subset \mathsf{T}$ be a stretch and $w \in \mathsf{T}$ the child of v_l. Then, we merge the stretch into the vertex v_1 (with label bs). Hence, there is a single vertex of degree 2 left in between vertices with higher degree, see Figure 2.3. We identify each vertex $v' \in \mathsf{T}'$ with its corresponding vertex $v \in \mathsf{T}$ due to this construction. We can distinguish the vertices of T' into $V_{\mathsf{n}} := \{v' \in \mathsf{T}' \mid v' \text{ with label } \mathsf{n}\}$ and $V_{\mathsf{s}} := \{v' \in \mathsf{T}' \mid v' \text{ with label } \mathsf{bs}\}$. This

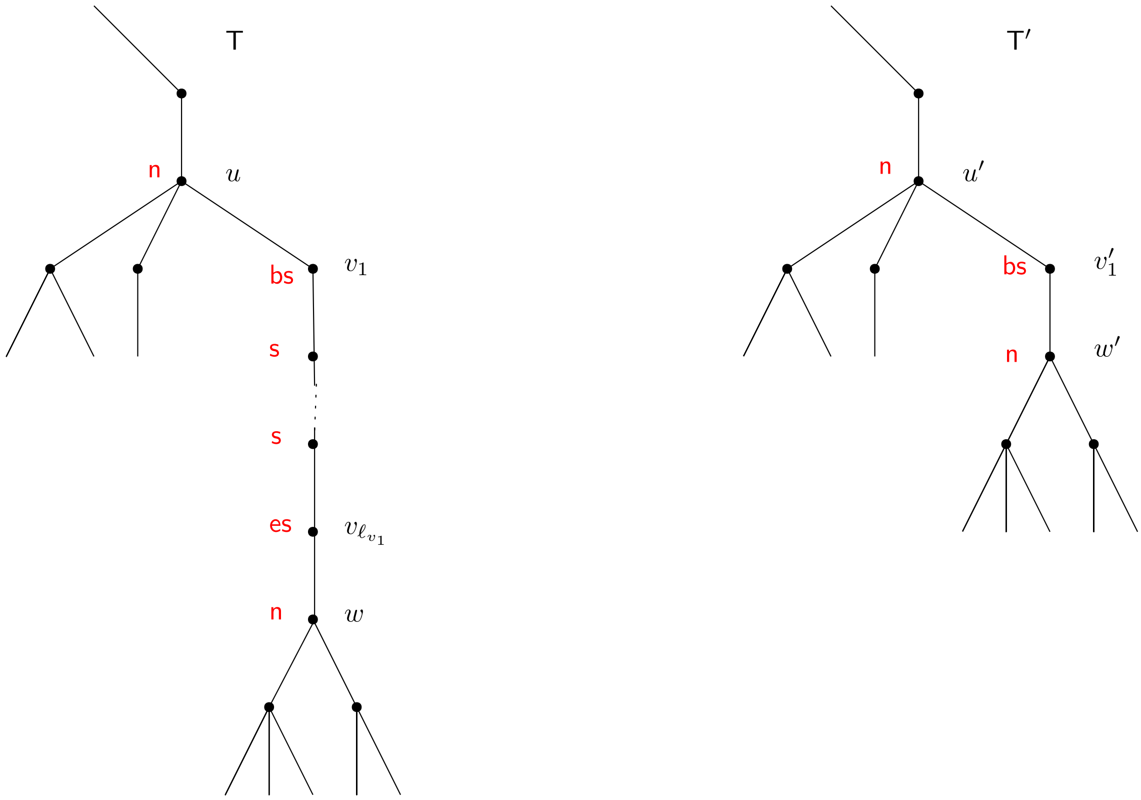

Figure 2.3: A stretch $[v_1, \dots, v_\ell]$ and its transformation to one vertex v'_1 in T'.

modified state space T' corresponds to the first two stages, namely $\mathsf{PER} \times \mathsf{GW}_{\mathsf{bg}}$, in the construction of $\mathsf{ST} \times \mathsf{PER} \times \mathsf{GW}_{\mathsf{bg}}$. In other words, it has the same law as $\mathbb{T}_{\mathsf{p} \times \mathsf{bg}}$. Moreover, the third step, i.e. ST, in the construction of the measure is encoded in the length of each stretch.

We introduce the following quantities. Let $\nu'(w')$ be the number of visits to w' and $\nu'_n(w')$ the number of particles in w' at time n. Then, for a fixed realization T' let $\mathbb{E}^{\mathsf{T}'}_{v'}[\nu(w')]$ be the expected number of visits to $w' \in \mathsf{T}'$, when the frog started in $v' \in \mathsf{T}'$. We also denote this as

$$m^{\mathsf{T}'}_{\mathrm{FM}'}(v', w') := \mathbb{E}^{\mathsf{T}'}_{v'}[\nu(w')].$$

The expected number $m^{\mathsf{T}'}_{\mathrm{FM}'}(v', w')$ depends on the state space T' and we can look at the expected value

$$m^{\mathsf{ST}}_{\mathrm{FM}'}(v', w') := \mathbb{E}_{\mathsf{ST}}[m^{\mathsf{T}'}_{\mathrm{FM}'}(v', w')]$$

with respect to ST for $v', w' \in \mathsf{T}'$. Note here, that the measure ST has no impact on the underlying tree but only on the number of frogs and the exit measure from the stretches. Moreover, it holds that

$$m^{\mathsf{ST}}_{\mathrm{FM}'}(o', o') < \infty \tag{2.2}$$

implies

$$m^{\mathsf{T}'}_{\mathrm{FM}'}(o', o') < \infty. \tag{2.3}$$

Construction of dominating BMC' **on** T'

In the next step we are going to define a branching Markov chain BMC' on T' such that

$$\mathbb{E}[\nu_{\mathrm{BMC}'}] < \infty \;\; \Rightarrow \;\; \mathbb{E}_{\mathrm{FM}'}[\nu'] < \infty\,, \tag{2.4}$$

where $\nu_{\mathrm{BMC}'}$ is the number of returns to the root of the BMC'.

We recall that the length L of a stretch in the original tree T is geometrically distributed; $L \sim geo(p_1)$. Let $L_v, v \in \mathsf{T}$, denote this random stretch attached to a vertex v with label bs. The presence of arbitrarily long stretches prevents the existence of transient BMC on T, see Lemma 2.6.4. For this reason, let $N \in \mathbb{N}$ (to be chosen later) and define the tree T_N as a copy of T where each stretch of length larger than N is replaced by a stretch of length N. We define a BMC, called BMC_N, on T_N, with driving measure SRW and the offspring distribution μ fulfills that $\mu_k(v) = \mathbb{P}[\eta(v) + 1 = k]$ for each $v \in \mathsf{T}_N$. The BMC_N defined on T_N defines naturally a branching Markov chain BMC'_N on T', where once a particle enters a stretch, it produces offspring particles according to the exit-measures. This quantity is given by $F_{\ell+1}(1, 0|\mu)$ and $F_{\ell+1}(1, \ell + 1|\mu)$ defined as in Lemma 2.6.1 and 2.6.7, where ℓ is the length of the stretch. The aim is now to find some N such that BMC'_N is still transient and dominates (in ST-expectation) the frog model FM'.

In order to find such a domination we compare the mean number of returns "pathwise" in FM' and BMC'_N. More precisely, we want to express the quantity $\nu'_n(o')$ in terms of frogs following a specific path. Let p' be a path starting and ending at o'. A path of length $n \in \mathbb{N}$ looks like $\mathsf{p}' = [o', p'_1, p'_2, \ldots, p'_{n-1}, o']$ with $p'_i \in \mathsf{T}'$ and $p'_i \sim p'_{i+1}$ for each i. Let θ_k denote the k-th cut of a path, that is $\theta_k(\mathsf{p}') := [p'_k, \ldots, o']$. We call a frog sleeping at some $p_i, 1 \le i \le n-1$, activated by frogs following the path p' ($\mathsf{affb}_{\mathsf{p}'}$), if inductively the frog was activated from a frog in p_{i-1} that was activated by frogs following the path p' or started at p_1 and followed p'. We denote by $\mathsf{affb}_{\mathsf{p}'}(v', i)$ for the event that the ith frog in v' is $\mathsf{affb}_{\mathsf{p}'}$. Additionally, for $i, j \in \mathbb{N}$ let $S_j(v', i)$ denote the position of the i-th frog initially placed at $v' \in \mathsf{T}'$ after j time steps after waking up. (Here we assume an arbitrary enumeration of the frogs at each vertex.) Using this, $\nu'_n(o')$ is equal to

$$\left| \bigcup_{|\mathsf{p}'|=n} \bigcup_{p'_i \in \mathsf{p}'} \bigcup_{r \in \mathbb{N}} A(p'_i, r, \mathsf{p}) \right| .$$

where

$$A(p'_i, r, \mathsf{p}') := \Big\{ \exists k : \{S_j(p'_i, r)\}_{j=0}^{n-k} = \theta_k(\mathsf{p}') \,\text{and}\, \mathsf{affb}_{\mathsf{p}'}(p'_i, r) \Big\} .$$

Now, we can rewrite

$$m^{\mathsf{ST}}_{\mathrm{FM}'}(o', o') = \mathbb{E}_{\mathsf{ST}}\left[\mathbb{E}^{\mathsf{T}'}_{o'}\left[\nu'\right] \right] = \mathbb{E}_{\mathsf{ST}}\left[\mathbb{E}^{\mathsf{T}'}_{o'}\left[\sum_{n=1}^{\infty} \nu'_n(o') \right] \right]$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \mathbb{E}_{\mathsf{ST}} \left[\mathbb{E}_{o'}^{\mathsf{T}'} \left[\sum_{|\mathsf{p}'|=n} \left| \left\{ \bigcup_{p_i' \in \mathsf{p}} \bigcup_{r \in \mathbb{N}} A(p_i', r, \mathsf{p}') \right\} \right| \right] \right] \\
&= \sum_{n=1}^{\infty} \sum_{|\mathsf{p}'|=n} \mathbb{E}_{\mathsf{ST}} \left[\mathbb{E}_{o'}^{\mathsf{T}'} \left[\left| \left\{ \bigcup_{p_i' \in \mathsf{p}'} \bigcup_{r \in \mathbb{N}} A(p_i', r, \mathsf{p}') \right\} \right| \right] \right] \qquad (2.5)
\end{aligned}
$$

by using the monotone convergence theorem. For a given path p' the term

$$
\nu_{\mathsf{ST}}'(\mathsf{p}') := \mathbb{E}_{\mathsf{ST}} \left[\mathbb{E}_{o'}^{\mathsf{T}'} \left[\left| \left\{ \bigcup_{p_i' \in \mathsf{p}'} \bigcup_{r \in \mathbb{N}} A(p_i', r, \mathsf{p}') \right\} \right| \right] \right]
$$

equals the expected number of frogs that were activated following the path and that follow the paths after their activation. In the same way as for the frog process we can define the expected number of particles $\nu_{\mathrm{BMC}}(\mathsf{p}')$ for a BMC following a path p'. In the remaining part of the proof we construct a branching Markov chain BMC$'$ such that

$$
\nu_{\mathsf{ST}}'(\mathsf{p}') \leq \mathbb{E}\left[\nu_{\mathrm{BMC}'}(\mathsf{p}')\right] \qquad (2.6)
$$

for all paths p'. Transience of the BMC then implies transiences of the frog model. The paths p' are concatenations of three different types of vertex sequences. *Type* 1 is a sequence that does not see any stretches. A sequence of *type* 2 traverses a stretch, whereas a sequence of *type* 3 visits a stretch but does not traverse it. We will split each path p' into these three types and give upper bounds for (2.5) for each type separately. We have to take into account that multiple visits of the same sequence of vertices are not independent from each other. Here the frogs face in every visit the same length of a stretch. Hence, while taking the expectation over the length of the stretches, multiple visits of the same vertices have to be considered at the same time. Therefore we give upper bounds of (2.5) for each combination of multiple visits. Then we combine the results for a final upper bound of a mixed path.

For this purpose we consider for the BMC the mean number of particles created in stretches in T_N. We consider the situation described in Figure 2.3. Let ℓ be the length of a stretch generated according to ST. Such a stretch appears in T_N with probability $p_1^{\ell-1}(1-p_1)$ if $\ell < N-1$ and with probability p^{N-1} if $\ell = N-1$. We denote by $m_{\mathrm{BMC}'}^{\mathsf{T}'}(p_i', p_{i+1}')$ the expected number of particles arriving in p_{i+1}' while starting in p_i'. Again we can look at the expectation with respect to ST

$$
m_{\mathrm{BMC}'}^{\mathsf{ST}}(p_i', p_{i+1}') = \mathbb{E}_{\mathsf{ST}}\left[m_{\mathrm{BMC}'}^{\mathsf{T}'}(p_i', p_{i+1}')\right],
$$

where ST impacts only the number of created particles and not the underlying tree. We define the vertices u and w as absorbing and denote by $\eta_N(u)$ (resp. $\eta_N(w)$) the number of particles absorbed in u (resp. in w), see also Section 2.6.1.

Only visits of type 1: We assume that $\mathsf{p}' = [p_0', p_1', p_2', \dots, p_{n-1}', p_n']$ only consists of sequences of type 1. Using the Markov property we can bound

$$
\nu'(\mathsf{p}') \leq \prod_{i=0}^{n-1} m_{\mathrm{FM}'}^{\mathsf{ST}}(p_i', p_{i+1}') = \prod_{i=0}^{n-1} m_{\mathrm{BMC}'}^{\mathsf{ST}}(p_i', p_{i+1}'), \qquad (2.7)
$$

due to the choice of the BMC$'$, see the paragraph after Equation (2.4).

Multiple visits of a stretch in sequences of type 2: We assume that the path also has some sequences of type 2, see Figure 2.4. An important observation is that every

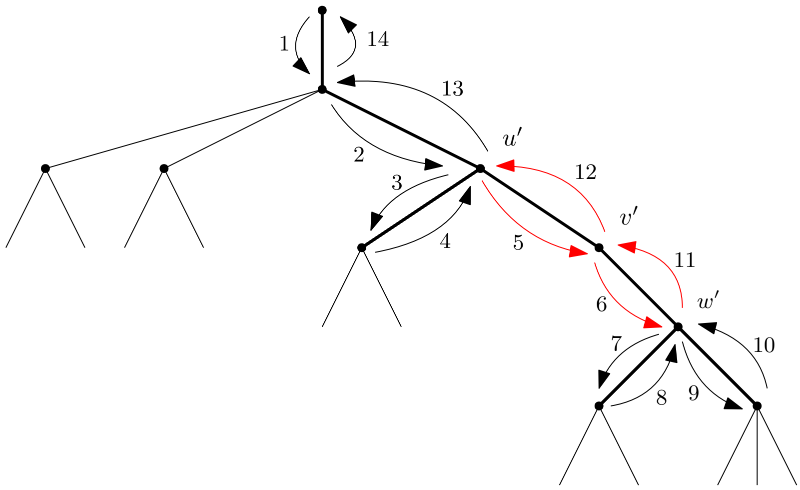

Figure 2.4: A typical path with a sequence of type 2.

path from o' to o' that traverses a stretch in one direction has to traverse it in the other direction as well. Hence, such a path in T' of length n has for example the form

$$\mathsf{p}' = [o', \underbrace{p'_1, \ldots, p'_i, u'}_{\text{degree}\geq 3}, v', \underbrace{w', p'_{i+4}, \ldots, p'_j, w'}_{\text{degree}\geq 3}, v', \underbrace{u', p'_{j+4}, \ldots, p'_{n-1}}_{\text{degree}\geq 3}, o'],$$

where v' has degree 2. We start with the case that a sequence of vertices is visited only twice. The case of more visits will be an immediate consequence.

In order to bound the expected number of frogs along this path we define $m^{\mathsf{T}'}_{\mathrm{FM}'}(u' \to v' \to w')$ as the expected number of frogs that follow the path $[u', v', w']$ in FM$'$ starting with one frog in u'. The modified frog model FM$'$ is defined such that all frogs in the stretch are woken up and distributed at the end of the stretches if the starting vertex of the stretch is visited. In the case of traversing a stretch, this is dominated by the following modification: if the frog jumps on v_1 from u the first time we start a BMC in v_1 with offspring distribution $\eta + 1$ and absorbing states u and w. The mean number of frogs absorbed in u and w can be calculated using Lemmata 2.6.1 and 2.6.7. This dominates FM$'$ since we consider a path traversing the stretch. This means that all vertices in the stretch were visited in the new model, since some particles arrived in w' and we can couple the sleeping frogs in FM$'$ with the created particles in the stretch. We conclude by Lemma 2.6.1 that

$$m^{\mathsf{T}'}_{\mathrm{FM}'}(u' \to v' \to w') \leq \frac{1}{deg(u')} F_{\ell+1}(1, \ell+1 \mid \bar{\eta}+1), \tag{2.8}$$

where $\ell = \ell_{v'} + 1$ is the length of the total stretch (including the initial point of the stretch). To take into account that at the second traversal of the stretch no sleeping frogs are left in the stretch we define $m^{\mathsf{T}',2+}_{\mathrm{FM}'}(w' \to v' \to u')$ as the expected number of particles following $[w', v', u']$ with no frogs between w' und u'. Hence,

$$\begin{aligned} m^{\mathsf{T}'}_{\mathrm{FM}'}(u' \to v' \to w') m^{\mathsf{T}',2+}_{\mathrm{FM}'}(w' \to v' \to u') &\le \frac{1}{deg(u')} F_{\ell+1}(1, \ell+1 \mid \bar{\eta}+1) \frac{1}{deg(w')} \frac{1}{\ell+1} \\ &\le \frac{1}{deg(u')} F_{\ell+1}(1, \ell+1 \mid \bar{\eta}+1)^2 \frac{1}{deg(w')}. \end{aligned}$$

Note that the last term equals the mean number of particles in BMC'_ℓ along the path $[u', v', w', v', u']$. Let φ such that $(\bar{\eta}+1)^{-1} = \cos\varphi$, then the function

$$f(\ell) = F_\ell(1, \ell \mid \bar{\mu}) \frac{1}{\ell} = \frac{\sin\varphi}{\ell \sin \ell\varphi}$$

is monotone decreasing in ℓ. We can now integrate with respect to ST to obtain:

$$\begin{aligned} &\mathbb{E}_{\mathsf{ST}}\left[m^{\mathsf{T}'}_{\mathrm{FM}'}(u' \to v' \to w') m^{\mathsf{T}',2+}_{\mathrm{FM}'}(w' \to v' \to u') \;\middle|\; L \le N \right] \\ &\le \frac{1}{deg(u')} \mathbb{E}_{\mathsf{ST}}\left[F_{L+1}(1, L+1 \mid \bar{\eta}+1)^2 \;\middle|\; L \le N \right] \frac{1}{deg(w')} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E}_{\mathsf{ST}}\left[m^{\mathsf{T}'}_{\mathrm{FM}'}(u' \to v' \to w') m^{\mathsf{T}',2+}_{\mathrm{FM}'}(w' \to v' \to u') \;\middle|\; L > N \right] \\ &\le \frac{1}{deg(u')} f(N+1) \frac{1}{deg(w')} \\ &\le \frac{1}{deg(u')} F_{N+1}(1, N+1 \mid \bar{\eta}+1)^2 \frac{1}{deg(w')}. \end{aligned}$$

If we have more visits of type 2, there are no new frogs waking up and we have as transition probability through the stretch $\frac{1}{\ell+1}$ for each visit. In the case of k visits we obtain

$$\begin{aligned} m^{\mathsf{T}'}_{\mathrm{FM}'}(u' \to v' \to w') & m^{\mathsf{T}',2+}_{\mathrm{FM}'}(w' \to v' \to u')^k m^{\mathsf{T}',2+}_{\mathrm{FM}'}(u' \to v' \to w')^{k-1} \\ &= \left(\frac{1}{deg(u')}\right)^k F_{\ell+1}(1, \ell+1 \mid \bar{\eta}+1) \left(\frac{1}{deg(w')}\right)^k \left(\frac{1}{\ell+1}\right)^{2k-1} \\ &\le \left(\frac{1}{deg(u')}\right)^k F_{\ell+1}(1, \ell+1 \mid \bar{\eta}+1)^{2k} \left(\frac{1}{deg(w')}\right)^k \end{aligned} \tag{2.9}$$

We notice again that $f_k(\ell) := F_\ell(1, \ell \mid \bar{\eta}+1)\,(1/\ell)^{2k-1}$ is monotone decreasing in ℓ and we can integrate with respect to ST in the same manner as above.

Define $m^{\mathsf{ST}}_{\mathrm{BMC}'}(u' \to v' \to w')$ as the expected number of particles that follow the path $[u', v', w']$ in BMC' starting with one particle in u'. Using the aforegoing estimate, we can bound $\mathbb{E}^{\mathsf{ST}}[m^{\mathsf{T}'}_{\mathrm{FM}'}(u' \to v' \to w') m^{\mathsf{T}',2+}_{\mathrm{FM}'}(w' \to v' \to u')^k m^{\mathsf{T}',2+}_{\mathrm{FM}'}(u' \to v' \to w')^{k-1}]$ by

$\mathbb{E}_{\mathsf{ST}}[m_{\mathrm{BMC}'}^{\mathsf{T}'}(u',w')^k m_{\mathrm{BMC}'}^{\mathsf{T}'}(w',u')^k]$. Moreover the stretches are independently generated. We obtain by induction for different sequences of type 2 that:

$$\nu'(\mathsf{p}') \leq \prod_{i=0}^{n-1} m_{\mathrm{BMC}'}^{\mathsf{ST}}(p'_i, p'_{i+1}). \tag{2.10}$$

Multiple visits of a stretch in sequences of type 2 and 3: We handle this situation in three steps. In the first we assume, that a sequence of vertices is only visited once in the manner of type 3. Secondly, we treat a sequence of a path which visits a stretch more than once in the manner of type 3. Lastly, we study sequences which are visited by type 2 and type 3 sequences. There, we have to distinguish between the type of the first visit of the sequence.

We start with the first part. We assume that the path p of length n contains a sequence of type 3, that is $p'_{i_j} = v'$,$i_j \in \{1,\ldots,n\}$, of degree 2 and $p_{i_j-1} = p_{i_j+1} = u'$, see Figure 2.5. This means that the frogs in FM' did not pass the stretch completely. We call these

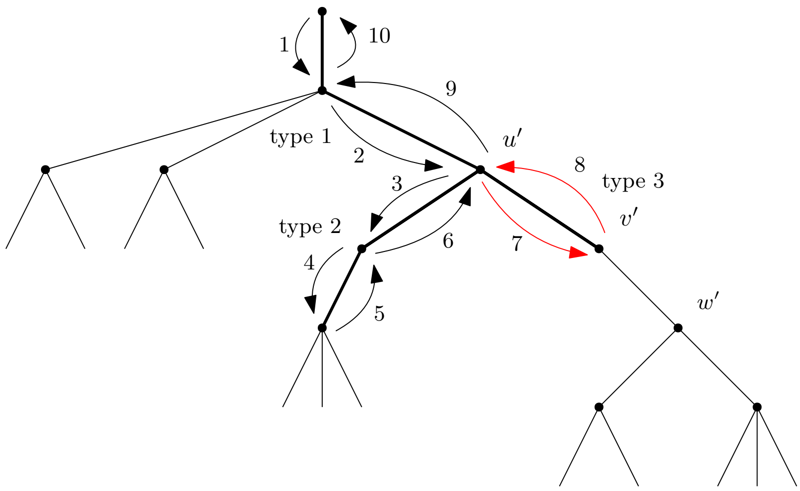

Figure 2.5: A typical path with sequences of type 1,2 and 3.

parts of the path stretchbits. A typical path p in this case can be for example

$$\mathsf{p}' = [o', \underbrace{p'_1, \ldots, p'_{i_1-2}}_{\text{type 1,2}}, \overbrace{u', v', u'}^{\text{type 3}}, \underbrace{p'_{i_1+2}, \ldots, p'_{n-1}}_{\text{type 1,2}}, o'] \,.$$

We define $m_{\mathrm{FM}'}^{\mathsf{T}'}(u' \to v' \to u')$ as the expected number of frogs that follow the path $[u', v', u']$ in FM' starting with one frog in u'. Then

$$m_{\mathrm{FM}'}^{\mathsf{T}'}(u' \to v' \to u') \leq \frac{1}{deg(u')}\left(\frac{\ell\bar{\eta}}{2} + \frac{\ell}{\ell+1}\right). \tag{2.11}$$

Recall that the distribution of the total stretch length $L = \ell_{v_1} + 1$ is exponential:

$$\mathbb{P}(L = \ell) = p_1^{\ell-1}(1-p_1),\ \forall\ \ell \geq 1.$$

Hence, integrating (2.11) with respect to ST yields

$$\begin{aligned} m_{\mathrm{FM}'}^{\mathsf{ST}}(u' \to v' \to u') &\leq \frac{1}{deg(u')} \sum_{\ell=1}^{\infty} \left(\frac{\ell \bar{\eta}}{2} + \frac{\ell}{\ell+1} \right) p_1^{\ell-1}(1-p_1) \\ &= \frac{1}{deg(u')} \left(\frac{\bar{\eta}}{2(1-p_1)} + \sum_{\ell=1}^{\infty} \left(\frac{\ell}{\ell+1} \right) p_1^{\ell-1}(1-p_1) \right). \end{aligned}$$

Let $d = \min\{i \geq 2 : p_i > 0\}$. A stretch of length ℓ is equivalent to an unbranched path of length $\ell+1$ in Section 2.6.2. As we only allow a maximum stretch length N in case of BMC_N, we obtain at maximum an unbranched path of length $N+1$. Then, using Lemma 2.6.5, Theorem 2.6.6, and Lemma 2.6.3 the spectral radius $\rho(P_{N+1})$ on the absorbing stretch piece of length $N+1$ satisfies

$$\rho(P_{N+1}) < \cos \left(\frac{\arccos\left(\frac{2\sqrt{d}}{d+1}\right)}{N+1} \right). \tag{2.12}$$

Furthermore,

$$m_{\mathrm{BMC}'}^{\mathsf{T}'}(u' \to v' \to u') = \frac{1}{deg(u')} F_{\ell+1}(1, 0|\bar{\mu}) \tag{2.13}$$

We now choose

$$\bar{\mu} = \cos \left(\frac{\arccos\left(\frac{2\sqrt{d}}{d+1}\right)}{N+1} - \varepsilon \right)^{-1}$$

for some sufficiently small $\varepsilon > 0$ and define

$$g(\ell) = F_\ell\,(1, 0|\bar{\mu}) < \infty.$$

Observe here that, since $\bar{\mu} < \frac{1}{\rho(\mathsf{T}')}$, the BMC' with mean offspring $\bar{\mu}$ is not only transient but it also holds that $\mathbb{E}_{\mathrm{BMC}'}[\nu] < \infty$, see Chapter 5.C in [69]. Now, integrating equation (2.13) with respect to ST yields

$$m_{\mathrm{BMC}'}^{\mathsf{ST}}(u' \to v' \to u') = \frac{1}{deg(u')} \sum_{\ell=1}^{N-1} g(\ell+1) p_1^{\ell-1}(1-p_1) + g(N+1) p_1^{N-1}$$

We now look for $\bar{\eta}$ sufficiently small and N sufficiently large such that

$$\left(\frac{\bar{\eta}}{2} \left(\frac{1}{1-p_1} \right) + \sum_{\ell=1}^{\infty} \left(\frac{\ell}{\ell+1} \right) p_1^{\ell-1}(1-p_1) \right)$$

$$< \sum_{\ell=1}^{N-1} g(\ell+1) p_1^{\ell-1}(1-p_1) + g(N+1) p_1^{N-1}. \tag{2.14}$$

In order to achieve this last inequality, it suffices to find an N such that

$$\sum_{\ell=1}^{\infty} \left(\frac{\ell}{\ell+1}\right) p_1^{\ell-1}(1-p_1) < \sum_{\ell=1}^{N-1} g(\ell+1) p_1^{\ell-1}(1-p_1) + g(N+1) p_1^{N-1}. \tag{2.15}$$

By Lemma 2.6.8 we can bound the right hand side from below by

$$\sum_{\ell=1}^{N-1} \frac{\ell}{\ell+1} \left(1 + \frac{(1+2\ell)\,\varphi^2}{3!}\right) p_1^{\ell-1}(1-p_1) + g(N+1) p_1^N$$

where $\varphi = \arccos(1/\bar{\mu})$. This reduces (2.15) to:

$$\sum_{\ell=N}^{\infty} \left(\frac{\ell}{\ell+1}\right) p_1^{\ell-1}(1-p_1) < \sum_{\ell=1}^{N-1} \frac{\ell}{\ell+1} \left(\frac{(1+2\ell)\,\varphi^2}{3!}\right) p_1^{\ell-1}(1-p_1) + g(N+1) p_1^{N-1}\,. \tag{2.16}$$

The left hand side of (2.16) decays exponentially in N while the first part of the right hand side has polynomial decay in N having the choice of φ in mind. Therefore, there exists some N such that (2.16) is verified.

We continue with the second part, where a sequence of the path faces multiple type 3 visits. If a frog makes a second type 3 visit to an already woken up stretch, this frog encounters no new frogs and returns to u' almost surely. This follows for every other visit of type 3. Hence, conditioning the frog upon not making another type 3 visit to a stretch has no influence on the possible frogs returning to the root and consequently on transience and recurrence. We will call this model FM$''$. But we notice that the path measure changes when we change to FM$''$:

$$\mathsf{P}[u' \to y' \,|\, \text{no visit to}\, v'] = \frac{1}{deg(u')-1}$$

where y' is any neighbour of u' apart from v'. Since the path measure of BMC$_N$ is unchanged we have to compare

$$m_{\text{BMC}'}^{\mathsf{T}'}(u', y') = \frac{\bar{\mu}}{deg(u')}$$

and

$$m_{\text{FM}''}^{\mathsf{T}'}(u', y') = \frac{1}{deg(u')-1}$$

as u' was visited already by assumption and obtain

$$\frac{1}{deg(u')-1} \le \frac{\bar{\mu}}{deg(u')} \iff \frac{deg(u')}{deg(u')-1} \le \bar{\mu}\,.$$

We conclude for the mean offspring $\bar{\mu}$ of BMC_N that a necessary condition for our majorization is

$$\frac{d_{min}+1}{d_{min}} \leq \bar{\mu} \tag{2.17}$$

with $d_{min} := \min\{k \geq 2 : p_k > 0\}$ is a necessary condition for our majorization. Using the new model FM$''$ we are left with only the first visit of type 3 to the stretch. As we have seen before, there is a N such that (2.16) holds.

Now, we will treat the third part, where we allow multiple visits of type 2 and 3 to a sequence of vertices. We want to erase again multiple visits of type 3 of a stretch and assume, that (2.17) holds, such that the BMC_N dominates the conditioned path. Then it remains to deal with either a first visit of type 2 or a first visit of type 3 and multiple visits of type 2. If the first visit is of type 2, we can bound the frog model by using (2.9) additionally to (2.17).

If the first visit is of type 3, and we have apart from other visits of type 3 (which will be erased and bounded using (2.17)) k visits and returns of type 2, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathsf{ST}}\left[m^{\mathsf{T}'}_{\mathrm{FM}'}(u'\to v'\to u')m^{\mathsf{T}',2+}_{\mathrm{FM}'}(w'\to v'\to u')^k m^{\mathsf{T}',2+}_{\mathrm{FM}'}(u'\to v'\to w')^k\right]\\
&=\left(\frac{1}{deg(u')}\right)^{k+1}\mathbb{E}_{\mathsf{ST}}\left[\left(\frac{\ell\bar{\eta}}{2}+\left(\frac{\ell}{\ell+1}\right)\right)\left(\frac{1}{\ell+1}\right)^{2k}\right]\left(\frac{1}{deg(w')}\right)^{k}.
\end{aligned}$$

For the upcoming equations we omit the factors of the transitions probabilities from u' to v' and from w' to v'. These probabilities are the same for the BMC and do not play a role for the comparison with the frog model. Then we get:

$$\begin{aligned}
&\sum_{\ell=1}^{\infty}\frac{\ell\bar{\eta}}{2}\left(\frac{1}{\ell+1}\right)^{2k}p_1^{\ell-1}(1-p_1)+\sum_{\ell=1}^{\infty}\left(\frac{\ell}{\ell+1}\right)\left(\frac{1}{\ell+1}\right)^{2k}p_1^{\ell-1}(1-p_1)\\
&\leq\frac{\bar{\eta}}{2}\left(\frac{1-p_1}{p_1^2}\right)\sum_{\ell=1}^{\infty}\frac{p_1^{\ell+1}}{(\ell+1)^{2k-1}}+\sum_{\ell=1}^{\infty}\left(\frac{\ell}{\ell+1}\right)\left(\frac{1}{\ell+1}\right)^{2k}p_1^{\ell}(1-p_1).
\end{aligned}\tag{2.18}$$

For the BMC we have the following identities as before:

$$\begin{aligned}
&\mathbb{E}_{\mathsf{ST}}\left[m^{\mathsf{T}'}_{\mathrm{BMC}'}(u'\to v'\to u')m^{\mathsf{T}'}_{\mathrm{BMC}'}(u'\to v'\to w')^k m^{\mathsf{T}'}_{\mathrm{BMC}'}(w'\to v'\to u')^k\right]\\
&=\left(\frac{1}{deg(u')}\right)^{k+1}\mathbb{E}_{\mathsf{ST}}\left[F_{\ell+1}(1,0|\bar{\mu})^2F_{\ell+1}(\ell,\ell+1|\bar{\mu})^{2k}\right]\left(\frac{1}{deg(w')}\right)^{k}
\end{aligned}$$

By Lemma 2.6.8 (and again omitting the transitions probabilities) this is greater or equal to

$$\sum_{\ell=1}^{N-1}\frac{\ell}{\ell+1}\left(\frac{1}{\ell+1}\right)^{2k}\left(1+\frac{(1+2\ell)\,\varphi^2}{3!}\right)p_1^{\ell-1}(1-p_1)$$

$$+\frac{N}{N+1}\left(\frac{1}{N+1}\right)^{2k}\left(1+\frac{(1+2N)\,\varphi^2}{3!}\right)p_1^N. \tag{2.19}$$

We want to show that we can choose for each p_1 and N an η such that the following holds for all $k \geq 1$:

$$\frac{\bar{\eta}}{2}\left(\frac{1-p_1}{p_1^2}\right)\left(\sum_{\ell=1}^{N-1}\frac{p_1^{\ell+1}}{(\ell+1)^{2k-1}}+\sum_{\ell=N}^{\infty}\frac{p_1^{\ell+1}}{(\ell+1)^{2k-1}}\right) \tag{2.20}$$
$$+\sum_{\ell=1}^{N-1}\left(\frac{\ell}{\ell+1}\right)\left(\frac{1}{\ell+1}\right)^{2k}p_1^{\ell-1}(1-p_1) \tag{2.21}$$
$$+\sum_{\ell=N}^{\infty}\left(\frac{\ell}{\ell+1}\right)\left(\frac{1}{\ell+1}\right)^{2k}p_1^{\ell-1}(1-p_1) \tag{2.22}$$
$$\leq \sum_{\ell=1}^{N-1}\frac{\ell}{\ell+1}\left(\frac{1}{\ell+1}\right)^{2k}p_1^{\ell-1}(1-p_1) \tag{2.23}$$
$$+\sum_{\ell=1}^{N-1}\frac{\ell}{\ell+1}\left(\frac{1}{\ell+1}\right)^{2k}\left(\frac{(1+2\ell)\,\varphi^2}{3!}\right)p_1^{\ell-1}(1-p_1) \tag{2.24}$$
$$+\frac{N}{N+1}\left(\frac{1}{N+1}\right)^{2k}p_1^{N-1}+\frac{N}{N+1}\left(\frac{1}{N+1}\right)^{2k}\left(\frac{(1+2N)\,\varphi^2}{3!}\right)p_1^{N-1}. \tag{2.25}$$

The second part of the left hand side, (2.21), is equal to the first part, (2.23), on the right hand side. Next we compare the third part of the left, (2.22), to the third part on the right, (2.25). We notice that the function $\left(\frac{\ell}{\ell+1}\right)\left(\frac{1}{\ell+1}\right)^{2k}$ is monotonously decreasing in ℓ and thus

$$\sum_{\ell=N}^{\infty}\left(\frac{\ell}{\ell+1}\right)\left(\frac{1}{\ell+1}\right)^{2k}p_1^{\ell-1}(1-p_1)\leq\left(\frac{N}{(N+1)^{2k+1}}\right)\sum_{\ell=N}^{\infty}p_1^{\ell-1}(1-p_1)$$
$$=\left(\frac{N}{(N+1)^{2k+1}}\right)p_1^{N-1}.$$

Now, we consider the remaining term on the left hand side, (2.20), and the second of the right hand side, (2.24). We start with giving an upper bound for the second sum in (2.20):

$$\sum_{\ell=N}^{\infty}\frac{p_1^{\ell+1}}{(\ell+1)^{2k-1}}\leq\left(\frac{1}{N+1}\right)^{2k-1}\sum_{\ell=N-1}^{\infty}\frac{p_1^{\ell}}{(1-p_1)}=\left(\frac{1}{N+1}\right)^{2k-1}\frac{p_1^{N-1}}{(1-p_1)}.$$

The second term of the right hand side, (2.24), can be transformed into

$$\sum_{\ell=1}^{N-1}\frac{\ell}{\ell+1}\left(\frac{1}{\ell+1}\right)^{2k}\left(\frac{(1+2\ell)\,\varphi^2}{3!}\right)p_1^{\ell-1}(1-p_1)$$

$$\geq \frac{1-p_1}{p_1^2}\left(\frac{\varphi^2}{3!(N+1)^2}\right)\sum_{l=1}^{N-1}\left(\frac{2\ell^2}{(\ell+1)^2}\right)\left(\frac{p_1^{\ell+1}}{(\ell+1)^{2k-1}}\right)$$
$$\geq \frac{1-p_1}{p_1^2}\left(\frac{(\arccos(1/\bar{\mu}))^2}{3!(N+1)^2}\right)\frac{1}{2}\sum_{\ell=1}^{N-1}\left(\frac{p_1^{\ell+1}}{(\ell+1)^{2k-1}}\right).$$

We have that (2.20) $<$ (2.24) if

$$\frac{\bar{\eta}}{2}\left(\frac{1-p_1}{p_1^2}\right)\left(\sum_{\ell=1}^{N-1}\frac{p_1^{\ell+1}}{(\ell+1)^{2k-1}}+\left(\frac{1}{N+1}\right)^{2k-1}\frac{p_1^{N-1}}{(1-p_1)}\right)$$
$$\leq \left(\frac{1-p_1}{p_1^2}\right)\left(\frac{(\arccos(1/\bar{\mu}))^2}{3!(N+1)^2}\right)\frac{1}{2}\sum_{\ell=1}^{N-1}\left(\frac{p_1^{\ell+1}}{(\ell+1)^{2k-1}}\right).$$

For all choices of p_1 and $N \in \mathbb{N}$ we can now find $\bar{\eta}$ sufficiently small such that the latter inequality is verified for all $k \in \mathbb{N}$.

Summary

We summarize all the conditions on η and $\bar{\mu}$ such that we can find a dominating transient BMC for a given frog model FM in the case when stretches come up:

1. $1 + \bar{\eta} < \bar{\mu}$;
2. $\frac{d_{min}+1}{d_{min}} \leq \bar{\mu}$;
3. Choosing η such that $\bar{\eta}$ is small enough such that there exists a N such that (2.14) holds;
4. Choosing η such that for given p_1 and the previously selected N the inequality (2.20)-(2.25) holds;
5. $\bar{\mu} < \left(\cos\frac{\arccos\left(\frac{2\sqrt{d_{min}}}{d_{min}+1}\right)}{N+1}\right)^{-1}$.

In other words, for every $p_1 > 0$ there exists some N such that if

$$\frac{d_{min}+1}{d_{min}} < \left(\cos\frac{\arccos\left(\frac{2\sqrt{d_{min}}}{d_{min}+1}\right)}{N+1}\right)^{-1} \tag{2.26}$$

there exists some small $\bar{\eta} > 0$ and some BMC$'$ with mean offspring larger than 1 such that $\mathbb{E}[\nu_{\mathrm{BMC}'}] < \infty$ and

$$\nu'_{\mathsf{ST}}(\mathsf{p}') \leq \mathbb{E}[\nu_{\mathrm{BMC}'}(\mathsf{p}')] \tag{2.27}$$

for all paths p' and GW-a.a. trees T'. Finally, we found that $\nu' < \infty$ FM-a.s. for GW-a.a. trees and hence $\nu < \infty$ FM-a.s. for GW-a.a. trees. The existence of the constant c_η follows from the 0–1-law of transience.

The existence of a transient phase is guaranteed since for all N there exists d_{min} fulfilling (2.26) as (2.26) is equal to

$$N + 1 < \frac{\arccos\left(\frac{2\sqrt{d_{min}}}{d_{min}+1}\right)}{\arccos\left(\frac{d_{min}}{d_{min}+1}\right)}$$

and the right hand side converges to ∞ for $d_{min} \to \infty$. □

2.5.3 Bushes and possible stretches

It is left to prove the main theorem of this chapter where we allow $p_0 > 0$. The proof starts with the following modification: Once a frog visits a vertex $v \in \mathsf{T}$ with bushes attached, all frogs in the bushes are woken up and placed at v. This is equivalent to changing the number of frogs in v and conditioning the frogs not to enter the bush. The erasure of the bushes does not change the transience behaviour of the process. Following this procedure, we end up with trees with stretches and without bushes and we can then apply the proof of Proposition 2.5.2.

Proof (Theorem 2.1.3) We assume that $p_0 > 0$ and start with explaining how we remove the bushes.

Removing bushes from T

Every infinite GW-tree can be seen as a multitype GW-tree $\bar{\mathbb{T}}$ with types g and b, see Section 2.2. We denote by T a realization of GW conditioned to be infinite. Moreover we recall that our GW-tree has bounded offspring: there is a $K = d_{max} < \infty$ such that $Y_i^{(n)} \leq d_{max}$ for all $i, n \in \mathbb{N}$. Therefore, every vertex which is part of a geodesic stretch can have at most $K - 1$ finite bushes attached.

To start with, we modify the original frog model FM. If a frog visits a vertex $v \in \mathsf{T}$ with attached bushes for the first time, then immediately all frogs from the bushes attached to v wake up and are placed at v. As $K + 1$ is the maximum degree of the tree, we know that there are at most $K - 1$ bushes attached to a vertex of type g. More formally, let $v_i, i \in \{1, \dots, k\}$ and $k \leq K - 1$, the vertices of type b adjacent to v and let G_{v_i} denote the random bush starting with root $v_i \in \mathsf{T}$. Then, there will be $\eta^*(v) := \sum_{i=1}^{k} \sum_{w \in G_{v_i}} \eta(w)$ frogs in vertex v with attached bushes and $\eta^*(u) := \eta_u$ frogs in a vertex of type g with no attached bushes. The bushes G_{v_i} are i.i.d. distributed like a subcritical GW-process with generating function $\tilde{f}$, see Section 2.2, and the expected size of G_{v_i} is finite. Conditioning the frog model on not entering bushes we obtain different transition probabilities for each frog. Let v be a vertex with neighboured bushes, $v_1, \dots, v_k$, $k \leq K - 1$ the attached roots of bushes and $w_1, \ldots, w_d$, $d \leq K + 1 - k$ its neighbours of type g. Then we obtain

$$\mathsf{P}[v \to w_i | \text{ not entering a bush}] = \frac{1}{d} \tag{2.28}$$

as new transition probabilities. This coincides with the probability of the first exit towards a neighbour w_i of type g starting in v. The new model actually lives on a new state space $\widehat{\mathsf{T}}$ that arises from T by erasing all bushes, see also Figure 2.6. Then, we identify the frog

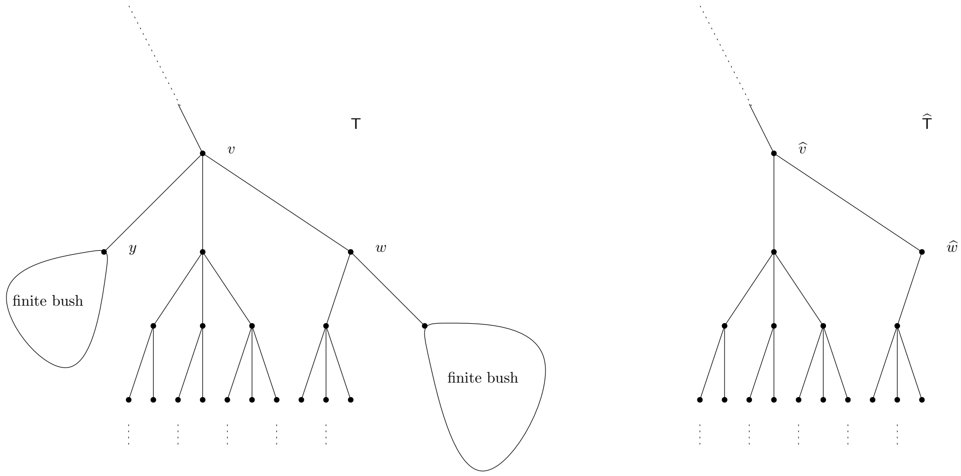

Figure 2.6: Construction of $\widehat{\mathsf{T}}$ from T by deleting all bushes.

configuration by $\eta^*(\widehat{v}) = \eta^*(v)$ of the two models on T and $\widehat{\mathsf{T}}$ and obtain the new frog model $\mathrm{FM}((\eta^*(\widehat{v}))_{\widehat{v}\in\widehat{\mathsf{T}}}, \widehat{\mathsf{T}})$. We keep here the whole sequence of random variables in the frog configuration to point out that the random variables are not identically distributed.

We denote by ν the number of visits to the root in FM and by $\widehat{\nu}$ the number of visits to the root in $\mathrm{FM}((\eta^*(\widehat{v}))_{\widehat{v}\in\widehat{\mathsf{T}}}, \widehat{\mathsf{T}})$. Coupling the frog configuration at each vertex as in Lemma 2.5.1 we find for each frog in FM a corresponding frog in $\mathrm{FM}((\eta^*(\widehat{v}))_{\widehat{v}\in\widehat{\mathsf{T}}}, \widehat{\mathsf{T}})$. Using a coupling as in the proof of Lemma 2.5.1 we obtain that

$$\widehat{\nu} \succeq \nu. \tag{2.29}$$

Thus transience of $\mathrm{FM}((\eta^*(\widehat{v}))_{\widehat{v}\in\widehat{\mathsf{T}}}, \widehat{\mathsf{T}})$ implies transience of FM.

Construction of a dominating BMC

Removing all bushes, we have to be aware that a sequence of vertices with only one child of type g will create new stretches, see Figure 2.7. Hence, we need to go on by using Proposition 2.5.2. But the newly appeared stretches can be unbalanced in the sense that some vertices were former neighbours to bushes and have the corresponding offspring and some not. This would inhibit the number of frogs emerging to the ends of the stretch to be equally distributed. Therefore, we modify the frog model in the following way: a vertex

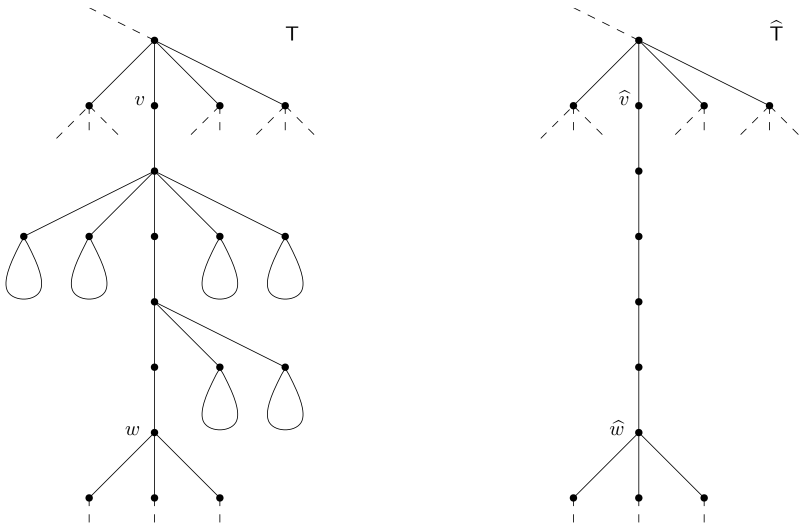

Figure 2.7: Creating a stretch during the modification from T to $\widehat{\mathsf{T}}$.

can have an offspring of at most K. Therefore, every vertex which is part of a stretch could have at most $K-1$ finite bushes attached. We set $\widehat{\eta}(\widehat{v}) := \sum_{i=1}^{K-1} \sum_{\widehat{w} \in G_{\widehat{v}_i}} \eta(\widehat{w})$ with $G_{\widehat{v}_i}$ being finite bushes generated according to $\mathbb{T}^{sub}$ for each vertex $\widehat{v} \in \widehat{\mathsf{T}}$ and notice that $(\widehat{\eta}(\widehat{v}))_{\widehat{v} \in \widehat{\mathsf{T}}}$ is a sequence of i.i.d. random variables and we call their common measure $\widehat{\eta}$. Then, the model $\mathrm{FM}((\eta^*(\widehat{v}))_{\widehat{v} \in \widehat{\mathsf{T}}}, \widehat{\mathsf{T}})$ is dominated by $\widehat{\mathrm{FM}} := \mathrm{FM}(\widehat{\mathsf{T}}, \widehat{\eta})$, as there are only more particles in the new model and we can couple the two processes such that every visit in $\mathrm{FM}((\eta^*(\widehat{v}))_{\widehat{v} \in \widehat{\mathsf{T}}}, \widehat{\mathsf{T}})$ has a corresponding visit in $\widehat{\mathrm{FM}}$.

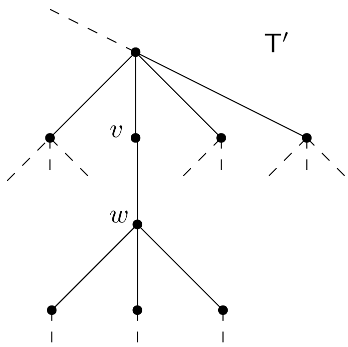

Figure 2.8: The modification of the stretch in Figure 2.7 in the step from $\widehat{\mathsf{T}}$ to T'.

In the same manner as in Proposition 2.5.2 we want to couple $\widehat{\mathrm{FM}}$ with a modified model FM' doing the same steps as in Proposition 2.5.2: if a frog enters a stretch all frogs from the stretch are woken up and placed according to their exit measures at the two ends of the stretch. This results in the modified state space T' by merging the stretches into one vertex like in Proposition 2.5.2, see Figure 2.8. As there are $\sum_{i=1}^{K-1} \sum_{\widehat{w} \in G_{\widehat{v}_i}} \eta(\widehat{w})$ frogs placed on each vertex, from a stretch of length $\ell_{\widehat{v}}$ leave on average

$$\mathbb{E}^{\mathsf{T}'}_{v'}[F_{u'}] = \frac{\ell_{\widehat{v}}\mathbb{E}[G](K-1)\bar{\eta}}{2} + \frac{\ell_{\widehat{v}}}{\ell_{\widehat{v}}+1},$$
$$\mathbb{E}^{\mathsf{T}'}_{v'}[F_{w'}] = \frac{\ell_{\widehat{v}}(K-1)\mathbb{E}[G]\bar{\eta}}{2} + \frac{1}{\ell_{\widehat{v}}+1}$$

frogs to the two ends of the stretch. Here, the length ℓ of the stretch is distributed according to $geo(\widehat{p}_1)+1$, where $\widehat{p}_1$ is the probability of having only one child of type g. For the construction of a dominating BMC let again $N \in \mathbb{N}$ and define the tree $\widehat{\mathsf{T}}_N$ as a copy of $\widehat{\mathsf{T}}$, where each stretch of length larger than N is replaced by a stretch of length N. On this tree we define again $\widehat{\mathrm{BMC}}_N$, on $\widehat{\mathsf{T}}_N$, with driving measure SRW and the offspring distribution μ is equal to the distribution which fulfills

$$\mu_k(\widehat{v}) = \mathbb{P}\left[\sum_{i=1}^{K-1}\sum_{\widehat{w}\in G_{\widehat{y}_i}} \eta(\widehat{w}) + 1 = k\right]$$

for any $\widehat{v} \in \widehat{\mathsf{T}_N}$. Its mean offspring is denoted by $\bar{\mu}$. We recall that T' is the tree, where the stretches of maximum length N are compressed to a single vertex (similar to Proposition 2.5.2). Then $\widehat{\mathrm{BMC}}_N$ defines naturally a $\mathrm{BMC}' = \widehat{\mathrm{BMC}}'_N$ on T': Once a particle enters a former stretch, it produces offspring particles according to the exit-measures.

To find an $N \in \mathbb{N}$ such that BMC' is dominating for FM' we proceed like in the proof of Proposition 2.5.2 with the difference that in average to both sides of a geodesic stretch of length ℓ exit

$$\frac{\ell\mathbb{E}[G](K-1)\bar{\eta}}{2}$$

frogs instead of $\frac{\bar{\eta}\ell}{2}$ frogs. The frog which is waking up the stretch leaves the stretch to each side with the same probability as before. Moreover the length of the stretch is now distributed according to $geo(\widehat{p}_1)+1$ and the probability that a vertex is dedicated as a starting vertex of a stretch is $ber(\widehat{p}_1)$-distributed, as well.

The BMC' has to fulfill the transience criterion Theorem 2.3.1, as well. We notice, that $\widehat{\mathsf{T}}_N$ corresponds to the tree T_N from the construction of the dominating Branching Markov chain in the proof of Proposition 2.5.2 and

$$\rho(\widehat{\mathsf{T}}_N) = \left(\cos\frac{\arccos\left(\frac{2\sqrt{d_{min}}}{d_{min}+1}\right)}{N+1}\right)$$

with $d_{min} = \min\{k \geq 2 : p_k > 0\}$. All together, using the same line of arguments as in Proposition 2.5.2, we have the following conditions on η and $\bar{\mu}$ such that there exists a dominating BMC':

1. $1 + \mathbb{E}[G]\bar{\eta}(K-1) < \bar{\mu}$;

2. $\frac{d_{min}+1}{d_{min}} \le \bar{\mu}$ where $d_{min} = \min\{k \ge 2 : \widehat{p}_k > 0\}$;
3. Choosing η such that $\bar{\eta}$ is small enough such that there exists an N such that

$$\left(\frac{\bar{\eta}\,\mathbb{E}[G](K-1)}{2} \left(\frac{1}{1-\widehat{p}_1} + 1 \right) + \sum_{\ell=1}^{\infty} \left(\frac{\ell}{\ell+1} \right) \widehat{p}_1^{\ell-1}(1-\widehat{p}_1) \right)$$
$$< \sum_{\ell=1}^{N-1} g(\ell+1)\widehat{p}_1^{\ell-1}(1-\widehat{p}_1) + g(N)\widehat{p}_1^{N}.$$

4. Choosing η such that for given $\widehat{p}_1$ and the previously selected N equation

$$\bar{\eta}\,\mathbb{E}[G](K-1) \left(\frac{3!(N+1)^2}{(\arccos(1/\bar{\mu})^2} \right) \left(\sum_{\ell=1}^{N-1} \frac{\widehat{p}_1^{\ell+1}}{(l+1)^{2k-1}} + \left(\frac{1}{N+1} \right) \frac{\widehat{p}_1^{N-1}}{(1-\widehat{p}_1)} \right)$$
$$\le \sum_{l=1}^{N-1} \left(\frac{\widehat{p}_1^{l+1}}{(l+1)^{2k-1}} \right)$$

holds;

5. $\bar{\mu} < \left(\cos \frac{\arccos\left(\frac{2\sqrt{d_{min}}}{d_{min}+1} \right)}{N+1} \right)^{-1}$. □

We can conclude similar to Proposition 2.5.2.

2.6 Some properties of Galton–Watson trees and branching random walks

2.6.1 The relation with generating functions

At various places we have used generating functions. They are a crucial tool in the study of BMC, e.g., see [5], [7], [26], [46], and [69]. Let M be a subset of the state space and modify the BMC in a way such that particles are absorbed in M and once they have arrived in M, they keep on producing one offspring a.s. In other words, particles arriving in M are *frozen*. Set $Z_\infty(M) \in \mathbb{N} \cup \{\infty\}$ as the total number of frozen particles in M at time "∞". For $M \subseteq \Gamma$, we define the first visiting generating function:

$$F(x, M|z) := \sum_{n \ge 0} \mathbb{P}\big[Z_n \in M, \forall m \le n-1 : Z_m \notin M \mid X_0 = x\big] z^n,$$

where Z_n is the original SRW and $\mathbb{P}$ its corresponding probability measure The following lemma will be used several times in our proofs; a short proof can be found for example in [7, Lemma 4.2].

Lemma 2.6.1 *Let $\bar{\mu}$ be the mean offspring of the BMC. For any $M \subseteq \Gamma$, we have*

$$\mathbb{E}\big[Z_\infty(M)\big] = F(e, M|\bar{\mu}).$$

2.6.2 Spectral radius of trees

In order to study recurrence and transience of a BMC it is essential to understand the spectral radius of the underlying Markov chain. In this section, we collect several results on the spectral radius of SRW on trees.

Definition 2.6.2 The isoperimetric constant $\iota(T)$ of a tree with edges E and vertices V is defined by

$$\iota(T) := \inf \left\{ \frac{\mid \delta_E F \mid}{Vol(F)} : F \subset X \text{ finite} \right\}$$

where $\delta_E F = E(F, X \setminus F)$ is the set of edges connecting F with $T \setminus F$ and $Vol(F) = \sum_{x \in F} deg(x)$.

For the isoperimetric constant it holds that, $\iota(T) = 0$ if and only if the spectral radius $\rho(T)$ of the simple random walk equal to 1, see Theorem 10.3 in [68].

There is a more precise statement on finite approximation of the spectral radius, e.g., see [4] and [50]. Consider an infinite irreducible Markov chain (X, P) and write $\rho(P)$ for its spectral radius. A subset $Y \subset X$ is called irreducible if the sub-stochastic operator

$$P_Y = (p_Y(x, y))_{x,y \in Y}$$

defined by $p_Y(x, y) := p(x, y)$ for all $x, y \in Y$ is irreducible. It is rather straightforward to show the next characterization.

Lemma 2.6.3 *Let (X, P) be an irreducible Markov chain. Then,*

$$\rho(P) = \sup_Y \rho(P_Y), \tag{2.30}$$

where the supremum is over finite and irreducible subsets $Y \subset X$. Furthermore, $\rho(P_F) < \rho(P_G)$ if $F \subsetneq G$.

We compare this also to the Perron-Frobenius theorem, see for example [60], especially for the last inequality. A first observation is the following result, see [69, Lemma 9.86]. We say that a stretch (or unbranched path) of length N in a tree T is a path $[v_0, v_1, \ldots, v_N]$ of distinct vertices such that $deg(v_k) = 2$ for $k = 1, \ldots, N-1$.

Lemma 2.6.4 *Let T be a locally finite tree T. If T contains stretches of arbitrary length, then $\rho(T) = 1$.*

Moreover, we can give a precise characterization of the spectral radius of a simple random walk on a GW-tree

Lemma 2.6.5 *Let $\rho(\mathsf{T})$ be the spectral radius of the simple random walk on a Galton–Watson tree T with offspring distribution $(p_i)_{i \geq 0}$. Then,*

- *if $p_0 + p_1 > 0$ we have $\rho(\mathsf{T}) = 1$ for GW-a.a. realizations T;*

- *if* $p_0 + p_1 = 0$ *we have* $\rho(\mathsf{T}) = \rho(T_{d+1}) = \frac{2\sqrt{d}}{d+1} < 1$ *for* GW*-a.a. infinite realizations* T,

where $d = \min\{i : p_i > 0\}$ *and* T_{d+1} *is the homogeneous tree with offspring* d.

Proof If T is finite, the simple random walk is recurrent and it holds that $\rho(\mathsf{T}) = 1$, see Section 1 in [68]. Now, let us assume that T is infinite. In the case where $p_1 > 0$ the tree contains, for every choice of $N \in \mathbb{N}$, GW-a.s. a stretch of length N; this is a consequence of the lemma of Borel–Cantelli. Using Lemma 2.6.4 we conclude that $\rho(\mathsf{T}) = 1$. Now, we assume that $p_1 = 0$ but $p_0 > 0$. In this case the tree T contains, for every choice of $N \in \mathbb{N}$, a finite bush of N generations, which we call bush B_N. For such a bush B_N it holds that $\frac{|\delta_E B_N|}{Vol(B_N)} \leq \frac{1}{2N}$. Again, by finding an arbitrary large bush we obtain $\iota(\mathsf{T}) = 0$ and consequently using Theorem 10.3 in [68] we conclude $\rho(\mathsf{T}) = 1$. In the case $p_0 + p_1 = 0$ Corollary 9.85 in [69] implies that $\rho(\mathsf{T}) \leq \rho(T_{d+1}) = \frac{2\sqrt{d}}{d+1}$ where d is the smallest offspring of the Galton–Watson tree and T_{d+1} denotes the homogeneous tree with offspring d. The remaining equality follows by finding arbitrarily large balls of T_{d+1} as copies in T as above and applying Lemma 2.6.3. $\square$

We construct a new tree $\widetilde{T}$ by replacing each edge e of T with a stretch of length $k = k(e)$. We call $\widetilde{T}$ a subdivision of T and $\max_e\{k(e)\}$ the maximal subdivision length of $\widetilde{T}$. We write $T_{(N)}$ for the subdivision of T where $k(e) = N$ for all edges e in T. We state a particular case of Theorem 9.89 in [69].

Theorem 2.6.6 *Let* T *be a locally finite tree and denote* $\rho(T)$ *(resp.* $\rho(T_{(N)})$*) the spectral radius of the SRW on* T *(resp.* $T_{(N)}$*). Then,*

a)
$$\rho(T_{(N)}) = \cos\frac{\arccos\rho(T)}{N}; \tag{2.31}$$

b) if $\widetilde{T}$ *is an arbitrary subdivision of* T *of maximal subdivision length* N *then*
$$\rho(T) \leq \rho(\widetilde{T}) \leq \rho(T_{(N)}). \tag{2.32}$$

2.6.3 Absorbing BMC on finite paths

We consider the SRW, $(Z_n)_{n\geq 0}$, on an unbranched path of length N with absorbing states v_0 and v_N. In other words, we consider the ruin problem (or birth-death chain) on $[N] := \{0, 1, \ldots, N\}$ defined through the transition kernel $P_N = (p_N(x,y))_{x,y\in[N]}$: $p_N(0,0) = p_N(0,N) = 1$ and $p_N(x,x+1) = p_N(x,x-1) = 1/2$ for $1 \leq x \leq N-1$. We set $\rho(P_N)$ for the spectral radius of the reducible class $\{1,\ldots,N-1\}$. Let

$$f_N^{(n)}(x,y) := \mathbb{P}[Z_n = y, Z_k \neq y \ \forall\ 0 \leq k < n | Z_0 = x] \tag{2.33}$$

and define the first visit generating function

$$F_N(x,y|z) := \sum_{n=0}^{\infty} f_N^{(n)}(x,y) z^n. \tag{2.34}$$

The convergence radius of the power series equals $R_N = 1/\rho(P_N)$.

We give the following expressions of the generating function F_N for two particular pairs of values of x and y, see Example 5.6 in [69].

Lemma 2.6.7 *Let $1 \leq z \leq R_N$ and φ such that $1/z = \cos\varphi$. Then,*

$$F_N\left(1, N \middle| \frac{1}{\cos\varphi}\right) = \frac{\sin\varphi}{\sin N\varphi} \text{ and } F_N\left(N-1, N \middle| \frac{1}{\cos\varphi}\right) = \frac{\sin(N-1)\varphi}{\sin N\varphi}. \tag{2.35}$$

We present lower bounds of these generating functions; the index shift is done to improve the presentation of the proofs in the main part.

Lemma 2.6.8 *Let $1 \leq z \leq R_N$ and φ such that $1/z = \cos\varphi$. Then,*

$$F_{N+1}\left(N, N+1 \middle| \frac{1}{\cos\varphi}\right) \geq \frac{N}{N+1}\left(1 + \frac{(1+2N)\,\varphi^2}{3!}\right) \tag{2.36}$$

$$F_{N+1}\left(1, N+1 \middle| \frac{1}{\cos\varphi}\right) \geq \frac{1}{N+1}\left(1 + \frac{\left(2N+N^2\right)\varphi^2}{3!}\right) \tag{2.37}$$

Proof For proving the above approximations we will use the infinite product expansion

$$\sin(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right),\ z \in \mathbb{C}$$

and the power series expansion

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},\ z \in \mathbb{C}$$

of the sine. Now,

$$F_N\left(N, N+1 \middle| \frac{1}{\cos\varphi}\right) = \frac{\sin(N)\varphi}{\sin(N+1)\varphi} = \frac{N\varphi}{(N+1)\varphi}\frac{\prod_{n=1}^{\infty}\left(1 - \frac{(N\varphi)^2}{n^2\pi^2}\right)}{\prod_{n=1}^{\infty}\left(1 - \frac{((N+1)\varphi)^2}{n^2\pi^2}\right)} \tag{2.38}$$

$$= \frac{N}{N+1}\frac{\prod_{n=1}^{\infty}\left(1 - \frac{N^2\varphi^2+2N\varphi^2+\varphi^2}{n^2\pi^2} + \frac{2N\varphi^2+\varphi^2}{n^2\pi^2}\right)}{\prod_{n=1}^{\infty}\left(1 - \frac{N^2\varphi^2+2N\varphi^2+\varphi^2}{n^2\pi^2}\right)} \tag{2.39}$$

$$\geq \frac{N}{N+1}\left(\prod_{n=1}^{\infty}\left(1 + \frac{2N\varphi^2+\varphi^2}{n^2\pi^2}\right)\right). \tag{2.40}$$

Defining $z = i\varphi\sqrt{1+2N}$ we obtain by using the product expansion and afterwards the power series expansion, that

$$\begin{aligned}
\frac{N}{N+1}\left(\prod_{n=1}^{\infty}\left(1+\frac{2N\varphi^2+\varphi^2}{n^2\pi^2}\right)\right) &= \frac{N}{N+1}\frac{\sin(z)}{z} \\
&\geq \frac{N}{N+1}\left(1+\frac{\varphi^2(1+2N)}{3!}\right).
\end{aligned}$$

The second part follows the exact same line as the first part of the proof. □

Chapter 3

The relation between quenched and annealed Lyapunov exponents in random potential on trees

Our subject of interest is a simple symmetric random walk on the integers which faces a random risk to be killed. This risk is described by random potentials, which are defined by a sequence of independent and identically distributed non-negative random variables. To determine the risk of taking a walk in these potentials we consider the decay of the Green function. There are two possible tools to describe this decay: The quenched Lyapunov exponent and the annealed Lyapunov exponent. It turns out that on the integers and on regular trees we can state a precise relation between these two.

3.1 Random walks with random killing

We consider a *simple random walk* $(S_n)_{n\geq 0}$ on the integers $\mathbb{Z}$ with starting point x. At each point of time it jumps independently of all the steps before with probability $\frac{1}{2}$ to the right or to the left. The path measure of the random walk will be denoted by P_x and the expectation value with respect to P_x by E_x. The simple symmetric random walk is a Markov process. Moreover, it is spatially homogeneous.

Furthermore, we attach to each site $x \in \mathbb{Z}$ a so called *random potential* $\omega(x)$ which influences the movement of the random walk. We assume that $\omega := (\omega(x))_{x\in\mathbb{Z}}$ is a sequence of nonnegative random variables which are independently and identically distributed (i.i.d.) by the common measure ν on $[0,\infty)$. From this we obtain the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ described by

$$\Omega := [0,\infty)^{\mathbb{Z}}$$

with its usual Borelian product σ-algebra $\mathcal{F}$ and the product measure

$$\mathbb{P} := \bigotimes_{x\in\mathbb{Z}} \nu \,.$$

The expectation value derived with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$. We assume that the potentials are not concentrated at 0, that is $\nu \neq \delta_0$, to avoid the trivial case. Considering the standard shift $T_i : \Omega \to \Omega$, $(\omega(x))_{x\in\mathbb{Z}} \mapsto ((\omega(x-i)_{x\in\mathbb{Z}})$ for $i \in \mathbb{Z}$ it is clear that $\mathbb{P}$ is shift invariant.

The random potentials represent a certain risk of dying for the random walk. For a fixed realization of the environment $\omega = (\omega(x))_{x\in\mathbb{Z}}$ the random walk dies with probability

$$q(x) := 1 - \exp(-\omega(x))$$

at each site $x \in \mathbb{Z}$ it reaches. If the random walk survives the site, it will uniformly choose the next site of its journey. Thus, the random walk will jump with probability $(1-q)/2$ to the right or to the left, see Figure 3.1. So, given the potentials, it becomes dangerous to take a walk in these environments.

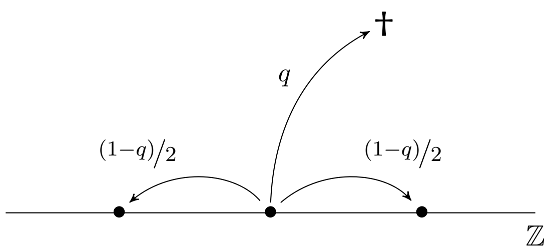

Figure 3.1: Simple random walk with random killing where $q(x) := 1 - \exp(-\omega(x))$ for a fixed realization of the potentials ω.

One measurement for the risk is the *Green function.* For $x, y \in \mathbb{Z}$ and a realization of the potential ω we define it for the random walk with random killing by

$$g(x,y,\omega) := \sum_{m>0} \mathsf{E}_x \left[\exp\left(-\sum_{i=0}^{m} \omega(S_i) \right) \mathbf{1}_{\{S_m = y\}} \right].$$

This is the expected number of visits in y before the random walk starting at x dies. We want to study the decay of g. In other words: how risky is it to walk around in this environment over long distances, that is if $|x-y| \to \infty$? Moreover g is still a random variable in ω and we may ask the same for the averaged Green function with respect to $\mathbb{P}$.

3.2 Quenched and annealed Lyapunov exponents

The Lyapunov exponents give us a precise description of the decay of Green's function. We start by shortly introducing them. Therefore we are in need of another two-point function, which is closely related to g.

First of all we define the stopping times $\tau_y, y \in \mathbb{Z}$, given by

$$\tau_y := \inf\{n \geq 0 \,:\, S_n = y\}$$

for each $y \in \mathbb{Z}$. This is the first point of time where the original random walk $(S_n)_{n\geq 0}$ hits $y \in \mathbb{Z}$.

Definition 3.2.1 For any realization of the potentials $\omega \in \Omega$ and $x, y \in \mathbb{Z}$ we define the two-point-functions

$$\begin{aligned} e(x,y,\omega) :=& \mathsf{E}_x \left[\exp\left(-\sum_{k=0}^{\tau_y - 1} \omega(S_k) \right),\, \tau_y < \infty \right], \\ a(x,y,\omega) :=& -\ln e(x,y,\omega)\,. \end{aligned}$$

The quantity $e(x,y,\omega)$ represents the probability of the random walk reaching y after it started at x and before it dies due to a fixed potential ω. As before, $e(x,y,\cdot)$ is a random variable in ω. Consequently, the expected probability of surviving a journey from x to y will be of interest to us hereafter. This procedure of taking the average is also called annealing the environment.

Definition 3.2.2 Let $x, y \in \mathbb{Z}$. We define

$$\begin{aligned} f(x,y) :=& \mathbb{E}[e(x,y,\omega)]\,, \\ b(x,y) :=& -\ln f(x,y)\,. \end{aligned}$$

To answer the above stated question we observe the behaviour of e and f in the long-run, that is when the distance of x and y tends to infinity. Looking for a precise description we turn to *Lyapunov exponents*. There are two ways of dealing with the random potential – the *quenched* and the *annealed* case.

In the quenched case we look at the exponential decay of the survival rate for a frozen realization of the potentials.

Proposition 3.2.3 *We suppose that ν has finite expectation. Then, for all $x \in \mathbb{Z}$ there exists the limit*

$$\alpha(x) := \lim_{n\to\infty} \frac{1}{n} a(0,nx,\omega) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[a(0,nx,\omega)] = \inf_{n \in N} \frac{1}{n} \mathbb{E}[a(0,nx,\omega)]$$

$\mathbb{P}$-almost surely and in L^1. Moreover, it holds that

$$\alpha(x) = \mathbb{E}[a(0,x,\omega)] = |x|\, \mathbb{E}[a(0,1,\omega)] \tag{3.1}$$

so that $\alpha(x)$ is a non-random norm.

The limit $\alpha(x)$ for $x \in \mathbb{Z}$ is called **quenched Lyapunov exponent**. The proof of the existence we find in [70, Proposition 4]. In this paper Zerner introduced the quenched Lyapunov exponents for simple symmetric random walks on $\mathbb{Z}^d$, $d \geq 1$. Just representation (3.1) is special on the integers, see [70, Proposition 10]. On the integers the function a is not only subadditive, but also additive. This property has its origin in the path properties of the simple symmetric random walk on the integers. The random walk can step just one site to the right or to the left at once (without involving the random potentials). That is, it cannot jump across a site, which implies

$$\{\tau_z < \infty\} \subseteq \{\tau_y < \infty\} \tag{3.2}$$

for all sites y between x and z. As this property is essential to the upcoming part we sketch the proof according to [70, Proposition 10].

Lemma 3.2.4 *Let ω be a realization of the potential and $x, y, z \in \mathbb{Z}$, with y between x and z. Then $a(x, z, \omega) = a(x, y, \omega) + a(y, z, \omega)$.*

Proof Let $\omega \in \Omega$ be a realization and without loss of generality $x \leq y \leq z \in \mathbb{Z}$. By (3.2) we can modify function e in the following way:

$$\begin{aligned} e(x, z, \omega) &= \mathsf{E}_x \left[\exp\left(-\sum_{i=0}^{\tau_z - 1} \omega(S_i) \right) \cdot \mathbf{1}_{\{\tau_z < \infty\}} \right] \\ &= \mathsf{E}_x \left[\exp\left(-\sum_{n=0}^{\tau_z - 1} \omega(S_i) \right) \cdot \mathbf{1}_{\{\tau_z < \infty\}} \cdot \mathbf{1}_{\{\tau_y < \infty\}} \right] . \end{aligned}$$

This, together with the tower property and the strong Markov property for the stopping time τ_y yields

$$e(x, z, \omega) = e(x, y, \omega)\, e(y, z, \omega) \tag{3.3}$$

which implies the additivity of a. □

The additivity of a allows to use Birkhoff's and Neumann's ergodic theorems in the proof of Propostion 3.2.3 instead of the subadditive limit theorem in higher dimensions. This provides formula (3.1) and proves Proposition 3.2.3.

In a so called shape-theorem [70, Theorem 8] Zerner proved that the quenched Lyapunov exponents describe the exponential decay of the Green function as follows:

$$\lim_{|x| \to \infty} \frac{-\ln g(0, x, \omega)}{\alpha(x)} = \lim_{|x| \to \infty} \frac{-\ln e(0, x, \omega)}{\alpha(x)} = 1$$

for $\mathbb{P}$ almost all $\omega \in \Omega$ and in $L^1(\mathbb{P})$. This is just the basic result concerning the decay of g, respectively e. For further reading we refer again to [70] and to [49] and [38].

In the case of the annealed Lyapunov exponents we have similar results. Here the expectation of the survival rate with respect to the potentials is taken before looking at its decay in the long run.

Proposition 3.2.5 *We suppose that ν has finite expectation. Then for all $x \in \mathbb{Z}$ there exists the limit*

$$\beta(x) := \lim_{n\to\infty} \frac{1}{n} b(0, nx) = \inf_{n\in N} \frac{1}{n} b(0, nx)$$

and β is a norm.

We call $\beta(x)$ for each $x \in \mathbb{Z}$ the **annealed Lyapunov exponent**. Flury has proven the existence of β also on higher dimensional lattices in [16]. We find there analogue shape theorems for the averaged Green function, for example

$$\lim_{|x|\to\infty} \frac{-\ln \mathbb{E}[g(0, x, \omega)]}{\beta(x)} = 1\,.$$

Indeed, in [16] a slightly more general case is treated, allowing other influences of the random potential on the random walk besides the extinction probability $1 - \exp(-\omega(x))$ at each site $x \in \mathbb{Z}$.

3.3 Main results of Chapter 3

The main result of this chapter is Theorem 3.4.2. It is a variational formula giving the annealed exponent as a minimization problem involving quenched exponents and entropy. This is the discrete analogon of Theorem 1.9 in [61]. The main tool for proving it is the additivity of a, see Lemma 3.2.4. Thus, this formula does not hold for random walks on $\mathbb{Z}^d$ where $d > 1$. Moreover, one bound in the proof comes by a change-of-measure argument and the other by Varadhan's Theorem. In section 3.5 we will introduce Lyapunov exponents on infinite regular trees. We will see that there – as a corollary of the aforegoing Section – the variational formula holds as well. Additionally, we will see that we can generalize the result for non-symmetric simple random walks.

In addition to the aforementioned literature we recommend [62] for a general overview and the background on Lyapunov exponents to the reader. There, mainly Lyapunov exponents for Brownian Motion moving in Poissonian Potential are treated. Some further current results related to Lyapunov exponents appear in [58] and [56]. Instead of Lyapunov exponents the authors observe quenched free energy and quenched point-to-point free energy in a more general situation in random environment. More precisely it is a generalization of random walks in random potential and random walk in random environment. As a result they develop a variational formula for the quenched free energy also using entropy.

Furthermore there are lots of related results for random walks in random environment. Zerner provides an introduction to Lyapunov exponents for RWRE in [71]. In [9] random walks in random environment especially on the integers are discussed and they provide a relation of some rate functions for the quenched to the annealed random environment.

3.4 The relation between quenched and annealed Lyapunov exponents on the integers

We have already seen that quenched and annealed Lyapunov exponents differ in their treatment of the random potential. Using the quenched approach, we observe the exponential decay for a typical realization of the random potentials. While using the annealed approach, we look at the environment as averaged. We now aim to describe this difference in greater detail. By applying Jensen's inequality it is easy to conclude that for the quenched and annealed Lyapunov exponents it holds

$$\alpha(x) \geq \beta(x)$$

for each $x \in \mathbb{Z}$. This relation also holds for simple symmetric random walks on $\mathbb{Z}^d$ with $d \geq 2$. But it turns out that in the case of a random walk on the integers we can prove an explicit formula using entropy for this relation. Before stating the main result we recall some definitions concerning entropy.

We consider the canonical projection $\pi_I : \Omega \to \Omega^I$ where I is a finite subset of $\mathbb{Z}$ and $\Omega^I := [0,\infty)^I$. The corresponding product σ-algebra is denoted by $\mathcal{F}_I$. For any probability measure $\mathbb{Q}$ on Ω we consider its restriction $\mathbb{Q}^I$ to $\mathcal{F}_I$. Now let $\tilde{\mathbb{Q}}$ be a second probability measure on Ω, which is absolutely continuous with respect to $\mathbb{Q}$ on $\mathcal{F}_I$ for a given finite intervall $I \subset \mathbb{Z}$. Hence we can define the Radon-Nikodym derivative f_I on $\mathcal{F}_I$ which is a positive and $\mathcal{F}_I$-measurable function.

Definition 3.4.1 (Relative Entropy) Let $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ be two probability measures on Ω and $I \subset \mathbb{Z}$ a finite intervall. Then we call

$$H_I(\tilde{\mathbb{Q}}|\mathbb{Q}) := \begin{cases} \int_\Omega f_I \ln f_I \, d\mathbb{Q} & \text{if } \tilde{\mathbb{Q}} \ll \mathbb{Q} \text{ on } \mathcal{F}_I \\ \infty & \text{else} \end{cases}$$

the **relative entropy of $\tilde{\mathbb{Q}}^I$ with respect to $\mathbb{Q}^I$** where $f_I := \frac{d\tilde{\mathbb{Q}}^I}{d\mathbb{Q}^I}$ denotes the Radon-Nikodym derivative.

In most of the literature on Information Theory, the relative entropy is denoted $D(\tilde{\mathbb{Q}}\|\mathbb{Q})$ and also known as the Kullback-Leibler divergence. Equivalently, if $\tilde{\mathbb{Q}} \ll \mathbb{Q}$ on $\mathcal{F}_I$ the relative entropy can be expressed by $H_I(\tilde{\mathbb{Q}}|\mathbb{Q}) = \int_\Omega \ln f_I \, d\tilde{\mathbb{Q}}$. As the function $x \ln x$ is strictly convex we see by Jensen's inequality that H_I is nonnegative and zero if and only if the two measures coincide. If $\tilde{\mathbb{Q}}$ and $\mathbb{Q}$ are shift invariant for $(T_i)_{i\in\mathbb{Z}}$, the relative entropy $H_I(\tilde{\mathbb{Q}}|\mathbb{Q})$ is shift invariant as well. We denote by $\mathcal{M}_1^t(\Omega)$ all shift invariant probability measures on Ω. Furthermore, under the additional assumption that $\mathbb{Q}$ is a product measure on Ω, the relative entropy becomes strongly superadditive, see [20, Proposition 15.10]. Thus under these assumptions the subadditive limit theorem guarantees the existence of

$$H(\tilde{\mathbb{Q}}|\mathbb{Q}) := \lim_{n\to\infty} \frac{1}{|I_n|} H_{I_n}(\tilde{\mathbb{Q}}|\mathbb{Q}) = \sup_{I_n} \frac{1}{|I_n|} H_{I_n}(\tilde{\mathbb{Q}}|\mathbb{Q}) \tag{3.4}$$

with $(I_n)_{n\in\mathbb{N}}$ a sequence of intervals satisfying $I_n \subseteq I_{n+1}$ for each $n \in \mathbb{N}$ and $|I_n|\to\infty$ when $n \to \infty$. We call $H(\tilde{\mathbb{Q}}|\mathbb{Q})$ the *specific relative entropy of* $\tilde{\mathbb{Q}}$ *with respect to* $\mathbb{Q}$. The distribution of the random potentials $\mathbb{P}$ fulfills all the above assumptions and the specific relative entropy $H(\mathbb{Q}|\mathbb{P})$ is well-defined for any shift invariant probability measure $\mathbb{Q}$ on Ω.

Additionally, in relation with the quenched Lyapunov exponent, we define for $r < 0$ the functions $F_r : \Omega \to [0, \infty]$ by

$$F_r(\omega) := -\ln \mathsf{E}_0 \left[\exp\left(-\sum_{i=0}^{\tau_1-1} \omega(S_i)\right), \tau_1 < \tau_r, \tau_1 < \infty\right]$$

and their counterpart F by omitting the inequality $\tau_1 < \tau_r$. It is easy to see that $F(\omega) = -\ln e(0,1,\omega) = a(0,1,\omega)$ and $0 \leq F \leq \omega(0) + \ln 2$. Consequently we have

$$\alpha(1) = \mathbb{E}[F(\omega)]\,. \tag{3.5}$$

Keeping in mind that $H(\mathbb{P}|\mathbb{P}) = 0$ and formula (3.5), the description of the relation between annealed and quenched Lyapunov exponents is the next variational formula:

Theorem 3.4.2 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be the probability space defined above where* $\mathbb{P} = \otimes_{x\in\mathbb{Z}}\nu$. *Moreover, we assume that* ν *has finite expectation. Then*

$$\beta(1) = \inf_{\mathbb{Q}}\{\mathbb{E}^{\mathbb{Q}}[F(\omega)] + H(\mathbb{Q}|\mathbb{P})\}$$

and the infimum runs over all shift invariant probability measures $\mathbb{Q}$ *on* Ω.

There is the following interpretation of this formula: When the environment is averaged out, the walk will most of the time experience a typical configuration. But with an exponentially small probability a large deviation event occurs and the walk goes through a more favorable environment that allows it to survive longer. The rate of survival in this configuration is then given by the quenched Lyapunov exponents (corresponding to the distribution of the rare event), but one has to pay the cost of benefiting from a favorable landscape, and this is given by the specific entropy (of the new distribution relative to the original one).

A similar relation for Brownian motion moving in Poissonian potential was proven in [61, Theorem 1.9]. We will follow Sznitman's ideas and split our proof in two parts proving the upper and lower bound

$$\beta(1) \leq \inf_{\mathbb{Q}\in\mathcal{M}_1^t(\Omega)}\{E^{\mathbb{Q}}[F(\omega)] + H(\mathbb{Q}|\mathbb{P})\} \tag{3.6}$$

$$\beta(1) \geq \inf_{\mathbb{Q}\in\mathcal{M}_1^t(\Omega)}\{E^{\mathbb{Q}}[F(\omega)] + H(\mathbb{Q}|\mathbb{P})\} \tag{3.7}$$

separately as well.

3.4.1 Proof of the upper bound

Proof *Upper bound* As a consequence of the multiplication property (3.3) of e and the spatial homogeneousity we connect the definition of f with F. The latter follows from the observation that $\{\tau_1 < \tau_r, \tau_1 < \infty\} \subseteq \{\tau_1 < \infty\}$.

$$\begin{aligned} f(0,n+1) &= \mathbb{E}\left[\prod_{k=0}^{n} e(k,k+1,\omega)\right] = \mathbb{E}\left[\prod_{k=0}^{n} e\left(0,1,T_{-k}(\omega)\right)\right] \\ &= \mathbb{E}\left[\exp\left(\sum_{k=0}^{n} -F \circ T_{-k}(\omega)\right)\right] \qquad (3.8) \\ &\geq \mathbb{E}\left[\exp\left(\sum_{k=0}^{n} -F_r \circ T_{-k}(\omega)\right)\right]. \qquad (3.9) \end{aligned}$$

Now let us consider an arbitrary shift invariant probability measure $\mathbb{Q} \in \mathcal{M}_1^t(\Omega)$. Thus $H(\mathbb{Q}|\mathbb{P})$ is well defined and we assume additionally

$$H(\mathbb{Q}|\mathbb{P}) < \infty \text{ and } \mathbb{E}^{\mathbb{Q}}[F] < \infty\,.$$

This implies that $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{F}_I$ for each finite intervall $I \subset \mathbb{Z}$ and we define the corresponding Radon-Nikodym derivatives f_I on $\mathcal{F}_I$, $I \subset \mathbb{Z}$. It is easy to see that each f_I is strictly positive $\mathbb{Q}$-almost surely and for each finite $I \subset \mathbb{Z}$ it holds that

$$\int_\Omega g\, d\mathbb{P} = \int_\Omega \frac{g}{f_I}\, d\mathbb{Q} \qquad (3.10)$$

for any $\mathcal{F}_I$ measurable function $g : \Omega \to \mathbb{R}$. The movement of the random walk in (3.9) is restricted by the definition of F_r to $A := [r{+}1, n{+}1]$ and the function $\exp\left(\sum_{k=0}^{n} -F_r \circ T_{-k}(\omega)\right)$ is $\mathcal{F}_A$-measurable. Then, by (3.9),(3.10) and Jensen's inequality we obtain:

$$\begin{aligned} f(0,n+1) &\geq \mathbb{E}\left[\exp\left(\sum_{k=0}^{n} -F_r \circ T_{-k}(\omega)\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sum_{k=0}^{n} -F_r \circ T_{-k}(\omega)\right) \cdot \frac{1}{f_A}\right] \\ &\geq \exp\left(\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{n} -F_r \circ T_{-k} - \ln f_A\right]\right) \\ &= \exp\left(\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{n} -F_r \circ T_{-k}\right] - H_A(\mathbb{Q}|\mathbb{P})\right). \end{aligned}$$

Taking the $n+1$-th root, the negative logarithm and the limit n to infinity on both sides yields

$$\beta(1) \leq \mathbb{E}^{\mathbb{Q}}[F_r] + H(\mathbb{Q}|\mathbb{P})\,.$$

Obviously, $\mathbf{1}_{\tau_1<\tau_r}$ converges monotonously from below to $\mathbf{1}$ for $r\to-\infty$ and we can replace F_r by F. Because of the finite expectation of ν there is at least $\mathbb{P}$ such that the right side of the last inequality is finite and we may deduce

$$\beta(1)\le\inf_{\mathbb{Q}\in\mathcal{M}_1^t(\Omega)}\{E^{\mathbb{Q}}[F(\omega)]+H(\mathbb{Q}|\mathbb{P})\}\,.$$

□

3.4.2 Proof of the lower bound

In order to prove the lower bound (3.7) we need to make use of some statements from the theory of large deviations, more specifically from process level large deviations theory. A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of probability measures on a Polish space E satisfies a *large deviation principle with rate function* J *and normalization* r_n if the following two inequalities hold:

$$\limsup_{n\to\infty}\frac{1}{r_n}\log\mu_n(F)\le-\inf_{x\in F}J(x)\qquad\forall F\subset E\,\text{closed}\tag{3.11}$$

$$\liminf_{n\to\infty}\frac{1}{r_n}\log\mu_n(G)\ge-\inf_{x\in G}J(x)\qquad\forall G\subset E\,\text{open}\tag{3.12}$$

where $J:E\to[0,\infty]$ is a lower semi-continuous function and $\{r_n\}_{n\in\mathbb{N}}\subset\mathbb{R}_+$ is a sequence of positive real numbers with $r_n\nearrow\infty$. We abreviate by writing that LDP(μ_n,r_n,J) holds if all these requirements are satisfied. When the sets $\{x\in E: J(x)\le c\}$ are compact for all $c\in[0;\infty)$ the rate function J is said to be *good*. Process level large deviations theory is concerned with the asymptotics of the distributions of the empirical measures of a whole process. Therefore we consider the random potentials $(\omega(i))_{i\in\mathbb{Z}}$ as a process with state space $[0,\infty)$. For this process the n-th empirical measure $R_n:\Omega\to\mathcal{M}_1(\Omega)$ is defined by

$$R_n(\omega):=\frac{1}{|I_n|}\sum_{i\in I_n}\delta_{T_{-i}(\omega)}$$

with the normalizing sequence of intervalls $(I_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ given by $I_n:=\{i\in\mathbb{Z}:-n<i<n\}$ for each $n\in\mathbb{N}$. Every probability measure $\mathbb{Q}$ on Ω induces a distribution $\mu_n\in\mathcal{M}_1(\mathcal{M}_1(\Omega))$ of the empirical measure R_n. The distributions of the empirical measures $(R_n)_{n\in\mathbb{N}}$ satisfy the LDP with good rate function $J:\mathcal{M}_1(\Omega)\to[0,\infty]$ defined by

$$J(\mathbb{Q}):=\begin{cases}H(\mathbb{Q}|\mathbb{P}) & \text{if }\ \mathbb{Q}\in\mathcal{M}_1^t(\Omega)\\ \infty & \text{else}\end{cases}\tag{3.13}$$

and the normalizing sequence $(|I_n|)_{n\in\mathbb{N}}$, see [57, Theorem 6.13].

Basically the following proof is an application of a version of Varadhan's theorem.

Proof *Lower bound* Let $\Phi:\mathcal{M}_1(\Omega)\to[-\infty,0]$ be the function defined by

$$\Phi(\mathbb{Q}):=\mathbb{E}^{\mathbb{Q}}[-F]\,.$$

We have already seen that F is a positive function. Hence $-F$ is bounded from above. Moreover, F is a continuous function with respect to the product topology on Ω. This is a consequence of the continuity of parameter dependent integrals. Recall that the weak convergence topology on $\mathcal{M}_1(\Omega)$ is the coarsest topology such that for each bounded and continuous $f \in C_b(\Omega)$ the map $\mathcal{M}_1(\Omega) \to \mathbb{R}$, $\rho \mapsto \int_\Omega f d\rho$ is continuous. Let $\rho \in \mathcal{M}_1(\Omega)$ and let $(\rho_n)_{n\in\mathbb{N}} \in \mathcal{M}_1(\Omega)$ be a sequence in $\mathcal{M}_1(\Omega)$ which converges weakly to ρ. Then it holds that

$$\limsup_{n\to\infty} \mathbb{E}^{\rho_n}[-F] \leq \mathbb{E}^{\rho}[-F]$$

as $-F$ is continuous and bounded from above. Consequently Φ is an upper-semi-continuous function with respect to the weak convergence topology.

By (3.13) we know that $LDP(\mu_n, |I_n|, J)$ holds with rate function J and as $-F$ is negative, the set $\{\mathbb{Q} \in \mathcal{M}_1(\Omega) : \Phi(\mathbb{Q}) \geq L\}$ is empty for $L > 0$.

Thus applying [12, Lemma 2.1.8] we conclude

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{|I_n|} \ln \int\limits_{\mathcal{M}_1(\Omega)} & \exp\left[|I_n|\Phi(\mathbb{Q})\right] d\mu_n(\mathbb{Q}) \\ & \leq \sup_{\{\mathbb{Q}\in\mathcal{M}_1(\Omega)\}} \left[\Phi(\mathbb{Q}) - J(\mathbb{Q})\right]\end{aligned}$$

and by some transformations and the shift invariance of $\mathbb{P}$ we obtain

$$\begin{aligned}\int\limits_{\mathcal{M}_1(\Omega)} \exp\left(|I_n|\Phi(\mathbb{Q})\right) d\mu_n(\mathbb{Q}) &= \mathbb{E}^{\mathbb{P}}\left[\exp\left((2n-1)\mathbb{E}^{R_n}[-F]\right)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\sum_{i=-(n-1)}^{n-1} F \circ T_{-i}(\omega)\right)\right] \\ &= \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\sum_{i=0}^{2n-1} F \circ T_{-i}(\omega)\right)\right].\end{aligned}$$

Moreover, by (3.8) it holds that

$$\beta(1) \geq \liminf_{n\to\infty} -\frac{1}{2n-1} \ln \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\sum_{k=0}^{2n-1} F \circ T_{-k}(\omega)\right)\right]$$

and we summarize for the annealed Lyapunov exponent:

$$\beta(1) \geq \inf_{\{\mathbb{Q}\in\mathcal{M}_1(\Omega)\}} \left[\mathbb{E}^{\mathbb{Q}}[F] + J(\mathbb{Q})\right].$$

Due to the positivity of F we see that $\mathbb{E}^{\mathbb{Q}}[F] \geq 0$ for all $\mathbb{Q} \in \mathcal{M}_1(\Omega)$. By the definition of J and the finite expectation of ν the infimum will not be reached for a not shift invariant measure. We conclude

$$\beta(1) \geq \inf_{\{\mathbb{Q}\in\mathcal{M}_1^t(\Omega)\}} \left[\mathbb{E}^{\mathbb{Q}}[F] + H(\mathbb{Q}|\mathbb{P})\right]. \qquad \square$$

3.5 Lyapunov exponents on trees

The aforegoing result could be proven mainly because of the additivity of a. This additivity property does not hold for simple symmetric random walks on the higher dimensional lattices $\mathbb{Z}^d$ with $d \geq 2$. Could there nevertheless be other structures on which the random walk moves, where we gain again the additivity of a? One possible answer is a d-regular tree.
In this section we will see how we can apply our main result to Lyapunov exponents on a d-regular tree. First of all we give a short introduction to regular trees and random walks on trees. For a more precise introduction in the context of random walks, see e.g. [69]. Second we will introduce Lyapunov exponents on an infinite tree.

3.5.1 Random walk with random killing on infinite d-regular trees

Let $d \geq 2$ and let T_d denote the d-regular infinite tree. We call $V(T_d)$ the set of vertices of the tree and $E(T_d)$ the set of edges. Then we define a symmetric nearest neighbour random walk $(Z_n)_{n\in\mathbb{N}}$ on T_d by choosing a starting point

$$Z_0 = v$$

for $v \in V(T_d)$ and the transition probabilities given by

$$p(x,y) = \begin{cases} 1/d & \text{if } x \sim y \\ 0 & \text{else}\,. \end{cases} \tag{3.14}$$

The relation $x \sim y$ for $x, y \in V(T_d)$ means that these two vertices are neighbours, i.e. $[x, y]$ is an edge in $E(T_d)$. Thus $(Z_n)_{n\in\mathbb{N}}$ is a spatially homogeneous Markov chain adapted to T_d. We will denote its path measure by P_v^T and the corresponding expectation value by E_v^T.

As we have done before on the integers we attach to each vertex $x \in V(T_d)$ a random potential $\omega(x)$ and assume that $\omega := (\omega(x))_{x\in V(T_d)}$ is a family of nonnegative random variables which are i.i.d. by ν on $[0,\infty)$. From this we obtain again the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ described by

$$\Omega := [0,\infty)^{V(T_d)}$$

with its usual product σ-algebra $\mathcal{F}$ and the product measure

$$\mathbb{P} := \bigotimes_{x\in V(T_d)} \nu\,.$$

The expectation value with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}^{\mathbb{P}}$. We assume again that ν has finite expectation.

As T_d is a tree, it contains no cycles and we have for each $v, z \in V(T_d)$ a unique shortest path $\pi = [v, ..., z]$ from v to z. The distance between two vertices is defined as the length of this shortest path:

$$d(x,z) := |[x, ..., z]|\,.$$

We call a sequence of distinct vertices $\pi = [\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots] \subset V(T_d)$ which satisfy $x_j \sim x_{j+1}$ for all $j \in \mathbb{Z}$ a *geodesic*. If we have an one-sided infinite sequence $\pi = [x_0, x_1, x_2, \ldots] \subset V(T_d)$ with $x_j \sim x_{j+1}$ for all $j \geq 1$ we call it *ray*.

Now we are interested in the riskiness of walking around on this tree equipped with the random potentials. More precisely we want to observe how risky journeys along a fixed geodesic are. Here $\tau_x, x \in V(T_d)$, is defined by

$$\tau_x := \inf\{n \geq 0 \,:\, Z_n = x\}$$

for each $x \in V(T_d)$.

The two-point-functions defined in the following section are the counterparts to our well know functions e, f, a and b from the first part of the chapter.

Definition 3.5.1 Let $x, y \in V(T_d)$ be two vertices of the tree and ω a realization of the random potentials. We define:

$$\begin{aligned}
F_q(x,y,\omega) &:= \mathsf{E}_x^T\left[\exp\left(-\sum_{k=0}^{\tau_y-1}\omega(Z_k)\right), \tau_y < \infty\right] \\
A(x,y,\omega) &:= -\ln F_q(x,y,\omega) \\
F_a(x,y) &:= \mathbb{E}^{\mathbb{P}}\left[\mathsf{E}_x^T\left[\exp\left(-\sum_{k=0}^{\tau_y-1}\omega(Z_k)\right), \tau_y < \infty\right]\right] \\
B(x,y) &:= -\ln F_a(x,y)\,.
\end{aligned}$$

The two functions F_q and F_a denote the probability that the random walk reaches y after starting at x, for the quenched environment where the potentials are frozen and the averaged environment. In contrast to e and f the random walk here is driven by the different path measure $\mathbb{P}_x^T$ on the tree.

Let us now fix a geodesic $\pi^* = [\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots]$ and look at the behaviour of $F_q(x_0, x_i, \omega)$ and $F_a(x_0, x_i)$ in the long run, that is if $d(x_0, x_i) \to \infty$. Having a closer look on the structure of the tree we see that each trajectory of the random walk from x_i to x_j contains the path $[x_i, x_j] \subset \pi^*$. That is, the random walk has to pass all these points on its journey at least once. Only the excursions to the branches beside the geodesic vary. For each $x, y \in V(T_d)$ a branch $T_{x,y}$ of the tree is defined by

$$T_{x,y} := \{v \in V(T_d) : y \in \pi(x,v)\}\,.$$

This allows a modification of the model in the following way: We can combine all the risk, which the random walk has to face during its excursions into a modified potential for each point of the geodesic and in the end we identify the geodesic with the integers. For the latter let $x_0 \in V(T_d)$ be a fixed starting point of the random walk on the geodesic. Then we can split π^* into two rays

$$\pi^+ = [x_0, x_1, x_2, \ldots] \text{ and } \pi^- = [x_0, x_{-1}, x_{-2}, \ldots]\,.$$

We identify the geodesic with the integers via

$$\lambda : \pi^* \to \mathbb{Z},\, x_i \mapsto \begin{cases} d(x_i, x_0) & \text{if } x_i \in \pi^+ \\ -d(x_i, x_0) & \text{if } x_i \in \pi^- \end{cases} \tag{3.15}$$

and vice versa. For modifying the potential we have to spend more effort. Moving on an infinite tree contains, besides the given potentials, the additional risk of leaving the geodesic and getting lost in the corresponding branch of the tree. Getting lost is understood as the random walk not returning to π^* again in finite time and disappearing at infinity of the tree within such a branch.

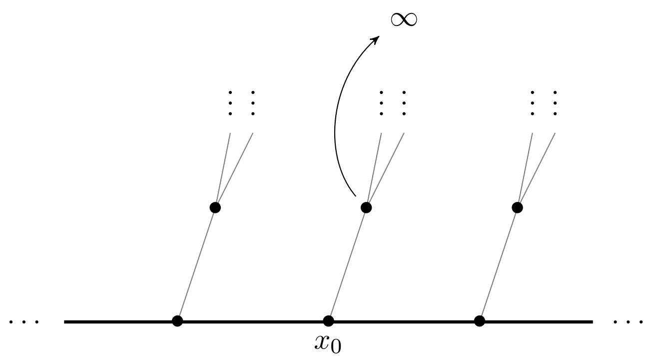

Figure 3.2: A 3-regular tree oriented on a fixed geodesic (black).

Consequently there are three possibilities how the random walk can die at a point x_i of the geodesic: Firstly due to the given potential $\omega(x_i)$, secondly due to the potentials on the way of a finite excursion into a branch, thirdly because the random walk can disappear to infinity. We want to combine these three risks in a new sequence of potentials. To do so we define for $x_i \in \pi^*$ the new stopping time

$$\sigma_{x_i} := \inf\{n \geq 1 : Z_n = x_{i+1} \text{ or } Z_n = x_{i-1}\}\,. \tag{3.16}$$

This is the first time the random walk on the tree hits the left or right neighbour of vertex x_i on the geodesic. This stopping time is finite with a positive probability: Using the generating function technique described in [69, Chapter 9], it holds

$$\begin{aligned} L(z) :=& \sum_{n=1}^{\infty} \mathsf{P}^T_{x_i}[\sigma_{x_i} = n]\, z^n \\ =& \frac{2}{d} z + \frac{d-2}{d} z\, F(z)\, L(z) \end{aligned}$$

where $F(z) := \sum_{k=1}^{\infty} \mathsf{P}^T_x[\tau_y = k]\, z^k$ for each $x, y \in V(T_d)$ with $y \sim x$. As the tree is regular and the random walk is space homogeneous, $F(z)$ is equal for each pair of neighbours.

$F(1)$ is the probability that the random walk reaches a given neighbour of the starting point in finite time. It is well known, that $F(1) = \frac{1}{d-1}$ and we conclude that

$$0 < \mathsf{P}^T_{x_i}[\sigma_{x_i} < \infty] = L(1) = \frac{2(d-1)}{(d-1)^2+1} \leq 1\,.$$

Furthermore we define for each $x_i \in \pi^*$ the new random variable

$$h(x_i,\omega) := \mathsf{E}^T_{x_i}\left[\exp\left(-\sum_{k=0}^{\sigma_{x_i}-1}\omega(Z_k)\right),\, \sigma_{x_i} < \infty\right],$$

which is the expected probability that the random walk starting at x_i survives its finite excursions to some branches before proceeding along the fixed geodesic π^*. Thus the probability to die at a vertex x_i, namely $1-h(x_i,\omega)$, comprises the three types of possible risks outlined above. In order to use the aforegoing theory we perform a slight modification of h to obtain the final new random potentials on the geodesic.

Proposition 3.5.2 *Let $\pi^* \subset V(T_d)$ be a fixed geodesic. At each vertex $x_i \in \pi^*$ the random walk moving along π^* survives with probability $e^{-\rho(x_i,\omega)}$ where $\rho = (\rho(x_i,\omega))_{x_i\in\pi^*}$ are positive i.i.d. random variables on Ω given by*

$$\rho(x_i,\omega) := -\ln h(x_i,\omega)\,. \tag{3.17}$$

Moreover, each $\rho(x_i,\omega)$ has finite expectation w.r.t. $\mathbb{P}$.

Proof We have seen before that the random walk survives each site on its journey with probability $e^{-\rho(x_i,\omega)}$. As $h(x_i,\omega)$ is $(0,1]$-valued, ρ is nonnegative. Additionally, $\omega \mapsto \rho(x_i,\omega)$ is for each $x_i \in \pi^*$ a continuous function with respect to the product topology on Ω and consequently measurable. Due to the definition of σ_{x_i} the movement of the random walk within the event which defines $h(x_i,\omega)$ is restricted to the union of $\{x_i\}$ with the branches $T_{x_i,y}$ with $y \neq \{x_{i+1},x_{i-1}\}$. This, together with the identical distribution of $(\omega(i))_{i\in T_d}$, implies that $(\rho(x_i,\omega))_{x_i\in\pi}$ are i.i.d. by the image measure $\tilde{\nu}$ defined by

$$\tilde{\nu}[B] = \nu\left[\{\omega : \rho(x_i,\omega) \in B\}\right] \tag{3.18}$$

for all Borel-sets $B \in \mathcal{B}([0,\infty))$. For the last statement we observe that $\left\{\sigma_{x_i} = 1\right\} \subseteq \left\{\sigma_{x_i} < \infty\right\}$ which implies

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}[\rho(x_i,\omega)] &= \mathbb{E}^{\mathbb{P}}\left[-\ln \mathsf{E}^T_{x_i}\left[\exp\left(-\sum_{k=0}^{\sigma_{x_i}-1}\omega(Z_k)\right)\mathbf{1}_{\{\sigma_{x_i}<\infty\}}\right]\right] \\
&\leq \mathbb{E}^{\mathbb{P}}\left[-\ln \mathsf{E}^T_{x_i}\left[\exp\left(-\sum_{k=0}^{\sigma_{x_i}-1}\omega(Z_k)\right)\mathbf{1}_{\{\sigma_{x_i}=1\}}\right]\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[-\ln \mathsf{E}^T_{x_i}\left[\exp\left(-\omega(x_i)\right)\mathbf{1}_{\{\sigma_{x_i}=1\}}\right]\right]
\end{aligned}$$

$$= \mathbb{E}^{\mathbb{P}}[\omega(x_i)] - \ln\left(\frac{2}{d}\right).$$

As the random potentials ω are supposed to have finite expectation, this holds for ρ as well. □

We obtain for the new potentials ρ the slightly modified probability space $(\tilde{\Omega} := [0,\infty)^{\mathbb{Z}}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}} := \otimes_{\mathbb{Z}}\tilde{\nu})$ where $\tilde{\mathcal{F}}$ is the usual borelian product-σ-algebra and $\tilde{\nu}$ defined as in (3.18). If ρ is given, we just have to know how often the random walk visits the different sites of the geodesic. As the random walk $(Z_n)_{n\in\mathbb{N}}$ is a symmetric random walk, it holds that

$$\mathsf{P}^T_{x_i}[Z_{\sigma_{x_i}} = x_{i+1} | \sigma_{x_i} < \infty] = \frac{1}{2} = \mathsf{P}^T_{x_i}[Z_{\sigma_{x_i}} = x_{i-1} | \sigma_{x_i} < \infty].$$

Therefore conditionally upon the events $[\sigma_{x_i} < \infty]$ the sequence of random variables $(\tilde{S}_n)_{n\in\mathbb{N}}$ defined by

$$\tilde{S}_0 := x_0 \text{ and } \tilde{S}_n := Z_{\sigma_{\tilde{S}_{n-1}}}$$

is a simple symmetric random walk on the geodesic π^* with starting point x_0. This, together with (3.15) enables us to define the Lyapunov exponents along a geodesic in the well-known manner for the random walk $(\tilde{S}_n)_{n\in\mathbb{N}}$ and the random potentials $\rho(\omega)$. Always (3.15) in mind, we will still denote the sites of the geodesic by its original name instead of integer numbers. Let $\tilde{a}$ and $\tilde{b}$ be defined as in Definitions 3.2.1 and 3.2.2 but for $(\tilde{S}_n)_{n\in\mathbb{N}}$ and ρ. Then the finite expectation of ρ guarantees the existence of the Lyapunov exponents $\tilde{\alpha}$ and $\tilde{\beta}$ as in Propositions 3.2.3 and 3.2.5. Moreover we see the following identity where $\tilde{\tau}_x$ denotes the first point of time where the random walk $(\tilde{S}_n)_{n\geq 0}$ hits $x \in \pi^*$:

$$\begin{aligned}
\tilde{a}(x_0, x_i, \rho) &:= -\ln \mathsf{E}^{\tilde{S}}_{x_0}\left[\exp\left(-\sum_{k=0}^{\tilde{\tau}_{x_i}-1} \rho(\tilde{S}_k)\right), \tilde{\tau}_{x_i} < \infty\right] \\
&= -\ln \mathsf{E}^{\tilde{S}}_{x_0}\left[\prod_{k=0}^{\tilde{\tau}_{x_i}-1} \mathsf{E}^T_{\tilde{S}_k}\left[\exp\left(-\sum_{m=0}^{\sigma_{\tilde{S}_k}-1} \omega(Z_m)\right) \mathbf{1}_{\sigma_{\tilde{S}_k}<\infty\}}\right] \mathbf{1}_{\{\tilde{\tau}_{x_i}<\infty\}}\right] \\
&= -\ln \mathsf{E}^{\tilde{S}}_{x_0}\left[\mathsf{E}^T_{x_0}\left[\exp\left(-\sum_{m=0}^{\sigma_{\tilde{S}_{\tilde{\tau}_{x_i}-1}}-1} \omega(Z_m)\right) \mathbf{1}_{\{\sigma_{\tilde{S}_{\tilde{\tau}_{x_i}-1}}<\infty\}}\right] \mathbf{1}_{\tilde{\tau}_{x_i}<\infty\}}\right] \\
&= -\ln \mathsf{E}^{\tilde{S}}_{x_0}\left[\mathsf{E}^T_{x_0}\left[\exp\left(-\sum_{m=0}^{\sigma_{x_{i-1}}-1} \omega(Z_m)\right) \mathbf{1}_{\{\sigma_{x_{i-1}}<\infty\}}\mathbf{1}_{\{Z_{\sigma_{x_{i-1}}}=x_i\}}\right]\right. \\
&\qquad \left. + \mathsf{E}^T_{x_0}\left[\exp\left(-\sum_{m=0}^{\sigma_{x_{i+1}}-1} \omega(Z_m)\mathbf{1}_{\{\sigma_{x_{i+1}}<\infty\}}\mathbf{1}_{\{Z_{\sigma_{x_{i+1}}}=x_i\}}\right] \mathbf{1}_{\tilde{\tau}_{x_i}<\infty\}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\ln \mathsf{E}_{x_0}^{\tilde{S}} \left[\mathsf{E}_{x_0}^{T} \left[\exp\left(-\sum_{m=0}^{\tau_{x_i}-1} \omega(Z_m) \right) \mathbf{1}_{\{\tau_{x_i}<\infty\}} \right] \mathbf{1}_{\{\tilde{\tau}_{x_i}<\infty\}} \right] \\
&= -\ln \mathsf{E}_{x_0}^{T} \left[\exp\left(-\sum_{m=0}^{\tau_{x_i}-1} \omega(Z_m) \right), \tau_{x_i} < \infty \right] \\
&= A(x_0, x_i, \omega)
\end{aligned}
$$

for all $x_i \in \pi^*$. The same holds for $\tilde{b}$ and B. This provides the existence of the Lyapunov exponents along π^*: Let $x_i \in \pi^*$, then

$$
\begin{aligned}
\tilde{\alpha}(x_i) &= \lim_{n\to\infty} \frac{1}{n} A(x_0, x_{ni}, \omega) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[A(x_0, x_{ni}, \omega)] \\
&= \inf_{n\in N} \frac{1}{n} \mathbb{E}[A(x_0, x_{ni}, \rho)]
\end{aligned}
$$

exists $\tilde{\mathbb{P}}$-a.s. and $L^1(\tilde{\mathbb{P}})$ and furthermore we have

$$
\tilde{\beta}(x_i) = \lim_{n\to\infty} \frac{1}{n} B(x_0, x_{ni}) = \inf_{n\in N} \frac{1}{n} B(x_0, x_{ni}) \,.
$$

Due to the properties of ρ in Proposition 3.5.2 we can apply Theorem 3.4.2 as well and see that our proven relation does hold for Lyapunov exponents on d-regular trees:

$$
\tilde{\beta}(x_1) = \inf_{\mathbb{Q}} \{ \mathbb{E}\left[A(x_0, x_1, \rho)\right] + H(\mathbb{Q}|\tilde{\mathbb{P}}) \} \,,
$$

where the infimum runs over all shift invariant probability measures on $\tilde{\Omega}$.

3.5.2 Non-symmetric random walk on trees

Let us go back to the integers for a moment. From now on we consider a nearest neighbour walk which is not symmetric. That is, in each step it jumps independently of all the steps before with probability p to the right and with probability $1-p$ to the left. It is easy to see that Propostion 3.2.3, Proposition 3.2.5 and Theorem 3.4 hold for this non-symmetric nearest neighbour random walk as well.

There is also a non-symmetric random walk counterpart on infinite regular trees. To define this we fix a root $o \in V(T_d)$ and a geodesic Θ and consider a particular representation of infinite regular trees (see Figure 3.3). This representation displays the generations H_i, $i \in \mathbb{Z}$, of a tree. We write

$$
\mathsf{h}(x) := i \text{ if } x \in H_i
$$

for each vertex $x \in V(T_d)$. For two vertices $x, y \in V(T_d)$ we call y a *predecessor* of x if $\mathsf{h}(y) = \mathsf{h}(x) - 1$ and a *child* if $\mathsf{h}(y) = \mathsf{h}(x) + 1$. Note that each H_i is infinite.

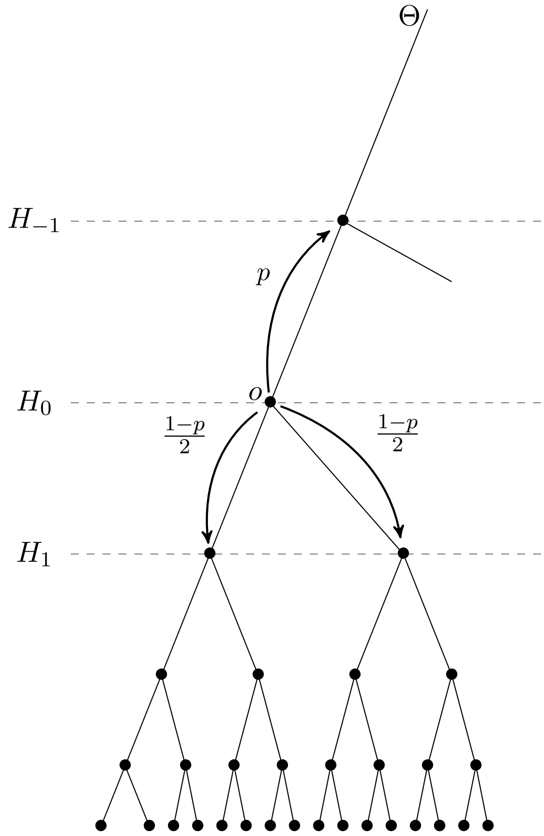

Figure 3.3: A 3-regular infinite tree.

Then we define the random walk with drift $(Z_n)_{n\in\mathbb{N}}$ in the following way: The random walk jumps from its current position to the predecessor with probability p and to a child with probability $1-p$. This yields the following transition probabilities for $u, v \in V(T_d)$

$$\mathsf{P}^p[Z_{n+1} = v \mid Z_n = u] = \begin{cases} p & \text{if } u \sim v \text{ and } \mathsf{h}(v) = h(u) - 1 \\ \frac{1-p}{d-1} & \text{if } u \sim v \text{ and } \mathsf{h}(v) = h(u) + 1 \\ 0 & \text{else} \end{cases} \tag{3.19}$$

for each $n \in \mathbb{N}$. The second line is due to the fact that one vertex has $d-1$ children in the next generation.

There are two kinds of infinite geodesics: the first has two ends downwards (like the thick geodesic in Figure 3.4), the second has one vertex in each generation (like the thick geodesic in Figure 3.5).

Firstly we take a geodesic $\pi = [\ldots, x_{-1}, x_0, x_1, \ldots]$ of the latter form, that is $\mathsf{h}(x_{i+1}) = \mathsf{h}(x_i) + 1$ for each $i \in \mathbb{Z}$. Then it holds that

$$\mathsf{P}^p_{x_i}[Z_{\sigma_{x_i}} = x_{i+1} | \sigma_{x_i} < \infty] = \frac{p}{p + \frac{1-p}{d-1}} = 1 - \mathsf{P}^p_{x_i}[Z_{\sigma_{x_i}} = x_{i-1} | \sigma_{x_i} < \infty]$$

where $\mathsf{P}^p_{x_i}$ is the path measure of the non-symmetric random walk induced by (3.19) with starting point x_i, and the stopping times σ_{x_i} are defined as in (3.16) with respect to P^p. Consequently the new sequence of random variables $(\tilde{S}_n)_{n\in\mathbb{N}}$ defined by $\tilde{S}_0 := x_0$ and $\tilde{S}_n := Z_{\sigma_{\tilde{S}_{n-1}}}$ is a non-symmetric random walk on π. This together with Proposition 3.5.2 provides again the existence of the quenched and annealed Lyapunov exponents and also their relation using relative entropy.

Let us consider now a geodesic $\pi = [\ldots, x_{-1}, x_0, x_1, \ldots]$ whose two ends point downwards. Such a geodesic has an highest point $x_k \in \pi$ like in Figure 3.4 and it holds

$$\mathsf{h}(x_{i+1}) = \begin{cases} \mathsf{h}(x_i) + 1 & \text{if } i \geq k \\ \mathsf{h}(x_i) - 1 & \text{if } i < k\,. \end{cases}$$

Looking at the transition probabilities of the random walk at x_k we see

$$\mathsf{P}^p[Z_{i+1} = x_{k+1} \mid Z_i = x_k] = \frac{1-p}{d-1} = \mathsf{P}^p[Z_{i+1} = x_{k-1} \mid Z_i = x_k]$$

respectively

$$\mathsf{P}^p[Z_{i+1} = x_k \mid Z_i = x_{k+1}] = p = \mathsf{P}^p[Z_{i+1} = x_k \mid Z_i = x_{k-1}]$$

for the two different neighbours of x_k (see Figure 3.4). At this vertex the direction of the drift changes and the immediate identification with the integers does not work. We need a slight modification of the aforegoing setting. Let the starting point of the random walk be x_0. We observe the travelling risk in the direction of $x_i \in \pi$ for $i > 0$.

If $k \leq 0$ the vertex x_k is equal to x_0 or not on the path $[x_0, x_i]$ for any $i > 0$ and we perform the following modification: Firstly we cut the geodesic at x_k and take the ray which includes x_i. Then we add to the ray the predecessor of x_k, which we call $\tilde{x}_{k-1}$. Adding the predecessor of $\tilde{x}_{k-1}$ and successively all the next predecessors, we obtain the modified geodesic $\pi' = [\ldots, \tilde{x}_{k-2}, \tilde{x}_{k-1}, x_k, x_{k+1}, \ldots]$ like in Figure 3.5. This modification doesn't change $A(x_0, x_{ni}, \omega)$ and $B(x_0, x_{ni})$ for all $n \in \mathbb{N}$ but the new geodesic π' can be identified with the integers as before.

If $k > 0$ the point x_k is on the path $[x_0, x_{mi}]$ for a $m \in \mathbb{N}$. But we know that it holds for each $n \geq m$

$$\begin{aligned} F_q(x_0, x_{ni}, \omega) &= F_q(x_0, x_k, \omega)\, F_q(x_k, x_{ni}, \omega) \\ &=: C(\omega) \cdot F_q(x_k, x_{ni}, \omega) \\ F_a(x_0, x_{ni}) &\geq \mathbb{E}[C(\omega)] \cdot F_a(x_k, x_{ni}) \end{aligned}$$

using the additivity property (Lemma 3.2.4) and the FKG-inequality. Consequently, we see

$$A(x_0, x_{ni}, \omega) = -\ln C(\omega) + A(x_k, x_{ni}, \omega) \tag{3.20}$$

$$B(x_0, x_{ni}) \leq -\ln \mathbb{E}[C(\omega)] + B(x_k, x_{ni})\,. \tag{3.21}$$

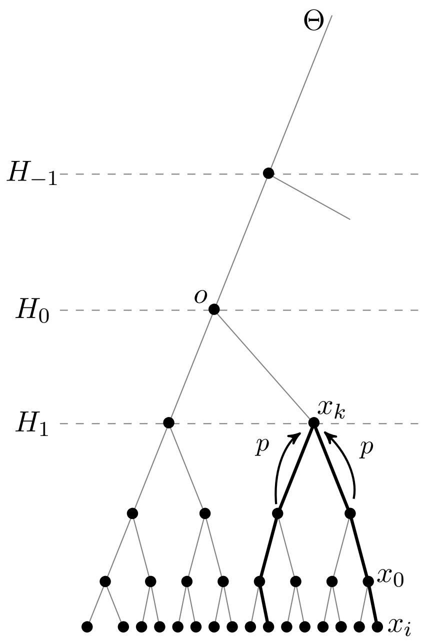

Figure 3.4: Original geodesic π

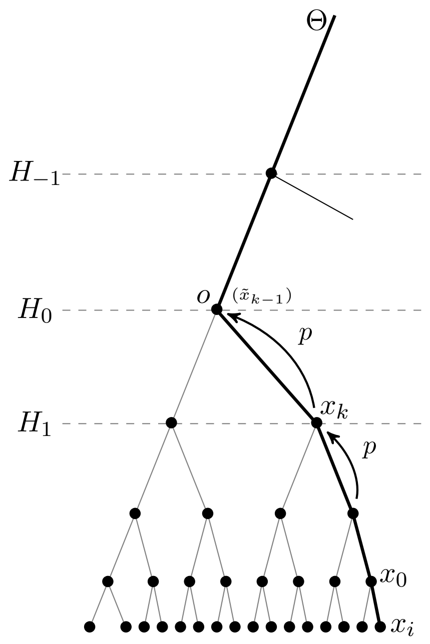

Figure 3.5: Modified geodesic π'

But because it is more probable to survive shorter journeys, it holds

$$B(x_0, x_{ni}) \geq B(x_k, x_{ni}) \tag{3.22}$$

as well. In $A(x_k, x_{ni}, \omega)$ and $B(x_k, x_{ni})$ the random walk starts at x_k and travels to the direction of x_{ni}. Hence we are in the aforegoing case of an uniform drift on the direct path from x_k to x_{ni} and we can identify the geodesic with the integers by cutting it and adding the predecessors. Thus, the Lyapunov exponents with starting point x_k

$$\tilde{\alpha}(x_i) := \lim_{n\to\infty} \frac{1}{n} A(x_k, x_{ni}, \omega)$$
$$\tilde{\beta}(x_i) := \lim_{n\to\infty} \frac{1}{n} B(x_k, x_{ni})$$

exist due to the same arguments as in Section 3.5.1. This, together with (3.20),(3.21) and (3.22) implies

$$\tilde{\alpha}(x_i) = \lim_{n\to\infty} \frac{1}{n} A(x_0, x_{ni}, \omega)$$
$$\tilde{\beta}(x_i) = \lim_{n\to\infty} \frac{1}{n} B(x_0, x_{ni})$$

and by coincidence of the limits it also holds that

$$\tilde{\beta}(x_1) = \inf_{\mathbb{Q}} \{\mathbb{E}\left[A(x_0, x_1, \rho)\right] + H(\mathbb{Q}|\tilde{\mathbb{P}})\},$$

where the infimum runs over all shift invariant probability measures on $\tilde{\Omega}$.

Now we have seen, how we can apply the variational formula for a random walk on the integers to a random walk on homogeneous trees. It may be of interest to observe other structures apart from lattices and homogenous trees and look for similar variational formulas. One subject of interest could be the infinite ladder: here the random walk has to pass at least one out of two vertices for reaching the next rung. Hence, the stopping time of reaching one of these two vertices is finite if the random walk is at the next rung. This could lead to a helpful multidimensional variation of the additivity of a.

Chapter 4

The cost of asymmetrizing vertex-transitive cubic graphs

A set S of vertices in a graph G with nontrivial automorphism group is called *asymmetrizing* if the identity mapping is the only automorphism of G that preserves S as a set. If such sets exist, then their minimum cardinality is the *asymmetrizing cost* $\rho(G)$ of G. We show that the cost is at most 5 for finite connected, cubic vertex-transitive graphs G that are different from K_4, $K_{3,3}$, the cube and the Petersen graph, unless G is a so-called split graph or highly symmetrically connected. For infinite connected graphs G the cost is finite if the automorphism group of G is countable, otherwise infinite. If it is infinite, then one can define the density of asymmetrizing colorings. For vertex-transitive graphs it is well-known that it can be zero or one half. Here we show that it can be strictly between 0 and one half for vertex-transitive graphs with two ends.

4.1 Automorphism breaking

All our graphs are undirected. If they are infinite we will require that their vertices have finite degrees. Such graphs are called *locally finite*. Evidently connected, locally finite infinite graphs are countable. Nonetheless, their automorphism groups can be uncountable, as the example of the 3-regular tree shows.

A vertex coloring of a graph G is *asymmetric* if the identity is the only automorphism of G that preserves it. The smallest number of colors needed is the *asymmetrizing number* or *distinguishing number* $\mathrm{D}(G)$ of G. One says such a coloring *breaks* the automorphisms of G. When $\mathrm{D}(G) = 2$ each of the two colors induces a set of vertices which is preserved only by the identity automorphism. Such sets are called *asymmetrizing*.

Asymmetric colorings go back at least to 1977, when Babai [3] showed that every k-regular tree, where $k \geq 2$ is an arbitrary cardinal, has an asymmetric 2-coloring. In 1991 the asymmetrization of trees was taken up again by Polat and Sabidussi [54], and Polat [52, 53]. The first paper considers trees of any cardinality, but also contains an algorithm that determines the number of inequivalent asymmetrizing sets of finite trees. The other two papers extend, and in some sense complete, the investigation. Polat and Sabidussi do not consider motion, which means that we cannot directly use their results.

Independently Albertson and Collins [1] introduced the term *distinguishing coloring* for an asymmetrizing coloring. Their paper spawned numerous other publications on the subject, for example [8, 29, 32, 43].

The asymmetrizing coloring number of connected, finite or infinite graphs of maximal degree 3 was investigated in [27]. It was shown that the asymmetrizing coloring number of these graphs is at most 3, and all graphs with asymmetrizing coloring number 3 were characterized. They consist of five infinite classes of tree like graphs that are not vertex-transitive and four finite vertex-transitive graphs, namely K_4, $K_{3,3}$, the cube and the Petersen graph.

The subject of our investigations are connected vertex-transitive cubic graphs of asymmetrizing coloring number 2. For graphs G with finite asymmetrizing sets we are interested in the smallest size of such sets, if all asymmetrizing sets are infinite, then we look for asymmetrizing sets that are sparse in a certain sense. The methods we use are quite different from those of [27]. In fact, for arc-regular cubic graphs we rely heavily on results of Tutte [64, 65] and Djoković and Miller [13].

4.2 Cost, density and summary of results of Chapter 4

If a graph G has asymmetrizing coloring number 2, then its set $V(G)$ of vertices can be partitioned into two sets $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that the stabilizer of either one is the trivial automorphism. In other words, if $\alpha \in \mathrm{Aut}(G)$ and $\alpha(V_i) = V_i$ for $i \in \{1, 2\}$, then $\alpha = \mathrm{id}$. Either of the sets V_1 or V_2 is an asymmetrizing set in the sense that the identity is the only automorphism that preserves it as a set. The smallest possible size of such a set is the *asymmetrizing cost* of G. It was introduced in [6] as *2-distinguishing cost* and is also called *Boutin-Imrich cost*. We denote it by $\rho(G)$.

If we use only the colors black and white, and always black for a minimum asymmetrizing set, then $\rho(G)$ is the minimal number of black vertices needed to break all automorphisms. $\rho(G)$ is our main topic of investigation.

Note that $\rho(G)$ can be finite for infinite graphs. In fact, in [6] it was shown that $\rho(G)$ is finite for connected, locally finite infinite graphs G if and only if $\mathrm{Aut}(G)$ is countable. If ρ is infinite one can still try to choose sparse asymmetrizing sets. This leads to the introduction of the density of sets of vertices. It was first introduced in [30].

Let S be a set of vertices of a graph G, $v \in G$, and let $B(v, n) := \{w \in G : d(v, w) \leq n\}$ denote the ball of radius n with center v, then

$$\delta_v(S) := \limsup_{n\to\infty} \frac{| B(v, n) \cap S |}{| B(v, n) |}$$

is the *density* of S at v. If $\delta_v(S)$ exists for all vertices, which is the case for locally finite graphs, then the density of S is defined as $\delta(S) = \sup\{\delta_v(S) : v \in V(G)\}$.

The infimum of $\delta(S)$ over all asymmetrizing sets S is then the *asymmetrizing density* $\delta(G)$ of G.

For finite G the density $\delta(G)$ is the quotient of the size of a smallest asymmetrizing set by $|V(G)|$.

4.2.1 Summary of results of Chapter 4

Our investigations take into account several graph parameters. One of them is the minimum length of the cycles of a graph G. It is called the *girth* of G and denoted by $g(G)$. In particular graphs of girth 4 play an important role.

Another parameter is the number of arc orbits. The orbit of an arc vw with $v, w \in G$ under the action of $\mathrm{Aut}(G)$ is the set

$$O(vw) = \{xy \mid x = \alpha(v), y = \alpha(w), \alpha \in \mathrm{Aut}(G)\}.$$

By vertex-transitivity every vertex has to be incident to at least one edge from every arc orbit, hence the number of arc orbits in a vertex-transitive cubic graph is 1,2 or 3.

If it is 3, and if we fix a vertex v, then all neighbors of v are also fixed. For a connected graph this implies that all vertices are fixed if one is fixed. If we color one vertex of such a graph black and leave all others white, then this is an asymmetrizing coloring.

If the number of arc orbits is 2, then one orbit consists of isolated edges that meet every vertex, and thus form a so-called *matching*, whereas the edges of the other orbit form a subgraph where every vertex has degree two. By vertex-transitivity such an orbit consists of cycles of the same lengths or of double rays.

An intriguing subcase occurs when the cycles that are not in the matching orbit form characteristic cycles by themselves, in a way that will be explained in detail in Section 4.4.3, such that each cycle in a characteristic cycle allows an automorphism of order 2 that leaves the other cycles of this characteristic cycle fixed. Such graphs are called *symmetrically connected* in Definition 4.4.15. A special case are the SPX graphs $\mathrm{SPX}(2, n, m)$ defined in Definition 4.4.6.

A simple example is depicted in Figure 4.1. It is clear how the quadrangles in the figure can be arranged in cyclic order or on a double ray, and that each quadrangle allows an automorphism of order 2 that leaves all other vertices fixed. It the arrangement is a double ray, we speak of a *chain of quadrangles*.

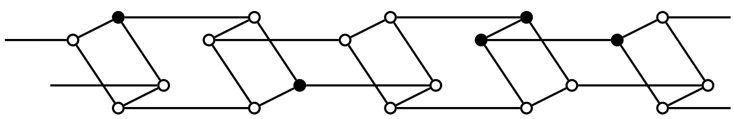

Figure 4.1: A chain of quadrangles with an asymmetrizing coloring.

The next theorem summarizes our results for finite graphs.

Theorem 4.2.1 *Let G be a finite connected, cubic, vertex-transitive graph different from K_4, $K_{3,3}$, the cube and the Petersen graph.*

1. *G has one arc orbit. Then* $1 \leq \rho(G) \leq 5$.

2. *G has two arc orbits. If it is an* SPX *graph* SPX$(2, n, m)$*, then its 2-distinguishing density is* $\max(3, \lceil \frac{m}{n} \rceil)$.

   If G is symmetrically connected, then its 2-distinguishing density $\delta(G)$ is at most $1/16$.

   In all other cases the 2-distinguishing cost $\rho(G)$ is at most 4.

3. *G has three arc orbits. Then* $\rho(G) = 1$.

Before we state our results for infinite graphs we define what we mean by polynomial growth. We say a connected, infinite locally finite graph G has *polynomial growth* with growth rate k, if there is a vertex v and a polynomial p of degree k such that

$$|B(v, n)| < p(n)$$

for all natural numbers n. It is easily seen that this implies that all functions $|B(w, n)|$ are bounded by polynomials of degree k, independent of the choice of $w \in V(G)$. We say that the growth of G is *linear* if $k = 1$, and *quadratic* if $k = 2$. A one- or two-sided infinite path has linear growth, planar grids have quadratic growth.

Now our result for infinite graphs.

Theorem 4.2.2 *Let G be an infinite connected, cubic, vertex-transitive graph. Then we can show:*

1. $g(G) = 3$ *or* 5*. Then* $1 \leq \rho(G) \leq 3$.

2. $g(G) = 4$*. If G is an* SPX *graph* SPX$(2, n)$*, then $\rho(G)$ is infinite and* $\delta(G) = (n2^{n+1})^{-1}$*. In this case G has linear growth.*

   If G is not an SPX *graph, then* $1 \leq \rho(G) \leq 3$*, or $\rho(G)$ is infinite and* $0 \leq \delta(G) \leq 1/16$.

3. $g(G) > 5$*. If G is not symmetrically connected, then* $1 \leq \rho(G) \leq 3$*, otherwise $\rho(G)$ is infinite and* $\delta(G) \leq 1/16$.

4.2.2 Structure of the chapter

The sections cover the cases of one and two arc orbits, and are subdivided according to girth. An overview of the structure is presented by the diagram in Figure 4.2.

For the case of three arc orbits we recall that it is trivial and therefore does not need a separate section.

In the case of two arc orbits girth 4 plays a special role, both for finite and infinite graphs. Graphs with this property have such a high symmetry that in several cases we obtain a density that is greater than zero. To show this we use a method, which we call folding and de-folding of a graph. This is basically a reduction of quadrangles to edges

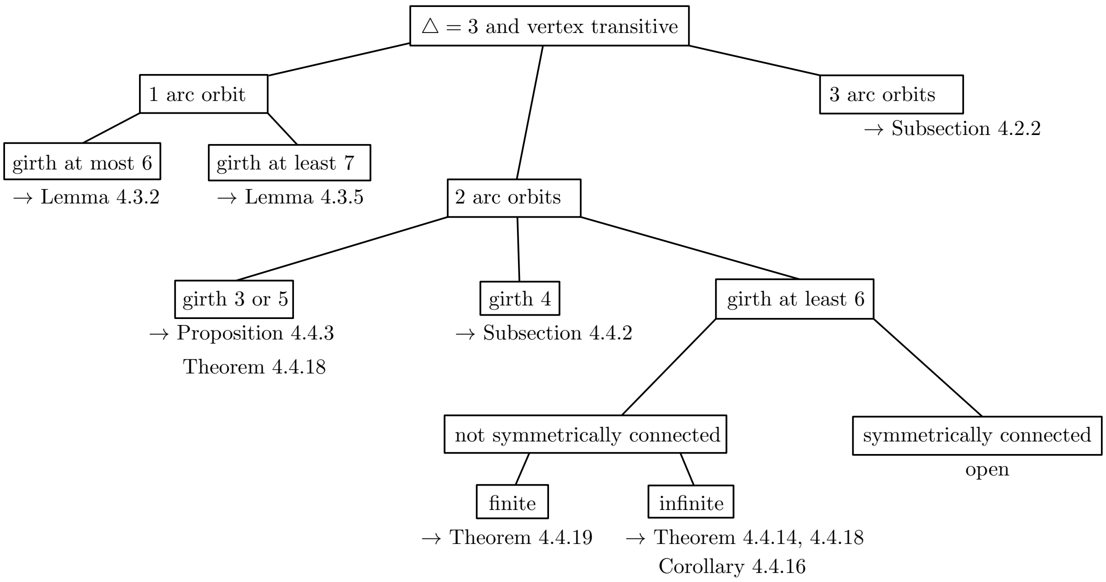

Figure 4.2: An overview of Sections 4.2, 4.3 and 4.4.

and the other way round. The arguments are quite different from those of the other parts of the chapter and presented in Section 4.4.2.

At the end of the chapter we present an alternative way to [21] to showing that every connected, vertex-transitive cubic graph with girth 5 is either 2-distinguishable or the Petersen graph. This section will be heavily based on structural observations of such graphs.

We conclude this section with remarks about density, growth, ends of graphs, the size of the automorphism group and related results. They are relevant for infinite graphs and can be skipped initially.

4.2.3 Density

The last two parts of the section deal with the relationship between the density of a graph, its growth rate and its end-structure.

We defined the density of a graph as the infimum of $\delta(S)$ taken over all asymmetrizing sets S, where $\delta(S) = \sup\{\delta_v(S) : v \in V(G)\}$. We shall show that for certain classes of graphs $\delta_v(S)$ is independent of the choice of v.

We begin with a result from [30] about connected graphs G for which there exists a constant c such that

$$|B(v, n+1)| \leq c \cdot |B(v, n)| \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

For connected graphs G satisfying Equation 4.1 it was shown in [30, Lemma 1] that a set S has zero density $\delta_v(S)$ at each vertex v, if $\delta_w(S) = 0$ for some $w \in V(G)$.

Clearly connected vertex-transitive graphs satisfy Equation 4.1, when they are cubic we can take $c = 3$. Hence, such a graph has zero asymmetrizing density zero if there exists an asymmetrizing set S with zero density at some vertex v.

In [30] it was further shown that large classes of graphs have asymmetrizing density zero, among then infinite trees without leaves and so called tree-like graphs.

For vertex-transitive graphs of polynomial growth we can make similar observations also for strictly positive density.

Lemma 4.2.3 *Let G be a vertex-transitive graph of polynomial growth with growth rate k and S an asymmetrizing set. If $|B(v,n)| \sim cn^k$ for a constant c, each $v \in V(G)$ and all $n \in \mathbb{N}$, then $\delta_v(S) = a \geq 0$ implies that $\delta_w(S) = a$ for all $w \in V(G)$.*

Proof The assumption $|B(v,n)| \sim cn^k$ is equivalent to

$$\lim_{n\to\infty} \frac{|B(v,n)|}{cn^k} = 1. \tag{4.2}$$

Let $\delta_v(S) = a$, $w \in G$, and $d := d(v,w)$. Clearly

$$|B(v,n) \cap S| \leq |B(w,d+n) \cap S| \leq |B(v,n+2d) \cap S|,$$

which is equivalent to

$$\frac{|B(v,n) \cap S|\,|B(v,n)|}{|B(v,n)|\,|B(v,n+2d)|} \leq \frac{|B(w,n+d) \cap S|\,|B(w,n+d)|}{|B(w,n+d)|\,|B(v,n+2d)|} \leq \frac{|B(v,n+2d) \cap S|}{|B(v,n+2d)|}\,.$$

By (4.2) we know that the left and right sides of the inequality converge to a, because $|B(w,n+d)|/|B(v,n+2d)|$, respectively $|B(v,n)|/|B(v,n+2d)|$, converge to 1 as $n \to \infty$. Hence

$$\delta_w(S) = \lim_{n\to\infty} \frac{|B(w,n) \cap S|}{|B(w,n)|} = a\,.$$

We shall apply Lemma 4.2.3 to vertex transitive graphs of linear growth.

4.2.4 Ends of graphs and exponential growth

We now introduce ends of graphs. Ends are an important concept in the investigation of infinite graphs. Our definition follows Halin [22], but the concept is due to Freudenthal [17]. Ends are equivalence classes of rays, also known as *rays.* One says two rays R_1, R_2 are *equivalent* if there is a third ray R_3 that meets both R_1 and R_2 infinitely often. It is easily checked that this is an equivalence relation, and the classes of equivalent rays are called *ends.*

A finite graph has no ends, a ray has one end, a double ray, that is, a two-sided infinite path, has two ends, planar grids one end, and the infinite regular trees T_d of degree $d > 2$ infinitely many ends.

It is also well known and follows from [23] that the number of ends of a connected, vertex-transitive graph is 0, 1, 2, or infinite.

Vertex transitive graphs with infinitely many ends do not have polynomial growth, their growth is exponential. One says a graph G has *exponential growth*, if there is a constant c such that

$$|B(v, n)| > c^n$$

for a vertex $v \in V(G)$ and all $n \in \mathbb{N}$. In connected, locally finite graphs exponential growth does not depend on the choice of v even if G is not vertex-transitive, although the value of c may depend on the choice of v.

At this point a few words about the infinite regular tree T_3 of degree 3 are appropriate. It is cubic, vertex-transitive, has exponential growth, and uncountable automorphism group. Its asymmetrizing coloring number is 2. As we already mentioned this was shown 1977 by Babai [3]. Later it was reproved numerous times.

Let us recall that $\rho(T_3)$ cannot be finite by a result of Boutin, Imrich [6], because $\mathrm{Aut}(T_3)$ is uncountable. That $\delta(T_3) = 0$ was shown in [30]. In the latter paper large classes of graphs with asymmetrizing density zero are exhibited. All of these graphs have infinite motion, that is, every non-identity automorphism moves infinitely many vertices. Vertex transitivity does not play a role in these considerations.

As shown in [30] it is relatively easy to construct graphs with nonzero asymmetrizing density. These examples are not vertex-transitive.

It is much more involved to construct vertex-transitive graphs with nonzero asymmetrizing density. The present thesis seems to be the first that exhibits such graphs. The examples have linear growth, two ends, uncountable automorphism group and finite motion, that is, each non-identity automorphism moves only finitely many vertices.

4.3 Graphs with one arc orbit

If there is only one arc orbit it is easy to see that there exists an automorphism φ to any two arbitrarily chosen edges uv and xy such that $\varphi(u) = x$ and $\varphi(v) = y$. Such graphs are called *arc-transitive*. We shall prove the following theorem.

Theorem 4.3.1 *Let G be an arc transitive cubic graph. If it has finite girth, then $\rho(G) \leq 4$, otherwise it is the T_3, which has infinite asymmetrizing cost and asymmetrizing density 0.*

If a cubic graph has no cycles, then it is the infinite tree T_3 with infinite cost and density zero, see [30].

It thus remains to prove the theorem for graphs with finite girth. In order to do this we divide the theorem into two parts, namely Lemma 4.3.2 for girth at most 6 and Lemma 4.3.5 for girth at least 7. The methods of proof are entirely different.

Lemma 4.3.2 *Let G be an arc transitive cubic graph of girth at most 6. Then $\rho(G) \leq 4$.*

Proof Because we forbid multiple edges the smallest girth is 3. If a symmetric graph G has a triangle and $v \in V(G)$ with neighbors x, y, z, then there must be an edge between any two of them and G is the K_4, which has no asymmetric 2-coloring.

For girth 4, let G be a symmetric graph of girth 4 and $v \in V(G)$ with the neighbors x, y, z. Clearly, any two of the edges vx, vy, vz must span a square. Let the squares be $vxay$, $vybz$ and $vzcx$. If, say, $a = b$, then x is in the two squares $vxay$ and $vxaz$. Let u be the third neighbor of y.

If $u = c$, then every vertex is in three squares, has three neighbors, and G is the $K_{3,3}$ spanned by $\{x, y, z, v, a, c\}$ and has no asymmetric 2-coloring.

If $u \neq c$, then it cannot be in a square with v, because the other 2 neighbors of y, namely v, a, have degree 3. Hence we can assume that a,b,c are pairwise distinct.

We know, that y has to be in three squares. By construction it is in the squares $vxay$ and $vybz$, hence the edges ya and yb must be in a square. Let w be the third neighbor to a. The third square containing y clearly must contain the edges by, ya and aw. Thus w is adjacent to b. By the same argument w is also adjacent to c. Therefore G is the cube, which has no asymmetric 2-coloring.

For girth 5 we invoke a result of Glover and Marušić [21], who showed that there are only two edge-transitive cubic graphs of girth 5, namely the Petersen graph and the pentagon dodecahedron. The Petersen graph does not have an asymmetric 2-coloring, and the asymmetrizing cost for the pentagon dodecahedron is 3, as is easily seen.

We conclude the proof with the observation that the Heawood graph is the only finite or infinite cubic graph of girth 6, see [28, Theorem 27]. It is the dual of the triangulation of the torus with underlying graph K_7. As shown in [28] its asymmetrizing cost is 5.

The Heawood graph is also known as Tutte's 6-cage [64]. Tutte showed that it is the only finite 4-arc regular graph of girth 6. The cited paper is the first on arc-transitive cubic graphs.

For girth ≥ 7 we will heavily rely on Tutte's results in [64, 65], as well as on those of Djokovic and Miller [13], who extended them to infinite graphs.

Following Tutte [64] we call a sequence of vertices $v_0, v_1, \ldots, v_s \in V(G)$ an *s-arc* if $v_i v_{i-1} \in E(G)$ for $1 \leq i \leq s$, but $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. Then G is *s-arc-transitive* if $\mathrm{Aut}(G)$ is transitive on the set of all s-arcs on G. A 1-arc-transitive graph is also called *symmetric*. Moreover, we call G *s-arc-regular* if for any two s-arcs $v_0 v_1 \ldots v_s$ and $w_0 w_1 \ldots w_s$ there is a unique automorphism φ which maps $v_0 v_1 \ldots v_s$ into $w_0 w_1 \ldots w_s$, respecting the order of the vertices.

For symmetric cubic finite graphs Tutte [65] proved the following theorem.

Theorem 4.3.3 (Tutte 1959) *Let G be a finite connected, symmetric cubic graph. Then G is s-arc-regular for some $s \leq 5$.*

Djokovic and Miller [13] extended it to infinite graphs.

Theorem 4.3.4 (Djokovic and Miller 1980) *Every infinite connected symmetric cubic graph is s-arc-regular for some $1 \leq s \leq 5$ with the exception of the infinite cubic tree.*

For the girth of s-arc-regular cubic graphs we will use the bound

$$2s \leq g(G) + 2 \tag{4.3}$$

from [64].

Lemma 4.3.5 *Let G be an arc transitive cubic graph of girth at least 7. Then $\rho(G) \leq 4$.*

Proof By symmetry we can invoke Theorems 4.3.3 and 4.3.4. They imply that our graphs are s-arc-regular for some $s \leq 5$.

If $s = 0$, then G is vertex-regular. It therefore suffices to color exactly one vertex black to break all automorphisms, and thus $\rho(G) = 1$.

If $s = 1, 2$ or 3 we choose a path $uxvw$ in G. This is possible because the girth is > 6. We color u, v, w black as visualized in Figure 4.3. Each color preserving automorphism φ either fixes u, because it is the only black vertex without black neighbors. As v and w have different distances from u, they are also fixed. Hence φ fixes the s-arcs ux, uxv and $uxvw$, where $s = 1, 2, 3$, respectively. By s-arc regularity φ is the identity.

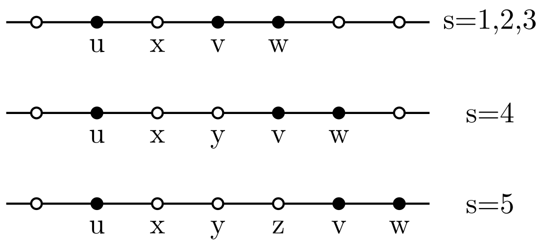

Figure 4.3: Colorings of s-arcs.

Now, let $s = 4$. For girth $g > 7$ we choose a path $uxyvw$ and color u, v and w black as in Figure 4.3. Then we argue as before to prove that the 4-arc $uxyvw$ is fixed by all color preserving automorphisms. If the girth is 7 this coloring allows that both v and w have distance 3 from u. In this case it suffices to color y black to fix the 4-arc $uxyvw$ by all color preserving automorphisms.

For $s = 5$ we first observe that the girth is at least 8 by Equation 4.3. We choose a path $uxyzvw$ of length 5 and color u, v and w black. If the girth is different from 9 this coloring fixes the 5-arc $uxyzvw$. If the girth is 9, then v, w could be interchanged by color preserving automorphisms. To avoid this we also color z black. This fixes $uxyzvw$ by the same arguments as before. By 5-arc regularity this is an asymmetrizing coloring.

Clearly the cost of our colorings is at most 4, which proves the lemma. Together with Theorem 4.3.4 and Lemma 4.3.2 this completes the proof of Theorem 4.3.1.

4.4 Graphs with two arc orbits

It is left to consider vertex-transitive cubic graphs with two arc orbits. Here every vertex has two edges in the same orbit C and one in the other orbit D. We can assume that the edges in orbit C form cycles which are connected by edges in D. Let g be the girth of these cycles in C. By the vertex-transitivity, two of those cycles can be either connected by g edges (and the graph will be finite) orconnected by k edges, where $k|g$, see Figure 4.4.

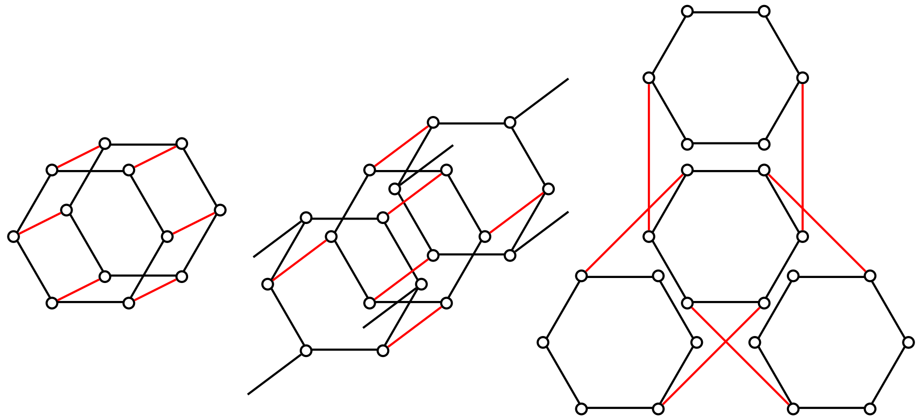

Figure 4.4: Possible connections of hexagons.

If $k = g$ the graph is a prism. In this case, and if $k > 2$, we notice, that fixing one cycle is enough to fix the whole graph, as moving a cycle that is not fixed influences the fixed cycle, as well. The smallest girth for which there exists a $k \in \mathbb{N}$ such that $2 < k < g$, is $g \geq 6$. In this case the so called standard coloring of Figure 4.13, with three vertices suffices to break all automorphisms. The only interesting cases left are $k = 1, 2$. Here, the cycles can either be connected like a tree, or form cycles of girth m by themselves.

Definition 4.4.1 An (m, g)**-cycle** is a cycle consisting of m cycles with girth g. Moreover, we define a path formed by cycles of girth g a g**-cycle-path**.

The graphs where two g-cycles are connected by either 1 or 2 edges are interesting. Here it is not enough to fix only one cycle. Every cycle connected to the fixed cycle can still be flipped. Firstly, we treat the case, where the g-cycles form a tree. Afterwards we have to distinguish with respect to the size of the girth. We start with girth $g = 3, 5$ and continue with girth $g = 4$. In this case we will find graphs with positive density. Then, we finish with $g \geq 6$. Now, we start with g-cycles forming a tree.

Theorem 4.4.2 *Let G be an infinite vertex-transitive cubic graph with girth $g \geq 3$ where the g-cycles are connected by one or two edges. If there is no (m, g)-cycle for any m and if every g-cycle is connected to more than two g-cycles, then $\delta(G) = 0$.*

Proof Let G be a graph fulfilling the assumptions. We select one g-cycle as a root and call it R. We observe that a coloring which fixes vertices up to a certain distance to R, does not fix the cycles which are further away, like in a tree. Indeed, contracting the g-cycles to one vertex we obtain either a tree G' of degree g or $\frac{g}{2}$. Here, we can use the tree-coloring of [30] such that only the identity preserves the coloring, and translate this to G, by coloring one vertex in a g-cycle which corresponds to a colored vertex in G'. This yields $\delta(G) = 0$.

The condition, that every g-cycle has at least 3 neighboring g-cycles guarantees, that the contracted graph G' is not a double ray. This can happen in the case of quadrangles connected by two edges and will be discussed later on.

4.4.1 Girth 3 and 5

Now, let $g = 3, 5$. Due to the fact of having two arc orbits, these are graphs, where triangles resp. pentagons are connected by one edge.

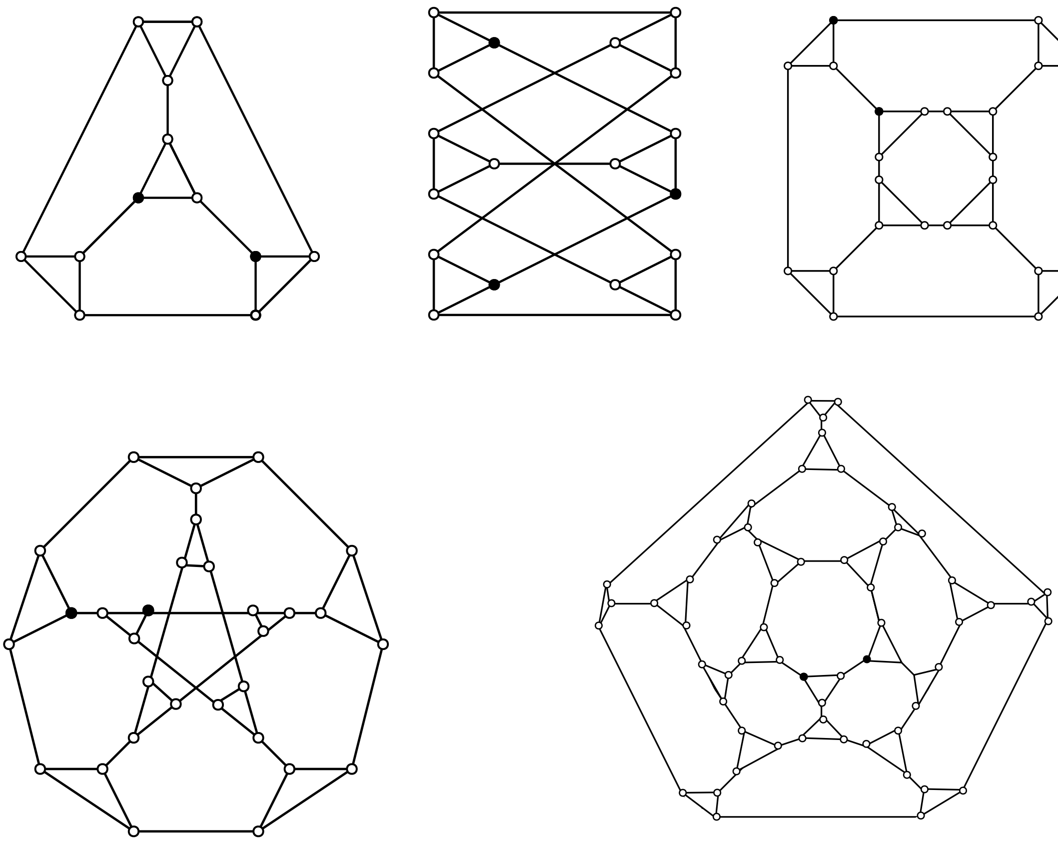

Figure 4.5: Distinguishing colorings of selected graphs consisting of triangles that are connected by one edge.

Proposition 4.4.3 *A vertex-transitive cubic graph G which consists of triangles connected by one edge has at most $\rho(G) = 3$ or $\delta(G) = 0$.*

Proof Let G be a graph fulfilling the assumptions. This graph has two arc orbits namely orbit E_1 with two edges at each vertex forming the triangles and orbit E_2 of the edges connecting the triangles. Firstly, we contract every triangle to a single vertex. This yields again a cubic graph G'. The edges of G' are all edges from E_2. Hence, all edges are in the same arc orbit and G' is symmetric, as well. The only symmetric cubic graphs allowing girth at most 5 are the K_4, $K_{3,3}$, the cube, the Petersen graph and the pentagon dodecahedron. For these special cases we present the coloring with 2 and 3 black vertices in Figure 4.5.

Moreover, we know by Theorem 4.3.3 and 4.3.4 that G' is s-arc-regular with $s \leq 5$ apart from the infinite cubic tree. In case of the tree, we use Theorem 4.4.2. Next, we will treat the cases of girth 6 and girth at least 7, separately. Allowing only girth at least 7, we observe that for each s the coloring from Figure 4.6 allows only the identity as an automorphism which preserves the coloring and projects the s-arc onto itself.

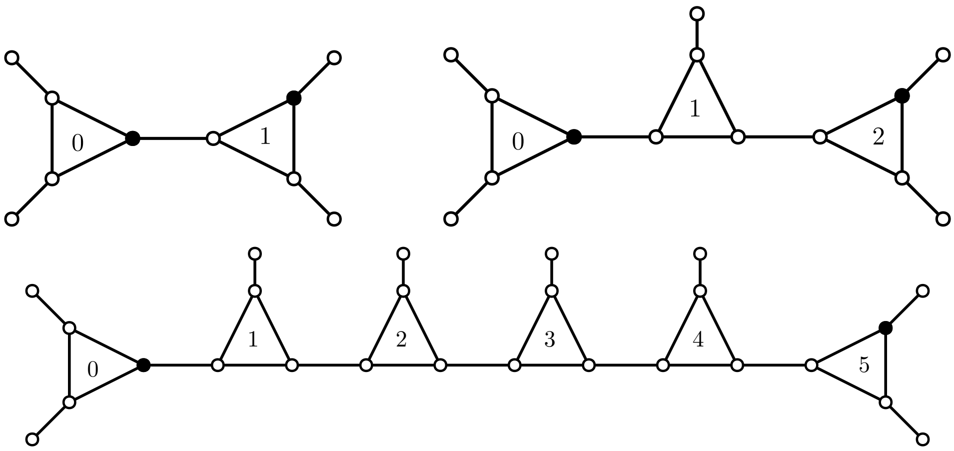

Figure 4.6: Distinguishing colorings of a 1,2 and 5-arc in a graph of triangles connected by one edge and m-girth at least 7.

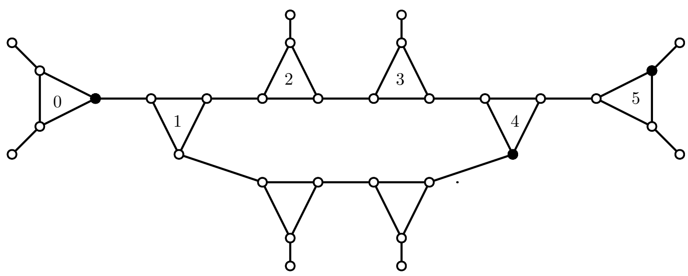

Figure 4.7: Distinguishing coloring of a 5-arc in a graph of triangles connected by one edge and m-girth 6.

In the case of m-girth 6 and $s = 5$ it could happen that the neighboring triangles to the colored triangles are part of a hexagon. Hence, there would be two paths connecting the colored triangles which can be mapped onto each other and preserve the coloring at the same time. Therefore, we need to color a triangle adjacent to one of the other colored triangles along the arc, see Figure 4.7. For $s \leq 4$ forming a hexagon is not a problem and two vertices are sufficient to break all automorphisms.

Graphs with girth $g = 5$, where the g-cycles are connected by one edge, will be discussed in the last section, as a reduction of 10-cycles connected by two edges in Theorem 4.4.18.

4.4.2 Girth 4

Here our subject of investigation are graphs with two arc orbits, one of which consists of quadrangles, and the other of disjoint edges, which form a matching. We first consider the case when there are two edges from the set of matching edges between pairs of adjacent quadrangles. Suppose the quadrangles $abcd$ and $uvwz$ are adjacent, and the edges between them are ax and by, where $x, y \in \{u, v, w, z\}$. If x, y are not adjacent, then the other two matching edges originating in $uvwz$ cannot originate from adjacent vertices, but this means that $abcd$ cannot be mapped into $uvwz$, which contradicts the transitivity assumption. Hence xy is an edge. Then the only possible graphs are the cube, the prism, the Moebius ladder, or the infinite ladder. None of these graphs has two arc orbits, where one consists of quadrangles and the other of a matching.

Hence, we can assume that the edges between $abcd$ and $uvwz$ are between opposite vertices of the quadrangles. It is easy to see that the only possible graphs in this case are the ring of at least three quadrangles, see Figure 4.8, or the chain of quadrangles of Figure 4.1. As the colorings in the figures indicate, the asymmetrization cost for the ring is the number of quadrangles, and for the chain of quadrangles, which has uncountable automorphism group, the asymmetrizing density is $\frac{1}{4}$.

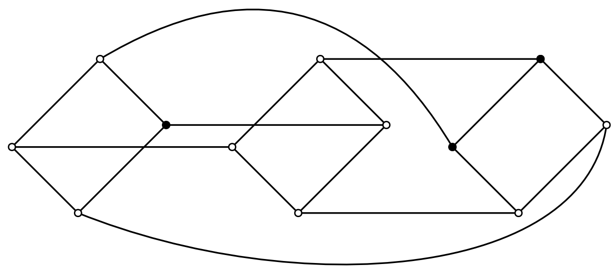

Figure 4.8: Chain of three quadrangles

Therefore we can assume from now on that there is at most one edge between two quadrangles. For such graphs, and for the chains or rings of quadrangles from above, we define a transformation, which we call *folding*, that reduces them to a smaller graphs. If G folds onto G' we wish to use the information about G' for the construction of asymmetrizing colorings of G.

Given a graph G that is either a chain or ring of quadrangles or a graph with at most one edge between adjacent quadrangles, we partition $V(G)$ into the sets of opposite vertices in the quadrangles, and then form the quotient graph G' of G with respect to this partition. The new vertices are connected by an edge if there is at least one edge between their preimages in G. We call this a folding, because it can be envisaged as an operation on the squares, where we first identify a pair of opposite vertices in each square. This folds the squares into a paths of length 2. Then the paths are folded into single edges. These new edges are disjoint and form a matching.

The edges incident to opposite vertices of the squares remain distinct after folding, but share one endpoint in G', compare Figure 4.9. It means that they form a subgraph where each vertex has degree 2, that is, a subgraph of cycles of equal lengths. Here we also admit cycles of length 2, that is, double edges, and cycles of infinite length, that is, double rays.

Cycles of length 2 occur when we fold a ring or chain of quadrangles, see Figure 4.10. Double rays appear when G consists of graphs arranged in a tree-like manner.

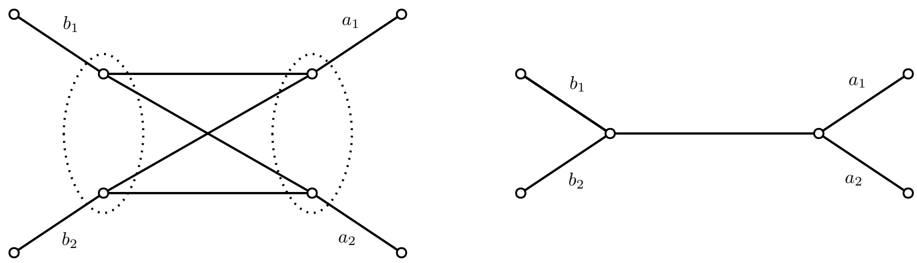

Figure 4.9: Folding of quadrangles

Figure 4.10: Chain of single and double edges.

Lemma 4.4.4 *Let G be a cubic vertex-transitive graph with exactly two arc orbits, one consisting of quadrangles and the other of isolated edges, and G' the corresponding graph after folding. Then G' is vertex-transitive as well and the subgroup of the automorphism group of G' that is induced by $Aut(G)$ contains two arc orbits, one consisting of a matching, and the other one of cycles or double rays.*

Proof G' is formed from G by identifying opposite vertices of each quadrangle and by replacing the four edges between the identified vertices by a single edge. Clearly each automorphism α of G induces an automorphism of G', say $\varphi(\alpha)$, because it preserves pairs of opposite vertices of quadrangles. As $Aut(G)$ acts transitively on the pairs of opposite vertices of the quadrangles the group $\varphi(Aut(G)) \subseteq Aut(G')$ acts transitively on G'.

Clearly $\varphi(Aut(G))$ acts transitively on the images of the quadrangles and transitively on the images of the matching edges in G, but $\varphi(Aut(G))$ does not map images of quadrangles into images of matching edges.

If G consists of quadrangles that are arranged in a tree like manner, then G' has only one arc orbit, despite the fact that $\varphi(Aut(G))$ has two arc orbits. Clearly G' cannot be folded again.

To find a condition when G' can be folded again, let us consider the case when the images of the matching edges of G form quadrangles. Then no edge of the matching edges of G' can be in a four-cycle, because then there would have to be two edges between two neighboring quadrangles of G, and the origins of the edges would have to be adjacent. But this case we have already excluded earlier. Hence, the matching edges of G' form an orbit under $Aut(G')$ if the images of the matching edges of G consist of quadrangles. In this case we can also fold G'.

Now we show that asymmetrizing colorings of G' induce asymmetrizing colorings of G.

Lemma 4.4.5 *Let G be a cubic vertex-transitive graph with an arc orbit consisting of quadrangles, and G' be its corresponding graph after folding. Then any asymmetrizing coloring of G' induces an asymmetrizing coloring of G of the same cost or density.*

Proof First we clarify how an asymmetrizing coloring c' of G' induces one of G. Let $v' \in G'$ be a colored vertex and v_1, v_2 be its its preimages in G. Then we choose randomly one of the preimages and color it. Let c be this coloring of G.

Suppose an automorphism α of G preseves c. As α preserves the set of opposite vertices of the quadrangles in G, which are the preimages of the vertices in G', it induces an automorphism $\varphi(\alpha)$ of G'. Moreover, if a preimage v_1, v_2 is mapped into u_1, u_2 by α, then either both pairs contain exactly one colored vertex, or both pairs contain only uncolored vertices. But then $\varphi(\alpha)$ preserves c' and must be the identity mapping, which means that all pairs of vertices of G are stabilized by α and that pairs that have just one colored vertex are fixed pointwise.

This means that we have to consider the possible interchange of the two uncolored opposite vertices u_1, u_2 in a quadrangle. Let u' be the image of $\{u_1, u_2\}$ under folding. Clearly an interchange of u_1,u_2 would induce an interchange of the two edges incident with u', that is, the images of the matching edges incident with u_1, resp. u_2 in G. But this is prohibited as G' as c' is asymmetrizing.

The assertion about the cost is trivial, the one about the density follows easily.

As a simple application let us have a look at the graph G consisting of quadrangles that are arranged in a tree-like manner. By folding we obtain an infinite cubic tree G'. We know that such trees have asymmetrizing 2-colorings of density zero. Any such coloring induces an asymmetrizing 2-coloring of G, and it is easy to see that the density is still zero.

Another, maybe more direct way to prove this will be followed in similar cases in Section 4.4.3.

Finite Graphs with two arc orbits consisting of a set of squares and a matching

We first consider graphs with two arc orbits consisting of a set of squares and a matching that can be reduced to a ring of m single and double edges. As we do not allow triple edges, $m \geq 2$. If G can be folded onto such a ring by n foldings, we denote it by $P(n,m)$. As the processes of folding and defolding yield unique graphs, up to isomorphisms, $P(n,m)$ is uniquely defined. Also, the graphs $P(1,m)$ are the rings of m quadrangles.

$P(1,2)$ is the cube and not 2-distinguishable, $P(2,2)$ is the Cartesian product of a C_8 by and edge, it has two arc orbits, one consisting of two cycles of length 8, so it is not in the class of graphs considered here, but it has 2-distinguishing cost 3. We can still apply defolding and all defolded graphs, that is, all $P(n,2)$, where $n \geq 2$, will have 2-distinguishing cost 3.

Hence we are only interested in the case when $m \geq 3$. In the sequel we will show that the $P(n,m)$ graphs with $m \geq 3$ and $1 \leq n \leq m-1$ are so called Split Praeger–Xu graphs, SPX–graphs for short. We will define them now and determine their 2-distinguishing costs.

For the graphs $P(m,m)$, $m > 2$, we will show that they have 2-distinguishing cost 1, and hence three arc orbits.

Definition 4.4.6 Let n and m be positive integers with $m > 3$ and $1 \leq n \leq m-1$. The graph $\mathrm{SPX}(2,n,m)$ has vertex-set $\mathbb{Z}_2^n \times \mathbb{Z}_m \times \{+,-\}$ and edge-set

$$\{\{(i_0,i_1,\ldots,i_{n-1},x,+),(i_1,i_2,\ldots,i_n,x+1,-)\} \mid i_j \in \mathbb{Z}_2, x \in \mathbb{Z}_m\}$$
$$\cup \{\{(i_0,i_1,\ldots,i_{n-1},x,+),(i_0,i_1,\ldots,i_{n-1},x,-)\} \mid i_j \in \mathbb{Z}_2, x \in \mathbb{Z}_m\}.$$

These are cubic, bipartite graphs. For $\mathrm{SPX}(2,2,m)$, where m is large, compare Figure 4.11.

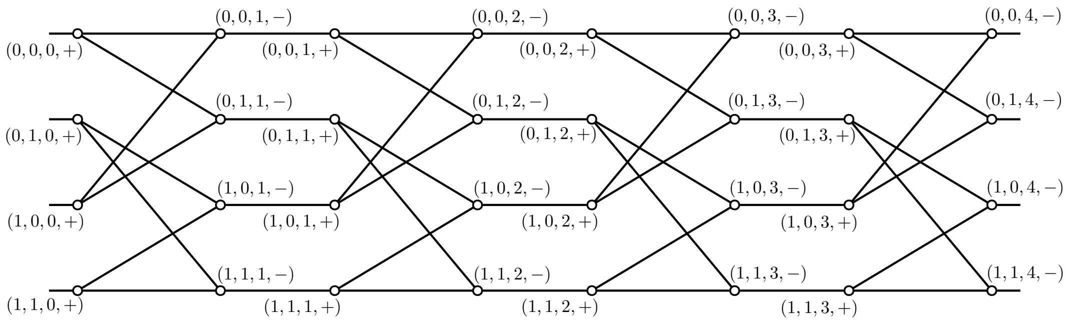

Figure 4.11: Part of $\mathrm{SPX}(2,2,m)$ for large m

In [48] it is shown that the wreath product $W = \mathbb{Z}_2^m \rtimes D_m$ acts on the vertex set of $\mathrm{SPX}(2,n,m)$, that is, on $V(\mathrm{SPX}(2,n,m) = \mathbb{Z}_2^n \times \mathbb{Z}_m \times \{+,-\}$ via the following action: for $g = (g_0,\ldots,g_{m-1},h) \in W$, with $g_0,\ldots,g_{m-1} \in \mathbb{Z}_2$ and $h \in D_r$, set

$$(v_0,v_1,\ldots,v_{n-1},x,\pm)^g = \begin{cases} (v_0+g_x, v_1+g_{x+1},\ldots,v_{n-1}+g_{x+n-1},x^h,\pm) & \text{if } h \in \mathbb{Z}_m, \\ (v_0+g_x, v_1+g_{x+1},\ldots,v_{n-1}+g_{x+n-1},x^h,\mp) & \text{otherwise,} \end{cases}$$

where the subscripts are to be understood modulo m and where x^h denotes the image of x under h. Clearly the action is vertex transitive and faithful, that is, any two different group elements act differently on $V(\mathrm{SPX}(2, n, m)$.

In fact, by [48, Lemma 2.8] W is the full group of automorphisms of $\mathrm{SPX}(2, n, m)$ if $m \geq 5$ and $1 \leq n \leq m - 1$.

If we choose g such that $g_0, g_1, \ldots, g_{n-2}, g_n, \ldots, g_{m-1}$ and h are equal to 0, then

$$(v_0, v_1, \ldots, v_{n-1}, x, \pm)^g = (v_0, v_1, \ldots, v_{n-1}, x, \pm)$$

for $n \leq x \leq m$. Calling the subgraph of $\mathrm{SPX}(2, n, m)$ that is spanned be the vertices with the same x the *x-th column,* this means that g fixes all vertices in the last $m - n$ columns. Because $n \leq m - 1$, at least one column is fixed pointwise. In other words, at least one vertex is moved in columns 0 to $n - 1$, and all vertices in the other columns are fixed pointwise.

In [48] it is observed that the subgraphs spanned by the vertices whose $n + 1$-st coordinates are x and $x + 1$ consist of disjoint quadrangles. We call these subgraphs columns of quadrangles, not to be confused with the columns of matching edges. If we fold an $\mathrm{SPX}(2, n, m)$ graph, where $n \geq 2$, then we obtain an $\mathrm{SPX}(2, n - 1, m)$ graph. Clearly $\mathrm{SPX}(2, 1, m)$ is a ring of quadrangles and, folding $\mathrm{SPX}(2, 1, m)$, we reach a ring consisting of m double and m single edges. Because defolding is unique, the Split Praeger–Xu graphs $\mathrm{SPX}(2, n, m)$ are exactly our $P(n, m)$ graphs.

Theorem 4.4.7 *Let G be an $\mathrm{SPX}(2, n, m)$ graph where $m \geq 5$ and $1 \leq n \leq m - 1$. Then G admits an asymmetrizing 2-coloring with $\rho(G) = \lceil \frac{m}{n} \rceil$, unless $\lceil \frac{m}{n} \rceil = 2$. Then $\rho(G) = 3$.*

Proof By [48, Lemma 2.8] the group W defined above is $\mathrm{Aut}(\mathrm{SPX}(2, n, m))$. Its action on the columns is that of D_m. We choose a vertex v_0^- in the half column $(0, -)$ and its images under the action of $n, 2n, \ldots, (\lfloor \frac{m}{n} \rfloor - 1)n \in \mathbb{Z}_m < D_m$. We color these vertices black, together with the vertex v_0^+ that is adjacent to v_0^- in column $(0, -)$. Notice that v_0^- is fixed when v_0^+ is fixed, and that setting v_0^+ black prevents shifting and switching of the columns by an element of order 2 in D_m, if $\rho(G) = \lceil \frac{m}{n} \rceil \neq 2$.

The only color preserving automorphisms stabilize the columns (and half columns). Suppose g is an automorphism moving a vertex in half column $(x, \pm)$. We can assume that $0 \leq x < \lceil \frac{m}{n} \rceil$.

It is of the form $g = (g_0, \ldots, g_{m-1}, 0)$ and its action is

$$(v_0, v_1, \ldots, v_{n-1}, x, \pm)^g = (v_0 + g_x, v_1 + g_{x+1}, \ldots, v_{n-1} + g_{x+n-1}, x, \pm).$$

Hence there is an i such that $v_i \neq v_i + g_{x+i}$, which means that $g_{x+i} \neq 0$. But then $g_{(x+j)+i-j} = g_{x+i} \neq 0$ for $-i \leq j < n - i$ and $g(j) = (g_0, \ldots, g_{m-1}, j)$ moves $(x, \pm)$ into $(x + j, \pm)$ for $-i \leq j < n - i$.

If $x = 0$ we can thus find a j such that $g(j)$ moves $(0, \pm)$ into a column that has no colored vertex, which is not possible, because $(0, \pm)$ contains v_0. If $x \neq 0$ there is a $g(j)$ that moves $(x, \pm)$, which contains a colored vertex, into $(0, \pm)$, which is also not possible.

Hence our coloring breaks $\mathrm{Aut}(\mathrm{SPX}(2, n, m))$.

Proposition 4.4.8 *The 2-distinguishing cost of the* SPX$(2,n,m)$ *graphs that are not covered by Theorem 4.4.7 is 3.*

Proof We have to treat the cases $m = 3$ and 4.

SPX$(2,1,3)$ has distinguishing cost 3, as depicted in Figure 4.8. By Lemma 4.4.5 this implies that $\rho(\mathrm{SPX}(2,2,3) \leq 3$. As we know that it cannot be 2, we infer that it is $\rho(\mathrm{SPX}(2,2,3) = 3$.

SPX$(2,1,4)$ has distinguishing cost 4, as is easily seen by extending the argument for SPX$(2,1,3)$. $\rho(\mathrm{SPX}(2,2,4) = 3$, which can be checked directly. Then $\rho(\mathrm{SPX}(2,3,4) = 3$ by the same arguments as before.

Proposition 4.4.9 *The distinguishing cost of the* $P(n,m)$ *graphs for* $n \geq m \geq 3$ *is 1.*

Proof $P(m,m)$, $m \geq 3$, is obtained from SPX$(2,m-1,m)$ by defolding. It has the same structure as SPX$(2,m,m)$ if we relax the condition that $n \leq m-1$ in the definition of SPX graphs. Also W acts on SPX$(2,m,m)$.

Let us consider the case $m = 3$ first. Suppose we color $(1,1,1,0,+)$ in $G = \mathrm{SPX}(2,3,3)$ black. Since G has two arc orbits there must be an automorphism that maps $(1,1,1,1,-)$ into $(1,1,0,1,-)$, and $(1,1,1,1,+)$ into $(1,1,0,1,+)$. Thus the set of neighbors of the vertex $(1,1,1,1,+)$ into the set of neighbors $(1,1,0,1,+)$, that is the set of vertices $\{(1,1,1,2,-),(1,1,0,2,-)\}$ into $\{(1,0,1,2,-),(1,0,0,2,-)\}$, and hence mapping the set $\{(1,1,1,2,+),(1,1,0,2,+)\}$ into $\{(1,0,1,2,+,(1,0,0,2,+)\}$. But this is not possible, because $(1,1,1,2,+) = (1,1,1,0,+)$, and $(1,1,1,0,+)$ is fixed.

But then $P(3,3)$ has three arc orbits, 2-distinguishing cost 1, and thus all $P(n,3)$ graphs for $n > 3$ by Lemma 4.4.5.

Similarly we show that $\gamma(P(m,m)) = 1$ for $m > 3$, and hence this also holds for all $P(n,m)$ with $n \geq m > 3$.

We wish to point out the argument for $P(3,3)$ does not work for $P(2,2)$, because SPX$(2,2,2)$ has quadrangles that are not in the columns of quadrangles.

This covers all finite graphs with two arc orbits, one of which consisting of quadrangles, that can be reduced to a ring of double and single edges. However, such a sequence of foldings can also end in other graphs. And this is what we will discuss now.

The sequence of foldings could end in graphs of girth 3 or 5. If it is 3, then it is easy to show that between two triangles in the reduced graph there can be at most one edge. By Proposition 4.4.3 these graphs have 2-distinguishing cost at most 3, and thus also all the graphs reached by sequences of defoldings.

If the girth is 5, then it can be the Petersen graph, the pentagondodecahedron, or a graph where there is at most one edge between two pentagons. The Petersen graph has distinguishing number 3. By [28] it can be distinguished by coloring one vertex red and two selected other vertices black. But the preimage of the Petersen graph, and a fortiori, for all graphs reached by a sequence of unfoldings, have distinguishing number 2 and cost 1, see Figure 4.12. For the pentagondodecahedron we have shown that its 2-distinguishing

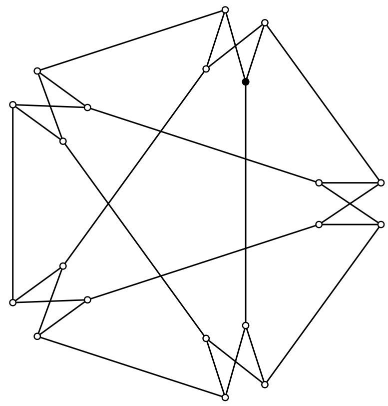

Figure 4.12: The preimage of a Petersen graph under folding.

cost is 3. Similar to the Petersen graph, preimages of the pentagpentagondodecahedron have distinguishing cost 1.

If the sequence of foldings ends in a graph of girth at least 6 we invoke Theorem 4.4.19 to see that they have 2-distinguishing cost 2, unless the graph reached is symmetrically connected, see Definition 4.4.15. We have no examples of finite symmetrically connected graphs, but if they do exist they have an orbit of cycles of at least length 8, with two edges between any two connected ones, compare Figure 4.16. If we distinguish one cycle with three black vertices, then the neighboring cycles can be fixed by one black vertex each. We continue this way until only one cycle in the orbit remains. It must be fixed, because all of its neighbors in other cycles are fixed. Thus the density of this coloring is at most $\frac{1}{8}$, and that of any graph folded to it at most $\frac{1}{16}$. Of course this extends to infinite symmetrically connected graphs.

For the finite case we have thus shown:

Theorem 4.4.10 *Let G be a finite graph with two arc orbits, where one induces a set of squares and the other a matching. If G is an* SPX*-graph* $\mathrm{SPX}(2, n, m)$*, then its 2-distinguishing density is* $\max(3, \lceil \frac{m}{n} \rceil)$*. If G is symmetrically connected, then its 2-distinguishing cost is at most $\frac{1}{16}$, and in all other cases $\rho(G)$ is at most 3.*

Infinite Split Praeger–Xu graphs

We now extend the definition of Split Praeger–Xu graphs to infinite graphs, show that they have uncountable automorphism groups, and determine their 2-distinguishing density.

If we replace $\mathbb{Z}_m$ in Definition 4.4.6 by $\mathbb{Z}$, the we obtain infinite graphs, for which we introduce the notation $\mathrm{SPX}(2, n)$. Their vertex sets are $\mathbb{Z}_2^n \times \mathbb{Z} \times \{+, -\}$.

Theorem 4.4.11 *Each* $\mathrm{SPX}(2, n)$ *graph admits an asymmetrizing 2-coloring of density* $\delta(G) = \frac{1}{n2^{n+1}}$.

Proof Let W be the group $\mathbb{Z}_2^\infty \rtimes D_\infty$ with the following action on $V(\mathrm{SPX}(2,n))$: for $g = (\ldots, g_{-1}, g_0, g_1, \ldots, h) \in W$, with $\ldots, g_{-1}, g_0, g_1, \ldots \in \mathbb{Z}_2$ and $h \in D_\infty$, let

$$(v_0, v_1, \ldots, v_{n-1}, x, \pm)^g = \begin{cases} (v_0 + g_x, v_1 + g_{x+1}, \ldots, v_{n-1} + g_{x+n-1}, x^h, \pm) & \text{if } h \in \mathbb{Z}_\infty, \\ (v_0 + g_x, v_1 + g_{x+1}, \ldots, v_{n-1} + g_{x+n-1}, x^h, \mp) & \text{otherwise.} \end{cases}$$

As in the finite case one sees that the action is faithful, vertex transitive, and that the set of columns is stabilized. By the same arguments as before one also sees that there are group elements that move at least one vertex in columns 0 to $n-1$, but fix all other vertices. Hence there are automorphisms that move at least one vertex in columns kn to $(k+1)n-1$ and fix all other vertices. Let A be the set of these automorphism. A has infinitely many elements, the product of the elements in any subset of A is well defined, and different subsets yield different products. Hence the number of automorphism of $\mathrm{SPX}(2,n)$ is uncountable. By a result of [6] this means that $\mathrm{SPX}(2,n)$ has no finite asymmetrizing set.

Although W stabilizes the set of columns, we have not shown this for $\mathrm{Aut}(\mathrm{SPX}(2,n))$ yet. To see this, we first observe that folding $\mathrm{SPX}(2,n)$ results in $\mathrm{SPX}(2,n-1)$, and that the folding preserves columns. Furthermore, any automorphism of $\mathrm{SPX}(2,n)$ induces an automorphism of $\mathrm{SPX}(2,n-1)$ by Lemma 4.4.4. Hence, if columns are not preserved in $\mathrm{SPX}(2,n)$, then they are not preserved in $\mathrm{SPX}(2,n-1)$, and consequently not in $\mathrm{SPX}(2,1)$, which is not the case. Therefore the set of columns is preserved.

We now choose an integer k that is a multiple of n larger than 5 and consider the columns $(ik, \pm)$, $i \in \mathbb{N}$. There are 2^n edges in each column, that is, a finite number, and there are only $2^n!$ ways to order them. By the vertex transitivity of $\mathrm{SPX}(2,n)$ there are automorphism φ_i that map column $(0,\pm)$ into column $(ik,\pm)$ at least two of them preserve the order of the edges in the columns, say φ_r and φ_s, where $r < s$. But then $\varphi_s\varphi_r^{-1}$ maps $(0,\pm)$ into $(s-r,\pm)$. We now identify the half column $(s-r,+)$ with the half column $(0,-)$ to obtain a graph H isomorphic to $\mathrm{SPX}(2,n,(s-r)n)$.

Clearly the action of the vertex stabilizer of v_0 on the first n columns of $\mathrm{SPX}(2,n)$ is the same as that of the vertex stabilizer of v_0 in H on its first n columns. And, for this case we have shown, that fixing v_0 (and the order of the columns) fixes each element of the first n columns.

Similarly, using vertex transitivity, that fixing a vertex in the half column $(jn,+)$ fixes all vertices in the n columns $(jn,+), \ldots, ((j+1)n-1,+)$. Coloring v_0 and one vertex each in the half columns $(jn,+)$, where $n \neq 0$, prevents translation of the columns and inverting their order, so all the black vertices remain fixed, and by the above remark also all vertices in their columns and the following $n-1$ columns. This fixes all vertices of $\mathrm{SPX}(2,n)$.

Because the number of vertices in n columns is $n2^{n+1}$, the density of this coloring is $\frac{1}{n2^{n+1}}$.

If one folds an $\mathrm{SPX}(2,n)$ graph, where $n > 1$, one obtains the graph $\mathrm{SPX}(2,n-1)$, and if one folds $\mathrm{SPX}(2,1)$ the result is a chain of single and double edges. The $\mathrm{SPX}(2,n)$

are thus exactly the graphs with two arc orbits consisting of a set of quadrangles and a matching that can be reduced to a chain of single and double edges by a finite number of foldings.

Ends of graphs

An infinite vertex-transitive graph can have either 1, 2 of infinitely many ends. The ones with two ends are exactly those with linear growth.

We show now that infinite vertex-transitive graphs with two ends that consists of a set of quadrangles and a matching either have finite cost, or are SPX$(2, n)$ graphs with strictly positive density.

Theorem 4.4.12 *Let G be an infinite graph with two ends consisting of quadrangles connected by one edge. Then $\rho(G) = 3$, $\delta(G) = 0$, or $\delta(G) = \frac{1}{n2^{n+1}}$ for some $n \geq 1$.*

Proof If G has two ends, we know by [31] that the graph has the structure of finitely many layers of infinite paths. Such graphs are called *strips* by Jung and Watkins [36]. Hence, performing the folding we reach in finitely many steps either a graph with girth not equal to 4, or the infinite chain with alternating single and double edges.

If we reach an infinite chain with alternating single and double edges, then $G =$ SPX$(2, n)$, for some $n \geq 1$, and the asymmetrizing density of G is $\frac{1}{n2^{n+1}}$.

Otherwise, let H be the graph obtained after a finite number of foldings. From Subsections 4.4.1 and 4.4.3, where we discuss graphs with girth 3 or at least girth 5 we infer that they have 2-distinguishing cost at most 3 or 2-distinguishing density 0. By Lemma 4.4.5 these colorings can be extended to asymmetrizing colorings of G with the same cost or density.

If the graph has 1 or infinitely many ends, we cannot assure that the sequence of foldings will stop in finitely many steps. This leads to the following problem:

Open Problem: Are there other graphs consisting of quadrangles connected by one edge with strictly positive density?

By observations about the edge transitivity of the contracted graph, where we contract all quadrangles to a single vertex, we tend to assume that the only cubic graphs with positive density are the SPX$(2, n)$ graphs.

4.4.3 Girth at least 6

Now we assume that the minimal cycle has length at least six. In general it is possible to distinguish every vertex in a cycle with girth $g \geq 6$ by three vertices with the following coloring: Take two adjacent vertices v, w of the cycle and color them black. Additionally color a third vertex z with $d(v, z) = 2$ or $d(w, z) = 2$ and $z \neq w \neq v$ black (Figure 4.13). It is easy to see, that only the identity preserves this labeling. But we will prove that in some cases taking into account the whole structure of the graph it is enough to color just two vertices to make all vertices distinguishable.

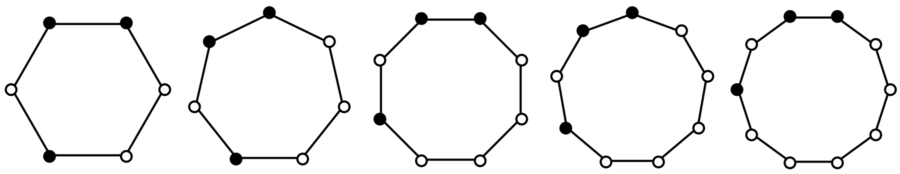

Figure 4.13: The standard coloring for fixing a cycle with $g \geq 6$.

Firstly, we will consider graphs, where the g-cycles are connected by two edges. In the second step, we show, how we get from the coloring in this case to graphs where the g-cycles are connected by one edge. As the g-cycles are connected by two edges, g has to be of the form $4n+2$ resp. $4n$ for $n \geq 2$. We are separating the cases $4n$ and $4n+2$ and start with the case $4n+2$, which will be a bit easier to handle, because the $4n$ case will show some problems caused by possibly too much symmetry. Moreover, we assume that we have $(m, 4n+2)$-cycles. Otherwise, we can apply Theorem 4.4.2.

The basic situation is now that we have $4n+2$ big cycles connected with each other by edges starting at opposite sites. These cycles form $(m, 4n+2)$-cycles. We want to color only 2 vertices to achieve symmetry breaking. We do this as follows:

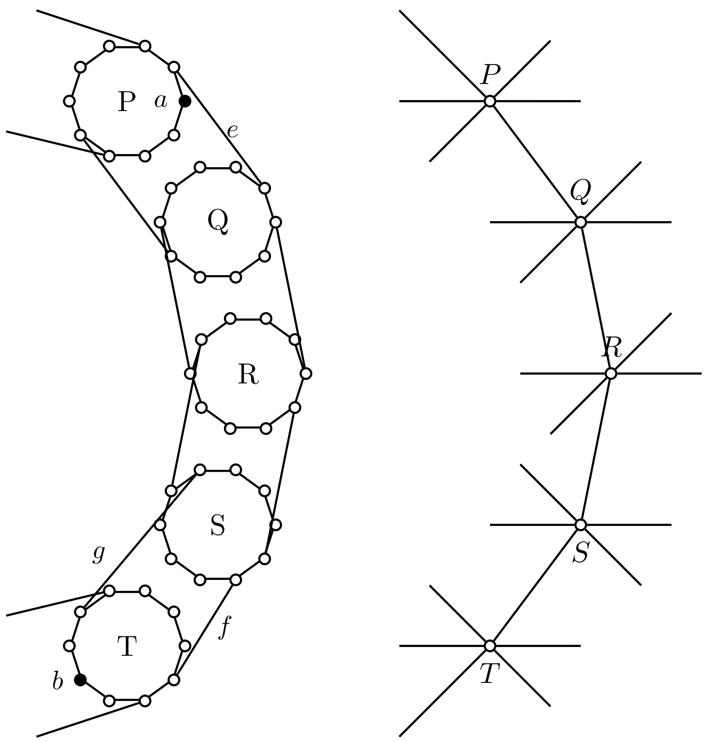

Figure 4.14: A part of an $(m, 10)$-cycle of decagons with its asymmetrizing coloring and the corresponding part after the contraction of the decagons.

Coloring 1: Choose one of the $(m, 4n+2)$-cycles and look only onto the participating $(4n+2)$-cycles and the connections between them. Now choose one of the $(4n+2)$-cycles and color a vertex next to a vertex which has an edge e to the next $(4n+2)$-cycle in the $(m, 4n+2)$-cycle. We call this vertex a. From a we choose the $(4n+2)$-cycle that has a distance of $\lfloor m/2 \rfloor - 1$ in terms of cycles and going into the direction of e. In this $(4n+2)$-cycle we color a vertex that has a distance of 2 to one of the edges f, g which lead back to a, see Figure 4.14.

In the following we will show that this coloring is sufficient to fix the whole graph. The proof will be split into two parts. Firstly, we prove that the chosen $(m, 4n+2)$-cycle with all its vertices is fixed. Then, we show that the whole graph is fixed if there is one fixed $(m, 4n+2)$-cycle. One important tool for showing this, is looking at the **contracted graph**, where we contract all $(4n+2)$-cycles to a single vertex. Double edges will be merged to one edge and the resulting graph has degree $2n+1$, is edge- and vertex-transitive and has girth m. The case of $(4n+2)$-cycles is less complex than treating $4n$-cycles as $2n+1$ is always odd and gives a certain amount of asymmetry which allows to use the uniqueness of shortest paths.

Girth equal to $4n+2$ for $n \geq 2$

Lemma 4.4.13 *Using Coloring 1 there is no automorphism which preserves the coloring and moves a vertex in the colored* $(m, 4n+2)$*-cycle.*

Proof We choose an $(m, 4n+2)$-cycle and apply Coloring 1. We call the two colored vertices a and b. Now, we consider all the paths between a and b which run through the least amount of $(4n+2)$-cycles. We immediately see that all these paths run through exactly the same $(4n+2)$-cycles by the following argument. We consider the contracted graph, where all $(4n+2)$-cycles are contracted to one vertex. In this graph this path is unique. This contracted path has length $\lfloor m/2 \rfloor - 1$ by construction. If there would be another path with this length between the two points we would find a cycle with fewer than m vertices – a contradiction to m being minimal. We conclude that all the $(4n+2)$-cycles containing these paths can be at most mapped onto one another and to none else.

a and b cannot be mapped to each other, because the start and end of the paths have different length in their corresponding $(4n+2)$-cycle. Thus, the order of the $(4n+2)$-cycles stays the same in the graph and can at most be mapped onto itself. The only possibility for one of these $(4n+2)$-cycles is to be reflected along the axis fixing a pair of vertices that have edges to one of the other $(4n+2)$-cycles in the m-cycle while the other pair of vertices to the other $(4n+2)$-cycle gets mapped to each other. But by the fact that we have $(4n+2)$-cycles it follows that this is not possible, because the paths passing to the next cycle have always different lengths if one wants the path to start at the two different incoming edges and leave at the same one – $(4n+2)/2$ is always odd.

It follows now that all the paths between a and b have to be fixed completely as well, and that all the $\lfloor m/2 \rfloor$ involved $(4n+2)$-cycles are fixed.

Let X and Y be the two neighboring $(4n+2)$-cycles of the ends of the fixed $(4n+2)$-cycle-path along the m-cycle. Those two cycles can only be reflected on the axis given by the two vertices connected to the fixed $(4n+2)$-cycle-path. But this reflection would imply that the $(4n+2)$-cycle-path between X and Y of length $\lfloor m/2 \rfloor - 2$ would have to be mapped to another path of this length between X and Y. This would yield a $(m', 4n+2)$-cycle including X and Y of size

$$m' = 2\left(\lfloor \frac{m}{2} \rfloor - 2\right) + 2 = 2\lfloor \frac{m}{2} \rfloor - 2 < m,$$

a contradiction. Thus, this reflection is not possible and it follows that there is no automorphism which preserves Coloring 1 and moves one of the other vertices in the $(m, 4n+2)$-cycle.

For the next proof, we need the following observation. For now, we have one $(m, 4n+2)$-cycle completely fixed. Consequently, all the neighboring $(4n+2)$-cycles are fixed in their position and only reflection along the axis of the connecting edges to the fixed $(m, 4n+2)$-cycle is possible. As $(4n+2)/2$ is always odd, the reflection results in pairwise changing of the other edge-pairs. Looking at the contracted graph, this corresponds to the pairwise exchange of edges. We call this the **property of joint edges**.

Theorem 4.4.14 *Using Coloring 1 the only remaining color-preserving automorphism of a vertex-transitive graph containing an $(m, 4n+2)$-cycle is the identity.*

Proof By the preceding lemma we have one $(m, 4n+2)$-cycle, which is completely fixed. Now, we look at the contracted graph, which is edge-transitive. All the neighbors of the m-cycle in the contracted graph are fixed as well with the possibility of exchanging pairwise the adjacent edges (property of joint edges).

Now, we want to show that such edge exchanging is actually not possible. We are going to look at m-cycles that have to pass through these adjacent edges, and see that a possible pairwise exchange of the edges will lead to a contradiction.

Call one of the vertices of the fixed m-cycle c and one of the neighbors outside the cycle f. We assume that the mentioned pairwise exchange of adjacent edges can be done at f. Due to the edge-transitivity there has to pass an m-cycle through the edge cf which passes through another edge adjacent to f, call this edge e_1. By the fact that cf is fixed we get through the assumed possible exchange that there has to pass another m-cycle through the joint edge e_2 of e_1 and cf. Both these mentioned m-cycles have to pass through the same edge at c since all the edges at c are fixed. All of this implies that there are actually at least 2 of these m-cycles passing through every edge, in fact the amount of m-cycles always has to be even by repeating this argument whenever one finds an odd number of m-cycle running through an edge. Now we have two adjacent edges contributing to two different cycles and we even get, by extending above argument, that if there passes a single m-cycles through two adjacent edges it has to be actually two such m-cycles and these cycles are joint in the way of joint edges, see Figure 4.15. We now get back to our already fixed m-cycle. Starting from there we will see that there are infinitely many different m-cycles in a bounded area, leading to a contradiction.

So far we only know of a single m-cycle passing through all the edges of our fixed m-cycle. Hence, we can choose two adjacent edges of the cycle and see that through these

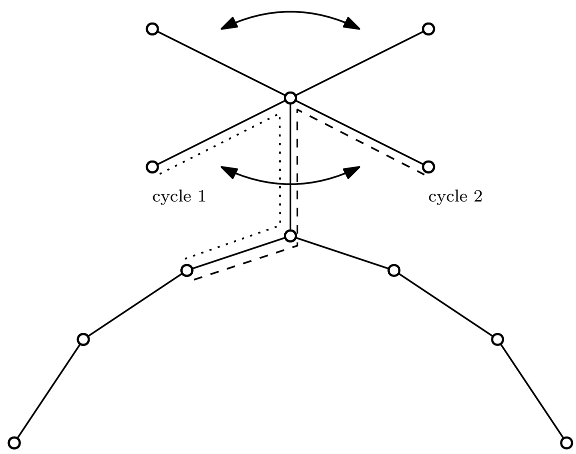

Figure 4.15: Cycles induced by the property of joint edges.

there has to pass at least another m-cycle. Since our starting m-cycle is fixed, one of these new m-cycles has to be completely fixed, as well, with all its vertices even in the not contracted graph.

By the fact that we assume that the joint edges at f can be exchanged we know that this newly selected m-cycle cannot pass through the edge cf, otherwise all the edges at f would be fixed, leading to a contradiction.

But this m-cycle we are looking at right now still has to leave our starting cycle at some vertex through an edge that is not part of the starting m-cycle. Call the two edges at this vertex x_1 and x_2. There is now only one m-cycle going through x_1 and x_2. By the observations above there has to pass another m-cycle through these. Now we can continue infinitely many times like this and always find a new fixed m-cycle. This happens since none of these can run through cf and we always have an edge adjacent to the starting m-cycle that has an odd number off m-cycles running through it. Ultimately we obtain a contradiction by finding infinitely many different m-cycles on a bounded ball of size $\frac{m}{2}$ around the starting m-cycle. Thus, the edges at f are actually fixed and this holds for all neighbors of the m-cycle, as well.

Using similar argumentation of joint edges and infinitely many m-cycles we get the same result for all the neighbors of our newly fixed vertices.

Finally, by continuing in this way inductively, sphere by sphere around the starting m-cycle, we can fix the entire graph.

Girth $4n$ for $n \geq 2$

Now, we continue with the more difficult case of $(m, 4n)$-cycles. Here we have two subcases depending on the connections inbetween the $4n$-cycles. The first subcase follows the same argumentation as the $4n + 2$ case. The other subcase of symmetrically connected graphs will turn out to be more difficult and will not be treated fully in this dissertation. We will give an example, where we obtain a graph with nonzero density and explain our conjecture. Now, we will explain what symmetrically connected means.

Definition 4.4.15 The $4n$-cycles in an $(m, 4n)$-cycle are connected **symmetrically** if one pair of opposite vertices is connected to one neighboring $4n$-cycle and the pair of edges with distance n to the first pair is connected to the other neighboring $4n$-cycle.

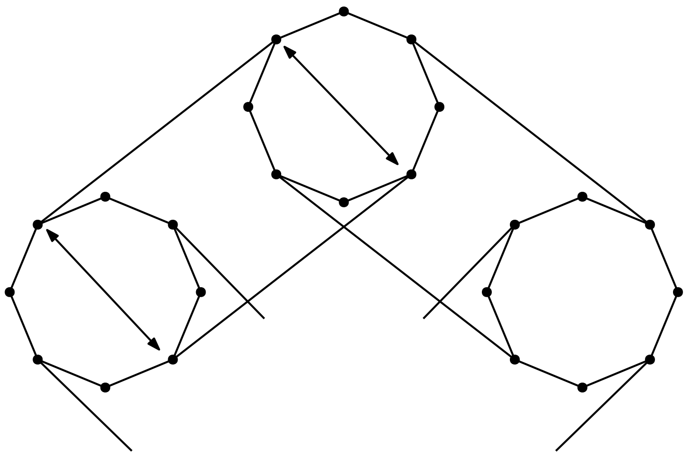

Figure 4.16: Symmetrically connected 8-cycles with the demonstration of a flip without influence to the other 8-cycles.

Corollary 4.4.16 *Assume that the smallest cycle is of size* $4n$ *with* $n \geq 2$ *and that the* $4n$*-cycles are not symmetrically connected inside the corresponding* $(m, 4n)$*-cycle. Then, using Coloring 1, the identity is the only remaining color-preserving automorphism.*

Proof Since there are no symmetrical connections, we can find again a unique $4n$-cycle-path as in Lemma 4.4.13 after applying Coloring 1. Continuing with the same arguments as in Theorem 4.4.14, we obtain a completely fixed graph.

In the case of symmetrical connections, the $4n$-cycles in the interior of the mentioned $4n$-cycle-path in the aforegoing proof can be flipped along the axis of their incoming and outgoing edges without influencing the other cycles. Hence, the uniqueness of the connecting path between the colored vertices is lost.

If there are two cycles in the graph, which are symmetrically connected, then there is at least one $(m, 4n)$-cycle, where all cycles are symmetrically connected.

Lemma 4.4.17 *Let there be an* $(m, 4n)$*-cycle where every* $4n$*-cycles can be mapped to each other by rotating the* $(m, 4n)$*-cycle, and let there be one* $4n$*-cycle* S *that is symmetrically connected to its neighboring* $4n$*-cycles* R *and* T *inside the* $(m, 4n)$*-cycle. Then all* $4n$*-cycles in the* $(m, 4n)$*-cycle are symmetrically connected to each other.*

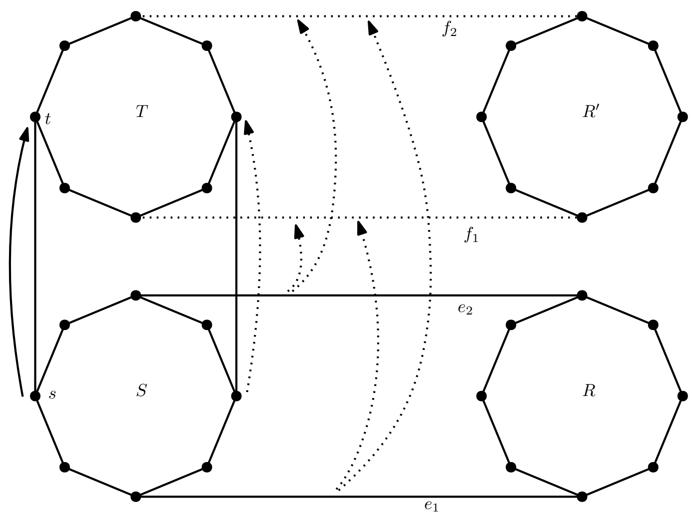

Figure 4.17: The symmetrcial connection of S induces the symmetrical connection of T and the existence of such edges f_1 and f_2 connecting T to R'.

Proof Let s be a vertex with an edge from S to T and let t be the corresponding vertex in T. By mapping s to t, we see that S and T get exchanged as cycles, and R gets mapped to the other neighbor R' of T, see Figure 4.17. Since S has symmetrical connections in the $(m, 4n)$-cycle by our assumption, this gets inherited to its image, which is T. Thus, the connections of T inside this $(m, 4n)$-cycle are also symmetrical and T is symmetrically connected to R'. By induction this holds for the whole $(m, 4n)$-cycle.

The smallest possible girth of such graphs is 8. For breaking all automorphisms in one $(m, 8)$-cycle it suffices to color in every other 8-cycle one vertex black to prohibit the flipping of the 8-cycles. This yield a density of at most $\frac{1}{16}$. For bigger girthes we obtain similarly a density of less than $\frac{1}{16}$. One symmetrically connected graph with positive density is depicted in Figure 4.18. This graph arises from the $SPX(2, 2)$ by substituting every quadrangle by an 8-cycle and choose the connecting edges such that the 8-cycles are symmetrically connected. Using Theorem 4.4.11 and identifying opposite edges with each other we see that the density of this graph is $\frac{1}{32}$. We strongly believe that all symmetrically connected graphs with positive density are constructed in a similar way and form a sort of stripe. As soon as the graphs are more intertwined or form a cactus graph, we could show in many cases that the graph has distinguishing density 0. But we will leave this as an open problem to the reader.

Open problem: Are there any other symmetrically connected graphs with girth at least 8 and connected by double edges with positive density?

One connecting edge

After studying $(m, 4n)$-cycles where the cycles were connected by two edges, we turn to $(m, 2n)$-cycles, resp. $(m, 2n + 1)$-cycles, where the cycles are connected by one edge. We

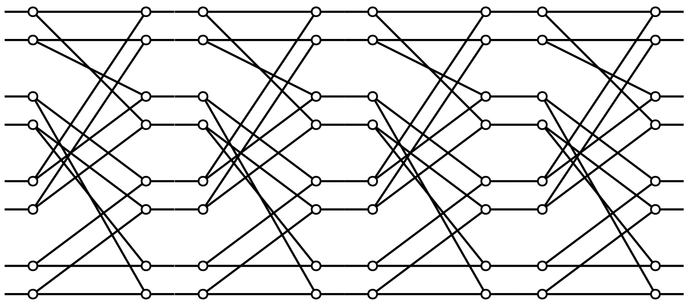

Figure 4.18: Replacing each quadrangle by an symmetrically connected 8-cycle in a $SPX(2,2)$ graph.

assume $n > 2$ in the even case and $n \geq 2$ in the odd. Similarly to the two edge cases we get costs of 2 in the case of $(m, 2n+1)$-cycles and in the even case if the cycles are not symmetrically connected, see Figure 4.19. This is an immediate consequence of the two

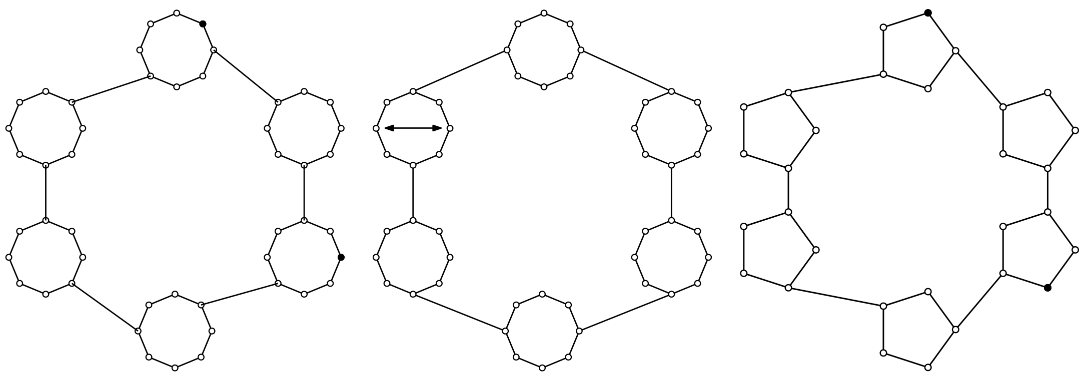

Figure 4.19: The non symmetrical and symmetrical connection in the even case and the standard colorings on the case of $\rho = 2$.

edge cases, as it uses the same arguments. The problematic feature of the symmetrical connections is again, that the reflection along the axis in one $2n$-cycle does not influence the rest of the cycle and a path entering and exiting the cycle through these edges keeps the same length under reflection. We believe here as well, that we need an infinite number of colored vertices to break all automorphisms. This will follow from the open problem in the last section and we leave this as well as an open problem.

For the remaining cases of single edges we go along the line of the case $4n+2$ for double connections Theorem 4.4.14.

Theorem 4.4.18 *Let G be a graph with $(m, 2n+1)$-cycles with $n \geq 2$, or without symmetrically connected $(m, 2n)$-cycles. Then, the only automorphism that preserves Coloring 1 is the identity.*

Proof The proof goes along the case of $(m, 4n+2)$-cycles respectively without symmetrically connected $(m, 2n)$-cycles. The path between the colored vertices of Coloring 1 is unique and this fixes the whole m-cycle. There is again the property of joint edges, which yields inductively that the whole graph is fixed by Coloring 1, as in the proof of Theorem 4.4.14.

Finite graphs

We can now determine the 2-distinguishing cost for finite graphs with two arc orbits and girth at least six were the g-cycles are not symmetrically connected.

Theorem 4.4.19 *Let G be a finite graph, with two arc orbits, $g(G) \geq 6$ and the g-cycles are not symmetrically connected. Then $\rho(G) = 2$.*

Proof Let G be a finite graph which consists of one of the following structures: $(m, 4n+2)$-cycles with double connections, $(m, 2n+1)$-cycles with single connections, $(m, 4n)$-cycles with not symmetrically connected double connections or $(m, 2n)$-cycles with not symmetrically connected single edges. Then G has $\rho(G) = 2$ using the standard coloring as for infinite graphs. This follows immediately with the same arguments as for its infinite versions.

If G is a finite graph with two arc orbits where the $(m, 2n)$- resp. $(m, 4n)$-cycles are symmetrically connected, we cannot state a precise asymmetrizing cost. We conjecture, that the cost is related to the size of the graph itself.

Open problem: What is the 2-asymmetrizing cost of a finite graph with two arc orbits where the cycles are symmetrically connected?

4.5 A classification of vertex-transitive cubic graphs with girth 5 and their 2-distinguishability

We conclude with a direct proof that every connected, vertex-transitive cubic graph with girth equal to 5 is either 2-distinguishable or the Petersen graph. The proof is different from the proof in [21] and also covers infinite graphs. This will be done by observing special structural properties of vertex-transitive graphs with girth equal to 5.

Lemma 4.5.1 *Let G be a connected, vertex-transitive, cubic graph that contains a pentagon, but neither squares nor triangles. Then it is the Petersen graph, the pentagondodecahedron or consists of distinct pentagons that are joined by at most one edge. Apart from the Petersen graph, these graphs are 2-distinguishable.*

Proof In the first case let G be a graph which contains two pentagons having two joint edges, see Figure 4.20. As no shorter cycles are permitted, vertices a,c and d have at least one neighbour, which is not part of the original two pentagons. Now, we see the

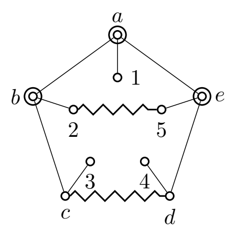

Figure 4.20: Illustration of Property 1.

following property in Figure 4.20: The neighbours of each neighbour of a, namely b and e, are pairwise connected. We call this property **Property 1**.

Due to vertex-transitivity, this property of a has to be true for all other vertices of the graph. But each vertex has 3 neighbours and we cannot be sure for which pair of neighbours of the other vertex Property 1 holds. Therefore we have to consider three cases, see Figure 4.21: In case one a and e are already connected. Moreover b has already

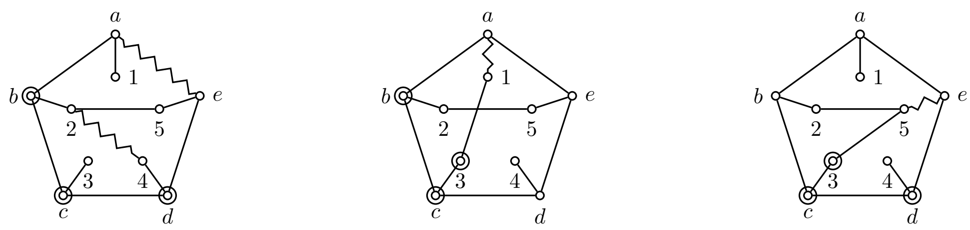

Figure 4.21: Three possibilities of applying Property 1.

three neighbours. Hence 2 and 4 have to be connected as well. In the second situation a has to be connected to one neighbour of a neighbour of 3. But a has already three neighbours. This together with the fact that e is already of degree three, as well, we obtain the edge from 3 to 1. The same considerations yield the edge from 3 to 5 in the third case. More edges can not be drawn immediately as the situation in Figure 4.22 can't be excluded.

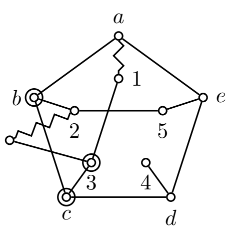

Figure 4.22: This edge satisfies Property 1, as well.

All these graphs are isomorphic to two different graphs, see Figure 4.23. Choosing in the left graph of Figure 4.23 vertex e to observe Property 1 and in the right graph vertex d, we see again two possible outcomes, see Figure 4.24.

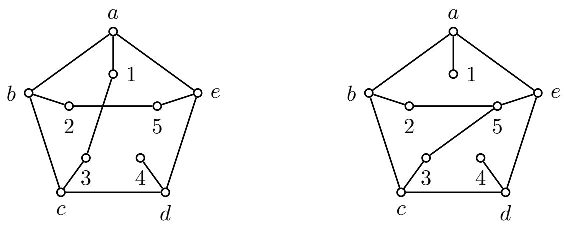

Figure 4.23: One edge more after applying Property 1 once.

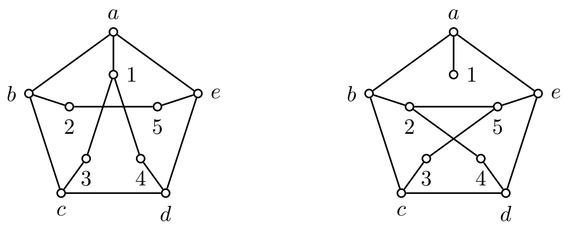

Figure 4.24: Two edges more after applying Property 1 twice.

Having a closer look at a in the left graph we see a new property. We know b and e are the neighbours of a which fulfill Property 1. Then the neighbours of the third neighbour of a (1), namely 3 and 4, have to be connected to at least one connected pair of neighbours of b and e. We call this property **Property 2**. In the right graph Property 2 must be valid for all vertices, as well. Choosing b as a new central point in the left graph we obtain the edge 24 and choosing e in the right graph, we obtain the edge 14, as the neighbor b of a has already degree 3 and determines the connecting edge for Property 2. These graphs are again isomorphic to one and the same graph, see Figure 4.25.

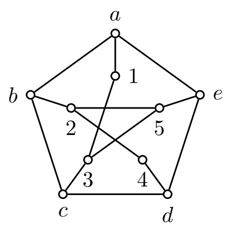

Figure 4.25: Applying Property 2.

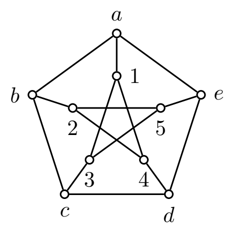

Figure 4.26: The Petersen Graph

Applying here Property 1 and Property 2 to vertex a, we obtain the Petersen Graph. This graph is not 2-distinguishable, see [28].

Secondly, let G be a graph which contains two pentagons with one joint edge. This yields by vertex transitivity, that every vertex has to be part of two pentagons. As it is not allowed that the graph contains shorter cycles then cycles of length 5 we can determine all the vertices and edges, like in Figure 4.27. Our aim is to show that not just the edge cd is a edge in between two pentagons, but all incident edges to d have to be joint edges of two pentagons.

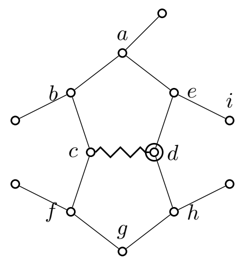

Figure 4.27: Two incident pentagons.

Assume that cd is the only double edge, so dh and de are just contained in one pentagon. Then, by vertex transitivity, this holds for every other vertex. But again, we do not know which incident edge of another vertex will be a double-edge. Considering e we have two possibilities, see Figure 4.28: Either ae is a double edge or ei. By our assumption, all the other incident edges of e and a respectively i cannot be double-edges. Looking at the right graph in Figure 4.28, we notice that this leads to a contradiction as the neighbors of i have to be connected either to a neighbor of h or a, such that ei would be a double edge of two pentagons. In the left graph we imagine the possible pentagon formed out of a, e and i and their two neighbors. By vertex transitivity, b has to be part of two pentagons, as well. Therefore, the third edge attached to b has to be a double edge. But all neighboring vertices of b have already degree 3 and we would need the pentagon going through the neighbor of a or f. But this yields more than one double edge incident to a or c and contradicts our assumption that a vertex can have only one double edge attached.

Figure 4.28: Contradiction in assuming just one incident double-edge

We conclude, that d cannot have only one incident double-edge and we assume, that d has two incident double-edges. Applying vertex transitivity, this holds for every other vertex. With very similar arguments as in the case of one double-edge a contradiction to vertex transitivity will show up and we conclude, that d, and so every vertex, has three incident double-edges and is part of three different pentagons. Excluding the case of the Peterson graph we obtain immediately two new edges, see Figure 4.29. Completing this observation with all vertices, we obtain the pentagondodecahedron and this is 2-distinguishable using the coloring in Figure 4.29.

This leaves the case where our graph consists of pentagons which are connected by at

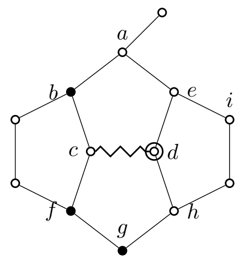

Figure 4.29: Two more edges.

most one edge. A connection via two edges is not possible. If we assume that two pentagons are connected by two edges, every vertex of a pentagon needs a partner vertex which is connected to the same pentagon but two is not a divisor of five. The asymmetrizing coloring and the cost of asymmetrizing of these graphs is presented in Section 4.4.1.

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