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Asymptotic properties of random outerplanar graphs

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Abstract

We consider random cacti graphs and random outerplanar graphs. More precisely, we investigate the component structure of these graphs for different values of the average degree. We show that there is a drastic change of the component structure, a so-called phase transition, when the average degree is around one. At that point the component structure changes from several components of small size to one unique largest component.

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1 Introduction and main results

1.1 Introduction

The starting point for the theory of random graphs was when Erdős and Rényi [8] introduced in 1959 the so-called *Erdős-Rényi* graph $G(n,m)$, a graph chosen uniformly at random from the class $\mathcal{G}(n,m)$ of all vertex-labelled graphs on vertex set $\{1,\dots,n\}$ with $m = m(n)$ edges. Since then, a lot of questions of the following type were considered (see e.g. [3, 12, 15]). What properties does $G(n,m)$ satisfy *with high probability* (whp for short), i.e. with probability tending to one as n tends to infinity?

A property, which was extensively studied, is the component structure of a graph. More precisely, it was investigated how the component structure of $G(n,m)$ changes, when $m = m(n)$ varies and whether there are ranges of m, where this change is very significant. Such dramatic changes are called *phase transitions*. For example, Erdős and Rényi [9] considered the component structure of $G(n,m)$ for different ranges of $\alpha := \frac{m}{n}$. They showed that whp

(i) if $\alpha \to c < \frac{1}{2}$, then all components have at most a logarithmic number of vertices;

(ii) if $\alpha \to \frac{1}{2}$, then the largest component has $\Theta\left(n^{\frac{2}{3}}\right)$ vertices;

(iii) if $\alpha \to c > \frac{1}{2}$, then the largest component has linearly many vertices, while all other components have at most a logarithmic number of vertices.

They called this transition from $O(\log n)$ through $\Theta\left(n^{\frac{2}{3}}\right)$ to $\Theta(n)$ the 'double jump'.

Bollobás [2] and Łuczak [20] showed later that this phase transition is actually 'smooth' by looking more closely at the critical range $m = \frac{n}{2} + o(n)$. In order to state their results we introduce the following notation. For a given graph G we denote by $|G|$ the number of vertices in G. Moreover, we order its components according to the number of vertices. Then we denote them by $H_1 = H_1(G), H_2 = H_2(G), \dots$ in such a way that $|H_i| \geq |H_j|$, whenever $i \geq j$. In addition, we use the asymptotic notation, which we formally define in Section 3.2.

Theorem 1.1 ([2, 20])**.** *Let $m = \frac{n}{2} + s$, where $s = s(n) = o(n)$ and let $G = G(n,m)$. Then for every $i \in \mathbb{N}$ the following holds whp.*

(i) If $\frac{s^3}{n^2} \to -\infty$, then H_i is a tree and

$$|H_i| = \left(\frac{1}{2} + o(1)\right)\frac{n^2}{s^2}\log\frac{|s|^3}{n^2}.$$

(ii) *If* $\frac{s^3}{n^2} \to c \in \mathbb{R}$, *then*

$$|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right).$$

(iii) *If* $\frac{s^3}{n^2} \to \infty$, *then*

$$|H_1| = (4 + o(1))\, s.$$

For $i \geq 2$, *we have* $|H_i| = o\left(n^{\frac{2}{3}}\right)$.

This drastic change of the component structure at $m = \frac{n}{2} + O\left(n^{\frac{2}{3}}\right)$ is called the *emergence of the giant component*.

These results raised the question whether there are also phase transitions in other classes of random graphs. Kang and Łuczak [16] considered this question for the class $\mathcal{P}(n,m)$ of all vertex-labelled planar graphs with n vertices and $m = m(n)$ edges. They showed that the component structure of a graph chosen uniformly at random from $\mathcal{P}(n,m)$, denoted by $P(n,m)$, features two phase transitions. The first one, analogous to $G(n,m)$, takes places at $m = \frac{n}{2} + O\left(n^{\frac{2}{3}}\right)$, when the giant component emerges. The second one is at $m = n + O\left(n^{\frac{3}{5}}\right)$, when the giant component covers almost all vertices. Kang, Moßhammer, and Sprüssel [17] extended these results to graphs on orientable surfaces. For simplicity, we state their results only for planar graphs.

Theorem 1.2 ([16, 17])**.** *Let* $m = \frac{n}{2} + s$, *where* $s = s(n) = o(n)$ *and let* $G = P(n,m)$. *For every* $i \in \mathbb{N}$ *whp the following holds.*

(i) *If* $\frac{s^3}{n^2} \to -\infty$, *then* H_i *is a tree and*

$$|H_i| = \left(\frac{1}{2} + o(1)\right) \frac{n^2}{s^2} \log \frac{|s|^3}{n^2}.$$

(ii) *If* $\frac{s^3}{n^2} \to c \in \mathbb{R}$, *then*

$$|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right).$$

(iii) *If* $\frac{s^3}{n^2} \to \infty$, *then*

$$|H_1| = 2s + O_p\left(n^{\frac{2}{3}}\right).$$

For $i \geq 2$, *we have* $|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right)$.

Theorem 1.3 ([16, 17])**.** *Let* $m = \alpha n$, *where* $\alpha = \alpha(n)$ *converges to a constant in* $\left(\frac{1}{2}, 1\right)$ *and let* $G = P(n,m)$. *Then*

$$|H_1| = (2\alpha - 1)n + O_p\left(n^{\frac{2}{3}}\right).$$

For $i \geq 2$, *we have* $|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right)$.

Theorem 1.4 ([16, 17]). *Let $m = n + t$, where $t = t(n) = o(n)$ and let $G = P(n, m)$. Then we have $n - |H_1| = O_p(r(n))$ and whp $n - |H_1| = \Omega(r(n))$, where*

$$r(n) := \begin{cases} |t| & \text{if } \frac{t^5}{n^3} \to -\infty \\ n^{\frac{3}{5}} & \text{if } \frac{t^5}{n^3} \to c \in \mathbb{R} \\ n^{\frac{3}{2}} t^{-\frac{3}{2}} & \text{if } \frac{t^5}{n^3} \to \infty \text{ and } t = o\left(n (\log n)^{-\frac{2}{3}}\right). \end{cases}$$

1.2 Main results

Kang, Moßhammer, and Sprüssel [17] used the core-kernel approach to prove Theorems 1.2-1.4. This method works for a wide variety of graph classes (see Theorem 2.3). We will show that the family of cacti graphs (see Section 3.1 for a formal definition) is an example of such a class.

Theorem 1.5. *Theorems 1.2-1.4 are also true for the class of cacti graphs.*

Later we will explicitly state Theorem 1.5 in Theorems 4.1-4.3. Surprisingly, we will see that there is no straightforward extension of the core-kernel approach from [17] to the case of outerplanar graphs. We will instead use the direct core approach to show that the giant component in a random outerplanar graph emerges also at $m = \frac{n}{2} + O\left(n^{\frac{2}{3}}\right)$.

Theorem 1.6. *Theorem 1.2 is also true for the class of outerplanar graphs.*

This theorem will be explicitly stated in Theorem 5.1.

1.3 Outline

This thesis is organised as follows. In the next chapter we describe the main idea of our proofs. Then, we formally define cacti and outerplanar graphs and state properties of these graph classes. After that, we gather bounds and asymptotic formulas, which we will use in our proofs. In Chapters 4-5 we prove Theorems 1.5-1.6. Finally, we discuss some possible generalisations of our ideas in Chapter 6.

2 Proof strategy

In this chapter we present the main ideas of our proofs. We start with the core-kernel approach, which was used in [17] to obtain Theorems 1.2-1.4. We will see that we can use that concept for cacti graphs too (see Section 2.3), while we have to modify it for outerplanar graphs (see Sections 2.4-2.5), so as to apply the direct core approach without using the kernel. We refer to Section 3.1 for formal definitions and properties of these two graph classes. Throughout this chapter, let G be a graph.

2.1 Decomposition

The main idea is to extract the part of G, that is responsible for the planarity. We start by looking at the components and introduce the following definitions.

Definition 2.1. A component of G is called

- a *tree* if it has no cycle;
- *unicyclic* if it has precisely one cycle;
- *complex* if it has at least two cycles.

The complex part Q_G of G is the union of all complex components. Moreover, we call G *complex* if all its components are complex.

Equivalently, a component is complex if and only if it has more edges than vertices. We will see that this difference between the number of edges and vertices plays an important role later.

Definition 2.2. Let H be a complex component of G with n vertices and m edges. Then we call $m - n$ the *excess* of H. The excess of G, denoted by $\ell = \ell(G)$ or by $\mathrm{ex}(G)$, is the sum of the excesses of all complex components of G.

We note that trees and unicyclic components are always planar. Thus, it suffices to look at the complex part if we ask whether G is planar or not.

2.2 Core and kernel

Now we decompose the complex part Q_G further. We observe that deleting a vertex of degree one in Q_G does not change the property of being (non-) planar. If we do that

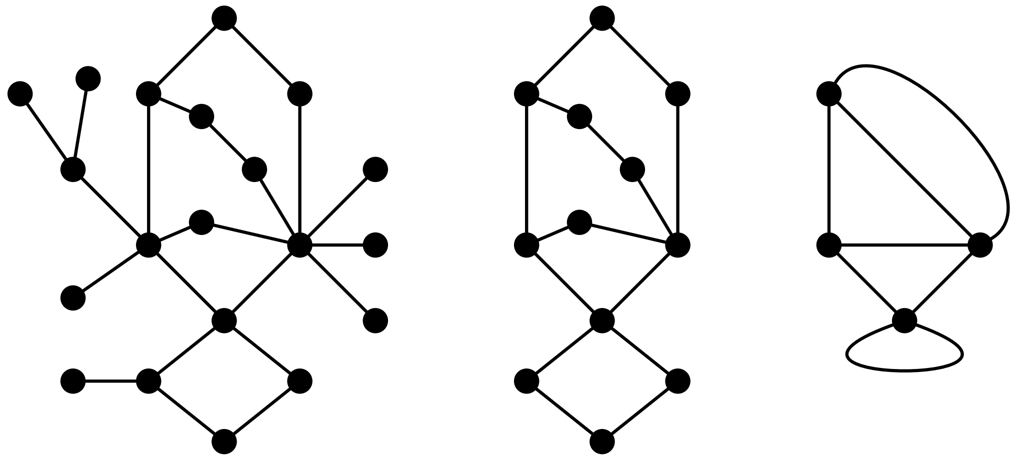

Figure 2.1: A planar graph G together with its core C_G in the middle and the kernel K_G on the right-hand side.

recursively, we end up with a graph of minimum degree at least two, which is called the core, or more precisely the 2-core.

Definition 2.3. The maximal subgraph of Q_G with minimum degree at least two is called the *core* and is denoted by C_G.

In the last decomposition step we consider vertices of degree two to obtain the kernel.

Definition 2.4. The *kernel* K_G of G is the graph, which is obtained from the core C_G by the following operation. We consider maximal paths in C_G, which consist only of vertices of degree two. Then we replace all such paths P by an edge between the vertices of degree at least three that are adjacent to the end vertices of P.

We note that by this operation loops and multiple edges can appear. So, the kernel is not necessarily a simple graph. Now a basic, but important, observation is the following lemma.

Lemma 2.1. *G is planar if and only if the kernel K_G is planar.*

We can easily extend that result to cacti graphs. On the other hand, we will see that this is not the case for outerplanar graphs, where we will obtain only a weaker statement.

2.3 Core-kernel approach for cacti graphs

Now we extend the aforementioned concept to cacti graphs.

Lemma 2.2. *G is a cactus if and only if its kernel K_G is a cactus.*

Proof. We use the characterisation of cacti graphs from Theorem 3.4. Then trees and unicyclic components are obviously cacti graphs. Moreover, deleting vertices of degree one does not create or destroy a cycle. The same is true for the decomposition step from the core to the kernel. Thus, we obtain

$$G \text{ is a cactus} \iff Q_G \text{ is a cactus} \iff C_G \text{ is a cactus} \iff K_G \text{ is a cactus.}$$

□

Kang, Moßhammer, and Sprüssel proved in [17] that Theorems 1.2-1.4 hold for a wide variety of graph classes. In short, the core-kernel approach is applicable if an analogous result to Lemma 2.2 holds. Before we formulate that general theorem, we need some definitions.

Definition 2.5. Let $\mathcal{X}$ be a class of (multi-) graphs. We write $\mathcal{X}(n,m)$ for the subclass of $\mathcal{X}$ containing all graphs with n vertices and m edges. In addition, we denote by $X(n,m)$ a graph chosen uniformly at random from $\mathcal{X}(n,m)$.

Definition 2.6. Let M be a multigraph and $i \in \mathbb{N}$. We denote by $e_i(M)$ the number of unordered pairs $\{x, y\}$ such that there are precisely i edges between x and $y \neq x$. Similarly, let $f_i(M)$ be the number of vertices x such that there are exactly i loops at x and $f(M)$ the total number of loops. Then the *weight* (or *compensation factor*) of M is given by

$$w(M) := 2^{-f(M)} \prod_{i \geq 1} (i!)^{-e_i(M)-f_i(M)}.$$

For a class $\mathcal{M}$ of multigraphs we set

$$|\mathcal{M}(n,m)| := \sum_{M \in \mathcal{M}(n,m)} w(M).$$

This concept was first introduced by Janson, Knuth, Łuczak, and Pittel [14] and is used to count the number of cores (see [17, p.11-12] for details). In the following we always assume that multigraphs are weighted.

Remark 2.1. We will use the weight of a multigraph only for the case of cubic cacti multigraphs. Then the weight simplifies to

$$w(M) = 2^{-f(M)-e_2(M)}.$$

Definition 2.7. We call a class $\mathcal{X}$ of graphs *weakly addable* if the following holds. Whenever we add an edge between two components of a graph $X \in \mathcal{X}$, then the resulting graph is also in $\mathcal{X}$.

Recall that we denote by $H_i = H_i(G)$ the i-th largest component of a graph G.

Theorem 2.3 ([17]). *Let $\mathcal{X}$ be a graph class and $\mathcal{Y}$ be a class of multigraphs of minimum degree at least three. Moreover, let $\mathcal{Y}(2n, 3n)$ be the subclass of $\mathcal{Y}$ containing all graphs with $2n$ vertices and $3n$ edges and let $Y(2n, 3n)$ be a graph chosen uniformly at random from $\mathcal{Y}(2n, 3n)$. Suppose that*

(i) a graph lies in $\mathcal{X}$ if and only if its kernel is in $\mathcal{Y}$;

(ii) there are constants $c, \gamma > 0$ and $k \in \mathbb{R}$ such that for $n \to \infty$

$$|\mathcal{Y}(2n, 3n)| = (1 + o(1))cn^k\gamma^{2n}(2n)!;$$

(iii) there is a constant $0 < q \leq 1$ with

$$\mathbb{P}[\, Y(2n, 3n) \text{ is connected}\,] \xrightarrow{n\to\infty} q;$$

(iv) $|H_1(Y(2n, 3n))| = 2n - O_p(1)$ and for each $i \in \mathbb{N}$, the probability that $|H_1(Y(2n, 3n))| = 2n - 2i$ is bounded away from both 0 and 1;

(v) $\mathcal{X}$ is weakly addable and closed under taking minors.

Then Theorems 1.2-1.4 hold for $\mathcal{X}$.

In Chapter 4 we will show that the class of cacti graphs fulfils the conditions of Theorem 2.3, whence Theorem 1.5 follows.

2.4 Direct core approach for outerplanar graphs

In contrast to other classes such as planar graphs, series-parallel graphs, and even cacti graphs, Statement (i) in Theorem 2.3 *does not hold* for outerplanar graphs. To illustrate that we look at the following example.

Example 2.1. We consider the complete bipartite graph $K_{2,3}$. It is well-known that this graph is not outerplanar (see e.g. [5]). In contrast, we see in Figure 2.2 that its kernel is outerplanar.

Therefore, we shall use the direct core approach without decomposing the core into the kernel. We start with the following observation.

Lemma 2.4. *G is outerplanar if and only if its core C_G is outerplanar.*

Proof. Trees and unicyclic components are obviously outerplanar. In addition, if we delete a vertex of degree one in a (non-) outerplanar graph it stays (non-) outerplanar. □

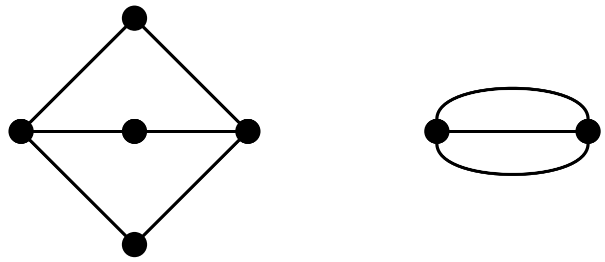

Figure 2.2: $K_{2,3}$ and its kernel.

Next, we want to use Lemma 2.4 to count the number of outerplanar graphs with n vertices and m edges. We illustrate that concept only for outerplanar graphs, but it is straightforward that it works also for planar and cacti graphs. To that end, we need some further definitions and notations.

Definition 2.8. We denote by

- $\mathcal{A}$ the class of all outerplanar graphs;
- $\mathcal{Q}$ the class of all complex outerplanar graphs (i.e. complex parts of graphs in $\mathcal{A}$);
- $\mathcal{C}$ the class of all complex outerplanar graphs with minimum degree at least two (i.e. cores of graphs in $\mathcal{A}$);
- $\mathcal{U}$ the class of all graphs without complex components;
- $\mathcal{G}$ the class of all graphs.

Definition 2.9. Let G be a graph with n vertices and m edges. We denote by

- $n_Q = n_Q(G)$ the number of vertices in Q_G;
- $n_C = n_C(G)$ the number of vertices in Q_C;
- $n_U = n_U(G) := n - n_Q$ the number of vertices in G outside the complex part Q_G;
- $m_U = m_U(G) := m - n_Q - \ell$ the number of edges in G outside Q_G.

Now we introduce our decomposition to obtain relations between the number of graphs in the above defined classes. We start with the decomposition into the complex part and into non-complex components. Reversely, we can construct all graphs in $\mathcal{A}(n, m)$ with a fixed value of n_Q and ℓ in the following way.

- Choose the labels of the n_Q vertices in the complex part, for which there exist $\binom{n}{n_Q}$ possibilities;
- Choose a complex part with n_Q vertices and $n_Q + \ell$ edges, for which we have $|\mathcal{Q}(n_Q, n_Q + \ell)|$ possibilities;

- Choose a graph without complex components with n_U vertices and m_U edges, for which there are $|\mathcal{U}(n_U, m_U)|$ possibilities.

Summing up over all possible values for n_Q and ℓ yields the following equation.

$$|\mathcal{A}(n,m)| = \sum_{n_Q,\ell} \binom{n}{n_Q} |\mathcal{Q}(n_Q, n_Q + \ell)| \cdot |\mathcal{U}(n_U, m_U)| \tag{2.1}$$

Later we will analyse the above sum. To that end, we will need some information about the term $|\mathcal{U}(n_U, m_U)|$. We get that from the following known theorem, which gives us an estimate for the probability

$$\rho(n,m) := |\mathcal{U}(n,m)| \binom{\binom{n}{2}}{m}^{-1}$$

that the Erdős-Rényi graph $G(n,m)$ has no complex components.

Lemma 2.5 ([4, 14, 17]). *Let $m = \frac{n}{2} + s$ with $s = s(n) < \frac{n}{2}$. Then there is a constant $c > 0$ such that for*

$$f(n,m) := c \left(\frac{2}{e}\right)^{2m-n} \frac{m^{m+\frac{1}{2}} n^{n-2m+\frac{1}{2}}}{(n-m)^{n-m+\frac{1}{2}}},$$

we have

(i) $\rho(n,m) \to 1$, *if* $\frac{s^3}{n^2} \to -\infty$;

(ii) for each $a \in \mathbb{R}$, there exists a constant $b > 0$ such that $\rho(n,m) \geq b$, whenever $s \leq a n^{\frac{2}{3}}$;

(iii) $\rho(n,m) \leq n^{-\frac{1}{2}} f(n,m)$, *if* $0 < s \leq \frac{n^{\frac{3}{4}}}{2}$;

(iv) $\rho(n,m) \leq f(n,m)$, *if* $s > 0$.

Statements (i) and (ii) in Lemma 2.5 are proven in [4], (iii) in [14] and (iv) in [17]. Next, we consider the decomposition step from the complex part to the core. We recall that we delete recursively vertices of degree one in the complex part to obtain the core. The reversed operation of that is to replace every vertex in the core by a rooted tree. Moreover, we note that the excess in Q_G is the same as that in C_G. Thus, we get all graphs in $\mathcal{Q}(n_Q, n_Q + \ell)$ with a fixed number of vertices in the core n_C as follows.

- Choose the labels for the n_C vertices of the core, for which there exist $\binom{n_Q}{n_C}$ possibilities;
- Choose a core with n_C vertices and $n_C + \ell$ edges, for which we have $|\mathcal{C}(n_C, n_C + \ell)|$ possibilities;
- Choose n_C rooted trees with n_Q vertices in total. According to Cayley's formula (see e.g. [22]) we have for that $n_C n_Q^{n_Q - n_C - 1}$ possibilities.

Now we sum up over all values for n_C to obtain

$$|\mathcal{Q}(n_Q, n_Q+\ell)| = \sum_{n_C} \binom{n_Q}{n_C} |\mathcal{C}(n_C, n_C+\ell)| n_C n_Q^{n_Q-n_C-1}. \tag{2.2}$$

In the sums of (2.1) and (2.2) we did not specify precisely in which sets the summation indices lie. But it is convenient to consider only terms, which are non-zero. This leads to the definition of admissible values. From now on we always assume that we sum only over admissible values.

Definition 2.10. We call a set of parameters *admissible* if there exists at least one graph having these values for the corresponding parameters.

2.5 Main contributions in counting formulas

The next step is to find in the sums (2.1) and (2.2) those terms, which are significantly larger than the other ones. In order to make that more precise, we use the following terminology.

Definition 2.11. For each $n \in \mathbb{N}$ let $I_0(n), I(n) \subseteq \mathbb{N}$ be finite index sets such that $I_0(n) \subseteq I(n)$. In addition, let $s_n(i) \geq 0$ for each $i \in I(n)$. Then the *main contribution* to the sums

$$\sum_{i \in I(n)} s_n(i)$$

is provided by $i \in I_0(n)$ if

$$\sum_{i \in I(n) \setminus I_0(n)} s_n(i) = o\left(\sum_{i \in I(n)} s_n(i)\right)$$

for $n \to \infty$. In that case we also say that the terms provided by $i \in I(n) \setminus I_0(n)$ are *negligible*.

Now the goal is to find sets I_{n_Q}, I_ℓ and I_{n_C} such that the main contributions to (2.1) and (2.2) are provided by $n_Q \in I_{n_Q}, \ell \in I_\ell$ and $n_C \in I_{n_C}$. Having such sets we immediately get results about the structure of a random outerplanar graph $G = A(n, m)$. Namely, that whp $n_Q(G) \in I_{n_Q}, \ell(G) \in I_\ell$ and $n_C(G) \in I_{n_C}$. To get strong results, we aim to find sets, which are as small as possible. Afterwards we use this concentration information and a double counting argument (see Lemma 5.19) to deduce the component structure of G. The main challenge, which remains, is to determine I_{n_Q}, I_ℓ and I_{n_C}. In [16] and [17] this was done by further decomposing the core into the kernel and count the number of kernels. We have seen in Section 2.4 that this concept is not applicable for outerplanar graphs, whence we need a different way to find I_{n_Q}, I_ℓ and I_{n_C}. In the following we present our strategy how to find these sets in the case of outerplanar graphs.

In order to illustrate the main idea of the analysis of the sums (2.1) and (2.2), we consider the generic sums

$$S_n = \sum_{i \in I(n)} s_n(i)$$

from Definition 2.11. The goal is to find 'small' sets $I_0(n)$ such that the main contribution to S_n is provided by $i \in I_0(n)$ or equivalently 'large' sets $I_1(n)$ such that the terms provided by $i \in I_1(n)$ are negligible in S_n. Our method to find these sets $I_1(n)$ is mainly based on the following observation.

Lemma 2.6. *For each $n \in \mathbb{N}$ let $I_1(n), I(n) \subseteq \mathbb{N}$ be finite index sets such that $I_1(n) \subseteq I(n)$ and let $s_n(i) \geq 0$ for each $i \in I(n)$. In addition, for each $n \in \mathbb{N}$ let $f_n : I_1(n) \to I(n)$ be a function. We assume that there are a function ε with $\varepsilon(n) = o(1)$ and a constant $M > 0$ such that for all $n \in \mathbb{N}, i \in I_1(n)$ and $j \in I(n)$*

$$\frac{s_n(i)}{s_n\left(f_n(i)\right)} \leq \varepsilon(n), \tag{2.3}$$

$$\text{and} \qquad \left|f_n^{-1}\left(\{j\}\right)\right| \leq M. \tag{2.4}$$

Then the terms provided by $i \in I_1(n)$ are negligible in $\sum_{i \in I(n)} s_n(i)$.

Lemma 2.6 is obvious, because

$$\sum_{i \in I_1(n)} s_n(i) \leq \varepsilon(n) \sum_{i \in I_1(n)} s_n(f_n(i)) \leq \varepsilon(n) M \sum_{i \in I(n)} s_n(i) = o(1) \sum_{i \in I(n)} s_n(i).$$

In most cases when we apply Lemma 2.6 the functions f_n will be of the form $f_n(i) = i + g(n)$ for some function $g : \mathbb{N} \to \mathbb{Z}$ or of the form $f_n(i) = \lfloor ci \rfloor$ for some constant $c > 0$. We note that such functions f_n always fulfil (2.4) for some $M > 0$. Thus, it remains to find a function ε with $\varepsilon = o(1)$ such that (2.3) is satisfied. For simplicity, we demonstrate our method of doing that only for the case when $f_n(i) = i - g(n)$ for some function g with $g(n) > 0$. Moreover, we assume that $I(n) = \{a_n, a_n + 1, \ldots, b_n\}$ for some $a_n < b_n$. We observe that

$$\frac{s_n(i)}{s_n(f_n(i))} = \frac{s_n(i)}{s_n(i - g(n))} = \prod_{k=i-g(n)}^{i-1} \frac{s_n(k+1)}{s_n(k)}. \tag{2.5}$$

Thus, we aim to find good upper bounds for $\frac{s_n(k+1)}{s_n(k)}$. We commonly state these bounds in the form $\exp(h(n))$ for some function $h : \mathbb{N} \to \mathbb{R}$. Then if we assume that

$$\frac{s_n(k+1)}{s_n(k)} \leq \exp(h(n)) \qquad \forall n \in \mathbb{N}, \forall k \in I(n) \backslash \{b_n\}, \tag{2.6}$$

we get with (2.5)

$$\frac{s_n(i)}{s_n(f_n(i))} \leq \exp\left(g(n)h(n)\right).$$

If we find such functions g and h with

$$g(n)h(n) \to -\infty \tag{2.7}$$

for $n \to \infty$, then we can apply Lemma 2.6. Mostly we first determine a function h such that (2.6) holds and then choose g accordingly such that (2.7) is satisfied. In addition, we note that in most cases we can weaken condition (2.6) in such a way that it suffices if

$$\frac{s_n(k+1)}{s_n(k)} \leq \exp(h(n)) \qquad \forall n \in \mathbb{N}, \forall k \in \widetilde{I}(n)$$

is fulfilled for some subsets $\widetilde{I}(n) \subseteq I(n)$. We can summarise the above idea as follows. The key for a good analysis of the sum $\sum_{i \in I(n)} s_n(i)$ is to have good bounds for the fractions $\frac{s_n(k+1)}{s_n(k)}$.

Now we describe how we find these bounds for the sums in (2.1) and (2.2). We start with the sum in (2.2) (see Lemmas 5.2-5.4). For admissible n_Q, ℓ and n_C we set

$$r(n_C) = r(n_C, n_Q, \ell) := \binom{n_Q}{n_C} |\mathcal{C}(n_C, n_C + \ell)| \, n_C n_Q^{n_Q - n_C - 1}.$$

Then for given n_Q and ℓ the sum in (2.2) becomes

$$|\mathcal{Q}(n_Q, n_Q + \ell)| = \sum_{n_C} r(n_C).$$

In order to find good bounds for $\frac{r(n_C+1)}{r(n_C)}$ we estimate $\frac{|\mathcal{C}(n_C+1, n_C+1+\ell)|}{|\mathcal{C}(n_C, n_C+\ell)|}$ with the following idea (see Lemma 5.2). Let $H \in \mathcal{C}(n_C, n_C + \ell)$ and an edge e of H, which is not a chord, be given. Then we obtain a graph $H' \in \mathcal{C}(n_C + 1, n_C + 1 + \ell)$ if we subdivide e by one vertex and label this new vertex with $n_C + 1$. We will see that there are two main reasons, why this construction gives reasonably good bounds. Firstly, there are only few chords in a typical core (see Lemma 3.3 and Theorem 5.18), whence 'most' of the edges can be subdivided. Secondly, a vertex in the core has typically degree two. Hence, 'most' of the graphs in $\mathcal{C}(n_C + 1, n_C + 1 + \ell)$ can be constructed in that way.

In the next step we consider the sum in (2.1) and shall determine I_{n_Q} and I_ℓ (see Lemmas 5.5-5.17). To simplify notation, for admissible n_Q and ℓ we set

$$s(n_Q, \ell) = s(n_Q, \ell, n, m) := \binom{n}{n_Q} |\mathcal{Q}(n_Q, n_Q + \ell)| \cdot |\mathcal{U}(n_U, m_U)|,$$

whence the sum in (2.1) becomes

$$|\mathcal{A}(n,m)| = \sum_{n_Q,\ell} s(n_Q,\ell).$$

Now the idea is to look at the fractions $\frac{s(n_Q+1,\ell)}{s(n_Q,\ell)}$ and $\frac{s\left(n_Q,\lfloor c\ell\rfloor\right)}{s(n_Q,\ell)}$ for a constant $c>0$. To get bounds for the term $|\mathcal{U}(n_U,m_U)|$, we will use Lemma 2.5. Thus, it remains to find estimates for $\frac{|\mathcal{Q}(n_Q+1,n_Q+1+\ell)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|}$ and $\frac{|\mathcal{Q}(n_Q,n_Q+\lfloor c\ell\rfloor)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|}$.

We will estimate $\frac{|\mathcal{Q}(n_Q,n_Q+\lfloor c\ell\rfloor)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|}$ without considering $\frac{|\mathcal{Q}(n_Q,n_Q+\ell+1)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|}$. Instead we will use that the number of complex outerplanar graphs is bounded from below by the number of complex cacti graphs and from above by the number of complex planar graphs, i.e.

$$\frac{|\mathcal{Q}(n_Q,n_Q+\lfloor c\ell\rfloor)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|} \le \frac{|\mathcal{Q}_P(n_Q,n_Q+\lfloor c\ell\rfloor)|}{|\mathcal{Q}_C(n_Q,n_Q+\ell)|}, \tag{2.8}$$

where $\mathcal{Q}_P(n_Q,n_Q+\ell)$ denotes the class of all complex planar graphs with n_Q vertices and $n_Q+\ell$ edges and $\mathcal{Q}_C(n_Q,n_Q+\ell)$ the class of all complex cacti graphs with n_Q vertices and $n_Q+\ell$ edges. We get estimates for $|\mathcal{Q}_C(n_Q,n_Q+\ell)|$ and $|\mathcal{Q}_P(n_Q,n_Q+\ell)|$ by using ideas from [17] (see Lemmas 4.8-4.9). At first glance the estimate in (2.8) seems to be quite rough. Therefore, we provide in the following lines some insight why we still get a reasonably good bound in that way. Obviously, we have $|\mathcal{Q}_C(n_Q,n_Q+\ell)| \le |\mathcal{Q}(n_Q,n_Q+\ell)| \le |\mathcal{Q}_P(n_Q,n_Q+\ell)|$. Now we use Lemmas 4.8-4.9 to compare these two bounds. We get that there is a constant c such that

$$\frac{|\mathcal{Q}_P(n_Q,n_Q+\ell)|}{|\mathcal{Q}_C(n_Q,n_Q+\ell)|} \le \Theta(1)c^{\ell}, \tag{2.9}$$

where we used that for m as in Theorem 1.2 we have whp $\ell = o\left(n_Q^{\frac{1}{3}}\right)$ (see Theorem 5.18). Thus, we make a multiplicative error of at most $\Theta(1)c^{\ell}$ if we use $|\mathcal{Q}_P(n_Q,n_Q+\ell)|$ as an estimate for $|\mathcal{Q}(n_Q,n_Q+\ell)|$. We observe that the possible error increases at most by the constant factor c if we increase ℓ by one. In order to get a better understanding that this error is quite 'small' for our considerations, we do the following. We investigate how fast $s(n_Q,\ell)$, or more precisely an estimate for $s(n_Q,\ell)$, changes in the variable ℓ. We denote by $u(n_Q,\ell)$ the upper bound of $s(n_Q,\ell)$, which we get in the following way. We bound $|\mathcal{Q}(n_Q,n_Q+\ell)|$ by $|\mathcal{Q}_P(n_Q,n_Q+\ell)|$ and Lemma 4.9. In addition, we bound $|\mathcal{U}(n_U,m_U)|$ with Lemma 2.5 (see (5.17) and (5.23) for a formal definition of $u(n_Q,\ell)$ for the cases $n_U \ge 2m_U$ and $n_U < 2m_U$, respectively). We will see in (5.22) and (5.24) that

$$\frac{u(n_Q,\ell+1)}{u(n_Q,\ell)} = \Theta(1)\frac{n_Q^{\frac{3}{2}}}{\ell^{\frac{3}{2}}}\frac{1}{n}.$$

We note that this fraction tends to infinity in the case $\ell = o\left(\frac{n_Q}{n^{\frac{2}{3}}}\right)$ and to zero in the case $\ell = \omega\left(\frac{n_Q}{n^{\frac{2}{3}}}\right)$. Thus, $u(n_Q, \ell)$ decays very fast outside the range $\ell = \Theta\left(\frac{n_Q}{n^{\frac{2}{3}}}\right)$. We observe that this decay is 'much faster' than the growth of the error in (2.9), whence the estimate in (2.8) is tight enough for our considerations. We will also see that as a consequence the 'typical' values for n_Q and ℓ satisfy $\ell = \Theta\left(\frac{n_Q}{n^{\frac{2}{3}}}\right)$ (see Lemma 5.9 and Theorem 5.18).

Another indicator why that estimate is tight enough is the following observation. The only difference between the structure of an outerplanar graph and a cacti graph is that an outerplanar graph can have chords (see Section 3.1). In other words, an outerplanar graph is a cacti graph if and only if it has no chords. We will see that the number of chords is bounded by the excess (see Lemma 3.3) and that for m as in Theorem 1.2 the excess is typically 'small' (see Theorem 5.18). Thus, a 'typical' outerplanar graph has only a very 'small' number of chords, or roughly speaking it is 'almost' a cacti graph. Hence, we expect that the number of outerplanar graphs is only 'slightly larger' than the number of cacti graphs.

For the fraction $\frac{|\mathcal{Q}(n_Q+1,n_Q+1+\ell)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|}$ (see Lemmas 5.5-5.6) we define for $i \in \{0,1\}$

$$r_i(n_C) = r_i(n_C, n_Q, \ell) := \binom{n_Q+i}{n_C} |\mathcal{C}(n_C, n_C+\ell)|\, n_C (n_Q+i)^{n_Q+i-n_C-1}.$$

With this notation we have

$$\frac{|\mathcal{Q}(n_Q+1, n_Q+1+\ell)|}{|\mathcal{Q}(n_Q, n_Q+\ell)|} = \frac{\sum_{n_C} r_1(n_C)}{\sum_{n_C} r_0(n_C)}. \tag{2.10}$$

From the analysis of (2.2) we already know sets I_0, I_1 such that the main contributions to $\sum_{n_C} r_0(n_C)$ and $\sum_{n_C} r_1(n_C)$ are provided by $n_C \in I_0$ and $n_C \in I_1$, respectively. We will see that we may assume $I := I_0 = I_1$ (see Lemmas 5.3-5.4). Then we will get a good bound for (2.10) if for $n_C \in I$ we estimate the fraction

$$\frac{r_1(n_C)}{r_0(n_C)} = (n_Q+1)\frac{n_Q+1}{n_Q-n_C+1}\left(\frac{n_Q+1}{n_Q}\right)^{n_Q-n_C-1}. \tag{2.11}$$

It is important to point out that the right-hand side in (2.11) is a rational and exponential function in the two variables n_Q and n_C, whence the analysis becomes quite simple.

3 Preliminaries

3.1 Outerplanar and cacti graphs

For a graph G we denote by $V(G)$ the set of vertices and by $E(G)$ the set of edges. Throughout this thesis, all graphs are vertex-labelled, so we let $V(G) = \{1, \ldots, n\}$. In addition, we denote by $|G|$ the number of vertices in G. In this chapter we look at two special subclasses of planar graphs: outerplanar graphs and cacti graphs.

Definition 3.1 ([6, p.115])**.** A graph is called *outerplanar* if it has a drawing in the plane in which every vertex lies on the boundary of the outer face.

We gather some properties of outerplanar graphs, which we will use later. First we are interested in the block structure and we start with a well-known theorem.

Theorem 3.1. *An outerplanar graph is biconnected if and only if it has a unique Hamiltonian cycle.*

A proof can be found for example in [21]. In the following we will always assume that the unique Hamiltonian cycle lies on the outer face. Using that statement we can immediately deduce the block structure of an outerplanar graph.

Proposition 3.1. *A block of an outerplanar graph is either*

- *an isolated vertex;*
- *a bridge or;*
- *a dissection of a convex polygon.*

Definition 3.2. Let G be an outerplanar graph. An edge of G is called a *chord* if it lies in a biconnected block B, but not in the unique Hamiltonian cycle of B.

Next, we consider a result, which bounds the number of edges in an outerplanar graph.

Theorem 3.2. *An outerplanar graph with $n \geq 2$ vertices has at most $2n - 3$ edges.*

Proof. This follows along the lines of the proof of the corresponding statement that a planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges (see e.g. [6, p.97]). One just needs the additional argument that there is one face, containing all n vertices. □

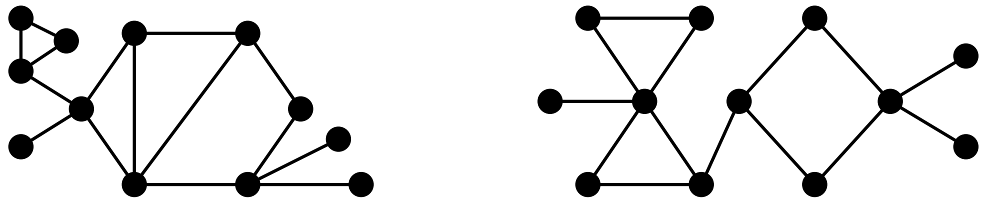

Figure 3.1: Two outerplanar graphs. The right one is also a cactus, while the left one is not a cactus.

In Chapter 5 the number of chords in an outerplanar graph plays a crucial role. There we will need the following result, which bounds the number of chords.

Lemma 3.3. *Let G be an outerplanar graph and $\ell = \ell(G)$ be the excess of G. Moreover, we denote by $b = b(G)$ the number of chords in G. Then we have*

$$b \leq \ell.$$

Proof. We recall that the core C_G has $|C_G|$ vertices and $|C_G| + \ell$ edges and that all chords lie in the core. Now let H be the graph, which is obtained from C_G by deleting all chords. We note that H has $|C_G|$ vertices, $|C_G| + \ell - b$ edges and minimum degree at least two. Thus, we obtain $2\,|C_G| \leq 2(|C_G| + \ell - b)$, which shows the assertion. □

Finally, we look at special outerplanar graphs, namely those without chords. We call them cacti graphs. There are two possible ways to define them formally.

Definition 3.3. A graph is called a *cactus* if each block is either

- an isolated vertex;
- a bridge or;
- a cycle.

Theorem 3.4. *G is a cactus if and only if no edge of G lies in more than one cycle.*

Proof. „$\Longrightarrow$“ We assume to the contrary that G is a cactus and there is an edge which lies in two cycles $C_1 \neq C_2$. Then $C_1 \cup C_2$ is biconnected, whence there is a block B, containing $C_1 \cup C_2$. But then B violates the definition of a cactus.
„$\Longleftarrow$“ We assume that G is not a cactus. Then there is a block B which is biconnected, but not a cycle. Thus, B must contain an edge, which lies in more than one cycle. □

3.2 Bounds and asymptotic formulas

In order to express asymptotic properties of a random graph, such as the number of vertices in the largest component, we use the following notations (see e.g. [13]). We recall that an event occurs *with high probability* (abbreviated as whp) if it occurs with probability tending to one as n tends to infinity.

Definition 3.4. Let $X = (X_n)_{n \in \mathbb{N}}$ be a sequence of random variables. For a function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$, $c_1, c_2 > 0$ and $n \in \mathbb{N}$, we consider the two inequalities

$$|X_n| \leq c_1 f(n), \tag{3.1}$$
$$|X_n| \geq c_2 f(n). \tag{3.2}$$

We say that

- $X_n = O(f)$ whp if there is a $c_1 > 0$ such that (3.1) holds whp;
- $X_n = o(f)$ whp if for every $c_1 > 0$ (3.1) holds whp;
- $X_n = \Omega(f)$ whp if there is a $c_2 > 0$ such that (3.2) holds whp;
- $X_n = \omega(f)$ whp if for every $c_2 > 0$ (3.2) holds whp;
- $X_n = \Theta(f)$ whp if $X_n = O(f)$ and $X_n = \Omega(f)$ whp;
- $X_n = O_p(f)$ if for every $\delta > 0$ there are $c_1 > 0$ and $N \in \mathbb{N}$ such that (3.1) holds for all $n \geq N$ with a probability of at least $1 - \delta$;
- $X_n = \Theta_p(f)$ if for every $\delta > 0$ there are $c_1, c_2 > 0$ and $N \in \mathbb{N}$ such that (3.1) and (3.2) hold for all $n \geq N$ with a probability of at least $1 - \delta$.

In our proofs we will frequently use the following asymptotic formulas and bounds.

Lemma 3.5 (Stirling's formula, [11])**.** *For $n \to \infty$ we have*

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right).$$

Lemma 3.6 (Chebyshev's inequality, [18])**.** *Let X be a random variable and $\varepsilon > 0$. Then we have*

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq \varepsilon\right] \leq \frac{\mathbb{V}[X]}{\varepsilon^2}.$$

The next formulas give bounds for $\exp(x)$ and $1 + x$. They are mainly based on the well-known formulas (see e.g. [19])

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
$$\log\left(\frac{1}{1-x}\right) = \sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{for } x \in (-1, 1).$$

Lemma 3.7. *Let $\delta \in (0,1)$. Then for $x \in [-\delta, \delta]$*

$$1 + x = \exp\left(x - \frac{x^2}{2} + O\left(x^3\right)\right).$$

Proof. We have

$$\log(1+x) = x - \frac{x}{2} + \sum_{k=3}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x}{2} + O\left(x^3 \sum_{k=0}^{\infty} \delta^k\right) = x - \frac{x}{2} + O\left(x^3\right).$$

□

Lemma 3.8. *Let $x \in \mathbb{R}$. Then*

$$1 + x \leq \exp(x).$$

Proof. The statement is trivial for $x \leq -1$ or $x \geq 0$. For $x \in (-1,0)$ we use that $\sum_{k=2}^{N} \frac{x^k}{k!} \geq 0$ for each $N \geq 2$. □

Lemma 3.9. *Let $x \in [0,2]$. Then*

$$1 + x \geq \exp\left(\frac{x}{2}\right).$$

Proof. We have

$$\exp\left(\frac{x}{2}\right) = 1 + \frac{x}{2} + \sum_{k=2}^{\infty} \frac{x^k}{2^k k!} \leq 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 \sum_{k=2}^{\infty} \frac{1}{k!} \leq 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 \leq 1 + x.$$

□

Lemma 3.10. *Let $x \in [-1,0]$. Then*

$$1 + \frac{x}{2} \geq \exp(x).$$

Proof. The assertion follows because of

$$\exp(x) = 1 + x + \sum_{k=2}^{\infty} \frac{(-1)^k |x|^k}{k!} \leq 1 + x + \frac{x^2}{2} \leq 1 + x - \frac{x}{2} = 1 + \frac{x}{2}.$$

□

Lemma 3.11. *Let $x \geq 0$. Then*

$$1 - x \leq \exp\left(-x - \frac{x^2}{2}\right).$$

Proof. For $x \geq 1$ the statement is trivial, otherwise we have

$$\log(1-x) = -x - \frac{x^2}{2} - \sum_{k=3}^{\infty} \frac{x^k}{k} \leq -x - \frac{x^2}{2}.$$

□

Next, we state two bounds for the falling factorial $(k)_i = \frac{k!}{(k-i)!}$.

Lemma 3.12. *Let $k, i \in \mathbb{N}$ with $i \leq k$ be given. Then we have*

$$k^i \exp\left(-\frac{i^2}{2(k-i)}\right) \leq (k)_i \leq k^i \exp\left(-\frac{i(i-1)}{2k}\right).$$

Proof. We use Lemma 3.8 to obtain

$$(k)_i = k^i \prod_{j=1}^{i-1}\left(1 - \frac{j}{k}\right) \leq k^i \prod_{j=1}^{i-1} \exp\left(-\frac{j}{k}\right) = k^i \exp\left(-\frac{i(i-1)}{2k}\right).$$

For the lower bound we use again Lemma 3.8 to get for $1 \leq j \leq i-1$

$$\frac{k}{k-j} = 1 + \frac{j}{k-j} \leq \exp\left(\frac{j}{k-j}\right) \leq \exp\left(\frac{j}{k-i}\right).$$

Using that yields

$$\begin{aligned}(k)_i = k^i \prod_{j=1}^{i-1}\left(1 - \frac{j}{k}\right) &\geq k^i \prod_{j=1}^{i-1} \exp\left(-\frac{j}{k-i}\right) \\ = k^i \exp\left(-\frac{(i-1)i}{2(k-i)}\right) &\geq k^i \exp\left(-\frac{i^2}{2(k-i)}\right).\end{aligned}$$

□

Corollary 3.13. *Let $k \in \mathbb{N}$ and $i = i(k) \leq k$. Then for $k \to \infty$*

$$(k)_i = k^i \exp\left(-\frac{i^2}{2k} + O\left(\frac{i^3}{k(k-i)}\right) + O\left(\frac{i}{k}\right)\right).$$

Proof. We use Lemma 3.12 to obtain

$$\begin{aligned}(k)_i &\geq k^i \exp\left(-\frac{i^2}{2k}\right) \exp\left(\frac{i^2}{2k} - \frac{i^2}{2(k-i)}\right) = k^i \exp\left(-\frac{i^2}{2k} - \frac{i^3}{2k(k-i)}\right), \\ (k)_i &\leq k^i \exp\left(-\frac{i^2}{2k}\right) \exp\left(\frac{i^2}{2k} - \frac{i(i-1)}{2k}\right) = k^i \exp\left(-\frac{i^2}{2k} + \frac{i}{2k}\right).\end{aligned}$$

□

Finally, we analyse the term $\binom{\binom{n}{2}}{m}$, which represents the number of different graphs with n vertices and m edges.

Lemma 3.14. *Let $n \in \mathbb{N}$ and $m = m(n) = \Theta(n)$. Then for $n \to \infty$*

$$\binom{\binom{n}{2}}{m} = \frac{n^{2m}}{(2m)^{m+\frac{1}{2}}\sqrt{\pi}} \exp\left(m - \frac{m}{n} - \frac{m^2}{n^2} + O\left(\frac{1}{n}\right)\right).$$

Proof. We use Lemmas 3.5, 3.7 and Corollary 3.13 to get

$$\begin{aligned}
\binom{\binom{n}{2}}{m} &= \frac{\binom{n}{2}_m}{m!} = \frac{\binom{n}{2}^m}{m!} \exp\left(-\frac{m^2}{2\binom{n}{2}} + O\left(\frac{m^3}{\binom{n}{2}\left(\binom{n}{2}-m\right)}\right) + O\left(\frac{m}{\binom{n}{2}}\right)\right) \\
&= \binom{n}{2}^m \frac{1}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + O\left(\frac{1}{m}\right)\right)} \exp\left(-\frac{m^2}{n^2-n} + O\left(\frac{1}{n}\right)\right) \\
&= \frac{n^{2m}}{2^{m+\frac{1}{2}}\sqrt{\pi} m^{m+\frac{1}{2}}} \left(\frac{n-1}{n}\right)^m \exp\left(m - \frac{m^2}{n^2} + O\left(\frac{1}{n}\right)\right) \\
&= \frac{n^{2m}}{2^{m+\frac{1}{2}}\sqrt{\pi} m^{m+\frac{1}{2}}} \exp\left(m\left[-\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right]\right) \exp\left(m - \frac{m^2}{n^2} + O\left(\frac{1}{n}\right)\right) \\
&= \frac{n^{2m}}{(2m)^{m+\frac{1}{2}}\sqrt{\pi}} \exp\left(m - \frac{m}{n} - \frac{m^2}{n^2} + O\left(\frac{1}{n}\right)\right).
\end{aligned}$$

□

Lemma 3.15. *Let $n \in \mathbb{N}$ and $m = m(n) = O(n)$. Then for $n \to \infty$*

$$\binom{\binom{n}{2}}{m}\binom{\binom{n-1}{2}}{m-1}^{-1} = \frac{n^2}{2m} \exp\left(\frac{2m}{n} + O\left(\frac{1}{n}\right)\right).$$

Proof. We have

$$\begin{aligned}
\binom{\binom{n}{2}}{m}\binom{\binom{n-1}{2}}{m-1}^{-1} &= \frac{\binom{n}{2} - m + 1}{m} \prod_{i=0}^{m-2} \frac{\binom{n}{2} - i}{\binom{n-1}{2} - i} \\
&= \frac{n^2}{2m}\left(1 + O\left(\frac{1}{n}\right)\right) \prod_{i=0}^{m-2} \frac{\binom{n}{2} - i}{\binom{n-1}{2} - i}.
\end{aligned}$$

Now we estimate the product in the last line by Lemma 3.7 to obtain

$$\begin{aligned}
\prod_{i=0}^{m-2} \frac{\binom{n}{2} - i}{\binom{n-1}{2} - i} \geq \left(\frac{\binom{n}{2}}{\binom{n-1}{2}}\right)^{m-1} &= \left(1 + \frac{2}{n-2}\right)^{m-1} \\
= \exp\left((m-1)\left[\frac{2}{n-2} + O\left(\frac{1}{n^2}\right)\right]\right) &= \exp\left(\frac{2m}{n} + O\left(\frac{1}{n}\right)\right).
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}\prod_{i=0}^{m-2} \frac{\binom{n}{2} - i}{\binom{n-1}{2} - i} \leq \left(\frac{\binom{n}{2} - m}{\binom{n-1}{2} - m} \right)^{m-1} &= \left(1 + \frac{2n-2}{(n-1)(n-2) - 2m} \right)^{m-1} \\ &= \exp\left((m-1) \left[\frac{2n-2}{n^2 - 3n + 2 - 2m} + O\left(\frac{1}{n^2} \right) \right] \right) \\ &= \exp\left(\frac{2m}{n} + O\left(\frac{1}{n} \right) \right),\end{aligned}$$

which completes the proof. □

4 Phase transitions in cacti graphs

In Sections 4.1-4.3 we will prove Theorem 1.5 by showing that cacti graphs fulfil all conditions of Theorem 2.3. Then, in Sections 4.4-4.5 we will collect further properties of cacti graphs, which we will use in Chapter 5. We start by stating Theorem 1.5 explicitly. To that end, we denote by $\mathcal{A}_C(n,m)$ the class of all cacti graphs on n vertices and m edges and by $A_C(n,m)$ a graph chosen uniformly at random from $\mathcal{A}_C(n,m)$. We also recall that $H_i = H_i(G)$ denotes the i-th largest component of a graph G and $|H_i|$ the number of vertices in H_i.

Theorem 4.1. *Let $m = \frac{n}{2} + s$, where $s = s(n) = o(n)$ and let $G = A_C(n,m)$. For every $i \in \mathbb{N}$ whp the following holds.*

(i) If $\frac{s^3}{n^2} \to -\infty$, then H_i is a tree and

$$|H_i| = \left(\frac{1}{2} + o(1)\right) \frac{n^2}{s^2} \log \frac{|s|^3}{n^2}.$$

(ii) If $\frac{s^3}{n^2} \to c \in \mathbb{R}$, then

$$|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right).$$

(iii) If $\frac{s^3}{n^2} \to \infty$, then

$$|H_1| = 2s + O_p\left(n^{\frac{2}{3}}\right).$$

For $i \geq 2$, we have $|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right)$.

Theorem 4.2. *Let $m = \alpha n$, where $\alpha = \alpha(n)$ converges to a constant in $\left(\frac{1}{2}, 1\right)$ and let $G = A_C(n,m)$. Then*

$$|H_1| = (2\alpha - 1)n + O_p\left(n^{\frac{2}{3}}\right).$$

For $i \geq 2$, we have $|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right)$.

Theorem 4.3. *Let $m = n + t$, where $t = t(n) = o(n)$ and let $G = A_C(n,m)$. Then we have $n - |H_1| = O_p(r(n))$ and whp $n - |H_1| = \Omega(r(n))$, where*

$$r(n) := \begin{cases} |t| & \text{if } \frac{t^5}{n^3} \to -\infty \\ n^{\frac{3}{5}} & \text{if } \frac{t^5}{n^3} \to c \in \mathbb{R} \\ n^{\frac{3}{2}} t^{-\frac{3}{2}} & \text{if } \frac{t^5}{n^3} \to \infty \text{ and } t = o\left(n(\log n)^{-\frac{2}{3}}\right). \end{cases}$$

4.1 Cubic cacti multigraphs

Now we estimate the number of cubic cacti multigraphs, i.e. cacti multigraphs, in which every vertex has degree three. To that end, we use techniques from analytic combinatorics, such as generating functions and singularity analysis (see e.g. [7, 10]). The estimate, which we get by the following lemma, immediately implies that cacti graphs satisfy Statement (ii) in Theorem 2.3. In addition, it is also the starting point to show that Statements (iii) and (iv) are fulfilled.

Definition 4.1. Let $\mathcal{K}$ be the class of all cubic cacti multigraphs and $\mathcal{K}_c$ the subclass of all *connected* graphs in $\mathcal{K}$.

Lemma 4.4. *There exist constants* $c_0, c_1, \gamma > 0$ *such that for* $n \to \infty$

$$|\mathcal{K}(2n,3n)| = (1+o(1))c_0 n^{-\frac{5}{2}}\gamma^{2n}(2n)!,$$

and

$$|\mathcal{K}_c(2n,3n)| = (1+o(1))c_1 n^{-\frac{5}{2}}\gamma^{2n}(2n)!.$$

Proof. Let $\mathcal{K}_c^\circ$ be the class of cubic connected cacti multigraphs, where one vertex is marked. Moreover, let $\mathcal{B}$ be the class of connected cacti multigraphs, where all but one vertex have degree three and the exceptional vertex has degree two. We denote by $B(z), K(z), K_c(z)$ and $K_c^\circ(z)$ the exponential generating functions of the classes $\mathcal{B}, \mathcal{K}, \mathcal{K}_c$ and $\mathcal{K}_c^\circ$, respectively. We now look for relations between these generating functions.

We start by partition the elements of $\mathcal{K}_c^\circ$ into four disjoint subclasses. To that end, we consider the marked vertex v and distinguish the following cases (see also Figure 4.1).
Case 1: There is a loop at v.
Case 2: There is a double edge at v.
Case 3: v lies in a cycle (of length $k \geq 3$).
Case 4: v lies in no cycle and there is no loop or double edge at v.
With that we obtain

$$\begin{aligned} K_c^\circ(z) &= \frac{1}{2}zB(z) + \frac{1}{2}z^2B(z)^2 + \sum_{k\geq 3}\frac{1}{2}z^kB(z)^k + \frac{1}{6}zB(z)^3 \\ &= \frac{zB(z)}{2(1-zB(z))} + \frac{zB(z)^3}{6}. \end{aligned} \tag{4.1}$$

We note that the factors $\frac{1}{2}$ in the first two terms count the weight of the loop and the double edge at v, respectively. We do the same case distinction also for the class $\mathcal{B}$ (see Figure 4.2). In that case we consider the unique vertex v of degree two to obtain

$$\begin{aligned} B(z) &= \frac{z}{2} + \frac{z^2}{2}B(z) + \sum_{k\geq 3}\frac{z^k}{2}B(z)^{k-1} + \frac{z}{2}B(z)^2 \\ &= \frac{z}{2(1-zB(z))} + \frac{z}{2}B(z)^2. \end{aligned} \tag{4.2}$$

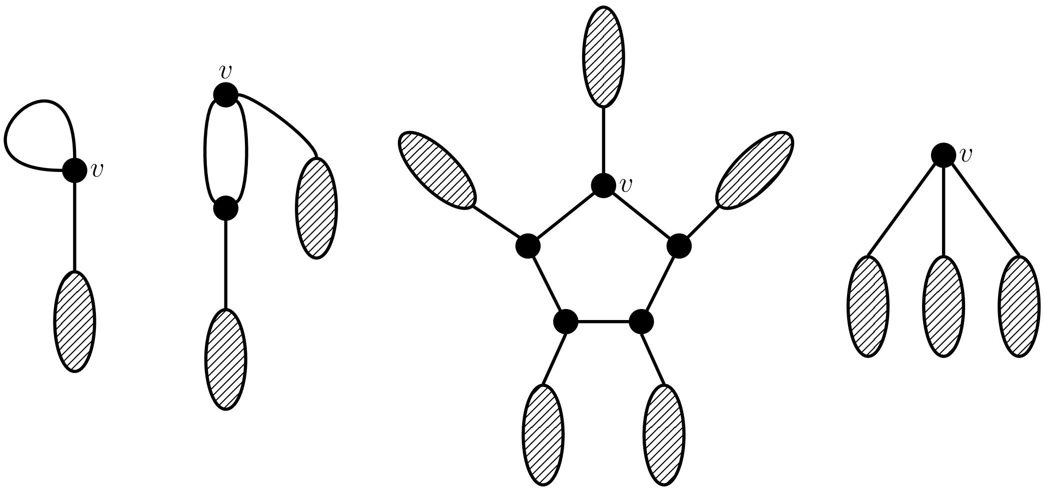

Figure 4.1: Decomposition of graphs in $\mathcal{K}_c^\circ$ according to cases 1-4. The ellipses represent graphs of $\mathcal{B}$.

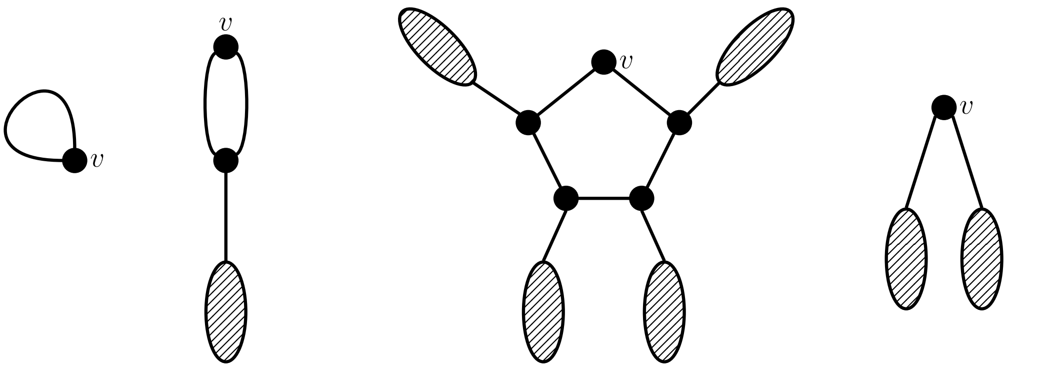

Figure 4.2: Decomposition of graphs in $\mathcal{B}$ according to cases 1-4. The ellipses represent graphs of $\mathcal{B}$.

We observe that the even coefficients in $B(z)$ are all zero, whence

$$B(z) = \sum_{i\geq 1} b_{2i-1} z^{2i-1}$$

for some $b_{2i-1} \in \mathbb{N}$. In order to simplify the following computation, we define

$$\widetilde{B}(u) := \sum_{i\geq 1} b_{2i-1} u^i.$$

Then (4.2) translates to

$$\widetilde{B}(u) = \frac{u}{2\left(1-\widetilde{B}(u)\right)} + \frac{1}{2}\widetilde{B}(u)^2 = G\left(u, \widetilde{B}(u)\right),$$

where $G(z,w)=\frac{z}{2(1-w)}+\frac{w^2}{2}$. Next, we apply Theorem VII.3 in [10] to $\widetilde{B}(u)$. Checking the conditions for $G(z,w)$ and solving the system

$$\begin{aligned} G(r,s) &= s\\ G_w(r,s) &= 1 \end{aligned}$$

yields for $u \to r$

$$\widetilde{B}(u) = s - \rho\sqrt{1-\frac{u}{r}} + O\left(1-\frac{u}{r}\right),$$

where $s = 1-\frac{\sqrt{3}}{3}$, $r=\frac{2\sqrt{3}}{9}$ and $\rho=\frac{\sqrt{2}}{3}$. Moreover, r is the unique dominant singularity of $\widetilde{B}(u)$, due to the aperiodicity of $\widetilde{B}(u)$.

Now we consider $K_c^\circ(z)$ and observe that all odd coefficients are zero. Therefore, we define $\widetilde{K}_c^\circ(u) := K_c^\circ(\sqrt{u})$. The same is true for $K_c(z)$ and we set $\widetilde{K}_c(u) := K_c(\sqrt{u})$. Next, we multiply (4.2) with $B(z)$ and subtract that from (4.1) to obtain

$$K_c^\circ(z) = B(z)^2 - \frac{z}{3}B(z)^3.$$

Multiplying this equation with z^2 and plugging in $\sqrt{u}$ for z gives

$$u \cdot \widetilde{K}_c^\circ(u) = \widetilde{B}(u)^2 - \frac{1}{3}\widetilde{B}(u)^3.$$

As r is the unique dominant singularity of $\widetilde{B}$, it is also the unique dominant singularity of $\widetilde{K}_c^\circ(u)$ and we have the following singular expansions for $u \to r$

$$\widetilde{K}_c^\circ(u) = k_0 + k_1\sqrt{1-\frac{u}{r}} + O\left(1-\frac{u}{r}\right),$$

where $k_0, k_1 \in \mathbb{R}$. Next, we observe

$$K_c(z) = \int \frac{K_c^\circ(z)}{z}\mathrm{d}z,$$

whence we get for $u \to r$

$$\widetilde{K}_c(u) = k_2 + k_3\left(1-\frac{u}{r}\right) + k_4\left(1-\frac{u}{r}\right)^{\frac{3}{2}} + O\left(\left(1-\frac{u}{r}\right)^2\right), \tag{4.3}$$

where $k_2, k_3, k_4 \in \mathbb{R}$. Moreover, we get that r is the unique dominant singularity of $\widetilde{K}_c(u)$. Thus, we can apply Theorems VI.1 and VI.3 in [10] to obtain that there exists a constant $c_1 > 0$ such that for $n \to \infty$

$$[u^n]\,\widetilde{K}_c(u) = c_1 r^{-n} n^{-\frac{5}{2}}\left(1+o(1)\right).$$

Due to the definition of $\widetilde{K}_c(u)$ we obtain with $\gamma = r^{-\frac{1}{2}}$

$$\left[z^{2n}\right] K_c(z) = [u^n]\, \widetilde{K}_c(u) = c_1 \gamma^{2n} n^{-\frac{5}{2}} \left(1 + o(1)\right),$$

which shows the second statement.

Next, we show the first statement. We observe that $K(z) = \exp\left(K_c(z)\right)$ and define $\widetilde{K}(u) := K(\sqrt{u})$. Then also $\widetilde{K}(u) = \exp\left(\widetilde{K}_c(u)\right)$ holds. Now we use (4.3) and the Taylor expansion of exp to get $k_5, k_6, k_7 \in \mathbb{R}$ such that for $u \to r$

$$\widetilde{K}(u) = k_5 + k_6 \left(1 - \frac{u}{r}\right) + k_7 \left(1 - \frac{u}{r}\right)^{\frac{3}{2}} + O\left(\left(1 - \frac{u}{r}\right)^2\right).$$

Hence, we get as before that there is a $c_0 > 0$ such that for $n \to \infty$

$$\left[z^{2n}\right] K(z) = [u^n]\, \widetilde{K}(u) = c_0 \gamma^{2n} n^{-\frac{5}{2}} \left(1 + o(1)\right).$$

This concludes the proof. □

Remark 4.1. For the proof of Lemma 4.4 one can also use a more general method, introduced by Bodirsky, Kang, Löffler, and McDiarmid [1].

4.2 Random cubic cacti multigraphs

Recall that $\mathcal{K}$ denotes the class of all cubic cacti multigraphs. Now we denote by $K(2n, 3n)$ a graph chosen uniformly at random from $\mathcal{K}(2n, 3n)$ and we consider its component structure. We use the idea of Lemma 2 in [16] to show the following statement, which will be the main argument that cacti graphs fulfil Statement (iv) of Theorem 2.3.

Lemma 4.5. *For $n \in \mathbb{N}$ let $G_n = K(2n, 3n)$.*

(i) Uniformly over all $n, j \in \mathbb{N}$ with $j < \frac{n}{2}$ we have

$$\mathbb{P}\left[\ |H_1(G_n)| = 2n - 2j\ \right] = \Theta\left(j^{-\frac{5}{2}}\right).$$

(ii) There exists a constant $M > 0$ such that for all $n, i \in \mathbb{N}$

$$\mathbb{P}\left[\ |H_1(G_n)| \leq 2n - 2i\ \right] \leq M i^{-\frac{3}{2}}.$$

Proof. Let $n, j \in \mathbb{N}$ with $j < \frac{n}{2}$ be given. We define

$$\mathcal{K}_j := \left\{G \in \mathcal{K} \mid |H_1(G)| = |G| - 2j\right\}.$$

Now we can construct all graphs in $\mathcal{K}_j(2n, 3n)$ in the following way.

- Choose $(2n-2j)$ labels for the vertices in the largest component, for which there are $\binom{2n}{2j}$ possibilities.
- Choose a graph from $\mathcal{K}_c(2n-2j, 3n-3j)$ for the largest component. According to Lemma 4.4 we have $(1+o(1))\, c_1(n-j)^{-\frac{5}{2}}\gamma^{2n-2j}(2n-2j)!$ possibilities for doing that.
- Choose a graph from $\mathcal{K}(2j, 3j)$, which can be done in $(1+o(1))\, c_0 j^{-\frac{5}{2}}\gamma^{2j}(2j)!$ different ways.

Hence, we get

$$|\mathcal{K}_j(2n, 3n)| = (1+o(1))c_0c_1(n-j)^{-\frac{5}{2}}j^{-\frac{5}{2}}\gamma^{2n}(2n)!.$$

Together with Lemma 4.4 this implies

$$\begin{aligned}\mathbb{P}[\ |H_1(G_n)| = 2n-2j\ ] &= \frac{|\mathcal{K}_j(2n,3n)|}{|\mathcal{K}(2n,3n)|}\\ &= (1+o(1))\frac{c_1(n-j)^{-\frac{5}{2}}j^{-\frac{5}{2}}}{n^{-\frac{5}{2}}} = \Theta\left(j^{-\frac{5}{2}}\right),\end{aligned}$$

which shows (i). Using that we get a constant $M_1 > 0$ such that for all $n, j \in \mathbb{N}$ with $j < \frac{n}{2}$

$$\mathbb{P}[\ |H_1(G_n)| = 2n-2j\ ] \le M_1 j^{-\frac{5}{2}}.$$

Thus, we obtain

$$\sum_{i\le j<\frac{n}{2}} \mathbb{P}[\ |H_1(G_n)| = 2n-2j\ ] \le M_1 \sum_{i\le j<\frac{n}{2}} j^{-\frac{5}{2}} \le M_1\left(i^{-\frac{5}{2}} + \int_i^\infty x^{-\frac{5}{2}}\mathrm{d}x\right) \le 2M_1 i^{-\frac{3}{2}}. \tag{4.4}$$

Next, we want to bound $\mathbb{P}[\ |H_1(G_n)| \le n\ ]$. To that end, we define

$$\mathcal{K}_{\le}(2n,3n) := \{G \in \mathcal{K}(2n,3n) \mid |H_1(G)| \le n\}.$$

Now let $G \in \mathcal{K}_{\le}(2n,3n)$ be given. Then we can partition the vertex set of G into two disjoint sets A and B such that $\frac{2n}{3} \le |A|, |B| \le \frac{4n}{3}$ and there are no edges between A and B. Hence, we obtain every graph of $\mathcal{K}_{\le}(2n,3n)$ at least once by the following construction.

- Choose i such that $\frac{2n}{3} \le 2i \le \frac{4n}{3}$.
- Choose $2i$ labels out of $\{1,\dots,2n\}$, for which there are $\binom{2n}{2i}$ possibilities.
- Choose a graph from $\mathcal{K}(2i, 3i)$, for which there are $(1+o(1))c_0 i^{-\frac{5}{2}}\gamma^{2i}(2i)!$ possibilities by Lemma 4.4.

- Choose a graph from $\mathcal{K}(2n-2i, 3n-3i)$, for which there are $(1+o(1))c_0(n-i)^{-\frac{5}{2}}\gamma^{2n-2i}(2n-2i)!$ possibilities.

Thus, we get that there is a $M_2 > 0$ such that

$$\begin{aligned}|\mathcal{K}_{\leq}(2n,3n)| &\leq \sum_{\frac{n}{3}\leq i\leq \frac{2n}{3}} \binom{2n}{2i}(2i)!(2n-2i)!(1+o(1))c_0^2 i^{-\frac{5}{2}}(n-i)^{-\frac{5}{2}}\gamma^{2n} \\ &\leq M_2(2n)!\gamma^{2n}\sum_{\frac{n}{3}\leq i\leq \frac{2n}{3}}(n-i)^{-\frac{5}{2}}i^{-\frac{5}{2}} \leq M_2(2n)!\gamma^{2n}\sum_{\frac{n}{3}\leq i\leq \frac{2n}{3}}\left(\frac{n}{3}\right)^{-\frac{5}{2}}\left(\frac{n}{3}\right)^{-\frac{5}{2}} \\ &\leq 3^5 M_2(2n)!\gamma^{2n}n^{-4}.\end{aligned}$$

Using that and Lemma 4.4 gives that there is a $M_3 > 0$ such that

$$\mathbb{P}\left[\,|H_1(G_n)| \leq n\,\right] = \frac{|\mathcal{K}_{\leq}(2n,3n)|}{|\mathcal{K}(2n,3n)|} \leq (1+o(1))\frac{3^5 M_2}{c_0}n^{-\frac{3}{2}} \leq M_3 n^{-\frac{3}{2}}. \tag{4.5}$$

Finally, combining (4.4) and (4.5) yields

$$\begin{aligned}\mathbb{P}\left[\,|H_1(G_n)| \leq 2n-2i\,\right] &\leq \mathbb{P}\left[\,|H_1(G_n)| \leq n\,\right] + \mathbb{P}\left[\,n < |H_1(G_n)| \leq 2n-2i\,\right] \\ &\leq M_3 n^{-\frac{3}{2}} + 2M_1 i^{-\frac{3}{2}} \leq (M_3 + 2M_1)\, i^{-\frac{3}{2}}.\end{aligned}$$

This shows the statement if we choose $M := M_3 + 2M_1$. □

4.3 Proof of main theorems

Now we have all ingredients to show that cacti graphs feature the same phase transitions as planar graphs.

Proof of Theorems 4.1-4.3. We show that cacti graphs fulfil the conditions of Theorem 2.3. To that end, let $\mathcal{X}$ be the class of all cacti graphs and $\mathcal{Y}$ the class of all cacti multigraphs with minimum degree at least three. Now (i) holds due to Lemma 2.2. Next, we use Lemma 4.4. Statement (ii) follows directly from it. For (iii) we observe that

$$\mathbb{P}\left[\,Y(2n,3n) \text{ is connected}\,\right] = \frac{|\mathcal{K}_c(2n,3n)|}{|\mathcal{K}(2n,3n)|} \xrightarrow{n\to\infty} \frac{c_1}{c_0} \in (0,1]\,. \tag{4.6}$$

For (iv) let $\delta > 0$ be given. Then we set $M_\delta = \left\lceil M^{\frac{2}{3}}\delta^{-\frac{2}{3}}\right\rceil$ and obtain by Lemma 4.5(ii)

$$\mathbb{P}\left[\,2n - |H_1(Y(2n,3n))| \geq 2M_\delta\,\right] \leq M\,(M_\delta)^{-\frac{3}{2}} \leq \delta.$$

This shows $|H_1(Y(2n,3n))| = 2n - O_p(1)$. Moreover, for fixed $i \in \mathbb{N}$ we get by Lemma 4.5(i) that $\mathbb{P}\left[\,|H_1(Y(2n,3n))| = 2n-2i\,\right]$ is bounded away from 0. In addition, it is bounded away from 1 by (4.6). Finally (v) follows directly from Theorem 3.4. □

4.4 Random cacti graphs

In the next two sections we will state some results about cacti graphs, which we will use in Chapter 5. Most of them we obtain by a natural extension of results from [17]. We start with the following definition.

Definition 4.2. We denote by

- $\mathcal{Q}_C$ the class of all complex cacti graphs;
- $\mathcal{Q}_P$ the class of all complex planar graphs.

For admissible n_Q and ℓ we set

$$s_C(n_Q,\ell) = s_C(n_Q,\ell,n,m) := \binom{n}{n_Q}|\mathcal{Q}_C(n_Q,n_Q+\ell)|\cdot|\mathcal{U}(n_U,m_U)|,$$
$$s_P(n_Q,\ell) = s_P(n_Q,\ell,n,m) := \binom{n}{n_Q}|\mathcal{Q}_P(n_Q,n_Q+\ell)|\cdot|\mathcal{U}(n_U,m_U)|.$$

Lemma 4.6. *We will use in Chapter 5 the simple observation that*

$$|\mathcal{A}_C(n,m)| \le |\mathcal{A}(n,m)|,$$
$$|\mathcal{Q}_C(n_Q,n_Q+\ell)| \le |\mathcal{Q}(n_Q,n_Q+\ell)| \le |\mathcal{Q}_P(n_Q,n_Q+\ell)|,$$
$$s_C(n_Q,\ell) \le s(n_Q,\ell) \le s_P(n_Q,\ell).$$

We now state a result about the number $n_Q = n_Q(G)$ of vertices in the complex part and the excess $\mathrm{ex}(G)$ of a random cacti graph G.

Theorem 4.7. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Moreover, let $G = A_C(m,n)$. Then we have whp*

$$n_Q = 2s + O_p\left(n^{\frac{2}{3}}\right),$$
$$ex(G) = \Theta\left(\frac{s}{n^{\frac{2}{3}}}\right).$$

Proof. Using the ideas of the proof of Theorem 5.1 in [17] yields the statement. □

4.5 Asymptotic number of cacti graphs

In Chapter 5 we shall use Lemma 4.6. To that end, we state bounds for $|\mathcal{A}_C(n,m)|$, $|\mathcal{Q}_C(n_Q,n_Q+\ell)|$ and $|\mathcal{Q}_P(n_Q,n_Q+\ell)|$. The following result will be used in the proofs of Lemmas 5.8-5.9.

Lemma 4.8. *There are $a, \varepsilon, \rho > 0$ and $b \in \mathbb{R}$ such that for all admissible n_Q and ℓ with $\ell \leq \varepsilon n_Q$*

$$|\mathcal{Q}_C(n_Q, n_Q + \ell)| \geq a n_Q^{n_Q + \frac{3\ell - 1}{2}} \rho^\ell \ell^{-\frac{3\ell}{2} - 2} \exp\left(b\sqrt{\frac{\ell^3}{n_Q}}\right).$$

Proof. Following along the lines of the proofs of Lemma 4.9(ii) and Corollary 4.11 in [17] yields the assertion. □

We get a similar result for $|\mathcal{Q}_P(n_Q, n_Q+\ell)|$ if we combine Lemma 4.9(i) and Corollary 4.11 in [17].

Lemma 4.9 ([17])**.** *There are $a_1, \rho_1 > 0$ and $b_1 \in \mathbb{R}$ such that for all admissible n_Q and ℓ*

$$|\mathcal{Q}_P(n_Q, n_Q + \ell)| \leq a_1 n_Q^{n_Q + \frac{3\ell - 1}{2}} \rho_1^\ell \ell^{-\frac{3\ell}{2} - 3} \exp\left(b_1\sqrt{\frac{\ell^3}{n_Q}}\right).$$

We conclude this section by giving an estimate for the number of cacti graphs. The following lemma is a slightly stronger result than Theorem 1.8 in [17].

Lemma 4.10. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there are $c > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$*

$$|\mathcal{A}_C(n, m)| \geq \frac{n^{n-\frac{1}{2}}}{(n-2s)^{\frac{n}{2}-s}} \exp\left(\frac{n}{2} - s + c \cdot \frac{s}{n^{\frac{2}{3}}}\right).$$

Proof. Due to (2.1) we have $|\mathcal{A}_C(n, m)| = \sum_{n_Q, \ell} s_C(n_Q, \ell)$. Now we estimate $s_C(n_Q, \ell)$ for $n_Q \in I_1 := \left[2s - c_1 n^{\frac{2}{3}}, 2s + c_1 n^{\frac{2}{3}}\right]$ and $\ell \in I_2 := \left[c_2 \frac{s}{n^{\frac{2}{3}}}, c_3 \frac{s}{n^{\frac{2}{3}}}\right]$, where we choose $c_1, c_2, c_3 > 0$ later. By Lemma 2.5(ii) we get that

$$|\mathcal{U}(n_U, m_U)| = \Theta(1)\binom{\binom{n_U}{2}}{m_U}.$$

Now we use Lemmas 3.5, 3.14 and 4.8 to obtain

$$\begin{aligned}
s_C(n_Q, \ell) &\geq \Theta(1)\frac{n^{n+\frac{1}{2}}}{n_Q^{n_Q+\frac{1}{2}} n_U^{n_U+\frac{1}{2}}} \cdot n_Q^{n_Q + \frac{3\ell-1}{2}} \rho^\ell \ell^{-\frac{3\ell}{2}-2} \binom{\binom{n_U}{2}}{m_U} \\
&= \Theta(1)\frac{n^{n+\frac{1}{2}} n_Q^{\frac{3\ell}{2}-1} \rho^\ell}{n_U^{n_U+\frac{1}{2}} \ell^{\frac{3\ell}{2}+2}} \cdot \frac{n_U^{2m_U}}{(2m_U)^{m_U+\frac{1}{2}}} \exp(m_U) \\
&= \Theta(1)\left(\frac{n-2s}{2m_U}\right)^{\frac{n}{2}-s} \left(\frac{n-n_Q}{2m_U}\right)^{2s-n_Q} \frac{n^{n-\frac{1}{2}}}{(n-2s)^{\frac{n}{2}-s}} \left(\frac{2n_Q^{\frac{3}{2}} \rho m_U}{\ell^{\frac{3}{2}} n_U^2 \ell^{\frac{2}{\ell}}}\right)^\ell \frac{\exp(m_U)}{n_Q}.
\end{aligned} \tag{4.7}$$

Now we observe that we can choose $0 < c_2 < c_3$ such that for all $n_Q \in I_1$ and $\ell \in I_2$

$$\left(\frac{2n_Q^{\frac{3}{2}} \rho m_U}{\ell^{\frac{3}{2}} n_U^2 \ell^{\frac{2}{\ell}}} \right) \geq 2.$$

Plugging in that in (4.7) and using Lemma 3.7 yields

$$\begin{aligned} s_C(n_Q, \ell) \geq & \Theta(1) \exp\left(\left(\frac{n}{2} - s \right) \left[\frac{n_Q - 2s + \ell}{m_U} - \frac{(n_Q - 2s + \ell)^2}{2m_U^2} \right] \right) \\ & \times \exp\left((2s - n_Q) \frac{n_Q - 2s + 2\ell}{n + 2s - 2n_Q - 2\ell} \right) \frac{n^{n-\frac{1}{2}}}{(n-2s)^{\frac{n}{2}-s}} \frac{2^\ell}{n_Q} \exp\left(m_U\right). \end{aligned}$$

To simplify notation we set $n_Q = 2s + p$, whence $p = O\left(n^{\frac{2}{3}}\right)$. Then we have

$$\begin{aligned} s_C(n_Q, \ell) &\geq \Theta(1) \exp\left(\frac{(n-2s)\, p}{n - 2s - 2p} - p - \frac{p^2}{n - 2s - 2p} \left(1 + \frac{n - 2s}{n - 2s - 2p} \right) \right) \\ &\quad \times \exp\left(\frac{(n-2s)\, \ell}{n - 2s - 2p} - \ell \right) \frac{n^{n-\frac{1}{2}}}{(n-2s)^{\frac{n}{2}-s}} \frac{2^\ell}{n_Q} \exp\left(m - 2s\right) \\ &= \Theta(1) \exp\left(O\left(\frac{p^3}{n^2} \right) \right) \frac{n^{n-\frac{1}{2}}}{(n-2s)^{\frac{n}{2}-s}} \frac{2^\ell}{n_Q} \exp\left(m - 2s\right) \\ &= \Theta(1) \frac{n^{n-\frac{1}{2}}}{(n-2s)^{\frac{n}{2}-s}} \frac{1}{s} \exp\left(m - 2s + \log 2 \cdot \ell\right). \end{aligned}$$

As this estimate holds uniformly over all $n_Q \in I_1$ and $\ell \in I_2$, we finally get

$$\begin{aligned} \sum_{n_Q, \ell} s_C(n_Q, \ell) &\geq \sum_{n_Q \in I_1, \ell \in I_2} s_C(n_Q, \ell) \\ &\geq \Theta(1) n^{\frac{2}{3}} \frac{s}{n^{\frac{2}{3}}} \frac{n^{n-\frac{1}{2}}}{(n-2s)^{\frac{n}{2}-s}} \frac{1}{s} \exp\left(m - 2s + c_2 \log 2 \cdot \frac{s}{n^{\frac{2}{3}}} \right) \\ &= \Theta(1) \frac{n^{n-\frac{1}{2}}}{(n-2s)^{\frac{n}{2}-s}} \exp\left(\frac{n}{2} - s + c_2 \log 2 \cdot \frac{s}{n^{\frac{2}{3}}} \right), \end{aligned}$$

which shows the statement. □

5 Phase transitions in outerplanar graphs

In this chapter we will prove Theorem 1.6 by using the ideas, presented in Chapter 2. We recall that we denoted by $\mathcal{A}$ the class of all outerplanar graphs and by $A(n,m)$ a graph chosen uniformly at random from $\mathcal{A}(n,m)$. In addition, $H_i = H_i(G)$ denotes the i-th largest component of a graph G and $|H_i|$ the number of vertices in H_i. With these notations we can state Theorem 1.6 explicitly.

Theorem 5.1. *Let $m = \frac{n}{2} + s$, where $s = s(n) = o(n)$ and let $G = A(n,m)$. For every $i \in \mathbb{N}$ whp the following holds.*

(i) If $\frac{s^3}{n^2} \to -\infty$, then H_i is a tree and

$$|H_i| = \left(\frac{1}{2} + o(1)\right) \frac{n^2}{s^2} \log \frac{|s|^3}{n^2}.$$

(ii) If $\frac{s^3}{n^2} \to c \in \mathbb{R}$, then

$$|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right).$$

(iii) If $\frac{s^3}{n^2} \to \infty$, then

$$|H_1| = 2s + O_p\left(n^{\frac{2}{3}}\right).$$

For $i \geq 2$, we have $|H_i| = \Theta_p\left(n^{\frac{2}{3}}\right)$.

We start by recalling some notations from Chapter 2. For a graph G we denote by

- $n_Q = n_Q(G)$ the number of vertices in the complex part;
- $n_C = n_C(G)$ the number of vertices in the core;
- $\ell = \ell(G)$ the excess;
- $n_U = n_U(G)$ the number of vertices outside the complex part;
- $m_U = m_U(G)$ the number of edges outside the complex part.

Moreover, we denote by

- $\mathcal{Q}$ the class of all complex outerplanar graphs;
- $\mathcal{C}$ the class of all outerplanar cores;

- $\mathcal{U}$ the class of all graphs without complex components.

Next, we recall the main idea from Chapter 2. There we obtained the following two equations by decomposing outerplanar graphs

$$|\mathcal{A}(n,m)| = \sum_{n_Q,\ell} s(n_Q,\ell), \tag{5.1}$$

$$|\mathcal{Q}(n_Q,n_Q+\ell)| = \sum_{n_C} r(n_C), \tag{5.2}$$

where

$$r(n_C) = \binom{n_Q}{n_C} |\mathcal{C}(n_C,n_C+\ell)|\, n_C n_Q^{n_Q-n_C-1},$$
$$s(n_Q,\ell) = \binom{n}{n_Q} |\mathcal{Q}(n_Q,n_Q+\ell)| \cdot |\mathcal{U}(n_U,m_U)|.$$

The goal is to find 'small' intervals I_{n_Q}, I_ℓ and I_{n_C} such that the main contributions to the sums in (5.1) and (5.2) are provided by $n_Q \in I_{n_Q}$, $\ell \in I_\ell$ and $n_C \in I_{n_C}$. In other words, we want to find the 'typical' number of vertices in the complex part, the 'typical' excess and the 'typical' number of vertices in the core of a random outerplanar graph. Then, we use this structural information about a random outerplanar graph to deduce the component structure.

In order to realise that idea we proceed as follows. In the first section we determine I_{n_C}. Then, Sections 5.2-5.4 are devoted to determine I_{n_Q} and I_ℓ in the following way. We first estimate $\frac{|\mathcal{Q}(n_Q+1,n_Q+1+\ell)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|}$. Then, we show that for each $K \in \mathbb{N}$ the terms provided by $\ell < K$ are negligible in $\sum_{n_Q,\ell} s(n_Q,\ell)$. Using that we find I_{n_Q} and I_ℓ in Section 5.4. In Section 5.5 we show that the complex part of a random outerplanar graph has whp a unique largest component, which is significantly larger than all other complex components. Finally, we prove Theorem 5.1 in Section 5.6 by showing that all non-complex components are 'small'.

5.1 Outerplanar cores

In this section we determine a 'small' interval I_{n_C} by analysing $\frac{r(n_C+1)}{r(n_C)}$. We will see that this fraction is significantly larger than one if $n_C = o\left(\sqrt{n_Q\ell}\right)$ and significantly smaller than one if $n_C = \omega\left(\sqrt{n_Q\ell}\right)$. Using that we will deduce that the main contribution to $\sum_{n_C} r(n_C)$ is provided by $n_C = \Theta\left(\sqrt{n_Q\ell}\right)$. In order to find a good estimate for $\frac{r(n_C+1)}{r(n_C)}$, we just need good bounds for $\frac{|\mathcal{C}(n_C+1,n_C+1+\ell)|}{|\mathcal{C}(n_C,n_C+\ell)|}$, since all other involved quantities in $r(n_C)$ are well-understood. We obtain these bounds from the following statement.

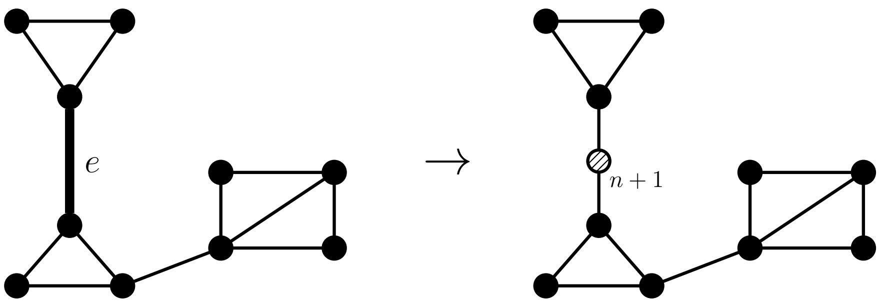

Figure 5.1: An extension via (C1).

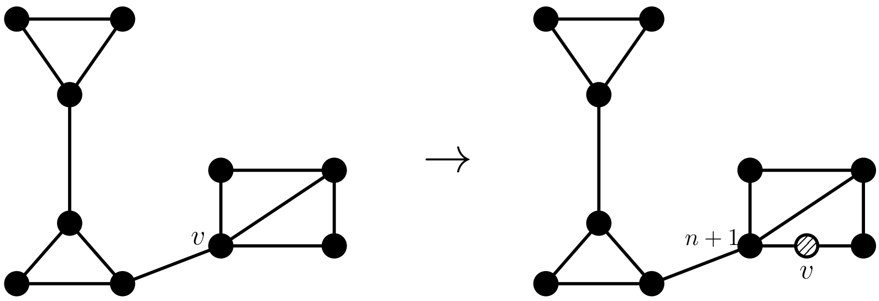

Figure 5.2: An extension via (C2).

Lemma 5.2. *(i) For all admissible $n = n_C$ and ℓ we have*

$$\frac{|\mathcal{C}(n+1, n+1+\ell)|}{|\mathcal{C}(n, n+\ell)|} \geq n + \frac{\ell}{80}.$$

(ii) If in addition $n - 8\ell \geq 0$, then

$$\frac{|\mathcal{C}(n+1, n+1+\ell)|}{|\mathcal{C}(n, n+\ell)|} \leq (n+\ell)\frac{n+1}{n+1-8\ell}.$$

Proof. The idea is to construct graphs in $\mathcal{C}(n+1, n+1+\ell)$ starting from graphs in $\mathcal{C}(n, n+\ell)$. To that end, we say that $H' \in \mathcal{C}(n+1, n+1+\ell)$ is an extension of $H \in \mathcal{C}(n, n+\ell)$ (or H can be extended to H') if one of the following two conditions holds.

(C1) There is an edge e in H such that subdividing e by one vertex and labelling this new vertex with $n+1$ leads to H'.

(C2) There is a vertex v in H, whose degree lies in $\{3, 4, 5, 6\}$, such that we can obtain H' from H in the following way. We relabel the vertex v with label $n+1$ and subdivide one of the incident edges by one vertex. The new vertex then receives the label of v.

We now show that for a fixed $H' \in \mathcal{C}(n+1, n+1+\ell)$ there are only a few graphs, which can be extended to H'. We consider first the case, when H' is an extension via (C1), i.e. there is some $H \in \mathcal{C}(n, n+\ell)$, which can be extended to H' by construction (C1). Then the vertex $n+1$ in H' has degree two, whence H' cannot be an extension via (C2). Moreover, we observe that there is at most one $H \in \mathcal{C}(n, n+\ell)$, which can be extended to H'. Next, we assume that H' is an extension via (C2). Then the number of graphs, which can be extended to H', is bounded from above by the degree of the vertex $n+1$, which is at most six.

In the next step we estimate the number of extensions of a fixed graph $H \in \mathcal{C}(n, n+\ell)$. We observe that if we choose an edge in H, which is not a chord, then (C1) leads to a graph in $\mathcal{C}(n+1, n+1+\ell)$. So it seems that the number of chords, which we denote by $b = b(H)$, plays an important role. We know by Lemma 3.3 that $b \leq \ell$.

Next, we partition the elements of $\mathcal{C}(n, n+\ell)$ into two disjoint subclasses, depending on the number of chords:

$$\mathcal{C}_1(n, n+\ell) = \left\{ H \in \mathcal{C}(n, n+\ell) \;\middle|\; b(H) \leq \frac{19}{20}\ell \right\};$$
$$\mathcal{C}_2(n, n+\ell) = \left\{ H \in \mathcal{C}(n, n+\ell) \;\middle|\; b(H) > \frac{19}{20}\ell \right\}.$$

Now every graph in $\mathcal{C}_1(n, n+\ell)$ has at least $n + \frac{\ell}{20}$ extensions via (C1). In addition, a graph in $\mathcal{C}_2(n, n+\ell)$ has at least n extensions via (C1). Thus, we obtain at least $|\mathcal{C}_1(n, n+\ell)| \left(n + \frac{\ell}{20}\right) + |\mathcal{C}_2(n, n+\ell)| \cdot n$ different graphs in $\mathcal{C}(n+1, n+1+\ell)$ if we extend graphs from $\mathcal{C}(n, n+\ell)$ via (C1).

Next, we consider a graph $H \in \mathcal{C}_2(n, n+\ell)$. We want to bound the number of vertices, whose degree lies in $\{3, 4, 5, 6\}$, from below. To that end, we define H_B as that graph, which we obtain from H after the following two operations. First we delete all edges, which are not chords and then we delete all isolated vertices. Since H_B is outerplanar, we get by using Theorem 3.2 that H_B has at least $\frac{b+3}{2} \geq \frac{b}{2} > \frac{19\ell}{40}$ vertices. All these vertices have degree at least one in H_B and therefore at least degree three in H. On the other hand, H has n vertices, $n+\ell$ edges and minimum degree at least 2, whence there are at most $\frac{2\ell}{5}$ vertices with degree at least seven. Thus, H has at least $\frac{19\ell}{40} - \frac{2\ell}{5} = \frac{3\ell}{40}$ vertices, whose degree lies in $\{3, 4, 5, 6\}$. We can use each such vertex v to extend H via (C2), since there are at least two edges, incident to v, which are not chords. Hence, we obtain at least $\frac{1}{6} \cdot |\mathcal{C}_2(n, n+\ell)| \cdot \frac{3\ell}{40} = \frac{\ell}{80} \cdot |\mathcal{C}_2(n, n+\ell)|$ different graphs in $\mathcal{C}(n+1, n+1+\ell)$ if we extend graphs from $\mathcal{C}(n, n+\ell)$ via (C2). Putting things together we obtain

$$\begin{aligned} |\mathcal{C}(n+1, n+1+\ell)| &\geq |\mathcal{C}_1(n, n+\ell)| \left(n + \frac{\ell}{20}\right) + |\mathcal{C}_2(n, n+\ell)| \cdot \left(n + \frac{\ell}{80}\right) \\ &\geq |\mathcal{C}(n, n+\ell)| \cdot \left(n + \frac{\ell}{80}\right). \end{aligned}$$

This shows part (i).

For (ii) we only consider extensions via (C1), whence we omit the term (C1) in the following. We will show that 'most' graphs in $\mathcal{C}(n+1, n+1+\ell)$ can be obtained by an extension. To that end, we estimate the probability that $G' = C(n+1, n+1+\ell)$ is an extension. We observe that G' is an extension if and only if the vertex $n+1$ has degree two and the two neighbours of it are not adjacent. Therefore, we introduce the following notation. A vertex v is called *good* if v has degree two and the two neighbours are not adjacent. In contrast, we call v *bad* if v has degree two, but is not good. The probability that a fixed vertex $i \in \{1, \dots, n+1\}$ is good in G' does not depend on i. This leads to the idea to bound the number of good vertices in a fixed $H' \in \mathcal{C}(n+1, n+1+\ell)$ from below. As the minimum degree of H' is at least two, there are at most 2ℓ vertices of degree at least three. Since H' is complex, every bad vertex has a neighbour of degree at least three. But the sum of all degrees of vertices with degree at least three is at most $2(n+1+\ell) - 2(n+1-2\ell) = 6\ell$. Thus, there are at most 6ℓ bad vertices, whence we conclude that there are at least $(n+1) - 2\ell - 6\ell = n+1-8\ell$ good vertices. As $H' \in \mathcal{C}(n+1, n+1+\ell)$ was arbitrary, this shows that the probability that G' is an extension is at least $\frac{n+1-8\ell}{n+1}$. Moreover, there are at most $(n+\ell)|\mathcal{C}(n, n+\ell)|$ extensions, whence we obtain

$$|\mathcal{C}(n+1, n+1+\ell)| \leq (n+\ell)|\mathcal{C}(n, n+\ell)| \cdot \frac{n+1}{n+1-8\ell}.$$

□

Now we use Lemma 5.2 to show that we can choose $I_{n_C} = \Theta\left(\sqrt{n_Q \ell}\right)$ if $\ell \to \infty$. We will see later that the assumption $\ell \to \infty$ is not a restriction for our considerations. The next lemma proves that the terms smaller than $\Theta\left(\sqrt{n_Q \ell}\right)$ are negligible. The lemma after that shows that the same is true for the terms larger than $\Theta\left(\sqrt{n_Q \ell}\right)$.

Lemma 5.3. *There are $a, b, c > 0$ such that for all admissible n_Q and ℓ*

$$\sum_{n_C \leq c\sqrt{n_Q \ell}} r(n_C) \leq a \cdot \exp(-b\ell) \sum_{n_C} r(n_C).$$

Proof. Let n_Q and ℓ be admissible. For the case $\ell = 0$ the statement is trivial if we just choose $a \geq 1$. Thus, we may assume $n_Q, \ell \neq 0$. Using Lemma 5.2(i) we obtain

$$\begin{aligned} \frac{r(n_C+1)}{r(n_C)} &= \frac{n_Q - n_C}{n_C + 1} \cdot \frac{n_C + 1}{n_C n_Q} \cdot \frac{|\mathcal{C}(n_C+1, n_C+1+\ell)|}{|\mathcal{C}(n_C, n_C+\ell)|} \\ &\geq \frac{n_Q - n_C}{n_C \cdot n_Q} \left(n_C + \frac{\ell}{80}\right) = \left(1 - \frac{n_C}{n_Q}\right)\left(1 + \frac{\ell}{80 n_C}\right). \end{aligned}$$

Now let $c_1 \in \left(0, \frac{1}{2}\right)$. Observing that $\left(1 - \frac{n_C}{n_Q}\right)\left(1 + \frac{\ell}{80 n_C}\right)$ is monotonically decreasing in n_C, we obtain for $n_C \le c_1\sqrt{n_Q \ell}$

$$\frac{r\left(n_C+1\right)}{r(n_C)} \ge \left(1 - \frac{c_1\sqrt{\ell}}{\sqrt{n_Q}}\right)\left(1 + \frac{\sqrt{\ell}}{80 c_1 \sqrt{n_Q}}\right) \overset{n_Q \ge \ell}{\ge} 1 + \sqrt{\frac{\ell}{n_Q}} \cdot \frac{1 - 80c_1^2 - c_1}{80 c_1}.$$

Now we can choose c_1 so small that we get for all $n_C \le 2c_1\sqrt{n_Q \ell}$

$$\frac{r\left(n_C+1\right)}{r(n_C)} \ge 1 + \sqrt{\frac{\ell}{n_Q}}. \tag{5.3}$$

If $\overline{n_C} := \left\lfloor c_1\sqrt{n_Q \ell}\right\rfloor$ is not admissible, then all $n_C \le \overline{n_C}$ are not admissible. Then the statement is fulfilled if we choose $c \le c_1$. Now we assume that $\overline{n_C}$ is admissible. Using Lemma 3.9 in (5.3) yields that for all $n_C \in I := \left[\frac{3c_1}{2}\sqrt{n_Q \ell}, 2c_1\sqrt{n_Q \ell}\right]$

$$\frac{r(n_C)}{r(\overline{n_C})} = \prod_{i=\overline{n_C}}^{n_C - 1} \frac{r(i+1)}{r(i)} \ge \exp\left(\frac{\sqrt{\ell}}{2\sqrt{n_Q}}\right)^{n_C - \overline{n_C}} \ge \exp\left(\frac{c_1}{2}\ell\right). \tag{5.4}$$

Now we observe that there is a $K > 0$ such that $\left\lfloor 2c_1\sqrt{n_Q \ell}\right\rfloor - \left\lceil \frac{3c_1}{2}\sqrt{n_Q \ell}\right\rceil > \frac{c_1}{3}\sqrt{n_Q \ell}$, whenever $n_Q \ell \ge K$. Thus, we obtain in that case

$$\begin{aligned}
\sum_{n_C \le c_1\sqrt{n_Q \ell}} r(n_C) &\overset{(5.3)}{\le} r(\overline{n_C}) \frac{1}{1 - \left(1 - \frac{1}{2}\sqrt{\frac{\ell}{n_Q}}\right)} \\
&\overset{(5.4)}{\le} \frac{6}{c_1\sqrt{n_Q \ell}} \cdot \exp\left(-\frac{c_1}{2}\ell\right) \sum_{n_C \in I} r(n_C) \cdot \sqrt{\frac{n_Q}{\ell}} \\
&\le \frac{6}{c_1} \exp\left(-\frac{c_1}{2}\ell\right) \sum_{n_C} r(n_C).
\end{aligned}$$

Now we choose $a := \frac{6}{c_1}$, $b := \frac{c_1}{2}$ and $c := \min\left\{c_1, \frac{1}{2\sqrt{K}}\right\}$. Then the statement holds for the case $n_Q \ell \ge K$. Otherwise, if $n_Q \ell < K$, then we get $c\sqrt{n_Q \ell} \le \frac{1}{2\sqrt{K}}\sqrt{K} = \frac{1}{2}$. Due to the assumption $\ell \ne 0$, $n_C = 0$ is not admissible, whence the statement follows also for that case. □

Lemma 5.4. *(i) For all admissible n_Q, ℓ and $c \ge 14$, we have*

$$\sum_{n_C \ge c\sqrt{n_Q \ell}} r(n_C) \le \exp\left(-\frac{c}{2}\ell\right) \sum_{n_C} r(n_C).$$

(ii) For every $\delta > 0$ there is a $N \in \mathbb{N}$ such that

$$\sum_{n_C \geq n_Q^\delta \sqrt{n_Q \ell}} r(n_C) \leq \exp\left(-\frac{1}{2}\ell n_Q^\delta\right) \sum_{n_C} r(n_C)$$

for all admissible n_Q and ℓ with $n_Q \geq N$.

Proof. The idea is similar as in the proof of Lemma 5.3. First we observe that both statements are trivial in the case $\ell = 0$. Thus, we may assume $\ell \neq 0$. For an admissible n_C with $n_C \geq 8\ell$, we know by Lemma 5.2(ii)

$$\begin{aligned}\frac{r(n_C+1)}{r(n_C)} &= \frac{n_Q - n_C}{n_C n_Q} \cdot \frac{|\mathcal{C}(n_C+1, n_C+1+\ell)|}{|\mathcal{C}(n_C, n_C+\ell)|} \leq \frac{n_Q - n_C}{n_C n_Q}(n_C+\ell)\frac{n_C+1}{n_C+1-8\ell} \\ &\leq \frac{n_Q - n_C}{n_C n_Q}(n_C+\ell)\frac{n_C}{n_C-8\ell} = \left(1 - \frac{n_C}{n_Q}\right)\left(1 + \frac{9\ell}{n_C - 8\ell}\right).\end{aligned}$$

Now we fix $c \geq 14$. Observing that $\left(1 - \frac{n_C}{n_Q}\right)\left(1 + \frac{9\ell}{n_C-8\ell}\right)$ is monotonically decreasing in n_C, we obtain for $n_C \geq (c-3)\sqrt{n_Q\ell} \geq 8\ell$

$$\begin{aligned}\frac{r(n_C+1)}{r(n_C)} &\leq \left(1 - \frac{(c-3)\sqrt{\ell}}{\sqrt{n_Q}}\right)\left(1 + \frac{9\ell}{(c-3)\sqrt{n_Q\ell} - 8\sqrt{n_Q\ell}}\right) \\ &\leq \left(1 - \sqrt{\frac{\ell}{n_Q}} \cdot \frac{(c-3)(c-11)-9}{c-11}\right) \leq 1 - \frac{c}{2}\sqrt{\frac{\ell}{n_Q}}. \end{aligned} \tag{5.5}$$

If $\lceil (c-3)\sqrt{n_Q\ell}\rceil$ is not admissible, then all $n_C \geq c\sqrt{n_Q\ell}$ are not admissible and the statement is trivial. Thus, we may assume that $\lceil (c-3)\sqrt{n_Q\ell}\rceil$ is admissible. Using (5.5) and Lemma 3.8 we get with $\overline{n_C} := \lceil (c-2)\sqrt{n_Q\ell}\rceil$ and $I := \left[(c-3)\sqrt{n_Q\ell}, \overline{n_C}\right]$

$$\begin{aligned}\sum_{n_C \geq c\sqrt{n_Q\ell}} r(n_C) &\leq r(\overline{n_C}) \sum_{n_C \geq c\sqrt{n_Q\ell}} \left(1 - \frac{c}{2}\sqrt{\frac{\ell}{n_Q}}\right)^{n_C - \overline{n_C}} \\ &\leq r(\overline{n_C})\left(1 - \frac{c}{2}\sqrt{\frac{\ell}{n_Q}}\right)^{\sqrt{n_Q\ell}} \frac{1}{\frac{c}{2}\sqrt{\frac{\ell}{n_Q}}} \leq r(\overline{n_C})\exp\left(-\frac{c}{2}\ell\right)\frac{2\sqrt{n_Q}}{c\sqrt{\ell}} \\ &\leq \exp\left(-\frac{c}{2}\ell\right)\frac{2\sqrt{n_Q}}{c\sqrt{\ell}}\frac{1}{2\sqrt{n_Q\ell}}\sum_{n_C\in I} r(n_C) \leq \exp\left(-\frac{c}{2}\ell\right)\sum_{n_C} r(n_C),\end{aligned}$$

which shows (i). For (ii) we set $c = n_Q^\delta$, which is larger than 14 for sufficiently large n_Q. Then (i) implies

$$\sum_{n_C \geq n_Q^\delta\sqrt{n_Q\ell}} r(n_C) \leq \exp\left(-\frac{n_Q^\delta}{2}\ell\right)\sum_{n_C} r(n_C),$$

as desired. □

5.2 Complex outerplanar graphs

The next step is to find I_{n_Q} and I_ℓ such that the main contribution to $\sum_{n_Q,\ell} s(n_Q,\ell)$ is provided by $n_Q \in I_{n_Q}$ and $\ell \in I_\ell$. We recall that we have

$$s(n_Q,\ell) = \binom{n}{n_Q} |\mathcal{Q}(n_Q, n_Q+\ell)| \cdot |\mathcal{U}(n_U, m_U)|.$$

One idea, which we will use to find I_{n_Q} and I_ℓ, is to consider the fraction $\frac{s(n_Q+1,\ell)}{s(n_Q,\ell)}$ in a similar way as we analysed $\frac{r(n_C+1)}{r(n_C)}$ in Section 5.1. To that end, we need estimates for $\frac{|\mathcal{Q}(n_Q+1,n_Q+1+\ell)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|}$, which we obtain by the following two lemmas.

Lemma 5.5. *There are $a_2, \varepsilon > 0$ and $K \in \mathbb{N}$ such that for all admissible n_Q and ℓ with $K \le \ell \le \varepsilon n_Q$*

$$\frac{|\mathcal{Q}(n_Q+1, n_Q+1+\ell)|}{|\mathcal{Q}(n_Q, n_Q+\ell)|} \ge (n_Q+1)\exp\left(1 + a_2\frac{\ell}{n_Q}\right).$$

Proof. Let admissible n_Q and ℓ with $K \le \ell \le \varepsilon n_Q$ be given, where we choose $K \in \mathbb{N}$ and $\varepsilon > 0$ later. We recall that for $i \in \{0,1\}$

$$r_i(n_C) = r_i(n_C, n_Q, \ell) = \binom{n_Q+i}{n_C} |\mathcal{C}(n_C, n_C+\ell)|\, n_C (n_Q+i)^{n_Q+i-n_C-1}.$$

Now we compare the sums $\sum_{n_C} r_1(n_C)$ and $\sum_{n_C} r_0(n_C)$. The admissible n_C for $\sum_{n_C} r_1(n_C)$ are precisely those n_C, which are admissible for $\sum_{n_C} r_0(n_C)$ or n_Q+1. Thus, the two sums have 'almost' the same number of terms, which leads to the idea to consider $\frac{r_1(n_C)}{r_0(n_C)}$. We have

$$\frac{r_1(n_C)}{r_0(n_C)} = (n_Q+1)\left(1 + \frac{n_C}{n_Q-n_C+1}\right)\left(1+\frac{1}{n_Q}\right)^{n_Q-n_C-1}. \tag{5.6}$$

Using (5.6) we get that $\frac{r_1(n_C)}{r_0(n_C)}$ is monotonically increasing in n_C, since

$$\frac{\frac{r_1(n_C+1)}{r_0(n_C+1)}}{\frac{r_1(n_C)}{r_0(n_C)}} = \frac{n_Q-n_C+1}{n_Q-n_C}\cdot\frac{n_Q}{n_Q+1} \ge \frac{n_Q+1}{n_Q}\cdot\frac{n_Q}{n_Q+1} = 1.$$

We now show the lower bound of the statement. To that end, we use Lemmas 5.3-5.4 to get $a, b_1, c_1, b_2, c_2 > 0$ such that

$$\sum_{n_C \leq c_1\sqrt{n_Q\ell}} r_1(n_C) \leq a \exp\left(-b_1\ell\right) \sum_{n_C} r_1(n_C),$$
$$\sum_{n_C \geq c_2\sqrt{n_Q\ell}} r_1(n_C) \leq \exp\left(-b_2\ell\right) \sum_{n_C} r_1(n_C).$$

We set $\overline{n_C} = \left\lceil c_1\sqrt{n_Q\ell}\right\rceil$, $I_1 := \{n_C \mid \overline{n_C} \leq n_C \leq n_Q\}$ and $I_2 := \{n_C \mid n_C < \overline{n_C}\}$. Now we choose $\varepsilon > 0$ small enough that $n_Q + 1 > c_2\sqrt{n_Q\ell}$, whenever $\ell \leq \varepsilon n_Q$. Then we get

$$\sum_{n_C\in I_1} r_1(n_C) \geq \sum_{n_C} r_1(n_C) \left(1 - a\exp(-b_1\ell) - \exp(-b_2\ell)\right). \tag{5.7}$$

Next, we define for $i \in \{0, 1\}$

$$A_i := \sum_{n_C\in I_1} r_i(n_C)$$
$$B_i := \sum_{n_C\in I_2} r_i(n_C).$$

Then we have

$$\frac{\sum_{n_C} r_1(n_C)}{\sum_{n_C} r_0(n_C)} \geq \frac{A_1 + B_1}{A_0 + B_0}.$$

In the next step we estimate $\frac{A_1}{A_0}$ and $\frac{B_1}{B_0}$. Observing that A_1 and A_0 have the same number of terms and using the monotonicity of $\frac{r_1(n_C)}{r_0(n_C)}$ yields

$$\frac{A_1}{A_0} \geq \min\left\{\frac{r_1(n_C)}{r_0(n_C)} \;\middle|\; n_C \in I_1\right\} \geq \frac{r_1(\overline{n_C})}{r_0(\overline{n_C})}.$$

Now we use (5.6), Lemma 3.7 and choose $\varepsilon > 0$ small enough to obtain

$$\begin{aligned}
\frac{r_1(\overline{n_C})}{r_0(\overline{n_C})} &= (n_Q+1)\exp\left(\frac{\overline{n_C}}{n_Q - \overline{n_C} + 1} - \frac{\overline{n_C}^2}{2\left(n_Q - \overline{n_C} + 1\right)^2} + O\left(\frac{\overline{n_C}^3}{\left(n_Q - \overline{n_C} + 1\right)^3}\right)\right)\\
&\quad\times \exp\left(\left(n_Q - \overline{n_C} - 1\right)\left[\frac{1}{n_Q} - \frac{1}{2n_Q^2} + O\left(\frac{1}{n_Q^3}\right)\right]\right)\\
&= (n_Q+1)\exp\left(1 + \frac{1}{n_Q - \overline{n_C} + 1}\left[-\frac{3}{2} + \frac{\overline{n_C}^2}{n_Q} - \frac{\overline{n_C}^2}{2(n_Q - \overline{n_C} + 1)}\right]\right)\\
&\quad\times \exp\left(O\left(\frac{\overline{n_C}}{n_Q^2}\right) + O\left(\frac{\overline{n_C}^3}{n_Q^2}\right)\right).
\end{aligned}$$

Now we can choose $\varepsilon > 0$ small and $K \in \mathbb{N}$ large enough such that

$$-\frac{3}{2} + \frac{\overline{n_C}^2}{n_Q} - \frac{\overline{n_C}^2}{2(n_Q - \overline{n_C} + 1)} \geq \frac{\overline{n_C}^2}{3n_Q} \geq \frac{c_1^2}{3}\ell,$$
$$O\left(\frac{\overline{n_C}}{n_Q^2}\right) + O\left(\frac{\overline{n_C}^3}{n_Q^3}\right) \geq -\frac{c_1^2}{6} \cdot \frac{\ell}{n_Q}.$$

Hence, we get

$$\frac{A_1}{A_0} \geq \frac{r_1(\overline{n_C})}{r_0(\overline{n_C})} \geq (n_Q+1)\exp\left(1 + \frac{c_1^2}{6} \cdot \frac{\ell}{n_Q}\right) = (n_Q+1)\exp\left(1 + c_3 \cdot \frac{\ell}{n_Q}\right), \tag{5.8}$$

where $c_3 := \frac{c_1^2}{6}$. In a similar way we estimate $\frac{B_1}{B_0}$. For n_Q large enough (which can be ensured by choosing K large enough) we obtain

$$\begin{aligned}\frac{B_1}{B_0} &\geq \min\left\{\frac{r_1(n_C)}{r_0(n_C)} \,\middle|\, n_C \in I_2\right\} \geq \frac{r_1(0)}{r_0(0)} = (n_Q+1)\left(1 + \frac{1}{n_Q}\right)^{n_Q - 1} \\ &= (n_Q+1)\exp\left(1 - \frac{3}{2n_Q} + O\left(\frac{1}{n_Q^2}\right)\right) \geq (n_Q+1)\exp\left(1 - \frac{2}{n_Q}\right).\end{aligned} \tag{5.9}$$

We note that the above computation also holds if $n_C = 0$ is not admissible. Next, we use (5.7) to obtain a $b_3 > 0$ such that $A_1 \exp(-b_3\ell) \geq B_1$. Now we use Lemmas 3.9-3.10 and (2.10), (5.8), (5.9) to get for suitable chosen $K \in \mathbb{N}$ and $\varepsilon > 0$

$$\begin{aligned}\frac{|\mathcal{Q}(n_Q+1, n_Q+1+\ell)|}{|\mathcal{Q}(n_Q, n_Q+\ell)|} &= \frac{\sum_{n_C} r_1(n_C)}{\sum_{n_C} r_0(n_C)} \geq \frac{A_1 + B_1}{A_0 + B_0} \\ &\geq \frac{A_1 + B_1}{\frac{A_1 \exp\left(-1 - c_3 \frac{\ell}{n_Q}\right)}{n_Q+1} + \frac{B_1 \exp\left(-1 + \frac{2}{n_Q}\right)}{n_Q+1}} \\ &\geq (n_Q+1)e\frac{1 + \exp(-b_3\ell)}{\exp\left(-c_3\frac{\ell}{n_Q}\right) + \exp\left(\frac{2}{n_Q} - b_3\ell\right)} \\ &\geq (n_Q+1)e\frac{1 + \exp(-b_3\ell)}{1 - c_3\frac{\ell}{2n_Q} + \left(1 + \frac{4}{n_Q}\right)\exp(-b_3\ell)} \\ &\geq (n_Q+1)e\left(1 + \frac{c_3\ell}{8n_Q}\right) \\ &\geq (n_Q+1)\exp\left(1 + \frac{c_3\ell}{16n_Q}\right),\end{aligned}$$

which shows the statement for $a_2 := \frac{c_3}{16}$. □

Lemma 5.6. *There are $a_3, \varepsilon > 0$ and $K \in \mathbb{N}$ such that for all admissible n_Q and ℓ with $K \le \ell \le \varepsilon n_Q$*

$$\frac{|\mathcal{Q}(n_Q+1, n_Q+1+\ell)|}{|\mathcal{Q}(n_Q, n_Q+\ell)|} \le (n_Q+1)\exp\left(1 + a_3\frac{\ell}{n_Q}\right).$$

Proof. We use the same notations as in Lemma 5.5 and distinguish two cases.
Case 1: $\ell \ge \log n_Q$. Using Lemma 5.4 we obtain for $\overline{n_C} := \lfloor 14\sqrt{(n_Q+1)\,\ell}\rfloor$

$$\frac{\sum_{n_C} r_1(n_C)}{\sum_{n_C} r_0(n_C)} \le \frac{\sum_{n_C \le \overline{n_C}} r_1(n_C)}{\sum_{n_C \le \overline{n_C}} r_0(n_C)} \cdot \frac{1}{1-\exp(-7\ell)}. \tag{5.10}$$

Now we choose $\varepsilon > 0$ small enough such that $14\sqrt{(n_Q+1)\,\ell} < n_Q + 1$, whenever $\ell \le \varepsilon n_Q$. Then $\sum_{n_C \le \overline{n_C}} r_1(n_C)$ and $\sum_{n_C \le \overline{n_C}} r_0(n_C)$ have the same number of terms. From now on we assume that $K \in \mathbb{N}$ is large and $\varepsilon > 0$ is small enough such that the following computations hold. With the same estimate, which led to (5.8), we get an $a_3 > 0$ such that

$$\frac{\sum_{n_C \le \overline{n_C}} r_1(n_C)}{\sum_{n_C \le \overline{n_C}} r_0(n_C)} \le \max\left\{\frac{r_1(n_C)}{r_0(n_C)} \,\middle|\, n_C \le \overline{n_C}\right\} \le \frac{r_1(\overline{n_C})}{r_0(\overline{n_C})} \le (n_Q+1)\exp\left(1 + \frac{a_3}{2}\frac{\ell}{n_Q}\right).$$

Then we use (5.10) and Lemma 3.10 to get

$$\begin{aligned}\frac{\sum_{n_C} r_1(n_C)}{\sum_{n_C} r_0(n_C)} &\le (n_Q+1)\exp\left(1 + \frac{a_3}{2}\frac{\ell}{n_Q}\right)\frac{1}{1-\exp\left(-7\log n_Q\right)}\\ &\le (n_Q+1)\exp\left(1 + \frac{a_3}{2}\frac{\ell}{n_Q}\right)\frac{1}{\exp\left(-2n_Q^{-7}\right)}\\ &\le (n_Q+1)\exp\left(1 + a_3\frac{\ell}{n_Q}\right).\end{aligned}$$

Case 2: $\ell < \log n_Q$. In that case we set $\overline{n_C} := \left\lfloor n_Q^{\frac{1}{3}}\sqrt{n_Q\ell}\right\rfloor$. We use Lemma 5.4(ii) with $\delta = \frac{1}{3}$ to obtain

$$\frac{\sum_{n_C} r_1(n_C)}{\sum_{n_C} r_0(n_C)} \le \frac{\sum_{n_C \le \overline{n_C}} r_1(n_C)}{\sum_{n_C \le \overline{n_C}} r_0(n_C)} \cdot \frac{1}{1-\exp\left(-\frac{1}{3}\ell n_Q^{\frac{1}{3}}\right)}. \tag{5.11}$$

For n_Q large enough we have $\overline{n_C} < n_Q+1$, whence $\sum_{n_C \le \overline{n_C}} r_1(n_C)$ and $\sum_{n_C \le \overline{n_C}} r_0(n_C)$ have the same number of terms. Now we estimate $\frac{r_1(n_C)}{r_0(n_C)}$ for $n_C \le \overline{n_C}$. Using (5.6) and Lemma 3.8 yields

$$\begin{aligned}\frac{r_1(n_C)}{r_0(n_C)} &\le (n_Q+1)\exp\left(\frac{n_C}{n_Q-n_C+1} + \frac{n_Q-n_C-1}{n_Q}\right)\\ &\le (n_Q+1)\exp\left(1 + \frac{2n_C^2}{n_Q^2}\right).\end{aligned} \tag{5.12}$$

Next, we partition the index set $\{n_C \mid n_C \le \overline{n_C}\}$ into smaller sets. For $i = 0, 1$ and $j = 1, \ldots, r = \left\lceil n_Q^{\frac{1}{3}} \right\rceil$ we define

$$I_j := \left\{ n_C \;\middle|\; (j-1)\sqrt{n_Q \ell} < n_C \le j\sqrt{n_Q \ell}, n_C \le \overline{n_C} \right\},$$
$$S_j^{(i)} := \sum_{n_C \in I_j} r_i(n_C).$$

With this notation we get

$$\frac{\sum_{n_C \le \overline{n_C}} r_1(n_C)}{\sum_{n_C \le \overline{n_C}} r_0(n_C)} = \frac{S_1^{(1)} + \ldots + S_r^{(1)}}{S_1^{(0)} + \ldots + S_r^{(0)}}.$$

Using (5.12) and Lemma 3.9 yields

$$\frac{S_j^{(1)}}{S_j^{(0)}} \le (n_Q + 1) \exp\left(1 + \frac{2j^2 \ell}{n_Q}\right) \le (n_Q + 1) e \left(1 + \frac{4j^2 \ell}{n_Q}\right). \tag{5.13}$$

In addition, we get from Lemma 5.4(i) for $j \ge 15$

$$S_j^{(0)} \le \frac{\exp\left(-\frac{j-1}{2} \cdot \ell\right)}{1 - \exp\left(-\frac{j-1}{2} \cdot \ell\right)} \left(S_1^{(0)} + \ldots + S_{j-1}^{(0)}\right) \le \exp\left(-\frac{j}{3} \cdot \ell\right) \left(S_1^{(0)} + \ldots + S_{j-1}^{(0)}\right). \tag{5.14}$$

Now we show by induction on $j = 14, \ldots, r$ that

$$\frac{S_1^{(1)} + \ldots + S_j^{(1)}}{S_1^{(0)} + \ldots + S_j^{(0)}} \le (n_Q + 1) e \left(1 + \frac{\ell}{n_Q} \left[4 \cdot 14^2 + \sum_{k=15}^{j} 4k^2 \exp\left(-\frac{k}{3}\ell\right)\right]\right).$$

The induction basis $j = 14$ follows immediately from (5.13). For the induction step $j \to (j+1)$ we set $X = S_1^{(1)} + \ldots + S_j^{(1)}$ and $Y = S_1^{(0)} + \ldots + S_j^{(0)}$. Using the induction hypothesis and (5.13) gives

$$\begin{aligned} \frac{S_1^{(1)} + \ldots + S_{j+1}^{(1)}}{S_1^{(0)} + \ldots + S_{j+1}^{(0)}} &= \frac{X + S_{j+1}^{(1)}}{Y + S_{j+1}^{(0)}} \le \frac{X + S_{j+1}^{(0)} (n_Q + 1) e \left(1 + \frac{4(j+1)^2 \ell}{n_Q}\right)}{Y + S_{j+1}^{(0)}} \\ &\le (n_Q + 1) e \cdot \frac{Y \left(1 + \frac{\ell}{n_Q}\left[4 \cdot 14^2 + \sum_{k=15}^{j} 4k^2 \exp\left(-\frac{k}{3}\ell\right)\right]\right) + S_{j+1}^{(0)} \left(1 + \frac{4(j+1)^2 \ell}{n_Q}\right)}{Y + S_{j+1}^{(0)}}. \end{aligned} \tag{5.15}$$

If $\left(1+\frac{4(j+1)^2\ell}{n_Q}\right) \le \left(1+\frac{\ell}{n_Q}\left[4\cdot 14^2+\sum_{k=15}^{j} 4k^2\exp\left(-\frac{k}{3}\ell\right)\right]\right)$, then (5.15) shows immediately the statement for $j+1$. Otherwise, we use (5.14) to get

$$\begin{aligned}
&\frac{S_1^{(1)}+\ldots+S_{j+1}^{(1)}}{S_1^{(0)}+\ldots+S_{j+1}^{(0)}}\\
&\le (n_Q+1)e\cdot\frac{1+\frac{\ell}{n_Q}\left[4\cdot 14^2+\sum_{k=15}^{j}4k^2\exp\left(-\frac{k}{3}\ell\right)\right]+\exp\left(-\frac{j+1}{3}\ell\right)\left(1+\frac{4(j+1)^2\ell}{n_Q}\right)}{1+\exp\left(-\frac{j+1}{3}\ell\right)}\\
&\le (n_Q+1)e\left(1+\frac{\ell}{n_Q}\left[4\cdot 14^2+\sum_{k=15}^{j+1}4k^2\exp\left(-\frac{k}{3}\ell\right)\right]\right),
\end{aligned}$$

which concludes the induction. Next, we observe that

$$\sum_{k=15}^{\infty}4k^2\exp\left(-\frac{k}{3}\ell\right)\le\sum_{k=15}^{\infty}4k^2\exp\left(-\frac{k}{3}\right)<\infty.$$

Therefore, the above result for $j=r$ implies together with (5.11) and Lemma 3.8 that there is an $a_3>0$ such that for n_Q large enough

$$\begin{aligned}
\frac{|\mathcal{Q}(n_Q+1,n_Q+1+\ell)|}{|\mathcal{Q}(n_Q,n_Q+\ell)|}&=\frac{\sum_{n_C}r_1(n_C)}{\sum_{n_C}r_0(n_C)}\\
&\le (n_Q+1)e\left(1+\frac{a_3}{2}\cdot\frac{\ell}{n_Q}\right)\frac{1}{1-\exp\left(-\frac{1}{3}\ell n_Q^{\frac{1}{3}}\right)}\\
&\le (n_Q+1)e\left(1+\frac{a_3}{2}\cdot\frac{\ell}{n_Q}\right)^2\\
&\le (n_Q+1)\exp\left(1+a_3\frac{\ell}{n_Q}\right),
\end{aligned}$$

as desired. □

5.3 Negligible terms

Later we will use Lemmas 5.5-5.6 to find I_{n_Q} and I_ℓ. These lemmas are only applicable in the case $\ell \ge K$. In this section we show that almost all outerplanar graphs satisfy this inequality or in other words that the terms provided by $\ell < K$ are negligible in $\sum_{n_Q,\ell} s(n_Q,\ell)$. Before we show that we prove the weaker statement that the terms provided by $n_Q < K$ and $\ell < K$ are negligible in $\sum_{n_Q,\ell} s(n_Q,\ell)$.

Lemma 5.7. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then for each $K \in \mathbb{N}$*

$$\sum_{n_Q, \ell < K} s(n_Q, \ell) = o\left(\sum_{n_Q, \ell} s(n_Q, \ell)\right).$$

Proof. Let $n_Q, \ell < K$ be given. For n large enough, we have $m_U > \frac{n_U}{2}$, whence by Lemma 2.5(iii) and (iv) we have

$$|\mathcal{U}(n_U, m_U)| \leq \binom{\binom{n_U}{2}}{m_U} c \left(\frac{2}{e}\right)^{2m_U - n_U} \frac{m_U^{m_U + \frac{1}{2}} n_U^{n_U - 2m_U + g(n_Q, \ell)}}{(n_U - m_U)^{n_U - m_U + \frac{1}{2}}},$$

where

$$g(n_Q, \ell) := \begin{cases} \frac{1}{2} & \text{if } m_U \geq \frac{n_U}{2} + \frac{n_U^{\frac{3}{4}}}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we use that and Lemma 3.14 to get

$$\begin{aligned} s(n_Q, \ell) &= n^{n_Q} \Theta(1) \, |\mathcal{U}(n_U, m_U)| \\ &\leq \Theta(1) \, n^{n_Q} \frac{n_U^{2m_U}}{(2m_U)^{m_U + \frac{1}{2}}} \exp(m_U) \, c \left(\frac{2}{e}\right)^{2m_U - n_U} \frac{m_U^{m_U + \frac{1}{2}} n_U^{n_U - 2m_U + g(n_Q, \ell)}}{(n_U - m_U)^{n_U - m_U + \frac{1}{2}}} \\ &= \Theta(1) \frac{n^{n + g(n_Q, \ell)}}{2^{n_U - m_U + \frac{1}{2}}} \frac{\exp(n_U - m_U)}{(n_U - m_U)^{n_U - m_U + \frac{1}{2}}} \\ &= \Theta(1) n^{n + g(n_Q, \ell) - \ell - \frac{1}{2}} \frac{\exp\left(\frac{n}{2} - s\right)}{(n - 2s)^{\frac{n}{2} - s}}. \end{aligned}$$

As there is only a bounded number of n_Q and ℓ with $n_Q, \ell < K$, it suffices to show $s(n_Q, \ell) = o\left(\sum_{n_Q, \ell} s(n_Q, \ell)\right)$ for all $n_Q, \ell < K$. If $\ell > 0$ or $g(n_Q, \ell) = 0$, then this follows immediately from Lemma 4.10. Otherwise, we have $\ell = n_Q = 0$ and $g(0, 0) = \frac{1}{2}$. Then by the definition of g we obtain $s \geq \frac{n^{\frac{3}{4}}}{2}$. Thus, we have $n^{\frac{1}{2}} = o\left(\exp\left(\frac{s}{n^{\frac{2}{3}}}\right)\right)$, whence we can use again Lemma 4.10 to obtain $s(0, 0) = o\left(\sum_{n_Q, \ell} s(n_Q, \ell)\right)$. □

Lemma 5.8. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then for each $K \in \mathbb{N}$*

$$\sum_{\substack{n_Q \\ \ell < K}} s(n_Q, \ell) = o\left(\sum_{n_Q, \ell} s(n_Q, \ell)\right).$$

Proof. Due to Lemma 5.7 it suffices to show

$$\sum_{n_Q \geq M, \ell < K} s(n_Q, \ell) = o\left(\sum_{n_Q, \ell} s(n_Q, \ell)\right),$$

for some $M \in \mathbb{N}$. Using Lemmas 4.6 and 4.8-4.9 we get that there is a $M \in \mathbb{N}$ such that uniformly over all $n_Q \geq M$ and $\ell < K$

$$s(n_Q, \ell) = \Theta(1) s_C(n_Q, \ell).$$

Hence, we obtain with Theorem 4.7

$$\sum_{n_Q \geq M, \ell < K} s(n_Q, \ell) = \Theta(1) \sum_{n_Q \geq M, \ell < K} s_C(n_Q, \ell) = o\left(\sum_{n_Q, \ell} s_C(n_Q, \ell)\right) = o\left(\sum_{n_Q, \ell} s(n_Q, \ell)\right).$$

□

5.4 Main contributions

Recall that for admissible n_Q and ℓ we have

$$s(n_Q, \ell) = s(n_Q, \ell, n, m) = \binom{n}{n_Q} |\mathcal{Q}(n_Q, n_Q + \ell)| \cdot |\mathcal{U}(n_U, m_U)|.$$

In this section we will determine I_{n_Q} and I_ℓ by analysing the fractions $\frac{s(n_Q+1,\ell)}{s(n_Q,\ell)}$ and $\frac{s(n_Q,\lfloor c\ell \rfloor)}{s(n_Q,\ell)}$. To that end, we need an estimate for $|\mathcal{U}(n_U, m_U)|$. We observe by Lemma 2.5 that $\rho(n_U, m_U)$ stays close to one, as long as $n_U \geq 2m_U$. Thus, we will use in that case $\binom{\binom{n_U}{2}}{m_U}$ as an estimate for $|\mathcal{U}(n_U, m_U)|$. In contrast, $\rho(n_U, m_U)$ starts becoming quite small if $n_U < 2m_U$. Hence, we will use in that case the stronger bounds, which are given by Lemma 2.5(iii) and (iv). Thus, we define

$$S_1 := \sum_{n_U \geq 2m_U} s(n_Q, \ell),$$
$$S_2 := \sum_{n_U < 2m_U} s(n_Q, \ell),$$

whence we obtain

$$\sum_{n_Q, \ell} s(n_Q, \ell) = S_1 + S_2.$$

Now the idea is to find for both sums S_1 and S_2 terms, which provide the main contribution. We start by analysing S_1 in the following way. We first consider the sum $\sum_{n_U \geq 2m_U} s(\overline{n_Q}, \ell)$ for a fixed value $\overline{n_Q}$ and determine terms, which provide the main contribution to that sum. By doing that we deduce that the main contribution to S_1 is provided by n_Q and ℓ with $\ell = \Theta\left(\frac{n_Q}{n^{\frac{2}{3}}}\right)$ or in other words by n_Q and ℓ with $n_Q = \Theta\left(n^{\frac{2}{3}}\ell\right)$ (see Lemma 5.9). In the next step we fix the value $\overline{\ell}$ and consider the sum $\sum_{n_U \geq 2m_U} s(n_Q, \overline{\ell})$. We will see in Lemma 5.10 that for some values of $\overline{\ell}$ the terms provided by $n_Q = \Theta\left(n^{\frac{2}{3}}\overline{\ell}\right)$ are negligible in $\sum_{n_U \geq 2m_U} s(n_Q, \overline{\ell})$. As an immediate consequence we obtain that the terms provided by such $\overline{\ell}$ are negligible in S_1. Using that we finally determine terms, which provide the main contribution to S_1, in Lemma 5.11. Afterwards we analyse S_2 in a similar fashion (see Lemmas 5.12-5.16).

Lemma 5.9. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there are $c_1, c_2 > 0$ such that the main contribution to $S_1 = \sum_{n_U \geq 2m_U} s(n_Q, \ell)$ is provided by n_Q and ℓ, which satisfy*

$$c_1 \frac{n_Q}{n^{\frac{2}{3}}} \leq \ell \leq c_2 \frac{n_Q}{n^{\frac{2}{3}}}. \tag{5.16}$$

Proof. Due to Lemma 5.8 it suffices to consider only terms $s(n_Q, \ell)$ for $\ell \geq K$, where K is as in Lemma 5.6. Thus, we assume in the following always that $\ell \geq K$ is satisfied. For admissible n_Q and ℓ we define

$$t(n_Q, \ell) := \binom{n}{n_Q} |\mathcal{Q}(n_Q, n_Q + \ell)| \binom{\binom{n_U}{2}}{m_U}.$$

By Lemma 2.5(ii) we get $t(n_Q, \ell) = \Theta(1) s(n_Q, \ell)$. Thus, terms, which provide the main contribution to $T_1 := \sum_{n_U \geq 2m_U} t(n_Q, \ell)$, provide also the main contribution to S_1. Hence, we consider in the following T_1 instead of S_1. Next, we define

$$u(n_Q, \ell) := \binom{n}{n_Q} a_1 n_Q^{n_Q + \frac{3\ell - 1}{2}} \rho_1^{\ell} \ell^{-\frac{3\ell}{2} - 3} \exp\left(b_1 \sqrt{\frac{\ell^3}{n_Q}}\right) \binom{\binom{n_U}{2}}{m_U}, \tag{5.17}$$

where a_1, b_1 and ρ_1 are such that Lemma 4.9 holds. Then we get $t(n_Q, \ell) \leq u(n_Q, \ell)$ by Lemma 4.6. Now we consider the fraction $\frac{u(n_Q, \ell+1)}{u(n_Q, \ell)}$.

$$\begin{aligned}\frac{u(n_Q, \ell+1)}{u(n_Q, \ell)} &= n_Q^{\frac{3}{2}} \rho_1 \exp\left(b_1 \sqrt{\frac{(\ell+1)^3}{n_Q}} - b_1 \sqrt{\frac{\ell^3}{n_Q}}\right) \frac{(\ell+1)^{-\frac{3\ell+9}{2}}}{\ell^{-\frac{3\ell+6}{2}}} \cdot \frac{m_U}{\binom{n_U}{2} - m_U + 1} \\ &= \Theta(1) \frac{n_Q^{\frac{3}{2}}}{\ell^{\frac{3}{2}}} \cdot \frac{m_U}{n^2} \leq \Theta(1) \frac{n_Q^{\frac{3}{2}}}{\ell^{\frac{3}{2}}} \cdot \frac{1}{n}. \end{aligned} \tag{5.18}$$

Thus, for every $\delta > 0$ there is a $c_3 > 0$ such that

$$\frac{u(n_Q, \ell+1)}{u(n_Q, \ell)} < \delta, \tag{5.19}$$

whenever $\ell \geq c_3 \frac{n_Q}{n^{\frac{2}{3}}}$. Now let $\varepsilon \in (0,1)$ as in Lemma 4.8 and for a given ℓ we set $\overline{\ell} := \lfloor \varepsilon \ell \rfloor$. Then by Lemma 4.8 there exists an $\alpha > 0$ such that for all admissible n_Q and ℓ

$$\frac{t(n_Q, \overline{\ell})}{u(n_Q, \overline{\ell})} \geq \alpha^{\ell}. \tag{5.20}$$

Now we choose $\delta > 0$ such that $\frac{\alpha}{\delta^{1-\varepsilon}} > 1$. Moreover, we assume $\ell \geq \frac{2c_3}{\varepsilon} \frac{n_Q}{n^{\frac{2}{3}}}$. Then we combine (5.19) and (5.20) to obtain for $n \to \infty$

$$\frac{t(n_Q, \overline{\ell})}{t(n_Q, \ell)} \geq \frac{t(n_Q, \overline{\ell})}{u(n_Q, \ell)} = \frac{u(n_Q, \overline{\ell})}{u(n_Q, \ell)} \cdot \frac{t(n_Q, \overline{\ell})}{u(n_Q, \overline{\ell})} \geq \delta^{-\ell(1-\varepsilon)} \alpha^{\ell} \to \infty,$$

since $\ell \geq \frac{2c_3}{\varepsilon} \frac{n_Q}{n^{\frac{2}{3}}} \to \infty$. Now we observe that if $n_Q + 2c_3 \cdot \frac{n_Q}{n^{\frac{2}{3}}} - 2 \geq 2s$, then $n_Q + 2\overline{\ell} \geq 2s$, whence the term $t(n_Q, \overline{\ell})$ occurs in T_1. Hence, the above computation shows that the terms provided by

$$\begin{aligned} n_Q + 2c_3 \cdot \frac{n_Q}{n^{\frac{2}{3}}} - 2 &\geq 2s, \\ \ell &\geq \frac{2c_3}{\varepsilon} \frac{n_Q}{n^{\frac{2}{3}}} \end{aligned}$$

are negligible in T_1. For the case $n_Q + 2c_3 \cdot \frac{n_Q}{n^{\frac{2}{3}}} - 2 < 2s$, we consider $\frac{u(n_Q+2, \ell-1)}{u(n_Q, \ell)}$ instead of $\frac{u(n_Q+1, \ell)}{u(n_Q, \ell)}$ in the above computation. Then we obtain that there is a $c_4 > 0$ such that the terms provided by

$$\begin{aligned} n_Q + 2c_3 \cdot \frac{n_Q}{n^{\frac{2}{3}}} - 2 &< 2s, \\ \ell &\geq c_4 \frac{n_Q}{n^{\frac{2}{3}}} \end{aligned}$$

are negligible in T_1. Hence, the main contribution to T_1 is provided by n_Q and ℓ with $\ell \leq c_2 \frac{n_Q}{n^{\frac{2}{3}}}$, where $c_2 := \max\left\{\frac{2c_3}{\varepsilon}, c_4\right\}$. From now on we always assume $\ell \leq c_2 \frac{n_Q}{n^{\frac{2}{3}}} = o(n_Q)$. Hence, Lemma 5.6 is applicable. Together with Lemmas 3.11 and 3.15 this yields

$$\begin{aligned}\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} &\leq \frac{n-n_Q}{n_Q+1}(n_Q+1)\exp\left(1+a_3\frac{\ell}{n_Q}\right)\binom{\binom{n_U-1}{2}}{m_U-1}\binom{\binom{n_U}{2}}{m_U}^{-1} \\ &= n_U\exp\left(1+a_3\frac{\ell}{n_Q}\right)\frac{2m_U}{n_U^2}\exp\left(-\frac{2m_U}{n_U}+O\left(\frac{1}{n_U}\right)\right) \\ &= \frac{2m_U}{n_U}\exp\left(1+a_3\frac{\ell}{n_Q}-\frac{2m_U}{n_U}+O\left(\frac{1}{n}\right)\right) \\ &= \left(1-\frac{n_Q+2\ell-2s}{n-n_Q}\right)\exp\left(a_3\frac{\ell}{n_Q}+\frac{n_Q+2\ell-2s}{n-n_Q}+O\left(\frac{1}{n}\right)\right) \\ &\leq \exp\left(\frac{a_3c_2}{n^{\frac{2}{3}}}-\frac{(n_Q+2\ell-2s)^2}{2(n-n_Q)^2}+O\left(\frac{1}{n}\right)\right). \qquad (5.21)\end{aligned}$$

Thus, for every $\beta>0$ there is a $\mu\in(0,1)$ such that

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)}\leq\mu,$$

whenever $n_Q+2\ell-2s\geq\beta n$ and n large enough. Now let n_Q and ℓ be such that $n_Q+2\ell-2s\geq 3\beta n$. Then we obtain for $n\to\infty$

$$\frac{t(n_Q,\ell)}{t\left(\left\lfloor\frac{n_Q}{2}\right\rfloor,\ell\right)}\leq\mu^{\left\lfloor\frac{n_Q}{2}\right\rfloor}\to 0.$$

As $\left\lfloor\frac{n_Q}{2}\right\rfloor$ and ℓ are still admissible and fulfil $\left\lfloor\frac{n_Q}{2}\right\rfloor+2\ell\geq 2s$, we conclude that the terms provided by $n_Q+2\ell-2s\geq 3\beta n$ are negligible in T_1. Hence, we may assume from now on $n_Q+2\ell-2s<3\beta n$. If we choose $\beta>0$ small enough, then we get $m_U=\Theta(n)$ and thus in (5.18)

$$\frac{u(n_Q,\ell+1)}{u(n_Q,\ell)}=\Theta(1)\frac{n_Q^{\frac{3}{2}}}{\ell^{\frac{3}{2}}}\frac{1}{n}. \qquad (5.22)$$

Now the same computation as before shows that there is a $c_1>0$ such that the main contribution to T_1 is provided by n_Q and ℓ with $\ell\geq c_1\frac{n_Q}{n^{\frac{2}{3}}}$. □

Lemma 5.10. *Let $m=m(n)=\frac{n}{2}+s$, where $s=s(n)=o(n)$ with $\frac{s^3}{n^2}\to\infty$. Then there is a constant $L>0$ such that the main contribution to $S_1=\sum_{n_U\geq 2m_U}s(n_Q,\ell)$ is provided by n_Q and ℓ, which satisfy*

$$\ell<L\frac{s}{n^{\frac{2}{3}}}.$$

Proof. We use the notations from the proof of Lemma 5.9. Now let n_Q and ℓ such that $n_Q \geq \frac{1}{4c_2}\ell n^{\frac{2}{3}}$. Then we get with (5.21)

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} \leq \exp\left(\frac{4a_3c_2}{n^{\frac{2}{3}}} - \frac{(n_Q+2\ell-2s)^2}{2(n-n_Q)^2} + O\left(\frac{1}{n}\right)\right)$$
$$\leq \exp\left(\frac{4a_3c_2}{n^{\frac{2}{3}}} - \frac{s^2}{2n^2}\left(\frac{L}{4c_2}-2\right)^2 + O\left(\frac{1}{n}\right)\right).$$

Now we can choose $L > 0$ large enough such that for sufficiently large n we have

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} \leq \exp\left(-\frac{s^2}{n^2}\right),$$

whenever $n_Q \geq \frac{1}{4c_2}\ell n^{\frac{2}{3}}$. Now we choose $L > 0$ large enough such that for all admissible n_Q and ℓ with $n_Q \geq \frac{1}{2c_2}\ell n^{\frac{2}{3}}$, also $\lceil \frac{n_Q}{2} \rceil$ and ℓ are admissible and satisfy $\lceil \frac{n_Q}{2} \rceil + 2\ell \geq 2s$. Then we obtain for $n \to \infty$

$$\frac{t(n_Q,\ell)}{t\left(\lceil \frac{n_Q}{2} \rceil, \ell\right)} \leq \exp\left(-\frac{s^2}{n^2}\right)^{\lfloor \frac{n_Q}{2} \rfloor} \to 0.$$

Hence, the terms provided by $n_Q \geq \frac{1}{2c_2}\ell n^{\frac{2}{3}}$ and $\ell \geq L\frac{s}{n^{\frac{2}{3}}}$ are negligible in T_1. As all n_Q and ℓ, which fulfil (5.16), also satisfy $n_Q \geq \frac{1}{2c_2}\ell n^{\frac{2}{3}}$, we deduce that the main contribution to T_1 is provided by n_Q and ℓ, which fulfil (5.16) and $\ell < L\frac{s}{n^{\frac{2}{3}}}$. □

Lemma 5.11. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there are $c_3, c_4 > 0$ such that for every function h with $h(n) = \omega(1)$ the main contribution to $S_1 = \sum_{n_U \geq 2m_U} s(n_Q, \ell)$ is provided by*

$$n_Q \in \left[2s - h(n)n^{\frac{2}{3}}, 2s + h(n)n^{\frac{2}{3}}\right],$$
$$\ell \in \left[c_3\frac{s}{n^{\frac{2}{3}}}, c_4\frac{s}{n^{\frac{2}{3}}}\right].$$

Proof. We use again the notation from the proofs of Lemmas 5.9-5.10. First we assume $\ell < L\frac{s}{n^{\frac{2}{3}}}$. Then we obtain by (5.21)

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} \leq \exp\left(\frac{a_3Ls}{n^{\frac{2}{3}}n_Q} - \frac{(n_Q+2\ell-2s)^2}{2(n-n_Q)^2} + O\left(\frac{1}{n}\right)\right).$$

Thus, there is a $L_2 > 0$ such that for n large enough

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} \leq \exp\left(-n^{-\frac{2}{3}}\right),$$

whenever $n_Q \geq 2s + L_2 n^{\frac{2}{3}}$. This implies that for every function h with $h(n) = \omega(1)$ the terms provided by $\ell < \frac{Ls}{n^{\frac{2}{3}}}$ and $n_Q \geq 2s + h(n)n^{\frac{2}{3}}$ are negligible in T_1. Thus, the main contribution to T_1 is provided by n_Q and ℓ, which satisfy (5.16) and $n_Q \leq 2s + h(n)n^{\frac{2}{3}}$. Such n_Q and ℓ then also fulfil $n_Q > 2s - h(n)n^{\frac{2}{3}}$ and $\ell = \Theta\left(\frac{s}{n^{\frac{2}{3}}}\right)$ for n large enough. □

Next, we analyse the sum $S_2 = \sum_{n_U < 2m_U} s(n_Q, \ell)$. To that end, we define for admissible n_Q and ℓ

$$t(n_Q,\ell) := \binom{n}{n_Q} |\mathcal{Q}(n_Q, n_Q+\ell)| \binom{\binom{n_U}{2}}{m_U} c \left(\frac{2}{e}\right)^{2m_U - n_U} \frac{m_U^{m_U+\frac{1}{2}} n_U^{n_U - 2m_U + g(n_Q)}}{(n_U - m_U)^{n_U - m_U + \frac{1}{2}}},$$

where

$$g(n_Q) := \begin{cases} \frac{1}{2} & \text{if } n_Q \leq 2s - n^{\frac{2}{3}}\widetilde{h}(n), \\ 0 & \text{otherwise,} \end{cases}$$

and $\widetilde{h}$ is a function with $\widetilde{h}(n) = \omega(1)$ such that for all admissible n_Q and ℓ we have $s(n_Q,\ell) \leq t(n_Q,\ell)$. Such a function $\widetilde{h}$ exists by Lemma 2.5. Now we determine terms, which provide the main contribution to $T_2 := \sum_{n_U < 2m_U} t(n_Q,\ell)$ (see Lemmas 5.12-5.15). We do that in the same way as before when we analysed S_1. Then, in Lemma 5.16 we will see that for those terms, which provide the main contribution to T_2, the estimate $s(n_Q,\ell) \leq t(n_Q,\ell)$ is 'tight', i.e. $\frac{t(n_Q,\ell)}{s(n_Q,\ell)} = O(1)$. Thus, these terms also provide the main contribution to S_2.

Lemma 5.12. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there are $c_1, c_2 > 0$ such that for all functions h_0 and h_1 with $h_0(n) = \omega(1)$ and $h_1(n) = \omega(1)$ the main contribution to $T_2 = \sum_{n_U < 2m_U} t(n_Q,\ell)$ is provided by $(n_Q,\ell) \in I_1 \cup I_2$, where*

$$I_1 := \left\{ (n_Q,\ell) \;\middle|\; \frac{1}{h_0(n)} \frac{n_Q}{n^{\frac{2}{3}}} \leq \ell \leq 1 + h_0(n) \frac{n_Q}{n^{\frac{2}{3}}}, n_Q \leq h_1(n) n^{\frac{2}{3}} \right\},$$
$$I_2 := \left\{ (n_Q,\ell) \;\middle|\; c_1 \frac{n_Q}{n^{\frac{2}{3}}} \leq \ell \leq c_2 \frac{n_Q}{n^{\frac{2}{3}}}, n_Q > h_1(n) n^{\frac{2}{3}} \right\}.$$

Proof. As in the proof of Lemma 5.11 we bound $t(n_Q,\ell)$ from above by

$$u(n_Q,\ell) := \frac{t(n_Q,\ell)}{|\mathcal{Q}(n_Q,n_Q+\ell)|} \binom{n}{n_Q} a_1 n_Q^{n_Q + \frac{3\ell-1}{2}} \rho_1^{\ell} \ell^{-\frac{3\ell}{2}-3} \exp\left(b_1 \sqrt{\frac{\ell^3}{n_Q}}\right) \binom{\binom{n_U}{2}}{m_U}. \quad (5.23)$$

Then we get by a similar computation as in (5.18)

$$\frac{u(n_Q,\ell+1)}{u(n_Q,\ell)} = \Theta(1) \frac{n_Q^{\frac{3}{2}}}{\ell^{\frac{3}{2}}} \frac{1}{n}. \quad (5.24)$$

Now the argument, which we used in the proof of Lemma 5.11 to obtain (5.16), yields the assertion. The reason why we get a slightly weaker result in that case is that we cannot guarantee $\frac{n_Q}{n^{\frac{2}{3}}} \to \infty$ as before. □

Lemma 5.13. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there is a function h_3 with $h_3(n) = \omega(1)$ such that the main contribution to $T_2 = \sum_{n_U < 2m_U} t(n_Q, \ell)$ is provided by n_Q and ℓ, which satisfy*

$$\ell > h_3(n).$$

Proof. We use the notation from Lemma 5.12. We observe that we can choose h_0 such that for all $(n_Q, \ell) \in I_1 \cup I_2$ Lemma 5.5 is applicable. Then we get by a similar computation as in (5.21)

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} \geq \exp\left(a_2\frac{\ell}{n_Q} + O\left(\frac{1}{n}\right)\right)(n-n_Q)^{g(n_Q+1)-g(n_Q)}. \tag{5.25}$$

Now we define $\overline{n_Q} := \left\lfloor 2s - n^{\frac{2}{3}}\widetilde{h}(n)\right\rfloor$, then $g(n_Q+1) = g(n_Q)$ for all $n_Q \neq \overline{n_Q}$. Next, we assume $\ell \leq h_3(n)$, where we choose h_3 later. We know by Lemma 5.8 that $\ell \geq K$ and if we choose K large enough, then for sufficiently large n we get by (5.25)

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} \geq \exp\left(\frac{1}{n_Q}\right) \tag{5.26}$$

for all $n_Q < \overline{n_Q}$. Now let h_4 be a function with $h_4(n) = \omega(1)$. We set

$$h_5(n) := \left\lceil n^{\frac{2}{3}} h_1(n) h_3(n) h_4(n)\right\rceil.$$

Then we observe that we can choose $\widetilde{h}, h_1, h_3, h_4$ such that for all $n \in \mathbb{N}$

$$\ell n^{\frac{2}{3}} h_1(n) + h_5(n) + 2h_3(n) < \overline{n_Q}.$$

Let now n_Q and ℓ such that $n_Q \leq \ell n^{\frac{2}{3}} h_1(n)$. Then we get with (5.26) for $n \to \infty$

$$\frac{t(n_Q+h_5(n),\ell)}{t(n_Q,\ell)} \geq \exp\left(\sum_{i=n_Q}^{n_Q+h_5(n)-1}\frac{1}{i}\right) = \exp\left(\Theta(1)\log\left(\frac{n_Q+h_5(n)}{n_Q}\right)\right) \to \infty.$$

Moreover, for all considered n_Q we have $n_Q + h_5(n) + 2\ell < 2s$, whence the terms provided by $\ell \leq h_3(n)$ and $n_Q \leq \ell n^{\frac{2}{3}} h_1(n)$ are negligible in T_2. Next, we observe that for n large enough all $(n_Q, \ell) \in I_1 \cup I_2$ with $\ell \leq h_3(n)$ also satisfy $n_Q \leq \ell n^{\frac{2}{3}} h_1(n)$. Thus, the main contribution to T_2 is provided by $(n_Q, \ell) \in I_1 \cup I_2$ with $\ell > h_3(n)$. This concludes the proof. □

Lemma 5.14. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there is a $\delta > 0$ such that the main contribution to $T_2 = \sum_{n_U < 2m_U} t(n_Q, \ell)$ is provided by n_Q and ℓ, which satisfy*

$$\ell > \delta \frac{s}{n^{\frac{2}{3}}}.$$

Proof. We use the notation from Lemmas 5.12-5.13. We assume $h_3(n) < \ell \leq \delta \frac{s}{n^{\frac{2}{3}}}$, where we choose $\delta > 0$ later and h_3 is as in Lemma 5.13. Now we use (5.25) to obtain for n large enough

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} \geq \exp\left(\frac{a_2}{2}\frac{\ell}{n_Q}\right) \geq \exp\left(\frac{a_2}{2}\frac{h_3(n)}{n_Q}\right), \tag{5.27}$$

whenever $n_Q < \overline{n_Q}$. Next, we observe that we can choose $\delta, \widetilde{h}$ such that for all $n \in \mathbb{N}$

$$\frac{\delta s}{c_1} + \lceil s \rceil + 2\delta \frac{s}{n^{\frac{2}{3}}} < \overline{n_Q}.$$

Now let n_Q and ℓ such that $n_Q \leq \frac{\ell n^{\frac{2}{3}}}{c_1}$. Then we obtain by (5.27) for $n \to \infty$

$$\frac{t\left(n_Q+\lceil s \rceil,\ell\right)}{t(n_Q,\ell)} \leq \exp\left(\sum_{i=n_Q}^{n_Q+\lceil s \rceil-1} \frac{a_2}{2}\frac{h_3(n)}{i}\right) = \exp\left(\Theta(1)h_3(n)\log\left(\frac{n_Q+s}{n_Q}\right)\right) \to \infty.$$

This shows that the terms provided by $h_3(n) < \ell \leq \delta \frac{s}{n^{\frac{2}{3}}}$ and $n_Q \leq \frac{\ell n^{\frac{2}{3}}}{c_1}$ are negligible in T_2. Thus, the main contribution to T_2 is provided by $(n_Q, \ell) \in I_1 \cup I_2$ with $\ell > \delta \frac{s}{n^{\frac{2}{3}}}$. □

Lemma 5.15. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there are $c_3, c_4 > 0$ such that for every function h with $h(n) = \omega(1)$ the main contribution to $T_2 = \sum_{n_U < 2m_U} t(n_Q, \ell)$ is provided by*

$$n_Q \in \left[2s - h(n)n^{\frac{2}{3}}, 2s\right],$$
$$\ell \in \left[c_3 \frac{s}{n^{\frac{2}{3}}}, c_4 \frac{s}{n^{\frac{2}{3}}}\right].$$

Proof. Again we use the notation from Lemmas 5.12-5.14. Now let n_Q and ℓ such that $\ell > \delta \frac{s}{n^{\frac{2}{3}}}$, where δ is as in Lemma 5.14. We observe that if we choose h_0 small enough, then for all $(n_Q, \ell) \in I_1 \cup I_2$ we have $\ell \leq n^{\frac{1}{2}}$. Hence, we assume from now on $\ell \leq n^{\frac{1}{2}}$. Due to $n_Q \leq 2s$ and (5.25) there is an $a_4 > 0$ such that for n large enough

$$\frac{t(n_Q+1,\ell)}{t(n_Q,\ell)} \geq \exp\left(a_4 \frac{1}{n^{\frac{2}{3}}}\right) \tag{5.28}$$

for all $n_Q \neq \overline{n_Q}$ and

$$\frac{t\left(\overline{n_Q}+1,\ell\right)}{t\left(\overline{n_Q},\ell\right)} \geq n^{-\frac{1}{2}}.$$

Now let n_Q and ℓ such that $n_Q + \left\lceil n^{\frac{1}{2}} \right\rceil < 2s - 2\ell$. Then we obtain for $n \to \infty$

$$\frac{t\left(n_Q + \left\lceil n^{\frac{1}{2}} \right\rceil, \ell\right)}{t(n_Q,\ell)} \geq \exp\left(a_4 \frac{1}{n^{\frac{2}{3}}} \cdot n^{\frac{1}{2}}\right) n^{-\frac{1}{2}} \to \infty.$$

We note that due to $\ell \leq n^{\frac{1}{2}}$ all n_Q with $n_Q < 2s - 4n^{\frac{1}{2}}$ also satisfy $n_Q + \left\lceil n^{\frac{1}{2}} \right\rceil < 2s - 2\ell$. Thus, the above computation shows that the terms provided by (n_Q, ℓ) with $\ell > \delta \frac{s}{n^{\frac{2}{3}}}$ and $n_Q < 2s - 4n^{\frac{1}{2}}$ are negligible in T_2. Next, we consider the case $n_Q \geq 2s - 4n^{\frac{1}{2}}$. Then for n large enough $n_Q > \overline{n_Q}$ holds. Now let h be some function with $h(n) = \omega(1)$ and we assume $n_Q + \left\lceil h(n) n^{\frac{2}{3}} \right\rceil < 2s - 2\ell$. Then we get by using (5.28) for $n \to \infty$

$$\frac{t\left(n_Q + \left\lceil h(n) n^{\frac{2}{3}} \right\rceil, \ell\right)}{t(n_Q,\ell)} \geq \exp\left(a_4 \frac{1}{n^{\frac{2}{3}}} n^{\frac{2}{3}} h(n)\right) \to \infty.$$

This shows that the terms provided by $\ell > \delta \frac{s}{n^{\frac{2}{3}}}$ and $n_Q + h(n) n^{\frac{2}{3}} < 2s - 2\ell$ are negligible in T_2. Thus, the main contribution to T_2 is provided by $(n_Q, \ell) \in I_1 \cup I_2$ with $n_Q \geq 2s - 2\ell - h(n) n^{\frac{2}{3}}$ and $\delta \frac{s}{n^{\frac{2}{3}}} \leq \ell$. If we choose h, h_1 small enough, all such pairs lie in I_2, whence $\ell = \Theta\left(\frac{s}{n^{\frac{2}{3}}}\right)$ follows. □

Lemma 5.16. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there are $c_3, c_4 > 0$ such that for every function h with $h(n) = \omega(1)$ the main contribution to $S_2 = \sum_{n_U < 2m_U} s(n_Q, \ell)$ is provided by*

$$n_Q \in \left[2s - h(n) n^{\frac{2}{3}}, 2s\right],$$
$$\ell \in \left[c_3 \frac{s}{n^{\frac{2}{3}}}, c_4 \frac{s}{n^{\frac{2}{3}}}\right].$$

Proof. Due to Lemma 5.15 there are $c_3, c_4, k > 0$ such that

$$\sum_{n_Q \in I_3, \ell \in I_4} t(n_Q, \ell) = \Theta(1) \sum_{n_U < 2m_U} t(n_Q, \ell),$$
$$\sum_{(n_Q,\ell) \notin I} t(n_Q, \ell) = o(1) \cdot \sum_{n_U < 2m_U} t(n_Q, \ell),$$

where

$$\begin{aligned} I_3 &:= \left[2s - kn^{\frac{2}{3}}, 2s\right], \\ I_4 &:= \left[c_3 \frac{s}{n^{\frac{2}{3}}}, c_4 \frac{s}{n^{\frac{2}{3}}}\right], \\ I &:= \left\{(n_Q, \ell) \;\middle|\; 2s - h(n)n^{\frac{2}{3}} \le n_Q \le 2s, \ell \in I_4\right\}. \end{aligned}$$

The idea is now to show that for $n_Q \in I_3$ and $\ell \in I_4$ the term $s(n_Q, \ell)$ is only 'slightly' smaller than $t(n_Q, \ell)$. We have

$$\frac{t(n_Q,\ell)}{s(n_Q,\ell)} = \binom{\binom{n_U}{2}}{m_U} c \left(\frac{2}{e}\right)^{2m_U - n_U} \frac{m_U^{m_U+\frac{1}{2}} n_U^{n_U - 2m_U + g(n_Q)}}{(n_U - m_U)^{n_U - m_U + \frac{1}{2}}} \cdot \frac{1}{|\mathcal{U}(n_U, m_U)|}.$$

We note that $\frac{m_U - \frac{n_U}{2}}{n_U^{\frac{2}{3}}} \le k$ for n large enough. Thus, we get by Lemma 2.5(ii) that

$$\binom{\binom{n_U}{2}}{m_U} \cdot \frac{1}{|\mathcal{U}(n_U, m_U)|} = \Theta(1).$$

In addition, we observe that for n large enough $g(n_Q) = 0$ for all $n_Q \in I_3$. Next, we show that for $m, n \in \mathbb{N}$ with $0 \le m - \frac{n}{2} \le kn^{\frac{2}{3}}$

$$\left(\frac{2}{e}\right)^{2m-n} \frac{m^{m+\frac{1}{2}} n^{n-2m}}{(n-m)^{n-m+\frac{1}{2}}} = O(1).$$

Using Lemma 3.7 we obtain

$$\begin{aligned} &\left(\frac{2}{e}\right)^{2m-n} \frac{m^{m+\frac{1}{2}} n^{n-2m}}{(n-m)^{n-m+\frac{1}{2}}} = \left(\frac{1}{e}\right)^{2m-n} \left(\frac{m}{n-m}\right)^{m+\frac{1}{2}} \left(\frac{2(n-m)}{n}\right)^{2m-n} \\ &= \exp\left(n - 2m + \left(m + \frac{1}{2}\right)\left[\frac{2m-n}{n-m} - \frac{(2m-n)^2}{2(n-m)^2} + O\left(\frac{(2m-n)^3}{n^3}\right)\right]\right) \\ &\quad \times \exp\left(-(2m-n)\left[\frac{2m-n}{n} + O\left(\frac{(2m-n)^2}{n^2}\right)\right]\right) \\ &= \exp\left(\frac{(2m-n)}{2n(n-m)^2}\left[-m(2m-n)^2 + O(1)\right]\right) = O(1). \end{aligned}$$

Hence, we obtain that for $n_Q \in I_3$ and $\ell \in I_4$

$$\frac{t(n_Q,\ell)}{s(n_Q,\ell)} = \Theta(1).$$

Then we obtain

$$\sum_{(n_Q,\ell)\notin I} s(n_Q,\ell) \le \sum_{(n_Q,\ell)\notin I} t(n_Q,\ell) = o(1)\cdot \sum_{n_U<2m_U} t(n_Q,\ell) = o(1)\cdot \sum_{n_Q\in I_3,\ell\in I_4} t(n_Q,\ell)$$
$$= o(1)\cdot \sum_{n_Q\in I_3,\ell\in I_4} s(n_Q,\ell) = o(1)\cdot \sum_{n_U<2m_U} s(n_Q,\ell).$$

□

Finally, we get I_{n_Q} and I_ℓ by using Lemmas 5.11 and 5.16.

Lemma 5.17. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Then there are $c_1, c_2 > 0$ such that for every function h with $h(n) = \omega(1)$ the main contribution to $\sum_{n_Q,\ell} s(n_Q,\ell)$ is provided by*

$$n_Q \in I_{n_Q} := \left[2s - h(n)n^{\frac{2}{3}}, 2s + h(n)n^{\frac{2}{3}}\right],$$
$$\ell \in I_\ell := \left[c_1\frac{s}{n^{\frac{2}{3}}}, c_2\frac{s}{n^{\frac{2}{3}}}\right].$$

Proof. Combining Lemmas 5.11 and 5.16 yields the assertion. □

5.5 Random outerplanar graphs

Now we use the intervals I_{n_Q}, I_ℓ and I_{n_C} to determine the typical number n_Q of vertices in the complex part, the typical number n_C of vertices in the core and the typical excess $\mathrm{ex}(G)$ in a random outerplanar graph G.

Theorem 5.18. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Moreover, let $G = A(n,m)$. Then whp*

$$n_Q = 2s + O_p\left(n^{\frac{2}{3}}\right),$$
$$n_C = \Theta\left(\frac{s}{n^{\frac{1}{3}}}\right),$$
$$ex(G) = \Theta\left(\frac{s}{n^{\frac{2}{3}}}\right).$$

Proof. The assertions about n_Q and $\ell = \mathrm{ex}(G)$ immediately follow from Lemma 5.17. Hence, we obtain $\ell \to \infty$ whp, whence we can combine Lemmas 5.3-5.4 to get whp

$$n_C = \Theta\left(\sqrt{n_Q\ell}\right) = \Theta\left(\sqrt{s\frac{s}{n^{\frac{2}{3}}}}\right) = \Theta\left(\frac{s}{n^{\frac{1}{3}}}\right).$$

□

Now we use the structural information from Theorem 5.18 to deduce the component structure of $A(n,m)$. Intuitively we expect that the largest component in a random outerplanar graph should be complex. Thus, we consider first the complex part. We show in the next lemma that the complex part has indeed whp a component, which is significantly larger than all the other ones. Then, in the next section we conclude the proof of Theorem 5.1 by showing that all non-complex components are 'small'.

Lemma 5.19. *Let $m = m(n) = \frac{n}{2} + s$, where $s = s(n) = o(n)$ with $\frac{s^3}{n^2} \to \infty$. Moreover, let $G = A(n,m)$. Then*

$$n_Q - |H_1(Q_G)| = O_p\left(n^{\frac{2}{3}}\right).$$

Proof. It suffices to show that whp

$$n_Q - |H_1(Q_G)| = o\left(n^{\frac{2}{3}}\widetilde{h}(n)\right)$$

for each function $\widetilde{h}$ with $\widetilde{h}(n) = \omega(1)$ and $\widetilde{h}(n) = o\left(\frac{s}{n^{\frac{2}{3}}}\right)$. Due to Theorem 5.18 we know that there are $c_1, c_2 > 0$ such that whp

$$s \le n_Q \le 3s, \quad n_C \le c_1 \frac{s}{n^{\frac{1}{3}}}, \quad \ell \le c_2 \frac{s}{n^{\frac{2}{3}}}. \tag{5.29}$$

Thus, there is a function h with $h(n) = \omega(1)$ such that (5.29) holds with a probability of at least $1 - \frac{1}{h(n)}$. In particular we can choose h such that $h(n) = o\left(\widetilde{h}(n)\right)$. Now we define

$$\mathcal{E}(n,m) := \left\{ G \in \mathcal{A}(n,m) \;\middle|\; n_Q(G) - |H_1(Q_G)| \ge \frac{\widetilde{h}(n) n^{\frac{2}{3}}}{h(n)^{\frac{1}{3}}} \right\}.$$

We will show by a double counting argument that for large n, we have

$$|\mathcal{E}(n,m)| \le \frac{1}{h(n)^{\frac{1}{3}}} |\mathcal{A}(n,m)|, \tag{5.30}$$

which shows the statement. We assume to the contrary that there are infinitely many $n \in \mathbb{N}$ such that (5.30) is violated. In the following we only consider such n. We know that the number of graphs in $\mathcal{A}(n,m)$, which do not satisfy (5.29), is at most

$$|\mathcal{A}(n,m)| \frac{1}{h(n)} < |\mathcal{E}(n,m)| \frac{1}{h(n)^{\frac{2}{3}}}.$$

Thus, $G' = E(n,m)$ satisfies (5.29) whp. Now we consider the following operation, which constructs starting from a graph $H' \in \mathcal{E}(n,m)$ a graph $H \in \mathcal{A}(n,m)$ (see Figure 5.3). We add an edge between two different components in $Q_{H'}$ and delete an edge

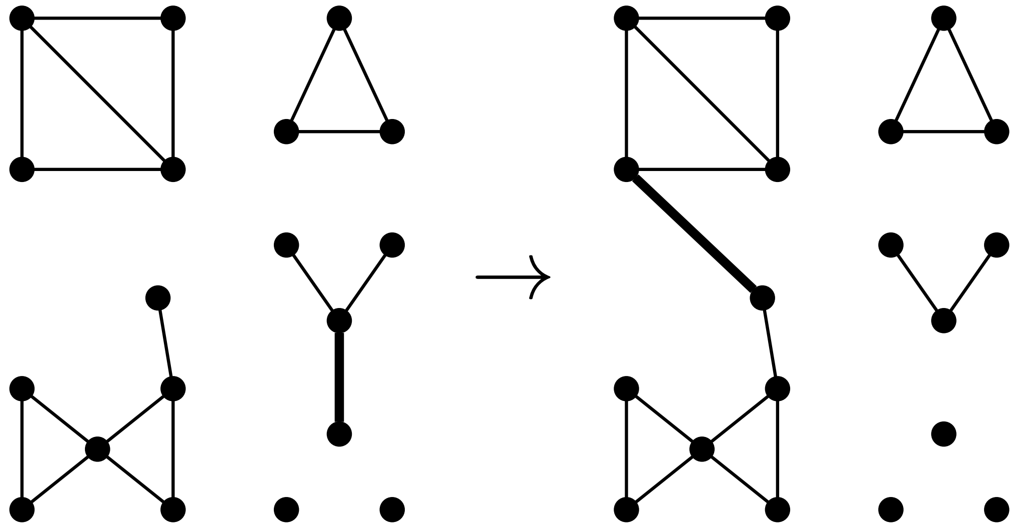

Figure 5.3: The construction used in the proof of Lemma 5.19: We add an edge between two different components in $Q_{H'}$ and delete an edge outside $Q_{H'}$.

outside $Q_{H'}$. We observe that the constructed graph H is still outerplanar and lies therefore in $\mathcal{A}(n,m)$. Now we estimate the number of choices for this operation for a fixed $H' \in \mathcal{E}(n,m)$, which satisfies (5.29). To that end, we partition the vertex set of H' into two disjoint sets A and B. If n is large enough, we can do that due to the definition of $\mathcal{E}(n,m)$ in such a way that

$$|A| \geq \frac{s}{2}, \quad |B| \geq \frac{\widetilde{h}(n) n^{\frac{2}{3}}}{h(n)^{\frac{1}{3}}}.$$

Hence, we have at least

$$|A| \cdot |B| \geq \frac{s\widetilde{h}(n) n^{\frac{2}{3}}}{2h(n)^{\frac{1}{3}}}$$

choices, for the edge we want to add. The number of edges which we can delete is for n large enough bounded from below by

$$m_U(H') = \frac{n}{2} + s - n_Q(H') - \ell(H') \geq \frac{n}{2} + s - 3s - c_2 \frac{s}{n^{\frac{2}{3}}} \geq \frac{n}{3}.$$

So in total there are at least $\frac{s\widetilde{h}(n) n^{\frac{5}{3}}}{6h(n)^{\frac{1}{3}}}$ choices for our operation. If we perform the construction in all graphs $H' \in \mathcal{E}(n,m)$, which satisfy (5.29), we obtain at least

$$\frac{s\widetilde{h}(n) n^{\frac{5}{3}}}{6h(n)^{\frac{1}{3}}} \cdot (1 + o(1)) \left|\mathcal{E}(n,m)\right| = \Theta\left(\frac{s\widetilde{h}(n) n^{\frac{5}{3}}}{h(n)^{\frac{1}{3}}}\right) \left|\mathcal{E}(n,m)\right|$$

(not necessarily different) graphs in $\mathcal{A}(n,m)$. Next, we prove that we also obtain 'many' different graphs by this operation. To that end, we first show that 'most' of the obtained graphs satisfy

$$s \leq n_Q \leq 3s, \quad n_C \leq 2c_1 \frac{s}{n^{\frac{1}{3}}}, \quad \ell \leq 2c_2 \frac{s}{n^{\frac{2}{3}}}. \tag{5.31}$$

Now we assume that we start with a graph $H' \in \mathcal{E}(n,m)$, which fulfils (5.29), and perform the construction step to obtain $H \in \mathcal{A}(n,m)$. We denote by uv the edge, which was added. We have $n_Q(H) = n_Q(H')$ and $\ell(H) = \ell(H') + 1$. Thus, we obtain for n large enough $s \leq n_Q(H) \leq 3s$ and $\ell(H) \leq 2c_2 \frac{s}{n^{\frac{2}{3}}}$.

Next, we look how the number of vertices in the core changes. To that end, we recall the construction step from the core to the complex part. There we replace every vertex of the core C_H by a rooted tree. Thus, every vertex w of the complex part lies in a rooted tree, which we denote by T_w. In addition, let r_w be the root of T_w and P_w the unique path from r_w to w. With this notation we can describe the vertices in C_H, namely

$$V(C_H) = V(C_{H'}) \cup V(P_u) \cup V(P_v).$$

Thus, we obtain

$$n_C(H) \leq n_C(H') + |T_u| + |T_v| \leq c_1 \frac{s}{n^{\frac{1}{3}}} + |T_u| + |T_v|.$$

In the next step we show that $|T_u|$ and $|T_v|$ are typically 'small'. To that end, we fix a core C with $|C| =: \overline{n_C} \leq c_1 \frac{s}{n^{\frac{1}{3}}}$ and a number $\overline{n_Q} \in [s, 3s]$. Then we choose uniformly at random a complex graph with $\overline{n_Q}$ vertices, which can be constructed from C. We obtain this graph by adding $\overline{n_C}$ rooted trees with in total $\overline{n_Q}$ vertices to C. Now let w be a vertex of C and T_w the tree, which is added to w. Then we have $\mathbb{E}\left[|T_w|\right] = \frac{\overline{n_Q}}{\overline{n_C}} = O\left(n^{\frac{1}{3}}\right)$ and $\mathbb{V}\left[|T_w|\right] = O\left(n^{\frac{1}{3}}\right)$. We use Chebyshev's inequality to obtain

$$\mathbb{P}\left[\ |T_w| > \frac{c_1}{2}\frac{s}{n^{\frac{1}{3}}}\ \right] \leq \frac{\mathbb{V}\left[|T_w|\right]}{\left(\frac{c_1}{2}\frac{s}{n^{\frac{1}{3}}} - \mathbb{E}\left[|T_w|\right]\right)^2} = \frac{O\left(n^{\frac{1}{3}}\right)}{\Theta\left(\frac{s^2}{n^{\frac{2}{3}}}\right)} = O\left(\frac{n}{s^2}\right).$$

Thus, we can use the union bound to obtain

$$\mathbb{P}\left[\ \exists w : |T_w| > \frac{c_1}{2}\frac{s}{n^{\frac{1}{3}}}\ \right] \leq \overline{n_C} \cdot O\left(\frac{n}{s^2}\right) \leq O\left(\frac{n^{\frac{2}{3}}}{s}\right) = o(1).$$

Hence, whp every added tree has at most $\frac{c_1}{2}\frac{s}{n^{\frac{1}{3}}}$ vertices, whence almost all obtained graphs H satisfy $n_C(H) \leq 2c_1 \frac{s}{n^{\frac{1}{3}}}$. We conclude that by our construction we obtain at least

$$\Theta\left(\frac{s\widetilde{h}(n)n^{\frac{5}{3}}}{h(n)^{\frac{1}{3}}}\right)|\mathcal{E}(n,m)| \tag{5.32}$$

(not necessarily) different graphs in $\mathcal{A}(n,m)$, which fulfil (5.31).

In the next step we show that a 'large' number of these graphs are different. To that end, we fix a graph $H \in \mathcal{A}(n,m)$, which satisfy (5.31) and consider the reverse operation of our construction. We delete a bridge in the complex part Q_H such that the two new obtained components stay complex and add an edge in the non-complex part such that no new complex component is created. We observe that the deleted edge must lie in the core C_H, whence there are at most

$$n_C(H) + \ell(H) \leq 2c_1\frac{s}{n^{\frac{1}{3}}} + c_2\frac{s}{n^{\frac{2}{3}}} = O\left(\frac{s}{n^{\frac{1}{3}}}\right)$$

choices for that edge. The number of possible edges, which can be added, is obviously bonded by n^2. Hence, H can be obtained at most $O\left(n^{\frac{5}{3}}s\right)$ times by our construction. This shows together with (5.32) that we obtain at least

$$\Theta\left(\frac{s\widetilde{h}(n)n^{\frac{5}{3}}}{h(n)^{\frac{1}{3}}}\right)|\mathcal{E}(n,m)| \cdot \frac{1}{O\left(n^{\frac{5}{3}}s\right)} = \Omega\left(\frac{\widetilde{h}(n)}{h(n)^{\frac{1}{3}}}\right)|\mathcal{E}(n,m)|$$

different graphs in $\mathcal{A}(n,m)$. Thus, we obtain

$$|\mathcal{A}(n,m)| = \Omega\left(\frac{\widetilde{h}(n)}{h(n)^{\frac{1}{3}}}\right)|\mathcal{E}(n,m)| = \Omega\left(\frac{\widetilde{h}(n)}{h(n)^{\frac{1}{3}}}\right)\frac{1}{h(n)^{\frac{1}{3}}}|\mathcal{A}(n,m)| = \omega(1)\,|\mathcal{A}(n,m)|\,,$$

which is a contradiction. Hence, we have shown (5.30), which concludes the proof. □

5.6 Proof of main theorem

Now we can show that outerplanar graphs feature the same first phase transition as planar graphs.

Proof of Theorem 5.1. For (i) we assume that $\frac{s^3}{n^2} \to -\infty$. Then by Lemma 2.5(i) the Erdős-Rényi graph $G(n,m)$ has whp no complex components and is therefore outerplanar. Thus, the statement follows immediately from Theorem 1.1(i). Now let $\frac{s^3}{n^2} \to c \in \mathbb{R}$. We use Lemma 2.5(ii) to obtain that the probability that $G(n,m)$ is outerplanar is bounded away from 0. Then Theorem 1.1(ii) yields the statement (ii).

Finally, we assume $\frac{s^3}{n^2} \to \infty$ and let $G = A(n,m)$. We first consider the components of the complex part Q_G. Combining Theorem 5.18 and Lemma 5.19 we get that the largest component in the complex part has $2s + O_p\left(n^{\frac{2}{3}}\right)$ vertices, while all other components have $O_p\left(n^{\frac{2}{3}}\right)$ vertices. Thus, it suffices to show that for each $i \in \mathbb{N}$ the i-th largest non-complex component of G has $\Theta_p\left(n^{\frac{2}{3}}\right)$ vertices. By Theorem 5.18 there is for every $\delta > 0$ a $c > 0$ such that with a probability of at least $1 - \delta$

$$n_U \in \left[n - 2s - cn^{\frac{2}{3}}, n - 2s + cn^{\frac{2}{3}}\right],$$
$$m_U \in \left[m - 2s - cn^{\frac{2}{3}}, m - 2s + cn^{\frac{2}{3}}\right].$$

In such a case also $-\frac{3}{2}cn^{\frac{2}{3}} \leq \frac{n_U}{2} - m_U \leq \frac{3}{2}cn^{\frac{2}{3}}$ holds and therefore $\frac{\left(\frac{n_U}{2} - m_U\right)^3}{n_U^2}$ is bounded. Hence, by Lemma 2.5(ii) the probability that $G(n_U, m_U)$ is outerplanar is bounded away from 0. Thus, we can use Theorem 1.1(ii) to obtain that the i-th largest non-complex component has $\Theta_p(n^{\frac{2}{3}})$ vertices. This shows the assertion (iii). □

6 Discussion

Roughly speaking, the class of outerplanar graphs lies between the class of cacti graphs and the class of planar graphs. As both classes feature the same two phase transitions even for $m > \frac{n}{2}$, we believe that they also occur in outerplanar graphs.

Conjecture 6.1. *Theorems 1.3-1.4 are also true for the class of outerplanar graphs.*

We even believe that this conjecture can be proven in a very similar way as Theorem 1.6. For that we would need analogous results to Lemmas 5.7-5.17, 5.19 and Theorem 5.18. It seems that the main challenge there will be to generalise Lemmas 5.9-5.16, while all other statements should be easily extendable. For the case $m = \alpha n$ with $\alpha \to c \in \left(\frac{1}{2}, 1\right)$ one can try to modify the proofs of Lemmas 5.9-5.16 in such a way that they also hold if s is replaced by $\frac{2\alpha-1}{2}n$. That does not seem possible in the case $m = n + o(n)$. Nevertheless, we believe that the main ingredients of that proof, which are Lemmas 2.5, 4.6 and 5.5-5.6 still suffice to show the statement also for that case.

The main difference between the proof of Theorem 1.2 in [17] and the proof of Theorem 5.1 in Chapter 5 is the following. The planar graphs in [17] are decomposed until the kernel is reached, while we stop for outerplanar graphs already at the core. This raises the question whether it is also possible to show Theorem 1.2 if one uses only the decomposition until the core. Surprisingly, most of the statements of Chapter 5 can be translated to the planar case. The starting point of our proof was Lemma 5.2, which can be extended to planar graphs. We note that the proof becomes even simpler in that case, since subdividing an edge in a planar graph leads always to a planar graph. Due to that reason one can also slightly improve the bounds.

Then the computations in the proofs of Lemmas 5.3-5.6 are also valid for planar graphs. Next, we consider Lemmas 5.7-5.17. The main argument, which we used there, was based on estimates for $\frac{|\mathcal{Q}(n_Q+1, n_Q+1+\ell)|}{|\mathcal{Q}(n_Q, n_Q+\ell)|}$ and $\frac{|\mathcal{Q}(n_Q, n_Q+\lfloor c\ell \rfloor)|}{|\mathcal{Q}(n_Q, n_Q+\ell)|}$. To estimate the first fraction we used Lemmas 5.5-5.6, which are due to the above discussion also true for planar graphs. For the second fraction we used estimates for $|\mathcal{Q}_C(n_Q, n_Q+\ell)|$ and $|\mathcal{Q}_P(n_Q, n_Q+\ell)|$, which we got from [17]. There of course a decomposition into the kernel was used to obtain these bounds. So we cannot use that idea if we want to stop our decomposition already at the core. Thus, we need to solve the following problem.

Problem 6.2. *For $c > 0$ and admissible n_Q and ℓ, derive a good estimate for*

$$\frac{|\mathcal{Q}_P(n_Q, n_Q+\lfloor c\ell \rfloor)|}{|\mathcal{Q}_P(n_Q, n_Q+\ell)|}.$$

We note that the estimate of $\frac{|\mathcal{Q}(n_Q, n_Q+\lfloor c\ell \rfloor)|}{|\mathcal{Q}(n_Q, n_Q+\ell)|}$, which we used, does not seem to be very tight. Thus, we believe that already a rough estimate in Problem 6.2 is sufficient. Having such an estimate, one could complete the proof for planar graphs in the same way as in Chapter 5. That is possible, because Theorem 5.18 and Lemma 5.19 can easily be extended to planar graphs. To sum up the above discussion, we can use the ideas presented in Chapter 5 to show Theorem 1.2 if we are able to solve Problem 6.2.

Finally, we compare the estimate for $|\mathcal{Q}_P(n_Q, n_Q+\ell)|$, which was used in [17] (see Lemma 4.9) with our approximation of $\frac{|\mathcal{Q}_P(n_Q+1, n_Q+1+\ell)|}{|\mathcal{Q}_P(n_Q, n_Q+\ell)|}$ from Lemmas 5.5-5.6. For simplicity, we assume $\ell = \omega(1)$ and $\ell = o(n_Q)$. If we could use the upper bound of Lemma 4.9 as an approximation for $|\mathcal{Q}_P(n_Q, n_Q+\ell)|$ we would get by using Lemma 3.7

$$\begin{aligned}
&\frac{|\mathcal{Q}_P(n_Q+1, n_Q+1+\ell)|}{|\mathcal{Q}_P(n_Q, n_Q+\ell)|} \\
&\approx \frac{a_1 \left(n_Q+1\right)^{n_Q+1+\frac{3\ell-1}{2}} \rho_1^\ell \ell^{-\frac{3\ell}{2}-3} \exp\left(b_1 \sqrt{\frac{\ell^3}{n_Q+1}}\right)}{a_1 n_Q^{n_Q+\frac{3\ell-1}{2}} \rho_1^\ell \ell^{-\frac{3\ell}{2}-3} \exp\left(b_1 \sqrt{\frac{\ell^3}{n_Q}}\right)} \\
&= (n_Q+1)\left(1+\frac{1}{n_Q}\right)^{n_Q+\frac{3\ell-1}{2}} \exp\left(b_1 \ell^{\frac{3}{2}}\left(\frac{1}{\sqrt{n_Q+1}}-\frac{1}{\sqrt{n_Q}}\right)\right) \\
&= (n_Q+1)\exp\left(\left(n_Q+\frac{3\ell-1}{2}\right)\left[\frac{1}{n_Q}-\frac{1}{2n_Q^2}+O\left(\frac{1}{n_Q^3}\right)\right]+O\left(\left(\frac{\ell}{n_Q}\right)^{\frac{3}{2}}\right)\right) \\
&= (n_Q+1)\exp\left(1+\frac{3\ell-2}{2n_Q}+O\left(\left(\frac{\ell}{n_Q}\right)^{\frac{3}{2}}\right)\right) \\
&= (n_Q+1)\exp\left(1+\left(\frac{3}{2}+o(1)\right)\frac{\ell}{n_Q}\right). \qquad (6.1)
\end{aligned}$$

In comparison to (6.1) we obtain by Lemmas 5.5-5.6 which are also true for planar graphs that

$$\frac{|\mathcal{Q}_P(n_Q+1, n_Q+1+\ell)|}{|\mathcal{Q}_P(n_Q, n_Q+\ell)|} = (n_Q+1)\exp\left(1+\Theta(1)\frac{\ell}{n_Q}\right). \qquad (6.2)$$

We observe that the estimates (6.1) and (6.2) match and that (6.1) is slightly stronger. But it is important to note that we derived the first one by a heuristic computation. Thus, it is not clear whether (6.1) holds for all admissible n_Q and ℓ. In contrast, (6.2) is true for *all* admissible n_Q and ℓ.

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