EXPLICATIO SIGNORUM.

$\overline{\mathfrak{ab}}$	denotet	complexum *omnium* punctorum cum punctis $\mathfrak{a}$, $\mathfrak{b}$ in recta sitorum.
$\mathfrak{a}\overline{\mathfrak{b}}$	«	rectæ $\overline{\mathfrak{ab}}$ in $\mathfrak{a}$ bifariam sectæ dimidium illud, quod punctum $\mathfrak{b}$ complectitur.
$\overline{\mathfrak{abc}}$	«	complexum *omnium* punctorum, quæ cum punctis $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ (non in eadem recta sitis) in eodem plano sunt.
$\mathfrak{ab}\overline{\mathfrak{c}}$	«	plani $\overline{\mathfrak{abc}}$ per $\overline{\mathfrak{ab}}$ bifariam secti dimidium, punctum $\mathfrak{c}$ complectens.
$\mathfrak{abc}$	«	portionum, in quas $\overline{\mathfrak{abc}}$ per complexum rectarum $\mathfrak{b}\overline{\mathfrak{a}}$, $\mathfrak{b}\overline{\mathfrak{c}}$ dividitur, *minorem*; sive *angulum*, cuius $\mathfrak{b}\overline{\mathfrak{a}}$, $\mathfrak{b}\overline{\mathfrak{c}}$ crura sunt.
$\mathfrak{abcd}$	«	(si $\mathfrak{d}$ in $\mathfrak{abc}$ sit et $\overline{\mathfrak{ba}}$, $\overline{\mathfrak{cd}}$ se invicem non secent) portionem ipsius $\mathfrak{abc}$ inter $\mathfrak{b}\overline{\mathfrak{a}}$, $\mathfrak{bc}$, $\mathfrak{c}\overline{\mathfrak{d}}$ comprehensam; $\mathfrak{bacd}$ vero portionem plani $\overline{\mathfrak{abc}}$ inter $\overline{\mathfrak{ab}}$, $\overline{\mathfrak{cd}}$ sitam.
R	«	angulum rectum.
$\mathfrak{ab} \stackrel{\frown}{=} \mathfrak{cd}$	«	$\mathfrak{cab} = \mathfrak{acd}$.
$\equiv$	«	congruens.*
$x \rightsquigarrow a$	«	x tendere ad limitem a.
$\bigcirc r$	«	peripheriam circuli radii r.
$\odot r$	«	aream circuli radii r.

* Sit fas, signo hocce, quo summus Geometra *GAUSS numeros congruos* insignivit, congruentiam geometricam quoque denotare: nulla ambiguitate exinde metuenda.

§. 1.

Si rectam $\mathfrak{a}\overline{\mathfrak{m}}$ non secet plani eiusdem recta $\mathfrak{b}\overline{\mathfrak{n}}$, at secet quævis $\mathfrak{b}\overline{\mathfrak{p}}$ (in $\mathfrak{abn}$): designetur hoc per Fig. 1.

$$\mathfrak{bn} \,|||\, \mathfrak{am}.$$

Dari talem $\mathfrak{b}\overline{\mathfrak{n}}$, et quidem *unicam*, e quovis puncto $\mathfrak{b}$ (extra $\overline{\mathfrak{a}\overline{\mathfrak{m}}}$), atque

$$\mathfrak{bam} + \mathfrak{abn} \text{ non } > 2R$$

esse patet; nam $\mathfrak{bc}$ circa $\mathfrak{b}$ mota, donec

$$\mathfrak{bam} + \mathfrak{abc} = 2R$$

fiat, $\mathfrak{b}\overline{\mathfrak{c}}$ aliquando *primo* non secat $\mathfrak{a}\overline{\mathfrak{m}}$, estque tunc $\mathfrak{bc} \,|||\, \mathfrak{am}$.

Nec non patet esse $\mathfrak{bn} \,|||\, \mathfrak{em}$, ubivis sit $\mathfrak{e}$ in $\overline{\mathfrak{a}\overline{\mathfrak{m}}}$ (supponendo in omnibus talibus casibus esse $\mathfrak{am} > \mathfrak{ae}$).

Et si, puncto $\mathfrak{c}$ in $\mathfrak{a}\overline{\mathfrak{m}}$ abeunte in infinitum, semper sit $\mathfrak{cd} = \mathfrak{cb}$: erit semper

$$\mathfrak{cdb} = (\mathfrak{cbd} < \mathfrak{nbc});$$

ast $\mathfrak{nbc} \to 0$; adeoque et $\mathfrak{adb} \to 0$.

§. 2.

Si $\mathfrak{bn} \,|||\, \mathfrak{am}$; *est quoque* $\mathfrak{cn} \,|||\, \mathfrak{am}$. Fig. 2.

Nam sit $\mathfrak{d}$ ubicunque in $\mathfrak{macn}$. Si $\mathfrak{c}$ in $\mathfrak{b}\overline{\mathfrak{n}}$ sit; $\mathfrak{b}\widetilde{\mathfrak{d}}$ secat $\mathfrak{a}\overline{\mathfrak{m}}$ (propter $\mathfrak{bn} \,|||\, \mathfrak{am}$), adeoque et $\mathfrak{c}\widetilde{\mathfrak{d}}$ secat $\mathfrak{a}\overline{\mathfrak{m}}$; si vero $\mathfrak{c}$ in $\mathfrak{b}\overline{\mathfrak{p}}$ fuerit; sit $\mathfrak{bq} \,|||\, \mathfrak{cd}$: cadit $\mathfrak{b}\overline{\mathfrak{q}}$ in $\mathfrak{abn}$ (§. 1.) secatque $\mathfrak{a}\overline{\mathfrak{m}}$, adeoque et $\mathfrak{c}\widetilde{\mathfrak{d}}$ secat $\mathfrak{a}\overline{\mathfrak{m}}$. Quævis $\mathfrak{c}\widetilde{\mathfrak{d}}$ igitur (in $\mathfrak{acn}$) secat in utroque casu $\mathfrak{a}\overline{\mathfrak{m}}$ absque eo, ut $\mathfrak{c}\overline{\mathfrak{n}}$ ipsam $\mathfrak{a}\overline{\mathfrak{m}}$ secet. Est ergo semper $\mathfrak{cn} \,|||\, \mathfrak{am}$.

§. 3.

Fig. 2. *Si tam* $\mathfrak{br}$ *quam* $\mathfrak{cs}$ *sit* $|||\,\mathfrak{am}$, *et* $\mathfrak{c}$ *non sit in* $\widetilde{\mathfrak{br}}$; *tum* $\mathfrak{b}\bar{\mathfrak{r}}$, $\mathfrak{c}\bar{\mathfrak{s}}$ *se invicem haud secant.*

Si enim $\mathfrak{b}\bar{\mathfrak{r}}$, $\mathfrak{c}\bar{\mathfrak{s}}$ punctum $\mathfrak{d}$ commune haberent; (per §. 2.) essent $\mathfrak{dr}$ et $\mathfrak{ds}$ simul $|||\,\mathfrak{am}$, caderetque (§. 1.) $\mathfrak{d}\bar{\mathfrak{s}}$ in $\mathfrak{d}\bar{\mathfrak{r}}$ et $\mathfrak{c}$ in $\widetilde{\mathfrak{br}}$ (contra hyp.).

§. 4.

Fig. 3. *Si* $\mathfrak{man} > \mathfrak{mab}$; *pro quovis puncto* $\mathfrak{b}$ *ipsius* $\mathfrak{a}\bar{\mathfrak{b}}$ *datur tale* $\mathfrak{c}$ *in* $\mathfrak{a}\bar{\mathfrak{m}}$, *ut sit* $\mathfrak{bcm} = \mathfrak{nam}$.

Nam datur (per §. 1.) $\mathfrak{bdm} > \mathfrak{nam}$, adeoque $\mathfrak{mdp} = \mathfrak{man}$, caditque $\mathfrak{b}$ in $\mathfrak{nadp}$. Si igitur $\mathfrak{nam}$ iuxta $\mathfrak{am}$ feratur, usquequo $\mathfrak{a}\bar{\mathfrak{n}}$ in $\mathfrak{d}\bar{\mathfrak{p}}$ veniat; aliquando $\mathfrak{a}\bar{\mathfrak{n}}$ per $\mathfrak{b}$ transiisse, et aliquod $\mathfrak{bcm} = \mathfrak{nam}$ esse oportet.

§. 5.

Fig. 1. *Si* $\mathfrak{bn}\,|||\,\mathfrak{am}$, *datur tale punctum* $\mathfrak{f}$ *in* $\overline{\mathfrak{am}}$, *ut sit* $\mathfrak{fm} \bumpeq \mathfrak{bn}$.

Nam (per §. 1.) datur $\mathfrak{bcm} > \mathfrak{cbn}$; et si $\mathfrak{ce} = \mathfrak{cb}$, adeoque $\mathfrak{ec} \bumpeq \mathfrak{bc}$; patet esse $\mathfrak{bem} < \mathfrak{ebn}$. Feratur $\mathfrak{p}$ per $\mathfrak{ec}$, angulo $\mathfrak{bpm}$ semper u, et angulo $\mathfrak{pbn}$ semper v dicto; patet u esse prius ei simultaneo v minus, posterius vero esse maius. Crescit vero u a $\mathfrak{bem}$ usque $\mathfrak{bcm}$ *continuo*; cum (per §. 4.) *nullus* angulus $> \mathfrak{bem}$ et $< \mathfrak{bcm}$ detur, cui u aliquando $=$ non fiat; pariter decrescit v ab $\mathfrak{ebn}$ usque $\mathfrak{cbn}$ continuo: datur itaque in $\mathfrak{ec}$ tale $\mathfrak{f}$, ut $\mathfrak{bfm} = \mathfrak{fbn}$ sit.

§. 6.

Si $\mathfrak{bn}\,|||\,\mathfrak{am}$, *atque ubivis sit* $\mathfrak{e}$ *in* $\overline{\mathfrak{am}}$ *et* $\mathfrak{g}$ *in* $\widetilde{\mathfrak{bn}}$: *tum* $\mathfrak{gn}\,|||\,\mathfrak{em}$ *et* $\mathfrak{em}\,|||\,\mathfrak{gn}$.

Nam (per §. 1.) est $\mathfrak{bn}\,|||\,\mathfrak{em}$, et hinc (per §. 2.) $\mathfrak{gn}\,|||\,\mathfrak{em}$.

Si porro $\mathfrak{fm} \bumpeq \mathfrak{bn}$ (§. 5.); tum $\mathfrak{mfbn} = \mathfrak{nbfm}$, adeoque (cum $\mathfrak{bn}\,|||\,\mathfrak{fm}$ sit) etiam $\mathfrak{fm}\,|||\,\mathfrak{bn}$, et (per præc.) $\mathfrak{em}\,|||\,\mathfrak{gn}$.

§. 7.

Si tam bn *quam* cp *sit* ||| am, *et* c *non sit in* $\overline{bn}$: *est etiam* bn ||| cp. Fig. 4.

Nam bñ, cp̃ se invicem non secant (§. 3.); sunt vero am, bn, cp aut in plano, aut non; atque in casu primo am aut in bncp est, aut non.

Si am, bn, cp in plano sint, et am in bncp cadat; tum quævis bq̃ (in nbc) secat $\overline{am}$ in aliquo puncto d (quia bn ||| am); porro cum dm ||| cp sit (§. 6.), patet dq̃ secare cp̃, adeoque esse bn ||| cp.

Si vero bn, cp in eadem plaga ipsius am sint; tum aliqua earum ex. gr. cp *intra* duas reliquas $\overline{bn}$, $\overline{am}$ cadit; quævis bq̃ (in nba) autem secat am̃, adeoque et ipsam cp̃. Est itaque bn ||| cp.

Si mab, mac *angulum* efficiant: tum cbn cum abn nonnisi bñ, am̃ vero (in abn) cum bñ, adeoque nbc quoque cum am̃, nihil commune habent. Per quamvis bd̃ (in nba) autem positum bcd̃ secat am̃, quia (propter bn ||| am) bd̃ secat am̃. Moto itaque bcd̃ circa bc, donec ipsam am̃ *prima vice* deserat, postremo cadet bcd̃ in bcñ. Eadem ratione cadet idem in bcp̃; cadit igitur bn in bcp. Porro si br ||| cp; tum (quia etiam am ||| cp) pari ratione cadit br in bam; nec non (propter br ||| cp) in bcp. Itaque br̃ ipsis mab, pcb commune, nempe ipsum bñ est, atque hinc bn ||| cp.

Si igitur cp ||| am, et b extra $\overline{cam}$ sit: tum sectio ipsorum bam, bcp, nempe bñ est ||| tam ad am, quam ad cp.*

§. 8.

Si bn ||| et ≏ cp (vel brevius bn ||| ≏ cp), *atque* am (*in* nbcp) *rectam* bc *perpendiculariter bisecet*; *tum* bn ||| am. Fig. 5.

Si enim bñ secaret $\overline{am}$, etiam cp̃ secaret $\overline{am}$ in eodem puncto (cum mabn = macp), quod et ipsis bñ, cp̃ commune esset, quamvis bn ||| cp sit. Quævis bq̃ (in cbn) vero secat cp̃; adeoque secat bq̃ etiam am̃. Consequenter bn ||| am.

* Casu tertio *praemisso* duo priores, adinstar casus secundi §. 10. brevius ac elegantius simul absolvi possunt. (Ed. I. Tom. I. Errata Appendicis).

§. 9.

Fig. 6. *Si* $\mathfrak{bn} \,|||\, \mathfrak{am}$, $\mathfrak{map} \perp \mathfrak{mab}$, *atque angulus, quem* $\mathfrak{nbd}$ cum $\mathfrak{nba}$ (in ea plaga ipsius $\mathfrak{mabn}$, ubi $\mathfrak{map}$ est) *facit, sit* $< R$: *tum* $\mathfrak{map}$ et $\mathfrak{nbd}$ *se invicem secant.*

Nam sit

$$\mathfrak{bam} = R,\ \mathfrak{ac} \perp \mathfrak{bn}$$

(sive in $\mathfrak{b}$ cadat $\mathfrak{c}$, sive non), et

$$\mathfrak{ce} \perp \mathfrak{bn} \text{ (in } \mathfrak{nb\tilde{d}});$$

erit (per hyp.) $\mathfrak{ace} < R$, et $\mathfrak{af}\ (\perp \mathfrak{ce})$ in $\mathfrak{ace}$ cadet. Sit $\mathfrak{a\tilde{p}}$ sectio (punctum $\mathfrak{a}$ commune habentium) $\mathfrak{b\tilde{f}}$ et $\mathfrak{am\tilde{p}}$; erit

$$\mathfrak{bap} = \mathfrak{bam} = R$$

(cum sit $\mathfrak{bam} \perp \mathfrak{map}$). Si denique $\mathfrak{ab\tilde{f}}$ in $\mathfrak{ab\tilde{n}}$ ponatur ($\mathfrak{a}$ et $\mathfrak{b}$ manentibus); cadet $\mathfrak{a\tilde{p}}$ in $\mathfrak{a\tilde{m}}$; atque cum

$$\mathfrak{ac} \perp \mathfrak{bn} \quad \text{et} \quad \mathfrak{af} < \mathfrak{ac}$$

sit, patet $\mathfrak{af}$ *intra* $\mathfrak{b\tilde{n}}$ terminari, adeoque $\mathfrak{bf}$ in $\mathfrak{abn}$ cadere. Secat autem $\mathfrak{b\tilde{f}}$ ipsam $\mathfrak{a\tilde{p}}$ in *hoc* situ (quia $\mathfrak{bn} \,|||\, \mathfrak{am}$), adeoque etiam in situ *primo* $\mathfrak{a\tilde{p}}$ et $\mathfrak{b\tilde{f}}$ se invicem secant; estque punctum sectionis ipsis $\mathfrak{ma\tilde{p}}$ et $\mathfrak{nb\tilde{d}}$ commune: secant itaque $\mathfrak{ma\tilde{p}}$ et $\mathfrak{nb\tilde{d}}$ se invicem.

Facile exhinc sequitur $\mathfrak{ma\tilde{p}}$ et $\mathfrak{nb\tilde{d}}$ se mutuo secare, si summa internorum, quos cum $\mathfrak{mabn}$ efficiunt, $< 2R$ sit.

§. 10.

Fig. 7. *Si tam* $\mathfrak{bn}$ *quam* $\mathfrak{cp}$ *sit* $|||\!\simeq \mathfrak{am}$; *est etiam* $\mathfrak{bn} \,|||\!\simeq \mathfrak{cp}$.

Nam $\mathfrak{mab}$ et $\mathfrak{mac}$ aut *angulum* efficiunt, aut in plano sunt.

Si prius; bisecet $\overline{\mathfrak{qdf}}$ rectam $\mathfrak{ab}$ perpendiculariter; erit $\mathfrak{dq} \perp \mathfrak{ab}$, adeoque $\mathfrak{dq} \,|||\, \mathfrak{am}$ (§. 8.); pariter si $\overline{\mathfrak{ers}}$ bisecet rectam $\mathfrak{ac}$ perpendiculariter, est $\mathfrak{er} \,|||\, \mathfrak{am}$; unde $\mathfrak{dq} \,|||\, \mathfrak{er}$ (§. 7.). Facile hinc (per §. 9.) consequitur, $\overline{\mathfrak{qdf}}$

et $\widetilde{ers}$ se mutuo secare, et sectionem $\widetilde{fs}$ esse $\mathbin{|||}\mathfrak{dq}$ (§. 7.), atque (propter $bn \mathbin{|||} \mathfrak{dq}$) esse etiam

$$fs \mathbin{|||} bn.$$

Est porro (pro quovis puncto ipsius $\widetilde{fs}$)

$$fb = fa = fc,$$

caditque $\widetilde{fs}$ in planum $\widetilde{tgf}$, rectam bc perpendiculariter bisecans. Est vero (per §. 7.) (cum sit $fs \mathbin{|||} bn$) etiam

$$gt \mathbin{|||} bn.$$

Pari modo demonstratur $gt \mathbin{|||} cp$ esse. Interim gt bisecat rectam bc perpendiculariter; adeoque $tgbn = tgcp$ (§. 1.) et

$$bn \mathbin{|||} \bumpeq cp.$$

Si bn, am, cp in plano sint; sit (*extra* hoc planum cadens) $fs \mathbin{|||} \bumpeq am$; tum (per præc.) $fs \mathbin{|||} \bumpeq$ tam ad bn quam ad cp, adeoque et $bn \mathbin{|||} \bumpeq cp$.

§. 11.

Complexus puncti a, atque *omnium* punctorum, quorum quodvis b tale est, ut si $bn \mathbin{|||} am$ sit, sit etiam $bn \bumpeq am$; dicatur F: sectio vero ipsius F cum quovis plano rectam am complectente nominetur L.

In quavis recta, quæ $\mathbin{|||} am$ est, F gaudet puncto, et nonnisi uno; atque patet L per am dividi in duas partes congruentes; dicatur $\widetilde{am}$ *axis* ipsius L; patet etiam, in quovis plano rectam am complectente, pro *axe* $\widetilde{am}$ unicum L dari. Quodvis eiusmodi L, dicatur *L ipsius* $\widetilde{am}$ (in plano, de quo agitur, intelligendo). Patet per L circa am revolutum, F describi, cuius $\widetilde{am}$ *axis* vocetur, et vicissim F *axi* $\widetilde{am}$ *attribuatur.*

§. 12.

Si b *ubivis in L ipsius* $\widetilde{am}$ *fuerit, et* $bn \mathbin{|||} \bumpeq am$ (§. 11.); *tum L ipsius* $\widetilde{am}$ *et L ipsius* $\widetilde{bn}$ *coincidunt.*

Nam dicatur L ipsius $b\bar{n}$ distinctionis ergo l; sitque c ubivis in l, et $cp \,|||\, \bumpeq bn$ (§. 11.); erit (cum et $bn \,|||\, \bumpeq am$ sit) $cp \,|||\, \bumpeq am$ (§. 10.), adeoque c etiam in L cadet. Et si c ubivis in L sit, et $cp \,|||\, \bumpeq am$; tum $cp \,|||\, \bumpeq bn$ (§. 10.); caditque c etiam in l (§. 11.). Itaque L et l sunt eadem; ac quævis $b\bar{n}$ est etiam axis ipsius L, et inter omnes axes ipsius L, $\bumpeq$ est.

Idem de F eodem modo patet.

§. 13.

Fig. 8. *Si* $bn \,|||\, am$, $cp \,|||\, dq$, *et* $bam + abn = 2R$ *sit*; *tum etiam* $dcp + cdq = 2R$.

Sit enim $ea = eb$ et $efm = dcp$ (§. 4.); erit (cum

$$bam + abn = 2R = abn + abg$$

sit)

$$ebg = eaf;$$

adeoque si etiam $bg = af$ sit,

$$\triangle\, ebg = \triangle\, eaf, \quad beg = aef,$$

cadetque g in $f\bar{e}$. Est porro $gfm + fgn = 2R$ (quia $egb = efa$). Est etiam $gn \,|||\, fm$ (§. 6.); itaque si $mfrs = pcdq$, tum $rs \,|||\, gn$ (§. 7.), et r in vel extra fg cadit (si cd non $= fg$, ubi res iam patet).

I. In casu primo est frs non $> (2R - rfm = fgn)$, quia $rs \,|||\, fm$; ast cum $rs \,|||\, gn$ sit, est etiam frs non $< fgn$; adeoque $frs = fgn$, et

$$rfm + frs = gfm + fgn = 2R.$$

Itaque et $dcp + cdq = 2R$.

II. Si r extra fg cadat; tunc $ngr = mfr$, sitque $mfgn = nghl = lhko$ et ita porro, usquequo $fk =$ vel prima vice $> fr$ fiat. Est heic $ko \,|||\, hl \,|||\, fm$ (§. 7.). Si k in r cadat; tum ko in rs cadit (§. 1.); adeoque

$$rfm + frs = kfm + fko = kfm + fgn = 2R;$$

si vero r in hk cadat, tum (per I.) est

$$rhl + hrs = 2R = rfm + frs = dcp + cdq.$$

§. 14.

Si $\mathfrak{bn} \parallel\!| \mathfrak{am}$, $\mathfrak{cp} \parallel\!| \mathfrak{dq}$, *et* $\mathfrak{bam} + \mathfrak{abn} < 2R$ *sit*; *tum etiam* $\mathfrak{dcp} + \mathfrak{cdq} < 2R$.

Si enim $\mathfrak{dcp} + \mathfrak{cdq}$ non esset $<$, adeoque (per §. 1.) esset $= 2R$; tum (per §. 13.) etiam $\mathfrak{bam} + \mathfrak{abn} = 2R$ esset (contra hyp.).

§. 15.

Perpensis §§. 13. et 14. *Systema Geometriae hypothesi veritatis Axiomatis Euclidei XI. insistens dicatur* Σ; *et hypothesi contrariae superstructum sit S. Omnia, quae expresse non dicentur, in* Σ *vel in S esse*; *absolute enuntiari*, i. e. *illa*, *sive* Σ *sive S reipsa sit*, *vera asseri intelligatur.*

§. 16.

Si $\mathfrak{am}$ *sit axis alicuius L*; *tum L in* Σ *recta* $\perp \mathfrak{am}$ *est.* Fig. 5.

Nam sit e quovis puncto $\mathfrak{b}$ ipsius L axis $\mathfrak{bn}$; erit in Σ

$$\mathfrak{bam} + \mathfrak{abn} = 2\mathfrak{bam} = 2R,$$

adeoque $\mathfrak{bam} = R$. Et si $\mathfrak{c}$ quodvis punctum in $\overset{\frown}{\mathfrak{ab}}$ sit, atque $\mathfrak{cp} \parallel\!| \mathfrak{am}$; est (per §. 13.) $\mathfrak{cp} \triangleq \mathfrak{am}$, adeoque $\mathfrak{c}$ in L (§. 11.).

In S vero nulla 3 *puncta* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *ipsius L vel F in recta sunt.*

Nam aliquis axium $\mathfrak{am}$, $\mathfrak{bn}$, $\mathfrak{cp}$ (ex. gr. $\mathfrak{am}$) intra duos reliquos cadit; et tunc (per §. 14.) tam $\mathfrak{bam}$ quam $\mathfrak{cam} < R$.

§. 17.

L est etiam in S linea, et F superficies. Fig. 7.

Nam (per §. 11.) quodvis planum ad axem $\mathfrak{a\overline{m}}$ (per punctum aliquod ipsius F) perpendiculare secat ipsum F in peripheria circuli, cuius planum (per §. 14.) ad nullum alium axem $\mathfrak{b\overline{n}}$ perpendiculare est. Revolvatur F circa $\mathfrak{bn}$; manebit (per §. 12.) quodvis punctum ipsius F in F, et sectio ipsius F cum plano ad $\mathfrak{b\overline{n}}$ non perpendiculari describet superficiem: atqui *F* (per §. 12.), quæcunque puncta a, b fuerint in eo, ita *sibi* congruere poterit, ut a in b cadat; est igitur *F superficies uniformis.*

Patet hinc (per §§. 11. et 12.) *L* esse *lineam uniformem.**

§. 18.

Fig. 7. *Cuiusvis plani, per punctum* a *ipsius F ad axem* am *oblique positi, sectio cum F in S peripheria circuli est.*

Nam sint a, b, c 3 puncta huius sectionis, et bn, cp axes; facient ambn, amcp angulum; nam secus planum (ex §. 16.) per a, b, c determinatum ipsam am complecteretur (contra hyp.). Plana igitur, rectas ab, ac perpendiculariter bisecantia se mutuo secant (§. 10.) in aliquo axe $\overline{fs}$ (ipsius *F*), atque fb = fa = fc. Sit ah ⊥ fs, et revolvatur fah circa fs; describet a peripheriam radii ha, per b et c euntem, et *simul* in *F* et $\overline{abc}$ sitam; nec *F* et $\overline{abc}$ præter ○ ha quidquam commune habent (§. 16.).

Patet etiam extremitate portionis fa lineæ *L* (tanquam radio) in *F* circa f mota ipsam ○ ha describi.

§. 19.

Fig. 5. *Perpendicularis* bt *ad axem* bn *ipsius L* (in planum ipsius *L* cadens) *est in S tangens ipsius L.*

Nam *L* in $\overline{bt}$ præter b nullo puncto gaudet (§. 14.), si vero bq in tbn cadat, tum centrum sectionis plani per bq ad tbn perpendicularis cum *F* ipsius $\overline{bn}$ (§. 18.) manifesto in $\overline{bq}$ locatur, et si bq diameter sit, patet $\overline{bq}$ lineam *L* ipsius $\overline{bn}$ in q secare.

§. 20.

Per quaevis duo puncta in F linea L determinatur (§§. 11. et 18.); atque (cum ex §§. 16. et 19. *L* perpendicularis ad omnes suos axes sit)

* Demonstrationem ad *S* restringere haud necesse est; quum facile ita proponatur, ut absolute (pro *S* et *Σ*) valeat. (Ed. I. Tom. I. Errata Appendicis).

quivis angulus Llineus in F angulo planorum ad F per crura perpendicularium aequalis est.

§. 21.

Duae lineae Lformes $\overline{ap}$, $\overline{bd}$ *in eodem F, cum tertia Lformi* ab *summam internorum* $< 2R$ *efficientes, se mutuo secant* (per $\overline{ap}$ in F intelligendo L per a, p ductum, per $\overline{ap}$ vero dimidium illud eius ex a incipiens, in quod p cadit). Fig. 6.

Nam si am, bn axes ipsius F sint; tum $\overline{amp}$, $\overline{bnd}$ secant se invicem (§. 9.); atque F secat eorundem sectionem (per §§. 7. et 11.); adeoque et $\overline{ap}$, $\overline{bd}$ se mutuo secant.

Patet exhinc Axioma XI. et omnia, quæ in Geometria Trigonometriaque (plana) asseruntur, *absolute* constare in F, rectarum vices lineis L subeuntibus: idcirco functiones trigonometricæ abhinc eodem sensu accipientur, quo in Σ veniunt; et peripheria circuli, cuius radius Lformis $= r$ in F, est $= 2\pi r$, et pariter $\odot r$ (in F) $= \pi r^2$ (per π intelligendo $\frac{1}{2} \bigcirc 1$ in F, sive notum 3·1415926 . . .).

§. 22.

Si $\overline{ab}$ *fuerit L ipsius* $\overline{am}$, *et* c *in* $\overline{am}$; *atque angulus* cab (e recta $\overline{am}$ et Lformi linea $\overline{ab}$ compositus) *feratur prius iuxta* $\overline{ab}$, *tum iuxta* $\overline{ba}$ *semper porro in infinitum: erit via* $\overline{cd}$ *ipsius* c *linea L ipsius* $\overline{cm}$. Fig. 9.

Nam (posteriore l dicta) sit punctum quodvis d in $\overline{cd}$, $\overline{dn} \parallel\!| \overline{cm}$, et b punctum ipsius L in $\overline{dn}$ cadens; erit $\overline{bn} \simeq \overline{am}$, et ac $=$ bd, adeoque $\overline{dn} \simeq \overline{cm}$, consequ. d in l. Si vero d in l et $\overline{dn} \parallel\!| \overline{cm}$, atque b punctum ipsius L ipsi $\overline{dn}$ commune sit; erit $\overline{am} \simeq \overline{bn}$ et $\overline{cm} \simeq \overline{dn}$, unde manifesto bd $=$ ac, cadetque d in viam puncti c, et sunt l et $\overline{cd}$ eadem. Designetur tale l per $l \parallel L$.

§. 23.

Si linea Lformis cdf $\parallel$ abe (§. 22.), et ab $=$ be, atque $\overline{am}$, $\overline{bn}$, $\overline{ep}$ sint axes; erit manifesto cd $=$ df; et si quælibet 3 puncta a, b, e fuerint ipsius Fig. 9.

$\widetilde{\mathfrak{ab}}$, ac $\mathfrak{ab} = n \, . \, \mathfrak{cd}$: erit quoque $\mathfrak{ae} = n \, . \, \mathfrak{cf}$; adeoque (manifesto etiam pro $\mathfrak{ab}$, $\mathfrak{ae}$, $\mathfrak{dc}$ incommensurabilibus)

$$\mathfrak{ab} : \mathfrak{cd} = \mathfrak{ae} : \mathfrak{cf},$$

estque $\mathfrak{ab} : \mathfrak{cd}$ ab $\mathfrak{ab}$ *independens* et per $\mathfrak{ac}$ *prorsus determinatum.* Denotetur quotus iste, nempe $\mathfrak{ab} : \mathfrak{cd}$ litera maiore eiusdem nominis (puta per X), quo $\mathfrak{ac}$ litera minuscula (ex. gr. x) insignitur.

§. 24.

Quaecunque x et y fuerint; *est* $Y = X^{\frac{y}{x}}$ (§. 23.).

Nam aut erit alterum (ipsorum x, y) multiplum alterius (ex. gr. y ipsius x), aut non.

Si $y = nx$; sit $x = \mathfrak{ac} = \mathfrak{cg} = \mathfrak{gh}$ &c, usquequo $\mathfrak{ah} = y$ fiat; sit porro $\mathfrak{cd} \parallel \mathfrak{gk} \parallel \mathfrak{hl}$; erit (§. 23.)

$$X = \mathfrak{ab} : \mathfrak{cd} = \mathfrak{cd} : \mathfrak{gk} = \mathfrak{gk} : \mathfrak{hl};$$

adeoque

$$\frac{\mathfrak{ab}}{\mathfrak{hl}} = \left(\frac{\mathfrak{ab}}{\mathfrak{cd}}\right)^n,$$

sive

$$Y = X^n = X^{\frac{y}{x}}.$$

Si x, y multipla ipsius i sint, puta

$$x = mi \quad \text{et} \quad y = ni;$$

est (per præc.)

$$X = I^m, \quad Y = I^n,$$

consequ.

$$Y = X^{\frac{n}{m}} = X^{\frac{y}{x}}.$$

Idem ad casum incommensurabilitatis ipsorum x, y facile extenditur. Si vero fuerit $q = y - x$; erit manifesto $Q = Y : X$.

Nec non manifestum est, in Σ pro quovis x esse $X = 1$, in S vero $X > 1$ esse, atque pro *quibusvis* $\mathfrak{ab}$, $\mathfrak{abe}$ dari tale $\mathfrak{cdf} \parallel \mathfrak{abe}$, ut sit $\mathfrak{cdf} = \mathfrak{ab}$, unde $\mathfrak{ambn} = \mathfrak{amep}$ erit, etsi hoc illius qualevis multiplum sit; quod singulare quidem est, sed absurditatem ipsius S evidenter non probat.

§. 25.

In quovis rectilineo triangulo sunt peripheriae radiorum lateribus aequalium, uti sinus angulorum oppositorum. Fig. 10.

Sit enim $\mathfrak{abc} = R$, et $\mathfrak{am} \perp \mathfrak{bac}$, atque sint $\mathfrak{bn}$, $\mathfrak{cp} \parallel\!| \mathfrak{am}$; erit $\mathfrak{cab} \perp \mathfrak{ambn}$, adeoque (cum $\mathfrak{cb} \perp \mathfrak{ba}$ sit) $\mathfrak{cb} \perp \mathfrak{ambn}$, consequ. $\mathfrak{cpbn} \perp \mathfrak{ambn}$. Secet F *ipsius* $\overline{\mathfrak{cp}}$ rectas $\overline{\mathfrak{bn}}$, $\overline{\mathfrak{am}}$ (respective) in $\mathfrak{d}$, $\mathfrak{e}$, et fascias $\mathfrak{cpbn}$, $\mathfrak{cpam}$, $\mathfrak{bnam}$ in lineis Lformibus $\mathfrak{cd}$, $\mathfrak{ce}$, $\mathfrak{de}$; erit (§. 20.) $\mathfrak{cde}$ = angulo ipsorum $\mathfrak{ndc}$, $\mathfrak{nde}$, adeoque $= R$; atque pari ratione est $\mathfrak{ced} = \mathfrak{cab}$.

Est autem (per §. 21.) in Llineo triangulo $\mathfrak{ced}$ (heic radio semper $=1$ posito)

$$\mathfrak{ec} : \mathfrak{dc} = 1 : \sin.\mathfrak{dec} = 1 : \sin.\mathfrak{cab}.$$

Est quoque (per §. 21.)

$$\begin{aligned} \mathfrak{ec} : \mathfrak{dc} &= \bigcirc\,\mathfrak{ec} : \bigcirc\,\mathfrak{dc} \text{ (in } F\text{)} \\ &= \bigcirc\,\mathfrak{ac} : \bigcirc\,\mathfrak{bc} \text{ (§. 18.)}; \end{aligned}$$

adeoque est etiam

$$\bigcirc\,\mathfrak{ac} : \bigcirc\,\mathfrak{bc} = 1 : \sin.\mathfrak{cab};$$

unde assertum pro quovis triangulo liquet.

§. 26.

In quovis sphaerico triangulo sunt sinus laterum, uti sinus angulorum iisdem oppositorum. Fig. 11.

Nam sit $\mathfrak{abc} = R$, et $\mathfrak{ced}$ perpendiculare ad sphæræ radium $\mathfrak{oa}$; erit $\mathfrak{ced} \perp \mathfrak{aob}$, et (cum etiam $\mathfrak{boc} \perp \mathfrak{boa}$ sit) $\mathfrak{cd} \perp \mathfrak{ob}$. In triangulis $\mathfrak{ceo}$, $\mathfrak{cdo}$ vero est (per §. 25.)

$$\begin{aligned} \bigcirc\,\mathfrak{ec} : \bigcirc\,\mathfrak{oc} : \bigcirc\,\mathfrak{dc} &= \sin.\mathfrak{coe} : 1 : \sin.\mathfrak{cod} \\ &= \sin.\mathfrak{ac} : 1 : \sin.\mathfrak{bc}; \end{aligned}$$

interim (§. 25.) etiam

$$\bigcirc\, ec : \bigcirc\, dc = \sin. cde : \sin. ced;$$

itaque

$$\sin. ac : \sin. bc = \sin. cde : \sin. ced;$$

est vero $cde = R = cba$, atque $ced = cab$. Consequenter

$$\sin. ac : \sin. bc = 1 : \sin. a.$$

E quo promanans Trigonometria sphaerica ab Axiomate XI. independenter stabilita est.

§. 27.

Fig. 12. *Si* ac, bd *sint* $\perp ab$, *et feratur* cab *iuxta* $\overline{ab}$; *erit* (*via puncti* c *dicta heic* cd)

$$cd : ab = \sin. u : \sin. v.$$

Nam sit $de \perp ca$; est in triangulis ade, adb (per §. 25.)

$$\bigcirc\, ed : \bigcirc\, ad : \bigcirc\, ab = \sin. u : 1 : \sin. v.$$

Revoluto $bacd$ circa ac, describetur $\bigcirc\, ab$ per b, $\bigcirc\, ed$ per d; et via dictæ cd denotetur heic per $\odot\, cd$. Sit porro polygonum quodvis $bfg\ldots$ ipsi $\odot\, ab$ inscriptum; nascetur per plana ex omnibus lateribus bf, fg &, ad $\odot\, ab$ perpendicularia, in $\odot\, cd$ quoque figura polygonalis totidem laterum; et demonstrari (ad instar §. 23.) potest, esse

$$cd : ab = dh : bf = hk : fg = \cdots,$$

adeoque

$$dh + hk + \cdots : bf + fg + \cdots = cd : ab$$

Quovis laterum bf, fg, . . . ad limitem o tendente, manifesto

$$bf + fg + \cdots \frown \bigcirc\, ab \quad \text{et} \quad dh + hk + \cdots \frown \bigcirc\, ed.$$

Itaque etiam

$$\bigcirc\, ed : \bigcirc\, ab = cd : ab.$$

Erat vero

$$\bigcirc\, ed : \bigcirc\, ab = \sin. u : \sin. v.$$

Consequ.

$$cd : ab = \sin. u : \sin. v.$$

Remoto ac a bd in infinitum, manet

$$cd : ab$$

adeoque etiam

$$\sin. u : \sin. v$$

constans; u vero $\frown R$ (§. 1.), et si $dm \; ||| \; bn$ sit, $v \frown z$; unde fit

$$cd : ab = 1 : \sin. z.$$

Via dicta cd denotabitur per $cd \parallel ab$.

§. 28.

Si $bn \; ||| \; \bumpeq am$, *et* c *in* $a\tilde{m}$, *atque* $ac = x$ *sit*: *erit* X (§. 23.) Fig. 13.

$$= \sin. u : \sin. v.$$

Nam si cd et ae sint $\perp bn$ et $bf \perp am$; erit (ad instar §. 27.)

$$\bigcirc bf : \bigcirc cd = \sin. u : \sin. v.$$

Est autem evidenter $bf = ae$: quamobrem

$$\bigcirc ea : \bigcirc dc = \sin. u : \sin. v.$$

In superficiebus vero F formibus ipsorum am et cm (ipsum $ambn$ in ab et cg secantibus) est (per §. 21.)

$$\bigcirc ea : \bigcirc dc = ab : cg = X.$$

Est itaque etiam

$$X = \sin. u : \sin. v.$$

§. 29.

Si $bam = R$, $ab = y$, *et* $bn \; ||| \; am$ *sit*; *erit in* S Fig. 14.

$$Y = \cot. \frac{1}{2} u.$$

Nam si fuerit $\mathfrak{ab}=\mathfrak{ac}$, et $\mathfrak{cp} \parallel \mathfrak{am}$ (adeoque $\mathfrak{bn} \,|||\, \bumpeq \mathfrak{cp}$), atque $\mathfrak{pcd}=\mathfrak{qcd}$; datur (§. 19.) $\mathfrak{ds} \perp \overset{\frown}{\mathfrak{cd}}$, ut $\mathfrak{ds} \,|||\, \mathfrak{cp}$, adeoque (§. 1.) $\mathfrak{dt} \,|||\, \mathfrak{cq}$ sit. Si porro $\mathfrak{be} \perp \overset{\frown}{\mathfrak{ds}}$; erit (§. 7.) $\mathfrak{ds} \,|||\, \mathfrak{bn}$, adeoque (§. 6.) $\mathfrak{bn} \,|||\, \mathfrak{es}$, et (cum $\mathfrak{dt} \,|||\, \mathfrak{cq}$ sit) $\mathfrak{bq} \,|||\, \mathfrak{et}$; consequ. (§. 1.) $\mathfrak{ebn} = \mathfrak{ebq}$.

Repræsententur, $\mathfrak{bcf}$ ex L ipsius $\mathfrak{bn}$, et $\mathfrak{fg}$, $\mathfrak{dh}$, $\mathfrak{ck}$ et $\mathfrak{el}$ ex L formibus lineis ipsorum $\mathfrak{ft}$, $\mathfrak{dt}$, $\mathfrak{cq}$ et $\mathfrak{et}$; erit evidenter (§. 22.)

$$\mathfrak{hg} = \mathfrak{df} = \mathfrak{dk} = \mathfrak{hc};$$

itaque

$$\mathfrak{cg} = 2\mathfrak{ch} = 2v.$$

Pariter patet

$$\mathfrak{bg} = 2\mathfrak{bl} = 2z$$

esse. Est vero

$$\mathfrak{bc} = \mathfrak{bg} - \mathfrak{cg};$$

quapropter

$$y = z - v,$$

adeoque (§. 24.)

$$Y = Z : V.$$

Est demum (§. 28.)

$$Z = 1 : \sin.\frac{1}{2}u \quad \text{et} \quad V = 1 : \sin.\left(R - \frac{1}{2}u\right),$$

consequ.

$$Y = \cot.\frac{1}{2}u.$$

§. 30.

Fig. 15. Verumtamen facile (ex §. 25.) patet, resolutionem problematis *Trigonometriae planae* in S, peripheriæ per radium expressæ indigere; hoc vero rectificatione ipsius L obtineri potest.

Sint $\mathfrak{ab}$, $\mathfrak{cm}$, $\mathfrak{c'm'} \perp \overset{\frown}{\mathfrak{ac}}$, atque $\mathfrak{b}$ ubivis in $\overset{\frown}{\mathfrak{ab}}$; erit (§. 25.)

$$\sin.u : \sin.v = \bigcirc p : \bigcirc y$$

et

$$\sin.u' : \sin.v' = \bigcirc p : \bigcirc y';$$

adeoque

$$\frac{\sin.u}{\sin.v}\bigcirc \quad = \frac{\sin.u'}{\sin.v'}\bigcirc y'.$$

Est vero (per §. 27.)

$$\sin. v : \sin. v' = \cos. u : \cos. u';$$

consequ.

$$\frac{\sin. u}{\cos. u} \bigcirc y = \frac{\sin. u'}{\cos. u'} \bigcirc y',$$

seu

$$\bigcirc y : \bigcirc y' = \text{tang.}\, u' : \text{tang.}\, u = \text{tang.}\, w : \text{tang.}\, w'.$$

Sint porro $\mathfrak{cn} \parallel\!| \mathfrak{ab}$, $\mathfrak{c'n'} \parallel\!| \mathfrak{ab}$ et $\mathfrak{cd}$, $\mathfrak{c'd'}$ lineæ L formes ad $\widetilde{\mathfrak{ab}}$ perpendiculares; erit (§. 21.) etiam

$$\bigcirc y : \bigcirc y' = r : r',$$

adeoque

$$r : r' = \text{tang.}\, w : \text{tang.}\, w'.$$

Crescat iam p ab $\mathfrak{a}$ incipiendo in infinitum; tum $w \sim z$ et $w' \sim z'$; quapropter etiam

$$r : r' = \text{tang.}\, z : \text{tang.}\, z'.$$

Constans $r : \text{tang.}\, z$ (ab r *independens*) dicatur i; dum $y \sim 0$, est

$$\left(\frac{r}{y} = \frac{i\, \text{tang.}\, z}{y}\right) \sim 1,$$

adeoque

$$\frac{y}{\text{tang.}\, z} \sim i.$$

Ex §. 29. fit

$$\text{tang.}\, z = \frac{1}{2}(Y - Y^{-1});$$

itaque

$$\frac{2y}{Y - Y^{-1}} \sim i,$$

seu (§. 24.)

$$\frac{2y I^{\frac{y}{i}}}{I^{\frac{2y}{i}} - 1} \sim i.$$

Notum autem est, expressionis istius (dum $y \sim 0$) limitem esse $\frac{i}{\text{log. nat.}\, I}$; est ergo

$$\frac{i}{\text{log. nat.}\, I} = i \quad \text{et} \quad I = e = 2{\cdot}7182818\ldots,$$

quæ quantitas insignis hic quoque elucet. Si nempe abhinc i illam rectam denotet, cuius $I = e$ sit, erit $r = i$ tang. z. Erat autem (§. 21.) $\bigcirc y = 2\pi r$; est igitur

$$\bigcirc y = 2\pi i \text{ tang. } z = \pi i (Y - Y^{-1}) =$$
$$= \pi i (e^{\frac{y}{i}} - e^{-\frac{y}{i}}) = \frac{\pi y}{\text{log. nat. } Y} (Y - Y^{-1})$$

(per §. 24.).

§. 31.

Fig. 16. Ad resolutionem omnium triangulorum rectangulorum rectilineorum trigonometricam (e qua omnium triangulorum resolutio in promtu est) in S 3 æquationes sufficiunt: nempe (a, b cathetos, c hypotenusam, et α, β angulos cathetis oppositos denotantibus) æquatio relationem exprimens *primo* inter a, c, α, *secundo* inter a, α, β, *tertio* inter a, b, c; nimirum ex his *reliquae* 3 per eliminationem prodeunt.

I. Ex §§. 25. et 30. est

$$1 : \sin. \alpha = (C - C^{-1}) : (A - A^{-1}) = (e^{\frac{c}{i}} - e^{-\frac{c}{i}}) : (e^{\frac{a}{i}} - e^{-\frac{a}{i}});$$

(æquatio pro α, c, a).

II. Ex §. 27. sequitur (si $\beta\mathfrak{m} \parallel \gamma\mathfrak{n}$ sit)

$$\cos. \alpha : \sin. \beta = 1 : \sin. u;$$

ex §. 29. autem fit

$$1 : \sin. u = \frac{1}{2}(A + A^{-1});$$

itaque

$$\cos. \alpha : \sin. \beta = \frac{1}{2}(A + A^{-1}) = \frac{1}{2}(e^{\frac{a}{i}} + e^{-\frac{a}{i}});$$

(æquatio pro α, β, a).

III. Si $\alpha\alpha' \perp \beta\alpha\gamma$, atque $\beta\beta'$ et $\gamma\gamma'$ fuerint $\parallel \alpha\alpha'$, (§. 27.), atque $\beta'\alpha'\gamma' \perp \alpha\alpha'$; erit manifesto (uti in §. 27.)

$$\frac{\beta\beta'}{\gamma\gamma'} = \frac{1}{\sin. u} = \frac{1}{2}(A + A^{-1}),$$
$$\frac{\gamma\gamma'}{\alpha\alpha'} = \frac{1}{2}(B + B^{-1}),$$

ac

$$\frac{\beta\beta'}{\alpha\alpha'} = \frac{1}{2}(C + C^{-1});$$

consequ.

$$\frac{1}{2}(C + C^{-1}) = \frac{1}{2}(A + A^{-1})\frac{1}{2}(B + B^{-1}),$$

sive

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2}(e^{\frac{a}{i}} + e^{-\frac{a}{i}})(e^{\frac{b}{i}} + e^{-\frac{b}{i}});$$

(æquatio pro a, b, c).

Si $\gamma\alpha\delta = R$, et $\beta\delta \perp \alpha\delta$ sit; erit

$$\bigcirc c : \bigcirc a = 1 : \sin.\alpha,$$

et

$$\bigcirc c : \bigcirc (d = \beta\delta) = 1 : \cos.\alpha,$$

adeoque ($\bigcirc x^2$ pro quovis x factum $\bigcirc x . \bigcirc x$ denotante) manifesto

$$\bigcirc a^2 + \bigcirc d^2 = \bigcirc c^2.$$

Est vero (per §. 27. et II.)

$$\bigcirc d = \bigcirc b . \frac{1}{2}(A + A^{-1}),$$

consequ.

$$(e^{\frac{c}{i}} - e^{-\frac{c}{i}})^2 = \frac{1}{4}(e^{\frac{a}{i}} + e^{-\frac{a}{i}})^2 (e^{\frac{b}{i}} - e^{-\frac{b}{i}})^2 + (e^{\frac{a}{i}} - e^{-\frac{a}{i}})^2;$$

alia æquatio pro a, b, c (cuius membrum *secundum* facile ad formam *symmetricam* seu *invariabilem* reducitur).

Denique ex

$$\frac{\cos.\alpha}{\sin.\beta} = \frac{1}{2}(A + A^{-1})$$

atque

$$\frac{\cos.\beta}{\sin.\alpha} = \frac{1}{2}(B + B^{-1})$$

fit (per III.)

$$\cot.\alpha \cot.\beta = \frac{1}{2}(e^{\frac{c}{i}} + e^{-\frac{c}{i}});$$

(æquatio pro α, β, c).

§. 32.

Restat adhuc modum *problemata* in S resolvendi breviter ostendere, quo (per exempla magis obvia) peracto, demum quid theoria hæcce præstet, candide dicetur.

g. 17. I. Sit $\overset{\frown}{\mathfrak{ab}}$ linea in plano, et $y=f(x)$ æquatio eius (pro coordinatis perpendicularibus), et quodvis incrementum ipsius z dicatur dz, atque incrementa ipsorum x, y, et areæ u, eidem dz respondentia, respective per dx, dy, du denotentur; sitque $\mathfrak{bh} \parallel \mathfrak{cf}$, et exprimatur (ex §§. 31. et 27.) $\frac{\mathfrak{bh}}{dx}$ per y, ac quæratur ipsius $\frac{dy}{dx}$ *limes* tendente dx ad limitem 0, (quod, ubi eiusmodi limes quæritur, subintelligatur): innotescet exinde etiam limes ipsius $\frac{dy}{\mathfrak{bh}}$, adeoque tg. $\mathfrak{hbg}$; eritque, (cum $\mathfrak{hbc}$ manifesto nec $>$ nec $<$ adeoque $=R$ sit), *tangens* in $\mathfrak{b}$ ipsius $\mathfrak{bg}$ per y determinata.

II. Demonstrari potest, esse

$$\frac{dz^2}{dy^2+\mathfrak{bh}^2} \sim 1.$$

Hinc *limes* ipsius $\frac{dz}{dx}$, et inde z integratione (per x expressum) reperitur.

Et potest lineæ cuiusvis *in concreto datae* æquatio in S inveniri, ex. gr. ipsius L.

Si enim $\mathfrak{am}$ axis ipsius L sit; tum quævis $\mathfrak{cb}$ ex $\mathfrak{am}$ secat L (cum per §. 19 quævis recta ex $\mathfrak{a}$ præter $\mathfrak{am}$ ipsum L secet); est vero (si $\mathfrak{bn}$ axis sit)

$$X=1:\text{sin. } \mathfrak{cbn} \quad (\text{§. } 28.),$$

atque

$$Y=\text{cot.}\frac{1}{2}\,\mathfrak{cbn} \quad (\text{§. } 29.),$$

unde fit

$$Y=X+\sqrt{X^2-1}$$

seu

$$e^{\frac{y}{i}}=e^{\frac{x}{i}}+\sqrt{e^{\frac{2x}{i}}-1}$$

æquatio quæsita. Erit hinc

$$\frac{dy}{dx} \backsim X(X^2-1)^{-\frac{1}{2}};$$

atqui

$$\frac{\mathfrak{bh}}{dx} = 1 : \sin.\mathfrak{cbn} = X;$$

adeoque

$$\frac{dy}{\mathfrak{bh}} \backsim (X^2-1)^{-\frac{1}{2}};$$

$$1 + \frac{dy^2}{\mathfrak{bh}^2} \backsim X^2(X^2-1)^{-1},$$

$$\frac{dz^2}{\mathfrak{bh}^2} \backsim X^2(X^2-1)^{-1},$$

$$\frac{dz}{\mathfrak{bh}} \backsim X(X^2-1)^{-\frac{1}{2}}$$

atque

$$\frac{dz}{dx} \backsim X^2(X^2-1)^{-\frac{1}{2}};$$

unde per integrationem invenitur

$$z = i(X^2-1)^{\frac{1}{2}} = i\cot.\mathfrak{cbn}$$

(uti §. 30.).

III. Manifesto

$$\frac{du}{dx} \backsim \frac{\mathfrak{hfcbh}}{dx},$$

quod (nonnisi ab y dependens) iam primum per y exprimendum est; unde u integrando prodit.

Si $\mathfrak{ab} = p$, $\mathfrak{ac} = q$, et $\mathfrak{cd} = r$, atque $\mathfrak{cabdc} = s$ sit; poterit (uti in II.) ostendi esse Fig. 12.

$$\frac{ds}{dq} \backsim r,$$

quod

$$= \frac{1}{2}\, p\left(e^{\frac{q}{i}} + e^{-\frac{q}{i}}\right),$$

atque integrando

$$s = \frac{1}{2}\, pi\left(e^{\frac{q}{i}} - e^{-\frac{q}{i}}\right).$$

Potest hoc absque integratione quoque deduci.

Aequatione e. g. circuli (ex §. 31, III.), rectæ (ex §. 31, II.), sectionis coni (per præc.) expressis; poterunt areæ quoque his lineis clausæ exprimi.

Palam est, superficiem t ad figuram planam p (in distantia q) $\|$ lam esse ad p in ratione potentiarum secundarum linearum homologarum, sive uti

$$\frac{1}{4}\left(e^{\frac{q}{i}}+e^{-\frac{q}{i}}\right)^2 : 1.$$

Porro computum soliditatis pari modo tractatum, facile patet duas integrationes requirere (cum et differentiale ipsum hic nonnisi per integrationem determinetur); et ante omnia solidum a p et t ac complexu omnium rectarum ad p perpendicularium, fines ipsorum p, t connectentium, clausum quærendum esse. Reperitur solidum istud (tam per integrationem quam sine ea)

$$=\frac{1}{8}pi\left(e^{\frac{2q}{i}}-e^{-\frac{2q}{i}}\right)+\frac{1}{2}pq.$$

Superficies quoque corporum is S determinari possunt, nec non *curvaturae*, *evolutae*, *evolventesque* linearum qualiumvis &. Quod curvaturam attinet; ea in S aut ipsius L est, aut per radium circuli, aut *distantiam* curvæ ad rectam $\|$ læ ab hac recta, determinatur; cum e præcedentibus facile ostendi possit, præter L, lineas circulares, ac rectæ $\|$ las, nullas in plano alias lineas uniformes dari.

IV. Pro circulo est (uti in III.)

$$\frac{d\odot x}{dx} \backsim \bigcirc x,$$

unde (per §. 30.) integrando fit

$$\odot x = \pi i^2\left(e^{\frac{x}{i}}-2+e^{-\frac{x}{i}}\right).$$

Fig. 9. V. Pro area $\mathfrak{cabdc}=u$ (linea L formi $\mathfrak{ab}=r$, huic $\|$ la $\mathfrak{cd}=y$, ac rectis $\mathfrak{ac}$, $\mathfrak{bd}=x$ clausa) est

$$\frac{du}{dx} \backsim y,$$

atque (§. 24.)

$$y = re^{-\frac{x}{i}};$$

adeoque (integrando)

$$u = ri\left(1 - e^{-\frac{x}{i}}\right).$$

Crescente x in infinitum, fiet in S $e^{-\frac{x}{i}} \backsim 0$, adeoque $u \backsim ri$. Per *quantitatem* ipsius $\mathfrak{mabn}$ in posterum limes iste intelligetur.

Simili modo invenitur, quod si p sit figura in F; spatium a p et complexu axium e terminis ipsius p ductorum clausum $= \frac{1}{2} pi$ sit.

VI. Si angulus ad centrum segmenti z sphæræ sit $2u$, peripheria circuli maximi sit p, et arcus $\mathfrak{fc}$ (anguli u) $= x$; erit (§. 25.) Fig. 10.

$$1 : \sin. u = p : \bigcirc \mathfrak{bc},$$

et hinc

$$\bigcirc \mathfrak{bc} = p \sin. u.$$

Interim est

$$x = \frac{pu}{2\pi}, \quad \text{ac} \quad dx = \frac{pdu}{2\pi}.$$

Est porro

$$\frac{dz}{dx} \backsim \bigcirc \mathfrak{bc},$$

et hinc

$$\frac{dz}{du} \backsim \frac{p^2}{2\pi} \sin. u,$$

unde (integrando)

$$z = \frac{\text{sin. vers. } u}{2\pi} p^2.$$

Cogitetur F in quod p (per meditullium $\mathfrak{f}$ segmenti transiens) cadit; planis $\overline{\mathfrak{fem}}$, $\overline{\mathfrak{cem}}$ per $\mathfrak{af}$, $\mathfrak{ac}$ ad F perpendiculariter positis, ipsumque in $\mathfrak{feg}$, $\mathfrak{ce}$ secantibus; et considerentur L formis $\mathfrak{cd}$ (ex $\mathfrak{c}$ ad $\mathfrak{feg}$ perpendicularis) nec non L formis $\mathfrak{cf}$; erit (§. 20.)

$$\mathfrak{cef} = u,$$

et (§. 21.)

$$\frac{\mathfrak{fd}}{p} = \frac{\text{sin. vers. } u}{2\pi},$$

adeoque

$$z = \mathfrak{fd} . p.$$

Ast (§. 21.)

$$p = \pi . \mathfrak{fdg},$$

itaque

$$z = \pi . \mathfrak{fd} . \mathfrak{fdg}.$$

Est autem (§. 21.)

$$\mathfrak{fd} . \mathfrak{fdg} = \mathfrak{fc} . \mathfrak{fc};$$

consequ.

$$z = \pi . \mathfrak{fc} . \mathfrak{fc} = \odot\, \mathfrak{fc} \text{ in } F.$$

Fig. 14. Sit iam $\mathfrak{bj} = \mathfrak{cj} = r$; erit (§. 30.)

$$2r = i\,(Y - Y^{-1}),$$

adeoque (§. 21.)

$$\odot\, 2r \ (\text{in } F) = \pi i^2 \,(Y - Y^{-1})^2.$$

Est quoque (IV.)

$$\odot\, 2y = \pi i^2\,(Y^2 - 2 + Y^{-2});$$

igitur $\odot\, 2r$ (in F) $= \odot\, 2y$, adeoque *et superficies z segmenti sphaerici aequatur circulo, chorda* $\mathfrak{fc}$ *tanquam radio descripto.*

Hinc tota sphæræ superficies

$$= \odot\, \mathfrak{fg} = \mathfrak{fdg} . p = \frac{p^2}{\pi},$$

suntque superficies sphaerarum, uti secundae potentiae peripheriarum earundem maximarum.

VII. Soliditas sphæræ radii x in S reperitur simili modo

$$= \frac{1}{2}\,\pi i^3 (X^2 - X^{-2}) - 2\pi i^2 x;$$

Fig. 12. superficies per revolutionem lineæ $\mathfrak{cd}$ circa $\mathfrak{ab}$ orta

$$= \frac{1}{2}\,\pi i p\,(Q^2 - Q^{-2}),$$

et corpus per $\mathfrak{cabdc}$ descriptum

$$= \frac{1}{4}\,\pi i^2 p\,(Q - Q^{-1})^2.$$

Quomodo vero omnia a (IV.) hucusque tractata etiam absque integratione perfici possint, brevitatis studio supprimitur.

Demonstrari potest, *omnis expressionis literam i continentis* (adeoque *hypothesi*, quod *detur i*, innixæ) *limitem*, *crescente i in infinitum*, *exprimere quantitatem plane pro* Σ (adeoque pro hypothesi *nullius i*), *siquidem non eveniant aequationes identicae.* Cave vero intelligas putari, *systema ipsum variari* posse (quod omnino *in se et per se determinatum* est) sed tantum *hypothesin*, quod *successive* fieri potest, donec non ad absurdum perducti fuerimus. *Posito* igitur, quod in *tali* expressione litera i pro casu, si S esset reipsa, *illam* quantitatem unicam designet, cuius $I = e$ sit; si vero *revera* Σ fuerit, *limes dictus* loco expressionis accipi *cogitetur*: manifesto *omnes* expressiones ex *hypothesi realitatis* ipsius S oriundæ (hoc sensu) *absolute valent*, etsi *prorsus ignotum sit*, *num* Σ *sit*, *aut non sit.*

Ita e. g. ex expressione in §. 30. obtenta facile (et quidem *tam* differentiationis auxilio, quam *absque* eo) valor notus pro Σ prodit

$$\bigcirc x = 2\pi x;$$

ex I. (§. 31.) rite tractato, sequitur

$$1 : \sin. \alpha = c : a;$$

ex II. vero

$$\frac{\cos. \alpha}{\sin. \beta} = 1, \quad \text{adeoque} \quad \alpha + \beta = R;$$

æquatio *prima* in III. fit identica, adeoque *valet* pro Σ, quamvis nihil in eo *determinet*; ex *secunda* autem fluit

$$c^2 = a^2 + b^2.$$

Aequationes notae fundamentales trigonometriae planae in Σ.

Porro inveniuntur (ex §. 32.) pro Σ area et corpus in III., utrumque

$$= pq;$$

ex IV.

$$\odot x = \pi x^2;$$

ex VII. sphæra radii x

$$=\frac{4}{3}\pi x^3$$

&c.

Sunt quoque theoremata ad finem (VI.) enuntiata manifesto *inconditionate vera.*

§. 33.

Superest adhuc, quid theoria ista sibi velit, (in §. 32. promissum) exponere.

I. Num Σ aut S aliquod *reipsa* sit, indecisum manet.

II. Omnia ex hypothesi *falsitatis* Ax. XI. deducta (semper *sensu* §. 32. intelligendo) *absolute* valent, adeoque *hoc sensu nulli hypothesi innituntur*. Habetur idcirco *trigonometria plana a priori*, in qua *solum* systema *verum ignotum* adeoque solummodo *absolutae* magnitudines expressionum incognitæ manent, per *unicum* vero casum notum, manifesto totum systema figeretur. Trigonometria sphærica autem in §. 26. absolute stabilitur. (Habeturque Geometria, Geometriæ planæ in Σ prorsus analoga in F).

III. Si *constaret* Σ esse, nihil hoc respectu amplius incognitum esset; si vero *constaret non esse* Σ, tunc (§. 31.) (e. g.) e lateribus x, y et angulo rectilineo ab iis intercepto, in *concreto datis* manifesto in se et per se impossibile esset triangulum absolute resolvere (i. e.) a priori determinare angulos ceteros et *rationem lateris tertii* ad duo data; nisi X, Y determinentur, ad quod *in concreto* haberi aliquod a oporteret, cuius A notum esset; atque tum i *unitas naturalis longitudinum* esset, (sicuti e est basis logarithmorum naturalium). Si existentia huius i constiterit; quomodo ad usum saltem quam exactissime construi possit, ostendetur.

IV. Sensu in I. et II. exposito patet, omnia in spatio methodo recentiorum Analytica (intra iustos fines valde laudanda) absolvi posse.

V. Denique lectoribus benevolis haud ingratum futurum est; pro casu illo, quodsi non Σ sed S reipsa esset, circulo æquale rectilineum construi.

§. 34.

Ex $\mathfrak{d}$ *ducitur* $\mathfrak{dm} \,|||\, \mathfrak{an}$ *modo sequente.* Fig. 12.

Fiat ex $\mathfrak{d}$

$$\mathfrak{db} \perp \mathfrak{an};$$

erigatur e puncto quovis aliquo $\mathfrak{a}$ rectæ $\widetilde{\mathfrak{ab}}$

$$\mathfrak{ac} \perp \mathfrak{an} \text{ (in } \mathfrak{dba}\text{)},$$

et demittatur

$$\mathfrak{de} \perp \mathfrak{ac};$$

erit (§. 27.)

$$\bigcirc\,\mathfrak{ed} : \bigcirc\,\mathfrak{ab} = 1 : \sin. z,$$

siquidem *fuerit* $\mathfrak{dm} \,|||\, \mathfrak{bn}$.

Est vero $\sin. z$ non > 1, adeoque $\mathfrak{ab}$ non $> \mathfrak{de}$. Descriptus igitur quadrans radio ipsi $\mathfrak{de}$ æquali ex $\mathfrak{a}$ in $\mathfrak{bac}$, gaudebit puncto aliquo $\mathfrak{b}$ vel $\mathfrak{o}$ cum $\mathfrak{b}\tilde{\mathfrak{d}}$ communi. Priore in casu manifesto $z = R$; in posteriore vero erit (§. 25.)

$$(\bigcirc\,\mathfrak{ao} = \bigcirc\,\mathfrak{ed}) : \bigcirc\,\mathfrak{ab} = 1 : \sin. \mathfrak{aob},$$

adeoque

$$z = \mathfrak{aob}.$$

Si itaque fiat $z = \mathfrak{aob}$, erit $\mathfrak{dm} \,|||\, \mathfrak{bn}$.

§. 35.

Si fuerit S reipsa; *ducetur recta ad anguli acuti crus unum perpendicularis, quae ad alterum* $|||$ *sit, hoc modo.* Fig. 18.

Sit $\mathfrak{am} \perp \mathfrak{bc}$, et accipiatur $\mathfrak{ab} = \mathfrak{ac}$ tam parvum (per §. 19.), ut si ducatur $\mathfrak{bn} \,|||\, \mathfrak{am}$ (§. 34.), sit $\mathfrak{abn} >$ angulo dato. Ducatur porro $\mathfrak{cp} \,|||\, \mathfrak{am}$ (§. 34.), fiantque $\mathfrak{nbq}$, $\mathfrak{pcd}$ utrumque æquale angulo dato; et $\mathfrak{b}\tilde{\mathfrak{q}}$, $\mathfrak{c}\tilde{\mathfrak{d}}$ se mutuo secabunt. Secet enim $\mathfrak{b}\tilde{\mathfrak{q}}$, (quod *per constr.* in $\mathfrak{nbc}$ cadit) ipsam $\mathfrak{c}\tilde{\mathfrak{p}}$ in $\mathfrak{e}$; erit (propter $\mathfrak{bn} \bumpeq \mathfrak{cp}$) $\mathfrak{ebc} < \mathfrak{ecb}$, adeoque $\mathfrak{ec} < \mathfrak{eb}$. Sint

$$\mathfrak{ef} = \mathfrak{ec}, \quad \mathfrak{efr} = \mathfrak{ecd}, \quad \text{et} \quad \mathfrak{fs} \,|||\, \mathfrak{ep};$$

cadet $\mathfrak{fs}$ in $\mathfrak{bfr}$. Nam cum $\mathfrak{bn} \,|||\, \mathfrak{cp}$, adeoque $\mathfrak{bn} \,|||\, \mathfrak{ep}$, atque $\mathfrak{bn} \,|||\, \mathfrak{fs}$ sit;

erit (§. 14.)

$$\mathfrak{fbn} + \mathfrak{bfs} < 2R = \mathfrak{fbn} + \mathfrak{bfr};$$

itaque $\mathfrak{bfs} < \mathfrak{bfr}$. Quamobrem $\mathfrak{f}\overline{\mathfrak{r}}$ secat $\mathfrak{e}\overline{\mathfrak{p}}$, adeoque $\mathfrak{c}\overline{\mathfrak{d}}$ quoque ipsam $\mathfrak{e}\overline{\mathfrak{q}}$ in puncto aliquo $\mathfrak{d}$.

Sit iam $\mathfrak{dg} = \mathfrak{dc}$, atque $\mathfrak{dgt} = \mathfrak{dcp} = \mathfrak{gbn}$; erit (cum $\mathfrak{cd} \bumpeq \mathfrak{gd}$ sit)

$$\mathfrak{bn} \bumpeq \mathfrak{gt} \bumpeq \mathfrak{cp}.$$

Si fuerit lineæ Lformis ipsius $\mathfrak{bn}$, punctum in $\mathfrak{b}\overline{\mathfrak{q}}$ cadens $\mathfrak{k}$ (§. 19.), et axis $\mathfrak{kl}$; erit

$$\mathfrak{bn} \bumpeq \mathfrak{kl},$$

adeoque

$$\mathfrak{bkl} = \mathfrak{bgt} = \mathfrak{dcp};$$

sed etiam

$$\mathfrak{kl} \bumpeq \mathfrak{cp}:$$

cadit ergo $\mathfrak{k}$ manifesto in $\mathfrak{g}$, estque $\mathfrak{gt} \, ||| \, \mathfrak{bn}$. Si vero $\mathfrak{ho}$ ipsum $\mathfrak{bg}$ perpendiculariter bisecet; erit $\mathfrak{ho} \, ||| \, \mathfrak{bn}$ constructum.

§. 36.

Fig. 10. Si fuerint data recta $\mathfrak{c}\overline{\mathfrak{p}}$ et planum $\overline{\mathfrak{mab}}$, atque fiat $\mathfrak{cb} \perp \overline{\mathfrak{mab}}$, (in $\overline{\mathfrak{bcp}}$) $\mathfrak{bn} \perp \mathfrak{bc}$, et $\mathfrak{cq} \, ||| \, \mathfrak{bn}$ (§. 34.); *sectio ipsius* $\mathfrak{c}\overline{\mathfrak{p}}$ (si hæc in $\mathfrak{bcq}$ cadat) *cum* $\mathfrak{b}\overline{\mathfrak{n}}$ (in $\overline{\mathfrak{cbn}}$), adeoque *cum* $\overline{\mathfrak{mab}}$ reperitur. Et si fuerint data duo plana $\overline{\mathfrak{pcq}}$, $\overline{\mathfrak{mab}}$, et sit $\mathfrak{cb} \perp \overline{\mathfrak{mab}}$, $\mathfrak{cr} \perp \overline{\mathfrak{pcq}}$, atque (in $\overline{\mathfrak{bcr}}$) $\mathfrak{bn} \perp \mathfrak{bc}$, $\mathfrak{cs} \perp \mathfrak{cr}$; cadent $\mathfrak{bn}$ in $\overline{\mathfrak{mab}}$, et $\mathfrak{cs}$ in $\overline{\mathfrak{pcq}}$; et sectione ipsarum $\overline{\mathfrak{bn}}$, $\overline{\mathfrak{cs}}$ (si detur) reperta, erit perpendicularis in $\mathfrak{pcq}$ per eandem ad $\mathfrak{c}\overline{\mathfrak{s}}$ ducta manifesto *sectio ipsorum* $\overline{\mathfrak{mab}}$, $\overline{\mathfrak{pcq}}$.

§. 37.

Fig. 7. *In* $\overline{\mathfrak{am}} \, ||| \, \mathfrak{bn}$ reperitur *tale* $\mathfrak{a}$, *ut sit* $\mathfrak{am} \bumpeq \mathfrak{bn}$; si (per §. 34.) construatur extra $\overline{\mathfrak{nbm}}$ $\mathfrak{gt} \, ||| \, \mathfrak{bn}$, et fiant $\mathfrak{bg} \perp \mathfrak{gt}$, $\mathfrak{gc} = \mathfrak{gb}$, atque $\mathfrak{cp} \, ||| \, \mathfrak{gt}$; ponaturque $\mathfrak{tg}\overline{\mathfrak{d}}$ ita, ut efficiat cum $\mathfrak{tg}\overline{\mathfrak{b}}$ angulum illi æqualem, quem $\mathfrak{pc}\overline{\mathfrak{a}}$ cum $\mathfrak{pc}\overline{\mathfrak{b}}$ facit; atque quæratur (per §. 36.) sectio $\overline{\mathfrak{dq}}$ ipsorum $\mathfrak{tg}\overline{\mathfrak{d}}$, $\mathfrak{nb}\overline{\mathfrak{a}}$; fiatque

$\mathfrak{ba} \perp \mathfrak{dq}$. Erit enimvero ob triangulorum *L* lineorum in *F* ipsius $\mathfrak{bn}$ exortorum similitudinem (§. 21.) manifesto $\mathfrak{db} = \mathfrak{da}$, et $\mathfrak{am} \bumpeq \mathfrak{bn}$.

Facile hinc patet (*L* lineis per *solos terminos* datis) reperiri posse etiam *terminos* proportionis quartum ac medium, atque omnes constructiones geometricas, quæ in Σ in plano fiunt, hoc modo in *F absque XI. Axiomate* perfici posse. Ita e. g. $4R$ in quotvis partes æquales geometrice dividi potest, si sectionem istam in Σ perficere licet.

§. 38.

Si construatur (per §. 37.) e. g. $\mathfrak{nbq} = \frac{1}{3} R$, et fiat (per §. 35.) in *S* ad Fig. 14.
$\mathfrak{b}\bar{\mathfrak{q}}$ perpendicularis $\mathfrak{am} \,|||\, \mathfrak{bn}$, atque determinetur (per §. 37.) $\mathfrak{jm} \bumpeq \mathfrak{bn}$; erit, si $\mathfrak{ja} = x$ sit, (§. 28.)

$$X = 1 : \sin. \frac{1}{3} R = 2,$$

atque x *geometrice* constructum.

Et potest $\mathfrak{nbq}$ ita computari, ut $\mathfrak{ja}$ ab i quovis dato minus discrepet, cum nonnisi $\sin. \mathfrak{nbq} = \frac{1}{e}$ esse debeat.

§. 39.

Si fuerint (in plano) $\mathfrak{pq}$ et $\mathfrak{st}$, $\|$ rectæ $\mathfrak{mn}$ (§. 27.), et $\mathfrak{ab}$, $\mathfrak{cd}$ sint per- Fig. 19.
pendiculares ad $\mathfrak{mn}$ æquales; manifesto est

$$\triangle\, \mathfrak{dec} = \triangle\, \mathfrak{bea},$$

adeoque anguli (forsan mixtilinei) $\mathfrak{ecp}$, $\mathfrak{eat}$ congruent, atque

$$\mathfrak{ec} = \mathfrak{ea}.$$

Si porro $\mathfrak{cf} = \mathfrak{ag}$, erit

$$\triangle\, \mathfrak{acf} = \triangle\, \mathfrak{cag},$$

et utrumque *quadrilateri* $\mathfrak{fagc}$ dimidium est. Si $\mathfrak{fagc}$, $\mathfrak{hagk}$ duo eiusmodi quadrilatera fuerint ad $\mathfrak{ag}$, inter $\mathfrak{pq}$ et $\mathfrak{st}$; æqualitas eorum (uti apud EUCLIDEM), nec non triangulorum $\mathfrak{agc}$, $\mathfrak{agh}$ eidem $\mathfrak{ag}$ insistentium, verticesque in $\overline{pq}$ habentium, æqualitas patet. Est porro

$$acf = cag, \quad gcq = cga,$$

atque

$$acf + acg + gcq = 2R$$

(§. 32.), adeoque etiam

$$cag + acg + cga = 2R;$$

itaque in quovis eiusmodi triangulo acg summa trium angulorum $= 2R$.

Sive in ag (quæ || mn) ceciderit autem *recta* ag, sive non; triangulorum *rectilineorum* agc, agh *tam ipsorum*, *quam summarum angulorum ipsorumdem*, *aequalitas* in aperto est.

§. 40.

Fig. 20. *Aequalia triangula* abc, abd *(abhinc rectilinea) uno latere aequali gaudentia*, *summas angulorum aequales habent.*

Nam dividat mn bifariam tam ac quam bc, et sit pq (per c) || mn; cadet d in $\overline{pq}$. Nam si $\overline{bd}$ ipsum $\overline{mn}$ in puncto e, adeoque (§. 39.) ipsum $\overline{pq}$ ad distantiam $ef = eb$ secet; erit

$$\triangle\, abc = \triangle\, abf,$$

adeoque et

$$\triangle\, abd = \triangle\, abf,$$

unde d in f cadit: si vero $\overline{bd}$ ipsum $\overline{mn}$ non secuerit, sit c punctum, ubi perpendicularis rectam ab bisecans ipsum $\overline{pq}$ secat, atque $gs = ht$ ita, ut $\overline{st}$ *productam* $\overline{bd}$ in puncto aliquo k secet (quod fieri posse modo simili patet, ut §. 4.); sint porro $sl = sa$, lo || st, atque o sectio ipsorum $\overline{bk}$ et $\overline{lo}$; esset tum (§. 39.)

$$\triangle\, abl = \triangle\, abo,$$

adeoque

$$\triangle\, abc > \triangle\, abd$$

(contra hyp.).

§. 41.

Aequalia triangula abc, def *aequalibus angulorum summis gaudent.* Fig. 21.

Nam secet mn tam ac quam bc, ita pq tam df quam fe bifariam, et sit rs || mn, atque to || pq; erit perpendicularis ag ad rs aut æqualis perpendiculari dh ad to, aut altera e. g. dh erit maior: in quovis casu ○ df e centro a cum gs punctum aliquod k commune habet, eritque (§. 39.)

$$\triangle\, abk = \triangle\, abc = \triangle\, def.$$

Est vero △ akb (per §. 40.) triangulo dfe, ac (per §. 39.) triangulo abc *aequiangulum.* Sunt igitur etiam triangula abc, def æquiangula.

In *S converti* quoque theorema potest. Sint enim triangula abc, def reciproce æquiangula, atque △ bal = △ def; erit (per præc.) alterum alteri, adeoque etiam △ abc triangulo abl æquiangulum, et hinc manifesto

$$bcl + blc + cbl = 2R.$$

Atqui (ex §. 31.) cuiusvis trianguli angulorum summa in *S* est $< 2R$: cadit igitur l in c.

§. 42.

Si fuerit complementum summae angulorum trianguli abc *ad* $2R$ Fig. 22.

$$u,$$

trianguli def *vero*

$$v;$$

est

$$\triangle\, abc : \triangle\, def = u : v.$$

Nam si quodvis triangulorum acg, gch, hcb, dfk, kfe sit $= p$, atque

$$\triangle\, abc = mp, \quad \triangle\, def = np;$$

sitque s summa angulorum cuiusvis trianguli, quod $= p$ est: erit manifesto

$$2R-u=ms-(m-1)2R=2R-m(2R-s),$$

et

$$u=m(2R-s),$$

et pariter

$$v=n(2R-s).$$

Est igitur

$$\triangle\,\mathfrak{abc}:\triangle\,\mathfrak{def}=m:n=u:v.$$

Ad casum incommensurabilitatis triangulorum $\mathfrak{abc}$, $\mathfrak{def}$ quoque extendi facile patet.

Eodem modo demonstratur triangula in superficie sphærica esse uti *excessus* summarum angulorum eorundem supra $2R$. Si duo anguli trianguli sphærici recti fuerint, tertius z erit excessus dictus; est autem triangulum istud (peripheria maxima p dicta) manifesto

$$=\frac{z}{2\pi}\,\frac{p^2}{2\pi}\quad(\S.\ 32.\ \mathrm{VI.});$$

consequenter quodvis triangulum, cuius angulorum excessus $=z$, est

$$=\frac{zp^2}{4\pi^2}.$$

§. 43.

Fig. 15. Iam *area* trianguli rectilinei in S per summam angulorum exprimetur.

Si $\mathfrak{ab}$ crescat in infinitum; erit (§. 42.)

$$\triangle\,\mathfrak{abc}:(R-u-v)$$

constans. Est vero

$$\triangle\,\mathfrak{abc}\backsim\mathfrak{bacn}\quad(\S.\ 32.\ \mathrm{V.})$$

et

$$R-u-v\backsim z\quad(\S.\ 1.);$$

adeoque

$$\mathfrak{bacn}:z=\triangle\,\mathfrak{abc}:(R-u-v)=\mathfrak{bac'n'}:z'.$$

Est porro manifesto

$$\mathfrak{bdcn}:\mathfrak{bd'c'n'}=r:r'=\text{tang.}\,z:\text{tang.}\,z'\quad(\S.\ 30.).$$

Pro $y' \frown 0$ autem est

$$\frac{\mathfrak{bd'c'n'}}{\mathfrak{bac'n'}} \frown 1,$$

nec non

$$\frac{\text{tang.}\, z'}{z'} \frown 1;$$

consequ.

$$\mathfrak{bdcn} : \mathfrak{bacn} = \text{tang.}\, z : z.$$

Erat vero (§. 32.)

$$\mathfrak{bdcn} = ri = i^2\, \text{tang.}\, z;$$

est igitur

$$\mathfrak{bacn} = zi^2.$$

Quovis triangulo, cuius angulorum summæ complementum ad $2R$ z est, in posterum breviter $\triangle$ dicto, erit idcirco

$$\triangle = zi^2.$$

Facile hinc liquet, quod si Fig. 14.

$$\mathfrak{or} \parallel\!| \mathfrak{am} \quad \text{et} \quad \mathfrak{ro} \parallel\!| \mathfrak{ab}$$

fuerint; *area* inter $\overset{\frown}{\mathfrak{or}}$, $\overset{\frown}{\mathfrak{st}}$, $\overset{\frown}{\mathfrak{bc}}$ comprehensa (quæ manifesto limes absolutus est areæ triangulorum rectilineorum sine fine crescentium, seu ipsius $\triangle$ pro $z \frown 2R$), sit

$$= \pi i^2 = \odot i \text{ in } F.$$

Limite isto per $\square$ denotato, erit porro (per §. 30.) Fig. 15.

$$\pi r^2 = \text{tang.}\, z^2 \square = \odot r \text{ in } F \text{ (§. 21.)}$$
$$= \odot s \text{ (per §. 32. VI.)},$$

si chorda $\mathfrak{dc}$ s dicatur. Si iam radio dato s, circuli in plano (sive radio L formi circuli in F) perpendiculariter bisecto, construatur (per §. 34.) $\mathfrak{db} \parallel\!| \bumpeq \mathfrak{cn}$; demissa perpendiculari $\mathfrak{ca}$ ad $\mathfrak{db}$, et erecta perpendiculari $\mathfrak{cm}$ ad $\mathfrak{ca}$; habebitur z; unde (per §. 37.) tang. z^2, radio L formi ad lubitum pro unitate assumto, *geometrice determinari potest per duas lineas uniformes eiusdem curvaturae* (quæ solis terminis datis, constructis axibus, manifesto tanquam rectæ commensurari, atque hoc respectu rectis æquivalentes spectari possunt).

Fig. 23 Porro construitur quadrilaterum ex. gr. regulare $= \square$, ut sequitur. Sit

$$\mathfrak{abc} = R,\ \mathfrak{bac} = \frac{1}{2} R,\ \mathfrak{acb} = \frac{1}{4} R, \text{ et } \mathfrak{bc} = x;$$

poterit X (ex §. 31. II.) per meras radices quadraticas exprimi, et (per §. 37.) construi: habitoque X, (per §. 38., sive etiam 29. et 35.) x ipsum determinari potest. Estque octuplum $\triangle\,\mathfrak{abc}$ manifesto $= \square$, atque *per hoc, circulus planus radii s, per figuram rectilineam, et lineas uniformes eiusdem generis (rectis, quoad comparationem inter se, aequivalentes) geometrice quadratus*; *circulus Fformis vero eodem modo complanatus*: *habeturque aut Axioma XI. Euclidis verum, aut quadratura circuli geometrica*; etsi hucusque indecisum manserit, quodnam ex his duobus revera locum habeat. Quoties tang. z^2 *vel* numerus integer *vel* fractio rationalis fuerit, cuius (ad simplicissimam formam reductæ) denominator *aut* numerus primus formæ $2^m + 1$ (cuius est etiam $2 = 2^0 + 1$) *aut* productum fuerit e quotcunque primis huius formæ, quorum (ipsum 2, qui solus quotvis vicibus occurrere potest, excipiendo) quivis *semel* ut factor occurrit: per theoriam polygonorum ill. *GAUSS* (præclarum nostri imo omnis ævi inventum), etiam ipsi tang. $z^2 \square = \odot s$ (et nonnisi pro talibus valoribus ipsius z) figuram rectilineam æqualem constituere licet. Nam *divisio* ipsius $\square$ (theoremate §. 42. facile ad quælibet polygona extenso) manifesto *sectionem* ipsius $2R$ requirit, quam (ut ostendi potest) unice sub dicta conditione geometrice perficere licet. In omnibus autem talibus casibus præcedentia facile ad scopum perducent. Et potest quævis figura rectilinea in polygonum regulare n laterum geometrice converti, siquidem n sub formam *GAUSSianam* cadat.

Superesset denique, (ut res omni numero absolvatur), impossibilitatem (absque suppositione aliqua) decidendi, num Σ aut aliquod (et quodnam) S sit, demonstrare: quod tamen occasioni magis idoneæ reservatur.