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An analytical approach to the fractional Laplacian

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Contents

Abstract

In this thesis, we give a brief introduction to the topic of the fractional Laplacian and present an analytical solution to the corresponding Dirichlet problem in the ball. All the methods used to prove the result only make use of elementary methods, so no pre-existing knowledge on the topic is necessary. We start by introducing the concept of the fractional Laplacian and all the necessary techniques that we will make use of. Next, we take functions that are well known from regular Laplacian analysis and redefine them in the context of the fractional Laplacian. With these tools, we are able to establish the main result, which can be found in Theorem 4.1, followed by an outlook on how to further pursue the topic beyond the concepts of this thesis.

In dieser Arbeit werden wir den fraktionellen Laplace-Operator einführen und eine analytische Lösung für das zugehörige Dirichlet-Problem im Ball präsentieren. Wir werden uns dabei auf elementare Techniken beschränken, etwaiges Vorwissen zu diesem Thema ist daher nicht notwendig. Nachdem wir uns mit dem Konzept des fraktionellen Laplace-Operators vertraut gemacht haben, werden wir einige zusätzliche Konzepte einführen, die wir später benötigen werden. Danach werden wir einige Funktionen, die aus der Analysis des regulären Laplace-Operators bereits bekannt sind, im Kontext des fraktionellen Laplace-Operators definieren. Mit diesen Hilfsmitteln sind wir schließlich in der Lage, das Hauptresultat, welches in Theorem 4.1 formuliert wird, zu zeigen. Abschließend geben wir einen Ausblick, wie die hier beschriebenen Konzepte und Ideen über den Umfang dieser Arbeit hinaus weiter vertieft werden können.

Introduction

Over the last decade, the analysis of pseudo-differential equations involving the so-called fractional Laplace operator $(-\Delta)^s$ for $s \in (0,1)$ has received a lot of attention. The applications for these kinds of equations are numerous; as a model for fractional diffusion (see [5, 8, 9, 19]) as well as an infinitesimal generator of Lévy processes (see [13, 20]), the fractional Laplacian is used in various different topics such as electro-magnetic fluids, ground-water solute transports, biology and finance.
In this thesis, we will consider the problem of analytically solving the equation

$$(-\Delta)^s u = 0$$

in the ball $B_\rho(0)$ for $\rho > 0$ with Dirichlet boundary conditions, understood in a suitable sense.
We first point out that there are a multitude of different ways to approach introducing $(-\Delta)^s$ as a local operator, which are not necessarily equivalent with each other, as is also shown in [15]. The operator obtained by the approaches that we are pursuing in this thesis is also referred to as *Riesz* fractional Laplacian or *integrated* fractional Laplacian, other approaches yield the so-called *spectral* fractional Laplacian or the *regional* fractional Laplacian. Since we will only deal with the integrated one in this thesis, we will omit the prefix and simply refer to $(-\Delta)^s$ as the fractional Laplacian from now on. For a more broad introduction that showcases alternative definitions for the global and local fractional Laplacian and the interplay between the resulting operators, we refer to [6, 14, 15].
One of the essential differences between the Dirichlet problem involving the fractional Laplacian as opposed to the regular Laplacian is that $(-\Delta)^s$ is non-local. As can be seen in [15], it is necessary to give boundary conditions not only on the sphere $\partial B_\rho(0)$, but on the entire exterior space $\mathbb{R}^n \backslash B_\rho(0)$. So for the full problem, we consider the equation

$$\begin{cases} (-\Delta)^s u & = 0 \text{ in } B_\rho(0) \\ \qquad u & = g \text{ in } \mathbb{R}^n \backslash B_\rho(0) \end{cases} \tag{0.1}$$

for a sufficiently smooth given function g.
This thesis is structured as follows: In Chapter 1, we introduce some fundamental tools that we require later on, Chapter 2 gives an introduction to the fractional Laplace operator, showing off several equivalent definitions. Afterwards, in Chapter 3, we will

take a look at several important functions known from regular Laplacian-analysis, which we will then generalize in a way that allows them to be used in the framework of the fractional Laplacian. The main result of this thesis is formulated in Theorem 4.1, and its respective proof can be found in Chapter 4. We will see how the functions introduced in the previous chapter are used to give an analytical solution to (0.1). As an interesting side result, we will also obtain a way of analytically solving the global fractional Poisson equation

$$(-\Delta)^s = f \tag{0.2}$$

for a sufficiently smooth function f, see Theorem 4.6.
The content of this thesis follows the observations and ideas made by [2] and [7], though we try to go a little more into detail on various statements and the respective proofs.

1. Fundamental definitions and concepts

For the entire thesis, let $n \in \mathbb{N}$ be arbitrary but fixed.

Starting off, we establish various basic concepts that we are going to make use of later on.

1.1. The Fourier Transform

One representation of the fractional Laplace operator, which we are going to use later, relies on the concept of Fourier transformation. In order to give a proper definition, we need the theorem of Plancherel as stated in [16, Theorem 3.12]:

Theorem 1.1 (Theorem of Plancherel)**.** *There is a unique operator $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with*

$$\langle \mathcal{F}f, \mathcal{F}g \rangle_{L^2(\mathbb{R}^n)} = \langle f, g \rangle_{L^2(\mathbb{R}^n)} \qquad \forall f, g \in L^2(\mathbb{R}^n),$$

such that

$$(\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} \, dx,$$

for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. It holds true that

$$(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$$

almost everywhere for all $f \in L^2(\mathbb{R}^n)$.

The above setting is quite general, we will instead mostly work with the following functional space:

Definition 1.2. *The Schwartz space of rapidly decaying functions is defined as*

$$\mathscr{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \;\middle|\; \sup_{x \in \mathbb{R}^n} |x^\alpha D_\beta f(x)| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0^n \right\},$$

with $\mathscr{S}^*(\mathbb{R}^n)$ denoting the topological dual space of $\mathscr{S}(\mathbb{R}^n)$.

Remark 1.1. *By equipping the Schwartz space with the family of seminorms*

$$[f]^N_{\mathscr{S}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \max_{|\alpha|,|\beta|<N} \left|x^\alpha D^\beta f(x)\right|$$

for every $N \in \mathbb{N}_0$, it becomes a locally convex topological space.

With this space, we can introduce the Fourier transform and its inverse in a well-defined way, see [16, Chapter 3] for details.

Definition 1.3. *For any $f \in \mathscr{S}(\mathbb{R}^n)$, we define the Fourier transform as*

$$\widehat{f}(\xi) := (\mathscr{F} f)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-i\xi x}dx$$

and the inverse Fourier transform as

$$\check{f}(x) := (\mathscr{F}^{-1} f)(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi)e^{ix\xi}d\xi.$$

Remark 1.2. *By substituting $\xi = 0$ and $x = 0$ in the respective definitions above, we obtain the identities*

$$\begin{aligned} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)dx &= \widehat{f}(0), \\ \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f}(\xi)d\xi &= f(0). \end{aligned}$$

1.2. The Gamma- and Beta-function

In order to solve certain integrals later on, we will need the concept of the gamma and beta functions and their respective properties. A more detailed introduction to the topic containing all the results below can be found in [18, Chapter 2]

Definition 1.4. *The gamma function is defined for $x \in \mathbb{R}$ as*

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt.$$

We give some elementary properties of the gamma function.

Proposition 1.5. *The following identities hold:*

$$\begin{aligned} \Gamma(n) &= (n-1)! && \text{for any } n \in \mathbb{N}, \\ \Gamma(t) &= (t-1)\Gamma(t-1) && \text{for any } t > 0, \\ \frac{2^{1-2t}\sqrt{\pi}}{\Gamma(t)} &= \frac{\Gamma(\frac{1}{2}+t)}{\Gamma(2t)} && \text{for any } t > 0, \\ \Gamma(s)\Gamma(1-s) &= \frac{\pi}{\sin(\pi s)} && \text{for any } s \in (0,1), \\ \Gamma\Big(\frac{1}{2}-s\Big)\Gamma\Big(\frac{1}{2}+s\Big) &= \frac{\pi}{\cos(\pi s)} && \text{for any } s \in (0,1). \end{aligned}$$

Definition 1.6. *The Beta function is defined for any $x, y > 0$ as*

$$B(x, y) := \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt.$$

The next proposition will establish a connection between the beta and the gamma function.

Proposition 1.7. *The beta function can equivalently be written as*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for any $x, y > 0$, see [12, page 908] for further details

1.3. Circle inversion with the center point x_0

An essential tool for transforming certain integrals is the so-called circle inversion. In the following, let $r > 0$ and $x_0 \in B_r(0)$.

Definition 1.8. *We define the inversion of a point $x \in \mathbb{R}^n \backslash \{x_0\}$ with center x_0 as*

$$K_{x_0}(x) := x_0 - \frac{r^2 - |x_0|^2}{|x - x_0|^2}(x - x_0). \tag{1.1}$$

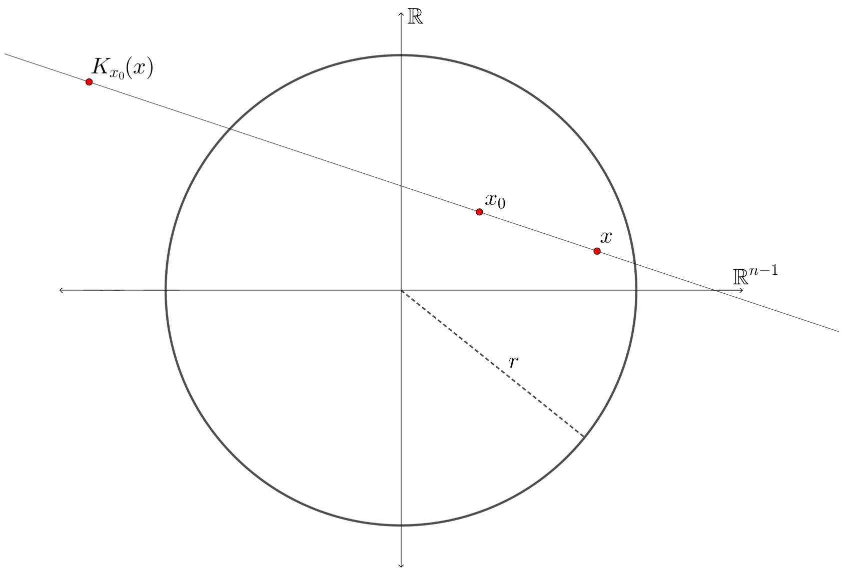

Figure 1.1.: Inversion of a point x with center x_0

To obtain a better understanding on how the above transformation works, we look at the following two observations.

Remark 1.3. *K_{x_0} is an involution on $\mathbb{R}^n \backslash \{x_0\}$. For any $x \in \mathbb{R}^n \backslash \{x_0\}$, the points $x_0, x, K_{x_0}(x)$ lie on one line, x_0 separates x and $K_{x_0}(x)$ and the identity*

$$|x_0|^2 + |x - x_0|\,|K_{x_0}(x) - x_0| = r^2 \tag{1.2}$$

holds.

Proof.
It is easy to see that the mapping K_{x_0} is bijective. Now let $x \in \mathbb{R}^n \backslash \{x_0\}$ be arbitrary and $x^* = K_{x_0}(x)$. From the definition of K_{x_0}, it is obvious that the three points lie on one line. To see that x_0 separates x and x^*, we need to check that $|x - x_0| \leqslant |x^* - x|$ and $|x^* - x_0| \leqslant |x^* - x|$ hold. Straightforward estimation yields

$$\begin{aligned} |x - x_0| &\leqslant \left(1 + \frac{r^2 - |x_0|^2}{|x - x_0|^2}\right)|x - x_0| = \left|-(x - x_0) - \frac{r^2 - |x_0|^2}{|x - x_0|^2}(x - x_0)\right| \\ &= \left|x_0 - \frac{r^2 - |x_0|^2}{|x - x_0|^2}(x - x_0) - x\right| = |x^* - x| \end{aligned}$$

as well as

$$|x^* - x_0| = \frac{r^2 - |x_0|^2}{|x - x_0|^2}\,|x - x_0| \leqslant \left(1 + \frac{r^2 - |x_0|^2}{|x - x_0|^2}\right)|x - x_0| = |x^* - x|\,.$$

Using the definition of K_{x_0}, we can easily prove identity (1.2):

$$\begin{aligned} |x_0|^2 + |x^* - x_0|\,|x - x_0| &= |x_0|^2 + \left|x_0 - \frac{r^2 - |x_0|^2}{|x - x_0|^2}(x - x_0) - x_0\right||x - x_0| \\ &= |x_0|^2 + \frac{r^2 - |x_0|^2}{|x - x_0|^2}\,|x - x_0|^2 = r^2. \end{aligned}$$

Finally, by using (1.2), we get

$$\begin{aligned} (K_{x_0} \circ K_{x_0})(x) &= K_{x_0}(x^*) = x_0 - \frac{r^2 - |x_0|^2}{|x^* - x_0|^2}(x^* - x_0) \\ &= x_0 + \frac{r^2 - |x_0|^2}{\frac{\left(r^2 - |x_0|^2\right)^2}{|x - x_0|^2}}\frac{r^2 - |x_0|^2}{|x - x_0|^2}(x - x_0) = x, \end{aligned}$$

showing that K_{x_0} is also an involution. $\square$

Proposition 1.9. *Let x^* and y^* be the inversions of $x \in \mathbb{R}^n \backslash \{x_0\}$ and $y \in \mathbb{R}^n \backslash \{x_0\}$ respectively. Then*

1. *If $x \in \partial B_r(0)$, then $x^* \in \partial B_r(0)$,*
2. *If $x \in B_r(0)$, then $x^* \in \mathbb{R}^n \backslash \partial B_r(0)$ and vice versa,*
3.

$$\frac{|x - x_0|^2}{(r^2 - |x_0|^2)(r^2 - |x|^2)} = \frac{1}{|x^*|^2 - r^2}, \tag{1.3}$$

4.

$$|\det(DK_{x_0}(x))| = \left(\frac{|x^* - x_0|}{|x - x_0|}\right)^n, \tag{1.4}$$

where $DK_{x_0}(x)$ denotes the Jacobian of K_{x_0} evaluated at x,

5.

$$|x^* - y^*| = \left(r^2 - |x_0|^2\right) \frac{|x - y|}{|x - x_0|\,|y - x_0|}. \tag{1.5}$$

Proof.
For the sake of simplicity and without loss of generality, we choose the center of the point inversion at zero, meaning $x_0 = 0$. The general case can be proven by using the Pythagorean theorem, see e.g. [7, Proposition A.3].

1. Let $x \in \partial B_r(0)$, meaning $|x| = r$. Then we have

$$|x^*| = \left|\frac{r^2}{|x|^2} x\right| = \frac{r^2}{|x|} = r$$

and therefore $x^* \in \partial B_r(0)$.

2. For $x \in B_r(0)$ we have $|x| < r$, which yields

$$|x^*| = \left|\frac{r^2}{|x|^2} x\right| = \frac{r^2}{|x|} > r.$$

Similarly, it can be shown that if $x \in \mathbb{R}^n \backslash B_r$, then $x^* \in B_r$.

3. By using (1.2), we can calculate

$$\frac{1}{|x^*|^2 - r^2} = \frac{1}{\frac{r^4}{|x|^2} - r^2} = \frac{1}{\frac{r^4 - r^2|x|^2}{|x|^2}} = \frac{|x|^2}{r^2(r^2 - |x|^2)},$$

which shows identity (1.3).

4. First, we evaluate the Jacobian of K_0:

$$DK_0(x) = \frac{r^2}{|x|^4}\begin{pmatrix} 2x_1^2 - |x|^2 & 2x_1x_2 & \dots & 2x_1x_n \\ 2x_1x_2 & 2x_2^2 - |x|^2 & \dots & 2x_2x_n \\ \vdots & & \ddots & \vdots \\ 2x_1x_n & 2x_2x_n & \dots & 2x_n^2 - |x|^2 \end{pmatrix},$$

where $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$. Since K_0 is invariant under rotation, we can assume without loss of generality that $x = |x|\, e_1$, which simplifies the Jacobian to

$$DK_0(x) = \frac{r^2}{|x|^4}\begin{pmatrix} |x|^2 & 0 & \dots & 0 \\ 0 & -|x|^2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -|x|^2 \end{pmatrix},$$

yielding

$$|\det(DK_0(x))| = \left|\frac{r^{2n}}{|x|^{4n}}(-1)^{n-1}\,|x|^{2n}\right| = \frac{1}{|x|^n}\frac{r^{2n}}{|x|^n} = \frac{|x^*|^n}{|x|^n}.$$

5. We write $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ and compute

$$\begin{aligned} |x^* - y^*| &= r^2\left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right| = \frac{r^2}{|x|^2\,|y|^2}\left|x\,|y|^2 - y\,|x|^2\right| \\ &= \frac{r^2}{|x|^2\,|y|^2}\sqrt{\sum_{i=1}^n \left(x_i\,|y|^2 - y_i\,|x|^2\right)^2} \\ &= \frac{r^2}{|x|^2\,|y|^2}\sqrt{\sum_{i=1}^n \left(x_i^2\,|y|^4 - 2x_iy_i\,|x|^2\,|y|^2 + y_i^2\,|x|^4\right)} \\ &= \frac{r^2}{|x|^2\,|y|^2}\sqrt{|x|^2\,|y|^4 - 2\,|x|^2\,|y|^2\sum_{i=1}^n x_iy_i + |y|^2\,|x|^4} \\ &= \frac{r^2}{|x|\,|y|}\sqrt{|y|^2 - 2\sum_{i=1}^n x_iy_i + |x|^2} \\ &= \frac{r^2}{|x|\,|y|}\sqrt{\sum_{i=1}^n (x_i - y_i)^2} = r^2\frac{|x-y|}{|x|\,|y|}, \end{aligned}$$

proving the last identity.

□

2. The fractional Laplacian and its framework

From now on, for this and the upcoming chapters, let $s \in (0,1)$ be arbitrary but fixed.

As we pointed out in the introduction, there are various different ways to define the fractional Laplace operator. Our first definition will work via the Cauchy principal value integral:

Definition 2.1. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. Then we define*

$$(-\Delta)^s u(x) := C(n,s)\ p.v. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \tag{2.1}$$

with

$$C(n,s) := \frac{2^{2s}\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(1-s)}. \tag{2.2}$$

Remark 2.1.

- *The singularity in the integral in* (2.1) *is in general not integrable, which means it has to be understood in the sense of a Cauchy principal value, namely*

$$p.v. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy := \lim_{\rho \to 0} \int_{\mathbb{R}^n \setminus B_\rho(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

- *The classical Laplacian can be expressed in a similar way, see [6, page 9].*

By shifting the singularity in the above integral from an arbitrary point x to the origin (see [17, Lemma 3.2] for a detailed proof), we obtain the following identity:

Proposition 2.2. *Let $(-\Delta)^s$ be the fractional Laplacian defined by* (2.1) *and let $u \in \mathscr{S}(\mathbb{R}^n)$. Then we have*

$$(-\Delta)^s u(x) = \frac{C(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}} dy. \tag{2.3}$$

The definition of the fractional Laplacian via (2.3) is motivated by the mathematical problem of modeling a random walk with arbitrarily long jumps, see [20].

Additionally, one can show that the fractional Laplacian can be equivalently defined via Fourier transformation.

Proposition 2.3. *Let* $(-\Delta)^s$ *be the fractional Laplacian defined by* (2.1) *and let* $u \in \mathscr{S}(\mathbb{R}^n)$*. Then we have*

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}\left(|\xi|^{2s}\,\widehat{u}(\xi)\right)(x). \tag{2.4}$$

We refer to [6, Lemma 3.1.1] for the proof. The constant $C(n,s)$ was chosen to guarantee the equivalencies between (2.1), (2.3) and (2.4), as can also be seen in [6].

Notice that this definition gives a visible connection between the regular and the fractional Laplacian in the following way:

Remark 2.2. *Recall that for* $u \in \mathscr{S}(\mathbb{R}^n)$*, the classical Laplacian can be written as*

$$\begin{aligned} -\Delta u(x) &= -\Delta(\mathscr{F}^{-1}(\hat{u}(x))) = -\Delta\left(\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\hat{u}(\xi)e^{ix\xi}d\xi\right) \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}|\xi|^2\,\hat{u}(\xi)e^{ix\xi}d\xi = \mathscr{F}^{-1}(|\xi|^2\,\hat{u}(\xi)), \end{aligned}$$

which makes $(-\Delta)^s$ *a natural generalization of the classical Laplacian with the limit cases*

$$\lim_{s\to 1}(-\Delta)^s u = -\Delta u \quad \text{and} \quad \lim_{s\to 0}(-\Delta)^s u = u,$$

see [17] for further details about the subject.

We underline that only functions defined on the whole space $\mathbb{R}^n$ can be applied to the fractional Laplacian, therefore the problem (0.1) has to be understood in the following way: For a function u in the ball B_ρ, $\rho > 0$, we define the following extension:

$$\widetilde{u}(x) := \begin{cases} u(x) & \text{for } x \in B_\rho(0) \\ 0 & \text{for } x \in \mathbb{R}^n \backslash B_\rho(0). \end{cases}$$

Whenever we write $(-\Delta)^s u$ for such a function, it will be understood as $(-\Delta)^s \widetilde{u}$.

The next regularity result will be useful later on. The respective proof can be found in [19, Proposition 2.1.7].

Proposition 2.4. *Let $\varepsilon > 0$ and $u \in C^{0,2s+\varepsilon}$. Then the mapping*

$$\begin{aligned} \mathbb{R}^n &\to \mathbb{R}^n, \\ x &\mapsto (-\Delta)^s u(x) \end{aligned} \tag{2.5}$$

is continuous.

As is the case of many other differential equations, it will be useful to define a solution to the problem in a weak sense. To do so, we introduce additional weighted functional spaces.

Definition 2.5. *For $s \in (0,1)$, we define the weighted L^1 space as*

$$L^1_s(\mathbb{R}^n) := \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx < \infty \right\},$$

along with the weighted L^1-norm

$$\|u\|_{L^1_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}} dx.$$

Remark 2.3. *It is possible to allow the fractional Laplacian to be defined for a broader set of functions. Indeed, for any $\varepsilon > 0$ and $x \in \mathbb{R}^n$, the term $(-\Delta)^s u(x)$, as given by (2.1), is well defined at x for any $u \in L^1_s(\mathbb{R}^n)$ that is either $C^{0,2s+\varepsilon}$ for $s < \frac{1}{2}$ or $C^{1,2s+\varepsilon-1}$ for $s \geqslant \frac{1}{2}$ in a neighborhood of x. A proof can be found in [19, Proposition 2.1.4].*

Definition 2.6. *For any $s \in (0,1)$, the weighted Schwartz space is defined as*

$$\mathscr{S}_s(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \,\middle|\, \forall \alpha \in \mathbb{N}_0^n, \sup_{x \in \mathbb{R}^n} \left\{ \left(1+|x|^{n+2s}\right) |D^\alpha f(x)| \right\} < \infty \right\},$$

with $\mathscr{S}_s^*(\mathbb{R}^n)$ denoting the topological dual space of $\mathscr{S}_s(\mathbb{R}^n)$.

Remark 2.4. *By equipping the weighted Schwartz space with the family of seminorms*

$$[f]^{|\alpha|}_{\mathscr{S}_s(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} (1+|x|^{n+2s}) |D^\alpha f(x)|,$$

for every $\alpha \in \mathbb{N}_0^n$,it becomes a locally convex topological space.

In order to establish a well-defined framework for a weak solution, we need the following statement:

Lemma 2.7. *Let $u \in \mathscr{S}(\mathbb{R}^n)$. Then $(-\Delta)^s u \in \mathscr{S}_s(\mathbb{R}^n)$.*

Proof.
We will prove the statement via induction over $|\alpha|$ for $\alpha \in \mathbb{N}_0^n$. Starting with the basis, let $\alpha = 0$, which means we need to prove

$$\left(1 + |x|^{n+2s}\right) |(-\Delta)^s u(x)| < \infty$$

for any $x \in \mathbb{R}^n$. This follows by showing that for any fixed $x \in \mathbb{R}^n \backslash B_1(0)$, the bound

$$|(-\Delta)^s u(x)| \leqslant \frac{c_{n,s}}{|x|^{n+2s}}$$

holds. To see this, we first use a Taylor expansion of u to obtain

$$\begin{aligned} u(x+y) &= u(x) + \nabla u(x)^\top \cdot y + \frac{1}{2} y^\top \cdot D^2 u(\xi_1) \cdot y, \\ u(x-y) &= u(x) - \nabla u(x)^\top \cdot y + \frac{1}{2} y^\top \cdot D^2 u(\xi_2) \cdot y, \end{aligned}$$

where $D^2 u(\xi_i)$ is the Hessian matrix of u evaluated at $\xi_i \in B_{|y|}(x)$ for $i \in \{1, 2\}$. We can now estimate

$$\begin{aligned} &|(-\Delta)^s u(x)| \\ &\leqslant \int_{|y|<\frac{|x|}{2}} \frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{n+2s}} dy + 2 \int_{|y|\geqslant\frac{|x|}{2}} \frac{|u(x) - u(x+y)|}{|y|^{n+2s}} dy \\ &\leqslant \int_{|y|<\frac{|x|}{2}} \frac{|D^2 u(\xi)|}{|y|^{n+2s-2}} + 2 |u(x)| \int_{|y|\geqslant\frac{|x|}{2}} \frac{1}{|y|^{n+2s}} dy + \frac{2^{n+2s+1}}{|x|^{n+2s}} \int_{|y|\geqslant\frac{|x|}{2}} |u(x-y)| \, dy \\ &\leqslant c_{n,s}^{(1)} \left|D^2 u(\xi)\right| + c_{n,s}^{(2)} |u(x)| + \frac{2^{n+2s+1}}{|x|^{n+2s}} \|u\|_{L^1(\mathbb{R}^n)} \\ &\leqslant c_{n,s}^{(1)} \left(\frac{|x|}{|\xi|}\right)^{n+2s} \frac{(1+|\xi|)^{n+2}}{|x|^{n+2s}} \left|D^2 u(\xi)\right| + c_{n,s}^{(2)} \frac{(1+|x|)^{n+2}}{|x|^{n+2s}} |u(x)| + \frac{2^{n+2s+1}}{|x|^{n+2s}} \|u\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

where $\xi \in B_{|y|}(x)$. We see that

$$|\xi| \geqslant |x| - |y| \geqslant |x| - \frac{|x|}{2} \geqslant \frac{|x|}{2}$$

and therefore

$$\frac{|x|}{|\xi|} \leqslant 2.$$

which ultimately yields

$$
\begin{aligned}
&|(-\Delta)^s u(x)| \\
&\leqslant \frac{1}{|x|^{n+2s}}\left(c_{n,s}^{(1)} 2^{n+2s}(1+|\xi|)^{n+2}\left|D^2u(\xi)\right| + c_{n,s}^{(2)}(1+|x|)^{n+2}|u(x)| + 2^{n+2s+1}\|u\|_{L^1(\mathbb{R}^n)}\right) \\
&\leqslant \frac{\widetilde{c}_{n,s}}{|x|^{n+2s}}\left(\sup_{z\in\mathbb{R}^n}\left\{(1+|z|)^{n+2}\left|D^2u(z)\right|\right\} + \sup_{z\in\mathbb{R}^n}\left\{(1+|z|)^{n+2}|u(z)|\right\} + \|u\|_{L^1(\mathbb{R}^n)}\right) \\
&= \frac{c_{n,s}}{|x|^{n+2s}},
\end{aligned}
$$

where we have made use of the fact that $u \in \mathscr{S}(\mathbb{R}^n)$.
With the basis proven, we now assume that

$$\left(1+|x|^{n+2s}\right)|D^\alpha(-\Delta)^s u(x)| < \infty$$

for a fixed $\alpha \in \mathbb{N}_0^n$ and let $k \in \{1, ...n\}$ be also fixed. Since $\partial_k u \in \mathscr{S}(\mathbb{R}^n)$, this estimate also holds true for $\partial_k u$. Additionally, by making use of (2.4), we have that

$$
\begin{aligned}
\partial_{x_k}(-\Delta)^s u(x) &= \partial_{x_k}\mathscr{F}^{-1}\left(|\xi|^{2s}\,\widehat{u}(\xi)\right)(x) = \mathscr{F}^{-1}\left(i\xi_k\,|\xi|^{2s}\,\widehat{u}(\xi)\right)(x) \\
&= \mathscr{F}^{-1}\left(|\xi|^{2s}\,\widehat{\partial}_{x_k}u(\xi)\right)(x) = (-\Delta)^s\partial_{x_k}u(x),
\end{aligned}
$$

which results in

$$\left(1+|x|^{n+2s}\right)|\partial_k D^\alpha(-\Delta)^s u(x)| = \left(1+|x|^{n+2s}\right)|D^\alpha(-\Delta)^s\partial_k u(x)| < \infty,$$

finishing the induction. □

The Poisson problem (0.2), which we will solve along the main result in Chapter 4, can now be considered in a distributional sense.

Definition 2.8. *Let $f \in \mathscr{S}^*(\mathbb{R}^n)$. We say that $u \in \mathscr{S}_s^*(\mathbb{R}^n)$ is a solution of (0.2) in the distributional sense if*

$$\langle u, (-\Delta)^s\varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)dx \qquad \text{for any } \varphi \in \mathscr{S}(\mathbb{R}^n),$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing of $\mathscr{S}_s^*(\mathbb{R}^n)$ and $\mathscr{S}_s(\mathbb{R}^n)$.

In addition, we want to keep the following estimate in mind:

Remark 2.5. *For any $u \in L_s^1(\mathbb{R}^n)$ and $v \in \mathscr{S}_s(\mathbb{R}^n)$, we have*

$$
\begin{aligned}
|\langle u, v\rangle| &\leqslant \int_{\mathbb{R}^n}|u(x)|\,|v(x)|\,dx = \int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+2s}}\left(1+|x|^{n+2s}\right)|v(x)|\,dx \\
&\leqslant \sup_{x\in\mathbb{R}^n}\left\{\left(1+|x|^{n+2s}\right)|v(x)|\right\}\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+2s}}dx \\
&= [v]^0_{\mathscr{S}_s(\mathbb{R}^n)}\|u\|_{L_s^1(\mathbb{R}^n)} < \infty.
\end{aligned}
$$

3. Important functions and properties

In this chapter we are going to give an introduction to three functions along with some of their most important properties. All the gathered results are going to be applied in Chapter 4.

3.1. The s-mean value property

The first function that we want to discuss will be a very useful tool in showing if a function is s-harmonic.

Definition 3.1. *For any $\rho > 0$ fixed, we define*

$$A_\rho(x) = \begin{cases} c(n,s) \cdot \dfrac{\rho^{2s}}{|x|^n \left(|x|^2 - \rho^2\right)^s} & \text{for } x \in \mathbb{R}^n \backslash \overline{B_\rho(0)} \\ 0 & \text{for } x \in \overline{B_\rho(0)}, \end{cases}$$

for all $x \in \mathbb{R}^n$, where

$$c(n,s) := \frac{\sin(\pi s)\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}+1}} \tag{3.1}$$

is a dimensional constant.

The choice of the constant $c(n,s)$ will become apparent in the following normalization property.

Lemma 3.2. *For any $\rho > 0$, we have*

$$\int_{|x| \geqslant \rho} A_\rho(x) dx = 1. \tag{3.2}$$

Proof.
We first use the transformation $r = \Phi(x) := |x|$ to change to polar coordinates

$$\begin{aligned}\int_{|x|\geqslant\rho} A_\rho(x)dx &= c(n,s)\int_{|x|\geqslant\rho} \frac{\rho^{2s}}{|x|^n\left(|x|^2-\rho^2\right)^s}dx \\ &= c(n,s)\int_\rho^\infty \frac{\rho^{2s}}{r^n(r^2-\rho^2)^s}\cdot\frac{1}{|\det(D\Phi)|}dr \\ &= c(n,s)\int_\rho^\infty \frac{\rho^{2s}}{r^n(r^2-\rho^2)^s}\cdot\frac{S_{n-1}}{r^{1-n}}dr \\ &= c(n,s)S_{n-1}\int_\rho^\infty \frac{\rho^{2s}}{r(r^2-\rho^2)^s}dr,\end{aligned}$$

where S_{n-1} is the measure of the $(n-1)$-dimensional unit sphere. By plugging in the values of both S_{n-1} and $c(n,s)$, we further get

$$\begin{aligned}\int_{|x|\geqslant\rho} A_\rho(x)dx &= \frac{\sin(\pi s)\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}+1}}\cdot\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_\rho^\infty \frac{\rho^{2s}}{r(r^2-\rho^2)^s}dr \\ &= \frac{2\sin(\pi s)}{\pi}\int_\rho^\infty \frac{1}{r(\frac{r^2}{\rho^2}-1)^s}dr.\end{aligned}$$

Substituting $t = \frac{r^2}{\rho^2} - 1$ along with (A.3) yields

$$\int_{|x|\geqslant\rho} A_\rho(x)dx = \frac{\sin(\pi s)}{\pi}\int_0^\infty \frac{1}{(t+1)t^s}dy = 1.$$

□

One might recall that a function is harmonic with respect to the regular Laplacian if and only if the so-called *mean value property* holds true. We now want to introduce a suitable counterpart for the fractional Laplacian.

Definition 3.3. *Let $x \in \mathbb{R}^n$ be arbitrary but fixed and $u \in L^1_s(\mathbb{R}^n)$ be continuous in a neighborhood of x. We say u has the s-mean value property at x if*

$$u(x) = (A_\rho * u)(x)$$

holds for any $\rho > 0$ arbitrarily small. If this property holds true for any $x \in \Omega \subseteq \mathbb{R}^n$, then u is said to have the s-mean value property in Ω.

The upcoming theorem is vital since it shows us that a sufficiently smooth function that has the s-mean value property, is also s-harmonic, which will give us a great alternative method for showing s-harmonicity of a function.

Theorem 3.4. *Let $\varepsilon > 0$ and $x \in \mathbb{R}^n$. In addition, let $u \in L^1_s(\mathbb{R}^n)$ and the following hold:*

1. $u \in C^{0,2s+\varepsilon}$ *in a neighborhood of* $x \in \mathbb{R}^n$ *for* $s < \frac{1}{2}$,

2. $u \in C^{1,2s-1+\varepsilon}$ *in a neighborhood of* $x \in \mathbb{R}^n$ *for* $s \geqslant \frac{1}{2}$.

If u *has the s-mean value property at* x, *then we have*

$$(-\Delta)^s u(x) = 0.$$

Proof.
Let $x \in \mathbb{R}^n$ and $\rho > 0$ be arbitrarily small. Then the s-mean value property of u along with (3.2) yields

$$\begin{aligned}
&\int_{|y|\geqslant\rho} \frac{u(x)-u(x-y)}{|y|^n\left(|y|^2-\rho^2\right)^s}dy \\
&= \frac{1}{\rho^{2s}}\left(u(x)\int_{|y|\geqslant\rho}\frac{\rho^{2s}}{|y|^n\left(|y|^2-\rho^2\right)^s}dy - \int_{|y|\geqslant\rho}\frac{\rho^{2s}}{|y|^n\left(|y|^2-\rho^2\right)^s}u(x-y)dy\right) \\
&= \frac{1}{c(n,s)\rho^{2s}}\left(u(x)\int_{|y|\geqslant\rho}A_\rho(y)dy - \int_{|y|\geqslant\rho}A_\rho(y)u(x-y)dy\right) \\
&= \frac{1}{c(n,s)\rho^{2s}}\left(u(x)-(A_\rho * u)(x)\right) = 0.
\end{aligned}$$

If we can show

$$\lim_{\rho\to 0}\int_{|y|\geqslant\rho}\frac{u(x)-u(x-y)}{|y|^{n+2s}}dy = \lim_{\rho\to 0}\int_{|y|\geqslant\rho}\frac{u(x)-u(x-y)}{|y|^n\left(|y|^2-\rho^2\right)^s}dy, \tag{3.3}$$

then we're done since

$$\begin{aligned}
(-\Delta)^s u(x) &= C(n,s)\lim_{\rho\to 0}\int_{|y|\geqslant\rho}\frac{u(x)-u(x-y)}{|y|^{n+2s}}dy \\
&= C(n,s)\lim_{\rho\to 0}\int_{|y|\geqslant\rho}\frac{u(x)-u(x-y)}{|y|^n\left(|y|^2-\rho^2\right)^s}dy = 0.
\end{aligned}$$

Let $R > \rho\sqrt{2}$ be fixed. We split the right hand side of (3.3) into two parts:

$$\begin{aligned}
&\int_{|y|\geqslant\rho}\frac{u(x)-u(x-y)}{|y|^n\left(|y|^2-\rho^2\right)^s}dy \\
&= \int_{|y|\geqslant R}\frac{u(x)-u(x-y)}{|y|^n\left(|y|^2-\rho^2\right)^s}dy + \int_{\rho\leqslant|y|<R}\frac{u(x)-u(x-y)}{|y|^n\left(|y|^2-\rho^2\right)^s}dy \\
&=: I_\rho(x) + \widetilde{I}_\rho(x).
\end{aligned}$$

We need to compute

$$\lim_{\rho\to 0} I_\rho(x) = \int_{|y|\geqslant R}\frac{u(x)-u(x-y)}{|y|^{n+2s}}dy, \tag{3.4}$$

$$\lim_{\rho\to 0}\widetilde{I}_\rho(x) = \lim_{\rho\to 0}\int_{\rho\leqslant|y|<R}\frac{u(x)-u(x-y)}{|y|^{n+2s}}dy \tag{3.5}$$

to prove identity (3.3). For $|y| \geqslant R > \rho\sqrt{2} > 0$, using $|y|^2 > 2\rho^2$, we can estimate

$$\frac{|y|^2}{|y|^2 - \rho^2} = \frac{1}{1 - \frac{\rho^2}{|y|^2}} < \frac{1}{1 - \frac{1}{2}} = 2,$$

which implies

$$\frac{1}{\left(|y|^2 - \rho^2\right)^s} < \frac{2^s}{|y|^{2s}}.$$

This, along with the fact that $u \in L^1_s(\mathbb{R}^n)$, allows us to obtain

$$\frac{u(x) - u(x-y)}{|y|^n \left(|y|^2 - \rho^2\right)^s} \leqslant 2^s \frac{|u(x) - u(x-y)|}{|y|^{n+2s}} \in L^1(\mathbb{R}^n \backslash B_R(0), dy).$$

The dominated convergence theorem now proves (3.4).
For $\rho \leqslant |y| < R$ we define

$$\begin{aligned} J_\rho(x) &:= \widetilde{I}_\rho(x) - \int_{\rho \leqslant |y| < R} \frac{u(x) - u(x-y)}{|y|^{n+2s}} dy \\ &= \int_{\rho \leqslant |y| < R} (u(x) - u(x-y)) \left(\frac{1}{|y|^n \left(|y|^2 - \rho^2\right)^s} - \frac{1}{|y|^{n+2s}} \right) dy \end{aligned}$$

and claim that $\lim_{\rho \to 0} J_\rho(x) = 0$. To show this, we consider two cases: For $s < \frac{1}{2}$, we use that $u \in C^{0,2s+\varepsilon}$ to estimate

$$\begin{aligned} |J_\rho(x)| &\leqslant \int_{\rho \leqslant |y| < R} |u(x) - u(x-y)| \left(\frac{1}{|y|^n \left(|y|^2 - \rho^2\right)^s} - \frac{1}{|y|^{n+2s}} \right) dy \\ &\leqslant c \int_{\rho \leqslant |y| < R} |y|^{2s+\varepsilon} \left(\frac{1}{|y|^n \left(|y|^2 - \rho^2\right)^s} - \frac{1}{|y|^{n+2s}} \right) dy. \end{aligned}$$

For $s \geqslant 1/2$ and $u \in C^{1,2s-1+\varepsilon}$ we use that

$$\begin{aligned} |u(x) - u(x-y) - y \cdot \nabla u(x)| &= \left| -\int_0^1 \frac{d}{dt} u(x-ty) dt - y \cdot \nabla u(x) \right| \\ &= \left| \int_0^1 y \left(\nabla u(x-ty) - \nabla u(x) \right) dt \right| \\ &\leqslant |y| \int_0^1 |\nabla u(x-ty) - \nabla u(x)| \, dt \\ &\leqslant c(s,\varepsilon) \, |y|^{2s+\varepsilon}. \end{aligned}$$

This, together with the fact that $\frac{y\cdot\nabla u(x)}{\left(|y|^2-\rho^2\right)^s|y|^n}$ and $\frac{y\cdot\nabla u(x)}{|y|^{n+2s}}$ are odd functions and hence vanish when integrated on the symmetrical domain $B_R\backslash B_\rho$, yields

$$\begin{aligned}
|J_\rho(x)| &= \left|\int_{\rho\leqslant|y|<R}\frac{u(x)-u(x-y)}{|y|^n\left(|y|^2-\rho^2\right)^s}-\frac{u(x)-u(x-y)}{|y|^{n+2s}}dy\right| \\
&= \left|\int_{\rho\leqslant|y|<R}\frac{u(x)-u(x-y)-y\cdot\nabla u(x)}{|y|^n\left(|y|^2-\rho^2\right)^s}-\frac{u(x)-u(x-y)-y\cdot\nabla u(x)}{|y|^{n+2s}}dy\right| \\
&\leqslant c(s,\varepsilon)\int_{\rho\leqslant|y|<R}|y|^{2s+\varepsilon}\left(\frac{1}{|y|^n\left(|y|^2-\rho^2\right)^s}-\frac{1}{|y|^{n+2s}}\right)dy.
\end{aligned}$$

Since we end up with the same estimate for both $s<1/2$ and $s\geqslant 1/2$, we obtain that

$$|J_\rho(x)| \leqslant c(s,\varepsilon)\int_{\rho\leqslant|y|<R}|y|^{2s+\varepsilon}\left(\frac{1}{|y|^n\left(|y|^2-\rho^2\right)^s}-\frac{1}{|y|^{n+2s}}\right)dy$$

holds for all $s\in(0,1)$. By passing to polar coordinates and making the change of variables $t:=r/\rho$, we can further estimate

$$\begin{aligned}
|J_\rho(x)| &\leqslant c(n,s,\varepsilon)\int_\rho^R r^{2s+\varepsilon}\left(\frac{1}{r^n\left(r^2-\rho^2\right)^s}-\frac{1}{r^{n+2s}}\right)r^{n-1}dr \\
&= c(n,s,\varepsilon)\int_\rho^R r^{2s+\varepsilon}\left(\frac{r^{2s}}{r^{1+2s}\left(r^2-\rho^2\right)^s}-\frac{1}{r^{1+2s}}\right)dr \\
&= c(n,s,\varepsilon)\int_\rho^R\frac{1}{r^{1-\varepsilon}}\left(\frac{r^{2s}}{\left(r^2-\rho^2\right)^s}-1\right)dr \\
&= c(n,s,\varepsilon)\int_1^{\frac{R}{\rho}}\frac{1}{(t\rho)^{1-\varepsilon}}\left(\frac{(t\rho)^{2s}}{\left((t\rho)^2-\rho^2\right)^s}-1\right)rdt \\
&= c(n,s,\varepsilon)\rho^\varepsilon\int_1^{\frac{R}{\rho}}\frac{1}{t^{1-\varepsilon}}\left(\frac{t^{2s}}{\left(t^2-1\right)^s}-1\right)\rho dt \\
&= c(n,s,\varepsilon)\rho^\varepsilon\int_1^{\frac{R}{\rho}}\frac{1}{t^{1-\varepsilon}}\left(\left(\frac{t}{t-1}\right)^s\left(\frac{t}{t+1}\right)^s-1\right)dt \\
&< c(n,s,\varepsilon)\rho^\varepsilon\int_1^{\frac{R}{\rho}}\frac{1}{t^{1-\varepsilon}}\left(\left(\frac{t}{t-1}\right)^s-1\right)dt \\
&= c(n,s,\varepsilon)\rho^\varepsilon\left(\int_1^{\sqrt{2}}\frac{1}{t^{1-\varepsilon}}\left(\left(\frac{t}{t-1}\right)^s-1\right)dt+\int_{\sqrt{2}}^{\frac{R}{\rho}}\frac{1}{t^{1-\varepsilon}}\left(\left(\frac{t}{t-1}\right)^s-1\right)dt\right).
\end{aligned}$$

The first integral is finite since

$$\begin{aligned}\int_1^{\sqrt{2}} \frac{1}{t^{1-\varepsilon}}\left(\left(\frac{t}{t-1}\right)^s - 1\right) dt &\leqslant \int_1^{\sqrt{2}} \frac{1}{t^{1-s-\varepsilon}}\left(\frac{1}{(t-1)^s} - \frac{1}{t^s}\right) dt \\ &\leqslant \tilde{c}(n,s)\int_1^{\sqrt{2}} \frac{1}{(t-1)^s} - \frac{1}{t^s} dt < \infty.\end{aligned}$$

For the second integral, we use that for all $t \geqslant \sqrt{2}$

$$\left(\frac{t}{t-1}\right)^s - 1 \leqslant \frac{s}{t\left(1-\frac{1}{\sqrt{2}}\right)^{s+1}},$$

which yields

$$\lim_{\rho\to 0}\int_{\sqrt{2}}^{\frac{R}{\rho}} \frac{1}{t^{1-\varepsilon}}\left(\left(\frac{t}{t-1}\right)^s - 1\right) dt \leqslant \bar{c}(s)\int_{\sqrt{2}}^{\infty} \frac{1}{t^{2-\varepsilon}} dt < \infty.$$

With this we have shown the claim $\lim_{\rho\to 0} J_\rho(x) = 0$, which proves (3.5) and thus (3.3), concluding the proof of the theorem. □

3.2. The Poisson Kernel

The Poisson Kernel is a well-known tool in potential theory to analytically solve the Laplace equation with Dirichlet boundary conditions in the ball. We will again give a more generalized definition that works for the context of the fractional Laplacian.

Definition 3.5. *Let $\rho > 0$, $x \in B_\rho(0)$ and $y \in \mathbb{R}^n \backslash \overline{B_\rho(0)}$. We define the Poisson-Kernel P_ρ by*

$$P_\rho(x,y) := \frac{c(n,s)}{|x-y|^n}\left(\frac{\rho^2 - |x|^2}{|y|^2 - \rho^2}\right)^s, \tag{3.6}$$

where $c(n,s)$ is the same dimensional constant chosen in (3.1).

As it was the case with A_ρ in the last section, the constant $c(n,s)$ was chosen for the sake of normalization.

Lemma 3.6. *For any $\rho > 0$ and $x \in B_\rho(0)$, we have*

$$\int_{|y|\geqslant\rho} P_\rho(x,y)dy = 1. \tag{3.7}$$

Proof.
In order to compute the integral, we will use hyperspherical coordinates with radius $r > 0$ and angles $\phi_1 \in [0, 2\pi]$, $\phi_2, ...\phi_{n-1} \in [0, \pi]$. Any $y \in \mathbb{R}^n \backslash \overline{B_\rho(0)}$ can then be written as

$$\begin{aligned}
y_n &= r\cos(\phi_{n-1}) \\
y_{n-1} &= r\sin(\phi_{n-1})\cos(\phi_{n-2}) \\
y_{n-2} &= r\sin(\phi_{n-1})\sin(\phi_{n-2})\cos(\phi_{n-3}) \\
&... \\
y_2 &= r\sin(\phi_{n-1})\sin(\phi_{n-2})\sin(\phi_{n-3}) \ ... \ \sin(\phi_2)\cos(\phi_1) \\
y_1 &= r\sin(\phi_{n-1})\sin(\phi_{n-2})\sin(\phi_{n-3}) \ ... \ \sin(\phi_2)\sin(\phi_1),
\end{aligned}$$

with the absolute value of the Jacobian given by

$$\left|\det \frac{\partial(y_i)}{\partial(r, \phi_j)}\right| = r^{n-1} \prod_{k=1}^{n-2} \sin^k(\phi_{k+1}).$$

Since $x \in B_\rho(0)$, we can make use of the spherical symmetry of $B_\rho(0)$, so we can assume without loss of generality that $x = |x|\, e_n$. Then by the law of cosines, the identity

$$|x - y|^2 = |x|^2 + r^2 - 2r\,|x| \cos\phi_{n-1}$$

holds, see Figure 3.1 for an illustration.

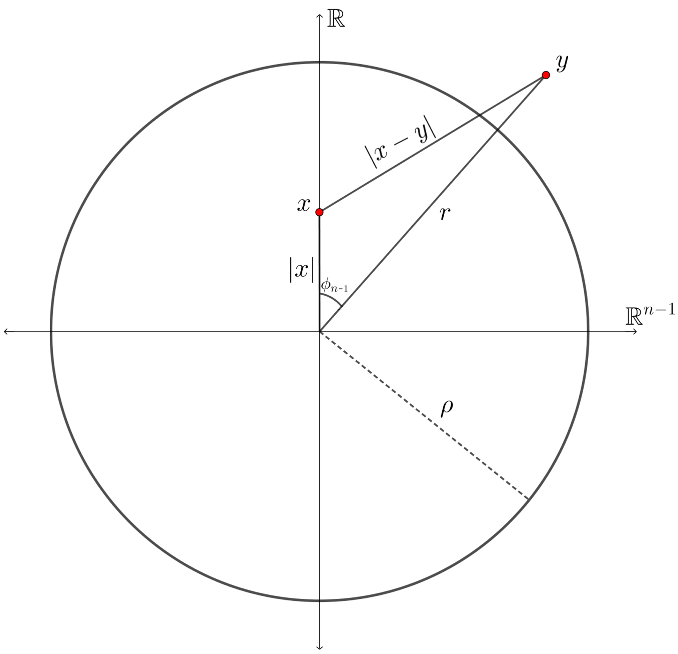

Figure 3.1.: Law of Cosines in $\mathbb{R}^n$

Also note that we have $|y| = r$, which can easily be seen by factorizing and making use of the Pythagorean trigonometric identity. Using the above change of coordinates, we further obtain

$$\begin{aligned}
&\int_{|y|\geqslant\rho} P_\rho(x,y)dy \\
&= c(n,s)\left(\rho^2-|x|^2\right)^s \int_{|y|\geqslant\rho} \frac{1}{\left(|y|^2-\rho^2\right)^s |x-y|^n} dy \\
&= c(n,s)\left(\rho^2-|x|^2\right)^s \int_\rho^\infty \int_0^{2\pi}\int_0^\pi ... \int_0^\pi \frac{r^{n-1}\prod_{k=1}^{n-2}\sin^k(\phi_{k+1})d\phi_{n-1}...d\phi_1 dr}{(r^2-\rho^2)^s\left(|x|^2+r^2-2r|x|\cos\phi_{n-1}\right)^{\frac{n}{2}}}.
\end{aligned}$$

We rename the integration variables and further calculate

$$\begin{aligned}
&\int_{|y|\geqslant\rho} P_\rho(x,y)dy \\
&= 2\pi c(n,s)\left(\rho^2-|x|^2\right)^s \prod_{k=1}^{n-3}\int_0^\pi \sin^k\phi \, d\phi \int_\rho^\infty\int_0^\pi \frac{r^{n-1}\sin^{n-2}\phi \, d\phi dr}{(r^2-\rho^2)^s\left(|x|^2+r^2-2r|x|\cos\phi\right)^{\frac{n}{2}}} \\
&= 2\pi c(n,s)\left(\frac{\rho^2}{|x|^2}-1\right)^s \prod_{k=1}^{n-3}\int_0^\pi \sin^k\phi \, d\phi \int_\rho^\infty\int_0^\pi \frac{r^{n-1}\sin^{n-2}\phi \, d\phi dr}{\left(\frac{r^2}{|x|^2}-\frac{\rho^2}{|x|^2}\right)^s |x|^n\left(1+\frac{r^2}{|x|^2}-\frac{2r\cos\phi}{|x|}\right)^{\frac{n}{2}}}.
\end{aligned}$$

Substituting $\tilde{r} = \frac{r}{|x|}$ and $\tilde{\rho} = \frac{\rho}{|x|}$ yields

$$\begin{aligned}
&\int_{|y|\geqslant\rho} P_\rho(x,y)dy \\
&= 2\pi c(n,s)\left(\tilde{\rho}^2-1\right)^s \prod_{k=1}^{n-3}\int_0^\pi \sin^k\phi \, d\phi \int_{\tilde{\rho}}^\infty\int_0^\pi \frac{(\tilde{r}|x|)^{n-1}\sin^{n-2}\phi \, d\phi \, |x| \, d\tilde{r}}{(\tilde{r}^2-\tilde{\rho}^2)^s|x|^n\left(1+\tilde{r}^2-2\tilde{r}\cos\phi\right)^{\frac{n}{2}}} \\
&= 2\pi c(n,s)\left(\tilde{\rho}^2-1\right)^s \prod_{k=1}^{n-3}\int_0^\pi \sin^k\phi \, d\phi \int_{\tilde{\rho}}^\infty \frac{\tilde{r}^{n-1}}{(\tilde{r}^2-\tilde{\rho}^2)^s}\int_0^\pi \frac{\sin^{n-2}\phi \, d\phi}{\left(1+\tilde{r}^2-2\tilde{r}\cos\phi\right)^{\frac{n}{2}}} d\tilde{r}.
\end{aligned}$$

Renaming $\tilde{r}$ and $\tilde{\rho}$ back to r and ρ, respectively, further results in

$$\begin{aligned}
&\int_{|y|\geqslant\rho} P_\rho(x,y)dy \\
&= 2\pi c(n,s)\left(\rho^2-1\right)^s \prod_{k=1}^{n-3}\int_0^\pi \sin^k\phi \, d\phi \int_\rho^\infty \frac{r^{n-1}}{(r^2-\rho^2)^s}\left(\int_0^\pi \frac{\sin^{n-2}\phi}{\left(1+r^2-2r\cos\phi\right)^{\frac{n}{2}}} d\phi\right) dr,
\end{aligned}$$

now with $r > 1$ and $\rho > 1$. We can now use (A.1), (A.4) and (A.5) along with the definition of $c(n,s)$ from (3.1) to finally calculate

$$\begin{aligned}
&\int_{|y|\geqslant\rho} P_\rho(x,y)dy \\
&= 2\pi c(n,s)\big(\rho^2-1\big)^s \prod_{k=1}^{n-3}\int_0^\pi \sin^k\phi\, d\phi \int_\rho^\infty \frac{r^{n-1}}{(r^2-\rho^2)^s}\frac{1}{(r^2-1)r^{n-2}}\left(\int_0^\pi \sin^{n-2}\phi\, d\phi\right)dr \\
&= c(n,s)\left(\pi\prod_{k=1}^{n-2}\int_0^\pi \sin^k\phi\, d\phi\right)\left(\big(\rho^2-1\big)^s\int_\rho^\infty \frac{2r}{(r^2-\rho^2)^s\,(r^2-1)}dr\right) \\
&= \frac{\Gamma(\frac{n}{2})\sin(\pi s)}{\pi^{\frac{n}{2}+1}}\cdot\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\cdot\frac{\pi}{\sin(\pi s)} = 1.
\end{aligned}$$

□

3.3. The fundamental solution

The third and last function that we want to discuss in the framework of the fractional Laplace operator is the fundamental solution. As we are going to see, the interplay between this function and A_ρ as well as P_ρ yields a handful of useful properties.

Definition 3.7. *Let $n \neq 2s$. Then for any $x \in \mathbb{R}^n\backslash\{0\}$, the fundamental solution is defined by*

$$\Psi(x) := \frac{a(n,s)}{|x|^{n-2s}},$$

where

$$a(n,s) := \frac{\Gamma(\frac{n}{2}-s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)}. \tag{3.8}$$

Remark 3.1. *The case $n = 2s$ is only possible, if $n = 1$ and $s = \frac{1}{2}$. For this specific case, we define the fundamental solution as*

$$\Psi(x) = a\big(1,\frac{1}{2}\big)\log(|x|),$$

with

$$a(1,\frac{1}{2}) = -\frac{1}{\pi}.$$

We recognize that Ψ is a weighted L^1-function.

Lemma 3.8. *For any $n \in \mathbb{N}$ and $s \in (0,1)$, we have $\Psi \in L^1_s(\mathbb{R}^n)$.*

Proof.
It is easy to see that $\Psi \in L^1_{loc}(\mathbb{R}^n)$. Now let $n \neq 2s$. We split the integral

$$\int_{\mathbb{R}^n} \frac{|\Psi(x)|}{1+|x|^{n+2s}}dx = a(n,s)\int_{\mathbb{R}^n} \frac{|x|^{2s-n}}{(1+|x|^{n+2s})}dx$$

into two parts:

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\Psi(x)|}{1+|x|^{n+2s}}dx &= a(n,s)\left(\int_{|x|<1} \frac{|x|^{2s-n}}{1+|x|^{n+2s}}dx + \int_{|x|\geqslant 1} \frac{|x|^{2s-n}}{1+|x|^{n+2s}}dx\right) \\ &=: a(n,s)\left(I_1+I_2\right).\end{aligned}$$

For the first integral, we use the estimate $1+|x|^{n+2s} \geqslant 1$ and pass to polar coordinates to get

$$I_1 = \int_{|x|<1} \frac{|x|^{2s-n}}{1+|x|^{n+2s}}dx \leqslant \int_{|x|<1} \frac{1}{|x|^{n-2s}}dx < \infty$$

by making use of Lemma A.1. For the second integral, the estimate $1+|x|^{n+2s} > |x|^{n+2s}$ along with a polar transformation an again Lemma A.1 yields

$$I_2 = \int_{|x|\geqslant 1} \frac{|x|^{2s-n}}{1+|x|^{n+2s}}dx < \int_{|x|\geqslant 1} \frac{1}{|x|^{n-2s}\,|x|^{n+2s}}dx = \int_{|x|\geqslant 1} \frac{1}{|x|^{2n}}dx < \infty.$$

For the case $n = 2s = 1$, using similar techniques yields

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\Psi(x)|}{1+|x|^2}dx &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{|\log(|x|)|}{1+|x|^2}dx \\ &= \frac{1}{\pi}\left(2\int_0^1 \frac{|\log(x)|}{1+x^2}dx + 2\int_1^\infty \frac{|\log(x)|}{1+x^2}dx\right) \\ &\leqslant \frac{1}{\pi}\left(2\int_0^1 |\log(x)|\,dx + 2\int_1^\infty \frac{|\log(x)|}{x^2}dx\right) \\ &= \frac{1}{\pi}(2+2) < \infty.\end{aligned}$$

□

Just as for the regular Laplacian, a global solution for the problem

$$(-\Delta)^s = f$$

for a sufficiently smooth function f can be established via convolution with Ψ. The next lemma will establish that the fractional Laplacian can be applied to the resulting function .

Lemma 3.9. *Let $f \in C_c(\mathbb{R}^n)$ be a continuous function with compact support, then $f * \Psi \in L^1_s(\mathbb{R}^n)$.*

Proof.
Since $f \in C_c(\mathbb{R}^n)$, there exists $R > 0$ such that supp $f \subseteq B_R(0)$. Now with the use of the Fubini Theorem, we can estimate

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{(f * \Psi)(x)}{1 + |x|^{n+2s}} dx &\leqslant \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+2s}} \left(\int_{|y|<R} |f(y)| \, |\Psi(x-y)| \, dy \right) dx \\
&= \int_{|y|<R} |f(y)| \left(\int_{\mathbb{R}^n} \frac{|\Psi(x-y)|}{1 + |x|^{n+2s}} dx \right) dy \\
&\leqslant \|f\|_{L^\infty(\mathbb{R}^n)} \int_{|y|<R} \left(\int_{\mathbb{R}^n} \frac{|\Psi(x-y)|}{1 + |x|^{n+2s}} dx \right) dy \\
&= \|f\|_{L^\infty(\mathbb{R}^n)} \int_{|y|<R} I_1(y) + I_2(y) dy,
\end{aligned}$$

with

$$\begin{aligned}
I_1(y) &= \int_{|y|<2R} \frac{|\Psi(x-y)|}{1 + |x|^{n+2s}} dx, \\
I_2(y) &= \int_{|y|\geqslant 2R} \frac{|\Psi(x-y)|}{1 + |x|^{n+2s}} dx.
\end{aligned}$$

The usage of the Fubini Theorem is justified since we are going to show that the above term is finite.
We will focus on the first integral. For $n \neq 2s$, we can estimate

$$\begin{aligned}
\int_{|y|<R} I_1(y) dy &= a(n,s) \int_{|y|<R} \left(\int_{|x|<2R} \frac{1}{|x-y|^{n-2s} \left(1 + |x|^{n+2s}\right)} dx \right) dy \\
&\leqslant a(n,s) \int_{|y|<R} \left(\int_{|x|<2R} \frac{1}{|x-y|^{n-2s}} dx \right) dy \\
&= a(n,s) \int_{|y|<R} \left(\int_{|\tilde{x}+y|<2R} \frac{1}{|\tilde{x}|^{n-2s}} d\tilde{x} \right) dy \\
&\leqslant a(n,s) \int_{|y|<R} \left(\int_{|x|<2R+|y|} \frac{1}{|\tilde{x}|^{n-2s}} d\tilde{x} \right) dy \\
&= c_{n,s} \int_{|y|<R} \left(\int_0^{2R+|y|} \frac{r^{n-1}}{r^{n-2s}} dr \right) dy \\
&= c_{n,s} \int_{|y|<R} \left(\int_0^{2R+|y|} \frac{1}{r^{1-2s}} dr \right) dy = \frac{c_{n,s}}{2s} \int_{|y|<R} (2R + |y|)^{2s} \, dy \\
&= \tilde{c}_{n,s} \int_0^R (2R+\rho)^{2s} \rho^{n-1} d\rho < \infty,
\end{aligned}$$

while for $n = 2s = 1$ we can use the triangle inequality to get

$$\begin{aligned}
\pi \int_{|y|<R} I_1(y)dy &= \int_{-R}^{R} \left(\int_{-2R}^{2R} \frac{|\log(|x-y|)|}{1+|x|^2} dx \right) dy \leqslant \int_{-R}^{R} \left(\int_{2R}^{2R} |\log(|x|+|y|)|\, dx \right) dy \\
&\leqslant \int_{-R}^{R} \left(\int_{-2R}^{2R} |\log(|x|+R)|\, dx \right) dy = 2R \int_{-2R}^{2R} |\log(x+R)|\, dx < \infty.
\end{aligned}$$

For the second integral, we will have to treat 3 separate cases. For $n > 2s$, we use the reverse triangle inequality and the fact that $1+|x|^{n+2s} > |x|^{n+2s}$ to obtain

$$\begin{aligned}
\int_{|y|<R} I_2(y)dy &= |a(n,s)| \int_{|y|<R} \left(\int_{\mathbb{R}/B_{2R}(0)} \frac{1}{|x-y|^{n-2s} \left(1+|x|^{n+2s}\right)} dx \right) dy \\
&< |a(n,s)| \int_{|y|<R} \left(\int_{\mathbb{R}/B_{2R}(0)} \frac{1}{(|x|-|y|)^{n-2s} |x|^{n+2s}} dx \right) dy \\
&= c_{n,s} \int_0^R \left(\int_{2R}^{\infty} \frac{r^{n-1}}{(r-\rho)^{n-2s} r^{n+2s}} dr \right) \rho^{n-1} d\rho \\
&\leqslant c_{n,s} \int_0^R \rho^{n-1} d\rho \int_{2R}^{\infty} \frac{1}{(r-R)^{n-2s} r^{1+2s}} dr \\
&\leqslant c_{n,s,R} \int_{2R}^{\infty} \frac{1}{(r-R)^{n-2s} (r-R)^{1+2s}} dr \\
&= c_{n,s,R} \int_{2R}^{\infty} \frac{1}{(r-R)^{n+1}} dr < \infty.
\end{aligned}$$

For $2s > n = 1$, we can again use the triangle inequality to get

$$\begin{aligned}
\int_{-R}^{R} I_2(y)dy &= |a(n,s)| \int_{-R}^{R} \left(\int_{|x|\geqslant 2R} \frac{|x-y|^{2s-1}}{1+|x|^{1+2s}} dx \right) dy \\
&< |a(n,s)| \int_{-R}^{R} \left(\int_{|x|\geqslant 2R} \frac{(|x|+|y|)^{2s-1}}{|x|^{1+2s}} dx \right) dy \\
&= \widetilde{c}_{n,s} \int_0^R \left(\int_{2R}^{\infty} \frac{(r+\rho)^{2s-1}}{r^{1+2s}} dr \right) d\rho \\
&= \widetilde{c}_{n,s} \int_{2R}^{\infty} \frac{1}{r^{1+2s}} \left(\int_0^R (r+\rho)^{2s-1}\, d\rho \right) dr < \infty.
\end{aligned}$$

And in the case $n = 2s = 1$ we can estimate

$$\begin{aligned}\pi \int_{-R}^{R} I_2(y)dy &= \int_{-R}^{R} \left(\int_{|x|\geqslant 2R} \frac{|\log(|x-y|)|}{1+|x|^2} dx \right) dy \\ &\leqslant \int_{-R}^{R} \left(\int_{|x|\geqslant 2R} \frac{|\log(|x|+|y|)|}{|x|^2} dx \right) dy \\ &\leqslant 2R \int_{|x|\geqslant 2R} \frac{|\log(|x|+R)|}{|x|^2} dx = c_{n,R} \int_{2R}^{\infty} \frac{|\log(t+R)|}{t^2} dx < \infty.\end{aligned}$$

In any case, we have shown that there exists a constant $c_{n,s,R} > 0$ such that

$$\int_{\mathbb{R}^n} \frac{(f * \Psi)(x)}{1+|x|^{n+2s}} dx \leqslant c_{n,s,R} \|f\|_{L^\infty(\mathbb{R}^n)} < \infty \tag{3.9}$$

holds, which proves $f * \Psi \in L^1_s(\mathbb{R}^n)$. □

Another fundamental property motivating the definition of Ψ is the following:

Theorem 3.10. *In the distributional sense, we have*

$$(-\Delta)^s \Psi = \delta_0, \tag{3.10}$$

where δ_0 is the Dirac delta centered at zero.

Proof.
See Chapter 4, page 46. □

With the basic framework of Ψ established, we will now focus on the important interplay between the fundamental solution and the two integral kernels A_ρ and P_ρ.

Lemma 3.11. *For any $\rho > 0$ and any $x \in \mathbb{R}^n \backslash B_\rho(0)$ we have*

$$\Psi(x) = \int_{|y|\geqslant \rho} A_\rho(y)\Psi(x-y)dy. \tag{3.11}$$

Proof.
Let $\rho > 0$ and $x \in \mathbb{R}^n \backslash B_\rho(0)$ be arbitrary but fixed. Now we first consider the case $n \neq 2s$. Plugging in the definitions of the fundamental solution and A_ρ, we get

$$\int_{|y|\geqslant \rho} A_\rho(y)\Psi(x-y)dy = a(n,s)c(n,s) \int_{|y|\geqslant \rho} \left(\frac{\rho^2}{|y|^2-\rho^2} \right)^s \frac{1}{|x-y|^{n-2s}} \frac{1}{|y|^n} dy.$$

Before we continue with the above term, we will focus solely on the integrand. For any $y \in \mathbb{R}^n \backslash B_\rho(0)$, we set $y^* = K_0(y)$ as well as $x^* := K_0(x)$, where K_0 is the point inversion at zero as defined in (1.1). From Proposition 1.9, we know that $x^*, y^* \in B_\rho(0)$, and by using (1.4), (1.2) and (1.3), (1.5) and then again (1.2), we can calculate

$$\begin{aligned}
\left(\frac{\rho^2}{|y|^2-\rho^2}\right)^s \frac{1}{|x-y|^{n-2s}}\frac{1}{|y|^n} & \left(\,|\det(DK_0(y))|\,\right)^{-1} \\
&= \left(\frac{\rho^2}{|y|^2-\rho^2}\right)^s \frac{1}{|x-y|^{n-2s}}\frac{1}{|y^*|^n} \\
&= \left(\frac{|y^*|^2}{\rho^2-|y^*|^2}\right)^s \frac{1}{|x-y|^{n-2s}}\frac{1}{|y^*|^n} \\
&= \left(\frac{|y^*|^2}{\rho^2-|y^*|^2}\right)^s \left(\frac{\rho^2}{|x|\,|y|\,|x^*-y^*|}\right)^{n-2s} \frac{1}{|y^*|^n} \\
&= \frac{1}{|x|^{n-2s}} \frac{1}{\left(\rho^2-|y^*|^2\right)^s |x^*-y^*|^{n-2s}} \left(\frac{\rho^2}{|y|\,|y^*|}\right)^{n-2s} \\
&= \frac{1}{|x|^{n-2s}} \frac{1}{\left(\rho^2-|y^*|^2\right)^s |x^*-y^*|^{n-2s}}.
\end{aligned}$$

Using this, we can simplify the above integral using a change of variables given by the point inversion transformation and obtain

$$\begin{aligned}
\int_{|y|\geqslant\rho} A_\rho(y)\Psi(x-y)dy &= \frac{a(n,s)c(n,s)}{|x|^{n-2s}} \int_{|y^*|<\rho} \frac{1}{\left(\rho^2-|y^*|^2\right)^s |x^*-y^*|^{n-2s}} dy^* \\
&= \frac{a(n,s)}{|x|^{n-2s}} = \Psi(x),
\end{aligned}$$

where we have also used identity (A.6) at the end. For the case $n = 2s = 1$, assuming without the loss of generality that $\rho = 1$, we need to calculate

$$\int_{|y|\geqslant 1} A_\rho(y)\Psi(x-y)dy = -\frac{1}{\pi^2}\int_{|y|\geqslant 1} \frac{\log(|x-y|)}{|y|\sqrt{y^2-1}}dy = -\frac{1}{\pi^2}\int_{|y|\geqslant 1} \frac{\log(|x-y|)}{y^2\sqrt{1-\frac{1}{y^2}}}dy.$$

We substitute $u := \frac{1}{x}$ and $v := \frac{1}{y}$ and obtain

$$\begin{aligned}
\int_{|y|\geqslant 1} A_\rho(y)\Psi(x-y)dy &= -\frac{1}{\pi^2}\int_{-1}^{1} \frac{\log(\left|\frac{1}{u}-\frac{1}{v}\right|)}{\frac{1}{v^2}\sqrt{1-v^2}}\frac{1}{v^2}dv \\
&= -\frac{1}{\pi^2}\int_{-1}^{1} \frac{\log(\left|\frac{v-u}{uv}\right|)}{\sqrt{1-v^2}}dv \\
&= -\frac{1}{\pi^2}\int_{-1}^{1} \frac{\log(|u-v|)-\log(|v|)+\log(|x|)}{\sqrt{1-v^2}}dv.
\end{aligned}$$

To solve these integrals, we use (A.2) and the fact that $|u| < 1$ to finally get

$$\begin{aligned}\int_{|y|>1} A_\rho(y)\Psi(x-y)dy &= -\frac{1}{\pi^2}\left(-\pi\log(2)+\pi\log(2)+\log(|x|)\int_{-1}^{1}\frac{1}{\sqrt{1-v^2}}dv\right)\\ &= -\frac{1}{\pi^2}\log(|x|)\pi = -\frac{1}{\pi}\log(|x|) = \Psi(x).\end{aligned}$$

□

On the other hand, we get the following identity for the Poisson kernel:

Lemma 3.12. *For any $\rho > 0$, let $x \in B_\rho(0)$ be fixed. Then for any $z \in \mathbb{R}^n \backslash B_\rho(0)$, the equality*

$$\Psi(x-z) = \int_{|y|\geqslant\rho} P_\rho(x,y)\Psi(y-z)dy \tag{3.12}$$

holds.

Proof.
Let $\rho > 0$ and both $x \in B_\rho(0)$ and $z \in \mathbb{R}^n\backslash B_\rho(0)$ be arbitrary but fixed. We again first consider the case $n \neq 2s$. Plugging in the definitions of the fundamental solution and the Poisson kernel, we obtain

$$\int_{|y|\geqslant\rho} P_\rho(x,y)\Psi(y-z)dy = a(n,s)c(n,s)\int_{|y|\geqslant\rho}\left(\frac{\rho^2-|x|^2}{|y|^2-\rho^2}\right)^s\frac{|y-z|^{2s-n}}{|x-y|^n}dy.$$

Starting with focusing on the first integrand, we set $y^* = K_x(y)$ as well as $z^* := K_x(z)$, where K_x is defined as in (1.1) From Proposition 1.9, we know that $x^*, y^* \in B_\rho(0)$, and by using (1.4), (1.3), (1.2) and (1.5), we can calculate

$$\begin{aligned}&\left(\frac{\rho^2-|x_0|^2}{|y|^2-\rho^2}\right)^s \frac{1}{|y-z|^{n-2s}}\frac{1}{|x-y|^n}\left(|\det(DK_x(y))|\right)^{-1}\\ &\quad= \left(\frac{\rho^2-|x|^2}{|y|^2-\rho^2}\right)^s\frac{1}{|y-z|^{n-2s}}\frac{1}{|y^*-x|^n}\\ &\quad= \left(\frac{\left(\rho^2-|x|^2\right)^2}{|y-x|^2\left(\rho^2-|y^*|^2\right)}\right)^s\frac{1}{|y-z|^{n-2s}}\frac{1}{|y^*-x|^n}\\ &\quad= \left(\frac{|y^*-x|^2}{\rho^2-|y^*|^2}\right)^s\left(\frac{\rho^2-|x|^2}{|y-x|\,|z-x|\,|y^*-z^*|}\right)^{n-2s}\frac{1}{|y^*-x|^n}\\ &\quad= \frac{1}{|z-x|^{n-2s}}\frac{1}{\left(\rho^2-|y^*|^2\right)^s|y^*-z^*|^{n-2s}}\left(\frac{\rho^2-|x|^2}{|y-x|\,|y^*-x|}\right)^{n-2s}\\ &\quad= \frac{1}{|z-x|^{n-2s}}\frac{1}{\left(\rho^2-|y^*|^2\right)^s|y^*-z^*|^{n-2s}}.\end{aligned}$$

Applying a change of variables and using (A.6) yields

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_\rho(0)} & P_\rho(x,y)\Psi(x-y)dy \\ &= \frac{a(n,s)c(n,s)}{|z-x|^{n-2s}}\int_{|y^*|<\rho}\frac{1}{\left(\rho^2-|y^*|^2\right)^s|y^*-z^*|^{n-2s}}dy^* \\ &= \frac{a(n,s)}{|x-z|^{n-2s}} = \Psi(x-z).\end{aligned}$$

For the case $n=2s=1$, we need to calculate

$$\int_{|y|>1} P_\rho(x,y)\Psi(y-z)dy = -\frac{1}{\pi^2}\int_{|y|\geqslant 1}\frac{\log(|y-z|)}{|y-x|}\sqrt{\frac{1-x^2}{y^2-1}}dy.$$

We substitute $u := \frac{zx-1}{z-x}$ and $v := \frac{yx-1}{y-x}$ and obtain

$$\begin{aligned}\int_{|y|>1} & P_\rho(x,y)\Psi(y-z)dy \\ &= -\frac{1}{\pi^2}\int_{|v|\leqslant 1}\log\left(\left|\frac{vx_0-1}{v-x_0}-\frac{ux_0-1}{u-x_0}\right|\right)\sqrt{\frac{1-x_0^2}{\frac{(vx_0-1)^2-(v-x_0)^2}{(v-x_0)^2}}}\frac{1}{v-x_0}dv \\ &= -\frac{1}{\pi^2}\int_{|v|\leqslant 1}\log\left(\left|\frac{v-u-x_0^2(v-u)}{(v-x_0)(u-x_0)}\right|\right)\sqrt{\frac{1-x_0^2}{v^2x_0^2-x_0^2-v^2+1}}dv \\ &= -\frac{1}{\pi^2}\int_{|v|\leqslant 1}\left(\log\left(\left|\frac{v-u}{v-x_0}\right|\right)+\log\left(\left|\frac{1-x_0^2}{u-x_0}\right|\right)\right)\sqrt{\frac{1-x_0^2}{(1-x_0^2)(1-v^2)}}dv \\ &= -\frac{1}{\pi^2}\int_{|v|\leqslant 1}\Big(\log(|v-u|)-\log(|v-x_0|)+\log(|x-z|)\Big)\frac{1}{\sqrt{1-v^2}}dv.\end{aligned}$$

By observing that $|u|\leqslant 1$ as well as $|x_0|\leqslant 1$, and applying (A.2), we finally get

$$\begin{aligned}\int_{|y|>1} P_\rho(x,y)\Psi(y-z)dy &= -\frac{1}{\pi^2}\left(-\pi\log(2)+\pi\log(2)+\log(|x-z|)\int_{|v|\leqslant 1}\frac{1}{\sqrt{1-v^2}}dv\right) \\ &= -\frac{1}{\pi^2}\log(|x-z|)\pi = -\frac{1}{\pi}\log(|x-z|) = \Psi(x-z).\end{aligned}$$

□

4. An analytical solution in the ball

Now that all the necessary concepts have been introduced, we are able to formulate the main theorem of this thesis:

Theorem 4.1. *Let $\rho > 0$, $g \in L^1_s(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and P_ρ be the Poisson kernel defined by* (3.6)*. Then the function*

$$u_g(x) = \begin{cases} \int_{|y| \geqslant \rho} P_\rho(x,y)g(y)dy & \text{if } |x| < \rho \\ g(x) & \text{if } |x| \geqslant \rho \end{cases}$$

is the unique pointwise continuous solution of the problem

$$\begin{cases} (-\Delta)^s u &= 0 \text{ in } B_\rho(0), \\ \qquad u &= g \text{ in } \mathbb{R}^n \backslash B_\rho(0). \end{cases} \tag{4.1}$$

Before we are able to prove this formula, a few additional statements are necessary. Starting off, the following proposition shows how the Fourier transform of the fundamental solution can be expressed.

Proposition 4.2. *Let $n > 2s$, let $g \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $\check{g} \in \mathcal{S}_s(\mathbb{R}^n)$. Then the equality*

$$(2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Psi(x)\check{g}(x)dx = \int_{\mathbb{R}^n} \frac{g(x)}{|x|^{2s}}dx \tag{4.2}$$

holds, where Ψ is the fundamental solution given by Definition 3.7.

Proof.
We will start by showing that identiy (4.2) is indeed well-defined. From Lemma 3.8, we know that $\Psi \in L^1_s(\mathbb{R}^n)$, which shows that the left hand side of (4.2) is finite thanks to Remark 2.5. The right hand side is also finite since

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|g(x)|}{|x|^{2s}}dx &= \int_{B_1(0)} \frac{|g(x)|}{|x|^{2s}}dx + \int_{|x| \geqslant 1} \frac{|g(x)|}{|x|^{2s}}dx \\ &\leqslant \sup_{x \in B_1(0)} |g(x)| \int_{B_1(0)} \frac{1}{|x|^{2s}}dx + \int_{|x| \geqslant 1} |g(x)|\, dx \\ &\leqslant c_{n,s} \sup_{x \in B_1(0)} |g(x)| + \|g\|_{L^1(\mathbb{R}^n)} < \infty \end{aligned}$$

by Lemma A.1. Next, we define

$$\widetilde{a}(n,s) := \frac{a_1(n,s)}{(2\pi)^{2s}\, a_2(n,s)},$$

with

$$\begin{aligned} a_1(n,s) &:= \int_0^\infty t^{\frac{n}{2}-s-1} e^{-\pi t} dt, \\ a_2(n,s) &:= \int_0^\infty t^{s-1} e^{-\pi t} dt. \end{aligned}$$

Both constants $a_1(n,s)$ and $a_2(n,s)$ are finite since $\frac{n}{2} - s - 1 > -1$ and $s - 1 > -1$. Additionally, by using the change of variables $\tau = \pi t$, we have

$$a_1(n,s) = \int_0^\infty \left(\frac{\tau}{\pi}\right)^{\frac{n}{2}-s-1} e^{-\tau}\frac{1}{\pi} d\tau = \pi^{s-\frac{n}{2}} \int_0^\infty \tau^{\frac{n}{2}-s-1} e^{-\tau} d\tau = \pi^{s-\frac{n}{2}} \Gamma\big(\frac{n}{2} - s\big),$$

as well as

$$a_2(n,s) = \int_0^\infty \left(\frac{\tau}{\pi}\right)^{s-1} e^{-\tau}\frac{1}{\pi} d\tau = \pi^{-s} \int_0^\infty \tau^{s-1} e^{-\tau} d\tau = \pi^{-s}\Gamma(s),$$

which yields the identity

$$\widetilde{a}(n,s) := \frac{\pi^{s-\frac{n}{2}}\Gamma(\frac{n}{2}-s)}{(2\pi)^{2s}\pi^{-s}\Gamma(s)} = \frac{\Gamma(\frac{n}{2}-s)}{2^{2s}\pi^{\frac{n}{2}}\Gamma(s)} = a(n,s).$$

With this in mind, we can rearrange (4.2) to get

$$(2\pi)^{\frac{n}{2}} \frac{a_1(n,s)}{(2\pi)^{2s}} \int_{\mathbb{R}^n} \frac{\breve{g}(x)}{|x|^{n-2s}} dx \quad = \quad a_2(n,s) \int_{\mathbb{R}^n} \frac{g(x)}{|x|^{2s}} dx. \tag{4.3}$$

We will now prove identity (4.3) by changing the left hand side. Using the change of variable $\tau = t/|x|^2$, we compute

$$\begin{aligned} (2\pi)^{\frac{n}{2}} \frac{a_1(n,s)}{(2\pi)^{2s}} &\int_{\mathbb{R}^n} \frac{\breve{g}(x)}{|x|^{n-2s}}\, dx \\ &= (2\pi)^{\frac{n}{2}-2s} \int_0^\infty t^{\frac{n}{2}-s-1} e^{-\pi t} dt \int_{\mathbb{R}^n} \frac{\breve{g}(x)}{|x|^{n-2s}} dx \\ &= (2\pi)^{\frac{n}{2}-2s} \int_{\mathbb{R}^n} \left(\int_0^\infty \left(\tau\,|x|^2\right)^{\frac{n}{2}-s-1} e^{-\pi\tau|x|^2} \frac{\breve{g}(x)}{|x|^{n-2s}}\, |x|^2\, d\tau \right) dx \\ &= (2\pi)^{\frac{n}{2}-2s} \int_0^\infty \tau^{\frac{n}{2}-s-1} \left(\int_{\mathbb{R}^n} e^{-\pi\tau|x|^2} \breve{g}(x) dx \right) d\tau. \end{aligned} \tag{4.4}$$

We now use the fact that the Fourier transform of the Gaussian distribution is given by

$$\mathscr{F}(e^{-\pi\tau|x|^2}) = \frac{1}{(2\pi\tau)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\pi\tau}}$$

for any $\tau > 0$. Applying the Plancharel identity for $f := e^{-\pi\tau|\cdot|^2}$ and $h := \breve{g}$ now yields

$$\int_{\mathbb{R}^n} e^{-\pi\tau|x|^2}\breve{g}(x)dx = \int_{\mathbb{R}^n} f(x)h(x)dx = \int_{\mathbb{R}^n} \widehat{f}(x)\widehat{h}(x)dx = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} \tau^{-\frac{n}{2}} e^{-\frac{|x|^2}{4\pi\tau}} g(x)dx.$$

Inserting this identity into (4.4) and using the change of variable $t = |x|^2 / (4\pi^2\tau)$ finally yields

$$\begin{aligned}
(2\pi)^{\frac{n}{2}} a_1(n,s)\int_{\mathbb{R}^n} \frac{\breve{g}(x)}{|x|^{n-2s}}dx &= (2\pi)^{\frac{n}{2}-2s}\int_0^\infty \tau^{\frac{n}{2}-s-1}\left(\int_{\mathbb{R}^n} e^{-\pi\tau|x|^2}\breve{g}(x)dx\right)d\tau \\
&= \frac{1}{(2\pi)^{2s}}\int_{\mathbb{R}^n}\left(\int_0^\infty \tau^{-s-1}e^{-\frac{|x|^2}{4\pi\tau}}d\tau\right)g(x)\ dx \\
&= \frac{1}{(2\pi)^{2s}}\int_{\mathbb{R}^n}\left(\int_0^\infty \left(\frac{|x|^2}{4\pi^2 t}\right)^{-s-1} e^{-\pi t}\frac{|x|^2}{4\pi^2 t^2}dt\right)g(x)\ dx \\
&= \frac{1}{(2\pi)^{2s}}\frac{(4\pi^2)^{s+1}}{4\pi^2}\int_0^\infty t^{s-1}e^{-\pi t}dt\int_{\mathbb{R}^n}\frac{g(x)}{|x|^{2s}}dx \\
&= a_2(n,s)\int_{\mathbb{R}^n}\frac{g(x)}{|x|^{2s}}dx,
\end{aligned}$$

which concludes the proof. □

The last proof relied on the fact that $n > 2s$. If $n \leqslant 2s$, it is no longer guaranteed that (4.2) and (4.3) are well defined. Therefore, additional properties are necessary.

Proposition 4.3. *Let $n = 1$, and $s \geqslant \frac{1}{2}$, let $g \in C(\mathbb{R}) \cap C^1((-\infty,0) \cup (0,\infty)) \cap L^1(\mathbb{R})$ with $\breve{g} \in \mathcal{S}_s(\mathbb{R})$. If there exist constants $c_1,\ c_2,\ c_3,\ c_4 > 0$ such that*

$$\begin{aligned}
|g(x)| &\leqslant c_1|x|^{2s} && \text{for } x \in \mathbb{R}, \\
|g(x)| &\leqslant \frac{c_2}{|x|} && \text{for } |x| > 1, \\
|g'(x)| &\leqslant \frac{c_3}{|x|^{1-2s}} && \text{for } 0 < |x| \leqslant 1, \\
|g'(x)| &\leqslant \frac{c_4}{|x|} && \text{for } |x| > 1,
\end{aligned} \tag{4.5}$$

then equality (4.2) holds.

Proof.
It follows again from Lemma 3.8 that the left hand side of (4.2) is well defined. For the right hand side, we make use of (4.5) and obtain

$$\int_{\mathbb{R}} \frac{|g(x)|}{|x|^{2s}} dx = \int_{-1}^{1} \frac{|g(x)|}{|x|^{2s}} dx + \int_{|x| \geqslant 1} \frac{|g(x)|}{|x|^{2s}} dx \leqslant 2c_1 + \|g\|_{L^1(\mathbb{R})} < \infty.$$

In addition, the constant $a(1,s)$ can be rewritten as

$$a(1,s) = \frac{2^{1-2s}\sqrt{\pi} \cdot \Gamma(\frac{1}{2}-s)}{\Gamma(s) \cdot 2\pi} = \frac{\Gamma(\frac{1}{2}+s)\Gamma(\frac{1}{2}-s)}{2\pi\Gamma(2s)} = \frac{1}{2\cos(\pi s)\Gamma(2s)}, \tag{4.6}$$

where we have made use of the properties of the Gamma function, see Proposition 1.5. Now let $s > \frac{1}{2}$. We have to prove

$$\int_{\mathbb{R}} \Psi(x)\breve{g}(x)dx = a(1,s) \lim_{R\to\infty} \int_{-R}^{R} |x|^{2s-1} \breve{g}(x)dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{g(x)}{|x|^{2s}} dx.$$

Therefore, for $R > 0$ we estimate

$$\begin{aligned}
\int_{-R}^{R} |x|^{2s-1} \breve{g}(x)dx &= \int_{-R}^{0} (-x)^{2s-1}\breve{g}(x)dx + \int_{0}^{R} x^{2s-1}\breve{g}(x)dx \\
&= \int_{0}^{R} x^{2s-1}\big(\breve{g}(-x) + \breve{g}(x)\big)dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{R} x^{2s-1} \left(\int_{\mathbb{R}} g(\xi) \left(e^{-ix\xi} + e^{ix\xi}\right) d\xi \right) dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{R} x^{2s-1} \left(\int_{\mathbb{R}} g(\xi) 2\cos(x\xi) d\xi \right) dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) \left(\int_{0}^{R} x^{2s-1} \cos(x\xi) dx \right) d\xi.
\end{aligned}$$

Focusing on the inner integral for any $\xi \in \mathbb{R}$, integration by parts and the change of variables $t = |\xi|\, x$ yields

$$\begin{aligned}
\int_{0}^{R} x^{2s-1} \cos(x\xi)dx &= x^{2s-1}\frac{\sin(x\xi)}{\xi}\bigg|_{0}^{R} - (2s-1)\int_{0}^{R} x^{2s-2}\frac{\sin(x\xi)}{\xi}dx \\
&= x^{2s-1}\frac{\sin(x\xi)}{\xi}\bigg|_{0}^{R} - (2s-1)\int_{0}^{R|\xi|} \left(\frac{t}{|\xi|}\right)^{2s-2} \frac{\sin\left(\frac{\xi}{|\xi|}t\right)}{\xi} \frac{1}{|\xi|}dt \\
&= R^{2s-1}\frac{\sin(R\xi)}{\xi} - \frac{2s-1}{|\xi|^{2s}} \int_{0}^{R|\xi|} \frac{\sin(t)}{t^{2-2s}} dt,
\end{aligned}$$

which yields

$$\int_{-R}^{R} |x|^{2s-1}\, \breve{g}(x)dx \;=\; \frac{2R^{2s-1}}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) \frac{\sin(R\xi)}{\xi} d\xi - \frac{2(2s-1)}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{g(\xi)}{|\xi|^{2s}} \left(\int_0^{R|\xi|} \frac{\sin(t)}{t^{2-2s}} dt \right) d\xi. \qquad (4.7)$$

We will proceed by evaluating both terms as R tends to infinity. We claim that

$$\lim_{R\to\infty} R^{2s-1} \int_{\mathbb{R}} g(\xi) \frac{\sin(R\xi)}{\xi} d\xi = 0 \qquad (4.8)$$

as well as

$$\lim_{R\to\infty} \int_{\mathbb{R}} \frac{g(\xi)}{|\xi|^{2s}} \left(\int_0^{R|\xi|} \frac{\sin(t)}{t^{2-2s}} dt \right) d\xi = -\Gamma(2s-1)\cos(\pi s) \int_{\mathbb{R}} \frac{g(\xi)}{|\xi|^{2s}} d\xi. \qquad (4.9)$$

Starting with the first identity, we use integration by parts to estimate

$$\begin{aligned} \left| \int_0^{\infty} \frac{g(\xi)}{\xi} \sin(R\xi) d\xi \right| &= \left| \frac{g(\xi)}{\xi} \frac{\cos(R\xi)}{R} \Big|_0^{\infty} - \int_0^{\infty} \frac{\xi g'(\xi) - g(\xi)}{\xi^2} \frac{\cos(R\xi)}{R} d\xi \right| \\ &\leqslant \frac{|g(\xi)|}{\xi} \frac{|\cos(R\xi)|}{R} \Big|_0^{\infty} + \int_0^{\infty} \frac{|g'(\xi)|}{\xi} \frac{|\cos(R\xi)|}{R} d\xi \\ &\quad + \int_0^{\infty} \frac{|g(\xi)|}{\xi^2} \frac{|\cos(R\xi)|}{R} d\xi =: I_1(R) + I_2(R) + I_3(R). \end{aligned}$$

By using (4.5), we get

$$\lim_{\xi\to 0} \frac{|g(\xi)|}{\xi} \frac{|\cos(R\xi)|}{R} \leqslant \lim_{\xi\to 0} \frac{1}{R} \frac{c_1 |\xi|^{2s}}{\xi} \leqslant \lim_{\xi\to 0} \frac{c_1}{R} |\xi|^{2s-1} = 0,$$

since $2s-1>0$. For all ξ with $|\xi|>1$ we have

$$\frac{|g(\xi)|}{\xi} \frac{|\cos(R\xi)|}{R} \leqslant \frac{c_2}{\xi^2},$$

hence

$$\lim_{\xi\to\infty} \frac{|g(\xi)|}{\xi} \frac{|\cos(R\xi)|}{R} = 0.$$

This implies $I_1(R) = 0$ for every $R>0$. For the second and third term, by changing variables $t = \xi R$, we estimate

$$\begin{aligned} I_2(R) &= \frac{1}{R}\left(\int_0^1 \frac{|g'(\xi)|}{\xi}\,|\cos(R\xi)|\,d\xi + \int_1^\infty \frac{|g'(\xi)|}{\xi}\,|\cos(R\xi)|\,d\xi\right) \\ &\leqslant \frac{1}{R}\left(c_3\int_0^1 \frac{1}{\xi^{2-2s}}\,|\cos(\xi R)|\,d\xi + c_4\int_1^\infty \frac{1}{\xi^2}\,|\cos(R\xi)|\,d\xi\right) \\ &= \frac{1}{R}\left(\frac{c_3}{R^{2s-1}}\int_0^R \frac{1}{t^{2-2s}}\,|\cos(t)|\,dt + c_4 R\int_R^\infty \frac{1}{t^2}\,|\cos(t)|\,dt\right) \\ &\leqslant \frac{1}{R}\left(\frac{c_3}{2s-1} + c_4\right) \leqslant \frac{c}{R} \end{aligned}$$

and

$$\begin{aligned} I_3(R) &= \frac{1}{R}\left(\int_0^1 \frac{|f(\xi)|}{\xi^2}\,|\cos(R\xi)|\,d\xi + \int_1^\infty \frac{|f(\xi)|}{\xi^2}\,|\cos(R\xi)|\,d\xi\right) \\ &\leqslant \frac{1}{R}\left(c_1\int_0^1 \frac{1}{\xi^{2-2s}}\,|\cos(R\xi)|\,d\xi + c_2\int_1^\infty \frac{1}{\xi^3}\,|\cos(R\xi)|\,d\xi\right) \\ &= \frac{1}{R}\left(\frac{c_1}{R^{2s-1}}\int_0^R \frac{1}{t^{2-2s}}\,|\cos(t)|\,dt + c_2 R^2\int_R^\infty \frac{1}{t^3}\,|\cos(t)|\,dt\right) \\ &\leqslant \frac{1}{R}\left(\frac{c_1}{2s-1} + \frac{c_2}{2}\right) \leqslant \frac{\tilde{c}}{R}. \end{aligned}$$

Similar estimations can be made for

$$\left|\int_{-\infty}^0 \frac{g(\xi)}{\xi}\sin(R\xi)d\xi\right| = \left|\int_0^\infty \frac{g(-\xi)}{\xi}\sin(R\xi)d\xi\right|,$$

which means there exists a constant $k > 0$ such that

$$\int_{\mathbb{R}} g(\xi)\frac{\sin(R\xi)}{\xi}d\xi \leqslant \frac{k}{R}.$$

This now yields

$$\lim_{R\to\infty} R^{2s-1}\int_{\mathbb{R}} g(\xi)\frac{\sin(R\xi)}{\xi}d\xi \leqslant \lim_{R\to\infty} R^{2s-1}\frac{k}{R} = \lim_{R\to\infty}\frac{k}{R^{2-2s}} = 0$$

which proves (4.8). In order to show (4.9), we first show that

$$\lim_{R\to\infty}\int_{\mathbb{R}} \frac{g(\xi)}{|\xi|^{2s}}\left(\int_0^{R|\xi|}\frac{\sin(t)}{t^{2-2s}}dt\right)d\xi = \int_{\mathbb{R}}\frac{g(\xi)}{|\xi|^{2s}}\left(\int_0^\infty \frac{\sin(t)}{t^{2-2s}}dt\right)d\xi. \tag{4.10}$$

We use integration by parts to estimate the difference

$$\begin{aligned}\left|\int_0^\infty \frac{\sin(t)}{t^{2-2s}}dt - \int_0^{R|\xi|}\frac{\sin(t)}{t^{2-2s}}dt\right| &= \left|\int_{R|\xi|}^\infty \frac{\sin(t)}{t^{2-2s}}dt\right| \\ &\leqslant \left.\frac{|\cos(t)|}{|t|^{2-2s}}\right|_{R|\xi|}^\infty + (2s-2)\int_{R|\xi|}^\infty \frac{|\cos(t)|}{|t|^{3-2s}}dt \\ &\leqslant \frac{1}{(R|\xi|)^{2-2s}} + \frac{1}{(R|\xi|)^{2-2s}} = \frac{2}{(R|\xi|)^{2-2s}},\end{aligned}$$

and thus there exists a $\widetilde{k}>0$ such that

$$\begin{aligned}&\left|\int_{\mathbb{R}} \frac{g(\xi)}{|\xi|^{2s}}\left(\int_0^\infty \frac{\sin(t)}{t^{2-2s}}dt - \int_0^{R|\xi|}\frac{\sin(t)}{t^{2-2s}}dt\right)d\xi\right| \\ &\qquad\leqslant \frac{2}{R^{2-2s}}\int_{\mathbb{R}}\frac{|g(\xi)|}{|\xi|^2}d\xi \\ &\qquad\leqslant \frac{2}{R^{2-2s}}\left(c_1\int_{-1}^1 \frac{1}{|\xi|^{2-2s}}d\xi + c_2\int_{|\xi|\geqslant 1}\frac{1}{|\xi|^3}d\xi\right) \leqslant \frac{\widetilde{k}}{R^{2-2s}},\end{aligned}$$

which vanishes as R goes to infinity, thus showing (4.10). This, together with Lemma A.8, now yields

$$\begin{aligned}\lim_{R\to\infty}\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|^{2s}}\left(\int_0^{R|\xi|}\frac{\sin(t)}{t^{2-2s}}dt\right)d\xi &= \int_{\mathbb{R}}\frac{g(\xi)}{|\xi|^{2s}}\left(\int_0^\infty \frac{\sin(t)}{t^{2-2s}}dt\right)d\xi \\ &= -\Gamma(2s-1)\cos(\pi s)\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|^{2s}}d\xi,\end{aligned}$$

implying (4.9). By using (4.6), (4.7), (4.8) and (4.9), we finally obtain

$$\begin{aligned}\int_{\mathbb{R}}\Psi(x)\breve{g}(x)dx &= a(1,s)\lim_{R\to\infty}\int_{-R}^R |x|^{2s-1}\,\breve{g}(x)dx \\ &= \lim_{R\to\infty}\frac{2a(1,s)R^{2s-1}}{\sqrt{2\pi}}\int_{\mathbb{R}} g(\xi)\frac{\sin(R\xi)}{\xi}d\xi \\ &\quad - \lim_{R\to\infty}\frac{2a(1,s)(2s-1)}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|^{2s}}\left(\int_0^{R|\xi|} t^{2s-2}\sin(t)dt\right)d\xi \\ &= \frac{2a(1,s)(2s-1)}{\sqrt{2\pi}}\Gamma(2s-1)\cos(\pi s)\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|^{2s}}d\xi \\ &= \frac{a(1,s)}{\sqrt{2\pi}}2\Gamma(2s)\cos(\pi s)\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|^{2s}}d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|^{2s}}d\xi,\end{aligned}$$

which proves the result for all $s > \frac{1}{2}$.
Now let $s = \frac{1}{2}$. Similar to our calculations to obtain (4.7), we get

$$\int_{-R}^{R} \log(|x|)\breve{g}(x)dx = \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi) \left(\int_0^R \log(x)\cos(\xi x)dx \right) d\xi.$$

Using integration by parts, we can observe that

$$\begin{aligned} \int_0^R \log(x)\cos(\xi x)dx &= \log(x)\frac{\sin(\xi x)}{\xi}\bigg|_0^R - \frac{1}{\xi}\int_0^R \frac{\sin(\xi x)}{x}dx \\ &= \log(R)\frac{\sin(\xi R)}{\xi} - \frac{1}{|\xi|}\int_0^{R|\xi|} \frac{\sin(t)}{t}dt, \end{aligned}$$

which implies

$$\int_{-R}^{R} \log(x)\breve{g}(x)dx = \frac{2\log(R)}{\sqrt{2\pi}} \int_{\mathbb{R}} g(\xi)\frac{\sin(\xi R)}{\xi}d\xi - \frac{2}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{g(\xi)}{|\xi|}\left(\int_0^{R|\xi|} \frac{\sin(t)}{t} \right) d\xi.$$

We now show that

$$\lim_{R\to\infty} \int_{\mathbb{R}} g(\xi)\frac{\sin(\xi R)}{\xi}d\xi = 0. \tag{4.11}$$

Splitting the integral yields

$$\int_0^\infty g(\xi)\frac{\sin(\xi R)}{\xi}d\xi = \int_0^{\frac{1}{R}} g(\xi)\frac{\sin(\xi R)}{\xi}d\xi + \int_{\frac{1}{R}}^\infty g(\xi)\frac{\sin(\xi R)}{\xi}d\xi.$$

The first term vanishes as R goes to infinity since by using (4.5) we get

$$\left| \int_0^{\frac{1}{R}} g(\xi)\frac{\sin(\xi R)}{\xi}d\xi \right| \leqslant \int_0^{\frac{1}{R}} |g(\xi)|\frac{\xi R}{\xi}d\xi \leqslant c_1 R\int_0^{\frac{1}{R}} \xi d\xi \leqslant \frac{c_1}{2R},$$

while for the second term, we use integration by parts to obtain

$$\begin{aligned} \left| \int_{\frac{1}{R}}^\infty \frac{g(\xi)}{\xi}\sin(\xi R)d\xi \right| \leqslant{}& \frac{|g(\xi)|}{\xi}\frac{|\cos(\xi R)|}{R}\bigg|_{\frac{1}{R}}^\infty + \int_{\frac{1}{R}}^\infty \frac{|g(\xi)|}{\xi^2}\frac{|\cos(\xi R)|}{R}d\xi \\ &+ \int_{\frac{1}{R}}^\infty \frac{|g'(\xi)|}{\xi}\frac{|\cos(\xi R)|}{R}d\xi = \widetilde{I}_1(R) + \widetilde{I}_2(R) + \widetilde{I}_3(R). \end{aligned}$$

For ξ sufficiently large, we have that

$$\frac{|g(\xi)|}{\xi}\frac{|\cos(\xi R)|}{R} \leqslant \frac{c_2}{R\xi^2},$$

which vanishes as ξ goes to infinity, while for all $|\xi| > 0$, we have

$$\frac{|g(\xi)|}{\xi}\frac{|\cos(\xi R)|}{R} \leqslant \frac{c_1}{R}.$$

Altogether, this yields

$$\lim_{R\to\infty} \log(R)\cdot I_1(R) = \lim_{R\to\infty}\left(\frac{\log(R)}{R}\lim_{\xi\to\infty}\frac{c_2}{\xi^2}\right) + \lim_{R\to\infty}\log(R)\frac{c_1}{R} = 0.$$

Next we use the change of variables $t = \xi R$ to obtain

$$\begin{aligned}
\log(R)\cdot I_2(R) &= \frac{\log(R)}{R}\left(\int_{\frac{1}{R}}^{1}\frac{|g(\xi)|}{\xi^2}|\cos(\xi R)|\,d\xi + \int_1^\infty \frac{|g(\xi)|}{\xi^2}|\cos(\xi R)|\,d\xi\right)\\
&\leqslant \frac{\log(R)}{R}\left(c_1\int_{\frac{1}{R}}^{1}\frac{|\cos(\xi R)|}{\xi}d\xi + c_2\int_1^\infty\frac{|\cos(\xi R)|}{\xi^3}d\xi\right)\\
&\leqslant \frac{\log(R)}{R}\left(c_1\int_1^R\frac{|\cos(t)|}{t}dt + R^2c_2\int_R^\infty\frac{|\cos(t)|}{t^3}dt\right)\\
&\leqslant \frac{\log(R)}{R}\left(c_1\int_1^R\frac{|\cos(t)|}{t}dt + R^2c_2\int_R^\infty\frac{|\cos(t)|}{t^3}dt\right)\\
&\leqslant c_1\frac{\log^2(R)}{R} + c_2\frac{\log(R)}{2R},
\end{aligned}$$

and hence

$$\lim_{R\to\infty}\log(R)\cdot I_2(R) = 0.$$

Similarly, we obtain

$$\begin{aligned}
\log(R)\cdot I_3(R) &= \frac{\log(R)}{R}\left(\int_{\frac{1}{R}}^{1}\frac{|g'(\xi)|}{\xi}|\cos(\xi R)|\,d\xi + \int_1^\infty \frac{|g'(\xi)|}{\xi}|\cos(\xi R)|\,d\xi\right)\\
&\leqslant \frac{\log(R)}{R}\left(c_3\int_{\frac{1}{R}}^{1}\frac{|\cos(\xi R)|}{\xi}d\xi + c_4\int_1^\infty\frac{|\cos(\xi R)|}{\xi^2}d\xi\right)\\
&\leqslant \frac{\log(R)}{R}\left(c_3\int_1^R\frac{|\cos(t)|}{t}dt + Rc_4\int_R^\infty\frac{|\cos(t)|}{t^2}dt\right)\\
&\leqslant \frac{\log(R)}{R}\left(c_3\int_1^R\frac{|\cos(t)|}{t}dt + Rc_4\int_R^\infty\frac{|\cos(t)|}{t^2}dt\right)\\
&\leqslant c_3\frac{\log^2(R)}{R} + c_4\frac{\log(R)}{R}
\end{aligned}$$

and thus

$$\lim_{R\to\infty} \log(R) \cdot I_3(R) = 0,$$

which ultimately shows (4.11) since the same bounds also hold for

$$\int_{-\infty}^{0} g(\xi)\frac{\sin(\xi R)}{\xi}d\xi = \int_{0}^{\infty} g(-\xi)\frac{\sin(\xi R)}{\xi}d\xi.$$

We have now shown that

$$\lim_{R\to\infty}\int_{-R}^{R} \log(|x|)\breve{g}(x)dx = \lim_{R\to\infty} -\frac{2}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{g(\xi)}{|\xi|}\left(\int_0^{R|\xi|}\frac{\sin(t)}{t}\right)d\xi.$$

Observe that (4.9) can also be applied for $s = \frac{1}{2}$ in the following sense:

$$\lim_{R\to\infty}\int_{\mathbb{R}} \frac{g(\xi)}{|\xi|}\left(\int_0^{R|\xi|}\frac{\sin(t)}{t}dt\right)d\xi = \frac{\pi}{2}\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|}d\xi.$$

This finally yields

$$\begin{aligned}\int_{\mathbb{R}} \Psi(x)\breve{g}(x)dx &= -\frac{1}{\pi}\lim_{R\to\infty}\int_{-R}^{R}\log(|x|)\breve{g}(x)dx \\ &= \frac{1}{\pi}\cdot\frac{2}{\sqrt{2\pi}}\cdot\frac{\pi}{2}\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|}d\xi = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{g(\xi)}{|\xi|}d\xi,\end{aligned}$$

which concludes the proof. □

With the help of identity (4.2), we are now able to give the proof to the claim

$$(-\Delta)^s\,\Psi = \delta_0$$

stated in the previous chapter.

Proof of Theorem 3.10.
In order to show (3.10) in the distributional sense, we have to check that

$$\langle \Psi, (-\Delta)^s\,\varphi\rangle = \varphi(0)$$

holds for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ be arbitrary but fixed and choose $g(x) := |x|^{2s}\,\widehat{\varphi}(x)$. For $n > 2s$, we want to apply Proposition 4.2 to g. It is easy to see that $g \in C(\mathbb{R}^n)$. Straightforward computation alongside the usage of Lemma A.1 also yields

$$\begin{aligned}\int_{\mathbb{R}^n} |g(x)|\,dx &\leqslant \int_{B_1(0)} |\widehat{\varphi}(x)|\,dx + \int_{|x|\geqslant 1} |x|^{2s}\,|\widehat{\varphi}(x)|\,dx \\ &\leqslant \sup_{x\in B_1(0)} |\widehat{\varphi}(x)| + \int_{|x|\geqslant 1} |x|^{2s-n-2}\,|x|^{n+2}\,|\widehat{\varphi}(x)|\,dx \\ &\leqslant \sup_{x\in B_1(0)} |\widehat{\varphi}(x)| + [\widehat{\varphi}]^{n+2}_{\mathscr{S}(\mathbb{R}^n)}\int_{|x|\geqslant 1}\frac{1}{|x|^{n+2-2s}}dx < \infty,\end{aligned}$$

which shows that $g \in L^1(\mathbb{R}^n)$. In addition we have $\mathscr{F}^{-1}g = \mathscr{F}^{-1}\left(|\xi|^{2s}\,\widehat{\varphi}(\xi)\right) = (-\Delta)^s\,\varphi \in \mathscr{S}_s(\mathbb{R}^n)$ (see Chapter 1).
For $2s \geqslant n = 1$, we will check the additional conditions from Proposition 4.3:

$$
\begin{aligned}
|g(x)| &\leqslant |x|^{2s}\,|\widehat{\varphi}(x)| \leqslant \|\widehat{\varphi}\|_{L^1(\mathbb{R})}\,|x|^{2s} = c_1\,|x|^{2s} \text{ for } x \in \mathbb{R},\\
|g(x)| &\leqslant \frac{|x|^3\,|\widehat{\varphi}(x)|}{|x|^{3-2s}} \leqslant \frac{[\widehat{\varphi}]^3_{\mathscr{S}(\mathbb{R})}}{|x|^{3-2s}} \leqslant \frac{c_2}{|x|} \text{ for } |x| > 1.
\end{aligned}
$$

Since $\varphi \in \mathscr{S}(\mathbb{R})$, we have $g \in C^1(\mathbb{R})$ and we can estimate

$$
\begin{aligned}
\left|\frac{d}{dx}g(x)\right| &\leqslant 2s\,|x|^{2s-1}\,|\widehat{\varphi}(x)| + |x|^{2s}\left|\frac{d}{dx}\widehat{\varphi}(x)\right|\\
&\leqslant \left(2s\|\widehat{\varphi}\|_{L^\infty(\mathbb{R})} + |x|\left\|\frac{d}{dx}\widehat{\varphi}\right\|_{L^\infty(\mathbb{R})}\right)|x|^{2s-1}\\
&\leqslant \frac{2s\|\widehat{\varphi}\|_{L^\infty(\mathbb{R})} + \left\|\frac{d}{dx}\widehat{\varphi}\right\|_{L^\infty(\mathbb{R})}}{|x|^{1-2s}} = \frac{c_3}{|x|^{1-2s}}
\end{aligned}
$$

for $0 < |x| \leqslant 1$ and

$$
\begin{aligned}
\left|\frac{d}{dx}g(x)\right| &\leqslant 2s\,|x|^{2s-1}\,|\widehat{\varphi}(x)| + |x|^{2s}\left|\frac{d}{dx}\widehat{\varphi}(x)\right|\\
&\leqslant \left(2s\,|x|^2\,|\widehat{\varphi}(x)| + |x|^3\left|\frac{d}{dx}\widehat{\varphi}(x)\right|\right)|x|^{2s-3}\\
&\leqslant \frac{2s[\widehat{\varphi}]^2_{\mathscr{S}(\mathbb{R})} + [\widehat{\varphi}]^3_{\mathscr{S}(\mathbb{R})}}{|x|^{3-2s}} = \frac{c_4}{|x|}
\end{aligned}
$$

for $|x| > 1$.

Since the conditions from Proposition 4.2 and Proposition 4.3 are met for $n > 2s$ and $n \leqslant 2s$ respectively, (4.2) holds for all $n \in \mathbb{N}$ and $s \in (0,1)$ which finally yields

$$
\begin{aligned}
\langle \Psi, (-\Delta)^s\,\varphi\rangle_s &= \int_{\mathbb{R}^n}\Psi(x)\mathscr{F}^{-1}\left(|\xi|^{2s}\,\widehat{\varphi}(\xi)\right)(x)dx = \int_{\mathbb{R}^n}\Psi(x)\breve{g}(x)dx\\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{g(x)}{|x|^{2s}}dx = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\widehat{\varphi}(x)dx = \varphi(0)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $s \in (0,1)$, where the last equality followed from Remark 1.2. □

From Proposition 4.2, we can also derive how the Fourier transform of the convolution between Ψ and a smooth function f can be established for $n > 2s$.

Proposition 4.4. *Let $n > 2s$, let $f \in C_c^\infty(\mathbb{R}^n)$ and let $g \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ with $\breve{g} \in \mathcal{S}_s(\mathbb{R}^n)$. Then the equality*

$$\int_{\mathbb{R}^n} (f * \Psi)(x)\breve{g}(x)dx = \int_{\mathbb{R}^n} \frac{\breve{f}(x)g(x)}{|x|^{2s}}dx \tag{4.12}$$

holds.

Proof.
We start by applying Fubini's theorem to the left hand side of (4.12) to obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} (f * \Psi)(x)\breve{g}(x)dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y)\Psi(y)dy\right)\breve{g}(x)dx \\
&= \int_{\mathbb{R}^n} \Psi(y)\left(\int_{\mathbb{R}^n} f(x-y)\breve{g}(x)dx\right)dy \\
&= \int_{\mathbb{R}^n} \Psi(y)\left(\int_{\mathbb{R}^n} f(x)\breve{g}(x+y)dx\right)dy. \qquad (4.13)
\end{aligned}$$

For $u \in C_c^\infty$ and $v \in \mathscr{S}_s(\mathbb{R}^n)$ we define the operation

$$u \hat{*} v(y) := \int_{\mathbb{R}^n} u(x)v(x+y)dx. \tag{4.14}$$

This operation is obviously well defined and the identity

$$\mathscr{F}(u\hat{*}v) = (2\pi)^{\frac{n}{2}}\, \breve{u} \cdot \hat{v}$$

holds since

$$\begin{aligned}
\mathscr{F}((u\hat{*}v)(x))(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (u\hat{*}v)(x)e^{-ix\cdot\xi}dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u(y)v(x+y)dy\right) e^{-ix\cdot\xi}dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y)v(x+y)e^{-i(x+y)\cdot\xi}e^{iy\cdot\xi}dxdy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(y)e^{iy\cdot\xi}\int_{\mathbb{R}^n} v(x+y)e^{-i(x+y)\cdot\xi}dx\ dy \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(y)e^{iy\cdot\xi}dy\int_{\mathbb{R}^n} v(z)e^{-iz\cdot\xi}dz = (2\pi)^{\frac{n}{2}}\, \breve{u}(\xi)\cdot\hat{v}(\xi).
\end{aligned}$$

We now define

$$h(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}}\mathscr{F}(f\hat{*}\breve{g})(\xi) = \breve{f}(\xi)\cdot g(\xi),$$

which allows us to write (4.13) as

$$\int_{\mathbb{R}^n} (f * \Psi)(x)\breve{g}(x)dx = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Psi(y)\breve{h}(y)dy.$$

We will show that $h \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $\mathscr{F}^{-1}h = f \hat{*} \breve{g} \in \mathscr{S}_s(\mathbb{R}^n)$ so that Proposition 4.2 can be applied. As a product of continuous functions, we obviously have $h \in C(\mathbb{R}^n)$. Additionally, we can estimate

$$\int_{\mathbb{R}^n} |h(\xi)|\, d\xi = \int_{\mathbb{R}^n} |\mathscr{F}(f \hat{*} \breve{g})(\xi)|\, d\xi \leqslant \int_{\mathbb{R}^n} \left|\breve{f}(\xi)\right| |g(\xi)|\, d\xi \leqslant \left\|\breve{f}\right\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty,$$

which shows $h \in L^1(\mathbb{R}^n)$. To show the last claim, let $R > 0$ such that $\operatorname{supp} f \subseteq B_R(0)$. We show that $(1 + |x|^{n+2s})\, |(f * \breve{g})(x)| < \infty$ for all $x \in \mathbb{R}$. For $|x| \leqslant 2R$, we have that

$$\begin{aligned} \left(1 + |x|^{n+2s}\right) |(f * \breve{g})(x)| &\leqslant \left(1 + (2R)^{n+2s}\right) \int_{B_R(x)} |f(x-y)\breve{g}(y)|\, dy \\ &\leqslant c_{n,s,R} \|f\|_{L^\infty(B_R(0))} \|\breve{g}\|_{L^\infty(B_{3R}(0))} < \infty \end{aligned}$$

since both f and g are bounded on a bounded domain. Now let $|x| > 2R$, and observe that for $\breve{g} \in S_s(\mathbb{R}^n)$, we have the bound $(1 + |x|^{n+2s})\, |\breve{g}(x)| \leqslant [\breve{g}]^0_{S_s(\mathbb{R}^n)}$ and therefore also $(1 + |x+y|^{n+2s})\, |\breve{g}(x+y)| \leqslant [\breve{g}]^0_{S_s(\mathbb{R}^n)}$ for every $y \in \mathbb{R}^n$. This yields the bound

$$|\breve{g}(x+y)| \leqslant \frac{[\breve{g}]^0_{S_s(\mathbb{R}^n)}}{1 + |x+y|^{n+2s}} \leqslant \frac{[\breve{g}]^0_{S_s(\mathbb{R}^n)}}{|x+y|^{n+2s}} \tag{4.15}$$

for every $y \in \mathbb{R}^n$. We can now estimate

$$\begin{aligned} \left(1 + |x|^{n+2s}\right) |(f \hat{*} \breve{g})(x)| &\leqslant |x|^{n+2s} \int_{|y|<R} |f(y)\breve{g}(x+y)|\, dy \\ &\leqslant \|f\|_{L^\infty(\mathbb{R}^n)} [\breve{g}]^0_{S_s(\mathbb{R}^n)} |x|^{n+2s} \int_{|y|<R} \frac{1}{|x+y|^{n+2s}} dy. \end{aligned}$$

It is easy to see that for $y \in B_R(0)$, we have $2|y| \leqslant 2R < |x|$, which further yields $|x+y| \geqslant |x| - |y| \geqslant \frac{|x|}{2}$ and therefore

$$\left(1 + |x|^{n+2s}\right) |(f \hat{*} \breve{g})(x)| \leqslant \|f\|_{L^\infty(\mathbb{R}^n)} [\breve{g}]^0_{S_s(\mathbb{R}^n)} \int_{|y|<R} 2\ dy < \infty.$$

In the same way it can also be proven that $\left(1 + |x|^{n+2s}\right) |(f * D^\alpha \breve{g})(x)| < \infty$ for all $\alpha \in \mathbb{N}_0^n$ since $D^\alpha f \bar{*} \breve{g} = f \bar{*} D^\alpha \breve{g}$ and $D^\alpha \breve{g} \in S_s(\mathbb{R}^n)$. Applying Proposition 4.2 finally yields

$$\begin{aligned}\int_{\mathbb{R}^n}(f*\Psi)(x)\breve{g}(x)dx &= \int_{\mathbb{R}^n}\Psi(y)\left(\int_{\mathbb{R}^n}f(x)\breve{g}(x+y)dx\right)dy\\ &= (2\pi)^{2s}\int_{\mathbb{R}^n}\Psi(y)\breve{h}(y)dy = (2\pi)^{2s}\int_{\mathbb{R}^n}\frac{h(x)}{(2\pi\,|x|)^{2s}}dx\\ &= \int_{\mathbb{R}^n}\frac{\breve{f}(x)\cdot g(x)}{|x|^{2s}}dx.\end{aligned}$$

□

We treat the case $n \leqslant 2s$ separately as it will make use of Proposition 4.3 instead.

Proposition 4.5. *Let* $n = 1$ *and* $s \geqslant \frac{1}{2}$, *let* $f \in C_c^\infty(\mathbb{R})$ *and let* $g \in C(\mathbb{R}) \cap C^1((-\infty, 0) \cup (0, \infty)) \cap L^1(\mathbb{R})$ *with* $\breve{g} \in \mathcal{S}_s(\mathbb{R})$. *If there exist constants* $d_1,\ d_2,\ d_3,\ d_4 > 0$ *such that*

$$\begin{aligned}|g(x)| &\leqslant d_1\,|x|^{2s} && \text{for } x \in \mathbb{R},\\ |g(x)| &\leqslant \frac{d_2}{|x|} && \text{for } |x| > 1,\\ |g'(x)| &\leqslant \frac{d_3}{|x|^{1-2s}} && \text{for } 0 < |x| \leqslant 1,\\ |g'(x)| &\leqslant \frac{d_4}{|x|} && \text{for } |x| > 1,\end{aligned}$$

then equality (4.12) *holds.*

Proof.
The proof is very similar to the proof of Proposition 4.4. We again set

$$h(\xi) := \frac{1}{\sqrt{2\pi}}\mathscr{F}(f\hat{*}\breve{g})(\xi) = \breve{f}(\xi)\cdot g(\xi),$$

where $\hat{*}$ is the operation defined in (4.14), and now we seek to apply Proposition 4.3 instead. We notice that $h \in C(\mathbb{R}) \cap C^1((-\infty, 0) \cup (0, \infty)) \cap L^1(\mathbb{R})$ and $\mathscr{F}^{-1}h = f\hat{*}\breve{g} \in \mathscr{S}_s(\mathbb{R}^n)$ as was shown in the last proof. This means we only need to check (4.5) to finish the proof. For $x \in \mathbb{R}$ we have

$$|h(x)| \leqslant |\breve{f}(x)|\,|g(x)| \leqslant \|\breve{f}\|_{L^\infty(\mathbb{R})}d_1\,|x|^{2s} =: c_1\,|x|^{2s},$$

while for x with $|x| \leqslant 1$ we can estimate

$$\begin{aligned}|h'(x)| &\leqslant |\breve{f}(x)|\,|g'(x)| + \left|\frac{d}{dx}\breve{f}(x)g(x)\right| \leqslant \|\breve{f}\|_{L^\infty(\mathbb{R})}d_3\,|x|^{2s-1} + \|\xi f(\xi)\|_{L^1(\mathbb{R})}d_2\,|x|^{2s}\\ &\leqslant \left(d_3\|\breve{f}\|_{L^\infty(\mathbb{R})} + d_2\|\xi f(\xi)\|_{L^\infty(\mathbb{R})}\right)|x|^{2s-1} =: c_3\,|x|^{2s-1}.\end{aligned}$$

For x with $|x| > 1$, we have

$$|h(x)| \leqslant |\check{f}(x)|\,|g(x)| \leqslant \|\check{f}\|_{L^\infty(\mathbb{R})}\frac{d_2}{|x|} =: \frac{c_2}{|x|}$$

and

$$\begin{aligned} |h'(x)| &\leqslant |\check{f}(x)|\,|g'(x)| + \left|\frac{d}{dx}\check{f}(x)g(x)\right| \leqslant |\check{f}(x)|\frac{d_4}{|x|} + \left|\int_{\mathbb{R}} f(\xi)(i\xi)e^{ix\xi}d\xi\right| |g(x)| \\ &\leqslant \|\check{f}\|_{L^\infty(\mathbb{R})}\frac{d_4}{|x|} + \|\xi f(\xi)\|_{L^1(\mathbb{R})}\frac{d_2}{|x|} = \frac{d_4\|\check{f}\|_{L^\infty(\mathbb{R})} + d_2\|\xi f(\xi)\|_{L^1(\mathbb{R})}}{|x|} =: \frac{c_4}{|x|}. \end{aligned}$$

This concludes the proof. □

The next result is really important in itself as it shows how the fundamental solution can be used to solve the global Poisson equation involving the fractional Laplacian.

Theorem 4.6. *Let $\varepsilon > 0$ and $f \in C_c^{0,2s+\varepsilon}(\mathbb{R}^n)$. Then the function $u := f * \Psi$ belongs to $L_s^1(\mathbb{R}^n) \cap C^{0,2s+\varepsilon}(\mathbb{R}^n)$ and solves*

$$(-\Delta)^s u = f, \tag{4.16}$$

both in the distributional sense as well as pointwise for all $x \in \mathbb{R}^n$.

Proof.
We have shown in Lemma 3.9 that $u \in L_s^1(\mathbb{R}^n)$. And thanks to [21, Theorem 9.3], we have $u \in C^{0,2s+\varepsilon}(\mathbb{R}^n)$. This ensures that (4.16) is well defined both in a distributional and pointwise sense. We will now prove that u is the distributional solution of (4.16) by showing that

$$\langle u, (-\Delta)^s\varphi\rangle_s = \langle f, \varphi\rangle_{L^2(\mathbb{R}^n)}$$

holds for every $\varphi \in \mathscr{S}(\mathbb{R}^n)$. We will start by proving this identity for $f \in C_c^\infty(\mathbb{R}^n)$. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ be arbitrary but fixed and define $g(\xi) := |x|^{2s}\,\widehat{\varphi}(\xi)$. Then f and g satisfy the conditions of both Proposition 4.4 and 4.5 (see the proof of Theorem 3.10 for further details). Thus, by (4.12), we get

$$\begin{aligned} \langle u, (-\Delta)^s\varphi\rangle_s &= \int_{\mathbb{R}^n}(f*\Psi)(x)(-\Delta)^s\varphi(x)dx = \int_{\mathbb{R}^n}(f*\Psi)(x)\mathscr{F}^{-1}\left(|\xi|^{2s}\,\widehat{\varphi}(\xi)\right)dx \\ &= \int_{\mathbb{R}^n}(f*\Psi)(x)\check{g}(x)dx = \int_{\mathbb{R}^n}\frac{\check{f}(x)g(x)}{|x|^{2s}}dx = \int_{\mathbb{R}^n}\check{f}(x)\widehat{\varphi}(x)dx \\ &= \left\langle \check{f}, \widehat{\varphi}\right\rangle_{L^2(\mathbb{R}^n)} = \langle f, \varphi\rangle_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where we used Plancherel's identity for the last equality, which is assured since f, $\check{f}$ and φ are sufficiently smooth. This shows that u is indeed the distributional solution

of (4.16) for $f \in C_c^\infty(\mathbb{R}^n)$.
Let now $\varepsilon > 0$, $f \in C_c^{0,2s+\varepsilon}$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ be arbitrary but fixed. Then there exists a sequence of functions $(f_k)_k \subseteq C_c^\infty(\mathbb{R}^n)$ with $\|f_k - f\|_{L^\infty(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0$, which also implies

$$\lim_{k\to\infty} \langle f_k, \varphi\rangle_{L^2(\mathbb{R}^n)} = \langle f, \varphi\rangle_{L^2(\mathbb{R}^n)}.$$

In addition, the functions defined by $u_k := \Psi * f_k$ all satisfy

$$\langle u_k, (-\Delta)^s \varphi\rangle_s = \langle f_k, \varphi\rangle_{L^2(\mathbb{R}^n)}. \tag{4.17}$$

Additionally, by using both Remark 2.5 and (3.9), there exists a constant $c_{n,s,R}$ such that

$$\begin{aligned}
\langle u - u_k, (-\Delta)^s \varphi\rangle_s &\leqslant [(-\Delta)^s\varphi]^0_{\mathscr{S}_s(\mathbb{R}^n)} \|u_k - u\|_{L^1_s(\mathbb{R}^n)} \\
&= [(-\Delta)^s\varphi]^0_{\mathscr{S}_s(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \frac{u(x) - u_k(x)}{1 + |x|^{n+2s}}\right)^{\frac{1}{2}} \\
&= [(-\Delta)^s\varphi]^0_{\mathscr{S}_s(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} \frac{(\Psi * (f - f_k))(x)}{1 + |x|^{n+2s}}\right)^{\frac{1}{2}} \\
&\leqslant c_{n,s,R} [(-\Delta)^s\varphi]^0_{\mathscr{S}_s(\mathbb{R}^n)} \|f - f_k\|_{L^\infty(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.
\end{aligned}$$

Together with (4.17), this yields

$$\langle u, (-\Delta)^s \varphi\rangle_s = \lim_{k\to\infty} \langle u_k, (-\Delta)^s \varphi\rangle_s = \lim_{k\to\infty} \langle f_k, \varphi\rangle_{L^2(\mathbb{R}^n)} = \langle f, \varphi\rangle_{L^2(\mathbb{R}^n)},$$

hence we have proven that u solves (4.16) in the distributional sense for all $f \in C_c^{0,2s+\varepsilon}(\mathbb{R}^n)$. To obtain pointwise solvability, we will first recall that Remark 2.3 ensures that $(-\Delta)^s u$ is well defined since $u \in C^{0,2s+\varepsilon}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Moreover, thanks to Proposition 2.4, we have that $\int_{\mathbb{R}^n} (-\Delta)^s u(x)\varphi(x)dx$ is well defined. In addition, by using Fubini's Theorem and changing variables, we obtain that for any $\varphi \in \mathscr{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} f(x)\varphi(x)dx = \int_{\mathbb{R}^n} u(x)(-\Delta)^s\varphi(x)dx = \int_{\mathbb{R}^n} (-\Delta)^s u(x)\varphi(x)dx.$$

Since both f and $(-\Delta)^s u$ are continuous, we conclude that (4.16) holds pointwise in $\mathbb{R}^n$ □

As an immediate consequence, we obtain another representation of a function belonging to $C_c^\infty(\mathbb{R}^n)$.

Corollary 4.7. *For any $u \in C_c^\infty(\mathbb{R}^n)$ there exists a function $f \in C^\infty(\mathbb{R}^n)$ such that*

$$(f * \Psi)(x) = u(x)$$

holds for all $x \in \mathbb{R}^n$. Additionally, there exists a constant $c_{n,s} > 0$ such that

$$|f(x)| \leqslant \frac{c_{n,s}}{|x|^{n+2s}}. \tag{4.18}$$

Proof.
Let $u \in C_c^\infty(\mathbb{R}^n)$ be arbitrary and define $f := (-\Delta)^s u \in C^\infty(\mathbb{R}^n)$. Then applying Theorem 4.6 yields

$$(f * \Psi)(x) = ((-\Delta)^s u * \Psi)(x) = u(x)$$

pointwise for all $x \in \mathbb{R}^n$. The bound (4.18) is shown in the proof of Lemma 2.7. □

We now have all the tools necessary to prove of the main result of this paper.

Proof of Theorem 4.1.
We will divide the proof into multiple steps:

Step 1: The solution is unique.

Step 2: $u_g \in C(\mathbb{R}^n)$.

Step 3: $u_g \in L_s^1(\mathbb{R}^n)$.

Step 4: u_g has the s-mean value property in $B_\rho(0)$ for $g \in C_c^\infty(\mathbb{R}^n)$.

Step 5: u_g has the s-mean value property in $B_\rho(0)$ for any $g \in L_s^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

We will prove all these steps one after another, which will ultimately prove the theorem.

Step 1: Let $u_1, u_2 \in C(\mathbb{R}^n)$ be two solutions of the Dirichlet problem (4.1). Then $u_1 - u_2 =: u \in C(\mathbb{R}^n)$ is a solution of the problem

$$\begin{cases} (-\Delta)^s u & = 0 \text{ in } B_\rho(0), \\ u & = 0 \text{ in } \mathbb{R}^n \backslash B_\rho(0). \end{cases}$$

By [6, Theorem 3.3.3], the solution is constant, and since u continuous and zero in $\mathbb{R}^n \backslash B_\rho$, we have $u_1 = u_2$ everywhere.

Step 2: Since u_g is obviously continuous in $B_\rho(0)$ and in $\mathbb{R}^n \backslash B_\rho(0)$, we need to check the continuity at the boundary ∂B_ρ. Therefore, let $y_0 \in \partial B_\rho$ and $\varepsilon > 0$ arbitrarily small but fixed. By the continuity of g, there exists a δ_ε such that

$$|g(y_0) - g(y)| < \varepsilon$$

for any $y \in B_{\delta_\varepsilon}(y_0)$. We fix $\mu > 0$ arbitrarily small such that $\mu < \frac{\delta_\varepsilon}{2}$ as well as $R > 2\rho$. Our aim is to show that

$$\lim_{\substack{x \to y_0 \\ x \in B_\mu(y_0) \cap B_\rho(0)}} (u_g(x) - u_g(y_0)) = 0, \tag{4.19}$$

as is shown in Figure 4.1.

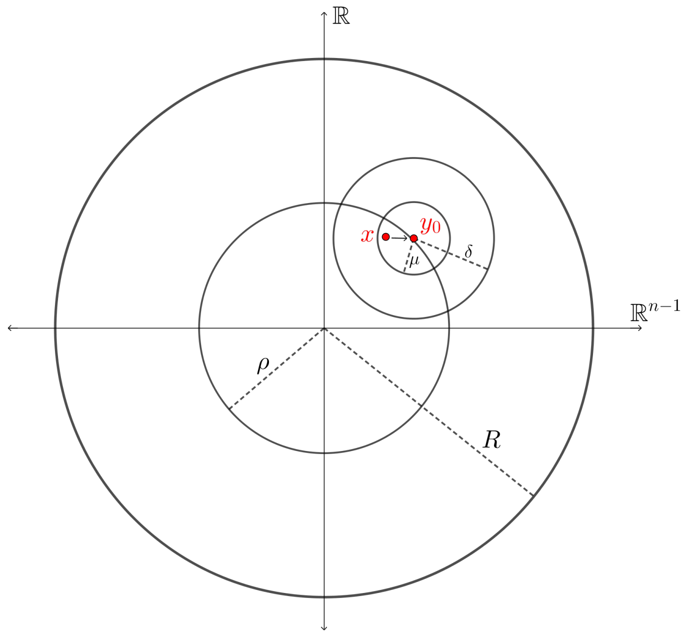

Figure 4.1.: Framework for proofing continuity

Now let $x \in B_\mu(y_0) \cap B_\rho(0)$ and estimate

$$\begin{aligned}
\left|u_g(x) - u_g(y_0)\right| &= \left| \int_{|y| \geqslant \rho} P_\rho(x,y) g(y) dy - g(y_0) \int_{|y| \geqslant \rho} P_\rho(x,y) dy \right| \\
&\leqslant \int_{|y| \geqslant \rho} P_\rho(x,y) \left|g(y) - g(y_0)\right| dy \\
&\leqslant \int_{\substack{|y| \geqslant \rho \\ |y-y_0| < \delta_\varepsilon}} P_\rho(x,y) \left|g(y) - g(y_0)\right| dy + \int_{\substack{|y| \geqslant \rho \\ |y-y_0| \geqslant \delta_\varepsilon}} P_\rho(x,y) \left|g(y) - g(y_0)\right| dy \\
&\leqslant \varepsilon \int_{|y| \geqslant \rho} P_\rho(x,y) dy + \int_{\substack{|y| > \rho \\ |y-y_0| \geqslant \delta_\varepsilon}} P_\rho(x,y) \left(|g(y)| + |g(y_0)|\right) dy \\
&= \varepsilon + \left(\rho^2 - |x|^2\right)^s \int_{\substack{|y| > \rho \\ |y-y_0| \geqslant \delta_\varepsilon}} \frac{|g(y)| + |g(y_0)|}{\left(|y|^2 - \rho^2\right)^s |x-y|^n} dy,
\end{aligned}$$

where we have made use of (3.7) twice. Before we continue with the above term, we observe that for any $x \in B_\rho(0)$ and $y \in \mathbb{R}^n \backslash B_R(0)$ we have

$$\frac{1}{\left(|y|^2 - \rho^2\right)^s |x-y|^n} \leqslant \frac{2^{n+s}}{|y|^{n+2s}}, \tag{4.20}$$

since by using the inequality $|x-y| \geqslant |y| - \rho$, we can estimate

$$\begin{aligned} \frac{|y|^{n+2s}}{\left(|y|^2 - \rho^2\right)^s |x-y|^n} &\leqslant \left(\frac{|y|}{|x-y|}\right)^n \left(\frac{|y|^2}{|y|^2-\rho^2}\right)^s \\ &\leqslant \left(\frac{|y|}{|y|-\rho}\right)^n \left(\frac{4}{3}\right)^s \leqslant 2^n \cdot 2^s = 2^{n+s}. \end{aligned}$$

Additionally, we notice that

$$\rho^2 - |x|^2 = (\rho + |x|)(\rho - |x|) < 2\rho\,|y_0 - x| < 2\rho\mu.$$

By using this, (4.20) and the fact that $|x-y| \geqslant \delta_\varepsilon - \mu > \frac{\delta_\varepsilon}{2}$ for any $y \in \mathbb{R}^n \backslash B_\delta(y_0)$ with $\rho < |y| < R$, we obtain

$$\begin{aligned} &\left|u_g(x) - u_g(y_0)\right| \\ &\leqslant \varepsilon + (2\rho\mu)^s \left(\int_{\substack{\rho<|y|<R \\ |y-y_0|\geqslant\delta_\varepsilon}} \frac{|g(y)| + |g(y_0)|}{\left(|y|^2-\rho^2\right)^s |x-y|^n} dy + \int_{|y|\geqslant R} \frac{|g(y)| + |g(y_0)|}{\left(|y|^2-\rho^2\right)^s |x-y|^n} dy \right) \\ &\leqslant \varepsilon + (2\rho\mu)^s \left(\frac{2^n}{\delta_\varepsilon^n} \int_{\substack{\rho<|y|<R \\ |y-y_0|\geqslant\delta_\varepsilon}} \frac{|g(y)| + |g(y_0)|}{\left(|y|^2-\rho^2\right)^s} dy + 2^{n+s} \int_{|y|\geqslant R} \frac{|g(y)| + |g(y_0)|}{|y|^{n+2s}} dy \right) \\ &\leqslant \varepsilon + (2\rho)^s \left(\frac{2^n}{\delta_\varepsilon^n} \bar{c}(\rho, R, s, g) + 2^{n+s} \|g\|_{L^1_s(\mathbb{R}^n)} + |g(y_0)| \frac{S_{n-1}}{2sR^{2s}} \right) \mu^s \\ &= \ \varepsilon + c(n, s, R, \rho, g, \delta_\varepsilon)\mu^s, \end{aligned}$$

where S_{n-1} is the measure of the $(n-1)$-dimensional unit sphere. This shows that the term $|u_g(x) - u_g(y_0)|$ tends to zero as both μ and ε tend to zero, which proves that the limit on the left hand side of (4.19) exists and is in fact zero, thus showing the continuity of u_g.

The aim of the next 3 steps is to show that all the requirements for Theorem 3.4 are met for any $x \in B_\rho(0)$. Theorem 3.4 then implies that $(-\Delta)^s u_g(x) = 0$ for all $x \in B_\rho(0)$.

Step 3: In order to show that $u_g \in L^1_s(\mathbb{R}^n)$, we need to show that

$$\int_{\mathbb{R}^n} \frac{|u_g(x)|}{1+|x|^{n+2s}} = \int_{|x|<\rho} \frac{|u_g(x)|}{1+|x|^{n+2s}} dx + \int_{|x|\geqslant\rho} \frac{|g(x)|}{1+|x|^{n+2s}} dx \tag{4.21}$$

is finite. The latter integral is finite since $g \in L^1_s(\mathbb{R}^n)$, so we now show that u_g is bounded in $B_\rho(0)$ to prove the finiteness of the former integral. Let $R > 2\rho$ and $x \in B_\rho(0)$, then by using (4.20), we can estimate

$$\begin{aligned} |u_g(x)| &\leqslant \int_{\rho \leqslant |y| < R} P_\rho(x,y)\,|g(y)|\,dy + \int_{|y| \geqslant R} P_\rho(x,y)\,|g(y)|\,dy \\ &\leqslant \sup_{\rho \leqslant |y| \leqslant R} |g(y)| + 2^{n+s} c(n,s) \left(\rho^2 - |x|^2\right)^s \int_{|y| \geqslant R} \frac{|g(y)|}{|y|^{n+2s}} dy \\ &\leqslant \sup_{\rho \leqslant |y| \leqslant R} |g(y)| + 2^{n+s} c(n,s) \rho^{2s} \int_{|y| \geqslant R} \frac{|g(y)|}{|y|^{n+2s}} dy < \infty. \end{aligned}$$

This shows that both terms in (4.21) are finite.

Step 4: Let $g \in C_c^\infty(\mathbb{R}^n)$ and $x \in B_\rho(0)$ be arbitrary but fixed. We will show that u_g has the s-mean value property at x, meaning that for all $0 < r < \rho - |x|$, the identity

$$\left(A_r * u_g\right)(x) = u_g(x) \tag{4.22}$$

holds. By Corollary 4.7, there exists a function $f \in C^\infty(\mathbb{R}^n)$ such that for all $y \in \mathbb{R}^n \backslash B_\rho(0)$ we have

$$\begin{aligned} g(y) &= \int_{\mathbb{R}^n} f(z)\Psi(y-z)dz \\ &= \int_{|z|<\rho} f(z)\Psi(y-z)dz + \int_{|t|>\rho} f(t)\Psi(y-t)dt. \end{aligned}$$

Using identity (3.12) we obtain

$$\begin{aligned} g(y) &= \int_{|z|<\rho} f(z) \left(\int_{|t|>\rho} P_\rho(z,t)\Psi(y-t)dt \right) dz + \int_{|t|>\rho} f(t)\Psi(y-t)dt \\ &= \int_{|t|>\rho} \left(\int_{|z|<\rho} f(z) P_\rho(z,t)dz \right) \Psi(y-t)dt + \int_{|t|>\rho} f(t)\Psi(y-t)dt \\ &= \int_{|t|>\rho} \left(\int_{|z|<\rho} f(z) P_\rho(z,t)dz + f(t) \right) \Psi(y-t)dt \\ &=: \int_{|t|>\rho} h(t)\Psi(y-t)dt. \end{aligned}$$

This together with (3.12) allows us to rewrite u_g as

$$\begin{aligned} u_g(x) &= \int_{|y|>\rho} P_\rho(x,y) g(y) dy = \int_{|y|>\rho} P_\rho(x,y) \left(\int_{|t|>\rho} h(t)\Psi(y-t)dt \right) dy \\ &= \int_{|t|>\rho} h(t) \left(\int_{|y|>\rho} P_\rho(x,y)\Psi(y-t)dy \right) dt \\ &= \int_{|t|>\rho} h(t)\Psi(x-t)dt. \end{aligned} \tag{4.23}$$

The use of Fubini's Theorem is justified since we have shown in step 3 that $|u_g(x)| < \infty$ for $x \in B_\rho(0)$. Now we can prove that u_g has the s-mean value property at x. Since for $r \leqslant \rho - |x|$ we have that $|x-t| \geqslant |t| - |x| \geqslant r$, which allows us to use (3.11) and obtain

$$\begin{aligned}
(A_r * u_g)(x) &= \int_{|y|>r} A_r(y) u_g(x-y) dy \\
&= \int_{|y|>r} A_r(y) \left(\int_{|t|>\rho} h(t) \Psi(x-y-t) dt \right) dy \\
&= \int_{|t|>\rho} h(t) \left(\int_{|y|>r} A_r(y) \Psi(x-t-y) dy \right) dt \\
&= \int_{|t|>\rho} h(t) \Psi(x-t) dt = u_g(x),
\end{aligned}$$

where we have also made use of equation (4.23) twice. This proves identity (4.22) holds for every $g \in C_c^\infty(\mathbb{R}^n)$.

Step 5: Now let $g \in L_s^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $r > 0$ be arbitrarily small. Our aim is that identity (4.22) still holds for this setting. In order to do so, let $(\nu_k)_{k\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ be a sequence of functions with $\nu_k(x) \in [0,1]$, $\nu_k = 1$ in $B_k(0)$ and $\nu_k = 0$ in $\mathbb{R}^n \backslash B_k(0)$, and $B_k \subsetneq B_{k+1}$. Then $g_k := \nu_k g \in C_c^\infty(\mathbb{R}^n)$ converges towards g pointwise in $\mathbb{R}^n$, uniformly on compact sets and in the $L_1^s(\mathbb{R}^n)$-norm. Now, thanks to step 4, we know that

$$(A_r * u_{g_k})(x) = u_{g_k}(x) \tag{4.24}$$

for any $k \in \mathbb{N}$. Moreover, we will show that

$$\lim_{k\to\infty} u_{g_k}(x) = u_g(x) \tag{4.25}$$

and

$$\lim_{k\to\infty} (A_r * u_{g_k})(x) = (A_r * u_g)(x) \tag{4.26}$$

for any $x \in B_\rho(0)$. By using these 3 identities, we have that

$$u_g(x) = \lim_{k\to\infty} u_{g_k}(x) = \lim_{k\to\infty} (A_r * u_{g_k})(x) = (A_r * u_g)(x), \tag{4.27}$$

proving that g indeed has the s-mean value property.
Starting with proving (4.25), let $x \in B_\rho(0)$ and choose $R > 2\rho$. Then by again

making use of (3.7) and (4.20) , we have that

$$\begin{aligned}
\left|u_g(x) \right. & - \left. u_{g_k}(x)\right| \\
& \leqslant \int_{|y|\geqslant\rho} P_\rho(x,y)\,|g(y)-g_k(y)|\,dy \\
& = c(n,s)\left(\rho^2-|x|^2\right)\int_{|y|\geqslant R}\frac{|g(y)-g_k(y)|}{\left(|y|^2-\rho^2\right)^s|x-y|^n}dy \\
& \quad + \int_{\rho\leqslant|y|<R} P_\rho(x,y)\,|g(y)-g_k(y)|\,dy \\
& \leqslant 2^{n+s}c(n,s)\left(\rho^2-|x|^2\right)\int_{|y|\geqslant R}\frac{|g(y)-g_k(y)|}{|y|^{n+2s}}dy \\
& \quad + \sup_{y\in\overline{B_R(0)}\setminus B_\rho(0)}|g(y)-g_k(y)|\int_{\rho\leqslant|y|<R}P_\rho(x,y)dy \\
& \leqslant 2^{n+s}\rho^2c(n,s)\int_{|y|\geqslant R}\frac{|g(y)-g_k(y)|}{|y|^{n+2s}}dy+\sup_{\rho\leqslant|y|\leqslant R}|g(y)-g_k(y)|\,,
\end{aligned}$$

which vanishes as k approaches infinity by the convergence in $L^1_s(\mathbb{R}^n)$ norm and the uniform convergence on compact sets of g_k to g.
The next step is to prove (4.26), so let $x\in B_\rho(0)$ and choose $R>2r$. Plugging in the definitions of A_ρ and P_ρ results in

$$\begin{aligned}
&\left|A_r * u_g(x) - A_r * u_{g_k}(x)\right| \\
&\qquad \leqslant \int_{|y|>r} A_r(y)\,|u_g(x-y)-u_{g_k}(x-y)|\,dy \\
&\qquad \leqslant \int_{\substack{|y|>r\\|x-y|\geqslant\rho}} A_r(y)\,|g(x-y)-g_k(x-y)|\,dy \\
&\qquad \quad + \int_{\substack{|y|>r\\|x-y|<\rho}} A_r(y)\left(\int_{|z|>\rho}P_\rho(x-y,z)\,|g(z)-g_k(z)|\,dz\right)dy \\
&\qquad =: I_1(x)+I_2(x).
\end{aligned}$$

We estimate the first integral by using (3.2) and (4.20), obtaining

$$\begin{aligned}
I_1(x) &= c(n,s)r^{2s}\int_{\substack{|y|>r\\|x-y|\geqslant\rho}}\frac{|g(x-y)-g_k(x-y)|}{\left(|y|^2-r^2\right)^s|y|^n}dy \\
&\leqslant \sup_{r\leqslant|y|\leqslant R}|g(x-y)-g_k(x-y)|\int_{r<|y|<R}A_r(y)dy \\
&\quad +2^{n+s}c(n,s)r^{2s}\int_{|y|>R}\frac{|g(x-y)-g_k(x-y)|}{|y|^{n+2s}}dy,
\end{aligned}$$

which again vanishes as k approaches infinity by the convergence in $L^1_s(\mathbb{R}^n)$ norm and the uniform convergence on compact sets of g_k to g. For the second integral, we choose an $\widetilde{R} > 2\rho$, then by using the bound (4.20) as well as identities (3.2) and (3.7), we estimate

$$\begin{aligned} I_2(x) &= \int_{\substack{|y|>r\\|x-y|<\rho}} A_r(y) \int_{\rho<|z|<R} P_\rho(x-y,z)\,|g(z)-g_k(z)|\,dz\;dy \\ &\quad + \int_{\substack{|y|>r\\|x-y|<\rho}} A_r(y) \int_{|z|>R} P_\rho(x-y,z)\,|g(z)-g_k(z)|\,dz\;dy \\ &\leqslant \sup_{\rho\leqslant|z|\leqslant R} |g(z)-g_k(z)| \int_{\substack{|y|>r\\|x-y|<\rho}} A_r(y) \int_{\rho<|z|<R} P_\rho(x-y,z) dz\;dy \\ &\quad + c(n,s) \int_{\substack{|y|>r\\|x-y|<\rho}} A_r(y)\rho^{2s} \int_{|z|>R} \frac{|g(z)-g_k(z)|}{\left(|z|^2-\rho^2\right)^s |(x-y)-z|^n} dz\;dy \\ &\leqslant \sup_{\rho\leqslant|z|\leqslant R} |g(z)-g_k(z)| + 2^{n+s}\rho^{2s} c(n,s) \int_{|z|>R} \frac{|g(z)-g_k(z)|}{|z|^{n+2s}} dz, \end{aligned}$$

which vanishes as k goes to infinity by the same argument we used for I_1. This proves (4.26), finishing the proof of step 5. Therefore u_g has the s-mean value property, which immediately implies the claim of the theorem.

5. Conclusion and further results

With the main result now proven, we have seen that many of the tools known from regular Laplacian analysis can be also used in the setting of the solution of the fractional Laplace operator. We were able to establish an analytical formula for the according Dirichlet problem for a sufficiently smooth given function on the exterior space. A reasonable next step would be to ask for the related Neumann problem and a suitable Dirichtlet-to-Neumann operator, though this comes with a few difficulties. In particular, the definition of a nonlocal fractional normal derivative has to be carefully introduced, a method of doing so was proposed by Dipierro, Ros-Oton, and Valdinoci in [10] and was further studied in [1].

Furthermore, we recall that there are other ways of defining the fractional Laplacian depending on the respective physical approach taken. While all these definitions turn out to be equivalent when looking at the operator in a global sense, these equivalencies no longer hold true when looking at the operator on a bounded domain Ω. Even though the fractional Laplacian turns out to be non-local no matter which definition is used, it turns out that for the Dirichlet problem with respect to the spectral fractional Laplacian $(-\Delta)^s_S$ and the regional fractional Laplacian $(-\Delta)^s_R$, it is sufficient to specify the boundary values. This means that for these operators, the well-posed Dirichlet problem takes the more familiar form of

$$\begin{cases} (-\Delta)^s_* u &= 0 \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega, \end{cases}$$

for a sufficiently smooth given function g, where $* \in \{S, R\}$. This results in very different analytical and numerical solving strategies from the beginning. The following paper goes into more detail about the analytical aspects of these operators: [11].

As one might expect, the studies on numerical solutions for semi-differential equations involving any type of the fractional Laplacian are vast. The definition of the integrated fractional Laplacian in particular allows the introduction of a fitting Sobolev space in a straightforward way as

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \;\middle|\; \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx\, dy < \infty \right\}.$$

This makes the operator amenable to a number of variational techniques, and we refer to [3, 5] for further information on the topic.

Appendix

A. Useful integral identities and estimations

Lemma A.1. *Let $k, \rho > 0$. Then, if $k < n$, it holds that*

$$\int_{|x|<\rho} \frac{1}{|x|^k} dx < \infty.$$

On the other hand, if $k > n$, we have that

$$\int_{|x|\geqslant\rho} \frac{1}{|x|^k} dx < \infty.$$

Proof.
For any $k < n$, we use the substitution $|x| = r$ to obtain

$$\int_{|x|<\rho} \frac{1}{|x|^k} dx = S_{n-1} \int_0^\rho \frac{r^{n-1}}{r^k} dr = S_{n-1} \int_0^\rho \frac{1}{r^{k-n+1}} dr,$$

where S_{n-1} is the measure of the $(n-1)$-dimensional unit sphere. Since $k - n + 1 < 1$, the above integral is obviously finite. The case for $k > n$ can be shown analogously.

□

The next lemma is found and proven as identity (A.25) in [7].

Lemma A.2. *For any $r > 1$, we have that*

$$\int_0^\pi \frac{\sin^{n-2}\phi}{(1 + r^2 - 2r\cos\phi)^{\frac{n}{2}}} d\phi = \frac{1}{(r^2 - 1)r^{n-2}} \int_0^\pi \sin^{n-2}\phi \; d\phi \tag{A.1}$$

The proof of the next result can be found in [4, page 549].

Lemma A.3. *For any $z \in B_1(0)$, we have*

$$\int_{-1}^1 \frac{\log(|z - v|)}{\sqrt{1 - v^2}} dv = -\pi \log(2). \tag{A.2}$$

Lemma A.4. *For any $s \in (0, 1)$, we have that*

$$\int_0^\infty \frac{1}{t^s(1 + t)} dt = \frac{\pi}{\sin(\pi s)}. \tag{A.3}$$

Proof.
We make use of the fundamental properties of the Gamma and Beta-function and get

$$\int_0^\infty \frac{1}{t^s(1+t)}dt = \int_0^\infty \frac{t^{(1-s)-1}}{(1+t)^{(1-s)+s}}dt = B(1-s,s) = \Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}.$$

□

Lemma A.5. *For any $\rho > 1$, we have that*

$$\left(\rho^2-1\right)^s \int_r^\infty \frac{2r}{\left(r^2-\rho^2\right)^s\left(r^2-1\right)}dr = \frac{\pi}{\sin(\pi s)}. \tag{A.4}$$

Proof.
We use the change of variable $t = \frac{r^2-\rho^2}{\rho^2-1}$ along with (A.3) to calculate

$$\left(\rho^2-1\right)^s \int_r^\infty \frac{2r}{\left(r^2-\rho^2\right)^s\left(r^2-1\right)}dr = \int_0^\infty \frac{1}{t^s\ (t+1)}dt = \frac{\pi}{\sin(\pi s)}.$$

□

Lemma A.6. *The identity*

$$\pi \prod_{k=1}^{n-2} \int_0^\pi \sin^k \phi\ d\phi = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{A.5}$$

holds

Proof.
We first start by calculating the integral on the left hand side for every $k \in \mathbb{N}^*$. We set

$$I_k := \int_0^\pi \sin^k \phi\ d\phi$$

and notice that $I_0 = \pi$ and $I_1 = 2$. Then, using integration by parts, we have that

$$\begin{aligned} I_k = \int_0^\pi \sin^{k-1}(\phi)\sin(\phi)\ d\phi &= -\sin^{k-1}(\phi)\cos(\phi)\Big|_0^\pi + (k-1)\int_0^\pi \sin^{k-2}(\phi)\cos^2(\phi)\ d\phi \\ &= (k-1)\int_0^\pi \sin^{k-2}(\phi)\left(1-\sin^2(\phi)\right)\ d\phi \\ &= (k-1)\left(\int_0^\pi \sin^{k-2}(\phi)\ d\phi - \int_0^\pi \sin^k(\phi)\ d\phi\right) \\ &= (k-1)\left(I_{k-2}-I_k\right), \end{aligned}$$

resulting in the recursion

$$I_k = \frac{k-1}{k} I_{k-2} \ , \ I_0 = \pi \ , \ I_1 = 2.$$

Solving this recursion gives

$$I_k = \begin{cases} \pi \prod_{i=1}^{\frac{k}{2}} \frac{2i-1}{2i} & \text{if } k \text{ even,} \\ 2 \prod_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{2i}{2i+1} & \text{if } k \text{ odd,} \end{cases}$$

for every $k \in \mathbb{N} \backslash \{1\}$. We will now solve (A.5) for the case that n is even. Elementary computations then yield

$$\begin{aligned} \pi \prod_{k=1}^{n-2} \int_0^\pi \sin^k \phi \, d\phi &= \pi \prod_{\substack{k=1 \\ k \text{ odd}}}^{n-3} I_k \prod_{\substack{k=2 \\ k \text{even}}}^{n-2} I_k = 2^{\frac{n}{2}-1} \pi^{\frac{n}{2}} \prod_{k=2}^{\frac{n}{2}-1} \prod_{i=1}^{k-1} \frac{2i}{2i+1} \cdot \prod_{k=2}^{\frac{n}{2}} \prod_{i=1}^{k-1} \frac{2i-1}{2i} \\ &= 2^{\frac{n}{2}-1} \pi^{\frac{n}{2}} \prod_{k=1}^{\frac{n}{2}-2} \left(\frac{2k}{2k+1} \right)^{\frac{n}{2}-k-1} \cdot \prod_{k=1}^{\frac{n}{2}-1} \left(\frac{2k-1}{2k} \right)^{\frac{n}{2}-k} \\ &= 2^{\frac{n}{2}-1} \pi^{\frac{n}{2}} \prod_{k=1}^{\frac{n}{2}-1} \frac{1}{2k} = \frac{\pi^{\frac{n}{2}}}{\prod_{k=1}^{\frac{n}{2}-1} k} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \end{aligned}$$

and very similar calculations show the same result for the case that n is odd. □

The next lemma is used to prove the identities (3.11) and (3.12).

Lemma A.7. *Let $r > 0$ and $x \in B_r(0)$. Then we have*

$$c(n, s) \int_{|y|<r} \frac{1}{\left(r^2 - |y|^2\right)^s |x-y|^{n-2s}} dy = 1, \tag{A.6}$$

where $c(n, s)$ is the constant defined by (3.1)

Proof.
Let $r > 0$ and $x \in B_r(0)$ be arbitrary but fixed. For any $y \in \mathbb{R}^n \backslash B_\rho(0)$, we set $y^* := K_x(y)$, where K_x is the point inversion at x as defined in (1.1). By making use of (1.4) and (1.3), we obtain

$$\begin{aligned} \frac{1}{\left(r^2 - |y|^2\right)^s |x-y|^{n-2s}} \left(|\det(DK_x(y))|\right)^{-1} &= \left(\frac{|x-y|}{r^2 - |y|^2} \right)^s \frac{1}{|x-y^*|^n} \\ &= \left(\frac{r^2 - |x|^2}{|y^*|^2 - r^2} \right)^s \frac{1}{|x-y^*|^n}, \end{aligned}$$

which, together with (3.7), gives us

$$\begin{aligned} c(n,s)\int_{|y|<r}\frac{1}{\left(r^2-|y|^2\right)^s|x-y|^{n-2s}}dy &= c(n,s)\int_{|y^*|\geqslant r}\left(\frac{r^2-|x|^2}{|y^*|^2-r^2}\right)^s\frac{1}{|x-y^*|^n}dy^* \\ &= \int_{|y^*|\geqslant r}P_r(x,y^*)dy^*=1. \end{aligned}$$

□

Lemma A.8. *For any $s\in(0,1)$ we have that*

$$\int_0^\infty t^{2s-2}\sin(t)dt=-\cos(\pi s)\Gamma(2s-1).$$

Proof.
The proof will be using some complex analysis. First, by using Euler's formula, we get

$$\int_0^\infty t^{2s-2}\sin(t)dt=-\int_0^\infty t^{2s-2}\Im(e^{-it})dt=-\Im\left(\int_0^\infty t^{2s-2}e^{-it}dt\right). \tag{A.7}$$

To evaluate the integral on the right hand side, let $r>0$ be arbitrary but fixed and define the domain $\Omega_r:=([0,r]\times[0,r])\cap B_r(0)$. Then we have that the contour integral $\int_{\partial\Omega_r}z^{2s-2}e^{-z}dz$ is 0 by Cauchy's Theorem since Ω_r is a star domain with no poles in its interior. By setting $\gamma_r:=\partial B_r(0)\cap([0,r]\times[0,r])$ (see Figure A.1), we can then split the contour integral into

$$\begin{aligned} 0 &= \int_{\partial\Omega_r}z^{2s-2}e^{-z}dz \\ &= \int_0^r t^{2s-2}e^{-t}dt+\int_{\gamma_r}z^{2s-2}e^{-z}dz-i\int_0^r(it)^{2s-2}e^{-it}dt. \end{aligned} \tag{A.8}$$

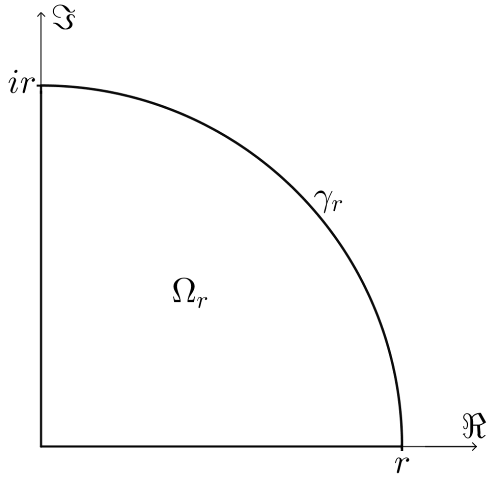

Figure A.1.: Contour integration along $\partial\Omega_r$

Sending r to infinity will yield an important correlation between (A.7) and (A.8). Therefore, we need to estimate the absolute value of $\int_{\gamma_r} z^{2s-2}e^{-z}dz$. By using polar coordinates $z = re^{i\varphi}$ and then changing variables to $t = \cos\varphi$, we estimate

$$\begin{aligned}
\left|\int_{\gamma_r} z^{2s-2}e^{-z}dz\right| &= \left|\int_0^{\pi/2} r^{2s-2}e^{i\varphi(2s-2)}e^{-re^{i\varphi}}ie^{i\varphi}d\varphi\right| \\
&= \left|\int_0^{\pi/2} r^{2s-1}e^{i(\varphi(2s-1)-r\sin\varphi)}e^{-r\cos\varphi}d\varphi\right| \\
&\leqslant r^{2s-1}\left|\int_0^{\pi/2} e^{-r\cos\varphi}d\varphi\right| = r^{2s-1}\left|\int_0^1 \frac{e^{-rt}}{\sqrt{1-t^2}}dt\right| \\
&\leqslant \frac{2r^{2s-1}}{\sqrt{3}}\left|\int_0^{1/2} e^{-rt}dt\right| + r^{2s-1}e^{-r/2}\left|\int_{1/2}^1 \frac{1}{\sqrt{1-t}}dt\right| \\
&\leqslant \frac{2}{\sqrt{3}r^{2-2s}}(e^{-r/2}-1) + r^{2s-1}e^{-r/2}\sqrt{2}.
\end{aligned}$$

The first term obviously tends to 0 as r goes to infinity. The same holds for the second term, though L'Hospital's rule is needed for $s \in (1/2, 1)$. Therefore, we have

$$\lim_{r\to\infty}\int_{\gamma_r} z^{2s-2}e^{-z}dz = 0.$$

Now we pass to the limit in (A.8) as r goes to infinity and obtain

$$\begin{aligned}
0 &= \int_0^\infty t^{2s-2}e^{-t}dt - i\int_0^\infty (it)^{2s-2}e^{-it}dt \\
&= \Gamma(2s-1) - i^{2s-1}\int_0^\infty t^{2s-2}e^{-it}dt,
\end{aligned}$$

which yields the identity

$$\int_0^\infty t^{2s-2}e^{-it}dt = i^{1-2s}\Gamma(2s-1).$$

By inserting this into (A.7) and using the fact that

$$\begin{aligned} i^{1-2s} &= \left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)^{1-2s} = \cos\left(\frac{\pi}{2}(1-2s)\right) + i\sin\left(\frac{\pi}{2}(1-2s)\right) \\ &= \cos\left(\frac{\pi}{2} - \pi s\right) + i\sin\left(\frac{\pi}{2} - \pi s\right) = \sin\left(\pi s\right) + i\cos\left(\pi s\right), \end{aligned}$$

we finally get

$$\begin{aligned} \int_0^\infty t^{2s-2}\sin(t)dt &= -\Im\left(\int_0^\infty t^{2s-2}e^{-it}dt\right) = -\Im\left(i^{1-2s}\Gamma(2s-1)\right) \\ &= -\Im\Big(\left(\sin\left(\pi s\right) + i\cos\left(\pi s\right)\right)\Big)\Gamma(2s-1) = -\cos(\pi s)\Gamma(2s-1). \end{aligned}$$

□

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