

Marco Zank

Inf-Sup Stable Space-Time Methods for Time-Dependent Partial Differential Equations

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Marco Zank

**Inf-Sup Stable Space-Time Methods for
Time-Dependent Partial Differential Equations**

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Partial Differential Equations**

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Marco Zank

Zusammenfassung

In dieser Arbeit werden Raum-Zeit-Variationsformulierungen und deren Diskretisierung mittels konformer, stückweise polynomieller Funktionen für die Wärmeleitungsgleichung und Wellengleichung in einem beschränkten Raum-Zeit-Zylinder Q mit Endzeitpunkt T betrachtet.

Für die Wärmeleitungsgleichung ist das Hauptresultat eine unbedingt stabile Galerkin-Bubnov-Raum-Zeit-Finite-Element-Methode mit stückweise linearen, stetigen Funktionen basierend auf einer Raum-Zeit-Variationsformulierung, welche in einem Unterraum eines anisotropen Sobolevraums formuliert wird. Diese Raum-Zeit-Variationsformulierung wird mithilfe von Fourierreihen und einer Transformation, welche ähnlich zur Hilberttransformation ist, analysiert. Daraus ergeben sich eine symmetrische und elliptische Variationsformulierung und infolgedessen eine symmetrische Galerkin-Diskretisierung für die erste Zeitableitung. Für die Wärmeleitungsgleichung wird unbedingte Stabilität für unstrukturierte Raum-Zeit-Netze bewiesen. Weiters werden Fehlerabschätzungen in $L^2(Q)$, in $H^1(Q)$ und in einer anisotropen Sobolevnorm für einen Tensorproduktansatz hergeleitet. Schließlich werden numerische Beispiele, welche die theoretischen Ergebnisse bestätigen, angegeben.

Für die Wellengleichung ist der Ausgangspunkt eine Raum-Zeit-Variationsformulierung in Teilräumen des Sobolevraums $H^1(Q)$. Diese Raum-Zeit-Variationsformulierung ist nicht inf-sup-stabil. Die Diskretisierung dieser Raum-Zeit-Variationsformulierung mittels einer konformen Raum-Zeit-Finite-Element-Methode mithilfe von stückweise linearen, stetigen Funktionen führt zu einer bedingten Stabilität des Verfahrens. Das heißt, für die Stabilität muss eine CFL-Bedingung zwischen der Orts- und Zeitmaschenweite erfüllt sein. Um die CFL-Bedingung zu vermeiden, wird für einen Tensorproduktansatz eine stabilisierte Raum-Zeit-Finite-Element-Methode mittels stückweise linearer, stetiger Funktionen hergeleitet. Für diese Formulierung werden unbedingte Stabilität in $L^2(Q)$ sowie Fehlerabschätzungen in $L^2(Q)$ und in $H^1(Q)$ bewiesen. Weiters werden numerische Beispiele, welche die theoretischen Ergebnisse bestätigen, angegeben. Zu guter Letzt werden Existenz- und Eindeigkeitssätze für die Wellengleichung als partielle Differentialgleichung im $L^2(Q)$ und in einem schwächeren Sinne als $L^2(Q)$ bewiesen. Die zugehörigen Lösungsoperatoren sind Isomorphismen, welche entsprechende inf-sup-Bedingungen garantieren.

Abstract

In this work, space-time variational formulations and their discretisations with conforming, piecewise polynomial functions for the heat and wave equation are considered in a bounded space-time cylinder Q with a terminal time T .

The main result for the heat equation is an unconditionally stable finite element method of Galerkin-Bubnov type with piecewise linear, continuous functions, which is based on a variational formulation in a subspace of an anisotropic Sobolev space. This space-time variational formulation is analysed with the help of Fourier series, and a kind of Hilbert transform is introduced. This leads to a symmetric and elliptic variational formulation and hence, to a symmetric Galerkin discretisation of the first-order time derivative. For the heat equation, unconditional stability for unstructured space-time meshes is proven. In addition, error estimates in $L^2(Q)$, in $H^1(Q)$ and in an anisotropic Sobolev norm for a tensor-product approach are derived. Finally, numerical examples, which confirm the theoretical results, are presented.

For the wave equation, a space-time variational formulation in a subspace of the Sobolev space $H^1(Q)$, which is not inf-sup stable, is used for a conforming space-time finite element method, which leads to a conditionally stable method, i.e. a CFL condition is required. For a tensor-product approach, a stabilised finite element method with piecewise linear, continuous functions is investigated, where unconditional stability in $L^2(Q)$ is proven. Furthermore, error estimates in $L^2(Q)$ and in $H^1(Q)$ are derived, and numerical examples, confirming the theoretical findings, are given. In addition, existence and uniqueness results for the wave equation as a partial differential equation in $L^2(Q)$ and in a weaker sense than $L^2(Q)$ are proven, including isomorphic solution operators and corresponding inf-sup conditions.

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1 INTRODUCTION

Standard approaches for the numerical solution of time-dependent partial differential equations are usually based on semi-discretisations in space and time, where the discretisation in space and time is split accordingly, see, e.g., [150] for parabolic partial differential equations, and [32, 33] for hyperbolic problems. Interpreting such approaches in a space-time sense, i.e. the time variable is considered as an additional spatial variable, these methods are related to tensor-product space-time methods, see, e.g., [16, 48–50, 85] for the parabolic case, and [19, 22, 55, 86, 164] for the hyperbolic one. An alternative is to discretise the time-dependent problem without separating the temporal and spatial variables, i.e. a space-time discretisation. This ansatz may lead to unstructured decompositions of the space-time domain. The approaches of unstructured meshes are considered, e.g., in [116, 142–144] for parabolic equations, and [42, 63, 111, 130, 140] for hyperbolic ones. More references are given in Chapter 3 for the parabolic equations, and in Chapter 4 for the hyperbolic problems. In general, the main advantages of space-time methods are space-time adaptivity, space-time parallelisation and the treatment of moving boundaries. At a first glance, a disadvantage is that a global linear system must be solved at once. Therefore, fast solvers and preconditioning are essential, which are not investigated in this work, see, e.g., [56]. In this thesis only direct solvers and the GMRES method are used. However, space-time approximation methods depend strongly on the space-time variational formulations on the continuous level. The focus of this thesis is on space-time variational formulations for the heat and wave equation, which result not only in inf-sup stable formulations but fit also very well to conforming space-time methods with piecewise polynomial functions. In addition, these space-time variational formulations might be useful for variational formulations and their analysis of boundary integral equations and boundary element methods.

To motivate space-time approximation methods, space-time adaptivity is investigated in the case of a space-time interpolation and in the case of an adaptive space-time boundary element method for the spatially one-dimensional wave equation.

1.1 Space-Time Interpolation

For the approximation of a function $u(x, t)$ depending on a spatial variable $x \in \Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, and on a time variable $t \in (0, T) \subset \mathbb{R}$, where Ω is a bounded Lipschitz domain and $T > 0$ is a terminal time, a better adaption of a sequence of arbitrary admissible and shape regular decompositions $(\mathcal{T}_N)_N$ of the space-time cylinder $Q := \Omega \times (0, T) \subset \mathbb{R}^{d+1}$ is possible in contrast to a tensor-product meshing of Q . As illustration, consider the rectangle

$$Q = (0, 3) \times (0, 6) \subset \mathbb{R}^2$$

with $\Omega = (0, 3)$ and $T = 6$ for the $C^2(\bar{Q})$ function $u_1: \bar{Q} \rightarrow \mathbb{R}$,

$$u_1(x, t) = \begin{cases} \frac{1}{2}(t-x-2)^3(x-t)^3, & x \leq t \text{ and } t-x \leq 2, \\ 0, & \text{else,} \end{cases} \quad (1.1)$$

which is plotted in Figure 1.1, and for the piecewise smooth function $u_2: \bar{Q} \rightarrow \mathbb{R}$,

$$u_2(x, t) = \begin{cases} \frac{1}{2} |\sin(\pi(x-t))|, & x \leq t, \\ 0, & \text{else,} \end{cases} \quad (1.2)$$

which is plotted in Figure 1.2.

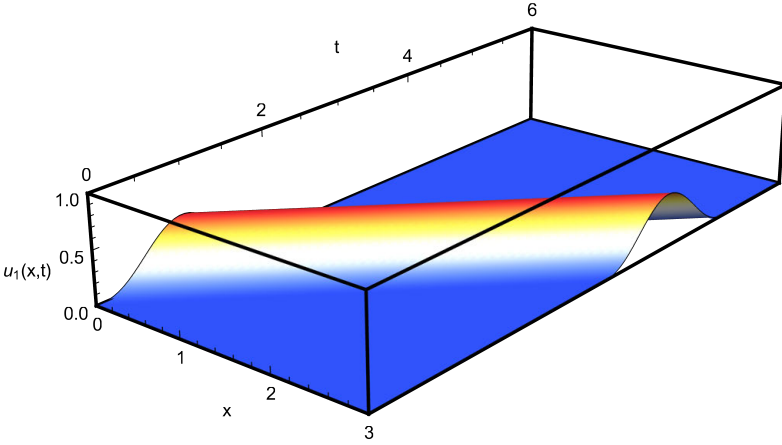


Figure 1.1: The smooth function u_1 of (1.1).

The given rectangle

$$\bar{Q} = \bar{\mathcal{T}}_N = \bigcup_{\ell=1}^N \bar{q}_\ell$$

is decomposed into N uniform space-time triangles $q_\ell \subset \mathbb{R}^2$ with mesh size h as given in Figure 1.3 for level 0, where \bar{M} is the number of vertices $\{(x_i, t_i)\}_{i=1}^{\bar{M}}$. The finite-dimensional space $S_h^1(Q) = \text{span}\{\psi_i\}_{i=1}^{\bar{M}} \subset H^1(Q)$ is the space of piecewise linear, continuous functions on these space-time triangles with the nodal basis functions ψ_i , and the space-time interpolation operator $I_h: C(\bar{Q}) \rightarrow S_h^1(Q)$ is defined by

$$I_h v(x, t) = \sum_{i=1}^{\bar{M}} v(x_i, t_i) \psi_i(x, t) \quad \text{for } (x, t) \in \bar{Q},$$

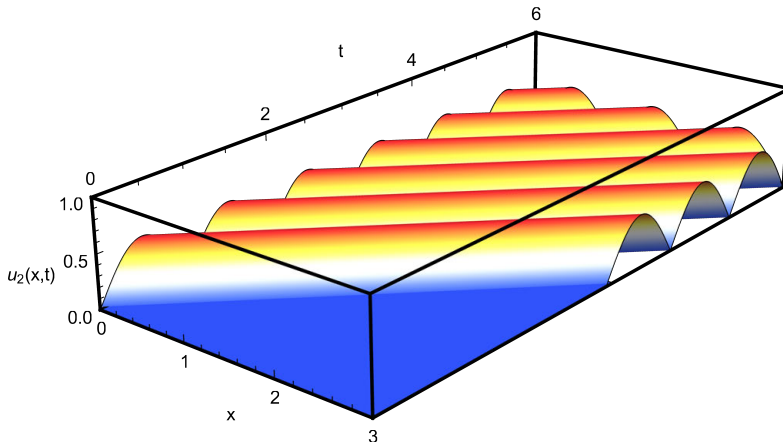


Figure 1.2: The piecewise smooth function u_2 of (1.2).

where $v \in C(\bar{Q})$ is a given continuous function, see Section 2.8 for details. Next, the interpolation errors in $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ for the functions u_1, u_2 are investigated for a sequence of uniform space-time meshes and for a sequence of adaptive space-time meshes. The uniform refinement strategy is depicted for the levels 0, 1, 2 in Figure 1.3. As adaptive refinement strategy, Dörfler marking [41] with parameter $\theta = 0.5$ for the norm $\|\cdot\|_{L^2(Q)}$ is used.

For the smooth function u_1 , the adaptive meshing is given in Figure 1.4. The uniform and the adaptive refinement strategies lead to optimal convergence rates with respect to $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$, see Table 1.1 and Figure 1.5. However, a comparison between the uniform and the adaptive schemes shows that the adaptive scheme needs considerably less degrees of freedom \tilde{M} for the same accuracy of the errors.

For the piecewise smooth function u_2 , the uniform refinement strategy results in reduced orders of convergence, see Table 1.2. With the adaptive refinement strategy, the optimal convergence rates are obtained, see Figure 1.7, and see Figure 1.6 for the meshes produced by the adaptive scheme.

To summarise, a main advantage of space-time methods is the space-time adaptivity, as depicted in Figure 1.4 and Figure 1.6, which is difficult to realise for standard approaches based on semi-discretisations, where the discretisation in space and time is split accordingly.

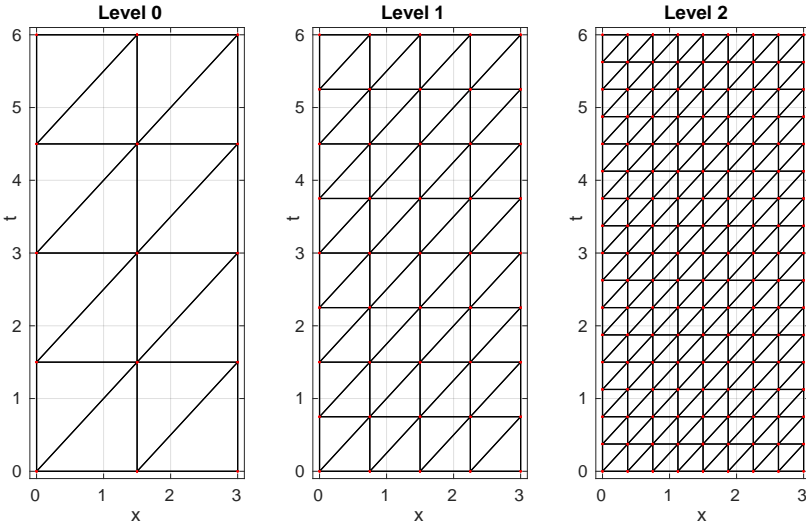
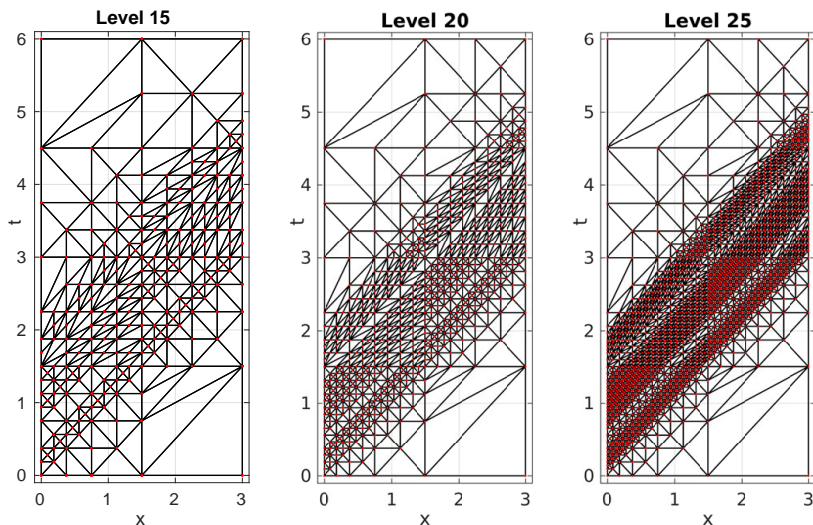


Figure 1.3: Uniform refinement strategy: Starting mesh, the meshes after one and two uniform refinement steps.

level	\vec{M}	N	$\ u_1 - I_h u_1\ _{L^2(Q)}$	eoc	$ u_1 - I_h u_1 _{H^1(Q)}$	eoc
0	15	16	5.122e-01	-	1.947e+00	-
1	45	64	2.302e-01	1.46	1.397e+00	0.60
2	153	256	5.797e-02	2.25	6.973e-01	1.14
3	561	1024	1.477e-02	2.11	3.537e-01	1.04
4	2145	4096	3.744e-03	2.05	1.788e-01	1.02
5	8385	16384	9.386e-04	2.03	8.957e-02	1.01
6	33153	65536	2.348e-04	2.02	4.481e-02	1.01
7	131841	262144	5.872e-05	2.01	2.241e-02	1.00
8	525825	1048576	1.468e-05	2.00	1.121e-02	1.00
9	2100225	4194304	3.670e-06	2.00	5.603e-03	1.00
10	8394753	16777216	9.176e-07	2.00	2.801e-03	1.00
11	33566721	67108864	2.294e-07	2.00	1.401e-03	1.00
12	134242305	268435456	5.735e-08	2.00	7.003e-04	1.00

Table 1.1: Interpolation errors for the function u_1 of (1.1) for $Q = (0,3) \times (0,6)$ for a uniform refinement strategy with the meshes of Figure 1.3.

Figure 1.4: Adaptive refinement strategy for the function u_1 in (1.1).

level	\bar{M}	N	$\ u_2 - I_h u_2\ _{L^2(Q)}$	eoc	$ u_2 - I_h u_2 _{H^1(Q)}$	eoc
0	15	16	7.373e-01	-	5.653e+00	-
1	45	64	7.423e-01	-0.01	5.437e+00	0.07
2	153	256	2.970e-01	1.50	4.021e+00	0.49
3	561	1024	1.044e-01	1.61	2.821e+00	0.55
4	2145	4096	3.613e-02	1.58	1.939e+00	0.56
5	8385	16384	1.257e-02	1.55	1.365e+00	0.51
6	33153	65536	4.404e-03	1.53	9.491e-01	0.53
7	131841	262144	1.549e-03	1.51	6.749e-01	0.49
8	525825	1048576	5.463e-04	1.51	4.717e-01	0.52
9	2100225	4194304	1.929e-04	1.50	3.364e-01	0.49
10	8394753	16777216	6.815e-05	1.50	2.355e-01	0.51
11	33566721	67108864	2.409e-05	1.50	1.681e-01	0.49
12	134242305	268435456	8.514e-06	1.50	1.177e-01	0.51

Table 1.2: Interpolation errors for the function u_2 in (1.2) for $Q = (0, 3) \times (0, 6)$ for a uniform refinement strategy with the meshes of Figure 1.3.

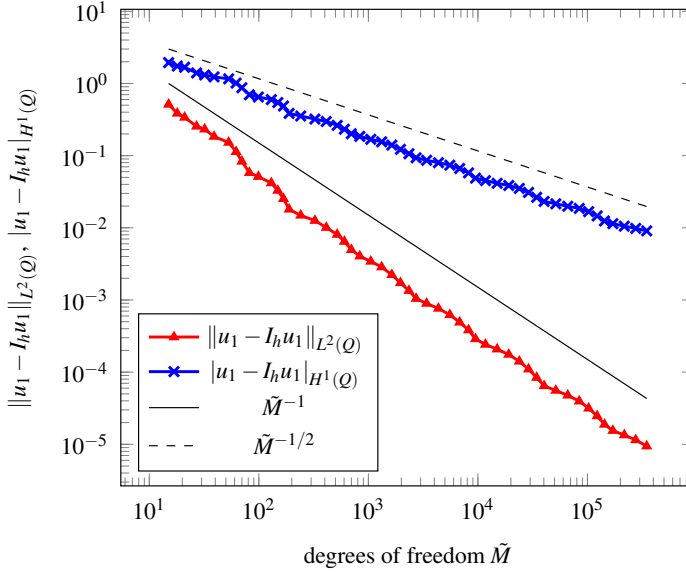


Figure 1.5: Interpolation errors for the function u_1 of (1.1) for $Q = (0,3) \times (0,6)$ for the adaptive refinement strategy with the meshes of Figure 1.4.

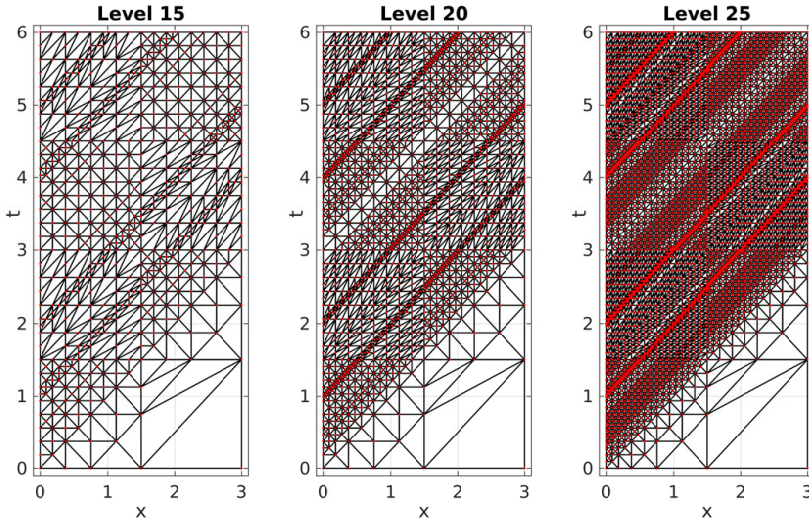
1.2 Boundary Element Method for the One-Dimensional Wave Equation

As a second example, an adaptive boundary element method for the spatially one-dimensional wave equation is investigated, see [161] for a summary. For details of the boundary element method, see [70, 131, 141]. As a model problem, consider the wave equation

$$\left. \begin{aligned} \partial_t u(x,t) - \partial_{xx} u(x,t) &= 0 & \text{for } (x,t) \in Q = (0,L) \times (0,T), \\ u(x,t) &= g(x,t) & \text{for } (x,t) \in \Sigma = \{0,L\} \times [0,T], \\ u(x,0) = \partial_t u(x,0) &= 0 & \text{for } x \in (0,L), \end{aligned} \right\} \quad (1.3)$$

where g is a given Dirichlet datum and $L > 0$, $T > 0$. Define

$$L^2(\Sigma) := L^2(0,T) \times L^2(0,T) = \left\{ v = \begin{pmatrix} v_0 \\ v_L \end{pmatrix} : v_0 \in L^2(0,T), v_L \in L^2(0,T) \right\}$$

Figure 1.6: Adaptive refinement strategy for the function u_2 in (1.2).

with the inner product

$$\langle v, w \rangle_{L^2(\Sigma)} := \langle v_0, w_0 \rangle_{L^2(0,T)} + \langle v_L, w_L \rangle_{L^2(0,T)} \quad \text{for } v, w \in L^2(\Sigma)$$

and introduce the Sobolev space

$$H_0^1(\Sigma) := \left\{ v = \begin{pmatrix} v_0 \\ v_L \end{pmatrix} : v_0 \in H^1(0,T), v_L \in H^1(0,T), v_0(0) = v_L(0) = 0 \right\}$$

with the inner product

$$\langle v, w \rangle_{H_0^1(\Sigma)} := \langle \partial_t v_0, \partial_t w_0 \rangle_{L^2(0,T)} + \langle \partial_t v_L, \partial_t w_L \rangle_{L^2(0,T)} \quad \text{for } v, w \in H_0^1(\Sigma),$$

see Section 2.2 for more details. In general, for $w \in L^2(0,T)$, set $w(t) := 0$ for $t < 0$ or $t > T$. The solution $u(x,t)$ of the wave equation (1.3) admits the representation

$$u = \tilde{\mathcal{V}} \partial_n u - \mathcal{W} g \quad \text{in } Q \tag{1.4}$$

with the single layer potential $\tilde{\mathcal{V}}$ and the double layer potential \mathcal{W} , where $\partial_n u$ denotes the unknown normal derivative of u on Σ . The single layer potential $\tilde{\mathcal{V}}$ is defined by

$$\tilde{\mathcal{V}} w(x,t) = \frac{1}{2} \int_0^{t-|x|} w_0(s) ds + \frac{1}{2} \int_0^{t-|x-L|} w_L(s) ds, \quad t \in [0,T], x \in (0,L),$$

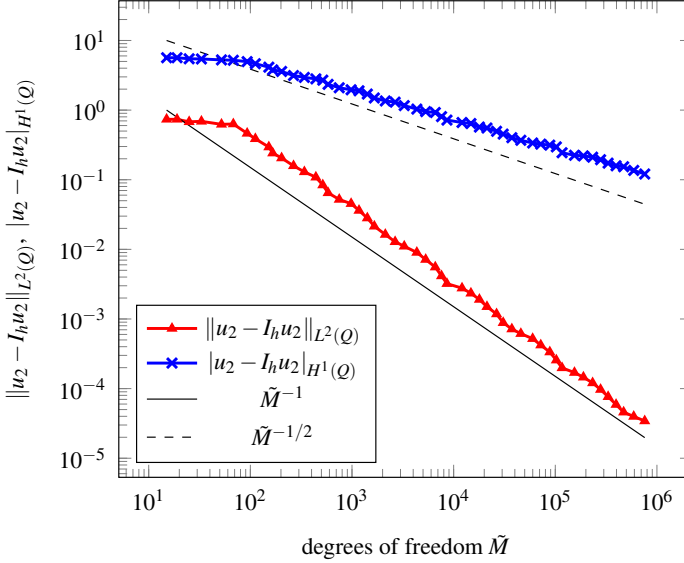


Figure 1.7: Interpolation errors for the function u_2 in (1.2) for $Q = (0, 3) \times (0, 6)$ for the adaptive refinement strategy with the meshes of Figure 1.6.

for a density $w = (w_0, w_L)^\top \in L^2(\Sigma)$ with $w = 0$ outside of Σ . The single layer operator $\mathcal{V}: L^2(\Sigma) \rightarrow H_{0,\perp}^1(\Sigma)$ is given by

$$\mathcal{V}w(t) := \frac{1}{2} \begin{pmatrix} \int_0^t w_0(s) ds + \int_0^{t-L} w_L(s) ds \\ \int_0^{t-L} w_0(s) ds + \int_0^t w_L(s) ds \end{pmatrix}, \quad t \in [0, T],$$

for a density $w = (w_0, w_L)^\top \in L^2(\Sigma)$ with $w = 0$ outside of Σ . Hence, it holds true that $\partial_t(\mathcal{V}w) \in L^2(\Sigma)$, i.e. $\partial_t \mathcal{V}: L^2(\Sigma) \rightarrow L^2(\Sigma)$. In [7], ellipticity and boundedness in $L^2(\Sigma)$ of the bilinear form $a_E(\cdot, \cdot): L^2(\Sigma) \times L^2(\Sigma) \rightarrow \mathbb{R}$,

$$a_E(w, v) := \langle \partial_t \mathcal{V}w, v \rangle_{L^2(\Sigma)} = \langle (\partial_t \mathcal{V}w)_0, v_0 \rangle_{L^2(0, T)} + \langle (\partial_t \mathcal{V}w)_L, v_L \rangle_{L^2(0, T)}$$

for $w, v \in L^2(\Sigma)$, are proven. Therefore, the following variational formulation is uniquely solvable:

Find $\partial_n u \in L^2(\Sigma)$ for a given $g \in H_0^1(\Sigma)$ such that

$$\forall v \in L^2(\Sigma): \quad a_E(\partial_n u, v) = \frac{1}{2} \langle \partial_t g, v \rangle_{L^2(\Sigma)} + \langle \partial_t(\mathcal{K}g), v \rangle_{L^2(\Sigma)}, \quad (1.5)$$

where the double layer operator \mathcal{K} is given for $g = 0$ outside of Σ by

$$\mathcal{K}g(t) = \mathcal{K} \begin{pmatrix} g_0 \\ g_L \end{pmatrix} (t) = -\frac{1}{2} \begin{pmatrix} g_L(t-L) \\ g_0(t-L) \end{pmatrix} \quad \text{for } t \in [0, T].$$

For a boundary element approximation, consider a decomposition of the lateral boundary

$$\Sigma = \bigcup_{i=1}^{N_0+N_L} \bar{\tau}_i$$

into $N_0 + N_L$ boundary elements τ_i with maximal mesh size $h = \max_i |\tau_i|$, where N_0 is the number of boundary elements for $x = 0$ and N_L is the number of boundary elements for $x = L$. The conforming ansatz space of piecewise constant functions

$$S_h^0(\Sigma) := S_{h_0}^0(0, T) \times S_{h_L}^0(0, T) = \text{span} \{ \phi_i^0 \}_{i=1}^{N_0+N_L} \subset L^2(\Sigma)$$

is used to define an approximate solution $w_h \in S_h^0(\Sigma)$. Then, the discretisation of (1.5) to find $w_h \in S_h^0(\Sigma) \subset L^2(\Sigma)$ such that

$$\forall v_h \in S_h^0(\Sigma) \subset L^2(\Sigma): \quad a_E(w_h, v_h) = \frac{1}{2} \langle \partial_t g, v_h \rangle_{L^2(\Sigma)} + \langle \partial_t(\mathcal{K}g), v_h \rangle_{L^2(\Sigma)} \quad (1.6)$$

is equivalent to the global linear system

$$V_h \underline{w} = \underline{g}$$

with the related system matrix $V_h \in \mathbb{R}^{(N_0+N_L) \times (N_0+N_L)}$, the right-hand side $\underline{g} \in \mathbb{R}^{N_0+N_L}$ and the vector of unknown coefficients $\underline{w} \in \mathbb{R}^{N_0+N_L}$ of $w_h \in S_h^0(\Sigma)$. Note that the system matrix V_h and the discretisation of the double layer operator \mathcal{K} are calculated analytically, whereas all other appearing integrals are computed by the usage of high-order integration rules. Since the bilinear form $a_E(\cdot, \cdot): L^2(\Sigma) \times L^2(\Sigma) \rightarrow \mathbb{R}$ is bounded and elliptic, the discrete variational formulation (1.6) is uniquely solvable and unconditionally stable. By Céa's Lemma and standard error estimates, there follows the a priori estimate

$$\|\partial_n u - w_h\|_{L^2(\Sigma)} \leq Ch^s \|\partial_n u\|_{H^s(\Sigma)} \quad (1.7)$$

for some $s \in [0, 1]$ and a constant $C > 0$, where $H^s(\Sigma) = H^s(0, T) \times H^s(0, T)$, see Section 2.2. An approximate solution $\tilde{u}_h \approx u$ in the space-time cylinder Q is given by inserting the approximate normal derivative $w_h \approx \partial_n u$ into the representation formula (1.4), i.e.

$$\tilde{u}_h := \tilde{V}w_h - \mathcal{W}g \quad \text{in } Q. \quad (1.8)$$

To derive an adaptive mesh refinement on Σ , an a posteriori error estimator [136] is used, which is based on the application of the normal derivative ∂_n to the approximate representation formula (1.8),

$$\tilde{w}_h := \partial_n \tilde{u}_h = \frac{1}{2} w_h + \mathcal{K}' w_h + \mathcal{D}g \quad \text{on } \Sigma.$$

Here, the adjoint double layer operator \mathcal{K}' and the hypersingular boundary integral operator \mathcal{D} are used. Hence, the local error estimators

$$\tilde{\eta}_i := \|\tilde{w}_h - w_h\|_{L^2(\tau_i)} \approx \|\partial_n u - w_h\|_{L^2(\tau_i)}$$

for $i = 1, \dots, N_0 + N_L$ are computable, where the adjoint double layer operator \mathcal{K}' and the hypersingular boundary integral operator \mathcal{D} are calculated analytically. For an adaptive refinement strategy, a parameter $\theta \in [0, 1]$ is chosen and all elements τ_i are refined, where

$$\tilde{\eta}_i \geq \theta \max_j \tilde{\eta}_j. \quad (1.9)$$

As numerical examples, consider $L = 3$ and $T = 6$, i.e. $Q = (0, 3) \times (0, 6)$, for the exact solutions u_1 and u_2 , which are given in (1.1) and in (1.2), with the smooth Dirichlet datum $g_1 := u_1|_\Sigma \in H_0^1(\Sigma)$ and the piecewise smooth Dirichlet datum $g_2 := u_2|_\Sigma \in H_0^1(\Sigma)$. In the case of the smooth Dirichlet datum $g_1 = u_1|_\Sigma$, the optimal order of convergence, i.e. $s = 1$ for the error estimate (1.7), is achieved by a uniform refinement strategy, see Table 1.3, and by the adaptive refinement strategy (1.9), see Figure 1.8. Furthermore, the $L^2(Q)$ error for the approximate solution (1.8) is given in Table 1.3. However, in the case of the piecewise smooth Dirichlet datum $g_2 = u_2|_\Sigma$, only reduced orders of convergence are obtained, when using a uniform refinement strategy, see Table 1.4 and Figure 1.9. Note that the full order of convergence is attained for the adaptive refinement strategy (1.9), see Figure 1.9. A resulting sequence of adaptive meshes is depicted in Figure 1.10, where different decompositions for $x = 0$ and $x = 3$ are used, i.e. a decomposition without time slabs.

Remark 1.2.1. *Acoustic scattering problems are often formulated in exterior domains, i.e. in an unbounded domain. The boundary element method is suited very well for such scattering problems, since only a meshing of the surface of the bounded interior domain is needed. The starting point of the boundary element method is the corresponding boundary integral equation. The standard approach of boundary integral equations for the wave equation uses the Laplace transform with respect to the time variable, see [20, 21, 71–73, 87, 132]. This Laplace transform method results in space-time variational formulations, where the related bilinear form is bounded and elliptic in different norms, i.e. the Lax-Milgram Theorem is not applicable in the space-time domain and related error estimates for a boundary element method are not optimal. See also [1, 57–62, 70, 121, 155, 156] for recent developments in this direction. In [6, 8, 69], an*

level	$N_0 + N_L$	$\ \partial_n u_1 - w_{1,h}\ _{L^2(\Sigma)}$	eoc	$\ u_1 - \tilde{u}_{1,h}\ _{L^2(Q)}$	eoc
0	2	1.33823e+00	-	7.26168e-01	-
1	4	1.14684e+00	0.22	4.49627e-01	0.69
2	8	1.10072e+00	0.06	3.24885e-01	0.47
3	16	8.06608e-01	0.45	1.46335e-01	1.15
4	32	4.02738e-01	1.00	3.75263e-02	1.96
5	64	2.04198e-01	0.98	9.55880e-03	1.97
6	128	1.03212e-01	0.98	2.49973e-03	1.94
7	256	5.17114e-02	1.00	5.95579e-04	2.07
8	512	2.58723e-02	1.00	1.56495e-04	1.93
9	1024	1.29381e-02	1.00	3.71371e-05	2.08
10	2048	6.46928e-03	1.00	9.78807e-06	1.92
11	4096	3.23467e-03	1.00	2.40515e-06	2.02
12	8192	1.61734e-03	1.00	5.99227e-07	2.00
13	16384	8.08670e-04	1.00	1.49398e-07	2.00
14	32768	4.04335e-04	1.00	3.77370e-08	1.99

Table 1.3: Numerical results for the boundary element method (1.6) for the function u_1 in (1.1) for $Q = (0, 3) \times (0, 6)$ for a uniform refinement strategy.

level	$N_0 + N_L$	$\ \partial_n u_2 - w_{2,h}\ _{L^2(\Sigma)}$	eoc	$\ u_2 - \tilde{u}_{2,h}\ _{L^2(Q)}$	eoc
0	2	3.95477e+00	-	2.59835e+00	-
1	4	3.33217e+00	0.25	5.78383e-01	2.17
2	8	3.11643e+00	0.10	4.73586e-01	0.29
3	16	3.16575e+00	-0.02	4.10036e-01	0.21
4	32	2.37997e+00	0.41	1.77812e-01	1.21
5	64	1.66423e+00	0.52	6.10341e-02	1.54
6	128	1.15613e+00	0.53	2.28464e-02	1.42
7	256	8.07589e-01	0.52	8.15019e-03	1.49
8	512	5.67073e-01	0.51	3.29593e-03	1.31
9	1024	3.99491e-01	0.51	1.33215e-03	1.31
10	2048	2.81940e-01	0.50	5.98854e-04	1.15
11	4096	1.99168e-01	0.50	2.74489e-04	1.13
12	8192	1.40764e-01	0.50	1.32773e-04	1.05
13	16384	9.95104e-02	0.50	6.47539e-05	1.04
14	32768	7.03558e-02	0.50	3.21587e-05	1.01

Table 1.4: Numerical results for the boundary element method (1.6) for the function u_2 in (1.2) for $Q = (0, 3) \times (0, 6)$ for a uniform refinement strategy.

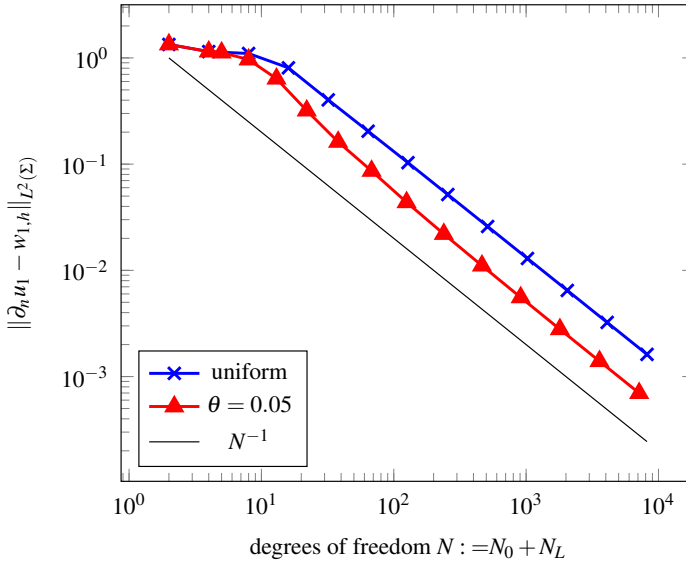


Figure 1.8: Numerical results for the boundary element method (1.6) for the function u_1 in (1.1) for $\mathcal{Q} = (0, 3) \times (0, 6)$ for the adaptive refinement strategy (1.9).

approach without the Laplace transform is considered for a screen problem in a two-dimensional spatial domain, see also [2–5] for further investigations. In addition, the work [77, 132] examines the boundary integral equations via semigroup theory and their discretisations via the convolution quadrature method [105, 106], see also [23], and [104] for a generalisation to variable time stepping. Note that this list of references is highly non-exhaustive. However, a complete analysis of space-time variational formulations for boundary integral equations of the wave equation seems to be still open. This motivates the investigations of space-time variational formulations for the wave equation in the interior and exterior of the space-time domain, see Chapter 4, since variational formulations of boundary integral equations are highly related to the variational formulations within the domain.

To summarise Section 1.1 and Section 1.2, one main advantage of space-time approximation methods, i.e. the space-time adaptivity, is realisable and leads to significantly lower numbers of the degrees of freedom for achieving a desired accuracy.

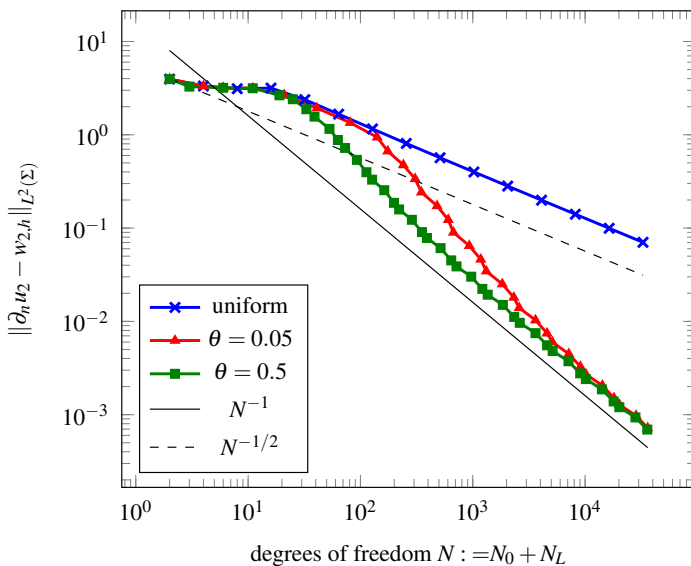


Figure 1.9: Numerical results for the boundary element method (1.6) for the function u_2 in (1.2) for $Q = (0, 3) \times (0, 6)$ for the adaptive refinement strategy (1.9).

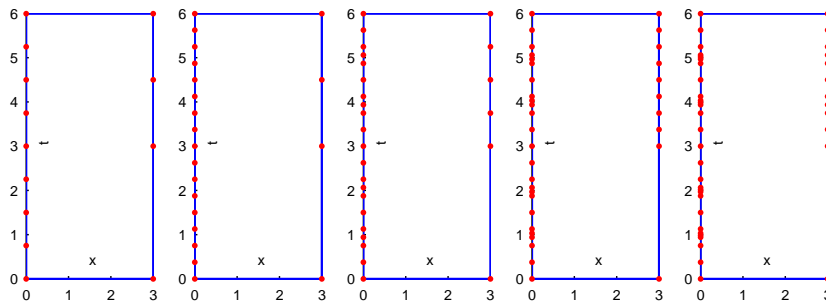


Figure 1.10: A sequence of adaptive meshes for the adaptive refinement strategy (1.9) for the function u_2 in (1.2).

Outline

The rest of this thesis is organised as follows: In Chapter 2 notations of distributions, Sobolev spaces and discretisation methods are fixed and their most important properties are repeated. In Chapter 3 the heat equation is examined, whereas in Chapter 4 the wave equation is investigated, where a more extensive overview of the literature and an outline of the different sections are given at the beginning of each chapter. In Chapter 5 a short summary of this thesis and an outlook for future work are given.

2 PRELIMINARIES

In this chapter, notations for function spaces, distributions and discretisation schemes are introduced and their most important properties are repeated. Furthermore, a short summary for variational methods is given.

In the whole thesis, $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded Lipschitz domain and $(0, T)$ is a time interval with the terminal time $T > 0$. The bounded space-time cylinder is defined as $Q := \Omega \times (0, T) \subset \mathbb{R}^{d+1}$, $\Sigma := \partial\Omega \times [0, T] \subset \mathbb{R}^{d+1}$ is the lateral boundary, and $\Sigma_0 := \Sigma \cup \Omega_0$ with $\Omega_0 := \Omega \times \{0\}$, $\Sigma_T := \Sigma \cup \Omega_T$ with $\Omega_T := \Omega \times \{T\}$ are parts of the boundary ∂Q of the space-time cylinder Q .

2.1 Distributions

As a reference for the theory of distributions see, e.g., [68, 138, 152]. In this work, $C_0^\infty(Q)$ is the set of infinitely differentiable real-valued functions with compact support in Q . The set $C_0^\infty(Q)$ endowed with the, usual for distributions, locally convex topology is denoted by $\mathcal{D}(Q)$ and is called the space of test functions on Q . The set of (Schwartz) distributions $\mathcal{D}'(Q)$ is given by all linear and sequentially continuous functionals on $\mathcal{D}(Q)$, see [138]. For a locally integrable function $v \in L_{\text{loc}}^1(Q)$, the distribution $T_v: \mathcal{D}(Q) \rightarrow \mathbb{R}$, defined by

$$T_v(\varphi) := \int_Q v(x, t) \varphi(x, t) dx dt \quad \text{for all } \varphi \in \mathcal{D}(Q),$$

is associated uniquely with that function $v \in L_{\text{loc}}^1(Q)$. Hence, the function $v \in L_{\text{loc}}^1(Q)$ and the related distribution $T_v: \mathcal{D}(Q) \rightarrow \mathbb{R}$ are identified. Throughout this work, $\square := \partial_{tt} - \Delta_x$ denotes the classical (pointwise) derivative for sufficiently smooth functions. Furthermore, let $\square_Q: \mathcal{D}'(Q) \rightarrow \mathcal{D}'(Q)$ be the distributional wave operator for distributions $\mathcal{D}'(Q)$, where for a distribution $T: \mathcal{D}(Q) \rightarrow \mathbb{R}$, derivatives are defined as usual:

$$\square_Q T(\varphi) = T(\square\varphi), \quad \varphi \in \mathcal{D}(Q).$$

In particular, for $Q_- := \Omega \times (-\infty, T) \subset \mathbb{R}^{d+1}$, let $\square_{Q_-}: \mathcal{D}'(Q_-) \rightarrow \mathcal{D}'(Q_-)$ be the distributional wave operator for distributions $\mathcal{D}'(Q_-)$.

The sets $C_0^\infty(0, T)$, $C_0^\infty(\Omega)$ and the spaces of test functions $\mathcal{D}(0, T)$, $\mathcal{D}(\Omega)$ are introduced analogously.

2.2 Sobolev Spaces in $(0, T)$

For an introduction to Sobolev spaces on intervals, see the references in the Section 2.3 and in addition, see [13, Kapitel 5] or [96, Chapter 8]. With the usual notations, the Hilbert space $H^s(0, T)$, $s \geq 0$, is the Sobolev space of real-valued functions endowed with the Sobolev-Slobodeckij inner product $\langle \cdot, \cdot \rangle_{H^s(0, T)}$ and the induced norm $\|\cdot\|_{H^s(0, T)}$. Analogously, for $s \geq 0$, $H^s(\mathbb{R})$ is the usual Sobolev space on the whole real line. Note that $H^s(0, T) \subset C[0, T]$ for $s > 1/2$, see [64, (1.4.4.6), page 27]. Hence, for $s \in (\frac{1}{2}, \frac{3}{2})$, one defines the closed subspaces

$$\begin{aligned} H_{0,0}^s(0, T) &:= \{v \in H^s(0, T) : v(0) = 0\}, \\ H_{0,T}^s(0, T) &:= \{v \in H^s(0, T) : v(T) = 0\}. \end{aligned}$$

In particular, for $s = 1$, the Sobolev spaces $H_0^1(0, T)$ and $H_{0,T}^1(0, T)$ are endowed with the inner products

$$\langle u, v \rangle_{H_0^1(0, T)} := \langle u, v \rangle_{H_{0,T}^1(0, T)} := \int_0^T \partial_t u(t) \partial_t v(t) dt$$

and with the induced norm

$$\|u\|_{H^1(0, T)} := \|\partial_t u\|_{L^2(0, T)} = \sqrt{\int_0^T |\partial_t u(t)|^2 dt}.$$

For $s = 1/2$, one defines via function space interpolation the Sobolev space

$$H_0^{1/2}(0, T) := [H_0^1(0, T), L^2(0, T)]_{1/2}$$

with the Hilbertian norm

$$\|u\|_{H_0^{1/2}(0, T)} := \sqrt{\|u\|_{H^{1/2}(0, T)}^2 + \int_0^T \frac{|u(t)|^2}{t} dt},$$

which is equivalent to the interpolation norm $\|\cdot\|_{[H_0^1(0, T), L^2(0, T)]_{1/2}}$, see [102, Théorème 11.7, page 72] and [102, Remarque 11.4, page 75]. Analogously, set

$$H_{0,T}^{1/2}(0, T) := [H_{0,T}^1(0, T), L^2(0, T)]_{1/2}$$

with the Hilbertian norm

$$\|u\|_{H_{0,T}^{1/2}(0, T)} := \sqrt{\|u\|_{H^{1/2}(0, T)}^2 + \int_0^T \frac{|u(t)|^2}{T-t} dt}.$$

The representations

$$\begin{aligned} H_0^{1/2}(0, T) &= \left\{ U_{|(0, T)} : U \in H^{1/2}(-\infty, T) \text{ with } U(t) = 0 \text{ for } t < 0 \right\} \\ &= \left\{ u \in H^{1/2}(0, T) : \int_0^T \frac{|u(t)|^2}{t} dt < \infty \right\} \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} H_0^{1/2}(0, T) &= \left\{ U_{|(0, T)} : U \in H^{1/2}(0, \infty) \text{ with } U(t) = 0 \text{ for } t > T \right\} \\ &= \left\{ u \in H^{1/2}(0, T) : \int_0^T \frac{|u(t)|^2}{T-t} dt < \infty \right\} \end{aligned} \quad (2.2)$$

hold true, see [102, Proposition 5.2, page 276] and [102, Remarque 11.4, page 75]. Because the test functions $C_0^\infty(0, T)$ are dense in $H^{1/2}(0, T)$, see [64, Theorem 1.4.2.4, page 25], the sets $H_0^{1/2}(0, T)$ and $H_0^{1/2}(0, T)$ are dense in $H^{1/2}(0, T)$. Note that the constant function $\mathbf{1}(t) := 1$ for $t \in (0, T)$ fulfils $\mathbf{1} \in H^{1/2}(0, T)$, $\mathbf{1} \notin H_0^{1/2}(0, T)$ and $\mathbf{1} \notin H_0^{1/2}(0, T)$ due to the representations (2.1) and (2.2). Because the set

$$C_0^\infty(0, T) = \{ \varphi_{|(0, T]} : \varphi \in C_0^\infty(0, \infty) \}$$

is dense in $H_0^1(0, T)$, it follows by interpolation arguments that the set $C_0^\infty(0, T)$ is dense in $H_0^{1/2}(0, T)$, see [102, Chapitre 1, Section 2.1, page 11], and analogously, the set

$$C_0^\infty[0, T) = \{ \varphi_{|[0, T)} : \varphi \in C_0^\infty(-\infty, T) \}$$

is dense in $H_0^{1/2}(0, T)$. It even holds true that the set $C_0^\infty(0, T)$ is dense in $H_0^{1/2}(0, T)$ and in $H_0^{1/2}(0, T)$, see Theorem 2.2.2.

Lemma 2.2.1. *The norm $\|\cdot\|_{H_0^{1/2}(0, T)}$, defined by*

$$\|u\|_{H_0^{1/2}(0, T)} := \inf \left\{ \|U\|_{H^{1/2}(\mathbb{R})} : U \in H^{1/2}(\mathbb{R}) \text{ with } U_{|(0, T)} = u, U(t) = 0 \text{ for } t < 0 \right\}$$

for $u \in H_0^{1/2}(0, T)$, is equivalent to $\|\cdot\|_{H_0^{1/2}(0, T)}$. Analogously, the norm $\|\cdot\|_{H_0^{1/2}(0, T)}$, defined by

$$\|u\|_{H_0^{1/2}(0, T)} := \inf \left\{ \|U\|_{H^{1/2}(\mathbb{R})} : U \in H^{1/2}(\mathbb{R}) \text{ with } U_{|[0, T)} = u, U(t) = 0 \text{ for } t > T \right\}$$

for $u \in H_0^{1/2}(0, T)$, is equivalent to $\|\cdot\|_{H_0^{1/2}(0, T)}$.

Proof. The proof is given only for the case of $H_0^{1/2}(0, T)$, the other case is proven analogously. By interpolation arguments, see [102, Proof of Proposition 5.2, page 276], the extension operator $\mathcal{E}_T: H_0^{1/2}(0, T) \rightarrow H^{1/2}(0, \infty)$, defined by

$$\mathcal{E}_T u(t) := \begin{cases} u(t), & t \in (0, T), \\ 0, & t \geq T, \end{cases}$$

for $u \in H_0^{1/2}(0, T)$, and the restriction operator $\mathcal{R}_T: H^{1/2}(0, \infty) \rightarrow H_0^{1/2}(0, T)$, given by

$$\mathcal{R}_T U(t) := U(t) - U(2T - t), \quad t \in (0, T),$$

for $U \in H^{1/2}(0, \infty)$, are bounded. For $u \in H_0^{1/2}(0, T)$, it holds true that $\mathcal{R}_T \mathcal{E}_T u = u$ and so, it follows that

$$\|u\|_{H_0^{1/2}(0, T)} = \|\mathcal{R}_T \mathcal{E}_T u\|_{H_0^{1/2}(0, T)} \leq C_{\mathcal{R}_T} \|\mathcal{E}_T u\|_{H^{1/2}(0, \infty)} \leq C_{\mathcal{R}_T} C_{\mathcal{E}_T} \|u\|_{H_0^{1/2}(0, T)},$$

i.e. the norms $\|\cdot\|_{H_0^{1/2}(0, T)}$ and $u \mapsto \|\mathcal{E}_T u\|_{H^{1/2}(0, \infty)}$ are equivalent. Since $\|\cdot\|_{H^{1/2}(0, \infty)}$ is equivalent to the norm

$$w \mapsto \inf \left\{ \|U\|_{H^{1/2}(\mathbb{R})} : U \in H^{1/2}(\mathbb{R}) \text{ with } U|_{(0, \infty)} = w \right\} \quad \text{for } w \in H^{1/2}(0, \infty),$$

see [160, Satz 5.3, page 100] with a natural extension by reflection in $t = 0$, the assertion follows. \square

Theorem 2.2.2. *The set $C_0^\infty(0, T)$ is dense in $H_0^{1/2}(0, T)$ and $H_0^{1/2}(0, T)$.*

Proof. The proof is given for $H_0^{1/2}(0, T)$. Because of the density of the set $C_0^\infty[0, T)$ in $H_0^{1/2}(0, T)$, it remains to prove the density of $C_0^\infty(0, T)$ in $C_0^\infty[0, T)$ with respect to the norm $\|\cdot\|_{H_0^{1/2}(0, T)}$. Therefore, fix an element $\varphi \in C_0^\infty[0, T)$ with $\text{supp}(\varphi) \subset [0, R]$, $T > R > 0$. Take an arbitrary extension $V \in C_0^\infty(\mathbb{R}) \subset H^{1/2}(\mathbb{R})$ with $V|_{[0, T)} = \varphi$ and $V(t) = 0$ for $t \geq T$, i.e. $V|_{[0, \infty)} \in C_0^\infty[0, \infty)$. The result [102, Lemme 11.1, page 60] yields for $X = Y = \mathbb{R}$ that there exist sequences $(\psi_n)_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R})$ and $(\varepsilon_n)_{n \in \mathbb{N}} \subset (0, T)$ such that $\psi_n(t) = 0$ for $t \in (-\varepsilon_n, \varepsilon_n)$, i.e. ψ_n vanishes in a neighbourhood of $t = 0$, and $\|\psi_n - V\|_{H^{1/2}(\mathbb{R})} \rightarrow 0$ as $n \rightarrow \infty$. Consider a cutoff function $\chi \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq \chi \leq 1$, $\chi|_{[0, R]} = 1$ and $\chi(t) = 0$ for $t > \frac{T+R}{2}$, see [160, Folgerung 1.2, page 18] for the existence of such function. Note that $\text{supp}(\chi) \subset (-\infty, T)$ and hence, $((\psi_n \cdot \chi)|_{(0, T)})_{n \in \mathbb{N}} \subset C_0^\infty(0, T)$ is the desired sequence. With Lemma 2.2.1 and the local property [102, Théorème 7.2, page 36] of $H^{1/2}(\mathbb{R})$, it

follows that

$$\begin{aligned}
 \|(\psi_n \cdot \chi)_{|(0,T)} - \varphi\|_{H_0^{1/2}(0,T)} &= \|(\psi_n \cdot \chi)_{|(0,T)} - \chi_{|(0,T)} \varphi\|_{H_0^{1/2}(0,T)} \\
 &\leq C_1 \|(\psi_n \cdot \chi)_{|(0,T)} - \chi_{|(0,T)} \varphi\|_{H_0^{1/2}(0,T)} \\
 &\leq C_1 \|\chi(\psi_n - V)\|_{H^{1/2}(\mathbb{R})} \\
 &\leq C_1 C_\chi \|\psi_n - V\|_{H^{1/2}(\mathbb{R})} \rightarrow 0 \quad \text{as } n \rightarrow \infty,
 \end{aligned}$$

where the constant $C_1 > 0$ comes from the norm equivalence of Lemma 2.2.1 and the constant $C_\chi > 0$ depends on the cutoff function χ and therefore, on φ . \square

The dual spaces $[H_0^1(0, T)]'$ and $[H_0^{1/2}(0, T)]'$ are characterised as completion of $L^2(0, T)$ with respect to the Hilbertian norms

$$\|g\|_{[H_0^1(0,T)]'} := \sup_{0 \neq v \in H_0^1(0,T)} \frac{|\langle g, v \rangle_{(0,T)}|}{\|v\|_{H^1(0,T)}}$$

and

$$\|f\|_{[H_0^{1/2}(0,T)]'} := \sup_{0 \neq w \in H_0^{1/2}(0,T)} \frac{|\langle f, w \rangle_{(0,T)}|}{\|w\|_{H^{1/2}(0,T)}},$$

where $\langle \cdot, \cdot \rangle_{(0,T)}$ denotes the duality pairing as extension of the inner product in $L^2(0, T)$, see [160, Satz 17.3, page 258]. In other words, for $[H_0^1(0, T)]'$ and $[H_0^{1/2}(0, T)]'$, there exist inner products $\langle \cdot, \cdot \rangle_{[H_0^1(0,T)]'}$ and $\langle \cdot, \cdot \rangle_{[H_0^{1/2}(0,T)]'}$, inducing the norms $\|\cdot\|_{[H_0^1(0,T)]'} = \sqrt{\langle \cdot, \cdot \rangle_{[H_0^1(0,T)]'}}$ and $\|\cdot\|_{[H_0^{1/2}(0,T)]'} = \sqrt{\langle \cdot, \cdot \rangle_{[H_0^{1/2}(0,T)]'}}$, i.e. with these abstract inner products, $[H_0^1(0, T)]'$ and $[H_0^{1/2}(0, T)]'$ are Hilbert spaces, see [158, Satz V.1.7, page 222].

Analogously, the dual spaces $[H_0^{1/2}(0, T)]'$ and $[H_0^1(0, T)]'$ are Hilbert spaces characterised as completion of $L^2(0, T)$ with respect to the Hilbertian norms

$$\|g\|_{[H_0^{1/2}(0,T)]'} := \sup_{0 \neq v \in H_0^{1/2}(0,T)} \frac{|\langle g, v \rangle_{(0,T)}|}{\|v\|_{H_0^{1/2}(0,T)}}$$

and

$$\|f\|_{[H_0^1(0,T)]'} := \sup_{0 \neq w \in H_0^1(0,T)} \frac{|\langle f, w \rangle_{(0,T)}|}{\|w\|_{H_0^1(0,T)}}. \quad (2.3)$$

2.3 Sobolev Spaces in Ω

For a general introduction to Sobolev spaces see, for example, the books [31, 64, 119, 160], and for function space interpolation, see [26, 102, 103, 153]. For $s \geq 0$, the usual Sobolev spaces of real-valued functions $H^s(\Omega)$, $H_0^s(\Omega)$ are endowed with the Sobolev-Slobodeckij inner product $\langle \cdot, \cdot \rangle_{H^s(\Omega)}$ and the norm $\|\cdot\|_{H^s(\Omega)}$. For the closed subspace $H_0^1(\Omega) \subset H^1(\Omega)$, the inner product

$$\langle u, v \rangle_{H_0^1(\Omega)} := \langle \nabla_x u, \nabla_x v \rangle_{L^2(\Omega)} = \int_{\Omega} \nabla_x u(x) \cdot \nabla_x v(x) dx, \quad u, v \in H_0^1(\Omega),$$

and the induced norm

$$\|u\|_{H_0^1(\Omega)} := \|u\|_{H_0^1(\Omega)} = \sqrt{\langle u, u \rangle_{H_0^1(\Omega)}} = \sqrt{\int_{\Omega} |\nabla_x u(x)|^2 dx}, \quad u \in H_0^1(\Omega),$$

are considered. For a type of Fourier series approach in Chapter 3 and Chapter 4, the eigenfunctions $\phi_i \in H_0^1(\Omega)$ with eigenvalues $\mu_i \in \mathbb{R}$, satisfying

$$-\Delta \phi_i = \mu_i \phi_i \quad \text{in } \Omega, \quad \phi_i = 0 \quad \text{on } \partial\Omega, \quad \|\phi_i\|_{L^2(\Omega)} = 1 \quad (2.4)$$

for $i \in \mathbb{N}$, are used, see [97, Theorem 4.1 in Chapter II, page 60]. Note that the eigenfunctions ϕ_i form an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H_0^1(\Omega)$. In addition, the eigenvalues μ_i satisfy

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots \quad \text{and} \quad \mu_i \rightarrow \infty \text{ as } i \rightarrow \infty.$$

Hence, for a function $u \in L^2(\Omega)$, it holds true that

$$\left\| u - \sum_{i=1}^M u_i \phi_i \right\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } M \rightarrow \infty,$$

i.e. $u = \sum_{i=1}^{\infty} u_i \phi_i$ in $L^2(\Omega)$, with the coefficients

$$u_i = \int_{\Omega} u(x) \phi_i(x) dx \in \mathbb{R},$$

and the $L^2(\Omega)$ norm is given by

$$\|u\|_{L^2(\Omega)} = \sqrt{\sum_{i=1}^{\infty} u_i^2}.$$

Analogously, for a function $u \in H_0^1(\Omega)$, it holds true that

$$\left\| \nabla_x u - \sum_{i=1}^M u_i \nabla_x \phi_i \right\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } M \rightarrow \infty,$$

i.e. $u = \sum_{i=1}^{\infty} u_i \phi_i$ in $H_0^1(\Omega)$, with the coefficients

$$u_i = \int_{\Omega} u(x) \phi_i(x) dx \in \mathbb{R}$$

and the $H^1(\Omega)$ seminorm is given by

$$|u|_{H^1(\Omega)} = \sqrt{\sum_{i=1}^{\infty} \mu_i u_i^2}.$$

The dual space $[H_0^1(\Omega)]'$ is a Hilbert space characterised as the completion of $L^2(\Omega)$ with respect to the Hilbertian norm

$$\|g\|_{[H_0^1(\Omega)]'} := \sup_{0 \neq v \in H_0^1(\Omega)} \frac{|\langle g, v \rangle_{\Omega}|}{|v|_{H^1(\Omega)}}, \quad (2.5)$$

where $\langle \cdot, \cdot \rangle_{\Omega}$ denotes the duality pairing as extension of the inner product in $L^2(\Omega)$, see Section 2.2.

2.4 Hilbert Tensor-Product and Bochner Spaces

For an introduction to the algebraic tensor-product \otimes and to the Hilbert tensor-product $\hat{\otimes}$, see [15, Chapter 12], [157, Unterkapitel 1.6], [128, Section II.4] or [152, Part III]. For Bochner spaces, see also [160, Kapitel IV], [162, Chapter 23], [82, Chapter 1 and 2], [139, Kapitel 10], [110, Chapter 2] and the recent work [12]. In this section, let H be a separable real Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be bounded Lipschitz domains with $d_1, d_2 \in \mathbb{N}$. Consider the Bochner space $L^2(\Omega_1; H)$ of classes of measurable vector-valued functions $U: \Omega_1 \rightarrow H$, i.e. for each element $U \in L^2(\Omega_1; H)$, it holds true that

$$U(y) \in H \quad \text{for almost all } y \in \Omega_1$$

such that

$$\|U\|_{L^2(\Omega_1; H)} := \sqrt{\int_{\Omega_1} \|U(y)\|_H^2 dy} < \infty.$$

The Bochner space $L^2(\Omega_1; H)$ is a Hilbert space with respect to the inner product

$$\langle U, V \rangle_{L^2(\Omega_1; H)} := \int_{\Omega_1} \langle U(y), V(y) \rangle_H dy.$$

The dual space $[L^2(\Omega_1; H)]'$ and the Bochner space $L^2(\Omega_1; H')$ are isometric, see [82, Corollary 1.3.22, page 54] and see also [162, Section 23.3]. Furthermore, the Bochner space $L^2(\Omega_1; H)$ and the Hilbert tensor-product $L^2(\Omega_1) \hat{\otimes} H$ are isometric, i.e.

$$L^2(\Omega_1; H) \simeq L^2(\Omega_1) \hat{\otimes} H \simeq H \hat{\otimes} L^2(\Omega_1),$$

see [15, Theorem 12.6.1, page 304].

For $m \in \mathbb{N}_0$, the Bochner Sobolev space is defined by

$$H^m(\Omega_1; H) := \{U \in L^2(\Omega_1; H) : \partial_y^\alpha U \in L^2(\Omega_1; H) \text{ for } |\alpha| \leq m\}$$

where ∂_y is the distributional derivative on Ω_1 with respect to y for vector-valued functions and $\alpha = (\alpha_1, \dots, \alpha_{d_1}) \in \mathbb{N}_0^{d_1}$ is a multi-index. With the inner product

$$\langle U, V \rangle_{H^m(\Omega_1; H)} := \int_{\Omega_1} \langle U(y), V(y) \rangle_H dy + \sum_{|\alpha| \leq m} \int_{\Omega_1} \langle \partial_y^\alpha U(y), \partial_y^\alpha V(y) \rangle_H dy$$

for $U, V \in H^m(\Omega_1; H)$, the Bochner Sobolev space $H^m(\Omega_1; H)$ is a Hilbert space. Furthermore, the Bochner Sobolev space $H^m(\Omega_1; H)$ and the Hilbert tensor-product $H^m(\Omega_1) \hat{\otimes} H$ are isometric, i.e.

$$H^m(\Omega_1; H) \simeq H^m(\Omega_1) \hat{\otimes} H \simeq H \hat{\otimes} H^m(\Omega_1), \quad (2.6)$$

see [15, Theorem 12.7.1, page 307].

As a first special case, for $m = 1$ and $\Omega_1 = (0, T)$, the Sobolev embedding theorem [110, Proposition 2.46, page 46] holds true, i.e.

$$H^1(0, T; H) \subset C([0, T]; H) \quad (2.7)$$

with a continuous embedding. Therefore, as in Section 2.2,

$$\begin{aligned} H_{0,0}^1(0, T; H) &:= \{V \in H^1(0, T; H) : V(0) = 0 \text{ in } H\}, \\ H_{0,T}^1(0, T; H) &:= \{V \in H^1(0, T; H) : V(T) = 0 \text{ in } H\} \end{aligned}$$

are closed subspaces of $H^1(0, T; H)$.

As a second special case, consider $H = H^p(\Omega_2)$ with $p \in \mathbb{N}_0$. Then, the space

$$\begin{aligned} H_{\otimes}^{p,m}(\Omega_2 \times \Omega_1) \\ := \left\{ u \in L^2(\Omega_2 \times \Omega_1) : \partial_x^{\alpha^2} \partial_y^{\alpha^1} u \in L^2(\Omega_2 \times \Omega_1) \text{ for } |\alpha^2| \leq p, |\alpha^1| \leq m \right\}, \end{aligned}$$

with the inner product

$$\langle u, v \rangle_{H_{\otimes}^{p,m}(\Omega_2 \times \Omega_1)} = \sum_{|\alpha^2| \leq p} \sum_{|\alpha^1| \leq m} \left\langle \partial_x^{\alpha^2} \partial_y^{\alpha^1} u, \partial_x^{\alpha^2} \partial_y^{\alpha^1} v \right\rangle_{L^2(\Omega_2 \times \Omega_1)},$$

where ∂_x and ∂_y denote the distributional derivatives with respect to x and y on Ω_2 and Ω_1 , is isometric to $H^m(\Omega_1; H^p(\Omega_2))$, i.e.

$$H_{\otimes}^{p,m}(\Omega_2 \times \Omega_1) \simeq H^m(\Omega_1; H^p(\Omega_2)) \simeq H^p(\Omega_2; H^m(\Omega_1)) \simeq H^p(\Omega_2) \hat{\otimes} H^m(\Omega_1), \quad (2.8)$$

see [15, Theorem 12.7.2, page 308] and (2.6). Thus, for $m = p = 0$, the relation (2.8) states

$$L^2(\Omega_2 \times \Omega_1) \simeq L^2(\Omega_2; L^2(\Omega_1)) \simeq L^2(\Omega_1; L^2(\Omega_2)) \simeq L^2(\Omega_2) \hat{\otimes} L^2(\Omega_1),$$

see also [128, Theorem II.10, page 52] and [139, last line, page 188]. Hence, for a separable Hilbert space $H \subset L^2(\Omega_2)$, the Bochner space $L^2(\Omega_1; H) \subset L^2(\Omega_1; L^2(\Omega_2))$ and the L^2 subspace

$$\{u \in L^2(\Omega_2 \times \Omega_1) : y \mapsto u(\cdot, y) \in L^2(\Omega_1; H)\}, \quad (2.9)$$

endowed with the inner product

$$\langle u, v \rangle := \int_{\Omega_1} \langle u(\cdot, y), v(\cdot, y) \rangle_H dy,$$

are isometric by the bijective isometry Φ given by $u(x, y) := (\Phi U)(x, y) := U(y)(x)$ for $(x, y) \in \Omega_2 \times \Omega_1$, $U \in L^2(\Omega_1; H)$. Therefore, for a separable Hilbert space $H \subset L^2(\Omega_2)$, the Bochner space $L^2(\Omega_1; H)$ is identified with the subspace (2.9) of $L^2(\Omega_2 \times \Omega_1)$, hence, one writes

$$L^2(\Omega_1; H) = \{u \in L^2(\Omega_2 \times \Omega_1) : y \mapsto u(\cdot, y) \in L^2(\Omega_1; H)\}.$$

Analogously, for a separable Hilbert space $H \subset L^2(\Omega_1)$, the Bochner space $L^2(\Omega_2; H)$ is identified with a subspace of $L^2(\Omega_2 \times \Omega_1)$, hence, one writes

$$L^2(\Omega_2; H) = \{u \in L^2(\Omega_2 \times \Omega_1) : x \mapsto u(x, \cdot) \in L^2(\Omega_2; H)\},$$

where this subspace of $L^2(\Omega_2 \times \Omega_1)$ is endowed with the inner product

$$\langle u, v \rangle := \int_{\Omega_2} \langle u(x, \cdot), v(x, \cdot) \rangle_H dx.$$

With these identifications, the anisotropic Sobolev spaces are defined for $0 \leq r \in \mathbb{R}$, $0 \leq s \in \mathbb{R}$ as

$$H^{r,s}(\Omega_2 \times \Omega_1) := L^2(\Omega_1; H^r(\Omega_2)) \cap L^2(\Omega_2; H^s(\Omega_1)) \subset L^2(\Omega_2 \times \Omega_1) \quad (2.10)$$

with the inner product

$$\langle u, v \rangle_{H^{r,s}(\Omega_2 \times \Omega_1)} := \int_{\Omega_1} \langle u(\cdot, y), v(\cdot, y) \rangle_{H^r(\Omega_2)} dy + \int_{\Omega_2} \langle u(x, \cdot), v(x, \cdot) \rangle_{H^s(\Omega_1)} dx.$$

For integers r and s , more explicit characterisations of the spaces $H^{r,s}(\Omega_2 \times \Omega_1)$ are given. For $p \in \mathbb{N}$, $m = 0$, it holds true that

$$\begin{aligned} H^{p,0}(\Omega_2 \times \Omega_1) &= \left\{ u \in L^2(\Omega_2 \times \Omega_1) : \partial_x^{\alpha^2} u \in L^2(\Omega_2 \times \Omega_1) \text{ for } |\alpha^2| \leq p \right\} \\ &= H_{\otimes}^{p,0}(\Omega_2 \times \Omega_1) = L^2(\Omega_1; H^p(\Omega_2)) \\ &\simeq H^p(\Omega_2; L^2(\Omega_1)) \simeq H^p(\Omega_2) \hat{\otimes} L^2(\Omega_1) \end{aligned}$$

with the inner product

$$\langle u, v \rangle_{H^{p,0}(\Omega_2 \times \Omega_1)} = \sum_{|\alpha^2| \leq p} \left\langle \partial_x^{\alpha^2} u, \partial_x^{\alpha^2} v \right\rangle_{L^2(\Omega_2 \times \Omega_1)}$$

and analogously, for $m \in \mathbb{N}$, $p = 0$, it holds true that

$$\begin{aligned} H^{0,m}(\Omega_2 \times \Omega_1) &= \left\{ u \in L^2(\Omega_2 \times \Omega_1) : \partial_y^{\alpha^1} u \in L^2(\Omega_2 \times \Omega_1) \text{ for } |\alpha^1| \leq m \right\} \\ &= H_{\otimes}^{0,m}(\Omega_2 \times \Omega_1) = L^2(\Omega_2; H^m(\Omega_1)) \\ &\simeq H^m(\Omega_1; L^2(\Omega_2)) \simeq L^2(\Omega_2) \hat{\otimes} H^m(\Omega_1) \end{aligned}$$

with the inner product

$$\langle u, v \rangle_{H^{0,m}(\Omega_2 \times \Omega_1)} = \sum_{|\alpha^1| \leq m} \left\langle \partial_y^{\alpha^1} u, \partial_y^{\alpha^1} v \right\rangle_{L^2(\Omega_2 \times \Omega_1)}.$$

As a last special case, consider the interval $\Omega_1 = (0, T)$. For $0 < s \in \mathbb{R} \setminus \mathbb{N}$, the space $H^s(0, T; L^2(\Omega_2))$ is defined via function space interpolation endowed with the inner product $\langle \cdot, \cdot \rangle_{H^s(0, T; L^2(\Omega_2))} := \langle \cdot, \cdot \rangle_{L^2(\Omega_2; H^s(0, T))}$ and with the to the interpolation norm equivalent norm $\|\cdot\|_{L^2(\Omega_2; H^s(0, T))}$, see [103, page 8]. So, in the following, the spaces $H^s(0, T; L^2(\Omega_2))$ and $L^2(\Omega_2; H^s(0, T))$ are identified, hence, one writes

$$L^2(\Omega_2; H^s(0, T)) = H^s(0, T; L^2(\Omega_2)). \quad (2.11)$$

2.5 Sobolev Spaces in Q

In this section, the notations and identifications of Section 2.2, Section 2.3 and Section 2.4 are used.

For an introduction to anisotropic Sobolev spaces, see [102, 103] and for a short summary see [35, Chapter 2].

For $0 \leq r \in \mathbb{R}$, $0 \leq s \in \mathbb{R}$, one defines as in (2.10), see also (2.11), the anisotropic Sobolev space

$$H^{r,s}(Q) := L^2(0, T; H^r(\Omega)) \cap H^s(0, T; L^2(\Omega)) \simeq (H^r(\Omega) \hat{\otimes} L^2(0, T)) \cap (L^2(\Omega) \hat{\otimes} H^s(0, T)),$$

which is a Hilbert space with respect to the inner product

$$\begin{aligned} \langle u, v \rangle_{H^{r,s}(Q)} &:= \int_0^T \langle u(\cdot, t), v(\cdot, t) \rangle_{H^r(\Omega)} dt + \int_{\Omega} \langle u(x, \cdot), v(x, \cdot) \rangle_{H^s(0, T)} dx \\ &= \langle u, v \rangle_{L^2(0, T; H^r(\Omega))} + \langle u, v \rangle_{H^s(0, T; L^2(\Omega))} \end{aligned}$$

for $u, v \in H^{r,s}(Q)$. For $r = s = 1$, note that

$$H^1(Q) = H^{1,1}(Q) \subset C([0, T]; L^2(\Omega)) \quad (2.12)$$

with a continuous embedding, see (2.7). The subspace

$$H_{0;0}^{1,1}(Q) := H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

is endowed with the inner product

$$\langle u, v \rangle_{H_{0;0}^{1,1}(Q)} := \int_0^T \int_{\Omega} (\partial_t u(x, t) \partial_t v(x, t) + \nabla_x u(x, t) \cdot \nabla_x v(x, t)) dx dt \quad (2.13)$$

and the induced norm

$$|u|_{H^1(Q)} := \sqrt{\langle u, u \rangle_{H_{0;0}^{1,1}(Q)}} = \left\{ \int_0^T \int_{\Omega} \left(|\partial_t u(x, t)|^2 + \sum_{m=1}^d |\partial_{x_m} u(x, t)|^2 \right) dx dt \right\}^{1/2}.$$

Note that in $H_{0;0}^{1,1}(Q)$, the seminorm $|\cdot|_{H^1(Q)}$ is a to $\|\cdot\|_{H^1(Q)}$ equivalent norm due to the Poincaré inequality. The subspaces

$$H_{0;0}^{1,1}(Q) := H_0^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \quad (2.14)$$

and

$$H_{0;0}^{1,1}(Q) := H_0^1(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$$

are endowed with the inner product (2.13) and the induced norm $|\cdot|_{H^1(Q)}$.

For functions defined in Ω , the standard trace operator

$$\gamma_0^{\text{int}}: H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

is linear and bounded, i.e. $\|\gamma_0^{\text{int}} z\|_{H^{1/2}(\partial\Omega)} \leq C_{\text{Tr}} \|z\|_{H^1(\Omega)}$ with a constant $C_{\text{Tr}} > 0$, where the Sobolev space $H^{1/2}(\partial\Omega)$ is the usual trace space, see [34, 64, 119, 160] for more details. The extended trace operator

$$\gamma_{0,x}^{\text{int}}: L^2(0, T; H^1(\Omega)) \rightarrow L^2(\Sigma) \quad (2.15)$$

satisfies the relation

$$\gamma_{0,x}^{\text{int}} v = v|_{\Sigma} \quad \text{for } v \in L^2(0, T; C(\overline{\Omega})),$$

the relation

$$\left\| \gamma_{0,x}^{\text{int}} v \right\|_{L^2(\Sigma)} = 0 \iff v \in L^2(0, T; H_0^1(\Omega)),$$

and, with the same constant $C_{\text{Tr}} > 0$ as γ_0^{int} , the boundedness estimate

$$\left\| \gamma_{0,x}^{\text{int}} v \right\|_{L^2(\Sigma)} \leq C_{\text{Tr}} \|v\|_{L^2(0, T; H^1(\Omega))},$$

see [12, Theorem 6.13, page 21]. Hence, the representations

$$\begin{aligned} H_{0; \cdot}^{1,1}(Q) &= \left\{ v \in H^1(Q) : \left\| \gamma_{0,x}^{\text{int}} v \right\|_{L^2(\Sigma)} = 0 \right\}, \\ H_{0;0}^{1,1}(Q) &= \left\{ v \in H^1(Q) : \left\| \gamma_{0,x}^{\text{int}} v \right\|_{L^2(\Sigma)} = \|v(\cdot, 0)\|_{L^2(\Omega)} = 0 \right\}, \\ H_{0;\cdot}^{1,1}(Q) &= \left\{ v \in H^1(Q) : \left\| \gamma_{0,x}^{\text{int}} v \right\|_{L^2(\Sigma)} = \|v(\cdot, T)\|_{L^2(\Omega)} = 0 \right\} \end{aligned}$$

are valid, i.e. $H_{0; \cdot}^{1,1}(Q)$, $H_{0;0}^{1,1}(Q)$ and $H_{0;\cdot}^{1,1}(Q)$ are closed subspaces of $H^1(Q)$.

The dual spaces $[H_{0;0}^{1,1}(Q)]'$ and $[H_{0;\cdot}^{1,1}(Q)]'$ are characterised as completion of $L^2(Q)$ with respect to the Hilbertian norms

$$\|g\|_{[H_{0;0}^{1,1}(Q)]'} := \sup_{0 \neq v \in H_{0;0}^{1,1}(Q)} \frac{|\langle g, v \rangle_Q|}{|v|_{H^1(Q)}}$$

and

$$\|f\|_{[H_{0;\cdot}^{1,1}(Q)]'} := \sup_{0 \neq w \in H_{0;\cdot}^{1,1}(Q)} \frac{|\langle f, w \rangle_Q|}{|w|_{H^1(Q)}},$$

where $\langle \cdot, \cdot \rangle_Q$ denotes the duality pairing as extension of the inner product in $L^2(Q)$, see Section 2.2.

For $s = 1/2$, one defines via function space interpolation the Sobolev space

$$H_0^{1/2}(0, T; L^2(\Omega)) := [H_0^1(0, T; L^2(\Omega)), L^2(0, T; L^2(\Omega))]_{1/2}$$

with the Hilbertian norm

$$\|u\|_{H_0^{1/2}(0,T;L^2(\Omega))} := \sqrt{\|u\|_{H^{1/2}(0,T;L^2(\Omega))}^2 + \int_0^T \frac{\|u(\cdot,t)\|_{L^2(\Omega)}^2}{t} dt}, \quad (2.16)$$

which is equivalent to the interpolation norm, see [102, Théorème 11.7, page 72] and [102, (5.8), page 276]. Analogously, set

$$H_0^{1/2}(0,T;L^2(\Omega)) := [H_0^1(0,T;L^2(\Omega)), L^2(0,T;L^2(\Omega))]_{1/2}$$

with the Hilbertian norm

$$\|u\|_{H_0^{1/2}(0,T;L^2(\Omega))} := \sqrt{\|u\|_{H^{1/2}(0,T;L^2(\Omega))}^2 + \int_0^T \frac{\|u(\cdot,t)\|_{L^2(\Omega)}^2}{T-t} dt}. \quad (2.17)$$

The representations

$$\begin{aligned} H_0^{1/2}(0,T;L^2(\Omega)) &= \left\{ v|_Q : v \in H^{1/2}(-\infty, T; L^2(\Omega)) \text{ with } v(\cdot, t) = 0 \text{ in } L^2(\Omega) \text{ for } t < 0 \right\} \\ &= \left\{ u \in H^{1/2}(0, T; L^2(\Omega)) : \int_0^T \frac{\|u(\cdot, t)\|_{L^2(\Omega)}^2}{t} dt < \infty \right\} \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} H_0^{1/2}(0, T; L^2(\Omega)) &= \left\{ v|_Q : v \in H^{1/2}(0, \infty; L^2(\Omega)) \text{ with } v(\cdot, t) = 0 \text{ in } L^2(\Omega) \text{ for } t > T \right\} \\ &= \left\{ u \in H^{1/2}(0, T; L^2(\Omega)) : \int_0^T \frac{\|u(\cdot, t)\|_{L^2(\Omega)}^2}{T-t} dt < \infty \right\} \end{aligned} \quad (2.19)$$

hold true, see [102, Proposition 5.2, page 276] and [102, Remarque 11.4, page 75]. Because the test functions

$$C_0^\infty(\Omega) \otimes C_0^\infty(0, T) = \text{span} \{ Q \ni (x, t) \mapsto \phi(x) \cdot \psi(t) \in \mathbb{R} : \phi \in C_0^\infty(\Omega), \psi \in C_0^\infty(0, T) \}$$

are dense in $L^2(\Omega) \hat{\otimes} H^{1/2}(0, T) \simeq H^{1/2}(0, T; L^2(\Omega))$, see [64, Theorem 1.4.2.4, page 25], [157, Satz 1.63, page 62] and for the tensor-product of functions see [152, Example II, page 407], the sets $H_0^{1/2}(0, T; L^2(\Omega))$ and $H_0^{1/2}(0, T; L^2(\Omega))$ are dense in $H^{1/2}(0, T; L^2(\Omega))$. Note that the constant function $\mathbf{1}(x, t) := 1$ for $(x, t) \in Q$ fulfils $\mathbf{1} \in H^{1/2}(0, T; L^2(\Omega))$, $\mathbf{1} \notin H_0^{1/2}(0, T; L^2(\Omega))$ and $\mathbf{1} \notin H_0^{1/2}(0, T; L^2(\Omega))$ due to the representations (2.18) and (2.19).

The first spaces needed for the heat equation are the anisotropic Sobolev spaces

$$\begin{aligned} H_{0;0}^{1,1/2}(\mathcal{Q}) &:= H_0^{1/2}(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \\ H_{0;0}^{1,1/2}(\mathcal{Q}) &:= H_0^{1/2}(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \end{aligned}$$

endowed with the Hilbertian norms

$$\begin{aligned} \|v\|_{H_{0;0}^{1,1/2}(\mathcal{Q})} &:= \sqrt{\|v\|_{H_0^{1/2}(0, T; L^2(\Omega))}^2 + \|\nabla_x v\|_{L^2(\mathcal{Q})}^2}, \\ \|w\|_{H_{0;0}^{1,1/2}(\mathcal{Q})} &:= \sqrt{\|w\|_{H_0^{1/2}(0, T; L^2(\Omega))}^2 + \|\nabla_x w\|_{L^2(\mathcal{Q})}^2}. \end{aligned} \quad (2.20)$$

The dual spaces $[H_{0;0}^{1,1/2}(\mathcal{Q})]'$ and $[H_{0;0}^{1,1/2}(\mathcal{Q})]'$ are characterised as completion of $L^2(\mathcal{Q})$ with respect to the Hilbertian norms

$$\|g\|_{[H_{0;0}^{1,1/2}(\mathcal{Q})]'} := \sup_{0 \neq v \in H_{0;0}^{1,1/2}(\mathcal{Q})} \frac{|\langle g, v \rangle_{\mathcal{Q}}|}{\|v\|_{H_{0;0}^{1,1/2}(\mathcal{Q})}}$$

and

$$\|f\|_{[H_{0;0}^{1,1/2}(\mathcal{Q})]'} := \sup_{0 \neq w \in H_{0;0}^{1,1/2}(\mathcal{Q})} \frac{|\langle f, w \rangle_{\mathcal{Q}}|}{\|w\|_{H_{0;0}^{1,1/2}(\mathcal{Q})}},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{Q}}$ denotes the duality pairing as extension of the inner product in $L^2(\mathcal{Q})$, see Section 2.2.

A second space needed for the heat equation is introduced. Therefore, one defines

$$\begin{aligned} W(\mathcal{Q}) &:= L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; [H_0^1(\Omega)]') \\ &\simeq (H_0^1(\Omega) \hat{\otimes} L^2(0, T)) \cap ([H_0^1(\Omega)]' \hat{\otimes} H^1(0, T)), \end{aligned} \quad (2.21)$$

which is a Hilbert space with the inner product

$$\langle u, v \rangle_{W(\mathcal{Q})} := \int_0^T \int_{\Omega} \nabla_x u(x, t) \cdot \nabla_x v(x, t) dx dt + \int_0^T \langle \partial_t u(\cdot, t), \partial_t v(\cdot, t) \rangle_{[H_0^1(\Omega)]'} dt,$$

see [160, Satz 25.4, page 380], and the norm is given by

$$\|u\|_{W(\mathcal{Q})} := \left(\int_0^T \int_{\Omega} |\nabla_x u(x, t)|^2 dx dt + \int_0^T \|\partial_t u(\cdot, t)\|_{[H_0^1(\Omega)]'}^2 dt \right)^{1/2}, \quad (2.22)$$

where the Hilbertian norm in the dual space $[H_0^1(\Omega)]'$ is given as in (2.5) for $t \in (0, T)$ by

$$\|\partial_t u(\cdot, t)\|_{[H_0^1(\Omega)]'} = \sup_{0 \neq z \in H_0^1(\Omega)} \frac{|\langle \partial_t u(\cdot, t), z \rangle_{\Omega}|}{\|z\|_{H^1(\Omega)}}.$$

Moreover, it holds true that

$$W(Q) \subset C([0, T]; L^2(\Omega)) \quad (2.23)$$

and this embedding is continuous, in other words, there exists a constant $C_{\text{em}} > 0$ such that

$$\forall u \in W(Q): \max_{t \in [0, T]} \|u(\cdot, t)\|_{L^2(\Omega)} = \max_{t \in [0, T]} \left(\int_{\Omega} |u(x, t)|^2 dx \right)^{1/2} \leq C_{\text{em}} \|u\|_{W(Q)},$$

see [162, Proposition 23.23, page 422]. In addition, the trace map

$$\gamma_{\Omega_0}: W(Q) \rightarrow L^2(\Omega), \quad \gamma_{\Omega_0} u(x) := u|_{\Omega_0}(x) = u(x, 0) \quad \text{for } x \in \Omega,$$

is surjective, see [102, Théorème 3.2, page 25], and bounded due to the continuous embedding (2.23) and

$$\|\gamma_{\Omega_0} u\|_{L^2(\Omega)} = \|u(\cdot, 0)\|_{L^2(\Omega)} \leq \max_{t \in [0, T]} \|u(\cdot, t)\|_{L^2(\Omega)} \leq C_{\text{em}} \|u\|_{W(Q)}$$

for $u \in W(Q)$ with the constant $C_{\text{em}} > 0$. Moreover, there exists a continuous, linear right inverse $\mathcal{E}_{\Omega_0}: L^2(\Omega) \rightarrow W(Q)$, satisfying for all $u_0 \in L^2(\Omega)$

$$\gamma_{\Omega_0} \mathcal{E}_{\Omega_0} u_0 = u_0 \text{ in } L^2(\Omega) \quad \text{and} \quad \|\mathcal{E}_{\Omega_0} u_0\|_{W(Q)} \leq C_{\text{ex}} \|u_0\|_{L^2(\Omega)} \quad (2.24)$$

with a constant $C_{\text{ex}} > 0$, see [102, Remarque 3.3, page 26]. Note that also Theorem 3.1.1 in Section 3.1 gives such an extension operator \mathcal{E}_{Ω_0} . Because of the embedding (2.23), the initial condition $v(\cdot, 0) = v_0$ in $L^2(\Omega)$ for a given $v_0 \in L^2(\Omega)$ is meaningful for functions $v \in W(Q) \subset C([0, T]; L^2(\Omega))$. Further, the subspace

$$W_0(Q) := \{v \in W(Q): v(\cdot, 0) = 0 \text{ in } L^2(\Omega)\} \subset W(Q) \quad (2.25)$$

is again a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_{W(Q)}$.

2.6 Discretisations in Time

For the given terminal time $T > 0$, the time interval $(0, T)$ is decomposed via the time steps

$$0 = t_0 < t_1 < t_2 < \dots < t_{N_t-1} < t_{N_t} = T,$$

where N_t denotes the number of time intervals $\tau_\ell = (t_{\ell-1}, t_\ell)$ for $\ell = 1, \dots, N_t$. In addition, the number of time steps t_ℓ is denoted by \bar{M}_t , i.e. $\bar{M}_t = N_t + 1$, and the local mesh sizes are given as $h_{t,\ell} = t_\ell - t_{\ell-1}$ for $\ell = 1, \dots, N_t$. Next, the global mesh size in time is defined by $h_t = \max_{\ell=1, \dots, N_t} h_{t,\ell}$, and the related finite element space

$$S_{h_t}^1(0, T) = \text{span}\{\varphi_k\}_{k=0}^{N_t}$$

of piecewise linear, continuous functions is introduced, where the usual nodal basis functions φ_k , $k = 0, \dots, N_t$, satisfy $\varphi_k(t_\ell) = \delta_{k\ell}$ for $k, \ell = 0, \dots, N_t$. In addition, the subspaces $S_{h_t,0}^1(0,T) \subset S_{h_t}^1(0,T)$ and $S_{h_t,0}^1(0,T) \subset S_{h_t}^1(0,T)$ fulfil the homogeneous initial or terminal conditions, i.e.

$$S_{h_t,0}^1(0,T) = S_{h_t}^1(0,T) \cap H_{0,0}^1(0,T) = \text{span}\{\varphi_k\}_{k=1}^{N_t}$$

and

$$S_{h_t,0}^1(0,T) = S_{h_t}^1(0,T) \cap H_{0,T}^1(0,T) = \text{span}\{\varphi_k\}_{k=0}^{N_t-1}.$$

Furthermore, $S_{h_t}^0(0,T)$ is the finite element space of piecewise constant functions. The mapping

$$Q_{h_t}^0: L^2(0,T) \rightarrow S_{h_t}^0(0,T)$$

denotes the L^2 projection on the piecewise constant finite element space $S_{h_t}^0(0,T)$, defined for $u \in L^2(0,T)$ by finding $Q_{h_t}^0 u \in S_{h_t}^0(0,T)$ such that

$$\langle Q_{h_t}^0 u, v_{h_t} \rangle_{L^2(0,T)} = \langle u, v_{h_t} \rangle_{L^2(0,T)} \quad (2.26)$$

for all $v_{h_t} \in S_{h_t}^0(0,T)$, satisfying the stability estimate

$$\|Q_{h_t}^0 u\|_{L^2(0,T)} \leq \|u\|_{L^2(0,T)}$$

and the representation

$$Q_{h_t}^0 u(t) = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} u(s) ds, \quad t \in \tau_\ell,$$

for $\ell = 1, \dots, N_t$.

For a continuous function $u \in C[0,T]$, the interpolation operator

$$I_{h_t}: C[0,T] \rightarrow S_{h_t}^1(0,T)$$

is defined by

$$I_{h_t} u(t) := \sum_{\ell=0}^{N_t} u(t_\ell) \varphi_\ell(t) \quad (2.27)$$

for $t \in [0,T]$, which is uniformly bounded with respect to the mesh size h_t as a mapping $I_{h_t}: H^1(0,T) \rightarrow S_{h_t}^1(0,T)$, i.e.

$$\forall u \in H^1(0,T): \quad \|I_{h_t} u\|_{H^1(0,T)} \leq C \|u\|_{H^1(0,T)}$$

with a constant $C > 0$ independent of h_t , see [51, Proposition 1.4, page 6], and in addition, it holds true that

$$\forall u \in H^1(0,T): \quad \|\partial_t I_{h_t} u\|_{L^2(0,T)} \leq \|\partial_t u\|_{L^2(0,T)}, \quad (2.28)$$

see [51, Proof of Proposition 1.4, page 6]. For $u \in H_0^1(0, T) \cap H^2(0, T)$, recall the standard error estimates for the piecewise linear interpolant

$$\|u - I_{h_t} u\|_{L^2(0, T)} \leq \frac{1}{\sqrt{8}} h_t \|\partial_t(u - I_{h_t} u)\|_{L^2(0, T)}, \quad (2.29)$$

$$\|u - I_{h_t} u\|_{H_0^1(0, T)} = \|\partial_t(u - I_{h_t} u)\|_{L^2(0, T)} \leq \frac{1}{\sqrt{3}} h_t \|\partial_{tt} u\|_{L^2(0, T)} \leq \frac{1}{\sqrt{3}} h_t \|\partial_t u\|_{H^1(0, T)}, \quad (2.30)$$

and therefore

$$\|u - I_{h_t} u\|_{L^2(0, T)} \leq \frac{1}{\sqrt{24}} h_t^2 \|\partial_{tt} u\|_{L^2(0, T)} \leq \frac{1}{\sqrt{24}} h_t^2 \|\partial_t u\|_{H^1(0, T)}. \quad (2.31)$$

An interpolation argument between (2.30) and (2.31) yields for $u \in H_0^1(0, T) \cap H^2(0, T)$

$$\|u - I_{h_t} u\|_{H_0^{1/2}(0, T)} \leq C h_t^{3/2} \|\partial_t u\|_{H^1(0, T)}, \quad (2.32)$$

where the constant $C > 0$ is independent of h_t , but dependent on the norm equivalence constants concerning $\|\cdot\|_{H_0^{1/2}(0, T)}$, see Theorem 3.4.2 and (3.33).

For a given function $u \in H_0^1(0, T)$, the H_0^1 projection $Q_{h_t}^1 u \in S_{h_t, 0}^1(0, T)$ is defined by

$$\langle \partial_t Q_{h_t}^1 u, \partial_t v_{h_t} \rangle_{L^2(0, T)} = \langle \partial_t u, \partial_t v_{h_t} \rangle_{L^2(0, T)} \quad (2.33)$$

for all $v_{h_t} \in S_{h_t, 0}^1(0, T)$, satisfying the stability estimate

$$\|\partial_t Q_{h_t}^1 u\|_{L^2(0, T)} \leq \|\partial_t u\|_{L^2(0, T)}.$$

In addition, for $s \in [0, 1]$, the standard error estimate

$$\|u - Q_{h_t}^1 u\|_{L^2(0, T)} \leq c h_t^{1+s} \|u\|_{H^{1+s}(0, T)} \quad (2.34)$$

holds true for $u \in H_0^1(0, T) \cap H^{1+s}(0, T)$ with a constant $c > 0$.

Next, time stepping schemes are introduced. For a given positive integer $M_x \in \mathbb{N}$, consider a first-order ordinary differential equation

$$\frac{d\mathbf{u}}{dt}(t) = \underline{F}(t, \mathbf{u}(t)) \quad \text{for } t \in [0, T],$$

where $\mathbf{u}(0) = \mathbf{u}_0 \in \mathbb{R}^{M_x}$ and $\underline{F}: [0, T] \times \mathbb{R}^{M_x} \rightarrow \mathbb{R}^{M_x}$ are the imposed initial condition and right-hand side. The right-hand side \underline{F} is assumed to be sufficiently smooth and Lipschitz continuous with respect to the second argument, i.e.

$$|\underline{F}(t, \mathbf{v}_1) - \underline{F}(t, \mathbf{v}_2)| \leq C_L |\mathbf{v}_1 - \mathbf{v}_2| \quad \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{M_x}, t \in [0, T]$$

with a Lipschitz constant $C_L > 0$. To approximate the function $\underline{u}: [0, T] \rightarrow \mathbb{R}^{M_x}$, time stepping methods are considered. These lead to approximations

$$\underline{u}(t_\ell) \approx \underline{U}^\ell \in \mathbb{R}^{M_x}$$

in each time step t_ℓ for $\ell = 0, \dots, N_t$. Therefore, the θ -method for $\theta \in [0, 1]$ is defined as

$$\underline{U}^{\ell+1} - \underline{U}^\ell = h_{t,\ell} \left((1 - \theta) \underline{F}(t_\ell, \underline{U}^\ell) + \theta \underline{F}(t_{\ell+1}, \underline{U}^{\ell+1}) \right) \quad (2.35)$$

for $\ell = 0, \dots, N_t$ with $\underline{U}^0 := \underline{u}_0 \in \mathbb{R}^{M_x}$. The θ -method simplifies to the explicit Euler method for $\theta = 0$ and to the implicit Euler method for $\theta = 1$. For $\theta = 1/2$, the Crank-Nicolson method is obtained. The explicit and implicit Euler method converge with order h_t , whereas the Crank-Nicolson method converges with order h_t^2 . Furthermore, the implicit Euler and the Crank-Nicolson methods are A-stable. On the other hand, the explicit Euler method is not A-stable. See [93, Section 7.4] and [75] for more details and proofs.

2.7 Discretisations in Space

Let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0, L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. For this situation, different discretisations in space are introduced as follows. The spatial domain Ω is decomposed as

$$\overline{\Omega} = \bigcup_{\ell=1}^{N_x} \overline{\omega}_\ell$$

with N_x spatial elements $\omega_\ell \subset \mathbb{R}^d$. The sequence $(\mathcal{T}_N)_N$ of decompositions is assumed to be admissible, shape regular and globally quasi-uniform. Here, the spatial elements ω_ℓ are intervals for $d = 1$, triangles or quadrilaterals for $d = 2$ and tetrahedra or hexahedra for $d = 3$. The local mesh sizes are given as

$$h_{x,\ell} = \left(\int_{\omega_\ell} dx \right)^{1/d} \quad \text{for } \ell = 1, \dots, N_x$$

and $h_x = \max_{\ell=1, \dots, N_x} h_{x,\ell}$ is the global mesh size. Furthermore, \tilde{M}_x is the number of vertices $\{x_i\}_{i=1}^{\tilde{M}_x}$ of the decomposition. The space

$$V_{h_x}(\Omega) = \text{span}\{\psi_i\}_{i=1}^{\tilde{M}_x} \subset H^1(\Omega)$$

is the space of piecewise linear, continuous functions $S_{h_x}^1(\Omega)$ on intervals ($d = 1$), triangles ($d = 2$), tetrahedra ($d = 3$), or $V_{h_x}(\Omega)$ is the space of piecewise linear/bilinear/trilinear, continuous functions $\mathcal{Q}_{h_x}^1(\Omega)$ on intervals ($d = 1$), quadrilaterals ($d = 2$), hexahedra ($d = 3$),

where the functions ψ_i are the usual nodal basis functions satisfying $\psi_i(x_k) = \delta_{ik}$ for $i, k = 1, \dots, \tilde{M}_x$. Recall that $S_{h_x}^1(0, L) = Q_{h_x}^1(0, L)$ on intervals. In addition, the subspace $V_{h_x,0}(\Omega) \subset V_{h_x}(\Omega)$ satisfies the homogeneous Dirichlet boundary condition, i.e.

$$V_{h_x,0}(\Omega) = V_{h_x}(\Omega) \cap H_0^1(\Omega).$$

After an ordering of the vertices $\{x_i\}_{i=1}^{\tilde{M}_x}$ in interior vertices $\{x_i\}_{i=1}^{M_x} \subset \Omega$ and boundary vertices $\{x_i\}_{i=M_x+1}^{\tilde{M}_x} \subset \partial\Omega$, this $H_0^1(\Omega)$ conforming subspace is written as

$$V_{h_x,0}(\Omega) = \text{span}\{\psi_i\}_{i=1}^{M_x}. \quad (2.36)$$

A function $U_{h_x} \in V_{h_x,0}(\Omega)$ admits the representation

$$U_{h_x}(x) = \sum_{i=1}^{M_x} U_i \psi_i(x)$$

for $x \in \bar{\Omega}$. In the remainder of this work, $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote mass and stiffness matrices defined via

$$M_{h_x}[i, j] = \langle \psi_j, \psi_i \rangle_{L^2(\Omega)} \quad (2.37)$$

for $i, j = 1, \dots, M_x$, and

$$A_{h_x}[i, j] = \langle \nabla_x \psi_j, \nabla_x \psi_i \rangle_{L^2(\Omega)} \quad (2.38)$$

for $i, j = 1, \dots, M_x$. The L^2 projection

$$Q_{h_x} : L^2(\Omega) \rightarrow V_{h_x,0}(\Omega)$$

on the piecewise linear, continuous functions, satisfying homogeneous Dirichlet boundary conditions, is given as the solution of the variational formulation to find $Q_{h_x}u \in V_{h_x,0}(\Omega)$ such that

$$\langle Q_{h_x}u, v_{h_x} \rangle_{L^2(\Omega)} = \langle u, v_{h_x} \rangle_{L^2(\Omega)} \quad (2.39)$$

for all $v_{h_x} \in V_{h_x,0}(\Omega)$, satisfying the stability estimate

$$\|Q_{h_x}u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)},$$

where $u \in L^2(\Omega)$ is a given function.

2.8 Discretisations in Space and Time

As in Section 2.7, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0, L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. Hence, the space-time cylinder

$Q = \Omega \times (0, T) \subset \mathbb{R}^{d+1}$ is polygonal for $d = 1$, or polyhedral for $d = 2$, or polychoral for $d = 3$. For this situation, different discretisations in space and time are introduced as follows.

First, consider a sequence $(\mathcal{T}_N)_N$ of admissible, shape regular and globally quasi-uniform decompositions

$$\bar{Q} = \overline{\mathcal{T}_N} = \bigcup_{\ell=1}^N \bar{q}_\ell$$

with N space-time elements $q_\ell \subset \mathbb{R}^{d+1}$, where q_ℓ is a triangle for $d = 1$ or a tetrahedron for $d = 2$ or a pentatope for $d = 3$, see [117, 142]. In addition, \tilde{M} is the number of vertices $\{(x_i, t_i)\}_{i=1}^{\tilde{M}}$ of the decomposition and $h = \max_{\ell=1, \dots, N} h_\ell$ is the global mesh size, where the local mesh sizes are given by

$$h_\ell = \left(\int_{q_\ell} dx dt \right)^{1/(d+1)} \quad \text{for } \ell = 1, \dots, N.$$

The space

$$S_h^1(Q) = \text{span}\{\psi_i\}_{i=1}^{\tilde{M}} \subset H^1(Q)$$

is the space of piecewise linear, continuous functions on triangles ($d = 1$), tetrahedra ($d = 2$) or pentatopes ($d = 3$), where $\{\psi_i\}_{i=1}^{\tilde{M}}$ is the nodal basis, i.e. $\psi_i(x_k, t_k) = \delta_{ki}$ for $k, i = 1, \dots, \tilde{M}$. A function $u_h \in S_h^1(Q)$ admits the representation

$$u_h(x, t) = \sum_{i=1}^{\tilde{M}} u_i \psi_i(x, t) \quad \text{for } (x, t) \in \bar{Q}.$$

For the space $S_h^1(Q)$, the space-time interpolation operator

$$I_h: C(\bar{Q}) \rightarrow S_h^1(Q)$$

is defined by

$$I_h u(x, t) = \sum_{i=1}^{\tilde{M}} u(x_i, t_i) \psi_i(x, t) \quad (2.40)$$

for $(x, t) \in \bar{Q}$.

Second, consider for a tensor-product ansatz a sequence $(\mathcal{T}_N)_N$ of admissible decompositions

$$\bar{Q} = \overline{\mathcal{T}_N} = \bar{\Omega} \times [0, T] = \left(\bigcup_{i=1}^{N_x} \bar{\omega}_i \right) \times \left(\bigcup_{\ell=1}^{N_t} \bar{\tau}_\ell \right) \quad (2.41)$$

with $N = N_x \cdot N_t$ space-time elements, where the time intervals $\tau_\ell = (t_{\ell-1}, t_\ell)$ are defined via the decomposition

$$0 = t_0 < t_1 < t_2 < \dots < t_{N_t-1} < t_{N_t} = T$$

of the time interval $(0, T)$ and where the spatial domain Ω is decomposed as

$$\bar{\Omega} = \bigcup_{i=1}^{N_x} \bar{\omega}_i.$$

Here, the spatial elements $\omega_i \subset \mathbb{R}^d$ are intervals for $d = 1$, triangles or quadrilaterals for $d = 2$ and tetrahedra or hexahedra for $d = 3$. The local mesh sizes are $h_{t,\ell} = t_\ell - t_{\ell-1}$ for $\ell = 1, \dots, N_t$ and

$$h_{x,i} = \left(\int_{\omega_i} dx \right)^{1/d} \quad \text{for } i = 1, \dots, N_x.$$

Furthermore, the global mesh size is given as $h = \max\{h_x, h_t\}$ with $h_t = \max_{\ell=1, \dots, N_t} h_{t,\ell}$ and $h_x = \max_{i=1, \dots, N_x} h_{x,i}$. It is always assumed that the sequence $(\mathcal{T}_N)_N$ of decompositions is shape regular and globally quasi-uniform. Next, consider the finite element space

$$Q_h^1(Q) := V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0, T) \quad (2.42)$$

of piecewise multilinear, continuous functions, where $V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ is the space of piecewise linear, continuous functions $S_{h_x,0}^1(\Omega)$ on intervals ($d = 1$), triangles ($d = 2$), tetrahedra ($d = 3$), or $V_{h_x,0}(\Omega)$ is the space of piecewise linear/bilinear/trilinear, continuous functions $Q_{h_x,0}^1(\Omega)$ on intervals ($d = 1$), quadrilaterals ($d = 2$), hexahedra ($d = 3$), fulfilling in both cases the homogeneous Dirichlet boundary conditions on the lateral boundary Σ , see (2.36). Recall that $S_{h_x,0}^1(0, L) = Q_{h_x,0}^1(0, L)$ on intervals. A function $u_h \in Q_h^1(Q)$ admits the representation

$$u_h(x, t) = \sum_{\ell=0}^{N_t} \underbrace{\sum_{j=1}^{M_x} u_j^\ell \psi_j(x)}_{=: U_{h_x,\ell}(x)} \varphi_\ell(t) = \sum_{\ell=0}^{N_t} U_{h_x,\ell}(x) \varphi_\ell(t) \quad \text{for } (x, t) \in \bar{Q}, \quad (2.43)$$

where φ_ℓ is a piecewise linear, continuous nodal basis function with respect to time and ψ_j is a piecewise linear/bilinear/trilinear, continuous nodal basis function with respect to space with $M_x := \dim V_{h_x,0}(\Omega)$. Furthermore, it holds true that $U_{h_x,\ell} \in V_{h_x,0}(\Omega)$ for $\ell = 0, \dots, N_t$.

The extended time interpolation operator

$$I_{h_t} : C([0, T]; L^2(\Omega)) \rightarrow L^2(\Omega) \otimes S_{h_t}^1(0, T)$$

is defined by

$$I_h u(x, t) = \sum_{\ell=0}^{N_t} u(x, t_\ell) \varphi_\ell(t) \quad (2.44)$$

for $(x, t) \in \bar{Q}$.

To derive space-time error estimates, different space-time projections are needed. As a first space-time projection, the $H_{0;0}^{1,1}(Q)$ projection

$$\mathcal{Q}_h^1: H_{0;0}^{1,1}(Q) \rightarrow \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$$

is introduced as the solution of the variational formulation to find $\mathcal{Q}_h^1 v \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ such that

$$\langle \partial_t \mathcal{Q}_h^1 v, \partial_t v_h \rangle_{L^2(Q)} + \langle \nabla_x \mathcal{Q}_h^1 v, \nabla_x v_h \rangle_{L^2(Q)} = \langle \partial_t v, \partial_t v_h \rangle_{L^2(Q)} + \langle \nabla_x v, \nabla_x v_h \rangle_{L^2(Q)} \quad (2.45)$$

for all $v_h \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$, where $v \in H_{0;0}^{1,1}(Q)$ is given. The stability estimate

$$\forall v \in H_{0;0}^{1,1}(Q): \quad |\mathcal{Q}_h^1 v|_{H^1(Q)} \leq |v|_{H^1(Q)}$$

and if Ω is sufficiently regular, for $s \in [0, 1]$, the standard error estimate

$$\|v - \mathcal{Q}_h^1 v\|_{L^2(Q)} \leq c h^{1+s} \|v\|_{H^{1+s}(Q)} \quad (2.46)$$

hold true for $v \in H_{0;0}^{1,1}(Q) \cap H^{1+s}(Q)$ with a constant $c > 0$.

As a second space-time projection, the H_0^1 - H_0^1 projection $\mathcal{Q}_{h_x}^1 \mathcal{Q}_{h_t}^1 v \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ is introduced analogously to [16, Section 2], where the function $v \in H_{0;0}^{1,1}(Q)$ is sufficiently smooth. First, for a given function $v \in L^2(0, T; H_0^1(\Omega))$, the extended H_0^1 projection $\mathcal{Q}_{h_x}^1 v \in V_{h_x,0}(\Omega) \otimes L^2(0, T)$ is defined by

$$\langle \nabla_x \mathcal{Q}_{h_x}^1 v, \nabla_x v_{h_x} \rangle_{L^2(Q)} = \langle \nabla_x v, \nabla_x v_{h_x} \rangle_{L^2(Q)} \quad (2.47)$$

for all $v_{h_x} \in V_{h_x,0}(\Omega) \otimes L^2(0, T)$. Note that $V_{h_x,0}(\Omega) \otimes L^2(0, T)$ is, as a tensor-product of the separable Hilbert spaces $(V_{h_x,0}(\Omega), \langle \nabla_x(\cdot), \nabla_x(\cdot) \rangle_{L^2(\Omega)})$ and $(L^2(0, T), \langle \cdot, \cdot \rangle_{L^2(0, T)})$, again a Hilbert space, where the inner product is given by $\langle \nabla_x(\cdot), \nabla_x(\cdot) \rangle_{L^2(Q)}$. Hence, by the Lax-Milgram Theorem, it follows the well-posedness of the extended H_0^1 projection

$$\mathcal{Q}_{h_x}^1: L^2(0, T; H_0^1(\Omega)) \rightarrow V_{h_x,0}(\Omega) \otimes L^2(0, T),$$

satisfying the stability estimate

$$\|\nabla_x \mathcal{Q}_{h_x}^1 v\|_{L^2(Q)} \leq \|\nabla_x v\|_{L^2(Q)}.$$

Furthermore, for $s \in [0, 1]$, the standard error estimate

$$\|v - Q_{h_x}^1 v\|_{L^2(Q)} \leq c h_x^{1+s} \|v\|_{L^2(0,T;H^{1+s}(\Omega))} \quad (2.48)$$

holds true for $v \in L^2(0, T; H_0^1(\Omega) \cap H^{1+s}(\Omega))$ with a constant $c > 0$, if Ω is sufficiently regular. Second, for a given function $v \in H_0^1(0, T; L^2(\Omega))$, fulfilling the homogeneous initial condition, the extended H_0^1 projection $Q_{h_t}^1 v \in L^2(\Omega) \otimes S_{h_t,0}^1(0, T)$ is defined by

$$\langle \partial_t Q_{h_t}^1 v, \partial_t v_{h_t} \rangle_{L^2(Q)} = \langle \partial_t v, \partial_t v_{h_t} \rangle_{L^2(Q)} \quad (2.49)$$

for all $v_{h_t} \in L^2(\Omega) \otimes S_{h_t,0}^1(0, T)$, satisfying the stability estimate

$$\|\partial_t Q_{h_t}^1 v\|_{L^2(Q)} \leq \|\partial_t v\|_{L^2(Q)},$$

where the well-posedness of the extended H_0^1 projection

$$Q_{h_t}^1 : H_0^1(0, T; L^2(\Omega)) \rightarrow L^2(\Omega) \otimes S_{h_t,0}^1(0, T)$$

is shown analogously as for $Q_{h_x}^1$. In addition, for $s \in [0, 1]$, the standard error estimate

$$\|v - Q_{h_t}^1 v\|_{L^2(Q)} \leq c h_t^{1+s} \|v\|_{H^{1+s}(0,T;L^2(\Omega))} \quad (2.50)$$

holds true for $v \in H_0^1(0, T; L^2(\Omega)) \cap H^{1+s}(0, T; L^2(\Omega))$ with a constant $c > 0$. Testing the variational formulation (2.49) with the test function

$$v_{h_t}(x, t) = \begin{cases} z(x) \cdot t & \text{for } (x, t) \in \Omega \times [0, t_\ell], \\ z(x) \cdot t_\ell & \text{for } (x, t) \in \Omega \times [t_\ell, T] \end{cases}$$

with an arbitrary function $z \in L^2(\Omega)$ for $\ell \in \{1, \dots, N_t\}$ yields

$$\int_{\Omega} z(x) Q_{h_t}^1 v(x, t_\ell) dx = \int_{\Omega} z(x) \int_0^{t_\ell} \partial_t Q_{h_t}^1 v(x, t) dt dx = \int_{\Omega} z(x) \int_0^{t_\ell} \partial_t v(x, t) dt dx = \int_{\Omega} z(x) v(x, t_\ell) dx$$

and hence, with the fundamental lemma of calculus of variations, there follows the interpolation property

$$Q_{h_t}^1 v(x, t_\ell) = v(x, t_\ell).$$

In other words, it holds true that

$$Q_{h_t}^1 = I_{h_t}$$

for functions in $H_0^1(0, T; L^2(\Omega))$, see (2.44).

Lemma 2.8.1. *The following properties of the projection operators $Q_{h_x}^1$ and $Q_{h_t}^1$ are true:*

1. For a function $v \in H^1(0, T; H_0^1(\Omega))$, it holds true that $Q_{h_x}^1 v \in V_{h_x, 0}(\Omega) \otimes H^1(0, T)$. If, in addition, the given function v satisfies $v \in H_0^1(0, T; H_0^1(\Omega))$, then it follows that $Q_{h_x}^1 v \in V_{h_x, 0}(\Omega) \otimes H_0^1(0, T)$.
2. For a function $v \in H_0^1(0, T; H^1(\Omega))$, it holds true that $Q_{h_t}^1 v \in H^1(\Omega) \otimes S_{h_t, 0}^1(0, T)$. If, in addition, the given function v satisfies $v \in H_0^1(0, T; H_0^1(\Omega))$, then it follows that $Q_{h_t}^1 v \in H_0^1(\Omega) \otimes S_{h_t, 0}^1(0, T)$.

Proof. For the first part, consider a representation for $(x, t) \in Q$

$$Q_{h_x}^1 v(x, t) = \sum_{j=1}^{M_x} \tilde{V}_j(t) \tilde{\Psi}_j(x), \quad (2.51)$$

where $\{\tilde{\Psi}_j\}_{j=1}^{M_x}$ is an orthonormal basis of $V_{h_x, 0}(\Omega)$ with respect to $\langle \nabla_x(\cdot), \nabla_x(\cdot) \rangle_{L^2(\Omega)}$ and $\tilde{V}_j \in L^2(0, T)$. To show that a weak derivative of \tilde{V}_j exists, fix an index $j \in \{1, \dots, M_x\}$. The definition of the extended H_0^1 projection $Q_{h_x}^1$ in (2.47) for $v_{h_x}(x, t) = \tilde{\Psi}_j(x)z(t)$ gives

$$\int_0^T \tilde{V}_j(t) z(t) dt = \int_0^T \langle \nabla_x Q_{h_x}^1 v(\cdot, t), \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} z(t) dt = \int_0^T \langle \nabla_x v(\cdot, t), \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} z(t) dt \quad (2.52)$$

for all $z \in C_0^\infty(0, T)$. So, the fundamental lemma of calculus of variations for (2.52) yields

$$\tilde{V}_j(t) = \langle \nabla_x v(\cdot, t), \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} \quad (2.53)$$

for $t \in (0, T)$. For $z = \partial_t \tilde{z}$ in (2.52), it follows that

$$\int_0^T \tilde{V}_j(t) \partial_t \tilde{z}(t) dt = \int_0^T \langle \nabla_x v(\cdot, t), \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} \partial_t \tilde{z}(t) dt = - \int_0^T \langle \partial_t \nabla_x v(\cdot, t), \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} \tilde{z}(t) dt$$

for all $\tilde{z} \in C_0^\infty(0, T)$, i.e. $\partial_t \tilde{V}_j(t) = \langle \partial_t \nabla_x v(\cdot, t), \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)}$. Furthermore, with the Cauchy-Schwarz inequality, it holds true that

$$\begin{aligned} \|\partial_t \tilde{V}_j\|_{L^2(0, T)}^2 &= \int_0^T \left(\langle \partial_t \nabla_x v(\cdot, t), \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} \right)^2 dt \\ &\leq \int_0^T \|\partial_t \nabla_x v(\cdot, t)\|_{L^2(\Omega)}^2 \|\nabla_x \tilde{\Psi}_j\|_{L^2(\Omega)}^2 dt = \|\partial_t \nabla_x v\|_{L^2(Q)}^2 \|\nabla_x \tilde{\Psi}_j\|_{L^2(\Omega)}^2 < \infty \end{aligned}$$

and so, the first assertion. If, in addition, $v(\cdot, 0) = 0$ in $H_0^1(\Omega)$, then the continuity of $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ and of the trace operator yields

$$\tilde{V}_j(t)|_{t=0} = \langle \nabla_x v(\cdot, t)|_{t=0}, \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} = 0.$$

The second part follows in an analogous way. \square

The next lemma shows that $Q_h^1 Q_{h_x}^1 v \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ is well-defined under regularity assumptions for v and that the operators in space and time commute, where the proof is analogous to [16, Lemma 2.1, page 261].

Lemma 2.8.2. *For a given function $v \in H_{0;0}^{1,1}(Q)$ with the regularity $\partial_t v \in L^2(0, T; H_0^1(\Omega))$ and $\partial_{x_m} v \in H_0^1(0, T; L^2(\Omega))$ for $m = 1, \dots, d$,*

1. *the relation $\partial_t Q_h^1 v = Q_{h_x}^1 \partial_t v \in V_{h_x,0}(\Omega) \otimes L^2(0, T)$,*
2. *the relation $\partial_{x_m} Q_{h_t}^1 v = Q_{h_t}^1 \partial_{x_m} v \in L^2(\Omega) \otimes S_{h_t,0}^1(0, T)$ for $m = 1, \dots, d$ and*
3. *the relation $Q_{h_t}^1 Q_{h_x}^1 v = Q_{h_x}^1 Q_{h_t}^1 v \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ hold true. In particular, the space-time projections $Q_{h_t}^1 Q_{h_x}^1 v$ and $Q_{h_x}^1 Q_{h_t}^1 v$ are well-defined.*

Furthermore, the error estimate

$$\|v - Q_{h_t}^1 Q_{h_x}^1 v\|_{L^2(Q)} \leq \|v - Q_{h_t}^1 v\|_{L^2(Q)} + \|v - Q_{h_x}^1 v\|_{L^2(Q)} + c h_x h_t \|\partial_t \nabla_x v\|_{L^2(Q)}$$

with a constant $c > 0$ is valid.

Proof. For the proof of the first relation, recall that

$$\partial_t Q_{h_x}^1 v \in V_{h_x,0}(\Omega) \otimes L^2(0, T)$$

by Lemma 2.8.1. Consider (2.47) for $\partial_t v \in L^2(0, T; H_0^1(\Omega))$ and with integration by parts, it follows that

$$\begin{aligned} \langle \nabla_x Q_{h_x}^1 \partial_t v, \nabla_x v_{h_x} \rangle_{L^2(Q)} &= \langle \nabla_x \partial_t v, \nabla_x v_{h_x} \rangle_{L^2(Q)} = -\langle \nabla_x v, \nabla_x \partial_t v_{h_x} \rangle_{L^2(Q)} \\ &= -\langle \nabla_x Q_{h_x}^1 v, \nabla_x \partial_t v_{h_x} \rangle_{L^2(Q)} = \langle \nabla_x \partial_t Q_{h_x}^1 v, \nabla_x v_{h_x} \rangle_{L^2(Q)} \\ &= \langle \nabla_x Q_{h_x}^1 \partial_t Q_{h_x}^1 v, \nabla_x v_{h_x} \rangle_{L^2(Q)} \end{aligned}$$

for all $v_{h_x} \in V_{h_x,0}(\Omega) \otimes C_0^\infty(0, T)$. Because of the density of $C_0^\infty(0, T)$ in $L^2(0, T)$, it holds true that

$$Q_{h_x}^1 \partial_t v = Q_{h_x}^1 \partial_t Q_{h_x}^1 v$$

with $\partial_t Q_{h_x}^1 v \in V_{h_x,0}(\Omega) \otimes L^2(0, T)$. So, the first relation is proven.

The proof of the second relation is analogous to the proof of the first relation.

For the third relation, by Lemma 2.8.1, it holds true that

$$\mathcal{Q}_{h_t}^1 v = \sum_{\ell=1}^{N_t} V^\ell \phi_\ell \in H_0^1(\Omega) \otimes S_{h_t,0}^1(0,T) \subset H_0^1(0,T;H_0^1(\Omega))$$

with coefficients $V^\ell \in H_0^1(\Omega)$ and so,

$$\mathcal{Q}_{h_x}^1 \mathcal{Q}_{h_t}^1 v \in V_{h_x,0}(\Omega) \otimes H_0^1(0,T) \subset H_0^1(0,T;H_0^1(\Omega))$$

is well-defined. With the representations as in (2.51) and in (2.53) for $\mathcal{Q}_{h_t}^1 v$, it follows for $(x,t) \in \bar{Q}$ that

$$\begin{aligned} \mathcal{Q}_{h_x}^1 \mathcal{Q}_{h_t}^1 v(x,t) &= \sum_{j=1}^{M_x} \tilde{V}_j(t) \tilde{\Psi}_j(x) = \sum_{j=1}^{M_x} \langle \nabla_x \mathcal{Q}_{h_t}^1 v(\cdot, t), \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} \tilde{\Psi}_j(x) \\ &= \sum_{\ell=1}^{N_t} \sum_{j=1}^{M_x} \langle \nabla_x V^\ell, \nabla_x \tilde{\Psi}_j \rangle_{L^2(\Omega)} \tilde{\Psi}_j(x) \phi_\ell(t) \end{aligned}$$

and so, $\mathcal{Q}_{h_x}^1 \mathcal{Q}_{h_t}^1 v \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$. Analogously, $\mathcal{Q}_{h_t}^1 \mathcal{Q}_{h_x}^1 v \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ is well-defined. With the help of the first relation, the second relation and the definitions (2.47), (2.49), it holds true that

$$\begin{aligned} \sum_{m=1}^d \langle \partial_t \partial_{x_m} \mathcal{Q}_{h_t}^1 \mathcal{Q}_{h_x}^1 v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} &= \sum_{m=1}^d \langle \partial_t \mathcal{Q}_{h_t}^1 \partial_{x_m} \mathcal{Q}_{h_x}^1 v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} \\ &= \sum_{m=1}^d \langle \partial_t \partial_{x_m} \mathcal{Q}_{h_x}^1 v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} \\ &= \sum_{m=1}^d \langle \partial_t \partial_{x_m} v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} \end{aligned}$$

and analogously,

$$\sum_{m=1}^d \langle \partial_t \partial_{x_m} \mathcal{Q}_{h_x}^1 \mathcal{Q}_{h_t}^1 v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} = \sum_{m=1}^d \langle \partial_t \partial_{x_m} v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)}$$

for all $v_h \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$. Hence, also the third relation is true.

The error estimate follows with the triangle inequality, the first and second relation and standard error estimates for $\mathcal{Q}_{h_t}^1$ and $\mathcal{Q}_{h_x}^1$ from

$$\begin{aligned} \|v - \mathcal{Q}_{h_t}^1 \mathcal{Q}_{h_x}^1 v\|_{L^2(Q)} &\leq \|v - \mathcal{Q}_{h_t}^1 v\|_{L^2(Q)} + \|v - \mathcal{Q}_{h_x}^1 v\|_{L^2(Q)} \\ &\quad + \underbrace{\|(v - \mathcal{Q}_{h_x}^1 v) - \mathcal{Q}_{h_t}^1 (v - \mathcal{Q}_{h_x}^1 v)\|_{L^2(Q)}}_{\leq c_1 h_t \|\partial_t (v - \mathcal{Q}_{h_x}^1 v)\|_{L^2(Q)} \leq c_1 c_2 h_t h_x \|\partial_t \nabla_x v\|_{L^2(Q)}} \end{aligned}$$

with constants $c_1, c_2 > 0$ independent of h_t and h_x . \square

Next, for a function $v \in C([0, T]; L^2(\Omega))$, investigate $I_{h_x} Q_{h_x} v \in Q_h^1(Q) \cap H_{0;1}^1(Q)$. Therefore, for a given function $v \in L^2(Q)$, the extended L^2 projection $Q_{h_x} v \in V_{h_x,0}(\Omega) \otimes L^2(0, T)$ is defined by

$$\langle Q_{h_x} v, v_{h_x} \rangle_{L^2(Q)} = \langle v, v_{h_x} \rangle_{L^2(Q)} \quad (2.54)$$

for all $v_{h_x} \in V_{h_x,0}(\Omega) \otimes L^2(0, T)$, satisfying the stability estimate

$$\|Q_{h_x} v\|_{L^2(Q)} \leq \|v\|_{L^2(Q)}, \quad (2.55)$$

where the well-posedness of the extended L^2 projection

$$Q_{h_x} : L^2(Q) \rightarrow V_{h_x,0}(\Omega) \otimes L^2(0, T)$$

is analysed as for the projection $Q_{h_x}^1$ given in (2.47). Furthermore, for $s \in [0, 1]$, the standard error estimate

$$\|v - Q_{h_x} v\|_{L^2(Q)} \leq c h_x^{1+s} \|v\|_{L^2(0, T; H^{1+s}(\Omega))}$$

holds true for $v \in L^2(0, T; H_0^1(\Omega) \cap H^{1+s}(\Omega))$ with a constant $c > 0$. The following properties of the projection operator Q_{h_x} are true:

Lemma 2.8.3. *For a function $v \in C([0, T]; L^2(\Omega))$, the relation*

$$Q_{h_x} v \in V_{h_x,0}(\Omega) \otimes C[0, T]$$

holds true. In addition, for a function $v \in H^r(0, T; L^2(\Omega))$ with $r \geq 0$, it holds true that

$$Q_{h_x} v \in V_{h_x,0}(\Omega) \otimes H^r(0, T) \quad \text{and} \quad \|Q_{h_x} v\|_{H^r(0, T; L^2(\Omega))} \leq \|v\|_{H^r(0, T; L^2(\Omega))}.$$

In particular, for $r = k \in \mathbb{N}$, the relation $\partial_t^k Q_{h_x} v = Q_{h_x} \partial_t^k v$ is valid.

Proof. The proof is analogous to the proof of Lemma 2.8.1. More precisely, take an orthonormal basis $\{\tilde{\psi}_j\}_{j=1}^{M_x}$ of $V_{h_x,0}(\Omega)$ with respect to $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ and write

$$Q_{h_x} v(x, t) = \sum_{j=1}^{M_x} \tilde{V}_j(t) \tilde{\psi}_j(x) \quad (2.56)$$

for $(x, t) \in Q$ with $\tilde{V}_j \in L^2(0, T)$. For the first assertion, it remains to show that $\tilde{V}_j \in C[0, T]$. For that reason, the equation (2.54) gives for $v_{h_x}(x, t) = \tilde{\psi}_j(x) z(t)$

$$\int_0^T \tilde{V}_j(t) z(t) dt = \langle Q_{h_x} v, \tilde{\psi}_j z \rangle_{L^2(Q)} = \langle v, \tilde{\psi}_j z \rangle_{L^2(Q)} = \int_0^T \langle v(\cdot, t), \tilde{\psi}_j \rangle_{L^2(\Omega)} z(t) dt \quad (2.57)$$

for each $z \in L^2(\Omega)$ and hence, the fundamental lemma of calculus of variations yields

$$\tilde{V}_j(t) = \langle v(\cdot, t), \tilde{\psi}_j \rangle_{L^2(\Omega)} = \int_{\Omega} v(x, t) \tilde{\psi}_j(x) dx \quad (2.58)$$

for $t \in (0, T)$. Because for almost all $x \in \Omega$ and all $t \in [0, T]$, the estimate

$$|v(x, t) \tilde{\psi}_j(x)| \leq \max_{s \in [0, T]} |v(x, s)| \cdot \max_{\tilde{x} \in \bar{\Omega}} |\psi_j(\tilde{x})| =: g(x)$$

holds true with $g \in L^2(\Omega)$, the theorem of continuity for parameter integrals [47, Satz 5.6, page 147] gives $\tilde{V}_j \in C[0, T]$ and hence, the first assertion.

For the last statement, an interpolation argument is used, see [102, Théorème 5.1, page 32 in Chapitre 1]. For $r = 0$, the assertion is trivial. Therefore, let $v \in H^k(0, T; L^2(\Omega))$ be given for $k \in \mathbb{N}$, i.e. $r = k$. Because of the representation (2.56), it remains to prove that $\partial_t^k \tilde{V}_j \in L^2(0, T)$. For $z = \partial_t^k \tilde{z}$ in (2.57), it follows that

$$\int_0^T \tilde{V}_j(t) \partial_t^k \tilde{z}(t) dt = \int_0^T \langle v(\cdot, t), \tilde{\psi}_j \rangle_{L^2(\Omega)} \partial_t^k \tilde{z}(t) dt = (-1)^k \int_0^T \langle \partial_t^k v(\cdot, t), \tilde{\psi}_j \rangle_{L^2(\Omega)} \tilde{z}(t) dt$$

for all $\tilde{z} \in C_0^\infty(0, T)$, i.e. $\partial_t^k \tilde{V}_j(t) = \langle \partial_t^k v(\cdot, t), \tilde{\psi}_j \rangle_{L^2(\Omega)}$. Furthermore, with the Cauchy-Schwarz inequality, it holds true that

$$\begin{aligned} \left\| \partial_t^k \tilde{V}_j \right\|_{L^2(0, T)}^2 &= \int_0^T \left(\langle \partial_t^k v(\cdot, t), \tilde{\psi}_j \rangle_{L^2(\Omega)} \right)^2 dt \\ &\leq \int_0^T \left\| \partial_t^k v(\cdot, t) \right\|_{L^2(\Omega)}^2 \underbrace{\left\| \tilde{\psi}_j \right\|_{L^2(\Omega)}^2}_{=1} dt = \left\| \partial_t^k v \right\|_{L^2(\Omega)}^2 < \infty. \end{aligned}$$

The relation $\partial_t^k Q_{h_x} v = Q_{h_x} \partial_t^k v$ is proven analogously to the relations of Lemma 2.8.2, and so, the assertion for $r = k$, where for the stability $\|Q_{h_x} v\|_{H^k(0, T; L^2(\Omega))} \leq \|v\|_{H^k(0, T; L^2(\Omega))}$, the stability (2.55) is used for $\partial_t^l Q_{h_x} v = Q_{h_x} \partial_t^l v$, $l = 0, \dots, k$. For arbitrary $r > 0$, the statement follows by interpolation. \square

For a given function $v \in C([0, T]; L^2(\Omega))$, Lemma 2.8.3 ensures that $I_{h_t} Q_{h_x} v \in Q_h^1(Q)$, given by

$$I_{h_t} Q_{h_x} v(x, t) = \sum_{\ell=0}^{N_t} Q_{h_x} v(x, t_\ell) \varphi_\ell(t) = \sum_{\ell=0}^{N_t} \sum_{j=1}^{M_x} V_\ell^j \psi_j(x) \varphi_\ell(t) \quad (2.59)$$

for $(x, t) \in \bar{Q}$, is well-defined.

Lemma 2.8.4. For a given function $v \in C([0, T]; L^2(\Omega))$, it holds true that

$$I_{h_t} Q_{h_x} v = Q_{h_x} I_{h_t} v \in Q_h^1(Q).$$

Furthermore, for $v \in L^2(0, T; H^s(\Omega)) \cap H^r(0, T; L^2(\Omega))$ with $s \in [0, 2]$ and $r \in (1/2, 2]$, the error estimate

$$\|v - I_{h_t} Q_{h_x} v\|_{L^2(Q)} \leq c_1 h_x^s \|v\|_{L^2(0, T; H^s(\Omega))} + c_2 h_t^r \|v\|_{H^r(0, T; L^2(\Omega))}$$

with constants $c_1, c_2 > 0$ independent of h_x and h_t is valid.

Proof. Take an orthonormal basis $\{\tilde{\psi}_j\}_{j=1}^{M_x}$ of $V_{h_x, 0}(\Omega)$ with respect to $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ and write with (2.58)

$$Q_{h_x} v(x, t) = \sum_{j=1}^{M_x} \tilde{V}_j(t) \tilde{\psi}_j(x) = \sum_{j=1}^{M_x} \langle v(\cdot, t), \tilde{\psi}_j \rangle_{L^2(\Omega)} \tilde{\psi}_j(x) \quad \text{for } (x, t) \in \bar{Q},$$

where $\tilde{V}_j \in C[0, T]$, see Lemma 2.8.3. With this representation, it follows for $(x, t) \in \bar{Q}$ that

$$I_{h_t} Q_{h_x} v(x, t) = \sum_{\ell=0}^{N_t} \sum_{j=1}^{M_x} \langle v(\cdot, t_\ell), \tilde{\psi}_j \rangle_{L^2(\Omega)} \tilde{\psi}_j(x) \phi_\ell(t)$$

and

$$Q_{h_x} I_{h_t} v(x, t) = \sum_{j=1}^{M_x} \langle I_{h_t} v(\cdot, t), \tilde{\psi}_j \rangle_{L^2(\Omega)} \tilde{\psi}_j(x) = \sum_{\ell=0}^{N_t} \sum_{j=1}^{M_x} \langle v(\cdot, t_\ell), \tilde{\psi}_j \rangle_{L^2(\Omega)} \tilde{\psi}_j(x) \phi_\ell(t),$$

i.e. $I_{h_t} Q_{h_x} v = Q_{h_x} I_{h_t} v \in Q_h^1(Q)$.

The error estimate follows with the triangle inequality, standard error estimates for I_{h_t} and Q_{h_x} and Lemma 2.8.3 from

$$\begin{aligned} \|v - I_{h_t} Q_{h_x} v\|_{L^2(Q)} &\leq \|v - Q_{h_x} v\|_{L^2(Q)} + \|Q_{h_x} v - I_{h_t} Q_{h_x} v\|_{L^2(Q)} \\ &\leq c_1 h_x^s \|v\|_{L^2(0, T; H^s(\Omega))} + c_2 h_t^r \|Q_{h_x} v\|_{H^r(0, T; L^2(\Omega))} \\ &\leq c_1 h_x^s \|v\|_{L^2(0, T; H^s(\Omega))} + c_2 h_t^r \|v\|_{H^r(0, T; L^2(\Omega))} \end{aligned}$$

with constants $c_1, c_2 > 0$ independent of h_x and h_t . □

Last, for a given function $v \in L^2(Q)$, the extended L^2 projection $Q_{h_t}^0 v \in L^2(\Omega) \otimes S_{h_t}^0(0, T)$ on the space $L^2(\Omega) \otimes S_{h_t}^0(0, T)$ of piecewise constant functions with respect to time is defined by

$$\langle Q_{h_t}^0 v, v_{h_t} \rangle_{L^2(Q)} = \langle v, v_{h_t} \rangle_{L^2(Q)} \quad (2.60)$$

for all $v_h \in L^2(\Omega) \otimes S_{h_t}^0(0, T)$, satisfying the stability estimate

$$\|Q_{h_t}^0 v\|_{L^2(Q)} \leq \|v\|_{L^2(Q)},$$

where the well-posedness of the extended L^2 projection

$$Q_{h_t}^0: L^2(Q) \rightarrow L^2(\Omega) \otimes S_{h_t}^0(0, T)$$

is analysed as for the projection $Q_{h_x}^1$ given in (2.47).

Lemma 2.8.5. *For $v \in L^2(Q)$, the extended L^2 projection $Q_{h_t}^0: L^2(Q) \rightarrow L^2(\Omega) \otimes S_{h_t}^0(0, T)$, defined in (2.60), satisfies the representation*

$$Q_{h_t}^0 v(x, t) = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} v(x, s) ds, \quad (x, t) \in \Omega \times \tau_\ell,$$

for $\ell = 1, \dots, N_t$. In particular, for a given function $v_h \in Q_h^1(Q)$ with the representation, see (2.43),

$$v_h(x, t) = \sum_{\ell=0}^{N_t} V_{h_x, \ell}(x) \phi_\ell(t), \quad (x, t) \in Q,$$

the relation

$$Q_{h_t}^0 v_h(x, t) = \sum_{\ell=0}^{N_t} V_{h_x, \ell}(x) Q_{h_t}^0 \phi_\ell(t), \quad (x, t) \in Q, \quad (2.61)$$

holds true, where the extended L^2 projection $Q_{h_t}^0: L^2(Q) \rightarrow L^2(\Omega) \otimes S_{h_t}^0(0, T)$, defined in (2.60), occurs on the left-hand side and the L^2 projection $Q_{h_t}^0: L^2(0, T) \rightarrow S_{h_t}^0(0, T)$, defined in (2.26), occurs on the right-hand side.

Proof. The proof of the first representation is analogous to the proofs of Lemma 2.8.1 or Lemma 2.8.3.

The proof of the second representation follows from the first representation. \square

2.9 Variational Methods

Let X and Y be real Banach spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Furthermore, let $a(\cdot, \cdot): X \times Y \rightarrow \mathbb{R}$ be a given continuous bilinear form and let $F: Y \rightarrow \mathbb{R}$ be a given continuous linear form. Consider the abstract variational problem to find $u \in X$ such that

$$\forall v \in Y: \quad a(u, v) = F(v). \quad (2.62)$$

A continuous, linear operator $A: X \rightarrow Y'$ is associated with $a(\cdot, \cdot): X \times Y \rightarrow \mathbb{R}$ by setting

$$\langle Au, v \rangle_{Y' \times Y} := a(u, v) \quad \text{for all } u \in X, v \in Y,$$

where $\langle \cdot, \cdot \rangle_{Y' \times Y}$ is the duality pairing.

Theorem 2.9.1 (Nečas). *Let $(X, \|\cdot\|_X)$ be a real Banach space, $(Y, \|\cdot\|_Y)$ be a real, reflexive Banach space and $a(\cdot, \cdot): X \times Y \rightarrow \mathbb{R}$ be a continuous bilinear form. Then, the following statements are equivalent:*

1. *For every given continuous linear form $F: Y \rightarrow \mathbb{R}$, a unique solution $u \in X$ of (2.62) exists, satisfying*

$$\|u\|_X \leq \tilde{c}_s \|F\|_{Y'}$$

with a constant $\tilde{c}_s > 0$.

2. *The continuous, linear operator $A: X \rightarrow Y'$ associated with $a(\cdot, \cdot): X \times Y \rightarrow \mathbb{R}$ is an isomorphism, i.e. A is bijective and $A^{-1}: Y' \rightarrow X$ is continuous.*
3. *The continuous bilinear form $a(\cdot, \cdot): X \times Y \rightarrow \mathbb{R}$ satisfies the following conditions:*

- *condition (N1), the so-called inf-sup condition:*

$$\exists c_s > 0: \quad \inf_{0 \neq u \in X} \sup_{0 \neq v \in Y} \frac{|a(u, v)|}{\|u\|_X \|v\|_Y} \geq c_s.$$

- *condition (N2):*

$$\forall v \in Y \setminus \{0\}: \exists u \in X: \quad a(u, v) \neq 0.$$

Moreover, it holds true that $\tilde{c}_s = \frac{1}{c_s}$.

Proof. If X, Y are real Hilbert spaces, a proof is contained in [118, Théorème 3.1, page 318] or in [29, Satz 3.6, page 119]. For the general case, see [51, Theorem 2.6, page 85]. \square

Remark 2.9.2. *For complex Hilbert spaces X and Y , a continuous sesquilinear form $a(\cdot, \cdot): X \times Y \rightarrow \mathbb{C}$ and a continuous linear form $F: Y \rightarrow \mathbb{C}$, a proof is included in [131, Theorem 2.1.44, page 36].*

3 HEAT EQUATION

The main focus of this chapter is on space-time variational formulations and conforming discretisations for parabolic problems. First, a highly non-exhaustive list of references, second, an overview of the sections of this chapter, where for each section the relevant literature is cited, and third, an outlook for possible extensions are given. Here, the model problem for a parabolic partial differential equation is the homogeneous Dirichlet problem for the heat equation,

$$\left. \begin{aligned} \partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) & \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &= 0 & \text{for } (x,t) \in \Sigma = \Gamma \times [0,T], \\ u(x,0) &= u_0(x) & \text{for } x \in \Omega, \end{aligned} \right\} \quad (3.1)$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, $T > 0$ is a terminal time, u_0 is a given initial condition and f is a given right-hand side. To compute an approximate solution of the heat equation (3.1), different numerical schemes including different approaches of the underlying mathematical framework are available. On the one hand, some of them are repeated in this chapter, but on the other hand, powerful tools like semigroup theory as in [91, 123] on the continuous part or on the discretisation side, any kind of discontinuous Galerkin methods [48–50, 56, 78, 85, 116, 134, 135] or finite difference methods [65, 97, 147] or boundary element methods are not in the scope of this work. For boundary integral equations and boundary element methods for the heat equation, see [14, 35, 40, 120] and in addition, see [39, 76, 107, 109, 125, 129, 149]. Furthermore, all approaches where the heat equation (3.1) is reformulated as a first-order system also in the spatial variables are excluded in this work, see, e.g., [25] and the references therein. In addition, see also the approaches in [10, 11, 38, 99, 100, 112, 137, 154].

Outline of Chapter 3

The remainder of this chapter examines the heat equation (3.1) as follows:

In Section 3.1 a pointwise spatial variational formulation coming from a so-called Galerkin method [36, 97, 98, 102, 160, 162] and time stepping schemes [65, 74, 85, 93, 150] are cited, see also [9, 16, 17, 43, 54, 79, 108, 148, 151, 159]. In Section 3.2 a space-time variational formulation with ansatz spaces of Bochner type, analysed via the inf-sup theory and including a stable space-time discretisation, is formulated, see [51, 137, 142–144]. In Section 3.3 an anisotropic space-time variational formulation [35, 98, 102, 103], which is obtained by a transposition and interpolation argument, is replied, and also an example for an unstable numerical scheme, which is derived by the (natural) usage of conforming, piecewise linear,

continuous ansatz and test functions, is given. Nevertheless, this anisotropic formulation leads to boundary integral equations in anisotropic Sobolev spaces, where the single layer and hypersingular boundary integral operators are elliptic, see [35, 40, 120]. For the last and main Section 3.4, see [145], a motivation is given by transmission problems. For transmission problems of the heat equation a coupling of finite and boundary element methods is a natural choice, i.e. the finite element method is used for the interior problem as in (3.1) and the boundary element method is used for the corresponding exterior problem. For the finite element part, the anisotropic variational formulation of Section 3.3 is not well-suited on the discretisation level because it seems that a stable finite element method is not available. On the one hand, the Bochner type variational formulation of Section 3.2 gives a stable finite element method, but on the other hand, from an analysis point of view, the boundary integral equations [35, 40, 120] and the variational formulation of Bochner type in Section 3.2 do not fit, i.e. the resulting trace spaces of the Bochner spaces are different from the anisotropic boundary spaces in Section 3.3. In other words, it seems that the analysis of a coupling of the corresponding discretisations in [35, 40, 120] and Section 3.2 is very difficult. To overcome the problem of non-fitting spaces, either the boundary integral equations are treated in trace spaces of the Bochner spaces of Section 3.2, or a stabilised finite element method of the anisotropic spaces of Section 3.3 is introduced. The second approach is the motivation of Section 3.4, where the main result is a symmetric and elliptic variational formulation and hence, a symmetric Galerkin discretisation of the first-order time derivative, see [145]. In addition, see [37, 52, 101]. In Section 3.4 the key ingredient is a type of Hilbert transform, where its fast realisation is not in the scope of this thesis. However, Section 3.4 is completed with error estimates and some numerical examples, which emphasise the theoretical results.

Outlook for Possible Extensions of Chapter 3

The results of this chapter for the model problem (3.1) can be transferred to a more general parabolic equation

$$\left. \begin{aligned} \partial_t u(x, t) - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{i,j}(x, t) \frac{\partial u}{\partial x_j}(x, t) \right) \\ + \sum_{i=1}^d a_i(x, t) \frac{\partial u}{\partial x_i}(x, t) + a(x, t) u(x, t) &= f(x, t) & \text{for } (x, t) \in Q, \\ u(x, t) &= g(x, t) & \text{for } (x, t) \in \Sigma, \\ u(x, 0) &= u_0(x) & \text{for } x \in \Omega, \end{aligned} \right\} \quad (3.2)$$

where $a_{i,j}$, a_i , a are given coefficients, f is a given right-hand side, g is a given Dirichlet datum and u_0 is a given initial condition, which have to satisfy smoothness and boundedness conditions, see, e.g., the classical references [36, 97, 98, 102, 103, 160, 162]. Since the derivations and the proofs of the corresponding results have to be done with great care, they are left for future work, including precise assumptions on the involved functions and function spaces.

3.1 Variational Formulation in Space and Pointwise in Time

In this section, a short overview of a pointwise in time variational formulation is given. Furthermore, numerical examples for lowest order in space, i.e. piecewise linear, continuous ansatz functions, combined with lowest order time stepping are presented.

The pointwise in time variational formulation of (3.1) is given with the notations of Section 2 as follows:

Find $u \in L^2(0, T; H_0^1(\Omega))$ with $\partial_t u \in L^2(0, T; [H_0^1(\Omega)]')$ and $u(\cdot, 0) = u_0$ in $L^2(\Omega)$ such that

$$\langle \partial_t u(\cdot, t), v \rangle_\Omega + \langle \nabla_x u(\cdot, t), \nabla_x v \rangle_{L^2(\Omega)} = \langle f(\cdot, t), v \rangle_\Omega \quad (3.3)$$

for almost all $t \in (0, T)$ and all $v \in H_0^1(\Omega)$, where $f \in L^2(0, T; [H_0^1(\Omega)]')$ and $u_0 \in L^2(\Omega)$ are the given right-hand side and the given initial condition. Here, ∂_t is the distributional derivative on $(0, T)$, i.e. equality (3.3) means that

$$-\int_0^T \langle u(\cdot, t), v \rangle_\Omega \frac{d\varphi}{dt}(t) dt + \int_0^T \langle \nabla_x u(\cdot, t), \nabla_x v \rangle_{L^2(\Omega)} \varphi(t) dt = \int_0^T \langle f(\cdot, t), v \rangle_\Omega \varphi(t) dt$$

for all $\varphi \in C_0^\infty(0, T)$. The variational formulation in (3.3) is examined in many books, for example, [102, Exemple 1, Chapitre 3, page 263], [160, Beispiel 28.1, Kapitel IV, page 409], [162, Section 23.8, Chapter 23, page 426] or [36, Mathematical Example 1, Chapter XVIII, page 524]. In these books, the following existence and uniqueness result is proven.

Theorem 3.1.1. *For given $f \in L^2(0, T; [H_0^1(\Omega)]')$ and $u_0 \in L^2(\Omega)$, there exists a unique solution u of the variational formulation (3.3), satisfying*

$$u \in L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \quad \partial_t u \in L^2(0, T; [H_0^1(\Omega)]'),$$

i.e. $u \in W(Q)$, and the stability estimate

$$\|u\|_{W(Q)} = \sqrt{\|u\|_{L^2(0, T; H_0^1(\Omega))}^2 + \|\partial_t u\|_{L^2(0, T; [H_0^1(\Omega)]')}^2} \leq c \left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0, T; [H_0^1(\Omega)]')} \right)$$

with a constant $c > 0$.

Proof. See the books [36, 102, 160, 162] as mentioned above. \square

For a discretisation scheme, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0, L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. With the notations

of Section 2.8, consider a discretisation of a tensor-product type (2.41) with the finite-dimensional space $\mathcal{Q}_h^1(Q) = V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0, T)$, see (2.42). Therefore, introduce for $x \in \Omega$ and $\ell \in \{0, \dots, N_t\}$ the approximation

$$U_{h_x,\ell}(x) := \sum_{i=1}^{M_x} U_i^\ell \psi_i(x) \approx u(x, t_\ell),$$

where $U_i^\ell \in \mathbb{R}$ are the unknown coefficients of the functions $U_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0, \dots, N_t\}$. Furthermore, set for $(x, t) \in Q$

$$u_h(x, t) := \sum_{\ell=0}^{N_t} \sum_{i=1}^{M_x} U_i^\ell \psi_i(x) \phi_\ell(t) = \sum_{\ell=0}^{N_t} U_{h_x,\ell}(x) \phi_\ell(t) \approx u(x, t), \quad (3.4)$$

i.e. $u_h \in \mathcal{Q}_h^1(Q)$.

For the pointwise in time variational formulation (3.3), a conforming discretisation in space with $V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ in combination with a θ -method (2.35) for $\theta = 1$ leads to the so-called implicit Euler Galerkin method to find $U_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0, \dots, N_t\}$ such that

$$U_{h_x,0} = \mathcal{Q}_{h_x} u_0$$

and for $\ell = 1, \dots, N_t$

$$\frac{1}{h_{t,\ell}} \langle U_{h_x,\ell} - U_{h_x,\ell-1}, v_{h_x} \rangle_{L^2(\Omega)} + \langle \nabla_x U_{h_x,\ell}, \nabla_x v_{h_x} \rangle_{L^2(\Omega)} = \frac{1}{h_{t,\ell}} \left\langle \int_{\tau_\ell}^{\tau_{\ell+1}} f(\cdot, s) ds, v_{h_x} \right\rangle_{\Omega} \quad (3.5)$$

for all $v_{h_x} \in V_{h_x,0}(\Omega)$, where $\mathcal{Q}_{h_x} : L^2(\Omega) \rightarrow V_{h_x,0}(\Omega)$ denotes the L^2 projection (2.39). This method is given in [74, (2.10), page 684] or in [79, (3.5), page 508] and differs from the methods [150, (1.47), page 16] or [65, (1.34), page 334] only in the right-hand side. The implicit Euler Galerkin method (3.5) is equivalent to the linear systems

$$M_{h_x} \underline{U}^0 = \underline{u}_0$$

and

$$(M_{h_x} + h_{t,\ell} A_{h_x}) \underline{U}^\ell = M_{h_x} \underline{U}^{\ell-1} + \underline{F}^\ell \quad (3.6)$$

for all $\ell = 1, \dots, N_t$, where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the mass matrix (2.37), $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the stiffness matrix (2.38) and the vectors $\underline{u}_0, \underline{F}^\ell \in \mathbb{R}^{M_x}$ are defined by

$$\underline{u}_0[i] := \langle u_0, \psi_i \rangle_{L^2(\Omega)}, \quad \underline{F}^\ell[i] := \left\langle \int_{\tau_\ell}^{\tau_{\ell+1}} f(\cdot, s) ds, \psi_i \right\rangle_{\Omega} \quad (3.7)$$

for $i = 1, \dots, M_x$ with the nodal basis functions ψ_i satisfying $V_{h_x,0}(\Omega) = \text{span}\{\psi_i\}_{i=1}^{M_x}$, see (2.36). The matrix $M_{h_x} + h_{t,\ell} A_{h_x}$ is positive definite and hence, the linear systems (3.6) are

uniquely solvable for all $\ell = 1, \dots, N_t$. Stability of the numerical scheme (3.5) holds true without any CFL condition because the implicit Euler method is unconditionally stable, see Section 2.6.

As a second discretisation method in time, the Crank-Nicolson method is considered, which follows from the θ -method (2.35) for $\theta = 1/2$. Hence, using the Crank-Nicolson method combined with a conforming discretisation in space with $V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for the pointwise in time variational formulation (3.3), there follows the so-called Crank-Nicolson Galerkin method to find $U_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0, \dots, N_t\}$ such that

$$U_{h_x,0} = Q_{h_x} u_0$$

and for $\ell = 1, \dots, N_t$

$$\begin{aligned} \frac{1}{h_{t,\ell}} \langle U_{h_x,\ell} - U_{h_x,\ell-1}, v_{h_x} \rangle_{L^2(\Omega)} + \frac{1}{2} \langle \nabla_x U_{h_x,\ell} + \nabla_x U_{h_x,\ell-1}, \nabla_x v_{h_x} \rangle_{L^2(\Omega)} \\ = \frac{1}{h_{t,\ell}} \left\langle \int_{\tau_\ell} f(\cdot, s) ds, v_{h_x} \right\rangle_{\Omega} \end{aligned} \quad (3.8)$$

for all $v_{h_x} \in V_{h_x,0}(\Omega)$, where $Q_{h_x}: L^2(\Omega) \rightarrow V_{h_x,0}(\Omega)$ denotes the L^2 projection (2.39). This method is given in [74, (2.11), page 684] and differs from the methods [150, (1.54), page 16] or [65, (1.34), page 334] only in the right-hand side. The Crank-Nicolson Galerkin method (3.8) is equivalent to the linear systems

$$M_{h_x} \underline{U}^0 = \underline{u}_0$$

and

$$\left(M_{h_x} + \frac{h_{t,\ell}}{2} A_{h_x} \right) \underline{U}^\ell = \left(M_{h_x} - \frac{h_{t,\ell}}{2} A_{h_x} \right) \underline{U}^{\ell-1} + \underline{F}^\ell \quad \text{for all } \ell = 1, \dots, N_t, \quad (3.9)$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the mass matrix (2.37), $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the stiffness matrix (2.38) and the vectors $\underline{u}_0, \underline{F}^\ell \in \mathbb{R}^{M_x}$ are given in (3.7). The matrix $M_{h_x} + \frac{h_{t,\ell}}{2} A_{h_x}$ is positive definite and hence, the linear systems (3.9) are uniquely solvable for all $\ell = 1, \dots, N_t$. Stability of the numerical scheme (3.8) holds true without any CFL condition because the Crank-Nicolson method is unconditionally stable, see Section 2.6.

Next, error estimates for the implicit Euler and the Crank-Nicolson Galerkin method are the aim. It seems that error estimates of the quantities $\|u(\cdot, t_\ell) - U_{h_x,\ell}\|_{L^2(\Omega)}$ for each $\ell = 0, \dots, N_t$ are standard, see [150, Theorem 1.6, page 16 or Theorem 1.5, page 15] or [65, Theorem 5.13, page 336], [85, 93] and also [43, 159] for some early references. However, here, error estimates in space-time norms $\|\cdot\|_{L^2(Q)}$, $|\cdot|_{H^1(Q)}$ are considered, see the work [9, 16, 17, 54, 74, 79, 108, 148, 151].

Theorem 3.1.2. *Let Ω be sufficiently regular and consider a constant step size $h_t = h_{t,\ell}$ for all $\ell = 1, \dots, N_t$ together with a sequence $(\mathcal{T}_N)_N$ of admissible, shape regular and globally quasi-uniform decompositions with maximal mesh size h_x . Furthermore, let the unique solution u of (3.3) be sufficiently smooth and let u_h be defined via (3.4), where the coefficients $U_{h_x,\ell}$ are calculated by the implicit Euler Galerkin method (3.5). Then, the space-time error estimate*

$$\|u - u_h\|_{L^2(Q)} \leq c_1(h_t + h_x^s) \|u\|_{L^2(0,T;H^s(\Omega))} + c_2 h_t^r \|u\|_{H^r(0,T;L^2(\Omega))}$$

holds true with $r \in (1/2, 1]$, $s \in [1, 2]$ and with constants $c_1 > 0$, $c_2 > 0$ independent of h_t and h_x .

Proof. This proof follows the ideas of the proof of [74, Theorem 3.1, page 684]. So, for $u \in C([0, T]; L^2(\Omega))$, one defines the function $I_{h_t} \mathcal{Q}_{h_x} u \in \mathcal{Q}_h^1(Q)$ as in (2.59), i.e.

$$I_{h_t} \mathcal{Q}_{h_x} u(x, t) = \sum_{\ell=0}^{N_t} \underbrace{\mathcal{Q}_{h_x} u(x, t_\ell)}_{=: \hat{U}_{h_x,\ell}(x)} \varphi_\ell(t) \quad \text{for } (x, t) \in \bar{Q}$$

with $\hat{U}_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$, satisfying

$$\langle \hat{U}_{h_x,\ell}, v_{h_x} \rangle_{L^2(\Omega)} = \langle u(\cdot, t_\ell), v_{h_x} \rangle_{L^2(\Omega)} \quad \text{for all } v_{h_x} \in V_{h_x,0}(\Omega)$$

for $\ell = 0, \dots, N_t$. With the triangle inequality, it holds true that

$$\|u - u_h\|_{L^2(Q)} \leq \|u - I_{h_t} \mathcal{Q}_{h_x} u\|_{L^2(Q)} + \|I_{h_t} \mathcal{Q}_{h_x} u - u_h\|_{L^2(Q)}.$$

The first term is estimated by standard error estimates of Lemma 2.8.4 and so, it remains to investigate the second term. Therefore, set

$$\eta_\ell := U_{h_x,\ell} - \hat{U}_{h_x,\ell} \in V_{h_x,0}(\Omega) \quad \text{for } \ell = 0, \dots, N_t,$$

where $\eta_0 = 0$. Hence, it holds true that

$$\begin{aligned} \|I_{h_t} \mathcal{Q}_{h_x} u - u_h\|_{L^2(Q)}^2 &= \sum_{\ell=1}^{N_t} \|I_{h_t} \mathcal{Q}_{h_x} u - u_h\|_{L^2(\Omega \times \tau_\ell)}^2 \\ &= \int_{\Omega} \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}}{3} (\eta_\ell(x)^2 + \eta_{\ell-1}(x)\eta_\ell(x) + \eta_{\ell-1}(x)^2) dx \\ &\leq h_t \sum_{\ell=1}^{N_t} \|\eta_\ell\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.10}$$

With [74, (4.5), page 689] for $\alpha = 0$ and [74, (4.4a), page 688] for $\alpha = \tilde{\alpha} = 0$, it follows from the last inequality (3.10) that

$$\begin{aligned} \|I_{h_t} Q_{h_x} u - u_h\|_{L^2(Q)} &\leq \left(\sum_{\ell=1}^{N_t} h_t \|\eta_\ell\|_{L^2(\Omega)}^2 \right)^{1/2} \\ &\leq c_1 (h_t + h_x^s) \|u\|_{L^2(0,T;H^s(\Omega))} + c_2 h_t^r \|u\|_{H^r(0,T;L^2(\Omega))} \end{aligned}$$

with $r \in (1/2, 1]$, $s \in [1, 2]$ and with constants $c_1 > 0$, $c_2 > 0$ coming from standard error estimates and inverse inequalities, see the proofs in [74] for details. \square

Theorem 3.1.3. *Let Ω be sufficiently regular and consider a constant step size $h_t = h_{t,\ell}$ for all $\ell = 1, \dots, N_t$ together with a sequence $(\mathcal{T}_N)_N$ of admissible, shape regular and globally quasi-uniform decompositions with maximal mesh size h_x . Furthermore, let the unique solution u of (3.3) be sufficiently smooth and let u_h be defined via (3.4), where the coefficients $U_{h_x,\ell}$ are calculated by the Crank-Nicolson Galerkin method (3.8). Then, the space-time error estimate*

$$\|u - u_h\|_{L^2(Q)} \leq c_1 (h_t^2 + h_x^s) \|u\|_{L^2(0,T;H^s(\Omega))} + c_2 h_t^r \|u\|_{H^r(0,T;L^2(\Omega))}$$

holds true with $r \in (1/2, 2]$, $s \in [1, 2]$ and with constants $c_1 > 0$, $c_2 > 0$ independent of h_t and h_x . Furthermore, for $Q_h^1(Q) \subset H_0^{1,1}(Q)$, assume the inverse inequality

$$\forall v_h \in Q_h^1(Q): \quad |v_h|_{H^1(Q)} \leq c_{\text{inv}} h^{-1} \|v_h\|_{L^2(Q)}$$

with a constant $c_{\text{inv}} > 0$ and $h = \max\{h_t, h_x\}$. Then, it holds true that

$$|u - u_h|_{H^1(Q)} \leq C c_{\text{inv}} h^\mu \|u\|_{H^{\mu+1}(Q)} + c_{\text{inv}} h^{-1} \|u_h - u\|_{L^2(Q)}$$

with $\mu \in [0, 1]$ and with a constant $C > 0$ independent of h .

Proof. For the $L^2(Q)$ error estimate, repeat the proof of Theorem 3.1.2 until (3.10). Then, with [74, (4.5), page 689] for $\alpha = 0$, see also [74, Lemma 4.4, page 690], and the first estimate of [74, Lemma 4.3, page 690] for $\alpha = \tilde{\alpha} = 0$, it follows from the inequality (3.10) that

$$\begin{aligned} \|I_{h_t} Q_{h_x} u - u_h\|_{L^2(Q)} &\leq \left(\sum_{\ell=1}^{N_t} h_t \|\eta_\ell\|_{L^2(\Omega)}^2 \right)^{1/2} \\ &\leq c_1 (h_t^2 + h_x^s) \|u\|_{L^2(0,T;H^s(\Omega))} + c_2 h_t^r \|u\|_{H^r(0,T;L^2(\Omega))}, \end{aligned}$$

with $r \in (1/2, 2]$, $s \in [1, 2]$ and with constants $c_1 > 0$, $c_2 > 0$ coming from standard error estimates and inverse inequalities, see the proofs in [74] for details.

For the $H^1(Q)$ error estimate, consider an $H_{0;1}^1(Q)$ projection to find $Q_{h,0}^1 u \in Q_h^1(Q)$ such that

$$\langle \partial_t Q_{h,0}^1 u, \partial_t v_h \rangle_{L^2(Q)} + \langle \nabla_x Q_{h,0}^1 u, \nabla_x v_h \rangle_{L^2(Q)} = \langle \partial_t u, \partial_t v_h \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v_h \rangle_{L^2(Q)}$$

for all $v_h \in Q_h^1(Q)$. Then it follows with the triangle inequality, standard error estimates for $Q_{h,0}^1$ and the inverse inequality in $Q_h^1(Q)$ that

$$\begin{aligned} |u - u_h|_{H^1(Q)} &\leq |u - Q_{h,0}^1 u|_{H^1(Q)} + |Q_{h,0}^1 u - u_h|_{H^1(Q)} \\ &\leq \tilde{C} h^\mu \|u\|_{H^{\mu+1}(Q)} + c_{\text{inv}} h^{-1} \|Q_{h,0}^1 u - u_h\|_{L^2(Q)} \\ &\leq \tilde{C} h^\mu \|u\|_{H^{\mu+1}(Q)} + c_{\text{inv}} h^{-1} \|Q_{h,0}^1 u - u\|_{L^2(Q)} + c_{\text{inv}} h^{-1} \|u_h - u\|_{L^2(Q)} \\ &\leq C c_{\text{inv}} h^\mu \|u\|_{H^{\mu+1}(Q)} + c_{\text{inv}} h^{-1} \|u_h - u\|_{L^2(Q)} \end{aligned}$$

for $\mu \in [0, 1]$ and hence, the assertion. \square

Remark 3.1.4. *Since in the proofs of Theorem 3.1.2 and Theorem 3.1.3, see [74] for more details, regularity results of related adjoint problems are used, one expects reduced orders for the error estimates, given in Theorem 3.1.2 and Theorem 3.1.3, if Ω is less regular.*

Corollary 3.1.5. *Let the assumptions of Theorem 3.1.2 and of Theorem 3.1.3 be fulfilled and let u be sufficiently smooth. Then, for the implicit Euler Galerkin method (3.5), the error estimate*

$$\|u - u_h\|_{L^2(Q)} \leq Ch$$

holds true, and for the Crank-Nicolson Galerkin method (3.8), the error estimates

$$\|u - u_h\|_{L^2(Q)} \leq Ch^2$$

and

$$|u - u_h|_{H^1(Q)} \leq Ch$$

hold true with a constant $C > 0$ independent of $h = \max\{h_t, h_x\}$.

Proof. These estimates follow immediately from Theorem 3.1.2 and of Theorem 3.1.3 for the maximal values of r, s and μ . \square

In the last part of this section, some numerical examples are presented. So, for the space-time cylinder

$$Q = \Omega \times (0, T) = (0, 1) \times (0, 2),$$

consider the solutions of (3.3)

$$\begin{aligned} u_1(x,t) &:= \sin(\pi x) (1+t) e^{-t/2}, \\ u_2(x,t) &:= \sin(\pi x) (T-t)^{3/4} t, \\ u_3(x,t) &:= \sin(\pi x) (t^2 + x^2)^{-3/8}, \\ u_4(x,t) &:= \sin(\pi x) ((T/4-t)^2 + x^2)^{-3/8} \end{aligned}$$

for $(x,t) \in \mathcal{Q}$. Note that $u_i \in H^1(\mathcal{Q})$ for $i = 1, 2, 3, 4$. The spatial interval $\Omega = (0, 1)$ is decomposed into N_x elements, i.e. intervals, and $\tilde{M}_x = N_x + 1$ vertices with the constant mesh size $h_x = 1/(\tilde{M}_x - 1) = 1/N_x$. For the time interval $(0, 2)$, there are \tilde{M}_t time steps introduced with the constant time step size $h_t = T/(\tilde{M}_t - 1)$. See Section 2 for more details. The appearing integrals for the initial condition and the right-hand side in (3.7) are calculated by the usage of high-order integration rules, and the degrees of freedom are denoted by

$$\text{dof} = (\tilde{M}_x - 2) \cdot (\tilde{M}_t - 1)$$

due to the homogeneous Dirichlet boundary condition and the initial condition.

In Table 3.1 and Table 3.2, the errors in $\|\cdot\|_{L^2(\mathcal{Q})}$ and in $|\cdot|_{H^1(\mathcal{Q})}$ are presented for the smooth function u_1 and for a uniform refinement strategy in space and time direction, i.e. $h_t \sim h_x$. Note that no CFL condition like $h_t \sim h_x^2$ is needed because the Crank-Nicolson method and the implicit Euler method are unconditionally stable, see Section 2.6. The error estimates of Theorem 3.1.3 and Theorem 3.1.2 are confirmed.

Table 3.3, Table 3.5 and Table 3.7 show that the position of singularities leads to different convergence behaviours for the Crank-Nicolson Galerkin method. For the function u_3 , the singularity is at $(0, 0)$ and hence, the initial condition $u_0 \in L^2(\Omega)$ is less regular, i.e. $u_0 \notin H_0^1(\Omega)$. This results in an observed convergence rate of $3/4$ in $\|\cdot\|_{L^2(\mathcal{Q})}$ and in no convergence in $|\cdot|_{H^1(\mathcal{Q})}$ although $u_3 \in H^1(\mathcal{Q})$, see Table 3.5. If the position of the singularity is at the terminal time T , as for the solution u_2 , or at the time $T/4$, as for the solution u_4 , then reduced orders of convergence are observed as expected, see Table 3.3, Table 3.7. For the implicit Euler method, analogous results are given in Table 3.4, Table 3.6, Table 3.8, where the position of the singularity for the functions u_2, u_3, u_4 plays no role.

\tilde{M}_x	\tilde{M}_t	dof	h_x	h_t	$\ u_1 - u_{1,h}\ _{L^2(Q)}$	eoc	$ u_1 - u_{1,h} _{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	2.5257e-01	-	1.6202e+00	-
5	5	12	0.25000	0.50000	6.3416e-02	1.54	8.2405e-01	0.75
9	9	56	0.12500	0.25000	1.5851e-02	1.80	4.1399e-01	0.89
17	17	240	0.06250	0.12500	3.9635e-03	1.90	2.0726e-01	0.95
33	33	992	0.03125	0.06250	9.9092e-04	1.95	1.0366e-01	0.98
65	65	4032	0.01562	0.03125	2.4773e-04	1.98	5.1836e-02	0.99
129	129	16256	0.00781	0.01562	6.1933e-05	1.99	2.5918e-02	0.99
257	257	65280	0.00391	0.00781	1.5483e-05	1.99	1.2959e-02	1.00
513	513	261632	0.00195	0.00391	3.8708e-06	2.00	6.4797e-03	1.00
1025	1025	1047552	0.00098	0.00195	9.6769e-07	2.00	3.2398e-03	1.00
2049	2049	4192256	0.00049	0.00098	2.4206e-07	2.00	1.6199e-03	1.00
4097	4097	16773120	0.00024	0.00049	6.0215e-08	2.01	8.0996e-04	1.00

Table 3.1: Numerical results of the Crank-Nicolson Galerkin method (3.8) for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for u_1 .

\tilde{M}_x	\tilde{M}_t	dof	h_x	h_t	$\ u_1 - u_{1,h}\ _{L^2(Q)}$	eoc	$ u_1 - u_{1,h} _{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	2.4176e-01	-	1.6141e+00	-
5	5	12	0.25000	0.50000	7.8469e-02	1.26	8.3436e-01	0.74
9	9	56	0.12500	0.25000	2.8683e-02	1.31	4.2342e-01	0.88
17	17	240	0.06250	0.12500	1.2317e-02	1.16	2.1337e-01	0.94
33	33	992	0.03125	0.06250	5.8288e-03	1.05	1.0719e-01	0.97
65	65	4032	0.01562	0.03125	2.8627e-03	1.01	5.3748e-02	0.98
129	129	16256	0.00781	0.01562	1.4228e-03	1.00	2.6918e-02	0.99
257	257	65280	0.00391	0.00781	7.0988e-04	1.00	1.3470e-02	1.00
513	513	261632	0.00195	0.00391	3.5463e-04	1.00	6.7383e-03	1.00
1025	1025	1047552	0.00098	0.00195	1.7725e-04	1.00	3.3699e-03	1.00
2049	2049	4192256	0.00049	0.00098	8.8607e-05	1.00	1.6852e-03	1.00
4097	4097	16773120	0.00024	0.00049	4.4300e-05	1.00	8.4263e-04	1.00

Table 3.2: Numerical results of the implicit Euler Galerkin method (3.5) for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for u_1 .

M_x	M_t	dof	h_x	h_t	$\ u_2 - u_{2,h}\ _{L^2(Q)}$	eoc	$ u_2 - u_{2,h} _{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	1.7974e-01	-	1.2812e+00	-
5	5	12	0.25000	0.50000	5.3647e-02	1.35	7.0582e-01	0.67
9	9	56	0.12500	0.25000	1.5830e-02	1.58	4.0519e-01	0.72
17	17	240	0.06250	0.12500	5.4556e-03	1.46	2.5515e-01	0.64
33	33	992	0.03125	0.06250	2.2538e-03	1.25	1.7871e-01	0.50
65	65	4032	0.01562	0.03125	1.0047e-03	1.15	1.3639e-01	0.39
129	129	16256	0.00781	0.01562	4.4819e-04	1.16	1.0957e-01	0.31
257	257	65280	0.00391	0.00781	1.9647e-04	1.19	9.0273e-02	0.28
513	513	261632	0.00195	0.00391	8.4780e-05	1.21	7.5223e-02	0.26
1025	1025	1047552	0.00098	0.00195	3.6195e-05	1.23	6.2996e-02	0.26
2049	2049	4192256	0.00049	0.00098	1.5351e-05	1.24	5.2874e-02	0.25
4097	4097	16773120	0.00024	0.00049	6.4854e-06	1.24	4.4422e-02	0.25

Table 3.3: Numerical results of the Crank-Nicolson Galerkin method (3.8) for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for u_2 .

M_x	M_t	dof	h_x	h_t	$\ u_2 - u_{2,h}\ _{L^2(Q)}$	eoc	$ u_2 - u_{2,h} _{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	3.7057e-01	-	1.6781e+00	-
5	5	12	0.25000	0.50000	1.9478e-01	0.72	1.0367e+00	0.54
9	9	56	0.12500	0.25000	1.0549e-01	0.80	6.1865e-01	0.67
17	17	240	0.06250	0.12500	5.6371e-02	0.86	3.7187e-01	0.70
33	33	992	0.03125	0.06250	2.9435e-02	0.92	2.3356e-01	0.66
65	65	4032	0.01562	0.03125	1.5118e-02	0.95	1.5857e-01	0.55
129	129	16256	0.00781	0.01562	7.6856e-03	0.97	1.1747e-01	0.43
257	257	65280	0.00391	0.00781	3.8823e-03	0.98	9.2856e-02	0.34
513	513	261632	0.00195	0.00391	1.9533e-03	0.99	7.6028e-02	0.29
1025	1025	1047552	0.00098	0.00195	9.8032e-04	0.99	6.3241e-02	0.27
2049	2049	4192256	0.00049	0.00098	4.9126e-04	1.00	5.2947e-02	0.26
4097	4097	16773120	0.00024	0.00049	2.4596e-04	1.00	4.4444e-02	0.25

Table 3.4: Numerical results of the implicit Euler Galerkin method (3.5) for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for u_2 .

\bar{M}_x	\bar{M}_t	dof	h_x	h_t	$\ u_3 - u_{3,h}\ _{L^2(Q)}$	eoc	$ u_3 - u_{3,h} _{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	3.8930e-01	-	2.8412e+00	-
5	5	12	0.25000	0.50000	1.5092e-01	1.06	2.2724e+00	0.25
9	9	56	0.12500	0.25000	7.2348e-02	0.95	2.0666e+00	0.12
17	17	240	0.06250	0.12500	3.9069e-02	0.85	2.0819e+00	-0.01
33	33	992	0.03125	0.06250	2.2136e-02	0.80	2.2423e+00	-0.10
65	65	4032	0.01562	0.03125	1.2839e-02	0.78	2.5212e+00	-0.17
129	129	16256	0.00781	0.01562	7.5378e-03	0.76	2.9090e+00	-0.21
257	257	65280	0.00391	0.00781	4.4532e-03	0.76	3.4053e+00	-0.23
513	513	261632	0.00195	0.00391	2.6393e-03	0.75	4.0171e+00	-0.24
1025	1025	1047552	0.00098	0.00195	1.5668e-03	0.75	4.7577e+00	-0.24
2049	2049	4192256	0.00049	0.00098	9.3084e-04	0.75	5.6463e+00	-0.25
4097	4097	16773120	0.00024	0.00049	5.5325e-04	0.75	6.7077e+00	-0.25

Table 3.5: Numerical results of the Crank-Nicolson Galerkin method (3.8) for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for u_3 .

\bar{M}_x	\bar{M}_t	dof	h_x	h_t	$\ u_3 - u_{3,h}\ _{L^2(Q)}$	eoc	$ u_3 - u_{3,h} _{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	3.1084e-01	-	2.9018e+00	-
5	5	12	0.25000	0.50000	1.7098e-01	0.67	2.2254e+00	0.30
9	9	56	0.12500	0.25000	1.0734e-01	0.60	1.7948e+00	0.28
17	17	240	0.06250	0.12500	6.2274e-02	0.75	1.4924e+00	0.25
33	33	992	0.03125	0.06250	3.4179e-02	0.85	1.2533e+00	0.25
65	65	4032	0.01562	0.03125	1.8147e-02	0.90	1.0550e+00	0.25
129	129	16256	0.00781	0.01562	9.4412e-03	0.94	8.8824e-01	0.25
257	257	65280	0.00391	0.00781	4.8483e-03	0.96	7.4759e-01	0.25
513	513	261632	0.00195	0.00391	2.4686e-03	0.97	6.2902e-01	0.25
1025	1025	1047552	0.00098	0.00195	1.2498e-03	0.98	5.2914e-01	0.25
2049	2049	4192256	0.00049	0.00098	6.3032e-04	0.99	4.4507e-01	0.25
4097	4097	16773120	0.00024	0.00049	3.1707e-04	0.99	3.7433e-01	0.25

Table 3.6: Numerical results of the implicit Euler Galerkin method (3.5) for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for u_3 .

\bar{M}_x	\bar{M}_t	dof	h_x	h_t	$\ u_4 - u_{4,h}\ _{L^2(Q)}$	eoc	$ u_4 - u_{4,h} _{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	6.2378e-01	-	2.5520e+00	-
5	5	12	0.25000	0.50000	1.7639e-01	1.41	2.7132e+00	-0.07
9	9	56	0.12500	0.25000	6.6921e-02	1.26	2.1509e+00	0.30
17	17	240	0.06250	0.12500	2.7324e-02	1.23	1.7729e+00	0.27
33	33	992	0.03125	0.06250	1.1359e-02	1.24	1.4811e+00	0.25
65	65	4032	0.01562	0.03125	4.7565e-03	1.24	1.2429e+00	0.25
129	129	16256	0.00781	0.01562	1.9991e-03	1.24	1.0444e+00	0.25
257	257	65280	0.00391	0.00781	8.4227e-04	1.24	8.7808e-01	0.25
513	513	261632	0.00195	0.00391	3.5559e-04	1.24	7.3836e-01	0.25
1025	1025	1047552	0.00098	0.00195	1.5046e-04	1.24	6.2093e-01	0.25
2049	2049	4192256	0.00049	0.00098	6.3843e-05	1.24	5.2221e-01	0.25
4097	4097	16773120	0.00024	0.00049	2.7189e-05	1.23	4.3923e-01	0.25

Table 3.7: Numerical results of the Crank-Nicolson Galerkin method (3.8) for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for u_4 .

\bar{M}_x	\bar{M}_t	dof	h_x	h_t	$\ u_4 - u_{4,h}\ _{L^2(Q)}$	eoc	$ u_4 - u_{4,h} _{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	5.8329e-01	-	2.5931e+00	-
5	5	12	0.25000	0.50000	2.5431e-01	0.93	2.9416e+00	-0.14
9	9	56	0.12500	0.25000	1.5258e-01	0.66	2.4423e+00	0.24
17	17	240	0.06250	0.12500	8.6590e-02	0.78	2.0575e+00	0.24
33	33	992	0.03125	0.06250	4.7030e-02	0.86	1.7383e+00	0.24
65	65	4032	0.01562	0.03125	2.4819e-02	0.91	1.4675e+00	0.24
129	129	16256	0.00781	0.01562	1.2856e-02	0.94	1.2372e+00	0.24
257	257	65280	0.00391	0.00781	6.5804e-03	0.96	1.0419e+00	0.25
513	513	261632	0.00195	0.00391	3.3420e-03	0.98	8.7684e-01	0.25
1025	1025	1047552	0.00098	0.00195	1.6887e-03	0.98	7.3767e-01	0.25
2049	2049	4192256	0.00049	0.00098	8.5041e-04	0.99	6.2046e-01	0.25
4097	4097	16773120	0.00024	0.00049	4.2730e-04	0.99	5.2183e-01	0.25

Table 3.8: Numerical results of the implicit Euler Galerkin method (3.5) for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for u_4 .

3.2 Space-Time Variational Formulation of Bochner Type

In this section, a short overview of space-time variational formulations of Bochner type is given and a stable space-time discretisation is formulated, see [51, 137, 142–144].

For the heat equation (3.1), a space-time variational formulation of Bochner type for the Hilbert space $W(Q)$ from (2.21) with the norm $\|\cdot\|_{W(Q)}$ from (2.22) is given as follows:

Find $u \in W(Q)$ such that

$$\langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} + \langle u(\cdot, 0), w \rangle_{L^2(\Omega)} = \langle f, v \rangle_Q + \langle u_0, w \rangle_{L^2(\Omega)} \quad (3.11)$$

for all $(v, w) \in L^2(0, T; H_0^1(\Omega)) \times L^2(\Omega) =: Y$, where $f \in L^2(0, T; [H_0^1(\Omega)]')$ and $u_0 \in L^2(\Omega)$ are the given right-hand side and the given initial condition. The space-time variational formulation of Bochner type (3.11) is equivalent to the pointwise in time variational formulation (3.3) because of the fundamental lemma of calculus, the density of $C_0^\infty(0, T)$ in $L^2(0, T)$ and so, the density of the algebraic tensor-product

$$H_0^1(\Omega) \otimes C_0^\infty(0, T) = \text{span} \{ Q \ni (x, t) \mapsto \phi(x) \cdot \psi(t) \in \mathbb{R} : \phi \in H_0^1(\Omega), \psi \in C_0^\infty(0, T) \}$$

in $H_0^1(\Omega) \hat{\otimes} L^2(0, T) \simeq L^2(0, T; H_0^1(\Omega))$. Hence, the unique solvability of the space-time variational formulation of Bochner type (3.11) follows from the pointwise in time variational formulation (3.3), i.e. from Theorem 3.1.1. An alternative proof of a uniqueness and existence result for the space-time variational formulation of Bochner type (3.11) uses the Nečas Theorem 2.9.1. Therefore, define the bilinear form $b(\cdot, \cdot) : W(Q) \times Y \rightarrow \mathbb{R}$ by

$$b(u, (v, w)) := \langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} + \langle u(\cdot, 0), w \rangle_{L^2(\Omega)} \quad (3.12)$$

for $u \in W(Q)$, $(v, w) \in Y$, where the Hilbert space Y is endowed with the inner product

$$\langle (v, w), (\hat{v}, \hat{w}) \rangle_Y := \int_0^T \int_\Omega \nabla_x v(x, t) \cdot \nabla_x \hat{v}(x, t) dx dt + \int_\Omega w(x) \hat{w}(x) dx \quad \text{for } (v, w), (\hat{v}, \hat{w}) \in Y.$$

Theorem 3.2.1. *The bilinear form (3.12) is continuous and fulfils the condition (N1) and the condition (N2), i.e. it holds true that:*

1. A constant $C > 0$ exists such that for all $u \in W(Q)$ and for all $(v, w) \in Y$

$$|b(u, (v, w))| \leq C \|u\|_{W(Q)} \|(v, w)\|_Y.$$

2. A constant $c_s > 0$ exists such that

$$\inf_{0 \neq u \in W(Q)} \sup_{0 \neq (v, w) \in Y} \frac{|b(u, (v, w))|}{\|u\|_{W(Q)} \|(v, w)\|_Y} \geq c_s.$$

3. For each element $(v, w) \in Y$, $(v, w) \neq 0$, an element $u \in W(Q)$ exists such that $b(u, (v, w)) \neq 0$.

Proof. For $u_0 = 0$, the proof is contained in the book [51, Proof of Theorem 6.6, page 282]. For the general case, see [137, Theorem 5.1, page 1300]. \square

The linear form $F: Y \rightarrow \mathbb{R}$ is given by

$$F(v, w) := \langle f, v \rangle_Q + \langle u_0, w \rangle_{L^2(\Omega)} \quad \text{for } (v, w) \in Y,$$

where its boundedness follows with the Cauchy-Schwarz inequality by

$$\begin{aligned} |F(v, w)| &\leq \|f\|_{L^2(0, T; [H_0^1(\Omega)]')} \|\nabla_x v\|_{L^2(Q)} + \|u_0\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \\ &\leq \sqrt{\|f\|_{L^2(0, T; [H_0^1(\Omega)]')}^2 + \|u_0\|_{L^2(\Omega)}^2} \|(v, w)\|_Y \end{aligned}$$

for all $(v, w) \in Y$. Hence, the variational formulation (3.11) is rewritten to find $u \in W(Q)$ such that

$$b(u, (v, w)) = F(v, w) \tag{3.13}$$

for all $(v, w) \in Y$. The following existence and uniqueness result holds true:

Theorem 3.2.2. *For each given $F = (f, u_0) \in Y'$, the variational formulation (3.13) and hence, the variational formulation (3.11) have a unique solution $u \in W(Q)$, satisfying*

$$\|u\|_{W(Q)} \leq C \sqrt{\|f\|_{L^2(0, T; [H_0^1(\Omega)]')}^2 + \|u_0\|_{L^2(\Omega)}^2}$$

with a constant $C > 0$. Furthermore, the solution operator

$$\mathcal{L}: Y' \rightarrow W(Q), \quad \mathcal{L}F = \mathcal{L}(f, u_0) := u,$$

is an isomorphism.

Proof. This follows with the Nečas Theorem 2.9.1 from Theorem 3.2.1. \square

In the remainder of this section, the initial condition u_0 is incorporated via homogenisation. So, the bilinear form

$$a(u, v) := \langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} \quad \text{for } u \in W(Q), v \in L^2(0, T; H_0^1(\Omega))$$

is introduced. The bilinear form $a(\cdot, \cdot)$ is bounded, i.e.

$$|a(u, v)| \leq \sqrt{2} \|u\|_{W(Q)} \|\nabla_x v\|_{L^2(Q)} \quad \text{for } u \in W(Q), v \in L^2(0, T; H_0^1(\Omega)).$$

Next, for the bilinear form

$$a(\cdot, \cdot): W_0(Q) \times L^2(0, T; H_0^1(\Omega)) \rightarrow \mathbb{R},$$

the requirements of the Nečas Theorem 2.9.1 are examined, where the Hilbert space $W_0(Q)$ is the subspace given in (2.25).

Theorem 3.2.3. *The bilinear form $a(\cdot, \cdot): W_0(Q) \times L^2(0, T; H_0^1(\Omega)) \rightarrow \mathbb{R}$ is continuous and fulfils the condition (N1) and the condition (N2), i.e. it holds true that:*

1. *For all $u \in W_0(Q)$ and for all $v \in L^2(0, T; H_0^1(\Omega))$, there is*

$$|a(u, v)| \leq \sqrt{2} \|u\|_{W(Q)} \|\nabla_x v\|_{L^2(Q)}.$$

2. *The inf-sup condition*

$$\inf_{0 \neq u \in W_0(Q)} \sup_{0 \neq v \in L^2(0, T; H_0^1(\Omega))} \frac{|a(u, v)|}{\|u\|_{W(Q)} \|\nabla_x v\|_{L^2(Q)}} \geq \frac{1}{2\sqrt{2}}$$

holds true.

3. *For each function $v \in L^2(0, T; H_0^1(\Omega))$, $v \neq 0$, an element $u \in W_0(Q)$ exists such that $a(u, v) \neq 0$.*

Proof. The proof is contained in the book [51, Proof of Theorem 6.6, page 282]. For the inf-sup constant, see [142, Theorem 2.1, page 5]. \square

For a given initial condition $u_0 \in L^2(\Omega)$ and a given right-hand side $f \in L^2(0, T; [H_0^1(\Omega)]')$, the variational formulation of the heat equation (3.1) is to find $u \in W(Q)$ with $u(\cdot, 0) = u_0$ in $L^2(\Omega)$ such that

$$a(u, v) = \langle f, v \rangle_Q \quad (3.14)$$

for all $v \in L^2(0, T; H_0^1(\Omega))$. By homogenisation, there follows the existence and uniqueness theorem:

Theorem 3.2.4. *Let the right-hand side $f \in L^2(0, T; [H_0^1(\Omega)]')$ and the initial condition $u_0 \in L^2(\Omega)$ be given. Then, the variational formulation (3.14) admits a unique solution $u \in W(Q)$ with $u(\cdot, 0) = u_0$ in $L^2(\Omega)$, satisfying*

$$\|u\|_{W(Q)} \leq 2\sqrt{2} \left(\|f\|_{L^2(0, T; [H_0^1(\Omega)]')} + \sqrt{2} C_{\text{ex}} \|u_0\|_{L^2(\Omega)} \right) + C_{\text{ex}} \|u_0\|_{L^2(\Omega)}$$

with a constant $C_{\text{ex}} > 0$ coming from the extension (2.24) of u_0 .

Proof. Consider the extension $\bar{u}_0 := \mathcal{E}_{\Omega_0} u_0 \in W(Q)$ from (2.24) with $\bar{u}_0(\cdot, 0) = u_0$ in $L^2(\Omega)$ satisfying $\|\bar{u}_0\|_{W(Q)} \leq C_{\text{ex}} \|u_0\|_{L^2(\Omega)}$ with a constant $C_{\text{ex}} > 0$. Next, investigate the variational formulation by homogenisation to find $\bar{u} \in W_0(Q)$ such that

$$a(\bar{u}, v) = \langle f, v \rangle_Q - a(\bar{u}_0, v) \quad (3.15)$$

for all $v \in L^2(0, T; H_0^1(\Omega))$. The right-hand side is bounded by

$$\begin{aligned} \left| \langle f, v \rangle_Q - a(\bar{u}_0, v) \right| &\leq \left(\|f\|_{L^2(0, T; [H_0^1(\Omega)]')} + \sqrt{2} \|\bar{u}_0\|_{W(Q)} \right) \|\nabla_x v\|_{L^2(Q)} \\ &\leq \left(\|f\|_{L^2(0, T; [H_0^1(\Omega)]')} + \sqrt{2} C_{\text{ex}} \|u_0\|_{L^2(\Omega)} \right) \|\nabla_x v\|_{L^2(Q)} \end{aligned} \quad (3.16)$$

for all $v \in L^2(0, T; H_0^1(\Omega))$. So, the Nečas Theorem 2.9.1 yields with Theorem 3.2.3 that there exists a unique solution $\bar{u} \in W_0(Q)$ of the variational formulation (3.15), satisfying with (3.16)

$$\|\bar{u}\|_{W(Q)} \leq 2\sqrt{2} \left(\|f\|_{L^2(0, T; [H_0^1(\Omega)]')} + \sqrt{2} C_{\text{ex}} \|u_0\|_{L^2(\Omega)} \right). \quad (3.17)$$

Set $u := \bar{u} + \bar{u}_0 \in W(Q)$.

Next, the independence of the extension \bar{u}_0 for u is examined. So, for a second extension $\hat{u}_0 \in W(Q)$, satisfying $\hat{u}_0(\cdot, 0) = u_0$ in $L^2(\Omega)$, there exists again $\hat{u} \in W_0(Q)$, satisfying the variational formulation (3.15). The difference $(\bar{u} + \bar{u}_0) - (\hat{u} + \hat{u}_0) \in W_0(Q)$ fulfils the homogeneous variational formulation

$$a((\bar{u} + \bar{u}_0) - (\hat{u} + \hat{u}_0), v) = 0 \quad \text{for all } v \in L^2(0, T; H_0^1(\Omega)).$$

Because of Theorem 3.2.3 and the Nečas Theorem 2.9.1, the to the bounded bilinear form $a(\cdot, \cdot): W_0(Q) \times L^2(0, T; H_0^1(\Omega)) \rightarrow \mathbb{R}$ related operator $A: W_0(Q) \rightarrow L^2(0, T; [H_0^1(\Omega)]')$ is an isomorphism. Hence, $A((\bar{u} + \bar{u}_0) - (\hat{u} + \hat{u}_0)) = 0$, i.e. $\bar{u} + \bar{u}_0 = \hat{u} + \hat{u}_0$ and therefore, the solution $u \in W(Q)$ is independent of the extension \bar{u}_0 for the initial condition u_0 .

With the triangle inequality, (3.17) and the continuity of the extension operator for u_0 , there follow the stability estimate

$$\begin{aligned} \|u\|_{W(Q)} &\leq \|\bar{u}\|_{W(Q)} + \|\bar{u}_0\|_{W(Q)} \\ &\leq 2\sqrt{2} \left(\|f\|_{L^2(0, T; [H_0^1(\Omega)]')} + \sqrt{2} C_{\text{ex}} \|u_0\|_{L^2(\Omega)} \right) + C_{\text{ex}} \|u_0\|_{L^2(\Omega)} \end{aligned}$$

and hence, the assertion. \square

For a discretisation scheme, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0, L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. As a conforming space-time discretisation, consider the space of piecewise linear, continuous functions $S_h^1(Q) \cap W_0(Q)$, see Section 2.8 for more details. For an arbitrary fixed extension $\bar{u}_0 \in W(Q)$ with $\bar{u}_0(\cdot, 0) = u_0$ in $L^2(\Omega)$ and $\|\bar{u}_0\|_{W(Q)} \leq \tilde{C}_{\text{ex}} \|u_0\|_{L^2(\Omega)}$ with a constant $\tilde{C}_{\text{ex}} > 0$ independent of u_0 , e.g., $\bar{u}_0 = \mathcal{E}_{\Omega_0} u_0$ from (2.24), the discrete variational formulation is to find $\bar{u}_h \in S_h^1(Q) \cap W_0(Q)$ such that

$$a(\bar{u}_h, v_h) = \langle f, v_h \rangle_Q - a(\bar{u}_0, v_h) \quad (3.18)$$

for all $v_h \in S_h^1(Q) \cap W_0(Q)$. Note that ansatz and test spaces are equal. Next, define the approximation

$$u_h := \bar{u}_h + \bar{u}_0 \in W(Q), \quad (3.19)$$

where in practice \bar{u}_0 is replaced by the space-time interpolant $I_h \bar{u}_0 \in S_h^1(Q)$ from (2.40), if \bar{u}_0 is smooth enough, see [141, page 246] or [51, Section 3.2.2, page 124] for the elliptic case.

The following stability and convergence theorem is contained in the work [142, Section 3, page 6].

Theorem 3.2.5. *Let the assumptions of Theorem 3.2.4 be satisfied with the unique solution $u \in W(Q)$ of (3.14). Further, let $\bar{u}_0 \in W(Q)$ be the extension from (3.18) of u_0 with $\bar{u}_0(\cdot, 0) = u_0$ in $L^2(\Omega)$ and $\|\bar{u}_0\|_{W(Q)} \leq \tilde{C}_{\text{ex}} \|u_0\|_{L^2(\Omega)}$ with a constant $\tilde{C}_{\text{ex}} > 0$ independent of u_0 . Then, a unique solution $\bar{u}_h \in S_h^1(Q) \cap W_0(Q)$ of the discrete variational formulation (3.18) exists, satisfying the stability estimate*

$$\|\nabla_x \bar{u}_h\|_{L^2(Q)} \leq 2\sqrt{2} \left(\|f\|_{L^2(0,T;[H_0^1(\Omega)]')} + \sqrt{2}\tilde{C}_{\text{ex}} \|u_0\|_{L^2(\Omega)} \right).$$

Furthermore, assume $\bar{u} = u - \bar{u}_0 \in H^s(Q)$ for some $s \in [1, 2]$. Then, for the approximation $u_h = \bar{u}_h + \bar{u}_0$ in (3.19), the error estimate

$$\|\nabla_x(u - u_h)\|_{L^2(Q)} \leq Ch^{s-1} \|\bar{u}\|_{H^s(Q)}$$

holds true with a constant $C > 0$.

Proof. The unique solvability of the discrete variational formulation (3.18) follows from the discrete inf-sup condition

$$\inf_{0 \neq u_h \in S_h^1(Q) \cap W_0(Q)} \sup_{0 \neq v_h \in S_h^1(Q) \cap W_0(Q)} \frac{a(u_h, v_h)}{\|\nabla_x u_h\|_{L^2(Q)} \|\nabla_x v_h\|_{L^2(Q)}} \geq \frac{1}{2\sqrt{2}}, \quad (3.20)$$

which is proven in [142, Theorem 3.5, page 7]. In addition, the discrete inf-sup condition (3.20) yields with the bound (3.16) the stability estimate

$$\|\nabla_x \bar{u}_h\|_{L^2(Q)} \leq 2\sqrt{2} \left(\|f\|_{L^2(0,T;[H_0^1(\Omega)]')} + \sqrt{2}\tilde{C}_{\text{ex}} \|u_0\|_{L^2(\Omega)} \right).$$

The error estimate follows from [142, Corollary 3.4, page 10] with

$$\|\nabla_x(u - u_h)\|_{L^2(Q)} = \|\nabla_x(\bar{u} - \bar{u}_h)\|_{L^2(Q)}$$

and hence, the assertion. \square

Numerical examples and further investigations are given in [142–144].

3.3 Space-Time Variational Formulation of Anisotropic Type

In this section, a short overview of a so-called space-time variational formulation of anisotropic type for a homogeneous initial condition, i.e. $u_0 = 0$, is given. The motivation comes from considering a FEM-BEM coupling for transmission problems of the heat equation. Because this variational formulation of anisotropic type arises by the treatment of boundary integral equations for the heat equation, the usage of this variational formulation of anisotropic type is natural for the finite element method. However, it seems that a stable conforming discretisation of this variational formulation of anisotropic type by piecewise linear, continuous functions is not available.

For the homogeneous Dirichlet problem of the heat equation

$$\left. \begin{aligned} \partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma = \Gamma \times [0,T], \\ u(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned} \right\}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$ and $T > 0$ is a given terminal time, the space-time variational formulation of anisotropic type is given as follows:

Find $u \in H_{0,0}^{1,1/2}(Q)$ such that

$$\langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} = \langle f, v \rangle_Q \quad (3.21)$$

for all $v \in H_{0,0}^{1,1/2}(Q)$, where $f \in [H_{0,0}^{1,1/2}(Q)]'$ is a given right-hand side, see Section 2.5 for the notations. The bilinear form $a(\cdot, \cdot): H_{0,0}^{1,1/2}(Q) \times H_{0,0}^{1,1/2}(Q) \rightarrow \mathbb{R}$,

$$a(u, v) := \langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)}$$

for $u \in H_{0,0}^{1,1/2}(Q)$, $v \in H_{0,0}^{1,1/2}(Q)$, is bounded, i.e. there exists a constant $C > 0$ such that

$$|a(u, v)| \leq C \|u\|_{H_{0,0}^{1,1/2}(Q)} \|v\|_{H_{0,0}^{1,1/2}(Q)}$$

for $u \in H_{0,0}^{1,1/2}(Q)$, $v \in H_{0,0}^{1,1/2}(Q)$, see [35, Lemma 2.6, page 505].

Remark 3.3.1. The bilinear form $\langle \partial_t u, v \rangle_{(0,T)}$ for $u, v \in C_0^\infty(0, T)$ has no continuous extension to

$$H_0^{1/2}(0, T) \times H_0^{1/2}(0, T)$$

or to

$$H^{1/2}(0, T) \times H^{1/2}(0, T),$$

i.e. the usage of the different ansatz and test spaces $H_{0;0}^{1,1/2}(Q)$, $H_{0;0}^{1,1/2}(Q)$ for the bilinear form

$$a(\cdot, \cdot): H_{0;0}^{1,1/2}(Q) \times H_{0;0}^{1,1/2}(Q) \rightarrow \mathbb{R}$$

is crucial. See [35, Remark 2.7, page 505] and [64, Proposition 1.4.4.8, page 32].

In [35], the following existence and uniqueness theorem is proven by a transposition and interpolation argument as in [102, 103], see also [98].

Theorem 3.3.2. *Let the right-hand side $f \in [H_{0;0}^{1,1/2}(Q)]'$ be given. Then, the variational formulation (3.21) has a unique solution $u \in H_{0;0}^{1,1/2}(Q)$, satisfying*

$$\|u\|_{H_{0;0}^{1,1/2}(Q)} \leq C \|f\|_{[H_{0;0}^{1,1/2}(Q)]'}$$

with a constant $C > 0$. Furthermore, the solution operator

$$\mathcal{L}: [H_{0;0}^{1,1/2}(Q)]' \rightarrow H_{0;0}^{1,1/2}(Q), \quad \mathcal{L}f := u,$$

is an isomorphism. In addition, the bilinear form

$$a(\cdot, \cdot): H_{0;0}^{1,1/2}(Q) \times H_{0;0}^{1,1/2}(Q) \rightarrow \mathbb{R}, \quad a(u, v) = \langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)},$$

is continuous and fulfils the condition (N1) and the condition (N2).

Proof. The existence and uniqueness of a solution $u \in H_{0;0}^{1,1/2}(Q)$ of the variational formulation (3.21) and that the solution operator $\mathcal{L}: [H_{0;0}^{1,1/2}(Q)]' \rightarrow H_{0;0}^{1,1/2}(Q)$ is an isomorphism follow from [35, Lemma 2.8, page 505]. The Nečas Theorem 2.9.1 yields the properties of the bilinear form $a(\cdot, \cdot): H_{0;0}^{1,1/2}(Q) \times H_{0;0}^{1,1/2}(Q) \rightarrow \mathbb{R}$. \square

Remark 3.3.3. *In Section 3.4 an alternative proof of Theorem 3.3.2 is given by the usage of Fourier series, see Theorem 3.4.19.*

For a discretisation scheme, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0, L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. With the notations of Section 2.8, a conforming space-time discretisation via the space of piecewise linear, continuous functions $S_h^1(Q)$ leads to the discrete ansatz and test spaces $S_h^1(Q) \cap H_{0;0}^{1,1/2}(Q)$ and $S_h^1(Q) \cap H_{0;0}^{1,1/2}(Q)$. Hence, the test space differs from that of the approach (3.18), i.e.

$$S_h^1(Q) \cap H_{0;0}^{1,1/2}(Q) \neq S_h^1(Q) \cap W_0(Q).$$

It turns out that the resulting finite element method, for the choice above of different ansatz and test spaces, is not stable. Therefore, only the time component is considered. That means, for a fixed $\mu > 0$, the ordinary differential equation

$$\partial_t u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0, T), \quad u(0) = 0,$$

and the variational formulation to find $u \in H_0^{1/2}(0, T)$ such that

$$\langle \partial_t u, v \rangle_{(0, T)} + \mu \langle u, v \rangle_{L^2(0, T)} = \langle f, v \rangle_{(0, T)} \quad (3.22)$$

for all $v \in H_0^{1/2}(0, T)$, where $f \in [H_0^{1/2}(0, T)]'$ is given, are investigated. Analogous to Theorem 3.3.2, there exists a unique solution $u \in H_0^{1/2}(0, T)$ of the variational formulation (3.22), see also Theorem 3.4.10. With the notations of Section 2.6, a conforming discretisation with piecewise linear, continuous functions $S_{h_t}^1(0, T) = \text{span}\{\varphi_k\}_{k=0}^{N_t}$ leads to the discrete variational formulation to find $u_{h_t} \in S_{h_t}^1(0, T) \cap H_0^{1/2}(0, T)$ such that

$$\langle \partial_t u_{h_t}, v_{h_t} \rangle_{L^2(0, T)} + \mu \langle u_{h_t}, v_{h_t} \rangle_{L^2(0, T)} = \langle f, v_{h_t} \rangle_{(0, T)} \quad (3.23)$$

for all $v_{h_t} \in S_{h_t}^1(0, T) \cap H_0^{1/2}(0, T)$. The resulting system matrix of (3.23) is given as

$$K_{h_t} = \frac{1}{2} \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ -1 & 0 & 1 & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & & & -1 & 0 & 1 & \\ & & & & & & & 1 \end{pmatrix} + \frac{\mu}{6} \begin{pmatrix} & & & & & & & & \\ & h_{t,1} & & & & & & & \\ 2h_{t,1} + 2h_{t,2} & & h_{t,2} & & & & & & \\ h_{t,2} & & 2h_{t,2} + 2h_{t,3} & & h_{t,3} & & & & \\ & & & \ddots & \ddots & & \ddots & & \\ & & & & & h_{t,N_t-1} & & 2h_{t,N_t-1} + 2h_{t,N_t} & h_{t,N_t} \end{pmatrix} \quad (3.24)$$

and hence, unique solvability of the discrete variational formulation (3.23) follows because $K_{h_t} \in \mathbb{R}^{N_t \times N_t}$ is a lower triangular matrix with positive diagonal elements.

For a uniform discretisation with mesh size h_t , the matrix K_{h_t} in (3.24) can be interpreted as a finite difference scheme

$$\begin{aligned} a_0 v_1 &= f_1, \\ a_1 v_1 + a_0 v_2 &= f_2, \\ a_2 v_{j-2} + a_1 v_{j-1} + a_0 v_j &= f_j \quad \text{for } j > 2, \end{aligned}$$

where

$$a_0 := \frac{1}{2} + \frac{\mu h_t}{6} = \frac{3 + \mu h_t}{6}, \quad a_1 := \frac{4\mu h_t}{6}, \quad a_2 := -\frac{1}{2} + \frac{\mu h_t}{6} = \frac{-3 + \mu h_t}{6}$$

with given $f_j \in \mathbb{R}$. The essential case $f_j = 0$ for $j > 2$ is examined. The solution of the homogeneous linear recurrence relation

$$a_2 v_{j-2} + a_1 v_{j-1} + a_0 v_j = 0 \quad \text{for } j > 2$$

is given for $j \geq 1$ by

$$v_j = A_0 \left(\frac{-2h_t\mu - \sqrt{9 + 3h_t^2\mu^2}}{3 + h_t\mu} \right)^{j-1} + A_1 \left(\frac{-2h_t\mu + \sqrt{9 + 3h_t^2\mu^2}}{3 + h_t\mu} \right)^{j-1},$$

where the coefficients $A_0, A_1 \in \mathbb{R}$ are determined by $f_1, f_2 \in \mathbb{R}$. Hence, in general, the sequence $(v_j)_{j \in \mathbb{N}}$ is unbounded as $j \rightarrow \infty$ independently of μ and h_t . In other words, the numerical scheme (3.23) is unstable for each $\mu > 0$ and each $h_t > 0$. For the heat equation, deriving a conforming discretisation by piecewise linear, continuous functions of the variational formulation (3.21), which results in a stable numerical scheme, is delicate and is not discussed in this thesis. On the other hand, an alternative approach with the help of a type of Hilbert transform is given in Section 3.4.

3.4 Space-Time Variational Formulation with a Type of Hilbert Transform

In this section, the space-time variational formulation of anisotropic type of Section 3.3 is examined with the help of a type of Hilbert transform, see [145]. Via a Fourier series ansatz a transformation operator \mathcal{H}_T is introduced, and existence and uniqueness of the space-time variational formulation of anisotropic type of Section 3.3 is proven directly, i.e. no transposition and interpolation argument is needed, see also [35, Remark 2.13, page 507]. For the resulting space-time variational formulation of this section, ansatz and test spaces are equal. Furthermore, the used analysis is developed on a finite time interval $(0, T)$ instead of considering an unbounded time interval $(0, \infty)$ as in [37, 52, 101]. Moreover, a conforming discretisation of the resulting variational formulation leads to an unconditionally stable finite element method, which is combinable with the boundary element method as in [35] via a FEM-BEM coupling. In the last part of this section, unconditional stability for unstructured space-time meshes, error estimates in $L^2(Q)$, in $H^1(Q)$ and in the anisotropic Sobolev space $H_0^{1/2}(0, T; L^2(\Omega))$ for a tensor-product approach are proven. Furthermore, numerical examples, which confirm the theoretical results, are presented.

3.4.1 Characterisation of $H_0^{1/2}(0, T)$ and $H_{,0}^{1/2}(0, T)$ via Fourier Series

In this subsection, the interpolation of function spaces as in [102, Chapitre 1, Section 2.1, page 11] is considered. Hence, all functions are complex-valued in this subsection, i.e. $H^s(0, T; \mathbb{C})$ and $H_0^{1/2}(0, T; \mathbb{C})$ are the complex-valued versions of the Sobolev spaces of Section 2.2. With the notations of [102, Chapitre 1, Section 2.1, page 11] let $Y := L^2(0, T; \mathbb{C})$ be the usual complex Hilbert space with the inner product

$$(u, v)_{L^2(0, T; \mathbb{C})} := \int_0^T u(t) \overline{v(t)} dt,$$

and let the complex Hilbert space $X := H_0^1(0, T; \mathbb{C})$ be endowed with the inner product

$$(u, v)_{H_0^1(0, T; \mathbb{C})} := \int_0^T \partial_t u(t) \overline{\partial_t v(t)} dt.$$

Clearly, X and Y are separable and X is a dense subset of Y with a compact embedding, see [13, Proof of Satz 5.12, page 148].

Next, an unbounded operator $\Lambda: Y \supset \text{dom}(\Lambda) \rightarrow Y$ with domain $\text{dom}(\Lambda) = X$ is constructed such that Λ is self-adjoint and positive in Y . Therefore, define the unbounded

sesquilinear form $\mathfrak{a}: \text{dom}(\mathfrak{a}) \times \text{dom}(\mathfrak{a}) \rightarrow \mathbb{C}$ by

$$\mathfrak{a}(u, v) := (u, v)_{H_0^1(0, T; \mathbb{C})} = \int_0^T \partial_t u(t) \overline{\partial_t v(t)} dt$$

for $u, v \in \text{dom}(\mathfrak{a}) := X = H_0^1(0, T; \mathbb{C}) \subset Y$. The sesquilinear form \mathfrak{a} is densely defined, symmetric, closed and lower semibounded in Y , i.e.

- $\text{dom}(\mathfrak{a})$ is dense in Y ,
- it holds true that $\mathfrak{a}(u, v) = \overline{\mathfrak{a}(v, u)}$ for all $u, v \in \text{dom}(\mathfrak{a})$,
- $(\text{dom}(\mathfrak{a}), (\cdot, \cdot)_{\mathfrak{a}})$ is a Hilbert space with the inner product $(u, v)_{\mathfrak{a}} := \mathfrak{a}(u, v)$,
- it holds true that

$$\mathfrak{a}(u, u) = \|u\|_{H_0^1(0, T; \mathbb{C})}^2 \geq \frac{\pi^2}{4T^2} \|u\|_{L^2(0, T; \mathbb{C})}^2$$

for all $u \in \text{dom}(\mathfrak{a})$ due to the Poincaré inequality, see Lemma 3.4.5 for the constant.

The Representation Theorem for Semibounded Forms, see [133, Theorem 10.7, page 228] and see also [90, Theorem 2.1, page 322], [157, Unterkapitel 4.2], yields that there exists a uniquely determined, lower semibounded, self-adjoint operator $S: Y \supset \text{dom}(S) \rightarrow Y$ such that

- it holds true that $\text{dom}(S) \subset \text{dom}(\mathfrak{a}) = X$,
- for all $u \in \text{dom}(S)$ and $v \in \text{dom}(\mathfrak{a})$, it holds true that

$$\mathfrak{a}(u, v) = (Su, v)_{L^2(0, T; \mathbb{C})}, \quad (3.25)$$

- it holds true that

$$\text{dom}(S) = \left\{ u \in X : \exists w_u \in Y : \forall v \in \text{dom}(\mathfrak{a}) : \mathfrak{a}(u, v) = (w_u, v)_{L^2(0, T; \mathbb{C})} \right\},$$

and the operator is given by $Su := w_u$ for $u \in \text{dom}(S)$,

- a lower bound for S is given by

$$(Su, u)_{L^2(0, T; \mathbb{C})} \geq \frac{\pi^2}{4T^2} \|u\|_{L^2(0, T; \mathbb{C})}^2$$

for all $u \in \text{dom}(S)$.

Lemma 3.4.1. *The unbounded operator $S: Y \supset \text{dom}(S) \rightarrow Y$ from above is given by*

$$Su = -\partial_{tt}u$$

and

$$\text{dom}(S) = \{u \in H^2(0, T; \mathbb{C}) : u(0) = 0, u'(T) = 0\}.$$

Proof. Let $u \in \text{dom}(S)$ be fixed. It is shown that u satisfies $u \in H^2(0, T; \mathbb{C})$ with $u(0) = 0$ and $u'(T) = 0$, i.e. u is contained in the right-hand side. First, because $\text{dom}(S) \subset \text{dom}(\mathfrak{a})$, it holds true that $u \in H_0^1(0, T; \mathbb{C})$ and so, $u(0) = 0$. Second, for $v \in C_0^\infty(0, T)$ in (3.25), it follows with integration by parts that

$$(Su, v)_{L^2(0, T; \mathbb{C})} = \mathfrak{a}(u, v) = -(u, \partial_{tt}v)_{L^2(0, T; \mathbb{C})},$$

hence, $Su = -\partial_{tt}u \in Y$ and so, $u \in H^2(0, T; \mathbb{C})$. Third, for $v \in C^\infty[0, T]$ with $v(0) = 0$ and $v(T) = 1$ in (3.25), it follows with integration by parts that

$$\mathfrak{a}(u, v) = (Su, v)_{L^2(0, T; \mathbb{C})} = -(\partial_{tt}u, v)_{L^2(0, T; \mathbb{C})} = -u'(T)\overline{v(T)} + \mathfrak{a}(u, v) \iff u'(T) = 0$$

and so, u is contained in the right-hand side.

Next, let u satisfy $u \in H^2(0, T; \mathbb{C})$ with $u(0) = 0$ and $u'(T) = 0$, i.e. u is contained in the right-hand side. Hence, $u \in X = H_0^1(0, T; \mathbb{C})$. The function u is contained in $\text{dom}(S)$, because for $w_u := -\partial_{tt}u \in L^2(0, T; \mathbb{C})$ with integration by parts, it holds true that

$$\mathfrak{a}(u, v) = -(\partial_{tt}u, v)_{L^2(0, T; \mathbb{C})} = (w_u, v)_{L^2(0, T; \mathbb{C})}$$

for all $v \in \text{dom}(\mathfrak{a})$. Thus, $Su = w_u = -\partial_{tt}u$. □

The Second Representation Theorem [90, Theorem 2.23, page 331] yields that the square root $\Lambda := S^{1/2}: Y \supset \text{dom}(\Lambda) \rightarrow Y$ fulfils $\text{dom}(\Lambda) = \text{dom}(\mathfrak{a}) = X = H_0^1(0, T; \mathbb{C})$ and

$$\mathfrak{a}(u, v) = (\Lambda u, \Lambda v)_{L^2(0, T; \mathbb{C})} \quad \text{for all } u, v \in X.$$

Recall that $\Lambda: Y \supset \text{dom}(\Lambda) \rightarrow Y$ is self-adjoint and positive in Y , because Λ is the unique square root of the self-adjoint and positive operator $S: Y \supset \text{dom}(S) \rightarrow Y$, see [157, Satz 8.22, page 303] or [133, Proposition 5.13, page 95].

Because of the compact embedding of X in Y , the operator $S: Y \supset \text{dom}(S) \rightarrow Y$ has a purely discrete spectrum, see [133, Proposition 10.6, page 227]. A simple calculation gives

$$V_k(t) := \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad \lambda_k := \frac{1}{T^2}\left(\frac{\pi}{2} + k\pi\right)^2 \quad \text{for } k \in \mathbb{N}_0, \quad (3.26)$$

which fulfil for $k \in \mathbb{N}_0$

$$-\partial_{tt}V_k(t) = \lambda_k V_k(t) \quad \text{for } t \in (0, T), \quad V_k(0) = 0, \quad \partial_t V_k(T) = 0,$$

i.e. $SV_k = \lambda_k V_k$ for $k \in \mathbb{N}_0$. These eigenfunctions V_k form an orthogonal basis in $L^2(0, T; \mathbb{C})$ satisfying

$$\int_0^T V_k(t)V_\ell(t)dt = \frac{T}{2} \delta_{k\ell} \quad \text{for } k, \ell \in \mathbb{N}_0$$

and in $H_0^1(0, T; \mathbb{C})$ with

$$\int_0^T \partial_t V_k(t)\partial_t V_\ell(t)dt = \lambda_k \int_0^T V_k(t)V_\ell(t)dt = \frac{1}{2T} \left(\frac{\pi}{2} + k\pi \right)^2 \delta_{k\ell} \quad \text{for } k, \ell \in \mathbb{N}_0.$$

Hence, by Parseval's identity, see [157, Entwicklungssatz 1.55, page 53] and [158, Satz V.4.9, page 254], it follows that for $u \in L^2(0, T; \mathbb{C})$, the expansion

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad u_k = \frac{2}{T} \int_0^T u(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt \quad (3.27)$$

holds true, and the norm is given by

$$\|u\|_{L^2(0, T; \mathbb{C})}^2 = \frac{2}{T} \sum_{k=0}^{\infty} \left| (u, V_k)_{L^2(0, T; \mathbb{C})} \right|^2 = \frac{T}{2} \sum_{k=0}^{\infty} |u_k|^2. \quad (3.28)$$

Furthermore, for the inner product, it follows that

$$(u, v)_{L^2(0, T; \mathbb{C})} = \int_0^T u(t)\overline{v(t)}dt = \frac{T}{2} \sum_{k=0}^{\infty} u_k \overline{v_k} \quad (3.29)$$

for $u, v \in L^2(0, T; \mathbb{C})$ with expansion coefficients u_k and v_k from (3.27). For a function $u \in H_0^1(0, T; \mathbb{C})$, the expansion (3.27) converges also in $H_0^1(0, T; \mathbb{C})$, i.e.

$$\partial_t u(t) = \frac{1}{T} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) u_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)$$

converges in $L^2(0, T; \mathbb{C})$, and the norm is given by

$$\|\partial_t u\|_{L^2(0, T; \mathbb{C})}^2 = \frac{2}{T} \sum_{k=0}^{\infty} \lambda_k \left| (u, V_k)_{L^2(0, T; \mathbb{C})} \right|^2 = \frac{1}{2T} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right)^2 |u_k|^2. \quad (3.30)$$

Before defining the interpolation space $H_0^{1/2}(0, T; \mathbb{C})$, note that for $\theta \in (0, 1)$ the powers $S^\theta: Y \supset \text{dom}(S^\theta) \rightarrow Y$ are given by the so-called Functional Calculus, see [133, Section 5.3] or [157, Unterkapitel 8.4], i.e.

$$\text{dom}(S^\theta) = \left\{ u \in Y : \frac{2}{T} \sum_{k=0}^{\infty} \lambda_k^{2\theta} \left| (u, V_k)_{L^2(0, T; \mathbb{C})} \right|^2 < \infty \right\}$$

and the equality

$$\|S^\theta u\|_{L^2(0, T; \mathbb{C})}^2 = \frac{2}{T} \sum_{k=0}^{\infty} \lambda_k^{2\theta} \left| (u, V_k)_{L^2(0, T; \mathbb{C})} \right|^2$$

holds true, see [133, Theorem 5.9, page 93].

Plugging these results in the definition of the interpolation spaces [102, Définition 2.1, page 12] gives with the expansion (3.27)

$$\begin{aligned} H_0^{1/2}(0, T; \mathbb{C}) &= [H_0^1(0, T; \mathbb{C}), L^2(0, T; \mathbb{C})]_{1/2} = \text{dom}(\Lambda^{1/2}) = \text{dom}(S^{1/4}) \\ &= \left\{ u \in L^2(0, T; \mathbb{C}) : \frac{2}{T} \sum_{k=0}^{\infty} \lambda_k^{1/2} \left| (u, V_k)_{L^2(0, T; \mathbb{C})} \right|^2 < \infty \right\} \\ &= \left\{ u \in L^2(0, T; \mathbb{C}) : u = \sum_{k=0}^{\infty} u_k V_k, \quad \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |u_k|^2 < \infty \right\} \end{aligned}$$

with the interpolation norm for $u \in H_0^{1/2}(0, T; \mathbb{C})$

$$\begin{aligned} \|u\|_{[H_0^1(0, T; \mathbb{C}), L^2(0, T; \mathbb{C})]_{1/2}} &= \sqrt{\|u\|_{L^2(0, T; \mathbb{C})}^2 + \|\Lambda^{1/2} u\|_{L^2(0, T; \mathbb{C})}^2} \\ &= \sqrt{\frac{2}{T} \sum_{k=0}^{\infty} \left(1 + \lambda_k^{1/2} \right) \left| (u, V_k)_{L^2(0, T; \mathbb{C})} \right|^2} \\ &= \sqrt{\frac{1}{2} \sum_{k=0}^{\infty} \left(T + \left(\frac{\pi}{2} + k\pi \right) \right) |u_k|^2}, \end{aligned} \quad (3.31)$$

see also [102, Proof of Théorème 16.2, page 112] and [15, Section 11.5] for such a construction. This motivates to define for $u, v \in H_0^{1/2}(0, T; \mathbb{C})$ with expansion (3.27) the norm

$$\|u\|_{H_0^{1/2}(0, T; \mathbb{C}), F} := \sqrt{\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |u_k|^2}, \quad (3.32)$$

as well as the inner product

$$(u, v)_{H_0^{1/2}(0, T; \mathbb{C}), F} := \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) u_k \overline{v_k},$$

where the subscript F stands for *Fourier series*.

Theorem 3.4.2. For all $u \in H_0^{1/2}(0, T; \mathbb{C})$, the norm equivalences

$$\|u\|_{H_0^{1/2}(0, T; \mathbb{C}), F} \leq \|u\|_{[H_0^1(0, T; \mathbb{C}), L^2(0, T; \mathbb{C})]_{1/2}} \leq \sqrt{1 + \frac{2T}{\pi}} \|u\|_{H_0^{1/2}(0, T; \mathbb{C}), F}$$

and

$$C_1 \|u\|_{H_0^{1/2}(0, T; \mathbb{C}), F} \leq \sqrt{\|u\|_{H^{1/2}(0, T; \mathbb{C})}^2 + \int_0^T \frac{|u(t)|^2}{t} dt} \leq C_2 \|u\|_{H_0^{1/2}(0, T; \mathbb{C}), F}$$

hold true with constants $C_1 > 0$ and $C_2 > 0$. Hence, $\left(H_0^{1/2}(0, T; \mathbb{C}), (\cdot, \cdot)_{H_0^{1/2}(0, T; \mathbb{C}), F}\right)$ is a Hilbert space.

Proof. From [102, Théorème 11.7, page 72] and [102, Remarque 11.4, page 75], it follows that the norm $u \mapsto \sqrt{\|u\|_{H^{1/2}(0, T; \mathbb{C})}^2 + \int_0^T \frac{|u(t)|^2}{t} dt}$ is equivalent to the interpolation norm $\|\cdot\|_{[H_0^1(0, T; \mathbb{C}), L^2(0, T; \mathbb{C})]_{1/2}}$, see also [102, Remarque 2.3, page 13]. It remains to prove that the norm $\|\cdot\|_{H_0^{1/2}(0, T; \mathbb{C}), F}$ and the interpolation norm $\|\cdot\|_{[H_0^1(0, T; \mathbb{C}), L^2(0, T; \mathbb{C})]_{1/2}}$ are equivalent. The first inequality $\|u\|_{H_0^{1/2}(0, T; \mathbb{C}), F} \leq \|u\|_{[H_0^1(0, T; \mathbb{C}), L^2(0, T; \mathbb{C})]_{1/2}}$ is trivial because of (3.31). The second inequality follows from

$$\begin{aligned} \|u\|_{[H_0^1(0, T; \mathbb{C}), L^2(0, T; \mathbb{C})]_{1/2}}^2 &= \frac{1}{2} \sum_{k=0}^{\infty} \left(T + \left(\frac{\pi}{2} + k\pi\right)\right) |u_k|^2 \\ &\leq \left(\frac{2T}{\pi} + 1\right) \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) |u_k|^2 \\ &= \left(\frac{2T}{\pi} + 1\right) \|u\|_{H_0^{1/2}(0, T; \mathbb{C}), F}^2, \end{aligned}$$

where the representation (3.31) is used again. \square

Remark 3.4.3. For the explicit calculation of boundedness constants, an interpolation argument, i.e. the Interpolation Theorem [30, Proposition 14.1.5, page 373] or [26, Theorem 3.1.2, page 40] or [153, Section 1.3] for the so-called K-Method of Interpolation with the interpolation norm $\|\cdot\|_{K_\theta(X; Y)}$ is used. Interpolating the Hilbert spaces $H_0^1(0, T; \mathbb{C})$ and $L^2(0, T; \mathbb{C})$ with the K-Method of Interpolation yields again $H_0^{1/2}(0, T; \mathbb{C})$ with the to

$$\begin{aligned}
& \|\cdot\|_{[H_0^1(0,T;\mathbb{C}),L^2(0,T;\mathbb{C})]_{1/2}} \text{ equivalent norm } \|\cdot\|_{K_{1/2}(H_0^1(0,T;\mathbb{C});L^2(0,T;\mathbb{C}))} \text{ fulfilling} \\
& \|u\|_{K_{1/2}(H_0^1(0,T;\mathbb{C});L^2(0,T;\mathbb{C}))}^2 = \frac{\pi}{2} \left[\|u\|_{[H_0^1(0,T;\mathbb{C}),L^2(0,T;\mathbb{C})]_{1/2}}^2 - \|u\|_{L^2(0,T;\mathbb{C})}^2 \right] \\
& = \frac{\pi}{2} \left\| \Lambda^{1/2} u \right\|_{L^2(0,T;\mathbb{C})}^2 \\
& = \frac{\pi}{2} \|u\|_{H_0^{1/2}(0,T;\mathbb{C}),F}^2 \tag{3.33}
\end{aligned}$$

for $u \in H_0^{1/2}(0,T;\mathbb{C})$ with (3.31), see [102, Proof of Théorème 15.1, page 108].

Next, the result of Theorem 3.4.2 is transferred to real-valued functions. Hence, for the real Hilbert space $H_0^{1/2}(0,T)$, see (2.1), the representation

$$H_0^{1/2}(0,T) = \left\{ u \in L^2(0,T) : u = \sum_{k=0}^{\infty} u_k V_k, \quad \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |u_k|^2 < \infty \right\}$$

holds true, and

$$\langle u, v \rangle_{H_0^{1/2}(0,T),F} := \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) u_k v_k$$

is an inner product, which induces a to $\|\cdot\|_{H_0^{1/2}(0,T)}$ equivalent norm, where $u_k, v_k \in \mathbb{R}$ are the expansion coefficients given by (3.27).

Analogously, the real Hilbert space $H_0^{1/2}(0,T)$ is investigated. Here, only the notations are introduced and some properties are stated. The eigenfunctions and eigenvalues

$$W_k(t) := \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad \hat{\lambda}_k := \frac{1}{T^2} \left(\frac{\pi}{2} + k\pi\right)^2, \quad k \in \mathbb{N}_0, \tag{3.34}$$

fulfil for $k \in \mathbb{N}_0$

$$-\partial_{tt} W_k(t) = \hat{\lambda}_k W_k(t) \quad \text{for } t \in (0,T), \quad \partial_t W_k(0) = 0, \quad W_k(T) = 0.$$

Note that $\lambda_k = \hat{\lambda}_k$ and $\partial_t V_k = \sqrt{\lambda_k} W_k$ for all $k \in \mathbb{N}_0$. These eigenfunctions W_k form an orthogonal basis in $L^2(0,T)$ satisfying

$$\int_0^T W_k(t) W_\ell(t) dt = \frac{T}{2} \delta_{k\ell} \quad \text{for } k, \ell \in \mathbb{N}_0$$

and in $H_0^1(0,T)$ with

$$\int_0^T \partial_t W_k(t) \partial_t W_\ell(t) dt = \lambda_k \int_0^T W_k(t) W_\ell(t) dt = \frac{1}{2T} \left(\frac{\pi}{2} + k\pi\right)^2 \delta_{k\ell} \quad \text{for } k, \ell \in \mathbb{N}_0.$$

Hence, by Parseval's identity, it follows that for $w \in L^2(0, T)$, the expansion

$$w(t) = \sum_{k=0}^{\infty} w_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad w_k = \frac{2}{T} \int_0^T w(t) \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt, \quad (3.35)$$

holds true, and the norm is given by

$$\|w\|_{L^2(0,T)}^2 = \frac{2}{T} \sum_{k=0}^{\infty} \left| (w, W_k)_{L^2(0,T)} \right|^2 = \frac{T}{2} \sum_{k=0}^{\infty} |w_k|^2.$$

Furthermore, for the inner product, it follows that

$$\langle w, z \rangle_{L^2(0,T)} = \int_0^T w(t)z(t)dt = \frac{T}{2} \sum_{k=0}^{\infty} w_k z_k \quad (3.36)$$

for $w, z \in L^2(0, T)$ with expansion coefficients w_k and z_k from (3.35). For $w \in H_0^1(0, T)$, the expansion (3.35) converges also in $H_0^1(0, T)$, i.e.

$$\partial_t w(t) = -\frac{1}{T} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) w_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)$$

converges in $L^2(0, T)$, and the norm is given by

$$\|\partial_t w\|_{L^2(0,T)}^2 = \frac{2}{T} \sum_{k=0}^{\infty} \lambda_k \left| (w, W_k)_{L^2(0,T)} \right|^2 = \frac{1}{2T} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^2 |w_k|^2.$$

For the real Hilbert space $H_0^{1/2}(0, T)$, see (2.2), the representation

$$H_0^{1/2}(0, T) = \left\{ u \in L^2(0, T) : u = \sum_{k=0}^{\infty} w_k W_k, \quad \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) |w_k|^2 < \infty \right\}$$

holds true, and

$$\langle w, z \rangle_{H_0^{1/2}(0,T),F} := \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) w_k z_k$$

is an inner product, which induces a to $\|\cdot\|_{H_0^{1/2}(0,T)}$ equivalent norm, where $w_k, z_k \in \mathbb{R}$ are the expansion coefficients given by (3.35).

Finally, representations of the dual spaces $[H_0^{1/2}(0, T)]'$ and $[H_0^1(0, T)]'$ are given. In Section 2.2 the dual space $[H_0^1(0, T)]'$ is characterised as a completion of $L^2(0, T)$ with respect to the Hilbertian norm $\|\cdot\|_{[H_0^1(0,T)]'}$, where $\|\cdot\|_{H_0^1(0,T)} = |\cdot|_{H^1(0,T)}$ is the norm in $H_0^1(0, T)$.

Analogously, in Section 2.2, the dual space $[H_0^{1/2}(0, T)]'$ is characterised as a completion of $L^2(0, T)$ with respect to the Hilbertian norm $\|\cdot\|_{[H_0^{1/2}(0, T)]'}$, where $\|\cdot\|_{H_0^{1/2}(0, T)}$ is the norm in $H_0^{1/2}(0, T)$, see (2.3). Here, a to $\|\cdot\|_{[H_0^{1/2}(0, T)]'}$ equivalent norm is given by

$$\|f\|_{[H_0^{1/2}(0, T)]', F} := \sup_{0 \neq w \in H_0^{1/2}(0, T)} \frac{|f(w)|}{\|w\|_{H_0^{1/2}(0, T), F}}$$

for $f \in [H_0^{1/2}(0, T)]'$. With the help of the expansion (3.35), the following lemma holds true, see also [96, Section 8.1].

Lemma 3.4.4. *For $f \in [H_0^{1/2}(0, T)]'$, the representation*

$$\|f\|_{[H_0^{1/2}(0, T)]', F}^2 = \frac{T^2}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-1} f_k^2$$

is valid, where $f_k := \frac{2}{T} f(W_k)$ with $W_k(t) = \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right)$. Furthermore, the dual space is given by

$$[H_0^{1/2}(0, T)]' = \left\{ g: H_0^{1/2}(0, T) \rightarrow \mathbb{R}: \quad g(w) = \frac{T}{2} \sum_{k=0}^{\infty} w_k g_k \text{ with } (g_k)_{k \in \mathbb{N}_0} \subset \mathbb{R} \right. \\ \left. \text{satisfying } \frac{T^2}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-1} g_k^2 < \infty \right\},$$

where the expansion coefficients w_k are from (3.35).

Analogously, for $f \in [H_0^1(0, T)]'$, the expansion

$$\|f\|_{[H_0^1(0, T)]'}^2 = \frac{T^3}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-2} f_k^2$$

is true, where $f_k := \frac{2}{T} f(W_k)$ with $W_k(t) = \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right)$.

Proof. Let $f \in [H_0^{1/2}(0, T)]'$ be fixed. For $w \in H_0^{1/2}(0, T)$, the representations

$$w(t) = \sum_{k=0}^{\infty} w_k \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) = \sum_{k=0}^{\infty} w_k W_k(t), \quad \|w\|_{H_0^{1/2}(0, T), F}^2 = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) w_k^2$$

and therefore, with the continuity of f ,

$$f(w) = \sum_{k=0}^{\infty} w_k f(W_k) = \frac{T}{2} \sum_{k=0}^{\infty} w_k f_k$$

hold true. Set $w_k^N := \left(\frac{\pi}{2} + k\pi\right)^{-1} f_k$ for $k = 0, \dots, N$ and $w_k^N = 0$ for $k > N$. Assume w.l.o.g. that $w^N := \sum_{k=0}^N w_k^N W_k \neq 0$. Thus, $w^N \in H_0^{1/2}(0, T)$ and it is obtained that

$$\begin{aligned} \left(\frac{T^2}{2} \sum_{k=0}^N \left(\frac{\pi}{2} + k\pi\right)^{-1} f_k^2\right)^{1/2} &= \frac{\sqrt{\frac{T^2}{2}} \sum_{k=0}^N \left(\frac{\pi}{2} + k\pi\right)^{-1} f_k^2}{\left(\sum_{k=0}^N \left(\frac{\pi}{2} + k\pi\right)^{-1} f_k^2\right)^{1/2}} = \frac{\sqrt{\frac{T^2}{2}} \sum_{k=0}^N w_k^N f_k}{\left(\sum_{k=0}^N \left(\frac{\pi}{2} + k\pi\right) (w_k^N)^2\right)^{1/2}} \\ &= \frac{f(w^N)}{\|w^N\|_{H_0^{1/2}(0, T), F}} \leq \sup_{0 \neq w \in H_0^{1/2}(0, T)} \frac{|f(w)|}{\|w\|_{H_0^{1/2}(0, T), F}}. \end{aligned}$$

Hence,

$$\|f\|_{[H_0^{1/2}(0, T)]', F}^2 \geq \frac{T^2}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-1} f_k^2$$

follows as $N \rightarrow \infty$.

On the other hand, the Cauchy-Schwarz inequality yields

$$\begin{aligned} \|f\|_{[H_0^{1/2}(0, T)]', F} &= \sup_{0 \neq w \in H_0^{1/2}(0, T)} \frac{|f(w)|}{\|w\|_{H_0^{1/2}(0, T), F}} \\ &= \sqrt{\frac{T^2}{2}} \sup_{0 \neq w \in H_0^{1/2}(0, T)} \frac{\sum_{k=0}^{\infty} w_k f_k \left(\frac{\pi}{2} + k\pi\right)^{-1/2} \left(\frac{\pi}{2} + k\pi\right)^{1/2}}{\left(\sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) w_k^2\right)^{1/2}} \\ &\leq \left(\frac{T^2}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-1} f_k^2\right)^{1/2}. \end{aligned}$$

Hence, the norm equality is proven and f is contained in the right-hand side.

To show that the right-hand side is a subset of $[H_0^{1/2}(0, T)]'$, one defines for a given sequence $(g_k)_{k \in \mathbb{N}_0} \subset \mathbb{R}$, satisfying $\frac{T^2}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-1} g_k^2 < \infty$, the element

$$g(w) := \frac{T}{2} \sum_{k=0}^{\infty} w_k g_k$$

for $w \in H_0^{1/2}(0, T)$ with the coefficients w_k from the expansion (3.35). The linear functional

$g: H_0^{1/2}(0, T) \rightarrow \mathbb{R}$ is well-defined and bounded due to

$$\begin{aligned} |g(w)| &= \left| \frac{T}{2} \sum_{k=0}^{\infty} w_k \left(\frac{\pi}{2} + k\pi \right)^{1/2} g_k \left(\frac{\pi}{2} + k\pi \right)^{-1/2} \right| \\ &\leq \underbrace{\left(\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) w_k^2 \right)^{1/2}}_{=\|w\|_{H_0^{1/2}(0,T),F}^2} \underbrace{\left(\frac{T^2}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right)^{-1} g_k^2 \right)^{1/2}}_{< \infty}. \end{aligned}$$

Hence, $g \in [H_0^{1/2}(0, T)]'$.

For $f \in [H_0^1(0, T)]'$, the proof is obtained in the same manner. \square

For $g \in L^2(0, T)$ with expansion coefficients g_k from expansion (3.35) with respect to W_k and for $w \in H_0^{1/2}(0, T)$ with expansion (3.35), note that

$$\langle g, w \rangle_{L^2(0,T)} = \frac{T}{2} \sum_{k=0}^{\infty} g_k w_k$$

and

$$\frac{T^2}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right)^{-1} g_k^2 \leq \frac{T^2}{2} \sum_{k=0}^{\infty} \frac{2}{\pi} g_k^2 = \frac{2T}{\pi} \|g\|_{L^2(0,T)}^2.$$

Hence, it follows that $g \in [H_0^{1/2}(0, T)]'$ with

$$\|g\|_{[H_0^{1/2}(0,T)]',F} \leq \sqrt{\frac{2T}{\pi}} \|g\|_{L^2(0,T)}$$

and there exists a unique continuous extension of $\langle \cdot, \cdot \rangle_{L^2(0,T)}$ on $[H_0^{1/2}(0, T)]' \times H_0^{1/2}(0, T)$, which is denoted as duality pairing $\langle \cdot, \cdot \rangle_{(0,T)}$. Thus, for $f \in [H_0^{1/2}(0, T)]'$ the usual notation $f(w) = \langle f, w \rangle_{(0,T)}$ for $w \in H_0^{1/2}(0, T)$ is used.

With the help of the representations (3.27) and (3.35), the following lemma states inequalities of Poincaré type with sharp constants.

Lemma 3.4.5. *The following inequalities are sharp:*

1. For $u \in H_0^{1/2}(0, T)$ and $z \in H_0^{1/2}(0, T)$, it holds true that

$$\|u\|_{L^2(0,T)} \leq \sqrt{\frac{2T}{\pi}} \|u\|_{H_0^{1/2}(0,T),F} \quad \text{and} \quad \|z\|_{L^2(0,T)} \leq \sqrt{\frac{2T}{\pi}} \|z\|_{H_0^{1/2}(0,T),F}.$$

2. For $u \in H_0^1(0, T)$ and $z \in H_0^1(0, T)$, it holds true that

$$\|u\|_{L^2(0, T)} \leq \frac{2T}{\pi} \|\partial_t u\|_{L^2(0, T)} \quad \text{and} \quad \|z\|_{L^2(0, T)} \leq \frac{2T}{\pi} \|\partial_t z\|_{L^2(0, T)}.$$

The inequalities remain valid for complex-valued functions.

Proof. The inequalities for $H_0^{1/2}(0, T)$ and $H_0^1(0, T)$ follow from the norm representations (3.28), (3.32) and (3.30) with

$$\|u\|_{L^2(0, T)}^2 = \frac{T}{2} \sum_{k=0}^{\infty} u_k^2 = \frac{2T}{\pi} \frac{1}{2} \sum_{k=0}^{\infty} \frac{\pi}{2} u_k^2 \leq \frac{2T}{\pi} \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) u_k^2 = \frac{2T}{\pi} \|u\|_{H_0^{1/2}(0, T), F}^2$$

and

$$\|u\|_{L^2(0, T)}^2 = \frac{4T^2}{\pi^2} \frac{1}{2T} \sum_{k=0}^{\infty} \left(\frac{\pi}{2}\right)^2 u_k^2 \leq \frac{4T^2}{\pi^2} \frac{1}{2T} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^2 u_k^2 = \frac{4T^2}{\pi^2} \|\partial_t u\|_{L^2(0, T)}^2,$$

which are sharp for functions u with expansion coefficients $u_0 \neq 0$ and $u_k = 0$ for $k \in \mathbb{N}$. Correspondingly, the inequalities for $H_0^{1/2}(0, T)$ and $H_0^1(0, T)$ are proven. \square

3.4.2 Transformation Operator \mathcal{H}_T

First, the distributional derivative ∂_t on $(0, T)$ for a fixed $u \in H_0^{1/2}(0, T)$ is investigated.

Theorem 3.4.6. For $u \in H_0^{1/2}(0, T)$, the distributional derivative ∂_t on $(0, T)$ satisfies $\partial_t u \in [H_0^{1/2}(0, T)]'$. More precisely, a uniquely determined element $g \in [H_0^{1/2}(0, T)]'$ exists, satisfying

$$\forall \varphi \in \mathcal{D}(0, T): \quad \partial_t T_u(\varphi) = \langle g, \varphi \rangle_{(0, T)},$$

where $T_u: \mathcal{D}(0, T) \rightarrow \mathbb{R}$, $T_u(\varphi) = \langle u, \varphi \rangle_{L^2(0, T)}$, is the to the function u related distribution, see Section 2.1.

In addition, it holds true that

$$\|\partial_t u\|_{[H_0^{1/2}(0, T)]', F} \leq \|u\|_{H_0^{1/2}(0, T), F}$$

and

$$\langle \partial_t u, w \rangle_{(0, T)} = \frac{1}{2} \sum_{k=0}^{\infty} u_k \left(\frac{\pi}{2} + k\pi\right) w_k \quad (3.37)$$

for all $w \in H_0^{1/2}(0, T)$ with expansion coefficients u_k from (3.27) and w_k from (3.35).

Proof. For $u \in H_0^{1/2}(0, T)$ with the representations

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) = \sum_{k=0}^{\infty} u_k V_k(t), \quad \|u\|_{H_0^{1/2}(0, T), F}^2 = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) u_k^2,$$

one defines the functional $g \in [H_0^{1/2}(0, T)]'$ via the sequence $(g_k)_{k \in \mathbb{N}_0} \subset \mathbb{R}$ with

$$g_k := \frac{1}{T} u_k \left(\frac{\pi}{2} + k\pi\right), \quad k \in \mathbb{N}_0,$$

satisfying

$$\frac{T^2}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-1} g_k^2 = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) u_k^2 = \|u\|_{H_0^{1/2}(0, T)}^2 < \infty,$$

see Lemma 3.4.4. Hence, $g \in [H_0^{1/2}(0, T)]'$ is well-defined. Next, consider the related distribution $T_u: \mathcal{D}(0, T) \rightarrow \mathbb{R}$ defined by

$$T_u(\varphi) = \langle u, \varphi \rangle_{L^2(0, T)}$$

for $\varphi \in \mathcal{D}(0, T)$. The distributional derivative ∂_t on $(0, T)$ is given by

$$\begin{aligned} \partial_t T_u(\varphi) &= -T_u(\partial_t \varphi) = -\langle u, \partial_t \varphi \rangle_{L^2(0, T)} = -\sum_{k=0}^{\infty} u_k \langle V_k, \partial_t \varphi \rangle_{L^2(0, T)} \\ &= \sum_{k=0}^{\infty} u_k \langle \partial_t V_k, \varphi \rangle_{L^2(0, T)} = \frac{1}{T} \sum_{k=0}^{\infty} u_k \left(\frac{\pi}{2} + k\pi\right) \langle W_k, \varphi \rangle_{L^2(0, T)} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} u_k \left(\frac{\pi}{2} + k\pi\right) \varphi_k = \frac{T}{2} \sum_{k=0}^{\infty} g_k \varphi_k = \langle g, \varphi \rangle_{(0, T)} \end{aligned}$$

for $\varphi \in \mathcal{D}(0, T)$ with expansion coefficients $\varphi_k = \frac{2}{T} \langle W_k, \varphi \rangle_{L^2(0, T)}$ from (3.35). Furthermore, with the last calculation and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \left| \langle g, \varphi \rangle_{(0, T)} \right| &= \left| \frac{1}{2} \sum_{k=0}^{\infty} u_k \left(\frac{\pi}{2} + k\pi\right) \varphi_k \right| \\ &\leq \sqrt{\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) u_k^2} \sqrt{\frac{1}{2} \sum_{\ell=0}^{\infty} \left(\frac{\pi}{2} + \ell\pi\right) \varphi_\ell^2} = \|u\|_{H_0^{1/2}(0, T), F} \|\varphi\|_{H_0^{1/2}(0, T), F}. \end{aligned}$$

Due to the density of $C_0^\infty(0, T)$ in $H_0^{1/2}(0, T)$, see Theorem 2.2.2, it holds true that

$$\|g\|_{[H_0^{1/2}(0, T)]', F} \leq \|u\|_{H_0^{1/2}(0, T), F}$$

and the element g is unique. The last equality of the assertion follows from the continuity of g and again from the density of $C_0^\infty(0, T)$ in $H_0^{1/2}(0, T)$. \square

The representation (3.37) motivates to define for $u \in L^2(0, T)$ with expansion (3.27) the function

$$(\mathcal{H}_T u)(t) := \sum_{\ell=0}^{\infty} u_{\ell} W_{\ell}(t) = \sum_{\ell=0}^{\infty} u_{\ell} \cos\left(\left(\frac{\pi}{2} + \ell\pi\right) \frac{t}{T}\right) \quad (3.38)$$

for $t \in (0, T)$. By construction, it holds true that $\mathcal{H}_T u \in L^2(0, T)$. Furthermore,

$$\mathcal{H}_T: L^2(0, T) \rightarrow L^2(0, T)$$

is bijective and norm preserving, i.e.

$$\|\mathcal{H}_T u\|_{L^2(0, T)} = \|u\|_{L^2(0, T)} \quad \text{for all } u \in L^2(0, T),$$

where the inverse transformation operator

$$\mathcal{H}_T^{-1}: L^2(0, T) \rightarrow L^2(0, T)$$

is given by

$$(\mathcal{H}_T^{-1} w)(t) = \sum_{k=0}^{\infty} w_k V_k(t) = \sum_{k=0}^{\infty} w_k \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right), \quad t \in (0, T),$$

for $w \in L^2(0, T)$ with expansion (3.35).

For $u \in H_0^{1/2}(0, T)$ with expansion (3.27), the function

$$(\mathcal{H}_T u)(t) = \sum_{\ell=0}^{\infty} u_{\ell} W_{\ell}(t) = \sum_{\ell=0}^{\infty} u_{\ell} \cos\left(\left(\frac{\pi}{2} + \ell\pi\right) \frac{t}{T}\right)$$

fulfils $\mathcal{H}_T u \in H_0^{1/2}(0, T)$ because the equality

$$\|\mathcal{H}_T u\|_{H_0^{1/2}(0, T), F} = \|u\|_{H_0^{1/2}(0, T), F}$$

holds true, i.e. $\mathcal{H}_T: H_0^{1/2}(0, T) \rightarrow H_0^{1/2}(0, T)$ is norm preserving. Furthermore,

$$\mathcal{H}_T: H_0^{1/2}(0, T) \rightarrow H_0^{1/2}(0, T)$$

is bijective. Analogously,

$$\mathcal{H}_T: H_0^1(0, T) \rightarrow H_0^1(0, T)$$

is norm preserving and bijective.

The representation (3.37) yields for $u, v \in H_0^{1/2}(0, T)$ and $w := \mathcal{H}_T v \in H_0^{1/2}(0, T)$

$$\langle \partial_t u, \mathcal{H}_T v \rangle_{(0, T)} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right) u_k v_k = \langle u, v \rangle_{H_0^{1/2}(0, T), F}$$

and

$$\langle \partial_t u, \mathcal{H}_T u \rangle_{(0,T)} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) u_k^2 = \|u\|_{H_0^{1/2}(0,T),F}^2. \quad (3.39)$$

Hence, the bilinear form $b(\cdot, \cdot): H_0^{1/2}(0,T) \times H_0^{1/2}(0,T) \rightarrow \mathbb{R}$,

$$b(u, v) := \langle \partial_t u, \mathcal{H}_T v \rangle_{(0,T)} = \langle u, v \rangle_{H_0^{1/2}(0,T),F}, \quad u, v \in H_0^{1/2}(0,T), \quad (3.40)$$

is bounded, elliptic and symmetric.

Next, some properties of the operator $\mathcal{H}_T: L^2(0,T) \rightarrow L^2(0,T)$ are given.

Lemma 3.4.7. *For $u \in L^2(0,T)$ and $w \in L^2(0,T)$, the equality*

$$\langle \mathcal{H}_T u, w \rangle_{L^2(0,T)} = \langle u, \mathcal{H}_T^{-1} w \rangle_{L^2(0,T)}$$

is valid.

Proof. For $u \in L^2(0,T)$ and $w \in L^2(0,T)$, the expansions

$$u(t) = \sum_{k=0}^{\infty} u_k \sin \left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right), \quad w(t) = \sum_{\ell=0}^{\infty} w_\ell \cos \left(\left(\frac{\pi}{2} + \ell\pi \right) \frac{t}{T} \right),$$

and

$$(\mathcal{H}_T u)(t) = \sum_{k=0}^{\infty} u_k \cos \left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right), \quad (\mathcal{H}_T^{-1} w)(t) = \sum_{\ell=0}^{\infty} w_\ell \sin \left(\left(\frac{\pi}{2} + \ell\pi \right) \frac{t}{T} \right)$$

hold true with expansion coefficients u_k from (3.27) and w_ℓ from (3.35). Hence, it follows with the representations (3.29), (3.36) that

$$\langle \mathcal{H}_T u, w \rangle_{L^2(0,T)} \stackrel{(3.36)}{=} \frac{T}{2} \sum_{k=0}^{\infty} u_k w_k \stackrel{(3.29)}{=} \langle u, \mathcal{H}_T^{-1} w \rangle_{L^2(0,T)}$$

and therefore, the assertion. \square

Lemma 3.4.8. *For all $v \in L^2(0,T)$, the inequality*

$$\langle v, \mathcal{H}_T v \rangle_{L^2(0,T)} \geq 0 \quad (3.41)$$

is valid.

Proof. By using the representations

$$v(t) = \sum_{k=0}^{\infty} v_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad (\mathcal{H}_T v)(t) = \sum_{\ell=0}^{\infty} v_\ell \cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right),$$

it follows with the continuity of the inner product $\langle \cdot, \cdot \rangle_{L^2(0,T)}$ that

$$\begin{aligned} \langle v, \mathcal{H}_T v \rangle_{L^2(0,T)} &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} v_k v_\ell \int_0^T \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) \cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right) dt \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} v_k v_\ell \int_0^T \left[\sin\left((k+\ell+1)\pi\frac{t}{T}\right) + \sin\left((k-\ell)\pi\frac{t}{T}\right) \right] dt \\ &= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} v_k v_\ell \left[-\frac{T}{(k+\ell+1)\pi} \cos\left((k+\ell+1)\pi\frac{t}{T}\right) \right]_0^T \\ &= \frac{T}{2\pi} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} v_k v_\ell \frac{1}{k+\ell+1} \left[1 - (-1)^{k+\ell+1} \right], \end{aligned}$$

where the second integral is ignored due to symmetry. When splitting k and ℓ into odd and even indices, i.e. $k = 2i, 2i+1$, $\ell = 2j, 2j+1$, this gives

$$\begin{aligned} \langle v, \mathcal{H}_T v \rangle_{L^2(0,T)} &= \frac{T}{\pi} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\frac{v_{2i} v_{2j}}{2i+2j+1} + \frac{v_{2i+1} v_{2j+1}}{2i+2j+3} \right] \\ &= \frac{T}{\pi} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[v_{2i} v_{2j} \int_0^1 x^{2i+2j} dx + v_{2i+1} v_{2j+1} \int_0^1 x^{2i+2j+2} dx \right] \\ &= \lim_{N \rightarrow \infty} \left\{ \frac{T}{\pi} \sum_{i=0}^N \sum_{j=0}^N \left[v_{2i} v_{2j} \int_0^1 x^{2i+2j} dx + v_{2i+1} v_{2j+1} \int_0^1 x^{2i+2j+2} dx \right] \right\} \\ &= \frac{T}{\pi} \lim_{N \rightarrow \infty} \left[\int_0^1 \left(\sum_{i=0}^N v_{2i} x^{2i} \right)^2 dx + \int_0^1 \left(\sum_{i=0}^N v_{2i+1} x^{2i+1} \right)^2 dx \right] \geq 0 \end{aligned}$$

and hence, the assertion follows. \square

Remark 3.4.9. The transformation \mathcal{H}_T is the counterpart on finite intervals $(0, T)$ of the Hilbert transform

$$\mathcal{H}: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

defined by

$$\mathcal{H}v(t) := \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{-\infty}^{t-\varepsilon} \frac{v(s)}{t-s} ds + \int_{t+\varepsilon}^{\infty} \frac{v(s)}{t-s} ds \right), \quad t \in \mathbb{R},$$

for a function $v \in L^2(\mathbb{R})$, see [92, 122]. The Hilbert transform \mathcal{H} has similar properties as the transformation \mathcal{H}_T , see [145].

3.4.3 Variational Formulation for $\partial_t u = f$

To get a first impression of the transformation \mathcal{H}_T , the simple initial value problem

$$\partial_t u(t) = f(t) \quad \text{for } t \in (0, T), \quad u(0) = 0, \quad (3.42)$$

is investigated. The corresponding variational formulation is to find $u \in H_0^{1/2}(0, T)$ such that

$$b(u, v) = \langle f, \mathcal{H}_T v \rangle_{(0, T)} \quad (3.43)$$

for all $v \in H_0^{1/2}(0, T)$, where $f \in [H_0^{1/2}(0, T)]'$ is a given right-hand side and the bilinear form

$$b(\cdot, \cdot): H_0^{1/2}(0, T) \times H_0^{1/2}(0, T) \rightarrow \mathbb{R}, \quad b(u, v) = \langle \partial_t u, \mathcal{H}_T v \rangle_{(0, T)},$$

is bounded, elliptic and symmetric, see (3.40). Hence, existence and uniqueness of a solution $u \in H_0^{1/2}(0, T)$ for the variational formulation (3.43) follow by the Lax-Milgram Theorem, since the right-hand side $f \in [H_0^{1/2}(0, T)]'$ satisfies

$$|\langle f, \mathcal{H}_T v \rangle_{(0, T)}| \leq \|f\|_{[H_0^{1/2}(0, T)]', F} \|\mathcal{H}_T v\|_{H_0^{1/2}(0, T), F} = \|f\|_{[H_0^{1/2}(0, T)]', F} \|v\|_{H_0^{1/2}(0, T), F} \quad (3.44)$$

for all $v \in H_0^{1/2}(0, T)$.

With the notations of Section 2.6, a conforming finite element discretisation of the variational formulation (3.43) is to find

$$u_{h_t} \in S_{h_t, 0}^1(0, T) = \text{span}\{\varphi_k\}_{k=1}^{N_t} \subset H_0^{1/2}(0, T)$$

such that

$$\langle \partial_t u_{h_t}, \mathcal{H}_T v_{h_t} \rangle_{L^2(0, T)} = \langle f, \mathcal{H}_T v_{h_t} \rangle_{(0, T)} \quad (3.45)$$

for all $v_{h_t} \in S_{h_t, 0}^1(0, T)$. Using standard arguments, e.g., [51, 141], there follow the unique solvability of (3.45) and the a priori error estimates

$$\|u - u_{h_t}\|_{H_0^{1/2}(0, T), F} \leq c h_t^{s-1/2} \|u\|_{H^s(0, T)} \quad \text{for } s \in (1/2, 2],$$

$$\|u - u_{h_t}\|_{L^2(0, T)} \leq c h_t^s \|u\|_{H^s(0, T)} \quad \text{for } s \in (1, 2]$$

and

$$\|u - u_{h_t}\|_{H_0^1(0, T)} \leq c h_t^{s-1} \|u\|_{H^s(0, T)} \quad \text{for } s \in (1, 2]$$

with a constant $c > 0$, when assuming $u \in H_0^{1/2}(0, T) \cap H^s(0, T)$. Note that the error estimate in the energy norm $\|u - u_{h_t}\|_{H_0^{1/2}(0, T), F}$ is a consequence of Céa's Lemma and the approximation property of $S_{h_t, 0}^1(0, T)$. The approximation property

$$\inf_{v_{h_t} \in S_{h_t, 0}^1(0, T)} \|v - v_{h_t}\|_{H_0^{1/2}(0, T), F} \leq c \cdot h_t^{3/2} \|v\|_{H^2(0, T)}$$

for $v \in H_0^1(0, T) \cap H^2(0, T)$ is derived by an interpolation argument, see (2.32). For $\|u - u_{h_t}\|_{L^2(0, T)}$, the Aubin-Nitsche trick is used, and for $\|u - u_{h_t}\|_{H_0^1(0, T)}$, an inverse inequality is required, i.e. for this situation a globally quasi-uniform mesh is needed. The Galerkin-Bubnov finite element formulation (3.45) is equivalent to the system of linear equations

$$K_{h_t} \underline{u} = \underline{f}$$

with a symmetric and positive definite stiffness matrix $K_{h_t} \in \mathbb{R}^{N_t \times N_t}$, defined by

$$K_{h_t}[j, k] = \langle \partial_t \varphi_k, \mathcal{H}_T \varphi_j \rangle_{L^2(0, T)} \quad \text{for } k, j = 1, \dots, N_t,$$

and the right-hand side $\underline{f} \in \mathbb{R}^{N_t}$, given by

$$\underline{f}[j] = \langle f, \mathcal{H}_T \varphi_j \rangle_{(0, T)} \quad \text{for } j = 1, \dots, N_t,$$

where high-order integration rules are used for the calculation. The evaluation of the transformed basis functions $\mathcal{H}_T \varphi_k$ can be done by using the definition (3.38). Although the piecewise linear basis functions φ_k have local support, the transformed basis functions $\mathcal{H}_T \varphi_k$ are global, see Figure 3.1 and Figure 3.2, and therefore, the stiffness matrix K_{h_t} is dense.

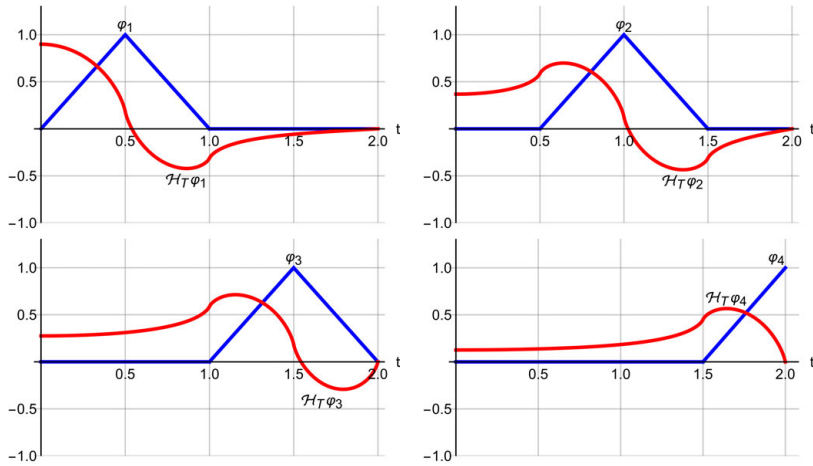
As a numerical example for (3.45), the solution

$$u(t) = \sin\left(\frac{9\pi}{4}t\right) \quad \text{for } t \in (0, 2) = (0, T)$$

of (3.42) with the right-hand side

$$f(t) = \frac{9\pi}{4} \cos\left(\frac{9\pi}{4}t\right) \quad \text{for } t \in (0, 2)$$

is considered. For the discretisation, a sequence of finite element spaces $S_{h_t, 0}^1(0, 2)$ of uniform mesh size $h_t = 2/N_t$, and $N_t = 2^{j+1}$, $j = 0, \dots, 10$, is introduced. Since the solution u is smooth, quadratic convergence in $L^2(0, 2)$ and linear convergence in $H^1(0, 2)$ are expected. This behaviour is confirmed by the numerical results as given in Table 3.9. In addition, the minimal and maximal eigenvalues of the symmetric stiffness matrix K_{h_t} , as well as the resulting spectral condition number of K_{h_t} , which behave as expected for a first-order differential operator, are given in Table 3.9.

Figure 3.1: Transformed basis functions $\mathcal{H}_T \varphi_k$, $k = 1, \dots, N_t$, $N_t = 4$.

N_t	$\ u - u_{h_t}\ _{L^2(0,2)}$	eoc	$\ \partial_t u - \partial_t u_{h_t}\ _{L^2(0,2)}$	eoc	$\lambda_{\min}(K_{h_t})$	$\lambda_{\max}(K_{h_t})$	$\kappa_2(K_{h_t})$
2	1.005e+00	-	7.059e+00	-	0.416617	0.960210	2.3
4	8.612e-01	0.2	5.880e+00	0.3	0.284445	1.116917	3.9
8	1.692e-01	2.3	3.660e+00	0.7	0.168755	1.128029	6.7
16	3.247e-02	2.4	1.826e+00	1.0	0.091472	1.132714	12.4
32	7.486e-03	2.1	9.051e-01	1.0	0.047463	1.133771	23.9
64	1.832e-03	2.0	4.512e-01	1.0	0.024147	1.134042	47.0
128	4.555e-04	2.0	2.254e-01	1.0	0.012174	1.134110	93.2
256	1.137e-04	2.0	1.127e-01	1.0	0.006112	1.134127	185.6
512	2.842e-05	2.0	5.634e-02	1.0	0.003062	1.134131	370.4
1024	7.103e-06	2.0	2.817e-02	1.0	0.001532	1.134133	740.1
2048	1.776e-06	2.0	1.409e-02	1.0	0.000767	1.134134	1479.4

Table 3.9: Numerical results for the Galerkin-Bubnov formulation (3.45).

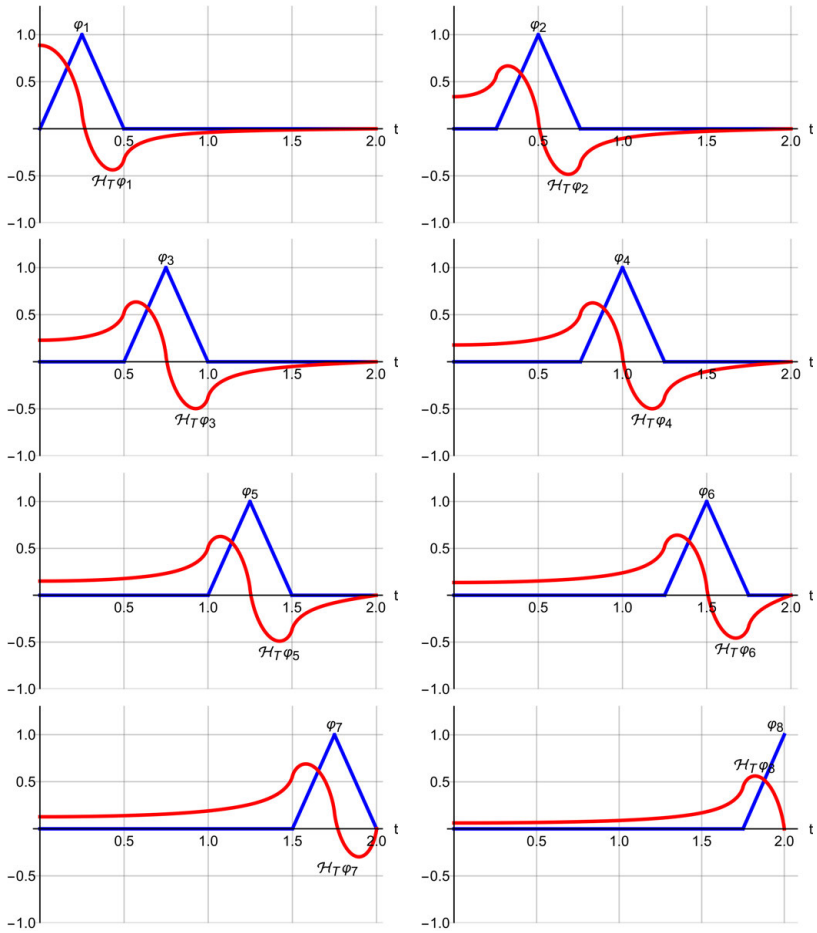


Figure 3.2: Transformed basis functions $\mathcal{H}_T \varphi_k$, $k = 1, \dots, N_I$, $N_I = 8$.

3.4.4 Variational Formulation for $\partial_t u + \mu u = f$

Instead of the initial value problem (3.42), consider for $\mu > 0$ the first-order linear equation

$$\partial_t u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0, T), \quad u(0) = 0, \quad (3.46)$$

and the related variational formulation to find $u \in H_0^{1/2}(0, T)$ such that

$$\langle \partial_t u, \mathcal{H}_T v \rangle_{(0, T)} + \mu \langle u, \mathcal{H}_T v \rangle_{L^2(0, T)} = \langle f, \mathcal{H}_T v \rangle_{(0, T)} \quad (3.47)$$

for all $v \in H_0^{1/2}(0, T)$, where $f \in [H_0^{1/2}(0, T)]'$ is given. The first-order ordinary differential equation (3.46) plays a central role in the analysis of the heat equation, see Subsection 3.4.5, where $\mu > 0$ corresponds to a Dirichlet eigenvalue of the Laplace operator, see (2.4).

Theorem 3.4.10. *Let $\mu > 0$ and the right-hand side $f \in [H_0^{1/2}(0, T)]'$ be given. Then, there exists a unique solution $u \in H_0^{1/2}(0, T)$ of the variational formulation (3.47), satisfying the stability estimate*

$$\|u\|_{H_0^{1/2}(0, T), F} \leq \|f\|_{[H_0^{1/2}(0, T)]', F}. \quad (3.48)$$

Proof. When combining (3.39) with Lemma 3.4.5 and (3.41), this gives

$$\left| \langle \partial_t u, \mathcal{H}_T v \rangle_{(0, T)} + \mu \langle u, \mathcal{H}_T v \rangle_{L^2(0, T)} \right| \leq \left(1 + \frac{2\mu T}{\pi} \right) \|u\|_{H_0^{1/2}(0, T), F} \|v\|_{H_0^{1/2}(0, T), F} \quad (3.49)$$

and

$$\langle \partial_t v, \mathcal{H}_T v \rangle_{(0, T)} + \mu \langle v, \mathcal{H}_T v \rangle_{L^2(0, T)} \geq \langle \partial_t v, \mathcal{H}_T v \rangle_{(0, T)} = \|v\|_{H_0^{1/2}(0, T), F}^2 \quad (3.50)$$

for all $u, v \in H_0^{1/2}(0, T)$, i.e. the bilinear form of the variational problem (3.47) is bounded and elliptic, implying unique solvability of (3.47) by the Lax-Milgram Theorem, including the stability estimate

$$\|u\|_{H_0^{1/2}(0, T), F} \leq \|f\|_{[H_0^{1/2}(0, T)]', F},$$

since the right-hand side f satisfies (3.44). □

A first regularity result is given in the next lemma.

Lemma 3.4.11. *Let $f \in L^2(0, T)$ be given. Then, the unique solution $u \in H_0^{1/2}(0, T)$ of (3.47) is given by*

$$u(t) = \int_0^t e^{\mu(s-t)} f(s) ds \quad (3.51)$$

for $t \in [0, T]$ and fulfils

$$\|\partial_t u\|_{L^2(0, T)} \leq \|f\|_{L^2(0, T)}.$$

Proof. By inserting $\int_0^t e^{\mu(s-t)} f(s) ds$ into the variational formulation (3.47), it follows that the unique solution $u \in H_0^{1/2}(0, T)$ is represented by (3.51) and that $u \in H_0^1(0, T)$. Furthermore, the ordinary differential equation (3.46) holds true for almost all $t \in (0, T)$. Multiplication of (3.46) by $\partial_t u \in L^2(0, T)$ and integration over $(0, T)$ yield

$$\langle f, \partial_t u \rangle_{L^2(0, T)} = \langle \partial_t u, \partial_t u \rangle_{L^2(0, T)} + \mu \langle u, \partial_t u \rangle_{L^2(0, T)} = \|\partial_t u\|_{L^2(0, T)}^2 + \frac{\mu}{2} u(T)^2.$$

The Cauchy-Schwarz inequality gives the assertion. \square

For the analysis of the heat equation, a μ explicit estimate for the solution u in $L^2(0, T)$ is required.

Lemma 3.4.12. *Let $u \in H_0^{1/2}(0, T)$ be the unique solution of the variational formulation (3.47), where $f \in [H_0^{1/2}(0, T)]'$ is given. Then, the inequality*

$$\|u\|_{L^2(0, T)}^2 \leq \frac{T}{2} \sum_{k=0}^{\infty} \frac{f_k^2}{\mu^2 + \frac{1}{T^2} \left(\frac{\pi}{2} + k\pi\right)^2} \quad (3.52)$$

is valid, where

$$f_k := \frac{2}{T} \langle f, W_k \rangle_{(0, T)}, \quad W_k(t) = \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right).$$

Proof. First, note that the right-hand side in (3.52) is finite due to

$$\frac{T}{2} \sum_{k=0}^{\infty} \frac{f_k^2}{\mu^2 + \frac{1}{T^2} \left(\frac{\pi}{2} + k\pi\right)^2} \leq \frac{2T}{\pi} \|f\|_{[H_0^{1/2}(0, T)]', F}^2 < \infty,$$

see Lemma 3.4.4.

Let $(f_n)_{n \in \mathbb{N}} \subset L^2(0, T)$ be a sequence with $\lim_{n \rightarrow \infty} \|f - f_n\|_{[H_0^{1/2}(0, T)]', F} = 0$, see (2.3). Let $u_n \in H_0^{1/2}(0, T)$ be the weak solution of the variational formulation (3.47) with right-hand side f_n . Hence, $u - u_n \in H_0^{1/2}(0, T)$ is the unique solution of (3.47) with right-hand side $f - f_n \in [H_0^{1/2}(0, T)]'$. Thus, the stability estimate (3.48) yields

$$\|u - u_n\|_{H_0^{1/2}(0, T), F} \leq \|f - f_n\|_{[H_0^{1/2}(0, T)]', F} \quad (3.53)$$

and therefore, $u_n \rightarrow u$ in $H_0^{1/2}(0, T)$ as $n \rightarrow \infty$. Write for $f_n \in L^2(0, T)$ the expansion (3.35) as

$$f_n(t) = \sum_{k=0}^{\infty} f_{n,k} \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right), \quad f_{n,k} = \frac{2}{T} \int_0^T f_n(t) \cos\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) dt.$$

First, with $\lambda_k = \frac{1}{T^2} \left(\frac{\pi}{2} + k\pi \right)^2$, the Cauchy-Schwarz inequality and Lemma 3.4.4, it follows that

$$\begin{aligned} \left| \frac{T}{2} \sum_{k=0}^{\infty} \frac{f_k^2}{\mu^2 + \lambda_k} - \frac{T}{2} \sum_{k=0}^{\infty} \frac{f_{n,k}^2}{\mu^2 + \lambda_k} \right| &\leq \frac{T}{2} \sum_{k=0}^{\infty} \frac{1}{\lambda_k} \underbrace{|f_k^2 - f_{n,k}^2|}_{=|f_k - f_{n,k}| |f_k + f_{n,k}|} \\ &\leq \left(\frac{T}{2} \sum_{k=0}^{\infty} \frac{|f_k - f_{n,k}|^2}{\sqrt{\lambda_k}} \right)^{1/2} \left(\frac{T}{2} \sum_{k=0}^{\infty} \frac{|f_k + f_{n,k}|^2}{\lambda_k^{3/2}} \right)^{1/2} \\ &\leq \|f - f_n\|_{[H_0^{1/2}(0,T)]',F} \frac{2T}{\pi} \|f + f_n\|_{[H_0^{1/2}(0,T)]',F} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. In other words, the equality

$$\lim_{n \rightarrow \infty} \frac{T}{2} \sum_{k=0}^{\infty} \frac{f_{n,k}^2}{\mu^2 + \lambda_k} = \frac{T}{2} \sum_{k=0}^{\infty} \frac{f_k^2}{\mu^2 + \lambda_k} \quad (3.54)$$

holds true.

Second, because of $f_n \in L^2(0, T)$, the representation (3.51)

$$\begin{aligned} u_n(t) &= \int_0^t e^{\mu(s-t)} f_n(s) ds = \sum_{k=0}^{\infty} f_{n,k} \cdot e^{-\mu t} \int_0^t e^{\mu s} \cos(\sqrt{\lambda_k} s) ds \\ &= \sum_{k=0}^{\infty} \frac{f_{n,k}}{\mu^2 + \lambda_k} \left[\sqrt{\lambda_k} \sin(\sqrt{\lambda_k} t) + \mu \cos(\sqrt{\lambda_k} t) - \mu e^{-\mu t} \right] \end{aligned}$$

holds true for $t \in [0, T]$, where the continuity of the inner product $\langle \cdot, \cdot \rangle_{L^2(0,t)}$ for fixed $t \in [0, T]$ is used. When computing all integrals, where again the continuity of the inner product $\langle \cdot, \cdot \rangle_{L^2(0,T)}$ is used, one obtains

$$\|u_n\|_{L^2(0,T)}^2 = \frac{T}{2} \sum_{k=0}^{\infty} \frac{f_{n,k}^2}{\mu^2 + \lambda_k} - \frac{1}{2} \mu \left[1 + e^{-2\mu T} \right] \left(\sum_{k=0}^{\infty} \frac{f_{n,k}}{\mu^2 + \lambda_k} \right)^2 \leq \frac{T}{2} \sum_{k=0}^{\infty} \frac{f_{n,k}^2}{\mu^2 + \lambda_k}.$$

Hence, the assertion follows as $n \rightarrow \infty$ with the help of (3.53) and (3.54). \square

Remark 3.4.13. From (3.52), it follows immediately the estimate

$$\|u\|_{L^2(0,T)}^2 \leq \frac{T^3}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right)^{-2} f_k^2 = \|f\|_{[H_0^1(0,T)]'}^2,$$

see Lemma 3.4.4 for the representation of the norm $\|\cdot\|_{[H_0^1(0,T)]'}$. Moreover, when the right-hand side satisfies $f \in L^2(0, T)$, the estimate (3.52) gives

$$\|u\|_{L^2(0,T)}^2 \leq \frac{T}{2\mu^2} \sum_{k=0}^{\infty} f_k^2 = \frac{1}{\mu^2} \|f\|_{L^2(0,T)}^2, \quad \text{i.e.} \quad \mu \|u\|_{L^2(0,T)} \leq \|f\|_{L^2(0,T)}.$$

With the notations of Section 2.6, the Galerkin-Bubnov discretisation of (3.47) is to find $u_{h_t} \in S_{h_t,0}^1(0,T)$ such that

$$\langle \partial_t u_{h_t}, \mathcal{H}_T v_{h_t} \rangle_{L^2(0,T)} + \mu \langle u_{h_t}, \mathcal{H}_T v_{h_t} \rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v_{h_t} \rangle_{(0,T)} \quad (3.55)$$

for all $v_{h_t} \in S_{h_t,0}^1(0,T)$. As for the initial value problem (3.42), the unique solvability of (3.55) follows by Céa's Lemma, including with (3.44) the stability estimate

$$\|u_{h_t}\|_{H_0^{1/2}(0,T),F} \leq \|f\|_{[H_0^{1/2}(0,T)]',F}$$

and with the help of (3.50), (3.49), an error estimate in the energy norm

$$\|u - u_{h_t}\|_{H_0^{1/2}(0,T),F} \leq c \cdot \left(1 + \frac{2\mu T}{\pi}\right) h_t^{s-1/2} \|u\|_{H^s(0,T)}$$

for $s \in (1/2, 2]$ and $u \in H_0^{1/2}(0,T) \cap H^s(0,T)$ with a constant $c > 0$ independent of μ and h_t . Hence, in general, a priori error estimates depend on μ and require a sufficiently small mesh size h_t to ensure convergence for large μ . In the following theorem, a refined error estimate in the energy norm $\|\cdot\|_{H_0^{1/2}(0,T),F}$ and an error estimate in $\|\cdot\|_{L^2(0,T)}$ are given.

Theorem 3.4.14. *Let $u \in H_0^{1/2}(0,T)$ and $u_{h_t} \in S_{h_t,0}^1(0,T)$ be the unique solutions of the variational formulations (3.47) and (3.55). If $u \in H_0^{1/2}(0,T) \cap H^2(0,T)$, then the error estimates*

$$\|u - u_{h_t}\|_{H_0^{1/2}(0,T),F} \leq C_{1/2}(h_t, \mu) \cdot h_t^{3/2} \|\partial_t u\|_{H^1(0,T)}$$

and

$$\|u - u_{h_t}\|_{L^2(0,T)} \leq C_0(h_t, \mu) \cdot h_t^2 \|\partial_t u\|_{H^1(0,T)} \quad (3.56)$$

hold true with

$$C_{1/2}(h_t, \mu) := \frac{1}{\sqrt{\pi} \sqrt[4]{18}} + \frac{1}{\sqrt{24}} \left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2 + 2(1 + 2\mu^2)T^2}}{2\sqrt[4]{6}} \mu h_t \right) \mu h_t$$

and

$$C_0(h_t, \mu) := C_{1/2}(h_t, \mu) \cdot \left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2 + 2(1 + 2\mu^2)T^2}}{2\sqrt[4]{6}} \mu h_t \right).$$

Proof. Using (3.50), the Galerkin orthogonality of the variational formulations (3.47), (3.55) and the norm invariance of \mathcal{H}_T , it holds true that

$$\begin{aligned} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F}^2 &\leq \langle \partial_t(u - u_{h_t}), \mathcal{H}_T(u - u_{h_t}) \rangle_{(0,T)} + \mu \langle u - u_{h_t}, \mathcal{H}_T(u - u_{h_t}) \rangle_{L^2(0,T)} \\ &= \langle \partial_t(u - u_{h_t}), \mathcal{H}_T(u - I_{h_t}u) \rangle_{(0,T)} + \mu \langle u - u_{h_t}, \mathcal{H}_T(u - I_{h_t}u) \rangle_{L^2(0,T)} \\ &\leq \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F} \|u - I_{h_t}u\|_{H_0^{1/2}(0,T),F} + \mu \|u - u_{h_t}\|_{L^2(0,T)} \|u - I_{h_t}u\|_{L^2(0,T)}. \end{aligned} \quad (3.57)$$

To estimate the error in $L^2(0, T)$, consider the adjoint problem

$$-\partial_t w(t) + \mu w(t) = u(t) - u_{h_t}(t) \quad \text{for } t \in (0, T), \quad w(T) = 0, \quad (3.58)$$

i.e. the function $w \in H_{0,0}^{1/2}(0, T)$ is the unique solution of the variational problem

$$-\langle \partial_t w, v \rangle_{(0,T)} + \mu \langle w, v \rangle_{L^2(0,T)} = \langle u - u_{h_t}, v \rangle_{L^2(0,T)} \quad (3.59)$$

for all $v \in H_0^{1/2}(0, T)$. Analogous to Lemma 3.4.11, the regularity result

$$\|\partial_t w\|_{L^2(0,T)} \leq \|u - u_{h_t}\|_{L^2(0,T)} \quad (3.60)$$

holds true and in addition, $\partial_t w \in H^1(0, T)$. Hence, from the differential equation in (3.58), one finds

$$\partial_{tt} w(t) = \mu \partial_t w(t) - \partial_t [u(t) - u_{h_t}(t)] \quad \text{for } t \in (0, T),$$

and therefore, with (3.60) and the Poincaré inequality of Lemma 3.4.5, the estimate

$$\begin{aligned} \|\partial_t w\|_{H^1(0,T)}^2 &= \|\partial_t w\|_{L^2(0,T)}^2 + \|\partial_{tt} w\|_{L^2(0,T)}^2 \\ &\leq (1 + 2\mu^2) \|\partial_t w\|_{L^2(0,T)}^2 + 2 \|\partial_t (u - u_{h_t})\|_{L^2(0,T)}^2 \\ &\leq (1 + 2\mu^2) \|u - u_{h_t}\|_{L^2(0,T)}^2 + 2 \|\partial_t (u - u_{h_t})\|_{L^2(0,T)}^2 \\ &\leq (1 + 2\mu^2) \frac{4T^2}{\pi^2} \|\partial_t (u - u_{h_t})\|_{L^2(0,T)}^2 + 2 \|\partial_t (u - u_{h_t})\|_{L^2(0,T)}^2 \\ &= \left(2 + \frac{4T^2}{\pi^2} + \frac{8T^2\mu^2}{\pi^2} \right) \|u - u_{h_t}\|_{H_0^1(0,T)}^2 \end{aligned} \quad (3.61)$$

follows. Since $\partial_t w \in H^1(0, T)$, an interpolation argument for the K -Method of Interpolation between (3.60) and (3.61) yields the estimate

$$\begin{aligned} \|\partial_t w\|_{K_{1/2}(H^1(0,T); L^2(0,T))} &\leq \sqrt[4]{2 + \frac{4T^2}{\pi^2} + \frac{8T^2\mu^2}{\pi^2}} \|u - u_{h_t}\|_{K_{1/2}(H_0^1(0,T); L^2(0,T))} \\ &= \sqrt{\frac{\pi}{2}} \sqrt[4]{2 + \frac{4T^2}{\pi^2} + \frac{8T^2\mu^2}{\pi^2}} \|u - u_{h_t}\|_{H_0^{1/2}(0,T), F} \end{aligned} \quad (3.62)$$

with the help of the norm equivalence (3.33).

For $v = u - u_{h_t} \in H_0^{1/2}(0, T)$ in (3.59) and with the Galerkin orthogonality for $u - u_{h_t}$, it follows that

$$\begin{aligned} \|u - u_{h_t}\|_{L^2(0,T)}^2 &= \langle u - u_{h_t}, u - u_{h_t} \rangle_{L^2(0,T)} \\ &= -\langle \partial_t w, u - u_{h_t} \rangle_{(0,T)} + \mu \langle w, u - u_{h_t} \rangle_{L^2(0,T)} \\ &= \langle \partial_t (u - u_{h_t}), w \rangle_{(0,T)} + \mu \langle u - u_{h_t}, w \rangle_{L^2(0,T)} \\ &= \langle \partial_t (u - u_{h_t}), w - I_{h_t} w \rangle_{(0,T)} + \mu \langle u - u_{h_t}, w - I_{h_t} w \rangle_{L^2(0,T)} \\ &\leq \|u - u_{h_t}\|_{H_0^{1/2}(0,T), F} \|w - I_{h_t} w\|_{H_0^{1/2}(0,T), F} \\ &\quad + \mu \|u - u_{h_t}\|_{L^2(0,T)} \|w - I_{h_t} w\|_{L^2(0,T)}. \end{aligned} \quad (3.63)$$

For the first summand in (3.63), it holds true that

$$\begin{aligned} \|w - I_{h_t} w\|_{H_0^{1/2}(0,T),F} &= \sqrt{\frac{2}{\pi}} \|w - I_{h_t} w\|_{K_{1/2}(H_0^1(0,T);L^2(0,T))} \\ &\leq \sqrt{\frac{2}{\pi}} \sqrt[4]{2} h_t^{1/2} \|\partial_t w\|_{L^2(0,T)} \\ &\leq \frac{\sqrt[4]{8}}{\sqrt{\pi}} h_t^{1/2} \|u - u_{h_t}\|_{L^2(0,T)}, \end{aligned}$$

where the first equality follows from a norm equivalence analogous to (3.33), the last inequality follows from (3.60), and the first estimate follows from an interpolation argument for the K -Method of Interpolation between

$$\|w - I_{h_t} w\|_{L^2(0,T)} \leq \frac{1}{\sqrt{8}} h_t \|\partial_t(w - I_{h_t} w)\|_{L^2(0,T)} \leq \frac{1}{\sqrt{2}} h_t \|\partial_t w\|_{L^2(0,T)} = \frac{1}{\sqrt{2}} h_t \|w\|_{H_0^1(0,T)}$$

and

$$\|w - I_{h_t} w\|_{H_0^1(0,T)} = \|\partial_t(w - I_{h_t} w)\|_{L^2(0,T)} \leq 2 \|\partial_t w\|_{L^2(0,T)} = 2 \|w\|_{H_0^1(0,T)}$$

with the stability (2.28) and an error estimate analogous to (2.29) since $w \in H_0^1(0,T)$.

For the second summand in (3.63), it follows with (3.62) that

$$\begin{aligned} \|w - I_{h_t} w\|_{L^2(0,T)} &\leq \frac{1}{\sqrt[4]{48}} h_t^{3/2} \|\partial_t w\|_{K_{1/2}(H^1(0,T);L^2(0,T))} \\ &\leq \frac{1}{\sqrt[4]{48}} \sqrt{\frac{\pi}{2}} \sqrt[4]{2 + \frac{4T^2}{\pi^2} + \frac{8T^2\mu^2}{\pi^2}} h_t^{3/2} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F} \\ &= \frac{\sqrt[4]{\pi^2 + 2(1 + 2\mu^2)T^2}}{2\sqrt[4]{6}} h_t^{3/2} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F}, \end{aligned}$$

where the first estimate follows from an interpolation argument for the K -Method of Interpolation between

$$\|w - I_{h_t} w\|_{L^2(0,T)} \leq \frac{1}{\sqrt{24}} h_t^2 \|\partial_{tt} w\|_{L^2(0,T)} \leq \frac{1}{\sqrt{24}} h_t^2 \|\partial_t w\|_{H^1(0,T)}$$

and

$$\|w - I_{h_t} w\|_{L^2(0,T)} \leq \frac{1}{\sqrt{8}} h_t \|\partial_t(w - I_{h_t} w)\|_{L^2(0,T)} \leq \frac{1}{\sqrt{2}} h_t \|\partial_t w\|_{L^2(0,T)}$$

with the stability (2.28) and an error estimate analogous to (2.29) since $\partial_t w \in H^1(0,T)$.

Therefore, the inequality (3.63) yields

$$\begin{aligned} \|u - u_{h_t}\|_{L^2(0,T)}^2 &\leq \frac{\sqrt[4]{8}}{\sqrt{\pi}} h_t^{1/2} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F} \|u - u_{h_t}\|_{L^2(0,T)} \\ &\quad + \mu \frac{\sqrt[4]{\pi^2 + 2(1 + 2\mu^2)T^2}}{2\sqrt[4]{6}} h_t^{3/2} \|u - u_{h_t}\|_{L^2(0,T)} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F}, \end{aligned}$$

i.e.

$$\|u - u_{h_t}\|_{L^2(0,T)} \leq \left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2 + 2(1 + 2\mu^2)T^2}}{2\sqrt[4]{6}} \mu h_t \right) h_t^{1/2} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F}. \quad (3.64)$$

When inserting this into (3.57), together with the estimate

$$\|u - I_{h_t}u\|_{H_0^{1/2}(0,T),F} = \sqrt{\frac{2}{\pi}} \|u - I_{h_t}u\|_{K_{1/2}(H_0^1(0,T);L^2(0,T))} \leq \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt[4]{72}} h_t^{3/2} \|\partial_t u\|_{H^1(0,T)}$$

derived with the help of (3.33) via an interpolation argument for the K -Method of Interpolation between (2.30) and (2.31), and again (2.31), this gives

$$\begin{aligned} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F}^2 &\leq \frac{1}{\sqrt{\pi}\sqrt[4]{18}} h_t^{3/2} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F} \|\partial_t u\|_{H^1(0,T)} \\ &\quad + \frac{1}{\sqrt{24}} \mu \left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2 + 2(1 + 2\mu^2)T^2}}{2\sqrt[4]{6}} \mu h_t \right) h_t^{5/2} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F} \|\partial_t u\|_{H^1(0,T)}, \end{aligned}$$

i.e.

$$\begin{aligned} \|u - u_{h_t}\|_{H_0^{1/2}(0,T),F} &\leq \left[\frac{1}{\sqrt{\pi}\sqrt[4]{18}} + \frac{1}{\sqrt{24}} \left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2 + 2(1 + 2\mu^2)T^2}}{2\sqrt[4]{6}} \mu h_t \right) \mu h_t \right] h_t^{3/2} \|\partial_t u\|_{H^1(0,T)} \end{aligned}$$

and thus, the first assertion is proven.

The $L^2(0, T)$ error estimate follows with the first assertion from (3.64). \square

To illustrate the error estimate (3.56), consider the given right-hand side f as

$$f(t) = 1 \quad \text{for } t \in (0, 2) = (0, T),$$

which results in the solution

$$u(t) = \frac{1}{\mu} [1 - e^{-\mu t}], \quad t \in (0, 2),$$

satisfying

$$\|\partial_t u\|_{L^2(0,2)} \leq \sqrt{\frac{\mu}{2}}.$$

As at the end of Subsection 3.4.3, a sequence of finite element spaces $S_{h_t,0}^1(0,2)$ of uniform mesh size $h_t = 2/N_t$, and $N_t = 2^{j+1}$, $j = 0, \dots, 10$, is introduced. Depending on μ , quadratic convergence is expected, but requiring a sufficiently small mesh size $h_t = 2/N_t$ for large μ . This well-known behaviour can be seen in Figure 3.3, Figure 3.4, Figure 3.5 and Figure 3.6, where the computed $L^2(0,2)$ error is plotted versus the error bound (3.56).

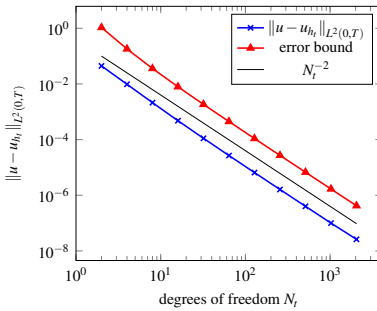


Figure 3.3: $L^2(0,T)$ error and error bound (3.56) for (3.47) for $\mu = 1$.

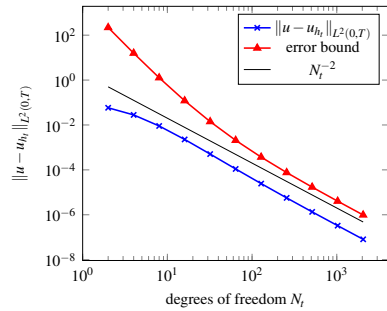


Figure 3.4: $L^2(0,T)$ error and error bound (3.56) for (3.47) for $\mu = 10$.

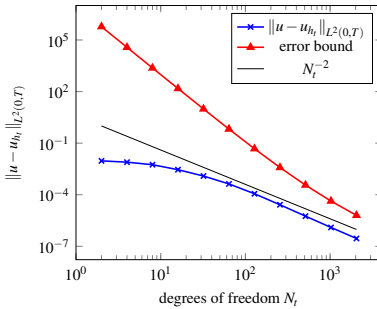


Figure 3.5: $L^2(0,T)$ error and error bound (3.56) for (3.47) for $\mu = 100$.

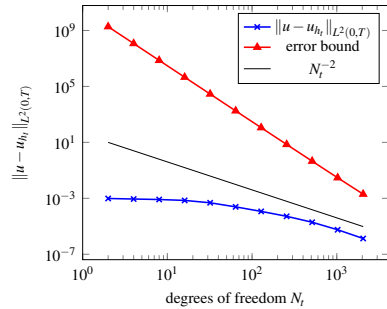


Figure 3.6: $L^2(0,T)$ error and error bound (3.56) for (3.47) for $\mu = 1000$.

Remark 3.4.15. *The Galerkin-Petrov finite element formulation (3.23) of Section 3.3 is uniquely solvable but unstable for any mesh size h_t and any $\mu > 0$, whereas the Galerkin-*

Bubnov finite element formulation (3.55) is uniquely solvable and stable independently of the mesh size h_i and $\mu > 0$.

3.4.5 Variational Formulation for the Heat Equation via Fourier Series

In this subsection, the ideas of Subsection 3.4.4 are transferred to the homogeneous Dirichlet problem of the heat equation

$$\left. \begin{aligned} \partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) & \text{for } (x,t) \in Q = \Omega \times (0, T), \\ u(x,t) &= 0 & \text{for } (x,t) \in \Sigma = \Gamma \times [0, T], \\ u(x,0) &= 0 & \text{for } x \in \Omega, \end{aligned} \right\} \quad (3.65)$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$ and $T > 0$ is a given terminal time. To write down the variational formulation (3.21) via a Fourier series approach, characterisations of the spaces $H_{0;0}^{1,1/2}(Q)$ and $H_{0;0}^{1,1/2}(Q)$ are given. Therefore, consider for $i \in \mathbb{N}$ the eigenfunctions ϕ_i and eigenvalues μ_i from (2.4), i.e.

$$-\Delta_x \phi_i = \mu_i \phi_i \quad \text{in } \Omega, \quad \phi_i = 0 \quad \text{on } \Gamma, \quad \|\phi_i\|_{L^2(\Omega)} = 1,$$

which form an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H_0^1(\Omega)$. Since the relation

$$L^2(Q) \simeq L^2(\Omega) \hat{\otimes} L^2(0, T) = \overline{L^2(\Omega) \otimes L^2(0, T)}^{\|\cdot\|_{L^2(Q)}}$$

holds true for tensor-products, see Section 2.4, the functions

$$Q \ni (x,t) \mapsto \phi_i(x) \cdot V_k(t) \in \mathbb{R}, \quad i \in \mathbb{N}, k \in \mathbb{N}_0,$$

form an orthogonal basis of $L^2(Q)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(Q)}$, see [128, Proposition 2, page 50], where the eigenfunctions V_k are given in (3.26). Hence, as an extension of the expansion (3.27), the representation

$$u(x,t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} V_k(t) \phi_i(x) = \sum_{i=1}^{\infty} U_i(t) \phi_i(x), \quad U_i(t) = \sum_{k=0}^{\infty} u_{i,k} V_k(t), \quad (3.66)$$

holds true for a function $u \in L^2(Q)$ with the coefficients

$$u_{i,k} = \frac{2}{T} \int_0^T \int_{\Omega} u(x,t) V_k(t) \phi_i(x) dx dt = \frac{2}{T} \int_0^T \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) \int_{\Omega} u(x,t) \phi_i(x) dx dt,$$

i.e. it holds true that

$$\left\| u - \sum_{i=1}^M \sum_{k=0}^N u_{i,k} V_k \cdot \phi_i \right\|_{L^2(Q)} \rightarrow 0 \quad \text{as } M \rightarrow \infty, N \rightarrow \infty$$

and

$$\left\| u - \sum_{i=1}^M U_i \cdot \phi_i \right\|_{L^2(Q)} \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

For the inner product, the representation

$$\langle u, \hat{u} \rangle_{L^2(Q)} = \sum_{i=1}^{\infty} \langle U_i, \hat{U}_i \rangle_{L^2(0,T)} = \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} \cdot \hat{u}_{i,k}$$

is valid for $u, \hat{u} \in L^2(Q)$ with the expansion (3.66), and if, in addition, $u, \hat{u} \in H_{0;1}^{1,1}(Q)$, it follows that

$$\begin{aligned} \langle u, \hat{u} \rangle_{H_{0;1}^{1,1}(Q)} &= \sum_{i=1}^{\infty} \left[\langle \partial_t U_i, \partial_t \hat{U}_i \rangle_{L^2(0,T)} + \mu_i \langle U_i, \hat{U}_i \rangle_{L^2(0,T)} \right] \\ &= \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T^2} \left(\frac{\pi}{2} + k\pi \right)^2 + \mu_i \right] u_{i,k} \cdot \hat{u}_{i,k} \end{aligned}$$

for the inner product (2.13). Correspondingly, the functions

$$Q \ni (x, t) \mapsto \phi_i(x) \cdot W_k(t) \in \mathbb{R}, \quad i \in \mathbb{N}, k \in \mathbb{N}_0,$$

form an orthogonal basis of $L^2(Q)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(Q)}$, where the eigenfunctions W_k are given in (3.34). Hence, as an extension of the expansion (3.35), the representation

$$z(x, t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} W_k(t) \phi_i(x) = \sum_{i=1}^{\infty} Z_i(t) \phi_i(x), \quad Z_i(t) = \sum_{k=0}^{\infty} z_{i,k} W_k(t), \quad (3.67)$$

holds true for $z \in L^2(Q)$ with the coefficients

$$z_{i,k} = \frac{2}{T} \int_0^T \int_{\Omega} z(x, t) W_k(t) \phi_i(x) dx dt = \frac{2}{T} \int_0^T \cos \left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \int_{\Omega} z(x, t) \phi_i(x) dx dt,$$

i.e. it holds true that

$$\left\| z - \sum_{i=1}^M \sum_{k=0}^N z_{i,k} W_k \cdot \phi_i \right\|_{L^2(Q)} \rightarrow 0 \quad \text{as } M \rightarrow \infty, N \rightarrow \infty$$

and

$$\left\| z - \sum_{i=1}^M Z_i \cdot \phi_i \right\|_{L^2(Q)} \rightarrow 0 \quad \text{as } M \rightarrow \infty.$$

Analogous to Subsection 3.4.1, the representations

$$H_0^{1/2}(0, T; L^2(\Omega)) = \left\{ u \in L^2(Q) : u(x, t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} V_k(t) \phi_i(x), \right. \\ \left. \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |u_{i,k}|^2 < \infty \right\}$$

with the inner product

$$\langle u, \hat{u} \rangle_{H_0^{1/2}(0, T; L^2(\Omega)), F} := \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) u_{i,k} \cdot \hat{u}_{i,k}$$

and

$$H_0^{1/2}(0, T; L^2(\Omega)) = \left\{ z \in L^2(Q) : z(x, t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} W_k(t) \phi_i(x), \right. \\ \left. \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |z_{i,k}|^2 < \infty \right\}$$

with the inner product

$$\langle z, \hat{z} \rangle_{H_0^{1/2}(0, T; L^2(\Omega)), F} := \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) z_{i,k} \cdot \hat{z}_{i,k}$$

hold true, where the induced norms $\|\cdot\|_{H_0^{1/2}(0, T; L^2(\Omega)), F}$ and $\|\cdot\|_{H_0^{1/2}(0, T; L^2(\Omega)), F}$ are equivalent to the norms $\|\cdot\|_{H_0^{1/2}(0, T; L^2(\Omega))}$ and $\|\cdot\|_{H_0^{1/2}(0, T; L^2(\Omega))}$ given in (2.16) and (2.17). So, the anisotropic space $H_{0,0}^{1,1/2}(Q) = H_0^{1/2}(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ is endowed with the inner product

$$\langle u, \hat{u} \rangle_{H_{0,0}^{1,1/2}(Q), F} := \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right] u_{i,k} \cdot \hat{u}_{i,k},$$

and analogously, $H_{0,0}^{1,1/2}(Q) = H_0^{1/2}(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ is endowed with

$$\langle z, \hat{z} \rangle_{H_{0,0}^{1,1/2}(Q), F} := \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right] z_{i,k} \cdot \hat{z}_{i,k}.$$

The transformation operator \mathcal{H}_T , given in (3.38), acts only with respect to the time variable t , i.e. for $u \in L^2(Q)$ with expansion (3.66), one defines

$$(\mathcal{H}_T u)(x, t) := \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} W_k(t) \phi_i(x) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} \cos \left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \phi_i(x) \quad (3.68)$$

for $(x, t) \in Q$. By construction, it holds true that $\mathcal{H}_T u \in L^2(Q)$. Furthermore,

$$\mathcal{H}_T: L^2(Q) \rightarrow L^2(Q)$$

is bijective and norm preserving, i.e.

$$\|\mathcal{H}_T u\|_{L^2(Q)} = \|u\|_{L^2(Q)} \quad \text{for all } u \in L^2(Q),$$

where the inverse transformation operator

$$\mathcal{H}_T^{-1}: L^2(Q) \rightarrow L^2(Q)$$

is given by

$$(\mathcal{H}_T^{-1} z)(x, t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} V_k(t) \phi_i(x) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} \sin\left(\left(\frac{\pi}{2} + k\pi\right) \frac{t}{T}\right) \phi_i(x), \quad (x, t) \in Q,$$

for $z \in L^2(Q)$ with the expansion (3.67). Analogously, the maps

$$\mathcal{H}_T: H_{0,0}^{1/2}(0, T; L^2(\Omega)) \rightarrow H_0^{1/2}(0, T; L^2(\Omega))$$

and

$$\mathcal{H}_T: H_{0,0}^{1,1/2}(Q) \rightarrow H_{0,0}^{1,1/2}(Q)$$

are norm preserving and bijective.

Finally, representations of the dual spaces $[H_{0,0}^{1,1/2}(Q)]'$ and $[H_0^{1/2}(0, T; L^2(\Omega))]'$ are given. In Section 2.5 the dual space $[H_{0,0}^{1,1/2}(Q)]'$ is characterised as a completion of $L^2(Q)$ with respect to the Hilbertian norm $\|\cdot\|_{[H_{0,0}^{1,1/2}(Q)]'}$, where $\|\cdot\|_{H_{0,0}^{1,1/2}(Q)}$ is the norm in $H_{0,0}^{1,1/2}(Q)$, see (2.20). A to $\|\cdot\|_{[H_{0,0}^{1,1/2}(Q)]'}$ equivalent Hilbertian norm is given by

$$\|f\|_{[H_{0,0}^{1,1/2}(Q)]', F} := \sup_{0 \neq z \in H_{0,0}^{1,1/2}(Q)} \frac{|\langle f, z \rangle_Q|}{\|z\|_{H_{0,0}^{1,1/2}(Q), F}}$$

for $f \in [H_{0,0}^{1,1/2}(Q)]'$, where $\langle \cdot, \cdot \rangle_Q$ denotes again the duality pairing as extension of the inner product in $L^2(Q)$. Moreover, a Hilbertian norm in $[H_0^{1/2}(0, T; L^2(\Omega))]'$ is

$$\|f\|_{[H_0^{1/2}(0, T; L^2(\Omega))]', F} := \sup_{0 \neq z \in H_0^{1/2}(0, T; L^2(\Omega))} \frac{|\langle f, z \rangle_Q|}{\|z\|_{H_0^{1/2}(0, T; L^2(\Omega)), F}}$$

for $f \in [H_0^{1/2}(0, T; L^2(\Omega))]'$.

Lemma 3.4.16. For $f \in [H_{0;0}^{1,1/2}(Q)]'$, the expansion

$$\|f\|_{[H_{0;0}^{1,1/2}(Q)]',F}^2 = \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i} f_{i,k}^2$$

is valid, where $f_{i,k} := \frac{2}{T} \langle f, W_k \phi_i \rangle_Q$ with $W_k(t) = \cos \left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right)$.

Analogously, for $f \in [H_0^{1/2}(0, T; L^2(\Omega))]'$, the equality

$$\|f\|_{[H_0^{1/2}(0, T; L^2(\Omega))]',F}^2 = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right)^{-1} f_{i,k}^2$$

holds true.

Proof. For $z \in H_{0;0}^{1,1/2}(Q)$ with (3.67), the representations

$$\begin{aligned} z(x, t) &= \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} \cos \left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \phi_i(x) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} W_k(t) \phi_i(x), \\ \|z\|_{H_{0;0}^{1,1/2}(Q),F}^2 &= \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right) z_{i,k}^2 \end{aligned}$$

and therefore, with the continuity of f ,

$$\langle f, z \rangle_Q = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} \langle f, W_k \phi_i \rangle_Q = \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} \cdot f_{i,k}$$

hold true. Set $z_{i,k}^N := \left(\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right)^{-1} f_{i,k}$ for $i = 1, \dots, N$, $k = 0, \dots, N$ and $z_{i,k}^N = 0$ for $i > N$ or $k > N$. Assume w.l.o.g. that $z^N \neq 0$. Thus, $z^N \in H_{0;0}^{1,1/2}(Q)$ and one obtains

$$\begin{aligned} \left(\frac{T}{2} \sum_{i=1}^N \sum_{k=0}^N \frac{1}{\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i} f_{i,k}^2 \right)^{1/2} &= \frac{\sqrt{\frac{T}{2}} \sum_{i=1}^N \sum_{k=0}^N \left(\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right)^{-1} f_{i,k}^2}{\left(\sum_{i=1}^N \sum_{k=0}^N \left(\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right)^{-1} f_{i,k}^2 \right)^{1/2}} \\ &= \frac{\sqrt{\frac{T}{2}} \sum_{i=1}^N \sum_{k=0}^N z_{i,k}^N f_{i,k}}{\left(\sum_{i=1}^N \sum_{k=0}^N \left(\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right) (z_{i,k}^N)^2 \right)^{1/2}} \\ &= \frac{\langle f, z^N \rangle_Q}{\|z^N\|_{H_{0;0}^{1,1/2}(Q),F}} \leq \sup_{0 \neq z \in H_{0;0}^{1,1/2}(Q)} \frac{|\langle f, z \rangle_Q|}{\|z\|_{H_{0;0}^{1,1/2}(Q),F}}. \end{aligned}$$

Hence,

$$\|f\|_{[H_{0,0}^{1,1/2}(Q)]',F}^2 \geq \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i} f_{i,k}^2$$

follows as $N \rightarrow \infty$.

On the other hand, the Cauchy-Schwarz inequality yields

$$\begin{aligned} \|f\|_{[H_{0,0}^{1,1/2}(Q)]',F} &= \sup_{0 \neq z \in H_{0,0}^{1,1/2}(Q)} \frac{|\langle f, z \rangle_Q|}{\|z\|_{H_{0,0}^{1,1/2}(Q),F}} \\ &= \sqrt{\frac{T}{2}} \sup_{0 \neq z \in H_{0,0}^{1,1/2}(Q)} \frac{\sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} f_{i,k} \left(\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right)^{1/2} \left(\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right)^{-1/2}}{\left(\sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right) z_{i,k}^2 \right)^{1/2}} \\ &\leq \left(\frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i} f_{i,k}^2 \right)^{1/2} \end{aligned}$$

and thus, the first assertion is proven.

For $f \in [H_0^{1/2}(0, T; L^2(\Omega))]',$ the proof is obtained in the same manner. \square

Analogous to the case of the ordinary differential equation, see Theorem 3.4.6, the distributional derivative ∂_t on Q of a function $u \in H_0^{1/2}(0, T; L^2(\Omega))$ is investigated.

Theorem 3.4.17. *For a function $u \in H_0^{1/2}(0, T; L^2(\Omega))$, the distributional derivative ∂_t on Q fulfils $\partial_t u \in [H_0^{1/2}(0, T; L^2(\Omega))]'.$ More precisely, a uniquely determined element $g \in [H_0^{1/2}(0, T; L^2(\Omega))]'$ exists, satisfying*

$$\forall \varphi \in \mathcal{D}(Q): \quad \partial_t T_u(\varphi) = \langle g, \varphi \rangle_Q,$$

where $T_u: \mathcal{D}(Q) \rightarrow \mathbb{R}, T_u(\varphi) = \langle u, \varphi \rangle_{L^2(Q)}$, is the to u related distribution, see Section 2.1.

In addition, it holds true that

$$\|\partial_t u\|_{[H_0^{1/2}(0, T; L^2(\Omega))]',F} \leq \|u\|_{H_0^{1/2}(0, T; L^2(\Omega)),F}$$

and

$$\langle \partial_t u, z \rangle_Q = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} \left(\frac{\pi}{2} + k\pi \right) z_{i,k} \quad (3.69)$$

for all $z \in H_0^{1/2}(0, T; L^2(\Omega))$ with expansion coefficients $u_{i,k}$ from (3.66) and $z_{i,k}$ from (3.67).

Proof. The proof is analogous to the proof of Theorem 3.4.6, since the algebraic tensor-product $C_0^\infty(\Omega) \otimes C_0^\infty(0, T) \subset C_0^\infty(Q)$ is dense in $H_{0,0}^{1/2}(0, T; L^2(\Omega))$. \square

Remark 3.4.18. The equation (3.69) leads, with the bijective transformation operator $\mathcal{H}_T: H_{0,0}^{1/2}(0, T; L^2(\Omega)) \rightarrow H_{0,0}^{1/2}(0, T; L^2(\Omega))$, to

$$\langle \partial_t u, \mathcal{H}_T \hat{u} \rangle_Q = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} \left(\frac{\pi}{2} + k\pi \right) \hat{u}_{i,k} = \langle u, \hat{u} \rangle_{H_{0,0}^{1/2}(0, T; L^2(\Omega)), F} \quad (3.70)$$

for all $u, \hat{u} \in H_{0,0}^{1/2}(0, T; L^2(\Omega))$ with expansion coefficients $u_{i,k}, \hat{u}_{i,k}$ from (3.66).

As in Section 3.3, the variational formulation of (3.65) is to find $u \in H_{0,0}^{1,1/2}(Q)$ such that

$$a(u, z) = \langle f, z \rangle_Q \quad (3.71)$$

for all $z \in H_{0,0}^{1,1/2}(Q)$, where $f \in [H_{0,0}^{1,1/2}(Q)]'$ is a given right-hand side and the bilinear form

$$a(\cdot, \cdot): H_{0,0}^{1,1/2}(Q) \times H_{0,0}^{1,1/2}(Q) \rightarrow \mathbb{R}$$

is defined by

$$a(u, z) := \langle \partial_t u, z \rangle_Q + \langle \nabla_x u, \nabla_x z \rangle_{L^2(Q)}$$

for $u \in H_{0,0}^{1,1/2}(Q)$, $z \in H_{0,0}^{1,1/2}(Q)$. Note that

$$\begin{aligned} \left| \langle \partial_t u, z \rangle_Q \right| &\leq \|\partial_t u\|_{[H_{0,0}^{1/2}(0, T; L^2(\Omega))]', F} \|z\|_{H_{0,0}^{1/2}(0, T; L^2(\Omega)), F} \\ &\leq \|u\|_{H_{0,0}^{1/2}(0, T; L^2(\Omega)), F} \|z\|_{H_{0,0}^{1/2}(0, T; L^2(\Omega)), F} \end{aligned}$$

for $u \in H_{0,0}^{1,1/2}(Q) \subset H_{0,0}^{1/2}(0, T; L^2(\Omega))$ and for $z \in H_{0,0}^{1,1/2}(Q) \subset H_{0,0}^{1/2}(0, T; L^2(\Omega))$ due to Theorem 3.4.17, i.e. $\langle \partial_t u, z \rangle_Q$ is well-defined and bounded on

$$H_{0,0}^{1/2}(0, T; L^2(\Omega)) \times H_{0,0}^{1/2}(0, T; L^2(\Omega)).$$

With this last estimate, it follows the boundedness of the bilinear form

$$a(\cdot, \cdot): H_{0,0}^{1,1/2}(Q) \times H_{0,0}^{1,1/2}(Q) \rightarrow \mathbb{R},$$

i.e. with the Cauchy-Schwarz inequality, it holds true that

$$\begin{aligned} |a(u, z)| &\leq \left| \langle \partial_t u, z \rangle_Q \right| + \left| \langle \nabla_x u, \nabla_x z \rangle_{L^2(Q)} \right| \\ &\leq \|u\|_{H_{0,0}^{1/2}(0, T; L^2(\Omega)), F} \|z\|_{H_{0,0}^{1/2}(0, T; L^2(\Omega)), F} + \|\nabla_x u\|_{L^2(Q)} \|\nabla_x z\|_{L^2(Q)} \\ &\leq \|u\|_{H_{0,0}^{1,1/2}(Q), F} \|z\|_{H_{0,0}^{1,1/2}(Q), F} \end{aligned}$$

for $u \in H_{0,0}^{1,1/2}(Q)$, $z \in H_{0,0}^{1,1/2}(Q)$.

Theorem 3.4.19. For a given $f \in [H_{0,0}^{1,1/2}(Q)]'$, a unique solution $u \in H_{0,0}^{1,1/2}(Q)$ of the variational formulation (3.71) exists, satisfying

$$\|u\|_{H_{0,0}^{1,1/2}(Q),F} \leq 2 \|f\|_{[H_{0,0}^{1,1/2}(Q)]',F}.$$

Furthermore, the solution operator

$$\mathcal{L}: [H_{0,0}^{1,1/2}(Q)]' \rightarrow H_{0,0}^{1,1/2}(Q), \quad \mathcal{L}f := u,$$

is an isomorphism. In addition, the bilinear form

$$a(\cdot, \cdot): H_{0,0}^{1,1/2}(Q) \times H_{0,0}^{1,1/2}(Q) \rightarrow \mathbb{R}$$

is continuous and fulfils the condition (N1), i.e.

$$\frac{1}{2} \leq \inf_{0 \neq v \in H_{0,0}^{1,1/2}(Q)} \sup_{0 \neq z \in H_{0,0}^{1,1/2}(Q)} \frac{|a(v, z)|}{\|v\|_{H_{0,0}^{1,1/2}(Q),F} \|z\|_{H_{0,0}^{1,1/2}(Q),F}}, \quad (3.72)$$

and the condition (N2), i.e. for each $z \in H_{0,0}^{1,1/2}(Q)$, $z \neq 0$, an element $v \in H_{0,0}^{1,1/2}(Q)$ exists such that $a(v, z) \neq 0$.

Proof. For the solution u of the variational problem (3.71), consider the ansatz (3.66)

$$u(x, t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} V_k(t) \phi_i(x) = \sum_{i=1}^{\infty} U_i(t) \phi_i(x), \quad U_i(t) = \sum_{k=0}^{\infty} u_{i,k} V_k(t),$$

where $U_i \in H_0^{1/2}(0, T)$ are unknown functions to be determined. When choosing, for a fixed $j \in \mathbb{N}$, $z(x, t) := Z(t) \phi_j(x)$ with an arbitrary $Z \in H_0^{1/2}(0, T)$ as test function, the variational formulation (3.71) leads to find $U_j \in H_0^{1/2}(0, T)$ such that

$$\langle \partial_t U_j, Z \rangle_{(0,T)} + \mu_j \langle U_j, Z \rangle_{L^2(0,T)} = \langle f, Z \phi_j \rangle_Q \quad (3.73)$$

for all $Z \in H_0^{1/2}(0, T)$. With Lemma 3.4.5, it holds true that

$$\begin{aligned} |\langle f, Z \phi_j \rangle_Q| &\leq \|f\|_{[H_{0,0}^{1,1/2}(Q)]',F} \|Z \phi_j\|_{H_{0,0}^{1,1/2}(Q),F} \\ &= \|f\|_{[H_{0,0}^{1,1/2}(Q)]',F} \sqrt{\|Z\|_{H_0^{1/2}(0,T),F}^2 + \mu_j \|Z\|_{L^2(0,T)}^2} \\ &\leq \sqrt{1 + \frac{2T\mu_j}{\pi}} \|f\|_{[H_{0,0}^{1,1/2}(Q)]',F} \|Z\|_{H_0^{1/2}(0,T),F} \end{aligned}$$

for all $Z \in H_0^{1/2}(0, T)$, and so,

$$\langle F_j, Z \rangle_{(0, T)} := \langle f, Z \phi_j \rangle_Q$$

fulfils $F_j \in [H_0^{1/2}(0, T)]'$. The unique solvability of (3.73) follows from the unique solvability of (3.47). So, there exists for every $j \in \mathbb{N}$ a unique solution $U_j \in H_0^{1/2}(0, T)$ of the variational formulation (3.73), satisfying

$$\begin{aligned} \|U_j\|_{H_0^{1/2}(0, T), F}^2 &= \langle \partial_t U_j, \mathcal{H}_T U_j \rangle_{(0, T)} \\ &\leq \langle \partial_t U_j, \mathcal{H}_T U_j \rangle_{(0, T)} + \mu_j \langle U_j, \mathcal{H}_T U_j \rangle_{L^2(0, T)} \\ &= \langle f, \phi_j \mathcal{H}_T U_j \rangle_Q. \end{aligned}$$

Hence, the coefficients $U_i \in H_0^{1/2}(0, T)$ are uniquely determined. Next, the convergence properties of the series expansion of u are investigated. Therefore, define the partial sums

$$u_M(x, t) := \sum_{j=1}^M U_j(t) \phi_j(x)$$

for $M \in \mathbb{N}$, and one concludes

$$\begin{aligned} \|u_M\|_{H_0^{1/2}(0, T; L^2(\Omega)), F}^2 &= \sum_{j=1}^M \|U_j\|_{H_0^{1/2}(0, T), F}^2 \leq \sum_{j=1}^M \langle f, \phi_j \mathcal{H}_T U_j \rangle_Q = \left\langle f, \sum_{j=1}^M \phi_j \mathcal{H}_T U_j \right\rangle_Q \\ &\leq \|f\|_{[H_{0,0}^{1,1/2}(Q)]', F} \| \mathcal{H}_T u_M \|_{H_{0,0}^{1,1/2}(Q), F} \\ &= \|f\|_{[H_{0,0}^{1,1/2}(Q)]', F} \|u_M\|_{H_{0,0}^{1,1/2}(Q), F}. \end{aligned}$$

Hence, using (3.52) for $f_{i,k} = \frac{2}{T} \langle F_i, W_k \rangle_{(0, T)} = \frac{2}{T} \langle f, \phi_i W_k \rangle_Q$, one obtains

$$\begin{aligned} \|u_M\|_{L^2(0, T; H_0^1(\Omega))}^2 &= \sum_{i=1}^M \mu_i \|U_i\|_{L^2(0, T)}^2 \leq \frac{T}{2} \sum_{i=1}^M \sum_{k=0}^{\infty} \frac{\mu_i}{\mu_i^2 + \frac{1}{T^2} \left(\frac{\pi}{2} + k\pi\right)^2} f_{i,k}^2 \\ &\leq T \sum_{i=1}^M \sum_{k=0}^{\infty} \frac{1}{\mu_i + \frac{1}{T} \left(\frac{\pi}{2} + k\pi\right)} f_{i,k}^2 \leq 2 \|f\|_{[H_{0,0}^{1,1/2}(Q)]', F}^2 \end{aligned}$$

with the help of the inequality

$$\frac{a}{a^2 + b^2} \leq \frac{a + b}{\frac{1}{2}(a + b)^2} = \frac{2}{a + b} \quad \text{for } 0 < a, b \in \mathbb{R}.$$

With this, it holds true that

$$\begin{aligned} \|u_M\|_{H_{0,0}^{1,1/2}(Q), F}^2 &= \|u_M\|_{H_0^{1/2}(0, T; L^2(\Omega)), F}^2 + \|u_M\|_{L^2(0, T; H_0^1(\Omega))}^2 \\ &\leq \|f\|_{[H_{0,0}^{1,1/2}(Q)]', F} \|u_M\|_{H_{0,0}^{1,1/2}(Q), F} + 2 \|f\|_{[H_{0,0}^{1,1/2}(Q)]', F}^2, \end{aligned}$$

and therefore, by solving the corresponding quadratic equation,

$$\|u_M\|_{H_{0,0}^{1,1/2}(Q),F} \leq 2\|f\|_{[H_{0,0}^{1,1/2}(Q)]',F}$$

follows for all $M \in \mathbb{N}$. The last inequality yields with $U_i(t) = \sum_{k=0}^{\infty} u_{i,k} V_k(t)$ the bound

$$\begin{aligned} \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right] u_{i,k}^2 &= \lim_{M \rightarrow \infty} \sum_{i=1}^M \left[\|U_i\|_{H_{0,0}^{1,1/2}(0,T),F}^2 + \mu_i \|U_i\|_{L^2(0,T)}^2 \right] \\ &= \lim_{M \rightarrow \infty} \|u_M\|_{H_{0,0}^{1,1/2}(Q),F}^2 \\ &\leq 4\|f\|_{[H_{0,0}^{1,1/2}(Q)]',F}^2 < \infty \end{aligned}$$

and thus, $u \in H_{0,0}^{1,1/2}(Q)$ with $\lim_{M \rightarrow \infty} u_M = u$ in $H_{0,0}^{1,1/2}(Q)$.

Next, it is shown that u is a solution of the variational formulation (3.71). This follows with the expansion $z = \sum_{j=1}^{\infty} Z_j \phi_j \in H_{0,0}^{1,1/2}(Q)$, $Z_j \in H_0^{1/2}(0,T)$, which is given by (3.67), from

$$\begin{aligned} a(u, z) &= \langle \partial_t u, z \rangle_Q + \langle \nabla_x u, \nabla_x z \rangle_{L^2(Q)} \\ &= \lim_{M \rightarrow \infty} \langle \partial_t u_M, z \rangle_Q + \lim_{M \rightarrow \infty} \langle \nabla_x u_M, \nabla_x z \rangle_{L^2(Q)} \\ &= \sum_{j=1}^{\infty} \langle \partial_t U_j \phi_j, z \rangle_Q + \sum_{j=1}^{\infty} \langle U_j \nabla_x \phi_j, \nabla_x z \rangle_{L^2(Q)} \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \langle \partial_t U_j \phi_j, Z_i \phi_i \rangle_Q + \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \langle U_j \nabla_x \phi_j, Z_i \nabla_x \phi_i \rangle_{L^2(Q)} \end{aligned}$$

and by using (3.73), from

$$a(u, z) = \sum_{j=1}^{\infty} \langle \partial_t U_j, Z_j \rangle_{(0,T)} + \sum_{j=1}^{\infty} \mu_j \langle U_j, Z_j \rangle_{L^2(0,T)} = \sum_{j=1}^{\infty} \langle f, Z_j \phi_j \rangle_Q = \langle f, z \rangle_Q.$$

The uniqueness of u is a consequence of the uniqueness of the coefficients U_j .

The remaining parts of the theorem follow from the Nečas Theorem 2.9.1. \square

The variational formulation (3.71) is equivalent to find $u \in H_{0,0}^{1,1/2}(Q)$ such that

$$a(u, \mathcal{H}_T v) = \langle \partial_t u, \mathcal{H}_T v \rangle_Q + \langle \nabla_x u, \nabla_x \mathcal{H}_T v \rangle_{L^2(Q)} \stackrel{!}{=} \langle f, \mathcal{H}_T v \rangle_Q \quad (3.74)$$

for all $v \in H_{0,0}^{1,1/2}(Q)$. Hence, unique solvability of the variational formulation (3.74) follows from the unique solvability of (3.71). In addition, the stability estimate (3.72) implies the stability estimate

$$\frac{1}{2} \|u\|_{H_{0,0}^{1,1/2}(Q),F} \leq \sup_{0 \neq v \in H_{0,0}^{1,1/2}(Q)} \frac{\langle \partial_t u, \mathcal{H}_T v \rangle_Q + \langle \nabla_x u, \nabla_x \mathcal{H}_T v \rangle_{L^2(Q)}}{\|v\|_{H_{0,0}^{1,1/2}(Q),F}}$$

for all $u \in H_{0,0}^{1,1/2}(Q)$.

To introduce approximate solutions, the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ is assumed to be an interval $\Omega = (0, L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. When using some conforming space-time finite element space $\mathcal{V}_h \subset H_{0,0}^{1,1/2}(Q)$, the Galerkin variational formulation of (3.74) is to find $u_h \in \mathcal{V}_h$ such that

$$a(u_h, \mathcal{H}_T v_h) = \langle f, \mathcal{H}_T v_h \rangle_Q \quad (3.75)$$

for all $v_h \in \mathcal{V}_h$. Note that ansatz and test spaces are equal.

Theorem 3.4.20. *Let $\mathcal{V}_h \subset H_{0,0}^{1,1/2}(Q)$ be a conforming space-time finite element space and let $f \in [H_{0,0}^{1,1/2}(Q)]'$ be a given right-hand side. Then, a unique solution $u_h \in \mathcal{V}_h$ of the Galerkin-Bubnov variational formulation (3.75) exists. If, in addition, the right-hand side fulfils $f \in [H_0^{1/2}(0, T; L^2(\Omega))]' \subset [H_{0,0}^{1,1/2}(Q)]'$, then the stability estimate*

$$\|u_h\|_{H_0^{1/2}(0, T; L^2(\Omega)),F} \leq \|f\|_{[H_0^{1/2}(0, T; L^2(\Omega))]',F}$$

is true.

Proof. Let $u_h^0 \in \mathcal{V}_h$ be any solution of the homogeneous variational formulation (3.75) with $f = 0$. With (3.70), $v_h = u_h^0 \in \mathcal{V}_h$ in (3.75) and Lemma 3.4.8, it follows that

$$\begin{aligned} \|u_h^0\|_{H_0^{1/2}(0, T; L^2(\Omega)),F}^2 &= \langle \partial_t u_h^0, \mathcal{H}_T u_h^0 \rangle_Q \\ &\leq \langle \partial_t u_h^0, \mathcal{H}_T u_h^0 \rangle_Q + \underbrace{\sum_{m=1}^d \int_{\Omega} \langle \partial_{x_m} u_h^0(x, \cdot), \partial_{x_m} \mathcal{H}_T u_h^0(x, \cdot) \rangle_{L^2(0, T)} dx}_{\geq 0} \\ &= \langle f, \mathcal{H}_T u_h^0 \rangle_Q \\ &= 0, \end{aligned} \quad (3.76)$$

which implies $u_h^0 = 0$ and thus, the uniqueness of a solution $u_h \in \mathcal{V}_h$ of the inhomogeneous variational formulation (3.75). Since ansatz and test spaces of the variational formulation (3.75) are equal, the unique solvability of the Galerkin-Bubnov variational formulation (3.75) follows.

If, in addition, the right-hand side fulfils $f \in [H_0^{1/2}(0, T; L^2(\Omega))] \subset [H_{0;0}^{1,1/2}(Q)]'$, then, with (3.76) for the unique solution $u_h \in \mathcal{V}_h$ of the Galerkin-Bubnov variational formulation (3.75), the inequality

$$\begin{aligned} \|u_h\|_{H_{0;0}^{1/2}(0, T; L^2(\Omega)), F}^2 &\leq \langle f, \mathcal{H}_T u_h \rangle_Q \\ &\leq \|f\|_{[H_0^{1/2}(0, T; L^2(\Omega))]', F} \|\mathcal{H}_T u_h\|_{H_{0;0}^{1/2}(0, T; L^2(\Omega)), F} \\ &= \|f\|_{[H_0^{1/2}(0, T; L^2(\Omega))]', F} \|u_h\|_{H_{0;0}^{1/2}(0, T; L^2(\Omega)), F} \end{aligned}$$

and hence, the stability estimate hold true. \square

A possible choice for a conforming space-time discretisation of (3.74) is the space of piecewise linear, continuous functions $\mathcal{V}_h = S_h^1(Q) \cap H_{0;0}^{1,1/2}(Q)$, see Section 2.8 for more details. However, to perform the temporal transformation \mathcal{H}_T easily, and to derive error estimates, based on the tensor-product structure, only a tensor-product space-time finite element space $\mathcal{V}_h = Q_h^1(Q) \cap H_{0;0}^{1,1/2}(Q)$ from (2.42) is considered in the remainder of this section. The Galerkin-Bubnov variational formulation of (3.74) is to find $u_h \in Q_h^1(Q) \cap H_{0;0}^{1,1/2}(Q)$ such that

$$a(u_h, \mathcal{H}_T v_h) = \langle f, \mathcal{H}_T v_h \rangle_Q \quad (3.77)$$

for all $v_h \in Q_h^1(Q) \cap H_{0;0}^{1,1/2}(Q)$. After an appropriate ordering of the degrees of freedom, the discrete variational formulation (3.77) is equivalent to the global linear system

$$K_h \underline{u} = \underline{F}$$

with the system matrix

$$K_h = A_{h_t}^{\mathcal{H}_T} \otimes M_{h_x} + M_{h_t}^{\mathcal{H}_T} \otimes A_{h_x} \in \mathbb{R}^{N_t \cdot M_x \times N_t \cdot M_x},$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote spatial mass and stiffness matrices given in (2.37) and (2.38), $M_{h_t}^{\mathcal{H}_T} \in \mathbb{R}^{N_t \times N_t}$ and $A_{h_t}^{\mathcal{H}_T} \in \mathbb{R}^{N_t \times N_t}$ are defined by

$$\begin{aligned} M_{h_t}^{\mathcal{H}_T}[\ell, k] &:= \langle \varphi_k, \mathcal{H}_T \varphi_\ell \rangle_{L^2(0, T)}, \\ A_{h_t}^{\mathcal{H}_T}[\ell, k] &:= \langle \partial_t \varphi_k, \mathcal{H}_T \varphi_\ell \rangle_{L^2(0, T)}, \end{aligned}$$

$\ell, k = 1, \dots, N_t$, and with the corresponding vector $\underline{F} \in \mathbb{R}^{N_t \cdot M_x}$ of the right-hand side.

The next aim is to derive error estimates in the space-time norms. First of all, for the unique solution $u \in H_{0;0}^{1,1/2}(Q)$ of the variational formulation (3.74) and for the unique solution $u_h \in Q_h^1(Q) \cap H_{0;0}^{1,1/2}(Q)$ of the discrete variational formulation (3.77), the Galerkin orthogonality

$$a(u - u_h, \mathcal{H}_T v_h) = 0 \quad (3.78)$$

holds true for all $v_h \in \mathcal{Q}_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)$. To derive an $L^2(Q)$ error estimate, a space-time projection

$$\mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 v \in \mathcal{Q}_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)$$

for a sufficiently smooth function $v \in H_{0,0}^{1,1/2}(Q)$ is introduced, see Section 2.8.

For a function $v \in H_{0,0}^{1,1/2}(0, T; L^2(\Omega))$, the $H_{0,0}^{1,1/2}$ projection $\mathcal{Q}_{h_t}^{1/2} v \in L^2(\Omega) \otimes S_{h_t,0}^1(0, T)$ is defined by

$$\langle \mathcal{Q}_{h_t}^{1/2} v, v_{h_t} \rangle_{H_{0,0}^{1,1/2}(0, T; L^2(\Omega)), F} = \langle v, v_{h_t} \rangle_{H_{0,0}^{1,1/2}(0, T; L^2(\Omega)), F} \quad (3.79)$$

for all $v_{h_t} \in L^2(\Omega) \otimes S_{h_t,0}^1(0, T)$. The properties of the $H_{0,0}^{1,1/2}$ projection $\mathcal{Q}_{h_t}^{1/2}$ are summarised in the following lemma.

Lemma 3.4.21. *Let $v \in H_{0,0}^{1,1/2}(0, T; L^2(\Omega))$ be a given function. For the $H_{0,0}^{1,1/2}$ projection $\mathcal{Q}_{h_t}^{1/2} v$, defined in (3.79), the following properties hold true:*

1. *The stability estimate*

$$\|\mathcal{Q}_{h_t}^{1/2} v\|_{H_{0,0}^{1,1/2}(0, T; L^2(\Omega)), F} \leq \|v\|_{H_{0,0}^{1,1/2}(0, T; L^2(\Omega)), F}$$

is true.

2. *If, in addition, $v \in H^s(0, T; L^2(\Omega))$ for some $s \in (1/2, 2]$, then the error estimates*

$$\|v - \mathcal{Q}_{h_t}^{1/2} v\|_{H_{0,0}^{1,1/2}(0, T; L^2(\Omega)), F} \leq c h_t^{s-1/2} \|v\|_{H^s(0, T; L^2(\Omega))}$$

and

$$\|v - \mathcal{Q}_{h_t}^{1/2} v\|_{L^2(Q)} \leq c h_t^s \|v\|_{H^s(0, T; L^2(\Omega))}$$

are valid with a constant $c > 0$.

Proof. First, the stability estimate follows from the Cauchy-Schwarz inequality with

$$v_{h_t} = \mathcal{Q}_{h_t}^{1/2} v \in L^2(\Omega) \otimes S_{h_t,0}^1(0, T)$$

in the variational formulation (3.79).

Second, with the Galerkin orthogonality

$$\langle v - \mathcal{Q}_{h_t}^{1/2} v, v_{h_t} \rangle_{H_{0,0}^{1,1/2}(0, T; L^2(\Omega)), F} = 0 \quad \text{for all } v_{h_t} \in L^2(\Omega) \otimes S_{h_t,0}^1(0, T),$$

the first error estimate is given by

$$\begin{aligned}
\|v - \mathcal{Q}_{h_t}^{1/2} v\|_{H_0^{1/2}(0,T;L^2(\Omega)),F}^2 &= \langle v - \mathcal{Q}_{h_t}^{1/2} v, v - \mathcal{Q}_{h_t}^{1/2} v \rangle_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
&= \langle v - \mathcal{Q}_{h_t}^{1/2} v, v \rangle_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
&= \langle v - \mathcal{Q}_{h_t}^{1/2} v, v - I_{h_t} v \rangle_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
&\leq \|v - \mathcal{Q}_{h_t}^{1/2} v\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} \|v - I_{h_t} v\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
&\leq \|v - \mathcal{Q}_{h_t}^{1/2} v\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} c h_t^{s-1/2} \|v\|_{H^s(0,T;L^2(\Omega))},
\end{aligned}$$

where I_{h_t} is the extended time interpolant (2.44) and $c > 0$ is the constant coming from standard interpolation error estimates.

The second error estimate is proven by an Aubin-Nitsche argument. Therefore, let the function $w \in H_0^{1/2}(0, T; L^2(\Omega))$ be the unique solution of

$$\langle w, z \rangle_{H_0^{1/2}(0,T;L^2(\Omega)),F} = \langle \partial_t w, \mathcal{H}_T z \rangle_Q \stackrel{!}{=} \langle \mathcal{H}_T(v - \mathcal{Q}_{h_t}^{1/2} v), \mathcal{H}_T z \rangle_{L^2(Q)} = \langle v - \mathcal{Q}_{h_t}^{1/2} v, z \rangle_{L^2(Q)} \quad (3.80)$$

for all $z \in H_0^{1/2}(0, T; L^2(\Omega))$, i.e.

$$\partial_t w(x, t) = \mathcal{H}_T(v - \mathcal{Q}_{h_t}^{1/2} v)(x, t) \quad \text{for } (x, t) \in Q.$$

For $z = v - \mathcal{Q}_{h_t}^{1/2} v \in H_0^{1/2}(0, T; L^2(\Omega))$ in (3.80), it follows with the Galerkin orthogonality and the first error estimate of this proof that

$$\begin{aligned}
\|v - \mathcal{Q}_{h_t}^{1/2} v\|_{L^2(Q)}^2 &= \langle w, v - \mathcal{Q}_{h_t}^{1/2} v \rangle_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
&= \langle w - \mathcal{Q}_{h_t}^{1/2} w, v - \mathcal{Q}_{h_t}^{1/2} v \rangle_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
&\leq \|w - \mathcal{Q}_{h_t}^{1/2} w\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} \|v - \mathcal{Q}_{h_t}^{1/2} v\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
&\leq c h_t^{1/2} \|\partial_t w\|_{L^2(Q)} h_t^{s-1/2} \|v\|_{H^s(0,T;L^2(\Omega))} \\
&= c h_t^s \|v - \mathcal{Q}_{h_t}^{1/2} v\|_{L^2(Q)} \|v\|_{H^s(0,T;L^2(\Omega))},
\end{aligned}$$

where the constant $c > 0$ comes from standard interpolation error estimates. □

Lemma 3.4.22. For a function $v \in H_0^{1/2}(0, T; H^1(\Omega))$, it holds true that

$$\mathcal{Q}_{h_t}^{1/2} v \in H^1(\Omega) \otimes S_{h_t,0}^1(0, T)$$

and if, in addition, $v \in H_0^{1/2}(0, T; H_0^1(\Omega))$, then

$$\mathcal{Q}_{h_t}^{1/2} v \in H_0^1(\Omega) \otimes S_{h_t,0}^1(0, T).$$

Proof. The proof is analogous to the proof of Lemma 2.8.1. \square

The next lemma shows that

$$\mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 v \in \mathcal{Q}_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)$$

is well-defined under regularity assumptions on the given function v and that the operators in space and time commute, where the extended H_0^1 projection $\mathcal{Q}_{h_x}^1$ is given in (2.47).

Lemma 3.4.23. *For a given function $v \in H_{0,0}^{1,1/2}(Q)$ with regularity $\partial_t v \in L^2(0, T; H_0^1(\Omega))$ and $\partial_{x_m} v \in H_0^{1/2}(0, T; L^2(\Omega))$ for $m = 1, \dots, d$,*

1. *the relation*

$$\partial_{x_m} \mathcal{Q}_{h_t}^{1/2} v = \mathcal{Q}_{h_t}^{1/2} \partial_{x_m} v \in L^2(\Omega) \otimes S_{h_t, 0}^1(0, T)$$

for $m = 1, \dots, d$ and

2. *the relation*

$$\mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 v = \mathcal{Q}_{h_x}^1 \mathcal{Q}_{h_t}^{1/2} v \in \mathcal{Q}_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)$$

hold true. In particular, the space-time projections $\mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 v$ and $\mathcal{Q}_{h_x}^1 \mathcal{Q}_{h_t}^{1/2} v$ are well-defined.

Furthermore, the error estimates

$$\|v - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 v\|_{L^2(Q)} \leq \|v - \mathcal{Q}_{h_t}^{1/2} v\|_{L^2(Q)} + \|v - \mathcal{Q}_{h_x}^1 v\|_{L^2(Q)} + c h_x h_t \|\partial_t \nabla_x v\|_{L^2(Q)}$$

and

$$\begin{aligned} \|v - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 v\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} &\leq \|v - \mathcal{Q}_{h_t}^{1/2} v\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} \\ &+ \|v - \mathcal{Q}_{h_x}^1 v\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} + c h_x h_t^{1/2} \|\partial_t \nabla_x v\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} \end{aligned}$$

with a constant $c > 0$ are valid.

Proof. The proof is analogous to the proof of Lemma 2.8.2. \square

Theorem 3.4.24. *Let the unique solution $u \in H_{0,0}^{1,1/2}(Q)$ of (3.74) satisfy the assumptions $\partial_t u \in L^2(0, T; H_0^1(\Omega))$ and $\partial_{x_m} u \in H_0^{1/2}(0, T; L^2(\Omega))$ for $m = 1, \dots, d$, and further, $\Delta_x u \in H_0^{1/2}(0, T; L^2(\Omega))$. Then, the solution $u_h \in \mathcal{Q}_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)$ of the Galerkin-Bubnov finite element discretisation (3.77) satisfies*

$$\begin{aligned} \|u - u_h\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} &\leq \|u - \mathcal{Q}_{h_t}^{1/2} u\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} + 2 \|u - \mathcal{Q}_{h_x}^1 u\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} \\ &+ c h_x h_t^{1/2} \|\partial_t \nabla_x u\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} + \left\| \Delta_x u - \mathcal{Q}_{h_t}^{1/2} \Delta_x u \right\|_{[H_0^{1/2}(0, T; L^2(\Omega))]', F} \end{aligned}$$

and

$$\begin{aligned} \|u - u_h\|_{L^2(\mathcal{Q})} &\leq \|u - \mathcal{Q}_{h_t}^{1/2} u\|_{L^2(\mathcal{Q})} + \|u - \mathcal{Q}_{h_x}^1 u\|_{L^2(\mathcal{Q})} + c h_x h_t \|\partial_t \nabla_x u\|_{L^2(\mathcal{Q})} \\ &\quad + \frac{2T}{\pi} \|\partial_t u - \mathcal{Q}_{h_x}^1 \partial_t u\|_{L^2(\mathcal{Q})} + \frac{2T}{\pi} \|\Delta_x u - \mathcal{Q}_{h_t}^{1/2} \Delta_x u\|_{L^2(\mathcal{Q})} \end{aligned}$$

with a constant $c > 0$ independent of h_t and h_x .

Proof. With the $H_0^{1/2}(0, T; L^2(\Omega))$ ellipticity of $a(\cdot, \cdot)$, the Galerkin orthogonality (3.78) of $a(\cdot, \cdot)$, the properties of the $H_0^{1/2}$ projection $\mathcal{Q}_{h_t}^{1/2}$, the properties of the H_0^1 projection $\mathcal{Q}_{h_x}^1$ and integration by parts, it holds true that

$$\begin{aligned} \left\| u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u \right\|_{H_0^{1/2}(0, T; L^2(\Omega)), F}^2 &\leq a(u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u, u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u) \\ &= a(u - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u, u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u) \\ &= \left\langle \partial_t (u - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u), \mathcal{H}_T(u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u) \right\rangle_{\mathcal{Q}} \\ &\quad + \left\langle \nabla_x (u - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u), \nabla_x \mathcal{H}_T(u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u) \right\rangle_{L^2(\mathcal{Q})} \end{aligned}$$

and further,

$$\begin{aligned} \left\| u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u \right\|_{H_0^{1/2}(0, T; L^2(\Omega)), F}^2 &= \left\langle \partial_t (u - \mathcal{Q}_{h_x}^1 u), \mathcal{H}_T(u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u) \right\rangle_{\mathcal{Q}} \\ &\quad + \left\langle \nabla_x (u - \mathcal{Q}_{h_x}^1 u), \nabla_x \mathcal{H}_T(u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u) \right\rangle_{L^2(\mathcal{Q})} \\ &= \left\langle \partial_t (u - \mathcal{Q}_{h_x}^1 u), \mathcal{H}_T(u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u) \right\rangle_{\mathcal{Q}} \\ &\quad - \left\langle \Delta_x (u - \mathcal{Q}_{h_t}^{1/2} u), \mathcal{H}_T(u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u) \right\rangle_{L^2(\mathcal{Q})}. \end{aligned} \quad (3.81)$$

The relation (3.81) yields

$$\begin{aligned} &\left\| u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u \right\|_{H_0^{1/2}(0, T; L^2(\Omega)), F}^2 \\ &\leq \|u - \mathcal{Q}_{h_x}^1 u\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} \left\| u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u \right\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} \\ &\quad + \left\| \Delta_x u - \mathcal{Q}_{h_t}^{1/2} \Delta_x u \right\|_{[H_0^{1/2}(0, T; L^2(\Omega))]', F} \left\| u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u \right\|_{H_0^{1/2}(0, T; L^2(\Omega)), F} \end{aligned}$$

and so

$$\begin{aligned}
& \|u - u_h\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
& \leq \|u - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} + \|u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
& \leq \|u - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} \\
& \quad + \|u - \mathcal{Q}_{h_x}^1 u\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} + \|\Delta_x u - \mathcal{Q}_{h_t}^{1/2} \Delta_x u\|_{[H_0^{1/2}(0,T;L^2(\Omega))]',F}.
\end{aligned}$$

This gives, with the second error estimate from Lemma 3.4.23, the $H_0^{1/2}(0,T;L^2(\Omega))$ error estimate.

It remains to prove the $L^2(Q)$ error estimate. With the Poincaré type inequality from Lemma 3.4.5, the relation (3.81) and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
\frac{\pi}{2T} \|u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)}^2 & \leq \|u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{H_0^{1/2}(0,T;L^2(\Omega)),F}^2 \\
& \leq \|\partial_t u - \mathcal{Q}_{h_x}^1 \partial_t u\|_{L^2(Q)} \|u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} \\
& \quad + \|\Delta_x u - \mathcal{Q}_{h_t}^{1/2} \Delta_x u\|_{L^2(Q)} \|u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|u - u_h\|_{L^2(Q)} & \leq \|u - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} + \|u_h - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} \\
& \leq \|u - \mathcal{Q}_{h_t}^{1/2} \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} \\
& \quad + \frac{2T}{\pi} \|\partial_t u - \mathcal{Q}_{h_x}^1 \partial_t u\|_{L^2(Q)} + \frac{2T}{\pi} \|\Delta_x u - \mathcal{Q}_{h_t}^{1/2} \Delta_x u\|_{L^2(Q)}
\end{aligned}$$

and with the L^2 error estimate from Lemma 3.4.23, the assertion follows. \square

Corollary 3.4.25. *Let the assumption of Theorem 3.4.24 be satisfied. If, in addition, the unique solution u of (3.74) is sufficiently smooth and the spatial H_0^1 projection $\mathcal{Q}_{h_x}^1$ fulfils the standard L^2 error estimate*

$$\|u - \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} \leq C h_x^2 \|u\|_{L^2(0,T;H^2(\Omega))}$$

with a constant $C > 0$, see (2.48), then, for the unique solution $u_h \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1/2}(Q)$ of the Galerkin-Bubnov finite element discretisation (3.77), the error estimates

$$\|u - u_h\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} \leq c h^{3/2}$$

and

$$\|u - u_h\|_{L^2(Q)} \leq ch^2$$

hold true with a constant $c > 0$ independent of the mesh size $h = \max\{h_t, h_x\}$.

Corollary 3.4.26. *Let the assumption of Theorem 3.4.24 be satisfied. Furthermore, let $u \in H^{1+s}(Q) \cap H_{0;0}^{1,1}(Q)$ be for some $s \in [0, 1]$ and let the $H_{0;0}^{1,1}(Q)$ projection Q_h^1 , given in (2.45), fulfil the standard error estimate*

$$\|u - Q_h^1 u\|_{L^2(Q)} \leq ch^{1+s} \|u\|_{H^{1+s}(Q)}$$

with a constant $c > 0$, see (2.46). Moreover, assume for $Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ the inverse inequality

$$\forall v_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q): \quad |v_h|_{H^1(Q)} \leq c_{\text{inv}} h^{-1} \|v_h\|_{L^2(Q)}$$

with a constant $c_{\text{inv}} > 0$ and $h = \max\{h_t, h_x\}$. Then, the error estimate

$$|u - u_h|_{H^1(Q)} \leq C c_{\text{inv}} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}} h^{-1} \|u_h - u\|_{L^2(Q)}$$

is valid with a constant $C > 0$ independent of h . If, in addition, the assumption of Corollary 3.4.25 is fulfilled, then, the error estimate

$$|u - u_h|_{H^1(Q)} \leq \tilde{C} h$$

holds true with a constant $\tilde{C} > 0$.

Proof. Note that

$$Q_h^1(Q) \cap H_{0;0}^{1,1}(Q) = Q_h^1(Q) \cap H_{0;0}^{1,1/2}(Q).$$

It follows with the triangle inequality, standard error estimates for Q_h^1 and the inverse inequality in $Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ that

$$\begin{aligned} |u - u_h|_{H^1(Q)} &\leq |u - Q_h^1 u|_{H^1(Q)} + |Q_h^1 u - u_h|_{H^1(Q)} \\ &\leq \tilde{C} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}} h^{-1} \|Q_h^1 u - u_h\|_{L^2(Q)} \\ &\leq \tilde{C} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}} h^{-1} \|Q_h^1 u - u\|_{L^2(Q)} + c_{\text{inv}} h^{-1} \|u_h - u\|_{L^2(Q)} \\ &\leq C c_{\text{inv}} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}} h^{-1} \|u_h - u\|_{L^2(Q)} \end{aligned}$$

with a constant $C > 0$ and hence, the assertion. \square

Remark 3.4.27. *The assumptions on the spatial H_0^1 projection $Q_{h_x}^1$ and on the $H_{0;0}^{1,1}(Q)$ projection Q_h^1 in Corollary 3.4.25 and Corollary 3.4.26 are fulfilled, if Ω is sufficiently regular. Thus, for less regular Ω , one expects reduced orders for the error estimates given in Corollary 3.4.25 and Corollary 3.4.26.*

Remark 3.4.28. Analogous space-time error estimates, as proven in Theorem 3.4.24, Corollary 3.4.25 and Corollary 3.4.26, are true for conforming tensor-product space-time finite element discretisations of (3.75) with piecewise polynomial, continuous functions of polynomial degree $p > 1$, i.e. the space-time error estimates

$$\begin{aligned} \|u - u_h\|_{H_0^{1/2}(0,T;L^2(\Omega)),F} &\leq c h^{p+1/2}, \\ \|u - u_h\|_{L^2(Q)} &\leq c h^{p+1}, \\ |u - u_h|_{H^1(Q)} &\leq c h^p \end{aligned}$$

are valid with a constant $c > 0$ for analogous assumptions as in Theorem 3.4.24, Corollary 3.4.25 and Corollary 3.4.26.

In the last part of this section, some numerical examples for the Galerkin-Bubnov variational formulation (3.77) are presented. Therefore, consider the space-time cylinder

$$Q = \Omega \times (0, T) = (0, 1) \times (0, 2)$$

for the finite element space $Q_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)$ with a uniform discretisation with mesh sizes

$$h_x = \frac{1}{N_x} \quad \text{and} \quad h_t = \frac{2}{N_t}$$

with

$$N_x = N_t = 2^j, \quad j = 1, \dots, 11,$$

see Section 2.8 for the notations. The number of the degrees of freedom is given as

$$\text{dof} = (N_x - 1) \cdot N_t.$$

The temporal transformation \mathcal{H}_T of the nodal basis functions of the finite-dimensional space $Q_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)$ is realised via the series representation (3.68) and the appearing integrals in (3.77) are calculated by the usage of high-order integration rules. In Table 3.10 the minimal and maximal singular values of the system matrix K_h , corresponding to (3.77), as well as the resulting spectral condition number of K_h are given. Note that the finite element stiffness matrix K_h is still positive definite, but not symmetric, and that no CFL condition is needed, see Theorem 3.4.20.

Remark 3.4.29. The to (3.77) related inf-sup constant

$$\inf_{0 \neq u_h \in Q_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)} \sup_{0 \neq v_h \in Q_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)} \frac{a(u_h, \mathcal{H}_T v_h)}{\|u_h\|_{H_{0,0}^{1,1/2}(Q),F} \|v_h\|_{H_{0,0}^{1,1/2}(Q),F}} =: c_S(h) \quad (3.82)$$

N_x	N_t	dof	h_x	h_t	$\sigma_{\max}(K_h)$	$\sigma_{\min}(K_h)$	$\kappa_2(K_h)$
2	2	2	0.50000000	1.00000000	2.27730420	0.90807052	2.5
4	4	12	0.25000000	0.50000000	5.67006464	0.28650743	19.8
8	8	56	0.12500000	0.25000000	7.29533890	0.11531994	63.3
16	16	240	0.06250000	0.12500000	7.80822886	0.04567559	170.9
32	32	992	0.03125000	0.06250000	7.94997290	0.01642703	484.0
64	64	4032	0.01562500	0.03125000	7.98720088	0.00472215	1691.4
128	128	16256	0.00781250	0.01562500	7.99675804	0.00120736	6623.3
256	256	65280	0.00390625	0.00781250	7.99918336	0.00030351	26355.9

Table 3.10: Stability behaviour of the Galerkin-Bubnov finite element discretisation (3.77) with uniform meshes for the space-time cylinder $Q = (0, 1) \times (0, 2)$.

seems to depend at least linearly on the mesh size h . As illustration, consider the inf-sup constant

$$\inf_{0 \neq u_h \in Q_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)} \sup_{0 \neq v_h \in Q_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)} \frac{a(u_h, \mathcal{H}_T v_h)}{\|\nabla_x u_h\|_{L^2(Q)} \|\nabla_x v_h\|_{L^2(Q)}} =: \tilde{c}_S(h), \quad (3.83)$$

satisfying

$$\tilde{c}_S(h) \geq c_S(h).$$

The inf-sup constant $\tilde{c}_S(h)$ is given as

$$\tilde{c}_S(h) = \sqrt{\lambda_{\min}},$$

where λ_{\min} is the minimal eigenvalue of the generalised eigenvalue problem [84, Subsection 3.6.6, page 124]

$$K_h^\top A_{h,0}^{-1} K_h \underline{u} = \lambda A_{h,0} \underline{u}$$

with the matrices

$$K_h[k, i] := a(\chi_i, \mathcal{H}_T \chi_k)$$

and

$$A_{h,0}[k, i] := \langle \nabla_x \chi_i, \nabla_x \chi_k \rangle_{L^2(Q)}$$

for $i, k = 1, \dots, \text{dof}$, where χ_i are the nodal basis functions of the space $Q_h^1(Q) \cap H_{0,0}^{1,1/2}(Q)$, i.e.

$$Q_h^1(Q) \cap H_{0,0}^{1,1/2}(Q) = \text{span}\{\chi_i\}_{i=1}^{\text{dof}}.$$

For a uniform discretisation with mesh sizes

$$h_x = \frac{1}{N_x} \quad \text{and} \quad h_t = \frac{2}{N_t}$$

with

$$N_x = N_t = 2^j, \quad j = 1, \dots, 8,$$

and

$$\text{dof} = (N_x - 1) \cdot N_t,$$

the inf-sup constant $\tilde{c}_S(h)$ of (3.83) is given in Table 3.11, where a linear dependency is observed. Hence, the inf-sup constant $c_S(h)$ of (3.82) seems to depend at least linearly on the mesh size $h = \max\{h_t, h_x\}$.

N_x	N_t	dof	h_x	h_t	$\tilde{c}_S(h)$
2	2	2	0.500000	1.000000	0.673637
4	4	12	0.250000	0.500000	0.375800
8	8	56	0.125000	0.250000	0.216577
16	16	240	0.062500	0.125000	0.117912
32	32	992	0.031250	0.062500	0.061679
64	64	4032	0.015625	0.031250	0.031558
128	128	16256	0.007813	0.015625	0.015963
256	256	65280	0.003906	0.007813	0.008028

Table 3.11: Optimal discrete inf-sup constant $\tilde{c}_S(h)$ of (3.83) with a uniform temporal mesh size h_t and a uniform spatial mesh size h_x for the space-time cylinder $Q = (0, 1) \times (0, 2)$.

Next, numerical examples concerning convergence rates are given. Therefore, consider the functions

$$u_1(x, t) := \sin\left(\frac{5\pi}{4}t\right) \sin(\pi x),$$

$$u_2(x, t) := t^{2/3} \sin(\pi x),$$

$$u_3(x, t) := x^{3/5}(x-1)t$$

for $(x, t) \in Q$,

$$Q = \Omega \times (0, T) = (0, 1) \times (0, 2),$$

as the solutions of (3.74). Since the solution u_1 is smooth, a quadratic convergence in $L^2(Q)$ and a linear convergence in $H^1(Q)$ are expected by Corollary 3.4.25 and by Corollary 3.4.26. This behaviour is confirmed by the numerical results given in Table 3.12.

The solutions u_2 and u_3 satisfy only

$$u_2 \in H^{7/6-\varepsilon}(Q) \quad \text{and} \quad u_3 \in H^{11/10-\varepsilon}(Q) \quad \text{for } \varepsilon > 0,$$

which leads to a reduced order of convergence in $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$, see Table 3.13 and Table 3.14.

N_x, N_t	dof	h_x	h_t	$\ u_1 - u_{1,h}\ _{L^2(Q)}$	eoc	$ u_1 - u_{1,h} _{H^1(Q)}$	eoc
2	2	0.500000	1.000000	0.91082337	-	4.48444176	-
4	12	0.250000	0.500000	0.15773958	2.53	1.89079374	1.25
8	56	0.125000	0.250000	0.02936109	2.43	0.84238860	1.17
16	240	0.062500	0.125000	0.00689515	2.09	0.41495827	1.02
32	992	0.031250	0.062500	0.00169574	2.02	0.20679353	1.00
64	4032	0.015625	0.031250	0.00042208	2.01	0.10331240	1.00
128	16256	0.007812	0.015625	0.00010539	2.00	0.05164563	1.00
256	65280	0.003906	0.007812	0.00002634	2.00	0.02582149	1.00
512	261632	0.001953	0.003906	0.00000658	2.00	0.01291058	1.00
1024	1047552	0.000977	0.001953	0.00000165	2.00	0.00645527	1.00
2048	4192256	0.000488	0.000977	0.00000041	2.01	0.00322763	1.00

Table 3.12: Numerical results of the Galerkin-Bubnov finite element discretisation (3.77) with uniform meshes for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for the function u_1 .

N_x, N_t	dof	h_x	h_t	$\ u_2 - u_{2,h}\ _{L^2(Q)}$	eoc	$ u_2 - u_{2,h} _{H^1(Q)}$	eoc
2	2	0.500000	1.000000	0.23826687	-	1.52140890	-
4	12	0.250000	0.500000	0.06709046	1.83	0.82368146	0.89
8	56	0.125000	0.250000	0.02030948	1.72	0.46578105	0.82
16	240	0.062500	0.125000	0.00707197	1.52	0.29871905	0.64
32	992	0.031250	0.062500	0.00280561	1.33	0.22207053	0.43
64	4032	0.015625	0.031250	0.00119675	1.23	0.18306107	0.28
128	16256	0.007812	0.015625	0.00052531	1.19	0.15842749	0.21
256	65280	0.003906	0.007812	0.00023285	1.17	0.13968147	0.18
512	261632	0.001953	0.003906	0.00010355	1.17	0.12397096	0.17
1024	1047552	0.000977	0.001953	0.00004610	1.17	0.11028663	0.17
2048	4192256	0.000488	0.000977	0.00002053	1.17	0.09819766	0.17

Table 3.13: Numerical results of the Galerkin-Bubnov finite element discretisation (3.77) with uniform meshes for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for the function u_2 .

N_x, N_t	dof	h_x	h_t	$\ u_3 - u_{3,h}\ _{L^2(Q)}$	eoc	$ u_3 - u_{3,h} _{H^1(Q)}$	eoc
2	2	0.500000	1.000000	0.15232131	-	1.13419849	-
4	12	0.250000	0.500000	0.06042186	1.33	0.91981810	0.30
8	56	0.125000	0.250000	0.02557510	1.24	0.79291178	0.21
16	240	0.062500	0.125000	0.01129138	1.18	0.70947658	0.16
32	992	0.031250	0.062500	0.00511443	1.14	0.64798432	0.13
64	4032	0.015625	0.031250	0.00234984	1.12	0.59815237	0.12
128	16256	0.007812	0.015625	0.00108778	1.11	0.55512396	0.11
256	65280	0.003906	0.007812	0.00050549	1.11	0.51657314	0.10
512	261632	0.001953	0.003906	0.00023536	1.10	0.48134128	0.10
1024	1047552	0.000977	0.001953	0.00010969	1.10	0.44881042	0.10
2048	4192256	0.000488	0.000977	0.00005115	1.10	0.41861673	0.10

Table 3.14: Numerical results of the Galerkin-Bubnov finite element discretisation (3.77) with uniform meshes for the space-time cylinder $Q = (0, 1) \times (0, 2)$ and for the function u_3 .

Outlook for Possible Extensions of Section 3.4

Since the constructions of the methods and the proofs in this section mainly rely on the treatment of the first-order temporal differential operator $\partial_t + \mu$ with a parameter $\mu > 0$, the results of this section for the model problem (3.65) can be transferred to a more general parabolic equation (3.2) under certain assumptions of the involved functions and function spaces, where details are left for future considerations.

4 WAVE EQUATION

The main focus of this chapter is on space-time variational formulations and conforming discretisations for hyperbolic problems. First, a highly non-exhaustive list of references, second, an overview of the sections of this chapter, where for each section the relevant literature is cited, and third, an outlook for possible extensions are given. Here, the model problem for a hyperbolic partial differential equation is the homogeneous Dirichlet problem for the wave equation,

$$\left. \begin{aligned} \partial_{tt}u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q = \Omega \times (0, T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma = \Gamma \times [0, T], \\ u(x,0) &= u_0(x) && \text{for } x \in \Omega, \\ \partial_t u(x,0) &= v_0(x) && \text{for } x \in \Omega, \end{aligned} \right\} \quad (4.1)$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, $T > 0$ is a terminal time, u_0, v_0 are given initial conditions and f is a given right-hand side. To compute an approximate solution of the wave equation (4.1), different numerical schemes including different approaches of the underlying mathematical framework are available. On the one hand, some of them are repeated in this chapter, but on the other hand, powerful tools like semigroup theory as in [91, 123] on the continuous part or on the discretisation side, any kind of discontinuous Galerkin methods [33, 42, 67, 86, 88, 89, 111, 130, 140] or finite difference methods [32, 33, 65, 97, 147] or boundary element methods, see Remark 1.2.1, are not in the scope of this work. Furthermore, all approaches where the wave equation (4.1) is reformulated as a first-order system in the spatial and/or time variables are excluded in this work, see [19, 22, 24, 42, 45, 46, 55, 83, 94, 163]. In addition, see also the approaches in [53, 63, 66, 80, 81, 85, 114, 115, 124].

Outline of Chapter 4

The remainder of this chapter examines the wave equation (4.1) as follows:

In Section 4.1 a pointwise spatial variational formulation coming from a so-called Galerkin method [36, 97, 102, 160, 162] and time stepping schemes [18, 22, 27, 28, 44, 95, 113, 126, 127] are cited. In Section 4.2 a space-time variational formulation [97] in a subspace of $H^1(Q)$ is examined, which fits very well to finite element methods with piecewise linear, continuous functions. This conforming finite element method is only conditionally stable, i.e. a CFL condition plays a decisive role for stability. To gain a deeper understanding of the CFL condition, an ordinary differential equation corresponding to the wave equation is analysed. For this ordinary differential equation, an unconditionally stable numerical scheme is introduced. By transferring this idea to the wave equation, a stabilised space-time finite

element method for the wave equation is obtained. This stabilisation is proposed in [164]. Finally, for this stabilised space-time finite element method, $L^2(Q)$ stability, $L^2(Q)$ and $H^1(Q)$ error estimates are proven, see [146]. In the last part of Section 4.2, numerical examples for a one-dimensional spatial domain and a two-dimensional spatial domain are given, including spatially adaptive refined meshes. In Section 4.3 and Section 4.4, existence and uniqueness results for the wave equation as a partial differential equation in $L^2(Q)$ and in a weaker sense than $L^2(Q)$ are derived, including isomorphic solution operators and corresponding inf-sup conditions.

Outlook for Possible Extensions of Chapter 4

The results of this chapter for the model problem (4.1) can be transferred to hyperbolic equations for vector fields, e.g., the Maxwell's equations, as well as to a more general hyperbolic equation for scalar functions

$$\left. \begin{aligned} \rho(x,t)\partial_{tt}u(x,t) - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a_{i,j}(x,t) \frac{\partial u}{\partial x_j}(x,t) \right) \\ + \sum_{i=1}^d a_i(x,t) \frac{\partial u}{\partial x_i}(x,t) + a_{d+1}(x,t) \frac{\partial u}{\partial t}(x,t) \\ + a(x,t)u(x,t) &= f(x,t) & \text{for } (x,t) \in Q, \\ u(x,t) &= g(x,t) & \text{for } (x,t) \in \Sigma, \\ u(x,0) &= u_0(x) & \text{for } x \in \Omega, \\ \partial_t u(x,0) &= v_0(x) & \text{for } x \in \Omega, \end{aligned} \right\} \quad (4.2)$$

where ρ , $a_{i,j}$, a_i , a are given coefficients, f is a given right-hand side, g is a given Dirichlet datum and u_0 , v_0 are given initial conditions, which have to satisfy smoothness and boundedness conditions, see, e.g., the classical references [36, 97, 102, 103, 160, 162, 164].

In particular, since the proofs of the results in Section 4.2, Section 4.3 and Section 4.4 mainly rely on the treatment of the second-order ordinary differential equation for the temporal differential operator $\partial_{tt} + \mu$ with a parameter $\mu > 0$, a generalisation to differential operators

$$\partial_{tt} + \mathcal{A}_x$$

with a spatial differential operator \mathcal{A}_x , acting on vector fields, is possible. Since the proofs have to be done with great care, they are left for future work, including precise assumptions on \mathcal{A}_x and the involved function spaces.

4.1 Variational Formulation in Space and Pointwise in Time

In this section, a short overview of a pointwise in time variational formulation is given. Furthermore, numerical examples for lowest order in space, i.e. piecewise linear, continuous ansatz functions, combined with lowest order time stepping are presented.

The pointwise in time variational formulation of (4.1) is given, with the notations of Section 2, as follows:

Find $u \in L^2(0, T; H_0^1(\Omega))$ with $\partial_t u \in L^2(Q)$, $\partial_{tt} u \in L^2(0, T; [H_0^1(\Omega)]')$, $u(\cdot, 0) = u_0$ in $H_0^1(\Omega)$ and $\partial_t u(\cdot, 0) = v_0$ in $L^2(\Omega)$ such that

$$\langle \partial_{tt} u(\cdot, t), v \rangle_{\Omega} + \langle \nabla_x u(\cdot, t), \nabla_x v \rangle_{L^2(\Omega)} = \langle f(\cdot, t), v \rangle_{L^2(\Omega)} \quad (4.3)$$

for almost all $t \in (0, T)$ and all $v \in H_0^1(\Omega)$, where $f \in L^2(Q)$, $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$ are the given right-hand side and the given initial conditions. Here, ∂_{tt} is the distributional derivative on $(0, T)$, i.e. equality (4.3) means that

$$\int_0^T \langle u(\cdot, t), v \rangle_{\Omega} \frac{d^2 \varphi}{dt^2}(t) dt + \int_0^T \langle \nabla_x u(\cdot, t), \nabla_x v \rangle_{L^2(\Omega)} \varphi(t) dt = \int_0^T \langle f(\cdot, t), v \rangle_{L^2(\Omega)} \varphi(t) dt$$

for all $\varphi \in C_0^\infty(0, T)$. The variational formulation in (4.3) is examined in many books, for example, [102, Théorème 8.1, Chapitre 3, page 287, and Théorème 8.2, Chapitre 3, page 296], [97, Theorem 4.2, Chapter IV, page 167], [160, Satz 29.1, Kapitel V, page 422], [162, Section 24.1, Chapter 24, page 453] or [36, Mathematical Example 1, Chapter XVIII, page 581]. In these books, the following existence and uniqueness result is proven.

Theorem 4.1.1. *For given $f \in L^2(Q)$, $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$, a unique solution u of the variational formulation (4.3) exists, satisfying*

$$\begin{aligned} u &\in L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; H_0^1(\Omega)), \\ \partial_t u &\in L^2(Q) \cap C([0, T]; L^2(\Omega)), \\ \partial_{tt} u &\in L^2(0, T; [H_0^1(\Omega)]'), \end{aligned}$$

and the stability estimate

$$\sqrt{\|u\|_{L^2(0, T; H_0^1(\Omega))}^2 + \|\partial_t u\|_{L^2(Q)}^2} \leq c \left(|u_0|_{H^1(\Omega)} + \|v_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q)} \right)$$

with a constant $c > 0$.

Proof. See the books [36, 97, 102, 160, 162] as mentioned above. \square

For a discretisation scheme, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0, L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. With the notations of Section 2.8, consider a discretisation of a tensor-product type (2.41) with the finite-dimensional space $\mathcal{Q}_h^1(\mathcal{Q}) = V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0, T)$, see (2.42). Therefore, introduce for $x \in \Omega$ and $\ell \in \{0, \dots, N_t\}$ the approximations

$$U_{h_x,\ell}(x) := \sum_{i=1}^{M_x} U_i^\ell \psi_i(x) \approx u(x, t_\ell)$$

and

$$\hat{U}_{h_x,\ell}(x) := \sum_{i=1}^{M_x} \hat{U}_i^\ell \psi_i(x) \approx \partial_t u(x, t_\ell),$$

where $U_i^\ell, \hat{U}_i^\ell \in \mathbb{R}$ are the unknown coefficients of $U_{h_x,\ell}, \hat{U}_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0, \dots, N_t\}$. Furthermore, set for $(x, t) \in \mathcal{Q}$

$$u_h(x, t) := \sum_{\ell=0}^{N_t} \sum_{i=1}^{M_x} U_i^\ell \psi_i(x) \varphi_\ell(t) = \sum_{\ell=0}^{N_t} U_{h_x,\ell}(x) \varphi_\ell(t) \approx u(x, t) \quad (4.4)$$

and

$$\hat{u}_h(x, t) := \sum_{\ell=0}^{N_t} \sum_{i=1}^{M_x} \hat{U}_i^\ell \psi_i(x) \varphi_\ell(t) = \sum_{\ell=0}^{N_t} \hat{U}_{h_x,\ell}(x) \varphi_\ell(t) \approx \partial_t u(x, t), \quad (4.5)$$

i.e. $u_h, \hat{u}_h \in \mathcal{Q}_h^1(\mathcal{Q})$.

For the pointwise in time variational formulation (4.3), a conforming discretisation in space with $V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ in combination with the Newmark scheme with Newmark parameters $\beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$ leads to the so-called Newmark Galerkin method to find the functions $U_{h_x,\ell}, \hat{U}_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0, \dots, N_t\}$ such that

$$U_{h_x,0} = \mathcal{Q}_{h_x} u_0, \quad \hat{U}_{h_x,0} = \mathcal{Q}_{h_x} v_0,$$

and for $\ell = 1, \dots, N_t$

$$\begin{aligned} \frac{1}{h_{t,\ell}^2} \langle U_{h_x,\ell} - U_{h_x,\ell-1} + h_{t,\ell} \hat{U}_{h_x,\ell-1}, v_{h_x} \rangle_{L^2(\Omega)} + \frac{1}{4} \langle \nabla_x U_{h_x,\ell} + \nabla_x U_{h_x,\ell-1}, \nabla_x v_{h_x} \rangle_{L^2(\Omega)} \\ = \frac{1}{4} \langle f(\cdot, t_\ell) + f(\cdot, t_{\ell-1}), v_{h_x} \rangle_{L^2(\Omega)} \end{aligned} \quad (4.6)$$

for all $v_{h_x} \in V_{h_x,0}(\Omega)$ and

$$\begin{aligned} \frac{1}{h_{t,\ell}} \langle \hat{U}_{h_x,\ell} - \hat{U}_{h_x,\ell-1}, \hat{v}_{h_x} \rangle_{L^2(\Omega)} + \frac{1}{2} \langle \nabla_x U_{h_x,\ell} + \nabla_x U_{h_x,\ell-1}, \nabla_x \hat{v}_{h_x} \rangle_{L^2(\Omega)} \\ = \frac{1}{2} \langle f(\cdot, t_\ell) + f(\cdot, t_{\ell-1}), \hat{v}_{h_x} \rangle_{L^2(\Omega)} \end{aligned} \quad (4.7)$$

for all $\hat{v}_{h_x} \in V_{h_x,0}(\Omega)$, where $Q_{h_x} : L^2(\Omega) \rightarrow V_{h_x,0}(\Omega)$ denotes the L^2 projection (2.39). This method is given in [127, (8.6-4), (8.6-5), (8.6-6), page 205], see also [18, 22, 27, 28, 44, 95, 113, 126]. The Newmark Galerkin method (4.6), (4.7) is equivalent to the linear systems

$$M_{h_x} \underline{U}^0 = \underline{u}_0, \quad M_{h_x} \hat{\underline{U}}^0 = \underline{v}_0,$$

and for all $\ell = 1, \dots, N_t$

$$\left(M_{h_x} + \frac{1}{4} h_{t,\ell}^2 A_{h_x} \right) \underline{U}^\ell = \left(M_{h_x} - \frac{1}{4} h_{t,\ell}^2 A_{h_x} \right) \underline{U}^{\ell-1} + h_{t,\ell} M_{h_x} \hat{\underline{U}}^{\ell-1} + \frac{1}{4} h_{t,\ell}^2 \left(\underline{F}^\ell + \underline{F}^{\ell-1} \right), \quad (4.8)$$

$$M_{h_x} \hat{\underline{U}}^\ell = M_{h_x} \hat{\underline{U}}^{\ell-1} - \frac{1}{2} h_{t,\ell} A_{h_x} \left(\underline{U}^\ell + \underline{U}^{\ell-1} \right) + \frac{1}{2} h_{t,\ell} \left(\underline{F}^\ell + \underline{F}^{\ell-1} \right), \quad (4.9)$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the mass matrix (2.37), $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the stiffness matrix (2.38) and the vectors $\underline{u}_0, \underline{v}_0, \underline{F}^\ell \in \mathbb{R}^{M_x}$ are defined by

$$\underline{u}_0[i] := \langle u_0, \psi_i \rangle_{L^2(\Omega)}, \quad \underline{v}_0[i] := \langle v_0, \psi_i \rangle_{L^2(\Omega)}, \quad \underline{F}^\ell[i] := \langle f(\cdot, t_\ell), \psi_i \rangle_{L^2(\Omega)} \quad (4.10)$$

for $i = 1, \dots, M_x$ with the nodal basis functions ψ_i satisfying $V_{h_x,0}(\Omega) = \text{span}\{\psi_i\}_{i=1}^{M_x}$. The matrix $M_{h_x} + \frac{1}{4} h_{t,\ell}^2 A_{h_x}$ is positive definite and hence, the linear systems (4.8), (4.9) are uniquely solvable for all $\ell = 1, \dots, N_t$. Stability of the numerical scheme (4.6), (4.7) holds true without any CFL condition because the Newmark method is unconditionally stable, see [22, 95, 113, 127]. Concerning error estimates, it seems that error estimates of the quantities $\|u(\cdot, t_\ell) - U_{h_x,\ell}\|_{L^2(\Omega)}$ for each $\ell = 0, \dots, N_t$ are standard, which are of optimal order $\mathcal{O}(h_t^2 + h_x^2)$, see [127, Chapitre 8], [113, Section 4.2] or [18, 27, 28, 44]. However, here, error estimates in space-time norms $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ for the approximate solution $u_h \in Q_h^1(Q)$, defined in (4.4), are considered. It seems that proofs of such error estimates are not available. Hence, the proofs of such statements are left for future work.

In the last part of this section, some numerical examples are presented. So, for the space-time cylinder $Q = \Omega \times (0, T) = (0, 1) \times (0, 10)$, consider the solution of (4.3)

$$u_1(x, t) = \sin(\pi x) \sin^2\left(\frac{5}{4}\pi t\right), \quad (x, t) \in Q,$$

see also the numerical examples of Section 4.2 for a comparison. The discretisation is done with respect to nonuniform meshes as shown in Figure 4.1, where a uniform refinement strategy is applied. The appearing integrals for the initial conditions and right-hand side in (4.10) are calculated by the usage of high-order integration rules, and the degrees of freedom are denoted by

$$\text{dof} = \dim Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$$

due to the homogeneous Dirichlet boundary condition and the initial conditions, see (2.14). In Table 4.1 the errors in the space-time norms $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ are presented for the smooth function u_1 , where the convergence rates are as expected. Note that no CFL condition is needed because the Newmark method is unconditionally stable.

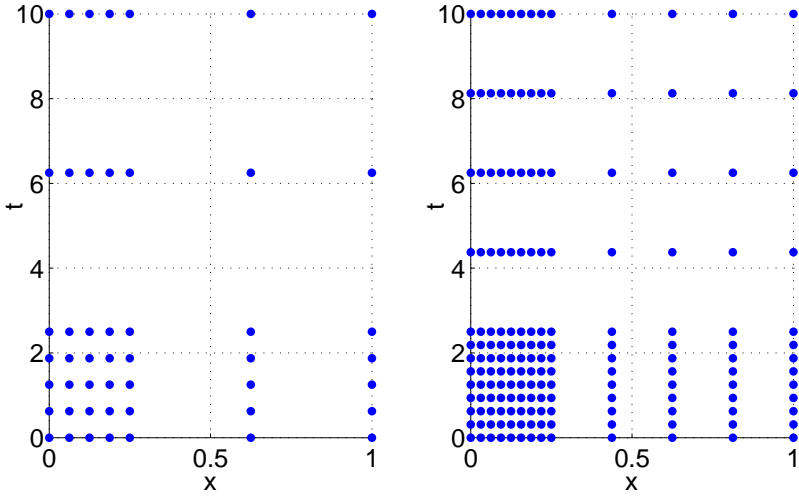


Figure 4.1: Nonuniform meshes: Starting mesh and the mesh after one uniform refinement step.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_1 - u_{1,h}\ _{L^2(Q)}$	eoc	$ u_1 - u_{1,h} _{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	7.088e+00	-	2.622e+01	-
132	0.18750	0.03125	1.87500	0.31250	2.921e+00	1.20	1.276e+01	0.97
552	0.09375	0.01562	0.93750	0.15625	4.728e+00	-0.67	1.811e+01	-0.49
2256	0.04688	0.00781	0.46875	0.07812	1.670e+00	1.48	8.558e+00	1.06
9120	0.02344	0.00391	0.23438	0.03906	6.701e-01	1.31	4.571e+00	0.90
36672	0.01172	0.00195	0.11719	0.01953	2.023e-01	1.72	1.813e+00	1.33
147072	0.00586	0.00098	0.05859	0.00977	5.326e-02	1.92	7.733e-01	1.23
589056	0.00293	0.00049	0.02930	0.00488	1.349e-02	1.98	3.652e-01	1.08
2357760	0.00146	0.00024	0.01465	0.00244	3.384e-03	1.99	1.797e-01	1.02
9434112	0.00073	0.00012	0.00732	0.00122	8.467e-04	2.00	8.950e-02	1.01
37742592	0.00037	0.00006	0.00366	0.00061	2.117e-04	2.00	4.471e-02	1.00

Table 4.1: Numerical results of the Newmark Galerkin method (4.6), (4.7) with nonuniform meshes for $Q = (0, 1) \times (0, 10)$ and for the function u_1 .

Remark 4.1.2. For a solution of (4.3)

$$u_2(x, t) = \sin(\pi x) t^2 (10 - t)^{3/4}, \quad (x, t) \in (0, 1) \times (0, 10),$$

as considered as numerical example in Section 4.2, the Newmark Galerkin method (4.6), (4.7) is not applicable because the corresponding right-hand side f has a singularity with respect to $t = T = 10$. However, an alternative is to adapt the treatment of the right-hand side f .

Remark 4.1.3. The Newmark Galerkin method (4.6), (4.7) fulfils a conservation of the total energy

$$E(t) := \frac{1}{2} \|\partial_t u(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla_x u(\cdot, t)\|_{L^2(\Omega)}^2, \quad t \in [0, T],$$

see [22, 95]. As illustration, consider a solution of the homogeneous wave equation, i.e.

$$u_3(x, t) = (\cos(\pi t) + \sin(\pi t)) \sin(\pi x), \quad (x, t) \in \mathcal{Q} = (0, 1) \times (0, 10),$$

with the total energy

$$E(t) = \frac{\pi^2}{2}, \quad t \in [0, 10].$$

Here, the initial conditions are

$$u_3(x, 0) = u_0(x) = \sin(\pi x), \quad \partial_t u_3(x, 0) = v_0(x) = \pi \sin(\pi x), \quad x \in \Omega.$$

For the solution u_3 and for the mesh as given in Figure 4.1, the discrete total energy

$$E_h(t) := \frac{1}{2} \|\hat{u}_h(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla_x u_h(\cdot, t)\|_{L^2(\Omega)}^2, \quad t \in [0, T],$$

is computed, where the approximation $\hat{u}_h \approx \partial_t u$, given in (4.5), is used. In Figure 4.2 the difference

$$E_h(t) - E(t) = E_h(t) - \frac{\pi^2}{2}, \quad t \in [0, 10],$$

is plotted pointwise for the approximate solution $u_{3,h}$ of the finest level of Table 4.2, where a conservation of the total energy is observed. In addition, the errors in the space-time norms $\|\cdot\|_{L^2(\mathcal{Q})}$ and $|\cdot|_{H^1(\mathcal{Q})}$ are given in Table 4.2, where the convergence rates are as expected.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_3 - u_{3,h}\ _{L^2(Q)}$	eoc	$ u_3 - u_{3,h} _{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	2.561e+00	-	1.066e+01	-
132	0.18750	0.03125	1.87500	0.31250	2.503e+00	0.03	9.821e+00	0.11
552	0.09375	0.01562	0.93750	0.15625	2.243e+00	0.15	9.780e+00	0.01
2256	0.04688	0.00781	0.46875	0.07812	2.581e+00	-0.20	1.152e+01	-0.23
9120	0.02344	0.00391	0.23438	0.03906	1.082e+00	1.25	4.845e+00	1.24
36672	0.01172	0.00195	0.11719	0.01953	3.012e-01	1.84	1.444e+00	1.74
147072	0.00586	0.00098	0.05859	0.00977	7.695e-02	1.97	4.605e-01	1.65
589056	0.00293	0.00049	0.02930	0.00488	1.933e-02	1.99	1.804e-01	1.35
2357760	0.00146	0.00024	0.01465	0.00244	4.839e-03	2.00	8.268e-02	1.12
9434112	0.00073	0.00012	0.00732	0.00122	1.210e-03	2.00	4.034e-02	1.03
37742592	0.00037	0.00006	0.00366	0.00061	3.026e-04	2.00	2.005e-02	1.01

Table 4.2: Numerical results of the Newmark Galerkin method (4.6), (4.7) with nonuniform meshes for $Q = (0, 1) \times (0, 10)$ and for the function u_3 .

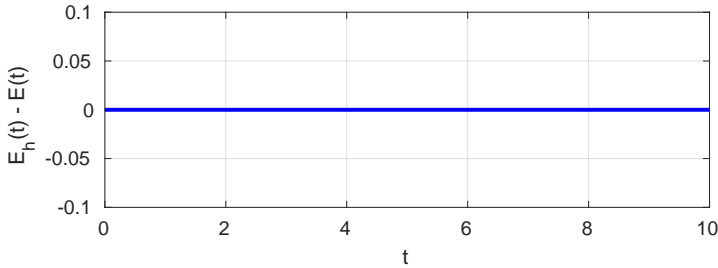


Figure 4.2: Difference of the total energy $E(t) = \frac{\pi^2}{2}$ and $E_h(t)$ of the Newmark Galerkin method (4.6), (4.7) with a nonuniform mesh for $Q = (0, 1) \times (0, 10)$ and for the function u_3 .

4.2 Space-Time Variational Formulation in $H^1(Q)$

In this section, a space-time variational formulation in subspaces of $H^1(Q)$ of the homogeneous Dirichlet problem for the wave equation,

$$\left. \begin{aligned} \partial_{tt}u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q = \Omega \times (0, T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma = \Gamma \times [0, T], \\ u(x,0) = \partial_t u(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned} \right\} \quad (4.11)$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, $T > 0$ is a terminal time and f is a given right-hand side, is examined and space-time finite element methods with piecewise linear, continuous functions are considered, see [146]. The unique solvability of this space-time variational formulation is given in [97], where the proof of the stability estimate is repeated in this section. A (natural) tensor-product approach by piecewise linear, continuous functions leads to a CFL condition

$$h_t \leq C h_x$$

with a constant $C > 0$, where h_t and h_x are the uniform mesh sizes in time and space. To gain a deeper understanding of the CFL condition, an ordinary differential equation corresponding to the wave equation is analysed. For this ordinary differential equation, an unconditionally stable numerical scheme is introduced. By transferring this idea to the wave equation, a stabilised space-time finite element method for the wave equation is obtained. In the last part of this section, unconditional stability in $L^2(Q)$, error estimates in the space-time norms $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$, and numerical examples are given.

4.2.1 Variational Formulation for $\partial_{tt}u + \mu u = f$

As a model problem for $\mu > 0$, consider the second-order linear equation,

$$\partial_{tt}u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0, T), \quad u(0) = \partial_t u(0) = 0, \quad (4.12)$$

and the variational formulation to find $u \in H_0^1(0, T)$ such that

$$a(u, w) = \langle f, w \rangle_{(0, T)} \quad (4.13)$$

for all $w \in H_0^1(0, T)$, where $T > 0$ and $f \in [H_0^1(0, T)]'$ are given. In (4.13), the bilinear form

$$a(\cdot, \cdot): H_0^1(0, T) \times H_0^1(0, T) \rightarrow \mathbb{R}$$

is defined by

$$a(u, w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(0, T)} + \mu \langle u, w \rangle_{L^2(0, T)} \quad (4.14)$$

for all $u \in H_0^1(0, T)$, $w \in H_0^1(0, T)$. Note that $\langle \cdot, \cdot \rangle_{(0, T)}$ denotes the duality pairing as extension of the inner product in $L^2(0, T)$, and the Sobolev spaces $H_0^1(0, T)$, $H_0^1(0, T)$, $[H_0^1(0, T)]'$ are introduced in Section 2.2.

Lemma 4.2.1. *The bilinear form $a(\cdot, \cdot): H_0^1(0, T) \times H_0^1(0, T) \rightarrow \mathbb{R}$ is bounded, i.e. it holds true that*

$$|a(u, w)| \leq \left(1 + \frac{4T^2\mu}{\pi^2}\right) |u|_{H^1(0, T)} |w|_{H^1(0, T)}$$

for all $u \in H_0^1(0, T)$, $w \in H_0^1(0, T)$.

Proof. The Cauchy-Schwarz inequality and the Poincaré inequality of Lemma 3.4.5 yield

$$\begin{aligned} |a(u, w)| &\leq |u|_{H^1(0, T)} |w|_{H^1(0, T)} + \mu \|u\|_{L^2(0, T)} \|w\|_{L^2(0, T)} \\ &\leq \left(1 + \frac{4T^2\mu}{\pi^2}\right) |u|_{H^1(0, T)} |w|_{H^1(0, T)} \end{aligned}$$

for $u \in H_0^1(0, T)$, $w \in H_0^1(0, T)$ and therefore, the assertion. \square

For $v \in H_0^1(0, T)$, one defines

$$(\overline{\mathcal{H}}_T v)(t) := v(T) - v(t), \quad t \in (0, T), \quad (4.15)$$

i.e. it holds true that $\overline{\mathcal{H}}_T v \in H_0^1(0, T)$. The operator

$$\overline{\mathcal{H}}_T: H_0^1(0, T) \rightarrow H_0^1(0, T)$$

is norm preserving, i.e.

$$\|v\|_{H_0^1(0, T)} = \|\partial_t v\|_{L^2(0, T)} = \|\partial_t \overline{\mathcal{H}}_T v\|_{L^2(0, T)} = \|\overline{\mathcal{H}}_T v\|_{H_0^1(0, T)},$$

and bijective, where the inverse operator

$$\overline{\mathcal{H}}_T^{-1}: H_0^1(0, T) \rightarrow H_0^1(0, T)$$

is given by

$$(\overline{\mathcal{H}}_T^{-1} w)(t) := w(0) - w(t), \quad t \in (0, T),$$

for $w \in H_0^1(0, T)$. Then the variational formulation (4.13) is equivalent to the variational formulation to find $u \in H_0^1(0, T)$ such that

$$a(u, \overline{\mathcal{H}}_T v) = -\langle \partial_t u, \partial_t \overline{\mathcal{H}}_T v \rangle_{L^2(0, T)} + \mu \langle u, \overline{\mathcal{H}}_T v \rangle_{L^2(0, T)} \stackrel{!}{=} \langle f, \overline{\mathcal{H}}_T v \rangle_{(0, T)} \quad (4.16)$$

for all $v \in H_0^1(0, T)$. Note that for the variational formulation (4.16), the ansatz and test spaces are equal. So, the existence and the uniqueness of a solution of (4.16) follow by a compact perturbation argument.

Theorem 4.2.2. For given $f \in [H_0^1(0, T)]'$, the variational formulation (4.13) admits a unique solution $u \in H_0^1(0, T)$, satisfying

$$|u|_{H^1(0, T)} \leq C(T, \mu) \|f\|_{[H_0^1(0, T)]'} \quad (4.17)$$

with a constant $C(T, \mu) > 0$ depending on T and μ .

Proof. By using the Riesz representation theorem, the variational formulation (4.16) is equivalent to the operator equation

$$\mathcal{A}u + \mu \mathcal{C}u = \hat{f},$$

where $\mathcal{A}: H_0^1(0, T) \rightarrow [H_0^1(0, T)]'$, defined via

$$\langle \mathcal{A}u, v \rangle := -\langle \partial_t u, \partial_t \overline{\mathcal{H}}_T v \rangle_{L^2(0, T)} = \langle \partial_t u, \partial_t v \rangle_{L^2(0, T)} \quad \text{for } u, v \in H_0^1(0, T),$$

is elliptic, and hence, invertible, and $\mathcal{C}: H_0^1(0, T) \rightarrow [H_0^1(0, T)]'$, defined via

$$\langle \mathcal{C}u, v \rangle := \langle u, \overline{\mathcal{H}}_T v \rangle_{L^2(0, T)} = \langle u, -v + v(T) \rangle_{L^2(0, T)} \quad \text{for } u, v \in H_0^1(0, T),$$

is compact, see [12, Satz 5.12, page 148]. Furthermore, the corresponding right-hand side $\hat{f}: H_0^1(0, T) \rightarrow \mathbb{R}$ is given by

$$\hat{f}(v) := \langle f, \overline{\mathcal{H}}_T v \rangle_{(0, T)}, \quad v \in H_0^1(0, T),$$

satisfying

$$|\hat{f}(v)| \leq \|f\|_{[H_0^1(0, T)]'} \|\overline{\mathcal{H}}_T v\|_{H_0^1(0, T)} = \|f\|_{[H_0^1(0, T)]'} \|v\|_{H_0^1(0, T)} = \|f\|_{[H_0^1(0, T)]'} |v|_{H^1(0, T)}$$

for all $v \in H_0^1(0, T)$, i.e. $\hat{f} \in [H_0^1(0, T)]'$. Hence, by applying the Fredholm alternative, it remains to ensure the injectivity of $\mathcal{A} + \mu \mathcal{C}$. Let $u \in H_0^1(0, T)$ be a solution of the homogeneous equation $(\mathcal{A} + \mu \mathcal{C})u = 0$, i.e.

$$\langle \partial_t u, \partial_t w \rangle_{L^2(0, T)} = \mu \langle u, w \rangle_{L^2(0, T)} \quad \text{for all } w \in H_0^1(0, T).$$

This is the weak formulation of the eigenvalue problem

$$-\partial_{tt} u(t) = \mu u(t) \quad \text{for } t \in (0, T), \quad u(0) = \partial_t u(0) = 0,$$

which only admits the trivial solution $u \equiv 0$. □

To examine the dependency of the constant $C(T, \mu)$ in (4.17) on T and μ , the continuous bilinear form $a(\cdot, \cdot): H_0^1(0, T) \times H_0^1(0, T) \rightarrow \mathbb{R}$ is investigated. Hence, some basic properties of the bilinear form $a(\cdot, \cdot): H_0^1(0, T) \times H_0^1(0, T) \rightarrow \mathbb{R}$ are shown.

Lemma 4.2.3. *The bilinear form $a(\cdot, \cdot): H_0^1(0, T) \times H_0^1(0, T) \rightarrow \mathbb{R}$ satisfies the condition (N1), i.e.*

$$\frac{2}{2 + \sqrt{\mu T}} |u|_{H^1(0, T)} \leq \sup_{0 \neq w \in H_0^1(0, T)} \frac{|a(u, w)|}{|w|_{H^1(0, T)}}$$

for all $u \in H_0^1(0, T)$.

Proof. Let $u \in H_0^1(0, T)$ be fixed and set for $t \in [0, T]$

$$\tilde{w}(t) := u(T) - u(t) + z(t)$$

with

$$z(t) := \sqrt{\mu} \int_t^T \sin(\sqrt{\mu}(s-t)) [u(s) - u(T)] ds.$$

It holds true that $\tilde{w}(T) = 0$ and $\tilde{w} \in H_0^1(0, T)$. Differentiation under the integral sign yields

$$\begin{aligned} \partial_t z(t) &= -\mu \int_t^T \cos(\sqrt{\mu}(s-t)) [u(s) - u(T)] ds, \\ \partial_{tt} z(t) &= -\mu^{3/2} \int_t^T \sin(\sqrt{\mu}(s-t)) [u(s) - u(T)] ds + \mu [u(t) - u(T)] \\ &= -\mu z(t) + \mu [u(t) - u(T)], \end{aligned}$$

i.e. the function $z \in H_0^1(0, T)$ is the unique solution of the adjoint equation

$$\partial_{tt} z(t) + \mu z(t) = \mu [u(t) - u(T)] \quad \text{for } t \in (0, T), \quad z(T) = \partial_t z(T) = 0.$$

Therefore, it holds true that

$$\begin{aligned} a(u, \tilde{w}) &= a(u, u(T) - u) + a(u, z) \\ &= \langle \partial_t u, \partial_t u \rangle_{L^2(0, T)} + \mu \langle u, u(T) - u \rangle_{L^2(0, T)} + \mu \langle u, u - u(T) \rangle_{L^2(0, T)} \\ &= |u|_{H^1(0, T)}^2. \end{aligned}$$

Integration by parts gives

$$\partial_t z(t) = -\mu \int_t^T \cos(\sqrt{\mu}(s-t)) [u(s) - u(T)] ds = \sqrt{\mu} \int_t^T \sin(\sqrt{\mu}(s-t)) \partial_t u(s) ds$$

and so, from the Cauchy-Schwarz inequality, it follows that

$$|z|_{H^1(0, T)} \leq \frac{\sqrt{\mu T}}{2} |u|_{H^1(0, T)}.$$

Finally, one gets the estimate

$$|\tilde{w}|_{H^1(0,T)} \leq |u|_{H^1(0,T)} + |z|_{H^1(0,T)} \leq \left(1 + \frac{\sqrt{\mu T}}{2}\right) |u|_{H^1(0,T)} = \frac{2 + \sqrt{\mu T}}{2} |u|_{H^1(0,T)}$$

to conclude

$$\sup_{0 \neq w \in H_0^1(0,T)} \frac{|a(u, w)|}{|w|_{H^1(0,T)}} \geq \frac{|a(u, \tilde{w})|}{|\tilde{w}|_{H^1(0,T)}} \geq \frac{2}{2 + \sqrt{\mu T}} |u|_{H^1(0,T)}$$

and so, the inf-sup condition follows. \square

Lemma 4.2.4. *The bilinear form $a(\cdot, \cdot): H_0^1(0, T) \times H_0^1(0, T) \rightarrow \mathbb{R}$ satisfies the condition (N2):*

For each function $0 \neq w \in H_0^1(0, T)$, an element $u \in H_0^1(0, T)$ with $a(u, w) \neq 0$ exists.

Proof. Let $0 \neq w \in H_0^1(0, T)$ be fixed. Set for $t \in [0, T]$

$$\tilde{u}(t) := \frac{1}{\sqrt{\mu}} \int_0^t w(s) \sin(\sqrt{\mu}(t-s)) ds.$$

It follows that $\tilde{u} \in H_0^1(0, T)$ satisfies (4.12) for the right-hand side $f = w \in H_0^1(0, T)$. Hence, one concludes

$$a(\tilde{u}, w) = \langle w, w \rangle_{L^2(0,T)} = \|w\|_{L^2(0,T)}^2 > 0$$

and therefore, the assertion. \square

With these properties of the bilinear form $a(\cdot, \cdot): H_0^1(0, T) \times H_0^1(0, T) \rightarrow \mathbb{R}$, the next existence and uniqueness theorem is proven, including an explicit dependency relation of the constant $C(T, \mu)$ on T and μ .

Theorem 4.2.5. *Let $f \in [H_0^1(0, T)]'$ be given. There exists a unique solution $u \in H_0^1(0, T)$ of the variational formulation (4.13). Furthermore,*

$$\mathcal{L}: [H_0^1(0, T)]' \rightarrow H_0^1(0, T), \quad \mathcal{L}f := u,$$

is an isomorphism, satisfying

$$|u|_{H^1(0,T)} = |\mathcal{L}f|_{H^1(0,T)} \leq \frac{2 + \sqrt{\mu T}}{2} \|f\|_{[H_0^1(0,T)]'}. \quad (4.18)$$

Proof. For the Hilbert spaces $(H_0^1(0, T), \langle \cdot, \cdot \rangle_{H_0^1(0, T)})$ and $(H_0^1(0, T), \langle \cdot, \cdot \rangle_{H_0^1(0, T)})$, use the Nečas Theorem 2.9.1 with the help of the previous lemmata to conclude the existence and uniqueness of a solution $u \in H_0^1(0, T)$ together with the a priori estimate. \square

Theorem 4.2.6. *There exist positive constants c_1, c_2 and further, a family of functions $u_{\mu, T} \in H_0^1(0, T)$ with*

$$\|\mathcal{L}^{-1}u_{\mu, T}\|_{[H_0^1(0, T)]'} \in \mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right) \quad \text{and} \quad |u_{\mu, T}|_{H^1(0, T)} \rightarrow c_1 \quad \text{as } \mu \rightarrow \infty,$$

and with

$$\|\mathcal{L}^{-1}u_{\mu, T}\|_{[H_0^1(0, T)]'} \in \mathcal{O}\left(\frac{1}{T}\right) \quad \text{and} \quad |u_{\mu, T}|_{H^1(0, T)} \rightarrow c_2 \quad \text{as } T \rightarrow \infty.$$

In particular, the inequality (4.18) is optimal with respect to the order of μ and T .

Proof. The asserted family of elements $u_{\mu, T} \in H_0^1(0, T)$ is given by

$$u_{\mu, T}(t) := \frac{1}{\sqrt{T^3}} \int_0^t s \sin(\sqrt{\mu}s) ds, \quad t \in [0, T],$$

where the initial conditions $u_{\mu, T}(0) = \partial_t u_{\mu, T}(0) = 0$ are fulfilled. One computes

$$|u_{\mu, T}|_{H^1(0, T)} = \sqrt{\frac{1}{6} - \frac{\sin(2\sqrt{\mu}T)}{4\sqrt{\mu}T} + \frac{\sin(2\sqrt{\mu}T)}{8\mu^{3/2}T^3} - \frac{\cos(2\sqrt{\mu}T)}{4\mu T^2}}$$

and hence,

$$|u_{\mu, T}|_{H^1(0, T)} \rightarrow \sqrt{\frac{1}{6}} \quad \text{as } \mu \rightarrow \infty \quad \text{or as } T \rightarrow \infty. \quad (4.19)$$

The corresponding right-hand side is

$$f_{\mu, T}(t) := \partial_{tt}u_{\mu, T}(t) + \mu u_{\mu, T}(t) = \frac{2}{\sqrt{T^3}} \sin(\sqrt{\mu}t), \quad t \in (0, T),$$

and by the usage of the Fourier series representation of $\|\cdot\|_{[H_0^1(0, T)]'}$, see Lemma 3.4.4, it follows that

$$\begin{aligned} \|\mathcal{L}^{-1}u_{\mu, T}\|_{[H_0^1(0, T)]'} &= \|f_{\mu, T}\|_{[H_0^1(0, T)]'} \\ &= \sqrt{\frac{T^3}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-2} f_{\mu, T, k}^2} \\ &= \sqrt{\frac{6}{\mu T^2} + \frac{\sin(2\sqrt{\mu}T) - 8\sin(\sqrt{\mu}T)}{\mu^{3/2}T^3}}. \end{aligned}$$

With this last equality, it holds true that $\|\mathcal{L}^{-1}u_{\mu,T}\|_{[H_0^1(0,T)]'} \in \mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ as $\mu \rightarrow \infty$ and $\|\mathcal{L}^{-1}u_{\mu,T}\|_{[H_0^1(0,T)]'} \in \mathcal{O}\left(\frac{1}{T}\right)$ as $T \rightarrow \infty$. \square

While for $f \in [H_0^1(0,T)]'$ the bound (4.18) shows an explicit dependence on $\sqrt{\mu}$, an estimate independent of μ is proven, when assuming $f \in L^2(0,T)$.

Lemma 4.2.7. *For a given right-hand side $f \in L^2(0,T)$, the unique solution $u \in H_0^1(0,T)$ of the variational formulation (4.13) satisfies*

$$\|u\|_{H^1(0,T)}^2 + \mu \|u\|_{L^2(0,T)}^2 \leq \frac{1}{2}T^2 \|f\|_{L^2(0,T)}^2.$$

In addition, the estimate above is optimal with respect to the order of the terminal time T and the order of the parameter μ .

Proof. For the solution $u \in H_0^1(0,T)$ and its first-order derivative, the representations

$$u(t) = \frac{1}{\sqrt{\mu}} \int_0^t \sin(\sqrt{\mu}(t-s)) f(s) ds, \quad t \in (0,T),$$

and

$$\partial_t u(t) = \int_0^t \cos(\sqrt{\mu}(t-s)) f(s) ds, \quad t \in (0,T),$$

hold true. With the calculations for $t \in (0,T)$

$$\begin{aligned} \partial_t u(t)^2 + \mu u(t)^2 &= \left[\int_0^t \cos(\sqrt{\mu}(t-s)) f(s) ds \right]^2 + \left[\int_0^t \sin(\sqrt{\mu}(t-s)) f(s) ds \right]^2 \\ &\leq \int_0^t \cos^2(\sqrt{\mu}(t-s)) ds \int_0^t f(s)^2 ds + \int_0^t \sin^2(\sqrt{\mu}(t-s)) ds \int_0^t f(s)^2 ds \\ &= t \int_0^t f(s)^2 ds \leq t \int_0^T f(s)^2 ds, \end{aligned}$$

it follows that

$$\begin{aligned} \|u\|_{H^1(0,T)}^2 + \mu \|u\|_{L^2(0,T)}^2 &= \int_0^T \left\{ \partial_t u(t)^2 + \mu u(t)^2 \right\} dt \\ &\leq \int_0^T t dt \int_0^T f(s)^2 ds = \frac{1}{2}T^2 \|f\|_{L^2(0,T)}^2, \end{aligned}$$

and so, the estimate.

For the optimality of this estimate, take the family $u_{\mu,T} \in H_0^1(0, T)$ of the proof of Theorem 4.2.6. The L^2 norm of $u_{\mu,T}$ is

$$\|u_{\mu,T}\|_{L^2(0,T)} = \sqrt{\frac{1}{6\mu} - \frac{5 \sin(2\sqrt{\mu}T)}{8\mu^{5/2}T^3} + \frac{\sin(2\sqrt{\mu}T)}{4\mu^{3/2}T} + \frac{1}{2\mu^2T^2} + \frac{3 \cos(2\sqrt{\mu}T)}{4\mu^2T^2}},$$

and hence, $\|u_{\mu,T}\|_{L^2(0,T)} \in \mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ as $\mu \rightarrow \infty$ and $\|u_{\mu,T}\|_{L^2(0,T)} \rightarrow \sqrt{\frac{1}{6\mu}}$ as $T \rightarrow \infty$. Furthermore, for the corresponding right-hand side $f_{\mu,T}(t) = \frac{2}{\sqrt{T^3}} \sin(\sqrt{\mu}t)$, the L^2 norm is given by

$$\|f_{\mu,T}\|_{L^2(0,T)} = \sqrt{\frac{2}{T^2} - \frac{\sin(2\sqrt{\mu}T)}{\sqrt{\mu}T^3}}$$

and so, with (4.19), the assertion follows. \square

With the notations of Section 2.6, a conforming Galerkin-Bubnov finite element discretisation of the equivalent variational formulation (4.16) is to find a function

$$u_{h_t} \in S_{h_t,0}^1(0, T) = \text{span}\{\varphi_k\}_{k=1}^{N_t} \subset H_0^1(0, T)$$

such that

$$a(u_{h_t}, \overline{\mathcal{H}}_T v_{h_t}) = -\langle \partial_t u_{h_t}, \partial_t \overline{\mathcal{H}}_T v_{h_t} \rangle_{L^2(0,T)} + \mu \langle u_{h_t}, \overline{\mathcal{H}}_T v_{h_t} \rangle_{L^2(0,T)} \stackrel{!}{=} \langle f, \overline{\mathcal{H}}_T v_{h_t} \rangle_{(0,T)} \quad (4.20)$$

for all $v_{h_t} \in S_{h_t,0}^1(0, T)$. Unique solvability of (4.20) and related error estimates follow as for the numerical solution of elliptic operator equations with compact perturbations, which is based on a discrete stability condition for a sufficiently small mesh size h_t . Using the optimal Poincaré constant of Lemma 3.4.5, it turns out that for a sufficiently small mesh size

$$h_t \leq \frac{\sqrt{3}\pi}{\sqrt{2}(2 + \sqrt{\mu}T)\mu T}, \quad (4.21)$$

the discrete stability condition

$$c(\mu, T) |u_{h_t}|_{H^1(0,T)} \leq \sup_{0 \neq v_{h_t} \in S_{h_t,0}^1(0,T)} \frac{a(u_{h_t}, \overline{\mathcal{H}}_T v_{h_t})}{|v_{h_t}|_{H^1(0,T)}}$$

holds true for all $u_{h_t} \in S_{h_t,0}^1(0, T)$ with a constant $c(\mu, T) > 0$, see [145] for a proof, implying the error estimate

$$|u - u_{h_t}|_{H^1(0,T)} \leq \tilde{c}(\mu, T) h_t \|u\|_{H^2(0,T)}$$

is given for $j \geq 1$ by

$$v_j = \begin{cases} A_0 \left(\frac{6-2h_t^2\mu + \sqrt{3}h_t\sqrt{\mu(h_t^2\mu-12)}}{6+h_t^2\mu} \right)^{j-1} + A_1 \left(\frac{6-2h_t^2\mu - \sqrt{3}h_t\sqrt{\mu(h_t^2\mu-12)}}{6+h_t^2\mu} \right)^{j-1}, & \text{if } h_t^2\mu > 12, \\ (-1)^{j-1}(A_0 + A_1(j-1)), & \text{if } h_t^2\mu = 12, \\ A_0 \cos\left((j-1)\arccos\frac{6-2h_t^2\mu}{6+h_t^2\mu}\right) + A_1 \sin\left((j-1)\arccos\frac{6-2h_t^2\mu}{6+h_t^2\mu}\right), & \text{if } h_t^2\mu < 12, \end{cases}$$

where the coefficients $A_0, A_1 \in \mathbb{R}$ are determined by $f_1, f_2 \in \mathbb{R}$. Hence, in general, the sequence $(v_j)_{j \in \mathbb{N}}$ is bounded as $j \rightarrow \infty$ if and only if

$$h_t^2\mu < 12 \quad \iff \quad h_t < \sqrt{\frac{12}{\mu}}, \quad (4.26)$$

see Table 4.6 for a numerical illustration.

From Remark 4.2.8, one concludes that the numerical scheme (4.22) is only conditionally stable. To overcome the mesh conditions (4.21) or (4.26), the numerical scheme in (4.22) is stabilised. Considering that the following technical lemmata are needed, where the trapezoidal rule is used analogously as in [164, Chapter 2]. In addition to $S_{h_t}^1(0, T)$, also the finite element space $S_{h_t}^0(0, T)$ of piecewise constant functions on the same time mesh is used, see Section 2.6.

Lemma 4.2.9. For all $f \in L^2(0, T)$, the equalities

$$\partial_t I_{h_t} \int_0^{\cdot} f(s) ds = Q_{h_t}^0 f = \partial_t I_{h_t} \int_T^{\cdot} f(s) ds \quad (4.27)$$

are valid, where $I_{h_t}: C[0, T] \rightarrow S_{h_t}^1(0, T)$ is the piecewise linear interpolation operator (2.27), and $Q_{h_t}^0: L^2(0, T) \rightarrow S_{h_t}^0(0, T)$ denotes the L^2 projection (2.26) on the piecewise constant finite element space $S_{h_t}^0(0, T)$.

Proof. On the element $\tau_\ell = (t_{\ell-1}, t_\ell)$, $\ell = 1, \dots, N_t$, it holds true that

$$\partial_t I_{h_t} \int_0^{\cdot} f(s) ds = \frac{1}{h_{t,\ell}} \left[\int_0^{t_\ell} f(s) ds - \int_0^{t_{\ell-1}} f(s) ds \right] = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} f(s) ds = Q_{h_t}^0 f$$

and

$$\partial_t I_{h_t} \int_T^{\cdot} f(s) ds = \frac{1}{h_{t,\ell}} \left[\int_T^{t_\ell} f(s) ds - \int_T^{t_{\ell-1}} f(s) ds \right] = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} f(s) ds = Q_{h_t}^0 f.$$

Hence, the assertion follows. \square

Lemma 4.2.10. For all $u_{h_t} \in S_{h_t,0}^1(0,T)$ and $w_{h_t} \in S_{h_t,0}^1(0,T)$, the representation

$$\langle u_{h_t}, w_{h_t} \rangle_{L^2(0,T)} = \frac{1}{12} \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} + \langle u_{h_t}, \mathcal{Q}_{h_t}^0 w_{h_t} \rangle_{L^2(0,T)} \quad (4.28)$$

holds true, where $\mathcal{Q}_{h_t}^0 : L^2(0,T) \rightarrow S_{h_t}^0(0,T)$ denotes the L^2 projection (2.26) on the piecewise constant finite element space $S_{h_t}^0(0,T)$.

Proof. With the error representation of the trapezoidal rule, one obtains on each finite element τ_ℓ , $\ell = 1, \dots, N_t$,

$$\begin{aligned} \mathcal{Q}_{h_t}^0 \int_T^{(\cdot)} w_{h_t}(s) ds &= \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} \int_T^t w_{h_t}(s) ds dt \\ &= \frac{1}{2} \left[\int_T^{t_{\ell-1}} w_{h_t}(s) ds + \int_T^{t_\ell} w_{h_t}(s) ds \right] - \frac{h_{t,\ell}^2}{12} \partial_t w_{h_t}|_{\tau_\ell} \\ &= \mathcal{Q}_{h_t}^0 I_{h_t} \int_T^{(\cdot)} w_{h_t}(s) ds - \frac{h_{t,\ell}^2}{12} \partial_t w_{h_t}|_{\tau_\ell}. \end{aligned}$$

Further, using integration by parts and (4.27), it follows that

$$\begin{aligned} \int_0^T \partial_t u_{h_t}(t) \left(I_{h_t} \int_T^{(\cdot)} w_{h_t}(s) ds \right) (t) dt &= - \int_0^T u_{h_t}(t) \left(\partial_t I_{h_t} \int_T^{(\cdot)} w_{h_t}(s) ds \right) (t) dt \\ &= - \int_0^T u_{h_t}(t) \mathcal{Q}_{h_t}^0 w_{h_t}(t) dt. \end{aligned}$$

With this, by using integration by parts and the local definition of the L^2 projection $\mathcal{Q}_{h_t}^0$, one concludes that

$$\langle u_{h_t}, w_{h_t} \rangle_{L^2(0,T)} = \int_0^T u_{h_t}(t) \left(\partial_t \int_T^{(\cdot)} w_{h_t}(s) ds \right) (t) dt = - \int_0^T \partial_t u_{h_t}(t) \left(\int_T^{(\cdot)} w_{h_t}(s) ds \right) (t) dt$$

as well as

$$\begin{aligned} \langle u_{h_t}, w_{h_t} \rangle_{L^2(0,T)} &= - \sum_{\ell=1}^{N_t} \int_{t_{\ell-1}}^{t_\ell} \partial_t u_{h_t}(t) \left(\mathcal{Q}_{h_t}^0 \int_T^{(\cdot)} w_{h_t}(s) ds \right) (t) dt \\ &= \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \int_{t_{\ell-1}}^{t_\ell} \partial_t u_{h_t}(t) \partial_t w_{h_t}(t) dt - \sum_{\ell=1}^{N_t} \int_{t_{\ell-1}}^{t_\ell} \partial_t u_{h_t}(t) \left(\mathcal{Q}_{h_t}^0 I_{h_t} \int_T^{(\cdot)} w_{h_t}(s) ds \right) (t) dt \end{aligned}$$

and further,

$$\begin{aligned}
\langle u_{h_t}, w_{h_t} \rangle_{L^2(0,T)} &= \frac{1}{12} \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} - \sum_{\ell=1}^{N_t} \int_{t_{\ell-1}}^{t_\ell} \partial_t u_{h_t}(t) \left(I_{h_t} \int_T^{\cdot} w_{h_t}(s) ds \right) (t) dt \\
&= \frac{1}{12} \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} - \int_0^T \partial_t u_{h_t}(t) \left(I_{h_t} \int_T^{\cdot} w_{h_t}(s) ds \right) (t) dt \\
&= \frac{1}{12} \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} + \langle u_{h_t}, \mathcal{Q}_{h_t}^0 w_{h_t} \rangle_{L^2(0,T)},
\end{aligned}$$

i.e. the representation (4.28). \square

Next, an alternative representation of the bilinear form $a(\cdot, \cdot)$ is given.

Corollary 4.2.11. *For $u_{h_t} \in S_{h_t,0}^1(0,T)$ and $w_{h_t} \in S_{h_t,0}^1(0,T)$, the equalities*

$$\begin{aligned}
a(u_{h_t}, w_{h_t}) &= -\langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \mu \langle u_{h_t}, w_{h_t} \rangle_{L^2(0,T)} \\
&= -\langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} \frac{\mu h_{t,\ell}^2}{12} \langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} + \mu \langle u_{h_t}, \mathcal{Q}_{h_t}^0 w_{h_t} \rangle_{L^2(0,T)} \\
&= \sum_{\ell=1}^{N_t} \left(\frac{\mu h_{t,\ell}^2}{12} - 1 \right) \langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} + \mu \langle u_{h_t}, \mathcal{Q}_{h_t}^0 w_{h_t} \rangle_{L^2(0,T)} \quad (4.29)
\end{aligned}$$

are valid.

Motivated by the representation (4.29), one defines the perturbed bilinear form

$$a_{h_t}(u_{h_t}, w_{h_t}) := -\langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \mu \langle u_{h_t}, \mathcal{Q}_{h_t}^0 w_{h_t} \rangle_{L^2(0,T)} \quad (4.30)$$

for $u_{h_t} \in S_{h_t,0}^1(0,T)$ and $w_{h_t} \in S_{h_t,0}^1(0,T)$, and consider the perturbed variational formulation to find $\tilde{u}_{h_t} \in S_{h_t,0}^1(0,T)$ such that

$$a_{h_t}(\tilde{u}_{h_t}, w_{h_t}) = \langle f, w_{h_t} \rangle_{(0,T)} \quad (4.31)$$

for all $w_{h_t} \in S_{h_t,0}^1(0,T)$.

Lemma 4.2.12. *The perturbed bilinear form (4.30) is bounded, i.e. it holds true that*

$$|a_{h_t}(u_{h_t}, w_{h_t})| \leq \left(1 + \frac{4}{\pi^2} \mu T^2 \right) |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)}$$

for all $u_{h_t} \in S_{h_t,0}^1(0,T)$, $w_{h_t} \in S_{h_t,0}^1(0,T)$.

2. For $\ell = j + 1, \dots, N_t$, the equalities

$$\langle \partial_t \bar{v}_{h_t}^j, \partial_t z_{h_t} \rangle_{L^2(\tau_\ell)} = \frac{1}{2} (z_\ell^2 - z_{\ell-1}^2)$$

and

$$\langle \bar{v}_{h_t}^j, \mathcal{Q}_{h_t}^0 z_{h_t} \rangle_{L^2(\tau_\ell)} = \frac{1}{2} \left(\int_{t_j}^{t_\ell} z_{h_t}(s) ds \right)^2 - \frac{1}{2} \left(\int_{t_j}^{t_{\ell-1}} z_{h_t}(s) ds \right)^2$$

are valid.

3. The estimate

$$|\bar{v}_{h_t}^j|_{H^1(0,T)} \leq \|z_{h_t}\|_{L^2(0,T)}$$

holds true.

Proof. For $z_{h_t} \in S_{h_t,0}^1(0,T)$, consider the piecewise linear interpolation of the antiderivative, i.e. for $t \in [0, T]$, one defines

$$\bar{v}_{h_t}(t) := \sum_{k=0}^{N_t} \left(\int_0^{t_k} z_{h_t}(s) ds \right) \varphi_k(t) = \left(I_{h_t} \int_0^{(\cdot)} z_{h_t}(s) ds \right) (t), \quad \bar{v}_{h_t} \in S_{h_t,0}^1(0,T).$$

The relation $\partial_t \bar{v}_{h_t} = \mathcal{Q}_{h_t}^0 z_{h_t}$ follows from (4.27). For a fixed index $j \in \{0, \dots, N_t - 1\}$, one defines

$$z_{h_t}^j(t) = \sum_{i=0}^{N_t} z_i^j \varphi_i(t), \quad z_i^j = \begin{cases} (-1)^{j-i} z_j & \text{for } i = 0, \dots, j, \\ z_i & \text{for } i = j + 1, \dots, N_t. \end{cases}$$

Note that $z_{h_t}^j \in S_{h_t,0}^1(0,T)$, and according to $z_{h_t}^j$ one introduces $\bar{v}_{h_t}^j$, satisfying $\partial_t \bar{v}_{h_t}^j = \mathcal{Q}_{h_t}^0 z_{h_t}^j$. In particular, for $j > 0$ and $t \in \tau_\ell$ for $\ell = 1, \dots, j$, it follows that

$$\partial_t \bar{v}_{h_t}^j(t) = \mathcal{Q}_{h_t}^0 z_{h_t}^j(t) = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} z_{h_t}^j(s) ds = \frac{1}{2} (z_{\ell-1}^j + z_\ell^j) = 0,$$

and due to $\bar{v}_{h_t}^j(0) = 0$, one concludes $\bar{v}_{h_t}^j(t) = 0$ for $t \in [0, t_j]$, i.e. the first assertion.

To prove the second assertion, one computes for $\ell = j + 1, \dots, N_t$

$$\begin{aligned} \langle \partial_t \bar{v}_{h_t}^j, \partial_t z_{h_t} \rangle_{L^2(\tau_\ell)} &= \langle \mathcal{Q}_{h_t}^0 z_{h_t}^j, \partial_t z_{h_t} \rangle_{L^2(\tau_\ell)} \\ &= \frac{1}{2} (z_{\ell-1}^j + z_\ell^j) (z_\ell - z_{\ell-1}) \\ &= \frac{1}{2} (z_{\ell-1} + z_\ell) (z_\ell - z_{\ell-1}) \\ &= \frac{1}{2} (z_\ell^2 - z_{\ell-1}^2) \end{aligned}$$

as well as

$$\begin{aligned}
\langle \bar{v}_{h_t}^j, \mathcal{Q}_{h_t}^0 z_{h_t} \rangle_{L^2(\tau_\ell)} &= \int_{t_{\ell-1}}^{t_\ell} \left(I_{h_t} \int_0^{(\cdot)} z_{h_t}^j(s) ds \right) (t) \mathcal{Q}_{h_t}^0 z_{h_t}(t) dt \\
&= \mathcal{Q}_{h_t}^0 z_{h_t}|_{\tau_\ell} \int_{t_{\ell-1}}^{t_\ell} \left[\int_0^{t_{\ell-1}} z_{h_t}^j(s) ds \varphi_{\ell-1}(t) + \int_0^{t_\ell} z_{h_t}^j(s) ds \varphi_\ell(t) \right] dt \\
&= \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} z_{h_t}(s) ds \frac{1}{2} h_{t,\ell} \left[\int_0^{t_{\ell-1}} z_{h_t}^j(s) ds + \int_0^{t_\ell} z_{h_t}^j(s) ds \right] \\
&= \frac{1}{2} \int_{t_{\ell-1}}^{t_\ell} z_{h_t}(s) ds \left[\int_{t_j}^{t_{\ell-1}} z_{h_t}(s) ds + \int_{t_j}^{t_\ell} z_{h_t}(s) ds \right]
\end{aligned}$$

and with completing the square,

$$\begin{aligned}
\langle \bar{v}_{h_t}^j, \mathcal{Q}_{h_t}^0 z_{h_t} \rangle_{L^2(\tau_\ell)} &= \frac{1}{2} \left(\int_{t_{\ell-1}}^{t_\ell} z_{h_t}(s) ds \right)^2 + \int_{t_{\ell-1}}^{t_\ell} z_{h_t}(s) ds \int_{t_j}^{t_{\ell-1}} z_{h_t}(s) ds \\
&= \frac{1}{2} \left(\int_{t_{\ell-1}}^{t_\ell} z_{h_t}(s) ds + \int_{t_j}^{t_{\ell-1}} z_{h_t}(s) ds \right)^2 - \frac{1}{2} \left(\int_{t_j}^{t_{\ell-1}} z_{h_t}(s) ds \right)^2 \\
&= \frac{1}{2} \left(\int_{t_j}^{t_\ell} z_{h_t}(s) ds \right)^2 - \frac{1}{2} \left(\int_{t_j}^{t_{\ell-1}} z_{h_t}(s) ds \right)^2.
\end{aligned}$$

Finally, from the L^2 stability of $\mathcal{Q}_{h_t}^0$, one concludes the third assertion, i.e.

$$\begin{aligned}
|\bar{v}_{h_t}^j|_{H^1(0,T)} &= |\bar{v}_{h_t}^j|_{H^1(t_j,T)} = \|\mathcal{Q}_{h_t}^0 z_{h_t}^j\|_{L^2(t_j,T)} \\
&= \|\mathcal{Q}_{h_t}^0 z_{h_t}\|_{L^2(t_j,T)} \leq \|\mathcal{Q}_{h_t}^0 z_{h_t}\|_{L^2(0,T)} \leq \|z_{h_t}\|_{L^2(0,T)}.
\end{aligned}$$

Hence, the lemma is proven. \square

Lemma 4.2.15. *The variational formulation to find $z_{h_t} \in S_{h_t,0}^1(0,T)$ such that*

$$a_{h_t}(v_{h_t}, z_{h_t}) = \langle g_0, v_{h_t} \rangle_{(0,T)} + \langle g_1, \partial_t v_{h_t} \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle g_2, \partial_t v_{h_t} \rangle_{L^2(\tau_\ell)} \quad (4.34)$$

for all $v_{h_t} \in S_{h_t,0}^1(0,T)$ is uniquely solvable, where the right-hand sides $g_0 \in [H_0^1(0,T)]'$ and $g_1, g_2 \in L^2(0,T)$ are given. Moreover, the stability estimate

$$\|z_{h_t}\|_{L^2(0,T)} \leq 2T \left\{ \|g_0\|_{[H_0^1(0,T)]'} + \|g_1\|_{L^2(0,T)} + h_t^2 \|g_2\|_{L^2(0,T)} \right\} \quad (4.35)$$

holds true for any mesh with maximal mesh size h_t .

Proof. The related finite element system matrix $\widetilde{K}_{h_t}^\top$ of the variational formulation (4.34) is upper triangular with positive diagonal elements, see (4.32), and hence, there exists a unique solution $z_{h_t} \in S_{h_t,0}^1(0,T)$ of (4.34).

For the estimate, consider for each index $j \in \{0, \dots, N_t - 1\}$ a function $\bar{v}_{h_t}^j \in S_{h_t,0}^1(0,T)$ as given in Lemma 4.2.14. Plugging these functions $\bar{v}_{h_t}^j$ into (4.34) and by using the properties of Lemma 4.2.14, this gives

$$\begin{aligned} a_{h_t}(\bar{v}_{h_t}^j, z_{h_t}) &= -\langle \partial_t \bar{v}_{h_t}^j, \partial_t z_{h_t} \rangle_{L^2(0,T)} + \mu \langle \bar{v}_{h_t}^j, \mathcal{Q}_{h_t}^0 z_{h_t} \rangle_{L^2(0,T)} \\ &= -\sum_{\ell=j+1}^{N_t} \langle \partial_t \bar{v}_{h_t}^j, \partial_t z_{h_t} \rangle_{L^2(\tau_\ell)} + \mu \sum_{\ell=j+1}^{N_t} \langle \bar{v}_{h_t}^j, \mathcal{Q}_{h_t}^0 z_{h_t} \rangle_{L^2(\tau_\ell)} \\ &= -\frac{1}{2} \sum_{\ell=j+1}^{N_t} (z_\ell^2 - z_{\ell-1}^2) + \frac{\mu}{2} \sum_{\ell=j+1}^{N_t} \left(\left(\int_{t_j}^{t_\ell} z_{h_t}(s) ds \right)^2 - \left(\int_{t_j}^{t_{\ell-1}} z_{h_t}(s) ds \right)^2 \right) \\ &= \frac{1}{2} z_j^2 + \frac{\mu}{2} \left(\int_{t_j}^T z_{h_t}(s) ds \right)^2 \end{aligned}$$

and so,

$$\begin{aligned} \langle g_0, \bar{v}_{h_t}^j \rangle_{(0,T)} + \langle g_1, \partial_t \bar{v}_{h_t}^j \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle g_2, \partial_t \bar{v}_{h_t}^j \rangle_{L^2(\tau_\ell)} &= a_{h_t}(\bar{v}_{h_t}^j, z_{h_t}) \\ &= \frac{1}{2} z_j^2 + \frac{\mu}{2} \left(\int_{t_j}^T z_{h_t}(s) ds \right)^2. \end{aligned}$$

This result yields

$$\begin{aligned} \|z_{h_t}\|_{L^2(0,T)}^2 &= \sum_{\ell=1}^{N_t} \|z_{h_t}\|_{L^2(\tau_\ell)}^2 = \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}}{3} (z_\ell^2 + z_\ell z_{\ell-1} + z_{\ell-1}^2) \leq \frac{1}{2} \sum_{\ell=1}^{N_t} h_{t,\ell} (z_\ell^2 + z_{\ell-1}^2) \\ &\leq \frac{1}{2} \sum_{j=1}^{N_t-1} h_{t,j} z_j^2 + \frac{1}{2} \sum_{j=0}^{N_t-1} h_{t,j+1} z_j^2 \\ &\leq \sum_{j=1}^{N_t-1} h_{t,j} \left\{ \langle g_0, \bar{v}_{h_t}^j \rangle_{(0,T)} + \langle g_1, \partial_t \bar{v}_{h_t}^j \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle g_2, \partial_t \bar{v}_{h_t}^j \rangle_{L^2(\tau_\ell)} \right\} \\ &\quad + \sum_{j=0}^{N_t-1} h_{t,j+1} \left\{ \langle g_0, \bar{v}_{h_t}^j \rangle_{(0,T)} + \langle g_1, \partial_t \bar{v}_{h_t}^j \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle g_2, \partial_t \bar{v}_{h_t}^j \rangle_{L^2(\tau_\ell)} \right\}. \end{aligned}$$

With the last inequality, the Cauchy-Schwarz inequality and the use of the properties of Lemma 4.2.14, one concludes

$$\begin{aligned} \|z_{h_t}\|_{L^2(0,T)}^2 &\leq \sum_{j=1}^{N_t-1} h_{t,j} \left\{ \|g_0\|_{[H_0^1(0,T)]'} + \|g_1\|_{L^2(0,T)} + h_t^2 \|g_2\|_{L^2(0,T)} \right\} \underbrace{|\bar{v}_{h_t}^j|_{H^1(0,T)}}_{\leq \|z_{h_t}\|_{L^2(0,T)}} \\ &\quad + \sum_{j=0}^{N_t-1} h_{t,j+1} \left\{ \|g_0\|_{[H_0^1(0,T)]'} + \|g_1\|_{L^2(0,T)} + h_t^2 \|g_2\|_{L^2(0,T)} \right\} \underbrace{|\bar{v}_{h_t}^j|_{H^1(0,T)}}_{\leq \|z_{h_t}\|_{L^2(0,T)}} \\ &\leq 2T \left\{ \|g_0\|_{[H_0^1(0,T)]'} + \|g_1\|_{L^2(0,T)} + h_t^2 \|g_2\|_{L^2(0,T)} \right\} \|z_{h_t}\|_{L^2(0,T)}, \end{aligned}$$

i.e. the assertion. \square

Corollary 4.2.16. *The variational formulation to find $v_{h_t} \in S_{h_t,0}^1(0,T)$ such that*

$$a_{h_t}(v_{h_t}, w_{h_t}) = \langle f_0, w_{h_t} \rangle_{(0,T)} + \langle f_1, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle f_2, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} \quad (4.36)$$

for all $w_{h_t} \in S_{h_t,0}^1(0,T)$ is uniquely solvable, where the right-hand sides $f_0 \in [H_0^1(0,T)]'$ and $f_1, f_2 \in L^2(0,T)$ are given. Moreover, the stability estimate

$$\|v_{h_t}\|_{L^2(0,T)} \leq 2T \left\{ \|f_0\|_{[H_0^1(0,T)]'} + \|f_1\|_{L^2(0,T)} + h_t^2 \|f_2\|_{L^2(0,T)} \right\} \quad (4.37)$$

holds true for any mesh with maximal mesh size h_t .

Proof. The related finite element system matrix \tilde{K}_{h_t} of the variational formulation (4.36) is lower triangular with positive diagonal elements, see (4.32), and hence, there exists a unique solution $v_{h_t} \in S_{h_t,0}^1(0,T)$ of (4.36).

The proof of the stability estimate is analogous to the proof of (4.35) with the help of a corresponding lemma analogous to Lemma 4.2.14. \square

Lemma 4.2.17. *For each $u_{h_t} \in S_{h_t,0}^1(0,T)$, the discrete inf-sup condition*

$$\frac{1}{1 + \frac{4}{\pi} \mu T^2} |u_{h_t}|_{H^1(0,T)} \leq \sup_{0 \neq w_{h_t} \in S_{h_t,0}^1(0,T)} \frac{|a_{h_t}(u_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}}$$

holds true.

Proof. For a fixed function $u_{h_t} \in S_{h_t,0}^1(0,T)$, let $w_{h_t} \in S_{h_t,0}^1(0,T)$ be the unique solution of (4.34) for $g_0 = g_2 = 0$ and $g_1 = \partial_t u_{h_t} \in L^2(0,T)$, i.e.

$$a_{h_t}(v_{h_t}, w_{h_t}) = \langle \partial_t u_{h_t}, \partial_t v_{h_t} \rangle_{L^2(0,T)} \quad (4.38)$$

is valid for all $v_{h_t} \in S_{h_t,0}^1(0,T)$. For the particular choice $v_{h_t}(t) = w_{h_t}(0) - w_{h_t}(t)$, $t \in [0,T]$, it holds true that $v_{h_t} \in S_{h_t,0}^1(0,T)$ and so, it follows that

$$\langle \partial_t w_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)} - \mu \langle w_{h_t} - w_{h_t}(0), \mathcal{Q}_{h_t}^0 w_{h_t} \rangle_{L^2(0,T)} = - \langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)}.$$

Hence, using the Cauchy-Schwarz inequality, the Poincaré inequality from Lemma 3.4.5 and the L^2 stability of the L^2 projection $\mathcal{Q}_{h_t}^0$, one concludes

$$\begin{aligned} |w_{h_t}|_{H^1(0,T)}^2 &= - \langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \mu \langle w_{h_t} - w_{h_t}(0), \mathcal{Q}_{h_t}^0 w_{h_t} \rangle_{L^2(0,T)} \\ &\leq |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)} + \mu \|w_{h_t} - w_{h_t}(0)\|_{L^2(0,T)} \|\mathcal{Q}_{h_t}^0 w_{h_t}\|_{L^2(0,T)} \\ &\leq |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)} + \frac{2}{\pi} \mu T |w_{h_t}|_{H^1(0,T)} \|w_{h_t}\|_{L^2(0,T)} \\ &\leq \left(1 + \frac{4}{\pi} \mu T^2\right) |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)}, \end{aligned}$$

where in the last step, the stability estimate $\|w_{h_t}\|_{L^2(0,T)} \leq 2T |u_{h_t}|_{H^1(0,T)}$ from (4.35) is used.

The choice $v_{h_t} = u_{h_t} \in S_{h_t,0}^1(0,T)$ in (4.38) and the estimate above yield

$$a_{h_t}(u_{h_t}, w_{h_t}) = |u_{h_t}|_{H^1(0,T)}^2 \geq \frac{1}{1 + \frac{4}{\pi} \mu T^2} |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)}$$

and hence, the discrete inf-sup condition follows. \square

Theorem 4.2.18. For given $f \in [H_{,0}^1(0,T)]'$, let the unique solution u of the variational formulation (4.13) satisfy $u \in H_{,0}^1(0,T) \cap H^s(0,T)$ for some $s \in [1,2]$. Then, there exists a unique solution $\tilde{u}_{h_t} \in S_{h_t,0}^1(0,T)$ of the Galerkin-Petrov finite element discretisation (4.31), satisfying the stability estimates

$$\begin{aligned} \|\tilde{u}_{h_t}\|_{L^2(0,T)} &\leq 2T \|f\|_{[H_{,0}^1(0,T)]'}, \\ |\tilde{u}_{h_t}|_{H^1(0,T)} &\leq \left(1 + \frac{4}{\pi} \mu T^2\right) \|f\|_{[H_{,0}^1(0,T)]'} \end{aligned}$$

and the error estimate

$$\begin{aligned} |u - \tilde{u}_{h_t}|_{H^1(0,T)} &\leq \left[1 + \left(1 + \frac{4}{\pi^2} \mu T^2\right) \left(1 + \frac{4}{\pi} \mu T^2\right)\right] C_1 h_t^{s-1} \|u\|_{H^s(0,T)} \\ &\quad + \frac{1}{12} \mu \left(1 + \frac{4}{\pi} \mu T^2\right) h_t^2 |u|_{H^1(0,T)}, \end{aligned}$$

where the constant $C_1 > 0$ is coming from standard interpolation error estimates.

Proof. First, the unique solvability of the variational formulation (4.31) and the first stability estimate follow from Corollary 4.2.16 for $f_0 = f$, $f_1 = f_2 = 0$. The second stability estimate is a consequence of the discrete inf-sup condition given in Lemma 4.2.17.

Second, for any $v_{h_t} \in S_{h_t,0}^1(0,T)$, it holds true that

$$|u - \tilde{u}_{h_t}|_{H^1(0,T)} \leq |u - v_{h_t}|_{H^1(0,T)} + |\tilde{u}_{h_t} - v_{h_t}|_{H^1(0,T)}$$

and it remains to estimate the second term. With the usage the discrete inf-sup condition of Lemma 4.2.17 and using the Galerkin orthogonality for the variational formulations (4.13) and (4.31), it follows that

$$\begin{aligned} \frac{1}{1 + \frac{4}{\pi}\mu T^2} |\tilde{u}_{h_t} - v_{h_t}|_{H^1(0,T)} &\leq \sup_{0 \neq w_{h_t} \in S_{h_t,0}^1(0,T)} \frac{|a_{h_t}(\tilde{u}_{h_t} - v_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}} \\ &= \sup_{0 \neq w_{h_t} \in S_{h_t,0}^1(0,T)} \frac{|a_{h_t}(\tilde{u}_{h_t}, w_{h_t}) - a_{h_t}(v_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}} \\ &= \sup_{0 \neq w_{h_t} \in S_{h_t,0}^1(0,T)} \frac{|a(u, w_{h_t}) - a_{h_t}(v_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}} \\ &= \sup_{0 \neq w_{h_t} \in S_{h_t,0}^1(0,T)} \frac{|a(u - v_{h_t}, w_{h_t}) + a(v_{h_t}, w_{h_t}) - a_{h_t}(v_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}}. \end{aligned}$$

Further, with the boundedness of the bilinear form $a(\cdot, \cdot)$ and the Poincaré inequality of Lemma 3.4.5, one concludes

$$\begin{aligned} a(u - v_{h_t}, w_{h_t}) &= -\langle \partial_t(u - v_{h_t}), \partial_t w_{h_t} \rangle_{L^2(0,T)} + \mu \langle u - v_{h_t}, w_{h_t} \rangle_{L^2(0,T)} \\ &\leq |u - v_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)} + \mu \|u - v_{h_t}\|_{L^2(0,T)} \|w_{h_t}\|_{L^2(0,T)} \\ &\leq \left(1 + \frac{4}{\pi^2}\mu T^2\right) |u - v_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)}. \end{aligned}$$

Moreover, using the representation (4.29), one estimates the consistency error by

$$\begin{aligned} |a(v_{h_t}, w_{h_t}) - a_{h_t}(v_{h_t}, w_{h_t})| &= \frac{1}{12}\mu \left| \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t v_{h_t}, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} \right| \\ &\leq \frac{1}{12}\mu h_t^2 |v_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)}. \end{aligned}$$

Hence, it follows that

$$\frac{1}{1 + \frac{4}{\pi}\mu T^2} |\tilde{u}_{h_t} - v_{h_t}|_{H^1(0,T)} \leq \left(1 + \frac{4}{\pi^2}\mu T^2\right) |u - v_{h_t}|_{H^1(0,T)} + \frac{1}{12}\mu h_t^2 |v_{h_t}|_{H^1(0,T)},$$

and therefore,

$$|u - \tilde{u}_{h_t}|_{H^1(0,T)} \leq \left[1 + \left(1 + \frac{4}{\pi^2} \mu T^2 \right) \left(1 + \frac{4}{\pi} \mu T^2 \right) \right] |u - v_{h_t}|_{H^1(0,T)} + \frac{1}{12} \mu \left(1 + \frac{4}{\pi} \mu T^2 \right) h_t^2 |v_{h_t}|_{H^1(0,T)}.$$

In particular, for the piecewise linear interpolation $v_{h_t} = I_{h_t} u$, it holds true that

$$\|u - I_{h_t} u\|_{H^1(0,T)} \leq C_1 h_t^{s-1} \|u\|_{H^s(0,T)}, \quad |I_{h_t} u|_{H^1(0,T)} \leq |u|_{H^1(0,T)},$$

see Section 2.6 and thus, the assertion. \square

Remark 4.2.19. *The proof of the error estimate in Theorem 4.2.18 is also given by [51, Lemma 2.27, page 95].*

Remark 4.2.20. *The discrete inf-sup constant of Lemma 4.2.17 depends on the parameter μ with order μ^{-1} . Numerical experiments for the optimal discrete inf-sup constant*

$$c_{H^1}(T, \mu) := \inf_{0 \neq v_{h_t} \in S_{h_t,0}^1(0,T)} \sup_{0 \neq w_{h_t} \in S_{h_t,0}^1(0,T)} \frac{|a_{h_t}(v_{h_t}, w_{h_t})|}{|v_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)}} \geq \frac{1}{1 + \frac{4}{\pi} \mu T^2} \quad (4.39)$$

show only a dependency of order $\mu^{-1/2}$, where a corresponding generalised eigenvalue problem is solved to compute $c_{H^1}(T, \mu)$, see [84, Subsection 3.6.6, page 124]. In Table 4.3 the optimal discrete inf-sup constant $c_{H^1}(T, \mu)$ is presented for $\mu \in \{125, 250, 500, 1000\}$ with $T = 10$ and a uniform mesh size $h_t = T/N_t$.

For the optimal discrete inf-sup constant with respect to $\|\cdot\|_{L^2(0,T)}$

$$c_{L^2}(T, \mu) := \inf_{0 \neq v_{h_t} \in S_{h_t,0}^1(0,T)} \sup_{0 \neq w_{h_t} \in S_{h_t,0}^1(0,T)} \frac{|a_{h_t}(v_{h_t}, w_{h_t})|}{\|v_{h_t}\|_{L^2(0,T)} |w_{h_t}|_{H^1(0,T)}} \geq \frac{1}{2T}, \quad (4.40)$$

where the last inequality follows from Corollary 4.2.16, numerical experiments with the same discretisations as for $c_{H^1}(T, \mu)$ confirm the independence of $c_{L^2}(T, \mu)$ from the parameter μ , see Table 4.4.

Numerical experiments for the bilinear form (4.14), i.e. without stabilisation, show a similar behaviour as for the perturbed bilinear form (4.30), provided the mesh size h_t is sufficiently small, i.e. $h_t < \sqrt{12/\mu}$.

Next, an $L^2(0,T)$ error estimate is stated, where the proof is based on the proof of [164, Theorem 3.1, page 175].

N_t	h_t	$c_{H^1}(10, 125)$	$c_{H^1}(10, 250)$	$c_{H^1}(10, 500)$	$c_{H^1}(10, 1000)$
4	2.5000000	1.12365e+01	2.21744e+01	4.40526e+01	8.78100e+01
8	1.2500000	9.77956e-01	1.62745e+00	2.96357e+00	5.65355e+00
16	0.6250000	3.33189e-01	4.22413e-01	6.05756e-01	6.80010e-01
32	0.3125000	1.12684e-01	1.35639e-01	1.79762e-01	2.31985e-01
64	0.1562500	4.89792e-02	4.96583e-02	5.60224e-02	6.97641e-02
128	0.0781250	3.33683e-02	2.72126e-02	2.47568e-02	2.50579e-02
256	0.0390625	2.91545e-02	2.16312e-02	1.66440e-02	1.36724e-02
512	0.0195312	2.83866e-02	2.02092e-02	1.46513e-02	1.08659e-02
1024	0.0097656	2.81193e-02	1.98896e-02	1.41544e-02	1.01405e-02
2048	0.0048828	2.80366e-02	1.98239e-02	1.40306e-02	9.98949e-03
4096	0.0024414	2.80160e-02	1.98080e-02	1.40038e-02	9.93791e-03
8192	0.0012207	2.80108e-02	1.98041e-02	1.39974e-02	9.92346e-03
16384	0.0006104	2.80096e-02	1.98031e-02	1.39958e-02	9.91987e-03
32768	0.0003052	2.80092e-02	1.98029e-02	1.39954e-02	9.91898e-03

Table 4.3: Optimal discrete inf-sup constant $c_{H^1}(T, \mu)$ of (4.39) for the perturbed bilinear form (4.30) for $\mu \in \{125, 250, 500, 1000\}$ with $T = 10$.

N_t	h_t	$c_{L^2}(10, 125)$	$c_{L^2}(10, 250)$	$c_{L^2}(10, 500)$	$c_{L^2}(10, 1000)$
4	2.5000000	1.40996e+01	2.77886e+01	5.51698e+01	1.09934e+02
8	1.2500000	2.65957e+00	4.41397e+00	8.01919e+00	1.52769e+01
16	0.6250000	1.68707e+00	2.25321e+00	3.30172e+00	3.74737e+00
32	0.3125000	8.80041e-01	1.24605e+00	1.79212e+00	2.45043e+00
64	0.1562500	4.89981e-01	6.40430e-01	8.88048e-01	1.26819e+00
128	0.0781250	3.59196e-01	4.07852e-01	4.91438e-01	6.42682e-01
256	0.0390625	3.26162e-01	3.37548e-01	3.62111e-01	4.07958e-01
512	0.0195312	3.14474e-01	3.19977e-01	3.26189e-01	3.37739e-01
1024	0.0097656	3.12185e-01	3.14389e-01	3.16821e-01	3.19974e-01
2048	0.0048828	3.11659e-01	3.12923e-01	3.14473e-01	3.14727e-01
4096	0.0024414	3.11531e-01	3.12570e-01	3.13677e-01	3.13615e-01
8192	0.0012207	3.11499e-01	3.12482e-01	3.13481e-01	3.13370e-01
16384	0.0006104	3.11491e-01	3.12460e-01	3.13432e-01	3.13310e-01
32768	0.0003052	3.11489e-01	3.12455e-01	3.13420e-01	3.13295e-01

Table 4.4: Optimal discrete inf-sup constant $c_{L^2}(T, \mu)$ of (4.40) for the perturbed bilinear form (4.30) for $\mu \in \{125, 250, 500, 1000\}$ with $T = 10$.

Theorem 4.2.21. *Let the unique solution u of (4.13) satisfy $u \in H_0^1(0, T) \cap H^s(0, T)$ for some $s \in [1, 2]$. Then, the unique solution $\tilde{u}_{h_t} \in S_{h_t, 0}^1(0, T)$ of the Galerkin-Petrov finite element discretisation (4.31) from Theorem 4.2.18 satisfies*

$$\|u - \tilde{u}_{h_t}\|_{L^2(0, T)} \leq c \left(1 + \frac{4}{\pi} T^2 \mu\right) h_t^s \|u\|_{H^s(0, T)} + \frac{\mu T h_t^2}{6} |u|_{H^1(0, T)}$$

with a constant $c > 0$ independent of μ and h_t .

Proof. With the representation (4.29) and the H_0^1 projection $\mathcal{Q}_{h_t}^1 u$ of (2.33), it holds true that

$$\begin{aligned} a_{h_t}(\tilde{u}_{h_t} - \mathcal{Q}_{h_t}^1 u, w_{h_t}) &= \underbrace{a_{h_t}(\tilde{u}_{h_t}, w_{h_t})}_{=a(u, w_{h_t})} - a_{h_t}(\mathcal{Q}_{h_t}^1 u, w_{h_t}) \\ &= a(u, w_{h_t}) - a(\mathcal{Q}_{h_t}^1 u, w_{h_t}) + \sum_{\ell=1}^{N_t} \frac{\mu h_{t, \ell}^2}{12} \langle \partial_t \mathcal{Q}_{h_t}^1 u, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} \\ &= - \underbrace{\langle \partial_t u, \partial_t w_{h_t} \rangle_{L^2(0, T)}}_{=\langle \partial_t \mathcal{Q}_{h_t}^1 u, \partial_t w_{h_t} \rangle_{L^2(0, T)}} + \mu \langle u, w_{h_t} \rangle_{L^2(0, T)} + \langle \partial_t \mathcal{Q}_{h_t}^1 u, \partial_t w_{h_t} \rangle_{L^2(0, T)} \\ &\quad - \mu \langle \mathcal{Q}_{h_t}^1 u, w_{h_t} \rangle_{L^2(0, T)} + \sum_{\ell=1}^{N_t} \frac{\mu h_{t, \ell}^2}{12} \langle \partial_t \mathcal{Q}_{h_t}^1 u, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} \\ &= \mu \langle u - \mathcal{Q}_{h_t}^1 u, w_{h_t} \rangle_{L^2(0, T)} + \sum_{\ell=1}^{N_t} \frac{\mu h_{t, \ell}^2}{12} \langle \partial_t \mathcal{Q}_{h_t}^1 u, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} \end{aligned}$$

for all $w_{h_t} \in S_{h_t, 0}^1(0, T)$ and so, $\tilde{u}_{h_t} - \mathcal{Q}_{h_t}^1 u \in S_{h_t, 0}^1(0, T)$ is the unique solution of the variational formulation (4.36) for

$$f_0 = \mu \cdot (u - \mathcal{Q}_{h_t}^1 u) \in L^2(0, T), \quad f_2 = \frac{\mu}{12} \partial_t \mathcal{Q}_{h_t}^1 u \in L^2(0, T) \quad \text{and} \quad f_1 = 0.$$

Therefore, the stability estimate (4.37), the Poincaré inequality from Lemma 3.4.5 and the stability of the H_0^1 projection give

$$\begin{aligned} \|\tilde{u}_{h_t} - \mathcal{Q}_{h_t}^1 u\|_{L^2(0, T)} &\leq 2T\mu \|u - \mathcal{Q}_{h_t}^1 u\|_{[H_0^1(0, T)]'} + \frac{\mu T h_t^2}{6} \|\partial_t \mathcal{Q}_{h_t}^1 u\|_{L^2(0, T)} \\ &\leq \frac{4}{\pi} T^2 \mu \|u - \mathcal{Q}_{h_t}^1 u\|_{L^2(0, T)} + \frac{\mu T h_t^2}{6} |u|_{H^1(0, T)}. \end{aligned}$$

With the last estimate, the triangle inequality and the error estimate (2.34) for the H_0^1 projection, it holds true that

$$\begin{aligned} \|u - \tilde{u}_{h_t}\|_{L^2(0, T)} &\leq \|u - \mathcal{Q}_{h_t}^1 u\|_{L^2(0, T)} + \|\tilde{u}_{h_t} - \mathcal{Q}_{h_t}^1 u\|_{L^2(0, T)} \\ &\leq c \left(1 + \frac{4}{\pi} T^2 \mu\right) h_t^s \|u\|_{H^s(0, T)} + \frac{\mu T h_t^2}{6} |u|_{H^1(0, T)} \end{aligned}$$

with a constant $c > 0$ independent of μ and h_t . Hence, the assertion follows. \square

As a numerical example for the Galerkin finite element methods (4.22) and (4.31), a uniform discretisation of the time interval $(0, T)$ with $T = 10$ and a uniform mesh size $h_t = T/N_t$ is introduced. For $\mu = 1000$, consider the solution

$$u(t) = \sin^2\left(\frac{5}{4}\pi t\right), \quad t \in [0, T],$$

where calculations of the appearing integrals for the related right-hand side in (4.22) and (4.31) are done by the usage of high-order integration rules.

In Table 4.5 the results for the stabilised variational formulation (4.31), which is unconditionally stable, are presented, where the error estimate in the energy norm of Theorem 4.2.18 is confirmed. In addition, the error in $L^2(0, T)$ is presented, where a quadratic convergence, as expected from Theorem 4.2.21, is observed.

N_t	h_t	$\ u - \tilde{u}_{h_t}\ _{L^2(0,10)}$	eoc	$\ u - \tilde{u}_{h_t}\ _{H^1(0,10)}$	eoc
4	2.5000000	1.7722e+00	-	9.0867e+00	-
8	1.2500000	6.0704e+00	-1.78	2.0130e+01	-1.15
16	0.6250000	1.2687e+00	2.26	9.4204e+00	1.10
32	0.3125000	5.7861e+00	-2.19	6.0121e+01	-2.67
64	0.1562500	3.3966e-01	4.09	6.1941e+00	3.28
128	0.0781250	7.6647e-02	2.15	2.2955e+00	1.43
256	0.0390625	2.0315e-02	1.92	9.4091e-01	1.29
512	0.0195312	5.2649e-03	1.95	4.1539e-01	1.18
1024	0.0097656	1.3365e-03	1.98	1.9803e-01	1.07
2048	0.0048828	3.3682e-04	1.99	9.7671e-02	1.02
4096	0.0024414	8.4229e-05	2.00	4.8663e-02	1.01
8192	0.0012207	2.1057e-05	2.00	2.4310e-02	1.00
16384	0.0006104	5.2644e-06	2.00	1.2152e-02	1.00
32768	0.0003052	1.3161e-06	2.00	6.0758e-03	1.00

Table 4.5: Numerical results for the stabilised variational formulation (4.31), $\mu = 1000$, $T = 10$.

In Table 4.6 the related results for the variational formulation (4.22) without stabilisation are presented, convergence is observed for a sufficiently small mesh size only. Note that $\sqrt{12/\mu} \approx 0.1095$.

4.2.2 Variational Formulation for the Wave Equation

Instead of the ordinary differential equation (4.12), the wave equation (4.11) is considered. The aim is to extend the results of Section 4.2.1 to the wave equation. So, for $u \in H_{0,0}^{1,1}(Q)$

N_t	h_t	$\ u - u_h\ _{L^2(0,10)}$	eoc	$\ u - u_h\ _{H^1(0,10)}$	eoc
4	2.5000000	7.0573e+01	-	9.8785e+01	-
8	1.2500000	1.6871e+03	-4.58	3.7166e+03	-5.23
16	0.6250000	9.1421e+07	-15.73	3.7247e+08	-16.61
32	0.3125000	2.3915e+15	-24.64	1.9496e+16	-25.64
64	0.1562500	1.6337e+22	-22.70	2.9536e+23	-23.85
128	0.0781250	3.1417e-02	78.78	1.7859e+00	77.13
256	0.0390625	9.2885e-03	1.76	8.2361e-01	1.12
512	0.0195312	2.4767e-03	1.91	3.9567e-01	1.06
1024	0.0097656	6.3105e-04	1.97	1.9532e-01	1.02
2048	0.0048828	1.5839e-04	1.99	9.7325e-02	1.00
4096	0.0024414	3.9633e-05	2.00	4.8620e-02	1.00
8192	0.0012207	9.9106e-06	2.00	2.4304e-02	1.00
16384	0.0006104	2.4778e-06	2.00	1.2152e-02	1.00
32768	0.0003052	6.1946e-07	2.00	6.0757e-03	1.00

Table 4.6: Numerical results for the variational formulation (4.22), $\mu = 1000$, $T = 10$.

and $w \in H_{0;0}^{1,1}(Q)$, one defines the bilinear form

$$a(u, w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)},$$

see Section 2.5 for the details of the Sobolev spaces. The boundedness of the bilinear form $a(\cdot, \cdot)$ is stated in the next lemma.

Lemma 4.2.22. *The bilinear form $a(\cdot, \cdot): H_{0;0}^{1,1}(Q) \times H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$ is bounded, i.e.*

$$|a(u, w)| \leq |u|_{H^1(Q)} |w|_{H^1(Q)}$$

for all $u \in H_{0;0}^{1,1}(Q)$, $w \in H_{0;0}^{1,1}(Q)$.

Proof. The assertion follows immediately from the Cauchy-Schwarz inequality. \square

The variational formulation of the wave equation (4.11) is to find $u \in H_{0;0}^{1,1}(Q)$ such that

$$a(u, w) = \langle f, w \rangle_{L^2(Q)} \tag{4.41}$$

for all $w \in H_{0;0}^{1,1}(Q)$, where $f \in L^2(Q)$ is given. Note that the initial condition $u(\cdot, 0) = 0$ is considered in the strong sense, whereas the initial condition $\partial_t u(\cdot, 0) = 0$ is incorporated in a weak sense.

For the analysis of (4.41), the adjoint problem to find $w \in H_{0,0}^{1,1}(Q)$ such that

$$a(u, w) = \langle g, u \rangle_{L^2(Q)} \quad (4.42)$$

for all $u \in H_{0,0}^{1,1}(Q)$, where $g \in L^2(Q)$ is a given right-hand side, occurs.

Next, an existence and uniqueness result for the variational formulation (4.41) is given. Such a result is contained in [97, Chapter IV], see also [145]. To summarise and to examine also the adjoint problem (4.42), the following theorem is stated.

Theorem 4.2.23. *Let $f, g \in L^2(Q)$ be given. There are unique solutions $u \in H_{0,0}^{1,1}(Q)$ of (4.41) and $w \in H_{0,0}^{1,1}(Q)$ of (4.42), satisfying*

$$|u|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|f\|_{L^2(Q)} \quad \text{and} \quad |w|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|g\|_{L^2(Q)}.$$

Proof. For the variational formulation (4.41), there exists a unique solution $u \in H_{0,0}^{1,1}(Q)$, see [97, Chapter IV, Theorem 3.1, page 157, and Theorem 3.2, page 160]. The estimate follows by a Fourier series ansatz as in [97, Section 7, Chapter IV], see [145]. When using the representation (3.66), any $u \in H_{0,0}^{1,1}(Q)$ admits the representation

$$u(x, t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} V_k(t) \phi_i(x) = \sum_{i=1}^{\infty} U_i(t) \phi_i(x), \quad (x, t) \in Q, \quad (4.43)$$

where $V_k(t)$ are the temporal eigenfunctions given in (3.26), and $\phi_i(x)$ are the spatial $L^2(\Omega)$ orthonormal eigenfunctions of the Laplacian with homogeneous Dirichlet boundary conditions, see (2.4). For the solution of the variational formulation (4.41), consider the ansatz (4.43), where the unknown functions $U_i \in H_0^1(0, T)$ are to be determined. When choosing, for a fixed $j \in \mathbb{N}$, $v(x, t) = V(t) \phi_j(x)$ with $V \in H_0^1(0, T)$ as test function, the variational formulation (4.41) results in finding $U_j \in H_0^1(0, T)$ such that

$$-\int_0^T \partial_t U_j(t) \partial_t V(t) dt + \mu_j \int_0^T U_j(t) V(t) dt = \int_0^T F_j(t) V(t) dt$$

for all $V \in H_0^1(0, T)$, where the functions $F_j \in L^2(0, T)$,

$$F_j(t) = \int_{\Omega} f(x, t) \phi_j(x) dx, \quad t \in (0, T),$$

are the coefficients of the Fourier expansion

$$f(x, t) = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} f_{j,k} V_k(t) \phi_j(x) = \sum_{j=1}^{\infty} F_j(t) \phi_j(x), \quad (x, t) \in Q,$$

see again (3.66). From this, one concludes

$$\begin{aligned} \|f\|_{L^2(\mathcal{Q})}^2 &= \int_0^T \int_{\Omega} |f(x,t)|^2 dx dt = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^T F_i(t) F_j(t) dt \int_{\Omega} \phi_i(x) \phi_j(x) dx \\ &= \sum_{j=1}^{\infty} \int_0^T |F_j(t)|^2 dt = \sum_{j=1}^{\infty} \|F_j\|_{L^2(0,T)}^2. \end{aligned}$$

Theorem 4.2.5 yields the uniqueness and the existence of the functions $U_j \in H_0^1(0,T)$. Hence, one obtains

$$\begin{aligned} \|u\|_{H^1(\mathcal{Q})}^2 &= \|u\|_{H_{0,0}^{1,1}(\mathcal{Q})}^2 = \int_0^T \int_{\Omega} \left[|\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2 \right] dx dt \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left[\int_0^T \partial_t U_i(t) \partial_t U_j(t) dt \int_{\Omega} \phi_i(x) \phi_j(x) dx \right. \\ &\quad \left. + \int_0^T U_i(t) U_j(t) dt \int_{\Omega} \nabla_x \phi_i(x) \cdot \nabla_x \phi_j(x) dx \right] \end{aligned}$$

and by using Lemma 4.2.7,

$$\begin{aligned} \|u\|_{H^1(\mathcal{Q})}^2 &= \sum_{i=1}^{\infty} \left[\int_0^T |\partial_t U_i(t)|^2 dt + \mu_i \int_0^T |U_i(t)|^2 dt \right] = \sum_{i=1}^{\infty} \left[\|U_i\|_{H^1(0,T)}^2 + \mu_i \|U_i\|_{L^2(0,T)}^2 \right] \\ &\leq \frac{1}{2} T^2 \sum_{i=1}^{\infty} \|F_i\|_{L^2(0,T)}^2 = \frac{1}{2} T^2 \|f\|_{L^2(\mathcal{Q})}^2. \end{aligned}$$

Analogous results hold true for the adjoint problem (4.42). \square

The variational formulation (4.41) is equivalent to find $u \in H_{0,0}^{1,1}(\mathcal{Q})$ such that

$$a(u, \overline{\mathcal{H}}_T v) = \langle f, \overline{\mathcal{H}}_T v \rangle_{L^2(\mathcal{Q})} \quad (4.44)$$

for all $v \in H_{0,0}^{1,1}(\mathcal{Q})$, where the transformation operator $\overline{\mathcal{H}}_T$, given in (4.15), acts only on the time variable t , i.e.

$$(\overline{\mathcal{H}}_T \hat{v})(x,t) = \hat{v}(x,T) - \hat{v}(x,t), \quad (x,t) \in \mathcal{Q}, \quad (4.45)$$

for $\hat{v} \in H_{0;0}^{1,1}(Q)$. Note that in (4.44) the ansatz and test spaces are equal and so, discretisation schemes of Galerkin-Bubnov type are possible.

The solution operators from Theorem 4.2.23 are not isomorphisms and hence, to derive from Theorem 4.2.23 an inf-sup condition, like

$$\forall u \in H_{0;0}^{1,1}(Q): \quad \sup_{0 \neq w \in H_{0;0}^{1,1}(Q)} \frac{|a(u, w)|}{|w|_{H^1(Q)}} \geq C_S |u|_{H^1(Q)} \quad (4.46)$$

with a constant $C_S > 0$, is not possible.

Theorem 4.2.24. *There does not exist a constant $C > 0$ such that each right-hand side $f \in L^2(Q)$ and the corresponding solution $u \in H_{0;0}^{1,1}(Q)$ of (4.41) satisfy*

$$|u|_{H^1(Q)} \leq C \|f\|_{[H_{0;0}^{1,1}(Q)]'}. \quad (4.47)$$

In particular, the inf-sup condition (4.46) does not hold true.

Proof. Consider the eigenfunctions $\phi_k \in H_0^1(\Omega)$ and eigenvalues $\mu_k > 0$ of the Dirichlet eigenvalue problem of the Laplacian, see (2.4). For $k \in \mathbb{N}$, take the eigenpair (ϕ_k, μ_k) and for $(x, t) \in Q$, set

$$u_k(x, t) := \phi_k(x) \int_0^t s \sin(\sqrt{\mu_k} s) ds,$$

$$f_k(x, t) := \partial_{tt} u_k(x, t) - \Delta_x u_k(x, t) = 2\phi_k(x) \sin(\sqrt{\mu_k} t).$$

The initial and boundary conditions $u_k(\cdot, 0) = \partial_t u_k(\cdot, 0) = 0$, $u_k|_{\Sigma} = 0$ and $u_k \in H_{0;0}^{1,1}(Q)$ are fulfilled. One computes as in Theorem 4.2.6 that

$$|u_k|_{H^1(Q)} \rightarrow \sqrt{\frac{T^3}{3}} \quad \text{and} \quad \|f_k\|_{[H_{0;0}^{1,1}(Q)]'} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

So, the first assertion is proven.

To show that the inf-sup condition (4.46) does not hold true, the bilinear form

$$a(\cdot, \cdot): H_{0;0}^{1,1}(Q) \times H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$$

is investigated. Because of Lemma 4.2.22, the bilinear form $a(\cdot, \cdot)$ is bounded. In addition, for $0 \neq w \in H_{0;0}^{1,1}(Q)$, there exists, according to Theorem 4.2.23, a unique solution $\tilde{u} \in H_{0;0}^{1,1}(Q)$ of (4.41) for $g = w \in L^2(Q)$, satisfying

$$\forall z \in H_{0;0}^{1,1}(Q): \quad a(\tilde{u}, z) = \langle w, z \rangle_{L^2(Q)}.$$

Hence, for $z = w$, it follows that $a(\tilde{u}, w) = \langle w, w \rangle_{L^2(Q)} > 0$, i.e. the condition $(\mathcal{N}2)$ holds true. If the inf-sup condition (4.46) would be true, the bilinear form $a(\cdot, \cdot)$ would fulfil all assumptions of the Nečas Theorem 2.9.1, which gives the estimate (4.47). But this would be contradictory to the first part of this proof. \square

To overcome this problem with the inf-sup condition (4.46), two different approaches are introduced in Section 4.3 and in Section 4.4. However, in the remainder of this section, conforming finite element discretisations for the variational formulation (4.41), resulting in Galerkin-Petrov schemes, are introduced and examined. Alternatively, the variational formulation (4.44) is discretised, which leads to a Galerkin-Bubnov scheme. In any case, for the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, it is assumed that $\Omega = (0, L)$ is an interval for $d = 1$, or Ω is polygonal for $d = 2$, or Ω is polyhedral for $d = 3$. In this thesis, only the tensor-product space-time finite element space

$$\mathcal{Q}_h^1(\mathcal{Q}) = V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0, T),$$

given in (2.42), is investigated for (4.41). Therefore, the Galerkin-Petrov finite element discretisation of the variational formulation (4.41) is to find

$$u_h \in \mathcal{Q}_h^1(\mathcal{Q}) \cap H_{0;0}^{1,1}(\mathcal{Q}) = V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0, T)$$

such that

$$a(u_h, w_h) = \langle f, w_h \rangle_{L^2(\mathcal{Q})} \quad (4.48)$$

for all $w_h \in \mathcal{Q}_h^1(\mathcal{Q}) \cap H_{0;0}^{1,1}(\mathcal{Q}) = V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0, T)$. The approximate function u_h admits the representation

$$u_h(x, t) = \sum_{\ell=1}^{N_t} \sum_{j=1}^{M_x} u_j^\ell \psi_j(x) \varphi_\ell(t) = \sum_{j=1}^{M_x} U_{h_t, j}(t) \psi_j(x), \quad U_{h_t, j}(t) = \sum_{\ell=1}^{N_t} u_j^\ell \varphi_\ell(t) \quad (4.49)$$

for $(x, t) \in \bar{\mathcal{Q}}$, see (2.43). After an appropriate ordering of the degrees of freedom, the discrete variational formulation (4.48) is equivalent to the global linear system

$$K_h \underline{u} = \underline{F}$$

with the system matrix

$$K_h = -A_{h_t} \otimes M_{h_x} + M_{h_t} \otimes A_{h_x} \in \mathbb{R}^{N_t \cdot M_x \times N_t \cdot M_x},$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote spatial mass and stiffness matrices given in (2.37) and (2.38), $M_{h_t} \in \mathbb{R}^{N_t \times N_t}$ and $A_{h_t} \in \mathbb{R}^{N_t \times N_t}$ denote temporal mass and stiffness matrices given in (4.25) and (4.24), and with the corresponding vector $\underline{F} \in \mathbb{R}^{N_t \cdot M_x}$ of the right-hand side.

Remark 4.2.25. *Note that the Galerkin-Petrov finite element discretisation (4.48) can be realised as a two-step method.*

Using a conforming semi-discretisation approach for the variational formulation (4.41) leads to find $\tilde{u}_{h_x} \in V_{h_x,0}(\Omega) \otimes H_{0;0}^1(0, T) \subset H_{0;0}^{1,1}(\mathcal{Q})$ such that

$$a(\tilde{u}_{h_x}, w_{h_x}) = \langle f, w_{h_x} \rangle_{L^2(\mathcal{Q})} \quad (4.50)$$

for all $w_{h_x} \in V_{h_x,0}(\Omega) \otimes H_0^1(0,T) \subset H_{0;0}^{1,1}(Q)$. The semi-discrete function \tilde{u}_{h_x} admits the representation

$$\tilde{u}_{h_x}(x,t) = \sum_{j=1}^{M_x} \tilde{U}_j(t) \psi_j(x), \quad (x,t) \in Q,$$

with the unknown temporal functions $\tilde{U}_j \in H_0^1(0,T)$ to be determined. With this representation, the semi-discrete variational formulation (4.50) is equivalent to the M_x equations to find

$$\tilde{\underline{U}} = (\tilde{U}_1, \dots, \tilde{U}_{M_x})^\top \in [H_0^1(0,T)]^{M_x}$$

such that

$$-\int_0^T M_{h_x} \partial_t \tilde{\underline{U}}(t) \partial_t W(t) dt + \int_0^T A_{h_x} \tilde{\underline{U}}(t) W(t) dt = \int_0^T \underline{f}(t) W(t) dt \quad (4.51)$$

for all $W \in H_0^1(0,T)$, where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote mass and stiffness matrices given in (2.37) and (2.38) together with the right-hand side

$$\underline{f} = (f_1, \dots, f_{M_x})^\top \in [L^2(0,T)]^{M_x},$$

defined by

$$f_j(t) := \int_{\Omega} f(x,t) \psi_j(x) dx, \quad t \in (0,T),$$

for $j = 1, \dots, M_x$.

By using the Cholesky decomposition

$$M_{h_x} = L_{h_x} L_{h_x}^\top$$

and for the symmetric, positive-definite matrix $\hat{A}_{h_x} := L_{h_x}^{-1} A_{h_x} L_{h_x}^{-\top}$, the decomposition

$$\hat{A}_{h_x} = V_{h_x} D_{h_x} V_{h_x}^\top, \quad D_{h_x} = \text{diag}(\hat{\mu}_k(\hat{A}_{h_x})), \quad V_{h_x} = (\underline{v}^1, \dots, \underline{v}^{M_x}), \quad \hat{A}_{h_x} \underline{v}^j = \hat{\mu}_k(\hat{A}_{h_x}) \underline{v}^j,$$

the variational formulation (4.51) is equivalent to find

$$V_{h_x}^\top L_{h_x}^\top \tilde{\underline{U}} = \underline{Z} = (Z_1, \dots, Z_{M_x})^\top \in [H_0^1(0,T)]^{M_x}$$

such that

$$-\int_0^T \partial_t \underline{Z}(t) \partial_t W(t) dt + \int_0^T D_{h_x} \underline{Z}(t) W(t) dt = \int_0^T V_{h_x}^\top L_{h_x}^{-1} \underline{f}(t) W(t) dt$$

for all $W \in H_0^1(0,T)$. The related approximation

$$\underline{Z}_{h_t} = (Z_{h_t,1}, \dots, Z_{h_t,M_x})^\top$$

is defined by finding, for $j = 1, \dots, M_x$, the functions

$$Z_{h_t, j} \in S_{h_t, 0}^1(0, T) = S_{h_t}^1(0, T) \cap H_{0,0}^1(0, T)$$

such that

$$-\langle \partial_t Z_{h_t, j}, \partial_t w_{h_t} \rangle_{L^2(0, T)} + \hat{\mu}_j(\hat{A}_{h_x}) \cdot \langle Z_{h_t, j}, w_{h_t} \rangle_{L^2(0, T)} = \langle g_j, w_{h_t} \rangle_{L^2(0, T)}$$

for all $w_{h_t} \in S_{h_t, 0}^1(0, T) = S_{h_t}^1(0, T) \cap H_{0,0}^1(0, T)$, where

$$\underline{g} = (g_1, \dots, g_{M_x})^\top = V_{h_x}^\top L_{h_x}^{-1} \underline{f}$$

is the transformed right-hand side. By construction, it holds true that

$$\underline{Z}_{h_t} = V_{h_x}^\top L_{h_x}^\top \underline{U}_{h_t},$$

where

$$\underline{U}_{h_t} = (U_{h_t, 1}, \dots, U_{h_t, M_x})^\top$$

is the vector of coefficients of the representation (4.49).

Stability and related error estimates for the finite element solutions $Z_{h_t, j} \in S_{h_t, 0}^1(0, T)$ follow for sufficiently small time mesh sizes h_t , see (4.21). However, as in Remark 4.2.8, for a uniform time mesh size h_t , the stability of the corresponding finite difference scheme holds true for

$$\hat{\mu}_j(\hat{A}_{h_x}) = \frac{[\underline{u}^j]^\top A_{h_x} \underline{u}^j}{[\underline{u}^j]^\top M_{h_x} \underline{u}^j} = \frac{\|\nabla_x u_{h_x}^j\|_{L^2(\Omega)}^2}{\|u_{h_x}^j\|_{L^2(\Omega)}^2} < \frac{12}{h_t^2} \quad \text{for } j = 1, \dots, M_x,$$

where $\underline{u}^j = L_{h_x}^{-\top} v^j \in \mathbb{R}^{M_x}$ are the transformed eigenvectors and $u_{h_x}^j \in V_{h_x, 0}(\Omega)$ are the related functions. With the inverse inequality

$$\forall v_{h_x} \in V_{h_x, 0}(\Omega): \quad \|\nabla_x v_{h_x}\|_{L^2(\Omega)}^2 \leq c_I h_x^{-2} \|v_{h_x}\|_{L^2(\Omega)}^2,$$

see [141, (9.19), page 217], with a constant $c_I > 0$, this condition is satisfied for

$$h_t < \sqrt{\frac{12}{c_I}} h_x, \tag{4.52}$$

i.e. a CFL condition is needed for stability. In the particular case $d = 1$, it holds true that $c_I = 12$, see the derivation of [141, (9.19), page 217] and therefore, stability follows for

$$h_t < h_x.$$

When $V_{h_x, 0}(\Omega) \subset H_0^1(\Omega)$ is also of tensor-product structure, i.e.

$$V_{h_x, 0}(\Omega) = \left(S_{h_{x_1}}^1(0, L_1) \otimes \dots \otimes S_{h_{x_d}}^1(0, L_d) \right) \cap H_0^1(\Omega),$$

for example, when considering $\Omega = (0, L_1) \times \dots \times (0, L_d) \subset \mathbb{R}^d$ with uniform mesh sizes h_{x_1}, \dots, h_{x_d} , one concludes $c_I = 12d$, and therefore, the stability condition

$$h_t < \frac{h_{x,\min}}{\sqrt{d}},$$

where $h_{x,\min} = \min\{h_{x_1}, \dots, h_{x_d}\}$.

As a numerical example, consider for $d = 2$ the spatial domain $\Omega = (0, 1)^2$ with uniform discretisations with mesh sizes $h_x = h_{x_1} = h_{x_2}$ and the exact solution

$$u(x_1, x_2, t) = t^2 \sin(\pi x_1) \sin(\pi x_2) \quad \text{for } (x_1, x_2, t) \in Q = \Omega \times (0, T)$$

with different terminal times $T \in \left\{ \frac{7}{10}, \frac{1}{\sqrt{2}}, \frac{3}{4}, 1, 2 \right\}$. Then stability follows when choosing

$$\frac{h_t}{h_x} < \frac{1}{\sqrt{2}} \approx 0.7071068. \quad (4.53)$$

In Table 4.7, Table 4.8, Table 4.9, Table 4.10 and Table 4.11, the $L^2(Q)$ error, the $H^1(Q)$ error and the maximal and minimal singular values of the related system matrix K_h of the Galerkin-Petrov formulation (4.48) are given, where the observed convergence rates are as expected, provided the CFL condition (4.53) is satisfied, i.e. the CFL condition (4.53) seems to be sharp. Here, the number of the degrees of freedom is denoted by

$$\text{dof} = \dim Q_h^1(Q) \cap H_{0;0}^{1,1}(Q) = \dim Q_h^1(Q) \cap H_{0;0}^{1,1}(Q).$$

dof	h_x	h_t	$\ u - u_h\ _{L^2(Q)}$	eoc	$ u - u_h _{H^1(Q)}$	eoc	$\sigma_{\max}(K_h)$	$\sigma_{\min}(K_h)$
2	0.500	0.3500	2.034e-02	-	4.204e-01	-	4.8e-01	4.7e-01
36	0.250	0.1750	4.737e-03	2.1	2.089e-01	1.0	9.4e-01	5.4e-02
392	0.125	0.0875	1.161e-03	2.0	1.040e-01	1.0	6.5e-01	7.0e-03
3600	0.062	0.0438	2.887e-04	2.0	5.193e-02	1.0	3.5e-01	1.2e-03
30752	0.031	0.0219	7.207e-05	2.0	2.595e-02	1.0	1.8e-01	2.3e-04
254016	0.016	0.0109	1.801e-05	2.0	1.298e-02	1.0	8.9e-02	3.6e-05

Table 4.7: Numerical results for the Galerkin-Petrov formulation (4.48) with uniform meshes for $Q = (0, 1)^2 \times (0, \frac{7}{10})$, satisfying the CFL condition (4.53).

From (4.52), one only expects conditional stability of (4.48). To stabilise the numerical scheme in (4.48), Zlotnik's idea [164], as in (4.29), is used again, and the following representation is proven.

dof	h_x	h_t	$\ u - u_h\ _{L^2(Q)}$	eoc	$ u - u_h _{H^1(Q)}$	eoc	$\sigma_{\max}(K_h)$	$\sigma_{\min}(K_h)$
2	0.500	0.3536	2.094e-02	-	4.303e-01	-	4.7e-01	4.7e-01
36	0.250	0.1768	4.888e-03	2.1	2.141e-01	1.0	9.3e-01	5.2e-02
392	0.125	0.0884	1.198e-03	2.0	1.066e-01	1.0	6.4e-01	6.1e-03
3600	0.062	0.0442	2.981e-04	2.0	5.323e-02	1.0	3.4e-01	7.6e-04
30752	0.031	0.0221	7.444e-05	2.0	2.661e-02	1.0	1.8e-01	9.6e-05
254016	0.016	0.0110	1.860e-05	2.0	1.330e-02	1.0	8.8e-02	1.2e-05

Table 4.8: Numerical results for the Galerkin-Petrov formulation (4.48) with uniform meshes for $Q = (0, 1)^2 \times (0, \frac{1}{\sqrt{2}})$ for the limit case of the CFL condition (4.53).

dof	h_x	h_t	$\ u - u_h\ _{L^2(Q)}$	eoc	$ u - u_h _{H^1(Q)}$	eoc	$\sigma_{\max}(K_h)$	$\sigma_{\min}(K_h)$
2	0.500	0.3750	2.476e-02	-	4.937e-01	-	5.0e-01	4.3e-01
36	0.250	0.1875	5.862e-03	2.1	2.469e-01	1.0	8.7e-01	4.1e-02
392	0.125	0.0938	1.443e-03	2.0	1.231e-01	1.0	6.0e-01	2.7e-03
3600	0.062	0.0469	3.594e-04	2.0	6.153e-02	1.0	3.2e-01	4.5e-05
30752	0.031	0.0234	8.977e-05	2.0	3.076e-02	1.0	1.7e-01	3.5e-08
254016	0.016	0.0117	2.244e-05	2.0	1.538e-02	1.0	8.3e-02	5.7e-14

Table 4.9: Numerical results for the Galerkin-Petrov formulation (4.48) with uniform meshes for $Q = (0, 1)^2 \times (0, \frac{3}{4})$, violating the CFL condition (4.53).

dof	h_x	h_t	$\ u - u_h\ _{L^2(Q)}$	eoc	$ u - u_h _{H^1(Q)}$	eoc	$\sigma_{\max}(K_h)$	$\sigma_{\min}(K_h)$
2	0.500	0.5000	5.418e-02	-	9.782e-01	-	7.2e-01	2.7e-01
36	0.250	0.2500	1.353e-02	2.0	4.986e-01	1.0	7.2e-01	1.5e-02
392	0.125	0.1250	3.381e-03	2.0	2.502e-01	1.0	4.6e-01	1.7e-04
3600	0.062	0.0625	8.453e-04	2.0	1.252e-01	1.0	2.4e-01	6.6e-08
30752	0.031	0.0312	2.113e-04	2.0	6.263e-02	1.0	1.2e-01	2.8e-14
254016	0.016	0.0156	8.621e+08	-41.9	4.635e+11	-42.8	6.2e-02	≈ 0

Table 4.10: Numerical results for the Galerkin-Petrov formulation (4.48) with uniform meshes for $Q = (0, 1)^2 \times (0, 1)$, violating the CFL condition (4.53).

dof	h_x	h_t	$\ u - u_h\ _{L^2(Q)}$	eoc	$ u - u_h _{H^1(Q)}$	eoc	$\sigma_{\max}(K_h)$	$\sigma_{\min}(K_h)$
2	0.500	1.0000	2.777e-01	-	5.638e+00	-	1.7e+00	1.8e-01
36	0.250	0.5000	7.355e-02	1.9	2.798e+00	1.0	1.5e+00	6.9e-03
392	0.125	0.2500	1.863e-02	2.0	1.404e+00	1.0	9.3e-01	2.4e-05
3600	0.062	0.1250	4.732e-03	2.0	7.028e-01	1.0	4.9e-01	7.0e-10
30752	0.031	0.0625	7.852e-01	-7.4	1.796e+02	-8.0	2.5e-01	≈ 0
254016	0.016	0.0312	1.710e+21	-70.9	7.642e+23	-71.8	1.2e-01	≈ 0

Table 4.11: Numerical results for the Galerkin-Petrov formulation (4.48) with uniform meshes for $Q = (0, 1)^2 \times (0, 2)$, violating the CFL condition (4.53).

Lemma 4.2.26. *For all $u_h \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$ and $w_h \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$, the bilinear form in (4.48) has the representation*

$$a(u_h, w_h) = -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{m=1}^d \langle \partial_{x_m} u_h, Q_{h_t}^0 \partial_{x_m} w_h \rangle_{L^2(Q)} + \sum_{m=1}^d \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \partial_{x_m} u_h, \partial_t \partial_{x_m} w_h \rangle_{L^2(\Omega \times \tau_\ell)}, \quad (4.54)$$

where $Q_{h_t}^0: L^2(Q) \rightarrow L^2(\Omega) \otimes S_{h_t}^0(0, T)$ denotes the extended L^2 projection (2.60) on the space $L^2(\Omega) \otimes S_{h_t}^0(0, T)$ of piecewise constant functions with respect to time.

Proof. Let $u_h \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$ and $w_h \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$ be given. With the representation (2.43), it follows for $(x, t) \in \bar{Q}$ that

$$u_h(x, t) = \sum_{\ell=1}^{N_t} \sum_{j=1}^{M_x} u_j^\ell \psi_j(x) \varphi_\ell(t) = \sum_{j=1}^{M_x} U_{h_t, j}(t) \psi_j(x), \quad U_{h_t, j}(t) = \sum_{\ell=1}^{N_t} u_j^\ell \varphi_\ell(t)$$

and

$$w_h(x, t) = \sum_{\ell=0}^{N_t-1} \sum_{j=1}^{M_x} w_j^\ell \psi_j(x) \varphi_\ell(t) = \sum_{j=1}^{M_x} W_{h_t, j}(t) \psi_j(x), \quad W_{h_t, j}(t) = \sum_{\ell=0}^{N_t-1} w_j^\ell \varphi_\ell(t).$$

Hence, for $m = 1, \dots, d$ and by using (4.28), it holds true that

$$\begin{aligned} \langle \partial_{x_m} u_h, \partial_{x_m} w_h \rangle_{L^2(Q)} &= \sum_{i=1}^{M_x} \sum_{j=1}^{M_x} \int_0^T U_{h_t, i}(t) W_{h_t, j}(t) dt \int_{\Omega} \partial_{x_m} \psi_i(x) \partial_{x_m} \psi_j(x) dx \\ &= \sum_{i=1}^{M_x} \sum_{j=1}^{M_x} \left[\frac{1}{12} \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t U_{h_t, i}, \partial_t W_{h_t, j} \rangle_{L^2(\tau_\ell)} + \langle U_{h_t, i}, Q_{h_t}^0 W_{h_t, j} \rangle_{L^2(0, T)} \right] \\ &\quad \cdot \int_{\Omega} \partial_{x_m} \psi_i(x) \partial_{x_m} \psi_j(x) dx \end{aligned}$$

and thus,

$$\langle \partial_{x_m} u_h, \partial_{x_m} w_h \rangle_{L^2(Q)} = \langle \partial_{x_m} u_h, Q_{h_t}^0 \partial_{x_m} w_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \partial_{x_m} u_h, \partial_t \partial_{x_m} w_h \rangle_{L^2(\Omega \times \tau_\ell)},$$

where the relation (2.61) is used. So, the assertion is proven. \square

Due to the representation (4.54), one defines for the functions $u_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ and $w_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ the perturbed bilinear form

$$\begin{aligned} a_h(u_h, w_h) &:= -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{m=1}^d \langle \partial_{x_m} u_h, Q_{h_t}^0 \partial_{x_m} w_h \rangle_{L^2(Q)} \\ &= -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{m=1}^d \langle Q_{h_t}^0 \partial_{x_m} u_h, \partial_{x_m} w_h \rangle_{L^2(Q)}. \end{aligned} \quad (4.55)$$

So, the perturbed variational problem is to find $\tilde{u}_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ such that

$$a_h(\tilde{u}_h, w_h) = \langle f, w_h \rangle_{L^2(Q)} \quad (4.56)$$

for all $w_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$. After an appropriate ordering of the degrees of freedom, the discrete variational formulation (4.56) is equivalent to the global linear system

$$\tilde{K}_h \underline{u} = \underline{F}$$

with the system matrix

$$\tilde{K}_h = -A_{h_t} \otimes M_{h_x} + \tilde{M}_{h_t} \otimes A_{h_x} \in \mathbb{R}^{N_t \cdot M_x \times N_t \cdot M_x},$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote spatial mass and stiffness matrices given in (2.37) and (2.38), $\tilde{M}_{h_t} \in \mathbb{R}^{N_t \times N_t}$ is the temporal perturbed mass matrix (4.33), and $A_{h_t} \in \mathbb{R}^{N_t \times N_t}$ is the temporal stiffness matrix (4.24), and with the corresponding vector $\underline{F} \in \mathbb{R}^{N_t \cdot M_x}$ of the right-hand side.

Remark 4.2.27. *Note that the Galerkin-Petrov finite element discretisation (4.56) can be realised as a two-step method, which differs from the Newmark Galerkin method (4.6), (4.7) of Section 4.1 only in the treatment of the input data.*

To prove the existence and uniqueness of a solution \tilde{u}_h of (4.56), the following lemma, which is analogous to Lemma 4.2.14, is shown.

Lemma 4.2.28. For a given $v_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$, represented by

$$v_h(x,t) = \sum_{\ell=0}^{N_t} V_{h_x,\ell}(x) \varphi_\ell(t), \quad V_{h_x,\ell}(x) = \sum_{j=1}^{M_x} v_j^\ell \Psi_j(x) \quad \text{for } (x,t) \in \bar{Q}$$

with $V_{h_x,0}(x) = 0$ for $x \in \Omega$ and $V_{h_x,\ell} \in V_{h_x,0}(\Omega)$, and for a fixed index $j \in \{1, \dots, N_t\}$, a function

$$\bar{z}_h^j \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$$

exists with the following properties:

1. For $(x,t) \in \bar{\Omega} \times [t_j, T]$, it holds true that $\bar{z}_h^j(x,t) = 0$.
2. For $\ell = 1, \dots, j$ and for $x \in \bar{\Omega}$, the equalities

$$\langle \partial_t \bar{z}_h^j(x, \cdot), \partial_t v_h(x, \cdot) \rangle_{L^2(\tau_\ell)} = \frac{1}{2} \left([V_{h_x,\ell-1}(x)]^2 - [V_{h_x,\ell}(x)]^2 \right)$$

and

$$\langle \partial_{x_m} \bar{z}_h^j(x, \cdot), \mathcal{Q}_{h_t}^0 \partial_{x_m} v_h(x, \cdot) \rangle_{L^2(\tau_\ell)} = \frac{1}{2} \left(\int_{t_{\ell-1}}^{t_j} \partial_{x_m} v_h(x,s) ds \right)^2 - \frac{1}{2} \left(\int_{t_\ell}^{t_j} \partial_{x_m} v_h(x,s) ds \right)^2$$

are valid for $m = 1, \dots, d$.

3. For $x \in \bar{\Omega}$, the estimate

$$\|\partial_t \bar{z}_h^j(x, \cdot)\|_{L^2(0,T)} \leq \|v_h(x, \cdot)\|_{L^2(0,T)}$$

holds true.

Proof. For $v_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$, one defines for $(x,t) \in \bar{Q}$

$$u_h^j(x,t) := \sum_{i=0}^{N_t} U_{h_x,i}^j(x) \varphi_i(t), \quad U_{h_x,i}^j := \begin{cases} V_{h_x,i}(x) & \text{for } i = 0, \dots, j, \\ (-1)^{j-i} V_{h_x,j}(x) & \text{for } i = j+1, \dots, N_t, \end{cases}$$

and further,

$$\bar{z}_h^j(x,t) := - \sum_{k=0}^{N_t} \left(\int_T^{t_k} u_h^j(x,s) ds \right) \varphi_k(t) = - \left(I_{h_t} \int_T^{(\cdot)} u_h^j(x,s) ds \right) (t),$$

where

$$I_{h_t} : C([0,T]; L^2(\Omega)) \rightarrow L^2(\Omega) \otimes S_{h_t}^1(0,T)$$

is the extended time interpolation operator, see (2.44). Note that $\bar{z}_h^j \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$. For $x \in \bar{\Omega}$, it follows from relation (4.27) that

$$\partial_t \bar{z}_h^j(x, \cdot) = -\mathcal{Q}_{h_t}^0 u_h^j(x, \cdot).$$

In particular, for $j < N_t$, $x \in \bar{\Omega}$, and for $t \in \tau_\ell$ for $\ell = j+1, \dots, N_t$, it holds true that

$$-\partial_t \bar{z}_h^j(x, t) = \mathcal{Q}_{h_t}^0 u_h^j(x, t) = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} u_h^j(x, s) ds = \frac{1}{2} \left(U_{h_{x,\ell-1}}^j(x) + U_{h_{x,\ell}}^j(x) \right) = 0,$$

and due to $\bar{z}_h^j(x, T) = 0$, one concludes $\bar{z}_h^j(x, t) = 0$ for $t \in [t_j, T]$, i.e. the first assertion.

To prove the second assertion, one computes for $x \in \bar{\Omega}$ and for $\ell = 1, \dots, j$

$$\begin{aligned} \langle \partial_t \bar{z}_h^j(x, \cdot), \partial_t v_h(x, \cdot) \rangle_{L^2(\tau_\ell)} &= \int_{t_{\ell-1}}^{t_\ell} \partial_t \bar{z}_h^j(x, t) \partial_t v_h(x, t) dt \\ &= -\frac{1}{2} \left(U_{h_{x,\ell-1}}^j(x) + U_{h_{x,\ell}}^j(x) \right) \int_{t_{\ell-1}}^{t_\ell} \partial_t v_h(x, t) dt \\ &= -\frac{1}{2} \left(U_{h_{x,\ell-1}}^j(x) + U_{h_{x,\ell}}^j(x) \right) \left(V_{h_{x,\ell}}(x) - V_{h_{x,\ell-1}}(x) \right) \\ &= \frac{1}{2} \left(V_{h_{x,\ell-1}}(x) + V_{h_{x,\ell}}(x) \right) \left(V_{h_{x,\ell-1}}(x) - V_{h_{x,\ell}}(x) \right) \\ &= \frac{1}{2} \left([V_{h_{x,\ell-1}}(x)]^2 - [V_{h_{x,\ell}}(x)]^2 \right). \end{aligned}$$

Moreover, for $m = 1, \dots, d$, it follows for $x \in \bar{\Omega}$ and for $\ell = 1, \dots, j$ that

$$\begin{aligned} \langle \partial_{x_m} \bar{z}_h^j(x, \cdot), \mathcal{Q}_{h_t}^0 \partial_{x_m} v_h(x, \cdot) \rangle_{L^2(\tau_\ell)} &= \mathcal{Q}_{h_t}^0 \partial_{x_m} v_h(x, \cdot) \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m} \bar{z}_h^j(x, t) dt \\ &= -\mathcal{Q}_{h_t}^0 \partial_{x_m} v_h(x, \cdot) \Big|_{\tau_\ell} \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m} \left[\int_T^{t_{\ell-1}} u_h^j(x, s) ds \varphi_{\ell-1}(t) + \int_T^{t_\ell} u_h^j(x, s) ds \varphi_\ell(t) \right] dt \\ &= -\frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m} v_h(x, t) dt \frac{1}{2} h_{t,\ell} \left[\int_T^{t_{\ell-1}} \partial_{x_m} u_h^j(x, s) ds + \int_T^{t_\ell} \partial_{x_m} u_h^j(x, s) ds \right] \end{aligned}$$

and further,

$$\begin{aligned} \langle \partial_{x_m} \bar{z}_h^j(x, \cdot), \mathcal{Q}_{h_t}^0 \partial_{x_m} v_h(x, \cdot) \rangle_{L^2(\tau_\ell)} &= -\frac{1}{2} \left[\int_{t_j}^{t_\ell} \partial_{x_m} v_h(x, t) dt - \int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h(x, t) dt \right] \\ &\quad \cdot \left[\int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h^j(x, s) ds + \int_{t_j}^{t_\ell} \partial_{x_m} v_h^j(x, s) ds \right] \\ &= \frac{1}{2} \left(\int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h^j(x, s) ds \right)^2 - \frac{1}{2} \left(\int_{t_j}^{t_\ell} \partial_{x_m} v_h^j(x, s) ds \right)^2. \end{aligned}$$

The L^2 stability of $\mathcal{Q}_{h_t}^0$ and using

$$\begin{aligned} \|\partial_t \bar{z}_h^j(x, \cdot)\|_{L^2(0, T)} &= \|\partial_t \bar{z}_h^j(x, \cdot)\|_{L^2(0, t_j)} = \|\mathcal{Q}_{h_t}^0 u_h^j(x, \cdot)\|_{L^2(0, t_j)} \\ &= \|\mathcal{Q}_{h_t}^0 v_h(x, \cdot)\|_{L^2(0, t_j)} \leq \|\mathcal{Q}_{h_t}^0 v_h(x, \cdot)\|_{L^2(0, T)} \leq \|v_h(x, \cdot)\|_{L^2(0, T)} \end{aligned}$$

for $x \in \bar{\Omega}$ yield the third property. \square

With the last lemma, the existence, uniqueness, and stability of a solution \tilde{u}_h of (4.56) are proven with the help of the next lemma.

Lemma 4.2.29. *Let $f_0 \in [H_{;0}^1(0, T; L^2(\Omega))]'$ and $f_1, f_2 \in L^2(Q)$ be given. The variational formulation to find $w_h \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ such that*

$$a_h(w_h, v_h) = \langle f_0, v_h \rangle_Q + \langle f_1, \partial_t v_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle f_2, \partial_t v_h \rangle_{L^2(\Omega \times \tau_\ell)} \quad (4.57)$$

for all $v_h \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ is uniquely solvable, and the stability estimate

$$\|w_h\|_{L^2(Q)} \leq 2T \left\{ \|f_0\|_{[H_{;0}^1(0, T; L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2 \|f_2\|_{L^2(Q)} \right\} \quad (4.58)$$

holds true.

Proof. Let $w_h^0 \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ be any solution of the homogeneous variational formulation (4.57) with $f_i \equiv 0$, and with the representation (2.43), i.e.

$$w_h^0(x, t) = \sum_{\ell=0}^{N_t} W_{h_{x,\ell}}^0(x) \varphi_\ell(t) \quad \text{for } (x, t) \in \bar{Q}, \quad W_{h_{x,\ell}}^0 \in V_{h_{x,0}}(\Omega), \quad W_{h_{x,0}}^0(x) = 0.$$

For each index $j \in \{1, \dots, N_t\}$, consider an element $\bar{z}_h^j \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ as given in Lemma 4.2.28. Plugging $v_h = \bar{z}_h^j$ into (4.57), it holds true that

$$\begin{aligned} 0 &= a_h(w_h^0, \bar{z}_h^j) = -\langle \partial_t w_h^0, \partial_t \bar{z}_h^j \rangle_{L^2(Q)} + \sum_{m=1}^d \langle Q_{h_t}^0 \partial_{x_m} w_h^0, \partial_{x_m} \bar{z}_h^j \rangle_{L^2(Q)} \\ &= -\sum_{\ell=1}^j \langle \partial_t w_h^0, \partial_t \bar{z}_h^j \rangle_{L^2(\Omega \times \tau_\ell)} + \sum_{m=1}^d \sum_{\ell=1}^j \langle Q_{h_t}^0 \partial_{x_m} w_h^0, \partial_{x_m} \bar{z}_h^j \rangle_{L^2(\Omega \times \tau_\ell)} \end{aligned}$$

and by using the properties of Lemma 4.2.28,

$$\begin{aligned} 0 &= a_h(w_h^0, \bar{z}_h^j) = -\int_{\Omega} \sum_{\ell=1}^j \left(\frac{1}{2} [W_{h_x, \ell-1}^0(x)]^2 - \frac{1}{2} [W_{h_x, \ell}^0(x)]^2 \right) dx \\ &\quad + \sum_{m=1}^d \int_{\Omega} \sum_{\ell=1}^j \left(\frac{1}{2} \left(\int_{t_{\ell-1}}^{t_j} \partial_{x_m} w_h^0(x, s) ds \right)^2 - \frac{1}{2} \left(\int_{t_\ell}^{t_j} \partial_{x_m} w_h^0(x, s) ds \right)^2 \right) dx \\ &= \frac{1}{2} \int_{\Omega} [W_{h_x, j}^0(x)]^2 dx + \frac{1}{2} \sum_{m=1}^d \int_{\Omega} \left(\int_0^{t_j} \partial_{x_m} w_h^0(x, s) ds \right)^2 dx. \end{aligned}$$

This result yields, with the Cauchy-Schwarz inequality and the use of the properties of Lemma 4.2.28,

$$\begin{aligned} \|w_h^0\|_{L^2(Q)}^2 &= \sum_{\ell=1}^{N_t} \|w_h^0\|_{L^2(\Omega \times \tau_\ell)}^2 \\ &= \int_{\Omega} \sum_{\ell=1}^{N_t} \frac{h_{t, \ell}}{3} \left([W_{h_x, \ell}^0(x)]^2 + W_{h_x, \ell}^0(x) W_{h_x, \ell-1}^0(x) + [W_{h_x, \ell-1}^0(x)]^2 \right) dx \\ &\leq \int_{\Omega} \sum_{j=1}^{N_t} \frac{h_{t, j}}{2} [W_{h_x, j}^0(x)]^2 dx + \int_{\Omega} \sum_{j=1}^{N_t-1} \frac{h_{t, j+1}}{2} [W_{h_x, j}^0(x)]^2 dx \leq 0, \end{aligned}$$

which implies $w_h^0 \equiv 0$. Therefore, by using

$$\dim Q_h^1(Q) \cap H_{0;0}^{1,1}(Q) = \dim Q_h^1(Q) \cap H_{0;0}^{1,1}(Q),$$

one concludes unique solvability of the variational formulation (4.57) for any right-hand sides $f_0 \in [H_{;0}^1(0, T; L^2(\Omega))]'$ and $f_1, f_2 \in L^2(Q)$. Following the approach as above, it holds true that

$$\langle f_0, \bar{z}_h^j \rangle_Q + \langle f_1, \partial_t \bar{z}_h^j \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h_{t, \ell}^2 \langle f_2, \partial_t \bar{z}_h^j \rangle_{L^2(\Omega \times \tau_\ell)} = a_h(w_h, \bar{z}_h^j) \geq \frac{1}{2} \int_{\Omega} [W_{h_x, j}^0(x)]^2 dx$$

and

$$\begin{aligned}
\|w_h\|_{L^2(Q)}^2 &\leq \int_{\Omega} \sum_{j=1}^{N_t} \frac{h_{t,j}}{2} [W_{h_{x,j}}(x)]^2 dx + \int_{\Omega} \sum_{j=1}^{N_t-1} \frac{h_{t,j+1}}{2} [W_{h_{x,j}}(x)]^2 dx \\
&\leq \sum_{j=1}^{N_t} h_{t,j} \left\{ \langle f_0, \bar{z}_h^j \rangle_Q + \langle f_1, \partial_t \bar{z}_h^j \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle f_2, \partial_t \bar{z}_h^j \rangle_{L^2(\Omega \times \tau_\ell)} \right\} \\
&\quad + \sum_{j=1}^{N_t-1} h_{t,j+1} \left\{ \langle f_0, \bar{z}_h^j \rangle_Q + \langle f_1, \partial_t \bar{z}_h^j \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle f_2, \partial_t \bar{z}_h^j \rangle_{L^2(\Omega \times \tau_\ell)} \right\} \\
&\leq \sum_{j=1}^{N_t} h_{t,j} \left\{ \|f_0\|_{[H_0^1(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2 \|f_2\|_{L^2(Q)} \right\} \|\partial_t \bar{z}_h^j\|_{L^2(Q)} \\
&\quad + \sum_{j=1}^{N_t-1} h_{t,j+1} \left\{ \|f_0\|_{[H_0^1(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2 \|f_2\|_{L^2(Q)} \right\} \|\partial_t \bar{z}_h^j\|_{L^2(Q)} \\
&\leq 2T \left\{ \|f_0\|_{[H_0^1(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2 \|f_2\|_{L^2(Q)} \right\} \|w_h\|_{L^2(Q)},
\end{aligned}$$

and hence, the stability estimate is proven. \square

Theorem 4.2.30. For $f \in L^2(Q)$, there exists a unique solution $\tilde{u}_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ of the Galerkin-Petrov finite element discretisation (4.56), satisfying the stability estimate

$$\|\tilde{u}_h\|_{L^2(Q)} \leq 2T \|f\|_{[H_0^1(0,T;L^2(\Omega))]'} \leq \frac{4}{\pi} T^2 \|f\|_{L^2(Q)}.$$

Proof. Setting $f_0 = f \in L^2(Q)$, $f_1 = f_2 = 0$ in (4.57) and the Poincaré inequality with respect to time, see Lemma 3.4.5, give the assertion. \square

Remark 4.2.31. Note that there exists a unique solution of the Galerkin-Petrov finite element discretisation (4.56) in the situation of Lemma 4.2.29, i.e. for a right-hand side f weaker than $L^2(Q)$.

To derive an estimate for the $L^2(Q)$ error $\|u - \tilde{u}_h\|_{L^2(Q)}$, the space-time projection

$$Q_h^1 Q_{h_x}^1 v \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q),$$

is used, which is well-defined for a sufficiently smooth function $v \in H_{0;0}^{1,1}(Q)$, for details see Lemma 2.8.2.

Theorem 4.2.32. Let $u \in H_{0;0}^{1,1}(Q)$ be the unique solution of the variational formulation (4.41), satisfying $\partial_t u \in L^2(0,T;H_0^1(\Omega))$ and $\partial_{x_m} u \in H_0^1(0,T;L^2(\Omega))$, $m = 1, \dots, d$, and

$\Delta_x u \in H_{0,0}^1(0, T; L^2(\Omega))$. Then, the unique solution $\tilde{u}_h \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$ of the Galerkin-Petrov finite element discretisation (4.56) satisfies the error estimate

$$\begin{aligned} \|u - \tilde{u}_h\|_{L^2(Q)} &\leq \|u - Q_{h_t}^1 u\|_{L^2(Q)} + \|u - Q_{h_x}^1 u\|_{L^2(Q)} + c h_x h_t \|\partial_t \nabla_x u\|_{L^2(Q)} \\ &\quad + 2T \left\{ \|\Delta_x(u - Q_{h_t}^1 u)\|_{[H_{0,0}^1(0, T; L^2(\Omega))]'} + \|\partial_t(Q_{h_x}^1 u - u)\|_{L^2(Q)} + \frac{h_t^2}{12} \|\partial_t \Delta_x u\|_{L^2(Q)} \right\} \end{aligned}$$

with a constant $c > 0$.

Proof. Since the solution $u \in H_{0,0}^{1,1}(Q)$ fulfils the assumptions of Lemma 2.8.2, the space-time projection

$$Q_{h_t}^1 Q_{h_x}^1 u \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$$

is well-defined. When using the representation (4.54), the properties of $Q_{h_t}^1 Q_{h_x}^1$ as given in Lemma 2.8.2, and applying integration by parts, it follows for all $w_h \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$ that

$$\begin{aligned} a_h(\tilde{u}_h - Q_{h_t}^1 Q_{h_x}^1 u, w_h) &= a_h(\tilde{u}_h, w_h) - a_h(Q_{h_t}^1 Q_{h_x}^1 u, w_h) \\ &= a(u, w_h) - a_h(Q_{h_t}^1 Q_{h_x}^1 u, w_h) \\ &= a(u, w_h) - a(Q_{h_t}^1 Q_{h_x}^1 u, w_h) + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q_{h_t}^1 Q_{h_x}^1 u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega \times \tau_\ell)} \\ &= -\langle \partial_t u, \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w_h \rangle_{L^2(Q)} + \langle \partial_t Q_{h_t}^1 Q_{h_x}^1 u, \partial_t w_h \rangle_{L^2(Q)} \\ &\quad - \langle \nabla_x Q_{h_t}^1 Q_{h_x}^1 u, \nabla_x w_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q_{h_t}^1 Q_{h_x}^1 u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega \times \tau_\ell)} \end{aligned}$$

and further,

$$\begin{aligned} a_h(\tilde{u}_h - Q_{h_t}^1 Q_{h_x}^1 u, w_h) &= -\langle \partial_t u, \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w_h \rangle_{L^2(Q)} + \langle \partial_t Q_{h_t}^1 u, \partial_t w_h \rangle_{L^2(Q)} \\ &\quad - \langle \nabla_x Q_{h_t}^1 u, \nabla_x w_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q_{h_t}^1 u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega \times \tau_\ell)} \\ &= \langle \partial_t(Q_{h_t}^1 u - u), \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x(u - Q_{h_t}^1 u), \nabla_x w_h \rangle_{L^2(Q)} \\ &\quad + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q_{h_t}^1 u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega \times \tau_\ell)} \\ &= \langle \partial_t(Q_{h_t}^1 u - u), \partial_t w_h \rangle_{L^2(Q)} + \langle -\Delta_x(u - Q_{h_t}^1 u), w_h \rangle_{L^2(Q)} \\ &\quad - \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \Delta_x Q_{h_t}^1 u, \partial_t w_h \rangle_{L^2(\Omega \times \tau_\ell)}. \end{aligned}$$

In particular, one observes that $\tilde{u}_h - \mathcal{Q}_{h_t}^1 \mathcal{Q}_{h_x}^1 u$ is the unique solution of (4.57) in the case

$$f_0 = -\Delta_x(u - \mathcal{Q}_{h_t}^1 u), \quad f_1 = \partial_t(\mathcal{Q}_{h_x}^1 u - u), \quad f_2 = -\frac{1}{12} \partial_t \Delta_x \mathcal{Q}_{h_t}^1 u.$$

Therefore, the stability estimate (4.58) and the stability of $\mathcal{Q}_{h_t}^1$ in $H_0^1(0, T)$ give

$$\begin{aligned} & \|\tilde{u}_h - \mathcal{Q}_{h_t}^1 \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} \\ & \leq 2T \left\{ \|\Delta_x(u - \mathcal{Q}_{h_t}^1 u)\|_{[H_0^1(0, T; L^2(\Omega))]' } + \|\partial_t(\mathcal{Q}_{h_x}^1 u - u)\|_{L^2(Q)} + \frac{h_t^2}{12} \|\partial_t \Delta_x \mathcal{Q}_{h_t}^1 u\|_{L^2(Q)} \right\} \\ & \leq 2T \left\{ \|\Delta_x(u - \mathcal{Q}_{h_t}^1 u)\|_{[H_0^1(0, T; L^2(\Omega))]' } + \|\partial_t(\mathcal{Q}_{h_x}^1 u - u)\|_{L^2(Q)} + \frac{h_t^2}{12} \|\partial_t \Delta_x u\|_{L^2(Q)} \right\}. \end{aligned}$$

With the last estimate, the triangle inequality, and the error estimate of Lemma 2.8.2, it holds true that

$$\begin{aligned} \|u - \tilde{u}_h\|_{L^2(Q)} & \leq \|u - \mathcal{Q}_{h_t}^1 \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} + \|\tilde{u}_h - \mathcal{Q}_{h_t}^1 \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} \\ & \leq \|u - \mathcal{Q}_{h_t}^1 u\|_{L^2(Q)} + \|u - \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} + c h_x h_t \|\partial_t \nabla_x u\|_{L^2(Q)} \\ & \quad + 2T \left\{ \|\Delta_x(u - \mathcal{Q}_{h_t}^1 u)\|_{[H_0^1(0, T; L^2(\Omega))]' } + \|\partial_t(\mathcal{Q}_{h_x}^1 u - u)\|_{L^2(Q)} + \frac{h_t^2}{12} \|\partial_t \Delta_x u\|_{L^2(Q)} \right\} \end{aligned}$$

with a constant $c > 0$ and so, the assertion. \square

Corollary 4.2.33. *Let the assumptions of Theorem 4.2.32 be satisfied. If, in addition, the unique solution u of (4.41) is sufficiently smooth and the spatial H_0^1 projection $\mathcal{Q}_{h_x}^1$ fulfils the standard L^2 error estimate*

$$\|u - \mathcal{Q}_{h_x}^1 u\|_{L^2(Q)} \leq C h_x^2 \|u\|_{L^2(0, T; H^2(\Omega))}$$

with a constant $C > 0$, see (2.48), then, for the unique solution $\tilde{u}_h \in \mathcal{Q}_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ of the Galerkin-Petrov finite element discretisation (4.56), the error estimate

$$\begin{aligned} \|u - \tilde{u}_h\|_{L^2(Q)} & \leq c h_x^2 \left(\|u\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t u\|_{L^2(0, T; H^2(\Omega))} \right) \\ & \quad + c h_x h_t \|\partial_t \nabla_x u\|_{L^2(Q)} + c h_t^2 \left(\|\partial_{tt} u\|_{L^2(Q)} + \|\partial_{tt} \Delta_x u\|_{L^2(Q)} + \|\partial_t \Delta_x u\|_{L^2(Q)} \right) \quad (4.59) \end{aligned}$$

holds true with a constant $c > 0$.

Proof. By using the error estimates (2.50) for the H_0^1 projection $\mathcal{Q}_{h_t}^1$ and (2.48) for the H_0^1 projection $\mathcal{Q}_{h_x}^1$, it follows from Theorem 4.2.32 the asserted error estimate. \square

Corollary 4.2.34. *Let the assumption of Theorem 4.2.32 be satisfied. Furthermore, let $u \in H^{1+s}(Q) \cap H_{0;0}^{1,1}(Q)$ for some $s \in [0, 1]$ and let the $H_{0;0}^{1,1}(Q)$ projection Q_h^1 , given in (2.45), fulfil the standard error estimate*

$$\|u - Q_h^1 u\|_{L^2(Q)} \leq c h^{1+s} \|u\|_{H^{1+s}(Q)}$$

with a constant $c > 0$, see (2.46). Moreover, assume for $Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ the inverse inequality

$$\forall v_h \in Q_h^1(Q) \cap H_{0;0}^{1,1}(Q): \quad |v_h|_{H^1(Q)} \leq c_{\text{inv}} h^{-1} \|v_h\|_{L^2(Q)}$$

with a constant $c_{\text{inv}} > 0$ and $h = \max\{h_t, h_x\}$. Then, the estimate

$$|u - \tilde{u}_h|_{H^1(Q)} \leq C c_{\text{inv}} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}} h^{-1} \|\tilde{u}_h - u\|_{L^2(Q)}$$

is valid with a constant $C > 0$ independent of h . If, in addition, the assumption of Corollary 4.2.33 is fulfilled, then, the error estimate

$$|u - \tilde{u}_h|_{H^1(Q)} \leq \tilde{C} h \tag{4.60}$$

holds true with a constant $\tilde{C} > 0$.

Proof. It follows with the triangle inequality, standard error estimates for Q_h^1 and the inverse inequality in $Q_h^1(Q) \cap H_{0;0}^{1,1}(Q)$ that

$$\begin{aligned} |u - \tilde{u}_h|_{H^1(Q)} &\leq |u - Q_h^1 u|_{H^1(Q)} + |Q_h^1 u - \tilde{u}_h|_{H^1(Q)} \\ &\leq \tilde{C} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}} h^{-1} \|Q_h^1 u - \tilde{u}_h\|_{L^2(Q)} \\ &\leq \tilde{C} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}} h^{-1} \|Q_h^1 u - u\|_{L^2(Q)} + c_{\text{inv}} h^{-1} \|\tilde{u}_h - u\|_{L^2(Q)} \\ &\leq C c_{\text{inv}} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}} h^{-1} \|\tilde{u}_h - u\|_{L^2(Q)} \end{aligned}$$

with a constant $C > 0$ and hence, the assertion. \square

Remark 4.2.35. *The assumptions on the spatial H_0^1 projection $Q_{h_x}^1$ and on the $H_{0;0}^{1,1}(Q)$ projection Q_h^1 in Corollary 4.2.33 and Corollary 4.2.34 are fulfilled, if Ω is sufficiently regular. Thus, for less regular Ω , one expects reduced orders for the error estimates given in Corollary 4.2.33 and Corollary 4.2.34.*

As a numerical example for the Galerkin-Petrov finite element method (4.56), consider the one-dimensional spatial domain $\Omega = (0, 1)$ and $T = 10$, i.e. the rectangular space-time domain

$$Q = \Omega \times (0, T) = (0, 1) \times (0, 10).$$

The discretisation is done with respect to nonuniform meshes as shown in Figure 4.1 of Section 4.1, where a uniform refinement strategy is applied. Note that these meshes do not fulfil the CFL condition (4.52). As exact solutions, the functions

$$u_1(x, t) = \sin(\pi x) \sin^2\left(\frac{5}{4}\pi t\right), \quad (x, t) \in Q,$$

$$u_2(x, t) = \sin(\pi x) t^2 (10 - t)^{3/4}, \quad (x, t) \in Q,$$

are chosen. The appearing integrals to compute the related right-hand side in (4.56) are calculated by using high-order quadrature rules and the number of the degrees of freedom is denoted by

$$\text{dof} = \dim Q_h^1(Q) \cap H_{0,0}^{1,1}(Q) = \dim Q_h^1(Q) \cap H_{0,0}^{1,1}(Q).$$

The numerical results for the smooth solution u_1 are given in Table 4.12, where unconditional stability and quadratic convergence in $\|\cdot\|_{L^2(Q)}$, as predicted by the error estimate (4.59), are observed. Moreover, linear convergence is seen, when measuring the error in $|\cdot|_{H^1(Q)}$, which confirms the error estimate (4.60).

For the singular solution u_2 , the related results are given in Table 4.13, where reduced orders of convergence in $\|\cdot\|_{L^2(Q)}$ and in $|\cdot|_{H^1(Q)}$ are observed. These convergence rates correspond to the reduced Sobolev regularity

$$u_2 \in H^{5/4-\varepsilon}(Q), \quad \varepsilon > 0.$$

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_1 - \tilde{u}_{1,h}\ _{L^2(Q)}$	eoc	$ u_1 - \tilde{u}_{1,h} _{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	3.579e+00	-	1.289e+01	-
132	0.18750	0.03125	1.87500	0.31250	1.975e+00	0.86	9.849e+00	0.39
552	0.09375	0.01562	0.93750	0.15625	9.213e-01	1.10	6.534e+00	0.59
2256	0.04688	0.00781	0.46875	0.07812	6.829e-01	0.43	5.210e+00	0.33
9120	0.02344	0.00391	0.23438	0.03906	2.466e-01	1.47	2.848e+00	0.87
36672	0.01172	0.00195	0.11719	0.01953	7.029e-02	1.81	1.435e+00	0.99
147072	0.00586	0.00098	0.05859	0.00977	1.819e-02	1.95	7.159e-01	1.00
589056	0.00293	0.00049	0.02930	0.00488	4.588e-03	1.99	3.576e-01	1.00
2357760	0.00146	0.00024	0.01465	0.00244	1.149e-03	2.00	1.788e-01	1.00
9434112	0.00073	0.00012	0.00732	0.00122	2.875e-04	2.00	8.938e-02	1.00
37742592	0.00037	0.00006	0.00366	0.00061	7.189e-05	2.00	4.469e-02	1.00

Table 4.12: Numerical results of the Galerkin-Petrov finite element discretisation (4.56) with nonuniform meshes for $Q = (0, 1) \times (0, 10)$ and for the function u_1 .

Remark 4.2.36. *The Galerkin-Petrov finite element method (4.56) fulfils a kind of conservation of the total energy*

$$E(t) := \frac{1}{2} \|\partial_t u(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla_x u(\cdot, t)\|_{L^2(\Omega)}^2, \quad t \in [0, T],$$

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_2 - \tilde{u}_{2,h}\ _{L^2(Q)}$	eoc	$ u_2 - \tilde{u}_{2,h} _{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	7.836e+01	-	3.173e+02	-
132	0.18750	0.03125	1.87500	0.31250	2.166e+01	1.86	1.191e+02	1.41
552	0.09375	0.01562	0.93750	0.15625	5.487e+00	1.98	5.225e+01	1.19
2256	0.04688	0.00781	0.46875	0.07812	1.777e+00	1.63	2.696e+01	0.95
9120	0.02344	0.00391	0.23438	0.03906	6.476e-01	1.46	1.593e+01	0.76
36672	0.01172	0.00195	0.11719	0.01953	3.001e-01	1.11	1.076e+01	0.57
147072	0.00586	0.00098	0.05859	0.00977	1.393e-01	1.11	8.077e+00	0.41
589056	0.00293	0.00049	0.02930	0.00488	6.156e-02	1.18	6.452e+00	0.32
2357760	0.00146	0.00024	0.01465	0.00244	2.650e-02	1.22	5.308e+00	0.28
9434112	0.00073	0.00012	0.00732	0.00122	1.126e-02	1.23	4.423e+00	0.26
37742592	0.00037	0.00006	0.00366	0.00061	4.758e-03	1.24	3.704e+00	0.26

Table 4.13: Numerical results of the Galerkin-Petrov finite element discretisation (4.56) with nonuniform meshes for $Q = (0, 1) \times (0, 10)$ and for the function u_2 .

since the Galerkin-Petrov finite element method (4.56) is related to the Newmark Galerkin method (4.6), (4.7), see Remark 4.2.27. As illustration, consider a solution of the homogeneous wave equation, i.e.

$$u_3(x, t) = (\cos(\pi t) + \sin(\pi t)) \sin(\pi x) \quad \text{for } (x, t) \in Q$$

with the total energy

$$E(t) = \frac{\pi^2}{2} \quad \text{for } t \in [0, 10],$$

where the space-time cylinder is given by

$$Q = (0, 1) \times (0, 10).$$

Here, the initial condition

$$u_3(x, 0) = u_0(x) = \sin(\pi x), \quad x \in \Omega,$$

is treated via homogenisation, while the initial condition

$$\partial_t u_3(x, 0) = v_0(x) = \pi \sin(\pi x), \quad x \in \Omega,$$

is incorporated in a weak sense. For the solution u_3 , the discrete total energy

$$E_h(t) := \frac{1}{2} \|\partial_t \tilde{u}_h(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla_x \tilde{u}_h(\cdot, t)\|_{L^2(\Omega)}^2, \quad t \in [0, T],$$

is computed. In Figure 4.3 the difference

$$E_h(t) - E(t) = \overline{E}_h(t) - \frac{\pi^2}{2} \quad \text{for } t \in [0, 10]$$

is plotted pointwise for the refinement level with uniform mesh sizes

$$h_t = \frac{10}{6 \cdot 2^{10}} \approx 0.00162760$$

and

$$h_x = \frac{1}{6 \cdot 2^{10}} \approx 0.00016276.$$

Note that $\partial_t \tilde{u}_h$ is piecewise constant in time. Probably due to the used space-time approximation, some oscillations within the finite element accuracy are observed, but no energy loss occurs, see also Figure 4.2.

For a comparison with the Newmark Galerkin method (4.6), (4.7) of Section 4.1, the errors in the space-time norms $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ are given in Table 4.14, where the convergence rates are as expected, when the nonuniform meshes as in Figure 4.1 are used.

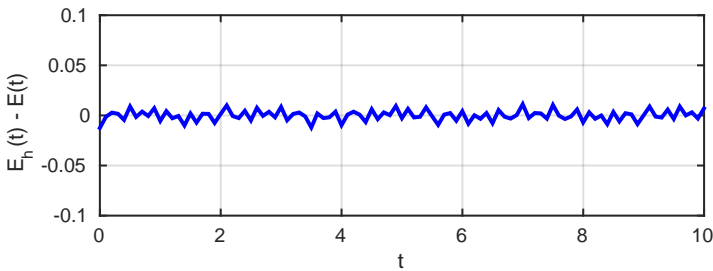


Figure 4.3: Difference of the total energy $E(t) = \frac{\pi^2}{2}$ and $E_h(t)$ of the Galerkin-Petrov finite element discretisation (4.56) with a uniform mesh for $Q = (0, 1) \times (0, 10)$ and for the function u_3 .

Remark 4.2.37. A comparison of Table 4.12 with Table 4.1 and Table 4.14 with Table 4.2 shows that the Newmark Galerkin method (4.6), (4.7) of Section 4.1 and the Galerkin-Petrov finite element discretisation give similar results, provided the right-hand side $f \in L^2(Q)$ has no singularity with respect to time. Note that the Newmark Galerkin method (4.6), (4.7) is not applicable to the solution u_2 .

Remark 4.2.38. The to (4.56) related inf-sup constant

$$\inf_{0 \neq u_h \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)} \sup_{0 \neq w_h \in Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)} \frac{a_h(u_h, w_h)}{|u_h|_{H^1(Q)} |w_h|_{H^1(Q)}} =: c_S(h) \quad (4.61)$$

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\ u_3 - \tilde{u}_{3,h}\ _{L^2(Q)}$	eoc	$ u_3 - \tilde{u}_{3,h} _{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	2.503e+00	-	1.051e+01	-
132	0.18750	0.03125	1.87500	0.31250	2.496e+00	0.00	9.798e+00	0.09
552	0.09375	0.01562	0.93750	0.15625	2.241e+00	0.15	9.774e+00	0.00
2256	0.04688	0.00781	0.46875	0.07812	2.580e+00	-0.20	1.152e+01	-0.23
9120	0.02344	0.00391	0.23438	0.03906	1.082e+00	1.24	4.846e+00	1.24
36672	0.01172	0.00195	0.11719	0.01953	3.013e-01	1.84	1.445e+00	1.74
147072	0.00586	0.00098	0.05859	0.00977	7.697e-02	1.97	4.606e-01	1.65
589056	0.00293	0.00049	0.02930	0.00488	1.934e-02	1.99	1.804e-01	1.35
2357760	0.00146	0.00024	0.01465	0.00244	4.841e-03	2.00	8.268e-02	1.12
9434112	0.00073	0.00012	0.00732	0.00122	1.211e-03	2.00	4.034e-02	1.03
37742592	0.00037	0.00006	0.00366	0.00061	3.027e-04	2.00	2.005e-02	1.01

Table 4.14: Numerical results of the Galerkin-Petrov finite element discretisation (4.56) with nonuniform meshes for $Q = (0, 1) \times (0, 10)$ and for the function u_3 .

seems to depend linearly on the mesh size $h = \max\{h_t, h_x\}$. The inf-sup constant $c_S(h)$ is given as

$$c_S(h) = \sqrt{\lambda_{\min}},$$

where λ_{\min} is the minimal eigenvalue of the generalised eigenvalue problem [84, Subsection 3.6.6, page 124]

$$\tilde{K}_h^\top A_{h,0}^{-1} \tilde{K}_h \underline{u} = \lambda A_{h,0} \underline{u}$$

with the matrices

$$\begin{aligned} \tilde{K}_h[k, i] &= a_h(\chi_i, \eta_k), \\ A_{h,0}[k, i] &= \langle \partial_t \chi_i, \partial_t \chi_k \rangle_{L^2(Q)} + \langle \nabla_x \chi_i, \nabla_x \chi_k \rangle_{L^2(Q)} \end{aligned}$$

and

$$A_{h,0}[k, i] = \langle \partial_t \eta_i, \partial_t \eta_k \rangle_{L^2(Q)} + \langle \nabla_x \eta_i, \nabla_x \eta_k \rangle_{L^2(Q)}$$

for $i, k = 1, \dots, \text{dof}$, where χ_i are the nodal basis functions of $Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$ and η_k are the nodal basis functions of $Q_h^1(Q) \cap H_{0,0}^{1,1}(Q)$, i.e.

$$Q_h^1(Q) \cap H_{0,0}^{1,1}(Q) = \text{span}\{\chi_i\}_{i=1}^{\text{dof}}$$

and

$$Q_h^1(Q) \cap H_{0,0}^{1,1}(Q) = \text{span}\{\eta_k\}_{k=1}^{\text{dof}}.$$

As illustration, consider the rectangular space-time domain

$$Q = \Omega \times (0, T) = (0, 1) \times (0, 2)$$

with uniform discretisations with mesh sizes $h_x = h_{x,\max} = h_{x,\min}$ and $h_t = h_{t,\max} = h_{t,\min}$, where a uniform refinement strategy is applied. The inf-sup constant $c_S(h)$ of (4.61) is given in Table 4.15, where a linear dependency on the mesh size h is observed.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$c_S(h)$
30	0.16667	0.16667	0.33333	0.33333	0.13867820
132	0.08333	0.08333	0.16667	0.16667	0.07504415
552	0.04167	0.04167	0.08333	0.08333	0.03971295
2256	0.02083	0.02083	0.04167	0.04167	0.02028705
9120	0.01042	0.01042	0.02083	0.02083	0.01012171
36672	0.00521	0.00521	0.01042	0.01042	0.00510211

Table 4.15: Optimal discrete inf-sup constant $c_S(h)$ of (4.61) for the perturbed bilinear form (4.55) with a uniform temporal mesh size h_t and a uniform spatial mesh size h_x for the space-time cylinder $Q = (0, 1) \times (0, 2)$.

In the remainder of this section, the two-dimensional spatial L-shaped domain

$$\Omega := (-1, 1)^2 \setminus ([0, 1] \times [-1, 0]) \subset \mathbb{R}^2 \quad (4.62)$$

and the terminal time $T = \frac{1}{4}$ are considered for the solutions

$$u_4(x_1, x_2, t) = \sin(\pi x_1) \sin(\pi x_2) \sin^2\left(\frac{5}{4}\pi t\right), \quad (x_1, x_2, t) \in Q,$$

$$u_5(x_1, x_2, t) = r(x_1, x_2)^{2/3} \cdot \sin\left(\frac{2}{3}\arg(x_1, x_2)\right) \cdot \sin(\pi t), \quad (x_1, x_2, t) \in Q,$$

where

$$(r(x_1, x_2), \arg(x_1, x_2)) \subset [0, \infty) \times [0, 2\pi)$$

are polar coordinates located in $\underline{Q} \in \mathbb{R}^2$ with the radial coordinate $r(x_1, x_2)$ and the angular coordinate $\arg(x_1, x_2)$. For the solution u_5 , the inhomogeneous Dirichlet boundary condition

$$u_5(x_1, x_2, t) = g(x_1, x_2, t), \quad (x_1, x_2, t) \in \Sigma,$$

is treated via homogenisation, and the second initial condition

$$\partial_t u_5(x_1, x_2, 0) = v_0(x_1, x_2), \quad (x_1, x_2) \in \Omega,$$

is incorporated in a weak sense. The spatial domain Ω is decomposed into uniform triangles with uniform mesh size $h_x = h_{x,\max} = h_{x,\min}$ as given in Figure 4.4 for level 0. The temporal domain

$$(0, 1/4) = (0, T)$$

is decomposed into nonuniform elements with the nodes

$$t_0 = 0.0, \quad t_1 = 0.125, \quad t_2 = 0.1875, \quad t_3 = 0.25 = T.$$

The appearing integrals to compute the related right-hand side in (4.56) and the vector, related to the second initial condition

$$\partial_t u_5(\cdot, \cdot, 0) = v_0,$$

are calculated by using high-order quadrature rules. The numerical results for the smooth solution u_4 , when a uniform refinement strategy is applied as in Figure 4.4, are given in Table 4.16, where unconditional stability and quadratic convergence in $\|\cdot\|_{L^2(Q)}$, as predicted by the error estimate (4.59), are observed. Moreover, linear convergence is seen, when measuring the error in $|\cdot|_{H^1(Q)}$, which confirms the error estimate (4.60). For the singular solution u_5 , the related results are given in Table 4.17, where a reduced order of convergence in $\|\cdot\|_{L^2(Q)}$ and in $|\cdot|_{H^1(Q)}$ are observed. However, for a fixed uniform time mesh with

$$h_t = \frac{1}{4000},$$

an adaptive meshing for the spatial domain Ω is considered with respect to the time element $\tau_{\hat{\ell}} = (t_{\hat{\ell}-1}, t_{\hat{\ell}})$ with

$$\hat{\ell} = \min \left(\arg \max_{\ell=1, \dots, N_t} \|u_5 - \tilde{u}_{5,h}\|_{L^2(\Omega \times \tau_{\ell})} \right),$$

i.e. the spatial decomposition

$$\bar{\Omega} = \bigcup_{i=1}^{N_x} \bar{\omega}_i \subset \mathbb{R}^2$$

is adaptively refined with respect to the local error indicator

$$\|u_5 - \tilde{u}_{5,h}\|_{L^2(\omega_i \times \tau_{\hat{\ell}})}, \quad i = 1, \dots, N_x,$$

where Dörfler marking [41] with parameter $\theta = 0.2$ is used.

This adaptive scheme seems to lead to optimal convergence rates in $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ with respect to the spatial variable, see Figure 4.5, Figure 4.6, and see Figure 4.7 for the meshes produced by the adaptive scheme. Since the stabilised method (4.56) is unconditionally stable, the usage of spatially adaptive refinement strategies is possible, which is confirmed by this example. Note that without the stabilisation such spatial refinement, as in Figure 4.7, is only possible for a sufficiently small temporal mesh size h_t due to the CFL condition (4.52). However, adaptive refinement strategies are left for future work. See also [115], for the approach of spatially graded meshes.

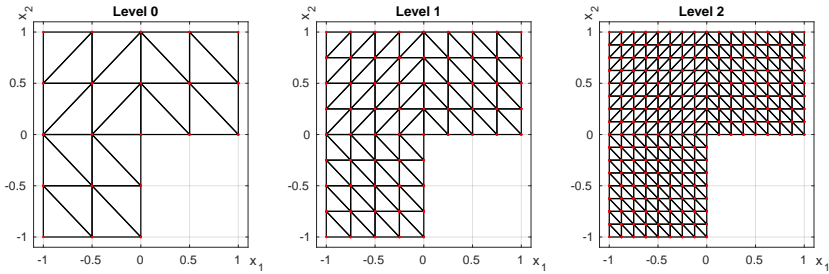


Figure 4.4: Uniform refinement strategy: Starting mesh, the meshes after one and two uniform refinement steps.

dof	$h_{x,max}$	$h_{x,min}$	$h_{t,max}$	$h_{t,min}$	$\ u_4 - \tilde{u}_{4,h}\ _{L^2(Q)}$	eoc	$ u_4 - \tilde{u}_{4,h} _{H^1(Q)}$	eoc
15	0.35355	0.35355	0.12500	0.06250	5.400e-02	-	7.381e-01	-
198	0.17678	0.17678	0.06250	0.03125	1.160e-02	1.79	3.280e-01	0.94
1932	0.08839	0.08839	0.03125	0.01562	2.627e-03	1.96	1.565e-01	0.97
16920	0.04419	0.04419	0.01562	0.00781	6.379e-04	1.96	7.719e-02	0.98
141360	0.02210	0.02210	0.00781	0.00391	1.582e-04	1.97	3.846e-02	0.98
1155168	0.01105	0.01105	0.00391	0.00195	3.948e-05	1.98	1.921e-02	0.99
9339072	0.00552	0.00552	0.00195	0.00098	9.865e-06	1.99	9.604e-03	1.00
75104640	0.00276	0.00276	0.00098	0.00049	2.466e-06	2.00	4.802e-03	1.00
602407680	0.00138	0.00138	0.00049	0.00024	6.165e-07	2.00	2.401e-03	1.00

Table 4.16: Numerical results of the Galerkin-Petrov finite element discretisation (4.56) for the L-shape (4.62) and $T = \frac{1}{4}$ for the function u_4 for a uniform refinement strategy.

dof	$h_{x,max}$	$h_{x,min}$	$h_{t,max}$	$h_{t,min}$	$\ u_5 - \tilde{u}_{5,h}\ _{L^2(Q)}$	eoc	$ u_5 - \tilde{u}_{5,h} _{H^1(Q)}$	eoc
15	0.35355	0.35355	0.12500	0.06250	4.897e-03	-	8.597e-02	-
198	0.17678	0.17678	0.06250	0.03125	1.729e-03	1.21	5.026e-02	0.62
1932	0.08839	0.08839	0.03125	0.01562	6.675e-04	1.25	3.016e-02	0.67
16920	0.04419	0.04419	0.01562	0.00781	2.737e-04	1.23	1.844e-02	0.68
141360	0.02210	0.02210	0.00781	0.00391	1.159e-04	1.21	1.140e-02	0.68
1155168	0.01105	0.01105	0.00391	0.00195	4.888e-05	1.23	7.106e-03	0.68
9339072	0.00552	0.00552	0.00195	0.00098	2.034e-05	1.26	4.448e-03	0.67
75104640	0.00276	0.00276	0.00098	0.00049	8.352e-06	1.28	2.792e-03	0.67
602407680	0.00138	0.00138	0.00049	0.00024	3.395e-06	1.30	1.755e-03	0.67

Table 4.17: Numerical results of the Galerkin-Petrov finite element discretisation (4.56) for the L-shape (4.62) and $T = \frac{1}{4}$ for the function u_5 for a uniform refinement strategy.

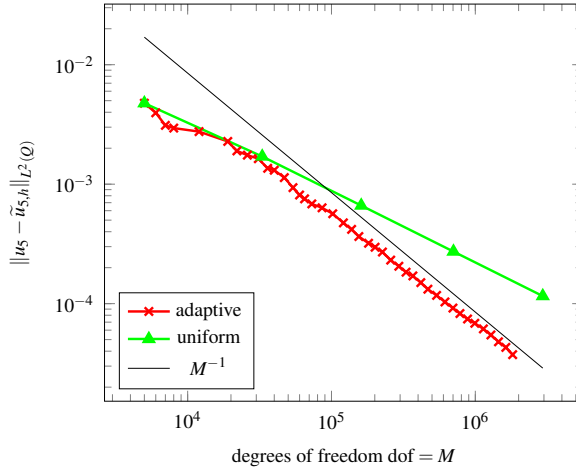


Figure 4.5: $L^2(Q)$ error of u_5 for (4.56) for the L-shape (4.62) and $T = \frac{1}{4}$ for $h_t = \frac{1}{4000}$ and for a spatially adaptive refinement strategy with the meshes of Figure 4.7.

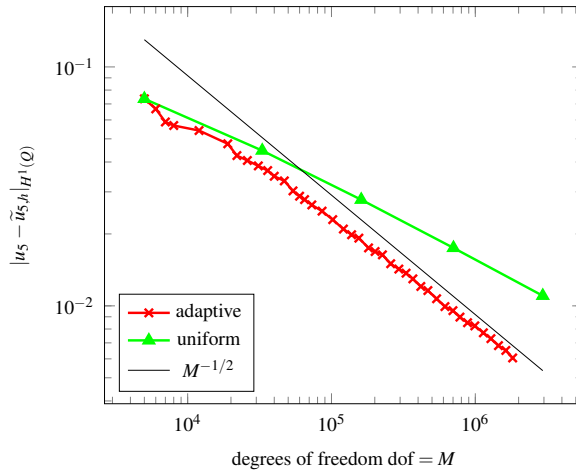


Figure 4.6: $H^1(Q)$ error of u_5 for (4.56) for the L-shape (4.62) and $T = \frac{1}{4}$ for $h_t = \frac{1}{4000}$ and for a spatially adaptive refinement strategy with the meshes of Figure 4.7.

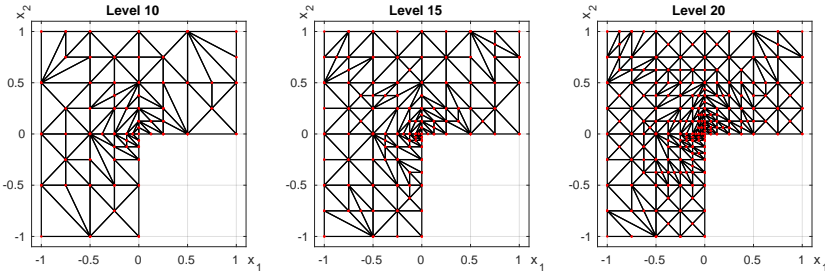


Figure 4.7: Spatially adaptive refinement strategy for the function u_5 .

Outlook for Possible Extensions of Section 4.2

Since the constructions of the methods and the proofs in this section mainly rely on the treatment of the second-order temporal differential operator $\partial_{tt} + \mu$ with a parameter $\mu > 0$, the results of this section for the model problem (4.11) can be transferred to hyperbolic equations for vector fields, e.g., the Maxwell's equations, as well as to a more general hyperbolic equation (4.2) for scalar functions with inhomogeneous initial and boundary conditions under certain assumptions of the involved functions and function spaces, where details are left for future considerations.

4.3 Space-Time Variational Formulation in a Strong Sense

In this section, the wave equation (4.11) is considered in $L^2(Q)$. Therefore, with the notations of Section 2.1, define

$$H^1(Q; \square) := \{v \in H^1(Q) : \square_Q v \in L^2(Q)\}$$

with the inner product

$$\langle u, v \rangle_{H^1(Q; \square)} := \langle u, v \rangle_{L^2(Q)} + \langle \partial_t u, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} + \langle \square_Q u, \square_Q v \rangle_{L^2(Q)}$$

and the induced norm

$$\|u\|_{H^1(Q; \square)} := \sqrt{\langle u, u \rangle_{H^1(Q; \square)}} = \sqrt{\|u\|_{H^1(Q)}^2 + \|\square_Q u\|_{L^2(Q)}^2}.$$

For a function $v \in H^1(Q; \square)$, the condition $\square_Q v \in L^2(Q)$ involves that there exists a function $f_v \in L^2(Q)$ with

$$\square_Q T_v(\varphi) = \int_Q f_v(x, t) \varphi(x, t) dx dt \quad \text{for all } \varphi \in \mathcal{D}(Q),$$

where

$$T_v : \mathcal{D}(Q) \rightarrow \mathbb{R}, \quad T_v(\varphi) = \int_Q v(x, t) \varphi(x, t) dx dt \quad \text{for } \varphi \in \mathcal{D}(Q),$$

is the distribution related to $v \in L^2(Q)$. Note that the function $f_v \in L^2(Q)$ is unique, because $C_0^\infty(Q)$ is dense in $L^2(Q)$.

Clearly, $(H^1(Q; \square), \langle \cdot, \cdot \rangle_{H^1(Q; \square)})$ is an inner product space.

Lemma 4.3.1. *The inner product space $(H^1(Q; \square), \langle \cdot, \cdot \rangle_{H^1(Q; \square)})$ is a Hilbert space.*

Proof. Consider a Cauchy sequence $(v_n)_{n \in \mathbb{N}} \subset H^1(Q; \square)$. Hence, $(v_n)_{n \in \mathbb{N}} \subset H^1(Q)$ is also a Cauchy sequence in $H^1(Q)$ and $(\square_Q v_n)_{n \in \mathbb{N}} \subset L^2(Q)$ is also a Cauchy sequence in $L^2(Q)$. So, there exist $v \in H^1(Q)$ with $\|v_n - v\|_{H^1(Q)} \rightarrow 0$ as $n \rightarrow \infty$ and $f \in L^2(Q)$ with $\|\square_Q v_n - f\|_{L^2(Q)} \rightarrow 0$ as $n \rightarrow \infty$. Let $T_v : \mathcal{D}(Q) \rightarrow \mathbb{R}$, $T_{v_n} : \mathcal{D}(Q) \rightarrow \mathbb{R}$, with

$$\forall \varphi \in \mathcal{D}(Q): \quad T_v(\varphi) = \int_Q v(x, t) \varphi(x, t) dx dt, \quad T_{v_n}(\varphi) = \int_Q v_n(x, t) \varphi(x, t) dx dt,$$

be the distributions related to the limit $v \in H^1(Q)$ and to $v_n \in H^1(Q; \square)$ for every $n \in \mathbb{N}$. It follows for all $\varphi \in \mathcal{D}(Q)$ that

$$\begin{aligned} \square_Q T_v(\varphi) &= T_v(\square\varphi) = \int_Q v(x,t) \square\varphi(x,t) dx dt = \lim_{n \rightarrow \infty} \int_Q v_n(x,t) \square\varphi(x,t) dx dt \\ &= \lim_{n \rightarrow \infty} T_{v_n}(\square\varphi) = \lim_{n \rightarrow \infty} \square_Q T_{v_n}(\varphi) = \lim_{n \rightarrow \infty} \int_Q \square_Q v_n(x,t) \varphi(x,t) dx dt \\ &= \int_Q f(x,t) \varphi(x,t) dx dt \end{aligned}$$

and so, it holds true that $\square_Q v = f \in L^2(Q)$. Hence, $v \in H^1(Q; \square)$. \square

Set

$$H_0^2(0, T; L^2(\Omega)) := \{v \in H^2(0, T; L^2(\Omega)) : v(\cdot, 0) = \partial_t v(\cdot, 0) = 0 \text{ in } L^2(\Omega)\}$$

and

$$H_0^1(\Omega; \Delta) := \{w \in H_0^1(\Omega) : \Delta_x w \in L^2(\Omega)\},$$

where Δ_x is the distributional Laplace operator for distributions $\mathcal{D}'(\Omega)$. Furthermore, define the subspace

$$\tilde{H}_0^2(Q; \square) := L^2(0, T; H_0^1(\Omega; \Delta)) \cap H_0^1(0, T; H_0^1(\Omega)) \cap H_0^2(0, T; L^2(\Omega)) \subset H^1(Q; \square)$$

and by completion the Hilbert space

$$\tilde{H}_0^1(Q; \square) := \overline{\tilde{H}_0^2(Q; \square)}^{\|\cdot\|_{H^1(Q; \square)}} \subset H^1(Q; \square)$$

endowed with the inner product $\langle \cdot, \cdot \rangle_{H^1(Q; \square)}$. That means,

$$\tilde{H}_0^1(Q; \square) = \{v \in H^1(Q; \square) : \exists (v_n)_{n \in \mathbb{N}} \subset \tilde{H}_0^2(Q; \square) \text{ with } \|v_n - v\|_{H^1(Q; \square)} \rightarrow 0\}.$$

Lemma 4.3.2. *It holds true that*

$$H_0^1(Q; \square) \subset H_{0;0}^{1,1}(Q),$$

i.e. for $u \in \tilde{H}_0^1(Q; \square)$, it follows that

$$\left\| \gamma_{0,x}^{\text{int}} u \right\|_{L^2(\Sigma)} = \|u(\cdot, 0)\|_{L^2(\Omega)} = 0,$$

where $\gamma_{0,x}^{\text{int}} : L^2(0, T; H^1(\Omega)) \rightarrow L^2(\Sigma)$ is the extended trace operator (2.15).

Proof. Let $u \in \tilde{H}_0^1(Q; \square)$ be fixed. Because of the completion, there exists an approximating sequence $(u_n)_{n \in \mathbb{N}} \subset \tilde{H}_0^2(Q; \square)$ with $\|u - u_n\|_{H^1(Q; \square)} \rightarrow 0$ as $n \rightarrow \infty$. With the constant $C_{\text{Tr}} > 0$ from the extended trace operator, see (2.15), it holds true that

$$\left\| \gamma_{0,x}^{\text{int}}(u - u_n) \right\|_{L^2(\Sigma)} \leq C_{\text{Tr}} \|u - u_n\|_{L^2(0,T;H^1(\Omega))} \leq C_{\text{Tr}} \|u - u_n\|_{H^1(Q; \square)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and hence,

$$\left\| \gamma_{0,x}^{\text{int}} u \right\|_{L^2(\Sigma)} = \lim_{n \rightarrow \infty} \underbrace{\left\| \gamma_{0,x}^{\text{int}} u_n \right\|_{L^2(\Sigma)}}_{=0} = 0.$$

With the continuous embedding (2.12), it follows that

$$\|u(\cdot, 0) - u_n(\cdot, 0)\|_{L^2(\Omega)} \leq C \|u - u_n\|_{H^1(Q)} \leq C \|u - u_n\|_{H^1(Q; \square)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and so,

$$\|u(\cdot, 0)\|_{L^2(\Omega)} = \lim_{n \rightarrow \infty} \underbrace{\|u_n(\cdot, 0)\|_{L^2(\Omega)}}_{=0} = 0.$$

Hence, the assertion is proven. \square

Lemma 4.3.3. For $u \in \tilde{H}_0^1(Q; \square)$, the estimate

$$\|u\|_{H^1(Q)} \leq \sqrt{1 + \frac{4T^2}{\pi^2}} \|u\|_{H^1(Q)}$$

holds true.

Proof. Let $u \in \tilde{H}_0^1(Q; \square)$ be fixed. Because of the completion, there exists an approximating sequence $(u_n)_{n \in \mathbb{N}} \subset \tilde{H}_0^2(Q; \square)$ with $\|u - u_n\|_{H^1(Q; \square)} \rightarrow 0$ as $n \rightarrow \infty$. Hence, it holds true that $u_n(x, \cdot) \in H_0^1(0, T)$ for almost all $x \in \Omega$. By using the Poincaré inequality with respect to time, see Lemma 3.4.5, it follows that

$$\|u_n\|_{L^2(Q)}^2 = \int_{\Omega} \int_0^T u_n(x, t)^2 dt dx \leq \frac{4T^2}{\pi^2} \int_{\Omega} \int_0^T \partial_t u_n(x, t)^2 dt dx = \frac{4T^2}{\pi^2} \|\partial_t u_n\|_{L^2(Q)}^2$$

and with

$$\begin{aligned} \|u_n\|_{H^1(Q)}^2 &= \|u_n\|_{L^2(Q)}^2 + \|\partial_t u_n\|_{L^2(Q)}^2 + \|\nabla_x u_n\|_{L^2(Q)}^2 \\ &\leq \left(1 + \frac{4T^2}{\pi^2}\right) \|\partial_t u_n\|_{L^2(Q)}^2 + \|\nabla_x u_n\|_{L^2(Q)}^2, \end{aligned}$$

the assertion by completion. \square

Lemma 4.3.4. For $u \in \tilde{H}_0^1(Q; \square)$, it holds true that

$$|u|_{H^1(Q)}^2 = 2 \int_0^T \int_0^t \langle \partial_t u(\cdot, s), \square_Q u(\cdot, s) \rangle_{L^2(\Omega)} ds dt \quad (4.63)$$

and hence,

$$|u|_{H^1(Q)} \leq 2T \|\square_Q u\|_{L^2(Q)}.$$

Proof. Let $u \in \tilde{H}_0^1(Q; \square)$ be fixed. Because of the completion, there exists an approximating sequence $(u_n)_{n \in \mathbb{N}} \subset \tilde{H}_0^2(Q; \square)$ with $\|u - u_n\|_{H^1(Q; \square)} \rightarrow 0$ as $n \rightarrow \infty$. With integration by parts for almost all $s \in (0, T)$, it holds true that

$$\begin{aligned} \left\langle \begin{pmatrix} \nabla_x u_n(\cdot, s) \\ \partial_t u_n(\cdot, s) \end{pmatrix}, \begin{pmatrix} \nabla_x \partial_t u_n(\cdot, s) \\ \Delta_x u_n(\cdot, s) \end{pmatrix} \right\rangle_{L^2(\Omega)} &= \langle \nabla_x u_n(\cdot, s), \nabla_x \partial_t u_n(\cdot, s) \rangle_{L^2(\Omega)} \\ &\quad + \langle \partial_t u_n(\cdot, s), \Delta_x u_n(\cdot, s) \rangle_{L^2(\Omega)} \\ &= \langle \nabla_x u_n(\cdot, s), \nabla_x \partial_t u_n(\cdot, s) \rangle_{L^2(\Omega)} \\ &\quad - \langle \nabla_x \partial_t u_n(\cdot, s), \nabla_x u_n(\cdot, s) \rangle_{L^2(\Omega)} \\ &\quad + \underbrace{\langle \gamma_0^{\text{int}} \partial_t u_n(\cdot, s), \partial_n u_n(\cdot, s) \rangle_{L^2(\partial\Omega)}}_{=0} \\ &= 0, \end{aligned}$$

where γ_0^{int} is the trace operator and ∂_n is the normal derivative, see Section 2.5.

The function u_n satisfies $\partial_t u_n(\cdot, 0) = 0$ in $L^2(\Omega)$ and $\nabla_x u_n(\cdot, 0) = \underline{0}$ in $[L^2(\Omega)]^d$ and so, it follows that

$$\begin{aligned} |u_n|_{H^1(Q)}^2 &= \|\partial_t u_n\|_{L^2(Q)}^2 + \|\nabla_x u_n\|_{L^2(Q)}^2 \\ &= \int_0^T \int_0^t \partial_s \left[\langle \partial_t u_n(\cdot, s), \partial_t u_n(\cdot, s) \rangle_{L^2(\Omega)} + \sum_{i=1}^d \langle \partial_{x_i} u_n(\cdot, s), \partial_{x_i} u_n(\cdot, s) \rangle_{L^2(\Omega)} \right] ds dt \\ &= 2 \int_0^T \int_0^t \left[\langle \partial_t u_n(\cdot, s), \partial_{tt} u_n(\cdot, s) \rangle_{L^2(\Omega)} + \sum_{i=1}^d \langle \partial_{x_i} u_n(\cdot, s), \partial_{x_i t} u_n(\cdot, s) \rangle_{L^2(\Omega)} \right] ds dt \end{aligned}$$

and together with the help of the last relation,

$$\begin{aligned}
 |u_n|_{\tilde{H}^1(Q)}^2 &= 2 \int_0^T \int_0^t \left\langle \begin{pmatrix} \nabla_x u_n(\cdot, s) \\ \partial_t u_n(\cdot, s) \end{pmatrix}, \begin{pmatrix} \nabla_x \partial_t u_n(\cdot, s) \\ \partial_{tt} u_n(\cdot, s) \end{pmatrix} \right\rangle_{L^2(\Omega)} \, ds dt \\
 &\quad - 2 \int_0^T \int_0^t \underbrace{\left\langle \begin{pmatrix} \nabla_x u_n(\cdot, s) \\ \partial_t u_n(\cdot, s) \end{pmatrix}, \begin{pmatrix} \nabla_x \partial_t u_n(\cdot, s) \\ \Delta_x u_n(\cdot, s) \end{pmatrix} \right\rangle_{L^2(\Omega)}}_{=0} \, ds dt \\
 &= 2 \int_0^T \int_0^t \left\langle \begin{pmatrix} \nabla_x u_n(\cdot, s) \\ \partial_t u_n(\cdot, s) \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_{tt} u_n(\cdot, s) - \Delta_x u_n(\cdot, s) \end{pmatrix} \right\rangle_{L^2(\Omega)} \, ds dt \\
 &= 2 \int_0^T \int_0^t \langle \partial_t u_n(\cdot, s), \partial_{tt} u_n(\cdot, s) - \Delta_x u_n(\cdot, s) \rangle_{L^2(\Omega)} \, ds dt.
 \end{aligned}$$

The completion procedure gives the equality (4.63). The Cauchy-Schwarz inequality yields the asserted estimate. \square

Corollary 4.3.5. *The inner product space $(\tilde{H}_0^1(Q; \square), \langle \square_Q(\cdot), \square_Q(\cdot) \rangle_{L^2(Q)})$ is a Hilbert space.*

Proof. The assertion follows immediately from Lemma 4.3.3 and Lemma 4.3.4. \square

In the following, $\tilde{H}_0^1(Q; \square)$ is endowed with the inner product $\langle \square_Q(\cdot), \square_Q(\cdot) \rangle_{L^2(Q)}$ and hence, with the norm $\|\square_Q(\cdot)\|_{L^2(Q)}$. The strong variational formulation of the wave equation (4.11) for given $f \in L^2(Q)$ is as follows:

Find $u \in \tilde{H}_0^1(Q; \square)$ such that

$$a_S(u, v) = \langle f, v \rangle_{L^2(Q)} \quad (4.64)$$

for all $v \in L^2(Q)$, where the bilinear form

$$a_S(\cdot, \cdot) : \tilde{H}_0^1(Q; \square) \times L^2(Q) \rightarrow \mathbb{R}$$

is defined by

$$a_S(u, v) := \langle \square_Q u, v \rangle_{L^2(Q)}$$

for $u \in \tilde{H}_0^1(Q; \square)$, $v \in L^2(Q)$.

Next, properties of the bilinear form $a_S(\cdot, \cdot) : \tilde{H}_0^1(Q; \square) \times L^2(Q) \rightarrow \mathbb{R}$ are shown and finally, unique solvability of the strong variational formulation (4.64) is proven.

Lemma 4.3.6. *The bilinear form $a_S(\cdot, \cdot): \tilde{H}_0^1(Q; \square) \times L^2(Q) \rightarrow \mathbb{R}$ is bounded, i.e.*

$$|a_S(u, v)| \leq \|\square_Q u\|_{L^2(Q)} \|v\|_{L^2(Q)}$$

for all $u \in \tilde{H}_0^1(Q; \square)$, $v \in L^2(Q)$.

Proof. The assertion follows immediately from the Cauchy-Schwarz inequality. \square

Lemma 4.3.7. *The bilinear form $a_S(\cdot, \cdot): \tilde{H}_0^1(Q; \square) \times L^2(Q) \rightarrow \mathbb{R}$ fulfils the condition $(\mathcal{N}1)$, i.e. the inf-sup condition*

$$\sup_{0 \neq v \in L^2(Q)} \frac{|a_S(u, v)|}{\|v\|_{L^2(Q)}} = \|\square_Q u\|_{L^2(Q)}$$

for all $u \in \tilde{H}_0^1(Q; \square)$.

Proof. The inf-sup condition follows by the representation of the norm $\|\cdot\|_{L^2(Q)}$. \square

Lemma 4.3.8. *The bilinear form $a_S(\cdot, \cdot): \tilde{H}_0^1(Q; \square) \times L^2(Q) \rightarrow \mathbb{R}$ fulfils the condition $(\mathcal{N}2)$:*

For each function $0 \neq v \in L^2(Q)$, an element $u \in \tilde{H}_0^1(Q; \square)$ with $a_S(u, v) \neq 0$ exists.

Proof. Let $v \in L^2(Q)$ be fixed. There exists an approximating sequence $(\hat{v}_n)_{n \in \mathbb{N}} \subset C_0^\infty(Q)$ such that $\|\hat{v}_n - v\|_{L^2(Q)} \rightarrow 0$ as $n \rightarrow \infty$. Consider the eigenfunctions $\phi_i \in H_0^1(\Omega)$ and eigenvalues $\mu_i > 0$ of the Dirichlet eigenvalue problem of the Laplacian, see (2.4). Write v and \hat{v}_n for $n \in \mathbb{N}$ as Fourier series

$$v(x, t) = \sum_{i=1}^{\infty} v_i(t) \phi_i(x)$$

and

$$\hat{v}_n(x, t) = \sum_{i=1}^{\infty} \hat{v}_{n,i}(t) \phi_i(x)$$

for $(x, t) \in Q$ with coefficients $v_i \in L^2(0, T)$, $\hat{v}_{n,i} \in L^2(0, T)$, see (3.66). It holds true that

$$\infty > \|v\|_{L^2(Q)}^2 = \sum_{i=1}^{\infty} \|v_i\|_{L^2(0, T)}^2$$

and for $n \in \mathbb{N}$

$$\infty > \|\hat{v}_n\|_{L^2(Q)}^2 = \sum_{i=1}^{\infty} \|\hat{v}_{n,i}\|_{L^2(0, T)}^2, \quad \infty > \|\nabla_x \hat{v}_n\|_{L^2(Q)}^2 = \sum_{i=1}^{\infty} \mu_i \|\hat{v}_{n,i}\|_{L^2(0, T)}^2.$$

Fix an index $n \in \mathbb{N}$. Define for $M \in \mathbb{N}$

$$u_{n,M}(x,t) = \sum_{i=1}^M \frac{1}{\sqrt{\mu_i}} \int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x), \quad (x,t) \in Q,$$

and

$$u_n(x,t) = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\mu_i}} \int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x), \quad (x,t) \in Q.$$

Since

$$\begin{aligned} \|u_n\|_{L^2(Q)}^2 &= \sum_{i=1}^{\infty} \frac{1}{\mu_i} \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \right]^2 dt \\ &\leq \sum_{i=1}^{\infty} \frac{1}{\mu_i} \int_0^T \int_0^t \hat{v}_{n,i}(s)^2 ds \int_0^t \sin^2(\sqrt{\mu_i}(t-s)) ds dt \\ &\leq \frac{T^2}{2} \sum_{i=1}^{\infty} \frac{1}{\mu_i} \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 \leq \frac{1}{\mu_1} \frac{T^2}{2} \|\hat{v}_n\|_{L^2(Q)}^2 < \infty, \end{aligned}$$

it holds true that $u_n \in L^2(Q)$ and $u_{n,M} \rightarrow u_n$ in $L^2(Q)$ as $M \rightarrow \infty$.

To compute the time derivative ∂_t of u_n , set for $M \in \mathbb{N}$

$$w_{n,M}(x,t) = \sum_{i=1}^M \int_0^t \hat{v}_{n,i}(s) \cos(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x), \quad (x,t) \in Q,$$

$$w_n(x,t) = \sum_{i=1}^{\infty} \int_0^t \hat{v}_{n,i}(s) \cos(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x), \quad (x,t) \in Q,$$

and compute

$$\begin{aligned} \|w_n\|_{L^2(Q)}^2 &= \sum_{i=1}^{\infty} \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \cos(\sqrt{\mu_i}(t-s)) ds \right]^2 dt \\ &\leq \frac{T^2}{2} \sum_{i=1}^{\infty} \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 = \frac{T^2}{2} \|\hat{v}_n\|_{L^2(Q)}^2 < \infty, \end{aligned}$$

i.e. $w_n \in L^2(Q)$ and $w_{n,M} \rightarrow w_n$ in $L^2(Q)$ as $M \rightarrow \infty$. It follows that

$$\begin{aligned}
 \int_0^T \int_{\Omega} u_n(x,t) \partial_t \varphi(x,t) dx dt &= \lim_{M \rightarrow \infty} \int_0^T \int_{\Omega} u_{n,M}(x,t) \partial_t \varphi(x,t) dx dt \\
 &= - \lim_{M \rightarrow \infty} \int_0^T \int_{\Omega} \partial_t u_{n,M}(x,t) \varphi(x,t) dx dt \\
 &= - \lim_{M \rightarrow \infty} \int_0^T \int_{\Omega} w_{n,M}(x,t) \varphi(x,t) dx dt \\
 &= - \int_0^T \int_{\Omega} w_n(x,t) \varphi(x,t) dx dt
 \end{aligned}$$

for all $\varphi \in C_0^\infty(Q)$, i.e. $\partial_t u_n = w_n \in L^2(Q)$. Analogously, the weak derivatives $\partial_{tt} u_n$, $\nabla_x u_n$, $\nabla_x \partial_t u_n$ and $\Delta_x u_n$ are derived, since one formally computes for $(x,t) \in Q$

$$\begin{aligned}
 \partial_{tt} u_n(x,t) &= - \sum_{i=1}^{\infty} \sqrt{\mu_i} \int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x) + \sum_{i=1}^{\infty} \hat{v}_{n,i}(t) \phi_i(x), \\
 \nabla_x u_n(x,t) &= \sum_{i=1}^{\infty} \frac{1}{\sqrt{\mu_i}} \int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \cdot \nabla_x \phi_i(x), \\
 \nabla_x \partial_t u_n(x,t) &= \sum_{i=1}^{\infty} \int_0^t \hat{v}_{n,i}(s) \cos(\sqrt{\mu_i}(t-s)) ds \cdot \nabla_x \phi_i(x), \\
 \Delta_x u_n(x,t) &= - \sum_{i=1}^{\infty} \sqrt{\mu_i} \int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x),
 \end{aligned}$$

where the term-by-term differentiation is allowed because of the estimates

$$\begin{aligned}
 \|\partial_{tt} u_n\|_{L^2(Q)}^2 &\leq 2 \sum_{i=1}^{\infty} \mu_i \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \right]^2 dt + 2 \|\hat{v}_n\|_{L^2(Q)}^2 \\
 &\leq T^2 \sum_{i=1}^{\infty} \mu_i \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 + 2 \|\hat{v}_n\|_{L^2(Q)}^2 \\
 &= T^2 \|\nabla_x \hat{v}_n\|_{L^2(Q)}^2 + 2 \|\hat{v}_n\|_{L^2(Q)}^2 < \infty,
 \end{aligned}$$

$$\begin{aligned} \|\nabla_x u_n\|_{L^2(Q)}^2 &= \sum_{i=1}^{\infty} \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \right]^2 dt \\ &\leq \frac{T^2}{2} \sum_{i=1}^{\infty} \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 = \frac{T^2}{2} \|\hat{v}_n\|_{L^2(Q)}^2 < \infty, \end{aligned}$$

$$\begin{aligned} \|\nabla_x \partial_t u_n\|_{L^2(Q)}^2 &= \sum_{i=1}^{\infty} \mu_i \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \cos(\sqrt{\mu_i}(t-s)) ds \right]^2 dt \\ &\leq \frac{T^2}{2} \sum_{i=1}^{\infty} \mu_i \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 = \frac{T^2}{2} \|\nabla_x \hat{v}_n\|_{L^2(Q)}^2 < \infty \end{aligned}$$

and

$$\begin{aligned} \|\Delta_x u_n\|_{L^2(Q)}^2 &= \sum_{i=1}^{\infty} \mu_i \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \sin(\sqrt{\mu_i}(t-s)) ds \right]^2 dt \\ &\leq \frac{T^2}{2} \sum_{i=1}^{\infty} \mu_i \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 = \frac{T^2}{2} \|\nabla_x \hat{v}_n\|_{L^2(Q)}^2 < \infty. \end{aligned}$$

Therefore, $u_n \in \tilde{H}_0^2(Q; \square)$.

Analogously, define for $(x, t) \in Q$

$$u(x, t) = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\mu_i}} \int_0^t v_i(s) \sin(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x)$$

and partial sums

$$u_M(x, t) = \sum_{i=1}^M \frac{1}{\sqrt{\mu_i}} \int_0^t v_i(s) \sin(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x)$$

for $M \in \mathbb{N}$. With the same arguments as above, it holds true that

$$\begin{aligned} \lim_{M \rightarrow \infty} u_M &= u \quad \text{in } L^2(Q), \\ \lim_{M \rightarrow \infty} \partial_t u_M &= \partial_t u \quad \text{in } L^2(Q), \\ \lim_{M \rightarrow \infty} \nabla_x u_M &= \nabla_x u \quad \text{in } [L^2(Q)]^d. \end{aligned}$$

For each $M \in \mathbb{N}$, one computes for $(x, t) \in Q$

$$\square u_M(x, t) = \square \left[\sum_{i=1}^M \frac{1}{\sqrt{\mu_i}} \int_0^t v_i(s) \sin(\sqrt{\mu_i}(t-s)) ds \cdot \phi_i(x) \right] = \sum_{i=1}^M v_i(t) \phi_i(x)$$

and hence, it follows that

$$\begin{aligned} \langle u, \square \varphi \rangle_{L^2(Q)} &= \int_0^T \int_{\Omega} u(x, t) \square \varphi(x, t) dx dt = \lim_{M \rightarrow \infty} \int_0^T \int_{\Omega} u_M(x, t) \square \varphi(x, t) dx dt \\ &= \lim_{M \rightarrow \infty} \int_0^T \int_{\Omega} \square u_M(x, t) \varphi(x, t) dx dt = \lim_{M \rightarrow \infty} \int_0^T \int_{\Omega} \sum_{i=1}^M v_i(t) \phi_i(x) \varphi(x, t) dx dt \\ &= \int_0^T \int_{\Omega} v(x, t) \varphi(x, t) dx dt = \langle v, \varphi \rangle_{L^2(Q)} \end{aligned}$$

for all $\varphi \in C_0^\infty(Q)$, i.e. $\square_Q u = v \in L^2(Q)$. The estimates

$$\|u_n - u\|_{H^1(Q)} \leq C \|\hat{v}_n - v\|_{L^2(Q)} \quad \text{and} \quad \|\square_Q(u_n - u)\|_{L^2(Q)} \leq \|\hat{v}_n - v\|_{L^2(Q)}$$

yield $\|u_n - u\|_{H^1(Q; \square)} \rightarrow 0$ as $n \rightarrow \infty$ and hence, $u \in H^1(Q; \square)$.

To summarise, the sequence $(u_n)_{n \in \mathbb{N}} \subset \tilde{H}_0^2(Q; \square)$ converges to $u \in H^1(Q; \square)$ in $\|\cdot\|_{H^1(Q; \square)}$ and hence, $u \in \tilde{H}_0^1(Q; \square)$ by the completion procedure. With $\square_Q u = v \in L^2(Q)$ and therefore,

$$a_S(u, v) \geq \|v\|_{L^2(Q)}^2 > 0,$$

the assertion follows. □

Theorem 4.3.9. *For each given $f \in L^2(Q)$, a unique solution $u \in \tilde{H}_0^1(Q; \square)$ of the variational formulation (4.64) exists. Furthermore,*

$$\mathcal{L}_S: L^2(Q) \rightarrow \tilde{H}_0^1(Q; \square), \quad \mathcal{L}_S f := u,$$

is an isomorphism satisfying

$$\|\square_Q u\|_{L^2(Q)} = \|\square_Q \mathcal{L}_S f\|_{L^2(Q)} = \|f\|_{L^2(Q)}.$$

Proof. With the help of the Nečas Theorem 2.9.1, the results in Lemma 4.3.6, Lemma 4.3.7 and Lemma 4.3.8 yield the existence and uniqueness of a solution $u \in \tilde{H}_0^1(Q; \square)$. In addition, with the variational formulation (4.64), the equalities

$$\|f\|_{L^2(Q)} = \sup_{0 \neq v \in L^2(Q)} \frac{|\langle f, v \rangle_{L^2(Q)}|}{\|v\|_{L^2(Q)}} = \sup_{0 \neq v \in L^2(Q)} \frac{|\langle \square_Q u, v \rangle_{L^2(Q)}|}{\|v\|_{L^2(Q)}} = \|\square_Q u\|_{L^2(Q)}$$

hold true and therefore, the assertion. □

Remark 4.3.10. *The following functions are given to get a first impression of the solution space $\tilde{H}_0^1(Q; \square)$.*

1. *Functions $u \in C^2(\bar{Q})$ with $u|_{\Sigma_0} = 0$, $\partial_t u|_{\Sigma_0} = 0$ and $\nabla_x u|_{\Omega_0} = \underline{0}$ are contained in $\tilde{H}_0^2(Q; \square) \subset \tilde{H}_0^1(Q; \square)$.*
2. *A function $u \in H^1(Q; \square)$ with*

$$\left\| \gamma_{0,x}^{\text{int}} u \right\|_{L^2(\Sigma)} = \|u(\cdot, 0)\|_{L^2(\Omega)} = 0 \quad \text{and} \quad \square_Q u = 0 \quad \text{in } Q$$

is in $\tilde{H}_0^1(Q; \square)$ if and only if $u = 0$ in Q . This follows immediately from the representation (4.63).

3. *Consider the smooth function*

$$u(x, t) = \sin(\pi x) \sin(\pi t) \quad \text{for } (x, t) \in (0, 1) \times (0, 1) = Q,$$

satisfying

$$\left\| \gamma_{0,x}^{\text{int}} u \right\|_{L^2(\Sigma)} = \|u(\cdot, 0)\|_{L^2(\Omega)} = 0 \quad \text{and} \quad \square_Q u = 0 \quad \text{in } Q.$$

The representation (4.63) yields that $u \notin \tilde{H}_0^1(Q; \square)$.

Outlook for Possible Extensions of Section 4.3

Since the constructions of the spaces and the proofs in this section mainly rely on the treatment of the second-order temporal differential operator $\partial_{tt} + \mu$ with a parameter $\mu > 0$, a generalisation of the results of this section to differential operators $\partial_{tt} + \mathcal{A}_x$, acting on vector fields or scalar fields, is possible, where the second-order spatial differential operator \mathcal{A}_x has to fulfil certain properties. To transfer existence and uniqueness results, as in Theorem 4.3.9, to the more general differential operator

$$\partial_{tt} + \mathcal{A}_x,$$

the analysis has to be done with great care, where the proofs are left for future work, including precise assumptions on \mathcal{A}_x and the involved function spaces.

4.4 Space-Time Variational Formulation in a Weak Sense

In this section, the wave equation (4.11) is considered in a weaker sense than $L^2(Q)$. Therefore, with the notations of Section 2.1 and Section 2.5, define

$$\mathcal{H}(Q) := \{v|_Q : v \in L^2(Q_-), v|_{\Omega \times (-\infty, 0)} = 0, \square_{Q_-} v \in [H_{0;0}^{1,1}(Q)]'\}$$

with the norm

$$\|v\|_{\mathcal{H}(Q)} := \sqrt{\|v\|_{L^2(Q)}^2 + \|\square_{Q_-} v\|_{[H_{0;0}^{1,1}(Q)]'}^2},$$

where

$$Q_- = \Omega \times (-\infty, T) \subset \mathbb{R}^{d+1}$$

is the unbounded domain with respect to time. For a function $v \in \mathcal{H}(Q)$, the condition $\square_{Q_-} v \in [H_{0;0}^{1,1}(Q)]'$ involves that there exists an element $f_v \in [H_{0;0}^{1,1}(Q)]'$ with

$$\square_{Q_-} T_v(\varphi) = \langle f_v, \varphi|_Q \rangle_Q \quad \text{for all } \varphi \in \mathcal{D}(Q_-),$$

where

$$T_v : \mathcal{D}(Q_-) \rightarrow \mathbb{R}, \quad T_v(\varphi) = \int_{Q_-} v(x,t) \varphi(x,t) dx dt = \int_Q v(x,t) \varphi(x,t) dx dt \quad \text{for } \varphi \in \mathcal{D}(Q_-),$$

is the distribution related to $v \in L^2(Q_-)$ and as in Section 2.5, $\langle \cdot, \cdot \rangle_Q$ denotes the duality pairing in $[H_{0;0}^{1,1}(Q)]' \times H_{0;0}^{1,1}(Q)$ as extension of the $L^2(Q)$ inner product. Note that $\varphi|_Q \in H_{0;0}^{1,1}(Q)$ for $\varphi \in \mathcal{D}(Q_-)$ and that $C_0^\infty(Q_-)|_Q$ is dense in $H_{0;0}^{1,1}(Q)$. Hence, the element $f_v \in [H_{0;0}^{1,1}(Q)]'$ is unique.

Clearly, $(\mathcal{H}(Q), \|\cdot\|_{\mathcal{H}(Q)})$ is a normed vector space and it is even a Banach space.

Lemma 4.4.1. *The normed vector space $(\mathcal{H}(Q), \|\cdot\|_{\mathcal{H}(Q)})$ is a Banach space.*

Proof. Consider a Cauchy sequence $(v_n)_{n \in \mathbb{N}} \subset \mathcal{H}(Q)$. Hence, $(v_n)_{n \in \mathbb{N}} \subset L^2(Q)$ is also a Cauchy sequence in $L^2(Q)$ and $(\square_{Q_-} v_n)_{n \in \mathbb{N}} \subset [H_{0;0}^{1,1}(Q)]'$ is also a Cauchy sequence in $[H_{0;0}^{1,1}(Q)]'$. So, there exist $v \in L^2(Q)$ with $\|v_n - v\|_{L^2(Q)} \rightarrow 0$ as $n \rightarrow \infty$ and $f \in [H_{0;0}^{1,1}(Q)]'$ with

$$\|\square_{Q_-} v_n - f\|_{[H_{0;0}^{1,1}(Q)]'} = \sup_{0 \neq w \in H_{0;0}^{1,1}(Q)} \frac{|\langle \square_{Q_-} v_n - f, w \rangle_Q|}{|w|_{H^1(Q)}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let $T_v : \mathcal{D}(Q_-) \rightarrow \mathbb{R}$, $T_{v_n} : \mathcal{D}(Q_-) \rightarrow \mathbb{R}$, with

$$T_v(\varphi) = \int_Q v(x,t) \varphi(x,t) dx dt, \quad T_{v_n}(\varphi) = \int_Q v_n(x,t) \varphi(x,t) dx dt \quad \text{for all } \varphi \in \mathcal{D}(Q_-),$$

be the distributions related to the limit $v \in L^2(Q)$ and to $v_n \in L^2(Q)$ for every $n \in \mathbb{N}$. It follows for all $\varphi \in \mathcal{D}(Q_-)$ that

$$\begin{aligned} \square_{Q_-} T_v(\varphi) &= T_v(\square\varphi) = \int_Q v(x,t) \square\varphi(x,t) dx dt = \lim_{n \rightarrow \infty} \int_Q v_n(x,t) \square\varphi(x,t) dx dt \\ &= \lim_{n \rightarrow \infty} T_{v_n}(\square\varphi) = \lim_{n \rightarrow \infty} \square_{Q_-} T_{v_n}(\varphi) = \lim_{n \rightarrow \infty} \langle \square_{Q_-} v_n, \varphi|_Q \rangle_Q = \langle f, \varphi|_Q \rangle_Q \end{aligned}$$

because $\varphi|_Q \in H_{0;0}^{1,1}(Q)$. So, it holds true that

$$\square_{Q_-} v = f \in [H_{0;0}^{1,1}(Q)]'.$$

Hence, $v \in \mathcal{H}(Q)$. □

Since the norm $\|\cdot\|_{\mathcal{H}(Q)}$ is a Hilbertian norm, see Section 2.5, $\mathcal{H}(Q)$ is even a Hilbert space with respect to an abstract inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}(Q)}$, which induces the norm $\|\cdot\|_{\mathcal{H}(Q)}$.

Lemma 4.4.2. *It holds true that*

$$H_{0;0}^{1,1}(Q) \subset \mathcal{H}(Q).$$

Furthermore, each function $u \in H_{0;0}^{1,1}(Q)$ fulfils

$$\langle \square_{Q_-} u, w \rangle_Q = a_{H^1}(u, w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)} \quad (4.65)$$

for all $w \in H_{0;0}^{1,1}(Q)$.

Proof. Let $u \in H_{0;0}^{1,1}(Q)$ be fixed. Set $v := u$ in Q and $v := 0$ in $Q_- \setminus Q$. Clearly, it holds true that $v \in L^2(Q_-)$ and $v|_{\Omega \times (-\infty, 0)} = 0$. It remains to prove that $\square_{Q_-} v \in [H_{0;0}^{1,1}(Q)]'$. Therefore, let $T_v: \mathcal{D}(Q_-) \rightarrow \mathbb{R}$, with

$$T_v(\varphi) = \int_{Q_-} v(x,t) \varphi(x,t) dx dt = \int_Q u(x,t) \varphi(x,t) dx dt \quad \text{for all } \varphi \in \mathcal{D}(Q_-),$$

be the distribution related to $v \in L^2(Q_-)$ and define $f_u \in [H_{0;0}^{1,1}(Q)]'$ by

$$\langle f_u, w \rangle_Q := a_{H^1}(u, w) = -\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)} \quad \text{for all } w \in H_{0;0}^{1,1}(Q).$$

Note that $f_u \in [H_{0;0}^{1,1}(Q)]'$ is bounded by the Cauchy-Schwarz inequality, satisfying the estimate $\|f_u\|_{[H_{0;0}^{1,1}(Q)]'} \leq |u|_{H^1(Q)}$, see Lemma 4.2.22. It follows for all $\varphi \in \mathcal{D}(Q_-)$ with integration by parts with respect to time and space that

$$\begin{aligned} \square_{Q_-} T_v(\varphi) &= T_v(\square\varphi) = \int_Q u(x,t) \square\varphi(x,t) dx dt = \int_Q u(x,t) (\partial_t - \Delta_x) \varphi(x,t) dx dt \\ &= -\langle \partial_t u, \partial_t \varphi|_Q \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \varphi|_Q \rangle_{L^2(Q)} = \langle f_u, \varphi|_Q \rangle_Q. \end{aligned}$$

Hence, it holds true that $\square_{Q_-} v = f_u \in [H_{0;0}^{1,1}(Q)]'$ and so, $u \in \mathcal{H}(Q)$. The equality (4.65) follows also from the last relation, because $C_0^\infty(Q_-)|_Q$ is dense in $H_{0;0}^{1,1}(Q)$. \square

Define by completion the Hilbert space

$$\mathcal{H}_0(Q) := \overline{H_{0;0}^{1,1}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}} \subset \mathcal{H}(Q)$$

endowed with the Hilbertian norm $\|\cdot\|_{\mathcal{H}(Q)}$. That means,

$$\mathcal{H}_0(Q) = \{v \in \mathcal{H}(Q) : \exists (v_n)_{n \in \mathbb{N}} \subset H_{0;0}^{1,1}(Q) \text{ with } \|v_n - v\|_{\mathcal{H}(Q)} \rightarrow 0\}.$$

Lemma 4.4.3. *For $u \in \mathcal{H}_0(Q)$, it holds true that*

$$\|\square_{Q_-} u\|_{[H_{0;0}^{1,1}(Q)]'} \geq \frac{\sqrt{2}}{T} \|u\|_{L^2(Q)}.$$

Proof. Let $0 \neq u \in \mathcal{H}(Q)$ be fixed. Because of the completion, there exists an approximating sequence $(u_n)_{n \in \mathbb{N}} \subset H_{0;0}^{1,1}(Q)$ with $\|u - u_n\|_{\mathcal{H}(Q)} \rightarrow 0$ as $n \rightarrow \infty$. Assume without loss of generality that $u_n \neq 0$ for all $n \in \mathbb{N}$. Because of Theorem 4.2.23, for each $n \in \mathbb{N}$, there exists a unique solution $\tilde{w}_n \in H_{0;0}^{1,1}(Q)$ of

$$\forall v \in H_{0;0}^{1,1}(Q) : a_{H^1}(v, \tilde{w}_n) = -\langle \partial_t \tilde{w}_n, \partial_t v \rangle_{L^2(Q)} + \langle \nabla_x \tilde{w}_n, \nabla_x v \rangle_{L^2(Q)} \stackrel{!}{=} \langle u_n, v \rangle_{L^2(Q)}, \quad (4.66)$$

satisfying $|\tilde{w}_n|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|u_n\|_{L^2(Q)}$. With equality (4.65), with $v = u_n \in H_{0;0}^{1,1}(Q)$ in the variational formulation (4.66) and with the stability estimate $|\tilde{w}_n|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|u_n\|_{L^2(Q)}$, it follows that

$$\begin{aligned} \|\square_{Q_-} u_n\|_{[H_{0;0}^{1,1}(Q)]'} &= \sup_{0 \neq w \in H_{0;0}^{1,1}(Q)} \frac{|\langle \square_{Q_-} u_n, w \rangle_Q|}{|w|_{H^1(Q)}} \geq \frac{|\langle \square_{Q_-} u_n, \tilde{w}_n \rangle_Q|}{|\tilde{w}_n|_{H^1(Q)}} = \frac{|a_{H^1}(u_n, \tilde{w}_n)|}{|\tilde{w}_n|_{H^1(Q)}} \\ &\geq \frac{\sqrt{2} \|u_n\|_{L^2(Q)}}{T \|u_n\|_{L^2(Q)}} = \frac{\sqrt{2}}{T} \|u_n\|_{L^2(Q)} \end{aligned}$$

and hence, the assertion by completion. \square

Corollary 4.4.4. *The inner product space $(\mathcal{H}_0(Q), \langle \square_{Q_-}(\cdot), \square_{Q_-}(\cdot) \rangle_{[H_{0;0}^{1,1}(Q)]'})$ is complete, i.e. a Hilbert space.*

Proof. The assertion follows immediately from Lemma 4.4.3. \square

In the following, $\mathcal{H}_0(Q)$ is endowed with the Hilbertian norm $\|\square_{Q_-}(\cdot)\|_{[H_{0;0}^{1,1}(Q)]'}$. The weak variational formulation for given $f \in [H_{0;0}^{1,1}(Q)]'$ is as follows:

Find $u \in \mathcal{H}_0(Q)$ such that

$$a_W(u, w) = \langle f, w \rangle_Q \quad (4.67)$$

for all $w \in H_{0;0}^{1,1}(Q)$, where the bilinear form $a_W(\cdot, \cdot): \mathcal{H}_0(Q) \times H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$ is defined by

$$a_W(u, w) := \langle \square_{Q_-} u, w \rangle_Q \quad \text{for } u \in \mathcal{H}_0(Q), w \in H_{0;0}^{1,1}(Q).$$

Next, properties of the bilinear form $a_W(\cdot, \cdot): \mathcal{H}_0(Q) \times H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$ are shown and finally, unique solvability of the weak variational formulation (4.67) is proven.

Lemma 4.4.5. *The bilinear form $a_W(\cdot, \cdot): \mathcal{H}_0(Q) \times H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$ is bounded, i.e.*

$$|a_W(u, w)| \leq \|\square_{Q_-} u\|_{[H_{0;0}^{1,1}(Q)]'} |w|_{H^1(Q)} \quad \text{for all } u \in \mathcal{H}_0(Q), w \in H_{0;0}^{1,1}(Q).$$

Proof. The assertion follows immediately by the definition of the space $\mathcal{H}_0(Q)$. \square

Lemma 4.4.6. *The bilinear form $a_W(\cdot, \cdot): \mathcal{H}_0(Q) \times H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$ fulfils the condition $(\mathcal{N}1)$, i.e. the inf-sup condition*

$$\sup_{0 \neq w \in H_{0;0}^{1,1}(Q)} \frac{|a_W(u, w)|}{|w|_{H^1(Q)}} = \|\square_{Q_-} u\|_{[H_{0;0}^{1,1}(Q)]'} \quad \text{for all } u \in \mathcal{H}_0(Q).$$

Proof. The inf-sup condition follows by the definition of the norm $\|\square_{Q_-}(\cdot)\|_{[H_{0;0}^{1,1}(Q)]'}$. \square

Lemma 4.4.7. *The bilinear form $a_W(\cdot, \cdot): \mathcal{H}_0(Q) \times H_{0;0}^{1,1}(Q) \rightarrow \mathbb{R}$ fulfils the condition $(\mathcal{N}2)$:*

For each $0 \neq w \in H_{0;0}^{1,1}(Q)$, an element $u \in \mathcal{H}_0(Q)$ with $a_W(u, w) \neq 0$ exists.

Proof. Let $0 \neq w \in H_{0;0}^{1,1}(Q)$ be a given function. Because of Theorem 4.2.23, there exists a unique solution $\tilde{u} \in H_{0;0}^{1,1}(Q)$ of

$$\forall z \in H_{0;0}^{1,1}(Q): \quad a_{H^1}(\tilde{u}, z) = -\langle \partial_t \tilde{u}, \partial_t z \rangle_{L^2(Q)} + \langle \nabla_x \tilde{u}, \nabla_x z \rangle_{L^2(Q)} \stackrel{!}{=} \langle w, z \rangle_{L^2(Q)}, \quad (4.68)$$

satisfying $\frac{\sqrt{2}}{T} |\tilde{u}|_{H^1(Q)} \leq \|w\|_{L^2(Q)}$. With the help of representation (4.65) and the variational formulation (4.68) for $z = w \in H_{0;0}^{1,1}(Q)$, it follows that

$$\begin{aligned} a_W(\tilde{u}, w) &= \langle \square_{Q_-} \tilde{u}, w \rangle_Q \\ &= a_{H^1}(\tilde{u}, w) = -\langle \partial_t \tilde{u}, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x \tilde{u}, \nabla_x w \rangle_{L^2(Q)} = \langle w, w \rangle_{L^2(Q)} > 0 \end{aligned}$$

and hence, the assertion. \square

Theorem 4.4.8. For each given $f \in [H_{0;0}^{1,1}(Q)]'$, a unique solution $u \in \mathcal{H}_0(Q)$ of the variational formulation (4.67) exists. Furthermore,

$$\mathcal{L}w : [H_{0;0}^{1,1}(Q)]' \rightarrow \mathcal{H}_0(Q), \quad \mathcal{L}wf := u,$$

is an isomorphism satisfying

$$\|\square_{Q_-} u\|_{[H_{0;0}^{1,1}(Q)]'} = \|\square_{Q_-} \mathcal{L}wf\|_{[H_{0;0}^{1,1}(Q)]'} = \|f\|_{[H_{0;0}^{1,1}(Q)]'}.$$

Proof. With the help of the Nečas Theorem 2.9.1, the results in Lemma 4.4.5, Lemma 4.4.6 and Lemma 4.4.7 yield the existence and uniqueness of a solution $u \in \mathcal{H}_0(Q)$. In addition, with the variational formulation (4.67), the equalities

$$\|f\|_{[H_{0;0}^{1,1}(Q)]'} = \sup_{0 \neq w \in H_{0;0}^{1,1}(Q)} \frac{|\langle f, w \rangle_Q|}{|w|_{H^1(Q)}} = \sup_{0 \neq w \in H_{0;0}^{1,1}(Q)} \frac{|aw(u, w)|}{|w|_{H^1(Q)}} = \|\square_{Q_-} u\|_{[H_{0;0}^{1,1}(Q)]'}$$

hold true and therefore, the assertion. \square

Remark 4.4.9. The following functions are given to get a first impression of the solution space $\mathcal{H}_0(Q)$.

1. Functions $u \in C^2(\overline{Q})$ with $u|_{\Sigma_0} = 0$ are contained in $H_{0;0}^{1,1}(Q) \subset \mathcal{H}_0(Q)$.
2. Consider the smooth function

$$u(x, t) = \sin(\pi x) \sin(\pi t) \quad \text{for } (x, t) \in (0, 1) \times (0, 1) = Q$$

satisfying $\gamma_0^{\text{int}} u = 0$ in Σ_0 and $\square_{Q_-} u = 0$ in Q . But there is

$$\square_{Q_-} u \neq 0$$

because the related distribution

$$T_u(\varphi) = \int_Q u(x, t) \varphi(x, t) dx dt \quad \text{for all } \varphi \in \mathcal{D}(Q_-)$$

fulfils with integration by parts for all $\varphi \in \mathcal{D}(Q_-)$

$$\begin{aligned} \square_{Q_-} T_u(\varphi) &= T_u(\square_{Q_-} \varphi) = \int_Q u(x, t) \square_{Q_-} \varphi(x, t) dx dt = \int_Q u(x, t) (\partial_{tt} - \partial_{xx}) \varphi(x, t) dx dt \\ &= -\langle \partial_t u, \partial_t \varphi \rangle_{L^2(Q)} + \langle \partial_x u, \partial_x \varphi \rangle_{L^2(Q)} = \pi \int_0^1 \sin(\pi x) \varphi(x, 0) dx. \end{aligned}$$

On the other hand, it holds true that $u \in H_{0;0}^{1,1}(Q) \subset \mathcal{H}_0(Q)$. In other words, the second initial condition

$$\partial_t u(\cdot, 0) = 0 \quad \text{in } \Omega$$

is not incorporated in the ansatz space $\mathcal{H}_0(Q)$.

Outlook for Possible Extensions of Section 4.4

Since the constructions of the spaces and the proofs in this section mainly rely on the treatment of the second-order temporal differential operator $\partial_{tt} + \mu$ with a parameter μ , a generalisation of the results of this section to differential operators $\partial_{tt} + \mathcal{A}_x$, acting on vector fields or scalar fields \underline{u} , is possible, where the second-order spatial differential operator \mathcal{A}_x has to fulfil certain properties, e.g., constant coefficients. In more detail, consider the more general ansatz and test spaces

$$\begin{aligned} X_0 &:= H_0^1(0, T; [L^2(\Omega)]^d) \cap L^2(0, T; V) \subset [L^2(Q)]^d, \\ X_{,0} &:= H_{,0}^1(0, T; [L^2(\Omega)]^d) \cap L^2(0, T; V) \subset [L^2(Q)]^d, \end{aligned}$$

endowed with the usual inner products, where $V \subset [L^2(\Omega)]^d$ is a given Hilbert space such that the set $[C_0^\infty(Q_-)]^d$ is dense in X_0 . Assume that the space-time variational formulation to find $\underline{u} \in X_0$, such that

$$\forall \underline{w} \in X_{,0}: -\langle \partial_t \underline{u}, \partial_t \underline{w} \rangle_{L^2(Q)} + \int_0^T a_{\mathcal{A}_x}(u(\cdot, t), w(\cdot, t)) dt = \langle \underline{f}, \underline{w} \rangle_{L^2(Q)}$$

admits a unique solution for each right-hand side $\underline{f} \in [L^2(Q)]^d$, satisfying the stability estimate

$$\|\underline{u}\|_{X_0} \leq c \|\underline{f}\|_{L^2(Q)}$$

with a constant $c > 0$. Further, assume that the to \mathcal{A}_x corresponding spatial bilinear form $a_{\mathcal{A}_x}(\cdot, \cdot): V \times V \rightarrow \mathbb{R}$ fulfils some properties, e.g., ellipticity and continuity. Then, existence and uniqueness results, as in Theorem 4.4.8, can be transferred to the more general differential operator

$$\partial_{tt} + \mathcal{A}_x.$$

Since the proofs have to be done with great care, they are left for future work, including precise assumptions on \mathcal{A}_x and V .

5 CONCLUSIONS AND OUTLOOK

In this work, space-time variational formulations and their discretisations with conforming, piecewise polynomial functions for the heat and wave equation are considered in a bounded space-time cylinder Q with a terminal time T .

The main result for the heat equation is an unconditionally stable finite element method of Galerkin-Bubnov type with piecewise linear, continuous functions, which is based on a variational formulation in a subspace of the anisotropic Sobolev space $H^{1,1/2}(Q)$. This space-time variational formulation is analysed with the help of Fourier series, and a kind of Hilbert transform \mathcal{H}_T is introduced. This leads to a symmetric and elliptic variational formulation and hence, to a symmetric Galerkin discretisation of the first-order time derivative ∂_t . For the heat equation, unconditional stability for unstructured space-time meshes, error estimates in $L^2(Q)$, in $H^1(Q)$ and in the anisotropic Sobolev space $H_0^{1,1/2}(0, T; L^2(\Omega))$ for a tensor-product approach are proven. Furthermore, numerical examples, which confirm the theoretical results, are presented. The main advantage of this formulation is the possibility of a combination with the standard boundary element method for the heat equation, i.e. a FEM-BEM coupling, see [39]. An investigation in such directions is a possible future work. In addition, proving a discrete inf-sup condition and error estimates for unstructured space-time meshes and a fast realisation of the kind of Hilbert transform \mathcal{H}_T given in (3.68) are also of interest in future.

For the wave equation, a space-time variational formulation in a subspace of the Sobolev space $H^1(Q)$, which is not inf-sup stable, is used for a conforming space-time finite element method, which leads to a conditionally stable method, i.e. a CFL condition is required. For a tensor-product approach, an unconditionally stable method with piecewise linear, continuous functions is investigated. An extension to a space-time approximation with unstructured space-time meshes remains open for the future. A first possibility is the use of locally refined meshes with hanging nodes, including related constraints to satisfy the continuity requirements of the ansatz space. A second possibility is to transfer the stabilisation to unstructured but admissible space-time meshes. In both cases, one constant source of difficulties is the situation of different initial and terminal conditions of the ansatz and test spaces $H_{0;0}^{1,1}(Q)$, $H_{0;0}^{1,1}(Q)$, which may lead to a nonsquare system of linear equations for discretisations based on unstructured space-time meshes. Using the transformation $\overline{\mathcal{H}}_T$ given in (4.45), i.e. a Galerkin-Bubnov scheme, is a possible way out.

Moreover, existence and uniqueness results for the wave equation as a partial differential equation in $L^2(Q)$ and in a weaker sense than $L^2(Q)$ are proven, including isomorphic solution operators and corresponding inf-sup conditions. These inf-sup stable space-time variational formulations in the strong or weak sense might be useful not only for other discretisation methods, e.g., wavelets, but also for the analysis of the related boundary integral equations.

For both equations, i.e. for the heat equation and for the wave equation, moving boundaries, space-time adaptive schemes, space-time parallelisations and especially fast solvers and preconditioning, which are based on space-time variational formulations given in this thesis, are left for future considerations. Finally, any extensions to more involved equations, e.g., Stokes equations or Maxwell's equations, are of interest in future, see also the outlooks for possible extensions in each chapter.

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