Svenja Hüning, M.Sc.

Geometric and algebraic analysis of subdivision processes

DISSERTATION

zur Erlangung des akademischen Grades
Doktorin der Naturwissenschaften

eingereicht an der

Technischen Universität Graz

Betreuer:
Univ.-Prof. Dr. Johannes Wallner

Institut für Geometrie

Graz, Mai 2019

EIDESSTATTLICHE ERKLÄRUNG
AFFIDAVIT

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst, andere als die angegebenen Quellen/Hilfsmittel nicht benutzt, und die den benutzten Quellen wörtlich und inhaltlich entnommenen Stellen als solche kenntlich gemacht habe. Das in TUGRAZonline hochgeladene Textdokument ist mit der vorliegenden Dissertation identisch.

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly indicated all material which has been quoted either literally or by content from the sources used. The text document uploaded to TUGRAZonline is identical to the present doctoral thesis.

Datum / *Date*

Unterschrift / *Signature*

Contents

Abstract

Subdivision schemes are iterative refinement rules used to generate smooth geometric objects, such as curves or surfaces.

In the first part of this thesis, we deal with nonlinear subdivision schemes, i.e., refinement algorithms applied to data lying in nonlinear spaces. Literature already provides a number of different approaches to transfer linear subdivision to nonlinear geometries. We use the so-called Riemannian analogue of a linear subdivision rule which is obtained from a linear rule by replacing affine averages by weighted geodesic averages and show that it is well defined on Cartan-Hadamard manifolds (i.e., simply connected, complete manifolds with nonpositive sectional curvature). Up to now, most convergence results for nonlinear analogues of linear schemes are limited to dense enough input data. In this thesis, we provide convergence results for Riemannian analogues of linear schemes on Cartan-Hadamard manifolds which are valid for all input data. In particular, we prove that if a linear subdivision scheme converges uniformly, then its Riemannian analogue on Cartan-Hadamard manifolds does so, too. Additionally, we analyse the Hölder continuity of the resulting limit functions.

On positively-curved Riemannian manifolds, the situation is appreciably different. This essentially follows from two points: The Riemannian analogue is no longer globally well defined and certain distance estimates on Cartan-Hadamard manifolds are not valid on positively-curved manifolds. As a first approach, we therefore restrict our analysis to the unit sphere and provide a strategy for showing convergence results for the Riemannian analogue of a linear subdivision scheme.

In the second part of this thesis, we focus on Hermite subdivision schemes and their ability of polynomial reproduction. In contrast to standard scalar subdivision schemes, a Hermite scheme operates on vector-valued input data, which is interpreted as function and its consecutive derivative values. A convergent subdivision scheme is said to reproduce polynomials if sampling the initial data from a polynomial yields the same polynomial in the limit. We provide algebraic conditions on the Hermite scheme, which fully characterise its ability to reproduce polynomials. This generalises similar conditions known from scalar subdivision. As a start, we consider schemes reproducing function values and first derivatives, afterwards we show an extension of this result to schemes of any order, i.e., we consider higher derivatives as well.

Acknowledgements

First, I wish to thank my advisor Johannes Wallner for introducing me to the topic of refinement algorithms. I am grateful for all the time he took for mathematical discussions and for his fruitful suggestions to improve this thesis. Moreover, he helped me a lot to develop my mathematical writing and presentation skills further.

During my research stay at the University of Florence Costanza Conti was an extremely friendly and caring host who made me feel at home. I would like to express my gratitude to her for the very productive collaboration from which I greatly benefited. Furthermore, I am thankful to her and Tomas Sauer for sharing their ideas concerning this thesis with me.

Many thanks also go to Wilfried Imrich for introducing me to the mathematical topic of symmetry breaking in graphs. I gained insight into an area which was totally new for me. It is a pleasure to work with him.

I really enjoy the atmosphere within the doctoral program 'Discrete Mathematics' and I thank my colleagues for interesting collaborations and mathematical discussions.

Furthermore, I would like to mention the people at the Institute of Geometry at TU Graz. They provide an extremely pleasant working atmosphere and it is a great pleasure to be a member of the group.

Finally, great thanks to all my friends who encouraged me to write this thesis.

1 Introduction

Subdivision rules are algorithms which produce limit curves by iteratively refining initial control polygons. De Rham first mentioned the idea in [18]. In the context of computer graphics the procedure was rediscovered in the 1970s by Chaikin [3]. Subdivision algorithms have become of interest in computer aided geometric design due to their local structure (which makes them easy to implement) and their capability to generate smooth curves [25]. For example the Lane-Riesenfeld algorithm can be used to compute B-spline curves [54, 25].

Subdivision algorithms are not limited to curve design, but are also used to generate smooth surfaces from data attached to control nets with irregular combinatorics. The Doo-Sabin scheme [29], the Catmull-Clark scheme [1] and the Loop scheme [56] are early contributions to subdivision with irregular meshes. We refer to the textbook [69] for a comprehensive overview on this topic.

Subdivision rules play an important role in geometric modelling [3, 30] and very much in computer graphics [69]. They are used to construct wavelets and find applications in multiresolution analysis [6, 17] as well as in approximation and interpolation theory [26, 19].

A linear *refinement rule* maps a sequence $(x_i)_{i\in\mathbb{Z}^s}$, $s \geqslant 1$, in a linear space to a sequence $(Sx_i)_{i\in\mathbb{Z}^s}$ where the new points $Sx_i = \sum_{j\in\mathbb{Z}^s} a_{i-2j}x_j$, $i \in \mathbb{Z}$, are given as affine linear combinations of finitely many old ones. The repeated application of this refinement rule determines a *subdivision scheme* S. The coefficients $(a_i)_{i\in\mathbb{Z}^s}$ are called the *mask* of the scheme.

In this thesis, all schemes are assumed to be *univariate*, i.e., $s = 1$, but the *multivariate* case is studied intensively, too [2, 38, 59].

We denote by S^k, $k \in \mathbb{N}$, the repeated application of the refinement rule and call a linear subdivision scheme S *convergent* if there exist piecewise linear functions f_k with $f_k\left(\frac{i}{2^k}\right) = \left(S^k x\right)_i$ which converge, uniformly on compact sets, to a limit.

For initial data lying in linear spaces subdivision rules are well studied regarding their properties of convergence and smoothness of the resulting limit curves, see for example [2, 25]. In this respect, relating the convergence of a subdivision scheme to algebraic conditions of the *symbol* of its mask, i.e., the Laurent polynomial $a(z) = \sum_{j\in\mathbb{Z}} a_j z^j$, was crucial [2]. Furthermore, the *derived scheme* which operates on divided differences of input data became another important tool in the study of refinement algorithms and their convergence [25, 28].

One might vary the coefficients of the mask in each iteration step which makes the subdivision rule *level-dependent*, see [30] for an introduction. Those schemes are also called *non-stationary* as opposed to *stationary* schemes which keep the same refinement rule throughout the whole subdivision process.

A well-known class of subdivision rules are *interpolatory* schemes which are characterised by the fact that they always preserve the data of the previous refinement level. For curve design we mention the 4-point scheme [26] and the 6-point Dubuc-Deslauriers scheme [19]. Their convergence and approximation order are well studied and the smoothness of resulting limit curves has been analysed. The Butterfly scheme was the first interpolatory scheme used on a triangular grid to generate surfaces [27].

Another interesting question is the following: If the initial data is sampled from a function, does a scheme reproduce the same function in the limit? Linear schemes reproducing polynomials are characterised in [9]. For exponential polynomials this question was considered for example in [31, 11]. Even if the function is not fully reproduced in the limit, one can study the approximation order of the subdivision scheme, for stationary [55] as well as for level-dependent schemes [13].

In this thesis, we contribute new results to two different areas within the wide field of subdivision schemes. The first part of the thesis focuses on the convergence analysis of subdivision rules applied to nonlinear data while the second part deals with the property of polynomial reproduction by Hermite subdivision schemes.

Introduction to the first part of this thesis

We are interested in the convergence analysis of nonlinear subdivision schemes which are obtained by adapting linear refinement rules to nonlinear geometries. Since data often comes from nonlinear geometries this has become an active field of research in the last years [21, 77, 40, 63, 34, 38]. For example, one can consider initial data lying in symmetric spaces, Lie groups or arbitrary Riemannian manifolds instead of only linear spaces. In particular, different methods to transfer linear subdivision schemes to nonlinear geometries have been introduced: the log-exp-analogue [70, 21], the projection analogue [82] and binary geodesic averaging [77]. After adapting linear schemes to a wider class of geometries, questions of their properties arise.

Before we discuss the convergence of nonlinear analogues of linear schemes in detail, we summarise some results on nonlinear refinement algorithms (assuming their convergence). They have been analysed regarding their approximation order and stability, see for example [47, 41, 39]. Furthermore, the smoothness of the limit curves of nonlinear analogues of linear schemes has been studied in many works, see for example [75, 42, 78]. However, it requires specific ways of transferring linear schemes to nonlinear geometries to show that the nonlinear limit is as least as smooth as its linear counterpart, see for example [83, 40].

If we assume that the nonlinear analogue of a linear scheme is convergent, then the smoothness of the resulting limit curves is fully studied. The situation for the convergence analysis of nonlinear schemes itself, however, is different and therefore still an active field of research. In [77] the so-called proximity conditions have been introduced to obtain convergence results for nonlinear analogues which are obtained from linear schemes by replacing binary averages by geodesic averages. Unfortunately, this procedure is in general not unique and the convergence results are limited to only 'dense enough' input data. However, there are results which apply to all input data if only special classes of subdivision schemes are considered, for example interpolatory schemes [76]. Furthermore, for special schemes convergence results could be proven for all input data by the repeated application of binary geodesic averaging [32, 33, 73]. As an example where geodesic subdivision has been used, we mention [67]. The author studied corner cutting schemes and a variation of the 4-point scheme on the sphere by an iterative computation of geodesic midpoints.

In this thesis we adapt linear schemes to Riemannian manifolds by replacing affine averages by the Riemannian center of mass. This method is known as the *Riemannian analogue* of a linear scheme, see [40]. It is said to be convergent if the subdivision rule considered in coordinate charts of the manifold converges. In [79] the convergence of the Riemannian analogue of a linear subdivision rule with nonnegative mask has been studied on Cartan-Hadamard manifolds (i.e., simply connected, complete Riemannian manifolds with nonpositive sectional curvature). More generally, convergence results have been obtained on Cartan-Hadamard spaces with the help of stochastic methods by interpreting the coefficients of the mask as probabilities [34, 35].

We present a convergence result for the Riemannian analogue which is valid for all affine invariant subdivision schemes with arbitrary mask on Cartan-Hadamard manifolds. Therefore, we prove an existence and uniqueness result of the Riemannian center of mass on Cartan-Hadamard manifolds. The existence and uniqueness analysis of the Riemannian center of mass, however, is not only important in the field of refinement algorithms and approximation theory but also in stochastics on manifolds, see for example [68].

In order to generalise further, we are interested in convergence results for all input data on positively-curved manifolds. Since the Riemannian center of mass is in general not globally well defined on positively-curved spaces and the estimate of distances cannot be directly transferred from Cartan-Hadamard manifolds, we need a new strategy to obtain convergence results. Therefore, we start by analysing subdivision schemes on a concrete positively-curved manifold, namely the unit sphere. It turns out that already on this elementary manifold the above mentioned difficulties become visible.

We aim for convergence results for the Riemannian analogues of linear subdivision schemes on the unit sphere without any sign restrictions on the mask. Therefore, the knowledge of explicit formulas for the gradient and the Hessian of the squared distance function on the sphere are important tools. As a first approach towards a general convergence analysis, we show that the Riemannian analogues of some well-known subdivision schemes such as the cubic Lane-Riesenfeld algorithm and the 4-point scheme converge on the unit sphere. Furthermore, we apply our strategy to a non-interpolatory scheme whose mask contains negative coefficients.

Introduction to the second part of this thesis

It is an active field of research to study subdivision schemes refining vector-valued sequences, rather than only *scalar* schemes whose input data consists of point sequences [61, 63, 59]. Therefore, the scalar coefficients of the mask are replaced by a matrix-valued sequence. Literature provides many results on the convergence of *vector subdivision* and the smoothness of the limit curves in the univariate case [61, 62] as well as in the multivariate case [5].

In the second part of this thesis, we focus on *Hermite subdivision schemes*. They are a special class of vector subdivision, as they are considered to refine vector-valued data consisting of function and its consecutive derivative values. The dimension of the vectors, i.e., the function value plus the number of considered derivatives, is called the *order* of the scheme. First analysed in [57], Hermite schemes nowadays find applications in various areas, for example approximation theory [4, 14, 44, 45, 71], Hermite-type multiwavelets [16, 51] and biomedical imaging [12, 74].

Hermite schemes due to the special structure of their input data include the smoothness analysis of the limit curve already in its convergence analysis. However, due to the interpretation of the vector entries as function and derivative values, they naturally become mildly level-dependent, in the sense that the matrix-valued mask is multiplied by so-called dilation matrices in each subdivision step.

To analyse the convergence of Hermite schemes the so-called spectral condition (or equivalently the sum rule [43, 45]) was introduced [23]. A Hermite scheme satisfying this condition admits a Taylor factorisation which links it to a stationary scheme and whose contractivity leads to a convergence criterion [59, 23]. Since it turns out that not all convergent Hermite subdivision schemes satisfy the spectral condition, further concepts such as the generalised Taylor operator have been introduced [58]. Quite recently spectral conditions have been extended to a wider approach, namely spectral chains. It is conjectured in [60] that spectral chains provide a necessary condition for the convergence of a Hermite subdivision scheme. Additionally, this concept is used to construct Hermite schemes of any regularity. Subdivision rules of Hermite type have been transferred to nonlinear geometries and analysed regarding their smoothness [63, 64].

A convergent Hermite scheme is said to reproduce polynomials, if sampling input data from a polynomial results in the same polynomial in the limit. In particular, the capability of polynomial reproduction implies that the scheme fulfils the spectral condition and therefore can be analysed regarding its convergence [59]. In [65] an overview of the relation between polynomial reproduction, the spectral condition and the sum rule is provided. Furthermore, polynomial reproduction is closely related to the approximation order of the scheme, in the linear situation [13] as well as for Hermite data [49].

Moreover, it has become of interest to study the capability of Hermite schemes to reproduce not only polynomials but also exponentials [12, 74]. Factorisations of Hermite schemes with respect to reproduction of exponential polynomials are studied in [7, 15]. The results are used for the convergence analysis, too [8, 50].

In the case of linear subdivision schemes, the question of polynomial reproduction is fully answered by providing algebraic conditions on the symbol of the scheme [9]. We extend those results to Hermite schemes and give purely algebraic conditions which fully characterise polynomial reproduction. We start by analysing schemes of order two, meaning we consider input data consisting of function values and first derivatives. With the help of this result we then give a characterisation for schemes of any order. The number of algebraic conditions needed only depends on the degree of the polynomials. Additionally, we demonstrate how to use our result to construct Hermite schemes that reproduce polynomials of a certain degree by only slightly increasing its *support* (i.e., the number of non-zero mask elements).

Organisation of the thesis

We start with an introduction of the basic concepts of linear subdivision. Then, we define the Riemannian analogue of a linear scheme and show its well-definedness on Cartan-Hadamard manifolds (i.e., simple connected, complete manifolds with nonpositive sectional curvature). We continue by proving that a Riemannian analogue of a linear scheme converges to a continuous limit function if the norm of the derived (resp. iterated derived) scheme is bounded from above by its dilation factor. We analyse the Hölder continuity of the resulting limit curves and show how to drop the assumption of simple connectivity of the underlying manifold.

In the next section, we analyse the convergence of the Riemannian analogue of a linear scheme on the unit sphere. To do so, we first deal with the question of well-definedness of the Riemannian center of mass on the unit sphere. Secondly, we introduce a strategy to estimate distances of refined data on the sphere using a second order Taylor approximation.

We continue with an introduction to Hermite subdivision schemes. Then, we define certain classes of polynomials and study their properties. These auxiliary polynomials are required to give purely algebraic conditions on the symbol of a Hermite scheme which characterise their property of polynomial reproduction. We first consider Hermite data consisting only of function values and first derivatives and then, in the last part, we extend the results to schemes of any order.

The first and the last part of this thesis consists of the following three publications with only small modifications:

- S. Hüning, J. Wallner, *Convergence of subdivision schemes on Riemannian manifolds with nonpositive sectional curvature*, Advances in Computational Mathematics, published online May 2019, DOI:10.1007/s10444-019-09693-x.

- C. Conti, S. Hüning, *An algebraic approach to polynomial reproduction of Hermite subdivision schemes*, Journal of Computational and Applied Mathematics, 349, 302-315, 2019, DOI:10.1016/j.cam.2018.08.009.

- S. Hüning, *Polynomial reproduction of Hermite subdivision schemes of any order*, submitted, 2019.

The results of Section 2.3 (convergence analysis of subdivision rules on the unit sphere) are not yet published.

The research leading to this thesis was supported by the Austrian Science Fund (FWF): W1230.

2 Subdivision in nonlinear geometries

2.1 Introduction

We introduce linear subdivision schemes as well as the notation used throughout the section. An overview how to transfer linear refinement algorithms to nonlinear geometries is provided. Our main focus is on the so-called Riemannian analogue of a linear scheme.

This chapter is based on the results presented in the publication

S. Hüning, J. Wallner, *Convergence of subdivision schemes on Riemannian manifolds with nonpositive sectional curvature*, Advances in Computational Mathematics, published online May 2019, DOI:10.1007/s10444-019-09693-x.

2.1.1 Linear subdivision

A *linear subdivision scheme* S maps a sequence of points $(x_i)_{i\in\mathbb{Z}}$ lying in a linear space to a new sequence of points $(Sx_i)_{i\in\mathbb{Z}}$ using the rule

$$Sx_i = \sum_{j\in\mathbb{Z}} a_{i-Nj}x_j.$$

Here $N \in \mathbb{N}$ is the *dilation factor*. We require $N \geqslant 2$, but the usual case is $N = 2$. Throughout the paper we assume that the sequence a_ℓ, $\ell \in \mathbb{Z}$, called the *mask* of the subdivision rule, has compact support. This means that $a_\ell \neq 0$ only for finitely many ℓ. It turns out that the condition

$$\sum_{j\in\mathbb{Z}} a_{i-Nj} = 1 \quad \text{for all } i \tag{2.1}$$

(*affine invariance*) is necessary for the convergence of linear subdivision schemes, see [33] and [2] for an overview. From now on, we make the assumption that all subdivision schemes are affine invariant.

To simplify notation, we initially consider only *binary* subdivision rules, i.e., rules with dilation factor $N = 2$. Then we can write the refinement rule in the following way:

$$(Sx)_{2i} = \sum_{j=-m}^{m+1} \alpha_j x_{i+j} \quad \text{and} \quad (Sx)_{2i+1} = \sum_{j=-m}^{m+1} \beta_j x_{i+j}, \tag{2.2}$$

with $m \in \mathbb{N}$ and coefficients α_j, β_j such that

$$\sum_{j=-m}^{m+1} \alpha_j = \sum_{j=-m}^{m+1} \beta_j = 1. \tag{2.3}$$

For example Chaikin's algorithm [3], which is given by the mask $(a_{-2}, \dots, a_1) = (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})$, can be written as

$$(Sx)_{2i} = \frac{3}{4}x_i + \frac{1}{4}x_{i+1} \quad \text{and} \quad (Sx)_{2i+1} = \frac{1}{4}x_i + \frac{3}{4}x_{i+1}. \tag{2.4}$$

Subdivision schemes satisfying $(Sx)_{2i} = x_i$ are called *interpolatory.* For example the well-known 4-point scheme is defined by

$$(Sx)_{2i} = x_i \quad \text{and} \quad (Sx)_{2i+1} = -\omega x_{i-1} + \Big(\frac{1}{2} + \omega\Big)x_i + \Big(\frac{1}{2} + \omega\Big)x_{i+1} - \omega x_{i+2}, \tag{2.5}$$

for some parameter ω, see [26]. The next example will be our main example throughout the section.

Example 2.1. We consider a non-interpolatory subdivision scheme with negative mask. Taking averages of the 4-point scheme with parameter $\omega = \frac{1}{16}$ and Chaikin's scheme yields

$$\begin{aligned}(Sx)_{2i} &= -\frac{1}{32}x_{i-1} + \frac{21}{32}x_i + \frac{13}{32}x_{i+1} - \frac{1}{32}x_{i+2},\\ (Sx)_{2i+1} &= -\frac{1}{32}x_{i-1} + \frac{13}{32}x_i + \frac{21}{32}x_{i+1} - \frac{1}{32}x_{i+2}.\end{aligned}$$

$\diamondsuit$

2.1.2 Adaption of subdivision to nonlinear geometries

In the last years, different ways to transfer linear schemes to nonlinear geometries have been studied [21, 33, 40]. Various methods to apply subdivision rules to data lying in Lie groups, symmetric spaces or Riemannian manifolds were introduced, an overview of concepts can be found in [40]. We shortly present some of them.

The *log-exp-analogue* of a linear scheme uses the intrinsic geometry of a Riemannian manifold. The idea is to lift the nonlinear data to suitable tangent spaces of the manifold by applying the inverse of the exponential map. Since the tangent spaces are linear spaces themselves, the refinement algorithm can be applied. Afterwards, using the exponential map, the refined data is dropped back down to the manifold, see [21, 70]. This method has been extended to Hermite subdivision schemes, too [63, 64].

Applying a geodesic averaging process instead of a linear one leads to convergence results for subdivision schemes on manifolds for all input data as shown in [32, 33].

Another well-studied and extrinsic method is the *projection analogue.* The main idea is to restrict the problem to surfaces which can be embedded in Euclidean spaces and use their linear structure [83, 37].

Our main focus, however, is on the so-called Riemannian analogue of a linear scheme which is introduced next.

Riemannian center of mass

We recall the extension of a linear subdivision scheme to manifold-valued data with the help of the Riemannian center of mass as shown in [40]. This generalisation of the concept of affine average is quite natural in the sense that we only replace the Euclidean distance by the Riemannian distance. The construction requires to introduce some notation. We denote the Riemannian inner product by $\langle \cdot, \cdot \rangle = |\cdot|^2$ on a Riemannian manifold M. The *Riemannian distance* $\operatorname{dist}(x, y)$ between two points $x, y \in M$ is given by

$$\operatorname{dist}(x, y) := \inf_{\gamma} \int_a^b |\dot{\gamma}(t)| \, dt,$$

where $\gamma : [a, b] \to M$ is a curve connecting points $\gamma(a) = x$ and $\gamma(b) = y$. Consider the weighted affine average

$$x^* = \sum_{j=0}^{n} \alpha_j x_j$$

of points $x_j \in \mathbb{R}^d$ w.r.t. weights $\alpha_j \in \mathbb{R}$, satisfying $\sum \alpha_j = 1$. It can be characterised as the unique minimum of the function

$$g_\alpha(x) = \sum_{j=0}^{n} \alpha_j \, |x - x_j|^2 \, .$$

We transfer this definition to Riemannian manifolds by replacing the Euclidean distance by the Riemannian distance. Let

$$f_\alpha(x) = \sum_{j=0}^{n} \alpha_j \operatorname{dist}(x, x_j)^2.$$

We call the minimiser of this function the *Riemannian center of mass* and denote it by

$$x^* = \operatorname{av}(\alpha, x).$$

Note that in general the Riemannian center of mass is only locally well defined. It will be one of the aims of this work to identify settings where the average is globally well defined. We extend the linear refinement rule (2.2) to manifold-valued data by defining

$$(Tx)_{2i} = \operatorname{av}(\alpha, x) \quad \text{and} \quad (Tx)_{2i+1} = \operatorname{av}(\beta, x). \tag{2.6}$$

Definition 2.2. We call T the *Riemannian analogue* of the linear subdivision scheme S.

2.2 Subdivision schemes on Cartan-Hadamard manifolds

A *Cartan-Hadamard manifold* is a complete, simply connected Riemannian manifold with nonpositive sectional curvature $K \leqslant 0$. In this section we prove a convergence result for Riemannian analogues of linear schemes on Cartan-Hadamard manifolds which can be applied to all input data. We study the Hölder continuity of the resulting limit curves and continue by showing how to use the universal covering of a manifold to drop the assumption of simple connectedness of the underlying manifold. We conclude by providing several examples.

2.2.1 The Riemannian center of mass on Cartan-Hadamard manifolds

We consider the question of the existence and uniqueness of the Riemannian analogue of a linear subdivision scheme on Cartan-Hadamard manifolds. Before, we recall some well-known facts about the concept of Jacobi fields since they are an important tool within in the proofs of this section.

Jacobi fields

We make use of the textbook of do Carmo [20, Section 5] as a guideline to introduce Jacobi fields here. For further details we refer to [20].

Let M be a Riemannian manifold and $\gamma : [0,1] \to M$ a geodesic. With Jacobi fields one can study the relation of geodesics being in a neighbourhood of γ. Therefore, we define a 1-parameter family of geodesics by

$$\begin{aligned} c : [0,1] \times (-\epsilon, \epsilon) &\to M \\ (u,s) &\mapsto c(u,s) \end{aligned}$$

with $c(u,0) = \gamma(u)$. Let $\dot{c}(u,s) := \frac{d}{ds}c(u,s)$. Then $J(u) = \dot{c}(u,0)$ is a *Jacobi field* along the geodesic γ. It is known that Jacobi fields are solutions of the linear 2nd order differential equation $\ddot{J} + R(\dot{\gamma}, J)\dot{\gamma} = 0$, with R denoting the Riemann curvature tensor. This differential equation is known as the *Jacobi equation.* For any given geodesic γ, there is a linear space of Jacobi vector fields (i.e., solutions of the Jacobi equation) whose dimension is twice the dimension of the manifold M. In particular, the behaviour of Jacobi fields is guided by the curvature of M.

Existence and Uniqueness

Cartan-Hadamard manifolds, and more generally manifolds with nonpositive sectional curvature, are a class of geometries where the Riemannian average can be made globally well defined. Let M be a Cartan-Hadamard manifold, i.e., a simply connected, complete Riemannian manifold with sectional curvature $K \leqslant 0$. To show well-definedness of geodesic averages we have to clarify the global existence and uniqueness of a minimiser of the function

$$f_\alpha(x) = \sum_{j=-m}^{m+1} \alpha_j \operatorname{dist}(x_j, x)^2, \quad \text{with} \quad \sum_j \alpha_j = 1 \tag{2.7}$$

and $x_j \in M$. A local answer to this question is not difficult, see for example [72]. The global well-definedness in case $\alpha_j \geqslant 0$ is shown in [53]. Hanne Hardering gave another proof of the global existence in [46]. We are mainly interested in the result she gave in Lemma 2.3. of [46] which we formulate as

Lemma 2.3 (H. Hardering, [46])**.** *The function f_α has at least one minimum. Moreover, there exists $r > 0$ (depending on the coefficients α_j and the distances of the points x_j from each other) such that all minima of f_α lie inside the compact ball $\overline{B_r(x_0)}$.*

To prove that the function f_α has a unique minimum we generalise a statement of H. Karcher [52]. It turns out that we can use arguments similar to his by splitting $\sum_{j=-m}^{m+1} \alpha_j \operatorname{dist}(x_j, x)^2$ into two sums depending on whether the corresponding coefficient is negative or not. Before we introduce the general notation used throughout the text, we illustrate the idea by means of Example 2.1.

Example 2.4. Consider the refinement rule defined by the coefficients α_j and β_j of Example 2.1. Define f_α according to (2.7) by

$$f_\alpha(x) = \sum_{j=-1}^{2} \alpha_j \operatorname{dist}(x_j, x)^2,$$

with $(\alpha_{-1}, \dots, \alpha_2) = (-\frac{1}{32}, \frac{21}{32}, \frac{13}{32}, -\frac{1}{32})$. We sort these coefficients in two groups depending on whether they are positive or not.

It is convenient to define $\alpha_+ = \frac{21}{32} + \frac{13}{32} = \frac{34}{32}$ and $\alpha_- = \left|-\frac{1}{32}\right| + \left|-\frac{1}{32}\right| = \frac{2}{32}$. We split the interval $[0, \alpha_+ + \alpha_-]$ in four subintervals whose length coincides with the values $|\alpha_j|$ (but in a different order). We define the function $\sigma : [0, \alpha_+ + \alpha_-] \to \{-1, 0, 1, 2\}$ by

$$\sigma(t) = \begin{cases} -1 & \text{for } t \in \left[0, \frac{1}{32}\right] \\ 2 & \text{for } t \in \left(\frac{1}{32}, \frac{2}{32}\right] \\ 0 & \text{for } t \in \left(\frac{2}{32}, \frac{23}{32}\right] \\ 1 & \text{for } t \in \left(\frac{23}{32}, \frac{36}{32}\right] \end{cases}$$

and see that

$$f_\alpha(x) = \sum_{j=-1}^{2} \alpha_j \operatorname{dist}(x_j, x)^2 = -\int_0^{\alpha_-} \operatorname{dist}(x_{\sigma(t)}, x)^2 dt + \int_{\alpha_-}^{\alpha_- + \alpha_+} \operatorname{dist}(x_{\sigma(t)}, x)^2 dt.$$

$\diamondsuit$

In the general case, we need the following notation to eventually rewrite the function in (2.7) as the sum of two integrals. We begin to sort our coefficients in two groups by defining two index sets

$$I_-^\alpha := \{j \mid \alpha_j < 0\}, \quad I_+^\alpha := \{j \mid \alpha_j \geqslant 0\}.$$

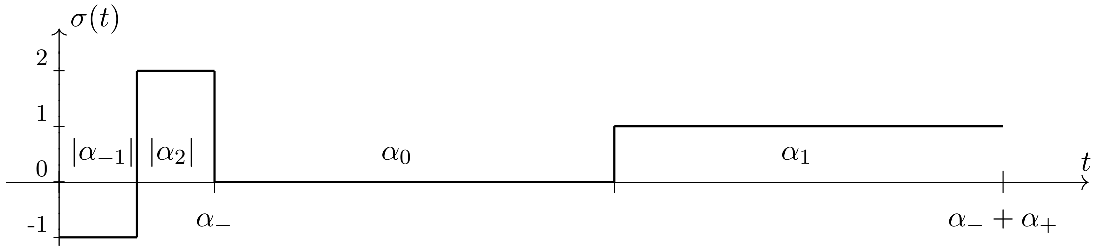

Figure 2.1: Construction of the index selection function σ on basis of the sequence $(\alpha_j)_{j=-1}^{2}$ with $\alpha_{-1}, \alpha_2 < 0$, $\alpha_0, \alpha_1 > 0$.

See Figure 2.1 for a schematically description of this procedure. We describe these sets as

$$I_-^\alpha = \{j_1, \ldots, j_n\}, \quad I_+^\alpha = \{j_{n+1}, \ldots, j_{2m+2}\},$$

with $j_1 < \ldots < j_n$ and $j_{n+1} < \ldots < j_{2m+2}$ for $n \in \{1, \ldots, 2m+2\}$ and $j_i \in \{-m, \ldots, m+1\}$. If $I_-^\alpha = \emptyset$, we set $n = 0$ and $I_+^\alpha = \{-m, \ldots, m+1\}$. Let

$$\alpha_+ = \sum_{j \in I_+^\alpha} \alpha_j, \quad \alpha_- = \sum_{j \in I_-^\alpha} |\alpha_j|, \quad \beta_+ = \sum_{j \in I_+^\beta} \beta_j, \quad \beta_- = \sum_{j \in I_-^\beta} |\beta_j|. \tag{2.8}$$

Assumption (2.3) implies that

$$\alpha_+ - \alpha_- = \beta_+ - \beta_- = 1. \tag{2.9}$$

We are now able to rewrite the function f_α in terms of two integrals

$$f_\alpha(x) = \sum_{j=-m}^{m+1} \alpha_j \ \operatorname{dist}(x_j, x)^2 = \Big(- \int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_- + \alpha_+} \Big) \operatorname{dist}\left(x_{\sigma(t)}, x\right)^2 dt \tag{2.10}$$

with the function $\sigma : [0, \alpha_+ + \alpha_-] \to \{-m, \ldots, m+1\}$ given as follows. It is constant in each of the successive intervals of length $|\alpha_{j_1}|, |\alpha_{j_2}|, \ldots, |\alpha_{j_{2m+2}}|$ which tile the interval $[0, \alpha_+ + \alpha_-]$. Its value in the k-th interval is given by the integer j_k. The values at subinterval boundaries are not relevant. For the sake of completeness we give the formal description of the function σ in the next

Remark 2.5. We define the functions

$$\begin{aligned} \sigma_1 : \{-m, \ldots, m+1\} &\to \mathbb{R} \\ t &\mapsto \sum_{i \leqslant j} |\alpha_i| \end{aligned}$$

and

$$\begin{aligned} \sigma_2 : [0, \alpha_+ + \alpha_-] &\to \{-m, \ldots, m+1\} \\ t &\mapsto \sup\{j | \sigma_j < t\} + 1. \end{aligned}$$

Moreover, let $\sigma_1(-m-1) := 0$. Then,

$$\sigma(t) := \begin{cases} \sigma_2\Big(t - 0 + \sum\limits_{\substack{j\in\{j_1,\dots,j_{2m+2}\},\\ j<j_1}} |\alpha_j|\Big) & \text{for } t \in [0, |\alpha_{j_1}|] \\ \sigma_2\Big(t - |\alpha_{j_1}| + \sum\limits_{\substack{j\in\{j_1,\dots,j_{2m+2}\},\\ j<j_2}} |\alpha_j|\Big) & \text{for } t \in \Big(|\alpha_{j_1}|, |\alpha_{j_1}| + |\alpha_{j_2}|\Big] \\ \qquad\vdots & \\ \sigma_2\Big(t - \sum\limits_{j\in I^{\alpha}_{-}\setminus\{j_n\}} |\alpha_j| + \sum\limits_{\substack{j\in\{j_1,\dots,j_{2m+2}\},\\ j<j_n}} |\alpha_j|\Big) & \text{for } t \in \Big(\sum\limits_{j\in I^{\alpha}_{-}\setminus\{j_n\}} |\alpha_j|, \alpha_-\Big] \\ & \\ \sigma_2\Big(t - \alpha_- + \sum\limits_{\substack{j\in\{j_1,\dots,j_{2m+2}\},\\ j<j_{n+1}}} |\alpha_j|\Big) & \text{for } t \in \big(\alpha_-, \alpha_- + \alpha_{j_{n+1}}\big] \\ \sigma_2\Big(t - \alpha_- - \alpha_{j_{n+1}} + \sum\limits_{\substack{j\in\{j_1,\dots,j_{2m+2}\},\\ j<j_{n+2}}} |\alpha_j|\Big) & \\ \qquad\qquad \text{for } t \in \big(\alpha_- + \alpha_{j_{n+1}}, \alpha_- + \alpha_{j_{n+1}} + \alpha_{j_{n+2}}\big] & \\ \qquad\vdots & \\ \sigma_2\Big(t - \alpha_- - \sum\limits_{j\in I^{\alpha}_{+}\setminus\{j_{2m+2}\}} \alpha_j + \sum\limits_{\substack{j\in\{j_1,\dots,j_{2m+2}\},\\ j<j_{2m+2}}} |\alpha_j|\Big) & \\ \qquad\qquad \text{for } t \in \Big(\alpha_- + \sum\limits_{j\in I^{\alpha}_{+}\setminus\{j_{2m+2}\}} \alpha_j \ , \ \alpha_- + \alpha_+\Big]. & \end{cases}$$

We note that the first part of the definition of σ represents the summands of (2.7) corresponding to coefficients of I^{α}_{-} whereas the second half represents the part corresponding to I^{α}_{+}.

Using the representation of the function f_α given in (2.10) we can state

Lemma 2.6. *On a Cartan-Hadamard manifold the gradient of the function f_α is given by the formula*

$$\frac{1}{2}\operatorname{grad} f_\alpha(x) = \int_0^{\alpha_-} \exp_x^{-1} x_{\sigma(t)} dt - \int_{\alpha_-}^{\alpha_-+\alpha_+} \exp_x^{-1} x_{\sigma(t)} dt,$$

where exp *denotes the Riemannian exponential map. Furthermore, we have*

$$\frac{d^2}{ds^2} f_\alpha(\gamma(s)) \geqslant 2\langle\dot\gamma(s), \dot\gamma(s)\rangle$$

for any geodesic $\gamma : [0,1] \to M$.

The proof of this lemma is mainly based on the proof of Theorem 1.2. in [52].

Proof. Recall the definition of f_α by (2.10). Let $\gamma : [0,1] \to M$ be a geodesic and

$$c_t(u,s) = \exp_{x_{\sigma(t)}}\big(u \cdot \exp^{-1}_{x_{\sigma(t)}} \gamma(s)\big).$$

For any s the geodesic $c_t(\cdot, s)$ connects $x_{\sigma(t)}$ with $\gamma(s)$. Those geodesics exist and are unique since M is Cartan-Hadamard. Additionally, let $c_t'(u,s) := \frac{d}{du} c_t(u,s)$ and $\dot c_t(u,s) := \frac{d}{ds} c_t(u,s)$. By construction, $\operatorname{dist}\big(x_{\sigma(t)}, \gamma(s)\big) = \|c_t'(u,s)\|$. For each t, s the vector field $J(u) = \dot c_t(u,s)$ along the geodesic $u \mapsto c_t(u,s)$ is a Jacobi field. Since

$$\begin{aligned} f_\alpha(\gamma(s)) &= \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big) \operatorname{dist}(x_{\sigma(t)}, \gamma(s))^2 dt \\ &= \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big) \langle c_t'(u,s), c_t'(u,s)\rangle dt \end{aligned}$$

we obtain

$$\frac{1}{2}\frac{d}{ds} f_\alpha(\gamma(s)) = \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\Big\langle \frac{\nabla}{\partial s} c_t'(u,s),\ c_t'(u,s)\Big\rangle\ dt.$$

Here $\frac{\nabla}{\partial s}$ denotes the covariant derivative along the curve $\gamma(s)$. In the following we use the facts that $\|c_t'(u,s)\|$ does not depend on s, $\frac{\nabla}{\partial s} c_t'(u,s) = \frac{\nabla}{\partial u} \dot c_t(u,s)$ and finally that $\frac{\nabla}{\partial u} c_t'(u,s) = 0$ since c is a geodesic. This leads to

$$\begin{aligned} &\Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\Big\langle \frac{\nabla}{\partial s} c_t'(u,s),\ c_t'(u,s)\Big\rangle\ dt \\ &\quad = \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big) \int_0^1 \Big\langle \frac{\nabla}{\partial s} c_t'(u,s),\ c_t'(u,s)\Big\rangle\ du\ dt \\ &\quad = \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big) \int_0^1 \Big\langle \frac{\nabla}{\partial u} \dot c_t(u,s),\ c_t'(u,s)\Big\rangle\ du\ dt \\ &\quad = \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big) \int_0^1 \frac{d}{du}\Big\langle \dot c_t(u,s).\ c_t'(u,s)\Big\rangle\ du\ dt. \end{aligned}$$

Since $\dot c_t(0,s) = 0$ we finally obtain

$$\frac{1}{2}\frac{d}{ds} f_\alpha(\gamma(s)) = \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\Big\langle \dot c_t(1,s),\ c_t'(1,s)\Big\rangle. \tag{2.11}$$

Observe that $c_t'(1,s) = -\exp^{-1}_{\gamma(s)} x_{\sigma(t)}$ (by definition of the exponential map) and $\dot c_t(1,s) = \dot\gamma(s)$ (by construction) are independent of t. Therefore,

$$\frac{1}{2}\frac{d}{ds} f_\alpha(\gamma(s)) = \Big\langle \dot\gamma(s),\ \Big(\int_0^{\alpha_-} - \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big) \exp^{-1}_{\gamma(s)} x_{\sigma(t)} dt\Big\rangle.$$

By the definition of the gradient we conclude that

$$\frac{1}{2}\operatorname{grad} f_\alpha(x) = \Big(\int_0^{\alpha_-} - \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big) \exp_x^{-1} x_{\sigma(t)} dt.$$

Using (2.9) we see that

$$\begin{aligned}\frac{1}{2}\frac{d^2}{ds^2} f_\alpha(\gamma(s)) &= \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\Big\langle \dot{c}_t(1,s),\ \frac{\nabla}{\partial u}\dot{c}_t(1,s)\Big\rangle\ dt \\ &= \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\Big\langle J(1), J'(1)\Big\rangle\ dt \\ &= \langle J(1), J'(1)\rangle \geqslant \langle \dot\gamma(s), \dot\gamma(s)\rangle.\end{aligned}$$

To obtain the inequality above we used the following relations between the Jacobi field and its derivative

$$J'(1)^{\tan} = J(1)^{\tan} \quad \text{and} \quad \langle J'(1)^{\mathrm{norm}}, J(1)\rangle \geqslant \langle J(1)^{\mathrm{norm}}, J(1)\rangle, \tag{2.12}$$

where $J^{\tan}$ (resp. J^{norm}) denotes the tangential (resp. normal) part of the Jacobi field; see Appendix A in [52] for more details. Here we used the fact that the sectional curvature of M is bounded form above by zero. □

Remark 2.7. We note that a direct further differentiation of (2.11) yields

$$\frac{d}{ds}\Big(\frac{1}{2}\frac{d}{ds} f_\alpha(\gamma(s))\Big) = \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\Big\langle \dot{c}_t(1,s),\ \frac{\nabla}{\partial u}\frac{d}{ds}c_t(1,s)\Big\rangle.$$

Thus,

$$\frac{\nabla}{\partial s}\frac{1}{2}\operatorname{grad} f_\alpha(\gamma(s)) = \Big(\int_0^{\alpha_-} - \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\frac{\nabla}{\partial u}\frac{d}{ds}c_t(1,s)dt.$$

This equality is used in the next section.

We sum up the results of the two lemmas above to state the main result of this section.

Theorem 2.8. *On a Cartan-Hadamard manifold M, the function*

$$f_\alpha(x) = \sum_{j=-m}^{m+1} \alpha_j \operatorname{dist}(x_j, x)^2$$

($\sum \alpha_j = 1$) with $x_j \in M$ has a unique minimum. This implies that the geodesic average is globally well defined on Cartan-Hadamard manifolds.

Proof. By Lemma 2.3 there exists a minimum of the function f_α and all its minima lie inside a compact ball. By the second part of Lemma 2.6 the function f_α is strictly convex, so the minimum is unique. □

2.2.2 Convergence result on Cartan-Hadamard manifolds

In this section, we prove that the Riemannian analogue of a linear subdivision scheme on a Cartan-Hadamard manifold converges for all input data, if the mask satisfies a contractivity condition with contractivity factor smaller than 1, see Theorems 2.11 and 2.15. The condition implying convergence involves derived schemes (and iterates of derived schemes) and is analogous to a well-known criterion which applies in the linear case. This kind of result was previously only known for schemes with nonnegative mask (see [79, Theorem 5]). It has already been conjectured in [40].

Contractivity condition

We begin by adapting Lemma 3 of [79].

Lemma 2.9. *Consider points x_j, coefficients α_j, β_j, for $j = -m, \dots, m+1$, and their center of mass $x^* = \mathrm{av}(\alpha, x)$, $x^{**} = \mathrm{av}(\beta, x)$ on a Cartan-Hadamard manifold. Moreover, we assume that (2.3) holds. Then,*

$$\mathrm{dist}(x^*, x^{**}) \leqslant \Big(\sum_{j=-m}^{m+1} \Big| \sum_{i \leqslant j} \alpha_i - \beta_i \Big| \Big) \cdot \max_{\ell} \mathrm{dist}(x_\ell, x_{\ell+1}).$$

To prove the next result we make use of the representation of f_α (resp. f_β) as in (2.10) in terms of the function σ (resp. τ). Before we give the proof of Lemma 2.9 we illustrate the idea by means of our main example:

Example 2.10. From Example 2.4 we know that

$$f_\alpha(x) = -\int_0^{\alpha_-} \mathrm{dist}(x_{\sigma(t)}, x)^2 dt + \int_{\alpha_-}^{\alpha_- + \alpha_+} \mathrm{dist}(x_{\sigma(t)}, x)^2 dt.$$

Similarly we obtain

$$f_\beta(x) = -\int_0^{\beta_-} \mathrm{dist}(x_{\tau(t)}, x)^2 dt + \int_{\beta_-}^{\beta_- + \beta_+} \mathrm{dist}(x_{\tau(t)}, x)^2 dt,$$

with $\beta_- = \frac{2}{32}$, $\beta_+ = \frac{34}{32}$ and

$$\tau(t) = \begin{cases} -1 & \text{for } t \in [0, \frac{1}{32}] \\ 2 & \text{for } t \in \left(\frac{1}{32}, \frac{2}{32}\right] \\ 0 & \text{for } t \in \left(\frac{2}{32}, \frac{15}{32}\right] \\ 1 & \text{for } t \in \left(\frac{15}{32}, \frac{36}{32}\right] . \end{cases}$$

In order to get the desired result in Lemma 2.9 we estimate the distance between the gradients of the functions f_α and f_β at the point $x^* = \mathrm{av}(\alpha, x)$ (as explained in more detail in the proof of the Lemma 2.9). To be able to do so, we make use of Lemma 2.6 and convert the resulting four integrals into two. In this case, we get

$$\Big\|\frac{1}{2}\operatorname{grad} f_\beta(x^*) - \frac{1}{2}\operatorname{grad} f_\alpha(x^*)\Big\| = \Big\|\sum_{j=-1}^{2}(\alpha_j-\beta_j)\exp_{x*}^{-1}x_j\Big\|$$
$$= \Big\| - \int_0^{\frac{8}{32}} \exp_{x*}^{-1} x_{\nu(t+\frac{8}{32})}dt + \int_0^{\frac{8}{32}} \exp_{x*}^{-1} x_{\nu(t)}dt\Big\|,$$

with

$$\nu(t) = \begin{cases} 0 & \text{for } t\in[0,\frac{8}{32}] \\ 1 & \text{for } t \in \left(\frac{8}{32},\frac{16}{32}\right]. \end{cases}$$

Note that the construction of the function ν is similar to the one of σ in (2.10). ♢

We are now ready to give the proof of Lemma 2.9 which follows the structure in [79] and the ideas of [52].

Proof of Lemma 2.9. To obtain a lower bound for the absolute value of the gradient of $\frac{1}{2}f_\alpha(x)$ we make use of Theorem 1.5. in [52] . Let γ be the geodesic starting from x^* and ending in x and let $c_t(u,s) = \exp_{x_{\sigma(t)}}\big(u\cdot \exp_{x_{\sigma(t)}}^{-1}\gamma(s)\big)$ be the family of geodesics from $x_{\sigma(t)}$ to $\gamma(s)$. We apply the Cauchy-Schwarz inequality and the fact that $\operatorname{grad} f_\alpha(x^*) = 0$ by definition of x^* to obtain

$$\Big\|\frac{1}{2}\operatorname{grad} f_\alpha(\gamma(1))\Big\|\cdot\Big\|\dot\gamma(1)\Big\| \geqslant \int_0^1 \frac{d}{ds}\Big\langle\frac{1}{2}\operatorname{grad} f_\alpha(\gamma(s)),\ \dot\gamma(s)\Big\rangle\, ds.$$

By Remark 2.7 we conclude

$$\Big\|\frac{1}{2}\operatorname{grad} f_\alpha(\gamma(1))\Big\|\cdot\Big\|\dot\gamma(1)\Big\| \geqslant \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\int_0^1 \Big\langle \frac{\nabla}{\partial u}\frac{d}{ds}c_t(1,s),\ \dot c_t(1,s)\Big\rangle\, ds\ dt,$$

with $\dot c_t(u,s) = \frac{d}{ds}c_t(u,s)$. As in the proof of Lemma 2.6 let $J(u) = \dot c_t(u,s)$ denote the Jacobi field along the curve $u\mapsto c_t(u,s)$. The dependence on s and t is not indicated in the notation. We have $J(1) = \dot\gamma(s)$ and $J'(1) = \frac{\nabla}{\partial u}\dot c_t(1,s)$. Using (2.9) we obtain

$$\Big\|\frac{1}{2}\operatorname{grad} f_\alpha(\gamma(1))\Big\|\cdot\Big\|\dot\gamma(1)\Big\| \geqslant \Big(-\int_0^{\alpha_-} + \int_{\alpha_-}^{\alpha_-+\alpha_+}\Big)\int_0^1 \langle J'(1), J(1)\rangle\ ds\ dt$$
$$= \langle J'(1), J(1)\rangle \geqslant \langle\dot\gamma(s),\dot\gamma(s)\rangle.$$

The last inequality follows in the same way as in the proof of Lemma 2.6. By the definition of the geodesic γ we have $\|\dot\gamma(s)\| = \operatorname{dist}(x,x^*)$ and conclude that

$$\Big\|\frac{1}{2}\operatorname{grad} f_\alpha(x)\Big\| \geqslant \operatorname{dist}(x,x^*). \tag{2.13}$$

By definition of x^* we have $\operatorname{grad} f_\alpha(x^*) = 0$. Together with Lemma 2.6 we obtain

$$\Big\|\frac{1}{2}\operatorname{grad} f_\beta(x^*)\Big\| = \Big\|\frac{1}{2}\operatorname{grad} f_\beta(x^*) - \frac{1}{2}\operatorname{grad} f_\alpha(x^*)\Big\| = \Big\|\sum_{j=-m}^{m+1}(\alpha_j - \beta_j)\ \exp_{x*}^{-1} x_j\Big\|.$$

We define the sequence $\delta = (\delta_j)_{j=-m,\dots,m+1}$ by $\delta_j = \alpha_j - \beta_j$. Let ν be the function constructed as σ in (2.10) with respect to the coefficients δ, i.e., the value of ν is constant in intervals of length $|\delta_j|$ and given by the corresponding index. Denote by δ_- (resp. δ_+) the sum of the absolute values of the negative (resp. nonnegative) coefficients of δ. Equation (2.3) implies that $\delta_- = \delta_+$. As in (2.10) we rewrite the sum above as an integral

$$\begin{aligned}\Big\|\sum_{j=-m}^{m+1}(\alpha_j - \beta_j)\ \exp_{x*}^{-1} x_j\Big\| &= \Big\| - \int_0^{\delta_-} \exp_{x*}^{-1} x_{\nu(t)} dt + \int_{\delta_-}^{\delta_-+\delta_+} \exp_{x*}^{-1} x_{\nu(t)} dt\Big\| \\ = \Big\|\int_0^{\delta_-}\big(-\exp_{x*}^{-1} x_{\nu(t)} + \exp_{x*}^{-1} x_{\nu(t+\delta_-)}\big)\ dt\Big\| &\leqslant \int_0^{\delta_-}\Big\|\exp_{x*}^{-1} x_{\nu(t+\delta_-)} - \exp_{x*}^{-1} x_{\nu(t)}\Big\|\ dt.\end{aligned}$$

With the help of (2.13) we conclude that

$$\begin{aligned}\operatorname{dist}(x^*, x^{**}) \leqslant \Big\|\frac{1}{2}\operatorname{grad} f_\beta(x^*)\Big\| &\leqslant \int_0^{\delta_-}\Big\|\exp_{x*}^{-1} x_{\nu(t+\delta_-)} - \exp_{x*}^{-1} x_{\nu(t)}\Big\|\ dt \\ \leqslant \int_0^{\delta_-}\operatorname{dist}(x_{\nu(t+\delta_-)}, x_{\nu(t)}) dt &\leqslant \int_0^{\delta_-} |\nu(t+\delta_-) - \nu(t)|\ dt \cdot \max_\ell \operatorname{dist}(x_\ell, x_{\ell+1}).\end{aligned}$$

To obtain the third inequality above we used the fact that on Cartan-Hadamard manifolds, the exponential map does not decrease distances, see for example [53].

It remains to show that

$$\int_0^{\delta_-} |\nu(t+\delta_-) - \nu(t)|\ dt = \sum_{j=-m}^{m+1}\Big|\sum_{i\leqslant j}\alpha_i - \beta_i\Big|.$$

To do that, we split the sequence of coefficients δ in two sequences η^1, η^2 defined by

$$\eta_j^1 := \begin{cases}\delta_j & \text{if } \delta_j \geqslant 0\\ 0 & \text{else}\end{cases} \qquad \text{and} \qquad \eta_j^2 := \begin{cases}|\delta_j| & \text{if } \delta_j < 0\\ 0 & \text{else.}\end{cases}$$

Similarly to the construction in the proof of Lemma 3 of [79], we consider the function ϵ_1 given by

$$\epsilon_1 : [0, \delta_-] \to \{-m, \dots, m+1\}, \quad \epsilon_1(t) := \sup\Big\{j\ \Big|\ \sum_{i\leqslant j}\eta_i^1 < t\Big\} + 1.$$

Analogously, we define ϵ_2 for the sequence η^2. We finally obtain

$$\begin{aligned}\int_0^{\delta_-} |\nu(t+\delta_-) - \nu(t)| \, dt &= \int_0^{\delta_-} |\epsilon_1(t) - \epsilon_2(t)| \, dt \\ &= \sum_{j=-m}^{m+1} \Big| \sum_{i \leqslant j} \eta_i^1 - \sum_{i \leqslant j} \eta_i^2 \Big| = \sum_{j=-m}^{m+1} \Big| \sum_{i \leqslant j} \alpha_i - \beta_i \Big|.\end{aligned}$$

This concludes the proof of Lemma 2.9. □

Recall that a linear, binary subdivision scheme S is given by $Sx_i = \sum_{j\in\mathbb{Z}} a_{i-2j} x_j$ with $\sum_{j\in\mathbb{Z}} a_{i-2j} = 1$ for all i. In order to obtain a convergence result for the Riemannian analogue T of S we have to estimate the distance between two consecutive points in the sequence $S^k x$. Let $\mu_j^{(r)} = \sum\limits_{i\leqslant j} a_{r-2i}$ and

$$\mu = \max_{r\in\{1,2\}} \sum_{j=-m}^{m+1} \left| \mu_j^{(r+1)} - \mu_j^{(r)} \right| . \tag{2.14}$$

Then, Lemma 2.9 implies that the subdivision rule T obeys a so-called *contractivity condition*

$$\operatorname{dist}(T^k x_{i+1}, T^k x_i) \leqslant \mu^k \cdot \sup_\ell \operatorname{dist}(x_\ell, x_{\ell+1}). \tag{2.15}$$

The factor μ is called *contractivity factor*. In Subsection 2.2.2 we show that the value of the contractivity factor μ in (2.14) is closely related to the norm of the derived scheme.

We make use of the result H. Hardering gave in [46, Lemma 2.3] again. In particular, she shows that all solutions of the minimisation problem stated in (2.7) lie inside a compact ball around x_0. The radius of this ball only depends on the chosen weights and the distances of x_i, $i = -m, \dots, m$, from x_0. In our setting, this means that the points of the refined sequence are not *too* far from the initial points. To be more precise it follows that there exists a constant $C > 0$ such that

$$\operatorname{dist}(Tx_{2i}, x_i) \leqslant C \cdot \sup_\ell \operatorname{dist}(x_\ell, x_{\ell+1}), \quad i \in \mathbb{Z}. \tag{2.16}$$

Subdivision schemes satisfying inequality (2.16) have been called *displacement-safe* by [33]. Together with (2.15) we conclude that

$$\operatorname{dist}(T^{k+1} x_{2i}, T^k x_i) \leqslant C\mu^k \varrho \quad \text{with} \quad \varrho := \sup_\ell \operatorname{dist}(x_\ell, x_{\ell+1}). \tag{2.17}$$

In the linear case (see [25]) a contractivity factor smaller than 1 itself leads to a convergence result, but this condition is not sufficient in the nonlinear case. Here we additionally need the fact that our schemes are displacement-safe as shown in [33] for manifold-valued subdivision schemes based on an averaging process. For interpolatory subdivision schemes, however, a contractivity factor smaller than 1 entails convergence of the scheme since (2.16) is satisfied anyway, see [33, 76].

We now state our convergence result which generalises the result of [79].

Theorem 2.11. *Consider a linear, binary, affine invariant subdivision scheme S. Denote by T the Riemannian analogue of S on a Cartan-Hadamard manifold M. Let μ be the contractivity factor defined by (2.14). If $\mu < 1$, then T converges to a continuous limit $T^\infty x$ for all input data x.*

Proof. Let $J = [a, b]$ be an interval and denote by $C(J, M)$ the continuous functions from J to M. We use $c|J$ for the restriction of a map c to an interval J. Denote by $c_k : \mathbb{R} \to M$ the broken geodesic which is the union of geodesic segments $c_k\big|[\frac{i}{2^k}, \frac{i+1}{2^k}]$ which connect successive points $T^k x_i$ and $T^k x_{i+1}$. We show that $\big(c_k|J\big)_{k\geqslant 0}$ is a Cauchy sequence in $C(J, M)$ for any J. The metric on $C(J, M)$ is given by $\operatorname{dist}(g, h) := \max_{t\in J} \operatorname{dist}(g(t), h(t))$. We now proceed as in the proof of Proposition 4 of [79]. Since T satisfies (2.15) and is displacement-safe it follows from the definition of the geodesics that

$$\operatorname{dist}(c_m, c_{m+1}) \leqslant \varrho\mu^m + C\varrho\mu^m + \varrho\mu^{m+1}. \tag{2.18}$$

Therefore,

$$\operatorname{dist}(c_m, c_n) \leqslant \big(\varrho + C\varrho + \varrho\mu\big)\frac{\mu^m - \mu^n}{1-\mu}$$

for all $m \leqslant n$. Thus, $\big(c_k\big|J\big)_{k\geqslant 0}$ is a Cauchy sequence in $C(J, M)$ for any interval $J = [a, b]$. Completeness of the space $C(J, M)$ implies existence of the limit function $T^\infty x$. □

Example 2.12. We compute the contractivity factor of the subdivision scheme introduced in Example 2.1. Using our previous results we get

$$\mu = \max\Big\{\frac{28}{32}, \frac{8}{32}\Big\} = \frac{28}{32} < 1. \tag{2.19}$$

Thus, the Riemannian analogue of the linear scheme converges on Cartan-Hadamard manifolds for all input data. Figure 2.2 illustrates the action of this subdivision scheme in the hyperbolic plane. ♢

Remark 2.13. So far we considered subdivision schemes with dilation factor $N = 2$. We note here that one can extend the convergence result given in Theorem 2.11 to subdivision schemes with arbitrary dilation factor. We still extend a linear subdivision scheme S to its nonlinear counterpart T by using the Riemannian analogue introduced in Subsection 2.1.2. Analogous to the binary case, we say that T satisfies a contractivity condition with contractivity factor μ if

$$\operatorname{dist}\big(T^k x_{i+1}, T^k x_i\big) \leqslant \mu^k \cdot \sup_\ell \operatorname{dist}\big(x_\ell, x_{\ell+1}\big), \ \ i \in \mathbb{Z}.$$

Also we say that T is displacement-safe if there exists a constant $C > 0$ such that

$$\operatorname{dist}\big((Tx)_{Ni}, x_i\big) \leqslant C \cdot \sup_\ell \operatorname{dist}\big(x_\ell, x_{\ell+1}\big), \ \ i \in \mathbb{Z}.$$

The convergence result now reads as follows.

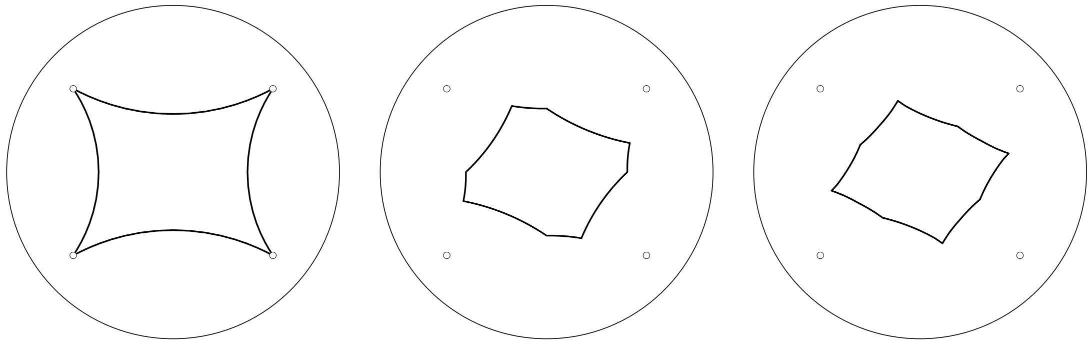

Figure 2.2: Subdivision algorithm of Example 2.1 with initial data $x_0 = (0.6, 0.5)$, $x_1 = (0.6, -0.5)$, $x_2 = (-0.6, -0.5)$ and $x_3 = (-0.6, 0.5)$ in the hyperbolic plane represented with the Poincaré disk model. From left to right: initial polygon, polygon after one refinement step, polygon after 4 refinement steps.

Theorem 2.14. *Let T be the Riemannian analogue of the linear subdivision rule S on a Cartan-Hadamard manifold M. Assume that (2.1) holds. Let $\mu_j^{(r)} = \sum_{i \leqslant j} a_{r-Ni}$ and*

$$\mu = \max_{r \in \{1,\dots,N\}} \sum_j \left| \mu_j^{(r+1)} - \mu_j^{(r)} \right| . \tag{2.20}$$

If $\mu < 1$, then T converges to a continuous limit $T^\infty x$ for all input data x.

The convergence proof in the case $N > 2$ is along the same lines as for $N = 2$.

Derived scheme

For every linear, affine invariant subdivision scheme S there exists the *derived scheme* S^* given by the rule $S^*\Delta = N\Delta S$ with $\Delta x_i = x_{i+1} - x_i$, see for example [40, Sec. 2.1]. In this section we show that the contractivity factor (2.20) is closely related to the norm

$$\|S^*\| := \max_{r \in \{1,\dots,N\}} \Big\{ \sum_j |a^*_{r-Nj}| \Big\}$$

of the derived scheme S^* with mask a^*. This result is not surprising since it holds in the linear case as well as for nonlinear subdivision schemes with nonnegative mask [79].

Theorem 2.15. *Let S be a linear, affine invariant subdivision rule with dilation factor N. Denote its derived scheme by S^*. If there exists an integer $m \geqslant 1$ such that $\frac{1}{N^m}\|S^{m*}\| < 1$, then the Riemannian analogue of S^m on a Cartan-Hadamard manifold converges for all input data.*

We can reuse the proof of Theorem 5 of [79] to show Theorem 2.15. We repeat it here for the reader's convenience.

Proof. Let $a^* = (a^*_j)_{j\in\mathbb{Z}}$ denote the mask of the derived scheme S^*. We consider the special input data $y = (y_j)_{j\in\mathbb{Z}}$ given by

$$y_j = \begin{cases} -1 & \text{if } j \leqslant 0 \\ 0 & \text{else.} \end{cases}$$

We obtain

$$\begin{aligned} \frac{1}{N}a^*_l &= \frac{1}{N}\sum\nolimits_k a^*_{l-Nk}(y_{k+1}-y_k) = \frac{1}{N}S^*(y_{l+1}-y_l) = \frac{1}{N}S^*\Delta y_l \\ &= \Delta S y_l = S y_{l+1} - S y_l = \sum\nolimits_{k\leqslant 0} a_{l-Nk} - a_{l+1-Nk}, \quad \text{and} \\ \frac{1}{N}a^*_{r-Nj} &= \sum\nolimits_{k\leqslant 0} a_{r-N(j+k)} - a_{r+1-N(j+k)} = \sum\nolimits_{i\leqslant j} a_{r-Ni} - a_{r+1-Ni}. \end{aligned}$$

By (2.20) we get

$$\sup_r \sum_j |\mu^{(r)}_j - \mu^{(r+1)}_j| = \frac{1}{N}\sup_r \sum_j |a^*_{r-Nj}| = \frac{1}{N}\|S^*\|.$$

Since the dilation factor of S^m is N^m, Theorem 2.14 gives the desired result. □

We have just seen that the contractivity factor (2.20) of the Riemannian analogue of a linear subdivision scheme S is given by

$$\mu = \frac{1}{N}\|S^*\|.$$

So in order to obtain a convergence result, it suffices to check if the norm of the derived scheme S^* is smaller than the dilation factor. Even if this is not the case we might get a convergence result by considering iterates of derived schemes S^{m*}, since the contractivity factor might decrease, see Subsection 2.2.5.

In [25] it is shown that if we ask for uniform convergence of a linear subdivision scheme S, the existence of an integer $m \geqslant 1$ such that $\frac{1}{N^m}\|S^{*m}\| < 1$ is equivalent to the convergence of the scheme. Thus, Theorem 2.15 states that if the linear subdivision scheme converges uniformly, so does a certain Riemannian analogue of this scheme on Cartan-Hadamard manifolds.

2.2.3 Hölder continuity

It has been shown in [76] that the limit function of an interpolatory subdivision scheme for manifold-valued data has Hölder continuity $-\frac{\log\mu}{\log 2}$. Here μ is a contractivity factor for the nonlinear analogue of the linear scheme. It depends only on the mask of the scheme. In [25] a similar inequality is proven for uniformly convergent subdivision schemes in linear spaces. We get the following related result.

Proposition 2.16. *Let T be the Riemannian analogue of a binary, affine invariant subdivision scheme S which has contractivity factor $\mu < 1$. Then, the limit curve $T^\infty x$ satisfies*

$$\operatorname{dist}\big(T^\infty x(t_1), T^\infty x(t_2)\big) \leqslant D|t_2 - t_1|^\iota,$$

with

$$D = 2 \cdot \Big(\frac{C\varrho + \varrho + \mu\varrho}{1-\mu} + \varrho\Big) \qquad \text{and} \qquad \iota = 1 - \frac{\log\|S^*\|}{\log 2},$$

for all $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < 1$ and all input data x, i.e., the limit curve is Hölder continuous with exponent ι.

Here the data-dependent constant ϱ is defined by the maximal distance of successive data points which contribute to the limit curve in the interval under consideration.

Proof. Assume that $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < 1$. Then, there exists an integer $k \in \mathbb{Z}$ such that $2^{-k-1} \leqslant |t_2 - t_1| \leqslant 2^{-k}$. As in the proof of Theorem 2.11 let c_k be the union of geodesic segments $c_k\big|[\frac{i}{2^k}, \frac{i+1}{2^k}]$ connecting the points $T^k x_i$ and $T^k x_{i+1}$. Together with (2.15) we obtain

$$\operatorname{dist}\big(c_{k+1}(t_1), c_{k+1}(t_2)\big) \leqslant 2 \sup_\ell \operatorname{dist}\big(T^{k+1}x_{\ell+1}, T^{k+1}x_\ell\big) \leqslant 2\mu^{k+1}\varrho.$$

Using (2.18) we have

$$\begin{aligned}\operatorname{dist}\big(T^\infty x(t), c_{k+1}(t)\big) &\leqslant \lim_{\ell\to\infty} \operatorname{dist}\big(c_\ell(t), c_{k+1}(t)\big)\\ &\leqslant \sum_{j=k+1}^{\infty} \operatorname{dist}\big(c_j(t), c_{j+1}(t)\big) = \frac{C\varrho + \varrho + \mu\varrho}{1-\mu}\mu^{k+1}\end{aligned}$$

for all $t \in \mathbb{R}$. Summarising the previous two observations leads to

$$\begin{aligned}&\operatorname{dist}\big(T^\infty x(t_1),\ T^\infty x(t_2)\big)\\ &\leqslant \operatorname{dist}\big(T^\infty x(t_1),\ c_{k+1}(t_1)\big) + \operatorname{dist}\big(c_{k+1}(t_1),\ c_{k+1}(t_2)\big) + \operatorname{dist}\big(c_{k+1}(t_2),\ T^\infty x(t_2)\big)\\ &\leqslant D\mu^{k+1}.\end{aligned}$$

Since $|t_2 - t_1| \leqslant 2^{-k}$, taking the logarithm shows that $\mu^{k+1} \leqslant \mu^{-\log_2(|t_2-t_1|)}$. We conclude that

$$\operatorname{dist}\big(T^\infty x(t_1),\ T^\infty x(t_2)\big) \leqslant D\Big(2^{\log_2(|t_2-t_1|)}\Big)^{-\log_2(\mu)} \leqslant D|t_2 - t_1|^\iota,$$

with $\iota = -\frac{\log\mu}{\log 2} = 1 - \frac{\log\|S^*\|}{\log 2}$. Here the last equality holds because $\mu = \frac{1}{2}\|S^*\|$. □

Example 2.17. For our main Example 2.1 we compute $\iota = -\log(\frac{28}{32})/\log 2 \approx 0.19$. This coincides with the result of the previous Proposition, since $\|S^*\| = \frac{7}{4}$ and thus, $\iota = 1 - \frac{\|S^*\|}{\log 2} \approx 0.19$. ♢

For subdivision schemes with arbitrary dilation factor we obtain

Proposition 2.18. *Let T be the Riemannian analogue of a linear subdivision scheme S on a Cartan-Hadamard manifold M satisfying (2.1). Moreover, we assume that T has contractivity factor $\mu < 1$. Then, the limit curve $T^\infty x$ satisfies*

$$\operatorname{dist}\big(T^\infty x(t_1), T^\infty x(t_2)\big) \leqslant D|t_2 - t_1|^\iota,$$

with

$$D = 2 \cdot \frac{C\varrho + \varrho + (N-1)\mu\varrho}{1-\mu} + N\varrho \qquad \text{and} \qquad \iota = 1 - \frac{\log \|S^*\|}{\log N}$$

for all $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < 1$ and all input data x. Here N is the dilation factor and the data-dependent constant ϱ is defined by the maximal distance of successive data points which contribute to the limit curve in the interval under consideration.

2.2.4 The case of manifolds which are not simply connected

We explain how to extend our previous results to a complete Riemannian manifold M with sectional curvature $K \leqslant 0$, i.e., we drop the assumption of simple connectedness. We use the fact that M has a so-called simply connected covering (*universal covering*) $\tilde{M}$. This is a simply connected manifold which projects onto M in a locally diffeomorphic way. The Riemannian metric on M is transported to $\tilde{M}$ by declaring the projection $\pi : \tilde{M} \to M$ a local isometry. An example is shown by Figure 2.3, where a strip of infinite length and width 1 wraps around the cylinder of height 1 infinitely many times. For the general theory of coverings, see for example [48]. Each data point x_j in M has a potentially large number of preimages $\pi^{-1}(x_j)$.

Re-definition of the Riemannian analogue of a linear scheme

So far our initial data always consisted of a sequence of points in M. Now we additionally choose a path $c(t)$ which connects the data points x_j in the correct order: we have $c(t_j) = x_j$ for suitable parameter values $\ldots < t_j < t_{j+1} < \ldots$. Such a path is not unique, see Figure 2.3. By well-known properties of the simply connected covering, this path can be uniquely lifted to a path $\tilde{c}(t)$ in $\tilde{M}$ which projects onto the original path $c(t)$, once a preimage $\tilde{x}_0$ with $\pi(\tilde{x}_0) = x_0$ has been chosen. This means that for all indices j we have

$$\tilde{c}(t_j) = \tilde{x}_j, \quad \text{with} \quad \pi(\tilde{x}_j) = x_j.$$

We can now simply apply the Riemannian analogue $\tilde{T}$ of the linear scheme S which operates on data from $\tilde{M}$, because $\tilde{M}$ is Cartan-Hadamard by construction. Note that there is no Riemannian analogue of S in M, since M is not simply connected and geodesic averages are not well defined in general. However, if our input data is a sequence x_j together with a connecting path as described above, we may let

$$Tx = \pi(\tilde{T}\tilde{x}) \quad \text{where } \tilde{x} \text{ arises from } x \text{ by lifting.}$$

We can still call T a natural Riemannian analogue of the linear subdivision scheme S.

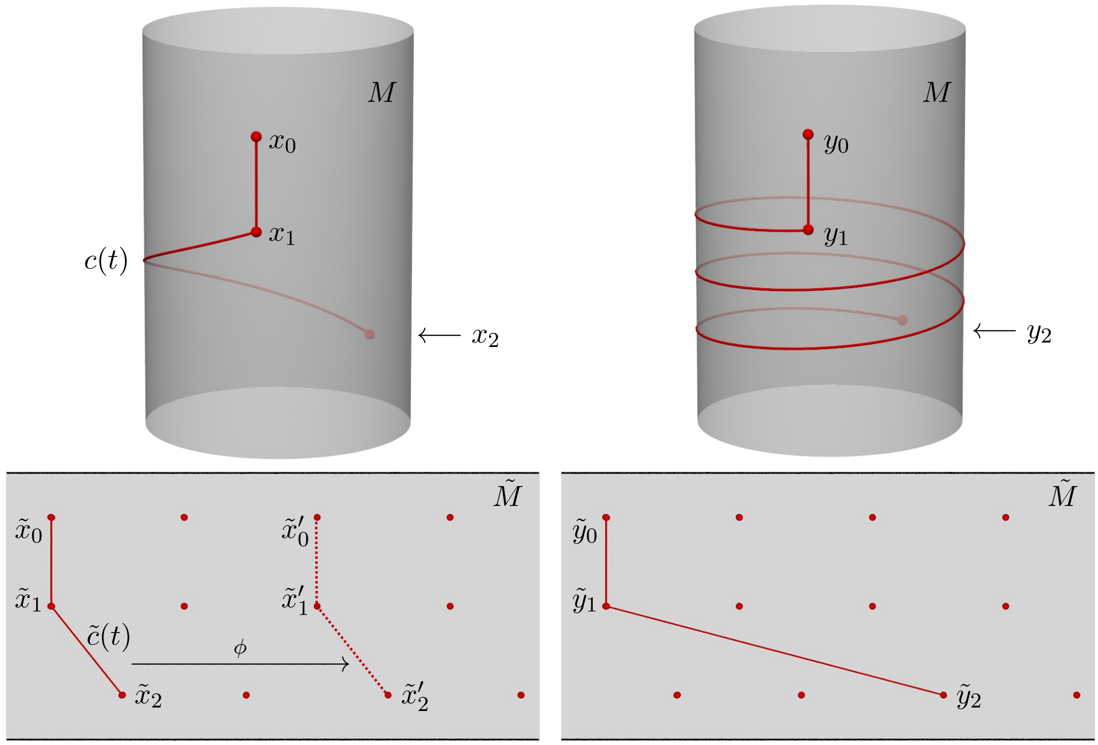

Figure 2.3: Top: initial data on a cylinder $M = S^1 \times [0,1]$ together with connecting paths. Bottom: their lift to the universal covering $\tilde{M}$, which is the strip $(-\infty, \infty) \times [0,1]$. The various possible liftings are mapped onto each other by a deck transformation ϕ.

Lemma 2.19. *For any given input data (x_j), the refined data $(Tx)_j$ computed by the Riemannian analogue T of a linear subdivision scheme S depends only on the homotopy class of the path $c(t)$ which is used to connect the data points.*

Proof. First, we show that Tx does not depend on the choice of the preimage $\tilde{x}_0$ in the covering space $\tilde{M}$: if another preimage $\tilde{x}'_0$ is chosen, there is an isometric *deck transformation* $\phi : \tilde{M} \to \tilde{M}$ which maps the original lifting to the new one and which commutes with the covering projection π. The action of $\tilde{T}$ is invariant under isometries, so $\pi(\tilde{T}\tilde{x}') = \pi(\tilde{T}\phi(\tilde{x})) = \pi(\phi(\tilde{T}\tilde{x})) = \pi(\tilde{T}\tilde{x})$. Further, it is well known that the lifted location $\tilde{x}_j$ of any individual data point x_j depends only on the homotopy class of the path c, cf. [48]. $\square$

With this modification of the notion of input data, our main result Theorem 2.15 now reads as follows.

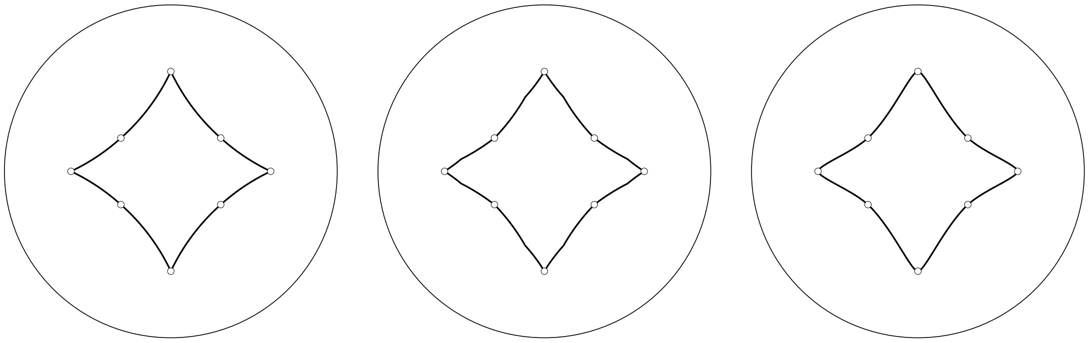

Figure 2.4: The 4-point scheme with $\omega = \frac{1}{16}$ in the hyperbolic plane represented with the Poincaré disk model. Left to right: initial polygon, polygon after one refinement step, polygon after 4 refinement steps.

Theorem 2.20. *Let M be a complete manifold with $K \leqslant 0$, and let S be a linear, affine invariant subdivision rule with dilation factor N. Denote by S^* its derived scheme. If there exists an integer $m \geqslant 1$ such that $\frac{1}{N^m}\|S^{m*}\| < 1$, then the Riemannian analogue of S^m on M produces continuous limits for all input data.*

2.2.5 Examples

We conclude this section with further examples.

4-point scheme

Consider the general 4-point scheme S introduced in (2.5). We would like to know for which values of $\omega \in (0, \infty)$ the Riemannian analogue T of S converges. The mask of the derived scheme is given by $a^*_{-2} = a^*_3 = -2\omega$, $a^*_{-1} = a^*_2 = 2\omega$ and $a^*_0 = a^*_1 = 1$. Thus, by Theorem 2.11, the contractivity factor is $\mu = 2|\omega| + \frac{1}{2}$ and T converges for arbitrary input data if $-\frac{1}{4} < \omega < \frac{1}{4}$. For $-\frac{1}{2} < \omega \leqslant 0$ this has already been known [34, 35]. In this case, the mask is nonnegative.

In particular, we obtain a contractivity factor of $\mu = \frac{5}{8}$ for the well-studied case of the 4-point scheme with $\omega = \frac{1}{16}$. By Proposition 2.16 we obtain a Hölder exponent of $\iota \approx 0.6781$. Figures 2.4 and 2.5 show an example of the 4-point scheme in the hyperbolic plane for $\omega = \frac{1}{16}$ resp. $\omega = 0.23$.

Now we consider two rounds of the 4-point scheme as one round of a subdivision scheme with dilation factor $N = 4$ which for simplicity is again called S. If $\omega = \frac{1}{16}$, our refinement rule is then given by

$$\begin{aligned}
(Sx)_{4i} &= x_i, \\
(Sx)_{4i+1} &= \tfrac{1}{16^2}\big(x_{i-2} - 18\,x_{i-1} + 216\,x_i + 66\,x_{i+1} - 9\,x_{i+2}\big), \\
(Sx)_{4i+2} &= \tfrac{1}{16^2}\big(-16\,x_{i-1} + 144\,x_i + 144\,x_{i+1} - 16\,x_{i+2}\big), \\
(Sx)_{4i+3} &= \tfrac{1}{16^2}\big(-9\,x_{i-1} + 66\,x_i + 216\,x_{i+1} - 18\,x_{i+2} + x_{i+3}\big).
\end{aligned}$$

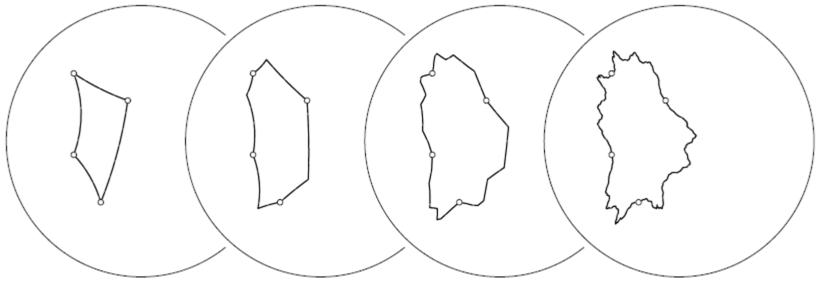

Figure 2.5: The 4-point scheme with $\omega = 0.23$ in the hyperbolic plane. Left to right: initial polygon, polygon after one refinement step, polygon after three refinement steps and limit curve. The limit curve is Hölder continuous with exponent 0.06.

The contractivity factor is

$$\mu = \max\left\{\frac{84}{16^2}, \frac{80}{16^2}\right\} = \frac{84}{16^2} \approx 0.3281.$$

Theorem 2.14 again confirms that a Riemannian analogue converges to a continuous limit function for all input data. Proposition 2.18 yields a Hölder exponent of $\iota \approx 0.80$.

Cubic Lane-Riesenfeld algorithm and the 4-point scheme

We consider a non-interpolatory subdivision scheme whose mask contains negative coefficients by combining the cubic Lane-Riesenfeld algorithm given by the mask

$$(Sx)_{2i} = \frac{1}{8}x_{i-1} + \frac{6}{8}x_i + \frac{1}{8}x_{i+1} \quad \text{and} \quad (Sx)_{2i+1} = \frac{1}{2}x_i + \frac{1}{2}x_{i+1}$$

with the 4-point scheme (2.5) with $\omega = \frac{1}{16}$. Taking averages of the cubic Lane-Riesenfeld algorithm and the 4-point scheme yields to the linear scheme

$$\begin{aligned}(Sx)_{2i} &= \frac{1}{16}x_{i-1} + \frac{14}{16}x_i + \frac{1}{16}x_{i+1},\\ (Sx)_{2i+1} &= -\frac{1}{32}x_{i-1} + \frac{17}{32}x_i + \frac{17}{32}x_{i+1} - \frac{1}{32}x_{i+2}\end{aligned}$$

which for simplicity is again called S. We compute a contractivity factor of $\mu = \frac{18}{32}$ for the Riemannian analogue of S which implies that the nonlinear analogue converges on complete Riemannian manifolds with nonpositive sectional curvature. We get a Hölder exponent of $\iota \approx 0.83$.

2.3 Subdivision on manifolds with positive sectional curvature

In the previous section, we analysed the Riemannian analogue of a linear subdivision scheme on Cartan-Hadamard manifolds. A key point in our studies was the assumption that the considered manifolds have nonpositive sectional curvature. Naturally, the next question is: What can we say about the convergence of the Riemannian analogue of a linear subdivision rule on positively-curved manifolds? Earlier works show that already the well-definedness of the Riemannian center of mass has to be studied carefully, especially if we do not restrict the mask to be nonnegative [52, 24]. It turns out that it is advantageous to first consider the well-known unit sphere as an example.

To prove the convergence of Riemannian analogues of linear schemes on the unit sphere we proceed as follows:

i) In Section 2.3.1, we start with an analysis of the Riemannian center of mass on positively-curved manifolds. Afterwards, we restrict ourselves to the unit sphere and provide a setting in which the Riemannian analogue of a linear subdivision scheme is well defined.

ii) In Section 2.3.2, we introduce a strategy to prove convergence results for the Riemannian analogue of a linear scheme on the unit sphere. The main idea is to estimate the length of a curve γ which joins a so-called reference point of a scheme with a point of its refined data. It requires technical details involving a second order Taylor approximation and estimates for the gradient and the Hessian of squared distance function on the unit sphere to give an upper bound on the length of γ. Throughout this part the cubic Lane-Riesenfeld algorithm serves as a main example to illustrate our results.

iii) In Section 2.3.3, we apply our strategy to show that the Riemannian analogues of some well-known linear subdivision schemes converge on the unit sphere.

2.3.1 Riemannian center of mass on manifolds with positive sectional curvature

Before we restrict ourselves to the unit sphere, we discuss the difficulties that arise by studying the Riemannian analogue of a linear subdivision scheme on positively-curved manifolds. Let M be a complete, simply connected Riemannian manifold with inner product $\langle\cdot,\cdot\rangle$ and sectional curvature $K > 0$. Denote by $B_r(x) = \{y \in M |\, \mathrm{dist}\,(x, y) < r\}$ the geodesic ball of radius $r > 0$ around $x \in M$ where *dist* again denotes the Riemannian distance.

Problem setting

To study the convergence of a Riemannian analogue T of a linear scheme S as given in Definition 2.2 we have to deal with the question if the function

$$f_\alpha(x) = \sum_{j=-m}^{m+1} \alpha_j \,\mathrm{dist}\,(x_j, x)^2\,, \quad \text{with} \quad \sum_j \alpha_j = 1 \tag{2.21}$$

admits a unique minimiser. Here $x_j \in M$ are fixed points on the manifold and α_j are real coefficients. Later, the points x_j correspond to the input data of a subdivision scheme while the coefficients α_j belong to its mask.

In contrast to Cartan-Hadamard manifolds, we cannot hope for global existence and uniqueness of the Riemannian center of mass. To see this, consider the north pole x_N resp. south pole x_S of the sphere and ask for their geodesic midpoint. Clearly, each point on the equator is a suitable choice and thus, a minimiser of $f(x) = \frac{1}{2}\operatorname{dist}(x_N, x)^2 + \frac{1}{2}\operatorname{dist}(x_S, x)^2$. One can show that locally there always exists a unique minimiser while globally there can be infinitely many.

A substantial number of contributions deals with the question of the effect of the sectional curvature, the distances between the points x_j and the choice of the coefficients on the existence of a unique minimiser, see for example [52, 24]. In [24] the authors provide explicit bounds on the input data (depending on the curvature and the chosen coefficients) to ensure the existence and uniqueness of a minimiser of (2.21) on manifolds with positive sectional curvature. In our setting, Corollary 9 of [24] reads as follows.

Lemma 2.21 (Dyer et al., [24])**.** *Let $x_j \in B_r(x)$, $j = -m, \dots, m+1$, for some $x \in M$ and $r > 0$. Then, the function f_α has a unique minimiser in $B_{r^*}(x)$, if*

i) $r < r^* < \min\{\frac{\iota_M}{2}, \frac{\pi}{4\sqrt{K}}\}$, *with ι_M denoting the injectivity radius of M,*

ii) $r^* > (1 + 2\alpha_-)r$,

iii) $r^* < \frac{\pi}{4\sqrt{K}}(1 + (1 + \frac{\pi}{2})\alpha_-)^{-1}$.

Here $\alpha_- := \sum_{\alpha_j < 0} |\alpha_j|$ *denotes the sum of the absolute values of the negative coefficients.*

Besides the fact that this existence and uniqueness result is a local answer compared to the one on Cartan-Hadamard manifolds, there is another crucial difference. Namely, the radius r^* (Lemma 2.21, i)) of the ball in which the unique minimiser lies is larger than the radius r of the ball containing the input data x_j.

So, locally Lemma 2.21 provides a setting in which the Riemannian analogue T of a linear subdivision scheme S is well defined. But the fact that the radius of the ball containing the refined data increases, leads to the question of how to control the distance between points in the sequence $(T^k x_i)_{i\in\mathbb{Z}}$, $k \geqslant 1$. As seen before, a convergence result for nonlinear subdivision schemes depends on the capability to control the distances of points in the sequence $(T^k x_i)_{i\in\mathbb{Z}}$ from each other as well as their distance to the input data. This can be seen e.g. in (2.15) and (2.17).

The distance estimate of refined data on Cartan-Hadamard manifolds is based on the fact that the exponential map does not decrease distances (see the proof of Lemma 2.9). This, however, is in general not true for positively-curved manifolds. We summarise the observations from above.

On manifolds with positive sectional curvature

i) we cannot hope for a convergence result which is valid for all input data.

ii) we obtain a local setting in which the Riemannian analogue of a linear subdivision scheme is well defined, see [24].

iii) we have to find a strategy to estimate distances between consecutive points of the refined data as well as their distance to the input data.

The Riemannian analogue of a linear subdivision scheme on the unit sphere

From now on, we restrict ourselves to the *unit sphere*, i.e., $M = S^n = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$ for $n \geqslant 2$. In particular, we have $K = 1$, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. We provide a setting on the unit sphere in which we can define the Riemannian analogue of a linear subdivision scheme. Therefore, we choose $x_j \in S^n$, $j = -m, \ldots, m+1$, such that $x_j \in B_r(x)$ for some $r > 0$ and $x \in S^n$. Since the sectional curvature $K = 1$ on the unit sphere and the injectivity radius is $\frac{\pi}{2}$, according to Lemma 2.21 the function f_α has a unique minimiser in $B_{r^*}(x)$, if

$$r^* > (1 + 2\alpha_-)\, r \geqslant r, \tag{2.22}$$

$$r^* < \frac{\pi}{4} \left(1 + \left(1 + \frac{\pi}{2}\right) \alpha_-\right)^{-1}. \tag{2.23}$$

In the special case of a scheme with only nonnegative coefficients, i.e., $\alpha_- = 0$, these conditions reduce to: If $r < \frac{\pi}{4}$, then there exists a radius r^* with $r < r^* < \frac{\pi}{4}$ such that the function f_α has a unique minimiser inside $B_{r^*}(x)$. If, however, we do have negative coefficients, due to conditions (2.22) and (2.23), we need to choose denser input data to ensure the existence of an area in which we have a unique minimiser. We summarise the results of [24] for our particular setting in

Proposition 2.22. *Let T be the Riemannian analogue of a linear subdivision scheme S, as in Definition 2.2, on the unit sphere S^n. We consider two cases:*

Case $\alpha_- = 0$:
Tx_i is well defined if the input data points x_i contributing to the computation of Tx_i lie within a ball of radius $r < \frac{\pi}{4}$.

Case $\alpha_- > 0$:
Tx_i is well defined if the input data points x_i contributing to the computation of Tx_i lie within a ball of radius r such that there exists an $r^* > r$ satisfying (2.22) and (2.23).

2.3.2 A strategy to prove convergence for Riemannian analogues of linear schemes on the unit sphere

First, we recall some facts about the squared distance function on the unit sphere and state a second order Taylor approximation of the function f_α defined in (2.21). Afterwards, we explain a strategy to estimate the distance between consecutive points Tx_i

and Tx_{i+1}, which belong to the sequence obtained after one refinement step of the Riemannian analogue T of a linear scheme S. Simultaneously, we bound their distance to the initial data. An iterative use of this method leads to the desired convergence result.

Throughout this part, we assume that the considered minima are well defined and unique.

The Riemannian distance function on the unit sphere

For the distance estimate we use explicit formulas for the gradient and the Hessian of the squared distance function $\operatorname{dist}(\cdot, y)^2$, $y \in M$. They have been computed by X. Pennec in [68, Supplement A] as an example of a more general analysis of Hessians of squared distance functions on manifolds. We introduce some notation and state results of [68] which we later use.

Let $T_xS^n = \{w \in \mathbb{R}^{n+1} \mid \langle w, x \rangle = 0\}$ denote the *tangent space* at a point $x \in S^n$. For two points x, $y \in S^n$, $x \neq -y$, their spherical distance is $\operatorname{dist}(x, y) = \arccos(\langle x, y \rangle)$ and the exponential map at $x \in S^n$ is given by

$$\begin{aligned} \exp_x : T_xS^n &\to S^n \\ w &\mapsto \cos(\|w\|)\, x + \frac{\sin(\|w\|)}{\|w\|} w. \end{aligned}$$

The inverse of the exponential map at x is well defined for all points on the sphere except the antipodal point of x. It is

$$\exp_x^{-1} : S^n \setminus \{-x\} \to T_xS^n \tag{2.24}$$

$$y \mapsto \frac{\operatorname{dist}(x, y)}{\sin(\operatorname{dist}(x, y))} (y - \cos(\operatorname{dist}(x, y))\, x). \tag{2.25}$$

Note that we will always tacitly assume that $\frac{s}{\sin(s)}$ means an analytic function which evaluates to 1 for $s = 0$. Let $w \in T_xS^n$ be a tangent vector. Then, $\exp_x(w)$ denotes the point on the manifold which is reached by the geodesic starting in x in direction w after time 1. We can therefore use the exponential map resp. its inverse to switch between the manifold and its tangent space at a fixed point $x \in S^n$, such that straight lines through the origin in the tangent space are mapped to geodesics on the sphere through x preserving the length of the curves. Let

$$g : S^n \to \mathbb{R}$$

be a function on the sphere and

$$\tilde{g} = g \circ \exp_x : T_xS^n \to \mathbb{R}$$

its composition with the exponential map, for some $x \in S^n$. Then, $\tilde{g}$ is a representation of g with respect to the coordinate chart $\exp_x^{-1}$. Since the first derivative of the exponential map is the identity we have

$$\operatorname{grad}(g)(x) = \operatorname{grad}(\tilde{g})(0) \tag{2.26}$$

for the gradient of g resp. $\tilde{g}$. In particular, it makes no difference if we consider g or its composition with $\exp_x$. The Hessian of $\tilde{g}$ can be computed since the function is defined on a linear space, namely the tangent space. We define the Hessian of g by

$$H(g)(x) := H(\tilde{g})(0). \tag{2.27}$$

If we talk about the the gradient and the Hessian of a function on the unit sphere, according to (2.26) and (2.27), we mean its composition with the exponential map defined on a linear space.

For any fixed $y \in M$ the gradient of the squared distance function is given by

$$\operatorname{grad}(\operatorname{dist}(\cdot, y)^2)(x) = -2\exp_x^{-1}(y). \tag{2.28}$$

Let $v := \frac{\exp_x^{-1}(y)}{\operatorname{dist}(x,y)}$, $y \neq x$, and $I \in \mathbb{R}^{(n+1)\times(n+1)}$ be the identity matrix. The Hessian of $\operatorname{dist}(\cdot, y)^2$ in the tangent space (i.e., the Hessian of its coordinate representation as explained above) has been computed in [68] as

$$H\left(\operatorname{dist}(\cdot, y)^2\right)(x) = 2\left(vv^T + \frac{\operatorname{dist}(x,y)}{\sin(\operatorname{dist}(x,y))}\cos(\operatorname{dist}(x,y))(I - xx^T - vv^T)\right). \tag{2.29}$$

Here x^T, v^T denote the *transpose* of x resp. v. This formula is valid for $y \neq x$. If $x = y$, we have $H\left(\operatorname{dist}(\cdot, x)^2\right)(x) = 2(I - xx^T)$ since $\lim_{x\to 0}\frac{x}{\sin(x)} = 1$ and $\cos(0) = 1$.

The eigenvalues of the Hessian are $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = \frac{2\operatorname{dist}(x,y)}{\tan(\operatorname{dist}(x,y))}$.
By linearity we obtain

$$\operatorname{grad}(f_\alpha)(x) = -2\sum_{j=-m}^{m+1}\alpha_j \exp_x^{-1}(x_j), \tag{2.30}$$

resp.

$$H(f_\alpha)(x) = 2\sum_{j=-m}^{m+1}\alpha_j\left(v_j v_j^T + \frac{\operatorname{dist}(x,x_j)}{\sin(\operatorname{dist}(x,x_j))}\cos(\operatorname{dist}(x,x_j))\left(I - xx^T - v_j v_j^T\right)\right) \tag{2.31}$$

with $v_j := \frac{\exp_x^{-1}(x_j)}{\operatorname{dist}(x,x_j)}$.

Taylor approximation of the squared distance function on the unit sphere

The second order Taylor expansion of the squared distance function helps to find an upper bound on the distances between the minimiser of f_α (as defined in (2.21)) and some input data $x_j \in S^n$, $j = -m, \dots, m+1$. We are interested in such an upper bound because in the convergence analysis of a Riemannian analogue T of a linear scheme S their distance represents the distance between initial data x_i and refined data Tx_i. The estimate results in a displacement-safe condition which we have already used for the convergence analysis of nonlinear schemes on Cartan-Hadamard manifolds, see (2.16).

Let $x^* \in S^n$ be the unique minimiser of f_α. Without loss of generality we choose coordinates such that $x^* = [0, \dots, 0, 1]^T$. Then, the first n unit vectors form a basis of $T_{x^*}S^n$. Now, we consider f_α as a function on the tangent space $T_{x^*}S^n$ and compute its Hessian with respect to the chosen coordinate system (actually, as before, its composition with the inverse of the exponential map, see (2.27)). Due to the particular coordinate system the gradient of f_α consists of the first n entries of the vector given in (2.30). The Hessian is given by the $n \times n$ submatrix of $H\left(f_\alpha\right)$ (2.31) obtained by deleting the last column and row. The second order Taylor approximation in x^* on the tangent space is given by

$$Tf_\alpha(x) = f_\alpha(x^*) + (x - x^*)^T \operatorname{grad}\left(f_\alpha\right)(x^*) + \frac{1}{2}(x - x^*)^T H\left(f_\alpha\right)(x^*)\left(x - x^*\right).$$

Differentiation leads to

$$\operatorname{grad}(Tf_\alpha)(x) = \operatorname{grad}(f_\alpha)(x^*) + H(f_\alpha)(x^*)x. \tag{2.32}$$

Since we are looking for minimisers of the function f_α, the idea is to consider

$$\operatorname{grad}\left(f_\alpha\right)(x^*) + H\left(f_\alpha\right)(x^*)x \overset{!}{=} 0. \tag{2.33}$$

If we assume that $H\left(f_\alpha\right)(x^*)$ is invertible, we deduce that

$$x = -H\left(f_\alpha\right)(x^*)^{-1} \operatorname{grad}\left(f_\alpha\right)(x^*). \tag{2.34}$$

This x is the unique stationary point of the second order Taylor approximation Tf_α.

We are now in a position to present the main ideas which lead to convergence results for Riemannian analogues of linear subdivision rules on the unit sphere.

Variable mask

We introduce a parameter $t \in [0, 1]$ and vary the coefficients α_j of a Riemannian analogue of a linear scheme such that they linearly depend on t. The idea is to choose *coefficient functions* $\alpha_j(t)$ such that at time $t = 0$ we exactly know the minimiser of $f_{\alpha(0)}$, call it the *reference point* $\bar{x}$, and at time $t = 1$ the minimiser of $f_{\alpha(1)}$ equals x^*. Thereby we assume that

$$\sum_{j=-m}^{m+1} \alpha_j(t) = 1 \quad \text{for any } t \in [0, 1]. \tag{2.35}$$

Consider the curve γ such that $\gamma(t)$ is the minimiser of $f_{\alpha(t)}$. Since γ then connects $\bar{x}$ and x^* we have $\operatorname{dist}\left(\bar{x}, x^*\right) \leqslant \int_0^1 \|\dot{\gamma}(t)\|\, dt$. The idea is to estimate $\|\dot{\gamma}(t)\|$ in order to find an upper bound on the distance between $\bar{x}$ and x^*. If, for example, we choose the reference point $\bar{x}$ to be one of our input data points, this strategy helps us to control the distance between the minimiser x^* of $f_{\alpha(1)}$ and the initial data. We will see that the choice of the reference point is crucial for our approach to work and has to be made individually for each scheme.

To make the described procedure more visible to the reader we illustrate it by means of a main example throughout this part of the thesis.

Example 2.23 (*cubic Lane-Riesenfeld, part I*)**.** We choose an input data sequence $(x_i)_{i\in\mathbb{Z}}$ on the sphere and consider the linear cubic Lane-Riesenfeld algorithm defined as

$$(Sx)_{2i} = \frac{1}{8}x_{i-1} + \frac{6}{8}x_i + \frac{1}{8}x_{i+1}, \tag{2.36}$$

$$(Sx)_{2i+1} = \frac{1}{2}x_i + \frac{1}{2}x_{i+1}, \tag{2.37}$$

for $i \in \mathbb{Z}$. Since the mask has nonnegative coefficients, Proposition 2.22 ensures that one refinement step for the Riemannian version T of S is well defined, if

$$\sup_\ell \operatorname{dist}\left(x_\ell, x_{\ell+1}\right) < \frac{\pi}{4}.$$

This assumption is even sufficient for the well-definedness of all subdivision steps of T, if we further can show that

$$\sup_\ell \operatorname{dist}\left(T^{k+1}x_\ell, T^{k+1}x_{\ell+1}\right) \leqslant \sup_\ell \operatorname{dist}\left(T^k x_\ell, T^k x_{\ell+1}\right), \quad \text{for all } k \geqslant 0.$$

We observe that Tx_{2i+1} is the geodesic midpoint of x_i and x_{i+1}. So, its distance to the input data can be bounded from above by half of the maximal distance of the input data. The more crucial part is to deal with the distance of the point Tx_{2i} obtained by (2.36) from x_i. Consider

$$Tx := \operatorname*{arg\,min}_{x\in S^n}\left(\frac{1}{8}\operatorname{dist}(x,x_{-1})^2 + \frac{6}{8}\operatorname{dist}(x,x_0)^2 + \frac{1}{8}\operatorname{dist}(x,x_1)^2\right) \tag{2.38}$$

for x_{-1}, x_0, $x_1 \in S^n$. Without loss of generality we assume that $x_0 = [0,\dots,0,1]^T$. With $m = 1$ and $\alpha_{-1} = \alpha_1 = \frac{1}{8}$, $\alpha_0 = \frac{3}{4}$ as well as $\alpha_2 = 0$, Tx is the minimiser of f_α. Note that we have to add the zero coefficient $\alpha_2 = 0$ here only to be compatible with our previous notation. In fact, it has no influence on the result and we therefore forget about it. We choose the time dependent coefficient functions as

$$\alpha_{-1}(t) = \alpha_1(t) = \frac{t}{8} \quad \text{and} \quad \alpha_0(t) = 1 - \frac{t}{4} \tag{2.39}$$

for all $t \in [0,1]$. Thus, at time $t = 0$ the reference point $\bar{x}$ of $f_{\alpha(0)}$ equals x_0 while at time $t = 1$ the minimiser of $f_{\alpha(1)}$ is exactly the point Tx.

This concludes preparations for the convergence analysis of the nonlinear analogue of the cubic Lane-Riesenfeld scheme. We continue with this example in 2.25. $\diamondsuit$

Estimating the distance to a minimiser

We explain how to estimate the distance between the reference point $\bar{x}$ (the minimiser of $f_{\alpha(0)}$) and the minimiser x^* of $f_{\alpha(1)}$. Recall that $x_j \in S^n$, $j = -m, \ldots, m+1$.

Assumption 1. Assume that

$$\operatorname{dist}(x_j, x_{j+1}) \leqslant r$$

for some constant $r > 0$ and all $j = -m, \ldots, m$. Further we choose r such that the minimiser of $f_{\alpha(t)}$ on the unit sphere is locally well defined for all $t \in [0,1]$.

Assumption 2. Let $r > 0$ be as in Assumption 1. Assume that

$$\|\dot{\gamma}(0)\| \leqslant rC_0$$

for some constant $C_0 > 0$.

Assumption 3. Let $r > 0$ and C_0 be as in Assumption 1 resp. 2. Assume that the following is true: If $\|\dot{\gamma}(t)\| \leqslant rC_0$ for all $t \in [0,1]$, then there exists a constant $C_1 < C_0$ such that $\|\dot{\gamma}(t)\| \leqslant rC_1 < rC_0$ for all $t \in [0,1]$.

Assumption 1 is necessary for the well-definedness of the Riemannian analogue of a linear scheme. Assumptions 2 & 3 help to estimate the distance between $\bar{x}$ and x^*.

Lemma 2.24. *Assume that Assumptions 1, 2 and 3 are satisfied for an $r > 0$ and constants C_0 and C_1. Let γ denote the curve which at time t is the minimiser of $f_{\alpha(t)}$. Then,*

$$\|\dot{\gamma}(t)\| \leqslant rC_1$$

for all $t \in [0,1]$.

Proof. Let $t^* = \sup\{t \in [0,1] \mid \|\dot{\gamma}(t)\| \leqslant rC_1\}$. Then,

$$\|\dot{\gamma}(t^*)\| \leqslant \lim_{t<t^*} \|\dot{\gamma}(t)\| \leqslant rC_1.$$

Assume that $t^* < 1$. Since $\|\dot{\gamma}(t)\|$ is continuous there exists an interval $J = (t^* - \epsilon, t^* + \epsilon)$, $\epsilon > 0$, with $\|\dot{\gamma}(\tilde{t})\| \leqslant rC_1$ for any $\tilde{t} \in J$. But this is a contradiction to t^* being maximal. $\square$

We illustrate the computation of $\dot{\gamma}(0)$ by means of our main example.

Example 2.25 (*cubic Lane-Riesenfeld, part II*)**.** The vector $\dot{\gamma}(0)$ estimates the direction pointing from x_0 towards Tx. First, we compute $H\left(f_{\alpha(0)}\right)(x_0)$ (this is the Hessian of the function $f_{\alpha(0)}$ composed with the exponential map, see (2.27)). By (2.44) we have

$\alpha_{-1}(0) = \alpha_1(0) = 0$ as well as $\alpha_0(0) = 1$. We observe that $\lim_{x\to 0} \frac{x}{\sin(x)} = 1$ and $\cos(\mathrm{dist}(x_0, x_0)) = 1$. Using (2.31) we deduce that

$$\begin{aligned} H\left(f_{\alpha(0)}\right)(x_0) &= 2\alpha_0(0)\left(v_0 v_0^T + \left(I - x_0 x_0^T - v_0 v_0^T\right)\right) \\ &= 2I \end{aligned}$$

in the chosen coordinate system. Remember that the second equality is based on the assumption $x_0 = [0, \dots, 0, 1]^T$. In particular, the inverse $H\left(f_{\alpha(0)}\right)(x_0)^{-1} = \frac{1}{2}I$ is well defined. By (2.30) we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \operatorname{grad}\left(f_{\alpha(t)}\right)(x_0) &= -2\left.\frac{d}{dt}\right|_{t=0} \sum_{j=-1}^{1} \alpha_j(t) \exp_{x_0}^{-1}(x_j) \\ &= -2\left(\frac{1}{8}\exp_{x_0}^{-1}(x_{-1}) + \frac{1}{8}\exp_{x_0}^{-1}(x_1)\right) \\ &= -\frac{1}{4}\left(\exp_{x_0}^{-1}(x_{-1}) + \exp_{x_0}^{-1}(x_1)\right), \end{aligned}$$

using the geometric fact $\exp_{x_0}^{-1}(x_0) = 0$. We conclude that

$$\dot{\gamma}(0) = \frac{1}{8}\left(\exp_{x_0}^{-1}(x_{-1}) + \exp_{x_0}^{-1}(x_1)\right).$$

Assuming that $\mathrm{dist}\left(x_j, x_{j+1}\right) \leqslant r$ for some $0 < r < \frac{\pi}{4}$ and $j = -1, 0$, the above shows that

$$\|\dot{\gamma}(0)\| \leqslant \frac{1}{4}r.$$

This is a first piece of information needed to establish constants C_0, C_1, and eventually prove convergence of the cubic Lane-Riesenfeld scheme on the unit sphere. ♢

We briefly summarize what we have seen so far and point out our next steps. Let S be a linear subdivision scheme and T its Riemannian analogue on the unit sphere.

i) We chose coefficient functions $\alpha_j(t)$ related to the mask of S and considered the curve γ which joins a reference point $\bar{x}$ and the minimiser x^* of $f_{\alpha(1)}$. Since $\mathrm{dist}\left(\bar{x}, x^*\right) \leqslant \int_0^1 \|\dot{\gamma}(t)\|\, dt$ we wish to find an upper bound on $\|\dot{\gamma}(t)\|$ to estimate the distance between the reference point and the minimiser of $f_{\alpha(1)}$.

ii) We introduced three assumptions:
- Assumption 1 ensures the well-definedness of the Riemannian analogue T.
- Assumptions 2 & 3 provide a strategy to find an upper bound on $\|\dot{\gamma}(t)\|$, see Lemma 2.24.

iii) We can use Proposition 2.22 to find input data such that Assumption 1 is verified. As seen in Example 2.25, we can use (2.34) for the verification of Assumption 2. Thus, it remains to verify Assumption 3.

The remaining part of this section provides a strategy which helps to find constants C_0, C_1 such that Assumption 3 is satisfied.

Let $t \in [0,1]$ be fixed. The following computations are done in the tangent space $T_{\gamma(t)}S^n$ where for simplicity we always assume that $\gamma(t)$ is the north pole $[0,\dots,0,1]^T$ of the sphere. Of course, the coordinates of the x_j's change for different t, but since we only consider distances which are independent of the chosen coordinate system, we do not indicate the coordinate change in the notation of the input data. The Hessian $H\left(\operatorname{dist}(\cdot,y)^2\right)(\gamma(t))$ of the squared distance function in the chosen coordinate system has the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{2\operatorname{dist}(\gamma(t),x_j)}{\tan(\operatorname{dist}(\gamma(t),x_j))}$, see Section 2.3.2. In particular, we have $\operatorname{dist}(\gamma(t),x_j) < \frac{\pi}{2}$, $j = -m,\dots,m+1$, since the radius r^* of the ball containing the input data and the minimiser $\gamma(t)$ is less than $\frac{\pi}{4}$, see Section 2.3.1. That is why $\lambda_2 \leqslant \lambda_1$ as well as $0 < \lambda_2 \leqslant 2$. So, we know that the inverse of the Hessian is well defined. We compute the Taylor expansion of $f_{\alpha(t)}$ in $\gamma(t)$ as shown in (2.34) and consider the derivative at time t. We obtain

$$\begin{aligned}\dot\gamma(t) = &-\left.\frac{d}{ds}\right|_{s=t}\left(H\left(f_{\alpha(s)}\right)(\gamma(t))\right)^{-1}\operatorname{grad}\left(f_{\alpha(t)}\right)(\gamma(t)) \\ &-\left(H\left(f_{\alpha(t)}\right)(\gamma(t))\right)^{-1}\left.\frac{d}{ds}\right|_{s=t}\operatorname{grad}\left(f_{\alpha(s)}\right)(\gamma(t))\,.\end{aligned} \tag{2.40}$$

By definition of the curve γ, we conclude that $\operatorname{grad}\left(f_{\alpha(t)}\right)(\gamma(t)) = 0$. Thus,

$$\dot\gamma(t) = -\left(H\left(f_{\alpha(t)}\right)(\gamma(t))\right)^{-1}\left.\frac{d}{ds}\right|_{s=t}\operatorname{grad}\left(f_{\alpha(s)}\right)(\gamma(t)). \tag{2.41}$$

In the following two lemmas we estimate the spectral norm of the inverse of the Hessian and the norm of the derivative of the gradient in order to find an upper bound on $\|\dot\gamma(t)\|$.

Lemma 2.26. *Let* $\operatorname{dist}(x_j,x_{j+1}) \leqslant r$ *for some* $r > 0$*,* $j = -m,\dots,m$*. Assume that* $\|\dot\gamma(t)\| \leqslant C_0 r$ *for* $C_0 > 0$ *and all* $t \in [0,1]$*. Let* ℓ_j*,* $j = -m,\dots,m+1$*, be constants such that* $\operatorname{dist}(x_j,\bar x) \leqslant \ell_j$*. Then,*

$$\|\left(H\left(f_{\alpha(t)}\right)(\gamma(t))\right)^{-1}\| \leqslant \frac{1}{|2-L(t)|}$$

with $L(t) = \displaystyle\sum_{j=-m}^{m+1}|\alpha_j(t)|\left(2-2\frac{C_0 r t+\ell_j}{\tan(C_0 r t+\ell_j)}\right)$ *for all* $t \in [0,1]$*.*

Proof. In order to give an upper bound on the maximal eigenvalue of the inverse of the Hessian we first bound the eigenvalues of the Hessian itself. We have

$$\begin{aligned}\|H\left(f_{\alpha(t)}\right)(\gamma(t))\| &= \left\|\sum_{j=-m}^{m+1}\alpha_j(t)H\left(\operatorname{dist}(x_j,\cdot)^2\right)(\gamma(t))\right\| \\ &\leqslant 2\sum_{j=-m}^{m+1}|\alpha_j(t)|,\end{aligned}$$

since the maximal eigenvalue of the Hessian of the squared distance function is $\lambda_1 = 2$. In particular, the norm of any eigenvalue of the Hessian is bounded from above by $2\sum_{j=-m}^{m+1} |\alpha_j(t)|$. Furthermore, we see that

$$\begin{aligned} \|2I - H\left(f_{\alpha(t)}\right)(\gamma(t))\| &= \left\| \sum_{j=-m}^{m+1} \alpha_j(t) \left(2I - H\left(\operatorname{dist}(x_j,\cdot)^2\right)(\gamma(t))\right) \right\| \\ &\leqslant \sum_{j=-m}^{m+1} |\alpha_j(t)| \left\|2I - H\left(\operatorname{dist}(x_j,\cdot)^2\right)(\gamma(t))\right\|. \end{aligned} \tag{2.42}$$

Denote by $\lambda_{2,j}(t)$ the smaller eigenvalue of $H\left(\operatorname{dist}(x_j,\gamma(t))^2\right)$. In fact,

$$\lambda_{2,j}(t) = \frac{2\operatorname{dist}(\gamma(t),x_j)}{\tan(\operatorname{dist}(\gamma(t),x_j))} < 2.$$

By Assumption 3 and since $\frac{s}{tan(s)}$ is positive and monotonically decreasing for $0 \leqslant s < \frac{\pi}{2}$ we deduce that

$$\lambda_{2,j}(t) \geqslant 2\frac{C_0rt + \ell_j}{\tan(C_0rt + \ell_j)}$$

for all $j = -m, \ldots, m+1$. By (2.42) we therefore obtain

$$\|2I - H\left(f_{\alpha(t)}\right)(\gamma(t))\| \leqslant L(t)$$

and the minimal eigenvalue of $H\left(f_{\alpha(t)}\right)(\gamma(t))$ is bounded from below by $|2-L(t)|$. Thus, the claim for the norm of inverse matrix follows. □

Lemma 2.27. *Let* $\operatorname{dist}(x_j, x_{j+1}) \leqslant r$ *for some* $r > 0$*,* $j = -m, \ldots, m$*. Assume that* $\|\dot\gamma(t)\| \leqslant C_0 r$ *for* $C_0 > 0$ *and all* $t \in [0,1]$*. Let* ℓ_j*,* $j = -m, \ldots, m+1$*, be constants such that* $\operatorname{dist}(x_j, \bar{x}) \leqslant \ell_j$*. Then,*

$$\left\| \frac{d}{ds}\bigg|_{s=t} \operatorname{grad}\left(f_{\alpha(s)}\right)(\gamma(t)) \right\| \leqslant 2\sum_{j=-m}^{m+1} |\dot\alpha_j(t)|\left(rC_0t + \ell_j\right)$$

for all $t \in [0,1]$*.*

Proof. Fix some $t \in [0,1]$. Since $\operatorname{dist}(\gamma(t), x_j) = \|\exp^{-1}_{\gamma(t)}(x_j)\|$ and, by assumption, $\|\dot\gamma(t)\| \leqslant C_0 r$ we deduce that

$$\begin{aligned} \|\exp^{-1}_{\gamma(t)}(x_j)\| &\leqslant \|\exp^{-1}_{\gamma(t)}(\bar{x})\| + \|\exp^{-1}_{\bar{x}}(x_j)\| \\ &\leqslant rC_0t + \ell_j. \end{aligned}$$

So,

$$\left\| \left.\frac{d}{ds}\right|_{s=t} \operatorname{grad}\left(f_{\alpha(s)}\right)(\gamma(t)) \right\| \leqslant 2 \sum_{j=-m}^{m+1} \left| \left.\frac{d}{ds}\right|_{s=t} \alpha_j(s) \right| \| \exp^{-1}_{\gamma(t)}(x_j) \|$$
$$\leqslant 2 \sum_{j=-m}^{m+1} |\dot{\alpha}_j(t)| \left(rC_0 t + \ell_j\right)$$

by (2.30). □

We summarise the results of the previous two lemmas in

Proposition 2.28. *Let* $\operatorname{dist}(x_j, x_{j+1}) \leqslant r$ *for some* $r > 0$, $j = -m, \ldots, m$. *Assume that* $\|\dot{\gamma}(t)\| \leqslant C_0 r$ *for* $C_0 > 0$ *and all* $t \in [0,1]$. *Let* ℓ_j, $j = -m, \ldots, m+1$, *be constants such that* $\operatorname{dist}(x_j, \bar{x}) \leqslant \ell_j$. *Then,*

$$\|\dot{\gamma}(t)\| \leqslant \frac{2}{|2 - L(t)|} \sum_{j=-m}^{m+1} |\dot{\alpha}_j(t)| \left(rC_0 t + \ell_j\right) \tag{2.43}$$

with $L(t) = \sum_{j=-m}^{m+1} |\alpha_j(t)| \left(2 - 2\frac{C_0 r t + \ell_j}{\tan\left(C_0 r t + \ell_j\right)}\right)$ *for all* $t \in [0,1]$.

Proof. By (2.41) we have

$$\|\dot{\gamma}(t)\| \leqslant \left\| \left(H\left(f_{\alpha(t)}\right)(\gamma(t))\right)^{-1} \left.\frac{d}{ds}\right|_{s=t} \operatorname{grad}\left(f_{\alpha(s)}\right)(\gamma(t)) \right\|$$
$$\leqslant \left\| \left(H\left(f_{\alpha(t)}\right)(\gamma(t))\right)^{-1} \right\| \left\| \left.\frac{d}{ds}\right|_{s=t} \operatorname{grad}\left(f_{\alpha(s)}\right)(\gamma(t)) \right\|$$

for all $t \in [0,1]$. The claim then follows by Lemma 2.26 and Lemma 2.27. □

We illustrate the results of Proposition 2.28 by means of our main example.

Example 2.29 (*cubic Lane-Riesenfeld, part III*)**.** Lemma 2.26 shows that

$$L(t) = \frac{t}{4}\left(2 - 2\frac{C_0 r t + r}{\tan\left(C_0 r t + r\right)}\right) + \left(1 - \frac{t}{4}\right)\left(2 - 2\frac{C_0 r t}{\tan\left(C_0 r t\right)}\right)$$

for all $t \in [0,1]$ with $\ell_0 = 0$ and $\ell_{-1} = \ell_1 = r$. Since this function is strictly increasing in the interval $[0,1]$ we have

$$L(t) \leqslant \frac{1}{4}\left(2 - 2\frac{C_0 r + r}{\tan\left(C_0 r + r\right)}\right) + \frac{3}{4}\left(2 - 2\frac{C_0 r}{\tan\left(C_0 r\right)}\right)$$

for all $t \in [0,1]$. We conclude that

$$\| \left(H\left(f_{\alpha(t)}\right)(\gamma(t))\right)^{-1} \| \leqslant \frac{1}{2-\left(2-\frac{1}{2}\frac{C_0 r+r}{\tan(C_0 r+r)}-\frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}\right)} = \frac{1}{\frac{1}{2}\frac{C_0 r+r}{\tan(C_0 r+r)}+\frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}}$$

which is an upper bound on the norm of the inverse of the Hessian which only depends on C_0 and r. This estimate is needed for the verification of Assumption 3 of our method.

Remember that we have chosen $\bar{x} = x_0$ as well as

$$\alpha_{-1}(t) = \alpha_1(t) = \frac{t}{8} \quad \text{and} \quad \alpha_0(t) = 1 - \frac{t}{4}, \quad t \in [0,1]. \tag{2.44}$$

Assume that $r < \frac{\pi}{4}$ is such that $\operatorname{dist}(x_j, x_0) \leqslant r$, for $j = -1, 1$. Since $\sum_{j=-1}^{1} |\dot{\alpha_j}(t)| = \frac{1}{2}$ Equation (2.43) reads as

$$\begin{aligned} \|\dot{\gamma}(t)\| &\leqslant \frac{2}{\frac{1}{2}\frac{C_0 r+r}{\tan(C_0 r+r)}+\frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}}\left(2\dot{\alpha}_1(t) r + \frac{1}{2} r C_0 t\right) \\ &= \frac{2}{\frac{1}{2}\frac{C_0 r+r}{\tan(C_0 r+r)}+\frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}}\left(\frac{1}{4} r + \frac{1}{2} r C_0 t\right) \end{aligned} \tag{2.45}$$

for all $t \in [0,1]$. Thus, we have obtained an upper bound on $\|\dot{\gamma}(t)\|$ and we are finally in a position to estimate the distance between x_0 and x^* as follows.

If we can guarantee that

$$\frac{2}{\frac{1}{2}\frac{C_0 r+r}{\tan(C_0 r+r)}+\frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}}\left(\frac{1}{4} r + \frac{1}{2} r C_0 t\right) \leqslant C_1 r \quad \text{for some } C_1 < C_0,$$

we have

$$\begin{aligned} \operatorname{dist}(x_0, x^*) &\leqslant \int_0^1 \|\dot{\gamma}(t)\| \, dt \\ &\leqslant \frac{2}{\frac{1}{2}\frac{C_0 r+r}{\tan(C_0 r+r)}+\frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}} \int_0^1 \frac{r}{4} + \frac{1}{2} r C_0 t \, dt \\ &= \frac{2}{\frac{1}{2}\frac{C_0 r+r}{\tan(C_0 r+r)}+\frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}}\left(\frac{r}{4} + \frac{r C_0}{4}\right). \end{aligned} \tag{2.46}$$

This is the last piece of preparations needed for our convergence analysis of the cubic Lane-Riesenfeld scheme on the unit sphere. The final convergence argument, presented in the next section, consists of specific choices of $r > 0$, C_0 and C_1. ♢

2.3.3 Convergence results on the unit sphere

We show convergence results for the Riemannian analogues of different linear subdivision schemes on the unit sphere. They are based on the distance estimates proved in the previous section.

Example 2.30 (*cubic Lane-Riesenfeld, part IV*)**.** We continue with the analysis of our main example. Before we study the convergence in full generality, we illustrate the idea for $r_0 = \frac{1}{4}$ and input data x_{-1}, x_0, $x_1 \in S^n$ such that $\operatorname{dist}(x_0, x_{-1}) \leqslant r_0$ as well as $\operatorname{dist}(x_0, x_1) \leqslant r_0$. Let $C_0 = 0.52 > \frac{1}{4}$, then $\|\dot\gamma(0)\| < rC_0$ for any $0 < r \leqslant \frac{1}{4}$, see Example 2.25. In particular, Assumption 2 is satisfied for all $0 < r \leqslant r_0$. Computations show that

$$\frac{1}{2}\frac{C_0 r_0 + r_0}{\tan(C_0 r_0 + r_0)} + \frac{3}{2}\frac{C_0 r_0}{\tan(C_0 r_0)} \approx 1.97.$$

Since $\frac{s}{\tan(s)}$ is positive and monotonically decreasing for $0 \leqslant s < \frac{\pi}{2}$, we conclude

$$\frac{2}{\frac{1}{2}\frac{C_0 r + r}{\tan(C_0 r + r)} + \frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}}\left(\frac{r}{4} + \frac{rC_0}{2}\right) \leqslant 1.02\left(\frac{r}{4} + 0.26r\right) = 0.51r$$

for any $0 < r \leqslant r_0$. By (2.45) we verified that under the assumption that $\|\dot\gamma(t)\| \leqslant C_0 r_0$ for all $t \in [0,1]$, we have $\|\dot\gamma(t)\| \leqslant C_1 r$ with $C_1 = 0.51 < C_0$ for all $t \in [0,1]$ and $0 < r \leqslant r_0$, i.e., we verified Assumption 3. By (2.46) we obtain

$$\operatorname{dist}(x_0, x^*) \leqslant 1.02\left(\frac{r}{4} + 0.13r\right) = 0.39r$$

for any $0 < r \leqslant r_0$.

We are now in a position to analyse the convergence of the Riemannian analogue T of the linear cubic Lane-Riesenfeld algorithm. Therefore, we choose input data $(x_i)_{i\in\mathbb{Z}}$ with

$$\sup_\ell \operatorname{dist}(x_\ell, x_{\ell+1}) < r_0.$$

Our previous computations together with the fact that the points of the sequence after one refinement step obtained by (2.37) are the geodesic midpoints of two consecutive input data points lead to

$$\begin{aligned}\operatorname{dist}(Tx_i, Tx_{i+1}) &\leqslant \frac{r_0}{2} + 0.39r_0 = 0.89r_0,\\ \operatorname{dist}(Tx_{2i}, x_i) &\leqslant 0.39r_0\end{aligned}$$

for all $i \in \mathbb{Z}$. Note that this conclusion highly depends on the mask and the choice of the reference point. We have shown that the distance of consecutive points in $(Tx_i)_{i\in\mathbb{Z}}$ is less than the maximal distance of consecutive input data points. In fact, we have already shown a bit more. Namely, for $r_0 = \frac{1}{4}$ and $C_0 = 0.52$ the following is true:

i) $\|\dot\gamma(0)\| \leqslant C_0 r$ for any $0 < r \leqslant \frac{1}{4}$.

ii) Under the assumption that $\|\dot{\gamma}(t)\| \leqslant C_0 r_0$ for all $t \in [0,1]$, the constant $C_1 = 0.51 < C_0$ satisfies $\|\dot{\gamma}(t)\| \leqslant C_1 r$ for all $t \in [0,1]$ and any $0 < r \leqslant \frac{1}{4}$.

Thus, we can iteratively apply our strategy to estimate distances we so far only used for the first refinement step. We deduce that

$$\operatorname{dist}\left(T^k x_i, T^k x_{i+1}\right) \leqslant 0.89^k r_0 = 0.89^k \sup_\ell \operatorname{dist}\left(x_\ell, x_{\ell+1}\right),$$
$$\operatorname{dist}\left(T x_{2i}, x_i\right) \leqslant 0.39 \sup_\ell \operatorname{dist}\left(x_\ell, x_{\ell+1}\right)$$

for all $k \geqslant 0$ and $i \in \mathbb{Z}$. The first inequality above implies that T admits a contractivity factor $\mu = 0.89 < 1$, while the second inequality ensures that T is displacement-safe. We proceed as in the proof of Theorem 2.11 (which also works on positively-curved manifolds) and conclude that the Riemannian analogue of the linear cubic Lane-Riesenfeld algorithm admits a continuous limit on the unit sphere, if the distance of consecutive input data points is bounded from above by $\frac{1}{4}$. Note that we can use the same proof as for Cartan-Hadamard manifolds here because we restrict ourselves to an area on the unit sphere in which all geodesics are unique and the Riemannian center of mass are well defined.

The previous paragraph deals with the special case of the fixed assumption on the input data $\sup_\ell \operatorname{dist}\left(x_\ell, x_{\ell+1}\right) < r_0$ with $r_0 = \frac{1}{4}$. Now, we analyse the general case of an upper bound r_0 $(0 < r_0 < \frac{\pi}{4})$. First, note that the well-definedness of T would no longer be guaranteed, if $r_0 \geqslant \frac{\pi}{4}$, see Proposition 2.22. Furthermore, if $C_0 > \frac{1}{4}$, Assumption 2 is satisfied for any $0 < r \leqslant r_0$. Consider the inequalities

$$\begin{aligned} & \frac{2}{\frac{1}{2}\frac{C_0 r + r}{\tan(C_0 r + r)} + \frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}} \left(\frac{1}{4} r + \frac{1}{2} r C_0\right) < r C_0 \\ \Leftrightarrow \quad & \frac{2}{\frac{1}{2}\frac{C_0 r + r}{\tan(C_0 r + r)} + \frac{3}{2}\frac{r C_0}{\tan(r C_0)}} \left(\frac{1}{4} + \frac{1}{2} C_0\right) < C_0 \end{aligned} \tag{2.47}$$

and

$$\begin{aligned} & \frac{2}{\frac{1}{2}\frac{C_0 r + r}{\tan(C_0 r + r)} + \frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}} \left(\frac{1}{4} r + \frac{1}{4} r C_0\right) < \frac{r}{2} \\ \Leftrightarrow \quad & \frac{2}{\frac{1}{2}\frac{C_0 r + r}{\tan(C_0 r + r)} + \frac{3}{2}\frac{C_0 r}{\tan(C_0 r)}} \left(\frac{1}{4} + \frac{1}{4} C_0\right) < \frac{1}{2}. \end{aligned} \tag{2.48}$$

We are looking for a constant $C_0 > \frac{1}{4}$ together with the largest possible value for $r < \frac{\pi}{4}$ such that (2.47) and (2.48) are both satisfied. Numerical examples show that for $r_0 = 0.6$ and $C_0 = 0.69$ we have

$$\frac{2}{\frac{1}{2}\frac{C_0 r_0 + r_0}{\tan(C_0 r_0 + r_0)} + \frac{3}{2}\frac{r_0 C_0}{\tan(r_0 C_0)}} \left(\frac{1}{4} + \frac{1}{2} C_0\right) \approx 0.68 < C_0, \tag{2.49}$$

$$\frac{2}{\frac{1}{2}\frac{C_0 r_0 + r_0}{\tan(C_0 r_0 + r_0)} + \frac{3}{2}\frac{C_0 r_0}{\tan(C_0 r_0)}} \left(\frac{1}{4} + \frac{1}{4} C_0\right) \approx 0.49 < \frac{1}{2}, \tag{2.50}$$

while for $r_0 > 0.6$, we do not always find a suitable constant C_0. In particular, this shows that (2.47) and (2.48) are satisfied for any $0 < r \leqslant 0.6$, again by the fact that $\frac{s}{\tan(s)}$ is positive and monotonically decreasing for $0 \leqslant s < \frac{\pi}{2}$.

By (2.49) Assumption 3 is satisfied for any $0 < r \leqslant 0.6$ and by (2.50) T admits a contractivity factor $\mu \approx 0.99 < 1$, i.e., $\operatorname{dist}\left(T^k x_i, T^k x_{i+1}\right) \leqslant \mu^k \sup_\ell \operatorname{dist}\left(x_\ell, x_{\ell+1}\right)$ for all $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. ♢

We summarize the conclusions of the example of the cubic Lane-Riesenfeld algorithm, parts I-IV, in the following

Corollary 2.31. *Let $(x_i)_{i\in\mathbb{Z}}$ be a sequence of points on the unit sphere. If*

$$\sup_\ell \operatorname{dist}(x_\ell, x_{\ell+1}) < 0.6,$$

then the Riemannian analogue of the linear cubic Lane-Riesenfeld algorithm converges to a continuous limit function on the unit sphere.

We remark that the constant C_0 might be chosen smaller for special input data as seen at the beginning of the example, but since we are first only interested in a convergence result, we do not specify the choice here any further.

Example: 4-point scheme

We analyse the 4-point scheme introduced in (2.5) for $\omega = \frac{1}{16}$. First, we focus on

$$\begin{aligned} Tx := \operatorname*{arg\,min}_{x\in S^n} \Big(&-\frac{1}{16}\operatorname{dist}(x, x_{-1})^2 + \frac{9}{16}\operatorname{dist}(x, x_0)^2 \\ &+ \frac{9}{16}\operatorname{dist}(x, x_1)^2 - \frac{1}{16}\operatorname{dist}(x, x_2)^2 \Big) \end{aligned} \tag{2.51}$$

for some $x_j \in S^n$, $j = -1, \ldots, 2$. Let $\operatorname{dist}(x_j, x_{j+1}) < r_0$ for an $r_0 > 0$ and $j = -1, 0, 1$. In particular, we have $\alpha_- = \frac{1}{8}$. Thus, Conditions (2.22), (2.23) and Proposition 2.22 imply that if $r_0 < \frac{0.59}{1.25} \cdot \frac{2}{3} \approx 0.31$, our input data lies inside a ball of small enough radius such that the minimiser Tx is well defined.

So, let $r_0 = 0.31$ and denote the geodesic midpoint of x_0 and x_1 by x_m. We use x_m as the reference point. Observe that due to our restrictions on the input data x_m is well defined. Let

$$\alpha_{-1}(t) = \alpha_2(t) = -\frac{t}{16} \quad \text{and} \quad \alpha_0(t) = \alpha_1(t) = \frac{1}{2} + \frac{t}{16} \tag{2.52}$$

be the coefficient functions for $t \in [0, 1]$ and γ be the curve connecting the minimisers of $f_{\alpha(t)}$.

We see that

$$\begin{aligned}
\Big\| \frac{d}{dt}\Big|_{t=0} \operatorname{grad}\left(f_{\alpha(t)}\right)(x_m) \Big\| &\leqslant 2 \Big\| \frac{d}{dt}\Big|_{t=0} \sum_{j=-1}^{2} \alpha_j(t) \exp_{x_m}^{-1}(x_j) \Big\| \\
&\leqslant 2 \left(\frac{1}{16} \exp_{x_m}^{-1}(x_{-1}) + \frac{1}{16} \exp_{x_m}^{-1}(x_0) + \frac{1}{16} \exp_{x_m}^{-1}(x_1) + \frac{1}{16} \exp_{x_m}^{-1}(x_2) \right) \\
&\leqslant 2 \left(\frac{2}{16} \cdot \frac{3r}{2} + \frac{2}{16} \cdot \frac{r}{2} \right) = 2 \cdot \frac{1}{4} r
\end{aligned}$$

for any $0 < r \leqslant r_0$. Moreover, we deduce that

$$L(t) = \frac{2t}{16} \left(2 - 2 \frac{C_0 rt + \frac{3}{2} r}{\tan(C_0 rt + \frac{3}{2} r)} \right) + 2 \left(\frac{1}{2} + \frac{t}{16} \right) \left(2 - 2 \frac{C_0 rt + \frac{1}{2} r}{\tan(C_0 rt + \frac{1}{2} r)} \right)$$

for a constant C_0, $L(t)$ as in Lemma 2.26 and all $t \in [0, 1]$. In particular,

$$L(0) = 2 - \frac{r}{\tan\left(\frac{1}{2} r\right)}.$$

Considered as a function in r, $L(0)$ is positive and monotonically increasing for $0 < r \leqslant r_0$. Thus,

$$L(0) \leqslant 2 - \frac{r_0}{\tan\left(\frac{1}{2} r_0\right)} \approx 0.02$$

and $\frac{2}{2-0.02} \cdot \frac{1}{4} r \approx \frac{1}{4} r$. Lemma 2.26, together with our previous computations, yields

$$\|\dot{\gamma}(0)\| \leqslant \frac{1}{4} r$$

for all $0 < r \leqslant r_0$ and Assumption 2 is satisfied for any constant $C_0 > \frac{1}{4}$. We assume that $\|\dot{\gamma}(t)\| < C_0 r$ for some $0 < r \leqslant r_0$ and all $t \in [0, 1]$. Again by monotonicity $L(t) \leqslant L(1)$ and by Proposition 2.28 we therefore have

$$\begin{aligned}
\|\dot{\gamma}(t)\| &\leqslant \frac{2}{|2 - L(1)|} \left(\frac{2}{16} \left(rC_0 t + \frac{3}{2} r \right) + \frac{2}{16} \left(rC_0 t + \frac{r}{2} \right) \right) \\
&= \frac{2}{|2 - L(1)|} \left(\frac{1}{4} rC_0 t + \frac{1}{4} r \right) \\
&\leqslant \frac{2}{|2 - L(1)|} \left(\frac{1}{4} rC_0 + \frac{1}{4} r \right)
\end{aligned}$$

for all $t \in [0, 1]$. This implies that

$$\begin{aligned}
\operatorname{dist}(m, Tx) &\leqslant \frac{2}{|2 - L(1)|} \int_0^1 \frac{1}{4} rC_0 t + \frac{1}{4} r \, dt \\
&= \frac{2}{|2 - L(1)|} \left(\frac{1}{8} rC_0 + \frac{1}{4} r \right)
\end{aligned}$$

and

$$\operatorname{dist}(x_0, Tx) \leqslant \operatorname{dist}(x_m, Tx) + \frac{r}{2}$$

for any $0 < r \leqslant r_0$. Now, we ask $\operatorname{dist}(x_m, Tx) + \frac{r}{2} < r$ to obtain a contractivity as well as displacement-safe condition in the end. Thus, we are looking for a constant $C_0 > \frac{1}{4}$ together with suitable choices for r such that

$$\frac{2}{|2 - L(1)|}\left(\frac{1}{4}C_0 + \frac{1}{4}\right) < C_0$$

and

$$\frac{2}{|2 - L(1)|}\left(\frac{1}{8}C_0 + \frac{1}{4}\right) < \frac{1}{2}.$$

Numerical computations show that if $C_0 = 0.45$, both inequalities are satisfied for any $0 < r \leqslant r_0 = 0.31$. Thus, the Riemannian analogue T of the linear 4-point scheme is displacement-safe and the maximal distance of consecutive points $T^k x_i$, $T^k x_{i+1}$ strictly decreases, if the iteration depth k goes to infinity. It therefore admits a contractivity factor less than 1 and we have shown

Corollary 2.32. *Let $(x_i)_{i\in\mathbb{Z}}$ be a sequence of points on the unit sphere. If*

$$\sup_{\ell} \operatorname{dist}(x_\ell, x_{\ell+1}) < 0.31,$$

then the Riemannian analogue of the linear 4-point scheme with parameter $\omega = \frac{1}{16}$ converges to a continuous limit function on the unit sphere.

While we have chosen an input data point as reference point in the first example, we have now seen that the choice of a geodesic midpoint of two initial data points yields to a convergence result. So far, we have considered one scheme whose mask contains only positive coefficients (cubic Lane-Riesenfeld algorithm) and one (4-point scheme) with the special property of being interpolatory. The next example shows that our strategy works for non-interpolatory schemes with negative coefficients, too.

Example: Combination of 4-point scheme and Chaikin's algorithm

We consider the linear scheme

$$(Sx)_{2i} = -\frac{1}{32}x_{i-1} + \frac{21}{32}x_i + \frac{13}{32}x_{i+1} - \frac{1}{32}x_{i+2}, \tag{2.53}$$

$$(Sx)_{2i+1} = -\frac{1}{32}x_{i-1} + \frac{13}{32}x_i + \frac{21}{32}x_{i+1} - \frac{1}{32}x_{i+2}, \tag{2.54}$$

$i \in \mathbb{Z}$, introduced in Example 2.1. Because of the symmetry of the two refinement rules it is sufficient to analyse

$$\begin{aligned} Tx := \operatorname*{arg\,min}_{x\in S^n} \Big(&-\frac{1}{32}\operatorname{dist}(x, x_{-1})^2 + \frac{21}{32}\operatorname{dist}(x, x_0)^2 \\ &+ \frac{13}{32}\operatorname{dist}(x, x_1)^2 - \frac{1}{32}\operatorname{dist}(x, x_2)^2 \Big) \end{aligned} \tag{2.55}$$

with $x_j \in S^n$, $j = -1, \dots, 2$. Let $\operatorname{dist}(x_j, x_{j+1}) < r_0$ for some $r_0 > 0$, $j = -1, 0, 1$. Since $\alpha_- = \frac{1}{16}$, Conditions (2.22), (2.23) and Proposition 2.22 imply that if $r_0 < \frac{0.68}{1.125} \cdot \frac{2}{3} \approx 0.4$, our input data lies inside a ball of small enough radius such that the minimiser Tx is well defined. So, let $r_0 = 0.4$. We choose our reference point $\bar{x} \in S^n$ to be the weighted geodesic average of x_0 and x_1 with weights $\beta_0 = 0.65$ and $\beta_1 = 0.35$. Define the coefficient functions as

$$\alpha_{-1}(t) = \alpha_2(t) = -\frac{t}{32}, \quad \alpha_0(t) = \frac{65}{100} + \frac{t}{160} \quad \text{and} \quad \alpha_1(t) = \frac{35}{100} + \frac{9}{160}t \tag{2.56}$$

for $t \in [0, 1]$ and let γ denote the curve connecting the minimisers of $f_{\alpha(t)}$. Then,

$$\begin{aligned}
\left\| \frac{d}{dt}\Big|_{t=0} \operatorname{grad}\left(f_{\alpha(t)}\right)(\bar{x}) \right\| &\leqslant 2 \left\| \frac{d}{dt}\Big|_{t=0} \sum_{j=-1}^{2} \alpha_j(t) \exp_{\bar{x}}^{-1}(x_j) \right\| \\
&\leqslant 2 \left(\frac{1}{32} \exp_{\bar{x}}^{-1}(x_{-1}) + \frac{1}{160} \exp_{\bar{x}}^{-1}(x_0) + \frac{9}{160} \exp_{\bar{x}}^{-1}(x_1) + \frac{1}{32} \exp_{\bar{x}}^{-1}(x_2) \right) \\
&\leqslant 2 \left(\frac{1}{32} \frac{135}{100} r + \frac{1}{160} \frac{35}{100} r + \frac{9}{160} \frac{65}{100} r + \frac{1}{32} \frac{165}{100} r \right) \approx 2 \cdot 0.13 r
\end{aligned}$$

for any $0 < r \leqslant r_0$. Moreover, we deduce that

$$\begin{aligned}
L(t) = {} & \frac{t}{32} \left(2 - 2 \frac{C_0 r t + 1.35 r}{\tan\left(C_0 r t + 1.35 r\right)} \right) + \left(0.65 + \frac{t}{160} \right) \left(2 - 2 \frac{C_0 r t + 0.35 r}{\tan\left(C_0 r t + 0.35 r\right)} \right) \\
& + \left(0.35 + \frac{9}{160} t \right) \left(2 - 2 \frac{C_0 r t + 0.65 r}{\tan(C_0 r t + 0.65 r)} \right) + \frac{t}{32} \left(2 - 2 \frac{C_0 r t + 1.65 r}{\tan(C_0 r t + 1.65 r)} \right)
\end{aligned}$$

for some constant C_0, $L(t)$ as in Lemma 2.26 and all $t \in [0, 1]$. Considered as a function in r, $L(0)$ is positive an monotonically increasing for $0 < r \leqslant r_0$. Thus, we conclude that

$$L(0) \leqslant 0.02$$

and $\frac{2}{2 - 0.02} \cdot 0.13 r \approx 0.13 r$. Lemma 2.26 and our previous computations show that

$$\|\dot{\gamma}(0)\| \leqslant 0.13 r$$

for all $0 < r \leqslant r_0$. So, Assumption 2 is satisfied for any constant $C_0 > 0.13$. We assume that $\|\dot{\gamma}(t)\| < C_0 r$ for some $0 < r \leqslant r_0$ and all $t \in [0, 1]$. Again by monotonicity $L(t) \leqslant L(1)$ and by Proposition 2.28 we therefore deduce that

$$\begin{aligned}
\|\dot{\gamma}(t)\| &\leqslant \frac{2}{|2 - L(1)|} \Big(\frac{1}{32} \left(r C_0 t + 1.35 r \right) + \frac{1}{160} \left(r C_0 t + 0.35 r \right) \\
&\qquad + \frac{9}{160} \left(r C_0 t + 0.65 r \right) + \frac{1}{32} \left(r C_0 t + 1.65 r \right) \Big) \\
&= \frac{2}{|2 - L(1)|} \left(\frac{1}{8} r C_0 t + 0.13 r \right) \\
&\leqslant \frac{2}{|2 - L(1)|} \left(\frac{1}{8} r C_0 + 0.13 r \right)
\end{aligned}$$

for all $t \in [0,1]$. This implies that

$$\begin{aligned}\operatorname{dist}(\bar{x}, Tx) &\leqslant \frac{2}{|2 - L(1)|} \int_0^1 \frac{1}{8} r C_0 t + 0.13r \; dt \\ &= \frac{2}{|2 - L(1)|} \left(\frac{1}{16} r C_0 + 0.13r \right)\end{aligned}$$

and

$$\operatorname{dist}(x_0, Tx) \leqslant \operatorname{dist}(\bar{x}, Tx) + 0.35r$$

for any $0 < r \leqslant r_0$. Now, we ask $\operatorname{dist}(\bar{x}, Tx) + 0.35r < \frac{r}{2}$ because this yields to a contractivity as well as displacement-safe condition later. Thus, we are looking for a constant $C_0 > 0.13$ together with suitable choices for r such that

$$\frac{2}{|2 - L(1)|} \left(\frac{1}{8} C_0 + 0.13 \right) < C_0$$

and

$$\frac{2}{|2 - L(1)|} \left(\frac{1}{16} C_0 + 0.13 \right) < \frac{1}{2} - 0.35 = 0.15.$$

If $C_0 = 0.16$, numerical computations show that both inequalities are satisfied for any $0 < r \leqslant r_0 = 0.4$. The iterative application of the estimate above shows that the Riemannian analogue T admits a contractivity factor less than 1 and is displacement-safe. Therefore, we have shown

Corollary 2.33. *Let $(x_i)_{i \in \mathbb{Z}}$ be a sequence of points on the unit sphere. If*

$$\sup_{\ell \in \mathbb{Z}} \operatorname{dist}(x_\ell, x_{\ell+1}) < 0.4,$$

then the Riemannian analogue of the linear subdivision scheme defined in (2.53) converges to a continuous limit function on the unit sphere.

2.4 Conclusion and outlook

Convergence results for the Riemannian analogue of a linear scheme with nonnegative mask coefficients have been studied on Cartan-Hadamard spaces in the univariate as well as in the multivariate setting [79, 34, 35]. We have extended the convergence results of [79] to univariate schemes with arbitrary mask on Cartan-Hadamard manifolds, see Theorem 2.15.

Less results are known for nonlinear analogues of linear subdivision schemes on manifolds with positive sectional curvature. We have introduced a strategy to prove convergence results and have applied it to several examples. Unfortunately, the results depend on a *well* chosen reference point which is different for each scheme, see Section 2.3.3.

Future research

- Ideally, of course, future research leads to a general convergence result for the Riemannian analogue of a linear scheme which only depends on the mask and the bounds on the curvature K of the underlying manifold.

- The subdivision schemes we analysed on the unit sphere so far, all have dilation factor 2. We guess that our strategy works for schemes with higher dilation factor, too, even so, the choices of reference points might become more crucial.

- We studied the convergence of nonlinear analogues of univariate, linear subdivision schemes with arbitrary mask. We are quite confident that similar results could be obtained in the multivariate setting where the known results are restricted to schemes with nonnegative masks, [34, 35].

  Up to now, convergence results for nonlinear subdivision schemes applied to data on meshes with irregular combinatorics are based on proximity conditions, and therefore are limited to 'dense enough' input data, [80, 81]. It would be of interest to give convergence criteria which apply to all input data.

3 Polynomial reproduction of Hermite schemes

In this part of the thesis, we focus on the capability of a Hermite subdivision scheme to reproduce polynomials. Meaning, we are looking for conditions guaranteeing that Hermite subdivision schemes applied to initial data sampled from a polynomial function yield the same polynomial and its derivatives in the limit.

3.1 Hermite schemes of order 2

We present a characterisation of polynomial reproduction of Hermite schemes by means of algebraic conditions on the subdivision symbol (resp. its derivatives). Those conditions also provide the correct parametrisation of the scheme and can be used to construct Hermite schemes producing polynomials up to a certain degree. This work generalises the results present in [9] where only scalar schemes were considered. In a first step, we focus on Hermite schemes dealing with function values and first derivatives only.

The presented results are based on the publication

C. Conti, S. Hüning, *An algebraic approach to polynomial reproduction of Hermite subdivision schemes*, Journal of Computational and Applied Mathematics, 349, 302-315, 2019, DOI:10.1016/j.cam.2018.08.009.

We begin by introducing our notation and continue with the analysis of certain classes of polynomials in Section 3.1.2. Then, we state our algebraic conditions in Theorem 3.9. We conclude the section with some examples. In particular, we construct a Hermite schemes which reproduces polynomials up to degree 5 from a Hermite scheme which reproduces polynomials up to degree 3 by only slightly increasing the support of its mask.

3.1.1 Notation and background

A (univariate) *Hermite subdivision operator* $H_{\mathcal{A}}$, based on the *matrix mask* $\mathcal{A} = \{A_l \in \mathbb{R}^{d\times d}, l \in \mathbb{Z}\}$, with *order* $d \geqslant 2$, acts on a sequence of Hermite data $f_n = \{\mathbf{f}_n(j),\ j \in \mathbb{Z}\}$ as

$$\mathbf{D}^{n+1}\mathbf{f}_{n+1}(i) = \sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{D}^n\mathbf{f}_n(j) \quad \forall\ i \in \mathbb{Z}, \quad n \geqslant 0, \tag{3.1}$$

where $\mathbf{D} = diag\left(1, \frac{1}{2}, \ldots, \frac{1}{2^{d-1}}\right)$. The *Hermite subdivision scheme*, still denoted by $H_{\mathcal{A}}$, is the repeated application of $H_{\mathcal{A}}$ when starting with an Hermite-type initial vector sequence composed of function and derivative values. We associate to $H_{\mathcal{A}}$ the *matrix symbol*

$$\mathbf{A}(z) = \sum_{l\in\mathbb{Z}} A_l z^l$$

and *sub-symbols*

$$\mathbf{A}_e(z) = \sum_{l\in\mathbb{Z}} A_{2l} z^{2l}, \quad \mathbf{A}_o(z) = \sum_{l\in\mathbb{Z}} A_{2l+1} z^{2l+1},$$

which are related by the equation

$$\mathbf{A}(z) = \mathbf{A}_e(z) + \mathbf{A}_o(z).$$

Their derivatives are defined as

$$\mathbf{A}^{(k)}(z) := \sum_{l\in\mathbb{Z}} \prod_{r=0}^{k-1} (l-r) A_l z^{l-k},$$

and

$$\mathbf{A}_e^{(k)}(z) := \sum_{l\in\mathbb{Z}} \prod_{r=0}^{k-1} (2l-r) A_{2l} z^{2l-k}, \quad \mathbf{A}_o^{(k)}(z) := \sum_{l\in\mathbb{Z}} \prod_{r=0}^{k-1} (2l+1-r) A_{2l+1} z^{2l+1-k},$$

respectively.

We are interested in both *primal* and *dual* Hermite schemes. From a geometric point of view, primal Hermite subdivision schemes are those that at each iteration retain or modify the given vectors and create a 'new' vector in between two 'old' ones. Dual schemes, instead, discard all given vectors after creating two new ones in between any pair of them. This fact is algebraically connected with the choice of the parameter values t_i^n, $i \in \mathbb{Z}$, to which we attach the vectors generated by the Hermite scheme. More precisely, the primal parametrisation is such that $t_i^n = \frac{i}{2^n}$ while the dual one is given by $t_i^n = \frac{i-\frac{1}{2}}{2^n}$. Therefore, we consider in this paper the parametrisation $t_i^n = \frac{i+t}{2^n}$ which includes primal and dual cases. We simply say that τ is the *parametrisation of the scheme* (see [10], for example). See Figure 3.1 for an illustration.

We continue with the notion of reproduction for Hermite schemes.

Definition 3.1. A Hermite subdivision scheme $H_{\mathcal{A}}$ with parametrisation τ *reproduces a function* $g \in C^d(\mathbb{R})$ if for any initial vector sequence $f_0 = \{\mathbf{f}_0(j) = [g(j+\tau), \ldots, g^{(d)}(j+\tau)]^T, \ j \in \mathbb{Z}\}$ the sequence $f_n = \{\mathbf{f}_n(j), \ j \in \mathbb{Z}\}$ defined by (3.1) is $\mathbf{f}_n(j) = [g((j+\tau)/2^n), \ldots, g^{(d)}((j+\tau)/2^n)]^T$ for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$.

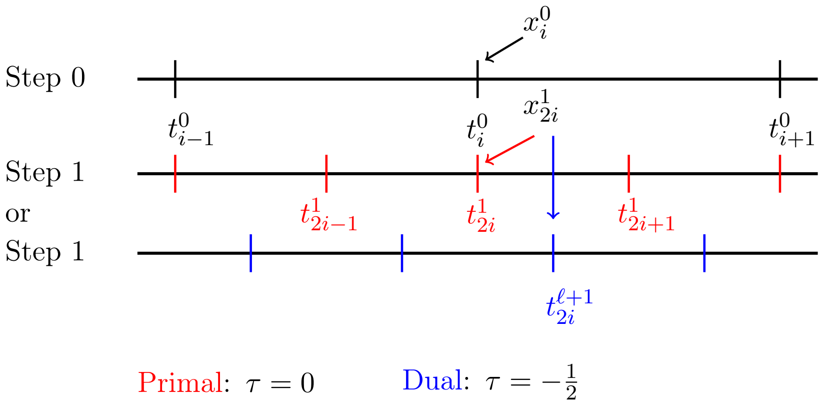

Figure 3.1: The figure illustrates the attachment of data points x_i^0 resp. x_{2i}^1 to parameter values $t_i^\ell \in \mathbb{R}$, $\ell = 0, 1$, for the first subdivision step. The primal parametrisation is shown in red, the dual one in blue.

3.1.2 Analysis of auxiliary polynomials

We study properties of auxiliary classes of polynomials which are needed to present the algebraic conditions characterising polynomial reproduction. In case of Hermite schemes of order $d = 2$, we need two different classes of polynomials. The first class already appears in [9]. Having defined the first class, the remaining class is closely related to it. We denote by $\prod_k$ the *set of polynomials* up to degree k.

Polynomials q_k

We start by defining the polynomials $q_k \in \prod_k$ as

$$q_0(x) := 1, \quad q_k(x) := \prod_{r=0}^{k-1}(2x - r), \quad k > 0. \tag{3.2}$$

Obviously, we can write them in terms of the monomial base of $\prod_k$, so that

$$q_k(-x) = \sum_{n=0}^{k} \gamma_n^k x^n, \quad \text{for some coefficients} \quad \gamma_n^k \in \mathbb{R}.$$

The reason why we expand $q_k(-x)$ instead of $q_k(x)$, will become clear later on. By definition $\gamma_k^k = (-1)^k 2^k$, hence $\gamma_k^k \neq 0$ for $k \geqslant 0$, while $\gamma_0^k = 0$ for all $k \geqslant 1$. For each $i \in \mathbb{Z}$ we define the polynomials

$$q_{0,i}(x) := 1, \quad q_{k,i}(x) := q_k\Big(x + \frac{i}{2}\Big) = \prod_{r=0}^{k-1}(2x + i - r), \quad k > 0, \tag{3.3}$$

which can also be written in terms of the monomial base as

$$q_{k,i}(-x) = \sum_{n=0}^{k} \gamma_n^{k,i} x^n, \quad \text{for some coefficients} \quad \gamma_n^{k,i} \in \mathbb{R}. \tag{3.4}$$

Obviously, $q_{k,0} = q_k$ and $\gamma_n^{k,0} = \gamma_n^k,\ n = 0, \ldots, k$.

Example 3.2. Computations show that

$$\begin{aligned}
q_{1,i}(x) &= 2x + i, \\
q_{2,i}(x) &= 4x^2 + (4i-2)x + i^2 - i, \\
q_{3,i}(x) &= 8x^3 + (12i-12)x^2 + (6i^2 - 12i + 4)x + i^3 - 3i^2 + 2i.
\end{aligned}$$

◇

In the next lemma we collect some relations between the coefficients of the polynomials $q_{k,i}$.

Lemma 3.3. *Let $i \in \mathbb{Z}$. For all $k \geqslant 1$ the coefficients of the polynomials $q_{k,i}$ in (3.4) satisfy*

$$\begin{aligned}
\gamma_0^{k,i} &= (i-(k-1))\gamma_0^{k-1,i}, \\
\gamma_n^{k,i} &= -2\gamma_{n-1}^{k-1,i} + (i-(k-1))\gamma_n^{k-1,i},\ n = 1, \ldots, k-1, \\
\gamma_k^{k,i} &= -2\gamma_{k-1}^{k-1,i}.
\end{aligned}$$

Proof. For $k = 1$ the claim is true by definition of the polynomials and its coefficients. For $k > 1$ it follows by (3.3) that $q_{k,i}(x) = q_{k-1,i}(x)(2x + i - (k-1))$. We obtain

$$\begin{aligned}
\sum_{n=0}^{k}(-1)^n\gamma_n^{k,i}x^n &= \sum_{n=0}^{k-1}(-1)^n\gamma_n^{k-1,i}x^n(2x+i-(k-1)) \\
&= 2\sum_{n=0}^{k-1}(-1)^n\gamma_n^{k-1,i}x^{n+1} + \sum_{n=0}^{k-1}(-1)^n(i-(k-1))\gamma_n^{k-1,i}x^n \\
&= 2\sum_{n=1}^{k}(-1)^{n-1}\gamma_{n-1}^{k-1,i}x^n + \sum_{n=0}^{k-1}(-1)^n(i-(k-1))\gamma_n^{k-1,i}x^n \\
&= 2(-1)^{k-1}\gamma_{k-1}^{k-1,i}x^k + \sum_{n=1}^{k-1}(2(-1)^{n-1}\gamma_{n-1}^{k-1,i} + (-1)^n(i-(k-1))\gamma_n^{k-1,i})x^n \\
&\quad + (i-(k-1))\gamma_0^{k-1,i}
\end{aligned}$$

by using (3.4). Comparison of the coefficients proves the lemma. □

Next, we study the relation of the coefficients of the polynomials $q_{k,i}$ (which do depend on $i \in \mathbb{Z}$) and those of the polynomials q_k, defined in (3.2) (which do not depend on $i \in \mathbb{Z}$). Let $\binom{r}{n} = \frac{r!}{(r-n)!n!}$ denote the binomial coefficient.

Lemma 3.4. *For $i \in \mathbb{Z}$ and $k \geqslant 0$, $\gamma_n^{k,i} = \sum_{r=n}^{k}(-1)^{r+n}\gamma_r^k\binom{r}{n}(\frac{i}{2})^{r-n}$, $n = 0,\ldots,k$. Moreover, $\gamma_k^{k,i} \neq 0$.*

Proof. Let $i \in \mathbb{Z}$. We have

$$\begin{aligned} q_k\Big(x+\frac{i}{2}\Big) &= \sum_{r=0}^{k}\gamma_r^k\Big(-x-\frac{i}{2}\Big)^r \\ &= \sum_{r=0}^{k}(-1)^r\gamma_r^k\sum_{n=0}^{r}\binom{r}{n}\Big(\frac{i}{2}\Big)^{r-n}x^n \\ &= \sum_{n=0}^{k}\sum_{r=n}^{k}(-1)^r\gamma_r^k\binom{r}{n}\Big(\frac{i}{2}\Big)^{r-n}x^n. \end{aligned}$$

By (3.4) a comparison of the coefficients leads to

$$\gamma_n^{k,i} = \sum_{r=n}^{k}(-1)^{r+n}\gamma_r^k\binom{r}{n}\Big(\frac{i}{2}\Big)^{r-n}, \qquad n = 0,\ldots,k. \tag{3.5}$$

Finally, since $\gamma_k^{k,i} = \gamma_k^k$ and $\gamma_k^k \neq 0$ the proof is complete. □

We conclude the subsection by stressing an important relation between the coefficients γ_n^k and the *Stirling numbers of the first kind* $\begin{bmatrix} k \\ n \end{bmatrix}$. Those numbers, well studied for example in [36], count the numbers of ways to arrange k elements into n cycles. From the initial conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$, $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0$, $n \geqslant 1$, they can be computed via the following recurrence relation:

$$\begin{bmatrix} k+1 \\ n \end{bmatrix} = k\begin{bmatrix} k \\ n \end{bmatrix} + \begin{bmatrix} k \\ n-1 \end{bmatrix}, \quad n \geqslant 1.$$

Following [36], the Stirling numbers of the first kind can also be described as coefficients in the expansion of the polynomial $x^{\underline{k}} = \prod_{r=0}^{k-1}(x-r)$ since $x^{\underline{k}} = \sum_{n=0}^{k}(-1)^{k-n}\begin{bmatrix} k \\ n \end{bmatrix}x^n$. By definition of the polynomials $q_k(-x)$ we therefore have

$$\gamma_n^k = (-1)^k 2^n \begin{bmatrix} k \\ n \end{bmatrix}, \quad n = 0,\ldots,k. \tag{3.6}$$

Polynomials $\tilde{q}_k$

We now define a second class of polynomials which is closely related to the polynomials $q_{k,i}$. First, we need to introduce the coefficients $\alpha_{k,\ell}$, for $\ell = 1,\ldots,k$, defined in a recursive way.

Definition 3.5. Let $k,\ \ell \in \mathbb{N},\ 1 \leqslant \ell \leqslant k$. We define the sequence of coefficients $\{\alpha_{k,\ell},\ \ell = 1,\dots,k\}$ as

$$\begin{aligned}
\alpha_{k,1} &:= 2k, \\
\alpha_{k,k-n+1} &:= (-1)^k 2^{-n+1}\Big(n\gamma_n^k - \sum_{j=1}^{k-n}(-1)^j \alpha_{k,j}\gamma_{n-1}^{k-j}\Big), \quad n = k-1,\dots,1.
\end{aligned}$$

For some explicit values of the coefficients, see Table 3.1. Due to (3.6), the recursive formula for $\alpha_{k,\ell}$ as given in Definition 3.5 above can be written as

$$\alpha_{k,k-n+1} = 2n \begin{bmatrix} k \\ n \end{bmatrix} - \sum_{j=1}^{k-n} \alpha_{k,j}^{1} \begin{bmatrix} k-j \\ n-1 \end{bmatrix}, \tag{3.7}$$

based on which we can prove the following Lemma.

Lemma 3.6. *Let $k,\ \ell \in \mathbb{N},\ 1 \leqslant \ell \leqslant k$. We have $\alpha_{k,\ell} = 2(\ell-1)!\binom{k}{\ell}$ for $\ell = 1,\dots,k$.*

Proof. We prove the statement by induction on ℓ. For $\ell = 1$, we have $\alpha_{k,1} = 2\binom{k}{1} = 2k$ which is true by Definition 3.5. For $k = 2$, we have $\alpha_{2,2} = 2\binom{2}{2} = 2$ which is also true. Now, assume that $k > 2$ and that the statement is true for some $\ell - 1$. We prove the statement for ℓ. Due to (3.7) we have

$$\begin{aligned}
\alpha_{k,\ell} &= 2(k+1-\ell)\begin{bmatrix} k \\ k+1-\ell \end{bmatrix} - \sum_{j=1}^{k-(k+1-\ell)} \alpha_{k,j}^{1} \begin{bmatrix} k-j \\ k+1-\ell-1 \end{bmatrix} \\
&= 2(k+1-\ell)\begin{bmatrix} k \\ k+1-\ell \end{bmatrix} - \sum_{j=1}^{\ell} 2(j-1)!\binom{k}{j}\begin{bmatrix} k-j \\ k-\ell \end{bmatrix} + 2(\ell-1)!\binom{k}{\ell},
\end{aligned}$$

where we use the induction hypothesis and the fact that $\begin{bmatrix} k-\ell \\ k-\ell \end{bmatrix} = 1$. It remains to show that

$$(k+1-\ell)\begin{bmatrix} k \\ k+1-\ell \end{bmatrix} = \sum_{j=1}^{\ell}(j-1)!\binom{k}{j}\begin{bmatrix} k-j \\ k-\ell \end{bmatrix}.$$

Using the fact that $\begin{bmatrix} j \\ 1 \end{bmatrix} = (j-1)!$ and the following identity, see [36, (6.29)],

$$\begin{bmatrix} n \\ 1+m \end{bmatrix}\binom{1+m}{\ell} = \sum_{j\in\mathbb{Z}} \begin{bmatrix} j \\ \ell \end{bmatrix}\begin{bmatrix} n-j \\ m \end{bmatrix}\binom{n}{j},$$

$k \backslash \ell$	1	2	3	4	5	6	7
1	2						
2	4	2					
3	6	6	4				
4	8	12	16	12			
5	10	20	40	60	48		
6	12	30	80	180	288	240	
7	14	42	140	420	1008	1680	1440

Table 3.1: Values of the coefficients $\alpha_{k,\ell},\ k = 1,\dots,7,\ \ell = 1,\dots,k$.

we conclude (in consideration of the terms that are actually equal to zero)

$$\begin{aligned}(k+1-\ell)\begin{bmatrix}k\\k+1-\ell\end{bmatrix} = \binom{k+1-\ell}{1}\begin{bmatrix}k\\k+1-\ell\end{bmatrix} &= \sum_{j=1}^{\ell}\begin{bmatrix}j\\1\end{bmatrix}\begin{bmatrix}k-j\\k-\ell\end{bmatrix}\binom{k}{j}\\ &= \sum_{j=1}^{\ell}(j-1)!\binom{k}{j}\begin{bmatrix}k-j\\k-\ell\end{bmatrix}.\end{aligned}$$

□

Based on the previous set of coefficients, for $i \in \mathbb{Z}$ and $k \geqslant 0$, we define $\tilde{q}_{k,i} \in \prod_{k-1}$ as

$$\tilde{q}_{0,i} := 0, \quad \tilde{q}_{k,i}(x) := \sum_{n=1}^{k}(-1)^n \alpha_{k,n} q_{k-n,i}(x), \quad k > 0, \quad i \in \mathbb{Z}. \tag{3.8}$$

As done before, for $k \geqslant 1$, we can write them in the form

$$\tilde{q}_{k,i}(-x) = \sum_{n=0}^{k-1}\tilde{\gamma}_n^{k,i}x^n, \quad \text{for some coefficients} \quad \tilde{\gamma}_n^{k,i} \in \mathbb{R}. \tag{3.9}$$

Lemma 3.7. *We have* $\tilde{\gamma}_{k-1}^{k,i} = k\gamma_k^{k,i} \neq 0$ *and* $\tilde{\gamma}_n^{k,i} = \sum_{j=1}^{k-n}(-1)^j\alpha_{k,j}\gamma_n^{k-j,i}$, *for* $k \in \mathbb{N}$, $i \in \mathbb{Z}$ *and* $n = 0,\dots,k-1$.

Proof. By definition of $\tilde{q}_{k,i}$ in (3.8), we obtain $\tilde{q}_{k,i}(-x) = \sum_{n=1}^{k}(-1)^n\alpha_{k,n}q_{k-n,i}(-x)$. Thus, the coefficient $\tilde{\gamma}_{k-1}^{k,i}$, which belongs to the x^{k-1} term, is given by

$$\tilde{\gamma}_{k-1}^{k,i} = -\alpha_{k,1}\gamma_{k-1}^{k-1,i} = k\gamma_k^{k,i}.$$

Here the last equality follows by Lemma 3.3. By Lemma 3.4 $\gamma_k^{k,i} \neq 0$. We compute

$$\begin{aligned}\tilde{q}_{k,i}(-x) &= \sum_{n=1}^{k}(-1)^n \alpha_{k,n} \sum_{j=0}^{k-n} \gamma_j^{k-n,i} x^j \\ &= \sum_{j=0}^{k-1}\left(\sum_{n=1}^{k-j}(-1)^n \alpha_{k,n}\gamma_j^{k-n,i}\right) x^j.\end{aligned}$$

This proves the second part of the lemma. □

We compare the coefficients of the polynomials $q_{k,i}$ and $\tilde{q}_{k,i}$ defined in (3.3) and (3.8).

Proposition 3.8. *The coefficients of the polynomials $q_{k,i}(x)$ and $\tilde{q}_{k,i}(x)$ satisfy the relation $\gamma_n^{k,i} = \frac{1}{n}\tilde{\gamma}_{n-1}^{k,i}$ for all $k \in \mathbb{N}$, $i \in \mathbb{Z}$ and $n = 1, \dots, k$.*

Proof. By Lemma 3.7 the claim of this lemma is equivalent to

$$\gamma_n^{k,i} = \frac{1}{n}\sum_{j=1}^{k-n+1}(-1)^j \alpha_{k,j}\gamma_{n-1}^{k-j,i}, \quad \text{for } i \in \mathbb{Z} \text{ and } n = 1, \dots, k. \tag{3.10}$$

First, let $n = k$. Using Lemma 3.3 and the fact that $\alpha_{k,1} = 2k$ we obtain

$$\gamma_k^{k,i} = -2\gamma_{k-1}^{k-1,i} = -\frac{1}{k}\alpha_{k,1}\gamma_{k-1}^{k-1,i}.$$

This proves (3.23) for $n = k$ and all $i \in \mathbb{Z}$. Now, let $n \in \{1, \dots, k-1\}$. We have

$$\alpha_{k,k-n+1} = (-1)^k 2^{-n+1}\Big(n\gamma_n^k - \sum_{j=1}^{k-n}(-1)^j \alpha_{k,j}\gamma_{n-1}^{k-j}\Big)$$

by Definition 3.5. Using the fact that $\gamma_{n-1}^{n-1} = (-1)^{n-1}2^{n-1}$ leads to

$$\begin{aligned}n\gamma_n^k &= (-1)^k 2^{n-1}\alpha_{k,k-n+1} + \sum_{j=1}^{k-n}(-1)^j \alpha_{k,j}\gamma_{n-1}^{k-j} \\ &= (-1)^{k-n+1}\alpha_{k,k-n+1}\gamma_{n-1}^{n-1} + \sum_{j=1}^{k-n}(-1)^j \alpha_{k,j}\gamma_{n-1}^{k-j} \\ &= \sum_{j=1}^{k-n+1}(-1)^j \alpha_{k,j}\gamma_{n-1}^{k-j}.\end{aligned}$$

Next, we show that this implies that (3.23) is true for any $i \in \mathbb{Z}$. Let $n \in \{1, \dots, k\}$. From above for any $r \in \{n, ..., k\}$ we just saw that

$$r\gamma_r^k = \sum_{j=1}^{k-r+1}(-1)^j \alpha_{k,j}\gamma_{r-1}^{k-j}.$$

Since $r\binom{r-1}{n-1} = n\binom{r}{n}$, the latter implies that

$$n\binom{r}{n}\gamma_r^k = \sum_{j=1}^{k-r+1}(-1)^j\alpha_{k,j}\binom{r-1}{n-1}\gamma_{r-1}^{k-j}.$$

Multiplying by $(-1)^{n+r}\left(\frac{i}{2}\right)^{r-n}$ on both sides and summing up for r from n to k gives

$$\begin{aligned}
n\sum_{r=n}^{k}(-1)^{r+n}\gamma_r^k\binom{r}{n}\left(\frac{i}{2}\right)^{r-n} &= \sum_{r=n}^{k}(-1)^{r+n}\left(\sum_{j=1}^{k-r+1}(-1)^j\alpha_{k,j}\binom{r-1}{n-1}\gamma_{r-1}^{k-j}\left(\frac{i}{2}\right)^{r-n}\right)\\
&= \sum_{r=n-1}^{k-1}\left(\sum_{j=1}^{k-r}(-1)^{r+n-1}(-1)^j\alpha_{k,j}\binom{r}{n-1}\gamma_r^{k-j}\left(\frac{i}{2}\right)^{r-(n-1)}\right)\\
&= \sum_{j=1}^{k-n+1}(-1)^j\alpha_{k,j}\left(\sum_{r=n-1}^{k-j}(-1)^{r+n-1}\binom{r}{n-1}\gamma_r^{k-j}\left(\frac{i}{2}\right)^{r-(n-1)}\right).
\end{aligned}$$

By (3.5) this implies that

$$\gamma_n^{k,i} = \frac{1}{n}\sum_{j=1}^{k-n+1}(-1)^j\alpha_{k,j}\gamma_{n-1}^{k-j,i},$$

which concludes the proof. □

3.1.3 Polynomial reproduction of Hermite schemes of order $d = 2$

We are finally in a position to give our algebraic conditions on the mask of a Hermite subdivision scheme of order $d = 2$ which ensures polynomial reproduction up to degree m. The main result is stated below but its proof is split into several Lemmas and Propositions. Let $\mathbf{e}_{s,d}$ denote the s-th canonical vector of $\mathbb{R}^d$ and $\mathbf{0}_d \in \mathbb{R}^d$ the zero vector.

Theorem 3.9. *Let $H_{\mathcal{A}}$ be a Hermite subdivision scheme with parametrisation τ. Then, $H_{\mathcal{A}}$ reproduces constants if and only if*

$$\boldsymbol{A}(-1)\boldsymbol{e}_{1,2} = \boldsymbol{0}_2, \tag{3.11}$$

$$\boldsymbol{A}(1)\boldsymbol{e}_{1,2} = 2\boldsymbol{e}_{1,2}. \tag{3.12}$$

Moreover, $H_{\mathcal{A}}$ reproduces polynomials up to degree $m \geqslant 1$ if and only if it reproduces constants and

$$\boldsymbol{A}^{(k)}(-1)\boldsymbol{e}_{1,2} + \sum_{\ell=1}^{k}\alpha_{k,\ell}\cdot\boldsymbol{A}^{(k-\ell)}(-1)\boldsymbol{e}_{2,2} = \boldsymbol{0}_2, \tag{3.13}$$

$$\boldsymbol{A}^{(k)}(1)\boldsymbol{e}_{1,2} + \sum_{\ell=1}^{k}\tilde{\alpha}_{k,\ell}^{1}\cdot\boldsymbol{A}^{(k-\ell)}(1)\boldsymbol{e}_{2,2} = \begin{bmatrix}2q_{k,2\tau}(-\frac{\tau}{2})\\ \tilde{q}_{k,2\tau}(-\frac{\tau}{2})\end{bmatrix}, \tag{3.14}$$

for all $k = 1,\dots,m$ with $\tilde{\alpha}_{k,\ell}^{1} = (-1)^{\ell}\alpha_{k,\ell}$, $\ell = 1,\dots,k$, and $\alpha_{k,\ell}$ as in Definition 3.5.

We prove the first part of Theorem 3.9 by presenting it as a separated Lemma. First, some important observations are made.

Remark 3.10. It is worthwhile to remark that:

1) Up to the reproduction of linear polynomials the algebraic conditions given in the theorem above are also given in [66], though presented in a different way.
2) The entries of the right-hand side (3.14) (with the convention $\prod_{r=0}^{-1} := 1$) are

$$2q_{k,2\tau}\left(-\frac{\tau}{2}\right) = 2\prod_{r=0}^{k-1}(\tau - r), \quad \tilde{q}_{k,2\tau}\left(-\frac{\tau}{2}\right) = \sum_{n=1}^{k}(-1)^n \alpha_{k,n}^1 \prod_{r=0}^{k-n-1}(\tau - r).$$

3) When $m = 1$ the previous result allows us to identify the correct parametrisation corresponding to the choice $\tau = \frac{1}{2}(\mathbf{A}^{(1)}(1)_{11} - 2\mathbf{A}^{(0)}(1)_{12})$.

Lemma 3.11. *A Hermite subdivision scheme $H_{\mathcal{A}}$ reproduces constants if and only if (3.11) and (3.12) are satisfied.*

Proof. Obviously, the reproduction of constants is equivalent to $\sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{e}_{1,2} = \mathbf{e}_{1,2}$ for all $i \in \mathbb{Z}$. Now, from (3.11) and (3.12) we have

$$\begin{aligned} 2\mathbf{e}_{1,2} &= (\mathbf{A}(1) + \mathbf{A}(-1))\mathbf{e}_{1,2} = 2\sum_{i\in\mathbb{Z}} A_{2i}\mathbf{e}_{1,2} \qquad \text{and} \\ 2\mathbf{e}_{1,2} &= (\mathbf{A}(1) - \mathbf{A}(-1))\mathbf{e}_{1,2} = 2\sum_{i\in\mathbb{Z}} A_{2i+1}\mathbf{e}_{1,2}, \end{aligned}$$

which are equivalent to the previous relation specialised for i even and i odd respectively. ☐

Note that the reproduction of constants does not depend on the chosen parametrisation. This is not surprising, since it is so in the scalar situation as well, see [9].

Lemma 3.12. *Let $m \geqslant 1$. Then, condition (3.13) is satisfied if and only if*

$$\begin{aligned} \boldsymbol{A}_e^{(k)}(1)\boldsymbol{e}_{1,2} + \sum_{\ell=1}^{k}(-1)^\ell \alpha_{k,\ell}\boldsymbol{A}_e^{(k-\ell)}(1)\boldsymbol{e}_{2,2} &= \frac{1}{2}\Big(\boldsymbol{A}^{(k)}(1)\boldsymbol{e}_{1,2} + \sum_{\ell=1}^{k}(-1)^\ell \alpha_{k,\ell}\boldsymbol{A}^{(k-\ell)}(1)\boldsymbol{e}_{2,2}\Big), \\ \boldsymbol{A}_o^{(k)}(1)\boldsymbol{e}_{1,2} + \sum_{\ell=1}^{k}(-1)^\ell \alpha_{k,\ell}\boldsymbol{A}_o^{(k-\ell)}(1)\boldsymbol{e}_{2,2} &= \frac{1}{2}\Big(\boldsymbol{A}^{(k)}(1)\boldsymbol{e}_{1,2} + \sum_{\ell=1}^{k}(-1)^\ell \alpha_{k,\ell}\boldsymbol{A}^{(k-\ell)}(1)\boldsymbol{e}_{2,2}\Big), \end{aligned}$$

for all $k = 1, \ldots, m$. Moreover, condition (3.11) is satisfied if and only if

$$\boldsymbol{A}_e(1)\boldsymbol{e}_{1,2} = \boldsymbol{A}_o(1)\boldsymbol{e}_{1,2} = \frac{1}{2}\boldsymbol{A}(1)\boldsymbol{e}_{1,2}.$$

Proof. Let $k \in \{1, \dots, m\}$. We have $\mathbf{A}^{(k)}(z) = \mathbf{A}_e^{(k)}(z) + \mathbf{A}_o^{(k)}(z)$ and therefore especially

$$\mathbf{A}^{(k)}(1) = \mathbf{A}_e^{(k)}(1) + \mathbf{A}_o^{(k)}(1), \tag{3.15}$$

and

$$\mathbf{A}^{(k)}(-1) = (-1)^k \mathbf{A}_e^{(k)}(1) + (-1)^{k+1} \mathbf{A}_o^{(k)}(1).$$

So, condition (3.13) is equivalent to

$$\mathbf{A}_e^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}_e^{(k-\ell)}(1)\mathbf{e}_{2,2} = \mathbf{A}_o^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}_o^{(k-\ell)}(1)\mathbf{e}_{2,2}. \tag{3.16}$$

Now, using (3.15) we write

$$\begin{aligned}
&\mathbf{A}^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}^{(k-\ell)}(1)\mathbf{e}_{2,2} \\
&= \mathbf{A}_e^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}_e^{(k-\ell)}(1)\mathbf{e}_{2,2} + \mathbf{A}_o^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}_o^{(k-\ell)}(1)\mathbf{e}_{2,2},
\end{aligned}$$

which, together with (3.16), proves the first part of the Lemma. Condition (3.11) is equivalent to $\mathbf{A}_e(1)\mathbf{e}_{1,2} = \mathbf{A}_o(1)\mathbf{e}_{1,2}$ and since $\mathbf{A}(1) = \mathbf{A}_e(1) + \mathbf{A}_o(1)$, the claim is proved. ☐

In the following we make use of the polynomials $q_{k,i}$ and $\tilde{q}_{k,i}$ introduced in the previous section. First, we unite them into the vector polynomial

$$Q_{k,i}(x) := \begin{bmatrix} q_{k,i}(x) \\ \tilde{q}_{k,i}(x) \end{bmatrix} \quad \text{with} \quad k \geqslant 0, \ i \in \mathbb{Z}.$$

Proposition 3.13. *Let $m \geqslant 1$. Then, conditions (3.13) and (3.14) are satisfied if and only if for all $i \in \mathbb{Z}$ and $\tau \in \mathbb{R}$,*

$$\sum_{j \in \mathbb{Z}} A_{i-2j} Q_{k,i+2\tau}(-j-\tau) = \begin{bmatrix} q_{k,i+2\tau}\left(\frac{-i-\tau}{2}\right) \\ \frac{1}{2}\tilde{q}_{k,i+2\tau}\left(\frac{-i-\tau}{2}\right) \end{bmatrix}, \qquad k = 1, \dots, m. \tag{3.17}$$

Especially, conditions (3.11) and (3.12) are satisfied if and only if $\sum_{j\in\mathbb{Z}} A_{i-2j} Q_{0,i}(-j) = \mathbf{e}_{1,2}$ for all $i \in \mathbb{Z}$.

Proof. Observe that by definition of the class of polynomials in (3.3) we obtain

$$\begin{aligned}
\mathbf{A}_e^{(k)}(1) &= \sum_{j\in\mathbb{Z}} q_{k,2(j-t)}(t) A_{2j} = \sum_{j\in\mathbb{Z}} q_{k,2t+2\tau}(-j-\tau) A_{2(t-j)}, \\
\mathbf{A}_o^{(k)}(1) &= \sum_{j\in\mathbb{Z}} q_{k,2(j-t)+1}(t) A_{2j+1} = \sum_{j\in\mathbb{Z}} q_{k,2t+2\tau+1}(-j-\tau) A_{2(t-j)+1},
\end{aligned}$$

for all $t \in \mathbb{Z}$ and $\tau \in \mathbb{R}$. Let $i \in 2\mathbb{Z}$ with $i = 2s$ for some $s \in \mathbb{Z}$. This observation together with Lemma 3.12 implies

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} A_{i-2j} Q_{k,i+2\tau}(-j-\tau) &= \sum_{j\in\mathbb{Z}} A_{2(s-j)} Q_{k,2s+2\tau}(-j-\tau) \\
&= \sum_{j\in\mathbb{Z}} q_{k,2s+2\tau}(-j-\tau) A_{2(s-j)} \mathbf{e}_{1,2} + \sum_{j\in\mathbb{Z}} \tilde{q}_{k,2s+2\tau}(-j-\tau) A_{2(s-j)} \mathbf{e}_{2,2} \\
&= \sum_{j\in\mathbb{Z}} q_{k,2s+2\tau}(-j-\tau) A_{2(s-j)} \mathbf{e}_{1,2} + \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} q_{k-\ell,2s+2\tau}(-j-\tau) A_{2(s-j)} \mathbf{e}_{2,2} \\
&= \mathbf{A}_e^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}_e^{(k-\ell)}(1) \mathbf{e}_{2,2} \\
&= \frac{1}{2}\Big(\mathbf{A}^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}^{(k-\ell)}(1) \mathbf{e}_{2,2} \Big) \\
&= \begin{bmatrix} q_{k,i+2\tau}\left(\frac{-i-\tau}{2}\right) \\ \frac{1}{2}\tilde{q}_{k,i+2\tau}\left(\frac{-i-\tau}{2}\right) \end{bmatrix},
\end{aligned}$$

showing the claim for i even. Similarly, for odd $i \in \mathbb{Z}$, $i = 2s+1$, we obtain that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} A_{i-2j} Q_{k,i+2\tau}(-j-\tau) &= \sum_{j\in\mathbb{Z}} A_{2(s-j)+1} Q_{k,2s+2\tau+1}(-j-\tau) \\
&= \sum_{j\in\mathbb{Z}} q_{k,2s+2\tau+1}(-j-\tau) A_{2(s-j)+1} \mathbf{e}_{1,2} + \sum_{j\in\mathbb{Z}} \tilde{q}_{k,2s+2\tau+1}(-j-\tau) A_{2(s-j)+1} \mathbf{e}_{2,2} \\
&= \sum_{j\in\mathbb{Z}} q_{k,2s+2\tau+1}(-j-\tau) A_{2(s-j)+1} \mathbf{e}_{1,2} \\
&\quad + \sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} q_{k-\ell,2s+2\tau+1}(-j-\tau) A_{2(s-j)+1} \mathbf{e}_{2,2} \\
&= \mathbf{A}_o^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}_o^{(k-\ell)}(1) \mathbf{e}_{2,2} \\
&= \frac{1}{2}\Big(\mathbf{A}^{(k)}(1)\mathbf{e}_{1,2} + \sum_{\ell=1}^{k} (-1)^\ell \alpha_{k,\ell} \mathbf{A}^{(k-\ell)}(1) \mathbf{e}_{2,2} \Big) \\
&= \begin{bmatrix} q_{k,i+2\tau}\left(\frac{-i-\tau}{2}\right) \\ \frac{1}{2}\tilde{q}_{k,i+2\tau}\left(\frac{-i-\tau}{2}\right) \end{bmatrix}.
\end{aligned}$$

The second part of the Proposition follows by Lemma 3.11 and $q_{0,i} = 1$ and $\tilde{q}_{0,i} = 0$. □

Note that the right-hand side (3.17) does not depend on $i \in \mathbb{Z}$ since $q_{k,i+2\tau}\left(\frac{-i-\tau}{2}\right) = q_{k,2\tau}\left(\frac{-\tau}{2}\right)$.

Proposition 3.14. *Let $H_{\mathcal{A}}$ be a Hermite subdivision scheme with parametrisation τ and $m \geqslant 0$. Then, $H_{\mathcal{A}}$ satisfies conditions* (3.11) – (3.12) *and conditions (3.13) – (3.14) for all $k = 1, \dots, m$, if and only if*

$$\sum_{j\in\mathbb{Z}} A_{i-2j}\boldsymbol{e}_{1,2} = \boldsymbol{e}_{1,2}, \quad i \in \mathbb{Z},$$

$$\sum_{j\in\mathbb{Z}} A_{i-2j}\begin{bmatrix}(j+\tau)^k\\ k(j+\tau)^{k-1}\end{bmatrix} = \frac{1}{2^k}\begin{bmatrix}(i+\tau)^k\\ k(i+\tau)^{k-1}\end{bmatrix}, \quad k = 1,\dots,m, \quad i \in \mathbb{Z}, \tag{3.18}$$

with the convention that (3.18) *(resp. (3.13) – (3.14)) is empty if $m = 0$.*

Proof. We prove the proposition by induction over m. The case $m = 0$ follows by Lemma 3.11. Assume that the statement is true for some $m-1$ and all $k = 1,\dots,m-1$. The proof uses the representations of the polynomials $q_{k,\ell}(x)$ and $\tilde{q}_{k,\ell}(x)$ as in (3.4) and (3.9). For $i \in \mathbb{Z}$, using Proposition 3.13 we obtain

$$\begin{aligned}
&\begin{bmatrix} q_{m,i+2\tau}\left(\frac{-i-\tau}{2}\right)\\ \frac{1}{2}\tilde{q}_{m,i+2\tau}\left(\frac{-i-\tau}{2}\right)\end{bmatrix} = \sum_{j\in\mathbb{Z}} A_{i-2j}Q_{m,i+2\tau}(-j-\tau)\\
&= \sum_{j\in\mathbb{Z}} A_{i-2j}q_{m,i+2\tau}(-j-\tau)\mathbf{e}_{1,2} + \sum_{j\in\mathbb{Z}} A_{i-2j}\tilde{q}_{m,i+2\tau}(-j-\tau)\mathbf{e}_{2,2}\\
&= \sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{n=0}^{m}\gamma_n^{m,i+2\tau}\begin{bmatrix}(j+\tau)^n\\ 0\end{bmatrix} + \sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{n=0}^{m-1}\tilde{\gamma}_n^{m,i+2\tau}\begin{bmatrix}0\\ (j+\tau)^n\end{bmatrix}\\
&= \gamma_m^{m,i+2\tau}\sum_{j\in\mathbb{Z}} A_{i-2j}\begin{bmatrix}(j+\tau)^m\\ m(j+\tau)^{m-1}\end{bmatrix}\\
&\quad + \underbrace{\sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{n=0}^{m-1}\gamma_n^{m,i+2\tau}\begin{bmatrix}(j+\tau)^n\\ 0\end{bmatrix} + \sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{n=0}^{m-2}\tilde{\gamma}_n^{m,i+2\tau}\begin{bmatrix}0\\ (j+\tau)^n\end{bmatrix}}_{(*)}.
\end{aligned}$$

Note that we used the relation $\tilde{\gamma}_{m-1}^{m,i+2\tau} = m\gamma_m^{m,i+2\tau}$ to obtain the last equality above, see Proposition 3.8.

Before we apply the induction hypothesis to $(*)$ we apply Proposition 3.8 again and get

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{n=0}^{m-1}\gamma_n^{m,i+2\tau}\begin{bmatrix}(j+\tau)^n\\0\end{bmatrix}+\sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{n=0}^{m-2}\tilde{\gamma}_n^{m,i+2\tau}\begin{bmatrix}0\\(j+\tau)^n\end{bmatrix}\\
&=\gamma_0^{m,i+2\tau}\sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{e}_{1,2}\\
&\quad+\sum_{n=0}^{m-2}\left(\gamma_{n+1}^{m,i+2\tau}\sum_{j\in\mathbb{Z}} A_{i-2j}\begin{bmatrix}(j+\tau)^{n+1}\\0\end{bmatrix}+\tilde{\gamma}_n^{m,i+2\tau}\sum_{j\in\mathbb{Z}} A_{i-2j}\begin{bmatrix}0\\(j+\tau)^n\end{bmatrix}\right)\\
&=\gamma_0^{m,i+2\tau}\sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{e}_{1,2}+\sum_{n=0}^{m-2}\gamma_{n+1}^{m,i+2\tau}\sum_{j\in\mathbb{Z}} A_{i-2j}\begin{bmatrix}(j+\tau)^{n+1}\\(n+1)(j+\tau)^n\end{bmatrix}.
\end{aligned}
$$

Now we use the assumption that the scheme reproduces constants for the first part of the right-hand side and apply the induction hypothesis for $k=1,\dots,m-1$ to the second part. Therefore,

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{n=0}^{m-1}\gamma_n^{m,i+2\tau}\begin{bmatrix}(j+\tau)^n\\0\end{bmatrix}+\sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{n=0}^{m-2}\tilde{\gamma}_n^{m,i+2\tau}\begin{bmatrix}0\\(j+\tau)^n\end{bmatrix}\\
&\qquad=\underbrace{\gamma_0^{m,i+2\tau}\mathbf{e}_{1,2}+\sum_{n=0}^{m-2}\frac{1}{2^{n+1}}\gamma_{n+1}^{m,i+2\tau}\begin{bmatrix}(i+\tau)^{n+1}\\(n+1)(i+\tau)^n\end{bmatrix}}_{(**)}.
\end{aligned}
$$

The next step is to rewrite the sum $(**)$ by first applying Proposition 3.8 and then using the definition of the polynomial $q_{m,i}$ (resp. $\tilde{q}_{m,i}$) as in (3.4) (resp. (3.9)). So,

$$
\begin{aligned}
&\gamma_0^{m,i+2\tau}\mathbf{e}_{1,2}+\sum_{n=0}^{m-2}\frac{1}{2^{n+1}}\gamma_{n+1}^{m,i+2\tau}\begin{bmatrix}(i+\tau)^{n+1}\\(n+1)(i+\tau)^n\end{bmatrix}\\
&\quad=\gamma_0^{m,i+2\tau}\mathbf{e}_{1,2}+\sum_{n=0}^{m-1}\gamma_{n+1}^{m,i+2\tau}\left(\frac{i+\tau}{2}\right)^{n+1}\mathbf{e}_{1,2}+\frac{1}{2}\sum_{n=0}^{m-1}\tilde{\gamma}_n^{m,i+2\tau}\left(\frac{i+\tau}{2}\right)^n\mathbf{e}_{2,2}\\
&\qquad-\gamma_m^{m,i+2\tau}\left(\frac{i+\tau}{2}\right)^m\mathbf{e}_{1,2}-\frac{1}{2}\tilde{\gamma}_{m-1}^{m,i+2\tau}\left(\frac{i+\tau}{2}\right)^{m-1}\mathbf{e}_{2,2}\\
&\quad=\gamma_0^{m,i+2\tau}\mathbf{e}_{1,2}+q_{m,i+2\tau}\Big(-\frac{i+\tau}{2}\Big)\mathbf{e}_{1,2}-\gamma_0^{m,i+2\tau}\mathbf{e}_{1,2}+\frac{1}{2}\tilde{q}_{m,i+2\tau}\Big(-\frac{i+\tau}{2}\Big)\mathbf{e}_{2,2}\\
&\qquad-\gamma_m^{m,i+2\tau}\left(\frac{i+\tau}{2}\right)^m\mathbf{e}_{1,2}-\frac{1}{2}\tilde{\gamma}_{m-1}^{m,i+2\tau}\left(\frac{i+\tau}{2}\right)^{m-1}\mathbf{e}_{2,2}\\
&\quad=q_{m,i+2\tau}\Big(-\frac{i+\tau}{2}\Big)\mathbf{e}_{1,2}+\frac{1}{2}\tilde{q}_{m,i+2\tau}\Big(-\frac{i+\tau}{2}\Big)\mathbf{e}_{2,2}-\gamma_m^{m,i+2\tau}\left(\frac{i+\tau}{2}\right)^m\mathbf{e}_{1,2}\\
&\qquad-\frac{1}{2}\tilde{\gamma}_{m-1}^{m,i+2\tau}\left(\frac{i+\tau}{2}\right)^{m-1}\mathbf{e}_{2,2}.
\end{aligned}
$$

We apply Proposition 3.8 to the right-hand side above to obtain

$$\begin{aligned}\gamma_0^{m,i+2\tau}\mathbf{e}_{1,2} + \sum_{n=0}^{m-2} \frac{1}{2^{n+1}}\gamma_{n+1}^{m,i+2\tau} \begin{bmatrix} (i+\tau)^{n+1} \\ (n+1)(i+\tau)^n \end{bmatrix} \\ = \begin{bmatrix} q_{m,i+2\tau}\left(\frac{-i-\tau}{2}\right) \\ \frac{1}{2}\tilde{q}_{m,i+2\tau}\left(\frac{-i-\tau}{2}\right) \end{bmatrix} - \gamma_m^{m,i+2\tau}\left(\frac{1}{2}\right)^m \begin{bmatrix} (i+\tau)^m \\ m(i+\tau)^{m-1} \end{bmatrix}.\end{aligned}$$

Summarising our previous computations leads to

$$\gamma_m^{m,i+2\tau}\sum_{j\in\mathbb{Z}} A_{i-2j} \begin{bmatrix} (j+\tau)^m \\ m(j+\tau)^{m-1} \end{bmatrix} = \gamma_m^{m,i+2\tau}\left(\frac{1}{2}\right)^m \begin{bmatrix} (i+\tau)^m \\ m(i+\tau)^{m-1} \end{bmatrix}.$$

Since $\gamma_m^{m,i+2\tau} \neq 0$, this is equivalent to

$$\sum_{j\in\mathbb{Z}} A_{i-2j} \begin{bmatrix} (j+\tau)^m \\ m(j+\tau)^{m-1} \end{bmatrix} = \left(\frac{1}{2}\right)^m \begin{bmatrix} (i+\tau)^m \\ m(i+\tau)^{m-1} \end{bmatrix},$$

which concludes the induction step. □

Remark 3.15. If $m = 1$ the sums in the proof above which are not defined are assumed to be zero. The conclusion of the proposition in this case is still true.

We are finally in a position to prove Theorem 3.9.

Proof of Theorem 3.9. We prove the statement by induction over the degree m of the polynomials. For $m = 0$ we refer to Lemma 3.11. So, for $m \geqslant 1$ assume the statement is true for some $m-1$ and all $k = 0, \dots, m-1$ and show that the Hermite subdivision scheme $H_{\mathcal{A}}$ reproduces polynomials of degree m. Let $p(x) = x^m + g(x)$ with $g(x) \in \prod_{m-1}$. By Definition 3.1 we have to show that for $n \in \mathbb{N}$ and $i \in \mathbb{Z}$,

$$\mathbf{f}_n(i) = \begin{bmatrix} p((i+\tau)/2^n) \\ p'((i+\tau)/2^n) \end{bmatrix} \quad \Longrightarrow \quad \mathbf{f}_{n+1}(i) = \begin{bmatrix} p((i+\tau)/2^{n+1}) \\ p'((i+\tau)/2^{n+1}) \end{bmatrix}.$$

Let $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. By (3.1) we have

$$\mathbf{D}^{n+1}\mathbf{f}_{n+1}(i) = \sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{D}^n \begin{bmatrix} ((j+\tau)/2^n)^m \\ m((j+\tau)/2^n)^{m-1} \end{bmatrix} + \sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{D}^n \begin{bmatrix} g((j+\tau)/2^n) \\ g'((j+\tau)/2^n) \end{bmatrix}.$$

This is equivalent to

$$2^{nm}\mathbf{D}^{n+1}\mathbf{f}_{n+1}(i) = \sum_{j\in\mathbb{Z}} A_{i-2j} \begin{bmatrix} (j+\tau)^m \\ m(j+\tau)^{m-1} \end{bmatrix} + 2^{nm}\sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{D}^n \begin{bmatrix} g((j+\tau)/2^n) \\ g'((j+\tau)/2^n) \end{bmatrix}.$$

Now, we apply Proposition 3.14 to the first summand of the right-hand side above and the induction hypothesis to the second. We obtain

$$2^{nm}\mathbf{D}^{n+1}\mathbf{f}_{n+1}(i) = \frac{1}{2^m} \begin{bmatrix} (i+\tau)^m \\ m(i+\tau)^{m-1} \end{bmatrix} + 2^{nm}\mathbf{D}^{n+1} \begin{bmatrix} g((i+\tau)/2^{n+1}) \\ g'((i+\tau)/2^{n+1}) \end{bmatrix}.$$

So, we see that

$$D^{n+1}\mathbf{f}_{n+1}(i) = \begin{bmatrix} 2^{-(n+1)m}(i+\tau)^m + g((i+\tau)/2^{n+1}) \\ 2^{-(n+1)m}m(i+\tau)^{m-1} + 2^{-(n+1)}g'((i+\tau)/2^{n+1}) \end{bmatrix}.$$

This is equivalent to

$$\mathbf{f}_{n+1}(i) = \begin{bmatrix} ((i+\tau)/2^{n+1})^m + g((i+\tau)/2^{n+1}) \\ m((i+\tau)/2^{n+1})^{m-1} + g'((i+\tau)/2^{n+1}) \end{bmatrix},$$

proving the claim. □

3.1.4 Examples of Hermite schemes of order $d = 2$

We give some examples to illustrate how to use the algebraic conditions obtained in Theorem 3.9. Moreover, we show how to use our results to modify known Hermite schemes such that they reproduce polynomials of higher degree.

Interpolatory scheme

We start with the interpolatory Hermite subdivision scheme $H_{\mathcal{A}}$ introduced by Merrien in [57]. The non-zero coefficients of its mask are given by

$$A_{-1} = \begin{bmatrix} \frac{1}{2} & \lambda \\ \frac{1}{2}(1-\mu) & \frac{\mu}{4} \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad A_1 = \begin{bmatrix} \frac{1}{2} & -\lambda \\ \frac{1}{2}(\mu-1) & \frac{\mu}{4} \end{bmatrix}.$$

It is known that the scheme reproduces polynomials of degree 1 for all λ, $\mu \in \mathbb{R}$. It reproduces Π_2 if and only if $\lambda = -\frac{1}{8}$. Moreover, it also reproduces polynomials of degree 3, if additionally $\mu = -\frac{1}{2}$.

We check our conditions of Theorem 3.9. Immediately, we get that $H_{\mathcal{A}}$ satisfies conditions (3.11) and (3.12) and therefore reproduces constants. For the following computations we use the values of the coefficients $\alpha_{k,\ell}$ as given in Table 3.1.

For $d = k = 1$ the equations (3.13) and (3.14) of Theorem 3.9 we get

$$\mathbf{A}^{(1)}(-1)\mathbf{e}_{1,2} + 2\mathbf{A}(-1)\mathbf{e}_{2,2} = \begin{bmatrix} 0 \\ -\mu-1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ -\frac{\mu}{2}+\frac{1}{2} \end{bmatrix} = \mathbf{0}_2,$$
$$\mathbf{A}^{(1)}(1)\mathbf{e}_{1,2} - 2\mathbf{A}(1)\mathbf{e}_{2,2} = \begin{bmatrix} 0 \\ \mu-1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ \frac{\mu}{2}+\frac{1}{2} \end{bmatrix} = -2\mathbf{e}_{2,2}.$$

Thus, the scheme reproduces Π_1 for all parameter values λ and μ. If we consider the case of quadratic polynomials, we obtain the additional equations

$$\mathbf{A}^{(2)}(-1)\mathbf{e}_{1,2} + 4\mathbf{A}^{(1)}(-1)\mathbf{e}_{2,2} + 2\mathbf{A}(-1)\mathbf{e}_{2,2} = \begin{bmatrix} -1-8\lambda \\ 0 \end{bmatrix}, \tag{3.19}$$
$$\mathbf{A}^{(2)}(1)\mathbf{e}_{1,2} - 4\mathbf{A}^{(1)}(1)\mathbf{e}_{2,2} + 2\mathbf{A}(1)\mathbf{e}_{2,2} = 2\mathbf{e}_{2,2}.$$

So, by (3.19) and the results of Theorem 3.9, we see that the subdivision scheme $H_{\mathcal{A}}$ reproduces quadratic polynomials if and only if $\lambda = -\frac{1}{8}$. We compute (3.13) and (3.14) for $d = k = 3$ and assume that $\lambda = -\frac{1}{8}$. We get

$$\begin{aligned} \mathbf{A}^{(3)}(-1)\mathbf{e}_{1,2} + 6\mathbf{A}^{(2)}(-1)\mathbf{e}_{2,2} + 6\mathbf{A}^{(1)}(-1)\mathbf{e}_{2,2} + 4\mathbf{A}(-1)\mathbf{e}_{2,2} &= \begin{bmatrix} 0 \\ -1-2\mu \end{bmatrix}, \\ \mathbf{A}^{(3)}(1)\mathbf{e}_{1,2} - 6\mathbf{A}^{(2)}(1)\mathbf{e}_{2,2} + 6\mathbf{A}^{(1)}(1)\mathbf{e}_{2,2} - 4\mathbf{A}(1)\mathbf{e}_{2,2} &= \begin{bmatrix} 0 \\ -5-2\mu \end{bmatrix}. \end{aligned}$$

According to Theorem 3.9, we conclude that the scheme reproduces cubic polynomials if and only if additionally $\mu = -\frac{1}{2}$. Theorem 3.9 also tells us that the Hermite scheme $H_{\mathcal{A}}$ does not reproduces polynomials of degree 4 since

$$\begin{aligned} &\mathbf{A}^{(4)}(-1)\mathbf{e}_{1,2} + 8\mathbf{A}^{(3)}(-1)\mathbf{e}_{2,2} + 12\mathbf{A}^{(2)}(-1)\mathbf{e}_{2,2} \\ &\qquad + 16\mathbf{A}^{(1)}(-1)\mathbf{e}_{2,2} + 12\mathbf{A}(-1)\mathbf{e}_{2,2} = \mathbf{e}_{1,2} \neq \mathbf{0}_2. \end{aligned}$$

Next, we use our algebraic conditions to obtain a modified Hermite subdivision scheme which reproduces polynomials of higher degree by only slightly increasing the support of the scheme. Consider the interpolatory Hermite subdivision scheme $H_{\bar{\mathcal{A}}}$ with non-zero coefficients

$$\bar{A}_{-3} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \ \bar{A}_{-1} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \ \bar{A}_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$
$$\bar{A}_1 = \begin{bmatrix} a_1 & -a_2 \\ -a_3 & a_4 \end{bmatrix}, \ \bar{A}_3 = \begin{bmatrix} b_1 & -b_2 \\ -b_3 & b_4 \end{bmatrix},$$

for some real numbers a_i, b_i, $i = 1, \dots, 4$. By Theorem 3.9 these coefficients have to satisfy the following linear system in order to reproduce polynomials up to degree 5

$$\begin{aligned} b_1 &= \frac{1}{128} - 3b_2, & b_4 &= \frac{1}{1408} - \frac{384}{1408}b_3, & a_1 &= \frac{1}{2} - b_1, \\ a_3 &= 24b_4 + 9b_3 + \frac{3}{4}, & a_4 &= \frac{1}{4} - b_4 - \frac{1}{2}a_3 - \frac{3}{2}b_3, & a_2 &= -\frac{1}{8} - 3b_2 - 2b_1. \end{aligned}$$

Choosing the values $b_3 = 0$ and $b_2 = \frac{1}{384}$ leads to $a_1 = \frac{1}{2}$, $a_2 = -17/128 \approx -0.13$, $a_3 = 135/176 \approx 0.77$ and $a_4 = -189/1408 \approx -0.13$. With this choice of coefficients the non-zero matrices $\bar{A}_{-1}, \bar{A}_0$ and $\bar{A}_1$ of the mask of the scheme $H_{\bar{\mathcal{A}}}$ are closely related to the corresponding ones of $H_{\mathcal{A}}$. See Figure 3.2 for the basic limit functions of the scheme $H_{\bar{\mathcal{A}}}$.

de Rham transform of a Hermite scheme

We now consider the de Rham transform of the interpolatory Hermite scheme $H_{\mathcal{A}}$ as introduced in [10, 22]. This scheme is a dual scheme, meaning that $\tau = -\frac{1}{2}$.

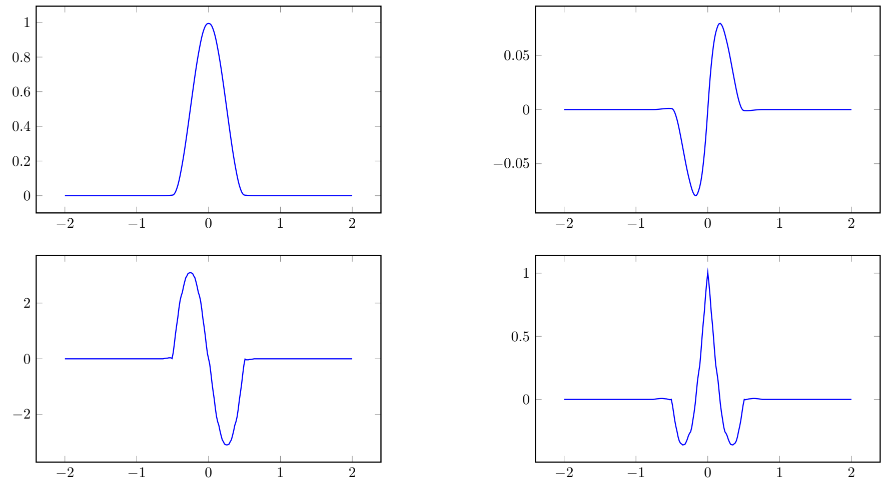

Figure 3.2: Left column: Basic limit function and derivative of the interpolatory Hermite scheme $H_{\bar{\mathcal{A}}}$ introduced in Section 3.1.4 for initial data $\mathbf{e}_{1,2}$ at 0 and $\mathbf{0}_2$ else. Right column: Basic limit function and derivative of the interpolatory Hermite scheme $H_{\bar{\mathcal{A}}}$ introduced in Section 3.1.4 for initial data $\mathbf{e}_{2,2}$ at 0 and $\mathbf{0}_2$ else.

For $\lambda, \mu \in \mathbb{R}$ the non-zero matrices of its mask (for simplicity again denoted by A_i) are given by

$$\begin{aligned}
A_{-2} &= \frac{1}{8}\begin{bmatrix} 2+4\lambda(1-\mu) & 4\lambda+2\lambda\mu \\ 4-2\mu-2\mu^2 & \mu^2+8\lambda(1-\mu) \end{bmatrix}, \\
A_{-1} &= \frac{1}{8}\begin{bmatrix} 6-4\lambda(1-\mu) & 8\lambda-2\lambda\mu \\ 4-2\mu-2\mu^2 & \mu^2-8\lambda(1-\mu)+2\mu \end{bmatrix}, \\
A_{0} &= \frac{1}{8}\begin{bmatrix} 6-4\lambda(1-\mu) & -8\lambda+2\lambda\mu \\ -4+2\mu+2\mu^2 & \mu^2-8\lambda(1-\mu)+2\mu \end{bmatrix}, \\
A_{1} &= \frac{1}{8}\begin{bmatrix} 2+4\lambda(1-\mu) & -4\lambda+-2\lambda\mu \\ -4+2\mu+2\mu^2 & \mu^2+8\lambda(1-\mu) \end{bmatrix}.
\end{aligned}$$

We see that the scheme reproduces constants since it satisfies (3.11) and (3.12). We obtain

$$\begin{aligned}
\mathbf{A}^{(1)}(-1)\mathbf{e}_{1,2}+2\mathbf{A}(-1)\mathbf{e}_{2,2} &= \frac{1}{8}\begin{bmatrix} 16\lambda(1-\mu) \\ 0 \end{bmatrix} + \frac{2}{8}\begin{bmatrix} -8\lambda+8\lambda\mu \\ 0 \end{bmatrix} = \mathbf{0}_2 \\
\mathbf{A}^{(1)}(1)\mathbf{e}_{1,2}-2\mathbf{A}(1)\mathbf{e}_{2,2} &= \frac{1}{8}\begin{bmatrix} -8 \\ -16+8\mu+8\mu^2 \end{bmatrix} - \frac{2}{8}\begin{bmatrix} 0 \\ 4\mu^2+4\mu \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.
\end{aligned}$$

Since $2q_{1,2\tau}\left(-\frac{\tau}{2}\right) = -1$ and $\tilde{q}_{1,2\tau}\left(-\frac{\tau}{2}\right) = -2$ we conclude that the scheme reproduces linear polynomials by Theorem 3.9. Next, we check if the scheme also reproduces polynomials of degree 2. By Table 3.1 we have $\alpha_{2,1} = 4$ and $\alpha_{2,2} = 2$. We compute $2q_{2,2\tau}(-\frac{\tau}{2}) = \frac{3}{2}$ and $\tilde{q}_{2,2\tau}(-\frac{\tau}{2}) = 4$. According to Theorem 3.9 we have to check if the scheme satisfies

$$\begin{aligned}\mathbf{A}^{(2)}(-1)\mathbf{e}_{1,2} + 4\mathbf{A}^{(1)}(-1)\mathbf{e}_{2,2} + 2\mathbf{A}(-1)\mathbf{e}_{2,2} &= \mathbf{0}_2,\\ \mathbf{A}^{(2)}(1)\mathbf{e}_{1,2} - 4\mathbf{A}^{(1)}(1)\mathbf{e}_{2,2} + 2\mathbf{A}(1)\mathbf{e}_{2,2} &= \begin{bmatrix}\frac{3}{2}\\ 4\end{bmatrix},\end{aligned}$$

in order to decide whether it reproduces $\prod_2$ or not. Computations lead to

$$\begin{aligned}\mathbf{A}^{(2)}(-1)\mathbf{e}_{1,2} + 4\mathbf{A}^{(1)}(-1)\mathbf{e}_{2,2} + 2\mathbf{A}(-1)\mathbf{e}_{2,2} &= \begin{bmatrix}0\\ 2-2\mu+16\lambda-16\lambda\mu\end{bmatrix},\\ \mathbf{A}^{(2)}(1)\mathbf{e}_{1,2} - 4\mathbf{A}^{(1)}(1)\mathbf{e}_{2,2} + 2\mathbf{A}(1)\mathbf{e}_{2,2} &= \begin{bmatrix}3+12\lambda\\ 4\end{bmatrix}.\end{aligned}$$

We conclude that the scheme reproduces polynomials up to degree 2 if and only if $\lambda = -\frac{1}{8}$. Similar computations for the case of cubic polynomials show that the choice of $\mu = -\frac{1}{2}$ leads to the reproduction of cubic polynomials.

3.2 Hermite schemes of any order

In this section, we extend Theorem 3.9 to Hermite schemes of any order d, meaning we use input data consisting of function values and its first $d-1$ derivatives. The crucial step to generalise the previous result is to define now d classes of polynomials (generalising $\tilde{q}_{k,i}$, see (3.8)). Therefore, we need suitable coefficients, according to Definition 3.5, which should be computable explicitly. Otherwise, it would be hard to use the obtained algebraic conditions in practise.

The new results of this section are based on the publication

S. Hüning, *Polynomial reproduction of Hermite subdivision schemes of any order*, submitted, 2019.

We use the same notation as in the previous section with one modification: The class of polynomials $\tilde{q}_{k,i}$ will be denoted by $q_{k,i,1}$ since this notation fits better with the generalisation. In Section 3.2.1 we define and analyse crucial auxiliary polynomials while in Section 3.2.2 the algebraic results are presented.

3.2.1 Analysis of auxiliary polynomials

To prove our main Theorem 3.19 we use the same strategy as presented in the previous section. We briefly recall the definition of the class of polynomials $q_{k,i}$ which is along the same lines as in Section 3.1.

Polynomials q_k

We recall the definition of the polynomials q_k by

$$q_0(x) := 1, \quad q_k(x) := \prod_{r=0}^{k-1}(2x - r), \quad k > 0.$$

In the monomial base we write

$$q_k(-x) = \sum_{n=0}^{k} \gamma_n^k x^n, \quad \text{for} \quad \gamma_n^k = 2^n(-1)^k \begin{bmatrix} k \\ n \end{bmatrix} \in \mathbb{R}.$$

Let $i \in \mathbb{Z}$. Consider the polynomials

$$q_{0,i}(x) := 1, \quad q_{k,i}(x) := q_k\Big(x + \frac{i}{2}\Big) = \prod_{r=0}^{k-1}(2x + i - r), \quad k > 0,$$

which can also be written in terms of the monomial base as

$$q_{k,i}(-x) = \sum_{n=0}^{k} \gamma_n^{k,i} x^n, \quad \text{for some coefficients} \quad \gamma_n^{k,i} \in \mathbb{R}.$$

Polynomials $q_{k,i,s-1}$

This section contains the crucial modifications needed to extend the main result of the previous section to Hermite schemes of any order. To be precise, we generalise the definitions of the polynomials $\tilde{q}_{k,i}$. Therefore, we first define some families of coefficients $\alpha_{k,s-1,s-1}$ which are a direct extension of those given in Definition 3.5. In particular, $\alpha_{k,s-1,1} = \alpha_{k,s-1}$.

We underline that, throughout this section we let

$$s, k \in \mathbb{N} \quad \text{with} \quad k \geqslant s-1, \quad s \geqslant 2 \quad \text{with} \quad i \in \mathbb{Z}. \tag{3.20}$$

Definition 3.16. We define

$$\begin{aligned}
\alpha_{k,s-1,s-1} \quad &:= 2^{s-1}\prod_{m=0}^{s-2}(k-m), \\
\alpha_{k,k-n+s-1,s-1} \quad &:= (-1)^k 2^{-n+s-1}\left(\prod_{m=0}^{s-2}(n-m)\gamma_n^k - \sum_{j=s-1}^{k-n+s-2}(-1)^j \alpha_{k,j,s-1}\gamma_{n-s+1}^{k-j}\right)
\end{aligned}$$

for $n = k-1, \ldots, s-1$.

Table 3.2 presents some values of $\alpha_{k,\ell,2}$. Since $\gamma_n^k = 2^n(-1)^k \begin{bmatrix} k \\ n \end{bmatrix}$ we can write

$k \backslash \ell$	2	3	4	5	6
2	8				
3	24	24			
4	48	96	88		
5	80	240	440	400	
6	120	480	1320	2400	2192

Table 3.2: Values of the coefficients $\alpha_{k,\ell,2}$ for $k = 2, \ldots, 6$ and $\ell = 2, \ldots, k$.

$\alpha_{k,k-n+s-1,s-1}$ as

$$\alpha_{k,k-n+s-1,s-1} = 2^{s-1} \prod_{m=0}^{s-2} (n-m) \begin{bmatrix} k \\ n \end{bmatrix} - \sum_{j=s-1}^{k-n+s-2} \alpha_{k,j,s-1} \begin{bmatrix} k-j \\ n-s+1 \end{bmatrix} \tag{3.21}$$

for $n = k-1, \ldots, s-1$.

Lemma 3.17. *For k, s as in (3.20), we have $\alpha_{k,\ell,s-1} = 2^{s-1}(s-1)! \begin{bmatrix} \ell \\ s-1 \end{bmatrix} \binom{k}{\ell}$ for all $\ell = s-1, \ldots, k$.*

Proof. The proof works by induction on ℓ. For $\ell = s-1$ the statement is true by Definition 3.16 and the fact that

$$2^{s-1}(s-1)! \begin{bmatrix} s-1 \\ s-1 \end{bmatrix} \binom{k}{s-1} = 2^{s-1}(s-1)! \binom{k}{s-1} = 2^{s-1} \prod_{m=0}^{s-2} (k-m).$$

Assume that the statement is true for some $\ell - 1$. We prove it for ℓ. Due to (3.21) we have

$$\begin{aligned} \alpha_{k,\ell,s-1} &= 2^{s-1} \prod_{m=0}^{s-2} (k+s-1-\ell-m) \begin{bmatrix} k \\ k+s-1-\ell \end{bmatrix} - \sum_{j=s-1}^{\ell-1} \alpha_{k,j}^{s-1} \begin{bmatrix} k-j \\ k-\ell \end{bmatrix} \\ &= 2^{s-1} \prod_{m=0}^{s-2} (k+s-1-\ell-m) \begin{bmatrix} k \\ k+s-1-\ell \end{bmatrix} \\ &\quad - \sum_{j=s-1}^{\ell} 2^{s-1}(s-1)! \begin{bmatrix} j \\ s-1 \end{bmatrix} \binom{k}{j} \begin{bmatrix} k-j \\ k-\ell \end{bmatrix} + 2^{s-1}(s-1)! \begin{bmatrix} \ell \\ s-1 \end{bmatrix} \binom{k}{\ell} \end{aligned}$$

where we use the induction hypothesis and the fact that $\begin{bmatrix} k-\ell \\ k-\ell \end{bmatrix} = 1$. We have to prove that

$$\prod_{m=0}^{s-2} (k+s-1-\ell-m) \begin{bmatrix} k \\ k+s-1-\ell \end{bmatrix} = \sum_{j=s-1}^{\ell} (s-1)! \begin{bmatrix} j \\ s-1 \end{bmatrix} \binom{k}{j} \begin{bmatrix} k-j \\ k-\ell \end{bmatrix}.$$

We have

$$\frac{1}{(s-1)!}\prod_{m=0}^{s-2}(k+s-1-\ell-m)\begin{bmatrix}k\\k+s-1-\ell\end{bmatrix}=\binom{k+s-1-\ell}{s-1}\begin{bmatrix}k\\k+s-1-\ell\end{bmatrix}$$
$$=\sum_{j=s-1}^{\ell}\begin{bmatrix}j\\s-1\end{bmatrix}\begin{bmatrix}k-j\\k-\ell\end{bmatrix}\binom{k}{j}$$

using the identity

$$\begin{bmatrix}n\\\ell+m\end{bmatrix}\binom{\ell+m}{\ell}=\sum_k\begin{bmatrix}k\\\ell\end{bmatrix}\begin{bmatrix}n-k\\m\end{bmatrix}\binom{n}{k}$$

given for example in [36, (6.29)]. □

For k, s as in (3.20), $i \in \mathbb{N}$, we introduce the polynomials

$$q_{k,i,s-1}(x) := \sum_{n=s-1}^{k}(-1)^n\alpha_{k,n,s-1}q_{k-n,i}(x) \tag{3.22}$$

and $q_{\ell,i,s-1} := 0$ for all $\ell = 0, \ldots, s-2$. In the monomial base we write

$$q_{k,i,s-1}(-x) = \sum_{n=0}^{k-(s-1)}\gamma_n^{k,i,s-1}x^n$$

for some coefficients $\gamma_n^{k,i,s-1} \in \mathbb{R}$.

Note that the coefficients $\gamma_n^{k,i}$ of the previous section correspond to $\gamma_n^{k,i,1}$ now.

Proposition 3.18. *The coefficients of the polynomials $q_{k,i,s-1}(x)$ and $q_{k,i}(x)$ satisfy the relation $\prod_{m=0}^{s-2}(n-m)\gamma_n^{k,i} = \gamma_{n-(s-1)}^{k,i,s-1}$ for all $i \in \mathbb{Z}$ and $n = s-1, \ldots, k$.*

Proof. The claim of the proposition is equivalent to

$$\prod_{m=0}^{s-2}(n-m)\gamma_n^{k,i} = \sum_{j=s-1}^{k-n+s-1}(-1)^j\alpha_{k,j,s-1}\gamma_{n-s+1}^{k-j,i} \quad \text{for } i \in \mathbb{Z} \text{ and } n = s-1, \ldots, k. \tag{3.23}$$

To see this, we first use the definition of $q_{k,i,s-1}$ as in (3.22) and then write $q_{k-n,i}$ in its canonical form. This leads to

$$q_{k,i,s-1}(-x) = \sum_{n=s-1}^{k}(-1)^n\alpha_{k,n,s-1}q_{k-n,i}(-x) = \sum_{j=0}^{k-s+1}\left(\sum_{n=s-1}^{k-j}(-1)^n\alpha_{k,n,s-1}\gamma_j^{k-n,i}\right)x^j.$$

Thus, we have $\gamma_j^{k,i,s-1} = \sum_{n=s-1}^{k-j}(-1)^n\alpha_{k,n,s-1}\gamma_j^{k-n,i}$ which proves that (3.23) is equivalent to the claim of the Proposition. First, let $n = k$. Using Lemma 3.3 we obtain

$$\gamma_k^{k,i} = (-2)^{s-1}\gamma_{k-s+1}^{k-s+1,i}.$$

Moreover, by the definition of $\alpha_{k,s-1,s-1}$ we have

$$\alpha_{k,s-1,s-1}\gamma_{k-s+1}^{k-s+1,i} = \prod_{m=0}^{s-2}(k-m)(-1)^{s-1}2^{s-1}\gamma_{k-s+1}^{k-s+1,i}.$$

This proves (3.23) for $n = k$ and all $i \in \mathbb{Z}$.

Now, let $n \in \{s-1, \dots, k-1\}$. Definition 3.16 and the fact that

$$\gamma_{n-s+1}^{n-s+1} = (-1)^{n-s+1}2^{n-s+1}$$

lead to

$$\begin{aligned}
\prod_{m=0}^{s-2}(n-m)\gamma_n^k &= (-1)^k 2^{n-s+1}\alpha_{k,k-n+s-1,s-1} + \sum_{j=s-1}^{k-n+s-2}(-1)^j\alpha_{k,j,s-1}\gamma_{n-s+1}^{k-j} \\
&= (-1)^{k-n+s-1}\alpha_{k,k-n+s-1,s-1}\gamma_{n-s+1}^{n-s+1} + \sum_{j=s-1}^{k-n+s-2}(-1)^j\alpha_{k,j,s-1}\gamma_{n-s+1}^{k-j} \\
&= \sum_{j=s-1}^{k-n+s-1}(-1)^j\alpha_{k,j,s-1}\gamma_{n-s+1}^{k-j}.
\end{aligned}$$

Next, we show that this implies that (3.23) is true for any $i \in \mathbb{Z}$. We still have $n \in \{s-1, \dots, k\}$. From above we know that for any $r \in \{n, ..., k\}$ we have

$$\prod_{m=0}^{s-2}(r-m)\gamma_r^k = \sum_{j=s-1}^{k-r+s-1}(-1)^j\alpha_{k,j,s-1}\gamma_{r-s+1}^{k-j}.$$

Since

$$\prod_{m=0}^{s-2}(n-m)\binom{r}{n} = \prod_{m=0}^{s-2}(r-m)\binom{r-s+1}{n-s+1},$$

the latter implies that

$$\prod_{m=0}^{s-2}(n-m)\binom{r}{n}\gamma_r^k = \sum_{j=s-1}^{k-r+s-1}(-1)^j\alpha_{k,j,s-1}\binom{r-s+1}{n-s+1}\gamma_{r-s+1}^{k-j}.$$

Multiplying by the term $(-1)^{n+r}\left(\frac{i}{2}\right)^{r-n}$ on both sides and summing up r from n to k leads to

$$\begin{aligned}
&\prod_{m=0}^{s-2}(n-m)\sum_{r=n}^{k}(-1)^{r+n}\gamma_r^k\binom{r}{n}\left(\frac{i}{2}\right)^{r-n}\\
&\quad=\sum_{r=n}^{k}(-1)^{r+n}\left(\sum_{j=s-1}^{k-r+s-1}(-1)^j\alpha_{k,j,s-1}\binom{r-s+1}{n-s+1}\gamma_{r-s+1}^{k-j}\left(\frac{i}{2}\right)^{r-n}\right)\\
&\quad=\sum_{r=n-s+1}^{k-s+1}\left(\sum_{j=s-1}^{k-r}(-1)^{r+n-s+1+j}\alpha_{k,j,s-1}\binom{r}{n-s+1}\gamma_r^{k-j}\left(\frac{i}{2}\right)^{r-(n-s+1)}\right)\\
&\quad=\sum_{j=s-1}^{k-n+s-1}(-1)^j\alpha_{k,j,s-1}\left(\sum_{r=n-s+1}^{k-j}(-1)^{r+n-s+1}\binom{r}{n-s+1}\gamma_r^{k-j}\left(\frac{i}{2}\right)^{r-(n-s+1)}\right).
\end{aligned}$$

By Lemma 3.4 this implies that

$$\prod_{m=0}^{s-2}(n-m)\gamma_n^{k,i}=\sum_{j=s-1}^{k-n+s-1}(-1)^j\alpha_{k,j,s-1}\gamma_{n-s+1}^{k-j,i},$$

which concludes the proof. □

3.2.2 Proof of the main theorem

We use the same proof technics as in the previous section to show our main theorem.

Theorem 3.19. *Let $H_{\mathcal{A}}$ denote a Hermite subdivision scheme of order $d \geqslant 2$. Then, $H_{\mathcal{A}}$ reproduces constants if and only if*

$$\boldsymbol{A}(-1)\boldsymbol{e}_{1,d}=\boldsymbol{0}_d, \tag{3.24}$$
$$\boldsymbol{A}(1)\boldsymbol{e}_{1,d}=2\boldsymbol{e}_{1,d}. \tag{3.25}$$

Moreover, $H_{\mathcal{A}}$ reproduces polynomials up to degree $m \geqslant 1$ if and only if it reproduces constants and

$$\boldsymbol{A}^{(k)}(-1)\boldsymbol{e}_{1,d}+\sum_{s=2}^{d}\left(\sum_{\ell=s-1}^{k}\alpha_{k,\ell,s-1}\cdot\boldsymbol{A}^{(k-\ell)}(-1)\boldsymbol{e}_{s,d}\right)=\boldsymbol{0}_d, \tag{3.26}$$
$$\boldsymbol{A}^{(k)}(1)\boldsymbol{e}_{1,d}+\sum_{s=2}^{d}\left(\sum_{\ell=s-1}^{k}\tilde{\alpha}_{k,\ell,s-1}\cdot\boldsymbol{A}^{(k-\ell)}(1)\boldsymbol{e}_{s,d}\right)=\boldsymbol{q}_d. \tag{3.27}$$

for all $k=1,\dots,m$ with $\tilde{\alpha}_{k,\ell,s-1}=(-1)^\ell\alpha_{k,\ell,s-1}$, $\ell=s-1,\dots,k$, with $\alpha_{k,\ell,s-1}$ as in Definition 3.16 and $\boldsymbol{q}_d := 2\boldsymbol{D}\Big(q_{k,2\tau}(\frac{-\tau}{2})\boldsymbol{e}_{1,d}+\sum_{s=2}^{d}q_{k,2\tau,s-1}\left(\frac{-\tau}{2}\right)\boldsymbol{e}_{s,d}\Big)$.

Note that the case $d = 2$ reduces exactly to Theorem 3.9.

Lemma 3.20. *A Hermite subdivision scheme $H_{\mathcal{A}}$ reproduces constants if and only if (3.24) and (3.25) are satisfied.*

Replacing $\mathbf{e}_{1,2}$ by $\mathbf{e}_{1,d}$ in the proof of Lemma 3.11 shows Lemma 3.20.

Lemma 3.21. *Let $m \geqslant 1$. Then, condition (3.26) is satisfied if and only if*

$$\begin{aligned}\boldsymbol{A}_e^{(k)}(1)\boldsymbol{e}_{1,d} &+ \sum_{s=2}^{d}\sum_{\ell=s-1}^{k}(-1)^{\ell}\alpha_{k,\ell,s-1}\boldsymbol{A}_e^{(k-\ell)}(1)\boldsymbol{e}_{s,d} \\ &= \frac{1}{2}\Big(\boldsymbol{A}^{(k)}(1)\boldsymbol{e}_{1,d} + \sum_{s=2}^{d}\sum_{\ell=s-1}^{k}(-1)^{\ell}\alpha_{k,\ell,s-1}\boldsymbol{A}^{(k-\ell)}(1)\boldsymbol{e}_{s,d}\Big),\end{aligned}$$

$$\begin{aligned}\boldsymbol{A}_o^{(k)}(1)\boldsymbol{e}_{1,d} &+ \sum_{s=2}^{d}\sum_{\ell=s-1}^{k}(-1)^{\ell}\alpha_{k,\ell,s-1}\boldsymbol{A}_o^{(k-\ell)}(1)\boldsymbol{e}_{s,d} \\ &= \frac{1}{2}\Big(\boldsymbol{A}^{(k)}(1)\boldsymbol{e}_{1,d} + \sum_{s=2}^{d}\sum_{\ell=s-1}^{k}(-1)^{\ell}\alpha_{k,\ell,s-1}\boldsymbol{A}^{(k-\ell)}(1)\boldsymbol{e}_{s,d}\Big),\end{aligned}$$

for all $k = 1, \dots, m$. Moreover, condition (3.24) is satisfied if and only if

$$\boldsymbol{A}_e(1)\boldsymbol{e}_{1,d} = \boldsymbol{A}_o(1)\boldsymbol{e}_{1,d} = \frac{1}{2}\boldsymbol{A}(1)\boldsymbol{e}_{1,d}.$$

Summing up s from 2 to d, replacing $\mathbf{e}_{1,2}$ by $\mathbf{e}_{1,d}$ and using the proof strategy presented in Lemma 3.12 show the statement above.

Consider the vector polynomial

$$Q_{k,i}(x) := \sum_{s=1}^{d} q_{k,i,s-1}(x)\mathbf{e}_{s,d} \quad \text{with} \quad k \geqslant 0,\ i \in \mathbb{Z}$$

which consists of our previously defined polynomials $q_{k,i,s-1}$. To simplify notation we require $q_{k,i,0} := q_{k,i}$.

Proposition 3.22. *Let $m \geqslant 1$. Then, conditions (3.26) and (3.27) are satisfied if and only if for all $i \in \mathbb{Z}$ and $\tau \in \mathbb{R}$,*

$$\sum_{j\in\mathbb{Z}} A_{i-2j}Q_{k,i+2\tau}(-j-\tau) = \boldsymbol{D}Q_{k,i+2\tau}\Big(\frac{-i-\tau}{2}\Big) \qquad k = 1, \dots, m.$$

Especially, conditions (3.24) and (3.25) are satisfied if and only if $\sum_{j\in\mathbb{Z}} A_{i-2j}Q_{0,i}(-j) = \boldsymbol{e}_{1,d}$ for all $i \in \mathbb{Z}$.

Proof. We have $\mathbf{A}_e^{(k)}(1) = \sum_{j\in\mathbb{Z}} q_{k,2t+2\tau}(-j-\tau)A_{2(t-j)}$ for all $t \in \mathbb{Z}$ and $\tau \in \mathbb{R}$. Let $i \in 2\mathbb{Z}$ with $i = 2t$ for some $t \in \mathbb{Z}$. With Lemma 3.21 we obtain

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} A_{i-2j} Q_{k,i+2\tau}(-j-\tau) \\
&\quad = \sum_{j\in\mathbb{Z}} q_{k,2t+2\tau}(-j-\tau)A_{2(t-j)}\mathbf{e}_{1,d} + \sum_{s=2}^{d}\sum_{j\in\mathbb{Z}} q_{k,2t+2\tau,s-1}(-j-\tau)A_{2(t-j)}\mathbf{e}_{s,d} \\
&\quad = \sum_{j\in\mathbb{Z}} q_{k,2t+2\tau}(-j-\tau)A_{2(t-j)}\mathbf{e}_{1,d} \\
&\qquad + \sum_{s=2}^{d}\sum_{\ell=s-1}^{k}\sum_{j\in\mathbb{Z}} (-1)^{\ell}\alpha_{k,\ell,s-1} q_{k-\ell,2t+2\tau}(-j-\tau)A_{2(t-j)}\mathbf{e}_{s,d} \\
&\quad = \frac{1}{2}\Big(\mathbf{A}^{(k)}(1)\mathbf{e}_{1,d} + \sum_{s=2}^{d}\sum_{\ell=s-1}^{k} (-1)^{\ell}\alpha_{k,\ell,s-1}\mathbf{A}^{(k-\ell)}(1)\mathbf{e}_{s,d}\Big) \\
&\quad = \mathbf{D}Q_{k,2\tau+i}\Big(\frac{-i-\tau}{2}\Big).
\end{aligned}
$$

For odd $i \in \mathbb{Z}$, $i = 2t+1$, the proof works analogously. □

Proposition 3.23. *Let $H_{\mathcal{A}}$ be a Hermite subdivision scheme of order $d \geqslant 2$ with parametrisation τ and $m \geqslant 0$. Then, $H_{\mathcal{A}}$ satisfies conditions* (3.24) – (3.25) *and conditions (3.26) – (3.27) for all $k = 1,\ldots,m$, if and only if*

$$
\sum_{j\in\mathbb{Z}} A_{i-2j}\boldsymbol{e}_{1,d} = \boldsymbol{e}_{1,d} \quad i \in \mathbb{Z},
$$

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{k}\boldsymbol{e}_{1,d} + \sum_{s=2}^{d}\sum_{j\in\mathbb{Z}} A_{i-2j}\prod_{\ell=0}^{s-2}(k-\ell)(j+\tau)^{k-s+1}\boldsymbol{e}_{s,d} \\
&= \frac{1}{2^{k}}\Big((i+\tau)^{k}\boldsymbol{e}_{1,d} + \sum_{s=2}^{d}\prod_{\ell=0}^{s-2}(k-\ell)(i+\tau)^{k-s+1}\boldsymbol{e}_{s,d}\Big) \quad k = 1,\ldots,m, \quad i \in \mathbb{Z},
\end{aligned}
\tag{3.28}
$$

with the convention that (3.28) *(resp. (3.26) – (3.27)) is empty if $m = 0$.*

Proof. The proof works by induction on m. The case $m = 0$ follows by Lemma 3.20. Assume that the statement is true for some $m-1$ and all $k = 1,\ldots,m-1$.

For $i \in \mathbb{Z}$, using Proposition 3.22 we obtain

$$\mathbf{D}Q_{m,i+2\tau}\Big(\frac{-i-\tau}{2}\Big) = \sum_{j\in\mathbb{Z}} A_{i-2j} Q_{m,i+2\tau}(-j-\tau)$$

$$= \sum_{j\in\mathbb{Z}} A_{i-2j} q_{m,i+2\tau}(-j-\tau)\mathbf{e}_{1,d} + \sum_{s=2}^{d}\sum_{j\in\mathbb{Z}} A_{i-2j} q_{m,i+2\tau,s-1}(-j-\tau)\mathbf{e}_{s,d}$$

$$= \sum_{j\in\mathbb{Z}} A_{i-2j} \sum_{n=0}^{m} \gamma_n^{m,i+2\tau}(j+\tau)^n \mathbf{e}_{1,d} + \sum_{s=2}^{d}\sum_{j\in\mathbb{Z}} A_{i-2j} \sum_{n=0}^{m-s+1} \gamma_n^{m,i+2\tau,s-1}(j+\tau)^n \mathbf{e}_{s,d}$$

$$= \gamma_m^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^m \mathbf{e}_{1,d} + \gamma_m^{m,i+2\tau} \sum_{s=2}^{d}\sum_{j\in\mathbb{Z}} A_{i-2j} \prod_{\ell=0}^{s-2}(m-\ell)(j+\tau)^{m-s+1}\mathbf{e}_{s,d}$$

$$\underbrace{+ \sum_{j\in\mathbb{Z}} A_{i-2j} \sum_{n=0}^{m-1} \gamma_n^{m,i+2\tau}(j+\tau)^n \mathbf{e}_{1,d} + \sum_{s=2}^{d}\sum_{j\in\mathbb{Z}} A_{i-2j} \sum_{n=0}^{m-s} \gamma_n^{m,i+2\tau,s-1}(j+\tau)^n \mathbf{e}_{s,d}}_{(*)}\,.$$

Here, we used the relation $\prod_{\ell=0}^{s-2}(m-\ell)\gamma_m^{m,i+2\tau} = \gamma_{m-s+1}^{m,i+2\tau,s-1}$ of Proposition 3.18. The idea now is to rewrite $(*)$ further before making use of the induction hypothesis. First, note that

$$\sum_{s=2}^{d}\sum_{n=0}^{m-s} \gamma_n^{m,i+2\tau,s-1} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^n \mathbf{e}_{s,d}$$

$$= \sum_{s=2}^{d}\sum_{b=s-1}^{m-1} \gamma_{b-s+1}^{m,i+2\tau,s-1} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{b-s+1}\mathbf{e}_{s,d}$$

$$= \sum_{s=2}^{d}\sum_{n=s-1}^{m-1} \prod_{\ell=0}^{s-2}(n-\ell)\gamma_n^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n-s+1}\mathbf{e}_{s,d}$$

$$= \sum_{s=2}^{d}\sum_{n=s-2}^{m-2} \prod_{\ell=0}^{s-2}(n+1-\ell)\gamma_{n+1}^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n-s+2}\mathbf{e}_{s,d}$$

$$= \sum_{s=2}^{d}\sum_{n=0}^{m-2} \prod_{\ell=0}^{s-2}(n+1-\ell)\gamma_{n+1}^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n-s+2}\mathbf{e}_{s,d}.$$

Here, the last equality is true because for any $n \in \{0,\dots,s-3\}$ the occurring summand is 0 because one of the factors in the product $\prod_{\ell=0}^{s-2}(n+1-\ell)$ is for sure 0.

By Proposition 3.18 we therefore conclude

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} A_{i-2j} \sum_{n=0}^{m-1} \gamma_n^{m,i+2\tau}(j+\tau)^n \mathbf{e}_{1,d} + \sum_{s=2}^{d}\sum_{j\in\mathbb{Z}} A_{i-2j} \sum_{n=0}^{m-s} \gamma_n^{m,i+2\tau,s-1}(j+\tau)^n \mathbf{e}_{s,d}\\
&= \gamma_0^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{e}_{1,d} + \sum_{n=0}^{m-2} \gamma_{n+1}^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n+1}\mathbf{e}_{1,d}\\
&\quad + \sum_{s=2}^{d}\sum_{n=0}^{m-2}\prod_{\ell=0}^{s-2}(n+1-\ell)\gamma_{n+1}^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n-s+2}\mathbf{e}_{s,d}.
\end{aligned}$$

We use the induction hypothesis for $k = 1, \dots, m-1$ and the fact that the scheme reproduces constants to obtain

$$\begin{aligned}
&\gamma_0^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{e}_{1,d} + \sum_{n=0}^{m-2} \gamma_{n+1}^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n+1}\mathbf{e}_{1,d}\\
&\quad + \sum_{s=2}^{d}\sum_{n=0}^{m-2}\prod_{\ell=0}^{s-2}(n+1-\ell)\gamma_{n+1}^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n-s+2}\mathbf{e}_{s,d}\\
&= \gamma_0^{m,i+2\tau} \sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{e}_{1,d} + \sum_{n=0}^{m-2} \gamma_{n+1}^{m,i+2\tau} \Bigg(\sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n+1}\mathbf{e}_{1,d}\\
&\quad + \sum_{s=2}^{d}\prod_{\ell=0}^{s-2}(n+1-\ell) \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^{n-s+2}\mathbf{e}_{s,d}\Bigg)\\
&= \underbrace{\gamma_0^{m,i+2\tau}\mathbf{e}_{1,d} + \sum_{n=0}^{m-2} \frac{1}{2^{n+1}}\gamma_{n+1}^{m,i+2\tau} \left((i+\tau)^{n+1}\mathbf{e}_{1,d} + \sum_{s=2}^{d}\prod_{\ell=0}^{s-2}(n+1-\ell)(i+\tau)^{n-s+2}\mathbf{e}_{s,d} \right)}_{(**)}.
\end{aligned}$$

Next, we rewrite the sum $(**)$ and then use the definitions of the polynomial $q_{m,i}$ resp. $q_{m,i,d-1}$.

So,

$$
\begin{aligned}
&\gamma_0^{m,i+2\tau}\mathbf{e}_{1,d} + \sum_{n=0}^{m-2}\frac{1}{2^{n+1}}\gamma_{n+1}^{m,i+2\tau}\left((i+\tau)^{n+1}\mathbf{e}_{1,d} + \sum_{s=2}^{d}\prod_{\ell=0}^{s-2}(n+1-\ell)\,(i+\tau)^{n-s+2}\,\mathbf{e}_{s,d}\right)\\
&= \gamma_0^{m,i+2\tau}\mathbf{e}_{1,d} + \sum_{n=0}^{m-1}\gamma_{n+1}^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{n+1}\mathbf{e}_{1,d} - \gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{m}\mathbf{e}_{1,d}\\
&\quad + \sum_{s=2}^{d}\sum_{n=s-2}^{m-2}\frac{1}{2^{n+1}}\gamma_{n+1}^{m,i+2\tau}\prod_{\ell=0}^{s-2}\Big(n+1-\ell\Big)\Big(i+\tau\Big)^{n-s+2}\mathbf{e}_{s,d}\\
&= \gamma_0^{m,i+2\tau}\mathbf{e}_{1,d} + \sum_{n=0}^{m-1}\gamma_{n+1}^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{n+1}\mathbf{e}_{1,d} - \gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{m}\mathbf{e}_{1,d}\\
&\quad + \sum_{s=2}^{d}\frac{1}{2^{s-1}}\sum_{n=s-2}^{m-2}\gamma_{n-s+2}^{m,i+2\tau,s-1}\Big(\frac{i+\tau}{2}\Big)^{n-s+2}\mathbf{e}_{s,d}\\
&= \gamma_0^{m,i+2\tau}\mathbf{e}_{1,d} + \sum_{n=0}^{m-1}\gamma_{n+1}^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{n+1}\mathbf{e}_{1,d} - \gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{m}\mathbf{e}_{1,d}\\
&\quad + \sum_{s=2}^{d}\frac{1}{2^{s-1}}\sum_{n=0}^{m-s+1}\gamma_{n}^{m,i+2\tau,s-1}\Big(\frac{i+\tau}{2}\Big)^{n}\mathbf{e}_{s,d} - \sum_{s=2}^{d}\frac{1}{2^{s-1}}\gamma_{m-s+1}^{m,i+2\tau,s-1}\Big(\frac{i+\tau}{2}\Big)^{m-s+1}\mathbf{e}_{s,d}\\
&= q_{m,i+2\tau}\Big(\frac{-i-\tau}{2}\Big)\mathbf{e}_{1,d} + \sum_{s=2}^{d}\frac{1}{2^{s-1}}q_{m,i+2\tau,s-1}\Big(\frac{-i-\tau}{2}\Big)\mathbf{e}_{s,d}\\
&\quad - \gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{m}\mathbf{e}_{1,d} - \sum_{s=2}^{d}\frac{1}{2^{s-1}}\gamma_{m-s+1}^{m,i+2\tau,s-1}\Big(\frac{i+\tau}{2}\Big)^{m-s+1}\mathbf{e}_{s,d}\\
&= q_{m,i+2\tau}\Big(\frac{-i-\tau}{2}\Big)\mathbf{e}_{1,d} + \sum_{s=2}^{d}\frac{1}{2^{s-1}}q_{m,i+2\tau,s-1}\Big(\frac{-i-\tau}{2}\Big)\mathbf{e}_{s,d}\\
&\quad - \gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{m}\mathbf{e}_{1,d} - \sum_{s=2}^{d}\frac{1}{2^{s-1}}\prod_{\ell=0}^{s-2}\Big(m-\ell\Big)\gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{m-s+1}\mathbf{e}_{s,d}\\
&= \mathbf{D}Q_{m,i+2\tau}\Big(\frac{-i-\tau}{2}\Big)\\
&\quad - \gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{m}\mathbf{e}_{1,d} - \sum_{s=2}^{d}\frac{1}{2^{s-1}}\prod_{\ell=0}^{s-2}\Big(m-\ell\Big)\gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^{m-s+1}\mathbf{e}_{s,d}.
\end{aligned}
$$

Summarising our computations above shows that $(*)$ is equal to the last term above.

Replacing $(*)$ by this expression leads to

$$\begin{aligned}
&\gamma_m^{m,i+2\tau}\sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^m \mathbf{e}_{1,d} + \gamma_m^{m,i+2\tau}\sum_{s=2}^{d}\sum_{j\in\mathbb{Z}} A_{i-2j}\prod_{\ell=0}^{s-2}(m-\ell)(j+\tau)^{m-s+1}\mathbf{e}_{s,d}\\
&= \gamma_m^{m,i+2\tau}\Big(\frac{i+\tau}{2}\Big)^m \mathbf{e}_{1,d} + \gamma_m^{m,i+2\tau}\sum_{s=2}^{d}\frac{1}{2^{s-1}}\prod_{\ell=0}^{s-2}(m-\ell)\Big(\frac{i+\tau}{2}\Big)^{m-s+1}\mathbf{e}_{s,d}\\
&= \frac{1}{2^m}\gamma_m^{m,i+2\tau}(i+\tau)^m\mathbf{e}_{1,d} + \frac{1}{2^m}\gamma_m^{m,i+2\tau}\sum_{s=2}^{d}\prod_{\ell=0}^{s-2}(m-\ell)(i+\tau)^{m-s+1}\mathbf{e}_{s,d}
\end{aligned}$$

Since $\gamma_m^{m,i+2\tau}\neq 0$, this concludes the induction step.

□

Proof of Theorem 3.19. We prove the statement by induction on m. For $m=0$, see Lemma 3.20. We choose $p(x)=x^m+g(x)$ with $g(x)$ a polynomial of degree $\leqslant m-1$. By (3.1) we have

$$\begin{aligned}
2^{nm}\mathbf{D}^{n+1}\mathbf{f}_{n+1}(i) &= \sum_{j\in\mathbb{Z}} A_{i-2j}\sum_{s=2}^{d}\prod_{\ell=0}^{s-2}(m-\ell)(j+\tau)^{m-s+1}\mathbf{e}_{s,d} + \sum_{j\in\mathbb{Z}} A_{i-2j}(j+\tau)^m\mathbf{e}_{1,d}\\
&\quad + 2^{nm}\sum_{j\in\mathbb{Z}} A_{i-2j}\mathbf{D}^n\sum_{s=1}^{d} g^{(s-1)}\Big(\frac{j+\tau}{2^n}\Big)\mathbf{e}_{s,d}.
\end{aligned}$$

Applying Proposition 3.23 and the induction hypothesis leads to

$$\begin{aligned}
\mathbf{D}^{n+1}\mathbf{f}_{n+1}(i) &= 2^{-(n+1)m}(i+\tau)^m\mathbf{e}_{1,d} + g\Big(\frac{i+\tau}{2^{n+1}}\Big)\mathbf{e}_{1,d}\\
&\quad + 2^{-(n+1)m}\sum_{s=2}^{d}\prod_{\ell=0}^{s-2}(m-\ell)(i+\tau)^{m-s+1}\mathbf{e}_{s,d} + \mathbf{D}^{n+1}\sum_{s=2}^{d} g^{(s-1)}\Big(\frac{i+\tau}{2^{n+1}}\Big)\mathbf{e}_{s,d}.
\end{aligned}$$

□

3.2.3 Example: Interpolatory scheme of order $d = 3$

Consider the primal and interpolatory Hermite scheme studied in [10]. The non-zero matrices of its mask are given by

$$A_{-1}=\mathbf{D}\begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \mu_1 & \mu_2 & \mu_3\\ \epsilon_1 & \epsilon_2 & \epsilon_3\end{bmatrix},\quad A_0=\mathbf{D},\quad A_1=\mathbf{D}\begin{bmatrix}\lambda_1 & -\lambda_2 & \lambda_3\\ -\mu_1 & \mu_2 & -\mu_3\\ \epsilon_1 & -\epsilon_2 & \epsilon_3\end{bmatrix},$$

with $\mathbf{D}=diag(1,\frac{1}{2},\frac{1}{4})$ and parameters $\lambda_i,\mu_i,\epsilon_i\in\mathbb{R}$. It is known that the scheme reproduces polynomials up to degree 3 if

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}, & \epsilon_1 &= 0, & \mu_2 &= \frac{1-\mu_1}{2},\\
\epsilon_3 &= \frac{1-\epsilon_2}{2}, & \lambda_3 &= \frac{-1-8\lambda_2}{16}, & \mu_3 &= \frac{2\mu_1-3}{24}.
\end{aligned}\tag{3.29}$$

We use our algebraic conditions presented in Theorem 3.19 to verify this fact.

$$\mathbf{A}(-1)\mathbf{e}_{1,3} = \begin{bmatrix} -2\lambda_1 + 1 \\ 0 \\ -2\epsilon_1 \end{bmatrix} = \mathbf{0}_3,$$
$$\mathbf{A}(1)\mathbf{e}_{1,3} = \begin{bmatrix} 2\lambda_1 + 1 \\ 0 \\ 2\epsilon_1 \end{bmatrix} = 2\mathbf{e}_{1,3}$$

in order to reproduce constants. This gives the first two relations of (3.29). By Table 3.1 we have $\alpha_{1,1} = 2$. By Theorem 3.19 we see that the scheme has to satisfy

$$\mathbf{A}^{(1)}(-1)\mathbf{e}_{1,3} + 2\mathbf{A}(-1)\mathbf{e}_{2,3} = \begin{bmatrix} 0 \\ -\mu_1 - 2\mu_2 + 1 \\ 0 \end{bmatrix} = \mathbf{0}_3,$$
$$\mathbf{A}^{(1)}(1)\mathbf{e}_{1,3} - 2\mathbf{A}(1)\mathbf{e}_{2,3} = \begin{bmatrix} 0 \\ -\mu_1 - 2\mu_2 - 1 \\ 0 \end{bmatrix} = -2\mathbf{e}_{2,3}$$

to reproduce linear polynomials. But this is the case if and only if $\mu_2 = \frac{1-\mu_1}{2}$. We go one step further and consider the reproduction of quadratic polynomials. Therefore, observe that $\alpha_{2,1,1} = 4$, $\alpha_{2,2,1} = 2$ and $\alpha_{2,2,2} = 8$, see Tables 3.1 and 3.2. Moreover, $q_{2,i,1}(-\frac{i}{2}) = 2$ and $q_{2,i,2}(-\frac{i}{2}) = 8$. So, we obtain the two conditions

$$\begin{aligned} &\mathbf{A}^{(2)}(-1)\mathbf{e}_{1,3} + 4\mathbf{A}^{(1)}(-1)\mathbf{e}_{2,3} + 2\mathbf{A}(-1)\mathbf{e}_{2,3} + 8\mathbf{A}(-1)\mathbf{e}_{3,3} \\ &\quad = \begin{bmatrix} -2\lambda_1 - 8\lambda_2 - 16\lambda_3 \\ -\mu_1 - 2\mu_2 + 1 \\ -\frac{1}{2}\epsilon_1 - 2\epsilon_2 - 4\epsilon_3 + 2 \end{bmatrix} = \mathbf{0}_3, \\ &\mathbf{A}^{(2)}(1)\mathbf{e}_{1,3} - 4\mathbf{A}^{(1)}(1)\mathbf{e}_{2,3} + 2\mathbf{A}(1)\mathbf{e}_{2,3} + 8\mathbf{A}(1)\mathbf{e}_{3,3} \\ &\quad = \begin{bmatrix} 2\lambda_1 + 8\lambda_2 + 16\lambda_3 \\ \mu_1 + 2\mu_2 + 1 \\ \frac{1}{2}\epsilon_1 + 2\epsilon_2 + 4\epsilon_3 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

which are satisfied if and only if $\epsilon_3 = \frac{1-\epsilon_2}{2}$ and $\lambda_3 = \frac{-1-8\lambda_2}{16}$ as given in (3.29). Now, we consider the algebraic conditions to reproduce polynomials of degree 3. Observe that $\alpha_{3,1,1} = \alpha_{3,2,1} = 6$, $\alpha_{3,3,1} = 4$ and $\alpha_{3,2,2} = \alpha_{3,3,2} = 24$. Therefore, we get

$$\begin{aligned} &\mathbf{A}^{(3)}(-1)\mathbf{e}_{1,3} + 6\mathbf{A}^{(2)}(-1)\mathbf{e}_{2,3} + 6\mathbf{A}^{(1)}(-1)\mathbf{e}_{2,3} + 4\mathbf{A}(-1)\mathbf{e}_{2,3} \\ &\quad + 24\mathbf{A}^{(1)}(-1)\mathbf{e}_{3,3} + 24\mathbf{A}(-1)\mathbf{e}_{3,3} = \begin{bmatrix} -6\lambda_1 - 24\lambda_2 - 48\lambda_3 \\ -3\mu_1 - 10\mu_2 - 24\mu_3 + 2 \\ -\frac{6}{4}\epsilon_1 - 6\epsilon_2 - 12\epsilon_3 + 6 \end{bmatrix} = \mathbf{0}_3, \end{aligned}$$

$$\begin{aligned}\mathbf{A}^{(3)}(1)\mathbf{e}_{1,3}-6\mathbf{A}^{(2)}(1)\mathbf{e}_{2,3}+6\mathbf{A}^{(1)}(1)\mathbf{e}_{2,3}-4\mathbf{A}(1)\mathbf{e}_{2,3}&\\ +24\mathbf{A}^{(1)}(1)\mathbf{e}_{3,3}-24\mathbf{A}(-1)\mathbf{e}_{3,3}&=\begin{bmatrix}-6\lambda_1-24\lambda_2-48\lambda_3\\ -3\mu_1-10\mu_2-24\mu_3-2\\ -\frac{6}{4}\epsilon_1-6\epsilon_2-12\epsilon_3-6\end{bmatrix}=\begin{bmatrix}0\\ -4\\ -12\end{bmatrix}\end{aligned}$$

which are satisfied if and only if $\mu_3=\frac{2\mu_2-3}{24}$. So, our algebraic conditions coincide with (3.29).

Bibliography

[1] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Comput. Aided Design*, 10(6):350–355, 1978.

[2] A. S. Cavaretta, W. Dahmen, and C. A. Michelli. *Stationary Subdivision*, volume 93. Memoirs of the American Mathematical Society, 1991.

[3] G. M. Chaikin. An algorithm for high speed curve generation. *Computer Graphics and Image Processing*, 3:346–349, 1974.

[4] M. Charina, C. Conti, K. Jetter, and G. Zimmermann. Scalar multivariate subdivision schemes and box splines. *Comput. Aided Geom. Design*, 28(5):285–306, 2011.

[5] M. Charina, C. Conti, and T. Sauer. Regularity of multivariate vector subdivision schemes. *Numer. Algorithms*, 39:97–113, 2005.

[6] A. Cohen and R. Ryan. *Wavelets and Multiscale Signal Processing.* Chapman & Hall, London, 1995.

[7] C. Conti, M. Cotronei, and T. Sauer. Factorization of Hermite subdivision operators preserving exponentials and polynomials. *Adv. Comput. Math.*, 42(5):1055–1079, 2016.

[8] C. Conti, M. Cotronei, and T. Sauer. Convergence of level-dependent Hermite subdivision schemes. *Appl. Numer. Math.*, 116:119–128, 2017.

[9] C. Conti and K. Hormann. Polynomial reproduction for univariate subdivision schemes of any arity. *J. Approx. Theory*, 163(4):413–437, 2011.

[10] C. Conti, J.-L. Merrien, and L. Romani. Dual Hermite subdivision schemes of de Rham-type. *BIT Num. Math.*, 54(4):955–977, 2014.

[11] C. Conti and L. Romani. Algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction. *J. Comput. Appl. Math.*, 236:543–556, 2011.

[12] C. Conti, L. Romani, and M. Unser. Ellipse-preserving Hermite interpolation and subdivision. *J. Math. Anal. Appl.*, 426(1):211–227, 2015.

[13] C. Conti, L. Romani, and J. Yoon. Approximation order and approximate sum rules in subdivision. *J. Approx. Theory*, 207:380–401, 2016.

[14] C. Conti and G. Zimmermann. Interpolatory rank-1 vector subdivision schemes. *Comput. Aided Geom. Design*, 21(4):341–351, 2004.

[15] M. Cotronei, C. Moosmüller, T. Sauer, and N. Sissouno. Level-dependent interpolatory Hermite subdivision schemes and wavelets. *Constr. Approx.*, 2018. DOI:10.1007/s00365-018-9444-4.

[16] M. Cotronei and N. Sissouno. A note on Hermite multiwavelets with polynomial and exponential vanishing moments. *Appl. Numer. Math.*, 120:21–34, 2017.

[17] I. Daubechies. *Ten Lectures on Wavelets*. SIAM, Philadelphia, 1992.

[18] G. de Rham. Sur une courbe plane. *J. de Math. Pures Appl.*, 35:25–42, 1956.

[19] G. Deslauriers and S. Dubuc. Symmetric iterative interpolation processes. *Constr. Approx.*, 5:49–68, 1989.

[20] M. P. do Carmo. *Riemannian Geometry*. Birkhäuser Verlag, 1992.

[21] D. L. Donoho. Wavelet-type representation of Lie-valued data. *Talk at the IMI "Approximation and Computation" meeting*, May 12-17 2001. Charleston, South Carolina.

[22] S. Dubuc and J.-L. Merrien. de Rham transform of a Hermite subdivision scheme. In M. Neamtu and L. L. Schumaker, editors, pages 121-132. *Approximation Theory XII: Nashville*, Nashboro Press, 2008.

[23] S. Dubuc and J.-L. Merrien. Hermite subdivision schemes and Taylor polynomials. *Constr. Approx.*, 29:219–245, 2009.

[24] R. Dyer, G. Vegter, and M. Wintraecken. Barycentric coordinate neighbourhoods in Riemannian manifolds. *https://arxiv.org/abs/1606.01585*, 2016.

[25] N. Dyn. Subdivision schemes in CAGD. *Adv. Numer. Anal.*, 2:36–104, 1992.

[26] N. Dyn, J. Gregory, and D. Levin. A 4-point interpolatory subdivision scheme for curve design. *Comput. Aided Geom. Design*, 4:257–268, 1987.

[27] N. Dyn, J. Gregory, and D. Levin. A Butterfly subdivision scheme for surface interpolation with tension control. *ACM Trans. Graph.*, 9:160–169, 1990.

[28] N. Dyn, J. Gregory, and D. Levin. Analysis of uniform binary subdivision schemes for curve design. *Constr. Approx.*, 7(2):127–148, 1991.

[29] N. Dyn, K. Hormann, M. A. Sabin, and Z. Shen. Polynomial reproduction by symmetric subdivision schemes. *J. Approx. Theory*, 155(1):28–42, 2008.

[30] N. Dyn and D. Levin. Subdivision schemes in geometric modelling. *Acta Numer.*, 11:73–144, 2002.

[31] N. Dyn, D. Levin, and A. Luzzatto. Exponential reproducing subdivision schemes. *Found. Comput. Math.*, 3:187–206, 2003.

[32] N. Dyn and N. Sharon. A global approach to the refinement of manifold data. *Math. Comp.*, 86(303):375–395, 2017.

[33] N. Dyn and N. Sharon. Manifold-valued subdivision schemes based on geodesic inductive averaging. *J. Comput. Appl. Math.*, 311:54–67, 2017.

[34] O. Ebner. Convergence of iterative schemes in metric spaces. *Proceedings of the American Mathematical Society*, 141:677–686, 2013.

[35] O. Ebner. Stochastic aspects of refinement schemes on metric spaces. *SIAM J. Numer. Anal.*, 52:717–734, 2014.

[36] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science.* Addison-Wesley Longman, Boston, 1994.

[37] P. Grohs. Smoothness equivalence properties of univariate subdivision schemes and their projection analogues. *Numer. Math.*, 113:163–180, 2009.

[38] P. Grohs. Smoothness of interpolatory multivariate subdivision in Lie groups. *IMA J. Numer. Anal.*, 29(3):760–772, 2009.

[39] P. Grohs. Approximation order from stability of nonlinear subdivision schemes. *J. Approx. Theory*, 162:1085–1094, 2010.

[40] P. Grohs. A general proximity analysis of nonlinear subdivision schemes. *SIAM J. Math. Anal.*, 42:729–750, 2010.

[41] P. Grohs. Stability of manifold-valued subdivision schemes and multiscale transformation. *Constr. Approx.*, 32(3):569–596, 2010.

[42] P. Grohs and J. Wallner. Log-exponential analogues of univariate subdivision schemes in Lie groups and their smoothness properties. In M. Neamtu and L. L. Schumaker, editors, *Approximation Theory XII: San Antonio 2007*, pages 181-190. Nasboro Press, 2008.

[43] B. Han. Vector cascade algorithms and refinable function vectors in Sobolev spaces. *J. Approx. Theory*, 124(1):44–88, 2003.

[44] B. Han and T. Yu. Face-based Hermite subdivision schemes. *J. Concr. Appl. Math.*, 4:435–450, 2003.

[45] B. Han, T. Yu, and Y. Xue. Noninterpolatory Hermite subdivision schemes. *Math. Comp.*, 74:1345–1367, 2005.

[46] H. Hardering. *Intrinsic Discretization Error Bounds for Geodesic Finite Elements.* PhD thesis, FU Berlin, 2015.

[47] S. Harizanov and P. Oswald. Stability of nonlinear subdivision and multiscale transforms. *Constr. Approx.*, 31(3):359–393, 2010.

[48] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[49] B. Jeong and J. Yoon. Construction of Hermite subdivision schemes reproducing polynomials. *J. Math. Anal. Appl.*, 451(1):565–582, 2017.

[50] B. Jeong and J. Yoon. Analysis of non-stationary Hermite subdivision schemes reproducing exponential polynomials. *J. Comput. Appl. Math.*, 349:452–469, 2019.

[51] R-Q. Jia and S-T. Liu. Wavelet bases of Hermite cubic splines on the interval. *Adv. Comput. Math.*, 25(1-3):23–39, 2006.

[52] H. Karcher. Riemannian center of mass and mollifier smoothing. *Comm. Pure Appl. Math.*, 30:509–541, 1977.

[53] S. Kobayashi and K. Nomizu. *Foundations of differential geometry Volume II.* Wiley, 1969.

[54] J. Lane and R. Riesenfeld. A theoretical development for the computer generation and display of piecewise polynomial surfaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2(1):35–46, 1980.

[55] A. Levin. Polynomial generation and quasi-interpolation in stationary non-uniform subdivision. *Adv. Comput. Math.*, 20:41–60, 2003.

[56] C. Loop. Smooth subdivision surfaces based on triangles. Master's thesis, University of Utah, Department of Mathematics, 1987.

[57] J.-L. Merrien. A family of Hermite interpolants by bisection algorithms. *Numer. Algorithms*, 2(2):187–200, 1992.

[58] J.-L. Merrien and T. Sauer. A generalized Taylor factorization for Hermite subdivision schemes. *J. Comput. Appl. Math.*, 236(4):565–574, 2011.

[59] J.-L. Merrien and T. Sauer. From Hermite to stationary subdivision schemes in one and several variables. *Adv. Comp. Math.*, 36:547–579, 2012.

[60] J.-L. Merrien and T. Sauer. Generalized Taylor operators and polynomial chains for Hermite subdivision schemes. *Numer. Math.*, 142(1):167–203, 2019.

[61] C. Micchelli and T. Sauer. Regularity of multiwavelets. *Adv. Comput. Math.*, 7(4):455–545, 1997.

[62] C. Micchelli and T. Sauer. On vector subdivision. *Mathematische Zeitschrift*, 229(4):621–674, 1998.

[63] C. Moosmüller. C^1 analysis of Hermite subdivision schemes on manifolds. *SIAM J. Numer. Anal.*, 54:3003–3031, 2016.

[64] C. Moosmüller. Hermite subdivision on manifolds via parallel transport. *Adv. Comput. Math.*, 43(5):1059–1074, 2017.

[65] C. Moosmüller. A note on spectral properties of Hermite subdivision operators. *Comput. Aided Geom. Design*, 69:1–10, 2019.

[66] C. Moosmüller and N. Dyn. Increasing the smoothness of vector and Hermite subdivision schemes. *IMA J. Numer. Anal.*, 39(2):579–606, 2019.

[67] B. Odehnal. Subdivision algorithms for ruled surfaces. *J. Geom. Graph.*, 12:1–18, 2008.

[68] X. Pennec. Barycentric subspace analysis on manifolds. *Ann. Statist.*, 46 (6A):2711–2746, 2018.

[69] J. Peters and U. Reif. *Sudivison surfaces.* Springer, 2018.

[70] I. Ur Rahman, I. Drori, V. C. Stodden, D. L. Donoho, and P. Schröder. Multiscale representations for manifold-valued data. *Multiscale Model. Sim.*, 4(4):1201–1232, 2005.

[71] L. Romani. A circle-preserving C^2 Hermite interpolatory subdivision scheme with tension control. *Comput. Aided Geom. Design*, 27(1):36–47, 2010.

[72] O. Sander. Geodesic finite elements of higher order. *IMA J. Appl. Math.*, 36:238–266, 2016.

[73] N. Sharon and U. Itai. Subdivision schemes for positive definite matrices. *Found. Comput. Math.*, 13:347–369, 2013.

[74] V. Uhlmann, R. Delgado-Gonzalo, C. Conti, L. Romani, and M. Unser. Exponential Hermite splines for the analysis of biomedical images. *ICASSP, IEEE International Conference on Acoustics, Speech and Signal Processing - Proceedings 6853874*, 317:1631–1634, 2014.

[75] J. Wallner. Smoothness analysis of subdivision schemes by proximity. *Constr. Approx.*, 24:289–318, 2004.

[76] J. Wallner. On convergent interpolatory subdivision schemes in Riemannian geometry. *Constr. Approx.*, 40:473–486, 2014.

[77] J. Wallner and N. Dyn. Convergence and C^1 analysis of subdivision schemes on manifolds by proximity. *Comput. Aided Geom. Design*, 22:593–622, 2005.

[78] J. Wallner, E. Nava Yazdani, and P. Grohs. Smoothness properties of Lie group subdivision schemes. *Multiscale Model. Sim.*, 6:493–505, 2007.

[79] J. Wallner, E. Nava Yazdani, and A. Weinmann. Convergence and smothness analysis of subdivision rules in Riemannian and symmetric spaces. *Adv. Comput. Math.*, 34:201–218, 2011.

[80] A. Weinmann. Nonlinear subdivision schemes on irregular meshes. *Constr. Approx.*, 31(3):395–415, 2010.

[81] A. Weinmann. Subdivision schemes with general dilation in the geometric and nonlinear setting. *J. Approx. Theory*, 164(1):105–137, 2012.

[82] G. Xie and T. Yu. Smoothness equivalence properties of manifold-valued data subdivision schemes based on the projection approach. *SIAM J. Numer. Anal.*, 45:1200–1225, 2007.

[83] G. Xie and T. Yu. Smoothness equivalence properties of interpolatory Lie group subdivision schemes. *IMA J. Appl. Math.*, 30(3):731–750, 2010.