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Inf-Sup Stable Space-Time Methods for Time-Dependent Partial Differential Equations

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Abstract

In this work, space-time variational formulations and their discretisations with conforming, piecewise polynomial functions for the heat and wave equation are considered in a bounded space-time cylinder Q with a finite time T.

The main result for the heat equation is an unconditionally stable finite element method of Galerkin-Bubnov type with piecewise linear, continuous functions, which is based on a variational formulation in a subspace of an anisotropic Sobolev space. This space-time variational formulation is analysed with the help of Fourier series, and a kind of Hilbert transform is introduced. This leads to a symmetric and elliptic variational formulation and hence, to a symmetric Galerkin discretisation of the first-order time derivative. For the heat equation, unconditional stability for unstructured space-time meshes is proven. In addition, error estimates in $L^2(Q)$, in $H^1(Q)$ and in an anisotropic Sobolev norm for a tensor-product approach are derived. Finally, numerical examples, which confirm the theoretical results, are presented.

For the wave equation, a space-time variational formulation in a subspace of the Sobolev space $H^1(Q)$, which is not inf-sup stable, is used for a conforming space-time finite element method, which leads to a conditionally stable method, i.e. a CFL condition is required. For a tensor-product approach, a stabilised finite element method with piecewise linear, continuous functions is investigated, where unconditional stability in $L^2(Q)$ is proven. Furthermore, error estimates in $L^2(Q)$ and in $H^1(Q)$ are derived, and numerical examples, confirming the theoretical findings, are given. In addition, existence and uniqueness results for the wave equation as a partial differential equation in $L^2(Q)$ and in a weaker sense than $L^2(Q)$ are proven, including isomorphic solution operators and corresponding inf-sup conditions.

Zusammenfassung

In dieser Arbeit werden Raum-Zeit-Variationsformulierungen und deren Diskretisierung mittels konformer, stückweise polynomieller Funktionen für die Wärmeleitungsgleichung und Wellengleichung in einem beschränkten Raum-Zeit-Zylinder Q mit Endzeitpunkt T betrachtet.

Für die Wärmeleitungsgleichung ist das Hauptresultat eine unbedingt stabile Galerkin-Bubnov-Raum-Zeit-Finite-Element-Methode mit stückweise linearen, stetigen Funktionen basierend auf einer Raum-Zeit-Variationsformulierung, welche in einem Unterraum eines anisotropen Sobolevraums formuliert wird. Diese Raum-Zeit-Variationsformulierung wird mithilfe von Fourierreihen und einer Transformation, welche ähnlich zur Hilberttransformation ist, analysiert. Daraus ergeben sich eine symmetrische und elliptische Variationsformulierung und infolgedessen eine symmetrische Galerkin-Diskretisierung für die erste Zeitableitung. Für die Wärmeleitungsgleichung wird unbedingte Stabilität für unstrukturierte Raum-Zeit-Netze bewiesen. Weiters werden Fehlerabschätzungen in $L^2(Q)$, in $H^1(Q)$ und in einer anisotropen Sobolevnorm für einen Tensorproduktansatz hergeleitet. Schließlich werden numerische Beispiele, welche die theoretischen Ergebnisse bestätigen, angegeben.

Für die Wellengleichung ist der Ausgangspunkt eine Raum-Zeit-Variationsformulierung in Teilräumen des Sobolevraums $H^1(Q)$. Diese Raum-Zeit-Variationsformulierung ist nicht inf-sup-stabil. Die Diskretisierung dieser Raum-Zeit-Variationsformulierung mittels einer konformen Raum-Zeit-Finite-Element-Methode mithilfe von stückweise linearen, stetigen Funktionen führt zu einer bedingten Stabilität des Verfahrens. Das heißt, für die Stabilität muss eine CFL-Bedingung zwischen der Orts- und Zeitmaschenweite erfüllt sein. Um die CFL-Bedingung zu vermeiden, wird für einen Tensorproduktansatz eine stabilisierte Raum-Zeit-Finite-Element-Methode mittels stückweise linearer, stetiger Funktionen hergeleitet. Für diese Formulierung werden unbedingte Stabilität in $L^2(Q)$ sowie Fehlerabschätzungen in $L^2(Q)$ und in $H^1(Q)$ bewiesen. Weiters werden numerische Beispiele, welche die theoretischen Ergebnisse bestätigen, angegeben. Zu guter Letzt werden Existenz- und Eindeutigkeitssätze für die Wellengleichung als partielle Differentialgleichung im $L^2(Q)$ und in einem schwächeren Sinne als $L^2(Q)$ bewiesen. Die zugehörigen Lösungsoperatoren sind Isomorphismen, welche entsprechende inf-sup-Bedingungen garantieren.

Danksagung

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CONTENTS

1 INTRODUCTION

Standard approaches for the numerical solution of time-dependent partial differential equations are usually based on semi-discretisations in space and time, where the discretisation in space and time is split accordingly, see, e.g., [150] for parabolic partial differential equations, and [32, 33] for hyperbolic problems. Interpreting such approaches in a space-time sense, i.e. the time variable is considered as an additional spatial variable, these methods are related to tensor-product space-time methods, see, e.g., [16,48–50,85] for the parabolic case, and [19,22,55,86,164] for the hyperbolic one. An alternative is to discretise the time-dependent problem without separating the temporal and spatial variables, i.e. a space-time discretisation. This ansatz may lead to unstructured decompositions of the space-time domain. The approaches of unstructured meshes are considered, e.g., in [116, 142–144] for parabolic equations, and [42, 63, 111, 130, 140] for hyperbolic ones. More references are given in Chapter 3 for the parabolic equations, and in Chapter 4 for the hyperbolic problems. In general, the main advantages of space-time methods are space-time adaptivity, space-time parallelisation and the treatment of moving boundaries. At a first glance, a disadvantage is that a global linear system must be solved at once. Therefore, fast solvers and preconditioning are essential, which are not investigated in this work, see, e.g., [56]. In this thesis only direct solvers and the GMRES method are used. However, space-time approximation methods depend strongly on the space-time variational formulations on the continuous level. The focus of this thesis are space-time variational formulations for the heat and wave equation, which result not only in inf-sup stable formulations but fit also very well to conforming space-time methods with piecewise polynomial functions. In addition, these space-time variational formulations might be useful for variational formulations and their analysis of boundary integral equations and boundary element methods.

To motivate space-time approximation methods, space-time adaptivity is investigated in the case of a space-time interpolation and in the case of an adaptive space-time boundary element method for the spatially one-dimensional wave equation.

1.1 Space-Time Interpolation

For the approximation of a function $u(x,t)$ depending on a spatial variable $x \in \Omega \subset \mathbb{R}^d$, $d = 1,2,3$, and on a time variable $t \in (0,T) \subset \mathbb{R}$, where Ω is a bounded Lipschitz domain and $T > 0$ is a finite time, a better adaption of a sequence of arbitrary admissible and shape regular decompositions $(\mathcal{T}_N)_N$ of the space-time cylinder $Q := \Omega \times (0,T) \subset \mathbb{R}^{d+1}$ is possible in contrast to a tensor-product meshing of Q. As illustration, consider the rectangle

$$Q = (0,3) \times (0,6) \subset \mathbb{R}^2$$

with $\Omega = (0,3)$ and $T = 6$ for the $C^2(\overline{Q})$ function $u_1\colon \overline{Q} \to \mathbb{R}$,

$$u_1(x,t) = \begin{cases} \frac{1}{2}(t-x-2)^3(x-t)^3, & x \le t \text{ and } t-x \le 2, \\ 0, & \text{else}, \end{cases} \tag{1.1}$$

which is plotted in Figure 1.1, and for the piecewise smooth function $u_2\colon \overline{Q} \to \mathbb{R}$,

$$u_2(x,t) = \begin{cases} \frac{1}{2}|\sin(\pi(x-t))|, & x \le t, \\ 0, & \text{else}, \end{cases} \tag{1.2}$$

which is plotted in Figure 1.2.

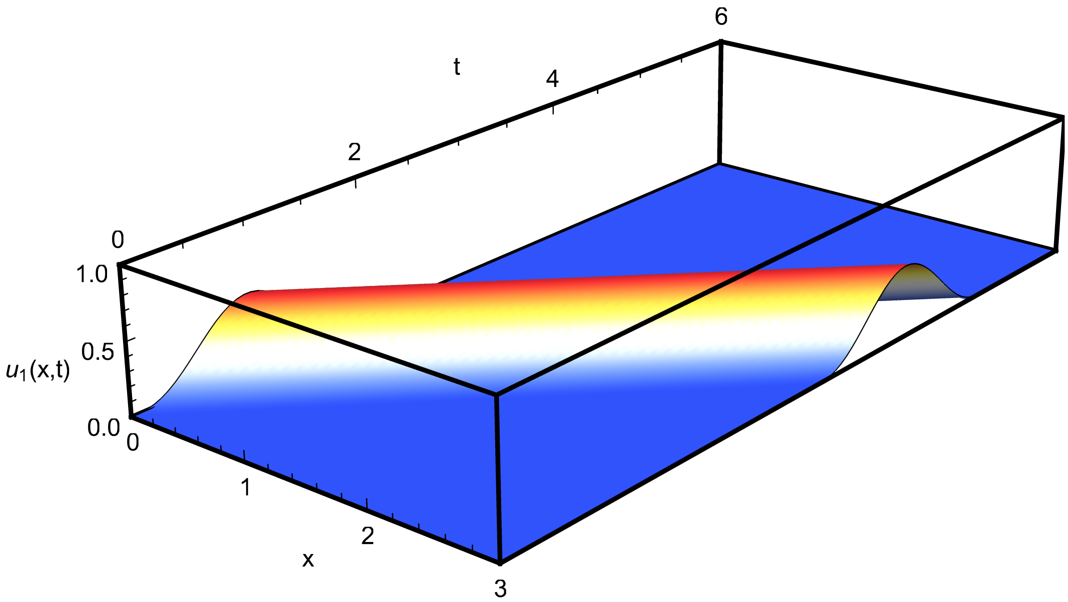

Figure 1.1: The smooth function u_1 of (1.1).

The given rectangle

$$\overline{Q} = \overline{\mathcal{T}_N} = \bigcup_{\ell=1}^{N} \overline{q}_\ell$$

is decomposed into N uniform space-time triangles $q_\ell \subset \mathbb{R}^2$ with mesh size h as given in Figure 1.3 for level 0, where $\tilde{M}$ is the number of vertices $\{(x_i,t_i)\}_{i=1}^{\tilde{M}}$. The finite-dimensional space $S_h^1(Q) = \operatorname{span}\{\psi_i\}_{i=1}^{\tilde{M}} \subset H^1(Q)$ is the space of piecewise linear, continuous functions on these space-time triangles with the nodal basis functions ψ_i, and the space-time interpolation operator $I_h\colon C(\overline{Q}) \to S_h^1(Q)$ is defined by

$$I_h v(x,t) = \sum_{i=1}^{\tilde{M}} v(x_i,t_i)\psi_i(x,t) \quad \text{for } (x,t) \in \overline{Q},$$

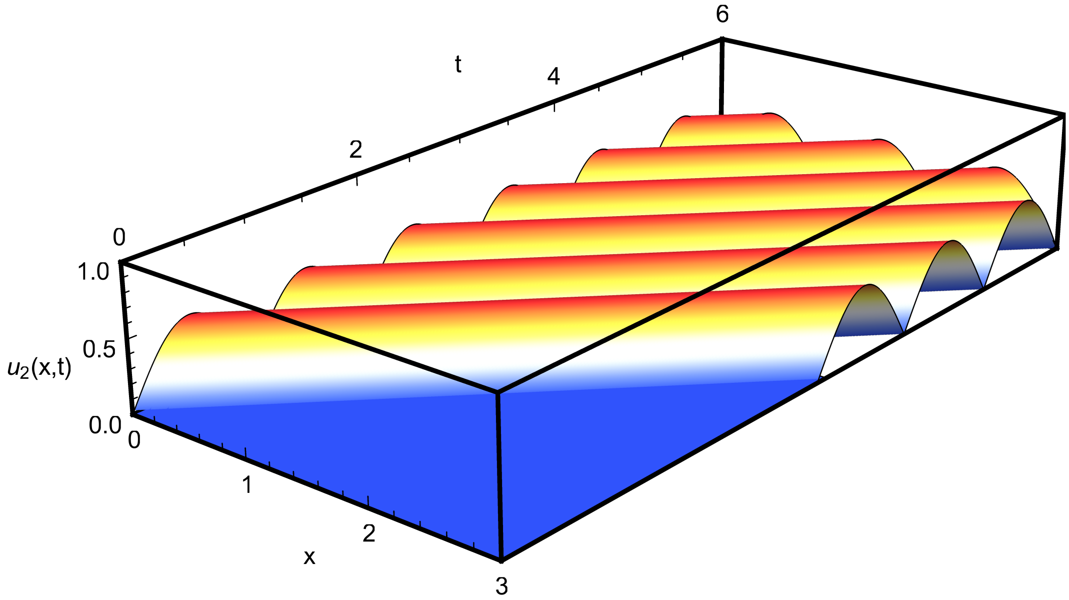

Figure 1.2: The piecewise smooth function u_2 of (1.2).

where $v \in C(\overline{Q})$ is a given continuous function, see Section 2.8 for details. Next, the interpolation errors in $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ for the functions u_1, u_2 are investigated for a sequence of uniform space-time meshes and for a sequence of adaptive space-time meshes. The uniform refinement strategy is depicted for the levels $0, 1, 2$ in Figure 1.3. As adaptive refinement strategy, Dörfler marking [41] with parameter $\theta = 0.5$ for the norm $\|\cdot\|_{L^2(Q)}$ is used.

For the smooth function u_1, the adaptive meshing is given in Figure 1.4. The uniform and the adaptive refinement strategies lead to optimal convergence rates with respect to $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$, see Table 1.1 and Figure 1.5. However, a comparison between the uniform and the adaptive schemes shows that the adaptive scheme needs considerably less degrees of freedom $\tilde{M}$ for the same accuracy of the errors.

For the piecewise smooth function u_2, the uniform refinement strategy results in reduced orders of convergence, see Table 1.2. With the adaptive refinement strategy, the optimal convergence rates are obtained, see Figure 1.7, and see Figure 1.6 for the meshes produced by the adaptive scheme.

To summarise, a main advantage of space-time methods is the space-time adaptivity, as depicted in Figure 1.4 and Figure 1.6, which is difficult to realise for standard approaches based on semi-discretisations, where the discretisation in space and time is split accordingly.

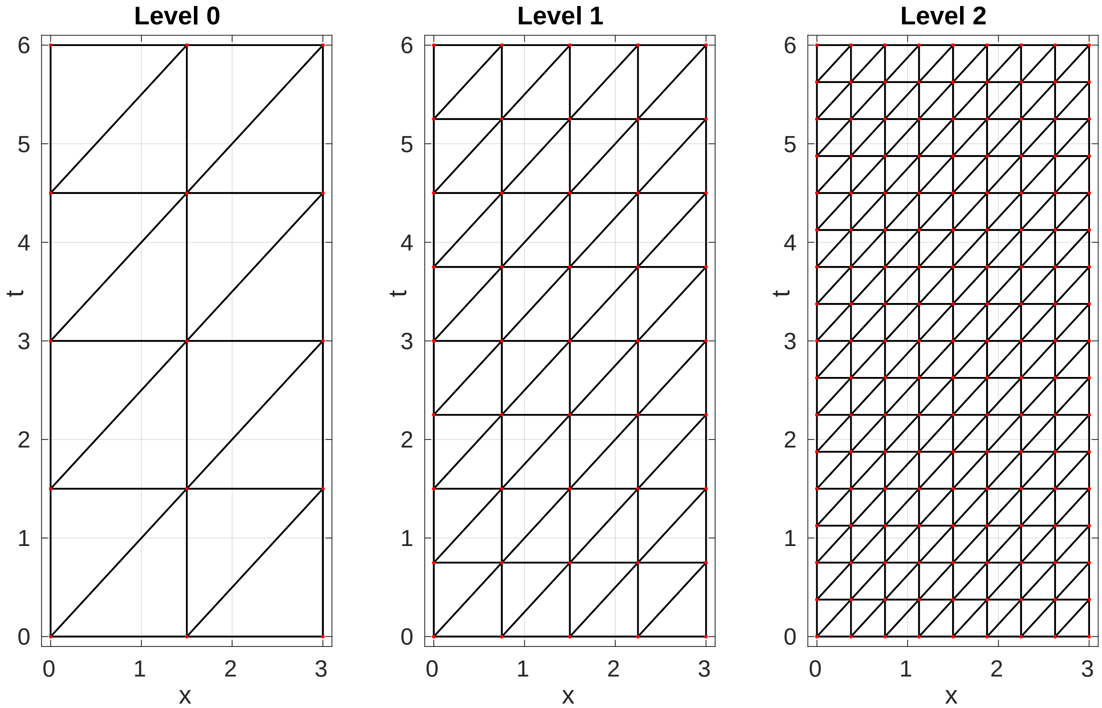

Figure 1.3: Uniform refinement strategy: Starting mesh, the meshes after one and two uniform refinement steps.

level	$\tilde{M}$	N	$\|u_1 - I_h u_1\|_{L^2(Q)}$	eoc	$\|u_1 - I_h u_1\|_{H^1(Q)}$	eoc
0	15	16	5.122e-01	-	1.947e+00	-
1	45	64	2.302e-01	1.46	1.397e+00	0.60
2	153	256	5.797e-02	2.25	6.973e-01	1.14
3	561	1024	1.477e-02	2.11	3.537e-01	1.04
4	2145	4096	3.744e-03	2.05	1.788e-01	1.02
5	8385	16384	9.386e-04	2.03	8.957e-02	1.01
6	33153	65536	2.348e-04	2.02	4.481e-02	1.01
7	131841	262144	5.872e-05	2.01	2.241e-02	1.00
8	525825	1048576	1.468e-05	2.00	1.121e-02	1.00
9	2100225	4194304	3.670e-06	2.00	5.603e-03	1.00
10	8394753	16777216	9.176e-07	2.00	2.801e-03	1.00
11	33566721	67108864	2.294e-07	2.00	1.401e-03	1.00
12	134242305	268435456	5.735e-08	2.00	7.003e-04	1.00

Table 1.1: Interpolation errors for the function u_1 of (1.1) for $Q = (0,3) \times (0,6)$ for a uniform refinement strategy with the meshes of Figure 1.3.

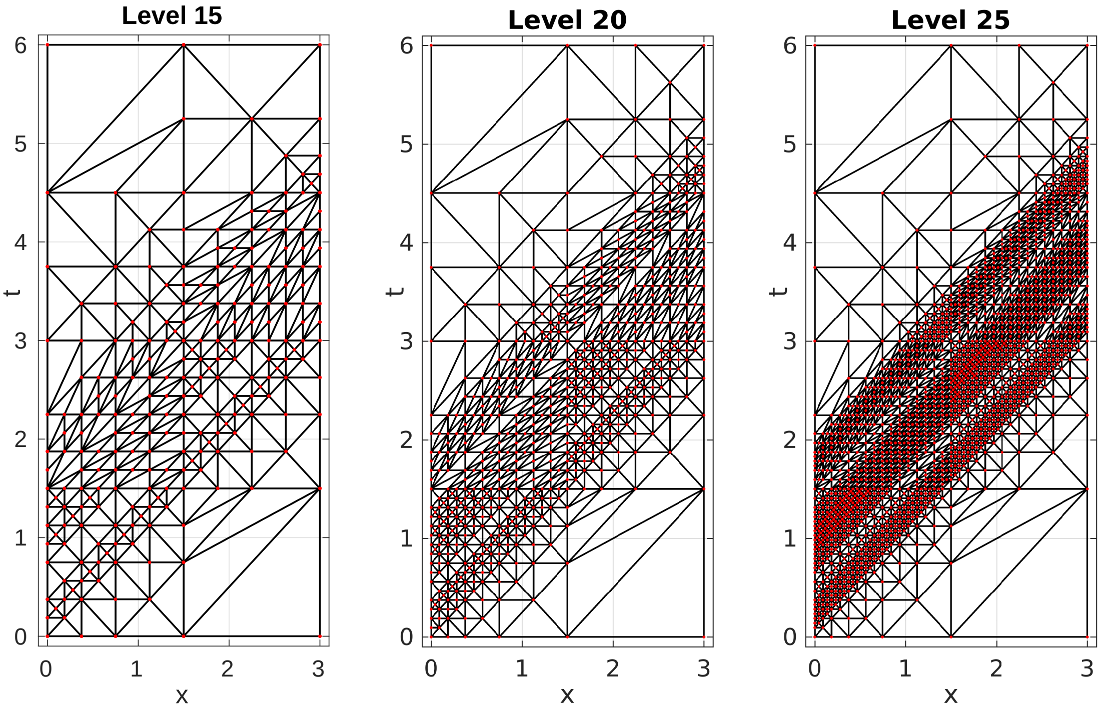

Figure 1.4: Adaptive refinement strategy for the function u_1 in (1.1).

level	$\tilde{M}$	N	$\|u_2 - I_h u_2\|_{L^2(Q)}$	eoc	$|u_2 - I_h u_2|_{H^1(Q)}$	eoc
0	15	16	7.373e-01	-	5.653e+00	-
1	45	64	7.423e-01	-0.01	5.437e+00	0.07
2	153	256	2.970e-01	1.50	4.021e+00	0.49
3	561	1024	1.044e-01	1.61	2.821e+00	0.55
4	2145	4096	3.613e-02	1.58	1.939e+00	0.56
5	8385	16384	1.257e-02	1.55	1.365e+00	0.51
6	33153	65536	4.404e-03	1.53	9.491e-01	0.53
7	131841	262144	1.549e-03	1.51	6.749e-01	0.49
8	525825	1048576	5.463e-04	1.51	4.717e-01	0.52
9	2100225	4194304	1.929e-04	1.50	3.364e-01	0.49
10	8394753	16777216	6.815e-05	1.50	2.355e-01	0.51
11	33566721	67108864	2.409e-05	1.50	1.681e-01	0.49
12	134242305	268435456	8.514e-06	1.50	1.177e-01	0.51

Table 1.2: Interpolation errors for the function u_2 in (1.2) for $Q = (0,3) \times (0,6)$ for a uniform refinement strategy with the meshes of Figure 1.3.

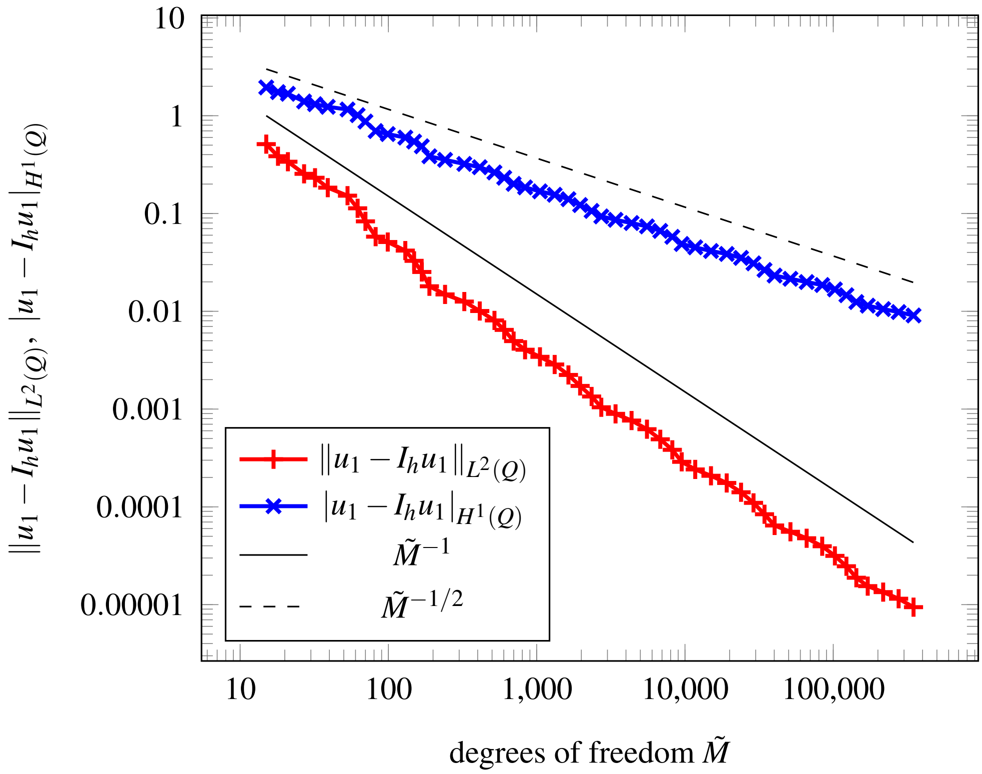

Figure 1.5: Interpolation errors for the function u_1 of (1.1) for $Q = (0,3) \times (0,6)$ for the adaptive refinement strategy with the meshes of Figure 1.4.

1.2 Boundary Element Method for the One-Dimensional Wave Equation

As a second example, an adaptive boundary element method for the spatially one-dimensional wave equation is investigated, see [161] for a summary. For details of the boundary element method, see [70, 131, 141]. As a model problem, consider the wave equation

$$\left.\begin{aligned} \partial_{tt} u(x,t) - \partial_{xx} u(x,t) &= 0 && \text{for } (x,t) \in Q = (0,L) \times (0,T), \\ u(x,t) &= g(x,t) && \text{for } (x,t) \in \Sigma = \{0,L\} \times [0,T], \\ u(x,0) = \partial_t u(x,0) &= 0 && \text{for } x \in (0,L), \end{aligned}\right\} \tag{1.3}$$

where g is a given Dirichlet datum and $L > 0$, $T > 0$. Define

$$L^2(\Sigma) := L^2(0,T) \times L^2(0,T) = \left\{ v = \begin{pmatrix} v_0 \\ v_L \end{pmatrix} : v_0 \in L^2(0,T),\, v_L \in L^2(0,T) \right\}$$

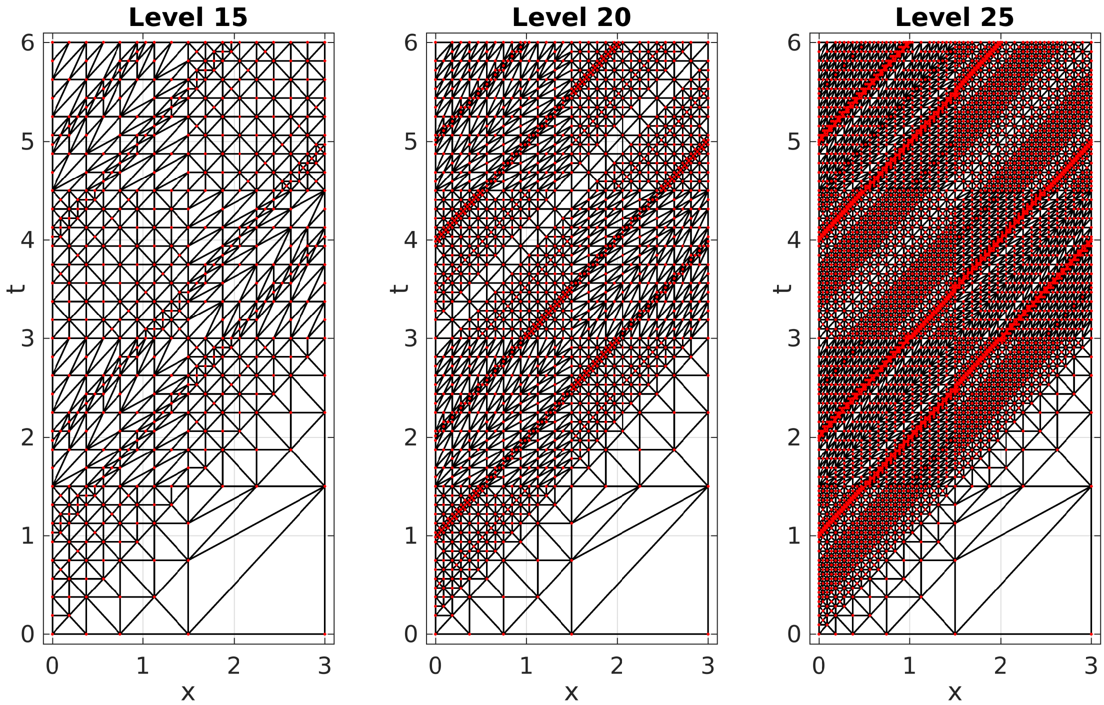

Figure 1.6: Adaptive refinement strategy for the function u_2 in (1.2).

with the inner product

$$\langle v,w\rangle_{L^2(\Sigma)} := \langle v_0,w_0\rangle_{L^2(0,T)} + \langle v_L,w_L\rangle_{L^2(0,T)} \quad \text{for } v,w\in L^2(\Sigma)$$

and introduce the Sobolev space

$$H^1_{0,}(\Sigma) := \left\{ v = \begin{pmatrix} v_0 \\ v_L \end{pmatrix} : v_0\in H^1(0,T),\, v_L\in H^1(0,T),\, v_0(0)=v_L(0)=0 \right\}$$

with the inner product

$$\langle v,w\rangle_{H^1_{0,}(\Sigma)} := \langle \partial_t v_0,\partial_t w_0\rangle_{L^2(0,T)} + \langle \partial_t v_L,\partial_t w_L\rangle_{L^2(0,T)} \quad \text{for } v,w\in H^1_{0,}(\Sigma),$$

see Section 2.2 for more details. In general, for $w\in L^2(0,T)$, set $w(t):=0$ for $t<0$ or $t>T$. The solution $u(x,t)$ of the wave equation (1.3) admits the representation

$$u = \tilde{\mathcal{V}}\partial_n u - \mathcal{W}g \quad \text{in } Q \tag{1.4}$$

with the single layer potential $\tilde{\mathcal{V}}$ and the double layer potential $\mathcal{W}$, where $\partial_n u$ denotes the unknown normal derivative of u on Σ. The single layer potential $\tilde{\mathcal{V}}$ is defined by

$$\tilde{\mathcal{V}}w(x,t) = \frac{1}{2}\int_0^{t-|x|} w_0(s)\,\mathrm{d}s + \frac{1}{2}\int_0^{t-|x-L|} w_L(s)\,\mathrm{d}s \quad \text{for } t\in[0,T],\, x\in(0,L)$$

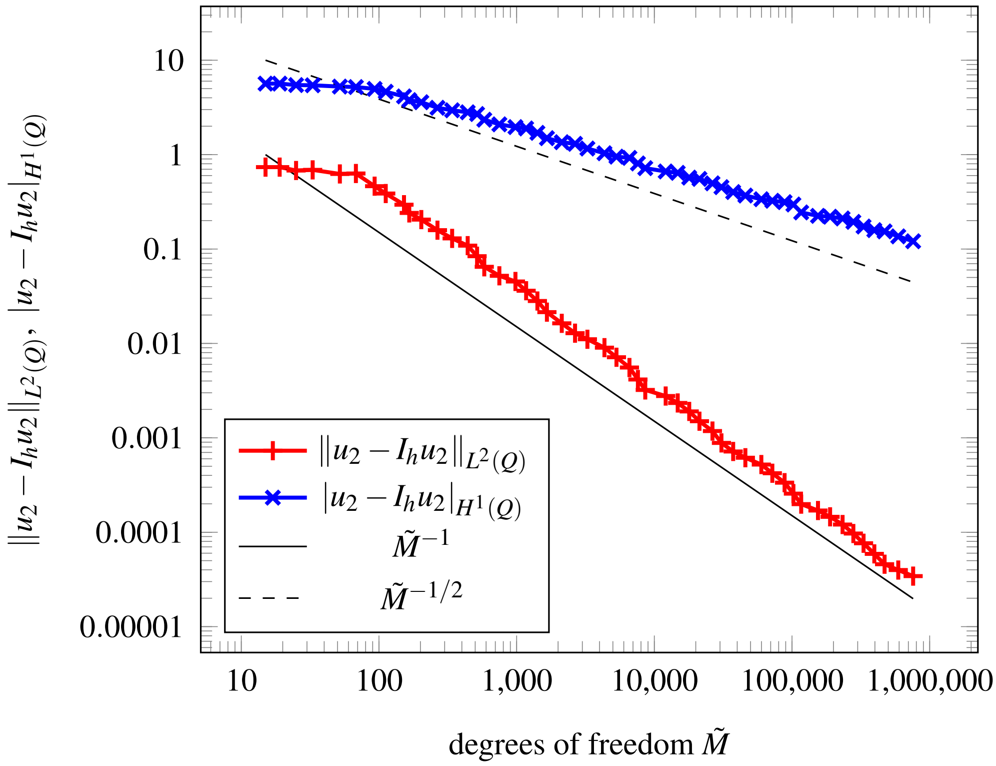

Figure 1.7: Interpolation errors for the function u_2 in (1.2) for $Q = (0,3) \times (0,6)$ for the adaptive refinement strategy with the meshes of Figure 1.6.

for a density $w = (w_0, w_L)^\top \in L^2(\Sigma)$ with $w = 0$ outside of Σ. The single layer operator $\mathcal{V}\colon\ L^2(\Sigma) \to H^1_{0,}(\Sigma)$ is given by

$$\mathcal{V}w(t) := \frac{1}{2} \begin{pmatrix} \int\limits_0^t w_0(s)\mathrm{d}s + \int\limits_0^{t-L} w_L(s)\mathrm{d}s \\ \int\limits_0^{t-L} w_0(s)\mathrm{d}s + \int\limits_0^t w_L(s)\mathrm{d}s \end{pmatrix}, \quad t \in [0,T],$$

for a density $w \in L^2(\Sigma)$. Hence, it holds $\partial_t(\mathcal{V}w) \in L^2(\Sigma)$, i.e. $\partial_t\mathcal{V}\colon\ L^2(\Sigma) \to L^2(\Sigma)$. In [7], ellipticity and boundedness in $L^2(\Sigma)$ of the bilinear form $a_E(\cdot,\cdot)\colon\ L^2(\Sigma) \times L^2(\Sigma) \to \mathbb{R}$,

$$a_E(w,v) := \langle \partial_t \mathcal{V}w, v\rangle_{L^2(\Sigma)} = \langle (\partial_t \mathcal{V}w)_0, v_0\rangle_{L^2(0,T)} + \langle (\partial_t \mathcal{V}w)_L, v_L\rangle_{L^2(0,T)}$$

for w, $v \in L^2(\Sigma)$, are proven. Therefore, the following variational formulation is uniquely solvable:

Find $\partial_n u \in L^2(\Sigma)$ for a given $g \in H^1_{0,}(\Sigma)$ such that

$$a_E(\partial_n u, v) = \frac{1}{2}\langle \partial_t g, v\rangle_{L^2(\Sigma)} + \langle \partial_t(\mathcal{K}g), v\rangle_{L^2(\Sigma)} \quad \forall v \in L^2(\Sigma), \tag{1.5}$$

where the double layer operator $\mathcal{K}$ is given for $g = 0$ outside of Σ by

$$\mathcal{K}g(t) = \mathcal{K}\begin{pmatrix} g_0 \\ g_L \end{pmatrix}(t) = -\frac{1}{2}\begin{pmatrix} g_L(t-L) \\ g_0(t-L) \end{pmatrix} \qquad \text{for } t \in [0,T].$$

For a boundary element approximation, consider a decomposition of the lateral boundary

$$\Sigma = \bigcup_{i=1}^{N_0+N_L} \overline{\tau}_i$$

into $N_0 + N_L$ boundary elements τ_i with maximal mesh size $h = \max_i |\tau_i|$, where N_0 is the number of boundary elements for $x = 0$ and N_L is the number of boundary elements for $x = L$. The conforming ansatz space of piecewise constant functions

$$S_h^0(\Sigma) := S_{h_0}^0(0,T) \times S_{h_L}^0(0,T) = \operatorname{span}\{\hat{\varphi}_i^0\}_{i=1}^{N_0+N_L} \subset L^2(\Sigma)$$

is used to define an approximate solution $w_h \in S_h^0(\Sigma)$. Then the discretisation of (1.5) to find $w_h \in S_h^0(\Sigma) \subset L^2(\Sigma)$ such that

$$a_E(w_h, v_h) = \frac{1}{2}\langle \partial_t g, v_h\rangle_{L^2(\Sigma)} + \langle \partial_t(\mathcal{K}g), v_h\rangle_{L^2(\Sigma)} \quad \forall v_h \in S_h^0(\Sigma) \subset L^2(\Sigma) \tag{1.6}$$

is equivalent to the global linear system

$$V_h \underline{w} = \underline{g}$$

with the related system matrix $V_h \in \mathbb{R}^{(N_0+N_L)\times(N_0+N_L)}$, the right-hand side $\underline{g} \in \mathbb{R}^{N_0+N_L}$ and the vector of unknown coefficients $\underline{w} \in \mathbb{R}^{N_0+N_L}$ of $w_h \in S_h^0(\Sigma)$. Note that the system matrix V_h and the discretisation of the double layer operator $\mathcal{K}$ are calculated analytically, whereas all other appearing integrals are computed by the usage of high-order integration rules. Since the bilinear form $a_E(\cdot,\cdot)\colon L^2(\Sigma) \times L^2(\Sigma) \to \mathbb{R}$ is bounded and elliptic, the discrete variational formulation (1.6) is uniquely solvable and unconditionally stable. By Céa's Lemma and standard error estimates, there follows the a priori estimate

$$\|\partial_n u - w_h\|_{L^2(\Sigma)} \leq C h^s \|\partial_n u\|_{H^s(\Sigma)} \tag{1.7}$$

for some $s \in [0,1]$ and a constant $C > 0$, where $H^s(\Sigma) = H^s(0,T) \times H^s(0,T)$, see Section 2.2. An approximate solution $\tilde{u}_h \approx u$ in the space-time cylinder Q is given by inserting the approximate normal derivative $w_h \approx \partial_n u$ into the representation formula (1.4), i.e.

$$\tilde{u}_h := \tilde{\mathcal{V}} w_h - \mathcal{W} g \quad \text{in } Q. \tag{1.8}$$

To derive an adaptive mesh refinement on Σ, an a posteriori error estimator [136] is used, which is based on the application of the normal derivative ∂_n to the approximate representation formula (1.8),

$$\tilde{w}_h := \partial_n \tilde{u}_h = \frac{1}{2} w_h + \mathcal{K}' w_h + \mathcal{D} g \quad \text{on } \Sigma.$$

Here, the adjoint double layer operator $\mathcal{K}'$ and the hypersingular boundary integral operator $\mathcal{D}$ are used. Hence, the local error estimators

$$\tilde{\eta}_i := \|\tilde{w}_h - w_h\|_{L^2(\tau_i)} \approx \|\partial_n u - w_h\|_{L^2(\tau_i)}$$

for $i = 1, \dots, N_0 + N_L$ are computable, where the adjoint double layer operator $\mathcal{K}'$ and the hypersingular boundary integral operator $\mathcal{D}$ are calculated analytically. For an adaptive refinement strategy, a parameter $\theta \in [0,1]$ is chosen and all elements τ_i are refined, where

$$\tilde{\eta}_i \geq \theta \max_j \tilde{\eta}_j. \tag{1.9}$$

As numerical examples, consider $L = 3$ and $T = 6$, i.e. $Q = (0,3) \times (0,6)$, for the exact solutions u_1 and u_2, which are given in (1.1) and in (1.2), with the smooth Dirichlet datum $g_1 := u_{1|\Sigma} \in H^1_{0,}(\Sigma)$ and the piecewise smooth Dirichlet datum $g_2 := u_{2|\Sigma} \in H^1_{0,}(\Sigma)$. In the case of the smooth Dirichlet datum $g_1 = u_{1|\Sigma}$, the optimal order of convergence, i.e. $s = 1$ for the error estimate (1.7), is achieved by a uniform refinement strategy, see Table 1.3, and by the adaptive refinement strategy (1.9), see Figure 1.8. Furthermore, the $L^2(Q)$ error for the approximate solution (1.8) is given in Table 1.3. However, in the case of the piecewise smooth Dirichlet datum $g_2 = u_{2|\Sigma}$, only reduced orders of convergence are obtained, when using a uniform refinement strategy, see Table 1.4 and Figure 1.9. Note that the full order of convergence is attained for the adaptive refinement strategy (1.9), see Figure 1.9. A resulting sequence of adaptive meshes is depicted in Figure 1.10, where different decompositions for $x = 0$ and $x = 3$ are used, i.e. a decomposition without time slabs.

Remark 1.2.1. *Acoustic scattering problems are often formulated in exterior domains, i.e. in an unbounded domain. The boundary element method is suited very well for such scattering problems, since only a meshing of the surface of the bounded interior domain is needed. The starting point of the boundary element method is the corresponding boundary integral equation. The standard approach of boundary integral equations for the wave equation uses the Laplace transform with respect to the time variable, see [20, 21, 71–73, 87, 132]. This Laplace transform method results in space-time variational formulations, where the related bilinear form is bounded and elliptic in different norms, i.e. the Lax-Milgram Theorem is not applicable in the space-time domain and related error estimates for a boundary element method are not optimal. See also [1, 57–62, 70, 121, 155, 156] for recent developments in this direction. In [6, 8, 69], an*

level	N_0+N_L	$\|\partial_n u_1 - w_{1,h}\|_{L^2(\Sigma)}$	eoc	$\|u_1 - \tilde{u}_{1,h}\|_{L^2(Q)}$	eoc
0	2	1.33823e+00	-	7.26168e-01	-
1	4	1.14684e+00	0.22	4.49627e-01	0.69
2	8	1.10072e+00	0.06	3.24885e-01	0.47
3	16	8.06608e-01	0.45	1.46335e-01	1.15
4	32	4.02738e-01	1.00	3.75263e-02	1.96
5	64	2.04198e-01	0.98	9.55880e-03	1.97
6	128	1.03212e-01	0.98	2.49973e-03	1.94
7	256	5.17114e-02	1.00	5.95579e-04	2.07
8	512	2.58723e-02	1.00	1.56495e-04	1.93
9	1024	1.29381e-02	1.00	3.71371e-05	2.08
10	2048	6.46928e-03	1.00	9.78807e-06	1.92
11	4096	3.23467e-03	1.00	2.40515e-06	2.02
12	8192	1.61734e-03	1.00	5.99227e-07	2.00
13	16384	8.08670e-04	1.00	1.49398e-07	2.00
14	32768	4.04335e-04	1.00	3.77370e-08	1.99

Table 1.3: Numerical results for the boundary element method (1.6) for the function u_1 in (1.1) for $Q = (0,3) \times (0,6)$ for a uniform refinement strategy.

level	N_0+N_L	$\|\partial_n u_2 - w_{2,h}\|_{L^2(\Sigma)}$	eoc	$\|u_2 - \tilde{u}_{2,h}\|_{L^2(Q)}$	eoc
0	2	3.95477e+00	-	2.59835e+00	-
1	4	3.33217e+00	0.25	5.78383e-01	2.17
2	8	3.11643e+00	0.10	4.73586e-01	0.29
3	16	3.16575e+00	-0.02	4.10036e-01	0.21
4	32	2.37997e+00	0.41	1.77812e-01	1.21
5	64	1.66423e+00	0.52	6.10341e-02	1.54
6	128	1.15613e+00	0.53	2.28464e-02	1.42
7	256	8.07589e-01	0.52	8.15019e-03	1.49
8	512	5.67073e-01	0.51	3.29593e-03	1.31
9	1024	3.99491e-01	0.51	1.33215e-03	1.31
10	2048	2.81940e-01	0.50	5.98854e-04	1.15
11	4096	1.99168e-01	0.50	2.74489e-04	1.13
12	8192	1.40764e-01	0.50	1.32773e-04	1.05
13	16384	9.95104e-02	0.50	6.47539e-05	1.04
14	32768	7.03558e-02	0.50	3.21587e-05	1.01

Table 1.4: Numerical results for the boundary element method (1.6) for the function u_2 in (1.2) for $Q = (0,3) \times (0,6)$ for a uniform refinement strategy.

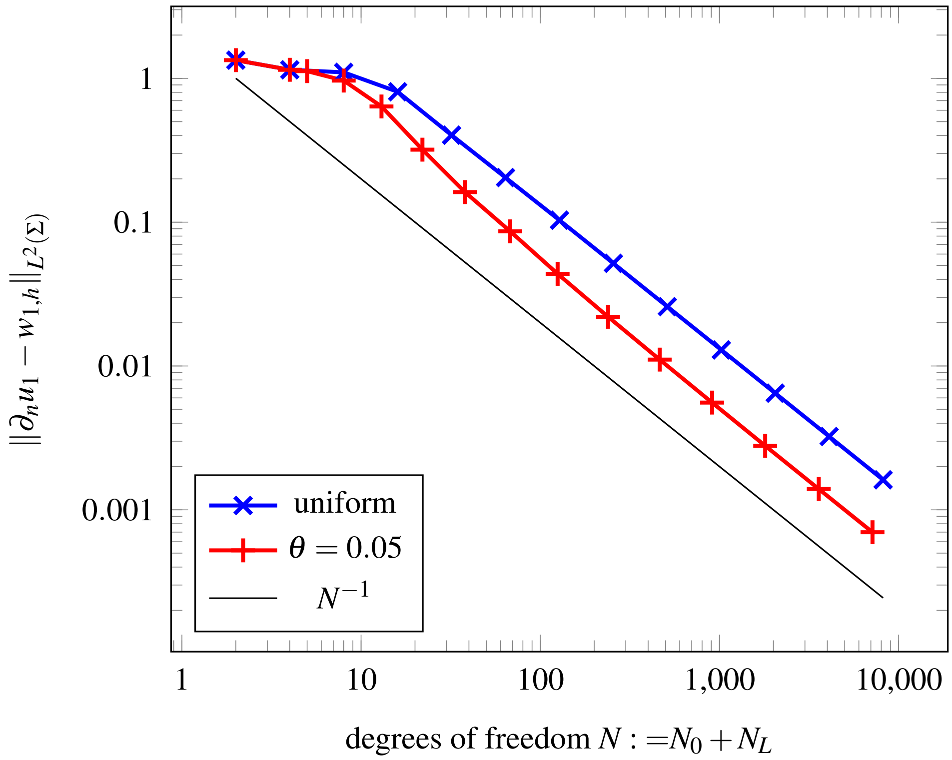

Figure 1.8: Numerical results for the boundary element method (1.6) for the function u_1 in (1.1) for $Q = (0,3) \times (0,6)$ for the adaptive refinement strategy (1.9).

approach without the Laplace transform is considered for a screen problem in a two-dimensional spatial domain, see also [2–5] for further investigations. In addition, the work [77, 132] examines the boundary integral equations via semigroup theory and their discretisations via the convolution quadrature method [105, 106], see also [23], and [104] for a generalisation to variable time stepping. Note that this list of references is highly non-exhaustive. However, a complete analysis of space-time variational formulations for boundary integral equations of the wave equation seems to be still open. This motivates the investigations of space-time variational formulations for the wave equation in the interior and exterior of the space-time domain, see Chapter 4, since variational formulations of boundary integral equations are highly related to the variational formulations within the domain.

To summarise Section 1.1 and Section 1.2, one main advantage of space-time approximation methods, i.e. the space-time adaptivity, is realisable and leads to significantly lower numbers of the degrees of freedom for achieving a desired accuracy.

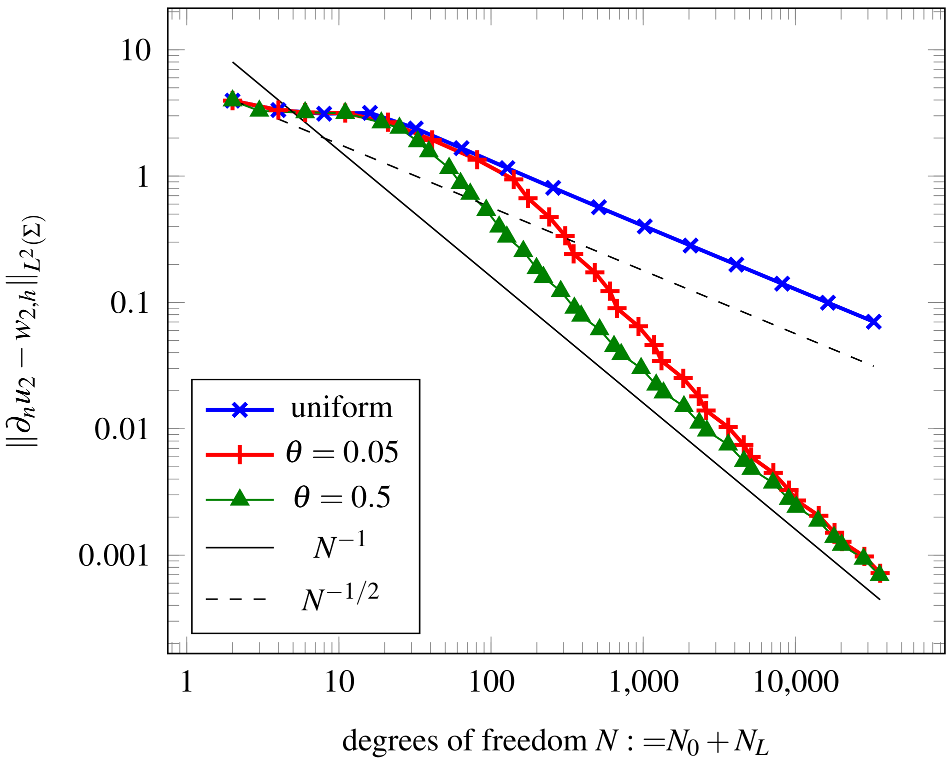

Figure 1.9: Numerical results for the boundary element method (1.6) for the function u_2 in (1.2) for $Q = (0,3) \times (0,6)$ for the adaptive refinement strategy (1.9).

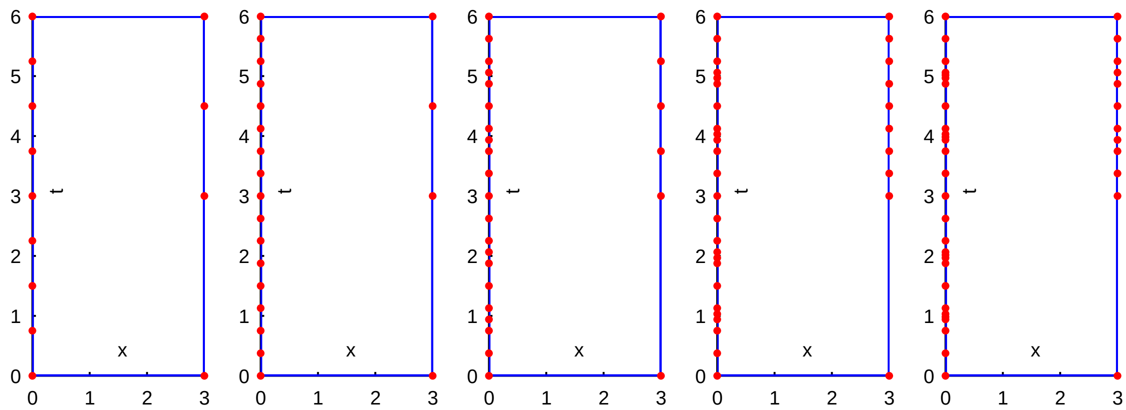

Figure 1.10: A sequence of adaptive meshes for the adaptive refinement strategy (1.9) for the function u_2 in (1.2).

Outline

The rest of this thesis is organised as follows: In Chapter 2 notations of distributions, Sobolev spaces and discretisation methods are fixed and their most important properties are repeated. In Chapter 3 the heat equation is examined, whereas in Chapter 4 the wave equation is investigated, where a more extensive overview of the literature and an outline of the different sections are given at the beginning of each chapter. In Chapter 5 a short summary of this thesis and an outlook for future work are given.

2 PRELIMINARIES

In this chapter, notations for function spaces, distributions and discretisation schemes are introduced and their most important properties are repeated. Furthermore, a short summary for variational methods is given.

In the whole thesis, $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded Lipschitz domain and $(0,T)$ is a time interval with the finite time $T > 0$. The bounded space-time cylinder is defined as $Q := \Omega \times (0,T) \subset \mathbb{R}^{d+1}$, $\Sigma := \partial\Omega \times [0,T] \subset \mathbb{R}^{d+1}$ is the lateral boundary, and $\Sigma_0 := \Sigma \cup \Omega_0$ with $\Omega_0 := \Omega \times \{0\}$, $\Sigma_T := \Sigma \cup \Omega_T$ with $\Omega_T := \Omega \times \{T\}$ are parts of the boundary ∂Q of the space-time cylinder Q.

2.1 Distributions

As a reference for the theory of distributions see, e.g., [68, 138, 152]. In this work, $C_0^\infty(Q)$ is the set of infinitely differentiable real-valued functions with compact support in Q. The set $C_0^\infty(Q)$ endowed with the, usual for distributions, locally convex topology is denoted by $\mathcal{D}(Q)$ and is called the space of test functions on Q. The set of (Schwartz) distributions $\mathcal{D}'(Q)$ is given by all linear and sequentially continuous functionals on $\mathcal{D}(Q)$, see [138]. For a locally integrable function $v \in L^1_{\text{loc}}(Q)$, the distribution $T_v \colon \mathcal{D}(Q) \to \mathbb{R}$, defined by

$$T_v(\varphi) := \int_Q v(x,t)\varphi(x,t)\,\mathrm{d}x\mathrm{d}t \qquad \text{for all } \varphi \in \mathcal{D}(Q),$$

is associated uniquely with that function $v \in L^1_{\text{loc}}(Q)$. Hence, the function $v \in L^1_{\text{loc}}(Q)$ and the related distribution $T_v \colon \mathcal{D}(Q) \to \mathbb{R}$ are identified. Throughout this work, $\Box := \partial_{tt} - \Delta_x$ denotes the classical (pointwise) derivative for sufficiently smooth functions. Furthermore, let $\Box_Q \colon \mathcal{D}'(Q) \to \mathcal{D}'(Q)$ be the distributional wave operator for distributions $\mathcal{D}'(Q)$, where for a distribution $T \colon \mathcal{D}(Q) \to \mathbb{R}$, derivatives are defined as usual:

$$\Box_Q T(\varphi) = T(\Box\varphi) \quad \forall \varphi \in \mathcal{D}(Q).$$

In particular, for $Q_- := \Omega \times (-\infty,T) \subset \mathbb{R}^{d+1}$, let $\Box_{Q_-} \colon \mathcal{D}'(Q_-) \to \mathcal{D}'(Q_-)$ be the distributional wave operator for distributions $\mathcal{D}'(Q_-)$.

The sets $C_0^\infty(0,T)$, $C_0^\infty(\Omega)$ and the spaces of test functions $\mathcal{D}(0,T)$, $\mathcal{D}(\Omega)$ are introduced analogously.

2.2 Sobolev Spaces in $(0,T)$

For an introduction to Sobolev spaces on intervals, see the references in the Section 2.3 and in addition, see [13, Kapitel 5] or [96, Chapter 8]. With the usual notations, the Hilbert space $H^s(0,T)$, $s \geq 0$, is the Sobolev space of real-valued functions endowed with the Sobolev-Slobodeckij inner product $\langle \cdot,\cdot \rangle_{H^s(0,T)}$ and the induced norm $\|\cdot\|_{H^s(0,T)}$. Analogously, $H^s(\mathbb{R})$ is the usual Sobolev space on the whole real line for $s \geq 0$. Note that $H^s(0,T) \subset C[0,T]$ for $s > 1/2$, see [64, (1.4.4.6), page 27]. Hence, for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$ one defines the subspaces

$$
\begin{aligned}
H^s_{0,}(0,T) &:= \{v \in H^s(0,T) \colon\ v(0) = 0\}, \\
H^s_{,0}(0,T) &:= \{v \in H^s(0,T) \colon\ v(T) = 0\}.
\end{aligned}
$$

In particular, for $s = 1$ the Sobolev spaces $H^1_{0,}(0,T)$ and $H^1_{,0}(0,T)$ are endowed with the inner products

$$
\langle u,v \rangle_{H^1_{0,}(0,T)} := \langle u,v \rangle_{H^1_{,0}(0,T)} := \int_0^T \partial_t u(t) \partial_t v(t) \mathrm{d}t,
$$

and with the induced norm

$$
|u|_{H^1(0,T)} := \|\partial_t u\|_{L^2(0,T)} = \sqrt{\int_0^T |\partial_t u(t)|^2 \,\mathrm{d}t}.
$$

For $s = 1/2$, one defines via function space interpolation the Sobolev space

$$
H^{1/2}_{0,}(0,T) := [H^1_{0,}(0,T), L^2(0,T)]_{1/2}
$$

with the Hilbertian norm

$$
\|u\|_{H^{1/2}_{0,}(0,T)} := \sqrt{\|u\|^2_{H^{1/2}(0,T)} + \int_0^T \frac{|u(t)|^2}{t} \mathrm{d}t},
$$

which is equivalent to the interpolation norm $\|\cdot\|_{[H^1_{0,}(0,T),L^2(0,T)]_{1/2}}$, see [102, Théorème 11.7, page 72] and [102, Remarque 11.4, page 75]. Analogously, set

$$
H^{1/2}_{,0}(0,T) := [H^1_{,0}(0,T), L^2(0,T)]_{1/2}
$$

with the Hilbertian norm

$$
\|u\|_{H^{1/2}_{,0}(0,T)} := \sqrt{\|u\|^2_{H^{1/2}(0,T)} + \int_0^T \frac{|u(t)|^2}{T-t} \mathrm{d}t}.
$$

There hold the representations

$$H^{1/2}_{0,}(0,T) = \Big\{U_{|(0,T)} \colon\ U \in H^{1/2}(-\infty,T) \text{ with } U(t) = 0 \text{ for } t < 0\Big\}$$
$$= \left\{u \in H^{1/2}(0,T) \colon\ \int\limits_0^T \frac{|u(t)|^2}{t}\mathrm{d}t < \infty\right\} \tag{2.1}$$

and

$$H^{1/2}_{,0}(0,T) = \Big\{U_{|(0,T)} \colon\ U \in H^{1/2}(0,\infty) \text{ with } U(t) = 0 \text{ for } t > T\Big\}$$
$$= \left\{u \in H^{1/2}(0,T) \colon\ \int\limits_0^T \frac{|u(t)|^2}{T-t}\mathrm{d}t < \infty\right\}, \tag{2.2}$$

see [102, Proposition 5.2, page 276] and [102, Remarque 11.4, page 75]. Because the test functions $C_0^\infty(0,T)$ are dense in $H^{1/2}(0,T)$, see [64, Theorem 1.4.2.4, page 25], the sets $H^{1/2}_{0,}(0,T)$ and $H^{1/2}_{,0}(0,T)$ are dense in $H^{1/2}(0,T)$. Note that the constant function $\mathbf{1}(t) := 1$ for $t \in (0,T)$ fulfils $\mathbf{1} \in H^{1/2}(0,T)$, $\mathbf{1} \notin H^{1/2}_{0,}(0,T)$ and $\mathbf{1} \notin H^{1/2}_{,0}(0,T)$ due to the representations (2.1) and (2.2). Because the set

$$C_0^\infty(0,T] = \big\{\varphi_{|(0,T]} \colon\ \varphi \in C_0^\infty(0,\infty)\big\}$$

is dense in $H^1_{0,}(0,T)$, it follows by interpolation arguments that the set $C_0^\infty(0,T]$ is dense in $H^{1/2}_{0,}(0,T)$, see [102, Chapitre 1, Section 2.1, page 11], and analogously, the set

$$C_0^\infty[0,T) = \big\{\varphi_{|[0,T)} \colon\ \varphi \in C_0^\infty(-\infty,T)\big\}$$

is dense in $H^{1/2}_{,0}(0,T)$. It even holds that the set $C_0^\infty(0,T)$ is dense in $H^{1/2}_{0,}(0,T)$ and in $H^{1/2}_{,0}(0,T)$, see Theorem 2.2.2.

Lemma 2.2.1. *The norm* $|||\cdot|||_{H^{1/2}_{0,}(0,T)}$*, defined by*

$$|||u|||_{H^{1/2}_{0,}(0,T)} := \inf\Big\{\|U\|_{H^{1/2}(\mathbb{R})} \colon U \in H^{1/2}(\mathbb{R}) \textit{ with } U_{|(0,T)} = u, U(t) = 0 \textit{ for } t < 0\Big\}$$

for $u \in H^{1/2}_{0,}(0,T)$*, is equivalent to* $\|\cdot\|_{H^{1/2}_{0,}(0,T)}$*. Analogously, the norm* $|||\cdot|||_{H^{1/2}_{,0}(0,T)}$*, defined by*

$$|||u|||_{H^{1/2}_{,0}(0,T)} := \inf\Big\{\|U\|_{H^{1/2}(\mathbb{R})} \colon U \in H^{1/2}(\mathbb{R}) \textit{ with } U_{|(0,T)} = u, U(t) = 0 \textit{ for } t > T\Big\}$$

for $u \in H^{1/2}_{,0}(0,T)$*, is equivalent to* $\|\cdot\|_{H^{1/2}_{,0}(0,T)}$*.*

Proof. The proof is given only for the case of $H^{1/2}_{,0}(0,T)$, the other case is proven analogously. By interpolation arguments, see [102, Proof of Proposition 5.2, page 276], the extension operator $\mathcal{E}_T\colon H^{1/2}_{,0}(0,T) \to H^{1/2}(0,\infty)$, defined by

$$\mathcal{E}_T u(t) := \begin{cases} u(t), & t \in (0,T), \\ 0, & t \geq T, \end{cases}$$

for $u \in H^{1/2}_{,0}(0,T)$, and the restriction operator $\mathcal{R}_T\colon H^{1/2}(0,\infty) \to H^{1/2}_{,0}(0,T)$, given by

$$\mathcal{R}_T U(t) := U(t) - U(2T-t), \quad t \in (0,T),$$

for $U \in H^{1/2}(0,\infty)$, are bounded. For $u \in H^{1/2}_{,0}(0,T)$, it holds $\mathcal{R}_T\mathcal{E}_T u = u$ and so, it follows

$$\|u\|_{H^{1/2}_{,0}(0,T)} = \|\mathcal{R}_T\mathcal{E}_T u\|_{H^{1/2}_{,0}(0,T)} \leq C_{\mathcal{R}_T}\|\mathcal{E}_T u\|_{H^{1/2}(0,\infty)} \leq C_{\mathcal{R}_T}C_{\mathcal{E}_T}\|u\|_{H^{1/2}_{,0}(0,T)},$$

i.e. the norms $\|\cdot\|_{H^{1/2}_{,0}(0,T)}$ and $u \mapsto \|\mathcal{E}_T u\|_{H^{1/2}(0,\infty)}$ are equivalent. Since $\|\cdot\|_{H^{1/2}(0,\infty)}$ is equivalent to the norm

$$w \mapsto \inf\left\{\|U\|_{H^{1/2}(\mathbb{R})} : U \in H^{1/2}(\mathbb{R}) \text{ with } U_{|(0,\infty)} = w\right\} \quad \text{for } w \in H^{1/2}(0,\infty),$$

see [160, Satz 5.3, page 100] with a natural extension by reflection in $t = 0$, the assertion follows. □

Theorem 2.2.2. *The set $C_0^\infty(0,T)$ is dense in $H^{1/2}_{0,}(0,T)$ and $H^{1/2}_{,0}(0,T)$.*

Proof. The proof is given for $H^{1/2}_{,0}(0,T)$. Because of the density of the set $C_0^\infty[0,T)$ in $H^{1/2}_{,0}(0,T)$, it remains to prove the density of $C_0^\infty(0,T)$ in $C_0^\infty[0,T)$ with respect to the norm $\|\cdot\|_{H^{1/2}_{,0}(0,T)}$. Therefore, fix an element $\varphi \in C_0^\infty[0,T)$ with $\operatorname{supp}(\varphi) \subset [0,R]$, $T > R > 0$. Take an arbitrary extension $V \in C_0^\infty(\mathbb{R}) \subset H^{1/2}(\mathbb{R})$ with $V_{|[0,T)} = \varphi$ and $V(t) = 0$ for $t \geq T$, i.e. $V_{|[0,\infty)} \in C_0^\infty[0,\infty)$. The result [102, Lemme 11.1, page 60] yields for $X = Y = \mathbb{R}$ that there exist sequences $(\psi_n)_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R})$ and $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,T)$ such that $\psi_n(t) = 0$ for $t \in (-\varepsilon_n, \varepsilon_n)$, i.e. ψ_n vanishes in a neighbourhood of $t = 0$, and $\|\psi_n - V\|_{H^{1/2}(\mathbb{R})} \to 0$ as $n \to \infty$. Consider a cutoff function $\chi \in C_0^\infty(\mathbb{R})$ satisfying $0 \leq \chi \leq 1$, $\chi_{|[0,R]} = 1$ and $\chi(t) = 0$ for $t > \frac{T+R}{2}$, see [160, Folgerung 1.2, page 18] for the existence of such function. Note that $\operatorname{supp}(\chi) \subset (-\infty, T)$ and hence, $((\psi_n \cdot \chi)_{|(0,T)})_{n\in\mathbb{N}} \subset C_0^\infty(0,T)$ is the desired sequence. With Lemma 2.2.1 and the local property [102, Théorème 7.2, page 36] of $H^{1/2}(\mathbb{R})$ it follows

$$\begin{aligned} \|(\psi_n \cdot \chi)_{|(0,T)} - \varphi\|_{H^{1/2}_{,0}(0,T)} &= \|(\psi_n \cdot \chi)_{|(0,T)} - \chi_{|(0,T)}\varphi\|_{H^{1/2}_{,0}(0,T)} \\ &\leq C_1 \left|\left|\left|(\psi_n \cdot \chi)_{|(0,T)} - \chi_{|(0,T)}\varphi\right|\right|\right|_{H^{1/2}_{,0}(0,T)} \\ &\leq C_1 \|\chi(\psi_n - V)\|_{H^{1/2}(\mathbb{R})} \\ &\leq C_1 C_\chi \|\psi_n - V\|_{H^{1/2}(\mathbb{R})} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where the constant $C_1 > 0$ comes from the norm equivalence of Lemma 2.2.1 and the constant $C_\chi > 0$ depends on the cutoff function χ and therefore, on φ. □

The dual spaces $[H^1_{0,}(0,T)]'$ and $[H^1_{,0}(0,T)]'$ are characterised as completion of $L^2(0,T)$ with respect to the Hilbertian norms

$$\|g\|_{[H^1_{0,}(0,T)]'} := \sup_{0 \neq v \in H^1_{0,}(0,T)} \frac{|\langle g, v\rangle_{(0,T)}|}{|v|_{H^1(0,T)}}$$

and

$$\|f\|_{[H^1_{,0}(0,T)]'} := \sup_{0 \neq w \in H^1_{,0}(0,T)} \frac{|\langle f, w\rangle_{(0,T)}|}{|w|_{H^1(0,T)}},$$

where $\langle \cdot, \cdot \rangle_{(0,T)}$ denotes the duality pairing as extension of the inner product in $L^2(0,T)$, see [160, Satz 17.3, page 258]. In other words, for $[H^1_{0,}(0,T)]'$ and $[H^1_{,0}(0,T)]'$, there exist inner products $\langle \cdot, \cdot \rangle_{[H^1_{0,}(0,T)]'}$ and $\langle \cdot, \cdot \rangle_{[H^1_{,0}(0,T)]'}$, inducing the norms $\|\cdot\|_{[H^1_{0,}(0,T)]'} = \sqrt{\langle \cdot, \cdot \rangle_{[H^1_{0,}(0,T)]'}}$ and $\|\cdot\|_{[H^1_{,0}(0,T)]'} = \sqrt{\langle \cdot, \cdot \rangle_{[H^1_{,0}(0,T)]'}}$, i.e. with these abstract inner products, $[H^1_{0,}(0,T)]'$ and $[H^1_{,0}(0,T)]'$ are Hilbert spaces, see [158, Satz V.1.7, page 222].

Analogously, the dual spaces $[H^{1/2}_{0,}(0,T)]'$ and $[H^{1/2}_{,0}(0,T)]'$ are Hilbert spaces characterised as completion of $L^2(0,T)$ with respect to the Hilbertian norms

$$\|g\|_{[H^{1/2}_{0,}(0,T)]'} := \sup_{0 \neq v \in H^{1/2}_{0,}(0,T)} \frac{|\langle g, v\rangle_{(0,T)}|}{\|v\|_{H^{1/2}_{0,}(0,T)}}$$

and

$$\|f\|_{[H^{1/2}_{,0}(0,T)]'} := \sup_{0 \neq w \in H^{1/2}_{,0}(0,T)} \frac{|\langle f, w\rangle_{(0,T)}|}{\|w\|_{H^{1/2}_{,0}(0,T)}}. \tag{2.3}$$

2.3 Sobolev Spaces in Ω

For a general introduction to Sobolev spaces see, for example, the books [31, 64, 119, 160], and for function space interpolation, see [26, 102, 103, 153]. For $s \geq 0$, the usual Sobolev spaces of real-valued functions $H^s(\Omega)$, $H^s_0(\Omega)$ are endowed with the Sobolev-Slobodeckij inner product $\langle \cdot, \cdot \rangle_{H^s(\Omega)}$ and the norm $\|\cdot\|_{H^s(\Omega)}$. For the subspace $H^1_0(\Omega) \subset H^1(\Omega)$, the inner product

$$\langle u, v \rangle_{H^1_0(\Omega)} := \langle \nabla_x u, \nabla_x v \rangle_{L^2(\Omega)} = \int_\Omega \nabla_x u(x) \cdot \nabla_x v(x) \mathrm{d}x, \quad u, v \in H^1_0(\Omega),$$

and the induced norm

$$|u|_{H^1(\Omega)} := \|u\|_{H_0^1(\Omega)} = \sqrt{\langle u, u\rangle_{H_0^1(\Omega)}} = \sqrt{\int_\Omega |\nabla_x u(x)|^2 \, \mathrm{d}x}, \quad u \in H_0^1(\Omega),$$

are considered. For a type of Fourier series approach in Chapter 3 and Chapter 4, the eigenfunctions $\phi_i \in H_0^1(\Omega)$ with eigenvalues $\mu_i \in \mathbb{R}$, satisfying

$$-\Delta\phi_i = \mu_i \phi_i \quad \text{in } \Omega, \quad \phi_i = 0 \quad \text{on } \partial\Omega, \quad \|\phi_i\|_{L^2(\Omega)} = 1 \tag{2.4}$$

for $i \in \mathbb{N}$, are used, see [97, Theorem 4.1 in Chapter II, page 60]. Note that the eigenfunctions ϕ_i form an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H_0^1(\Omega)$. In addition, the eigenvalues μ_i satisfy

$$0 < \mu_1 \le \mu_2 \le \mu_3 \le \dots \quad \text{and} \quad \mu_i \to \infty \text{ as } i \to \infty.$$

Hence, for a function $u \in L^2(\Omega)$, there holds

$$\left\| u - \sum_{i=1}^M u_i \phi_i \right\|_{L^2(\Omega)} \to 0 \quad \text{as } M \to \infty,$$

i.e. $u = \sum_{i=1}^\infty u_i \phi_i$ in $L^2(\Omega)$, with the coefficients

$$u_i = \int_\Omega u(x)\phi_i(x)\mathrm{d}x \in \mathbb{R},$$

and the $L^2(\Omega)$ norm is given by

$$\|u\|_{L^2(\Omega)} = \sqrt{\sum_{i=1}^\infty u_i^2}.$$

Analogously, for a function $u \in H_0^1(\Omega)$, there holds

$$\left\| \nabla_x u - \sum_{i=1}^M u_i \nabla_x \phi_i \right\|_{L^2(\Omega)} \to 0 \quad \text{as } M \to \infty,$$

i.e. $u = \sum_{i=1}^\infty u_i \phi_i$ in $H_0^1(\Omega)$, with the coefficients

$$u_i = \int_\Omega u(x)\phi_i(x)\mathrm{d}x \in \mathbb{R}$$

and the $H^1(\Omega)$ seminorm is given by

$$|u|_{H^1(\Omega)} = \sqrt{\sum_{i=1}^\infty \mu_i u_i^2}.$$

The dual space $[H_0^1(\Omega)]'$ is a Hilbert space characterised as the completion of $L^2(\Omega)$ with respect to the Hilbertian norm

$$\|g\|_{[H_0^1(\Omega)]'} := \sup_{0\neq v\in H_0^1(\Omega)} \frac{|\langle g,v\rangle_\Omega|}{|v|_{H^1(\Omega)}}, \tag{2.5}$$

where $\langle\cdot,\cdot\rangle_\Omega$ denotes the duality pairing as extension of the inner product in $L^2(\Omega)$, see Section 2.2.

2.4 Hilbert Tensor-Product and Bochner Spaces

For an introduction to the algebraic tensor-product $\otimes$ and to the Hilbert tensor-product $\hat{\otimes}$, see [15, Chapter 12], [157, Unterkapitel 1.6], [128, Section II.4] or [152, Part III]. For Bochner spaces, see also [160, Kapitel IV], [162, Chapter 23], [82, Chapter 1 and 2], [139, Kapitel 10], [110, Chapter 2] and the recent work [12]. In this section, let H be a separable real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be bounded Lipschitz domains with $d_1, d_2 \in \mathbb{N}$. Consider the Bochner space $L^2(\Omega_1;H)$ of classes of measurable vector-valued functions $U\colon\ \Omega_1 \to H$, i.e. for each element $U \in L^2(\Omega_1;H)$ it holds

$$U(y) \in H \quad \text{for almost all } y \in \Omega_1,$$

such that

$$\|U\|_{L^2(\Omega_1;H)} := \sqrt{\int_{\Omega_1} \|U(y)\|_H^2 \mathrm{d}y} < \infty.$$

The Bochner space $L^2(\Omega_1;H)$ is a Hilbert space with respect to the inner product

$$\langle U,V\rangle_{L^2(\Omega_1;H)} := \int_{\Omega_1} \langle U(y),V(y)\rangle_H \mathrm{d}y.$$

The dual space $[L^2(\Omega_1;H)]'$ and the Bochner space $L^2(\Omega_1;H')$ are isometric, see [82, Corollary 1.3.22, page 54] and see also [162, Section 23.3]. Furthermore, the Bochner space $L^2(\Omega_1;H)$ and the Hilbert tensor-product $L^2(\Omega_1)\hat{\otimes}H$ are isometric, i.e.

$$L^2(\Omega_1;H) \simeq L^2(\Omega_1)\hat{\otimes}H \simeq H\hat{\otimes}L^2(\Omega_1),$$

see [15, Theorem 12.6.1, page 304].

For $m \in \mathbb{N}_0$, the Bochner Sobolev space is defined by

$$H^m(\Omega_1;H) := \left\{U \in L^2(\Omega_1;H)\colon\ \partial_y^\alpha U \in L^2(\Omega_1;H) \text{ for } |\alpha| \le m\right\}$$

where ∂_y is the distributional derivative on Ω_1 with respect to y for vector-valued functions and $\alpha = (\alpha_1, \dots, \alpha_{d_1}) \in \mathbb{N}_0^{d_1}$ is a multi-index. With the inner product

$$\langle U, V \rangle_{H^m(\Omega_1;H)} := \int_{\Omega_1} \langle U(y), V(y) \rangle_H \mathrm{d}y + \sum_{|\alpha| \leq m} \int_{\Omega_1} \left\langle \partial_y^\alpha U(y), \partial_y^\alpha V(y) \right\rangle_H \mathrm{d}y$$

for $U, V \in H^m(\Omega_1; H)$, the Bochner Sobolev space $H^m(\Omega_1; H)$ is a Hilbert space. Furthermore, the Bochner Sobolev space $H^m(\Omega_1; H)$ and the Hilbert tensor-product $H^m(\Omega_1) \hat{\otimes} H$ are isometric, i.e.

$$H^m(\Omega_1; H) \simeq H^m(\Omega_1) \hat{\otimes} H \simeq H \hat{\otimes} H^m(\Omega_1), \tag{2.6}$$

see [15, Theorem 12.7.1, page 307].

As a first special case, for $m = 1$ and $\Omega_1 = (0, T)$, it holds the Sobolev embedding theorem [110, Proposition 2.46, page 46], i.e.

$$H^1(0, T; H) \subset C([0, T]; H) \tag{2.7}$$

with a continuous embedding. Therefore, as in Section 2.2,

$$H^1_{0,}(0, T; H) := \{V \in H^1(0, T; H) \colon\ V(0) = 0 \text{ in } H\},$$
$$H^1_{,0}(0, T; H) := \{V \in H^1(0, T; H) \colon\ V(T) = 0 \text{ in } H\}$$

are subspaces of $H^1(0, T; H)$.

As a second special case, consider $H = H^p(\Omega_2)$ with $p \in \mathbb{N}_0$. Then the space

$$H_\otimes^{p,m}(\Omega_2 \times \Omega_1) := \left\{ u \in L^2(\Omega_2 \times \Omega_1) \colon\ \partial_x^{\alpha^2} \partial_y^{\alpha^1} u \in L^2(\Omega_2 \times \Omega_1) \text{ for } \left|\alpha^2\right| \leq p, \left|\alpha^1\right| \leq m \right\},$$

with the inner product

$$\langle u, v \rangle_{H_\otimes^{p,m}(\Omega_2 \times \Omega_1)} = \sum_{|\alpha^2| \leq p} \sum_{|\alpha^1| \leq m} \left\langle \partial_x^{\alpha^2} \partial_y^{\alpha^1} u, \partial_x^{\alpha^2} \partial_y^{\alpha^1} v \right\rangle_{L^2(\Omega_2 \times \Omega_1)},$$

where ∂_x and ∂_y denote the distributional derivatives with respect to x and y on Ω_2 and Ω_1, is isometric to $H^m(\Omega_1; H^p(\Omega_2))$, i.e.

$$H_\otimes^{p,m}(\Omega_2 \times \Omega_1) \simeq H^m(\Omega_1; H^p(\Omega_2)) \simeq H^p(\Omega_2; H^m(\Omega_1)) \simeq H^p(\Omega_2) \hat{\otimes} H^m(\Omega_1), \tag{2.8}$$

see [15, Theorem 12.7.2, page 308] and (2.6). Thus, for $m = p = 0$, the relation (2.8) states

$$L^2(\Omega_2 \times \Omega_1) \simeq L^2(\Omega_2; L^2(\Omega_1)) \simeq L^2(\Omega_1; L^2(\Omega_2)) \simeq L^2(\Omega_2) \hat{\otimes} L^2(\Omega_1),$$

see also [128, Theorem II.10, page 52] and [139, last line, page 188]. Hence, for a separable Hilbert space $H \subset L^2(\Omega_2)$, the Bochner space $L^2(\Omega_1;H) \subset L^2(\Omega_1;L^2(\Omega_2))$ and the L^2 subspace

$$\left\{u \in L^2(\Omega_2 \times \Omega_1) : \ y \mapsto u(\cdot,y) \in L^2(\Omega_1;H)\right\}, \tag{2.9}$$

endowed with inner product

$$\langle u,v\rangle := \int_{\Omega_1} \langle u(\cdot,y), v(\cdot,y)\rangle_H \mathrm{d}y,$$

are isometric by the bijective isometry Φ given by $u(x,y) := (\Phi U)(x,y) := U(y)(x)$ for $(x,y) \in \Omega_2 \times \Omega_1, U \in L^2(\Omega_1;H)$. Therefore, for a separable Hilbert space $H \subset L^2(\Omega_2)$, the Bochner space $L^2(\Omega_1;H)$ is identified with the subspace (2.9) of $L^2(\Omega_2 \times \Omega_1)$, hence, one writes

$$L^2(\Omega_1;H) = \left\{u \in L^2(\Omega_2 \times \Omega_1) : \ y \mapsto u(\cdot,y) \in L^2(\Omega_1;H)\right\}.$$

Analogously, for a separable Hilbert space $H \subset L^2(\Omega_1)$ the Bochner space $L^2(\Omega_2;H)$ is identified with a subspace of $L^2(\Omega_2 \times \Omega_1)$, hence, one writes

$$L^2(\Omega_2;H) = \left\{u \in L^2(\Omega_2 \times \Omega_1) : \ x \mapsto u(x,\cdot) \in L^2(\Omega_2;H)\right\},$$

where this subspace of $L^2(\Omega_2 \times \Omega_1)$ is endowed with the inner product

$$\langle u,v\rangle := \int_{\Omega_2} \langle u(x,\cdot), v(x,\cdot)\rangle_H \mathrm{d}x.$$

With these identifications, the anisotropic Sobolev spaces are defined for $0 \le r \in \mathbb{R}$, $0 \le s \in \mathbb{R}$ as

$$H^{r,s}(\Omega_2 \times \Omega_1) := L^2(\Omega_1;H^r(\Omega_2)) \cap L^2(\Omega_2;H^s(\Omega_1)) \subset L^2(\Omega_2 \times \Omega_1) \tag{2.10}$$

with the inner product

$$\langle u,v\rangle_{H^{r,s}(\Omega_2\times\Omega_1)} := \int_{\Omega_1} \langle u(\cdot,y), v(\cdot,y)\rangle_{H^r(\Omega_2)} \mathrm{d}y + \int_{\Omega_2} \langle u(x,\cdot), v(x,\cdot)\rangle_{H^s(\Omega_1)} \mathrm{d}x.$$

For integers r and s, simpler characterisations of the spaces $H^{r,s}(\Omega_2 \times \Omega_1)$ are given. For $p \in \mathbb{N}$, $m = 0$, it holds

$$\begin{aligned} H^{p,0}(\Omega_2 \times \Omega_1) =& \left\{u \in L^2(\Omega_2 \times \Omega_1) : \ \partial_x^{\alpha^2} u \in L^2(\Omega_2 \times \Omega_1) \text{ for } \left|\alpha^2\right| \le p\right\} \\ =& H^{p,0}_{\otimes}(\Omega_2 \times \Omega_1) = L^2(\Omega_1;H^p(\Omega_2)) \\ \simeq& H^p(\Omega_2;L^2(\Omega_1)) \simeq H^p(\Omega_2)\hat{\otimes}L^2(\Omega_1) \end{aligned}$$

with the inner product

$$\langle u,v\rangle_{H^{p,0}(\Omega_2\times\Omega_1)} = \sum_{|\alpha^2|\le p} \left\langle \partial_x^{\alpha^2} u, \partial_x^{\alpha^2} v \right\rangle_{L^2(\Omega_2\times\Omega_1)}$$

and analogously, for $m \in \mathbb{N}$, $p = 0$, it holds

$$\begin{aligned} H^{0,m}(\Omega_2\times\Omega_1) =& \left\{ u \in L^2(\Omega_2\times\Omega_1) :\ \partial_y^{\alpha^1} u \in L^2(\Omega_2\times\Omega_1) \text{ for } \left|\alpha^1\right| \le m \right\} \\ =& H_{\otimes}^{0,m}(\Omega_2\times\Omega_1) = L^2(\Omega_2; H^m(\Omega_1)) \\ \simeq& H^m(\Omega_1; L^2(\Omega_2)) \simeq L^2(\Omega_2)\hat{\otimes} H^m(\Omega_1) \end{aligned}$$

with the inner product

$$\langle u,v\rangle_{H^{0,m}(\Omega_2\times\Omega_1)} = \sum_{|\alpha^1|\le m} \left\langle \partial_y^{\alpha^1} u, \partial_y^{\alpha^1} v \right\rangle_{L^2(\Omega_2\times\Omega_1)}.$$

As a last special case, consider the interval $\Omega_1 = (0,T)$. For $0 < s \in \mathbb{R}\setminus\mathbb{N}$, the space $H^s(0,T;L^2(\Omega_2))$ is defined via function space interpolation endowed with the inner product $\langle\cdot,\cdot\rangle_{H^s(0,T;L^2(\Omega_2))} := \langle\cdot,\cdot\rangle_{L^2(\Omega_2;H^s(0,T))}$ and with the to the interpolation norm equivalent norm $\|\cdot\|_{L^2(\Omega_2;H^s(0,T))}$, see [103, page 8]. So, in the following, the spaces $H^s(0,T;L^2(\Omega_2))$ and $L^2(\Omega_2;H^s(0,T))$ are identified, hence, one writes

$$L^2(\Omega_2;H^s(0,T)) = H^s(0,T;L^2(\Omega_2)). \tag{2.11}$$

2.5 Sobolev Spaces in Q

In this section, the notations and identifications of Section 2.2, Section 2.3 and Section 2.4 are used.

For an introduction to anisotropic Sobolev spaces, see [102, 103] and for a short summary see [35, Chapter 2].

For $0 \le r \in \mathbb{R}$, $0 \le s \in \mathbb{R}$, one defines as in (2.10), see also (2.11), the anisotropic Sobolev space

$$H^{r,s}(Q) := L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega)) \simeq \left(H^r(\Omega)\hat{\otimes} L^2(0,T)\right) \cap \left(L^2(\Omega)\hat{\otimes} H^s(0,T)\right),$$

which is a Hilbert space with respect to the inner product

$$\begin{aligned} \langle u,v\rangle_{H^{r,s}(Q)} :=& \int_0^T \langle u(\cdot,t), v(\cdot,t)\rangle_{H^r(\Omega)} \mathrm{d}t + \int_\Omega \langle u(x,\cdot), v(x,\cdot)\rangle_{H^s(0,T)} \mathrm{d}x \\ =& \langle u,v\rangle_{L^2(0,T;H^r(\Omega))} + \langle u,v\rangle_{H^s(0,T;L^2(\Omega))} \end{aligned}$$

for $u,v \in H^{r,s}(Q)$. Note that for $r = s = 1$, there hold

$$H^1(Q) = H^{1,1}(Q) \subset C([0,T];L^2(\Omega)) \tag{2.12}$$

with a continuous embedding, see (2.7). The subspace

$$H^{1,1}_{0;}(Q) := H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$$

is endowed with the inner product

$$\langle u,v\rangle_{H^{1,1}_{0;}(Q)} := \int_0^T \int_\Omega (\partial_t u(x,t)\partial_t v(x,t) + \nabla_x u(x,t)\cdot\nabla_x v(x,t))\,\mathrm{d}x\mathrm{d}t \tag{2.13}$$

and the induced norm

$$|u|_{H^1(Q)} := \sqrt{\langle u,u\rangle_{H^{1,1}_{0;}(Q)}} = \left\{ \int_0^T \int_\Omega \left(|\partial_t u(x,t)|^2 + \sum_{m=1}^d |\partial_{x_m} u(x,t)|^2 \right) \mathrm{d}x\mathrm{d}t \right\}^{1/2}.$$

Note that in $H^{1,1}_{0;}(Q)$, the seminorm $|\cdot|_{H^1(Q)}$ is a to $\|\cdot\|_{H^1(Q)}$ equivalent norm due to the Poincaré inequality. The subspaces

$$H^{1,1}_{0;0,}(Q) := H^1_{0,}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \tag{2.14}$$

and

$$H^{1,1}_{0;,0}(Q) := H^1_{,0}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$$

are endowed with the inner product (2.13) and the induced norm $|\cdot|_{H^1(Q)}$.

For functions defined in Ω, the standard trace operator

$$\gamma_0^{\mathrm{int}}\colon\ H^1(\Omega) \to H^{1/2}(\partial\Omega)$$

is bounded, i.e. $\|\gamma_0^{\mathrm{int}} z\|_{H^{1/2}(\partial\Omega)} \le C_{\mathrm{Tr}}\|z\|_{H^1(\Omega)}$ with a constant $C_{\mathrm{Tr}} > 0$, where the Sobolev space $H^{1/2}(\partial\Omega)$ is the usual trace space, see [34, 64, 119, 160] for more details. The extended trace operator

$$\gamma_{0,x}^{\mathrm{int}}\colon\ L^2(0,T;H^1(\Omega)) \to L^2(\Sigma) \tag{2.15}$$

satisfies the relation

$$\gamma_{0,x}^{\mathrm{int}} v = v_{|\Sigma} \quad \text{for } v \in L^2(0,T;C(\overline{\Omega})),$$

the relation

$$\left\|\gamma_{0,x}^{\mathrm{int}} v\right\|_{L^2(\Sigma)} = 0 \quad \Longleftrightarrow \quad v \in L^2(0,T;H^1_0(\Omega)),$$

and with the same constant $C_{\mathrm{Tr}} > 0$ as γ_0^{int}, the boundedness estimate

$$\left\|\gamma_{0,x}^{\mathrm{int}} v\right\|_{L^2(\Sigma)} \le C_{\mathrm{Tr}}\|v\|_{L^2(0,T;H^1(\Omega))},$$

see [12, Theorem 6.13, page 21]. Hence, the representations

$$H^{1,1}_{0;}(Q) = \left\{ v \in H^1(Q) \colon \left\| \gamma^{\text{int}}_{0,x} v \right\|_{L^2(\Sigma)} = 0 \right\},$$
$$H^{1,1}_{0;0,}(Q) = \left\{ v \in H^1(Q) \colon \left\| \gamma^{\text{int}}_{0,x} v \right\|_{L^2(\Sigma)} = \|v(\cdot,0)\|_{L^2(\Omega)} = 0 \right\},$$
$$H^{1,1}_{0;,0}(Q) = \left\{ v \in H^1(Q) \colon \left\| \gamma^{\text{int}}_{0,x} v \right\|_{L^2(\Sigma)} = \|v(\cdot,T)\|_{L^2(\Omega)} = 0 \right\}$$

are valid.

The dual spaces $[H^{1,1}_{0;0,}(Q)]'$ and $[H^{1,1}_{0;,0}(Q)]'$ are characterised as completion of $L^2(Q)$ with respect to the Hilbertian norms

$$\|g\|_{[H^{1,1}_{0;0,}(Q)]'} := \sup_{0 \neq v \in H^{1,1}_{0;0,}(Q)} \frac{|\langle g, v \rangle_Q|}{|v|_{H^1(Q)}}$$

and

$$\|f\|_{[H^{1,1}_{0;,0}(Q)]'} := \sup_{0 \neq w \in H^{1,1}_{0;,0}(Q)} \frac{|\langle f, w \rangle_Q|}{|w|_{H^1(Q)}},$$

where $\langle \cdot, \cdot \rangle_Q$ denotes the duality pairing as extension of the inner product in $L^2(Q)$, see Section 2.2.

For $s = 1/2$, one defines via function space interpolation the Sobolev space

$$H^{1/2}_{0,}(0,T;L^2(\Omega)) := [H^1_{0,}(0,T;L^2(\Omega)), L^2(0,T;L^2(\Omega))]_{1/2}$$

with the Hilbertian norm

$$\|u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} := \sqrt{\|u\|^2_{H^{1/2}(0,T;L^2(\Omega))} + \int_0^T \frac{\|u(\cdot,t)\|^2_{L^2(\Omega)}}{t} \mathrm{d}t}, \tag{2.16}$$

which is equivalent to the interpolation norm, see [102, Théorème 11.7, page 72] and [102, (5.8), page 276]. Analogously, set

$$H^{1/2}_{,0}(0,T;L^2(\Omega)) := [H^1_{,0}(0,T;L^2(\Omega)), L^2(0,T;L^2(\Omega))]_{1/2}$$

with the Hilbertian norm

$$\|u\|_{H^{1/2}_{,0}(0,T;L^2(\Omega))} := \sqrt{\|u\|^2_{H^{1/2}(0,T;L^2(\Omega))} + \int_0^T \frac{\|u(\cdot,t)\|^2_{L^2(\Omega)}}{T-t} \mathrm{d}t}. \tag{2.17}$$

There hold the representations

$$H^{1/2}_{0,}(0,T;L^2(\Omega)) = \left\{ v_{|Q}\colon\ v \in H^{1/2}(-\infty,T;L^2(\Omega)) \text{ with } v(\cdot,t)=0 \text{ in } L^2(\Omega) \text{ for } t<0 \right\}$$
$$= \left\{ u \in H^{1/2}(0,T;L^2(\Omega))\colon\ \int_0^T \frac{\|u(\cdot,t)\|^2_{L^2(\Omega)}}{t}\,\mathrm{d}t < \infty \right\} \tag{2.18}$$

and

$$H^{1/2}_{,0}(0,T;L^2(\Omega)) = \left\{ v_{|Q}\colon\ v \in H^{1/2}(0,\infty;L^2(\Omega)) \text{ with } v(\cdot,t)=0 \text{ in } L^2(\Omega) \text{ for } t>T \right\}$$
$$= \left\{ u \in H^{1/2}(0,T;L^2(\Omega))\colon\ \int_0^T \frac{\|u(\cdot,t)\|^2_{L^2(\Omega)}}{T-t}\,\mathrm{d}t < \infty \right\}, \tag{2.19}$$

see [102, Proposition 5.2, page 276] and [102, Remarque 11.4, page 75]. Because the test functions

$$C_0^\infty(\Omega)\otimes C_0^\infty(0,T) = \operatorname{span}\{\phi\cdot\psi \in C_0^\infty(Q)\colon\ \phi\in C_0^\infty(\Omega),\ \psi\in C_0^\infty(0,T)\} \subset C_0^\infty(Q)$$

are dense in $L^2(\Omega)\hat{\otimes}H^{1/2}(0,T) \simeq H^{1/2}(0,T;L^2(\Omega))$, see [64, Theorem 1.4.2.4, page 25], [157, Satz 1.63, page 62] and for the tensor-product of functions see [152, Example II, page 407], the sets $H^{1/2}_{0,}(0,T;L^2(\Omega))$ and $H^{1/2}_{,0}(0,T;L^2(\Omega))$ are dense in $H^{1/2}(0,T;L^2(\Omega))$. Note that the constant function $\mathbf{1}(x,t) := 1$ for $(x,t)\in Q$ fulfils $\mathbf{1}\in H^{1/2}(0,T;L^2(\Omega))$, $\mathbf{1}\notin H^{1/2}_{0,}(0,T;L^2(\Omega))$ and $\mathbf{1}\notin H^{1/2}_{,0}(0,T;L^2(\Omega))$ due to the representations (2.18) and (2.19).

The first spaces needed for the heat equation are the anisotropic Sobolev spaces

$$H^{1,1/2}_{0;0,}(Q) := H^{1/2}_{0,}(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega)),$$
$$H^{1,1/2}_{0;,0}(Q) := H^{1/2}_{,0}(0,T;L^2(\Omega))\cap L^2(0,T;H^1_0(\Omega))$$

endowed with the Hilbertian norms

$$\|v\|_{H^{1,1/2}_{0;0,}(Q)} := \sqrt{\|v\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega))} + \|\nabla_x v\|^2_{L^2(Q)}},$$
$$\|w\|_{H^{1,1/2}_{0;,0}(Q)} := \sqrt{\|w\|^2_{H^{1/2}_{,0}(0,T;L^2(\Omega))} + \|\nabla_x w\|^2_{L^2(Q)}}. \tag{2.20}$$

The dual spaces $[H^{1,1/2}_{0;0,}(Q)]'$ and $[H^{1,1/2}_{0;,0}(Q)]'$ are characterised as completion of $L^2(Q)$ with respect to the Hilbertian norms

$$\|g\|_{[H^{1,1/2}_{0;0,}(Q)]'} := \sup_{0\neq v\in H^{1,1/2}_{0;0,}(Q)} \frac{|\langle g,v\rangle_Q|}{\|v\|_{H^{1,1/2}_{0;0,}(Q)}}$$

and

$$\|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} := \sup_{0 \neq w \in H^{1,1/2}_{0;,0}(Q)} \frac{|\langle f, w\rangle_Q|}{\|w\|_{H^{1,1/2}_{0;,0}(Q)}},$$

where $\langle \cdot, \cdot \rangle_Q$ denotes the duality pairing as extension of the inner product in $L^2(Q)$, see Section 2.2.

A second space needed for the heat equation is introduced. Therefore, one defines

$$\begin{aligned} W(Q) :=& L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;[H^1_0(\Omega)]') \\ \simeq& \left(H^1_0(\Omega)\hat{\otimes}L^2(0,T)\right) \cap \left([H^1_0(\Omega)]'\hat{\otimes}H^1(0,T)\right), \end{aligned} \tag{2.21}$$

which is a Hilbert space with the inner product

$$\langle u, v\rangle_{W(Q)} := \int_0^T \int_\Omega \nabla_x u(x,t) \cdot \nabla_x v(x,t) \mathrm{d}x\mathrm{d}t + \int_0^T \langle \partial_t u(\cdot,t), \partial_t v(\cdot,t)\rangle_{[H^1_0(\Omega)]'} \mathrm{d}t,$$

see [160, Satz 25.4, page 380], and the norm is given by

$$\|u\|_{W(Q)} := \left(\int_0^T \int_\Omega |\nabla_x u(x,t)|^2 \,\mathrm{d}x\mathrm{d}t + \int_0^T \|\partial_t u(\cdot,t)\|^2_{[H^1_0(\Omega)]'} \mathrm{d}t \right)^{1/2}, \tag{2.22}$$

where the Hilbertian norm in the dual space $[H^1_0(\Omega)]'$ is given as in (2.5) for $t \in (0,T)$ by

$$\|\partial_t u(\cdot,t)\|_{[H^1_0(\Omega)]'} = \sup_{0 \neq z \in H^1_0(\Omega)} \frac{|\langle \partial_t u(\cdot,t), z\rangle_\Omega|}{|z|_{H^1(\Omega)}}.$$

Moreover, it holds

$$W(Q) \subset C([0,T];L^2(\Omega)) \tag{2.23}$$

and this embedding is continuous, in other words, there exists a constant $C_{\mathrm{em}} > 0$ such that

$$\max_{t \in [0,T]} \|u(\cdot,t)\|_{L^2(\Omega)} = \max_{t \in [0,T]} \left(\int_\Omega |u(x,t)|^2 \,\mathrm{d}x \right)^{1/2} \leq C_{\mathrm{em}} \|u\|_{W(Q)} \quad \text{for } u \in W(Q),$$

see [162, Proposition 23.23, page 422]. In addition, the trace map

$$\gamma_{\Omega_0} \colon W(Q) \to L^2(\Omega), \quad \gamma_{\Omega_0} u(x) := u_{|\Omega_0}(x) = u(x,0) \quad \text{for } x \in \Omega,$$

is surjective, see [102, Théorème 3.2, page 25], and bounded due to the continuous embedding (2.23) and

$$\left\|\gamma_{\Omega_0} u\right\|_{L^2(\Omega)} = \|u(\cdot,0)\|_{L^2(\Omega)} \leq \max_{t \in [0,T]} \|u(\cdot,t)\|_{L^2(\Omega)} \leq C_{\mathrm{em}} \|u\|_{W(Q)}$$

for $u \in W(Q)$ with the constant $C_{\mathrm{em}} > 0$. Moreover, there exists a continuous, linear right inverse $\mathcal{E}_{\Omega_0} \colon L^2(\Omega) \to W(Q)$, satisfying for all $u_0 \in L^2(\Omega)$

$$\gamma_{\Omega_0} \mathcal{E}_{\Omega_0} u_0 = u_0 \text{ in } L^2(\Omega) \quad \text{and} \quad \left\| \mathcal{E}_{\Omega_0} u_0 \right\|_{W(Q)} \leq C_{\mathrm{ex}} \|u_0\|_{L^2(\Omega)} \tag{2.24}$$

with a constant $C_{\mathrm{ex}} > 0$, see [102, Remarque 3.3, page 26]. Note that also Theorem 3.1.1 in Section 3.1 gives such an extension operator $\mathcal{E}_{\Omega_0}$. Because of the embedding (2.23), the initial condition $v(\cdot,0) = v_0$ in $L^2(\Omega)$ for a given $v_0 \in L^2(\Omega)$ is meaningful. Further, the subspace

$$W_{0,}(Q) := \left\{ v \in W(Q) \colon\ v(\cdot,0) = 0 \text{ in } L^2(\Omega) \right\} \subset W(Q) \tag{2.25}$$

is again a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_{W(Q)}$.

2.6 Discretisations in Time

For the given finite time $T > 0$, the time interval $(0,T)$ is decomposed via the time steps

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N_t - 1} < t_{N_t} = T,$$

where N_t denotes the number of time intervals $\tau_\ell = (t_{\ell-1}, t_\ell)$ for $\ell = 1, \ldots, N_t$. In addition, the number of time steps t_ℓ is denoted by $\tilde{M}_t$, i.e. $\tilde{M}_t = N_t + 1$, and the local mesh sizes are given as $h_{t,\ell} = t_\ell - t_{\ell-1}$ for $\ell = 1, \ldots, N_t$. Next, the global mesh size in time is defined by $h_t = \max_{\ell=1,\ldots,N_t} h_{t,\ell}$, and the related finite element space

$$S^1_{h_t}(0,T) = \operatorname{span}\{\varphi_k\}_{k=0}^{N_t}$$

of piecewise linear, continuous functions is introduced, where the usual nodal basis functions φ_k, $k = 0, \ldots, N_t$, satisfy $\varphi_k(t_\ell) = \delta_{k\ell}$ for $k, \ell = 0, \ldots, N_t$. In addition, the subspaces $S^1_{h_t,0,}(0,T) \subset S^1_{h_t}(0,T)$ and $S^1_{h_t,,0}(0,T) \subset S^1_{h_t}(0,T)$ fulfil the homogeneous initial or end conditions, i.e.

$$S^1_{h_t,0,}(0,T) = S^1_{h_t}(0,T) \cap H^1_{0,}(0,T) = \operatorname{span}\{\varphi_k\}_{k=1}^{N_t}$$

and

$$S^1_{h_t,,0}(0,T) = S^1_{h_t}(0,T) \cap H^1_{,0}(0,T) = \operatorname{span}\{\varphi_k\}_{k=0}^{N_t - 1}.$$

Furthermore, $S^0_{h_t}(0,T)$ is the finite element space of piecewise constant functions. The mapping

$$Q^0_{h_t} \colon\ L^2(0,T) \to S^0_{h_t}(0,T)$$

denotes the L^2 projection on the piecewise constant finite element space $S^0_{h_t}(0,T)$, defined for $u \in L^2(0,T)$ by finding $Q^0_{h_t} u \in S^0_{h_t}(0,T)$ such that

$$\left\langle Q^0_{h_t} u, v_{h_t} \right\rangle_{L^2(0,T)} = \langle u, v_{h_t} \rangle_{L^2(0,T)} \tag{2.26}$$

for all $v_{h_t} \in S^0_{h_t}(0,T)$, satisfying the stability estimate

$$\|Q^0_{h_t} u\|_{L^2(0,T)} \leq \|u\|_{L^2(0,T)}.$$

For a continuous function $u \in C[0,T]$, the interpolation operator

$$I_{h_t} : C[0,T] \to S^1_{h_t}(0,T)$$

is defined by

$$I_{h_t} u(t) := \sum_{\ell=0}^{N_t} u(t_\ell) \varphi_\ell(t) \tag{2.27}$$

for $t \in [0,T]$, which is uniformly bounded with respect to the mesh size h_t as a mapping $I_{h_t} : H^1(0,T) \to S^1_{h_t}(0,T)$, i.e.

$$\|I_{h_t} u\|_{H^1(0,T)} \leq C \|u\|_{H^1(0,T)} \quad \forall u \in H^1(0,T)$$

with a constant $C > 0$ independent of h_t, see [51, Proposition 1.4, page 6], and in addition, it holds

$$\|\partial_t I_{h_t} u\|_{L^2(0,T)} \leq \|\partial_t u\|_{L^2(0,T)} \quad \forall u \in H^1(0,T), \tag{2.28}$$

see [51, Proof of Proposition 1.4, page 6]. For $u \in H^1_{0,}(0,T) \cap H^2(0,T)$, recall the standard error estimates for the piecewise linear interpolant

$$\|u - I_{h_t} u\|_{L^2(0,T)} \leq \frac{1}{\sqrt{8}} h_t \, \|\partial_t (u - I_{h_t} u)\|_{L^2(0,T)}, \tag{2.29}$$

$$\|u - I_{h_t} u\|_{H^1_{0,}(0,T)} = \|\partial_t (u - I_{h_t} u)\|_{L^2(0,T)} \leq \frac{1}{\sqrt{3}} h_t \, \|\partial_{tt} u\|_{L^2(0,T)} \leq \frac{1}{\sqrt{3}} h_t \, \|\partial_t u\|_{H^1(0,T)}, \tag{2.30}$$

and therefore

$$\|u - I_{h_t} u\|_{L^2(0,T)} \leq \frac{1}{\sqrt{24}} h_t^2 \, \|\partial_{tt} u\|_{L^2(0,T)} \leq \frac{1}{\sqrt{24}} h_t^2 \, \|\partial_t u\|_{H^1(0,T)}. \tag{2.31}$$

An interpolation argument between (2.30) and (2.31) yields for $u \in H^1_{0,}(0,T) \cap H^2(0,T)$

$$\|u - I_{h_t} u\|_{H^{1/2}_{0,}(0,T)} \leq C \, h_t^{3/2} \, \|\partial_t u\|_{H^1(0,T)}, \tag{2.32}$$

where the constant $C > 0$ is independent of h_t, but dependent on the norm equivalence constants concerning $\|\cdot\|_{H^{1/2}_{0,}(0,T)}$, see Theorem 3.4.2 and (3.32).

For a given function $u \in H^1_{0,}(0,T)$, the $H^1_{0,}$ projection $Q^1_{h_t} u \in S^1_{h_t,0,}(0,T)$ is defined by

$$\left\langle \partial_t Q^1_{h_t} u, \partial_t v_{h_t} \right\rangle_{L^2(0,T)} = \langle \partial_t u, \partial_t v_{h_t} \rangle_{L^2(0,T)} \tag{2.33}$$

for all $v_{h_t} \in S^1_{h_t,0,}(0,T)$, satisfying the stability estimate

$$\|\partial_t Q^1_{h_t} u\|_{L^2(0,T)} \le \|\partial_t u\|_{L^2(0,T)}.$$

In addition, it holds for $s \in [0,1]$ the standard error estimate

$$\|u - Q^1_{h_t} u\|_{L^2(0,T)} \le c\, h_t^{1+s} \|u\|_{H^{1+s}(0,T)} \tag{2.34}$$

for $u \in H^1_{0,}(0,T) \cap H^{1+s}(0,T)$ with a constant $c > 0$.

Next, time stepping schemes are introduced. For a given positive integer $M_x \in \mathbb{N}$, consider a first-order ordinary differential equation

$$\frac{\mathrm{d}\underline{u}}{\mathrm{d}t}(t) = \underline{F}(t, \underline{u}(t)) \qquad \text{for } t \in [0,T],$$

where $\underline{u}(0) = \underline{u}_0 \in \mathbb{R}^{M_x}$ and $\underline{F}\colon\ [0,T] \times \mathbb{R}^{M_x} \to \mathbb{R}^{M_x}$ are the imposed initial condition and right-hand side. The right-hand side $\underline{F}$ is assumed to be sufficiently smooth and Lipschitz continuous with respect to the second argument, i.e.

$$|\underline{F}(t, \underline{v}_1) - \underline{F}(t, \underline{v}_2)| \le C_L\, |\underline{v}_1 - \underline{v}_2| \qquad \text{for all } \underline{v}_1, \underline{v}_2 \in \mathbb{R}^{M_x}, t \in [0,T]$$

with a Lipschitz constant $C_L > 0$. To approximate the function $\underline{u}\colon\ [0,T] \to \mathbb{R}^{M_x}$, time stepping methods are considered. These lead to approximations

$$\underline{u}(t_\ell) \approx \underline{U}^\ell \in \mathbb{R}^{M_x}$$

in each time step t_ℓ for $\ell = 0, \ldots, N_t$. Therefore, the θ-method for $\theta \in [0,1]$ is defined as

$$\underline{U}^{\ell+1} - \underline{U}^\ell = h_{t,\ell} \Big((1-\theta) \underline{F}(t_\ell, \underline{U}^\ell) + \theta \underline{F}(t_{\ell+1}, \underline{U}^{\ell+1}) \Big) \tag{2.35}$$

for $\ell = 0, \ldots, N_t$ with $\underline{U}^0 := \underline{u}_0 \in \mathbb{R}^{M_x}$. The θ-method simplifies to the explicit Euler method for $\theta = 0$ and to the implicit Euler method for $\theta = 1$. For $\theta = 1/2$, the Crank-Nicolson method is obtained. The explicit and implicit Euler method converge with order h_t, whereas the Crank-Nicolson method converges with order h_t^2. Furthermore, the implicit Euler and the Crank-Nicolson methods are A-stable. On the other hand, the explicit Euler method is not A-stable. See [93, Section 7.4] and [75] for more details and proofs.

2.7 Discretisations in Space

Let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0,L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. For this situation, different discretisations in space are introduced as follows. The spatial domain Ω is decomposed as

$$\overline{\Omega} = \bigcup_{\ell=1}^{N_x} \overline{\omega}_\ell$$

with N_x spatial elements $\omega_\ell \subset \mathbb{R}^d$. The sequence $(\mathcal{T}_N)_N$ of decompositions is assumed to be admissible, shape regular and globally quasi-uniform. Here, the spatial elements ω_ℓ are intervals for $d=1$, triangles or quadrilaterals for $d=2$ and tetrahedra or hexahedra for $d=3$. The local mesh sizes are given as

$$h_{x,\ell} = \left(\int_{\omega_\ell} \mathrm{d}x \right)^{1/d} \qquad \text{for } \ell = 1, \ldots, N_x$$

and $h_x = \max_{\ell=1,\ldots,N_x} h_{x,\ell}$ is the global mesh size. Furthermore, $\tilde{M}_x$ is the number of vertices $\{x_i\}_{i=1}^{\tilde{M}_x}$ of the decomposition. The space

$$V_{h_x}(\Omega) = \operatorname{span}\{\psi_i\}_{i=1}^{\tilde{M}_x} \subset H^1(\Omega)$$

is the space of piecewise linear, continuous functions $S^1_{h_x}(\Omega)$ on intervals ($d=1$), triangles ($d=2$), tetrahedra ($d=3$), or $V_{h_x}(\Omega)$ is the space of piecewise linear/bilinear/trilinear, continuous functions $Q^1_{h_x}(\Omega)$ on intervals ($d=1$), quadrilaterals ($d=2$), hexahedra ($d=3$), where the functions ψ_i are the usual nodal basis functions satisfying $\psi_i(x_k) = \delta_{ik}$ for $i,k = 1,\ldots,\tilde{M}_x$. Recall that $S^1_{h_x}(0,L) = Q^1_{h_x}(0,L)$ on intervals. In addition, the subspace $V_{h_x,0}(\Omega) \subset V_{h_x}(\Omega)$ satisfies the homogeneous Dirichlet boundary condition, i.e.

$$V_{h_x,0}(\Omega) = V_{h_x}(\Omega) \cap H^1_0(\Omega).$$

After an ordering of the vertices $\{x_i\}_{i=1}^{\tilde{M}_x}$ in interior vertices $\{x_i\}_{i=1}^{M_x} \subset \Omega$ and boundary vertices $\{x_i\}_{i=M_x+1}^{\tilde{M}_x} \subset \partial\Omega$, this $H^1_0(\Omega)$ conforming subspace is written as

$$V_{h_x,0}(\Omega) = \operatorname{span}\{\psi_i\}_{i=1}^{M_x}. \tag{2.36}$$

A function $U_{h_x} \in V_{h_x,0}(\Omega)$ admits the representation

$$U_{h_x}(x) = \sum_{i=1}^{M_x} U_i \psi_i(x)$$

for $x \in \overline{\Omega}$. In the remainder of this work, $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote mass and stiffness matrices defined via

$$M_{h_x}[i,j] = \langle \psi_j, \psi_i \rangle_{L^2(\Omega)} \tag{2.37}$$

for $i,j = 1,\ldots,M_x$, and

$$A_{h_x}[i,j] = \langle \nabla_x \psi_j, \nabla_x \psi_i \rangle_{L^2(\Omega)} \tag{2.38}$$

for $i,j = 1,\ldots,M_x$. The L^2 projection

$$Q_{h_x} \colon\ L^2(\Omega) \to V_{h_x,0}(\Omega)$$

on the piecewise linear, continuous functions, satisfying homogeneous Dirichlet boundary conditions, is given as the solution of the variational formulation to find $Q_{h_x}u \in V_{h_x,0}(\Omega)$ such that

$$\langle Q_{h_x}u, v_{h_x}\rangle_{L^2(\Omega)} = \langle u, v_{h_x}\rangle_{L^2(\Omega)} \tag{2.39}$$

for all $v_{h_x} \in V_{h_x,0}(\Omega)$, satisfying the stability estimate

$$\|Q_{h_x}u\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)},$$

where $u \in L^2(\Omega)$ is a given function.

2.8 Discretisations in Space and Time

As in Section 2.7, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0,L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. Hence, the space-time cylinder $Q = \Omega \times (0,T) \subset \mathbb{R}^{d+1}$ is polygonal for $d = 1$, or polyhedral for $d = 2$, or polychoral for $d = 3$. For this situation, different discretisations in space and time are introduced as follows.

First, consider a sequence $(\mathcal{T}_N)_N$ of admissible, shape regular and globally quasi-uniform decompositions

$$\overline{Q} = \overline{\mathcal{T}_N} = \bigcup_{\ell=1}^{N} \overline{q}_\ell$$

with N space-time elements $q_\ell \subset \mathbb{R}^{d+1}$, where q_ℓ is a triangle for $d = 1$ or a tetrahedron for $d = 2$ or a pentatope for $d = 3$, see [117, 142]. In addition, $\tilde{M}$ is the number of vertices $\{(x_i,t_i)\}_{i=1}^{\tilde{M}}$ of the decomposition and $h = \max_{\ell=1,\dots,N} h_\ell$ is the global mesh size, where the local mesh sizes are given by

$$h_\ell = \left(\int_{q_\ell} \mathrm{d}x\mathrm{d}t \right)^{1/(d+1)} \qquad \text{for } \ell = 1,\dots,N.$$

The space

$$S_h^1(Q) = \operatorname{span}\{\psi_i\}_{i=1}^{\tilde{M}} \subset H^1(Q)$$

is the space of piecewise linear, continuous functions on triangles ($d = 1$), tetrahedra ($d = 2$) or pentatopes ($d = 3$), where $\{\psi_i\}_{i=1}^{\tilde{M}}$ is the nodal basis, i.e. $\psi_i(x_k,t_k) = \delta_{ki}$ for $k,i = 1,\dots,\tilde{M}$. A function $u_h \in S_h^1(Q)$ admits the representation

$$u_h(x,t) = \sum_{i=1}^{\tilde{M}} u_i \psi_i(x,t) \qquad \text{for } (x,t) \in \overline{Q}.$$

For the space $S_h^1(Q)$, the space-time interpolation operator

$$I_h\colon\ C(\overline{Q}) \to S_h^1(Q)$$

is defined by

$$I_h u(x,t) = \sum_{i=1}^{\tilde{M}} u(x_i,t_i)\,\psi_i(x,t) \tag{2.40}$$

for $(x,t) \in \overline{Q}$.

Second, consider for a tensor-product ansatz a sequence $(\mathcal{T}_N)_N$ of admissible decompositions

$$\overline{Q} = \overline{\mathcal{T}_N} = \overline{\Omega} \times [0,T] = \left(\bigcup_{i=1}^{N_x} \overline{\omega}_i\right) \times \left(\bigcup_{\ell=1}^{N_t} \overline{\tau}_\ell\right) \tag{2.41}$$

with $N = N_x \cdot N_t$ space-time elements, where the time intervals $\tau_\ell = (t_{\ell-1}, t_\ell)$ are defined via the decomposition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N_t-1} < t_{N_t} = T$$

of the time interval $(0,T)$ and where the spatial domain Ω is decomposed as

$$\overline{\Omega} = \bigcup_{i=1}^{N_x} \overline{\omega}_i.$$

Here, the spatial elements $\omega_i \subset \mathbb{R}^d$ are intervals for $d = 1$, triangles or quadrilaterals for $d = 2$ and tetrahedra or hexahedra for $d = 3$. The local mesh sizes are $h_{t,\ell} = t_\ell - t_{\ell-1}$ for $\ell = 1,\ldots,N_t$ and

$$h_{x,i} = \left(\int_{\omega_i} \mathrm{d}x\right)^{1/d} \qquad \text{for } i = 1,\ldots,N_x.$$

Furthermore, the global mesh size is given as $h = \max\{h_x, h_t\}$ with $h_t = \max_{\ell=1,\ldots,N_t} h_{t,\ell}$ and $h_x = \max_{i=1,\ldots,N_x} h_{x,i}$. It is always assumed that the sequence $(\mathcal{T}_N)_N$ of decompositions is shape regular and globally quasi-uniform. Next, consider the finite element space

$$Q_h^1(Q) := V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0,T) \tag{2.42}$$

of piecewise multilinear, continuous functions, where $V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ is the space of piecewise linear, continuous functions $S_{h_x,0}^1(\Omega)$ on intervals ($d = 1$), triangles ($d = 2$), tetrahedra ($d = 3$), or $V_{h_x,0}(\Omega)$ is the space of piecewise linear/bilinear/trilinear, continuous functions $Q_{h_x,0}^1(\Omega)$ on intervals ($d = 1$), quadrilaterals ($d = 2$), hexahedra ($d = 3$), fulfilling in both cases the homogeneous Dirichlet boundary conditions on the lateral boundary Σ, see (2.36). Recall that $S^1_{h_x,0}(0,L) = Q^1_{h_x,0}(0,L)$ on intervals. A function $u_h \in Q^1_h(Q)$ admits the representation

$$u_h(x,t) = \sum_{\ell=0}^{N_t} \underbrace{\sum_{j=1}^{M_x} u_j^\ell \psi_j(x)}_{=:U_{h_x,\ell}(x)} \varphi_\ell(t) = \sum_{\ell=0}^{N_t} U_{h_x,\ell}(x)\varphi_\ell(t) \quad \text{for } (x,t) \in \overline{Q}, \tag{2.43}$$

where φ_ℓ is a piecewise linear, continuous nodal basis function with respect to time and ψ_j is a piecewise linear/bilinear/trilinear, continuous nodal basis function with respect to space with $M_x := \dim V_{h_x,0}(\Omega)$. Furthermore, it holds $U_{h_x,\ell} \in V_{h_x,0}(\Omega)$ for $\ell = 0,\dots,N_t$.

The extended time interpolation operator

$$I_{h_t}\colon\, C([0,T];L^2(\Omega)) \to L^2(\Omega) \otimes S^1_{h_t}(0,T)$$

is defined by

$$I_{h_t} u(x,t) = \sum_{\ell=0}^{N_t} u(x,t_\ell)\varphi_\ell(t) \tag{2.44}$$

for $(x,t) \in \overline{Q}$.

To derive space-time error estimates, different space-time projections are needed. As a first space-time projection, the $H^{1,1}_{0;0,}(Q)$ projection

$$Q^1_h\colon\, H^{1,1}_{0;0,}(Q) \to Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$$

is introduced as the solution of the variational formulation to find $Q^1_h v \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ such that

$$\left\langle \partial_t Q^1_h v, \partial_t v_h \right\rangle_{L^2(Q)} + \left\langle \nabla_x Q^1_h v, \nabla_x v_h \right\rangle_{L^2(Q)} = \langle \partial_t v, \partial_t v_h \rangle_{L^2(Q)} + \langle \nabla_x v, \nabla_x v_h \rangle_{L^2(Q)} \tag{2.45}$$

for all $v_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$, where $v \in H^{1,1}_{0;0,}(Q)$ is given. There hold the stability estimate

$$\left| Q^1_h v \right|_{H^1(Q)} \le |v|_{H^1(Q)} \quad \forall v \in H^{1,1}_{0;0,}(Q)$$

and if Ω is sufficiently regular, for $s \in [0,1]$ the standard error estimate

$$\|v - Q^1_h v\|_{L^2(Q)} \le c\, h^{1+s} \|v\|_{H^{1+s}(Q)} \tag{2.46}$$

for $v \in H^{1,1}_{0;0,}(Q) \cap H^{1+s}(Q)$ with a constant $c > 0$.

As a second space-time projection, the $H^1_{0,}$-H^1_0 projection $Q^1_{h_t} Q^1_{h_x} v \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ is introduced analogously to [16, Section 2], where the function $v \in H^{1,1}_{0;0,}(Q)$ is sufficiently smooth. First, for a given function $v \in L^2(0,T;H^1_0(\Omega))$, the extended H^1_0 projection $Q^1_{h_x} v \in V_{h_x,0}(\Omega) \otimes L^2(0,T)$ is defined by

$$\left\langle \nabla_x Q^1_{h_x} v, \nabla_x v_{h_x} \right\rangle_{L^2(Q)} = \langle \nabla_x v, \nabla_x v_{h_x} \rangle_{L^2(Q)} \tag{2.47}$$

for all $v_{h_x} \in V_{h_x,0}(\Omega) \otimes L^2(0,T)$. Note that $V_{h_x,0}(\Omega) \otimes L^2(0,T)$ is, as a tensor-product of the separable Hilbert spaces $\Big(V_{h_x,0}(\Omega), \langle \nabla_x(\cdot), \nabla_x(\cdot)\rangle_{L^2(\Omega)}\Big)$ and $\Big(L^2(0,T), \langle \cdot,\cdot\rangle_{L^2(0,T)}\Big)$, again a Hilbert space, where the inner product is given by $\langle \nabla_x(\cdot), \nabla_x(\cdot)\rangle_{L^2(Q)}$. Hence, by the Lax-Milgram Theorem, it follows the well-posedness of the extended H_0^1 projection

$$Q_{h_x}^1 \colon \; L^2(0,T;H_0^1(\Omega)) \to V_{h_x,0}(\Omega) \otimes L^2(0,T),$$

satisfying the stability estimate

$$\|\nabla_x Q_{h_x}^1 v\|_{L^2(Q)} \le \|\nabla_x v\|_{L^2(Q)}.$$

Furthermore, it holds for $s \in [0,1]$ the standard error estimate

$$\|v - Q_{h_x}^1 v\|_{L^2(Q)} \le c\, h_x^{1+s} \|v\|_{L^2(0,T;H^{1+s}(\Omega))} \tag{2.48}$$

for $v \in L^2(0,T;H_0^1(\Omega) \cap H^{1+s}(\Omega))$ with a constant $c > 0$, if Ω is sufficiently regular. Second, for a given function $v \in H_{0,}^1(0,T;L^2(\Omega))$, fulfilling the homogeneous initial condition, the extended $H_{0,}^1$ projection $Q_{h_t}^1 v \in L^2(\Omega) \otimes S_{h_t,0,}^1(0,T)$ is defined by

$$\big\langle \partial_t Q_{h_t}^1 v, \partial_t v_{h_t} \big\rangle_{L^2(Q)} = \langle \partial_t v, \partial_t v_{h_t} \rangle_{L^2(Q)} \tag{2.49}$$

for all $v_{h_t} \in L^2(\Omega) \otimes S_{h_t,0,}^1(0,T)$, satisfying the stability estimate

$$\|\partial_t Q_{h_t}^1 v\|_{L^2(Q)} \le \|\partial_t v\|_{L^2(Q)},$$

where the well-posedness of the extended $H_{0,}^1$ projection

$$Q_{h_t}^1 \colon \; H_{0,}^1(0,T;L^2(\Omega)) \to L^2(\Omega) \otimes S_{h_t,0,}^1(0,T)$$

is shown analogously as for $Q_{h_x}^1$. In addition, it holds for $s \in [0,1]$ the standard error estimate

$$\|v - Q_{h_t}^1 v\|_{L^2(Q)} \le c\, h_t^{1+s} \|v\|_{H^{1+s}(0,T;L^2(\Omega))} \tag{2.50}$$

for $v \in H_{0,}^1(0,T;L^2(\Omega)) \cap H^{1+s}(0,T;L^2(\Omega))$ with a constant $c > 0$. Testing the variational formulation (2.49) with the test function

$$v_{h_t}(x,t) = \begin{cases} z(x) \cdot t & \text{for } (x,t) \in \Omega \times [0,t_\ell), \\ z(x) \cdot t_\ell & \text{for } (x,t) \in \Omega \times [t_\ell, T] \end{cases}$$

with an arbitrary function $z \in L^2(\Omega)$ for $\ell \in \{1,\ldots,N_t\}$ yields

$$\int_\Omega z(x) Q_{h_t}^1 v(x,t_\ell)\mathrm{d}x = \int_\Omega z(x) \int_0^{t_\ell} \partial_t Q_{h_t}^1 v(x,t)\mathrm{d}t\mathrm{d}x = \int_\Omega z(x) \int_0^{t_\ell} \partial_t v(x,t)\mathrm{d}t\mathrm{d}x = \int_\Omega z(x) v(x,t_\ell)\mathrm{d}x$$

and hence, with the fundamental lemma of calculus of variations follows the interpolation property

$$Q^1_{h_t} v(x,t_\ell) = v(x,t_\ell).$$

In other words, it holds $Q^1_{h_t} = I_{h_t}$ for functions in $H^1_{0,}(0,T;L^2(\Omega))$, see (2.44).

Lemma 2.8.1. *The following properties of the projection operators $Q^1_{h_x}$ and $Q^1_{h_t}$ are true:*

1. *For a function $v \in H^1(0,T;H^1_0(\Omega))$, it holds that $Q^1_{h_x} v \in V_{h_x,0}(\Omega) \otimes H^1(0,T)$. If, in addition, v satisfies $v \in H^1_{0,}(0,T;H^1_0(\Omega))$, then it follows $Q^1_{h_x} v \in V_{h_x,0}(\Omega) \otimes H^1_{0,}(0,T)$.*
2. *For a function $v \in H^1_{0,}(0,T;H^1(\Omega))$, it holds that $Q^1_{h_t} v \in H^1(\Omega) \otimes S^1_{h_t,0,}(0,T)$ and if, in addition, v satisfies $v \in H^1_{0,}(0,T;H^1_0(\Omega))$, then $Q^1_{h_t} v \in H^1_0(\Omega) \otimes S^1_{h_t,0,}(0,T)$.*

Proof. For the first part, consider a representation for $(x,t) \in Q$

$$Q^1_{h_x} v(x,t) = \sum_{j=1}^{M_x} \tilde{V}_j(t) \tilde{\psi}_j(x), \tag{2.51}$$

where $\{\tilde{\psi}_j\}_{j=1}^{M_x}$ is an orthonormal basis of $V_{h_x,0}(\Omega)$ with respect to $\langle \nabla_x(\cdot), \nabla_x(\cdot)\rangle_{L^2(\Omega)}$ and $\tilde{V}_j \in L^2(0,T)$. To show that a weak derivative of $\tilde{V}_j$ exists, fix an index $j \in \{1,\ldots,M_x\}$. The definition of the extended H^1_0 projection $Q^1_{h_x}$ in (2.47) for $v_{h_x}(x,t) = \tilde{\psi}_j(x) z(t)$ gives

$$\int_0^T \tilde{V}_j(t) z(t)\,\mathrm{d}t = \int_0^T \left\langle \nabla_x Q^1_{h_x} v(\cdot,t), \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} z(t)\,\mathrm{d}t = \int_0^T \left\langle \nabla_x v(\cdot,t), \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} z(t)\,\mathrm{d}t \tag{2.52}$$

for all $z \in C^\infty_0(0,T)$. So, the fundamental lemma of calculus of variations for (2.52) yields

$$\tilde{V}_j(t) = \left\langle \nabla_x v(\cdot,t), \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \tag{2.53}$$

for $t \in (0,T)$. For $z = \partial_t \tilde{z}$ in (2.52), it follows

$$\int_0^T \tilde{V}_j(t) \partial_t \tilde{z}(t)\,\mathrm{d}t = \int_0^T \left\langle \nabla_x v(\cdot,t), \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \partial_t \tilde{z}(t)\,\mathrm{d}t = -\int_0^T \left\langle \partial_t \nabla_x v(\cdot,t), \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \tilde{z}(t)\,\mathrm{d}t$$

for all $\tilde{z} \in C^\infty_0(0,T)$, i.e. $\partial_t \tilde{V}_j(t) = \left\langle \partial_t \nabla_x v(\cdot,t), \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)}$. Furthermore, it holds with the Cauchy-Schwarz inequality

$$\begin{aligned} \|\partial_t \tilde{V}_j\|^2_{L^2(0,T)} &= \int_0^T \left(\left\langle \partial_t \nabla_x v(\cdot,t), \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \right)^2 \mathrm{d}t \\ &\le \int_0^T \|\partial_t \nabla_x v(\cdot,t)\|^2_{L^2(\Omega)} \left\|\nabla_x \tilde{\psi}_j\right\|^2_{L^2(\Omega)} \mathrm{d}t = \|\partial_t \nabla_x v\|^2_{L^2(Q)} \left\|\nabla_x \tilde{\psi}_j\right\|^2_{L^2(\Omega)} < \infty \end{aligned}$$

and so, the first assertion. If, in addition, $v(\cdot,0)=0$ in $H^1_0(\Omega)$, then the continuity of $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ and of the trace operator yields

$$\tilde{V}_j(t)_{|t=0} = \left\langle \nabla_x v(\cdot,t)_{|t=0}, \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} = 0.$$

The second part follows in an analogous way. □

The next lemma shows that $Q^1_{h_t} Q^1_{h_x} v \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ is well-defined under regularity assumptions for v and that the operators in space and time commute, where the proof is analogous to [16, Lemma 2.1, page 261].

Lemma 2.8.2. *For a given function $v \in H^{1,1}_{0;0,}(Q)$ with the regularity $\partial_t v \in L^2(0,T;H^1_0(\Omega))$ and $\partial_{x_m} v \in H^1_{0,}(0,T;L^2(\Omega))$ for $m=1,\dots,d$, there hold*

1. *the relation $\partial_t Q^1_{h_x} v = Q^1_{h_x} \partial_t v \in V_{h_x,0}(\Omega) \otimes L^2(0,T)$,*
2. *the relation $\partial_{x_m} Q^1_{h_t} v = Q^1_{h_t} \partial_{x_m} v \in L^2(\Omega) \otimes S^1_{h_t,0,}(0,T)$ for $m=1,\dots,d$ and*
3. *the relation $Q^1_{h_t} Q^1_{h_x} v = Q^1_{h_x} Q^1_{h_t} v \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$. In particular, the space-time projections $Q^1_{h_t} Q^1_{h_x} v$ and $Q^1_{h_x} Q^1_{h_t} v$ are well-defined.*

Furthermore, the error estimate

$$\|v - Q^1_{h_t} Q^1_{h_x} v\|_{L^2(Q)} \le \|v - Q^1_{h_t} v\|_{L^2(Q)} + \|v - Q^1_{h_x} v\|_{L^2(Q)} + c\, h_x h_t \|\partial_t \nabla_x v\|_{L^2(Q)}$$

with a constant $c>0$ is valid.

Proof. For the proof of the first relation, recall that

$$\partial_t Q^1_{h_x} v \in V_{h_x,0}(\Omega) \otimes L^2(0,T)$$

by Lemma 2.8.1. Consider (2.47) for $\partial_t v \in L^2(0,T;H^1_0(\Omega))$ and with integration by parts follows

$$\begin{aligned}
\left\langle \nabla_x Q^1_{h_x} \partial_t v, \nabla_x v_{h_x} \right\rangle_{L^2(Q)} &= \langle \nabla_x \partial_t v, \nabla_x v_{h_x} \rangle_{L^2(Q)} = -\langle \nabla_x v, \nabla_x \partial_t v_{h_x} \rangle_{L^2(Q)} \\
&= -\left\langle \nabla_x Q^1_{h_x} v, \nabla_x \partial_t v_{h_x} \right\rangle_{L^2(Q)} = \left\langle \nabla_x \partial_t Q^1_{h_x} v, \nabla_x v_{h_x} \right\rangle_{L^2(Q)} \\
&= \left\langle \nabla_x Q^1_{h_x} \partial_t Q^1_{h_x} v, \nabla_x v_{h_x} \right\rangle_{L^2(Q)}
\end{aligned}$$

for all $v_{h_x} \in V_{h_x,0}(\Omega) \otimes C^\infty_0(0,T)$. Because of the density of $C^\infty_0(0,T)$ in $L^2(0,T)$, it holds $Q^1_{h_x} \partial_t v = Q^1_{h_x} \partial_t Q^1_{h_x} v$ with $\partial_t Q^1_{h_x} v \in V_{h_x,0}(\Omega) \otimes L^2(0,T)$. So, the first relation is proven.

The proof of the second relation is analogous to the proof of the first relation.

For the third relation, note that it holds by Lemma 2.8.1 that

$$Q^1_{h_t} v = \sum_{\ell=1}^{N_t} V^\ell \varphi_\ell \in H^1_0(\Omega) \otimes S^1_{h_t,0,}(0,T) \subset H^1_{0,}(0,T;H^1_0(\Omega))$$

with coefficients $V^\ell \in H^1_0(\Omega)$ and so,

$$Q^1_{h_x} Q^1_{h_t} v \in V_{h_x,0}(\Omega) \otimes H^1_{0,}(0,T) \subset H^1_{0,}(0,T;H^1_0(\Omega))$$

is well-defined. With the representations as in (2.51) and in (2.53) for $Q^1_{h_t} v$, there follow for $(x,t) \in \overline{Q}$

$$\begin{aligned} Q^1_{h_x} Q^1_{h_t} v(x,t) &= \sum_{j=1}^{M_x} \tilde{V}_j(t) \tilde{\psi}_j(x) = \sum_{j=1}^{M_x} \left\langle \nabla_x Q^1_{h_t} v(\cdot,t), \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \tilde{\psi}_j(x) \\ &= \sum_{\ell=1}^{N_t} \sum_{j=1}^{M_x} \left\langle \nabla_x V^\ell, \nabla_x \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \tilde{\psi}_j(x) \varphi_\ell(t) \end{aligned}$$

and so, $Q^1_{h_x} Q^1_{h_t} v \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$. Analogously, $Q^1_{h_t} Q^1_{h_x} v \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ is well-defined. With the help of the first relation, the second relation and the definitions (2.47), (2.49), there hold

$$\begin{aligned} \sum_{m=1}^{d} \langle \partial_t \partial_{x_m} Q^1_{h_t} Q^1_{h_x} v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} &= \sum_{m=1}^{d} \langle \partial_t Q^1_{h_t} \partial_{x_m} Q^1_{h_x} v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} \\ &= \sum_{m=1}^{d} \langle \partial_t \partial_{x_m} Q^1_{h_x} v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} \\ &= \sum_{m=1}^{d} \langle \partial_t \partial_{x_m} v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} \end{aligned}$$

and analogously,

$$\sum_{m=1}^{d} \langle \partial_t \partial_{x_m} Q^1_{h_x} Q^1_{h_t} v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)} = \sum_{m=1}^{d} \langle \partial_t \partial_{x_m} v, \partial_t \partial_{x_m} v_h \rangle_{L^2(Q)}$$

for all $v_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$. Hence, also the third relation is true.

The error estimate follows with the triangle inequality, the first and second relation and standard error estimates for $Q^1_{h_t}$ and $Q^1_{h_x}$ from

$$\begin{aligned} \|v - Q^1_{h_t} Q^1_{h_x} v\|_{L^2(Q)} &\le \|v - Q^1_{h_t} v\|_{L^2(Q)} + \|v - Q^1_{h_x} v\|_{L^2(Q)} \\ &\quad + \underbrace{\|(v - Q^1_{h_x} v) - Q^1_{h_t}(v - Q^1_{h_x} v)\|_{L^2(Q)}}_{\le c_1 h_t \|\partial_t (v - Q^1_{h_x} v)\|_{L^2(Q)} \le c_1 c_2 h_t h_x \|\partial_t \nabla_x v\|_{L^2(Q)}} \end{aligned}$$

with constants $c_1, c_2 > 0$ independent of h_t and h_x. □

Next, for a function $v \in C([0,T];L^2(\Omega))$, investigate $I_{h_t} Q_{h_x} v \in Q_h^1(Q) \cap H_{0;}^{1,1}(Q)$. Therefore, for a given function $v \in L^2(Q)$, the extended L^2 projection $Q_{h_x} v \in V_{h_x,0}(\Omega) \otimes L^2(0,T)$ is defined by

$$\langle Q_{h_x} v, v_{h_x} \rangle_{L^2(Q)} = \langle v, v_{h_x} \rangle_{L^2(Q)} \tag{2.54}$$

for all $v_{h_x} \in V_{h_x,0}(\Omega) \otimes L^2(0,T)$, satisfying the stability estimate

$$\|Q_{h_x} v\|_{L^2(Q)} \le \|v\|_{L^2(Q)}, \tag{2.55}$$

where the well-posedness of the extended L^2 projection

$$Q_{h_x} \colon \; L^2(Q) \to V_{h_x,0}(\Omega) \otimes L^2(0,T)$$

is analysed as for the projection $Q_{h_x}^1$ given in (2.47). Furthermore, it holds for $s \in [0,1]$ the standard error estimate

$$\|v - Q_{h_x} v\|_{L^2(Q)} \le c\, h_x^{1+s} \|v\|_{L^2(0,T;H^{1+s}(\Omega))}$$

for $v \in L^2(0,T;H_0^1(\Omega) \cap H^{1+s}(\Omega))$ with a constant $c > 0$. The following properties of the projection operator Q_{h_x} are true:

Lemma 2.8.3. *For a function $v \in C([0,T];L^2(\Omega))$, it holds that $Q_{h_x} v \in V_{h_x,0}(\Omega) \otimes C[0,T]$. In addition, for a function $v \in H^r(0,T;L^2(\Omega))$, there hold $Q_{h_x} v \in V_{h_x,0}(\Omega) \otimes H^r(0,T)$ and $\|Q_{h_x} v\|_{H^r(0,T;L^2(\Omega))} \le \|v\|_{H^r(0,T;L^2(\Omega))}$ for $r \ge 0$. In particular, for $r = k \in \mathbb{N}$, the relation $\partial_t^k Q_{h_x} v = Q_{h_x} \partial_t^k v$ is valid.*

Proof. The proof is analogous to the proof of Lemma 2.8.1. More precisely, take an orthonormal basis $\{\tilde{\psi}_j\}_{j=1}^{M_x}$ of $V_{h_x,0}(\Omega)$ with respect to $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ and write

$$Q_{h_x} v(x,t) = \sum_{j=1}^{M_x} \tilde{V}_j(t) \tilde{\psi}_j(x) \tag{2.56}$$

for $(x,t) \in Q$ with $\tilde{V}_j \in L^2(0,T)$. For the first assertion, it remains to show that $\tilde{V}_j \in C[0,T]$. For that reason, the equation (2.54) gives for $v_{h_x}(x,t) = \tilde{\psi}_j(x) z(t)$

$$\int_0^T \tilde{V}_j(t) z(t) \mathrm{d}t = \left\langle Q_{h_x} v, \tilde{\psi}_j z \right\rangle_{L^2(Q)} = \left\langle v, \tilde{\psi}_j z \right\rangle_{L^2(Q)} = \int_0^T \left\langle v(\cdot,t), \tilde{\psi}_j \right\rangle_{L^2(\Omega)} z(t) \mathrm{d}t \tag{2.57}$$

for each $z \in L^2(\Omega)$ and hence, the fundamental lemma of calculus of variations yields

$$\tilde{V}_j(t) = \left\langle v(\cdot,t), \tilde{\psi}_j \right\rangle_{L^2(\Omega)} = \int_\Omega v(x,t) \tilde{\psi}_j(x) \mathrm{d}x \tag{2.58}$$

for $t \in (0,T)$. Because for almost all $x \in \Omega$ and all $t \in [0,T]$ it holds

$$\left|v(x,t)\tilde{\psi}_j(x)\right| \le \max_{s\in[0,T]} |v(x,s)| \cdot \max_{\tilde{x}\in\overline{\Omega}} \left|\psi_j(\tilde{x})\right| =: g(x)$$

with $g \in L^2(\Omega)$, the theorem of continuity for parameter integrals [47, Satz 5.6, page 147] gives $\tilde{V}_j \in C[0,T]$ and hence, the first assertion.

For the last statement, an interpolation argument is used, see [102, Théorème 5.1, page 32 in Chapitre 1]. For $r = 0$, the assertion is trivial. Therefore, let $v \in H^k(0,T;L^2(\Omega))$ be given for $k \in \mathbb{N}$, i.e. $r = k$. Because of the representation (2.56), it remains to prove that $\partial_t^k \tilde{V}_j \in L^2(0,T)$. For $z = \partial_t^k \tilde{z}$ in (2.57), it follows

$$\int_0^T \tilde{V}_j(t)\partial_t^k \tilde{z}(t)\mathrm{d}t = \int_0^T \left\langle v(\cdot,t), \tilde{\psi}_j\right\rangle_{L^2(\Omega)} \partial_t^k \tilde{z}(t)\mathrm{d}t = (-1)^k \int_0^T \left\langle \partial_t^k v(\cdot,t), \tilde{\psi}_j\right\rangle_{L^2(\Omega)} \tilde{z}(t)\mathrm{d}t$$

for all $\tilde{z} \in C_0^\infty(0,T)$, i.e. $\partial_t^k \tilde{V}_j(t) = \left\langle \partial_t^k v(\cdot,t), \tilde{\psi}_j\right\rangle_{L^2(\Omega)}$. Furthermore, it holds with the Cauchy-Schwarz inequality

$$\begin{aligned}\left\|\partial_t^k \tilde{V}_j\right\|^2_{L^2(0,T)} &= \int_0^T \left(\left\langle \partial_t^k v(\cdot,t), \tilde{\psi}_j\right\rangle_{L^2(\Omega)}\right)^2 \mathrm{d}t \\ &\le \int_0^T \left\|\partial_t^k v(\cdot,t)\right\|^2_{L^2(\Omega)} \underbrace{\|\tilde{\psi}_j\|^2_{L^2(\Omega)}}_{=1} \mathrm{d}t = \left\|\partial_t^k v\right\|^2_{L^2(Q)} < \infty.\end{aligned}$$

The relation $\partial_t^k Q_{h_x} v = Q_{h_x} \partial_t^k v$ is proven analogously to the relations of Lemma 2.8.2, and so, the assertion for $r = k$, where for the stability $\|Q_{h_x} v\|_{H^k(0,T;L^2(\Omega))} \le \|v\|_{H^k(0,T;L^2(\Omega))}$, the stability (2.55) is used for $\partial_t^l Q_{h_x} v = Q_{h_x} \partial_t^l v$, $l = 0,\dots,k$. For arbitrary $r > 0$, the statement follows by interpolation. □

For a given function $v \in C([0,T];L^2(\Omega))$, Lemma 2.8.3 ensures that $I_{h_t} Q_{h_x} v \in Q_h^1(Q)$, given by

$$I_{h_t} Q_{h_x} v(x,t) = \sum_{\ell=0}^{N_t} Q_{h_x} v(x,t_\ell)\varphi_\ell(t) = \sum_{\ell=0}^{N_t} \sum_{j=1}^{M_x} V_\ell^j \psi_j(x)\varphi_\ell(t) \tag{2.59}$$

for $(x,t) \in \overline{Q}$, is well-defined.

Lemma 2.8.4. *For a given function $v \in C([0,T];L^2(\Omega))$, there holds*

$$I_{h_t} Q_{h_x} v = Q_{h_x} I_{h_t} v \in Q_h^1(Q).$$

Furthermore, for $v \in L^2(0,T;H^s(\Omega)) \cap H^r(0,T;L^2(\Omega))$ *with* $s \in [0,2]$ *and* $r \in (1/2,2]$, *the error estimate*

$$\|v - I_{h_t} Q_{h_x} v\|_{L^2(Q)} \le c_1\, h_x^s \|v\|_{L^2(0,T;H^s(\Omega))} + c_2\, h_t^r \|v\|_{H^r(0,T;L^2(\Omega))}$$

with constants $c_1, c_2 > 0$ *independent of* h_x *and* h_t *is valid.*

Proof. Take an orthonormal basis $\{\tilde{\psi}_j\}_{j=1}^{M_x}$ of $V_{h_x,0}(\Omega)$ with respect to $\langle \cdot,\cdot \rangle_{L^2(\Omega)}$ and write with (2.58)

$$Q_{h_x} v(x,t) = \sum_{j=1}^{M_x} \tilde{V}_j(t) \tilde{\psi}_j(x) = \sum_{j=1}^{M_x} \left\langle v(\cdot,t), \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \tilde{\psi}_j(x) \quad \text{for } (x,t) \in \overline{Q},$$

where $\tilde{V}_j \in C[0,T]$, see Lemma 2.8.3. With this representation, there follow for $(x,t) \in \overline{Q}$

$$I_{h_t} Q_{h_x} v(x,t) = \sum_{\ell=0}^{N_t} \sum_{j=1}^{M_x} \left\langle v(\cdot,t_\ell), \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \tilde{\psi}_j(x) \varphi_\ell(t)$$

and

$$Q_{h_x} I_{h_t} v(x,t) = \sum_{j=1}^{M_x} \left\langle I_{h_t} v(\cdot,t), \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \tilde{\psi}_j(x) = \sum_{\ell=0}^{N_t} \sum_{j=1}^{M_x} \left\langle v(\cdot,t_\ell), \tilde{\psi}_j \right\rangle_{L^2(\Omega)} \tilde{\psi}_j(x) \varphi_\ell(t),$$

i.e. $I_{h_t} Q_{h_x} v = Q_{h_x} I_{h_t} v \in Q_h^1(Q)$.

The error estimate follows with the triangle inequality, standard error estimates for I_{h_t} and Q_{h_x} and Lemma 2.8.3 from

$$\begin{aligned}\|v - I_{h_t} Q_{h_x} v\|_{L^2(Q)} &\le \|v - Q_{h_x} v\|_{L^2(Q)} + \|Q_{h_x} v - I_{h_t} Q_{h_x} v\|_{L^2(Q)} \\ &\le c_1\, h_x^s \|v\|_{L^2(0,T;H^s(\Omega))} + c_2\, h_t^r \|Q_{h_x} v\|_{H^r(0,T;L^2(\Omega))} \\ &\le c_1\, h_x^s \|v\|_{L^2(0,T;H^s(\Omega))} + c_2\, h_t^r \|v\|_{H^r(0,T;L^2(\Omega))}\end{aligned}$$

with constants $c_1, c_2 > 0$ independent of h_x and h_t. □

2.9 Variational Methods

Let X and Y be real Banach spaces endowed with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Furthermore, let $a(\cdot,\cdot)\colon X \times Y \to \mathbb{R}$ be a given continuous bilinear form and let $F\colon Y \to \mathbb{R}$ be a given continuous linear form. Consider the abstract variational problem to find $u \in X$ such that

$$a(u,v) = F(v) \quad \forall v \in Y. \tag{2.60}$$

A continuous, linear operator $A\colon X \to Y'$ is associated with $a(\cdot,\cdot)\colon X \times Y \to \mathbb{R}$ by setting

$$\langle Au, v\rangle_{Y'\times Y} := a(u,v) \quad \text{for all } u \in X, v \in Y,$$

where $\langle\cdot,\cdot\rangle_{Y'\times Y}$ is the duality pairing.

Theorem 2.9.1 (Nečas)**.** *Let $(X, \|\cdot\|_X)$ be a real Banach space, $(Y, \|\cdot\|_Y)$ be a real, reflexive Banach space and $a(\cdot,\cdot)\colon X \times Y \to \mathbb{R}$ be a continuous bilinear form. Then the following statements are equivalent:*

1. *For every given continuous linear form $F\colon Y \to \mathbb{R}$, there exists a unique solution $u \in X$ of* (2.60)*, satisfying*
$$\|u\|_X \le \tilde{c}_s \|F\|_{Y'}$$
with a constant $\tilde{c}_s > 0$.
2. *The continuous, linear operator $A\colon X \to Y'$ associated with $a(\cdot,\cdot)\colon X \times Y \to \mathbb{R}$ is an isomorphism, i.e. A is bijective and $A^{-1}\colon Y' \to X$ is continuous.*
3. *For the continuous bilinear form $a(\cdot,\cdot)\colon X \times Y \to \mathbb{R}$, there hold:*
   - *There exists a constant $c_s > 0$ such that*
   $$\inf_{0 \ne u \in X} \sup_{0 \ne v \in Y} \frac{|a(u,v)|}{\|u\|_X \|v\|_Y} \ge c_s.$$
   - *For each $v \in Y$, $v \ne 0$, there exists an element $u \in X$ such that $a(u,v) \ne 0$.*

Moreover, it holds $\tilde{c}_s = \frac{1}{c_s}$.

Proof. If X, Y are real Hilbert spaces, a proof is contained in [118, Théorème 3.1, page 318] or in [29, Satz 3.6, page 119]. For the general case, see [51, Theorem 2.6, page 85]. □

Remark 2.9.2. *For complex Hilbert spaces X and Y, a continuous sesquilinear form $a(\cdot,\cdot)\colon X \times Y \to \mathbb{C}$ and a continuous linear form $F\colon Y \to \mathbb{C}$, a proof is included in [131, Theorem 2.1.44, page 36].*

3 HEAT EQUATION

The main focus of this chapter are space-time variational formulations and conforming discretisations for parabolic problems. First, a highly non-exhaustive list of references and second, an overview of the sections of this chapter are given, where for each section the relevant literature is cited. Here, the model problem for a parabolic partial differential equation is the homogeneous Dirichlet problem for the heat equation,

$$\left.\begin{aligned} \partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma = \Gamma \times [0,T], \\ u(x,0) &= u_0(x) && \text{for } x \in \Omega, \end{aligned}\right\} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, $T > 0$ is a finite time, u_0 is a given initial condition and f is a given right-hand side. To compute an approximate solution of the heat equation (3.1), different numerical schemes including different approaches of the underlying mathematical framework are available. On the one hand, some of them are repeated in this chapter, but on the other hand, powerful tools like semigroup theory as in [91, 123] on the continuous part or on the discretisation side, any kind of discontinuous Galerkin methods [48–50,56,78,85,116,134,135] or finite difference methods [65, 97, 147] or boundary element methods are not in the scope of this work. For boundary integral equations and boundary element methods for the heat equation, see [14, 35, 40, 120] and in addition, see [39, 76, 107, 109, 125, 129, 149]. Furthermore, all approaches where the heat equation (3.1) is reformulated as a first-order system also in the spatial variables are excluded in this work, see, e.g., [25] and the references therein. In addition, see also the approaches in [10, 11, 38, 99, 100, 112, 137, 154].

Outline of Chapter 3

The remainder of this chapter examines the heat equation (3.1) as follows:

In Section 3.1 a pointwise spatial variational formulation coming from a so-called Galerkin method [36, 97, 98, 102, 160, 162] and time stepping schemes [65, 74, 85, 93, 150] are cited, see also [9, 16, 17, 43, 54, 79, 108, 148, 151, 159]. In Section 3.2 a space-time variational formulation with ansatz spaces of Bochner type, analysed via the inf-sup theory and including a stable space-time discretisation, is formulated, see [51, 137, 142–144]. In Section 3.3 an anisotropic space-time variational formulation [35, 98, 102, 103], which is obtained by a transposition and interpolation argument, is replied, and also an example for an unstable numerical scheme, which is derived by the (natural) usage of conforming, piecewise linear, continuous ansatz and test functions, is given. Nevertheless, this anisotropic formulation leads to boundary integral equations in anisotropic Sobolev spaces, where the single layer and hypersingular boundary integral operators are elliptic, see [35, 40, 120]. For the last and main Section 3.4, see [145], a motivation is given by transmission problems. For transmission problems of the heat equation a coupling of finite and boundary element methods is a natural choice, i.e. the finite element method is used for the interior problem as in (3.1) and the boundary element method is used for the corresponding exterior problem. For the finite element part, the anisotropic variational formulation of Section 3.3 is not well-suited on the discretisation level because it seems that a stable finite element method is not available. On the one hand, the Bochner type variational formulation of Section 3.2 gives a stable finite element method, but on the other hand, from an analysis point of view, the boundary integral equations [35, 40, 120] and the variational formulation of Bochner type in Section 3.2 do not fit, i.e. the resulting trace spaces of the Bochner spaces are different from the anisotropic boundary spaces in Section 3.3. In other words, it seems that the analysis of a coupling of the corresponding discretisations in [35, 40, 120] and Section 3.2 is very difficult. To overcome the problem of non-fitting spaces, either the boundary integral equations are treated in trace spaces of the Bochner spaces of Section 3.2, or a stabilised finite element method of the anisotropic spaces of Section 3.3 is introduced. The second approach is the motivation of Section 3.4, where the main result is a symmetric and elliptic variational formulation and hence, a symmetric Galerkin discretisation of the first-order time derivative, see [145]. In addition, see [37, 52, 101]. In Section 3.4 the key ingredient is a type of Hilbert transform, where its fast realisation is not in the scope of this thesis. However, Section 3.4 is completed with error estimates and some numerical examples, which emphasise the theoretical results.

3.1 Variational Formulation in Space and Pointwise in Time

In this section, a short overview of a pointwise in time variational formulation is given. Furthermore, numerical examples for lowest order in space, i.e. piecewise linear, continuous ansatz functions, combined with lowest order time stepping are presented.

The pointwise in time variational formulation of (3.1) is given with the notations of Section 2 as follows:

Find $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;[H_0^1(\Omega)]')$ and $u(\cdot,0) = u_0$ in $L^2(\Omega)$ such that

$$\langle \partial_t u(\cdot,t), v\rangle_\Omega + \langle \nabla_x u(\cdot,t), \nabla_x v\rangle_{L^2(\Omega)} = \langle f(\cdot,t), v\rangle_\Omega \tag{3.2}$$

for almost all $t \in (0,T)$ and all $v \in H_0^1(\Omega)$, where $f \in L^2(0,T;[H_0^1(\Omega)]')$ and $u_0 \in L^2(\Omega)$ are the given right-hand side and the given initial condition. Here, ∂_t is the distributional derivative on $(0,T)$, i.e. equality (3.2) means that it holds

$$-\int_0^T \langle u(\cdot,t), v\rangle_\Omega \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t)\mathrm{d}t + \int_0^T \langle \nabla_x u(\cdot,t), \nabla_x v\rangle_{L^2(\Omega)} \varphi(t)\mathrm{d}t = \int_0^T \langle f(\cdot,t), v\rangle_\Omega \varphi(t)\mathrm{d}t$$

for all $\varphi \in C_0^\infty(0,T)$. The variational formulation in (3.2) is examined in many books, for example, [102, Exemple 1, Chapitre 3, page 263], [160, Beispiel 28.1, Kapitel IV, page 409], [162, Section 23.8, Chapter 23, page 426] or [36, Mathematical Example 1, Chapter XVIII,page 524]. In these books, the following existence and uniqueness result is proven.

Theorem 3.1.1. *For given $f \in L^2(0,T;[H_0^1(\Omega)]')$ and $u_0 \in L^2(\Omega)$, there exists a unique solution u of the variational formulation* (3.2)*, satisfying*

$$u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega)), \quad \partial_t u \in L^2(0,T;[H_0^1(\Omega)]'),$$

i.e. $u \in W(Q)$, and the stability estimate

$$\|u\|_{W(Q)} = \sqrt{\|u\|^2_{L^2(0,T;H_0^1(\Omega))} + \|\partial_t u\|^2_{L^2(0,T;[H_0^1(\Omega)]')}} \le c\left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;[H_0^1(\Omega)]')}\right)$$

with a constant $c > 0$.

Proof. See the books [36, 102, 160, 162] as mentioned above. □

For a discretisation scheme, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0,L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. With the notations of Section 2.8, consider a discretisation of a tensor-product type (2.41) with the finite-dimensional space $Q_h^1(Q) = V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0,T)$, see (2.42). Therefore, introduce for $x \in \Omega$ and $\ell \in \{0,\dots,N_t\}$ the approximation

$$U_{h_x,\ell}(x) := \sum_{i=1}^{M_x} U_i^\ell \psi_i(x) \approx u(x,t_\ell),$$

where $U_i^\ell \in \mathbb{R}$ are the unknown coefficients of the functions $U_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0,\dots,N_t\}$. Furthermore, set for $(x,t) \in Q$

$$u_h(x,t) := \sum_{\ell=0}^{N_t} \sum_{i=1}^{M_x} U_i^\ell \psi_i(x) \varphi_\ell(t) = \sum_{\ell=0}^{N_t} U_{h_x,\ell}(x) \varphi_\ell(t) \approx u(x,t), \tag{3.3}$$

i.e. $u_h \in Q_h^1(Q)$.

For the pointwise in time variational formulation (3.2), a conforming discretisation in space with $V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ in combination with a θ-method (2.35) for $\theta = 1$ leads to the so-called implicit Euler Galerkin method to find $U_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0,\dots,N_t\}$ such that

$$U_{h_x,0} = Q_{h_x} u_0$$

and for $\ell = 1,\dots,N_t$

$$\frac{1}{h_{t,\ell}} \langle U_{h_x,\ell} - U_{h_x,\ell-1}, v_{h_x} \rangle_{L^2(\Omega)} + \langle \nabla_x U_{h_x,\ell}, \nabla_x v_{h_x} \rangle_{L^2(\Omega)} = \frac{1}{h_{t,\ell}} \left\langle \int_{\tau_\ell} f(\cdot,s)\,\mathrm{d}s, v_{h_x} \right\rangle_\Omega \tag{3.4}$$

for all $v_{h_x} \in V_{h_x,0}(\Omega)$, where $Q_{h_x}\colon L^2(\Omega) \to V_{h_x,0}(\Omega)$ denotes the L^2 projection (2.39). This method is given in [74, (2.10), page 684] or in [79, (3.5), page 508] and differs from the methods [150, (1.47), page 16] or [65, (1.34), page 334] only in the right-hand side. The implicit Euler Galerkin method (3.4) is equivalent to the linear systems

$$M_{h_x} \underline{U}^0 = \underline{u}_0$$

and

$$(M_{h_x} + h_{t,\ell} A_{h_x})\, \underline{U}^\ell = M_{h_x} \underline{U}^{\ell-1} + \underline{F}^\ell \tag{3.5}$$

for all $\ell = 1,\dots,N_t$, where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the mass matrix (2.37), $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the stiffness matrix (2.38) and the vectors $\underline{u}_0$, $\underline{F}^\ell \in \mathbb{R}^{M_x}$ are defined by

$$\underline{u}_0[i] := \langle u_0, \psi_i \rangle_{L^2(\Omega)}, \quad \underline{F}^\ell[i] := \left\langle \int_{\tau_\ell} f(\cdot,s)\,\mathrm{d}s, \psi_i \right\rangle_\Omega \tag{3.6}$$

for $i = 1,\dots,M_x$ with the nodal basis functions ψ_i satisfying $V_{h_x,0}(\Omega) = \operatorname{span}\{\psi_i\}_{i=1}^{M_x}$, see (2.36). The matrix $M_{h_x} + h_{t,\ell} A_{h_x}$ is positive definite and hence, the linear systems (3.5) are uniquely solvable for all $\ell = 1, \ldots, N_t$. Stability of the numerical scheme (3.4) holds without any CFL condition because the implicit Euler method is unconditionally stable, see Section 2.6.

As a second discretisation method in time, the Crank-Nicolson method is considered, which follows from the θ-method (2.35) for $\theta = 1/2$. Hence, using the Crank-Nicolson method combined with a conforming discretisation in space with $V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for the pointwise in time variational formulation (3.2), there follows the so-called Crank-Nicolson Galerkin method to find $U_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0, \ldots, N_t\}$ such that

$$U_{h_x,0} = Q_{h_x} u_0$$

and for $\ell = 1, \ldots, N_t$

$$\begin{aligned} \frac{1}{h_{t,\ell}} \langle U_{h_x,\ell} - U_{h_x,\ell-1}, v_{h_x} \rangle_{L^2(\Omega)} + \frac{1}{2} \langle \nabla_x U_{h_x,\ell} + \nabla_x U_{h_x,\ell-1}, \nabla_x v_{h_x} \rangle_{L^2(\Omega)} \\ = \frac{1}{h_{t,\ell}} \left\langle \int_{\tau_\ell} f(\cdot, s) \mathrm{d}s, v_{h_x} \right\rangle_\Omega \end{aligned} \tag{3.7}$$

for all $v_{h_x} \in V_{h_x,0}(\Omega)$, where $Q_{h_x} : L^2(\Omega) \to V_{h_x,0}(\Omega)$ denotes the L^2 projection (2.39). This method is given in [74, (2.11), page 684] and differs from the methods [150, (1.54), page 16] or [65, (1.34), page 334] only in the right-hand side. The Crank-Nicolson Galerkin method (3.7) is equivalent to the linear systems

$$M_{h_x} \underline{U}^0 = \underline{u}_0$$

and

$$\left(M_{h_x} + \frac{h_{t,\ell}}{2} A_{h_x} \right) \underline{U}^\ell = \left(M_{h_x} - \frac{h_{t,\ell}}{2} A_{h_x} \right) \underline{U}^{\ell-1} + \underline{F}^\ell \quad \text{for all } \ell = 1, \ldots, N_t, \tag{3.8}$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the mass matrix (2.37), $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the stiffness matrix (2.38) and the vectors $\underline{u}_0$, $\underline{F}^\ell \in \mathbb{R}^{M_x}$ are given in (3.6). The matrix $M_{h_x} + \frac{h_{t,\ell}}{2} A_{h_x}$ is positive definite and hence, the linear systems (3.8) are uniquely solvable for all $\ell = 1, \ldots, N_t$. Stability of the numerical scheme (3.7) holds without any CFL condition because the Crank-Nicolson method is unconditionally stable, see Section 2.6.

Next, error estimates for the implicit Euler and the Crank-Nicolson Galerkin method are the aim. It seems that error estimates of the quantities $\|u(\cdot, t_\ell) - U_{h_x,\ell}\|_{L^2(\Omega)}$ for each $\ell = 0, \ldots, N_t$ are standard, see [150, Theorem 1.6, page 16 or Theorem 1.5, page 15] or [65, Theorem 5.13, page 336], [85, 93] and also [43, 159] for some early references. However, here, error estimates in space-time norms $\|\cdot\|_{L^2(Q)}$, $|\cdot|_{H^1(Q)}$ are considered, see the work [9, 16, 17, 54, 74, 79, 108, 148, 151].

Theorem 3.1.2. *Let Ω be sufficiently regular and consider a constant step size $h_t = h_{t,\ell}$ for all $\ell = 1,\dots,N_t$ together with a sequence $(\mathcal{T}_N)_N$ of admissible, shape regular and globally quasi-uniform decompositions with maximal mesh size h_x. Furthermore, let the unique solution u of* (3.2) *be sufficiently smooth and let u_h be defined via* (3.3)*, where the coefficients $U_{h_x,\ell}$ are calculated by the implicit Euler Galerkin method* (3.4)*. Then there holds the space-time error estimate*

$$\|u - u_h\|_{L^2(Q)} \le c_1 (h_t + h_x^s) \|u\|_{L^2(0,T;H^s(\Omega))} + c_2 h_t^r \|u\|_{H^r(0,T;L^2(\Omega))},$$

with $r \in (1/2, 1]$, $s \in [1,2]$ and with constants $c_1 > 0$, $c_2 > 0$ independent of h_t and h_x.

Proof. This proof follows the ideas of the proof of [74, Theorem 3.1, page 684]. So, for $u \in C([0,T]; L^2(\Omega))$, one defines the function $I_{h_t} Q_{h_x} u \in Q_h^1(Q)$ as in (2.59), i.e.

$$I_{h_t} Q_{h_x} u(x,t) = \sum_{\ell=0}^{N_t} \underbrace{Q_{h_x} u(x,t_\ell)}_{=:\hat{U}_{h_x,\ell}(x)} \varphi_\ell(t) \qquad \text{for } (x,t) \in \overline{Q}$$

with $\hat{U}_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$, satisfying

$$\left\langle \hat{U}_{h_x,\ell}, v_{h_x} \right\rangle_{L^2(\Omega)} = \langle u(\cdot, t_\ell), v_{h_x} \rangle_{L^2(\Omega)} \qquad \text{for all } v_{h_x} \in V_{h_x,0}(\Omega)$$

for $\ell = 0,\dots,N_t$. With the triangle inequality, it holds

$$\|u - u_h\|_{L^2(Q)} \le \|u - I_{h_t} Q_{h_x} u\|_{L^2(Q)} + \|I_{h_t} Q_{h_x} u - u_h\|_{L^2(Q)}.$$

The first term is estimated by standard error estimates of Lemma 2.8.4 and so, it remains to investigate the second term. Therefore, set

$$\eta_\ell := U_{h_x,\ell} - \hat{U}_{h_x,\ell} \in V_{h_x,0}(\Omega) \qquad \text{for } \ell = 0,\dots,N_t,$$

where $\eta_0 = 0$. Hence, it holds

$$\begin{aligned} \|I_{h_t} Q_{h_x} u - u_h\|^2_{L^2(Q)} &= \sum_{\ell=1}^{N_t} \|I_{h_t} Q_{h_x} u - u_h\|^2_{L^2(\Omega \times \tau_\ell)} \\ &= \int_\Omega \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}}{3} \left(\eta_\ell(x)^2 + \eta_{\ell-1}(x)\eta_\ell(x) + \eta_{\ell-1}(x)^2 \right) \mathrm{d}x \\ &\le h_t \sum_{\ell=1}^{N_t} \|\eta_\ell\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.9}$$

With [74, (4.5), page 689] for $\alpha = 0$ and [74, (4.4a), page 688] for $\alpha = \tilde{\alpha} = 0$, it follows from the last inequality (3.9) that

$$\|I_{h_t} Q_{h_x} u - u_h\|_{L^2(Q)} \leq \left(\sum_{\ell=1}^{N_t} h_t \|\eta_\ell\|^2_{L^2(\Omega)} \right)^{1/2}$$
$$\leq c_1 (h_t + h_x^s) \|u\|_{L^2(0,T;H^s(\Omega))} + c_2 h_t^r \|u\|_{H^r(0,T;L^2(\Omega))}$$

with $r \in (1/2, 1]$, $s \in [1,2]$ and with constants $c_1 > 0, c_2 > 0$ coming from standard error estimates and inverse inequalities, see the proofs in [74] for details. □

Theorem 3.1.3. *Let Ω be sufficiently regular and consider a constant step size $h_t = h_{t,\ell}$ for all $\ell = 1, \ldots, N_t$ together with a sequence $(\mathcal{T}_N)_N$ of admissible, shape regular and globally quasi-uniform decompositions with maximal mesh size h_x. Furthermore, let the unique solution u of* (3.2) *be sufficiently smooth and let u_h be defined via* (3.3)*, where the coefficients $U_{h_x,\ell}$ are calculated by the Crank-Nicolson Galerkin method* (3.7)*. Then there holds the space-time error estimate*

$$\|u - u_h\|_{L^2(Q)} \leq c_1 (h_t^2 + h_x^s) \|u\|_{L^2(0,T;H^s(\Omega))} + c_2 h_t^r \|u\|_{H^r(0,T;L^2(\Omega))},$$

with $r \in (1/2, 2]$, $s \in [1,2]$ and with constants $c_1 > 0, c_2 > 0$ independent of h_t and h_x. Furthermore, assume for $Q_h^1(Q) \subset H^{1,1}_{0;}(Q)$ the inverse inequality

$$|v_h|_{H^1(Q)} \leq c_{\mathrm{inv}} h^{-1} \|v_h\|_{L^2(Q)} \quad \forall v_h \in Q_h^1(Q)$$

with a constant $c_{\mathrm{inv}} > 0$ and $h = \max\{h_t, h_x\}$. Then it holds

$$|u - u_h|_{H^1(Q)} \leq C c_{\mathrm{inv}} h^{\mu} \|u\|_{H^{\mu+1}(Q)} + c_{\mathrm{inv}} h^{-1} \|u_h - u\|_{L^2(Q)}$$

with $\mu \in [0,1]$ and with a constant $C > 0$ independent of h.

Proof. For the $L^2(Q)$ error estimate, repeat the proof of Theorem 3.1.2 until (3.9). Then, with [74, (4.5), page 689] for $\alpha = 0$, see also [74, Lemma 4.4, page 690], and the first estimate of [74, Lemma 4.3, page 690] for $\alpha = \tilde{\alpha} = 0$, there follows from the inequality (3.9) that

$$\|I_{h_t} Q_{h_x} u - u_h\|_{L^2(Q)} \leq \left(\sum_{\ell=1}^{N_t} h_t \|\eta_\ell\|^2_{L^2(\Omega)} \right)^{1/2}$$
$$\leq c_1 (h_t^2 + h_x^s) \|u\|_{L^2(0,T;H^s(\Omega))} + c_2 h_t^r \|u\|_{H^r(0,T;L^2(\Omega))},$$

with $r \in (1/2, 2]$, $s \in [1,2]$ and with constants $c_1 > 0, c_2 > 0$ coming from standard error estimates and inverse inequalities, see the proofs in [74] for details.

For the $H^1(Q)$ error estimate, consider an $H^{1,1}_{0;}(Q)$ projection to find $Q^1_{h,0}u \in Q^1_h(Q)$ such that

$$\left\langle \partial_t Q^1_{h,0}u, \partial_t v_h \right\rangle_{L^2(Q)} + \left\langle \nabla_x Q^1_{h,0}u, \nabla_x v_h \right\rangle_{L^2(Q)} = \langle \partial_t u, \partial_t v_h \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v_h \rangle_{L^2(Q)}$$

for all $v_h \in Q^1_h(Q)$. Then it follows with the triangle inequality, standard error estimates for $Q^1_{h,0}$ and the inverse inequality in $Q^1_h(Q)$ that

$$\begin{aligned}
|u - u_h|_{H^1(Q)} &\leq \left|u - Q^1_{h,0}u\right|_{H^1(Q)} + \left|Q^1_{h,0}u - u_h\right|_{H^1(Q)} \\
&\leq \tilde{C} h^{\mu} \|u\|_{H^{\mu+1}(Q)} + c_{\text{inv}} h^{-1} \left\|Q^1_{h,0}u - u_h\right\|_{L^2(Q)} \\
&\leq \tilde{C} h^{\mu} \|u\|_{H^{\mu+1}(Q)} + c_{\text{inv}} h^{-1} \left\|Q^1_{h,0}u - u\right\|_{L^2(Q)} + c_{\text{inv}} h^{-1} \|u_h - u\|_{L^2(Q)} \\
&\leq C c_{\text{inv}} h^{\mu} \|u\|_{H^{\mu+1}(Q)} + c_{\text{inv}} h^{-1} \|u_h - u\|_{L^2(Q)}
\end{aligned}$$

for $\mu \in [0,1]$ and hence, the assertion. □

Remark 3.1.4. *Since in the proofs of Theorem 3.1.2 and Theorem 3.1.3, see [74] for more details, regularity results of related adjoint problems are used, one expects reduced orders for the error estimates, given in Theorem 3.1.2 and Theorem 3.1.3, if Ω is less regular.*

Corollary 3.1.5. *Let the assumptions of Theorem 3.1.2 and of Theorem 3.1.3 be fulfilled and let u be sufficiently smooth. Then, for the implicit Euler Galerkin method* (3.4)*, there holds the error estimate*

$$\|u - u_h\|_{L^2(Q)} \leq C h$$

and for the Crank-Nicolson Galerkin method (3.7)*, there hold the error estimates*

$$\|u - u_h\|_{L^2(Q)} \leq C h^2$$

and

$$|u - u_h|_{H^1(Q)} \leq C h$$

with a constant $C > 0$ independent of $h = \max\{h_t, h_x\}$.

Proof. These estimates follow immediately from Theorem 3.1.2 and of Theorem 3.1.3 for the maximal values of r, s and μ. □

In the last part of this section, some numerical examples are presented. So, for the space-time cylinder

$$Q = \Omega \times (0,T) = (0,1) \times (0,2),$$

consider the solutions of (3.2)

$$
\begin{aligned}
u_1(x,t) &:= \sin(\pi x)\,(1+t)\,\mathrm{e}^{-t/2},\\
u_2(x,t) &:= \sin(\pi x)\,(T-t)^{3/4}\,t,\\
u_3(x,t) &:= \sin(\pi x)(t^2+x^2)^{-3/8},\\
u_4(x,t) &:= \sin(\pi x)((T/4-t)^2+x^2)^{-3/8}
\end{aligned}
$$

for $(x,t) \in Q$. Note that $u_i \in H^1(Q)$ for $i = 1,2,3,4$. The spatial interval $\Omega = (0,1)$ is decomposed into N_x elements, i.e. intervals, and $\tilde{M}_x = N_x + 1$ vertices with the constant mesh size $h_x = 1/(\tilde{M}_x - 1) = 1/N_x$. For the time interval $(0,2)$, there are $\tilde{M}_t$ time steps introduced with the constant time step size $h_t = T/(\tilde{M}_t - 1)$. See Section 2 for more details. The appearing integrals for the initial condition and the right-hand side in (3.6) are calculated by the usage of high-order integration rules, and the degrees of freedom are denoted by

$$
\mathrm{dof} = (\tilde{M}_x - 2)\cdot(\tilde{M}_t - 1)
$$

due to the homogeneous Dirichlet boundary condition and the initial condition.

In Table 3.1 and Table 3.2, the errors in $\|\cdot\|_{L^2(Q)}$ and in $|\cdot|_{H^1(Q)}$ are presented for the smooth function u_1 and for a uniform refinement strategy in space and time direction, i.e. $h_t \sim h_x$. Note that no CFL condition like $h_t \sim h_x^2$ is needed because the Crank-Nicolson method and the implicit Euler method are unconditionally stable, see Section 2.6. The error estimates of Theorem 3.1.3 and Theorem 3.1.2 are confirmed.

Table 3.3, Table 3.5 and Table 3.7 show that the position of singularities leads to different convergence behaviours for the Crank-Nicolson Galerkin method. For the function u_3, the singularity is at $(0,0)$ and hence, the initial condition $u_0 \in L^2(\Omega)$ is less regular, i.e. $u_0 \notin H_0^1(\Omega)$. This results in an observed convergence rate of $3/4$ in $\|\cdot\|_{L^2(Q)}$ and in no convergence in $|\cdot|_{H^1(Q)}$ although $u_3 \in H^1(Q)$, see Table 3.5. If the position of the singularity is at the end time T, as for the solution u_2, or at the time $T/4$, as for the solution u_4, then reduced orders of convergence are observed as expected, see Table 3.3, Table 3.7. For the implicit Euler method, analogous results are given in Table 3.4, Table 3.6, Table 3.8, where the position of the singularity for the functions u_2, u_3, u_4 plays no role.

$\tilde{M}_x$	$\tilde{M}_t$	dof	h_x	h_t	$\|u_1 - u_{1,h}\|_{L^2(Q)}$	eoc	$\|u_1 - u_{1,h}\|_{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	2.5257e-01	-	1.6202e+00	-
5	5	12	0.25000	0.50000	6.3416e-02	1.54	8.2405e-01	0.75
9	9	56	0.12500	0.25000	1.5851e-02	1.80	4.1399e-01	0.89
17	17	240	0.06250	0.12500	3.9635e-03	1.90	2.0726e-01	0.95
33	33	992	0.03125	0.06250	9.9092e-04	1.95	1.0366e-01	0.98
65	65	4032	0.01562	0.03125	2.4773e-04	1.98	5.1836e-02	0.99
129	129	16256	0.00781	0.01562	6.1933e-05	1.99	2.5918e-02	0.99
257	257	65280	0.00391	0.00781	1.5483e-05	1.99	1.2959e-02	1.00
513	513	261632	0.00195	0.00391	3.8708e-06	2.00	6.4797e-03	1.00
1025	1025	1047552	0.00098	0.00195	9.6769e-07	2.00	3.2398e-03	1.00
2049	2049	4192256	0.00049	0.00098	2.4206e-07	2.00	1.6199e-03	1.00
4097	4097	16773120	0.00024	0.00049	6.0215e-08	2.01	8.0996e-04	1.00

Table 3.1: Numerical results of the Crank-Nicolson Galerkin method (3.7) for the space-time cylinder $Q = (0,1) \times (0,2)$ and for u_1.

$\tilde{M}_x$	$\tilde{M}_t$	dof	h_x	h_t	$\|u_1 - u_{1,h}\|_{L^2(Q)}$	eoc	$\|u_1 - u_{1,h}\|_{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	2.4176e-01	-	1.6141e+00	-
5	5	12	0.25000	0.50000	7.8469e-02	1.26	8.3436e-01	0.74
9	9	56	0.12500	0.25000	2.8683e-02	1.31	4.2342e-01	0.88
17	17	240	0.06250	0.12500	1.2317e-02	1.16	2.1337e-01	0.94
33	33	992	0.03125	0.06250	5.8288e-03	1.05	1.0719e-01	0.97
65	65	4032	0.01562	0.03125	2.8627e-03	1.01	5.3748e-02	0.98
129	129	16256	0.00781	0.01562	1.4228e-03	1.00	2.6918e-02	0.99
257	257	65280	0.00391	0.00781	7.0988e-04	1.00	1.3470e-02	1.00
513	513	261632	0.00195	0.00391	3.5463e-04	1.00	6.7383e-03	1.00
1025	1025	1047552	0.00098	0.00195	1.7725e-04	1.00	3.3699e-03	1.00
2049	2049	4192256	0.00049	0.00098	8.8607e-05	1.00	1.6852e-03	1.00
4097	4097	16773120	0.00024	0.00049	4.4300e-05	1.00	8.4263e-04	1.00

Table 3.2: Numerical results of the implicit Euler Galerkin method (3.4) for the space-time cylinder $Q = (0,1) \times (0,2)$ and for u_1.

$\tilde{M}_x$	$\tilde{M}_t$	dof	h_x	h_t	$\|u_2 - u_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}\|_{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	1.7974e-01	-	1.2812e+00	-
5	5	12	0.25000	0.50000	5.3647e-02	1.35	7.0582e-01	0.67
9	9	56	0.12500	0.25000	1.5830e-02	1.58	4.0519e-01	0.72
17	17	240	0.06250	0.12500	5.4556e-03	1.46	2.5515e-01	0.64
33	33	992	0.03125	0.06250	2.2538e-03	1.25	1.7871e-01	0.50
65	65	4032	0.01562	0.03125	1.0047e-03	1.15	1.3639e-01	0.39
129	129	16256	0.00781	0.01562	4.4819e-04	1.16	1.0957e-01	0.31
257	257	65280	0.00391	0.00781	1.9647e-04	1.19	9.0273e-02	0.28
513	513	261632	0.00195	0.00391	8.4780e-05	1.21	7.5223e-02	0.26
1025	1025	1047552	0.00098	0.00195	3.6195e-05	1.23	6.2996e-02	0.26
2049	2049	4192256	0.00049	0.00098	1.5351e-05	1.24	5.2874e-02	0.25
4097	4097	16773120	0.00024	0.00049	6.4854e-06	1.24	4.4422e-02	0.25

Table 3.3: Numerical results of the Crank-Nicolson Galerkin method (3.7) for the space-time cylinder $Q = (0,1) \times (0,2)$ and for u_2.

$\tilde{M}_x$	$\tilde{M}_t$	dof	h_x	h_t	$\|u_2 - u_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}\|_{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	3.7057e-01	-	1.6781e+00	-
5	5	12	0.25000	0.50000	1.9478e-01	0.72	1.0367e+00	0.54
9	9	56	0.12500	0.25000	1.0549e-01	0.80	6.1865e-01	0.67
17	17	240	0.06250	0.12500	5.6371e-02	0.86	3.7187e-01	0.70
33	33	992	0.03125	0.06250	2.9435e-02	0.92	2.3356e-01	0.66
65	65	4032	0.01562	0.03125	1.5118e-02	0.95	1.5857e-01	0.55
129	129	16256	0.00781	0.01562	7.6856e-03	0.97	1.1747e-01	0.43
257	257	65280	0.00391	0.00781	3.8823e-03	0.98	9.2856e-02	0.34
513	513	261632	0.00195	0.00391	1.9533e-03	0.99	7.6028e-02	0.29
1025	1025	1047552	0.00098	0.00195	9.8032e-04	0.99	6.3241e-02	0.27
2049	2049	4192256	0.00049	0.00098	4.9126e-04	1.00	5.2947e-02	0.26
4097	4097	16773120	0.00024	0.00049	2.4596e-04	1.00	4.4444e-02	0.25

Table 3.4: Numerical results of the implicit Euler Galerkin method (3.4) for the space-time cylinder $Q = (0,1) \times (0,2)$ and for u_2.

$\tilde{M}_x$	$\tilde{M}_t$	dof	h_x	h_t	$\|u_3 - u_{3,h}\|_{L^2(Q)}$	eoc	$\|u_3 - u_{3,h}\|_{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	3.8930e-01	-	2.8412e+00	-
5	5	12	0.25000	0.50000	1.5092e-01	1.06	2.2724e+00	0.25
9	9	56	0.12500	0.25000	7.2348e-02	0.95	2.0666e+00	0.12
17	17	240	0.06250	0.12500	3.9069e-02	0.85	2.0819e+00	-0.01
33	33	992	0.03125	0.06250	2.2136e-02	0.80	2.2423e+00	-0.10
65	65	4032	0.01562	0.03125	1.2839e-02	0.78	2.5212e+00	-0.17
129	129	16256	0.00781	0.01562	7.5378e-03	0.76	2.9090e+00	-0.21
257	257	65280	0.00391	0.00781	4.4532e-03	0.76	3.4053e+00	-0.23
513	513	261632	0.00195	0.00391	2.6393e-03	0.75	4.0171e+00	-0.24
1025	1025	1047552	0.00098	0.00195	1.5668e-03	0.75	4.7577e+00	-0.24
2049	2049	4192256	0.00049	0.00098	9.3084e-04	0.75	5.6463e+00	-0.25
4097	4097	16773120	0.00024	0.00049	5.5325e-04	0.75	6.7077e+00	-0.25

Table 3.5: Numerical results of the Crank-Nicolson Galerkin method (3.7) for the space-time cylinder $Q = (0,1) \times (0,2)$ and for u_3.

$\tilde{M}_x$	$\tilde{M}_t$	dof	h_x	h_t	$\|u_3 - u_{3,h}\|_{L^2(Q)}$	eoc	$\|u_3 - u_{3,h}\|_{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	3.1084e-01	-	2.9018e+00	-
5	5	12	0.25000	0.50000	1.7098e-01	0.67	2.2254e+00	0.30
9	9	56	0.12500	0.25000	1.0734e-01	0.60	1.7948e+00	0.28
17	17	240	0.06250	0.12500	6.2274e-02	0.75	1.4924e+00	0.25
33	33	992	0.03125	0.06250	3.4179e-02	0.85	1.2533e+00	0.25
65	65	4032	0.01562	0.03125	1.8147e-02	0.90	1.0550e+00	0.25
129	129	16256	0.00781	0.01562	9.4412e-03	0.94	8.8824e-01	0.25
257	257	65280	0.00391	0.00781	4.8483e-03	0.96	7.4759e-01	0.25
513	513	261632	0.00195	0.00391	2.4686e-03	0.97	6.2902e-01	0.25
1025	1025	1047552	0.00098	0.00195	1.2498e-03	0.98	5.2914e-01	0.25
2049	2049	4192256	0.00049	0.00098	6.3032e-04	0.99	4.4507e-01	0.25
4097	4097	16773120	0.00024	0.00049	3.1707e-04	0.99	3.7433e-01	0.25

Table 3.6: Numerical results of the implicit Euler Galerkin method (3.4) for the space-time cylinder $Q = (0,1) \times (0,2)$ and for u_3.

$\tilde{M}_x$	$\tilde{M}_t$	dof	h_x	h_t	$\|u_4 - u_{4,h}\|_{L^2(Q)}$	eoc	$\|u_4 - u_{4,h}\|_{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	6.2378e-01	-	2.5520e+00	-
5	5	12	0.25000	0.50000	1.7639e-01	1.41	2.7132e+00	-0.07
9	9	56	0.12500	0.25000	6.6921e-02	1.26	2.1509e+00	0.30
17	17	240	0.06250	0.12500	2.7324e-02	1.23	1.7729e+00	0.27
33	33	992	0.03125	0.06250	1.1359e-02	1.24	1.4811e+00	0.25
65	65	4032	0.01562	0.03125	4.7565e-03	1.24	1.2429e+00	0.25
129	129	16256	0.00781	0.01562	1.9991e-03	1.24	1.0444e+00	0.25
257	257	65280	0.00391	0.00781	8.4227e-04	1.24	8.7808e-01	0.25
513	513	261632	0.00195	0.00391	3.5559e-04	1.24	7.3836e-01	0.25
1025	1025	1047552	0.00098	0.00195	1.5046e-04	1.24	6.2093e-01	0.25
2049	2049	4192256	0.00049	0.00098	6.3843e-05	1.24	5.2221e-01	0.25
4097	4097	16773120	0.00024	0.00049	2.7189e-05	1.23	4.3923e-01	0.25

Table 3.7: Numerical results of the Crank-Nicolson Galerkin method (3.7) for the space-time cylinder $Q = (0,1) \times (0,2)$ and for u_4.

$\tilde{M}_x$	$\tilde{M}_t$	dof	h_x	h_t	$\|u_4 - u_{4,h}\|_{L^2(Q)}$	eoc	$\|u_4 - u_{4,h}\|_{H^1(Q)}$	eoc
3	3	2	0.50000	1.00000	5.8329e-01	-	2.5931e+00	-
5	5	12	0.25000	0.50000	2.5431e-01	0.93	2.9416e+00	-0.14
9	9	56	0.12500	0.25000	1.5258e-01	0.66	2.4423e+00	0.24
17	17	240	0.06250	0.12500	8.6590e-02	0.78	2.0575e+00	0.24
33	33	992	0.03125	0.06250	4.7030e-02	0.86	1.7383e+00	0.24
65	65	4032	0.01562	0.03125	2.4819e-02	0.91	1.4675e+00	0.24
129	129	16256	0.00781	0.01562	1.2856e-02	0.94	1.2372e+00	0.24
257	257	65280	0.00391	0.00781	6.5804e-03	0.96	1.0419e+00	0.25
513	513	261632	0.00195	0.00391	3.3420e-03	0.98	8.7684e-01	0.25
1025	1025	1047552	0.00098	0.00195	1.6887e-03	0.98	7.3767e-01	0.25
2049	2049	4192256	0.00049	0.00098	8.5041e-04	0.99	6.2046e-01	0.25
4097	4097	16773120	0.00024	0.00049	4.2730e-04	0.99	5.2183e-01	0.25

Table 3.8: Numerical results of the implicit Euler Galerkin method (3.4) for the space-time cylinder $Q = (0,1) \times (0,2)$ and for u_4.

3.2 Space-Time Variational Formulation of Bochner Type

In this section, a short overview of space-time variational formulations of Bochner type is given and a stable space-time discretisation is formulated, see [51, 137, 142–144].

For the heat equation (3.1), a space-time variational formulation of Bochner type for the Hilbert space $W(Q)$ from (2.21) with the norm $\|\cdot\|_{W(Q)}$ from (2.22) is given as follows:

Find $u \in W(Q)$ such that

$$\langle \partial_t u, v\rangle_Q + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} + \langle u(\cdot,0), w\rangle_{L^2(\Omega)} = \langle f, v\rangle_Q + \langle u_0, w\rangle_{L^2(\Omega)} \tag{3.10}$$

for all $(v,w) \in L^2(0,T;H_0^1(\Omega)) \times L^2(\Omega) =: Y$, where $f \in L^2(0,T;[H_0^1(\Omega)]')$ and $u_0 \in L^2(\Omega)$ are the given right-hand side and the given initial condition. The space-time variational formulation of Bochner type (3.10) is equivalent to the pointwise in time variational formulation (3.2) because of the fundamental lemma of calculus, the density of $C_0^\infty(0,T)$ in $L^2(0,T)$ and so, the density of the algebraic tensor-product

$$H_0^1(\Omega) \otimes C_0^\infty(0,T) = \operatorname{span}\left\{\phi \cdot \psi \colon\ \phi \in H_0^1(\Omega),\ \psi \in C_0^\infty(0,T)\right\}$$

in $H_0^1(\Omega)\hat{\otimes}L^2(0,T) \simeq L^2(0,T;H_0^1(\Omega))$. Hence, the unique solvability of the space-time variational formulation of Bochner type (3.10) follows from the pointwise in time variational formulation (3.2), i.e. from Theorem 3.1.1. An alternative proof of a uniqueness and existence result for the space-time variational formulation of Bochner type (3.10) uses the Nečas Theorem 2.9.1. Therefore, define the bilinear form $b(\cdot,\cdot)\colon\ W(Q) \times Y \to \mathbb{R}$ by

$$b(u,(v,w)) := \langle \partial_t u, v\rangle_Q + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} + \langle u(\cdot,0), w\rangle_{L^2(\Omega)} \tag{3.11}$$

for $u \in W(Q)$, $(v,w) \in Y$, where the Hilbert space Y is endowed with the inner product

$$\langle (v,w),(\hat{v},\hat{w})\rangle_Y := \int_0^T \int_\Omega \nabla_x v(x,t) \cdot \nabla_x \hat{v}(x,t) \mathrm{d}x\mathrm{d}t + \int_\Omega w(x)\hat{w}(x)\mathrm{d}x \quad \text{for } (v,w),(\hat{v},\hat{w}) \in Y.$$

Theorem 3.2.1. *The bilinear form* (3.11) *is continuous and fulfils an inf-sup condition and the surjectivity condition, i.e. there hold:*

1. *There exists a constant $C > 0$ such that for all $u \in W(Q)$ and for all $(v,w) \in Y$*

$$|b(u,(v,w))| \le C\|u\|_{W(Q)}\|(v,w)\|_Y.$$

2. *There exists a constant $c_s > 0$ such that*

$$\inf_{0 \ne u \in W(Q)} \sup_{0 \ne (v,w) \in Y} \frac{|b(u,(v,w))|}{\|u\|_{W(Q)}\|(v,w)\|_Y} \ge c_s.$$

3. *For each element* $(v,w) \in Y$, $(v,w) \neq 0$, *there exists an element* $u \in W(Q)$ *such that* $b(u,(v,w)) \neq 0$.

Proof. For $u_0 = 0$, the proof is contained in the book [51, Proof of Theorem 6.6, page 282]. For the general case, see [137, Theorem 5.1, page 1300]. □

The linear form $F\colon Y \to \mathbb{R}$ is given by

$$F(v,w) := \langle f,v\rangle_Q + \langle u_0,w\rangle_{L^2(\Omega)} \quad \text{for } (v,w) \in Y,$$

where its boundedness follows with the Cauchy-Schwarz inequality by

$$\begin{aligned}|F(v,w)| &\leq \|f\|_{L^2(0,T;[H_0^1(\Omega)]')}\|\nabla_x v\|_{L^2(Q)} + \|u_0\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \\ &\leq \sqrt{\|f\|^2_{L^2(0,T;[H_0^1(\Omega)]')} + \|u_0\|^2_{L^2(\Omega)}}\,\|(v,w)\|_Y\end{aligned}$$

for all $(v,w) \in Y$. Hence, the variational formulation (3.10) is rewritten to find $u \in W(Q)$ such that

$$b(u,(v,w)) = F(v,w) \quad \text{for all } (v,w) \in Y. \tag{3.12}$$

There holds the following existence and uniqueness result:

Theorem 3.2.2. *For each given* $F = (f,u_0) \in Y'$, *the variational formulation* (3.12) *and hence, the variational formulation* (3.10) *have a unique solution* $u \in W(Q)$, *satisfying*

$$\|u\|_{W(Q)} \leq C\sqrt{\|f\|^2_{L^2(0,T;[H_0^1(\Omega)]')} + \|u_0\|^2_{L^2(\Omega)}}$$

with a constant $C > 0$. *Furthermore, the solution operator*

$$\mathcal{L}\colon Y' \to W(Q), \quad \mathcal{L}F = \mathcal{L}(f,u_0) := u,$$

is an isomorphism.

Proof. This follows with the Nečas Theorem 2.9.1 from Theorem 3.2.1. □

In the remainder of this section, the initial condition u_0 is incorporated via homogenisation. So, the bilinear form

$$a(u,v) := \langle \partial_t u,v\rangle_Q + \langle \nabla_x u,\nabla_x v\rangle_{L^2(Q)} \quad \text{for } u \in W(Q),\, v \in L^2(0,T;H_0^1(\Omega))$$

is introduced. The bilinear form $a(\cdot,\cdot)$ is bounded, i.e.

$$|a(u,v)| \leq \sqrt{2}\|u\|_{W(Q)}\|\nabla_x v\|_{L^2(Q)} \quad \text{for } u \in W(Q),\, v \in L^2(0,T;H_0^1(\Omega)).$$

Next, the requirements of the Nečas Theorem 2.9.1 for the bilinear form

$$a(\cdot,\cdot)\colon W_{0,}(Q) \times L^2(0,T;H_0^1(\Omega)) \to \mathbb{R}$$

are examined, where the Hilbert space $W_{0,}(Q)$ is the subspace given in (2.25).

Theorem 3.2.3. *The bilinear form* $a(\cdot,\cdot)\colon\, W_{0,}(Q) \times L^2(0,T;H^1_0(\Omega)) \to \mathbb{R}$ *is continuous and fulfils an inf-sup condition and the surjectivity condition, i.e. there hold:*

1. *For all* $u \in W_{0,}(Q)$ *and for all* $v \in L^2(0,T;H^1_0(\Omega))$*, there is*
$$|a(u,v)| \le \sqrt{2}\|u\|_{W(Q)}\|\nabla_x v\|_{L^2(Q)}.$$
2. *There holds the inf-sup condition*
$$\inf_{0\neq u\in W_{0,}(Q)}\ \sup_{0\neq v\in L^2(0,T;H^1_0(\Omega))} \frac{|a(u,v)|}{\|u\|_{W(Q)}\|\nabla_x v\|_{L^2(Q)}} \ge \frac{1}{2\sqrt{2}}.$$
3. *For each function* $v \in L^2(0,T;H^1_0(\Omega))$*,* $v \neq 0$*, there exists an element* $u \in W_{0,}(Q)$ *such that* $a(u,v) \neq 0$*.*

Proof. The proof is contained in the book [51, Proof of Theorem 6.6, page 282]. For the inf-sup constant, see [142, Theorem 2.1, page 5]. □

For a given initial condition $u_0 \in L^2(\Omega)$ and a given right-hand side $f \in L^2(0,T;[H^1_0(\Omega)]')$, the variational formulation of the heat equation (3.1) is to find $u \in W(Q)$ with $u(\cdot,0) = u_0$ in $L^2(\Omega)$ such that
$$a(u,v) = \langle f,v\rangle_Q \tag{3.13}$$
for all $v \in L^2(0,T;H^1_0(\Omega))$. By homogenisation, there follows the existence and uniqueness theorem:

Theorem 3.2.4. *Let the right-hand side* $f \in L^2(0,T;[H^1_0(\Omega)]')$ *and the initial condition* $u_0 \in L^2(\Omega)$ *be given. Then the variational formulation* (3.13) *admits a unique solution* $u \in W(Q)$ *with* $u(\cdot,0) = u_0$ *in* $L^2(\Omega)$*, satisfying*
$$\|u\|_{W(Q)} \le 2\sqrt{2}\left(\|f\|_{L^2(0,T;[H^1_0(\Omega)]')} + \sqrt{2}C_{\mathrm{ex}}\|u_0\|_{L^2(\Omega)}\right) + C_{\mathrm{ex}}\|u_0\|_{L^2(\Omega)}$$
with a constant $C_{\mathrm{ex}} > 0$ *coming from the extension* (2.24) *of* u_0*.*

Proof. Consider the extension $\overline{u}_0 := \mathcal{E}_{\Omega_0} u_0 \in W(Q)$ from (2.24) with $\overline{u}_0(\cdot,0) = u_0$ in $L^2(\Omega)$ satisfying $\|\overline{u}_0\|_{W(Q)} \le C_{\mathrm{ex}}\|u_0\|_{L^2(\Omega)}$ with a constant $C_{\mathrm{ex}} > 0$. Next, investigate the variational formulation by homogenisation to find $\overline{u} \in W_{0,}(Q)$ such that
$$a(\overline{u},v) = \langle f,v\rangle_Q - a(\overline{u}_0,v) \tag{3.14}$$

for all $v \in L^2(0,T;H^1_0(\Omega))$. The right-hand side is bounded by

$$\begin{aligned}\left|\langle f,v\rangle_Q - a(\overline{u}_0,v)\right| &\leq \left(\|f\|_{L^2(0,T;[H^1_0(\Omega)]')} + \sqrt{2}\|\overline{u}_0\|_{W(Q)}\right)\|\nabla_x v\|_{L^2(Q)} \\ &\leq \left(\|f\|_{L^2(0,T;[H^1_0(\Omega)]')} + \sqrt{2}C_{\mathrm{ex}}\|u_0\|_{L^2(\Omega)}\right)\|\nabla_x v\|_{L^2(Q)} \qquad (3.15)\end{aligned}$$

for all $v \in L^2(0,T;H^1_0(\Omega))$. So, the Nečas Theorem 2.9.1 yields with Theorem 3.2.3 that there exists a unique solution $\overline{u} \in W_{0,}(Q)$ of the variational formulation (3.14), satisfying with (3.15)

$$\|\overline{u}\|_{W(Q)} \leq 2\sqrt{2}\left(\|f\|_{L^2(0,T;[H^1_0(\Omega)]')} + \sqrt{2}C_{\mathrm{ex}}\|u_0\|_{L^2(\Omega)}\right). \qquad (3.16)$$

Set $u := \overline{u} + \overline{u}_0 \in W(Q)$.

Next, the independence of the extension $\overline{u}_0$ for u is examined. So, for a second extension $\hat{u}_0 \in W(Q)$, satisfying $\hat{u}_0(\cdot,0) = u_0$ in $L^2(\Omega)$, there exists again $\hat{u} \in W_{0,}(Q)$, satisfying the variational formulation (3.14). The difference $(\overline{u}+\overline{u}_0) - (\hat{u}+\hat{u}_0) \in W_{0,}(Q)$ fulfils the homogeneous variational formulation

$$a((\overline{u}+\overline{u}_0) - (\hat{u}+\hat{u}_0), v) = 0 \quad \text{for all } v \in L^2(0,T;H^1_0(\Omega)).$$

Because of Theorem 3.2.3 and the Nečas Theorem 2.9.1, the to the bounded bilinear form $a(\cdot,\cdot)\colon W_{0,}(Q) \times L^2(0,T;H^1_0(\Omega)) \to \mathbb{R}$ related operator $A\colon W_{0,}(Q) \to L^2(0,T;[H^1_0(\Omega)]')$ is an isomorphism. Hence, $A\left((\overline{u}+\overline{u}_0) - (\hat{u}+\hat{u}_0)\right) = 0$, i.e. $\overline{u}+\overline{u}_0 = \hat{u}+\hat{u}_0$ and therefore, the solution $u \in W(Q)$ is independent of the extension $\overline{u}_0$ for the initial condition u_0.

With the triangle inequality, (3.16) and the continuity of the extension operator for u_0, there follow the stability estimate

$$\begin{aligned}\|u\|_{W(Q)} &\leq \|\overline{u}\|_{W(Q)} + \|\overline{u}_0\|_{W(Q)} \\ &\leq 2\sqrt{2}\left(\|f\|_{L^2(0,T;[H^1_0(\Omega)]')} + \sqrt{2}C_{\mathrm{ex}}\|u_0\|_{L^2(\Omega)}\right) + C_{\mathrm{ex}}\|u_0\|_{L^2(\Omega)}\end{aligned}$$

and hence, the assertion. □

For a discretisation scheme, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0,L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. As a conforming space-time discretisation, consider the space of piecewise linear, continuous functions $S^1_h(Q) \cap W_{0,}(Q)$, see Section 2.8 for more details. For an arbitrary fixed extension $\overline{u}_0 \in W(Q)$ with $\overline{u}_0(\cdot,0) = u_0$ in $L^2(\Omega)$ and $\|\overline{u}_0\|_{W(Q)} \leq \tilde{C}_{\mathrm{ex}}\|u_0\|_{L^2(\Omega)}$ with a constant $\tilde{C}_{\mathrm{ex}} > 0$ independent of u_0, e.g., $\overline{u}_0 = \mathcal{E}_{\Omega_0}u_0$ from (2.24), the discrete variational formulation is to find $\overline{u}_h \in S^1_h(Q) \cap W_{0,}(Q)$ such that

$$a(\overline{u}_h, v_h) = \langle f, v_h\rangle_Q - a(\overline{u}_0, v_h) \qquad (3.17)$$

for all $v_h \in S_h^1(Q) \cap W_{0,}(Q)$. Note that ansatz and test spaces are equal. Next, define the approximation

$$u_h := \overline{u}_h + \overline{u}_0 \in W(Q), \tag{3.18}$$

where in practice $\overline{u}_0$ is replaced by the space-time interpolant $I_h\overline{u}_0 \in S_h^1(Q)$ from (2.40), if $\overline{u}_0$ is smooth enough, see [141, page 246] or [51, Section 3.2.2, page 124] for the elliptic case.

The following stability and convergence theorem is contained in the work [142, Section 3, page 6].

Theorem 3.2.5. *Let the assumptions of Theorem 3.2.4 be satisfied with the unique solution $u \in W(Q)$ of* (3.13)*. Further, let $\overline{u}_0 \in W(Q)$ be the extension from* (3.17) *of u_0 with $\overline{u}_0(\cdot,0) = u_0$ in $L^2(\Omega)$ and $\|\overline{u}_0\|_{W(Q)} \le \tilde{C}_{\text{ex}}\|u_0\|_{L^2(\Omega)}$ with a constant $\tilde{C}_{\text{ex}} > 0$ independent of u_0. Then there exists a unique solution $\overline{u}_h \in S_h^1(Q) \cap W_{0,}(Q)$ of the discrete variational formulation* (3.17)*, satisfying the stability estimate*

$$\|\nabla_x \overline{u}_h\|_{L^2(Q)} \le 2\sqrt{2}\left(\|f\|_{L^2(0,T;[H_0^1(\Omega)]')} + \sqrt{2}\tilde{C}_{\text{ex}}\|u_0\|_{L^2(\Omega)}\right).$$

Furthermore, assume $\overline{u} = u - \overline{u}_0 \in H^s(Q)$ for some $s \in [1,2]$. Then it holds for the approximation $u_h = \overline{u}_h + \overline{u}_0$ in (3.18) *the error estimate*

$$\|\nabla_x(u - u_h)\|_{L^2(Q)} \le C h^{s-1}\|\overline{u}\|_{H^s(Q)}$$

with a constant $C > 0$.

Proof. The unique solvability of the discrete variational formulation (3.17) follows from the discrete inf-sup condition

$$\inf_{0 \ne u_h \in S_h^1(Q)\cap W_{0,}(Q)} \sup_{0 \ne v_h \in S_h^1(Q)\cap W_{0,}(Q)} \frac{a(u_h, v_h)}{\|\nabla_x u_h\|_{L^2(Q)}\|\nabla_x v_h\|_{L^2(Q)}} \ge \frac{1}{2\sqrt{2}}, \tag{3.19}$$

which is proven in [142, Theorem 3.5, page 7]. In addition, the discrete inf-sup condition (3.19) yields with the bound (3.15) the stability estimate

$$\|\nabla_x \overline{u}_h\|_{L^2(Q)} \le 2\sqrt{2}\left(\|f\|_{L^2(0,T;[H_0^1(\Omega)]')} + \sqrt{2}\tilde{C}_{\text{ex}}\|u_0\|_{L^2(\Omega)}\right).$$

The error estimate follows from [142, Corollary 3.4, page 10] with

$$\|\nabla_x(u - u_h)\|_{L^2(Q)} = \|\nabla_x(\overline{u} - \overline{u}_h)\|_{L^2(Q)}$$

and hence, the assertion. □

Numerical examples and further investigations are given in [142–144].

3.3 Space-Time Variational Formulation of Anisotropic Type

In this section, a short overview of a so-called space-time variational formulation of anisotropic type for a homogeneous initial condition, i.e. $u_0 = 0$, is given. The motivation comes from considering a FEM-BEM coupling for transmission problems of the heat equation. Because this variational formulation of anisotropic type arises by the treatment of boundary integral equations for the heat equation, the usage of this variational formulation of anisotropic type is natural for the finite element method. However, it seems that a stable conforming discretisation of this variational formulation of anisotropic type by piecewise linear, continuous functions is not available.

For the homogeneous Dirichlet problem of the heat equation

$$\left.\begin{array}{rcll} \partial_t u(x,t) - \Delta_x u(x,t) &=& f(x,t) & \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &=& 0 & \text{for } (x,t) \in \Sigma = \Gamma \times [0,T], \\ u(x,0) &=& 0 & \text{for } x \in \Omega, \end{array}\right\}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$ and $T > 0$ is a given finite time, the space-time variational formulation of anisotropic type is given as follows:

Find $u \in H^{1,1/2}_{0;0,}(Q)$ such that

$$\langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} = \langle f, v \rangle_Q \tag{3.20}$$

for all $v \in H^{1,1/2}_{0;,0}(Q)$, where $f \in [H^{1,1/2}_{0;,0}(Q)]'$ is a given right-hand side, see Section 2.5 for the notations. The bilinear form $a(\cdot,\cdot)\colon H^{1,1/2}_{0;0,}(Q) \times H^{1,1/2}_{0;,0}(Q) \to \mathbb{R}$,

$$a(u,v) := \langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)}$$

for $u \in H^{1,1/2}_{0;0,}(Q)$, $v \in H^{1,1/2}_{0;,0}(Q)$, is bounded, i.e. there exists a constant $C > 0$ such that

$$|a(u,v)| \leq C \|u\|_{H^{1,1/2}_{0;0,}(Q)} \|v\|_{H^{1,1/2}_{0;,0}(Q)}$$

for $u \in H^{1,1/2}_{0;0,}(Q)$, $v \in H^{1,1/2}_{0;,0}(Q)$, see [35, Lemma 2.6, page 505].

Remark 3.3.1. *The bilinear form $\langle \partial_t u, v \rangle_{(0,T)}$ for $u, v \in C_0^\infty(0,T)$ has no continuous extension to*

$$H^{1/2}_{0,}(0,T) \times H^{1/2}_{0,}(0,T)$$

or to

$$H^{1/2}(0,T) \times H^{1/2}(0,T),$$

i.e. the usage of the different ansatz and test spaces $H^{1,1/2}_{0;0,}(Q)$, $H^{1,1/2}_{0;,0}(Q)$ *for the bilinear form*

$$a(\cdot,\cdot)\colon\ H^{1,1/2}_{0;0,}(Q)\times H^{1,1/2}_{0;,0}(Q)\to\mathbb{R}$$

is crucial. See [35, Remark 2.7, page 505] and [64, Proposition 1.4.4.8, page 32].

In [35], the following existence and uniqueness theorem is proven by a transposition and interpolation argument as in [102, 103], see also [98].

Theorem 3.3.2. *Let the right-hand side* $f\in[H^{1,1/2}_{0;,0}(Q)]'$ *be given. Then the variational formulation* (3.20) *has a unique solution* $u\in H^{1,1/2}_{0;0,}(Q)$, *satisfying*

$$\|u\|_{H^{1,1/2}_{0;0,}(Q)}\le C\|f\|_{[H^{1,1/2}_{0;,0}(Q)]'}$$

with a constant $C>0$. *Furthermore, the solution operator*

$$\mathcal{L}\colon\ [H^{1,1/2}_{0;,0}(Q)]'\to H^{1,1/2}_{0;0,}(Q),\quad \mathcal{L}f:=u,$$

is an isomorphism. In addition, the bilinear form

$$a(\cdot,\cdot)\colon\ H^{1,1/2}_{0;0,}(Q)\times H^{1,1/2}_{0;,0}(Q)\to\mathbb{R},\quad a(u,v)=\langle\partial_t u,v\rangle_Q+\langle\nabla_x u,\nabla_x v\rangle_{L^2(Q)},$$

is continuous and fulfils an inf-sup condition and the surjectivity condition.

Proof. The existence and uniqueness of a solution $u\in H^{1,1/2}_{0;0,}(Q)$ of the variational formulation (3.20) and that the solution operator $\mathcal{L}\colon\ [H^{1,1/2}_{0;,0}(Q)]'\to H^{1,1/2}_{0;0,}(Q)$ is an isomorphism follow from [35, Lemma 2.8, page 505]. The Nečas Theorem 2.9.1 yields the properties of the bilinear form $a(\cdot,\cdot)\colon\ H^{1,1/2}_{0;0,}(Q)\times H^{1,1/2}_{0;,0}(Q)\to\mathbb{R}$. □

Remark 3.3.3. *In Section 3.4 an alternative proof of Theorem 3.3.2 is given by the usage of Fourier series, see Theorem 3.4.19.*

For a discretisation scheme, let the bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$ be an interval $\Omega=(0,L)$ for $d=1$, or polygonal for $d=2$, or polyhedral for $d=3$. With the notations of Section 2.8, a conforming space-time discretisation via the space of piecewise linear, continuous functions $S^1_h(Q)$ leads to the discrete ansatz and test spaces $S^1_h(Q)\cap H^{1,1/2}_{0;0,}(Q)$ and $S^1_h(Q)\cap H^{1,1/2}_{0;,0}(Q)$. Hence, the test space differs from that of the approach (3.17), i.e.

$$S^1_h(Q)\cap H^{1,1/2}_{0;,0}(Q)\neq S^1_h(Q)\cap W_{0,}(Q).$$

It turns out that the resulting finite element method, for the choice above of different ansatz and test spaces, is not stable. Therefore, only the time component is considered. That means, for a fixed $\mu > 0$, the ordinary differential equation

$$\partial_t u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0,$$

and the variational formulation to find $u \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u, v \rangle_{(0,T)} + \mu \langle u, v \rangle_{L^2(0,T)} = \langle f, v \rangle_{(0,T)} \tag{3.21}$$

for all $v \in H^{1/2}_{,0}(0,T)$, where $f \in [H^{1/2}_{,0}(0,T)]'$ is given, are investigated. Analogous to Theorem 3.3.2, there exists a unique solution $u \in H^{1/2}_{0,}(0,T)$ of the variational formulation (3.21), see also Theorem 3.4.10. With the notations of Section 2.6, a conforming discretisation with piecewise linear, continuous functions $S^1_{h_t}(0,T) = \text{span}\{\varphi_k\}_{k=0}^{N_t}$ leads to the discrete variational formulation to find $u_{h_t} \in S^1_{h_t}(0,T) \cap H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u_{h_t}, v_{h_t} \rangle_{L^2(0,T)} + \mu \langle u_{h_t}, v_{h_t} \rangle_{L^2(0,T)} = \langle f, v_{h_t} \rangle_{(0,T)} \tag{3.22}$$

for all $v_{h_t} \in S^1_{h_t}(0,T) \cap H^{1/2}_{,0}(0,T)$. The resulting system matrix of (3.22) is given as

$$K_{h_t} = \frac{1}{2} \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \end{pmatrix} + \frac{\mu}{6} \begin{pmatrix} h_{t,1} & & & & \\ 2h_{t,1} + 2h_{t,2} & h_{t,2} & & & \\ h_{t,2} & 2h_{t,2} + 2h_{t,3} & h_{t,3} & & \\ & \ddots & \ddots & \ddots & \\ & & h_{t,N_t-1} & 2h_{t,N_t-1} + 2h_{t,N_t} & h_{t,N_t} \end{pmatrix} \tag{3.23}$$

and hence, unique solvability of the discrete variational formulation (3.22) follows because $K_{h_t} \in \mathbb{R}^{N_t \times N_t}$ is a lower triangular matrix with positive diagonal elements.

For a uniform discretisation with mesh size h_t, the matrix K_{h_t} in (3.23) can be interpreted as a finite difference scheme

$$\begin{aligned} a_0 v_1 &= f_1, \\ a_1 v_1 + a_0 v_2 &= f_2, \\ a_2 v_{j-2} + a_1 v_{j-1} + a_0 v_j &= f_j \quad \text{for } j > 2, \end{aligned}$$

where

$$a_0 := \frac{1}{2} + \frac{\mu h_t}{6} = \frac{3+\mu h_t}{6}, \quad a_1 := \frac{4\mu h_t}{6}, \quad a_2 := -\frac{1}{2} + \frac{\mu h_t}{6} = \frac{-3+\mu h_t}{6}$$

with given $f_j \in \mathbb{R}$. The essential case $f_j = 0$ for $j > 2$ is examined. The solution of the homogeneous linear recurrence relation

$$a_2 v_{j-2} + a_1 v_{j-1} + a_0 v_j = 0 \quad \text{for } j > 2$$

is given for $j \geq 1$ by

$$v_j = A_0 \left(\frac{-2h_t\mu - \sqrt{9+3h_t^2\mu^2}}{3+h_t\mu} \right)^{j-1} + A_1 \left(\frac{-2h_t\mu + \sqrt{9+3h_t^2\mu^2}}{3+h_t\mu} \right)^{j-1},$$

where the coefficients $A_0, A_1 \in \mathbb{R}$ are determined by $f_1, f_2 \in \mathbb{R}$. Hence, in general, the sequence $(v_j)_{j\in\mathbb{N}}$ is unbounded as $j \to \infty$ independently of μ and h_t. In other words, the numerical scheme of (3.22) is unstable for each μ and each h_t. For the heat equation, deriving a conforming discretisation by piecewise linear, continuous functions of the variational formulation (3.20), which results in a stable numerical scheme, is delicate and is not discussed in this thesis. On the other hand, an alternative approach with the help of a type of Hilbert transform is given in Section 3.4.

3.4 Space-Time Variational Formulation with a Type of Hilbert Transform

In this section, the space-time variational formulation of anisotropic type of Section 3.3 is examined with the help of a type of Hilbert transform, see [145]. Via a Fourier series ansatz a transformation operator $\mathcal{H}_T$ is introduced, and existence and uniqueness of the space-time variational formulation of anisotropic type of Section 3.3 is proven directly, i.e. no transposition and interpolation argument is needed, see also [35, Remark 2.13, page 507]. For the resulting space-time variational formulation of this section, ansatz and test spaces are equal. Furthermore, the used analysis is developed on a finite time interval $(0,T)$ instead of considering an unbounded time interval $(0,\infty)$ as in [37, 52, 101]. Moreover, a conforming discretisation of the resulting variational formulation leads to an unconditionally stable finite element method, which is combinable with the boundary element method as in [35] via a FEM-BEM coupling. In the last part of this section, unconditional stability for unstructured space-time meshes, error estimates in $L^2(Q)$, in $H^1(Q)$ and in the anisotropic Sobolev space $H^{1/2}_{0,}(0,T;L^2(\Omega))$ for a tensor-product approach are proven. Furthermore, numerical examples, which confirm the theoretical results, are presented.

3.4.1 Characterisation of $H^{1/2}_{0,}(0,T)$ and $H^{1/2}_{,0}(0,T)$ via Fourier Series

In this subsection, the interpolation of functions spaces as in [102, Chapitre 1, Section 2.1, page 11] is considered. Hence, all functions are complex-valued in this subsection, i.e. $H^s(0,T;\mathbb{C})$ and $H^{1/2}_{0,}(0,T;\mathbb{C})$ are the complex-valued versions of the Sobolev spaces of Section 2.2. With the notations of [102, Chapitre 1, Section 2.1, page 11] let $Y := L^2(0,T;\mathbb{C})$ be the usual complex Hilbert space with inner product

$$(u,v)_{L^2(0,T;\mathbb{C})} := \int_0^T u(t)\overline{v(t)}\mathrm{d}t,$$

and let the complex Hilbert space $X := H^1_{0,}(0,T;\mathbb{C})$ be endowed with the inner product

$$(u,v)_{H^1_{0,}(0,T;\mathbb{C})} := \int_0^T \partial_t u(t)\overline{\partial_t v(t)}\mathrm{d}t.$$

Clearly, X and Y are separable and X is a dense subset of Y with a compact embedding, see [13, Proof of Satz 5.12, page 148].

Next, an unbounded operator $\Lambda\colon\ Y \supset \mathrm{dom}(\Lambda) \to Y$ with domain $\mathrm{dom}(\Lambda) = X$ is constructed such that Λ is self-adjoint and positive in Y. Therefore, define the unbounded sesquilinear form $\mathfrak{a}\colon \operatorname{dom}(\mathfrak{a}) \times \operatorname{dom}(\mathfrak{a}) \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) := (u,v)_{H^1_{0,}(0,T;\mathbb{C})} = \int_0^T \partial_t u(t)\overline{\partial_t v(t)}\mathrm{d}t$$

for $u,v \in \operatorname{dom}(\mathfrak{a}) := X = H^1_{0,}(0,T;\mathbb{C}) \subset Y$. The sesquilinear form $\mathfrak{a}$ is densely defined, symmetric, closed and lower semibounded in Y, i.e.

- $\operatorname{dom}(\mathfrak{a})$ is dense in Y,
- it holds $\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)}$ for all $u,v \in \operatorname{dom}(\mathfrak{a})$,
- $(\operatorname{dom}(\mathfrak{a}), (\cdot,\cdot)_{\mathfrak{a}})$ is a Hilbert space with the inner product $(u,v)_{\mathfrak{a}} := \mathfrak{a}(u,v)$,
- it holds
$$\mathfrak{a}(u,u) = \|u\|^2_{H^1_{0,}(0,T;\mathbb{C})} \geq \frac{\pi^2}{4T^2}\|u\|^2_{L^2(0,T;\mathbb{C})}$$
for all $u \in \operatorname{dom}(\mathfrak{a})$ due to the Poincaré inequality, see Lemma 3.4.5 for the constant.

The Representation Theorem for Semibounded Forms, see [133, Theorem 10.7, page 228] and see also [90, Theorem 2.1, page 322], [157, Unterkapitel 4.2], yields that there exists a uniquely determined, lower semibounded, self-adjoint operator $S\colon Y \supset \operatorname{dom}(S) \to Y$ such that

- it holds $\operatorname{dom}(S) \subset \operatorname{dom}(\mathfrak{a}) = X$,
- it holds for all $u \in \operatorname{dom}(S)$ and $v \in \operatorname{dom}(\mathfrak{a})$ that
$$\mathfrak{a}(u,v) = (Su,v)_{L^2(0,T;\mathbb{C})}, \tag{3.24}$$
- it holds
$$\operatorname{dom}(S) = \left\{u \in X\colon\ \exists w_u \in Y \text{ with } \mathfrak{a}(u,v) = (w_u,v)_{L^2(0,T;\mathbb{C})}\ \forall v \in \operatorname{dom}(\mathfrak{a})\right\},$$
and the operator is given by $Su := w_u$ for $u \in \operatorname{dom}(S)$,
- a lower bound for S is given by
$$(Su,u)_{L^2(0,T;\mathbb{C})} \geq \frac{\pi^2}{4T^2}\|u\|^2_{L^2(0,T;\mathbb{C})}$$
for all $u \in \operatorname{dom}(S)$.

Lemma 3.4.1. *The unbounded operator $S\colon\ Y \supset \operatorname{dom}(S) \to Y$ from above is given by*

$$Su = -\partial_{tt}u$$

and

$$\operatorname{dom}(S) = \left\{u \in H^2(0,T;\mathbb{C})\colon\ u(0) = 0,\, u'(T) = 0\right\}.$$

Proof. Let $u \in \mathrm{dom}(S)$ be fixed. It is shown that u satisfies $u \in H^2(0,T;\mathbb{C})$ with $u(0) = 0$ and $u'(T) = 0$, i.e. u is contained in the right-hand side. First, because $\mathrm{dom}(S) \subset \mathrm{dom}(\mathfrak{a})$, it holds $u \in H^1_{0,}(0,T;\mathbb{C})$ and so, $u(0) = 0$. Second, for $v \in C_0^\infty(0,T)$ in (3.24), it follows with integration by parts that

$$(Su,v)_{L^2(0,T;\mathbb{C})} = \mathfrak{a}(u,v) = -(u,\partial_{tt}v)_{L^2(0,T;\mathbb{C})},$$

hence, $Su = -\partial_{tt}u \in Y$ and so, $u \in H^2(0,T;\mathbb{C})$. Third, for $v \in C^\infty[0,T]$ with $v(0) = 0$ and $v(T) = 1$ in (3.24), it follows with integration by parts that

$$\mathfrak{a}(u,v) = (Su,v)_{L^2(0,T;\mathbb{C})} = -(\partial_{tt}u,v)_{L^2(0,T;\mathbb{C})} = -u'(T)\overline{v(T)} + \mathfrak{a}(u,v) \quad \Longleftrightarrow \quad u'(T) = 0$$

and so, u is contained in the right-hand side.

Now, let u satisfy $u \in H^2(0,T;\mathbb{C})$ with $u(0) = 0$ and $u'(T) = 0$, i.e. u is contained in the right-hand side. Hence, $u \in X = H^1_{0,}(0,T;\mathbb{C})$. The function u is contained in $\mathrm{dom}(S)$, because for $w_u := -\partial_{tt}u \in L^2(0,T;\mathbb{C})$, it holds with integration by parts

$$\mathfrak{a}(u,v) = -(\partial_{tt}u,v)_{L^2(0,T;\mathbb{C})} = (w_u,v)_{L^2(0,T;\mathbb{C})}$$

for all $v \in \mathrm{dom}(\mathfrak{a})$. Thus, $Su = w_u = -\partial_{tt}u$. □

The Second Representation Theorem [90, Theorem 2.23, page 331] yields that the square root $\Lambda := S^{1/2}\colon Y \supset \mathrm{dom}(\Lambda) \to Y$ fulfils $\mathrm{dom}(\Lambda) = \mathrm{dom}(\mathfrak{a}) = X = H^1_{0,}(0,T;\mathbb{C})$ and

$$\mathfrak{a}(u,v) = (\Lambda u, \Lambda v)_{L^2(0,T;\mathbb{C})} \quad \text{for all } u,v \in X.$$

Recall that $\Lambda\colon Y \supset \mathrm{dom}(\Lambda) \to Y$ is self-adjoint and positive in Y, because Λ is the unique square root of the self-adjoint and positive operator $S\colon Y \supset \mathrm{dom}(S) \to Y$, see [157, Satz 8.22, page 303] or [133, Proposition 5.13, page 95].

Because of the compact embedding of X in Y, the operator $S\colon Y \supset \mathrm{dom}(S) \to Y$ has a purely discrete spectrum, see [133, Proposition 10.6, page 227]. A simple calculation gives

$$V_k(t) := \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad \lambda_k := \frac{1}{T^2}\left(\frac{\pi}{2} + k\pi\right)^2 \quad \text{for } k \in \mathbb{N}_0, \tag{3.25}$$

which fulfil for $k \in \mathbb{N}_0$

$$-\partial_{tt}V_k(t) = \lambda_k V_k(t) \quad \text{for } t \in (0,T), \quad V_k(0) = 0, \quad \partial_t V_k(T) = 0,$$

i.e. $SV_k = \lambda_k V_k$ for $k \in \mathbb{N}_0$. These eigenfunctions V_k form an orthogonal basis in $L^2(0,T;\mathbb{C})$ satisfying

$$\int_0^T V_k(t)V_\ell(t)\mathrm{d}t = \frac{T}{2}\,\delta_{k\ell} \quad \text{for } k,\ell \in \mathbb{N}_0$$

and in $H^1_{0,}(0,T;\mathbb{C})$ with

$$\int_0^T \partial_t V_k(t)\partial_t V_\ell(t)\mathrm{d}t = \lambda_k \int_0^T V_k(t)V_\ell(t)\mathrm{d}t = \frac{1}{2T}\left(\frac{\pi}{2}+k\pi\right)^2 \delta_{k\ell} \quad \text{for } k,\ell \in \mathbb{N}_0.$$

Hence, by Parseval's identity, see [157, Entwicklungssatz 1.55, page 53] and [158, Satz V.4.9, page 254], it follows that for $u \in L^2(0,T;\mathbb{C})$, it holds the expansion

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad u_k = \frac{2}{T}\int_0^T u(t)\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)\mathrm{d}t \tag{3.26}$$

and the norm is given by

$$\|u\|^2_{L^2(0,T;\mathbb{C})} = \frac{2}{T}\sum_{k=0}^{\infty}\left|(u,V_k)_{L^2(0,T;\mathbb{C})}\right|^2 = \frac{T}{2}\sum_{k=0}^{\infty}|u_k|^2. \tag{3.27}$$

Furthermore, for the inner product, it follows

$$(u,v)_{L^2(0,T;\mathbb{C})} = \int_0^T u(t)\overline{v(t)}\mathrm{d}t = \frac{T}{2}\sum_{k=0}^{\infty} u_k\overline{v_k} \tag{3.28}$$

for $u,v \in L^2(0,T;\mathbb{C})$ with expansion coefficients u_k and v_k from (3.26). For a function $u \in H^1_{0,}(0,T;\mathbb{C})$, the expansion (3.26) converges also in $H^1_{0,}(0,T;\mathbb{C})$, i.e.

$$\partial_t u(t) = \frac{1}{T}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)u_k\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)$$

converges in $L^2(0,T;\mathbb{C})$, and the norm is given by

$$\|\partial_t u\|^2_{L^2(0,T;\mathbb{C})} = \frac{2}{T}\sum_{k=0}^{\infty}\lambda_k\left|(u,V_k)_{L^2(0,T;\mathbb{C})}\right|^2 = \frac{1}{2T}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^2|u_k|^2. \tag{3.29}$$

Before defining the interpolation space $H^{1/2}_{0,}(0,T;\mathbb{C})$, note that for $\theta \in (0,1)$ the powers $S^\theta\colon Y \supset \mathrm{dom}(S^\theta) \to Y$ are given by the so-called Functional Calculus, see [133, Section 5.3] or [157, Unterkapitel 8.4], i.e.

$$\mathrm{dom}(S^\theta) = \left\{u \in Y\colon \frac{2}{T}\sum_{k=0}^{\infty}\lambda_k^{2\theta}\left|(u,V_k)_{L^2(0,T;\mathbb{C})}\right|^2 < \infty\right\}$$

and it holds

$$\left\|S^\theta u\right\|^2_{L^2(0,T;\mathbb{C})} = \frac{2}{T}\sum_{k=0}^{\infty}\lambda_k^{2\theta}\left|(u,V_k)_{L^2(0,T;\mathbb{C})}\right|^2,$$

see [133, Theorem 5.9, page 93].

Plugging these results in the definition of the interpolation spaces [102, Définition 2.1, page 12] gives with the expansion (3.26)

$$\begin{aligned} H_{0,}^{1/2}(0,T;\mathbb{C}) &= [H_{0,}^1(0,T;\mathbb{C}), L^2(0,T;\mathbb{C})]_{1/2} = \operatorname{dom}(\Lambda^{1/2}) = \operatorname{dom}(S^{1/4}) \\ &= \left\{ u \in L^2(0,T;\mathbb{C}) \colon \frac{2}{T} \sum_{k=0}^{\infty} \lambda_k^{1/2} \left| (u, V_k)_{L^2(0,T;\mathbb{C})} \right|^2 < \infty \right\} \\ &= \left\{ u \in L^2(0,T;\mathbb{C}) \colon u = \sum_{k=0}^{\infty} u_k V_k, \quad \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |u_k|^2 < \infty \right\} \end{aligned}$$

with the interpolation norm for $u \in H_{0,}^{1/2}(0,T;\mathbb{C})$

$$\begin{aligned} \|u\|_{[H_{0,}^1(0,T;\mathbb{C}), L^2(0,T;\mathbb{C})]_{1/2}} &= \sqrt{\|u\|^2_{L^2(0,T;\mathbb{C})} + \left\|\Lambda^{1/2} u\right\|^2_{L^2(0,T;\mathbb{C})}} \\ &= \sqrt{\frac{2}{T} \sum_{k=0}^{\infty} \left(1 + \lambda_k^{1/2}\right) \left| (u, V_k)_{L^2(0,T;\mathbb{C})} \right|^2} \\ &= \sqrt{\frac{1}{2} \sum_{k=0}^{\infty} \left(T + \left(\frac{\pi}{2} + k\pi \right) \right) |u_k|^2}, \end{aligned} \tag{3.30}$$

see also [102, Proof of Théorème 16.2, page 112] and [15, Section 11.5] for such a construction. This motivates to define for $u, v \in H_{0,}^{1/2}(0,T;\mathbb{C})$ with expansion (3.26) the norm

$$\|u\|_{H_{0,}^{1/2}(0,T;\mathbb{C}),F} := \sqrt{\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |u_k|^2}, \tag{3.31}$$

as well as the inner product

$$(u, v)_{H_{0,}^{1/2}(0,T;\mathbb{C}),F} := \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) u_k \overline{v_k},$$

where the subscript F stands for *Fourier series*.

Theorem 3.4.2. *There hold the norm equivalences for all $u \in H_{0,}^{1/2}(0,T;\mathbb{C})$*

$$\|u\|_{H_{0,}^{1/2}(0,T;\mathbb{C}),F} \le \|u\|_{[H_{0,}^1(0,T;\mathbb{C}), L^2(0,T;\mathbb{C})]_{1/2}} \le \sqrt{1 + \frac{2T}{\pi}} \|u\|_{H_{0,}^{1/2}(0,T;\mathbb{C}),F}$$

and

$$C_1 \|u\|_{H_{0,}^{1/2}(0,T;\mathbb{C}),F} \le \sqrt{\|u\|^2_{H^{1/2}(0,T;\mathbb{C})} + \int_0^T \frac{|u(t)|^2}{t} \mathrm{d}t} \le C_2 \|u\|_{H_{0,}^{1/2}(0,T;\mathbb{C}),F}$$

with constants $C_1 > 0$ *and* $C_2 > 0$. *Hence,* $\left(H_{0,}^{1/2}(0,T;\mathbb{C}), (\cdot,\cdot)_{H_{0,}^{1/2}(0,T;\mathbb{C}),F}\right)$ *is a Hilbert space.*

Proof. From [102, Théorème 11.7, page 72] and [102, Remarque 11.4, page 75], it follows that the norm $u \mapsto \sqrt{\|u\|^2_{H^{1/2}(0,T;\mathbb{C})} + \int_0^T \frac{|u(t)|^2}{t}\mathrm{d}t}$ is equivalent to the interpolation norm $\|\cdot\|_{[H^1_{0,}(0,T;\mathbb{C}),L^2(0,T;\mathbb{C})]_{1/2}}$, see also [102, Remarque 2.3, page 13]. It remains to prove that the norm $\|\cdot\|_{H_{0,}^{1/2}(0,T;\mathbb{C}),F}$ and the interpolation norm $\|\cdot\|_{[H^1_{0,}(0,T;\mathbb{C}),L^2(0,T;\mathbb{C})]_{1/2}}$ are equivalent. The first inequality $\|u\|_{H_{0,}^{1/2}(0,T;\mathbb{C}),F} \le \|u\|_{[H^1_{0,}(0,T;\mathbb{C}),L^2(0,T;\mathbb{C})]_{1/2}}$ is trivial because of (3.30). The second inequality follows from

$$\begin{aligned}
\|u\|^2_{[H^1_{0,}(0,T;\mathbb{C}),L^2(0,T;\mathbb{C})]_{1/2}} &= \frac{1}{2}\sum_{k=0}^{\infty}\left(T + \left(\frac{\pi}{2}+k\pi\right)\right)|u_k|^2 \\
&\le \left(\frac{2T}{\pi}+1\right)\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)|u_k|^2 \\
&= \left(\frac{2T}{\pi}+1\right)\|u\|^2_{H_{0,}^{1/2}(0,T;\mathbb{C}),F},
\end{aligned}$$

where the representation (3.30) is used again. □

Remark 3.4.3. *For the explicit calculation of boundedness constants, an interpolation argument, i.e. the Interpolation Theorem [30, Proposition 14.1.5, page 373] or [26, Theorem 3.1.2, page 40] or [153, Section 1.3] for the so-called K-Method of Interpolation with the interpolation norm* $\|\cdot\|_{K_\theta(X;Y)}$, *is used. Interpolating the Hilbert spaces* $H^1_{0,}(0,T;\mathbb{C})$ *and* $L^2(0,T;\mathbb{C})$ *with the K-Method of Interpolation yields again* $H_{0,}^{1/2}(0,T;\mathbb{C})$ *with the to* $\|\cdot\|_{[H^1_{0,}(0,T;\mathbb{C}),L^2(0,T;\mathbb{C})]_{1/2}}$ *equivalent norm* $\|\cdot\|_{K_{1/2}(H^1_{0,}(0,T;\mathbb{C});L^2(0,T;\mathbb{C}))}$ *fulfilling*

$$\begin{aligned}
\|u\|^2_{K_{1/2}(H^1_{0,}(0,T;\mathbb{C});L^2(0,T;\mathbb{C}))} &= \frac{\pi}{2}\left[\|u\|^2_{[H^1_{0,}(0,T;\mathbb{C}),L^2(0,T;\mathbb{C})]_{1/2}} - \|u\|^2_{L^2(0,T;\mathbb{C})}\right] \\
&= \frac{\pi}{2}\left\|\Lambda^{1/2}u\right\|^2_{L^2(0,T;\mathbb{C})} \\
&= \frac{\pi}{2}\|u\|^2_{H_{0,}^{1/2}(0,T;\mathbb{C}),F}
\end{aligned} \tag{3.32}$$

for $u \in H_{0,}^{1/2}(0,T;\mathbb{C})$ *with* (3.30)*, see [102, Proof of Théorème 15.1, page 108].*

Now, the result of Theorem 3.4.2 is transferred to real-valued functions. Hence, for the real Hilbert space $H_{0,}^{1/2}(0,T)$, see (2.1), it holds the representation

$$H_{0,}^{1/2}(0,T) = \left\{ u \in L^2(0,T) \colon u = \sum_{k=0}^{\infty} u_k V_k, \quad \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)|u_k|^2 < \infty \right\}$$

and

$$\langle u, v \rangle_{H^{1/2}_{0,}(0,T),F} := \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)u_k v_k$$

is a inner product, which induces a to $\|\cdot\|_{H^{1/2}_{0,}(0,T)}$ equivalent norm, where $u_k, v_k \in \mathbb{R}$ are the expansion coefficients given by (3.26).

Analogously, the real Hilbert space $H^{1/2}_{,0}(0,T)$ is investigated. Here, only the notations are introduced and some properties are stated. The eigenfunctions and eigenvalues

$$W_k(t) := \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad \hat{\lambda}_k := \frac{1}{T^2}\left(\frac{\pi}{2}+k\pi\right)^2, \quad k \in \mathbb{N}_0, \tag{3.33}$$

fulfil for $k \in \mathbb{N}_0$

$$-\partial_{tt}W_k(t) = \hat{\lambda}_k W_k(t) \quad \text{for } t \in (0,T), \quad \partial_t W_k(0) = 0, \quad W_k(T) = 0.$$

Note that there hold $\lambda_k = \hat{\lambda}_k$ and $\partial_t V_k = \sqrt{\lambda_k} W_k$ for all $k \in \mathbb{N}_0$. These eigenfunctions W_k form an orthogonal basis in $L^2(0,T)$ satisfying

$$\int_0^T W_k(t)W_\ell(t)\mathrm{d}t = \frac{T}{2}\,\delta_{k\ell} \quad \text{for } k,\ell \in \mathbb{N}_0$$

and in $H^1_{,0}(0,T)$ with

$$\int_0^T \partial_t W_k(t)\partial_t W_\ell(t)\mathrm{d}t = \lambda_k \int_0^T W_k(t)W_\ell(t)\mathrm{d}t = \frac{1}{2T}\left(\frac{\pi}{2}+k\pi\right)^2 \delta_{k\ell} \quad \text{for } k,\ell \in \mathbb{N}_0.$$

Hence, by Parseval's identity, it follows that for $w \in L^2(0,T)$, it holds the expansion

$$w(t) = \sum_{k=0}^{\infty} w_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right), \quad w_k = \frac{2}{T}\int_0^T w(t)\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)\mathrm{d}t \tag{3.34}$$

and the norm is given by

$$\|w\|^2_{L^2(0,T)} = \frac{2}{T}\sum_{k=0}^{\infty}\left|(w, W_k)_{L^2(0,T)}\right|^2 = \frac{T}{2}\sum_{k=0}^{\infty}|w_k|^2.$$

Furthermore, for the inner product follows

$$\langle w, z \rangle_{L^2(0,T)} = \int_0^T w(t)z(t)\mathrm{d}t = \frac{T}{2}\sum_{k=0}^{\infty} w_k z_k \tag{3.35}$$

for $w,z \in L^2(0,T)$ with expansion coefficients w_k and z_k from (3.34). For $w \in H^1_{,0}(0,T)$, the expansion (3.34) converges also in $H^1_{,0}(0,T)$, i.e.

$$\partial_t w(t) = -\frac{1}{T}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big) w_k \sin\Big(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\Big)$$

converges in $L^2(0,T)$, and the norm is given by

$$\|\partial_t w\|^2_{L^2(0,T)} = \frac{2}{T}\sum_{k=0}^{\infty}\lambda_k \left|(w,W_k)_{L^2(0,T)}\right|^2 = \frac{1}{2T}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)^2 |w_k|^2.$$

For the real Hilbert space $H^{1/2}_{,0}(0,T)$, see (2.2), it holds the representation

$$H^{1/2}_{,0}(0,T) = \left\{ u \in L^2(0,T) \colon u = \sum_{k=0}^{\infty} w_k W_k, \quad \frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)|w_k|^2 < \infty \right\}$$

and

$$\langle w,z\rangle_{H^{1/2}_{,0}(0,T),F} := \frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big) w_k z_k$$

is a inner product, which induces a to $\|\cdot\|_{H^{1/2}_{,0}(0,T)}$ equivalent norm, where $w_k, z_k \in \mathbb{R}$ are the expansion coefficients given by (3.34).

Finally, representations of the dual spaces $[H^{1/2}_{,0}(0,T)]'$ and $[H^1_{,0}(0,T)]'$ are given. In Section 2.2 the dual space $[H^1_{,0}(0,T)]'$ is characterised as a completion of $L^2(0,T)$ with respect to the Hilbertian norm $\|\cdot\|_{[H^1_{,0}(0,T)]'}$, where $\|\cdot\|_{H^1_{,0}(0,T)} = |\cdot|_{H^1(0,T)}$ is the norm in $H^1_{,0}(0,T)$. Analogously, in Section 2.2, the dual space $[H^{1/2}_{,0}(0,T)]'$ is characterised as a completion of $L^2(0,T)$ with respect to the Hilbertian norm $\|\cdot\|_{[H^{1/2}_{,0}(0,T)]'}$, where $\|\cdot\|_{H^{1/2}_{,0}(0,T)}$ is the norm in $H^{1/2}_{,0}(0,T)$, see (2.3). Here, a to $\|\cdot\|_{[H^{1/2}_{,0}(0,T)]'}$ equivalent norm is given by

$$\|f\|_{[H^{1/2}_{,0}(0,T)]',F} := \sup_{0\neq w\in H^{1/2}_{,0}(0,T)} \frac{|f(w)|}{\|w\|_{H^{1/2}_{,0}(0,T),F}}$$

for $f \in [H^{1/2}_{,0}(0,T)]'$. With the help of the expansion (3.34), the following lemma holds, see also [96, Section 8.1].

Lemma 3.4.4. *For $f \in [H^{1/2}_{,0}(0,T)]'$, there holds*

$$\|f\|^2_{[H^{1/2}_{,0}(0,T)]',F} = \frac{T^2}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)^{-1} f_k^2,$$

where $f_k := \frac{2}{T} f(W_k)$ *with* $W_k(t) = \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)$. *Furthermore, the dual space is given by*

$$[H^{1/2}_{,0}(0,T)]' = \left\{ g\colon H^{1/2}_{,0}(0,T) \to \mathbb{R}\colon \quad g(w) = \frac{T}{2}\sum_{k=0}^{\infty} w_k g_k \text{ with } (g_k)_{k\in\mathbb{N}_0} \subset \mathbb{R} \right.$$
$$\left. \text{satisfying } \frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1} g_k^2 < \infty \right\},$$

where the expansion coefficients w_k *are from* (3.34).

Analogously, for $f \in [H^1_{,0}(0,T)]'$, *there holds*

$$\|f\|^2_{[H^1_{,0}(0,T)]'} = \frac{T^3}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-2} f_k^2$$

where $f_k := \frac{2}{T} f(W_k)$ *with* $W_k(t) = \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)$.

Proof. Let $f \in [H^{1/2}_{,0}(0,T)]'$ be fixed. For $w \in H^{1/2}_{,0}(0,T)$, there hold the representations

$$w(t) = \sum_{k=0}^{\infty} w_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) = \sum_{k=0}^{\infty} w_k W_k(t), \quad \|w\|^2_{H^{1/2}_{,0}(0,T),F} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right) w_k^2$$

and therefore, with the continuity of f,

$$f(w) = \sum_{k=0}^{\infty} w_k f(W_k) = \frac{T}{2}\sum_{k=0}^{\infty} w_k f_k.$$

Set $w_k^N := \left(\frac{\pi}{2}+k\pi\right)^{-1} f_k$ for $k = 0,\ldots,N$ and $w_k^N = 0$ for $k > N$. Assume w.l.o.g. that $w^N := \sum_{k=0}^N w_k^N W_k \neq 0$. Thus, $w^N \in H^{1/2}_{,0}(0,T)$ and it is obtained that

$$\begin{aligned}\left(\frac{T^2}{2}\sum_{k=0}^{N}\left(\frac{\pi}{2}+k\pi\right)^{-1} f_k^2\right)^{1/2} &= \frac{\sqrt{\frac{T^2}{2}}\sum_{k=0}^{N}\left(\frac{\pi}{2}+k\pi\right)^{-1} f_k^2}{\left(\sum_{k=0}^{N}\left(\frac{\pi}{2}+k\pi\right)^{-1} f_k^2\right)^{1/2}} = \frac{\sqrt{\frac{T^2}{2}}\sum_{k=0}^{N} w_k^N f_k}{\left(\sum_{k=0}^{N}\left(\frac{\pi}{2}+k\pi\right)(w_k^N)^2\right)^{1/2}} \\ &= \frac{f(w^N)}{\|w^N\|_{H^{1/2}_{,0}(0,T),F}} \le \sup_{0\neq w\in H^{1/2}_{,0}(0,T)} \frac{|f(w)|}{\|w\|_{H^{1/2}_{,0}(0,T),F}}.\end{aligned}$$

Hence,

$$\|f\|^2_{[H^{1/2}_{,0}(0,T)]',F} \ge \frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1} f_k^2$$

follows as $N \to \infty$.

On the other hand, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
\|f\|_{[H_{,0}^{1/2}(0,T)]',F} &= \sup_{0\neq w\in H_{,0}^{1/2}(0,T)} \frac{|f(w)|}{\|w\|_{H_{,0}^{1/2}(0,T),F}} \\
&= \sqrt{\frac{T^2}{2}} \sup_{0\neq w\in H_{,0}^{1/2}(0,T)} \frac{\sum\limits_{k=0}^{\infty} w_k f_k \left(\frac{\pi}{2}+k\pi\right)^{-1/2}\left(\frac{\pi}{2}+k\pi\right)^{1/2}}{\left(\sum\limits_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)w_k^2\right)^{1/2}} \\
&\le \left(\frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1} f_k^2\right)^{1/2}.
\end{aligned}$$

Hence, the norm equality is proven and f is contained in the right-hand side.

To show that the right-hand side is a subset of $[H_{,0}^{1/2}(0,T)]'$, one defines for a given sequence $(g_k)_{k\in\mathbb{N}_0} \subset \mathbb{R}$, satisfying $\frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1} g_k^2 < \infty$, the element

$$g(w) := \frac{T}{2}\sum_{k=0}^{\infty} w_k g_k$$

for $w \in H_{,0}^{1/2}(0,T)$ with the coefficients w_k from the expansion (3.34). The linear functional $g\colon H_{,0}^{1/2}(0,T) \to \mathbb{R}$ is well-defined and bounded due to

$$\begin{aligned}
|g(w)| &= \left|\frac{T}{2}\sum_{k=0}^{\infty} w_k\left(\frac{\pi}{2}+k\pi\right)^{1/2} g_k\left(\frac{\pi}{2}+k\pi\right)^{-1/2}\right| \\
&\le \underbrace{\left(\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)w_k^2\right)^{1/2}}_{=\|w\|^2_{H_{,0}^{1/2}(0,T),F}} \underbrace{\left(\frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1} g_k^2\right)^{1/2}}_{<\infty}.
\end{aligned}$$

Hence, $g \in [H_{,0}^{1/2}(0,T)]'$.

For $f \in [H_{,0}^{1}(0,T)]'$, the proof is of the same manner. □

Note that for $g \in L^2(0,T)$, with expansion coefficients g_k from expansion (3.34) with respect to W_k, there holds for $w \in H_{,0}^{1/2}(0,T)$ with expansion (3.34) that

$$\langle g, w\rangle_{L^2(0,T)} = \frac{T}{2}\sum_{k=0}^{\infty} g_k w_k$$

and
$$\frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1}g_k^2 \le \frac{T^2}{2}\sum_{k=0}^{\infty}\frac{2}{\pi}g_k^2 = \frac{2T}{\pi}\|g\|^2_{L^2(0,T)}.$$

Hence, it follows that $g \in [H^{1/2}_{,0}(0,T)]'$ with

$$\|g\|_{[H^{1/2}_{,0}(0,T)]',F} \le \sqrt{\frac{2T}{\pi}}\|g\|_{L^2(0,T)}$$

and there exists a unique continuous extension of $\langle\cdot,\cdot\rangle_{L^2(0,T)}$ on $[H^{1/2}_{,0}(0,T)]' \times H^{1/2}_{,0}(0,T)$, which is denoted as duality pairing $\langle\cdot,\cdot\rangle_{(0,T)}$. Thus, for $f \in [H^{1/2}_{,0}(0,T)]'$ the usual notation $f(w) = \langle f,w\rangle_{(0,T)}$ for $w \in H^{1/2}_{,0}(0,T)$ is used.

With the help of the representations (3.26) and (3.34), the following lemma states inequalities of Poincaré type with sharp constants.

Lemma 3.4.5. *The following inequalities are sharp:*

1. *For $u \in H^{1/2}_{0,}(0,T)$ and $z \in H^{1/2}_{,0}(0,T)$, there hold*

$$\|u\|_{L^2(0,T)} \le \sqrt{\frac{2T}{\pi}}\|u\|_{H^{1/2}_{0,}(0,T),F} \quad \text{and} \quad \|z\|_{L^2(0,T)} \le \sqrt{\frac{2T}{\pi}}\|z\|_{H^{1/2}_{,0}(0,T),F}.$$

2. *For $u \in H^1_{0,}(0,T)$ and $z \in H^1_{,0}(0,T)$, there hold*

$$\|u\|_{L^2(0,T)} \le \frac{2T}{\pi}\|\partial_t u\|_{L^2(0,T)} \quad \text{and} \quad \|z\|_{L^2(0,T)} \le \frac{2T}{\pi}\|\partial_t z\|_{L^2(0,T)}.$$

The inequalities remain valid for complex-valued functions.

Proof. The inequalities for $H^{1/2}_{0,}(0,T)$ and $H^1_{0,}(0,T)$ follow from the norm representations (3.27), (3.31) and (3.29) with

$$\|u\|^2_{L^2(0,T)} = \frac{T}{2}\sum_{k=0}^{\infty}u_k^2 = \frac{2T}{\pi}\frac{1}{2}\sum_{k=0}^{\infty}\frac{\pi}{2}u_k^2 \le \frac{2T}{\pi}\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)u_k^2 = \frac{2T}{\pi}\|u\|^2_{H^{1/2}_{0,}(0,T),F}$$

and

$$\|u\|^2_{L^2(0,T)} = \frac{4T^2}{\pi^2}\frac{1}{2T}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}\right)^2u_k^2 \le \frac{4T^2}{\pi^2}\frac{1}{2T}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^2u_k^2 = \frac{4T^2}{\pi^2}\|\partial_t u\|^2_{L^2(0,T)},$$

which are sharp for functions u with expansion coefficients $u_0 \ne 0$ and $u_k = 0$ for $k \in \mathbb{N}$. Correspondingly, the inequalities for $H^{1/2}_{,0}(0,T)$ and $H^1_{,0}(0,T)$ are proven. □

3.4.2 Transformation Operator $\mathcal{H}_T$

First, the distributional derivative ∂_t on $(0,T)$ for a fixed $u \in H_{0,}^{1/2}(0,T)$ is investigated.

Theorem 3.4.6. *For $u \in H_{0,}^{1/2}(0,T)$, there holds for the distributional derivative ∂_t on $(0,T)$ that $\partial_t u \in [H_{,0}^{1/2}(0,T)]'$. More precisely, there exists a uniquely determined element $g \in [H_{,0}^{1/2}(0,T)]'$, satisfying*

$$\partial_t T_u(\varphi) = \langle g, \varphi \rangle_{(0,T)} \quad \forall \varphi \in \mathcal{D}(0,T),$$

where $T_u \colon \mathcal{D}(0,T) \to \mathbb{R}$, $T_u(\varphi) = \langle u, \varphi \rangle_{L^2(0,T)}$, is the to u related distribution, see Section 2.1.

In addition, there hold

$$\|\partial_t u\|_{[H_{,0}^{1/2}(0,T)]',F} \leq \|u\|_{H_{0,}^{1/2}(0,T),F}$$

and

$$\langle \partial_t u, w \rangle_{(0,T)} = \frac{1}{2} \sum_{k=0}^{\infty} u_k \Big(\frac{\pi}{2} + k\pi \Big) w_k \tag{3.36}$$

for all $w \in H_{,0}^{1/2}(0,T)$ with expansion coefficients u_k from (3.26) *and w_k from* (3.34).

Proof. For $u \in H_{0,}^{1/2}(0,T)$ with the representations

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\Big(\Big(\frac{\pi}{2} + k\pi \Big) \frac{t}{T} \Big) = \sum_{k=0}^{\infty} u_k V_k(t), \quad \|u\|^2_{H_{0,}^{1/2}(0,T),F} = \frac{1}{2} \sum_{k=0}^{\infty} \Big(\frac{\pi}{2} + k\pi \Big) u_k^2,$$

one defines the functional $g \in [H_{,0}^{1/2}(0,T)]'$ via the sequence $(g_k)_{k \in \mathbb{N}_0} \subset \mathbb{R}$ with

$$g_k := \frac{1}{T} u_k \Big(\frac{\pi}{2} + k\pi \Big), \quad k \in \mathbb{N}_0,$$

satisfying

$$\frac{T^2}{2} \sum_{k=0}^{\infty} \Big(\frac{\pi}{2} + k\pi \Big)^{-1} g_k^2 = \frac{1}{2} \sum_{k=0}^{\infty} \Big(\frac{\pi}{2} + k\pi \Big) u_k^2 = \|u\|^2_{H_{0,}^{1/2}(0,T)} < \infty,$$

see Lemma 3.4.4. Hence, $g \in [H_{,0}^{1/2}(0,T)]'$ is well-defined. Next, consider the related distribution $T_u \colon \mathcal{D}(0,T) \to \mathbb{R}$ defined by

$$T_u(\varphi) = \langle u, \varphi \rangle_{L^2(0,T)}$$

for $\varphi \in \mathcal{D}(0,T)$. The distributional derivative ∂_t on $(0,T)$ is given by

$$\begin{aligned}\partial_t T_u(\varphi) &= -T_u(\partial_t \varphi) = -\langle u, \partial_t \varphi\rangle_{L^2(0,T)} = -\sum_{k=0}^{\infty} u_k \langle V_k, \partial_t \varphi\rangle_{L^2(0,T)} \\ &= \sum_{k=0}^{\infty} u_k \langle \partial_t V_k, \varphi\rangle_{L^2(0,T)} = \frac{1}{T}\sum_{k=0}^{\infty} u_k \Big(\frac{\pi}{2}+k\pi\Big)\langle W_k, \varphi\rangle_{L^2(0,T)} \\ &= \frac{1}{2}\sum_{k=0}^{\infty} u_k \Big(\frac{\pi}{2}+k\pi\Big)\varphi_k = \frac{T}{2}\sum_{k=0}^{\infty} g_k \varphi_k = \langle g, \varphi\rangle_{(0,T)}\end{aligned}$$

for $\varphi \in \mathcal{D}(0,T)$ with expansion coefficients $\varphi_k = \frac{2}{T}\langle W_k, \varphi\rangle_{L^2(0,T)}$ from (3.34). Furthermore, it holds with the last calculation and the Cauchy-Schwarz inequality that

$$\begin{aligned}\Big|\langle g, \varphi\rangle_{(0,T)}\Big| &= \left|\frac{1}{2}\sum_{k=0}^{\infty} u_k \Big(\frac{\pi}{2}+k\pi\Big)\varphi_k\right| \\ &\le \sqrt{\frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)u_k^2}\sqrt{\frac{1}{2}\sum_{\ell=0}^{\infty}\Big(\frac{\pi}{2}+\ell\pi\Big)\varphi_\ell^2} = \|u\|_{H^{1/2}_{0,}(0,T),F}\|\varphi\|_{H^{1/2}_{,0}(0,T),F}.\end{aligned}$$

Due to the density of $C_0^\infty(0,T)$ in $H^{1/2}_{,0}(0,T)$, see Theorem 2.2.2, it holds

$$\|g\|_{[H^{1/2}_{,0}(0,T)]',F} \le \|u\|_{H^{1/2}_{0,}(0,T),F}$$

and the element g is unique. The last equality of the assertion follows from the continuity of g and again from the density of $C_0^\infty(0,T)$ in $H^{1/2}_{,0}(0,T)$. □

The representation (3.36) motivates to define for $u \in L^2(0,T)$ with expansion (3.26) the function

$$(\mathcal{H}_T u)(t) := \sum_{\ell=0}^{\infty} u_\ell W_\ell(t) = \sum_{\ell=0}^{\infty} u_\ell \cos\Big(\Big(\frac{\pi}{2}+\ell\pi\Big)\frac{t}{T}\Big) \tag{3.37}$$

for $t \in (0,T)$. By construction, it holds $\mathcal{H}_T u \in L^2(0,T)$. Furthermore,

$$\mathcal{H}_T\colon\ L^2(0,T) \to L^2(0,T)$$

is bijective and norm preserving, i.e.

$$\|\mathcal{H}_T u\|_{L^2(0,T)} = \|u\|_{L^2(0,T)} \quad \text{for all } u \in L^2(0,T),$$

where the inverse transformation operator

$$\mathcal{H}_T^{-1}\colon\ L^2(0,T) \to L^2(0,T)$$

is given by

$$(\mathcal{H}_T^{-1}w)(t) = \sum_{k=0}^{\infty} w_k V_k(t) = \sum_{k=0}^{\infty} w_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad t \in (0,T),$$

for $w \in L^2(0,T)$ with expansion (3.34).

For $u \in H_{0,}^{1/2}(0,T)$ with expansion (3.26), the function

$$(\mathcal{H}_T u)(t) = \sum_{\ell=0}^{\infty} u_\ell W_\ell(t) = \sum_{\ell=0}^{\infty} u_\ell \cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right)$$

fulfils $\mathcal{H}_T u \in H_{,0}^{1/2}(0,T)$ because it holds

$$\|\mathcal{H}_T u\|_{H_{,0}^{1/2}(0,T),F} = \|u\|_{H_{0,}^{1/2}(0,T),F},$$

i.e. $\mathcal{H}_T\colon\ H_{0,}^{1/2}(0,T) \to H_{,0}^{1/2}(0,T)$ is norm preserving. Furthermore,

$$\mathcal{H}_T\colon\ H_{0,}^{1/2}(0,T) \to H_{,0}^{1/2}(0,T)$$

is bijective. Analogously,

$$\mathcal{H}_T\colon\ H_{0,}^{1}(0,T) \to H_{,0}^{1}(0,T)$$

is norm preserving and bijective.

The representation (3.36) yields for $u,v \in H_{0,}^{1/2}(0,T)$ and $w := \mathcal{H}_T v \in H_{,0}^{1/2}(0,T)$

$$\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2} + k\pi\right) u_k v_k = \langle u, v\rangle_{H_{0,}^{1/2}(0,T),F}$$

and

$$\langle \partial_t u, \mathcal{H}_T u\rangle_{(0,T)} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2} + k\pi\right) u_k^2 = \|u\|^2_{H_{0,}^{1/2}(0,T),F}. \tag{3.38}$$

Hence, the bilinear form $b(\cdot,\cdot)\colon\ H_{0,}^{1/2}(0,T) \times H_{0,}^{1/2}(0,T) \to \mathbb{R}$,

$$b(u,v) := \langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} = \langle u, v\rangle_{H_{0,}^{1/2}(0,T),F}, \quad u,v \in H_{0,}^{1/2}(0,T), \tag{3.39}$$

is bounded, elliptic and symmetric.

Next, some properties of the operator $\mathcal{H}_T\colon\ L^2(0,T) \to L^2(0,T)$ are given.

Lemma 3.4.7. *For $u \in L^2(0,T)$ and $w \in L^2(0,T)$, there holds*

$$\langle \mathcal{H}_T u, w\rangle_{L^2(0,T)} = \langle u, \mathcal{H}_T^{-1} w\rangle_{L^2(0,T)}.$$

Proof. For $u \in L^2(0,T)$ and $w \in L^2(0,T)$, there hold the expansions

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad w(t) = \sum_{\ell=0}^{\infty} w_\ell \cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right),$$

and

$$(\mathcal{H}_T u)(t) = \sum_{k=0}^{\infty} u_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad (\mathcal{H}_T^{-1} w)(t) = \sum_{\ell=0}^{\infty} w_\ell \sin\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right)$$

with expansion coefficients u_k from (3.26) and w_ℓ from (3.34). Hence, it follows with the representations (3.28), (3.35) that

$$\langle \mathcal{H}_T u, w\rangle_{L^2(0,T)} \overset{(3.35)}{=} \frac{T}{2}\sum_{k=0}^{\infty} u_k w_k \overset{(3.28)}{=} \langle u, \mathcal{H}_T^{-1} w\rangle_{L^2(0,T)}$$

and therefore, the assertion. □

Lemma 3.4.8. *There holds*

$$\langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} \geq 0 \tag{3.40}$$

for all $v \in L^2(0,T)$.

Proof. By using the representations

$$v(t) = \sum_{k=0}^{\infty} v_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad (\mathcal{H}_T v)(t) = \sum_{\ell=0}^{\infty} v_\ell \cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right),$$

it follows with the continuity of the inner product $\langle\cdot,\cdot\rangle_{L^2(0,T)}$ that

$$\begin{aligned}
\langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} v_k v_\ell \int_0^T \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)\cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right)\mathrm{d}t \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} v_k v_\ell \int_0^T \left[\sin\left((k+\ell+1)\pi\frac{t}{T}\right) + \sin\left((k-\ell)\pi\frac{t}{T}\right)\right]\mathrm{d}t \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} v_k v_\ell \left[-\frac{T}{(k+\ell+1)\pi}\cos\left((k+\ell+1)\pi\frac{t}{T}\right)\right]_0^T \\
&= \frac{T}{2\pi}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} v_k v_\ell \frac{1}{k+\ell+1}\left[1-(-1)^{k+\ell+1}\right],
\end{aligned}$$

where the second integral is ignored due to symmetry. When splitting k and ℓ into odd and even indices, i.e. $k = 2i, 2i+1,\ \ell = 2j, 2j+1$, this gives

$$\begin{aligned}
\langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} &= \frac{T}{\pi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{v_{2i}v_{2j}}{2i+2j+1} + \frac{v_{2i+1}v_{2j+1}}{2i+2j+3}\right] \\
&= \frac{T}{\pi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[v_{2i}v_{2j}\int_0^1 x^{2i+2j}\mathrm{d}x + v_{2i+1}v_{2j+1}\int_0^1 x^{2i+2j+2}\mathrm{d}x\right] \\
&= \lim_{N\to\infty}\left\{\frac{T}{\pi}\sum_{i=0}^{N}\sum_{j=0}^{N}\left[v_{2i}v_{2j}\int_0^1 x^{2i+2j}\mathrm{d}x + v_{2i+1}v_{2j+1}\int_0^1 x^{2i+2j+2}\mathrm{d}x\right]\right\} \\
&= \frac{T}{\pi}\lim_{N\to\infty}\left[\int_0^1\left(\sum_{i=0}^{N} v_{2i}x^{2i}\right)^2\mathrm{d}x + \int_0^1\left(\sum_{i=0}^{N} v_{2i+1}x^{2i+1}\right)^2\mathrm{d}x\right] \geq 0
\end{aligned}$$

and hence, the assertion follows. □

Remark 3.4.9. *The transformation $\mathcal{H}_T$ is the counterpart on finite intervals $(0,T)$ of the Hilbert transform $\mathcal{H}\colon\ L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by*

$$\mathcal{H}v(t) := \frac{1}{\pi}\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{t-\varepsilon}\frac{v(s)}{t-s}\mathrm{d}s + \int_{t+\varepsilon}^{\infty}\frac{v(s)}{t-s}\mathrm{d}s\right), \quad t\in\mathbb{R},$$

for a function $v \in L^2(\mathbb{R})$, see [92, 122]. The Hilbert transform $\mathcal{H}$ has similar properties as the transformation $\mathcal{H}_T$, see [145].

3.4.3 Variational Formulation for $\partial_t u = f$

To get a first impression of the transformation $\mathcal{H}_T$, the simple initial value problem

$$\partial_t u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0, \tag{3.41}$$

is investigated. The corresponding variational formulation is to find $u \in H^{1/2}_{0,}(0,T)$ such that

$$b(u,v) = \langle f, \mathcal{H}_T v\rangle_{(0,T)} \tag{3.42}$$

for all $v \in H^{1/2}_{0,}(0,T)$, where $f \in [H^{1/2}_{,0}(0,T)]'$ is a given right-hand side and the bilinear form

$$b(\cdot,\cdot)\colon\ H^{1/2}_{0,}(0,T)\times H^{1/2}_{0,}(0,T)\to\mathbb{R}, \quad b(u,v) = \langle\partial_t u, \mathcal{H}_T v\rangle_{(0,T)},$$

is bounded, elliptic and symmetric, see (3.39). Hence, existence and uniqueness of a solution $u \in H^{1/2}_{0,}(0,T)$ for the variational formulation (3.42) follow by the Lax-Milgram Theorem, since the right-hand side $f \in [H^{1/2}_{,0}(0,T)]'$ satisfies

$$\left|\langle f, \mathcal{H}_T v\rangle_{(0,T)}\right| \leq \|f\|_{[H^{1/2}_{,0}(0,T)]',F} \|\mathcal{H}_T v\|_{H^{1/2}_{,0}(0,T),F} = \|f\|_{[H^{1/2}_{,0}(0,T)]',F} \|v\|_{H^{1/2}_{0,}(0,T),F} \tag{3.43}$$

for all $v \in H^{1/2}_{0,}(0,T)$.

With the notations of Section 2.6, a conforming finite element discretisation of the variational formulation (3.42) is to find $u_{h_t} \in S^1_{h_t,0,}(0,T) = \operatorname{span}\{\varphi_k\}_{k=1}^{N_t} \subset H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u_{h_t}, \mathcal{H}_T v_{h_t}\rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v_{h_t}\rangle_{(0,T)} \tag{3.44}$$

for all $v_{h_t} \in S^1_{h_t,0,}(0,T)$. Using standard arguments, e.g., [51, 141], there follow the unique solvability of (3.44) and the a priori error estimates

$$\begin{aligned}
\|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} &\leq c\, h_t^{s-1/2} \|u\|_{H^s(0,T)} && \text{for } s \in (1/2, 2],\\
\|u - u_{h_t}\|_{L^2(0,T)} &\leq c\, h_t^{s} \|u\|_{H^s(0,T)} && \text{for } s \in (1/2, 2],\\
\|u - u_{h_t}\|_{H^1_{0,}(0,T)} &\leq c\, h_t^{s-1} \|u\|_{H^s(0,T)} && \text{for } s \in (1, 2]
\end{aligned}$$

with a constant $c > 0$, when assuming $u \in H^{1/2}_{0,}(0,T) \cap H^s(0,T)$. Note that the error estimate in the energy norm $\|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F}$ is a consequence of Céa's Lemma and the approximation property of $S^1_{h_t,0,}(0,T)$. The approximation property

$$\inf_{v_{h_t} \in S^1_{h_t,0,}(0,T)} \|v - v_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \leq c \cdot h_t^{3/2} \|v\|_{H^2(0,T)}$$

for $v \in H^1_{0,}(0,T) \cap H^2(0,T)$ is derived by an interpolation argument, see (2.32). For $\|u - u_{h_t}\|_{L^2(0,T)}$, the Aubin-Nitsche trick is used, and for $\|u - u_{h_t}\|_{H^1_{0,}(0,T)}$, an inverse inequality is required, i.e. for this situation a globally quasi-uniform mesh is needed. The Galerkin-Bubnov finite element formulation (3.44) is equivalent to the system of linear equations

$$K_{h_t} \underline{u} = \underline{f}$$

with a symmetric and positive definite stiffness matrix $K_{h_t} \in \mathbb{R}^{N_t \times N_t}$, defined by

$$K_{h_t}[j,k] = \langle \partial_t \varphi_k, \mathcal{H}_T \varphi_j\rangle_{L^2(0,T)} \quad \text{for } k, j = 1, \ldots, N_t,$$

and the right-hand side $\underline{f} \in \mathbb{R}^{N_t}$, given by

$$\underline{f}[j] = \langle f, \mathcal{H}_T \varphi_j\rangle_{(0,T)} \quad \text{for } j = 1, \ldots, N_t,$$

where high-order integration rules are used for the calculation. The evaluation of the transformed basis functions $\mathcal{H}_T\varphi_k$ can be done by using the definition (3.37). Although the piecewise linear basis functions φ_k have local support, the transformed basis functions $\mathcal{H}_T\varphi_k$ are global, see Figure 3.1 and Figure 3.2, and therefore, the stiffness matrix K_{h_t} is dense.

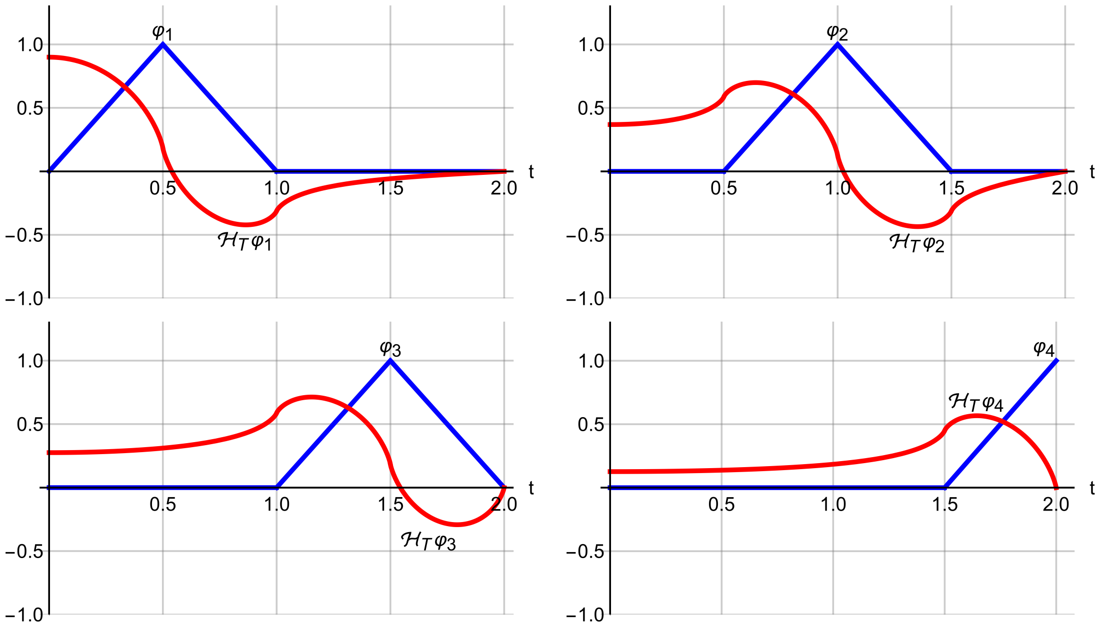

Figure 3.1: Transformed basis functions $\mathcal{H}_T\varphi_k$, $k = 1,\dots,N_t$, $N_t = 4$.

As a numerical example for (3.44), the solution $u(t) = \sin\left(\frac{9\pi}{4}t\right)$ for $t \in (0,2) = (0,T)$ of (3.41) with the right-hand side $f(t) = \frac{9\pi}{4}\cos\left(\frac{9\pi}{4}t\right)$ for $t \in (0,2)$ is considered. For the discretisation, a sequence of finite element spaces $S^1_{h_t,0,}(0,2)$ of uniform mesh size $h_t = 2/N_t$, and $N_t = 2^{j+1}$, $j = 0,\dots,7$, is introduced. Since the solution u is smooth, quadratic convergence in $L^2(0,2)$ and linear convergence in $H^1(0,2)$ are expected. This behaviour is confirmed by the numerical results as given in Table 3.9. In addition, the minimal and maximal eigenvalues of the symmetric stiffness matrix K_{h_t} as well as the resulting spectral condition number of K_{h_t}, which behave as expected for a first-order differential operator, are given in Table 3.9.

3.4.4 Variational Formulation for $\partial_t u + \mu u = f$

Instead of the initial value problem (3.41), consider for $\mu > 0$ the first-order linear equation

$$\partial_t u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0, \tag{3.45}$$

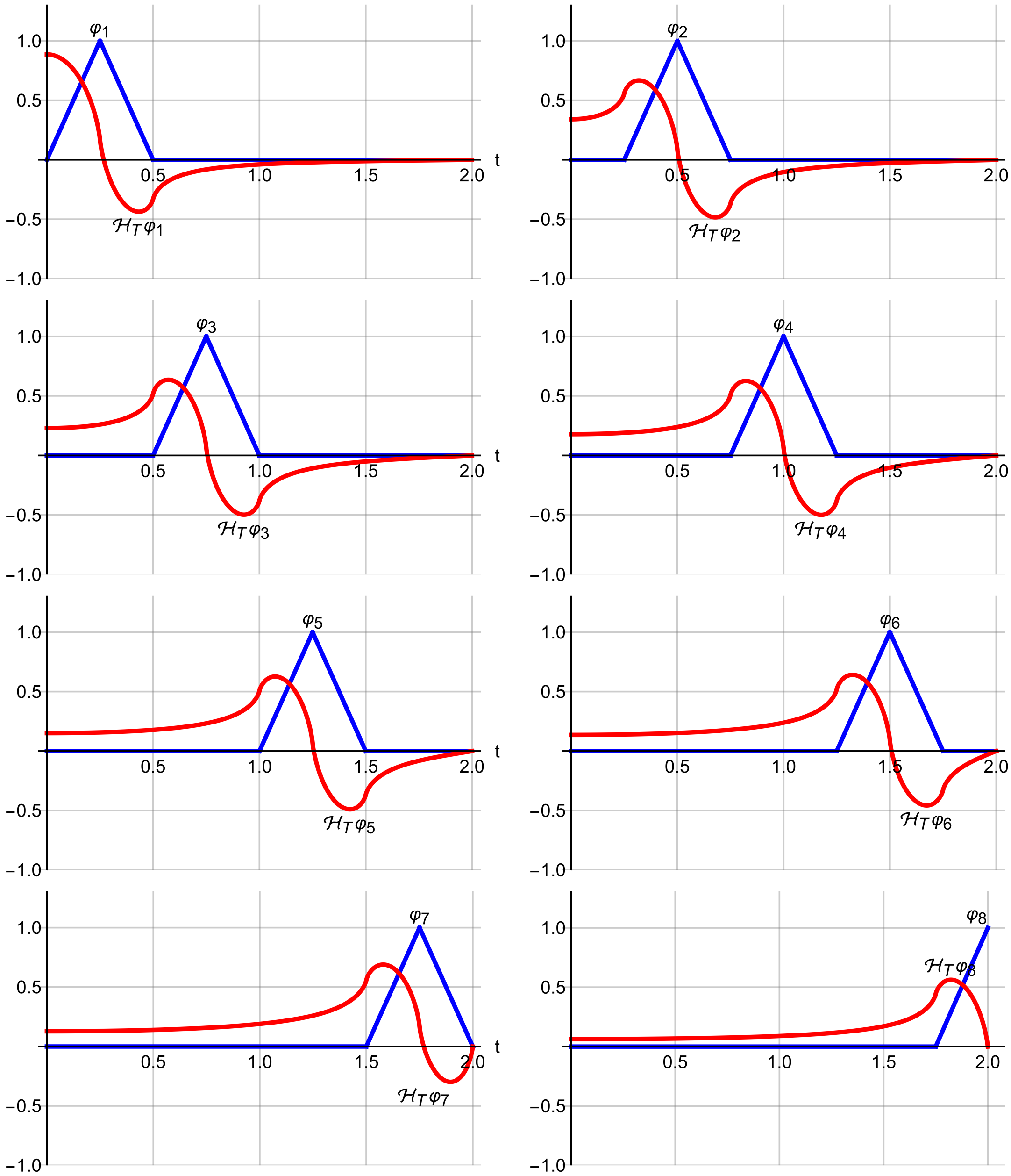

Figure 3.2: Transformed basis functions $\mathcal{H}_T\varphi_k$, $k = 1, \ldots, N_t$, $N_t = 8$.

N_t	$\|u-u_{h_t}\|_{L^2(0,2)}$	eoc	$\|\partial_t u-\partial_t u_{h_t}\|_{L^2(0,2)}$	eoc	$\lambda_{\min}(K_{h_t})$	$\lambda_{\max}(K_{h_t})$	$\kappa_2(K_{h_t})$
2	1.00473818	-	7.05949197	-	0.4166	0.9602	2.3
4	0.86127822	0.2	5.88004588	0.3	0.2844	1.1169	3.9
8	0.16924553	2.3	3.66044528	0.7	0.1688	1.1280	6.7
16	0.03246999	2.4	1.82612730	1.0	0.0915	1.1327	12.4
32	0.00748649	2.1	0.90514235	1.0	0.0475	1.1338	23.9
64	0.00183184	2.0	0.45124173	1.0	0.0241	1.1340	47.0
128	0.00045545	2.0	0.22543481	1.0	0.0122	1.1341	93.2
256	0.00011371	2.0	0.11269290	1.0	0.0061	1.1341	185.6

Table 3.9: Numerical results for the Galerkin-Bubnov formulation (3.44).

and the related variational formulation to find $u \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} + \mu \langle u, \mathcal{H}_T v\rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v\rangle_{(0,T)} \tag{3.46}$$

for all $v \in H^{1/2}_{0,}(0,T)$, where $f \in [H^{1/2}_{,0}(0,T)]'$ is given. The first-order ordinary differential equation (3.45) plays a central role in the analysis of the heat equation, see Subsection 3.4.5, where $\mu > 0$ corresponds to a Dirichlet eigenvalue of the Laplace operator, see (2.4).

Theorem 3.4.10. *Let $\mu > 0$ and the right-hand side $f \in [H^{1/2}_{,0}(0,T)]'$ be given. Then there exists a unique solution $u \in H^{1/2}_{0,}(0,T)$ of the variational formulation* (3.46)*, satisfying the stability estimate*

$$\|u\|_{H^{1/2}_{0,}(0,T),F} \leq \|f\|_{[H^{1/2}_{,0}(0,T)]',F}. \tag{3.47}$$

Proof. When combining (3.38) with Lemma 3.4.5 and (3.40), this gives

$$\left| \langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} + \mu \langle u, \mathcal{H}_T v\rangle_{L^2(0,T)} \right| \leq \left(1 + \frac{2\mu T}{\pi}\right) \|u\|_{H^{1/2}_{0,}(0,T),F} \|v\|_{H^{1/2}_{0,}(0,T),F} \tag{3.48}$$

and

$$\langle \partial_t v, \mathcal{H}_T v\rangle_{(0,T)} + \mu \langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} \geq \langle \partial_t v, \mathcal{H}_T v\rangle_{(0,T)} = \|v\|^2_{H^{1/2}_{0,}(0,T),F} \tag{3.49}$$

for all $u, v \in H^{1/2}_{0,}(0,T)$, i.e. the bilinear form of the variational problem (3.46) is bounded and elliptic, implying unique solvability of (3.46) by the Lax-Milgram Theorem, including the stability estimate

$$\|u\|_{H^{1/2}_{0,}(0,T),F} \leq \|f\|_{[H^{1/2}_{,0}(0,T)]',F},$$

since the right-hand side f satisfies (3.43). □

A first regularity result is given in the next lemma.

Lemma 3.4.11. *Let $f \in L^2(0,T)$ be given. Then the unique solution $u \in H_{0,}^{1/2}(0,T)$ of* (3.46) *is given by*

$$u(t) = \int_0^t e^{\mu(s-t)} f(s) \mathrm{d}s \tag{3.50}$$

for $t \in [0,T]$ and fulfils

$$\|\partial_t u\|_{L^2(0,T)} \le \|f\|_{L^2(0,T)}.$$

Proof. By inserting $\int_0^t e^{\mu(s-t)} f(s)\mathrm{d}s$ into the variational formulation (3.46), it follows that the unique solution $u \in H_{0,}^{1/2}(0,T)$ is represented by (3.50) and that $u \in H_{0,}^1(0,T)$. Furthermore, the ordinary differential equation (3.45) holds for almost all $t \in (0,T)$. Multiplication of (3.45) by $\partial_t u \in L^2(0,T)$ and integration via $(0,T)$ yield

$$\langle f, \partial_t u\rangle_{L^2(0,T)} = \langle \partial_t u, \partial_t u\rangle_{L^2(0,T)} + \mu \langle u, \partial_t u\rangle_{L^2(0,T)} = \|\partial_t u\|^2_{L^2(0,T)} + \frac{\mu}{2} u(T)^2.$$

The Cauchy-Schwarz inequality gives the assertion. $\square$

For the analysis of the heat equation, a μ dependent estimate for the solution u in $L^2(0,T)$ is required.

Lemma 3.4.12. *Let $u \in H_{0,}^{1/2}(0,T)$ be the unique solution of the variational formulation* (3.46)*, where $f \in [H_{,0}^{1/2}(0,T)]'$ is given. Then it holds*

$$\|u\|^2_{L^2(0,T)} \le \frac{T}{2} \sum_{k=0}^{\infty} \frac{f_k^2}{\mu^2 + \frac{1}{T^2}(\frac{\pi}{2} + k\pi)^2}, \tag{3.51}$$

where

$$f_k := \frac{2}{T} \langle f, W_k\rangle_{(0,T)}, \quad W_k(t) = \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right).$$

Proof. First, note that the right-hand side in (3.51) is finite due to

$$\frac{T}{2} \sum_{k=0}^{\infty} \frac{f_k^2}{\mu^2 + \frac{1}{T^2}(\frac{\pi}{2} + k\pi)^2} \le \frac{2T}{\pi} \|f\|^2_{[H_{,0}^{1/2}(0,T)]',F} < \infty,$$

see Lemma 3.4.4.

Let $(f_n)_{n\in\mathbb{N}} \subset L^2(0,T)$ be a sequence with $\lim_{n\to\infty} \|f - f_n\|_{[H_{,0}^{1/2}(0,T)]',F} = 0$, see (2.3). Let $u_n \in H_{0,}^{1/2}(0,T)$ be the weak solution of the variational formulation (3.46) with right-hand side f_n. Hence, $u - u_n \in H^{1/2}_{0,}(0,T)$ is the unique solution of (3.46) with right-hand side $f - f_n \in [H^{1/2}_{,0}(0,T)]'$. Thus, the stability estimate (3.47) yields

$$\|u - u_n\|_{H^{1/2}_{0,}(0,T),F} \leq \|f - f_n\|_{[H^{1/2}_{,0}(0,T)]',F} \tag{3.52}$$

and therefore, $u_n \to u$ in $H^{1/2}_{0,}(0,T)$ as $n \to \infty$. Write for $f_n \in L^2(0,T)$ the expansion (3.34) as

$$f_n(t) = \sum_{k=0}^{\infty} f_{n,k} \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad f_{n,k} = \frac{2}{T}\int_0^T f_n(t) \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) \mathrm{d}t.$$

First, with $\lambda_k = \frac{1}{T^2}\left(\frac{\pi}{2} + k\pi\right)^2$, the Cauchy-Schwarz inequality and Lemma 3.4.4, it follows

$$\begin{aligned}
\left|\frac{T}{2}\sum_{k=0}^{\infty}\frac{f_k^2}{\mu^2+\lambda_k} - \frac{T}{2}\sum_{k=0}^{\infty}\frac{f_{n,k}^2}{\mu^2+\lambda_k}\right| &\leq \frac{T}{2}\sum_{k=0}^{\infty}\frac{1}{\lambda_k}\underbrace{|f_k^2 - f_{n,k}^2|}_{=|f_k - f_{n,k}||f_k + f_{n,k}|} \\
&\leq \left(\frac{T}{2}\sum_{k=0}^{\infty}\frac{|f_k - f_{n,k}|^2}{\sqrt{\lambda_k}}\right)^{1/2}\left(\frac{T}{2}\sum_{k=0}^{\infty}\frac{|f_k + f_{n,k}|^2}{\lambda_k^{3/2}}\right)^{1/2} \\
&\leq \|f - f_n\|_{[H^{1/2}_{,0}(0,T)]',F}\,\frac{2T}{\pi}\,\|f + f_n\|_{[H^{1/2}_{,0}(0,T)]',F} \to 0
\end{aligned}$$

as $n \to \infty$. It other words, it holds

$$\lim_{n\to\infty}\frac{T}{2}\sum_{k=0}^{\infty}\frac{f_{n,k}^2}{\mu^2+\lambda_k} = \frac{T}{2}\sum_{k=0}^{\infty}\frac{f_k^2}{\mu^2+\lambda_k}. \tag{3.53}$$

Second, because of $f_n \in L^2(0,T)$, it holds for the solution u_n the representation (3.50) for $t \in [0,T]$

$$\begin{aligned}
u_n(t) &= \int_0^t e^{\mu(s-t)} f_n(s)\mathrm{d}s = \sum_{k=0}^{\infty} f_{n,k}\cdot e^{-\mu t}\int_0^t e^{\mu s}\cos(\sqrt{\lambda_k}s)\mathrm{d}s \\
&= \sum_{k=0}^{\infty}\frac{f_{n,k}}{\mu^2+\lambda_k}\left[\sqrt{\lambda_k}\sin(\sqrt{\lambda_k}t) + \mu\cos(\sqrt{\lambda_k}t) - \mu e^{-\mu t}\right],
\end{aligned}$$

where the continuity of the inner product $\langle\cdot,\cdot\rangle_{L^2(0,t)}$ for fixed $t \in [0,T]$ is used. When computing all integrals, where again the continuity of the inner product $\langle\cdot,\cdot\rangle_{L^2(0,T)}$ is used, one obtains

$$\|u_n\|^2_{L^2(0,T)} = \frac{T}{2}\sum_{k=0}^{\infty}\frac{f_{n,k}^2}{\mu^2+\lambda_k} - \frac{1}{2}\mu\left[1 + e^{-2\mu T}\right]\left(\sum_{k=0}^{\infty}\frac{f_{n,k}}{\mu^2+\lambda_k}\right)^2 \leq \frac{T}{2}\sum_{k=0}^{\infty}\frac{f_{n,k}^2}{\mu^2+\lambda_k}.$$

Now, the assertion follows as $n \to \infty$ with the help of (3.52) and (3.53). □

Remark 3.4.13. *From* (3.51)*, it follows immediately the estimate*

$$\|u\|^2_{L^2(0,T)} \le \frac{T^3}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-2} f_k^2 \;=\; \|f\|^2_{[H^1_{,0}(0,T)]'},$$

see Lemma 3.4.4 for the representation of the norm $\|\cdot\|_{[H^1_{,0}(0,T)]'}$*. Moreover, when it holds* $f \in L^2(0,T)$*, the estimate* (3.51) *gives*

$$\|u\|^2_{L^2(0,T)} \le \frac{T}{2\mu^2}\sum_{k=0}^{\infty} f_k^2 = \frac{1}{\mu^2}\|f\|^2_{L^2(0,T)}, \quad \textit{i.e.} \quad \mu\,\|u\|_{L^2(0,T)} \le \|f\|_{L^2(0,T)}.$$

With the notations of Section 2.6, the Galerkin-Bubnov discretisation of (3.46) is to find $u_{h_t} \in S^1_{h_t,0,}(0,T)$ such that

$$\langle \partial_t u_{h_t}, \mathcal{H}_T v_{h_t}\rangle_{L^2(0,T)} + \mu\langle u_{h_t}, \mathcal{H}_T v_{h_t}\rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v_{h_t}\rangle_{(0,T)} \tag{3.54}$$

for all $v_{h_t} \in S^1_{h_t,0,}(0,T)$. As for the initial value problem (3.41), the unique solvability of (3.54) follows by Céa's Lemma, including with (3.43) the stability estimate

$$\|u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \le \|f\|_{[H^{1/2}_{,0}(0,T)]',F}$$

and with the help of (3.49), (3.48), an error estimate in the energy norm

$$\|u-u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \le c\cdot\left(1+\frac{2\mu T}{\pi}\right) h_t^{s-1/2}\|u\|_{H^s(0,T)}$$

for $s \in (1/2,2]$ and $u \in H^{1/2}_{0,}(0,T)\cap H^s(0,T)$ with a constant $c>0$ independent of μ and h_t. Hence, in general, a priori error estimates depend on μ and require a sufficiently small mesh size h_t to ensure convergence for large μ. In the following theorem, a refined error estimate in the energy norm $\|\cdot\|_{H^{1/2}_{0,}(0,T),F}$ and an error estimate in $\|\cdot\|_{L^2(0,T)}$ are given.

Theorem 3.4.14. *Let* $u \in H^{1/2}_{0,}(0,T)$ *and* $u_{h_t} \in S^1_{h_t,0,}(0,T)$ *be the unique solutions of the variational formulations* (3.46) *and* (3.54)*. If* $u \in H^{1/2}_{0,}(0,T)\cap H^2(0,T)$*, then there hold the error estimates*

$$\|u-u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \le C_{1/2}(h_t,\mu)\cdot h_t^{3/2}\,\|\partial_t u\|_{H^1(0,T)}$$

and

$$\|u-u_{h_t}\|_{L^2(0,T)} \le C_0(h_t,\mu)\cdot h_t^2\,\|\partial_t u\|_{H^1(0,T)} \tag{3.55}$$

with

$$C_{1/2}(h_t,\mu) := \frac{1}{\sqrt{\pi}\sqrt[4]{18}} + \frac{1}{\sqrt{24}}\left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2+2(1+2\mu^2)T^2}}{2\sqrt[4]{6}}\mu h_t\right)\mu h_t$$

and

$$C_0(h_t,\mu) := C_{1/2}(h_t,\mu)\cdot\left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2+2(1+2\mu^2)T^2}}{2\sqrt[4]{6}}\,\mu h_t\right).$$

Proof. Using (3.49) and the Galerkin orthogonality of the variational formulations (3.46) and (3.54), it holds with the norm invariance of $\mathcal{H}_T$ that

$$\begin{aligned}\|u-u_{h_t}\|^2_{H^{1/2}_{0,}(0,T),F} &\le \langle\partial_t(u-u_{h_t}),\mathcal{H}_T(u-u_{h_t})\rangle_{(0,T)} + \mu\langle u-u_{h_t},\mathcal{H}_T(u-u_{h_t})\rangle_{L^2(0,T)}\\ &= \langle\partial_t(u-u_{h_t}),\mathcal{H}_T(u-I_{h_t}u)\rangle_{(0,T)} + \mu\langle u-u_{h_t},\mathcal{H}_T(u-I_{h_t}u)\rangle_{L^2(0,T)}\\ &\le \|u-u_{h_t}\|_{H^{1/2}_{0,}(0,T),F}\|u-I_{h_t}u\|_{H^{1/2}_{0,}(0,T),F} + \mu\,\|u-u_{h_t}\|_{L^2(0,T)}\|u-I_{h_t}u\|_{L^2(0,T)}.\end{aligned} \tag{3.56}$$

To estimate the error in $L^2(0,T)$, consider the adjoint problem

$$-\partial_t w(t) + \mu w(t) = u(t) - u_{h_t}(t) \quad \text{for } t\in(0,T), \quad w(T)=0, \tag{3.57}$$

i.e. the function $w \in H^{1/2}_{,0}(0,T)$ is the unique solution of the variational problem

$$-\langle\partial_t w, v\rangle_{(0,T)} + \mu\langle w,v\rangle_{L^2(0,T)} = \langle u-u_{h_t}, v\rangle_{L^2(0,T)} \tag{3.58}$$

for all $v\in H^{1/2}_{0,}(0,T)$. Analogous to Lemma 3.4.11, it holds the regularity result

$$\|\partial_t w\|_{L^2(0,T)} \le \|u-u_{h_t}\|_{L^2(0,T)} \tag{3.59}$$

and in addition, $\partial_t w \in H^1(0,T)$. Hence, from the differential equation in (3.57), one finds

$$\partial_{tt} w(t) = \mu\partial_t w(t) - \partial_t[u(t)-u_{h_t}(t)] \quad \text{for } t\in(0,T),$$

and therefore, with (3.59) and the Poincaré inequality of Lemma 3.4.5

$$\begin{aligned}\|\partial_t w\|^2_{H^1(0,T)} &= \|\partial_t w\|^2_{L^2(0,T)} + \|\partial_{tt} w\|^2_{L^2(0,T)}\\ &\le (1+2\mu^2)\|\partial_t w\|^2_{L^2(0,T)} + 2\|\partial_t(u-u_{h_t})\|^2_{L^2(0,T)}\\ &\le (1+2\mu^2)\|u-u_{h_t}\|^2_{L^2(0,T)} + 2\|\partial_t(u-u_{h_t})\|^2_{L^2(0,T)}\\ &\le (1+2\mu^2)\frac{4T^2}{\pi^2}\|\partial_t(u-u_{h_t})\|^2_{L^2(0,T)} + 2\|\partial_t(u-u_{h_t})\|^2_{L^2(0,T)}\\ &= \left(2+\frac{4T^2}{\pi^2}+\frac{8T^2\mu^2}{\pi^2}\right)\|u-u_{h_t}\|^2_{H^1_{0,}(0,T)}\end{aligned} \tag{3.60}$$

follows. Since $\partial_t w \in H^1(0,T)$, an interpolation argument for the K-Method of Interpolation between (3.59) and (3.60) yields the estimate

$$\begin{aligned}\|\partial_t w\|_{K_{1/2}(H^1(0,T);L^2(0,T))} &\le \sqrt[4]{2+\frac{4T^2}{\pi^2}+\frac{8T^2\mu^2}{\pi^2}}\|u-u_{h_t}\|_{K_{1/2}(H^1_{0,}(0,T);L^2(0,T))}\\ &= \sqrt{\frac{\pi}{2}}\sqrt[4]{2+\frac{4T^2}{\pi^2}+\frac{8T^2\mu^2}{\pi^2}}\|u-u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \qquad (3.61)\end{aligned}$$

with the help of the norm equivalence (3.32).

For $v = u - u_{h_t} \in H^{1/2}_{0,}(0,T)$ in (3.58) and with the Galerkin orthogonality for $u - u_{h_t}$, it follows

$$\begin{aligned}\|u-u_{h_t}\|^2_{L^2(0,T)} &= \langle u-u_{h_t}, u-u_{h_t}\rangle_{L^2(0,T)}\\ &= -\langle \partial_t w, u-u_{h_t}\rangle_{(0,T)} + \mu\langle w, u-u_{h_t}\rangle_{L^2(0,T)}\\ &= \langle \partial_t(u-u_{h_t}), w\rangle_{(0,T)} + \mu\langle u-u_{h_t}, w\rangle_{L^2(0,T)}\\ &= \langle \partial_t(u-u_{h_t}), w-I_{h_t}w\rangle_{(0,T)} + \mu\langle u-u_{h_t}, w-I_{h_t}w\rangle_{L^2(0,T)}\\ &\le \|u-u_{h_t}\|_{H^{1/2}_{0,}(0,T),F}\|w-I_{h_t}w\|_{H^{1/2}_{,0}(0,T),F}\\ &\quad + \mu\|u-u_{h_t}\|_{L^2(0,T)}\|w-I_{h_t}w\|_{L^2(0,T)}. \qquad (3.62)\end{aligned}$$

For the first summand in (3.62), it holds

$$\begin{aligned}\|w-I_{h_t}w\|_{H^{1/2}_{,0}(0,T),F} &= \sqrt{\frac{2}{\pi}}\|w-I_{h_t}w\|_{K_{1/2}(H^1_{,0}(0,T);L^2(0,T))}\\ &\le \sqrt{\frac{2}{\pi}}\sqrt[4]{2}\,h_t^{1/2}\,\|\partial_t w\|_{L^2(0,T)}\\ &\le \frac{\sqrt[4]{8}}{\sqrt{\pi}}h_t^{1/2}\,\|u-u_{h_t}\|_{L^2(0,T)},\end{aligned}$$

where the first equality follows from a norm equivalence analogous to (3.32), the last inequality follows from (3.59), and the first estimate follows from an interpolation argument for the K-Method of Interpolation between

$$\|w-I_{h_t}w\|_{L^2(0,T)} \le \frac{1}{\sqrt{8}}h_t\,\|\partial_t(w-I_{h_t}w)\|_{L^2(0,T)} \le \frac{1}{\sqrt{2}}h_t\,\|\partial_t w\|_{L^2(0,T)} = \frac{1}{\sqrt{2}}h_t\,\|w\|_{H^1_{,0}(0,T)}$$

and

$$\|w-I_{h_t}w\|_{H^1_{,0}(0,T)} = \|\partial_t(w-I_{h_t}w)\|_{L^2(0,T)} \le 2\|\partial_t w\|_{L^2(0,T)} = 2\|w\|_{H^1_{,0}(0,T)}$$

with the stability (2.28) and an error estimate analogous to (2.29) since $w \in H^1_{,0}(0,T)$.

For the second summand in (3.62), it follows with (3.61)

$$\begin{aligned}\|w - I_{h_t} w\|_{L^2(0,T)} &\leq \frac{1}{\sqrt[4]{48}} h_t^{3/2} \, \|\partial_t w\|_{K_{1/2}(H^1(0,T);L^2(0,T))} \\ &\leq \frac{1}{\sqrt[4]{48}} \sqrt{\frac{\pi}{2}} \sqrt[4]{2 + \frac{4T^2}{\pi^2} + \frac{8T^2\mu^2}{\pi^2}} \, h_t^{3/2} \, \|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \\ &= \frac{\sqrt[4]{\pi^2 + 2(1+2\mu^2)T^2}}{2\sqrt[4]{6}} \, h_t^{3/2} \, \|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F},\end{aligned}$$

where the first estimate follows from an interpolation argument for the K-Method of Interpolation between

$$\|w - I_{h_t} w\|_{L^2(0,T)} \leq \frac{1}{\sqrt{24}} h_t^2 \, \|\partial_{tt} w\|_{L^2(0,T)} \leq \frac{1}{\sqrt{24}} h_t^2 \, \|\partial_t w\|_{H^1(0,T)}$$

and

$$\|w - I_{h_t} w\|_{L^2(0,T)} \leq \frac{1}{\sqrt{8}} h_t \, \|\partial_t (w - I_{h_t} w)\|_{L^2(0,T)} \leq \frac{1}{\sqrt{2}} h_t \, \|\partial_t w\|_{L^2(0,T)}$$

with the stability (2.28) and an error estimate analogous to (2.29) since $\partial_t w \in H^1(0,T)$.

Therefore, the inequality (3.62) yields

$$\begin{aligned}\|u - u_{h_t}\|^2_{L^2(0,T)} \leq{}& \frac{\sqrt[4]{8}}{\sqrt{\pi}} h_t^{1/2} \, \|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \|u - u_{h_t}\|_{L^2(0,T)} \\ &+ \mu \frac{\sqrt[4]{\pi^2 + 2(1+2\mu^2)T^2}}{2\sqrt[4]{6}} \, h_t^{3/2} \, \|u - u_{h_t}\|_{L^2(0,T)} \|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F},\end{aligned}$$

i.e.

$$\|u - u_{h_t}\|_{L^2(0,T)} \leq \left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2 + 2(1+2\mu^2)T^2}}{2\sqrt[4]{6}} \, \mu h_t \right) h_t^{1/2} \, \|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F}. \quad (3.63)$$

When inserting this into (3.56), together with the estimate

$$\|u - I_{h_t} u\|_{H^{1/2}_{0,}(0,T),F} = \sqrt{\frac{2}{\pi}} \|u - I_{h_t} u\|_{K_{1/2}(H^1_{0,}(0,T);L^2(0,T))} \leq \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt[4]{72}} h_t^{3/2} \|\partial_t u\|_{H^1(0,T)}$$

derived with the help of (3.32) via an interpolation argument for the K-Method of Interpolation between (2.30) and (2.31), and again (2.31), this gives

$$\begin{aligned}\|u - u_{h_t}\|^2_{H^{1/2}_{0,}(0,T),F} \leq{}& \frac{1}{\sqrt{\pi}\sqrt[4]{18}} h_t^{3/2} \, \|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \|\partial_t u\|_{H^1(0,T)} \\ &+ \frac{1}{\sqrt{24}} \mu \left(\frac{\sqrt[4]{8}}{\sqrt{\pi}} + \frac{\sqrt[4]{\pi^2 + 2(1+2\mu^2)T^2}}{2\sqrt[4]{6}} \, \mu h_t \right) h_t^{5/2} \, \|u - u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \|\partial_t u\|_{H^1(0,T)},\end{aligned}$$

i.e.

$$\|u-u_{h_t}\|_{H^{1/2}_{0,}(0,T),F} \leq \left[\frac{1}{\sqrt{\pi}\sqrt[4]{18}}+\frac{1}{\sqrt{24}}\left(\frac{\sqrt[4]{8}}{\sqrt{\pi}}+\frac{\sqrt[4]{\pi^2+2(1+2\mu^2)T^2}}{2\sqrt[4]{6}}\mu h_t\right)\mu h_t\right] h_t^{3/2}\,\|\partial_t u\|_{H^1(0,T)}$$

and thus, the first assertion is proven.

The $L^2(0,T)$ error estimate follows with the first assertion from (3.63). □

To illustrate the error estimate (3.55), consider the given right-hand side f as $f(t)=1$ for $t\in(0,2)=(0,T)$, which results in the solution

$$u(t)=\frac{1}{\mu}\left[1-e^{-\mu t}\right], \quad t\in(0,2),$$

satisfying $\|\partial_{tt}u\|_{L^2(0,2)} \leq \sqrt{\frac{\mu}{2}}$. As at the end of Subsection 3.4.3, a sequence of finite element spaces $S^1_{h_t,0,}(0,2)$ of uniform mesh size $h_t=2/N_t$, and $N_t=2^{j+1}$, $j=0,\ldots,7$, is introduced. Depending on μ, quadratic convergence is expected, but requiring a sufficiently small mesh size $h_t=2/N_t$ for large μ. This well-known behaviour can be seen in Figure 3.3, Figure 3.4, Figure 3.5 and Figure 3.6, where the computed $L^2(0,2)$ error is plotted versus the error bound (3.55).

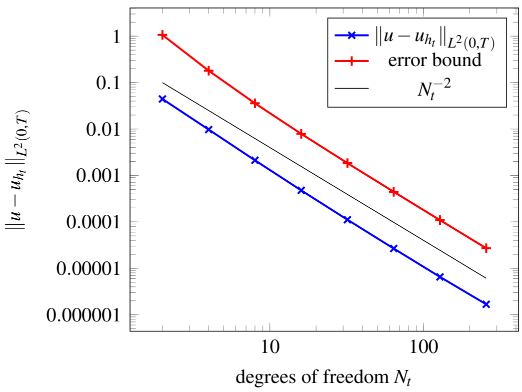

Figure 3.3: $L^2(0,T)$ error and error bound (3.55) for (3.46) for $\mu=1$.

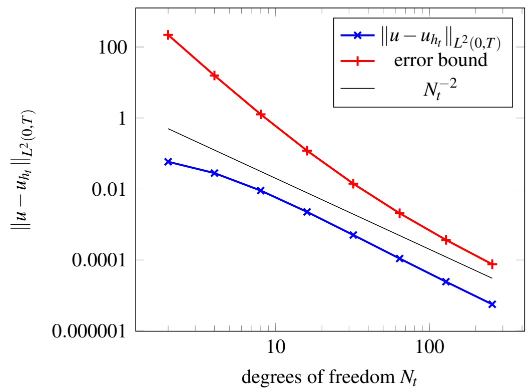

Figure 3.4: $L^2(0,T)$ error and error bound (3.55) for (3.46) for $\mu=10$.

Remark 3.4.15. *The Galerkin-Petrov finite element formulation* (3.22) *of Section 3.3 is uniquely solvable but unstable for any mesh size* h_t *and any* $\mu>0$, *whereas the Galerkin-Bubnov finite element formulation* (3.54) *is uniquely solvable and stable independently of the mesh size* h_t *and* $\mu>0$.

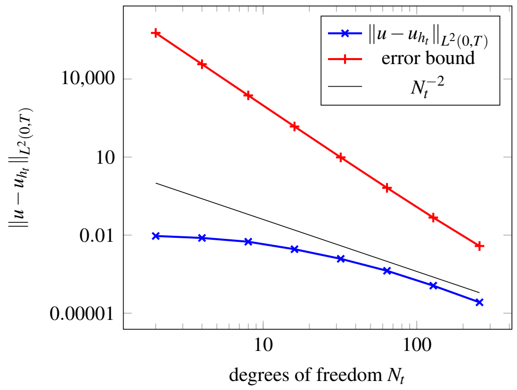

Figure 3.5: $L^2(0,T)$ error and error bound (3.55) for (3.46) for $\mu = 100$.

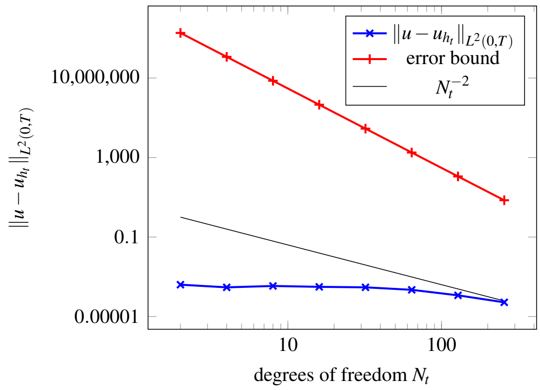

Figure 3.6: $L^2(0,T)$ error and error bound (3.55) for (3.46) for $\mu = 1000$.

3.4.5 Variational Formulation for the Heat Equation via Fourier Series

In this subsection, the ideas of Subsection 3.4.4 are transferred to the homogeneous Dirichlet problem of the heat equation

$$\left.\begin{aligned} \partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma = \Gamma \times [0,T], \\ u(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned}\right\} \tag{3.64}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$ and $T > 0$ is a given finite time. To write down the variational formulation (3.20) via a Fourier series approach, characterisations of the spaces $H^{1,1/2}_{0;0,}(Q)$ and $H^{1,1/2}_{0;,0}(Q)$ are given. Therefore, consider for $i \in \mathbb{N}$ the eigenfunctions ϕ_i and eigenvalues μ_i from (2.4), i.e.

$$-\Delta_x \phi_i = \mu \phi_i \quad \text{in } \Omega, \quad \phi_i = 0 \quad \text{on } \Gamma, \quad \|\phi_i\|_{L^2(\Omega)} = 1,$$

which form an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in $H^1_0(\Omega)$. Since the relation

$$L^2(Q) \simeq L^2(\Omega) \hat{\otimes} L^2(0,T) = \overline{L^2(\Omega) \otimes L^2(0,T)}^{\|\cdot\|_{L^2(Q)}}$$

holds for tensor-products, see Section 2.4, the functions

$$\phi_i \cdot V_k \in L^2(Q), \quad i \in \mathbb{N}, k \in \mathbb{N}_0,$$

form an orthogonal basis of $L^2(Q)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{L^2(Q)}$, see [128, Proposition 2, page 50], where the eigenfunctions V_k are given in (3.25). Hence, as an extension of the expansion (3.26), it holds for a function $u \in L^2(Q)$ the representation

$$u(x,t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} V_k(t) \phi_i(x) = \sum_{i=1}^{\infty} U_i(t) \phi_i(x), \quad U_i(t) = \sum_{k=0}^{\infty} u_{i,k} V_k(t) \tag{3.65}$$

with the coefficients

$$u_{i,k} = \frac{2}{T}\int_0^T\int_\Omega u(x,t)V_k(t)\phi_i(x)\mathrm{d}x\mathrm{d}t = \frac{2}{T}\int_0^T \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)\int_\Omega u(x,t)\phi_i(x)\mathrm{d}x\mathrm{d}t,$$

i.e. there hold

$$\left\| u - \sum_{i=1}^{M}\sum_{k=0}^{N} u_{i,k}V_k\cdot\phi_i \right\|_{L^2(Q)} \to 0 \quad \text{as } M\to\infty, N\to\infty$$

and

$$\left\| u - \sum_{i=1}^{M} U_i\cdot\phi_i \right\|_{L^2(Q)} \to 0 \quad \text{as } M\to\infty.$$

For the inner product, it holds

$$\langle u,\hat{u}\rangle_{L^2(Q)} = \sum_{i=1}^{\infty}\langle U_i,\hat{U}_i\rangle_{L^2(0,T)} = \frac{T}{2}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} u_{i,k}\cdot\hat{u}_{i,k}$$

for $u,\hat{u}\in L^2(Q)$ with the expansion (3.65), and if, in addition, $u,\hat{u}\in H^{1,1}_{0;}(Q)$, it follows

$$\begin{aligned}\langle u,\hat{u}\rangle_{H^{1,1}_{0;}(Q)} &= \sum_{i=1}^{\infty}\left[\langle\partial_t U_i,\partial_t\hat{U}_i\rangle_{L^2(0,T)} + \mu_i\langle U_i,\hat{U}_i\rangle_{L^2(0,T)}\right]\\ &= \frac{T}{2}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\left[\frac{1}{T^2}\left(\frac{\pi}{2}+k\pi\right)^2+\mu_i\right]u_{i,k}\cdot\hat{u}_{i,k}\end{aligned}$$

for the inner product (2.13). Correspondingly, the functions

$$\phi_i\cdot W_k\in L^2(Q),\quad i\in\mathbb{N}, k\in\mathbb{N}_0,$$

form an orthogonal basis of $L^2(Q)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{L^2(Q)}$, where the eigenfunctions W_k are given in (3.33). Hence, as an extension of the expansion (3.34), it holds for $z\in L^2(Q)$ the representation

$$z(x,t) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} z_{i,k}W_k(t)\phi_i(x) = \sum_{i=1}^{\infty} Z_i(t)\phi_i(x),\quad Z_i(t) = \sum_{k=0}^{\infty} z_{i,k}W_k(t) \tag{3.66}$$

with the coefficients

$$z_{i,k} = \frac{2}{T}\int_0^T\int_\Omega z(x,t)W_k(t)\phi_i(x)\mathrm{d}x\mathrm{d}t = \frac{2}{T}\int_0^T \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)\int_\Omega z(x,t)\phi_i(x)\mathrm{d}x\mathrm{d}t,$$

i.e. there hold

$$\left\| z - \sum_{i=1}^{M} \sum_{k=0}^{N} z_{i,k} W_k \cdot \phi_i \right\|_{L^2(Q)} \to 0 \quad \text{as } M \to \infty, N \to \infty$$

and

$$\left\| z - \sum_{i=1}^{M} Z_i \cdot \phi_i \right\|_{L^2(Q)} \to 0 \quad \text{as } M \to \infty.$$

Analogous to Subsection 3.4.1, there hold the representations

$$H^{1/2}_{0,}(0,T;L^2(\Omega)) = \Bigg\{ u \in L^2(Q) \colon u(x,t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} V_k(t) \phi_i(x), \\ \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |u_{i,k}|^2 < \infty \Bigg\}$$

with the inner product

$$\langle u, \hat{u} \rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} := \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) u_{i,k} \cdot \hat{u}_{i,k}$$

and

$$H^{1/2}_{,0}(0,T;L^2(\Omega)) = \Bigg\{ z \in L^2(Q) \colon z(x,t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} W_k(t) \phi_i(x), \\ \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) |z_{i,k}|^2 < \infty \Bigg\}$$

with the inner product

$$\langle z, \hat{z} \rangle_{H^{1/2}_{,0}(0,T;L^2(\Omega)),F} := \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi \right) z_{i,k} \cdot \hat{z}_{i,k},$$

where the induced norms $\|\cdot\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F}$ and $\|\cdot\|_{H^{1/2}_{,0}(0,T;L^2(\Omega)),F}$ are equivalent to the norms $\|\cdot\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))}$ and $\|\cdot\|_{H^{1/2}_{,0}(0,T;L^2(\Omega))}$ given in (2.16) and (2.17). So, the anisotropic space $H^{1,1/2}_{0;0,}(Q) = H^{1/2}_{0,}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ is endowed with the inner product

$$\langle u, \hat{u} \rangle_{H^{1,1/2}_{0;0,}(Q),F} := \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right] u_{i,k} \cdot \hat{u}_{i,k},$$

and analogously, $H^{1,1/2}_{0;,0}(Q) = H^{1/2}_{,0}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ is endowed with

$$\langle z, \hat{z} \rangle_{H^{1,1/2}_{0;,0}(Q),F} := \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right] z_{i,k} \cdot \hat{z}_{i,k}.$$

The transformation operator $\mathcal{H}_T$, given in (3.37), acts only with respect to the time variable t, i.e. for $u \in L^2(Q)$ with expansion (3.65), one defines

$$(\mathcal{H}_T u)(x,t) := \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} W_k(t) \phi_i(x) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} \cos\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \phi_i(x) \tag{3.67}$$

for $(x,t) \in Q$. By construction, it holds $\mathcal{H}_T u \in L^2(Q)$. Furthermore,

$$\mathcal{H}_T : \; L^2(Q) \to L^2(Q)$$

is bijective and norm preserving, i.e.

$$\|\mathcal{H}_T u\|_{L^2(Q)} = \|u\|_{L^2(Q)} \quad \text{for all } u \in L^2(Q),$$

where the inverse transformation operator

$$\mathcal{H}_T^{-1} : \; L^2(Q) \to L^2(Q)$$

is given by

$$(\mathcal{H}_T^{-1} z)(x,t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} V_k(t) \phi_i(x) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} z_{i,k} \sin\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \phi_i(x), \quad (x,t) \in Q,$$

for $z \in L^2(Q)$ with the expansion (3.66). Analogously, the maps

$$\mathcal{H}_T : \; H^{1/2}_{0,}(0,T;L^2(\Omega)) \to H^{1/2}_{,0}(0,T;L^2(\Omega))$$

and

$$\mathcal{H}_T : \; H^{1,1/2}_{0;0,}(Q) \to H^{1,1/2}_{0;,0}(Q)$$

are norm preserving and bijective.

Finally, representations of the dual spaces $[H^{1,1/2}_{0;,0}(Q)]'$ and $[H^{1/2}_{,0}(0,T;L^2(\Omega))]'$ are given. In Section 2.5 the dual space $[H^{1,1/2}_{0;,0}(Q)]'$ is characterised as a completion of $L^2(Q)$ with respect to the Hilbertian norm $\|\cdot\|_{[H^{1,1/2}_{0;,0}(Q)]'}$, where $\|\cdot\|_{H^{1,1/2}_{0;,0}(Q)}$ is the norm in $H^{1,1/2}_{0;,0}(Q)$, see (2.20). A to $\|\cdot\|_{[H^{1,1/2}_{0;,0}(Q)]'}$ equivalent Hilbertian norm is given by

$$\|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F} := \sup_{0 \neq z \in H^{1,1/2}_{0;,0}(Q)} \frac{|\langle f, z \rangle_Q|}{\|z\|_{H^{1,1/2}_{0;,0}(Q),F}}$$

for $f \in [H^{1,1/2}_{0;,0}(Q)]'$, where $\langle \cdot,\cdot \rangle_Q$ denotes again the duality pairing as extension of the inner product in $L^2(Q)$. Moreover, a Hilbertian norm in $[H^{1/2}_{,0}(0,T;L^2(\Omega))]'$ is

$$\|f\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F} := \sup_{0\neq z\in H^{1/2}_{,0}(0,T;L^2(\Omega))} \frac{|\langle f,z\rangle_Q|}{\|z\|_{H^{1/2}_{,0}(0,T;L^2(\Omega)),F}}$$

for $f \in [H^{1/2}_{,0}(0,T;L^2(\Omega))]'$.

Lemma 3.4.16. *For $f \in [H^{1,1/2}_{0;,0}(Q)]'$, it holds*

$$\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]',F} = \frac{T}{2}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} \frac{1}{\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i} f_{i,k}^2,$$

where $f_{i,k} := \frac{2}{T}\langle f, W_k\phi_i\rangle_Q$ with $W_k(t) = \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)$.

Analogously, for $f \in [H^{1/2}_{,0}(0,T;L^2(\Omega))]'$, there holds

$$\|f\|^2_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F} = \frac{1}{2}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1} f_{i,k}^2.$$

Proof. For $z \in H^{1,1/2}_{0;,0}(Q)$, there hold with (3.66) the representations

$$z(x,t) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} z_{i,k}\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)\phi_i(x) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} z_{i,k}W_k(t)\phi_i(x),$$

$$\|z\|^2_{H^{1,1/2}_{0;,0}(Q),F} = \frac{T}{2}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right) z_{i,k}^2$$

and therefore, with the continuity of f

$$\langle f,z\rangle_Q = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} z_{i,k}\langle f, W_k\phi_i\rangle_Q = \frac{T}{2}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} z_{i,k}\cdot f_{i,k}.$$

Set $z^N_{i,k} := \left(\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right)^{-1} f_{i,k}$ for $i = 1,\ldots,N,\ k = 0,\ldots,N$ and $z^N_{i,k} = 0$ for $i > N$ or

$k > N$. Assume w.l.o.g. that $z^N \neq 0$. Thus, $z^N \in H^{1,1/2}_{0;,0}(Q)$ and one obtains

$$\left(\frac{T}{2}\sum_{i=1}^{N}\sum_{k=0}^{N}\frac{1}{\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i}f_{i,k}^2\right)^{1/2} = \frac{\sqrt{\frac{T}{2}}\sum\limits_{i=1}^{N}\sum\limits_{k=0}^{N}\left(\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right)^{-1}f_{i,k}^2}{\left(\sum\limits_{i=1}^{N}\sum\limits_{k=0}^{N}\left(\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right)^{-1}f_{i,k}^2\right)^{1/2}}$$

$$= \frac{\sqrt{\frac{T}{2}}\sum\limits_{i=1}^{N}\sum\limits_{k=0}^{N} z_{i,k}^N f_{i,k}}{\left(\sum\limits_{i=1}^{N}\sum\limits_{k=0}^{N}\left(\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right)(z_{i,k}^N)^2\right)^{1/2}}$$

$$= \frac{\langle f, z^N\rangle_Q}{\|z^N\|_{H^{1,1/2}_{0;,0}(Q),F}} \leq \sup_{0\neq z\in H^{1,1/2}_{0;,0}(Q)}\frac{|\langle f,z\rangle_Q|}{\|z\|_{H^{1,1/2}_{0;,0}(Q),F}}.$$

Hence,

$$\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]',F} \geq \frac{T}{2}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i}f_{i,k}^2$$

follows as $N \to \infty$.

On the other hand, the Cauchy-Schwarz inequality yields

$$\|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F} = \sup_{0\neq z\in H^{1,1/2}_{0;,0}(Q)}\frac{|\langle f,z\rangle_Q|}{\|z\|_{H^{1,1/2}_{0;,0}(Q),F}}$$

$$= \sqrt{\frac{T}{2}}\sup_{0\neq z\in H^{1,1/2}_{0;,0}(Q)}\frac{\sum\limits_{i=1}^{\infty}\sum\limits_{k=0}^{\infty} z_{i,k}f_{i,k}\left(\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right)^{1/2}\left(\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right)^{-1/2}}{\left(\sum\limits_{i=1}^{\infty}\sum\limits_{k=0}^{\infty}\left(\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right)z_{i,k}^2\right)^{1/2}}$$

$$\leq \left(\frac{T}{2}\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i}f_{i,k}^2\right)^{1/2}$$

and thus, the first assertion is proven.

For $f \in [H^{1/2}_{,0}(0,T;L^2(\Omega))]'$, the proof is obtained in the same manner. □

Analogous to the case of the ordinary differential equation, see Theorem 3.4.6, the distributional derivative ∂_t on Q of a function $u \in H^{1/2}_{0,}(0,T;L^2(\Omega))$ is investigated.

Theorem 3.4.17. *For a function* $u \in H^{1/2}_{0,}(0,T;L^2(\Omega))$, *the distributional derivative* ∂_t *on* Q *fulfils* $\partial_t u \in [H^{1/2}_{,0}(0,T;L^2(\Omega))]'$. *More precisely, there exists a uniquely determined element* $g \in [H^{1/2}_{,0}(0,T;L^2(\Omega))]'$ *satisfying*

$$\partial_t T_u(\varphi) = \langle g, \varphi \rangle_Q \quad \forall \varphi \in \mathcal{D}(Q),$$

where $T_u \colon \mathcal{D}(Q) \to \mathbb{R}$, $T_u(\varphi) = \langle u, \varphi \rangle_{L^2(Q)}$, *is the to* u *related distribution, see Section 2.1. In addition, there hold*

$$\|\partial_t u\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F} \leq \|u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F}$$

and

$$\langle \partial_t u, z \rangle_Q = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} \left(\frac{\pi}{2} + k\pi \right) z_{i,k} \tag{3.68}$$

for all $z \in H^{1/2}_{,0}(0,T;L^2(\Omega))$ *with expansion coefficients* $u_{i,k}$ *from* (3.65) *and* $z_{i,k}$ *from* (3.66).

Proof. The proof is analogous to the proof of Theorem 3.4.6, since the algebraic tensor-product $C_0^\infty(\Omega) \otimes C_0^\infty(0,T) \subset C_0^\infty(Q)$ is dense in $H^{1/2}_{,0}(0,T;L^2(\Omega))$. □

Remark 3.4.18. *The equation* (3.68) *leads, with the bijective transformation operator* $\mathcal{H}_T \colon H^{1/2}_{0,}(0,T;L^2(\Omega)) \to H^{1/2}_{,0}(0,T;L^2(\Omega))$, *to*

$$\langle \partial_t u, \mathcal{H}_T \hat{u} \rangle_Q = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} \left(\frac{\pi}{2} + k\pi \right) \hat{u}_{i,k} = \langle u, \hat{u} \rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega),F} \tag{3.69}$$

for all $u, \hat{u} \in H^{1/2}_{0,}(0,T;L^2(\Omega))$ *with expansion coefficients* $u_{i,k}, \hat{u}_{i,k}$ *from* (3.65).

As in Section 3.3, the variational formulation of (3.64) is to find $u \in H^{1,1/2}_{0;0,}(Q)$ such that

$$a(u,z) = \langle f, z \rangle_Q \tag{3.70}$$

for all $z \in H^{1,1/2}_{0;,0}(Q)$, where $f \in [H^{1,1/2}_{0;,0}(Q)]'$ is a given right-hand side and the bilinear form $a(\cdot,\cdot) \colon H^{1,1/2}_{0;0,}(Q) \times H^{1,1/2}_{0;,0}(Q) \to \mathbb{R}$ is defined by

$$a(u,z) := \langle \partial_t u, z \rangle_Q + \langle \nabla_x u, \nabla_x z \rangle_{L^2(Q)}$$

for $u \in H^{1,1/2}_{0;0,}(Q)$, $z \in H^{1,1/2}_{0;,0}(Q)$. Note that for $u \in H^{1,1/2}_{0;0,}(Q) \subset H^{1/2}_{0,}(0,T;L^2(\Omega))$ and for $z \in H^{1,1/2}_{0;,0}(Q) \subset H^{1/2}_{,0}(0,T;L^2(\Omega))$, it holds

$$\begin{aligned}
\left| \langle \partial_t u, z \rangle_Q \right| &\leq \|\partial_t u\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F} \|z\|_{H^{1/2}_{,0}(0,T;L^2(\Omega)),F} \\
&\leq \|u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \|z\|_{H^{1/2}_{,0}(0,T;L^2(\Omega)),F}
\end{aligned}$$

due to Theorem 3.4.17, i.e. $\langle \partial_t u, z \rangle_Q$ is well-defined and bounded on

$$H_{0,}^{1/2}(0,T;L^2(\Omega)) \times H_{,0}^{1/2}(0,T;L^2(\Omega)).$$

With this last estimate, it follows the boundedness of the bilinear form

$$a(\cdot,\cdot)\colon\ H_{0;0,}^{1,1/2}(Q) \times H_{0;,0}^{1,1/2}(Q) \to \mathbb{R},$$

i.e. it holds with the Cauchy-Schwarz inequality

$$\begin{aligned} |a(u,z)| &\le \left| \langle \partial_t u, z \rangle_Q \right| + \left| \langle \nabla_x u, \nabla_x z \rangle_{L^2(Q)} \right| \\ &\le \|u\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F} \|z\|_{H_{,0}^{1/2}(0,T;L^2(\Omega)),F} + \|\nabla_x u\|_{L^2(Q)} \|\nabla_x z\|_{L^2(Q)} \\ &\le \|u\|_{H_{0;0,}^{1,1/2}(Q),F} \|z\|_{H_{0;,0}^{1,1/2}(Q),F} \end{aligned}$$

for $u \in H_{0;0,}^{1,1/2}(Q)$, $z \in H_{0;,0}^{1,1/2}(Q)$.

Theorem 3.4.19. *For a given right-hand side $f \in [H_{0;,0}^{1,1/2}(Q)]'$, there exists a unique solution $u \in H_{0;0,}^{1,1/2}(Q)$ of the variational formulation* (3.70)*, satisfying*

$$\|u\|_{H_{0;0,}^{1,1/2}(Q),F} \le 2\, \|f\|_{[H_{0;,0}^{1,1/2}(Q)]',F}.$$

Furthermore, the solution operator

$$\mathcal{L}\colon\ [H_{0;,0}^{1,1/2}(Q)]' \to H_{0;0,}^{1,1/2}(Q), \quad \mathcal{L}f := u,$$

is an isomorphism. In addition, the bilinear form $a(\cdot,\cdot)\colon\ H_{0;0,}^{1,1/2}(Q) \times H_{0;,0}^{1,1/2}(Q) \to \mathbb{R}$ is continuous and fulfils an inf-sup condition, i.e.

$$\frac{1}{2} \le \inf_{0 \ne v \in H_{0;0,}^{1,1/2}(Q)} \sup_{0 \ne z \in H_{0;,0}^{1,1/2}(Q)} \frac{|a(v,z)|}{\|v\|_{H_{0;0,}^{1,1/2}(Q),F} \|z\|_{H_{0;,0}^{1,1/2}(Q),F}}, \tag{3.71}$$

and the surjectivity condition, i.e. for each function $z \in H_{0;,0}^{1,1/2}(Q)$, $z \ne 0$, there exists an element $v \in H_{0;0,}^{1,1/2}(Q)$ such that $a(v,z) \ne 0$.

Proof. For the solution u of the variational problem (3.70), consider the ansatz (3.65)

$$u(x,t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} V_k(t) \phi_i(x) = \sum_{i=1}^{\infty} U_i(t) \phi_i(x), \quad U_i(t) = \sum_{k=0}^{\infty} u_{i,k} V_k(t),$$

where $U_i \in H^{1/2}_{0,}(0,T)$ are unknown functions to be determined. When choosing, for a fixed $j \in \mathbb{N}$, $z(x,t) := Z(t)\phi_j(x)$ with an arbitrary $Z \in H^{1/2}_{,0}(0,T)$ as test function, the variational formulation (3.70) leads to find $U_j \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t U_j, Z \rangle_{(0,T)} + \mu_j \langle U_j, Z \rangle_{L^2(0,T)} = \langle f, Z\phi_j \rangle_Q \tag{3.72}$$

for all $Z \in H^{1/2}_{,0}(0,T)$. With Lemma 3.4.5, it holds

$$\begin{aligned} \left| \langle f, Z\phi_j \rangle_Q \right| &\le \|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F} \left\| Z\phi_j \right\|_{H^{1,1/2}_{0;,0}(Q),F} \\ &= \|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F} \sqrt{\|Z\|^2_{H^{1/2}_{,0}(0,T),F} + \mu_j \|Z\|^2_{L^2(0,T)}} \\ &\le \sqrt{1 + \frac{2T\mu_j}{\pi}} \|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F} \|Z\|_{H^{1/2}_{,0}(0,T),F} \end{aligned}$$

for all $Z \in H^{1/2}_{,0}(0,T)$, and so,

$$\left\langle F_j, Z \right\rangle_{(0,T)} := \langle f, Z\phi_j \rangle_Q$$

fulfils $F_j \in [H^{1/2}_{,0}(0,T)]'$. The unique solvability of (3.72) follows from the unique solvability of (3.46). So, there exists for every $j \in \mathbb{N}$ a unique solution $U_j \in H^{1/2}_{0,}(0,T)$ of the variational formulation (3.72), satisfying

$$\begin{aligned} \|U_j\|^2_{H^{1/2}_{0,}(0,T),F} &= \langle \partial_t U_j, \mathcal{H}_T U_j \rangle_{(0,T)} \\ &\le \langle \partial_t U_j, \mathcal{H}_T U_j \rangle_{(0,T)} + \mu_j \langle U_j, \mathcal{H}_T U_j \rangle_{L^2(0,T)} \\ &= \langle f, \phi_j \mathcal{H}_T U_j \rangle_Q. \end{aligned}$$

Hence, the coefficients $U_i \in H^{1/2}_{0,}(0,T)$ are uniquely determined. Next, the convergence properties of the series expansion of u are investigated. Therefore, define the partial sums

$$u_M(x,t) := \sum_{j=1}^{M} U_j(t)\phi_j(x)$$

for $M \in \mathbb{N}$, and one concludes

$$\begin{aligned} \|u_M\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} &= \sum_{j=1}^{M} \|U_j\|^2_{H^{1/2}_{0,}(0,T),F} \le \sum_{j=1}^{M} \langle f, \phi_j \mathcal{H}_T U_j \rangle_Q = \left\langle f, \sum_{j=1}^{M} \phi_j \mathcal{H}_T U_j \right\rangle_Q \\ &\le \|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F} \|\mathcal{H}_T u_M\|_{H^{1,1/2}_{0;,0}(Q),F} \\ &= \|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F} \|u_M\|_{H^{1,1/2}_{0;0,}(Q),F}. \end{aligned}$$

Hence, using (3.51) for $f_{i,k} = \frac{2}{T}\langle F_i, W_k\rangle_{(0,T)} = \frac{2}{T}\langle f, \phi_i W_k\rangle_Q$, one obtains

$$\begin{aligned}\|u_M\|^2_{L^2(0,T;H_0^1(\Omega))} = \sum_{i=1}^M \mu_i \|U_i\|^2_{L^2(0,T)} &\le \frac{T}{2}\sum_{i=1}^M\sum_{k=0}^\infty \frac{\mu_i}{\mu_i^2 + \frac{1}{T^2}(\frac{\pi}{2}+k\pi)^2} f_{i,k}^2 \\ &\le T\sum_{i=1}^M\sum_{k=0}^\infty \frac{1}{\mu_i + \frac{1}{T}(\frac{\pi}{2}+k\pi)} f_{i,k}^2 \le 2\,\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]',F}\end{aligned}$$

with the help of the inequality

$$\frac{a}{a^2+b^2} \le \frac{a+b}{\frac{1}{2}(a+b)^2} = \frac{2}{a+b} \quad \text{for } 0 < a,b \in \mathbb{R}.$$

With this, it holds

$$\begin{aligned}\|u_M\|^2_{H^{1,1/2}_{0;0,}(Q),F} &= \|u_M\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} + \|u_M\|^2_{L^2(0,T;H_0^1(\Omega))} \\ &\le \|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F}\|u_M\|_{H^{1,1/2}_{0;0,}(Q),F} + 2\,\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]',F},\end{aligned}$$

and therefore, by solving the corresponding quadratic equation,

$$\|u_M\|_{H^{1,1/2}_{0;0,}(Q),F} \le 2\|f\|_{[H^{1,1/2}_{0;,0}(Q)]',F}$$

follows for all $M \in \mathbb{N}$. The last inequality yields with $U_i(t) = \sum_{k=0}^\infty u_{i,k}V_k(t)$ the bound

$$\begin{aligned}\frac{T}{2}\sum_{i=1}^\infty\sum_{k=0}^\infty \left[\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right]u_{i,k}^2 &= \lim_{M\to\infty}\sum_{i=1}^M\left[\|U_i\|^2_{H^{1/2}_{0,}(0,T),F} + \mu_i\|U_i\|^2_{L^2(0,T)}\right] \\ &= \lim_{M\to\infty}\|u_M\|^2_{H^{1,1/2}_{0;0,}(Q),F} \\ &\le 4\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]',F} < \infty\end{aligned}$$

and thus, $u \in H^{1,1/2}_{0;0,}(Q)$ with $\lim_{M\to\infty} u_M = u$ in $H^{1,1/2}_{0;0,}(Q)$.

Next, it is shown that u is a solution of the variational formulation (3.70). This follows with the expansion $z = \sum_{j=1}^\infty Z_j\phi_j \in H^{1,1/2}_{0;,0}(Q)$, $Z_j \in H^{1/2}_{,0}(0,T)$, which is given by (3.66), from

$$\begin{aligned}a(u,z) &= \langle \partial_t u, z\rangle_Q + \langle \nabla_x u, \nabla_x z\rangle_{L^2(Q)} \\ &= \lim_{M\to\infty}\langle \partial_t u_M, z\rangle_Q + \lim_{M\to\infty}\langle \nabla_x u_M, \nabla_x z\rangle_{L^2(Q)} \\ &= \sum_{j=1}^\infty \left\langle \partial_t U_j\phi_j, z\right\rangle_Q + \sum_{j=1}^\infty \langle U_j\nabla_x\phi_j, \nabla_x z\rangle_{L^2(Q)} \\ &= \sum_{j=1}^\infty\sum_{i=1}^\infty \left\langle \partial_t U_j\phi_j, Z_i\phi_i\right\rangle_Q + \sum_{j=1}^\infty\sum_{i=1}^\infty \langle U_j\nabla_x\phi_j, Z_i\nabla_x\phi_i\rangle_{L^2(Q)}\end{aligned}$$

and by using (3.72), from

$$a(u,z) = \sum_{j=1}^{\infty} \left\langle \partial_t U_j, Z_j \right\rangle_{(0,T)} + \sum_{j=1}^{\infty} \mu_j \langle U_j, Z_j \rangle_{L^2(0,T)} = \sum_{j=1}^{\infty} \left\langle f, Z_j \phi_j \right\rangle_Q = \langle f, z \rangle_Q.$$

The uniqueness of u is a consequence of the uniqueness of the coefficients U_j.

The remaining parts of the theorem follow from the Nečas Theorem 2.9.1. □

The variational formulation (3.70) is equivalent to find $u \in H^{1,1/2}_{0;0,}(Q)$ such that

$$a(u, \mathcal{H}_T v) = \langle \partial_t u, \mathcal{H}_T v \rangle_Q + \langle \nabla_x u, \nabla_x \mathcal{H}_T v \rangle_{L^2(Q)} \stackrel{!}{=} \langle f, \mathcal{H}_T v \rangle_Q \tag{3.73}$$

for all $v \in H^{1,1/2}_{0;0,}(Q)$. Hence, unique solvability of the variational formulation (3.73) follows from the unique solvability of (3.70). In addition, the stability estimate (3.71) implies the stability estimate

$$\frac{1}{2} \|u\|_{H^{1,1/2}_{0;0,}(Q),F} \leq \sup_{0 \neq v \in H^{1,1/2}_{0;0,}(Q)} \frac{\langle \partial_t u, \mathcal{H}_T v \rangle_Q + \langle \nabla_x u, \nabla_x \mathcal{H}_T v \rangle_{L^2(Q)}}{\|v\|_{H^{1,1/2}_{0;0,}(Q),F}}$$

for all $u \in H^{1,1/2}_{0;0,}(Q)$.

To introduce approximate solutions, the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ is assumed to be an interval $\Omega = (0,L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. When using some conforming space-time finite element space $\mathcal{V}_h \subset H^{1,1/2}_{0;0,}(Q)$, the Galerkin variational formulation of (3.73) is to find $u_h \in \mathcal{V}_h$ such that

$$a(u_h, \mathcal{H}_T v_h) = \langle f, \mathcal{H}_T v_h \rangle_Q \tag{3.74}$$

for all $v_h \in \mathcal{V}_h$. Note that ansatz and test spaces are equal.

Theorem 3.4.20. *Let $\mathcal{V}_h \subset H^{1,1/2}_{0;0,}(Q)$ be a conforming space-time finite element space and let $f \in [H^{1,1/2}_{0;,0}(Q)]'$ be a given right-hand side. Then there exists a unique solution $u_h \in \mathcal{V}_h$ of the Galerkin-Bubnov variational formulation* (3.74)*. If, in addition, the right-hand side fulfils $f \in [H^{1/2}_{,0}(0,T;L^2(\Omega))]' \subset [H^{1,1/2}_{0;,0}(Q)]'$, then the stability estimate*

$$\|u_h\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \leq \|f\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F}$$

is true.

Proof. Let $u_h^0 \in \mathcal{V}_h$ be any solution of the homogeneous variational formulation (3.74) with $f = 0$. With (3.69), $v_h = u_h^0 \in \mathcal{V}_h$ in (3.74) and Lemma 3.4.8, it follows

$$\begin{aligned}
\|u_h^0\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} &= \langle \partial_t u_h^0, \mathcal{H}_T u_h^0 \rangle_Q \\
&\leq \langle \partial_t u_h^0, \mathcal{H}_T u_h^0 \rangle_Q + \sum_{m=1}^{d} \int_\Omega \underbrace{\langle \partial_{x_m} u_h^0(x,\cdot), \partial_{x_m} \mathcal{H}_T u_h^0(x,\cdot) \rangle_{L^2(0,T)}}_{\geq 0} \mathrm{d}x \\
&= \langle f, \mathcal{H}_T u_h^0 \rangle_Q \qquad (3.75)\\
&= 0,
\end{aligned}$$

which implies $u_h^0 = 0$ and thus, the uniqueness of a solution $u_h \in \mathcal{V}_h$ of the inhomogeneous variational formulation (3.74). Since ansatz and test spaces of the variational formulation (3.74) are equal, the unique solvability of the Galerkin-Bubnov variational formulation (3.74) follows.

If, in addition, the right-hand side fulfils $f \in [H^{1/2}_{,0}(0,T;L^2(\Omega))]' \subset [H^{1,1/2}_{0;,0}(Q)]'$, then, for the unique solution $u_h \in \mathcal{V}_h$ of the Galerkin-Bubnov variational formulation (3.74), there hold with (3.75)

$$\begin{aligned}
\|u_h\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} &\leq \langle f, \mathcal{H}_T u_h \rangle_Q \\
&\leq \|f\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F} \|\mathcal{H}_T u_h\|_{H^{1/2}_{,0}(0,T;L^2(\Omega)),F} \\
&= \|f\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F} \|u_h\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F}
\end{aligned}$$

and hence, the stability estimate. □

A possible choice for a conforming space-time discretisation of (3.73) is the space of piecewise linear, continuous functions $\mathcal{V}_h = S_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$, see Section 2.8 for more details. However, to perform the temporal transformation $\mathcal{H}_T$ easily, and to derive error estimates, based on the tensor-product structure, only a tensor-product space-time finite element space $\mathcal{V}_h = Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$ from (2.42) is considered in the remainder of this section. The Galerkin-Bubnov variational formulation of (3.73) is to find $u_h \in Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$ such that

$$a(u_h, \mathcal{H}_T v_h) = \langle f, \mathcal{H}_T v_h \rangle_Q \qquad (3.76)$$

for all $v_h \in Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$.

The next aim is to derive error estimates in the space-time norms. First of all, it holds for the unique solution $u \in H^{1,1/2}_{0;0,}(Q)$ of the variational formulation (3.73) and for the unique solution $u_h \in Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$ of the discrete variational formulation (3.76) the Galerkin orthogonality

$$a(u-u_h, \mathcal{H}_T v_h) = 0 \tag{3.77}$$

for all $v_h \in Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$. To derive an $L^2(Q)$ error estimate, a space-time projection $Q_{h_t}^{1/2} Q_{h_x}^1 v \in Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$ for a sufficiently smooth function $v \in H^{1,1/2}_{0;0,}(Q)$ is introduced, see Section 2.8.

For a function $v \in H^{1/2}_{0,}(0,T;L^2(\Omega))$, the $H^{1/2}_{0,}$ projection $Q_{h_t}^{1/2} v \in L^2(\Omega) \otimes S^1_{h_t,0,}(0,T)$ is defined by

$$\langle Q_{h_t}^{1/2} v, v_{h_t} \rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} = \langle v, v_{h_t} \rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \tag{3.78}$$

for all $v_{h_t} \in L^2(\Omega) \otimes S^1_{h_t,0,}(0,T)$. The properties of the $H^{1/2}_{0,}$ projection $Q_{h_t}^{1/2}$ are summarised in the following lemma.

Lemma 3.4.21. *Let $v \in H^{1/2}_{0,}(0,T;L^2(\Omega))$ be a given function. For the $H^{1/2}_{0,}$ projection $Q_{h_t}^{1/2} v$, defined in* (3.78)*, there hold the following properties:*

1. *The stability estimate*

$$\|Q_{h_t}^{1/2} v\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \leq \|v\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F}$$

   is true.

2. *If, in addition, it holds $v \in H^s(0,T;L^2(\Omega))$ for some $s \in (1/2, 2]$, then the error estimates*

$$\|v - Q_{h_t}^{1/2} v\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \leq c\, h_t^{s-1/2} \|v\|_{H^s(0,T;L^2(\Omega))},$$
$$\|v - Q_{h_t}^{1/2} v\|_{L^2(Q)} \leq c\, h_t^s \|v\|_{H^s(0,T;L^2(\Omega))}$$

   with a constant $c > 0$ are valid.

Proof. First, the stability estimate follows from the Cauchy-Schwarz inequality with

$$v_{h_t} = Q_{h_t}^{1/2} v \in L^2(\Omega) \otimes S^1_{h_t,0,}(0,T)$$

in the variational formulation (3.78).

Second, with the Galerkin orthogonality

$$\langle v - Q_{h_t}^{1/2} v, v_{h_t} \rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} = 0 \quad \text{for all } v_{h_t} \in L^2(\Omega) \otimes S^1_{h_t,0,}(0,T),$$

the first error estimate is given by

$$\begin{aligned}
\|v - Q_{h_t}^{1/2} v\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} &= \langle v - Q_{h_t}^{1/2} v, v - Q_{h_t}^{1/2} v\rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&= \langle v - Q_{h_t}^{1/2} v, v\rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&= \langle v - Q_{h_t}^{1/2} v, v - I_{h_t} v\rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\leq \|v - Q_{h_t}^{1/2} v\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \|v - I_{h_t} v\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\leq \|v - Q_{h_t}^{1/2} v\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} c\, h_t^{s-1/2} \|v\|_{H^s(0,T;L^2(\Omega))},
\end{aligned}$$

where I_{h_t} is the extended time interpolant (2.44) and $c > 0$ is the constant coming from standard interpolation error estimates.

The second error estimate is proven by an Aubin-Nitsche argument. Therefore, let the function $w \in H^{1/2}_{0,}(0,T;L^2(\Omega))$ be the unique solution of

$$\langle w, z\rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} = \langle \partial_t w, \mathcal{H}_T z\rangle_Q \stackrel{!}{=} \langle \mathcal{H}_T(v - Q_{h_t}^{1/2} v), \mathcal{H}_T z\rangle_{L^2(Q)} = \langle v - Q_{h_t}^{1/2} v, z\rangle_{L^2(Q)} \tag{3.79}$$

for all $z \in H^{1/2}_{0,}(0,T;L^2(\Omega))$, i.e.

$$\partial_t w(x,t) = \mathcal{H}_T(v - Q_{h_t}^{1/2} v)(x,t) \quad \text{for } (x,t) \in Q.$$

For $z = v - Q_{h_t}^{1/2} v \in H^{1/2}_{0,}(0,T;L^2(\Omega))$ in (3.79), it follows with the Galerkin orthogonality and the first error estimate of this proof that

$$\begin{aligned}
\|v - Q_{h_t}^{1/2} v\|^2_{L^2(Q)} &= \langle w, v - Q_{h_t}^{1/2} v\rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&= \langle w - Q_{h_t}^{1/2} w, v - Q_{h_t}^{1/2} v\rangle_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\leq \|w - Q_{h_t}^{1/2} w\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \|v - Q_{h_t}^{1/2} v\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\leq c\, h_t^{1/2} \|\partial_t w\|_{L^2(Q)} h_t^{s-1/2} \|v\|_{H^s(0,T;L^2(\Omega))} \\
&= c\, h_t^s \|v - Q_{h_t}^{1/2} v\|_{L^2(Q)} \|v\|_{H^s(0,T;L^2(\Omega))},
\end{aligned}$$

where the constant $c > 0$ comes from standard interpolation error estimates. □

Lemma 3.4.22. *For a function $v \in H^{1/2}_{0,}(0,T;H^1(\Omega))$, there holds that*

$$Q_{h_t}^{1/2} v \in H^1(\Omega) \otimes S^1_{h_t,0,}(0,T)$$

and if, in addition, $v \in H^{1/2}_{0,}(0,T;H^1_0(\Omega))$, then

$$Q_{h_t}^{1/2} v \in H^1_0(\Omega) \otimes S^1_{h_t,0,}(0,T).$$

Proof. The proof is analogous to the proof of Lemma 2.8.1. □

The next lemma shows that $Q_{h_t}^{1/2} Q_{h_x}^1 v \in Q_h^1(Q) \cap H_{0;0,}^{1,1/2}(Q)$ is well-defined under regularity assumptions on the given function v and that the operators in space and time commute, where the extended H_0^1 projection $Q_{h_x}^1$ is given in (2.47).

Lemma 3.4.23. *For a given function $v \in H_{0;0,}^{1,1/2}(Q)$ with regularity $\partial_t v \in L^2(0,T;H_0^1(\Omega))$ and $\partial_{x_m} v \in H_{0,}^{1/2}(0,T;L^2(\Omega))$ for $m = 1,\dots,d$, there hold*

1. *the relation $\partial_{x_m} Q_{h_t}^{1/2} v = Q_{h_t}^{1/2} \partial_{x_m} v \in L^2(\Omega) \otimes S_{h_t,0,}^1(0,T)$ for $m = 1,\dots,d$ and*
2. *the relation $Q_{h_t}^{1/2} Q_{h_x}^1 v = Q_{h_x}^1 Q_{h_t}^{1/2} v \in Q_h^1(Q) \cap H_{0;0,}^{1,1/2}(Q)$. In particular, the space-time projections $Q_{h_t}^{1/2} Q_{h_x}^1 v$ and $Q_{h_x}^1 Q_{h_t}^{1/2} v$ are well-defined.*

Furthermore, the error estimates

$$\|v - Q_{h_t}^{1/2} Q_{h_x}^1 v\|_{L^2(Q)} \le \|v - Q_{h_t}^{1/2} v\|_{L^2(Q)} + \|v - Q_{h_x}^1 v\|_{L^2(Q)} + c\, h_x h_t \|\partial_t \nabla_x v\|_{L^2(Q)}$$

and

$$\begin{aligned}\|v - Q_{h_t}^{1/2} Q_{h_x}^1 v\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F} &\le \|v - Q_{h_t}^{1/2} v\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F} \\ &\quad + \|v - Q_{h_x}^1 v\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F} + c\, h_x h_t^{1/2} \|\partial_t \nabla_x v\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F}\end{aligned}$$

with a constant $c > 0$ are valid.

Proof. The proof is analogous to the proof of Lemma 2.8.2. □

Theorem 3.4.24. *Let the unique solution $u \in H_{0;0,}^{1,1/2}(Q)$ of* (3.73) *satisfy the assumptions $\partial_t v \in L^2(0,T;H_0^1(\Omega))$ and $\partial_{x_m} v \in H_{0,}^{1/2}(0,T;L^2(\Omega))$ for $m = 1,\dots,d$, and further, $\Delta_x u \in H_{0,}^{1/2}(0,T;L^2(\Omega))$. Then the solution $u_h \in Q_h^1(Q) \cap H_{0;0,}^{1,1/2}(Q)$ of the Galerkin-Bubnov finite element discretisation* (3.76) *satisfies*

$$\begin{aligned}\|u - u_h\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F} &\le \|u - Q_{h_t}^{1/2} u\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F} + 2\left\|u - Q_{h_x}^1 u\right\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F} \\ &\quad + c\, h_x h_t^{1/2} \|\partial_t \nabla_x u\|_{H_{0,}^{1/2}(0,T;L^2(\Omega)),F} + \left\|\Delta_x u - Q_{h_t}^{1/2} \Delta_x u\right\|_{[H_{,0}^{1/2}(0,T;L^2(\Omega))]',F}\end{aligned}$$

and

$$\begin{aligned}\|u - u_h\|_{L^2(Q)} &\le \|u - Q_{h_t}^{1/2} u\|_{L^2(Q)} + \|u - Q_{h_x}^1 u\|_{L^2(Q)} + c\, h_x h_t \|\partial_t \nabla_x u\|_{L^2(Q)} \\ &\quad + \frac{2T}{\pi} \left\|\partial_t u - Q_{h_x}^1 \partial_t u\right\|_{L^2(Q)} + \frac{2T}{\pi} \left\|\Delta_x u - Q_{h_t}^{1/2} \Delta_x u\right\|_{L^2(Q)}\end{aligned}$$

with a constant $c > 0$ independent of h_t and h_x.

Proof. With the $H^{1/2}_{0,}(0,T;L^2(\Omega))$ ellipticity of $a(\cdot,\cdot)$, the Galerkin orthogonality (3.77) of $a(\cdot,\cdot)$, the properties of the $H^{1/2}_{0,}$ projection $Q^{1/2}_{h_t}$, the properties of the H^1_0 projection $Q^1_{h_x}$ and integration by parts, it holds that

$$\begin{aligned}
\left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} &\le a(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u, u_h - Q^{1/2}_{h_t} Q^1_{h_x} u) \\
&= a(u - Q^{1/2}_{h_t} Q^1_{h_x} u, u_h - Q^{1/2}_{h_t} Q^1_{h_x} u) \\
&= \left\langle \partial_t (u - Q^{1/2}_{h_t} Q^1_{h_x} u), \mathcal{H}_T (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u) \right\rangle_Q \\
&\quad + \left\langle \nabla_x (u - Q^{1/2}_{h_t} Q^1_{h_x} u), \nabla_x \mathcal{H}_T (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u) \right\rangle_{L^2(Q)}
\end{aligned}$$

and further,

$$\begin{aligned}
\left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} &= \left\langle \partial_t (u - Q^1_{h_x} u), \mathcal{H}_T (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u) \right\rangle_Q \\
&\quad + \left\langle \nabla_x (u - Q^{1/2}_{h_t} u), \nabla_x \mathcal{H}_T (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u) \right\rangle_{L^2(Q)} \\
&= \left\langle \partial_t (u - Q^1_{h_x} u), \mathcal{H}_T (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u) \right\rangle_Q \\
&\quad - \left\langle \Delta_x (u - Q^{1/2}_{h_t} u), \mathcal{H}_T (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u) \right\rangle_{L^2(Q)}. \qquad (3.80)
\end{aligned}$$

The relation (3.80) yields

$$\begin{aligned}
&\left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\quad \le \left\|u - Q^1_{h_x} u\right\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\qquad + \left\|\Delta_x u - Q^{1/2}_{h_t} \Delta_x u\right\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F} \left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F}
\end{aligned}$$

and so

$$\begin{aligned}
&\|u - u_h\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\quad \le \left\|u - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} + \left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\quad \le \left\|u - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\
&\qquad + \left\|u - Q^1_{h_x} u\right\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} + \left\|\Delta_x u - Q^{1/2}_{h_t} \Delta_x u\right\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]',F}.
\end{aligned}$$

This gives, with the second error estimate from Lemma 3.4.23, the $H^{1/2}_{0,}(0,T;L^2(\Omega))$ error estimate.

It remains to prove the $L^2(Q)$ error estimate. With the Poincaré type inequality from Lemma 3.4.5, the relation (3.80) and the Cauchy-Schwarz inequality, there follows

$$\begin{aligned}\frac{\pi}{2T}\left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|^2_{L^2(Q)} &\leq \left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \\ &\leq \|\partial_t u - Q^1_{h_x}\partial_t u\|_{L^2(Q)} \left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{L^2(Q)} \\ &\quad + \left\|\Delta_x u - Q^{1/2}_{h_t}\Delta_x u\right\|_{L^2(Q)} \left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{L^2(Q)}.\end{aligned}$$

This implies

$$\begin{aligned}\|u - u_h\|_{L^2(Q)} &\leq \left\|u - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{L^2(Q)} + \left\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{L^2(Q)} \\ &\leq \left\|u - Q^{1/2}_{h_t} Q^1_{h_x} u\right\|_{L^2(Q)} \\ &\quad + \frac{2T}{\pi}\|\partial_t u - Q^1_{h_x}\partial_t u\|_{L^2(Q)} + \frac{2T}{\pi}\left\|\Delta_x u - Q^{1/2}_{h_t}\Delta_x u\right\|_{L^2(Q)}\end{aligned}$$

and with the L^2 error estimate from Lemma 3.4.23, the assertion follows. □

Corollary 3.4.25. *Let the assumption of Theorem 3.4.24 be satisfied. If, in addition, the unique solution u of* (3.73) *is sufficiently smooth and the spatial H^1_0 projection $Q^1_{h_x}$ fulfils the standard L^2 error estimate*

$$\|u - Q^1_{h_x} u\|_{L^2(Q)} \leq C h_x^2 \|u\|_{L^2(0,T;H^2(\Omega))}$$

with a constant $C > 0$, see (2.48)*, then, for the unique solution $u_h \in Q^1_h(Q) \cap H^{1,1/2}_{0;0,}(Q)$ of the Galerkin-Bubnov finite element discretisation* (3.76)*, there hold the error estimates*

$$\|u - u_h\|_{H^{1/2}_{0,}(0,T;L^2(\Omega)),F} \leq c\, h^{3/2}$$

and

$$\|u - u_h\|_{L^2(Q)} \leq c\, h^2$$

with a constant $c > 0$ independent of the mesh size $h = \max\{h_t, h_x\}$.

Corollary 3.4.26. *Let the assumption of Theorem 3.4.24 be satisfied. Furthermore, let $u \in H^{1+s}(Q) \cap H^{1,1}_{0;0,}(Q)$ be for some $s \in [0,1]$ and let the $H^{1,1}_{0;0,}(Q)$ projection Q^1_h, given in* (2.45)*, fulfil the standard error estimate*

$$\|u - Q^1_h u\|_{L^2(Q)} \leq c\, h^{1+s} \|u\|_{H^{1+s}(Q)}$$

with a constant $c > 0$, see (2.46). Moreover, assume for $Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ the inverse inequality

$$|v_h|_{H^1(Q)} \leq c_{\text{inv}}\, h^{-1} \|v_h\|_{L^2(Q)} \quad \forall v_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$$

with a constant $c_{\text{inv}} > 0$ and $h = \max\{h_t, h_x\}$. Then it holds

$$|u - u_h|_{H^1(Q)} \leq C\, c_{\text{inv}}\, h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}}\, h^{-1} \|u_h - u\|_{L^2(Q)}$$

with a constant $C > 0$ independent of h. If, in addition, the assumption of Corollary 3.4.25 is fulfilled, then it holds the error estimate

$$|u - u_h|_{H^1(Q)} \leq \tilde{C}\, h$$

with a constant $\tilde{C} > 0$.

Proof. Note that $Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q) = Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$. It follows with the triangle inequality, standard error estimates for Q_h^1 and the inverse inequality in $Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ that

$$\begin{aligned}
|u - u_h|_{H^1(Q)} &\leq \left|u - Q_h^1 u\right|_{H^1(Q)} + \left|Q_h^1 u - u_h\right|_{H^1(Q)} \\
&\leq \tilde{C} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}}\, h^{-1} \left\|Q_h^1 u - u_h\right\|_{L^2(Q)} \\
&\leq \tilde{C} h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}}\, h^{-1} \left\|Q_h^1 u - u\right\|_{L^2(Q)} + c_{\text{inv}}\, h^{-1} \|u_h - u\|_{L^2(Q)} \\
&\leq C\, c_{\text{inv}}\, h^s \|u\|_{H^{s+1}(Q)} + c_{\text{inv}}\, h^{-1} \|u_h - u\|_{L^2(Q)}
\end{aligned}$$

with a constant $C > 0$ and hence, the assertion. □

Remark 3.4.27. *The assumptions on the spatial H_0^1 projection $Q_{h_x}^1$ and on the $H^{1,1}_{0;0,}(Q)$ projection Q_h^1 in Corollary 3.4.25 and Corollary 3.4.26 are fulfilled, if Ω is sufficiently regular. Thus, for less regular Ω, one expects reduced orders for the error estimates given in Corollary 3.4.25 and Corollary 3.4.26.*

In the last part of this section, some numerical examples for the Galerkin-Bubnov variational formulation (3.76) are presented. Therefore, consider the space-time cylinder $Q = \Omega \times (0,T) = (0,1) \times (0,2)$ for the finite element space $Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$ with a uniform discretisation with mesh sizes $h_x = \frac{1}{N_x}$ and $h_t = \frac{2}{N_t}$ with $N_x = N_t = 2^j$, $j = 1, \ldots, 6$, see Section 2.8 for the notations. The number of the degrees of freedom is given as

$$\text{dof} = (N_x - 1) \cdot N_t.$$

The temporal transformation $\mathcal{H}_T$ of the nodal basis functions of the finite-dimensional space $Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$ is realised via the series representation (3.67) and the appearing integrals in (3.76) are calculated by the usage of high-order integration rules. In Table 3.10

N_x	N_t	dof	h_x	h_t	$\sigma_{\max}(A_h)$	$\sigma_{\min}(A_h)$	$\kappa_2(A_h)$
2	2	2	0.5000000	1.0000000	2.27730420	0.90807052	2.51
4	4	12	0.2500000	0.5000000	5.67006465	0.28650743	19.79
8	8	56	0.1250000	0.2500000	7.29533888	0.11531994	63.26
16	16	240	0.0625000	0.1250000	7.80822886	0.04567559	170.95
32	32	992	0.0312500	0.0625000	7.94997290	0.01642706	483.96
64	64	4032	0.0156250	0.0312500	7.98720088	0.00472216	1691.43

Table 3.10: Stability behaviour of the Galerkin-Bubnov finite element discretisation (3.76) with uniform meshes for the space-time cylinder $Q = (0,1) \times (0,2)$.

the minimal and maximal singular values of the system matrix A_h, corresponding to (3.76), as well as the resulting spectral condition number of A_h are given. Note that the finite element stiffness matrix A_h is still positive definite, but not symmetric, and that no CFL condition is needed, see Theorem 3.4.20.

Remark 3.4.28. *The to* (3.76) *related inf-sup constant*

$$\inf_{0\neq u_h\in Q_h^1(Q)\cap H^{1,1/2}_{0;0,}(Q)} \sup_{0\neq v_h\in Q_h^1(Q)\cap H^{1,1/2}_{0;0,}(Q)} \frac{a(u_h,\mathcal{H}_T v_h)}{\|u_h\|_{H^{1,1/2}_{0;0,}(Q),F}\|v_h\|_{H^{1,1/2}_{0;0,}(Q),F}} =: c_S(h) \tag{3.81}$$

seems to depend at least linearly on the mesh size h. As illustration, consider the inf-sup constant

$$\inf_{0\neq u_h\in Q_h^1(Q)\cap H^{1,1/2}_{0;0,}(Q)} \sup_{0\neq v_h\in Q_h^1(Q)\cap H^{1,1/2}_{0;0,}(Q)} \frac{a(u_h,\mathcal{H}_T v_h)}{\|\nabla_x u_h\|_{L^2(Q)}\|\nabla_x v_h\|_{L^2(Q)}} =: \tilde{c}_S(h), \tag{3.82}$$

satisfying

$$\tilde{c}_S(h) \geq c_S(h).$$

The inf-sup constant $\tilde{c}_S(h)$ is given as

$$\tilde{c}_S(h) = \sqrt{\lambda_{\min}},$$

where $\lambda_{\min}$ is the minimal eigenvalue of the generalised eigenvalue problem [84, Subsection 3.6.6, page 124]

$$A_h^\top K_{h,0,}^{-1} A_h \underline{u} = \lambda K_{h,0,} \underline{u}$$

with the matrices

$$A_h[k,i] := a(\chi_i, \mathcal{H}_T \chi_k), \qquad K_{h,0,}[k,i] := \langle \nabla_x \chi_i, \nabla_x \chi_k \rangle_{L^2(Q)} \qquad \text{for } i,k = 1,\dots,\text{dof},$$

where χ_i are the nodal basis functions of $Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q)$, i.e.

$$Q_h^1(Q) \cap H^{1,1/2}_{0;0,}(Q) = \operatorname{span}\{\chi_i\}_{i=1}^{\text{dof}}.$$

For a uniform discretisation with mesh sizes $h_x = \frac{1}{N_x}$ *and* $h_t = \frac{2}{N_t}$ *with* $N_x = N_t = 2^j$, $j = 1,\dots,6$, *and* $\mathrm{dof} = (N_x - 1)\cdot N_t$, *the inf-sup constant* $\tilde{c}_S(h)$ *of* (3.82) *is given in Table 3.11, where a linear dependency is observed. Hence, the inf-sup constant* $c_S(h)$ *of* (3.81) *seems to depend at least linearly on the mesh size* $h = \max\{h_t, h_x\}$.

N_x	N_t	dof	h_x	h_t	$\tilde{c}_S(h)$
2	2	2	0.500000	1.000000	0.673637
4	4	12	0.250000	0.500000	0.375800
8	8	56	0.125000	0.250000	0.216577
16	16	240	0.062500	0.125000	0.117912
32	32	992	0.031250	0.062500	0.061679
64	64	4032	0.015625	0.031250	0.031558

Table 3.11: Optimal discrete inf-sup constant $\tilde{c}_S(h)$ of (3.82) with a uniform temporal mesh size h_t and a uniform spatial mesh size h_x for the space-time cylinder $Q = (0,1)\times(0,2)$.

Next, numerical examples concerning convergence rates are given. Therefore, consider the functions

$$u_1(x,t) := \sin\left(\frac{5\pi}{4}t\right)\sin(\pi x),$$
$$u_2(x,t) := t^{2/3}\sin(\pi x),$$
$$u_3(x,t) := x^{3/5}(x-1)t$$

for $(x,t) \in Q = \Omega \times (0,T) = (0,1)\times(0,2)$ as the solutions of (3.73). Since the solution u_1 is smooth, a quadratic convergence in $L^2(Q)$ and a linear convergence in $H^1(Q)$ are expected by Corollary 3.4.25 and by Corollary 3.4.26. This behaviour is confirmed by the numerical results given in Table 3.12.

The solutions u_2 and u_3 satisfy only $u_2 \in H^{7/6-\varepsilon}(Q)$ and $u_3 \in H^{11/10-\varepsilon}(Q)$ for $\varepsilon > 0$, which leads to a reduced order of convergence in $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$, see Table 3.13 and Table 3.14.

N_x	N_t	dof	h_x	h_t	$\|u_1 - u_{1,h}\|_{L^2(Q)}$	eoc	$\|u_1 - u_{1,h}\|_{H^1(Q)}$	eoc
2	2	2	0.500000	1.000000	0.91080532	-	4.48436523	-
4	4	12	0.250000	0.500000	0.15774388	2.50	1.89083082	1.20
8	8	56	0.125000	0.250000	0.02936086	2.40	0.84239378	1.20
16	16	240	0.062500	0.125000	0.00689501	2.10	0.41495910	1.00
32	32	992	0.031250	0.062500	0.00169574	2.00	0.20679363	1.00
64	64	4032	0.015625	0.031250	0.00042203	2.00	0.10331237	1.00

Table 3.12: Numerical results of the Galerkin-Bubnov finite element discretisation (3.76) with uniform meshes for the space-time cylinder $Q = (0,1) \times (0,2)$ and for the function u_1.

N_x	N_t	dof	h_x	h_t	$\|u_2 - u_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}\|_{H^1(Q)}$	eoc
2	2	2	0.500000	1.000000	0.23792432	-	1.54418611	-
4	4	12	0.250000	0.500000	0.06665335	1.80	0.85581404	0.85
8	8	56	0.125000	0.250000	0.02005423	1.70	0.51060465	0.75
16	16	240	0.062500	0.125000	0.00693925	1.50	0.35279809	0.53
32	32	992	0.031250	0.062500	0.00275227	1.30	0.27858415	0.34
64	64	4032	0.015625	0.031250	0.00118292	1.20	0.23695846	0.23

Table 3.13: Numerical results of the Galerkin-Bubnov finite element discretisation (3.76) with uniform meshes for the space-time cylinder $Q = (0,1) \times (0,2)$ and for the function u_2.

N_x	N_t	dof	h_x	h_t	$\|u_3 - u_{3,h}\|_{L^2(Q)}$	eoc	$\|u_3 - u_{3,h}\|_{H^1(Q)}$	eoc
2	2	2	0.500000	1.000000	0.15401734	-	1.57948940	-
4	4	12	0.250000	0.500000	0.06178342	1.30	1.37041171	0.20
8	8	56	0.125000	0.250000	0.02645880	1.20	1.23225873	0.15
16	16	240	0.062500	0.125000	0.01181625	1.20	1.12878387	0.13
32	32	992	0.031250	0.062500	0.00542705	1.10	1.04367032	0.11
64	64	4032	0.015625	0.031250	0.00254779	1.10	0.96942251	0.11

Table 3.14: Numerical results of the Galerkin-Bubnov finite element discretisation (3.76) with uniform meshes for the space-time cylinder $Q = (0,1) \times (0,2)$ and for the function u_3.

4 WAVE EQUATION

The main focus of this chapter are space-time variational formulations and conforming discretisations for hyperbolic problems. First, a highly non-exhaustive list of references and second, an overview of the sections of this chapter are given, where for each section the relevant literature is cited. Here, the model problem for a hyperbolic partial differential equation is the homogeneous Dirichlet problem for the wave equation,

$$\left.\begin{array}{rcll} \partial_{tt}u(x,t) - \Delta_x u(x,t) &=& f(x,t) & \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &=& 0 & \text{for } (x,t) \in \Sigma = \Gamma \times [0,T], \\ u(x,0) &=& u_0(x) & \text{for } x \in \Omega, \\ \partial_t u(x,0) &=& v_0(x) & \text{for } x \in \Omega, \end{array}\right\} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, $T > 0$ is a finite time, u_0, v_0 are given initial conditions and f is a given right-hand side. To compute an approximate solution of the wave equation (4.1), different numerical schemes including different approaches of the underlying mathematical framework are available. On the one hand, some of them are repeated in this chapter, but on the other hand, powerful tools like semigroup theory as in [91, 123] on the continuous part or on the discretisation side, any kind of discontinuous Galerkin methods [33, 42, 67, 86, 88, 89, 111, 130, 140] or finite difference methods [32, 33, 65, 97, 147] or boundary element methods, see Remark 1.2.1, are not in the scope of this work. Furthermore, all approaches where the wave equation (4.1) is reformulated as a first-order system in the spatial and/or time variables are excluded in this work, see [19, 22, 24, 42, 45, 46, 55, 83, 94, 163]. In addition, see also the approaches in [53, 63, 66, 80, 81, 85, 114, 115, 124].

Outline of Chapter 4

The remainder of this chapter examines the wave equation (4.1) as follows:

In Section 4.1 a pointwise spatial variational formulation coming from a so-called Galerkin method [36,97,102,160,162] and time stepping schemes [18,22,27,28,44,95,113,126,127] are cited. In Section 4.2 a space-time variational formulation [97] in a subspace of $H^1(Q)$ is examined, which fits very well to finite element methods with piecewise linear, continuous functions. This conforming finite element method is only conditionally stable, i.e. a CFL condition plays a decisive role for stability. To gain a deeper understanding of the CFL condition, an ordinary differential equation corresponding to the wave equation is analysed. For this ordinary differential equation, an unconditionally stable numerical scheme is introduced. By transferring this idea to the wave equation, a stabilised space-time finite element method for the wave equation is obtained. This stabilisation is proposed in [164].

Finally, for this stabilised space-time finite element method, $L^2(Q)$ stability, $L^2(Q)$ and $H^1(Q)$ error estimates are proven, see [146]. In the last part of Section 4.2, numerical examples for a one-dimensional spatial domain and a two-dimensional spatial domain are given, including spatially adaptive refined meshes. In Section 4.3 and Section 4.4, existence and uniqueness results for the wave equation as a partial differential equation in $L^2(Q)$ and in a weaker sense than $L^2(Q)$ are derived, including isomorphic solution operators and corresponding inf-sup conditions.

4.1 Variational Formulation in Space and Pointwise in Time

In this section, a short overview of a pointwise in time variational formulation is given. Furthermore, numerical examples for lowest order in space, i.e. piecewise linear, continuous ansatz functions, combined with lowest order time stepping are presented.

The pointwise in time variational formulation of (4.1) is given, with the notations of Section 2, as follows:

Find $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(Q)$, $\partial_{tt} u \in L^2(0,T;[H_0^1(\Omega)]')$, $u(\cdot,0) = u_0$ in $H_0^1(\Omega)$ and $\partial_t u(\cdot,0) = v_0$ in $L^2(\Omega)$ such that

$$\langle \partial_{tt} u(\cdot,t), v\rangle_\Omega + \langle \nabla_x u(\cdot,t), \nabla_x v\rangle_{L^2(\Omega)} = \langle f(\cdot,t), v\rangle_{L^2(\Omega)} \tag{4.2}$$

for almost all $t \in (0,T)$ and all $v \in H_0^1(\Omega)$, where $f \in L^2(Q)$, $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$ are the given right-hand side and the given initial conditions. Here, ∂_{tt} is the distributional derivative on $(0,T)$, i.e. equality (4.2) means that it holds

$$\int_0^T \langle u(\cdot,t), v\rangle_\Omega \frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}(t)\mathrm{d}t + \int_0^T \langle \nabla_x u(\cdot,t), \nabla_x v\rangle_{L^2(\Omega)} \varphi(t)\mathrm{d}t = \int_0^T \langle f(\cdot,t), v\rangle_\Omega \varphi(t)\mathrm{d}t$$

for all $\varphi \in C_0^\infty(0,T)$. The variational formulation in (4.2) is examined in many books, for example, [102, Théorème 8.1, Chapitre 3, page 287, and Théorème 8.2, Chapitre 3, page 296], [97, Theorem 4.2, Chapter IV, page 167], [160, Satz 29.1, Kapitel V, page 422], [162, Section 24.1, Chapter 24, page 453] or [36, Mathematical Example 1, Chapter XVIII, page 581]. In these books, the following existence and uniqueness result is proven.

Theorem 4.1.1. *For given $f \in L^2(Q)$, $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$ exists a unique solution u of the variational formulation* (4.2)*, satisfying*

$$\begin{aligned} u &\in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];H_0^1(\Omega)), \\ \partial_t u &\in L^2(Q) \cap C([0,T];L^2(\Omega)), \\ \partial_{tt} u &\in L^2(0,T;[H_0^1(\Omega)]'), \end{aligned}$$

and the stability estimate

$$\sqrt{\|u\|^2_{L^2(0,T;H_0^1(\Omega))} + \|\partial_t u\|^2_{L^2(Q)}} \le c\left(|u_0|_{H^1(\Omega)} + \|v_0\|_{L^2(\Omega)} + \|f\|_{L^2(Q)}\right)$$

with a constant $c > 0$.

Proof. See the books [36, 97, 102, 160, 162] as mentioned above. □

For a discretisation scheme, let the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ be an interval $\Omega = (0,L)$ for $d = 1$, or polygonal for $d = 2$, or polyhedral for $d = 3$. With the notations of Section 2.8, consider a discretisation of a tensor-product type (2.41) with the finite-dimensional space $Q_h^1(Q) = V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0,T)$, see (2.42). Therefore, introduce for $x \in \Omega$ and $\ell \in \{0,\ldots,N_t\}$ the approximations

$$U_{h_x,\ell}(x) := \sum_{i=1}^{M_x} U_i^\ell \psi_i(x) \approx u(x,t_\ell)$$

and

$$\hat{U}_{h_x,\ell}(x) := \sum_{i=1}^{M_x} \hat{U}_i^\ell \psi_i(x) \approx \partial_t u(x,t_\ell),$$

where $U_i^\ell, \hat{U}_i^\ell \in \mathbb{R}$ are the unknown coefficients of $U_{h_x,\ell}, \hat{U}_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0,\ldots,N_t\}$. Furthermore, set for $(x,t) \in Q$

$$u_h(x,t) := \sum_{\ell=0}^{N_t} \sum_{i=1}^{M_x} U_i^\ell \psi_i(x) \varphi_\ell(t) = \sum_{\ell=0}^{N_t} U_{h_x,\ell}(x) \varphi_\ell(t) \approx u(x,t) \tag{4.3}$$

and

$$\hat{u}_h(x,t) := \sum_{\ell=0}^{N_t} \sum_{i=1}^{M_x} \hat{U}_i^\ell \psi_i(x) \varphi_\ell(t) = \sum_{\ell=0}^{N_t} \hat{U}_{h_x,\ell}(x) \varphi_\ell(t) \approx \partial_t u(x,t), \tag{4.4}$$

i.e. $u_h, \hat{u}_h \in Q_h^1(Q)$.

For the pointwise in time variational formulation (4.2), a conforming discretisation in space with $V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ in combination with the Newmark scheme with Newmark parameters $\beta = \frac{1}{4}, \ \gamma = \frac{1}{2}$ leads to the so-called Newmark Galerkin method to find the functions $U_{h_x,\ell}, \hat{U}_{h_x,\ell} \in V_{h_x,0}(\Omega) \subset H_0^1(\Omega)$ for $\ell \in \{0,\ldots,N_t\}$ such that

$$U_{h_x,0} = Q_{h_x} u_0, \quad \hat{U}_{h_x,0} = Q_{h_x} v_0,$$

and for $\ell = 1,\ldots,N_t$

$$\begin{aligned}\frac{1}{h_{t,\ell}^2}\langle U_{h_x,\ell} - U_{h_x,\ell-1} + h_{t,\ell}\hat{U}_{h_x,\ell-1}, v_{h_x}\rangle_{L^2(\Omega)} + \frac{1}{4}\langle \nabla_x U_{h_x,\ell} + \nabla_x U_{h_x,\ell-1}, \nabla_x v_{h_x}\rangle_{L^2(\Omega)} \\ = \frac{1}{4}\langle f(\cdot,t_\ell) + f(\cdot,t_{\ell-1}), v_{h_x}\rangle_{L^2(\Omega)}\end{aligned} \tag{4.5}$$

for all $v_{h_x} \in V_{h_x,0}(\Omega)$ and

$$\begin{aligned}\frac{1}{h_{t,\ell}}\langle \hat{U}_{h_x,\ell} - \hat{U}_{h_x,\ell-1}, \hat{v}_{h_x}\rangle_{L^2(\Omega)} + \frac{1}{2}\langle \nabla_x U_{h_x,\ell} + \nabla_x U_{h_x,\ell-1}, \nabla_x \hat{v}_{h_x}\rangle_{L^2(\Omega)} \\ = \frac{1}{2}\langle f(\cdot,t_\ell) + f(\cdot,t_{\ell-1}), \hat{v}_{h_x}\rangle_{L^2(\Omega)}\end{aligned} \tag{4.6}$$

for all $\hat{v}_{h_x} \in V_{h_x,0}(\Omega)$, where $Q_{h_x} : L^2(\Omega) \to V_{h_x,0}(\Omega)$ denotes the L^2 projection (2.39). This method is given in [127, (8.6-4), (8.6-5), (8.6-6), page 205], see also [18, 22, 27, 28, 44, 95, 113, 126]. The Newmark Galerkin method (4.5), (4.6) is equivalent to the linear systems

$$M_{h_x}\underline{U}^0 = \underline{u}_0, \quad M_{h_x}\underline{\hat{U}}^0 = \underline{v}_0,$$

and for all $\ell = 1, \ldots, N_t$

$$\left(M_{h_x} + \frac{1}{4}h_{t,\ell}^2 A_{h_x}\right)\underline{U}^\ell = \left(M_{h_x} - \frac{1}{4}h_{t,\ell}^2 A_{h_x}\right)\underline{U}^{\ell-1} + h_{t,\ell}M_{h_x}\underline{\hat{U}}^{\ell-1} + \frac{1}{4}h_{t,\ell}^2\left(\underline{F}^\ell + \underline{F}^{\ell-1}\right), \tag{4.7}$$

$$M_{h_x}\underline{\hat{U}}^\ell = M_{h_x}\underline{\hat{U}}^{\ell-1} - \frac{1}{2}h_{t,\ell}A_{h_x}\left(\underline{U}^\ell + \underline{U}^{\ell-1}\right) + \frac{1}{2}h_{t,\ell}\left(\underline{F}^\ell + \underline{F}^{\ell-1}\right), \tag{4.8}$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the mass matrix (2.37), $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ is the stiffness matrix (2.38) and the vectors $\underline{u}_0, \underline{v}_0, \underline{F}^\ell \in \mathbb{R}^{M_x}$ are defined by

$$\underline{u}_0[i] := \langle u_0, \psi_i \rangle_{L^2(\Omega)}, \quad \underline{v}_0[i] := \langle v_0, \psi_i \rangle_{L^2(\Omega)}, \quad \underline{F}^\ell[i] := \langle f(\cdot, t_\ell), \psi_i \rangle_{L^2(\Omega)} \tag{4.9}$$

for $i = 1, \ldots, M_x$ with the nodal basis functions ψ_i satisfying $V_{h_x,0}(\Omega) = \operatorname{span}\{\psi_i\}_{i=1}^{M_x}$. The matrix $M_{h_x} + \frac{1}{4}h_{t,\ell}^2 A_{h_x}$ is positive definite and hence, the linear systems (4.7), (4.8) are uniquely solvable for all $\ell = 1, \ldots, N_t$. Stability of the numerical scheme (4.5), (4.6) holds without any CFL condition because the Newmark method is unconditionally stable, see [22, 95, 113, 127]. Concerning error estimates, it seems that error estimates of the quantities $\|u(\cdot, t_\ell) - U_{h_x,\ell}\|_{L^2(\Omega)}$ for each $\ell = 0, \ldots, N_t$ are standard, which are of optimal order $\mathcal{O}(h_t^2 + h_x^2)$, see [127, Chapitre 8], [113, Section 4.2] or [18, 27, 28, 44]. However, here, error estimates in space-time norms $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ for the approximate solution $u_h \in Q_h^1(Q)$, defined in (4.3), are considered. It seems that proofs of such error estimates are not available. Hence, the proofs of such statements are left for future work.

In the last part of this section, some numerical examples are presented. So, for the space-time cylinder $Q = \Omega \times (0, T) = (0, 1) \times (0, 10)$, consider the solution of (4.2)

$$u_1(x, t) = \sin(\pi x)\sin^2\left(\frac{5}{4}\pi t\right), \quad (x, t) \in Q,$$

see also the numerical examples of Section 4.2 for a comparison. The discretisation is done with respect to nonuniform meshes as shown in Figure 4.1, where a uniform refinement strategy is applied. The appearing integrals for the initial conditions and right-hand side in (4.9) are calculated by the usage of high-order integration rules, and the degrees of freedom are denoted by

$$\mathrm{dof} = \dim Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q)$$

due to the homogeneous Dirichlet boundary condition and the initial conditions, see (2.14). In Table 4.1 the errors in the space-time norms $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ are presented for the smooth function u_1, where the convergence rates are as expected. Note that no CFL condition is needed because the Newmark method is unconditionally stable.

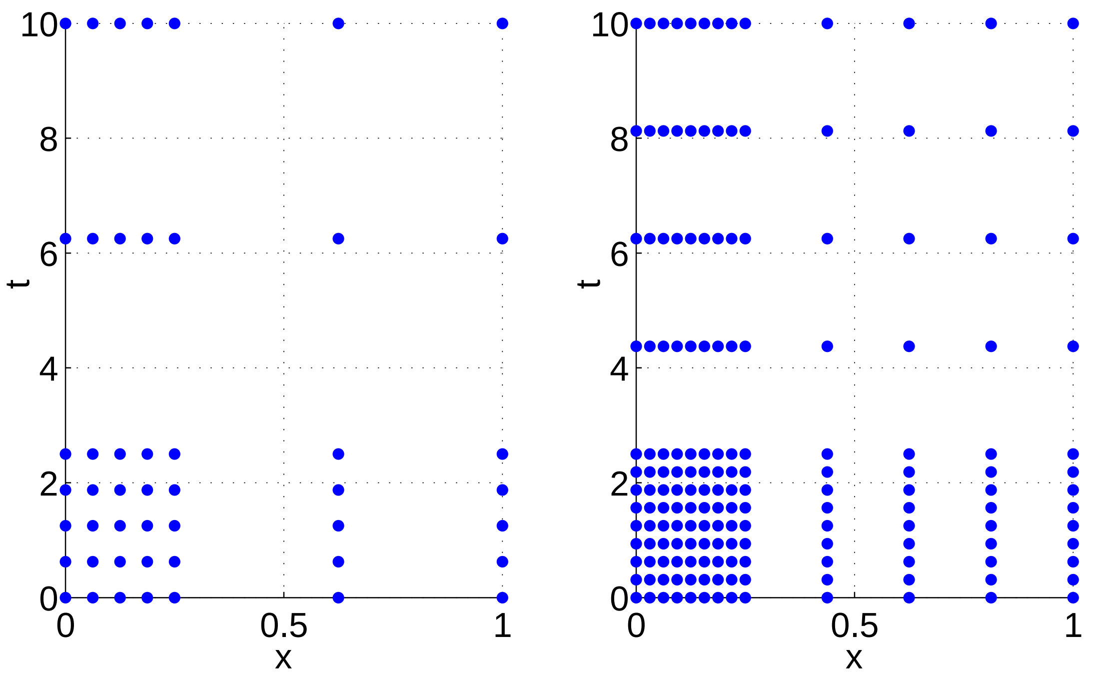

Figure 4.1: Nonuniform meshes: Starting mesh and the mesh after one uniform refinement step.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_1-u_{1,h}\|_{L^2(Q)}$	eoc	$\|u_1-u_{1,h}\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	7.088e+00	-	2.622e+01	-
132	0.18750	0.03125	1.87500	0.31250	2.921e+00	1.20	1.276e+01	0.97
552	0.09375	0.01562	0.93750	0.15625	4.728e+00	-0.67	1.811e+01	-0.49
2256	0.04688	0.00781	0.46875	0.07812	1.670e+00	1.48	8.558e+00	1.06
9120	0.02344	0.00391	0.23438	0.03906	6.701e-01	1.31	4.571e+00	0.90
36672	0.01172	0.00195	0.11719	0.01953	2.023e-01	1.72	1.813e+00	1.33
147072	0.00586	0.00098	0.05859	0.00977	5.326e-02	1.92	7.733e-01	1.23
589056	0.00293	0.00049	0.02930	0.00488	1.349e-02	1.98	3.652e-01	1.08
2357760	0.00146	0.00024	0.01465	0.00244	3.384e-03	1.99	1.797e-01	1.02
9434112	0.00073	0.00012	0.00732	0.00122	8.467e-04	2.00	8.950e-02	1.01
37742592	0.00037	0.00006	0.00366	0.00061	2.117e-04	2.00	4.471e-02	1.00

Table 4.1: Numerical results of the Newmark Galerkin method (4.5), (4.6) with nonuniform meshes for $Q=(0,1)\times(0,10)$ and for the function u_1.

Remark 4.1.2. *For a solution of* (4.2)

$$u_2(x,t) = \sin(\pi x)t^2(10-t)^{3/4}, \quad (x,t) \in (0,1) \times (0,10),$$

as considered as numerical example in Section 4.2, the Newmark Galerkin method (4.5)*,* (4.6) *is not applicable because the corresponding right-hand side f has a singularity with respect to $t = T = 10$. Hence, an alternative is to adapt the treatment of the right-hand side f.*

Remark 4.1.3. *The Newmark Galerkin method* (4.5)*,* (4.6) *fulfils a conservation of the total energy*

$$E(t) := \frac{1}{2}\|\partial_t u(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla_x u(\cdot,t)\|^2_{L^2(\Omega)}, \quad t \in [0,T],$$

see [22, 95]. As illustration, consider a solution of the homogeneous wave equation, i.e.

$$u_3(x,t) = (\cos(\pi t) + \sin(\pi t))\sin(\pi x), \quad (x,t) \in Q = (0,1) \times (0,10),$$

with the total energy

$$E(t) = \frac{\pi^2}{2}, \quad t \in [0,10].$$

Here, the initial conditions are

$$u_3(x,0) = u_0(x) = \sin(\pi x), \quad \partial_t u_3(x,0) = v_0(x) = \pi \sin(\pi x), \quad x \in \Omega.$$

For the solution u_3 and for the mesh as given in Figure 4.1, the discrete total energy

$$E_h(t) := \frac{1}{2}\|\hat{u}_h(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla_x u_h(\cdot,t)\|^2_{L^2(\Omega)}, \quad t \in [0,T],$$

is computed, where the approximation $\hat{u}_h \approx \partial_t u$, given in (4.4)*, is used. In Figure 4.2 the difference*

$$E_h(t) - E(t) = E_h(t) - \frac{\pi^2}{2}, \quad t \in [0,10],$$

is plotted pointwise for the approximate solution $u_{3,h}$ of the finest level of Table 4.2, where a conservation of the total energy is observed. In addition, the errors in the space-time norms $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ are given in Table 4.2, where the convergence rates are as expected.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_3 - u_{3,h}\|_{L^2(Q)}$	eoc	$\|u_3 - u_{3,h}\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	2.561e+00	-	1.066e+01	-
132	0.18750	0.03125	1.87500	0.31250	2.503e+00	0.03	9.821e+00	0.11
552	0.09375	0.01562	0.93750	0.15625	2.243e+00	0.15	9.780e+00	0.01
2256	0.04688	0.00781	0.46875	0.07812	2.581e+00	-0.20	1.152e+01	-0.23
9120	0.02344	0.00391	0.23438	0.03906	1.082e+00	1.25	4.845e+00	1.24
36672	0.01172	0.00195	0.11719	0.01953	3.012e-01	1.84	1.444e+00	1.74
147072	0.00586	0.00098	0.05859	0.00977	7.695e-02	1.97	4.605e-01	1.65
589056	0.00293	0.00049	0.02930	0.00488	1.933e-02	1.99	1.804e-01	1.35
2357760	0.00146	0.00024	0.01465	0.00244	4.839e-03	2.00	8.268e-02	1.12
9434112	0.00073	0.00012	0.00732	0.00122	1.210e-03	2.00	4.034e-02	1.03
37742592	0.00037	0.00006	0.00366	0.00061	3.026e-04	2.00	2.005e-02	1.01

Table 4.2: Numerical results of the Newmark Galerkin method (4.5), (4.6) with nonuniform meshes for $Q = (0,1) \times (0,10)$ and for the function u_3.

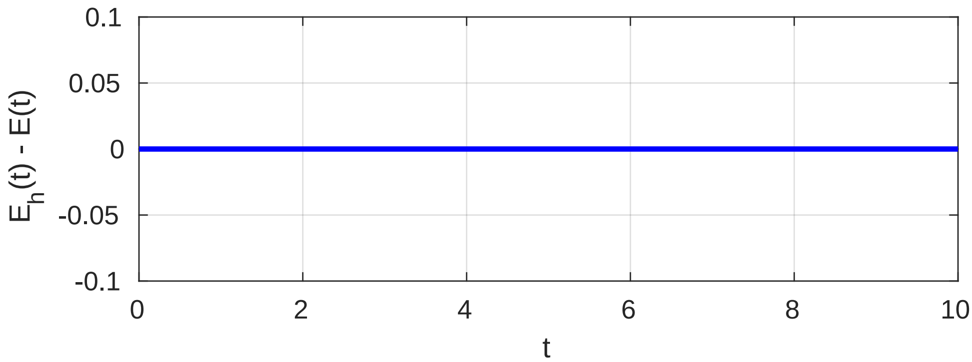

Figure 4.2: Difference of the total energy $E(t) = \frac{\pi^2}{2}$ and $E_h(t)$ of the Newmark Galerkin method (4.5), (4.6) with a nonuniform mesh for $Q = (0,1) \times (0,10)$ and for the function u_3.

4.2 Space-Time Variational Formulation in $H^1(Q)$

In this section, a space-time variational formulation in subspaces of $H^1(Q)$ of the homogeneous Dirichlet problem for the wave equation,

$$\left.\begin{aligned} \partial_{tt}u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q = \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma = \Gamma \times [0,T], \\ u(x,0) = \partial_t u(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned}\right\} \tag{4.10}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, $T > 0$ is a finite time and f is a given right-hand side, is examined and space-time finite element methods with piecewise linear, continuous functions are considered, see [146]. The unique solvability of this space-time variational formulation is given in [97], where the proof of the stability estimate is repeated in this section. A (natural) tensor-product approach by piecewise linear, continuous functions leads to a CFL condition

$$h_t \le C h_x$$

with a constant $C > 0$, where h_t and h_x are the uniform mesh sizes in time and space. To gain a deeper understanding of the CFL condition, an ordinary differential equation corresponding to the wave equation is analysed. For this ordinary differential equation, an unconditionally stable numerical scheme is introduced. By transferring this idea to the wave equation, a stabilised space-time finite element method for the wave equation is obtained. In the last part of this section, unconditional stability in $L^2(Q)$, error estimates in the space-time norms $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$, and numerical examples are given.

4.2.1 Variational Formulation for $\partial_{tt}u + \mu u = f$

As a model problem, consider the second-order linear equation for $\mu > 0$,

$$\partial_{tt}u(t) + \mu\, u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(0) = 0, \tag{4.11}$$

and the variational formulation to find $u \in H^1_{0,}(0,T)$ such that

$$a(u,w) = \langle f, w \rangle_{(0,T)} \tag{4.12}$$

for all $w \in H^1_{,0}(0,T)$, where $T > 0$ and $f \in [H^1_{,0}(0,T)]'$ are given. In (4.12), the bilinear form $a(\cdot,\cdot)\colon H^1_{0,}(0,T) \times H^1_{,0}(0,T) \to \mathbb{R}$ is defined by

$$a(u,w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(0,T)} + \mu \langle u, w \rangle_{L^2(0,T)} \tag{4.13}$$

for all $u \in H^1_{0,}(0,T)$, $w \in H^1_{,0}(0,T)$. Note that $\langle \cdot,\cdot \rangle_{(0,T)}$ denotes the duality pairing as extension of the inner product in $L^2(0,T)$, and the Sobolev spaces $H^1_{0,}(0,T)$, $H^1_{,0}(0,T)$, $[H^1_{,0}(0,T)]'$ are introduced in Section 2.2.

Lemma 4.2.1. *The bilinear form* $a(\cdot,\cdot)\colon\, H^1_{0,}(0,T)\times H^1_{,0}(0,T)\to\mathbb{R}$ *is bounded, i.e. it holds*

$$|a(u,w)| \le \left(1+\frac{4T^2\mu}{\pi^2}\right)|u|_{H^1(0,T)}|w|_{H^1(0,T)} \quad \textit{for all } u\in H^1_{0,}(0,T),\, w\in H^1_{,0}(0,T).$$

Proof. The Cauchy-Schwarz inequality and the Poincaré inequality of Lemma 3.4.5 yield

$$\begin{aligned}|a(u,w)| &\le |u|_{H^1(0,T)}\,|w|_{H^1(0,T)} + \mu\,\|u\|_{L^2(0,T)}\|w\|_{L^2(0,T)}\\ &\le \left(1+\frac{4T^2\mu}{\pi^2}\right)|u|_{H^1(0,T)}\,|w|_{H^1(0,T)}\end{aligned}$$

for $u\in H^1_{0,}(0,T)$, $w\in H^1_{,0}(0,T)$ and therefore, the assertion. □

For $v\in H^1_{0,}(0,T)$, one defines

$$(\overline{\mathcal{H}}_T v)(t) := v(T)-v(t), \quad t\in(0,T), \tag{4.14}$$

i.e. it holds $\overline{\mathcal{H}}_T v\in H^1_{,0}(0,T)$. The operator

$$\overline{\mathcal{H}}_T\colon\, H^1_{0,}(0,T)\to H^1_{,0}(0,T)$$

is norm preserving, i.e.

$$\|v\|_{H^1_{0,}(0,T)} = \|\partial_t v\|_{L^2(0,T)} = \left\|\partial_t\overline{\mathcal{H}}_T v\right\|_{L^2(0,T)} = \left\|\overline{\mathcal{H}}_T v\right\|_{H^1_{,0}(0,T)},$$

and bijective, where the inverse operator

$$\overline{\mathcal{H}}_T^{-1}\colon\, H^1_{,0}(0,T)\to H^1_{0,}(0,T)$$

is given by

$$(\overline{\mathcal{H}}_T^{-1} w)(t) := w(0)-w(t), \quad t\in(0,T),$$

for $w\in H^1_{,0}(0,T)$. Then the variational formulation (4.12) is equivalent to the variational formulation to find $u\in H^1_{0,}(0,T)$ such that

$$a(u,\overline{\mathcal{H}}_T v) = -\langle\partial_t u,\partial_t\overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} + \mu\langle u,\overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} \stackrel{!}{=} \langle f,\overline{\mathcal{H}}_T v\rangle_{(0,T)} \tag{4.15}$$

for all $v\in H^1_{0,}(0,T)$. Note that for the variational formulation (4.15), the ansatz and test spaces are equal. So, the existence and the uniqueness of a solution of (4.15) follow by a compact perturbation argument.

Theorem 4.2.2. *For given* $f \in [H^1_{,0}(0,T)]'$, *the variational formulation* (4.12) *admits a unique solution* $u \in H^1_{0,}(0,T)$, *satisfying*

$$|u|_{H^1(0,T)} \le C(T,\mu)\, \|f\|_{[H^1_{,0}(0,T)]'} \tag{4.16}$$

with a constant $C(T,\mu) > 0$ *depending on* T *and* μ.

Proof. By using the Riesz representation theorem, the variational formulation (4.15) is equivalent to the operator equation

$$\mathcal{A}u + \mu\mathcal{C}u = \hat{f},$$

where $\mathcal{A}\colon H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'$, defined via

$$\langle \mathcal{A}u, v\rangle := -\langle \partial_t u, \partial_t \overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} = \langle \partial_t u, \partial_t v\rangle_{L^2(0,T)} \quad \text{for } u,v \in H^1_{0,}(0,T),$$

is elliptic, and hence, invertible, and $\mathcal{C}\colon H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'$, defined via

$$\langle \mathcal{C}u, v\rangle := \langle u, \overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} = \langle u, -v + v(T)\rangle_{L^2(0,T)} \quad \text{for } u,v \in H^1_{0,}(0,T),$$

is compact, see [12, Satz 5.12, page 148]. Furthermore, the corresponding right-hand side $\hat{f}\colon H^1_{0,}(0,T) \to \mathbb{R}$ is given by

$$\hat{f}(v) := \langle f, \overline{\mathcal{H}}_T v\rangle_{(0,T)}, \quad v \in H^1_{0,}(0,T),$$

satisfying

$$\left|\hat{f}(v)\right| \le \|f\|_{[H^1_{,0}(0,T)]'} \left\|\overline{\mathcal{H}}_T v\right\|_{H^1_{,0}(0,T)} = \|f\|_{[H^1_{,0}(0,T)]'} \|v\|_{H^1_{0,}(0,T)} = \|f\|_{[H^1_{,0}(0,T)]'} |v|_{H^1(0,T)}$$

for all $v \in H^1_{0,}(0,T)$, i.e. $\hat{f} \in [H^1_{0,}(0,T)]'$. Hence, by applying the Fredholm alternative, it remains to ensure the injectivity of $\mathcal{A} + \mu\mathcal{C}$. Let $u \in H^1_{0,}(0,T)$ be a solution of the homogeneous equation $(\mathcal{A} + \mu\mathcal{C})u = 0$, i.e.

$$\langle \partial_t u, \partial_t w\rangle_{L^2(0,T)} = \mu \langle u, w\rangle_{L^2(0,T)} \quad \text{for all } w \in H^1_{,0}(0,T).$$

This is the weak formulation of the eigenvalue problem

$$-\partial_{tt} u(t) = \mu\, u(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(0) = 0,$$

which only admits the trivial solution $u \equiv 0$. □

To examine the dependency of the constant $C(T,\mu)$ in (4.16) on T and μ, the continuous bilinear form $a(\cdot,\cdot)\colon H^1_{0,}(0,T) \times H^1_{,0}(0,T) \to \mathbb{R}$ is investigated. Hence, some basic properties of the bilinear form $a(\cdot,\cdot)\colon H^1_{0,}(0,T) \times H^1_{,0}(0,T) \to \mathbb{R}$ are shown.

Lemma 4.2.3. *For the bilinear form* $a(\cdot,\cdot)\colon\ H^1_{0,}(0,T)\times H^1_{,0}(0,T)\to\mathbb{R}$*, there holds the inf-sup condition*

$$\frac{2}{2+\sqrt{\mu}T}\,|u|_{H^1(0,T)} \le \sup_{0\neq w\in H^1_{,0}(0,T)} \frac{|a(u,w)|}{|w|_{H^1(0,T)}} \quad \textit{for all } u\in H^1_{0,}(0,T).$$

Proof. Let $u\in H^1_{0,}(0,T)$ be fixed and set for $t\in[0,T]$

$$\widetilde{w}(t) := u(T)-u(t)+z(t)$$

with

$$z(t) := \sqrt{\mu}\int_t^T \sin\left(\sqrt{\mu}(s-t)\right)[u(s)-u(T)]\mathrm{d}s.$$

It holds $\widetilde{w}(T)=0$ and $\widetilde{w}\in H^1_{,0}(0,T)$. Differentiation under the integral sign yields

$$\begin{aligned}
\partial_t z(t) &= -\mu\int_t^T \cos\left(\sqrt{\mu}(s-t)\right)[u(s)-u(T)]\mathrm{d}s,\\
\partial_{tt} z(t) &= -\mu^{3/2}\int_t^T \sin\left(\sqrt{\mu}(s-t)\right)[u(s)-u(T)]\mathrm{d}s + \mu[u(t)-u(T)]\\
&= -\mu z(t) + \mu[u(t)-u(T)],
\end{aligned}$$

i.e. the function $z\in H^1_{,0}(0,T)$ is the unique solution of the adjoint equation

$$\partial_{tt} z(t) + \mu z(t) = \mu[u(t)-u(T)] \quad \text{for } t\in(0,T), \quad z(T)=\partial_t z(T)=0.$$

Therefore, it holds

$$\begin{aligned}
a(u,\widetilde{w}) &= a(u,u(T)-u) + a(u,z)\\
&= \langle \partial_t u, \partial_t u\rangle_{L^2(0,T)} + \mu\langle u, u(T)-u\rangle_{L^2(0,T)} + \mu\langle u, u-u(T)\rangle_{L^2(0,T)}\\
&= |u|^2_{H^1(0,T)}.
\end{aligned}$$

Integration by parts gives

$$\partial_t z(t) = -\mu\int_t^T \cos\left(\sqrt{\mu}(s-t)\right)[u(s)-u(T)]\mathrm{d}s = \sqrt{\mu}\int_t^T \sin\left(\sqrt{\mu}(s-t)\right)\partial_t u(s)\mathrm{d}s$$

and so, from the Cauchy-Schwarz inequality follows

$$|z|_{H^1(0,T)} \le \frac{\sqrt{\mu}T}{2}|u|_{H^1(0,T)}.$$

Finally, one gets the estimate

$$|\widetilde{w}|_{H^1(0,T)} \le |u|_{H^1(0,T)} + |z|_{H^1(0,T)} \le \left(1 + \frac{\sqrt{\mu}T}{2}\right)|u|_{H^1(0,T)} = \frac{2+\sqrt{\mu}T}{2}|u|_{H^1(0,T)}$$

to conclude

$$\sup_{0\neq w\in H^1_{,0}(0,T)} \frac{|a(u,w)|}{|w|_{H^1(0,T)}} \ge \frac{|a(u,\widetilde{w})|}{|\widetilde{w}|_{H^1(0,T)}} \ge \frac{2}{2+\sqrt{\mu T}}\,|u|_{H^1(0,T)}$$

and so, the inf-sup condition follows. □

Lemma 4.2.4. *For the bilinear form* $a(\cdot,\cdot)\colon\ H^1_{0,}(0,T)\times H^1_{,0}(0,T)\to\mathbb{R}$*, there holds the surjectivity condition:*

For each function $0\neq w\in H^1_{,0}(0,T)$*, there exists an element* $u\in H^1_{0,}(0,T)$ *with* $a(u,w)\neq 0$*.*

Proof. Let $0\neq w\in H^1_{,0}(0,T)$ be fixed. Set for $t\in[0,T]$

$$\widetilde{u}(t) := \frac{1}{\sqrt{\mu}}\int_0^t w(s)\sin\left(\sqrt{\mu}(t-s)\right)\mathrm{d}s.$$

It follows that $\widetilde{u}\in H^1_{0,}(0,T)$ satisfies (4.11) for the right-hand side $f = w\in H^1_{,0}(0,T)$. Hence, one concludes

$$a(\widetilde{u},w) = \langle w,w\rangle_{L^2(0,T)} = \|w\|^2_{L^2(0,T)} > 0$$

and therefore, the assertion. □

With these properties of the bilinear form $a(\cdot,\cdot)\colon\ H^1_{0,}(0,T)\times H^1_{,0}(0,T)\to\mathbb{R}$, the next existence and uniqueness theorem is proven, including an explicit dependency relation of the constant $C(T,\mu)$ on T and μ.

Theorem 4.2.5. *Let* $f\in[H^1_{,0}(0,T)]'$ *be given. There exists a unique solution* $u\in H^1_{0,}(0,T)$ *of the variational formulation* (4.12)*. Furthermore,*

$$\mathcal{L}\colon\ [H^1_{,0}(0,T)]'\to H^1_{0,}(0,T),\qquad \mathcal{L}f := u,$$

is an isomorphism, satisfying

$$|u|_{H^1(0,T)} = |\mathcal{L}f|_{H^1(0,T)} \le \frac{2+\sqrt{\mu}T}{2}\,\|f\|_{[H^1_{,0}(0,T)]'}. \tag{4.17}$$

Proof. For the Hilbert spaces $\left(H^1_{0,}(0,T), \langle\cdot,\cdot\rangle_{H^1_{0,}(0,T)}\right)$ and $\left(H^1_{,0}(0,T), \langle\cdot,\cdot\rangle_{H^1_{,0}(0,T)}\right)$, use the Nečas Theorem 2.9.1 with the help of the previous lemmata to conclude the existence and uniqueness of a solution $u \in H^1_{0,}(0,T)$ together with the a priori estimate. □

Theorem 4.2.6. *There exist positive constants* c_1, c_2 *and further, a family of functions* $u_{\mu,T} \in H^1_{0,}(0,T)$ *with* $\|\mathcal{L}^{-1}u_{\mu,T}\|_{[H^1_{,0}(0,T)]'} \in \mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ *and* $|u_{\mu,T}|_{H^1(0,T)} \to c_1$ *as* $\mu \to \infty$, *and with* $\|\mathcal{L}^{-1}u_{\mu,T}\|_{[H^1_{,0}(0,T)]'} \in \mathcal{O}\left(\frac{1}{T}\right)$ *and* $|u_{\mu,T}|_{H^1(0,T)} \to c_2$ *as* $T \to \infty$. *In particular, the inequality* (4.17) *is optimal with respect to the order of* μ *and* T.

Proof. The asserted family of elements $u_{\mu,T} \in H^1_{0,}(0,T)$ is given by

$$u_{\mu,T}(t) := \frac{1}{\sqrt{T^3}} \int_0^t s \sin(\sqrt{\mu}s)\,\mathrm{d}s, \quad t \in [0,T],$$

where the initial conditions $u_{\mu,T}(0) = \partial_t u_{\mu,T}(0) = 0$ are fulfilled. One computes

$$|u_{\mu,T}|_{H^1(0,T)} = \sqrt{\frac{1}{6} - \frac{\sin\left(2\sqrt{\mu}T\right)}{4\sqrt{\mu}T} + \frac{\sin\left(2\sqrt{\mu}T\right)}{8\mu^{3/2}T^3} - \frac{\cos\left(2\sqrt{\mu}T\right)}{4\mu T^2}}$$

and hence,

$$|u_{\mu,T}|_{H^1(0,T)} \to \sqrt{\frac{1}{6}} \quad \text{as } \mu \to \infty \quad \text{or as } T \to \infty. \tag{4.18}$$

The corresponding right-hand side is

$$f_{\mu,T}(t) := \partial_{tt} u_{\mu,T}(t) + \mu\, u_{\mu,T}(t) = \frac{2}{\sqrt{T^3}} \sin(\sqrt{\mu}t), \quad t \in (0,T),$$

and by the usage of the Fourier series representation of $\|\cdot\|_{[H^1_{,0}(0,T)]'}$, see Lemma 3.4.4, it follows

$$\begin{aligned}
\|\mathcal{L}^{-1}u_{\mu,T}\|_{[H^1_{,0}(0,T)]'} &= \|f_{\mu,T}\|_{[H^1_{,0}(0,T)]'} \\
&= \sqrt{\frac{T^3}{2} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^{-2} f^2_{\mu,T,k}} \\
&= \sqrt{\frac{6}{\mu T^2} + \frac{\sin\left(2\sqrt{\mu}T\right) - 8\sin\left(\sqrt{\mu}T\right)}{\mu^{3/2}T^3}}.
\end{aligned}$$

So, it holds $\|\mathcal{L}^{-1}u_{\mu,T}\|_{[H^1_{,0}(0,T)]'} \in \mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ as $\mu \to \infty$ and $\|\mathcal{L}^{-1}u_{\mu,T}\|_{[H^1_{,0}(0,T)]'} \in \mathcal{O}\left(\frac{1}{T}\right)$ as $T \to \infty$. □

While for $f \in [H^1_{,0}(0,T)]'$ the bound (4.17) shows an explicit dependence on $\sqrt{\mu}$, an estimate independent of μ is proven, when assuming $f \in L^2(0,T)$.

Lemma 4.2.7. *For a given right-hand side $f \in L^2(0,T)$, the unique solution $u \in H^1_{0,}(0,T)$ of the variational formulation* (4.12) *satisfies*

$$|u|^2_{H^1(0,T)} + \mu \, \|u\|^2_{L^2(0,T)} \le \frac{1}{2} T^2 \, \|f\|^2_{L^2(0,T)}.$$

In addition, the estimate above is optimal with respect to the order of the finite time T and the order of the parameter μ.

Proof. For the solution $u \in H^1_{0,}(0,T)$ and its first-order derivative, there hold the representations

$$u(t) = \frac{1}{\sqrt{\mu}} \int_0^t \sin\left(\sqrt{\mu}(t-s)\right) f(s)\mathrm{d}s, \quad t \in (0,T),$$

and

$$\partial_t u(t) = \int_0^t \cos\left(\sqrt{\mu}(t-s)\right) f(s)\mathrm{d}s, \quad t \in (0,T).$$

With the calculations for $t \in (0,T)$

$$\begin{aligned}
\partial_t u(t)^2 + \mu\, u(t)^2 &= \left[\int_0^t \cos\left(\sqrt{\mu}(t-s)\right) f(s)\mathrm{d}s\right]^2 + \left[\int_0^t \sin\left(\sqrt{\mu}(t-s)\right) f(s)\mathrm{d}s\right]^2 \\
&\le \int_0^t \cos^2\left(\sqrt{\mu}(t-s)\right)\mathrm{d}s \int_0^t f(s)^2\mathrm{d}s + \int_0^t \sin^2\left(\sqrt{\mu}(t-s)\right)\mathrm{d}s \int_0^t f(s)^2\mathrm{d}s \\
&= t \int_0^t f(s)^2\mathrm{d}s \le t \int_0^T f(s)^2\mathrm{d}s,
\end{aligned}$$

there follow

$$\begin{aligned}
|u|^2_{H^1(0,T)} + \mu \, \|u\|^2_{L^2(0,T)} &= \int_0^T \left\{\partial_t u(t)^2 + \mu\, u(t)^2\right\}\mathrm{d}t \\
&\le \int_0^T t\mathrm{d}t \int_0^T f(s)^2\mathrm{d}s = \frac{1}{2} T^2 \, \|f\|^2_{L^2(0,T)},
\end{aligned}$$

and so, the estimate.

For the optimality of this estimate, take the family $u_{\mu,T} \in H^1_{0,}(0,T)$ of the proof of Theorem 4.2.6. The L^2 norm of $u_{\mu,T}$ is

$$\|u_{\mu,T}\|_{L^2(0,T)} = \sqrt{\frac{1}{6\mu} - \frac{5\sin\left(2\sqrt{\mu}T\right)}{8\mu^{5/2}T^3} + \frac{\sin\left(2\sqrt{\mu}T\right)}{4\mu^{3/2}T} + \frac{1}{2\mu^2T^2} + \frac{3\cos\left(2\sqrt{\mu}T\right)}{4\mu^2T^2}},$$

and hence, $\|u_{\mu,T}\|_{L^2(0,T)} \in \mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ as $\mu \to \infty$ and $\|u_{\mu,T}\|_{L^2(0,T)} \to \sqrt{\frac{1}{6\mu}}$ as $T \to \infty$. Furthermore, for the corresponding right-hand side $f_{\mu,T}(t) = \frac{2}{\sqrt{T^3}}\sin(\sqrt{\mu}t)$, the L^2 norm is given by

$$\|f_{\mu,T}\|_{L^2(0,T)} = \sqrt{\frac{2}{T^2} - \frac{\sin(2\sqrt{\mu}T)}{\sqrt{\mu}T^3}}$$

and so, with (4.18), the assertion follows. □

With the notations of Section 2.6, a conforming Galerkin-Bubnov finite element discretisation of the equivalent variational formulation (4.15) is to find a function

$$u_{h_t} \in S^1_{h_t,0,}(0,T) = \operatorname{span}\{\varphi_k\}_{k=1}^{N_t} \subset H^1_{0,}(0,T)$$

such that

$$a(u_{h_t}, \overline{\mathcal{H}}_T v_{h_t}) = -\langle \partial_t u_{h_t}, \partial_t \overline{\mathcal{H}}_T v_{h_t}\rangle_{L^2(0,T)} + \mu\langle u_{h_t}, \overline{\mathcal{H}}_T v_{h_t}\rangle_{L^2(0,T)} \stackrel{!}{=} \langle f, \overline{\mathcal{H}}_T v_{h_t}\rangle_{(0,T)} \quad (4.19)$$

for all $v_{h_t} \in S^1_{h_t,0,}(0,T)$. Unique solvability of (4.19) and related error estimates follow as for the numerical solution of elliptic operator equations with compact perturbations, which is based on a discrete stability condition for a sufficiently small mesh size h_t. Using the optimal Poincaré constant of Lemma 3.4.5, it turns out that for a sufficiently small mesh size

$$h_t \le \frac{\sqrt{3}\pi}{\sqrt{2}(2+\sqrt{\mu}T)\mu T}, \quad (4.20)$$

there holds the discrete stability condition

$$c(\mu,T)\,|u_{h_t}|_{H^1(0,T)} \le \sup_{0\neq v_{h_t}\in S^1_{h_t,0,}(0,T)} \frac{a(u_{h_t}, \overline{\mathcal{H}}_T v_{h_t})}{|v_{h_t}|_{H^1(0,T)}}$$

for all $u_{h_t} \in S^1_{h_t,0,}(0,T)$ with a constant $c(\mu,T) > 0$, see [145] for a proof, implying the error estimate

$$|u - u_{h_t}|_{H^1(0,T)} \le \widetilde{c}(\mu,T)\, h_t\, \|u\|_{H^2(0,T)}$$

with a constant $\widetilde{c}(\mu,T) > 0$, when assuming $u \in H^2(0,T) \cap H^1_{0,}(0,T)$.

Alternatively, the Galerkin-Petrov finite element discretisation of the variational formulation (4.12) is to find $u_{h_t} \in S^1_{h_t,0,}(0,T) = \operatorname{span}\{\varphi_k\}_{k=1}^{N_t} \subset H^1_{0,}(0,T)$ such that

$$a(u_{h_t}, w_{h_t}) = -\langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \mu \langle u_{h_t}, w_{h_t} \rangle_{L^2(0,T)} \stackrel{!}{=} \langle f, w_{h_t} \rangle_{(0,T)} \tag{4.21}$$

for all $w_{h_t} \in S^1_{h_t,,0}(0,T) = \operatorname{span}\{\varphi_k\}_{k=0}^{N_t-1} \subset H^1_{,0}(0,T)$. The related system matrix of (4.21) is given by

$$K_{h_t} = -A_{h_t} + \mu M_{h_t} \in \mathbb{R}^{N_t \times N_t} \tag{4.22}$$

with the stiffness matrix

$$A_{h_t} = \begin{pmatrix} \frac{-1}{h_{t,1}} & & & & \\ \frac{1}{h_{t,1}} + \frac{1}{h_{t,2}} & \frac{-1}{h_{t,2}} & & & \\ \frac{-1}{h_{t,2}} & \frac{1}{h_{t,2}} + \frac{1}{h_{t,3}} & \frac{-1}{h_{t,3}} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{-1}{h_{t,N_t-1}} & \frac{1}{h_{t,N_t-1}} + \frac{1}{h_{t,N_t}} & \frac{-1}{h_{t,N_t}} \end{pmatrix} \tag{4.23}$$

and the mass matrix

$$M_{h_t} = \frac{1}{6} \begin{pmatrix} h_{t,1} & & & & \\ 2h_{t,1} + 2h_{t,2} & h_{t,2} & & & \\ h_{t,2} & 2h_{t,2} + 2h_{t,3} & h_{t,3} & & \\ & \ddots & \ddots & \ddots & \\ & & h_{t,N_t-1} & 2h_{t,N_t-1} + 2h_{t,N_t} & h_{t,N_t} \end{pmatrix}. \tag{4.24}$$

Hence, unique solvability of (4.21) follows because K_{h_t} is a lower triangular matrix with positive diagonal elements.

Remark 4.2.8. *For a uniform discretisation with mesh size h_t, one can interpret the matrix K_{h_t} in* (4.22) *as a finite difference scheme*

$$\begin{aligned} a_0 v_1 &= f_1, \\ a_1 v_1 + a_0 v_2 &= f_2, \\ a_0 v_{j-2} + a_1 v_{j-1} + a_0 v_j &= f_j \quad \textit{for } j > 2, \end{aligned}$$

where

$$a_0 := \frac{1}{h_t} + \frac{\mu h_t}{6} = \frac{6 + \mu h_t^2}{6h_t}, \qquad a_1 := \frac{-2}{h_t} + \frac{4\mu h_t}{6} = \frac{-6 + 2\mu h_t^2}{3h_t}$$

and with given $f_j \in \mathbb{R}$. The essential case $f_j = 0$ for $j > 2$ is examined. The solution of the homogeneous linear recurrence relation

$$a_0 v_{j-2} + a_1 v_{j-1} + a_0 v_j = 0 \quad \textit{for } j > 2$$

is given for $j \geq 1$ by

$$v_j = \begin{cases} A_0 \left(\frac{6-2h_t^2\mu+\sqrt{3}h_t\sqrt{\mu\left(h_t^2\mu-12\right)}}{6+h_t^2\mu} \right)^{j-1} + A_1 \left(\frac{6-2h_t^2\mu-\sqrt{3}h_t\sqrt{\mu\left(h_t^2\mu-12\right)}}{6+h_t^2\mu} \right)^{j-1}, & \text{if } h_t^2\mu > 12, \\ (-1)^{j-1}(A_0 + A_1(j-1)), & \text{if } h_t^2\mu = 12, \\ A_0 \cos\left((j-1)\arccos\frac{6-2h_t^2\mu}{6+h_t^2\mu}\right) + A_1 \sin\left((j-1)\arccos\frac{6-2h_t^2\mu}{6+h_t^2\mu}\right), & \text{if } h_t^2\mu < 12, \end{cases}$$

where the coefficients $A_0, A_1 \in \mathbb{R}$ are determined by $f_1, f_2 \in \mathbb{R}$. Hence, in general, the sequence $(v_j)_{j\in\mathbb{N}}$ is bounded as $j \to \infty$ if and only if

$$h_t^2\mu < 12 \qquad \Longleftrightarrow \qquad h_t < \sqrt{\frac{12}{\mu}}, \tag{4.25}$$

see Table 4.6 for a numerical illustration.

From Remark 4.2.8, one concludes that the numerical scheme (4.21) is only conditionally stable. To overcome the mesh conditions (4.20) or (4.25), the numerical scheme in (4.21) is stabilised. Considering that the following technical lemmata are needed, where the trapezoidal rule is used analogously as in [164, Chapter 2]. In addition to $S_{h_t}^1(0,T)$, also the finite element space $S_{h_t}^0(0,T)$ of piecewise constant functions on the same time mesh is used, see Section 2.6.

Lemma 4.2.9. *For all $f \in L^2(0,T)$, there holds*

$$\partial_t I_{h_t} \int_0^{(\cdot)} f(s)\mathrm{d}s = Q_{h_t}^0 f = \partial_t I_{h_t} \int_T^{(\cdot)} f(s)\mathrm{d}s, \tag{4.26}$$

where $I_{h_t} \colon C[0,T] \to S_{h_t}^1(0,T)$ is the piecewise linear interpolation operator (2.27)*, and $Q_{h_t}^0 \colon L^2(0,T) \to S_{h_t}^0(0,T)$ denotes the L^2 projection* (2.26) *on the piecewise constant finite element space $S_{h_t}^0(0,T)$.*

Proof. On the element $\tau_\ell = (t_{\ell-1}, t_\ell)$, $\ell = 1, \ldots, N_t$, there hold

$$\partial_t I_{h_t} \int_0^{(\cdot)} f(s)\mathrm{d}s = \frac{1}{h_{t,\ell}} \left[\int_0^{t_\ell} f(s)\mathrm{d}s - \int_0^{t_{\ell-1}} f(s)\mathrm{d}s \right] = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} f(s)\mathrm{d}s = Q_{h_t}^0 f$$

and

$$\partial_t I_{h_t} \int_T^{(\cdot)} f(s)\mathrm{d}s = \frac{1}{h_{t,\ell}} \left[\int_T^{t_\ell} f(s)\mathrm{d}s - \int_T^{t_{\ell-1}} f(s)\mathrm{d}s \right] = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} f(s)\mathrm{d}s = Q_{h_t}^0 f.$$

Hence, the assertion follows. $\square$

Lemma 4.2.10. *For all $u_{h_t} \in S^1_{h_t,0,}(0,T)$ and $w_{h_t} \in S^1_{h_t,,0}(0,T)$, there holds the representation*

$$\langle u_{h_t}, w_{h_t}\rangle_{L^2(0,T)} = \frac{1}{12}\sum_{\ell=1}^{N_t} h^2_{t,\ell}\langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(\tau_\ell)} + \langle u_{h_t}, Q^0_{h_t} w_{h_t}\rangle_{L^2(0,T)}, \tag{4.27}$$

where $Q^0_{h_t}\colon L^2(0,T) \to S^0_{h_t}(0,T)$ denotes the L^2 projection (2.26) *on the piecewise constant finite element space $S^0_{h_t}(0,T)$.*

Proof. With the error representation of the trapezoidal rule, one obtains on each finite element τ_ℓ, $\ell = 1,\ldots,N_t$,

$$\begin{aligned} Q^0_{h_t}\int_T^{(\cdot)} w_{h_t}(s)\,\mathrm{d}s &= \frac{1}{h_{t,\ell}}\int_{t_{\ell-1}}^{t_\ell}\int_T^t w_{h_t}(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{1}{2}\left[\int_T^{t_{\ell-1}} w_{h_t}(s)\,\mathrm{d}s + \int_T^{t_\ell} w_{h_t}(s)\,\mathrm{d}s\right] - \frac{h^2_{t,\ell}}{12}\,\partial_t w_{h_t|\tau_\ell} \\ &= Q^0_{h_t} I_{h_t}\int_T^{(\cdot)} w_{h_t}(s)\,\mathrm{d}s - \frac{h^2_{t,\ell}}{12}\,\partial_t w_{h_t|\tau_\ell}. \end{aligned}$$

Further, using integration by parts and (4.26), it follows

$$\begin{aligned} \int_0^T \partial_t u_{h_t}(t)\left(I_{h_t}\int_T^{(\cdot)} w_{h_t}(s)\,\mathrm{d}s\right)(t)\,\mathrm{d}t &= -\int_0^T u_{h_t}(t)\left(\partial_t I_{h_t}\int_T^{(\cdot)} w_{h_t}(s)\,\mathrm{d}s\right)(t)\,\mathrm{d}t \\ &= -\int_0^T u_{h_t}(t)\,Q^0_{h_t} w_{h_t}(t)\,\mathrm{d}t. \end{aligned}$$

With this, by using integration by parts and the local definition of the L^2 projection $Q^0_{h_t}$, one concludes that

$$\langle u_{h_t}, w_{h_t}\rangle_{L^2(0,T)} = \int_0^T u_{h_t}(t)\left(\partial_t\int_T^{(\cdot)} w_{h_t}(s)\,\mathrm{d}s\right)(t)\,\mathrm{d}t = -\int_0^T \partial_t u_{h_t}(t)\left(\int_T^{(\cdot)} w_{h_t}(s)\,\mathrm{d}s\right)(t)\,\mathrm{d}t$$

as well as

$$\begin{aligned}
\langle u_{h_t}, w_{h_t}\rangle_{L^2(0,T)} &= -\sum_{\ell=1}^{N_t} \int\limits_{t_{\ell-1}}^{t_\ell} \partial_t u_{h_t}(t) \left(Q_{h_t}^0 \int\limits_T^{(\cdot)} w_{h_t}(s)\mathrm{d}s \right)(t)\,\mathrm{d}t \\
&= \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \int\limits_{t_{\ell-1}}^{t_\ell} \partial_t u_{h_t}(t)\,\partial_t w_{h_t}(t)\,\mathrm{d}t - \sum_{\ell=1}^{N_t} \int\limits_{t_{\ell-1}}^{t_\ell} \partial_t u_{h_t}(t) \left(Q_{h_t}^0 I_{h_t} \int\limits_T^{(\cdot)} w_{h_t}(s)\,\mathrm{d}s \right)(t)\,\mathrm{d}t
\end{aligned}$$

and further,

$$\begin{aligned}
\langle u_{h_t}, w_{h_t}\rangle_{L^2(0,T)} &= \frac{1}{12}\sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(\tau_\ell)} - \sum_{\ell=1}^{N_t} \int\limits_{t_{\ell-1}}^{t_\ell} \partial_t u_{h_t}(t) \left(I_{h_t} \int\limits_T^{(\cdot)} w_{h_t}(s)\,\mathrm{d}s \right)(t)\,\mathrm{d}t \\
&= \frac{1}{12}\sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(\tau_\ell)} - \int\limits_0^T \partial_t u_{h_t}(t) \left(I_{h_t} \int\limits_T^{(\cdot)} w_{h_t}(s)\mathrm{d}s \right)(t)\,\mathrm{d}t \\
&= \frac{1}{12}\sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(\tau_\ell)} + \langle u_{h_t}, Q_{h_t}^0 w_{h_t}\rangle_{L^2(0,T)},
\end{aligned}$$

i.e. the representation (4.27). □

Now, an alternative representation of the bilinear form $a(\cdot,\cdot)$ is given.

Corollary 4.2.11. *For $u_{h_t} \in S_{h_t,0,}^1(0,T)$ and $w_{h_t} \in S_{h_t,,0}^1(0,T)$, there hold*

$$\begin{aligned}
a(u_{h_t}, w_{h_t}) &= -\langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(0,T)} + \mu \langle u_{h_t}, w_{h_t}\rangle_{L^2(0,T)} \\
&= -\langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} \frac{\mu h_{t,\ell}^2}{12} \langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(\tau_\ell)} + \mu \langle u_{h_t}, Q_{h_t}^0 w_{h_t}\rangle_{L^2(0,T)} \\
&= \sum_{\ell=1}^{N_t} \left(\frac{\mu h_{t,\ell}^2}{12} - 1 \right) \langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(\tau_\ell)} + \mu \langle u_{h_t}, Q_{h_t}^0 w_{h_t}\rangle_{L^2(0,T)}. \qquad (4.28)
\end{aligned}$$

Motivated by the representation (4.28), one defines the perturbed bilinear form

$$a_{h_t}(u_{h_t}, w_{h_t}) := -\langle \partial_t u_{h_t}, \partial_t w_{h_t}\rangle_{L^2(0,T)} + \mu \langle u_{h_t}, Q_{h_t}^0 w_{h_t}\rangle_{L^2(0,T)} \qquad (4.29)$$

for $u_{h_t} \in S_{h_t,0,}^1(0,T)$ and $w_{h_t} \in S_{h_t,,0}^1(0,T)$, and consider the perturbed variational formulation to find $\widetilde{u}_{h_t} \in S_{h_t,0,}^1(0,T)$ such that

$$a_{h_t}(\widetilde{u}_{h_t}, w_{h_t}) = \langle f, w_{h_t}\rangle_{(0,T)} \qquad (4.30)$$

for all $w_{h_t} \in S_{h_t,,0}^1(0,T)$.

Lemma 4.2.12. *The perturbed bilinear form* (4.29) *is bounded, i.e. it holds*

$$|a_{h_t}(u_{h_t}, w_{h_t})| \leq \left(1 + \frac{4}{\pi^2}\mu T^2\right) |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)}$$

for all $u_{h_t} \in S^1_{h_t,0,}(0,T)$, $w_{h_t} \in S^1_{h_t,,0}(0,T)$.

Proof. With the Cauchy-Schwarz inequality, the L^2 stability of $Q^0_{h_t}$, and the Poincaré inequality of Lemma 3.4.5, there follow

$$\begin{aligned} |a_{h_t}(u_{h_t}, w_{h_t})| &\leq |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)} + \mu \, \|u_{h_t}\|_{L^2(0,T)} \|Q^0_{h_t} w_{h_t}\|_{L^2(0,T)} \\ &\leq \left(1 + \frac{4}{\pi^2}\mu T^2\right) |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)} \end{aligned}$$

for $u_{h_t} \in S^1_{h_t,0,}(0,T)$, $w_{h_t} \in S^1_{h_t,,0}(0,T)$, and so, the assertion. □

The related system matrix of (4.30) is given by

$$\widetilde{K}_{h_t} = -A_{h_t} + \mu \widetilde{M}_{h_t} \in \mathbb{R}^{N_t \times N_t} \tag{4.31}$$

with the stiffness matrix A_{h_t} in (4.23) and the perturbed mass matrix

$$\widetilde{M}_{h_t} = \frac{1}{4} \begin{pmatrix} h_{t,1} & & & & \\ h_{t,1} + h_{t,2} & h_{t,2} & & & \\ h_{t,2} & h_{t,2} + h_{t,3} & h_{t,3} & & \\ & \ddots & \ddots & \ddots & \\ & & h_{t,N_t-1} & h_{t,N_t-1} + h_{t,N_t} & h_{t,N_t} \end{pmatrix}. \tag{4.32}$$

Hence, unique solvability of (4.30) follows because $\widetilde{K}_{h_t}$ is a lower triangular matrix with positive diagonal elements.

To prove a discrete stability condition for the perturbed bilinear form (4.29), the following lemma is needed, which is analogous to [164, Theorem 2.1, page 168].

Lemma 4.2.13. *For a given* $z_{h_t} \in S^1_{h_t,,0}(0,T)$*, represented by*

$$z_{h_t}(t) = \sum_{i=0}^{N_t} z_i \varphi_i(t) \quad \text{with } z_{N_t} = 0,$$

and a fixed index $j \in \{0, \ldots, N_t - 1\}$*, there exists a function* $\overline{v}^j_{h_t} \in S^1_{h_t,0,}(0,T)$ *with the following properties:*

1. *For* $t \in [0, t_j]$*, it holds* $\overline{v}^j_{h_t}(t) = 0$.

2. *For $\ell = j+1,\ldots,N_t$, there hold*

$$\langle \partial_t \overline{v}_{h_t}^j, \partial_t z_{h_t} \rangle_{L^2(\tau_\ell)} = \frac{1}{2}\left(z_\ell^2 - z_{\ell-1}^2\right)$$

and

$$\langle \overline{v}_{h_t}^j, Q_{h_t}^0 z_{h_t} \rangle_{L^2(\tau_\ell)} = \frac{1}{2}\left(\int_{t_j}^{t_\ell} z_{h_t}(s)\mathrm{d}s\right)^2 - \frac{1}{2}\left(\int_{t_j}^{t_{\ell-1}} z_{h_t}(s)\mathrm{d}s\right)^2.$$

3. *There holds the estimate*

$$|\overline{v}_{h_t}^j|_{H^1(0,T)} \leq \|z_{h_t}\|_{L^2(0,T)}.$$

Proof. For $z_{h_t} \in S^1_{h_t,,0}(0,T)$, consider the piecewise linear interpolation of the antiderivative, i.e. for $t \in [0,T]$, one defines

$$\overline{v}_{h_t}(t) := \sum_{k=0}^{N_t} \left(\int_0^{t_k} z_{h_t}(s)\mathrm{d}s\right) \varphi_k(t) = \left(I_{h_t} \int_0^{(\cdot)} z_{h_t}(s)\mathrm{d}s\right)(t), \quad \overline{v}_{h_t} \in S^1_{h_t,0,}(0,T).$$

The relation $\partial_t \overline{v}_{h_t} = Q_{h_t}^0 z_{h_t}$ follows from (4.26). For a fixed index $j \in \{0,\ldots,N_t-1\}$, one now defines

$$z_{h_t}^j(t) = \sum_{i=0}^{N_t} z_i^j \varphi_i(t), \quad z_i^j = \begin{cases} (-1)^{j-i} z_j & \text{for } i = 0,\ldots,j, \\ z_i & \text{for } i = j+1,\ldots,N_t. \end{cases}$$

Note that $z_{h_t}^j \in S^1_{h_t,,0}(0,T)$, and according to $z_{h_t}^j$ one introduces $\overline{v}_{h_t}^j$, satisfying $\partial_t \overline{v}_{h_t}^j = Q_{h_t}^0 z_{h_t}^j$. In particular, for $j > 0$ and $t \in \tau_\ell$ for $\ell = 1,\ldots,j$, then it follows

$$\partial_t \overline{v}_{h_t}^j(t) = Q_{h_t}^0 z_{h_t}^j(t) = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} z_{h_t}^j(s)\,\mathrm{d}s = \frac{1}{2}\left(z_{\ell-1}^j + z_\ell^j\right) = 0,$$

and due to $\overline{v}_{h_t}^j(0) = 0$, one concludes $\overline{v}_{h_t}^j(t) = 0$ for $t \in [0,t_j]$, i.e. the first assertion.

To prove the second assertion, one computes for $\ell = j+1,\ldots,N_t$

$$\begin{aligned} \langle \partial_t \overline{v}_{h_t}^j, \partial_t z_{h_t} \rangle_{L^2(\tau_\ell)} &= \langle Q_{h_t}^0 z_{h_t}^j, \partial_t z_{h_t} \rangle_{L^2(\tau_\ell)} \\ &= \frac{1}{2}\left(z_{\ell-1}^j + z_\ell^j\right)\left(z_\ell - z_{\ell-1}\right) \\ &= \frac{1}{2}\left(z_{\ell-1} + z_\ell\right)\left(z_\ell - z_{\ell-1}\right) \\ &= \frac{1}{2}\left(z_\ell^2 - z_{\ell-1}^2\right) \end{aligned}$$

as well as

$$\begin{aligned}
\langle \overline{v}^j_{h_t}, Q^0_{h_t} z_{h_t} \rangle_{L^2(\tau_\ell)} &= \int\limits_{t_{\ell-1}}^{t_\ell} \left(I_{h_t} \int\limits_0^{(\cdot)} z^j_{h_t}(s)\mathrm{d}s \right)(t)\, Q^0_{h_t} z_{h_t}(t)\,\mathrm{d}t \\
&= Q^0_{h_t} z_{h_t|\tau_\ell} \int\limits_{t_{\ell-1}}^{t_\ell} \left[\int\limits_0^{t_{\ell-1}} z^j_{h_t}(s)\mathrm{d}s\, \varphi_{\ell-1}(t) + \int\limits_0^{t_\ell} z^j_{h_t}(s)\mathrm{d}s\, \varphi_\ell(t) \right] \mathrm{d}t \\
&= \frac{1}{h_{t,\ell}} \int\limits_{t_{\ell-1}}^{t_\ell} z_{h_t}(s)\mathrm{d}s\, \frac{1}{2} h_{t,\ell} \left[\int\limits_0^{t_{\ell-1}} z^j_{h_t}(s)\mathrm{d}s + \int\limits_0^{t_\ell} z^j_{h_t}(s)\mathrm{d}s \right] \\
&= \frac{1}{2} \int\limits_{t_{\ell-1}}^{t_\ell} z_{h_t}(s)\mathrm{d}s \left[\int\limits_{t_j}^{t_{\ell-1}} z_{h_t}(s)\mathrm{d}s + \int\limits_{t_j}^{t_\ell} z_{h_t}(s)\mathrm{d}s \right]
\end{aligned}$$

and with completing the square,

$$\begin{aligned}
\langle \overline{v}^j_{h_t}, Q^0_{h_t} z_{h_t} \rangle_{L^2(\tau_\ell)} &= \frac{1}{2} \left(\int\limits_{t_{\ell-1}}^{t_\ell} z_{h_t}(s)\,\mathrm{d}s \right)^2 + \int\limits_{t_{\ell-1}}^{t_\ell} z_{h_t}(s)\,\mathrm{d}s \int\limits_{t_j}^{t_{\ell-1}} z_{h_t}(s)\,\mathrm{d}s \\
&= \frac{1}{2} \left(\int\limits_{t_{\ell-1}}^{t_\ell} z_{h_t}(s)\,\mathrm{d}s + \int\limits_{t_j}^{t_{\ell-1}} z_{h_t}(s)\,\mathrm{d}s \right)^2 - \frac{1}{2} \left(\int\limits_{t_j}^{t_{\ell-1}} z_{h_t}(s)\,\mathrm{d}s \right)^2 \\
&= \frac{1}{2} \left(\int\limits_{t_j}^{t_\ell} z_{h_t}(s)\,\mathrm{d}s \right)^2 - \frac{1}{2} \left(\int\limits_{t_j}^{t_{\ell-1}} z_{h_t}(s)\mathrm{d}s \right)^2 .
\end{aligned}$$

Finally, from the L^2 stability of $Q^0_{h_t}$, one concludes the third assertion, i.e.

$$\begin{aligned}
|\overline{v}^j_{h_t}|_{H^1(0,T)} = |\overline{v}^j_{h_t}|_{H^1(t_j,T)} &= \|Q^0_{h_t} z^j_{h_t}\|_{L^2(t_j,T)} \\
&= \|Q^0_{h_t} z_{h_t}\|_{L^2(t_j,T)} \le \|Q^0_{h_t} z_{h_t}\|_{L^2(0,T)} \le \|z_{h_t}\|_{L^2(0,T)}.
\end{aligned}$$

Hence, the lemma is proven. □

Lemma 4.2.14. *The variational formulation to find* $z_{h_t} \in S^1_{h_t,,0}(0,T)$ *such that*

$$a_{h_t}(v_{h_t}, z_{h_t}) = \langle g_0, v_{h_t} \rangle_{(0,T)} + \langle g_1, \partial_t v_{h_t} \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell} \langle g_2, \partial_t v_{h_t} \rangle_{L^2(\tau_\ell)} \tag{4.33}$$

for all $v_{h_t} \in S^1_{h_t,0,}(0,T)$ *is uniquely solvable, where the right-hand sides* $g_0 \in [H^1_{0,}(0,T)]'$ *and* $g_1, g_2 \in L^2(0,T)$ *are given. Moreover, the stability estimate*

$$\|z_{h_t}\|_{L^2(0,T)} \le 2T \left\{ \|g_0\|_{[H^1_{0,}(0,T)]'} + \|g_1\|_{L^2(0,T)} + h_t^2 \|g_2\|_{L^2(0,T)} \right\} \tag{4.34}$$

holds for any mesh with maximal mesh size h_t.

Proof. The related finite element system matrix $\widetilde{K}_{h_t}^\top$ of the variational formulation (4.33) is upper triangular with positive diagonal elements, see (4.31), and hence, there exists a unique solution $z_{h_t} \in S^1_{h_t,,0}(0,T)$ of (4.33).

For the estimate, consider for each index $j \in \{0,\dots,N_t-1\}$ a function $\overline{v}^j_{h_t} \in S^1_{h_t,0,}(0,T)$ as given in Lemma 4.2.13. Plugging these functions $\overline{v}^j_{h_t}$ into (4.33) and by using the properties of Lemma 4.2.13, this gives

$$\begin{aligned}
a_{h_t}(\overline{v}^j_{h_t}, z_{h_t}) &= -\langle \partial_t \overline{v}^j_{h_t}, \partial_t z_{h_t}\rangle_{L^2(0,T)} + \mu \langle \overline{v}^j_{h_t}, Q^0_{h_t} z_{h_t}\rangle_{L^2(0,T)} \\
&= -\sum_{\ell=j+1}^{N_t} \langle \partial_t \overline{v}^j_{h_t}, \partial_t z_{h_t}\rangle_{L^2(\tau_\ell)} + \mu \sum_{\ell=j+1}^{N_t} \langle \overline{v}^j_{h_t}, Q^0_{h_t} z_{h_t}\rangle_{L^2(\tau_\ell)} \\
&= -\frac{1}{2}\sum_{\ell=j+1}^{N_t}\left(z_\ell^2 - z_{\ell-1}^2\right) + \frac{\mu}{2}\sum_{\ell=j+1}^{N_t}\left(\left(\int_{t_j}^{t_\ell} z_{h_t}(s)\,\mathrm{d}s\right)^2 - \left(\int_{t_j}^{t_{\ell-1}} z_{h_t}(s)\,\mathrm{d}s\right)^2\right) \\
&= \frac{1}{2} z_j^2 + \frac{\mu}{2}\left(\int_{t_j}^{T} z_{h_t}(s)\,\mathrm{d}s\right)^2
\end{aligned}$$

and so,

$$\begin{aligned}
\langle g_0, \overline{v}^j_{h_t}\rangle_{(0,T)} + \langle g_1, \partial_t \overline{v}^j_{h_t}\rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle g_2, \partial_t \overline{v}^j_{h_t}\rangle_{L^2(\tau_\ell)} &= a_{h_t}(\overline{v}^j_{h_t}, z_{h_t}) \\
&= \frac{1}{2} z_j^2 + \frac{\mu}{2}\left(\int_{t_j}^{T} z_{h_t}(s)\,\mathrm{d}s\right)^2.
\end{aligned}$$

This result yields

$$\begin{aligned}
\|z_{h_t}\|^2_{L^2(0,T)} &= \sum_{\ell=1}^{N_t} \|z_{h_t}\|^2_{L^2(\tau_\ell)} = \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}}{3}\left(z_\ell^2 + z_\ell z_{\ell-1} + z_{\ell-1}^2\right) \le \frac{1}{2}\sum_{\ell=1}^{N_t} h_{t,\ell}\left(z_\ell^2 + z_{\ell-1}^2\right) \\
&\le \frac{1}{2}\sum_{j=1}^{N_t-1} h_{t,j} z_j^2 + \frac{1}{2}\sum_{j=0}^{N_t-1} h_{t,j+1} z_j^2 \\
&\le \sum_{j=1}^{N_t-1} h_{t,j}\left\{\langle g_0, \overline{v}^j_{h_t}\rangle_{(0,T)} + \langle g_1, \partial_t \overline{v}^j_{h_t}\rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle g_2, \partial_t \overline{v}^j_{h_t}\rangle_{L^2(\tau_\ell)}\right\} \\
&\quad + \sum_{j=0}^{N_t-1} h_{t,j+1}\left\{\langle g_0, \overline{v}^j_{h_t}\rangle_{(0,T)} + \langle g_1, \partial_t \overline{v}^j_{h_t}\rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h_{t,\ell}^2 \langle g_2, \partial_t \overline{v}^j_{h_t}\rangle_{L^2(\tau_\ell)}\right\}.
\end{aligned}$$

With the last inequality, the Cauchy-Schwarz inequality and the use of the properties of Lemma 4.2.13, one concludes

$$\|z_{h_t}\|^2_{L^2(0,T)} \le \sum_{j=1}^{N_t-1} h_{t,j} \left\{ \|g_0\|_{[H^1_{0,}(0,T)]'} + \|g_1\|_{L^2(0,T)} + h_t^2 \|g_2\|_{L^2(0,T)} \right\} \underbrace{|\overline{v}^j_{h_t}|_{H^1(0,T)}}_{\le \|z_{h_t}\|_{L^2(0,T)}}$$
$$+ \sum_{j=0}^{N_t-1} h_{t,j+1} \left\{ \|g_0\|_{[H^1_{0,}(0,T)]'} + \|g_1\|_{L^2(0,T)} + h_t^2 \|g_2\|_{L^2(0,T)} \right\} \underbrace{|\overline{v}^j_{h_t}|_{H^1(0,T)}}_{\le \|z_{h_t}\|_{L^2(0,T)}}$$
$$\le 2T \left\{ \|g_0\|_{[H^1_{0,}(0,T)]'} + \|g_1\|_{L^2(0,T)} + h_t^2 \|g_2\|_{L^2(0,T)} \right\} \|z_{h_t}\|_{L^2(0,T)},$$

i.e. the assertion. □

Corollary 4.2.15. *The variational formulation to find $v_{h_t} \in S^1_{h_t,0,}(0,T)$ such that*

$$a_{h_t}(v_{h_t}, w_{h_t}) = \langle f_0, w_{h_t} \rangle_{(0,T)} + \langle f_1, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell} \langle f_2, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} \tag{4.35}$$

for all $w_{h_t} \in S^1_{h_t,,0}(0,T)$ is uniquely solvable, where the right-hand sides $f_0 \in [H^1_{,0}(0,T)]'$ and $f_1, f_2 \in L^2(0,T)$ are given. Moreover, the stability estimate

$$\|v_{h_t}\|_{L^2(0,T)} \le 2T \left\{ \|f_0\|_{[H^1_{,0}(0,T)]'} + \|f_1\|_{L^2(0,T)} + h_t^2 \|f_2\|_{L^2(0,T)} \right\} \tag{4.36}$$

holds for any mesh with maximal mesh size h_t.

Proof. The related finite element system matrix $\widetilde{K}_{h_t}$ of the variational formulation (4.35) is lower triangular with positive diagonal elements, see (4.31), and hence, there exists a unique solution $v_{h_t} \in S^1_{h_t,0,}(0,T)$ of (4.35).

The proof of the stability estimate is analogous to the proof of (4.34) with the help of a corresponding lemma analogous to Lemma 4.2.13. □

Lemma 4.2.16. *For each $u_{h_t} \in S^1_{h_t,0,}(0,T)$, there holds the discrete inf-sup condition*

$$\frac{1}{1 + \frac{4}{\pi}\mu T^2} |u_{h_t}|_{H^1(0,T)} \le \sup_{0 \ne w_{h_t} \in S^1_{h_t,,0}(0,T)} \frac{|a_{h_t}(u_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}}.$$

Proof. For a fixed function $u_{h_t} \in S^1_{h_t,0,}(0,T)$, let $w_{h_t} \in S^1_{h_t,,0}(0,T)$ be the unique solution of (4.33) for $g_0 = g_2 = 0$ and $g_1 = \partial_t u_{h_t} \in L^2(0,T)$, i.e. it holds

$$a_{h_t}(v_{h_t}, w_{h_t}) = \langle \partial_t u_{h_t}, \partial_t v_{h_t} \rangle_{L^2(0,T)} \tag{4.37}$$

for all $v_{h_t} \in S^1_{h_t,0,}(0,T)$. For the particular choice $v_{h_t}(t) = w_{h_t}(0) - w_{h_t}(t)$, $t \in [0,T]$, it holds $v_{h_t} \in S^1_{h_t,0,}(0,T)$ and so, it follows

$$\langle \partial_t w_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)} - \mu \langle w_{h_t} - w_{h_t}(0), Q^0_{h_t} w_{h_t} \rangle_{L^2(0,T)} = -\langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)}.$$

Hence, using the Cauchy-Schwarz inequality, the Poincaré inequality from Lemma 3.4.5 and the L^2 stability of the L^2 projection $Q^0_{h_t}$, one concludes

$$\begin{aligned}
|w_{h_t}|^2_{H^1(0,T)} &= -\langle \partial_t u_{h_t}, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \mu \, \langle w_{h_t} - w_{h_t}(0), Q^0_{h_t} w_{h_t} \rangle_{L^2(0,T)} \\
&\leq |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)} + \mu \, \|w_{h_t} - w_{h_t}(0)\|_{L^2(0,T)} \|Q^0_{h_t} w_{h_t}\|_{L^2(0,T)} \\
&\leq |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)} + \frac{2}{\pi} \mu T |w_{h_t}|_{H^1(0,T)} \|w_{h_t}\|_{L^2(0,T)} \\
&\leq \left(1 + \frac{4}{\pi} \mu T^2\right) |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)},
\end{aligned}$$

where in the last step, the stability estimate $\|w_{h_t}\|_{L^2(0,T)} \leq 2T |u_{h_t}|_{H^1(0,T)}$ from (4.34) is used.

The choice $v_{h_t} = u_{h_t} \in S^1_{h_t,0,}(0,T)$ in (4.37) and the estimate above yield

$$a_{h_t}(u_{h_t}, w_{h_t}) = |u_{h_t}|^2_{H^1(0,T)} \geq \frac{1}{1 + \frac{4}{\pi} \mu T^2} |u_{h_t}|_{H^1(0,T)} |w_{h_t}|_{H^1(0,T)}$$

and hence, the discrete inf-sup condition follows. □

Theorem 4.2.17. *For given $f \in [H^1_{,0}(0,T)]'$, let the unique solution u of the variational formulation* (4.12) *satisfy $u \in H^1_{0,}(0,T) \cap H^s(0,T)$ for some $s \in [1,2]$. Then there exists a unique solution $\widetilde{u}_{h_t} \in S^1_{h_t,0,}(0,T)$ of the Galerkin-Petrov finite element discretisation* (4.30)*, satisfying the stability estimates*

$$\|\widetilde{u}_{h_t}\|_{L^2(0,T)} \leq 2T \|f\|_{[H^1_{,0}(0,T)]'},$$

$$|\widetilde{u}_{h_t}|_{H^1(0,T)} \leq \left(1 + \frac{4}{\pi} \mu T^2\right) \|f\|_{[H^1_{,0}(0,T)]'}$$

and the error estimate

$$\begin{aligned}
|u - \widetilde{u}_{h_t}|_{H^1(0,T)} \leq \left[1 + \left(1 + \frac{4}{\pi^2} \mu T^2\right)\left(1 + \frac{4}{\pi} \mu T^2\right)\right] C_1 \, h_t^{s-1} \, \|u\|_{H^s(0,T)} \\
+ \frac{1}{12} \mu \left(1 + \frac{4}{\pi} \mu T^2\right) h_t^2 |u|_{H^1(0,T)},
\end{aligned}$$

where the constant $C_1 > 0$ is coming from standard interpolation error estimates.

Proof. First, the unique solvability of the variational formulation (4.30) and the first stability estimate follow from Corollary 4.2.15 for $f_0 = f$, $f_1 = f_2 = 0$. The second stability estimate is a consequence of the discrete inf-sup condition given in Lemma 4.2.16.

Second, for any $v_{h_t} \in S^1_{h_t,0,}(0,T)$, it holds

$$|u - \widetilde{u}_{h_t}|_{H^1(0,T)} \leq |u - v_{h_t}|_{H^1(0,T)} + |\widetilde{u}_{h_t} - v_{h_t}|_{H^1(0,T)}$$

and it remains to bound the second term. With the usage the discrete inf-sup condition of Lemma 4.2.16 and using the Galerkin orthogonality for the variational formulations (4.12) and (4.30), it follows

$$\begin{aligned}
\frac{1}{1+\frac{4}{\pi}\mu T^2}|\widetilde{u}_{h_t} - v_{h_t}|_{H^1(0,T)} &\leq \sup_{0\neq w_{h_t}\in S^1_{h_t,,0}(0,T)} \frac{|a_{h_t}(\widetilde{u}_{h_t} - v_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}} \\
&= \sup_{0\neq w_{h_t}\in S^1_{h_t,,0}(0,T)} \frac{|a_{h_t}(\widetilde{u}_{h_t}, w_{h_t}) - a_{h_t}(v_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}} \\
&= \sup_{0\neq w_{h_t}\in S^1_{h_t,,0}(0,T)} \frac{|a(u, w_{h_t}) - a_{h_t}(v_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}} \\
&= \sup_{0\neq w_{h_t}\in S^1_{h_t,,0}(0,T)} \frac{|a(u - v_{h_t}, w_{h_t}) + a(v_{h_t}, w_{h_t}) - a_{h_t}(v_{h_t}, w_{h_t})|}{|w_{h_t}|_{H^1(0,T)}}.
\end{aligned}$$

Further, with the boundedness of the bilinear form $a(\cdot,\cdot)$ and the Poincaré inequality of Lemma 3.4.5, one concludes

$$\begin{aligned}
a(u - v_{h_t}, w_{h_t}) &= -\langle \partial_t(u - v_{h_t}), \partial_t w_{h_t}\rangle_{L^2(0,T)} + \mu\langle u - v_{h_t}, w_{h_t}\rangle_{L^2(0,T)} \\
&\leq |u - v_{h_t}|_{H^1(0,T)}|w_{h_t}|_{H^1(0,T)} + \mu\|u - v_{h_t}\|_{L^2(0,T)}\|w_{h_t}\|_{L^2(0,T)} \\
&\leq \left(1 + \frac{4}{\pi^2}\mu T^2\right)|u - v_{h_t}|_{H^1(0,T)}|w_{h_t}|_{H^1(0,T)}.
\end{aligned}$$

Moreover, using the representation (4.28), one estimates the consistency error by

$$\begin{aligned}
|a(v_{h_t}, w_{h_t}) - a_{h_t}(v_{h_t}, w_{h_t})| &= \frac{1}{12}\mu \left| \sum_{\ell=1}^{N_t} h^2_{t,\ell}\langle \partial_t v_{h_t}, \partial_t w_{h_t}\rangle_{L^2(\tau_\ell)} \right| \\
&\leq \frac{1}{12}\mu\, h_t^2\, |v_{h_t}|_{H^1(0,T)}|w_{h_t}|_{H^1(0,T)}.
\end{aligned}$$

Hence, there follow

$$\frac{1}{1+\frac{4}{\pi}\mu T^2}|\widetilde{u}_{h_t} - v_{h_t}|_{H^1(0,T)} \leq \left(1 + \frac{4}{\pi^2}\mu T^2\right)|u - v_{h_t}|_{H^1(0,T)} + \frac{1}{12}\mu\, h_t^2\, |v_{h_t}|_{H^1(0,T)},$$

and therefore,

$$|u-\widetilde{u}_{h_t}|_{H^1(0,T)} \le \left[1+\left(1+\frac{4}{\pi^2}\mu T^2\right)\left(1+\frac{4}{\pi}\mu T^2\right)\right]|u-v_{h_t}|_{H^1(0,T)} + \frac{1}{12}\mu\left(1+\frac{4}{\pi}\mu T^2\right)h_t^2\,|v_{h_t}|_{H^1(0,T)}.$$

In particular, for the piecewise linear interpolation $v_{h_t} = I_{h_t}u$, there hold

$$\|u-I_{h_t}u\|_{H^1(0,T)} \le C_1\, h_t^{s-1}\,\|u\|_{H^s(0,T)}, \quad |I_{h_t}u|_{H^1(0,T)} \le |u|_{H^1(0,T)},$$

see Section 2.6 and thus, the assertion. □

Remark 4.2.18. *The proof of the error estimate in Theorem 4.2.17 is also given by [51, Lemma 2.27, page 95].*

Remark 4.2.19. *The discrete inf-sup constant of Lemma 4.2.16 depends on the parameter μ with order μ^{-1}. Numerical experiments for the optimal discrete inf-sup constant*

$$c_{H^1}(T,\mu) := \inf_{0\neq v_{h_t}\in S^1_{h_t,0,}(0,T)}\ \sup_{0\neq w_{h_t}\in S^1_{h_t,,0}(0,T)} \frac{|a_{h_t}(v_{h_t},w_{h_t})|}{|v_{h_t}|_{H^1(0,T)}|w_{h_t}|_{H^1(0,T)}} \ge \frac{1}{1+\frac{4}{\pi}\mu T^2} \tag{4.38}$$

show only a dependency of order $\mu^{-1/2}$, where a corresponding generalised eigenvalue problem is solved to compute $c_{H^1}(T,\mu)$, see [84, Subsection 3.6.6, page 124]. In Table 4.3 the optimal discrete inf-sup constant $c_{H^1}(T,\mu)$ is presented for $\mu \in \{125, 250, 500, 1000\}$ with $T = 10$ and a uniform mesh size $h_t = T/N_t$.

For the optimal discrete inf-sup constant with respect to $\|\cdot\|_{L^2(0,T)}$

$$c_{L^2}(T,\mu) := \inf_{0\neq v_{h_t}\in S^1_{h_t,0,}(0,T)}\ \sup_{0\neq w_{h_t}\in S^1_{h_t,,0}(0,T)} \frac{|a_{h_t}(v_{h_t},w_{h_t})|}{\|v_{h_t}\|_{L^2(0,T)}|w_{h_t}|_{H^1(0,T)}} \ge \frac{1}{2T}, \tag{4.39}$$

where the last inequality follows from Corollary 4.2.15, numerical experiments with the same discretisations as for $c_{H^1}(T,\mu)$ confirm the independence of $c_{L^2}(T,\mu)$ from the parameter μ, see Table 4.4.

Numerical experiments for the bilinear form (4.13)*, i.e. without stabilisation, show a similar behaviour as for the perturbed bilinear form* (4.29)*, provided the mesh size h_t is sufficiently small, i.e. $h_t < \sqrt{12/\mu}$.*

Next, an $L^2(0,T)$ error estimate is stated, where the proof is based on the proof of [164, Theorem 3.1, page 175].

N_t	h_t	$c_{H^1}(10,125)$	$c_{H^1}(10,250)$	$c_{H^1}(10,500)$	$c_{H^1}(10,1000)$
4	2.5000000	1.12365e+01	2.21744e+01	4.40526e+01	8.78100e+01
8	1.2500000	9.77956e-01	1.62745e+00	2.96357e+00	5.65355e+00
16	0.6250000	3.33189e-01	4.22413e-01	6.05756e-01	6.80010e-01
32	0.3125000	1.12684e-01	1.35639e-01	1.79762e-01	2.31985e-01
64	0.1562500	4.89792e-02	4.96583e-02	5.60224e-02	6.97641e-02
128	0.0781250	3.33683e-02	2.72126e-02	2.47568e-02	2.50579e-02
256	0.0390625	2.91545e-02	2.16312e-02	1.66440e-02	1.36724e-02
512	0.0195312	2.83866e-02	2.02092e-02	1.46513e-02	1.08659e-02
1024	0.0097656	2.81193e-02	1.98896e-02	1.41544e-02	1.01405e-02
2048	0.0048828	2.80366e-02	1.98239e-02	1.40306e-02	9.98949e-03
4096	0.0024414	2.80160e-02	1.98080e-02	1.40038e-02	9.93791e-03
8192	0.0012207	2.80108e-02	1.98041e-02	1.39974e-02	9.92346e-03
16384	0.0006104	2.80096e-02	1.98031e-02	1.39958e-02	9.91987e-03
32768	0.0003052	2.80092e-02	1.98029e-02	1.39954e-02	9.91898e-03

Table 4.3: Optimal discrete inf-sup constant $c_{H^1}(T,\mu)$ of (4.38) for the perturbed bilinear form (4.29) for $\mu \in \{125,250,500,1000\}$ with $T = 10$.

N_t	h_t	$c_{L^2}(10,125)$	$c_{L^2}(10,250)$	$c_{L^2}(10,500)$	$c_{L^2}(10,1000)$
4	2.5000000	1.40996e+01	2.77886e+01	5.51698e+01	1.09934e+02
8	1.2500000	2.65957e+00	4.41397e+00	8.01919e+00	1.52769e+01
16	0.6250000	1.68707e+00	2.25321e+00	3.30172e+00	3.74737e+00
32	0.3125000	8.80041e-01	1.24605e+00	1.79212e+00	2.45043e+00
64	0.1562500	4.89981e-01	6.40430e-01	8.88048e-01	1.26819e+00
128	0.0781250	3.59196e-01	4.07852e-01	4.91438e-01	6.42682e-01
256	0.0390625	3.26162e-01	3.37548e-01	3.62111e-01	4.07958e-01
512	0.0195312	3.14474e-01	3.19977e-01	3.26189e-01	3.37739e-01
1024	0.0097656	3.12185e-01	3.14389e-01	3.16821e-01	3.19974e-01
2048	0.0048828	3.11659e-01	3.12923e-01	3.14473e-01	3.14727e-01
4096	0.0024414	3.11531e-01	3.12570e-01	3.13677e-01	3.13615e-01
8192	0.0012207	3.11499e-01	3.12482e-01	3.13481e-01	3.13370e-01
16384	0.0006104	3.11491e-01	3.12460e-01	3.13432e-01	3.13310e-01
32768	0.0003052	3.11489e-01	3.12455e-01	3.13420e-01	3.13295e-01

Table 4.4: Optimal discrete inf-sup constant $c_{L^2}(T,\mu)$ of (4.39) for the perturbed bilinear form (4.29) for $\mu \in \{125,250,500,1000\}$ with $T = 10$.

Theorem 4.2.20. *Let the unique solution u of* (4.12) *satisfy* $u \in H^1_{0,}(0,T) \cap H^s(0,T)$ *for some* $s \in [1,2]$. *Then the unique solution* $\widetilde{u}_{h_t} \in S^1_{h_t,0,}(0,T)$ *of the Galerkin-Petrov finite element discretisation* (4.30) *from Theorem 4.2.17 satisfies*

$$\|u - \widetilde{u}_{h_t}\|_{L^2(0,T)} \le c \left(1 + \frac{4}{\pi} T^2 \mu\right) h_t^s \|u\|_{H^s(0,T)} + \frac{\mu T h_t^2}{6} |u|_{H^1(0,T)}$$

with a constant $c > 0$ *independent of* μ *and* h_t.

Proof. With the representation (4.28) and the $H^1_{0,}$ projection $Q^1_{h_t} u$ of (2.33), there holds for all $w_{h_t} \in S^1_{h_t,,0}(0,T)$

$$\begin{aligned}
a_{h_t}(\widetilde{u}_{h_t} - Q^1_{h_t} u, w_{h_t}) &= \underbrace{a_{h_t}(\widetilde{u}_{h_t}, w_{h_t})}_{=a(u,w_{h_t})} - a_{h_t}(Q^1_{h_t} u, w_{h_t}) \\
&= a(u, w_{h_t}) - a(Q^1_{h_t} u, w_{h_t}) + \sum_{\ell=1}^{N_t} \frac{\mu h_{t,\ell}^2}{12} \langle \partial_t Q^1_{h_t} u, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} \\
&= -\underbrace{\langle \partial_t u, \partial_t w_{h_t} \rangle_{L^2(0,T)}}_{=\langle \partial_t Q^1_{h_t} u, \partial_t w_{h_t} \rangle_{L^2(0,T)}} + \mu \langle u, w_{h_t} \rangle_{L^2(0,T)} \\
&\quad + \langle \partial_t Q^1_{h_t} u, \partial_t w_{h_t} \rangle_{L^2(0,T)} - \mu \langle Q^1_{h_t} u, w_{h_t} \rangle_{L^2(0,T)} \\
&\quad + \sum_{\ell=1}^{N_t} \frac{\mu h_{t,\ell}^2}{12} \langle \partial_t Q^1_{h_t} u, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)} \\
&= \mu \langle u - Q^1_{h_t} u, w_{h_t} \rangle_{L^2(0,T)} + \sum_{\ell=1}^{N_t} \frac{\mu h_{t,\ell}^2}{12} \langle \partial_t Q^1_{h_t} u, \partial_t w_{h_t} \rangle_{L^2(\tau_\ell)}
\end{aligned}$$

and so, $\widetilde{u}_{h_t} - Q^1_{h_t} u \in S^1_{h_t,0,}(0,T)$ is the unique solution of the variational formulation (4.35) for $f_0 = \mu \cdot (u - Q^1_{h_t} u) \in L^2(0,T)$, $f_2 = \frac{\mu}{12} \partial_t Q^1_{h_t} u \in L^2(0,T)$ and $f_1 = 0$. Therefore, the stability estimate (4.36), the Poincaré inequality from Lemma 3.4.5 and the stability of the $H^1_{0,}$ projection give

$$\begin{aligned}
\|\widetilde{u}_{h_t} - Q^1_{h_t} u\|_{L^2(0,T)} &\le 2T\mu \left\| u - Q^1_{h_t} u \right\|_{[H^1_{,0}(0,T)]'} + \frac{\mu T h_t^2}{6} \left\| \partial_t Q^1_{h_t} u \right\|_{L^2(0,T)} \\
&\le \frac{4}{\pi} T^2 \mu \left\| u - Q^1_{h_t} u \right\|_{L^2(0,T)} + \frac{\mu T h_t^2}{6} |u|_{H^1(0,T)}.
\end{aligned}$$

With the last estimate, the triangle inequality and the error estimate (2.34) for the $H^1_{0,}$ projection, it holds

$$\begin{aligned}
\|u - \widetilde{u}_{h_t}\|_{L^2(0,T)} &\le \|u - Q^1_{h_t} u\|_{L^2(0,T)} + \|\widetilde{u}_{h_t} - Q^1_{h_t} u\|_{L^2(0,T)} \\
&\le c \left(1 + \frac{4}{\pi} T^2 \mu\right) h_t^s \|u\|_{H^s(0,T)} + \frac{\mu T h_t^2}{6} |u|_{H^1(0,T)}
\end{aligned}$$

with a constant $c > 0$ independent of μ and h_t. Hence, the assertion follows. □

As a numerical example for the Galerkin finite element methods (4.21) and (4.30), a uniform discretisation of the time interval $(0,T)$ with $T = 10$ and a uniform mesh size $h_t = T/N_t$ is introduced. For $\mu = 1000$, consider the solution $u(t) = \sin^2\left(\frac{5}{4}\pi t\right)$, where calculations of the appearing integrals for the related right-hand side in (4.21) and (4.30) are done by the usage of high-order integration rules.

In Table 4.5 the results for the stabilised variational formulation (4.30), which is unconditionally stable, are presented, where the error estimate in the energy norm of Theorem 4.2.17 is confirmed. In addition, the error in $L^2(0,T)$ is presented, where a quadratic convergence, as expected from Theorem 4.2.20, is observed.

N_t	h_t	$\|u-\widetilde{u}_{h_t}\|_{L^2(0,10)}$	eoc	$\lvert u-\widetilde{u}_{h_t}\rvert_{H^1(0,10)}$	eoc
4	2.5000000	1.7722e+00	-	9.0867e+00	-
8	1.2500000	6.0704e+00	-1.78	2.0130e+01	-1.15
16	0.6250000	1.2687e+00	2.26	9.4204e+00	1.10
32	0.3125000	5.7861e+00	-2.19	6.0121e+01	-2.67
64	0.1562500	3.3966e-01	4.09	6.1941e+00	3.28
128	0.0781250	7.6647e-02	2.15	2.2955e+00	1.43
256	0.0390625	2.0315e-02	1.92	9.4091e-01	1.29
512	0.0195312	5.2649e-03	1.95	4.1539e-01	1.18
1024	0.0097656	1.3365e-03	1.98	1.9803e-01	1.07
2048	0.0048828	3.3682e-04	1.99	9.7671e-02	1.02
4096	0.0024414	8.4229e-05	2.00	4.8663e-02	1.01
8192	0.0012207	2.1057e-05	2.00	2.4310e-02	1.00
16384	0.0006104	5.2644e-06	2.00	1.2152e-02	1.00
32768	0.0003052	1.3161e-06	2.00	6.0758e-03	1.00

Table 4.5: Numerical results for the stabilised variational formulation (4.30), $\mu = 1000$, $T = 10$.

In Table 4.6 the related results for the variational formulation (4.21) without stabilisation are presented, convergence is observed for a sufficiently small mesh size only. Note that $\sqrt{12/\mu} \approx 0.1095$.

4.2.2 Variational Formulation for the Wave Equation

Instead of the ordinary differential equation (4.11), the wave equation (4.10) is considered. The aim is to extend the results of Section 4.2.1 to the wave equation. So, for $u \in H^{1,1}_{0;0,}(Q)$ and $w \in H^{1,1}_{0;,0}(Q)$, one defines the bilinear form

$$a(u,w) := -\langle \partial_t u, \partial_t w \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w \rangle_{L^2(Q)},$$

N_t	h_t	$\|u-u_{h_t}\|_{L^2(0,10)}$	eoc	$\|u-u_{h_t}\|_{H^1(0,10)}$	eoc
4	2.5000000	7.0573e+01	-	9.8785e+01	-
8	1.2500000	1.6871e+03	-4.58	3.7166e+03	-5.23
16	0.6250000	9.1421e+07	-15.73	3.7247e+08	-16.61
32	0.3125000	2.3915e+15	-24.64	1.9496e+16	-25.64
64	0.1562500	1.6337e+22	-22.70	2.9536e+23	-23.85
128	0.0781250	3.1417e-02	78.78	1.7859e+00	77.13
256	0.0390625	9.2885e-03	1.76	8.2361e-01	1.12
512	0.0195312	2.4767e-03	1.91	3.9567e-01	1.06
1024	0.0097656	6.3105e-04	1.97	1.9532e-01	1.02
2048	0.0048828	1.5839e-04	1.99	9.7325e-02	1.00
4096	0.0024414	3.9633e-05	2.00	4.8620e-02	1.00
8192	0.0012207	9.9106e-06	2.00	2.4304e-02	1.00
16384	0.0006104	2.4778e-06	2.00	1.2152e-02	1.00
32768	0.0003052	6.1946e-07	2.00	6.0757e-03	1.00

Table 4.6: Numerical results for the variational formulation (4.21), $\mu = 1000, \quad T = 10$.

see Section 2.5 for the details of the Sobolev spaces. The boundedness of the bilinear form $a(\cdot,\cdot)$ is stated in the next lemma.

Lemma 4.2.21. *The bilinear form* $a(\cdot,\cdot)\colon H^{1,1}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}$ *is bounded, i.e.*

$$|a(u,w)| \leq |u|_{H^1(Q)}\,|w|_{H^1(Q)} \qquad \textit{for all } u \in H^{1,1}_{0;0,}(Q),\, w \in H^{1,1}_{0;,0}(Q).$$

Proof. The assertion follows immediately from the Cauchy-Schwarz inequality. □

The variational formulation of the wave equation (4.10) is to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$a(u,w) = \langle f, w\rangle_{L^2(Q)} \tag{4.40}$$

for all $w \in H^{1,1}_{0;,0}(Q)$, where $f \in L^2(Q)$ is given. Note that the initial condition $u(\cdot,0) = 0$ is considered in the strong sense, whereas the initial condition $\partial_t u(\cdot,0) = 0$ is incorporated in a weak sense.

For the analysis of (4.40), the adjoint problem to find $w \in H^{1,1}_{0;,0}(Q)$ such that

$$a(u,w) = \langle g, u\rangle_{L^2(Q)} \tag{4.41}$$

for all $u \in H^{1,1}_{0;0,}(Q)$, where $g \in L^2(Q)$ is a given right-hand side, occurs.

Next, an existence and uniqueness result for the variational formulation (4.40) is given. Such a result is contained in [97, Chapter IV], see also [145]. To summarise and to examine also to adjoint problem (4.41), the following theorem is stated.

Theorem 4.2.22. *Let $f, g \in L^2(Q)$ be given. There are unique solutions $u \in H^{1,1}_{0;0,}(Q)$ of* (4.40) *and $w \in H^{1,1}_{0;,0}(Q)$ of* (4.41)*, satisfying*

$$|u|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|f\|_{L^2(Q)} \qquad \textit{and} \qquad |w|_{H^1(Q)} \leq \frac{1}{\sqrt{2}} T \|g\|_{L^2(Q)}.$$

Proof. For the variational formulation (4.40), there exists a unique solution $u \in H^{1,1}_{0;0,}(Q)$, see [97, Chapter IV, Theorem 3.1, page 157, and Theorem 3.2, page 160]. The estimate follows by a Fourier series ansatz as in [97, Section 7, Chapter IV], see [145]. When using the representation (3.65), any $u \in H^{1,1}_{0;0,}(Q)$ admits the representation

$$u(x,t) = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} u_{i,k} V_k(t) \phi_i(x) = \sum_{i=1}^{\infty} U_i(t) \phi_i(x), \quad (x,t) \in Q, \tag{4.42}$$

where $V_k(t)$ are the temporal eigenfunctions given in (3.25), and $\phi_i(x)$ are the spatial $L^2(\Omega)$ orthonormal eigenfunctions of the Laplacian with homogeneous Dirichlet boundary conditions, see (2.4). For the solution of the variational formulation (4.40), consider the ansatz (4.42), where the unknown functions $U_i \in H^1_{0,}(0,T)$ are to be determined. When choosing, for a fixed $j \in \mathbb{N}$, $v(x,t) = V(t)\phi_j(x)$ with $V \in H^1_{,0}(0,T)$ as test function, the variational formulation (4.40) results in finding $U_j \in H^1_{0,}(0,T)$ such that

$$-\int_0^T \partial_t U_j(t) \partial_t V(t) \mathrm{d}t + \mu_j \int_0^T U_j(t) V(t) \mathrm{d}t = \int_0^T F_j(t) V(t) \mathrm{d}t$$

for all $V \in H^1_{,0}(0,T)$, where the functions $F_j \in L^2(0,T)$,

$$F_j(t) = \int_\Omega f(x,t) \phi_j(x) \mathrm{d}x, \quad t \in (0,T),$$

are the coefficients of the Fourier expansion

$$f(x,t) = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} f_{j,k} V_k(t) \phi_j(x) = \sum_{j=1}^{\infty} F_j(t) \phi_j(x), \quad (x,t) \in Q,$$

see again (3.65). From this, one concludes

$$\begin{aligned} \|f\|^2_{L^2(Q)} &= \int_0^T \int_\Omega |f(x,t)|^2 \, \mathrm{d}x\mathrm{d}t = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^T F_i(t) F_j(t) \mathrm{d}t \int_\Omega \phi_i(x) \phi_j(x) \mathrm{d}x \\ &= \sum_{j=1}^{\infty} \int_0^T \left| F_j(t) \right|^2 \mathrm{d}t = \sum_{j=1}^{\infty} \|F_j\|^2_{L^2(0,T)}. \end{aligned}$$

Theorem 4.2.5 yields the uniqueness and the existence of the functions $U_j \in H^1_{0,}(0,T)$. Hence, one obtains

$$|u|^2_{H^1(Q)} = \|u\|^2_{H^{1,1}_{0;0,}(Q)} = \int_0^T \int_\Omega \Big[|\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2 \Big] \mathrm{d}x\mathrm{d}t$$
$$= \sum_{i=1}^\infty \sum_{j=1}^\infty \Bigg[\int_0^T \partial_t U_i(t) \partial_t U_j(t) \mathrm{d}t \int_\Omega \phi_i(x)\phi_j(x)\mathrm{d}x + \int_0^T U_i(t)U_j(t)\mathrm{d}t \int_\Omega \nabla_x \phi_i(x) \cdot \nabla_x \phi_j(x)\mathrm{d}x \Bigg]$$

and by using Lemma 4.2.7,

$$|u|^2_{H^1(Q)} = \sum_{i=1}^\infty \Bigg[\int_0^T |\partial_t U_i(t)|^2 \mathrm{d}t + \mu_i \int_0^T |U_i(t)|^2 \mathrm{d}t \Bigg] = \sum_{i=1}^\infty \Big[|U_i|^2_{H^1(0,T)} + \mu_i \|U_i\|^2_{L^2(0,T)} \Big]$$
$$\leq \frac{1}{2} T^2 \sum_{i=1}^\infty \|F_i\|^2_{L^2(0,T)} = \frac{1}{2} T^2 \, \|f\|^2_{L^2(Q)}.$$

Analogous results hold for the adjoint problem (4.41). □

The variational formulation (4.40) is equivalent to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$a(u, \overline{\mathcal{H}}_T v) = \langle f, \overline{\mathcal{H}}_T v \rangle_{L^2(Q)} \tag{4.43}$$

for all $v \in H^{1,1}_{0;0,}(Q)$, where the transformation operator $\overline{\mathcal{H}}_T$, given in (4.14), acts only on the time variable t, i.e.

$$(\overline{\mathcal{H}}_T \hat{v})(x,t) = \hat{v}(x,T) - \hat{v}(x,t), \quad (x,t) \in Q, \tag{4.44}$$

for $\hat{v} \in H^{1,1}_{0;0,}(Q)$. Note that in (4.43) the ansatz and test spaces are equal and so, discretisation schemes of Galerkin-Bubnov type are possible.

The solution operators from Theorem 4.2.22 are not isomorphisms and hence, to derive from Theorem 4.2.22 an inf-sup condition, like

$$\sup_{0 \neq w \in H^{1,1}_{0;,0}(Q)} \frac{|a(u,w)|}{|w|_{H^1(Q)}} \geq C_S \, |u|_{H^1(Q)} \quad \forall u \in H^{1,1}_{0;0,}(Q) \tag{4.45}$$

with a constant $C_S > 0$, is not possible.

Theorem 4.2.23. *There does not exist a constant $C > 0$ such that each right-hand side $f \in L^2(Q)$ and the corresponding solution $u \in H^{1,1}_{0;0,}(Q)$ of* (4.40) *satisfy*

$$|u|_{H^1(Q)} \le C \|f\|_{[H^{1,1}_{0;,0}(Q)]'}. \tag{4.46}$$

In particular, the inf-sup condition (4.45) *does not hold.*

Proof. Consider the eigenfunctions $\phi_k \in H^1_0(\Omega)$ and eigenvalues $\mu_k > 0$ of the Dirichlet eigenvalue problem of the Laplacian, see (2.4). For $k \in \mathbb{N}$, take the eigenpair (ϕ_k, μ_k) and set for $(x,t) \in Q$

$$u_k(x,t) := \phi_k(x) \int_0^t s \sin(\sqrt{\mu_k} s) \, \mathrm{d}s,$$

$$f_k(x,t) := \partial_{tt} u_k(x,t) - \Delta_x u_k(x,t) = 2\phi_k(x) \sin(\sqrt{\mu_k} t).$$

The initial and boundary conditions $u_k(\cdot,0) = \partial_t u_k(\cdot,0) = 0$, $u_{k|\Sigma} = 0$ and $u_k \in H^{1,1}_{0;0,}(Q)$ are fulfilled. One computes as in Theorem 4.2.6 that

$$|u_k|_{H^1(Q)} \to \sqrt{\frac{T^3}{3}} \quad \text{and} \quad \|f_k\|_{[H^{1,1}_{0;,0}(Q)]'} \to 0 \quad \text{as } k \to \infty.$$

So, the first assertion is proven.

To show that the inf-sup condition (4.45) does not hold, the bilinear form

$$a(\cdot,\cdot)\colon\ H^{1,1}_{0;0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}$$

is investigated. Because of Lemma 4.2.21, the bilinear form $a(\cdot,\cdot)$ is bounded. In addition, for $0 \ne w \in H^{1,1}_{0;,0}(Q)$, there exists, according to Theorem 4.2.22, a unique solution $\widetilde{u} \in H^{1,1}_{0;0,}(Q)$ of (4.40) for $g = w \in L^2(Q)$, satisfying

$$a(\widetilde{u}, z) = \langle w, z \rangle_{L^2(Q)} \quad \forall z \in H^{1,1}_{0;,0}(Q).$$

Hence, for $z = w$, it follows $a(\widetilde{u}, w) = \langle w, w \rangle_{L^2(Q)} > 0$, i.e. a surjectivity condition holds. If the inf-sup condition (4.45) would be true, the bilinear form $a(\cdot,\cdot)$ would fulfil all assumptions of the Nečas Theorem 2.9.1, which gives the estimate (4.46). But this would be contradictory to the first part of this proof. □

To overcome this problem with the inf-sup condition (4.45), two different approaches are introduced in Section 4.3 and in Section 4.4. However, in the remainder of this section, conforming finite element discretisations for the variational formulation (4.40), resulting in Galerkin-Petrov schemes, are introduced and examined. Alternatively, the variational formulation (4.43) is discretised, which leads to a Galerkin-Bubnov scheme. In any case, for the bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, it is assumed that $\Omega = (0,L)$ is an interval for $d = 1$, or Ω is polygonal for $d = 2$, or Ω is polyhedral for $d = 3$. In this thesis, only the tensor-product space-time finite element space

$$Q_h^1(Q) = V_{h_x,0}(\Omega) \otimes S_{h_t}^1(0,T),$$

given in (2.42), is investigated for (4.40). Therefore, the Galerkin-Petrov finite element discretisation of the variational formulation (4.40) is to find

$$u_h \in Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q) = V_{h_x,0}(\Omega) \otimes S_{h_t,0,}^1(0,T)$$

such that

$$a(u_h, w_h) = \langle f, w_h \rangle_{L^2(Q)} \tag{4.47}$$

for all $w_h \in Q_h^1(Q) \cap H_{0;,0}^{1,1}(Q) = V_{h_x,0}(\Omega) \otimes S_{h_t,,0}^1(0,T)$. The approximate function u_h admits the representation

$$u_h(x,t) = \sum_{\ell=1}^{N_t} \sum_{j=1}^{M_x} u_j^\ell \psi_j(x) \varphi_\ell(t) = \sum_{j=1}^{M_x} U_{h_t,j}(t) \psi_j(x), \quad U_{h_t,j}(t) = \sum_{\ell=1}^{N_t} u_j^\ell \varphi_\ell(t) \tag{4.48}$$

for $(x,t) \in \overline{Q}$, see (2.43). After an appropriate ordering of the degrees of freedom, the discrete variational formulation (4.47) is equivalent to the global linear system

$$K_h \underline{u} = \underline{F}$$

with the system matrix

$$K_h = -M_{h_x} \otimes A_{h_t} + A_{h_x} \otimes M_{h_t} \in \mathbb{R}^{M_x \cdot N_t \times M_x \cdot N_t},$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote spatial mass and stiffness matrices given in (2.37) and (2.38), $M_{h_t} \in \mathbb{R}^{N_t \times N_t}$ and $A_{h_t} \in \mathbb{R}^{N_t \times N_t}$ denote temporal mass and stiffness matrices given in (4.24) and (4.23), and with the corresponding vector $\underline{F} \in \mathbb{R}^{M_x \cdot N_t}$ of the right-hand side.

Using a conforming semi-discretisation approach for the variational formulation (4.40) leads to find $\widetilde{u}_{h_x} \in V_{h_x,0}(\Omega) \otimes H_{0,}^1(0,T) \subset H_{0;0,}^{1,1}(Q)$ such that

$$a(\widetilde{u}_{h_x}, w_{h_x}) = \langle f, w_{h_x} \rangle_{L^2(Q)} \tag{4.49}$$

for all $w_{h_x} \in V_{h_x,0}(\Omega) \otimes H_{,0}^1(0,T) \subset H_{0;,0}^{1,1}(Q)$. The semi-discrete function $\widetilde{u}_{h_x}$ admits the representation

$$\widetilde{u}_{h_x}(x,t) = \sum_{j=1}^{M_x} \widetilde{U}_j(t) \psi_j(x), \quad (x,t) \in Q,$$

with the unknown temporal functions $\widetilde{U}_j \in H^1_{0,}(0,T)$ to be determined. With this representation, the semi-discrete variational formulation (4.49) is equivalent to the M_x equations to find

$$\underline{\widetilde{U}} = \left(\widetilde{U}_1, \ldots, \widetilde{U}_{M_x}\right)^\top \in [H^1_{0,}(0,T)]^{M_x}$$

such that

$$-\int_0^T M_{h_x} \partial_t \underline{\widetilde{U}}(t) \partial_t W(t) \mathrm{d}t + \int_0^T A_{h_x} \underline{\widetilde{U}}(t) W(t) \mathrm{d}t = \int_0^T \underline{f}(t) W(t) \mathrm{d}t \qquad (4.50)$$

for all $W \in H^1_{,0}(0,T)$, where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote mass and stiffness matrices given in (2.37) and (2.38) together with the right-hand side

$$\underline{f} = (f_1, \ldots, f_{M_x})^\top \in [L^2(0,T)]^{M_x},$$

defined by

$$f_j(t) := \int_\Omega f(x,t) \psi_j(x) \mathrm{d}x, \quad t \in (0,T),$$

for $j = 1, \ldots, M_x$.

By using the Cholesky decomposition

$$M_{h_x} = L_{h_x} L_{h_x}^\top$$

and for the symmetric, positive-definite matrix $\hat{A}_{h_x} := L_{h_x}^{-1} A_{h_x} L_{h_x}^{-\top}$, the decomposition

$$\hat{A}_{h_x} = V_{h_x} D_{h_x} V_{h_x}^\top, \quad D_{h_x} = \mathrm{diag}\left(\hat{\mu}_k(\hat{A}_{h_x})\right), \quad V_{h_x} = \left(\underline{v}^1, \ldots, \underline{v}^{M_x}\right), \quad \hat{A}_{h_x} \underline{v}^j = \hat{\mu}_k(\hat{A}_{h_x})\, \underline{v}^j,$$

the variational formulation (4.50) is equivalent to find

$$V_{h_x}^\top L_{h_x}^\top \underline{\widetilde{U}} = \underline{Z} = (Z_1, \ldots, Z_{M_x})^\top \in [H^1_{0,}(0,T)]^{M_x}$$

such that

$$-\int_0^T \partial_t \underline{Z}(t) \partial_t W(t) \mathrm{d}t + \int_0^T D_{h_x} \underline{Z}(t) W(t) \mathrm{d}t = \int_0^T V_{h_x}^\top L_{h_x}^{-1} \underline{f}(t) W(t) \mathrm{d}t$$

for all $W \in H^1_{,0}(0,T)$. The related approximation

$$\underline{Z}_{h_t} = (Z_{h_t,1}, \ldots, Z_{h_t,M_x})^\top$$

is defined by finding, for $j = 1, \ldots, M_x$, the functions

$$Z_{h_t,j} \in S^1_{h_t,0,}(0,T) = S^1_{h_t}(0,T) \cap H^1_{0,}(0,T)$$

such that

$$-\langle \partial_t Z_{h_t,j}, \partial_t w_{h_t} \rangle_{L^2(0,T)} + \hat{\mu}_j(\hat{A}_{h_x}) \cdot \langle Z_{h_t,j}, w_{h_t} \rangle_{L^2(0,T)} = \langle g_j, w_{h_t} \rangle_{L^2(0,T)}$$

for all $w_{h_t} \in S^1_{h_t,,0}(0,T) = S^1_{h_t}(0,T) \cap H^1_{,0}(0,T)$, where

$$\underline{g} = (g_1, \ldots, g_{M_x})^\top = V_{h_x}^\top L_{h_x}^{-1} \underline{f}$$

is the transformed right-hand side. By construction, it holds

$$\underline{Z}_{h_t} = V_{h_x}^\top L_{h_x}^\top \underline{U}_{h_t},$$

where

$$\underline{U}_{h_t} = (U_{h_t,1}, \ldots, U_{h_t,M_x})^\top$$

is the vector of coefficients of the representation (4.48).

Stability and related error estimates for the finite element solutions $Z_{h_t,j} \in S^1_{h_t,0,}(0,T)$ follow for sufficiently small time mesh sizes h_t, see (4.20). However, as in Remark 4.2.8, for a uniform time mesh size h_t, the stability of the corresponding finite difference scheme holds for

$$\hat{\mu}_j(\hat{A}_{h_x}) = \frac{[\underline{u}^j]^\top A_{h_x} \underline{u}^j}{[\underline{u}^j]^\top M_{h_x} \underline{u}^j} = \frac{\|\nabla_x u^j_{h_x}\|^2_{L^2(\Omega)}}{\|u^j_{h_x}\|^2_{L^2(\Omega)}} < \frac{12}{h_t^2} \quad \text{for } j = 1, \ldots, M_x,$$

where $\underline{u}^j = L_{h_x}^{-\top} \underline{v}^j \in \mathbb{R}^{M_x}$ are the transformed eigenvectors and $u^j_{h_x} \in V_{h_x,0}(\Omega)$ are the related functions. With the inverse inequality

$$\|\nabla_x v_{h_x}\|^2_{L^2(\Omega)} \leq c_I \, h_x^{-2} \, \|v_{h_x}\|^2_{L^2(\Omega)} \quad \forall v_{h_x} \in V_{h_x,0}(\Omega),$$

see [141, (9.19), page 217], with a constant $c_I > 0$, this condition is satisfied for

$$h_t < \sqrt{\frac{12}{c_I}} \, h_x, \tag{4.51}$$

i.e. a CFL condition is needed for stability. In the particular case $d = 1$, it holds $c_I = 12$, see the derivation of [141, (9.19), page 217] and therefore, stability follows for

$$h_t < h_x.$$

When $V_{h_x,0}(\Omega) \subset H^1_0(\Omega)$ is also of tensor-product structure, i.e.

$$V_{h_x,0}(\Omega) = \left(S^1_{h_{x_1}}(0,L_1) \otimes \ldots \otimes S^1_{h_{x_d}}(0,L_d) \right) \cap H^1_0(\Omega),$$

for example, when considering $\Omega = (0,L_1)\times\ldots\times(0,L_d)\subset\mathbb{R}^d$ with uniform mesh sizes $h_{x_1},\ldots,h_{x_d}$, one concludes $c_I = 12d$, and therefore, the stability condition

$$h_t < \frac{h_{x,\min}}{\sqrt{d}},$$

where $h_{x,\min} = \min\{h_{x_1},\ldots,h_{x_d}\}$.

As a numerical example, consider for $d=2$ the spatial domain $\Omega = (0,1)^2$ with uniform discretisations with mesh sizes $h_x = h_{x_1} = h_{x_2}$ and the exact solution

$$u(x_1,x_2,t) = t^2\sin(\pi x_1)\sin(\pi x_2) \quad \text{for } (x_1,x_2,t)\in Q = \Omega\times(0,T)$$

with different finite times $T \in \left\{\frac{7}{10}, \frac{1}{\sqrt{2}}, \frac{3}{4}, 1, 2\right\}$. Then stability follows when choosing

$$\frac{h_t}{h_x} < \frac{1}{\sqrt{2}} \approx 0.7071068. \tag{4.52}$$

In Table 4.7, Table 4.8, Table 4.9, Table 4.10 and Table 4.11, the $L^2(Q)$ error, $H^1(Q)$ error and the spectral condition number $\kappa_2(K_h)$ of the related system matrix K_h of the Galerkin-Petrov formulation (4.47) are given, where the observed convergence rates are as expected, provided the CFL condition (4.52) is satisfied, i.e. the CFL condition (4.52) seems to be sharp. Here, the degrees of freedom are denoted by

$$\text{dof} = \dim Q_h^1(Q)\cap H^{1,1}_{0;0,}(Q) = \dim Q_h^1(Q)\cap H^{1,1}_{0;,0}(Q).$$

dof	h_x	h_t	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_{H^1(Q)}$	eoc	$\kappa_2(K_h)$
2	0.5000000	0.3500000	2.0373e-02	-	3.8884e-01	-	1.0544e+00
36	0.2500000	0.1750000	4.7396e-03	2.1	1.9320e-01	1.0	4.8152e+01
392	0.1250000	0.0875000	1.1608e-03	2.0	9.6190e-02	1.0	2.5261e+02
3600	0.0625000	0.0437500	2.8868e-04	2.0	4.8036e-02	1.0	9.4412e+02
30752	0.0312500	0.0218750	7.2073e-05	2.0	2.4011e-02	1.0	3.9542e+03
254016	0.0156250	0.0109375	1.8012e-05	2.0	1.2004e-02	1.0	1.5656e+04

Table 4.7: Numerical results for the Galerkin-Petrov formulation (4.47) with uniform meshes for $Q = (0,1)^2\times(0,\frac{7}{10})$, satisfying the CFL condition (4.52).

From (4.51), one only expects conditional stability of (4.47). To stabilise the numerical scheme in (4.47), Zlotnik's idea [164], as in (4.28), is used again, and the following representation is proven.

dof	h_x	h_t	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_{H^1(Q)}$	eoc	$\kappa_2(K_h)$
2	0.5000000	0.3535534	2.0970e-02	-	3.9813e-01	-	1.0000e+00
36	0.2500000	0.1767767	4.8903e-03	2.1	1.9798e-01	1.0	5.1911e+01
392	0.1250000	0.0883883	1.1986e-03	2.0	9.8593e-02	1.0	2.4119e+02
3600	0.0625000	0.0441942	2.9813e-04	2.0	4.9240e-02	1.0	1.0614e+03
30752	0.0312500	0.0220971	7.4437e-05	2.0	2.4613e-02	1.0	4.8885e+03
254016	0.0156250	0.0110485	1.8603e-05	2.0	1.2305e-02	1.0	1.9849e+04

Table 4.8: Numerical results for the Galerkin-Petrov formulation (4.47) with uniform meshes for $Q=(0,1)^2\times(0,\frac{1}{\sqrt{2}})$ for the limit case of the CFL condition (4.52).

dof	h_x	h_t	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_{H^1(Q)}$	eoc	$\kappa_2(K_h)$
2	0.5000000	0.3750000	2.4795e-02	-	4.5709e-01	-	1.3456e+00
36	0.2500000	0.1875000	5.8640e-03	2.1	2.2835e-01	1.0	5.5029e+01
392	0.1250000	0.0937500	1.4435e-03	2.0	1.1387e-01	1.0	6.5707e+02
3600	0.0625000	0.0468750	3.5946e-04	2.0	5.6892e-02	1.0	2.0536e+04
30752	0.0312500	0.0234375	8.9775e-05	2.0	2.8440e-02	1.0	1.1774e+07
254016	0.0156250	0.0117188	2.2438e-05	2.0	1.4219e-02	1.0	3.1270e+12

Table 4.9: Numerical results for the Galerkin-Petrov formulation (4.47) with uniform meshes for $Q=(0,1)^2\times(0,\frac{3}{4})$, violating the CFL condition (4.52).

dof	h_x	h_t	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_{H^1(Q)}$	eoc	$\kappa_2(K_h)$
2	0.500000	0.500000	5.4253e-02	-	9.0857e-01	-	4.0000e+00
36	0.250000	0.250000	1.3532e-02	2.0	4.6108e-01	1.0	2.0727e+02
392	0.125000	0.125000	3.3817e-03	2.0	2.3115e-01	1.0	9.2901e+03
3600	0.062500	0.062500	8.4531e-04	2.0	1.1564e-01	1.0	1.0357e+07
30752	0.031250	0.031250	2.1132e-04	2.0	5.7831e-02	1.0	1.1044e+13
254016	0.015625	0.015625	4.7793e+08	-41.0	2.2237e+11	-41.8	1.6503e+28

Table 4.10: Numerical results for the Galerkin-Petrov formulation (4.47) with uniform meshes for $Q=(0,1)^2\times(0,1)$, violating the CFL condition (4.52).

dof	h_x	h_t	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_{H^1(Q)}$	eoc	$\kappa_2(K_h)$
2	0.500000	1.000000	2.7814e-01	-	5.2099e+00	-	1.4440e+01
36	0.250000	0.500000	7.3571e-02	1.9	2.5832e+00	1.0	4.7953e+02
392	0.125000	0.250000	1.8630e-02	2.0	1.2955e+00	1.0	8.5065e+04
3600	0.062500	0.125000	4.7326e-03	2.0	6.4816e-01	1.0	1.3222e+09
30752	0.031250	0.062500	4.8264e+00	-10.0	9.8287e+02	-10.6	4.5260e+19
254016	0.015625	0.031250	2.4079e+23	-75.4	8.8587e+25	-76.3	1.3550e+39

Table 4.11: Numerical results for the Galerkin-Petrov formulation (4.47) with uniform meshes for $Q=(0,1)^2\times(0,2)$, violating the CFL condition (4.52).

Lemma 4.2.24. *For all $u_h\in Q_h^1(Q)\cap H^{1,1}_{0;0,}(Q)$ and $w_h\in Q_h^1(Q)\cap H^{1,1}_{0;,0}(Q)$, the bilinear form in* (4.47) *has the representation*

$$a(u_h,w_h)=-\langle\partial_t u_h,\partial_t w_h\rangle_{L^2(Q)}+\sum_{m=1}^{d}\langle\partial_{x_m}u_h,Q^0_{h_t}\partial_{x_m}w_h\rangle_{L^2(Q)}$$
$$+\sum_{m=1}^{d}\sum_{\ell=1}^{N_t}\frac{h^2_{t,\ell}}{12}\langle\partial_t\partial_{x_m}u_h,\partial_t\partial_{x_m}w_h\rangle_{L^2(\Omega\times\tau_\ell)},\quad(4.53)$$

where $Q^0_{h_t}\colon L^2(0,T)\to S^0_{h_t}(0,T)$ denotes the L^2 projection (2.26) *with respect to time on the space $S^0_{h_t}(0,T)$ of piecewise constant functions.*

Proof. Let $u_h\in Q_h^1(Q)\cap H^{1,1}_{0;0,}(Q)$ and $w_h\in Q_h^1(Q)\cap H^{1,1}_{0;,0}(Q)$ be given. With the representation (2.43), there follow for $(x,t)\in\overline{Q}$

$$u_h(x,t)=\sum_{\ell=1}^{N_t}\sum_{j=1}^{M_x}u_j^\ell\psi_j(x)\varphi_\ell(t)=\sum_{j=1}^{M_x}U_{h_t,j}(t)\psi_j(x),\quad U_{h_t,j}(t)=\sum_{\ell=1}^{N_t}u_j^\ell\varphi_\ell(t)$$

and

$$w_h(x,t)=\sum_{\ell=0}^{N_t-1}\sum_{j=1}^{M_x}w_j^\ell\psi_j(x)\varphi_\ell(t)=\sum_{j=1}^{M_x}W_{h_t,j}(t)\psi_j(x),\quad W_{h_t,j}(t)=\sum_{\ell=0}^{N_t-1}w_j^\ell\varphi_\ell(t).$$

Hence, for $m=1,\dots,d$ and by using (4.27), there hold

$$\langle\partial_{x_m}u_h,\partial_{x_m}w_h\rangle_{L^2(Q)}=\sum_{i=1}^{M_x}\sum_{j=1}^{M_x}\int_0^T U_{h_t,i}(t)W_{h_t,j}(t)\,\mathrm{d}t\int_\Omega\partial_{x_m}\psi_i(x)\partial_{x_m}\psi_j(x)\,\mathrm{d}x$$
$$=\sum_{i=1}^{M_x}\sum_{j=1}^{M_x}\left[\frac{1}{12}\sum_{\ell=1}^{N_t}h^2_{t,\ell}\langle\partial_t U_{h_t,i},\partial_t W_{h_t,j}\rangle_{L^2(\tau_\ell)}+\langle U_{h_t,i},Q^0_{h_t}W_{h_t,j}\rangle_{L^2(0,T)}\right]$$
$$\cdot\int_\Omega\partial_{x_m}\psi_i(x)\partial_{x_m}\psi_j(x)\,\mathrm{d}x$$

and thus,

$$\langle \partial_{x_m} u_h, \partial_{x_m} w_h \rangle_{L^2(Q)} = \langle \partial_{x_m} u_h, Q^0_{h_t} \partial_{x_m} w_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \partial_{x_m} u_h, \partial_t \partial_{x_m} w_h \rangle_{L^2(\Omega\times\tau_\ell)}.$$

So, the assertion is proven. □

Due to the representation (4.53), one defines for the functions $u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ and $w_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)$ the perturbed bilinear form

$$\begin{aligned} a_h(u_h, w_h) &:= -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{m=1}^{d} \langle \partial_{x_m} u_h, Q^0_{h_t} \partial_{x_m} w_h \rangle_{L^2(Q)} \\ &= -\langle \partial_t u_h, \partial_t w_h \rangle_{L^2(Q)} + \sum_{m=1}^{d} \langle Q^0_{h_t} \partial_{x_m} u_h, \partial_{x_m} w_h \rangle_{L^2(Q)}. \end{aligned} \tag{4.54}$$

So, the perturbed variational problem is to find $\widetilde{u}_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ such that

$$a_h(\widetilde{u}_h, w_h) = \langle f, w_h \rangle_{L^2(Q)} \tag{4.55}$$

for all $w_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)$. After an appropriate ordering of the degrees of freedom, the discrete variational formulation (4.55) is equivalent to the global linear system

$$\widetilde{K}_h \underline{u} = \underline{F}$$

with the system matrix

$$\widetilde{K}_h = -M_{h_x} \otimes A_{h_t} + A_{h_x} \otimes \widetilde{M}_{h_t} \in \mathbb{R}^{M_x \cdot N_t \times M_x \cdot N_t},$$

where $M_{h_x} \in \mathbb{R}^{M_x \times M_x}$ and $A_{h_x} \in \mathbb{R}^{M_x \times M_x}$ denote spatial mass and stiffness matrices given in (2.37) and (2.38), $\widetilde{M}_{h_t} \in \mathbb{R}^{N_t \times N_t}$ is the temporal perturbed mass matrix (4.32), and $A_{h_t} \in \mathbb{R}^{N_t \times N_t}$ is the temporal stiffness matrix (4.23), and with the corresponding vector $\underline{F} \in \mathbb{R}^{M_x \cdot N_t}$ of the right-hand side.

To prove the existence and uniqueness of a solution $\widetilde{u}_h$ of (4.55), the following lemma, which is analogous to Lemma 4.2.13, is shown.

Lemma 4.2.25. *For a given $v_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$, represented by*

$$v_h(x,t) = \sum_{\ell=0}^{N_t} V_{h_x,\ell}(x)\varphi_\ell(t), \quad V_{h_x,\ell}(x) = \sum_{j=1}^{M_x} v_j^\ell \psi_j(x) \quad \textit{for } (x,t) \in \overline{Q}$$

with $V_{h_x,0}(x) = 0$ for $x \in \Omega$ and $V_{h_x,\ell} \in V_{h_x,0}(\Omega)$, and for a fixed index $j \in \{1,\dots,N_t\}$, there exists a function $\overline{z}_h^j \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)$ with the following properties:

1. *For* $(x,t) \in \overline{\Omega} \times [t_j, T]$, *it holds* $\overline{z}_h^j(x,t) = 0$.
2. *For* $\ell = 1, \dots, j$ *and for* $x \in \overline{\Omega}$, *there hold*

$$\langle \partial_t \overline{z}_h^j(x,\cdot), \partial_t v_h(x,\cdot) \rangle_{L^2(\tau_\ell)} = \frac{1}{2}\Big([V_{h_x,\ell-1}(x)]^2 - [V_{h_x,\ell}(x)]^2 \Big)$$

and

$$\langle \partial_{x_m} \overline{z}_h^j(x,\cdot), Q_{h_t}^0 \partial_{x_m} v_h(x,\cdot) \rangle_{L^2(\tau_\ell)} = \frac{1}{2} \left(\int_{t_{\ell-1}}^{t_j} \partial_{x_m} v_h(x,s) \mathrm{d}s \right)^2 - \frac{1}{2} \left(\int_{t_\ell}^{t_j} \partial_{x_m} v_h(x,s) \mathrm{d}s \right)^2$$

for $m = 1, \dots, d$.

3. *For* $x \in \overline{\Omega}$, *there holds the estimate*

$$\|\partial_t \overline{z}_h^j(x,\cdot)\|_{L^2(0,T)} \le \|v_h(x,\cdot)\|_{L^2(0,T)}.$$

Proof. For $v_h \in Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q)$, one defines for $(x,t) \in \overline{Q}$

$$u_h^j(x,t) := \sum_{i=0}^{N_t} U_{h_x,i}^j(x) \varphi_i(t), \quad U_{h_x,i}^j(x) := \begin{cases} V_{h_x,i}(x) & \text{for } i = 0, \dots, j, \\ (-1)^{j-i} V_{h_x,j}(x) & \text{for } i = j+1, \dots, N_t, \end{cases}$$

and further,

$$\overline{z}_h^j(x,t) := -\sum_{k=0}^{N_t} \left(\int_T^{t_k} u_h^j(x,s)\,\mathrm{d}s \right) \varphi_k(t) = -\left(I_{h_t} \int_T^{(\cdot)} u_h^j(x,s)\,\mathrm{d}s \right)(t),$$

where $I_{h_t}\colon\ C[0,T] \to S_{h_t}^1(0,T)$ is the interpolation operator with respect to time, see (2.27). Note that $\overline{z}_h^j \in Q_h^1(Q) \cap H_{0;,0}^{1,1}(Q)$. For $x \in \overline{\Omega}$, it follows from relation (4.26) that

$$\partial_t \overline{z}_h^j(x,\cdot) = -Q_{h_t}^0 u_h^j(x,\cdot).$$

In particular, for $j < N_t$, $x \in \overline{\Omega}$, and for $t \in \tau_\ell$ for $\ell = j+1, \dots, N_t$, then it holds

$$-\partial_t \overline{z}_h^j(x,t) = Q_{h_t}^0 u_h^j(x,t) = \frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} u_h^j(x,s)\,\mathrm{d}s = \frac{1}{2}\Big(U_{h_x,\ell-1}^j(x) + U_{h_x,\ell}^j(x) \Big) = 0,$$

and due to $\overline{z}_h^j(x,T) = 0$, one concludes $\overline{z}_h^j(x,t) = 0$ for $t \in [t_j, T]$, i.e. the first assertion.

To prove the second assertion, one computes for $x \in \overline{\Omega}$ and for $\ell = 1, \ldots, j$

$$\begin{aligned}
\langle \partial_t \overline{z}_h^j(x,\cdot), \partial_t v_h(x,\cdot)\rangle_{L^2(\tau_\ell)} &= \int_{t_{\ell-1}}^{t_\ell} \partial_t \overline{z}_h^j(x,t)\, \partial_t v_h(x,t)\,\mathrm{d}t \\
&= -\frac{1}{2}\Big(U_{h_x,\ell-1}^j(x) + U_{h_x,\ell}^j(x)\Big) \int_{t_{\ell-1}}^{t_\ell} \partial_t v_h(x,t)\,\mathrm{d}t \\
&= -\frac{1}{2}\Big(U_{h_x,\ell-1}^j(x) + U_{h_x,\ell}^j(x)\Big)\Big(V_{h_x,\ell}(x) - V_{h_x,\ell-1}(x)\Big) \\
&= \frac{1}{2}\left(V_{h_x,\ell-1}(x) + V_{h_x,\ell}(x)\right)\Big(V_{h_x,\ell-1}(x) - V_{h_x,\ell}(x)\Big) \\
&= \frac{1}{2}\Big([V_{h_x,\ell-1}(x)]^2 - [V_{h_x,\ell}(x)]^2\Big).
\end{aligned}$$

Moreover, for $m = 1, \ldots, d$, there follow for $x \in \overline{\Omega}$ and for $\ell = 1, \ldots, j$

$$\langle \partial_{x_m} \overline{z}_h^j(x,\cdot), Q_{h_t}^0 \partial_{x_m} v_h(x,\cdot)\rangle_{L^2(\tau_\ell)} = Q_{h_t}^0 \partial_{x_m} v_h(x,\cdot)_{|\tau_\ell} \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m} \overline{z}_h^j(x,t)\,\mathrm{d}t$$

$$\begin{aligned}
&= -Q_{h_t}^0 \partial_{x_m} v_h(x,\cdot)_{|\tau_\ell} \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m} \left[\int_T^{t_{\ell-1}} u_h^j(x,s)\mathrm{d}s\, \varphi_{\ell-1}(t) + \int_T^{t_\ell} u_h^j(x,s)\mathrm{d}s\, \varphi_\ell(t)\right] \mathrm{d}t \\
&= -\frac{1}{h_{t,\ell}} \int_{t_{\ell-1}}^{t_\ell} \partial_{x_m} v_h(x,t)\mathrm{d}t\, \frac{1}{2} h_{t,\ell} \left[\int_T^{t_{\ell-1}} \partial_{x_m} u_h^j(x,s)\mathrm{d}s + \int_T^{t_\ell} \partial_{x_m} u_h^j(x,s)\mathrm{d}s\right]
\end{aligned}$$

and further,

$$\begin{aligned}
\langle \partial_{x_m} \overline{z}_h^j(x,\cdot), Q_{h_t}^0 \partial_{x_m} v_h(x,\cdot)\rangle_{L^2(\tau_\ell)} &= -\frac{1}{2}\left[\int_{t_j}^{t_\ell} \partial_{x_m} v_h(x,t)\mathrm{d}t - \int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h(x,t)\mathrm{d}t\right] \\
&\quad \cdot \left[\int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h^j(x,s)\mathrm{d}s + \int_{t_j}^{t_\ell} \partial_{x_m} v_h^j(x,s)\mathrm{d}s\right] \\
&= \frac{1}{2}\left(\int_{t_j}^{t_{\ell-1}} \partial_{x_m} v_h^j(x,s)\mathrm{d}s\right)^2 - \frac{1}{2}\left(\int_{t_j}^{t_\ell} \partial_{x_m} v_h^j(x,s)\mathrm{d}s\right)^2.
\end{aligned}$$

The L^2 stability of $Q_{h_t}^0$ and using

$$\begin{aligned}
\|\partial_t \overline{z}_h^j(x,\cdot)\|_{L^2(0,T)} &= \|\partial_t \overline{z}_h^j(x,\cdot)\|_{L^2(0,t_j)} = \|Q_{h_t}^0 u_h^j(x,\cdot)\|_{L^2(0,t_j)} \\
&= \|Q_{h_t}^0 v_h(x,\cdot)\|_{L^2(0,t_j)} \le \|Q_{h_t}^0 v_h(x,\cdot)\|_{L^2(0,T)} \le \|v_h(x,\cdot)\|_{L^2(0,T)}
\end{aligned}$$

for $x \in \overline{\Omega}$ yield the third property. □

With the last lemma, the existence, uniqueness, and stability of a solution $\widetilde{u}_h$ of (4.55) are proven with the help of the next lemma.

Lemma 4.2.26. *Let $f_0 \in [H^1_{,0}(0,T;L^2(\Omega))]'$ and $f_1, f_2 \in L^2(Q)$ be given. The variational formulation to find $w_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ such that*

$$a_h(w_h, v_h) = \langle f_0, v_h \rangle_Q + \langle f_1, \partial_t v_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell} \langle f_2, \partial_t v_h \rangle_{L^2(\Omega\times\tau_\ell)} \tag{4.56}$$

for all $v_h \in Q^1_h(Q) \cap H^{1,1}_{0;,0}(Q)$ is uniquely solvable, and there holds the stability estimate

$$\|w_h\|_{L^2(Q)} \le 2T \left\{ \|f_0\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2 \|f_2\|_{L^2(Q)} \right\}. \tag{4.57}$$

Proof. Let $w^0_h \in Q^1_h(Q) \cap H^{1,1}_{0;0,}(Q)$ be any solution of the homogeneous variational formulation (4.56) with $f_i \equiv 0$, and with the representation (2.43), i.e.

$$w^0_h(x,t) = \sum_{\ell=0}^{N_t} W^0_{h_x,\ell}(x)\varphi_\ell(t) \quad \text{for } (x,t) \in \overline{Q}, \quad W^0_{h_x,\ell} \in V_{h_x,0}(\Omega), \quad W^0_{h_x,0}(x) = 0.$$

For each index $j \in \{1,\dots,N_t\}$, consider an element $\overline{z}^j_h \in Q^1_h(Q) \cap H^{1,1}_{0;,0}(Q)$ as given in Lemma 4.2.25. Plugging $v_h = \overline{z}^j_h$ into (4.56), there hold

$$\begin{aligned} 0 = a_h(w^0_h, \overline{z}^j_h) &= -\langle \partial_t w^0_h, \partial_t \overline{z}^j_h \rangle_{L^2(Q)} + \sum_{m=1}^{d} \langle Q^0_{h_t} \partial_{x_m} w^0_h, \partial_{x_m} \overline{z}^j_h \rangle_{L^2(Q)} \\ &= -\sum_{\ell=1}^{j} \langle \partial_t w^0_h, \partial_t \overline{z}^j_h \rangle_{L^2(\Omega\times\tau_\ell)} + \sum_{m=1}^{d} \sum_{\ell=1}^{j} \langle Q^0_{h_t} \partial_{x_m} w^0_h, \partial_{x_m} \overline{z}^j_h \rangle_{L^2(\Omega\times\tau_\ell)} \end{aligned}$$

and by using the properties of Lemma 4.2.25,

$$\begin{aligned} 0 = a_h(w^0_h, \overline{z}^j_h) = &-\int_\Omega \sum_{\ell=1}^{j} \left(\frac{1}{2}[W^0_{h_x,\ell-1}(x)]^2 - \frac{1}{2}[W^0_{h_x,\ell}(x)]^2 \right) \mathrm{d}x \\ &+ \sum_{m=1}^{d} \int_\Omega \sum_{\ell=1}^{j} \left(\frac{1}{2} \left(\int_{t_{\ell-1}}^{t_j} \partial_{x_m} w^0_h(x,s)\,\mathrm{d}s \right)^2 - \frac{1}{2} \left(\int_{t_\ell}^{t_j} \partial_{x_m} w^0_h(x,s)\,\mathrm{d}s \right)^2 \right) \mathrm{d}x \\ = &\frac{1}{2} \int_\Omega [W^0_{h_x,j}(x)]^2\,\mathrm{d}x + \frac{1}{2} \sum_{m=1}^{d} \int_\Omega \left(\int_0^{t_j} \partial_{x_m} w^0_h(x,s)\,\mathrm{d}s \right)^2 \mathrm{d}x. \end{aligned}$$

This result yields, with the Cauchy-Schwarz inequality and the use of the properties of Lemma 4.2.25,

$$\begin{aligned}
\|w_h^0\|^2_{L^2(Q)} &= \sum_{\ell=1}^{N_t} \|w_h^0\|^2_{L^2(\Omega\times\tau_\ell)} \\
&= \int_\Omega \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}}{3}\Big([W^0_{h_x,\ell}(x)]^2 + W^0_{h_x,\ell}(x)W^0_{h_x,\ell-1}(x) + [W^0_{h_x,\ell-1}(x)]^2\Big)\,\mathrm{d}x \\
&\le \int_\Omega \sum_{j=1}^{N_t} \frac{h_{t,j}}{2}[W^0_{h_x,j}(x)]^2\,\mathrm{d}x + \int_\Omega \sum_{j=1}^{N_t-1} \frac{h_{t,j+1}}{2}[W^0_{h_x,j}(x)]^2\,\mathrm{d}x \le 0,
\end{aligned}$$

which implies $w_h^0 \equiv 0$. Therefore, by using

$$\dim Q_h^1(Q)\cap H^{1,1}_{0;0,}(Q) = \dim Q_h^1(Q)\cap H^{1,1}_{0;,0}(Q),$$

one concludes unique solvability of the variational formulation (4.56) for any right-hand sides $f_0 \in [H^1_{,0}(0,T;L^2(\Omega))]'$ and $f_1, f_2 \in L^2(Q)$. Following the approach as above, there hold

$$\langle f_0, \bar{z}_h^j\rangle_Q + \langle f_1, \partial_t \bar{z}_h^j\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell}\langle f_2, \partial_t\bar{z}_h^j\rangle_{L^2(\Omega\times\tau_\ell)} = a_h(w_h, \bar{z}_h^j) \ge \frac{1}{2}\int_\Omega [W_{h_x,j}(x)]^2\,\mathrm{d}x$$

and

$$\begin{aligned}
\|w_h\|^2_{L^2(Q)} &\le \int_\Omega \sum_{j=1}^{N_t} \frac{h_{t,j}}{2}[W_{h_x,j}(x)]^2\,\mathrm{d}x + \int_\Omega \sum_{j=1}^{N_t-1} \frac{h_{t,j+1}}{2}[W_{h_x,j}(x)]^2\,\mathrm{d}x \\
&\le \sum_{j=1}^{N_t} h_{t,j}\left\{\langle f_0, \bar{z}_h^j\rangle_Q + \langle f_1, \partial_t\bar{z}_h^j\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell}\langle f_2, \partial_t\bar{z}_h^j\rangle_{L^2(\Omega\times\tau_\ell)}\right\} \\
&\quad + \sum_{j=1}^{N_t-1} h_{t,j+1}\left\{\langle f_0, \bar{z}_h^j\rangle_Q + \langle f_1, \partial_t\bar{z}_h^j\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} h^2_{t,\ell}\langle f_2, \partial_t\bar{z}_h^j\rangle_{L^2(\Omega\times\tau_\ell)}\right\} \\
&\le \sum_{j=1}^{N_t} h_{t,j}\Big\{\|f_0\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2\,\|f_2\|_{L^2(Q)}\Big\}\|\partial_t\bar{z}_h^j\|_{L^2(Q)} \\
&\quad + \sum_{j=1}^{N_t-1} h_{t,j+1}\Big\{\|f_0\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2\,\|f_2\|_{L^2(Q)}\Big\}\|\partial_t\bar{z}_h^j\|_{L^2(Q)} \\
&\le 2T\left\{\|f_0\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|f_1\|_{L^2(Q)} + h_t^2\,\|f_2\|_{L^2(Q)}\right\}\|w_h\|_{L^2(Q)},
\end{aligned}$$

and hence, the stability estimate is proven. □

Theorem 4.2.27. *For $f \in L^2(Q)$, there exists a unique solution $\widetilde{u}_h \in Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q)$ of the Galerkin-Petrov finite element discretisation* (4.55)*, satisfying the stability estimate*

$$\|\widetilde{u}_h\|_{L^2(Q)} \le 2T\|f\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} \le \frac{4}{\pi}T^2\|f\|_{L^2(Q)}.$$

Proof. Setting $f_0 = f \in L^2(Q)$, $\;f_1 = f_2 = 0$ in (4.56) and the Poincaré inequality with respect to time, see Lemma 3.4.5, give the assertion. □

Remark 4.2.28. *Note that there exists a unique solution of the Galerkin-Petrov finite element discretisation* (4.55) *in the situation of Lemma 4.2.26, i.e. for a right-hand side f weaker than $L^2(Q)$.*

To derive an estimate for the $L^2(Q)$ error $\|u - \widetilde{u}_h\|_{L^2(Q)}$, the space-time projection

$$Q_{h_t}^1 Q_{h_x}^1 v \in Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q),$$

is used, which is well-defined for a sufficiently smooth function $v \in H_{0;0,}^{1,1}(Q)$, for details see Lemma 2.8.2.

Theorem 4.2.29. *Let $u \in H_{0;0,}^{1,1}(Q)$ be the unique solution of the variational formulation* (4.40)*, satisfying $\partial_t u \in L^2(0,T;H_0^1(\Omega))$ and $\partial_{x_m} u \in H_{0,}^1(0,T;L^2(\Omega))$, $m = 1,\ldots,d$, and $\Delta_x u \in H_{0,}^1(0,T;L^2(\Omega))$. Then the unique solution $\widetilde{u}_h \in Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q)$ of the Galerkin-Petrov finite element discretisation* (4.55) *satisfies the error estimate*

$$\begin{aligned}\|u - \widetilde{u}_h\|_{L^2(Q)} &\le \|u - Q_{h_t}^1 u\|_{L^2(Q)} + \|u - Q_{h_x}^1 u\|_{L^2(Q)} + c\,h_x h_t \|\partial_t \nabla_x u\|_{L^2(Q)} \\ &\quad + 2T\Big\{\|\Delta_x(u - Q_{h_t}^1 u)\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|\partial_t(Q_{h_x}^1 u - u)\|_{L^2(Q)} + \frac{h_t^2}{12}\|\partial_t \Delta_x u\|_{L^2(Q)}\Big\}\end{aligned}$$

with a constant $c > 0$.

Proof. Since the solution $u \in H_{0;0,}^{1,1}(Q)$ fulfils the assumptions of Lemma 2.8.2, the space-time projection $Q_{h_t}^1 Q_{h_x}^1 u \in Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q)$ is well-defined. When using the representation (4.53), the properties of $Q_{h_t}^1 Q_{h_x}^1$ as given in Lemma 2.8.2, and applying integration by parts, it follows for all $w_h \in Q_h^1(Q) \cap H_{0;,0}^{1,1}(Q)$ that

$$\begin{aligned} a_h(\widetilde{u}_h - Q_{h_t}^1 Q_{h_x}^1 u, w_h) &= a_h(\widetilde{u}_h, w_h) - a_h(Q_{h_t}^1 Q_{h_x}^1 u, w_h) \\ &= a(u, w_h) - a_h(Q_{h_t}^1 Q_{h_x}^1 u, w_h) \\ &= a(u, w_h) - a(Q_{h_t}^1 Q_{h_x}^1 u, w_h) + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12}\langle \partial_t \nabla_x Q_{h_t}^1 Q_{h_x}^1 u, \partial_t \nabla_x w_h\rangle_{L^2(\Omega\times\tau_\ell)} \\ &= -\langle \partial_t u, \partial_t w_h\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w_h\rangle_{L^2(Q)} + \langle \partial_t Q_{h_t}^1 Q_{h_x}^1 u, \partial_t w_h\rangle_{L^2(Q)} \\ &\quad - \langle \nabla_x Q_{h_t}^1 Q_{h_x}^1 u, \nabla_x w_h\rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12}\langle \partial_t \nabla_x Q_{h_t}^1 Q_{h_x}^1 u, \partial_t \nabla_x w_h\rangle_{L^2(\Omega\times\tau_\ell)} \end{aligned}$$

and further,

$$\begin{aligned} a_h(\widetilde{u}_h - Q^1_{h_t} Q^1_{h_x} u, w_h) &= -\langle \partial_t u, \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x w_h \rangle_{L^2(Q)} + \langle \partial_t Q^1_{h_x} u, \partial_t w_h \rangle_{L^2(Q)} \\ &\quad - \langle \nabla_x Q^1_{h_t} u, \nabla_x w_h \rangle_{L^2(Q)} + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q^1_{h_t} u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega \times \tau_\ell)} \\ &= \langle \partial_t (Q^1_{h_x} u - u), \partial_t w_h \rangle_{L^2(Q)} + \langle \nabla_x (u - Q^1_{h_t} u), \nabla_x w_h \rangle_{L^2(Q)} \\ &\quad + \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \nabla_x Q^1_{h_t} u, \partial_t \nabla_x w_h \rangle_{L^2(\Omega \times \tau_\ell)} \\ &= \langle \partial_t (Q^1_{h_x} u - u), \partial_t w_h \rangle_{L^2(Q)} + \langle -\Delta_x (u - Q^1_{h_t} u), w_h \rangle_{L^2(Q)} \\ &\quad - \sum_{\ell=1}^{N_t} \frac{h_{t,\ell}^2}{12} \langle \partial_t \Delta_x Q^1_{h_t} u, \partial_t w_h \rangle_{L^2(\Omega \times \tau_\ell)}. \end{aligned}$$

In particular, one observes that $\widetilde{u}_h - Q^1_{h_t} Q^1_{h_x} u$ is the unique solution of (4.56) in the case

$$f_0 = -\Delta_x (u - Q^1_{h_t} u), \quad f_1 = \partial_t (Q^1_{h_x} u - u), \quad f_2 = -\frac{1}{12} \partial_t \Delta_x Q^1_{h_t} u.$$

Therefore, the stability estimate (4.57) and the stability of $Q^1_{h_t}$ in $H^1_{0,}(0,T)$ give

$$\begin{aligned} &\|\widetilde{u}_h - Q^1_{h_t} Q^1_{h_x} u\|_{L^2(Q)} \\ &\quad \le 2T \left\{ \|\Delta_x (u - Q^1_{h_t}) u\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|\partial_t (Q^1_{h_x} u - u)\|_{L^2(Q)} + \frac{h_t^2}{12} \|\partial_t \Delta_x Q^1_{h_t} u\|_{L^2(Q)} \right\} \\ &\quad \le 2T \left\{ \|\Delta_x (u - Q^1_{h_t}) u\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|\partial_t (Q^1_{h_x} u - u)\|_{L^2(Q)} + \frac{h_t^2}{12} \|\partial_t \Delta_x u\|_{L^2(Q)} \right\}. \end{aligned}$$

With the last estimate, the triangle inequality, and the error estimate of Lemma 2.8.2, there hold

$$\begin{aligned} \|u - \widetilde{u}_h\|_{L^2(Q)} &\le \|u - Q^1_{h_t} Q^1_{h_x} u\|_{L^2(Q)} + \|\widetilde{u}_h - Q^1_{h_t} Q^1_{h_x} u\|_{L^2(Q)} \\ &\le \|u - Q^1_{h_t} u\|_{L^2(Q)} + \|u - Q^1_{h_x} u\|_{L^2(Q)} + c\, h_x h_t \|\partial_t \nabla_x u\|_{L^2(Q)} \\ &\quad + 2T \left\{ \|\Delta_x (u - Q^1_{h_t} u)\|_{[H^1_{,0}(0,T;L^2(\Omega))]'} + \|\partial_t (Q^1_{h_x} u - u)\|_{L^2(Q)} + \frac{h_t^2}{12} \|\partial_t \Delta_x u\|_{L^2(Q)} \right\} \end{aligned}$$

with a constant $c > 0$ and so, the assertion. □

Corollary 4.2.30. *Let the assumptions of Theorem 4.2.29 be satisfied. If, in addition, the unique solution u of* (4.40) *is sufficiently smooth and the spatial H^1_0 projection $Q^1_{h_x}$ fulfils the standard L^2 error estimate*

$$\|u - Q^1_{h_x} u\|_{L^2(Q)} \le C h_x^2 \|u\|_{L^2(0,T;H^2(\Omega))}$$

with a constant $C > 0$, *see* (2.48), *then, for the unique solution* $\widetilde{u}_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ *of the Galerkin-Petrov finite element discretisation* (4.55), *it holds the error estimate*

$$\begin{aligned}\|u-\widetilde{u}_h\|_{L^2(Q)} &\le c\,h_x^2\left(\|u\|_{L^2(0,T;H^2(\Omega))}+\|\partial_t u\|_{L^2(0,T;H^2(\Omega))}\right)\\ &\quad + c\,h_x\,h_t\|\partial_t\nabla_x u\|_{L^2(Q)} + c\,h_t^2\left(\|\partial_{tt}u\|_{L^2(Q)}+\|\partial_{tt}\Delta_x u\|_{L^2(Q)}+\|\partial_t\Delta_x u\|_{L^2(Q)}\right) \end{aligned} \tag{4.58}$$

with a constant $c > 0$.

Proof. By using the error estimates (2.50) for the $H^1_{0,}$ projection $Q^1_{h_t}$ and (2.48) for the H^1_0 projection $Q^1_{h_x}$, it follows from Theorem 4.2.29 the asserted error estimate. □

Corollary 4.2.31. *Let the assumption of Theorem 4.2.29 be satisfied. Furthermore, let* $u \in H^{1+s}(Q) \cap H^{1,1}_{0;0,}(Q)$ *for some* $s \in [0,1]$ *and let the* $H^{1,1}_{0;0,}(Q)$ *projection* Q_h^1, *given in* (2.45), *fulfil the standard error estimate*

$$\|u - Q_h^1 u\|_{L^2(Q)} \le c\,h^{1+s}\|u\|_{H^{1+s}(Q)}$$

with a constant $c > 0$, *see* (2.46). *Moreover, assume for* $Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ *the inverse inequality*

$$|v_h|_{H^1(Q)} \le c_{\mathrm{inv}}\,h^{-1}\|v_h\|_{L^2(Q)} \quad \forall v_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$$

with a constant $c_{\mathrm{inv}} > 0$ *and* $h = \max\{h_t, h_x\}$. *Then it holds*

$$|u-\widetilde{u}_h|_{H^1(Q)} \le C\,c_{\mathrm{inv}}\,h^s\|u\|_{H^{s+1}(Q)} + c_{\mathrm{inv}}\,h^{-1}\|\widetilde{u}_h - u\|_{L^2(Q)}$$

with a constant $C > 0$ *independent of* h. *If, in addition, the assumption of Corollary 4.2.30 is fulfilled, then it holds the error estimate*

$$|u-\widetilde{u}_h|_{H^1(Q)} \le \widetilde{C}\,h \tag{4.59}$$

with a constant $\widetilde{C} > 0$.

Proof. It follows with the triangle inequality, standard error estimates for Q_h^1 and the inverse inequality in $Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ that

$$\begin{aligned}|u-\widetilde{u}_h|_{H^1(Q)} &\le \left|u - Q_h^1 u\right|_{H^1(Q)} + \left|Q_h^1 u - \widetilde{u}_h\right|_{H^1(Q)}\\ &\le \widetilde{C}h^s\|u\|_{H^{s+1}(Q)} + c_{\mathrm{inv}}\,h^{-1}\left\|Q_h^1 u - \widetilde{u}_h\right\|_{L^2(Q)}\\ &\le \widetilde{C}h^s\|u\|_{H^{s+1}(Q)} + c_{\mathrm{inv}}\,h^{-1}\left\|Q_h^1 u - u\right\|_{L^2(Q)} + c_{\mathrm{inv}}\,h^{-1}\|\widetilde{u}_h - u\|_{L^2(Q)}\\ &\le C\,c_{\mathrm{inv}}\,h^s\|u\|_{H^{s+1}(Q)} + c_{\mathrm{inv}}\,h^{-1}\|\widetilde{u}_h - u\|_{L^2(Q)}\end{aligned}$$

with a constant $C > 0$ and hence, the assertion. □

Remark 4.2.32. *The assumptions on the spatial H_0^1 projection $Q_{h_x}^1$ and on the $H_{0;0,}^{1,1}(Q)$ projection Q_h^1 in Corollary 4.2.30 and Corollary 4.2.31 are fulfilled, if Ω is sufficiently regular. Thus, for less regular Ω, one expects reduced orders for the error estimates given in Corollary 4.2.30 and Corollary 4.2.31.*

As a numerical example for the Galerkin-Petrov finite element method (4.55), consider the one-dimensional spatial domain $\Omega = (0,1)$ and $T = 10$, i.e. the rectangular space-time domain

$$Q = \Omega \times (0,T) = (0,1) \times (0,10).$$

The discretisation is done with respect to nonuniform meshes as shown in Figure 4.1 of Section 4.1, where a uniform refinement strategy is applied. Note that these meshes do not fulfil the CFL condition (4.51). As exact solutions, the functions

$$u_1(x,t) = \sin(\pi x)\sin^2\left(\frac{5}{4}\pi t\right), \quad (x,t) \in Q,$$
$$u_2(x,t) = \sin(\pi x)t^2(10-t)^{3/4}, \quad (x,t) \in Q,$$

are chosen. The appearing integrals to compute the related right-hand side in (4.55) are calculated by using high-order quadrature rules and the degrees of freedom are denoted by

$$\text{dof} = \dim Q_h^1(Q) \cap H_{0;0,}^{1,1}(Q) = \dim Q_h^1(Q) \cap H_{0;,0}^{1,1}(Q).$$

The numerical results for the smooth solution u_1 are given in Table 4.12, where unconditional stability and quadratic convergence in $\|\cdot\|_{L^2(Q)}$, as predicted by the error estimate (4.58), are observed. Moreover, linear convergence is seen, when measuring the error in $|\cdot|_{H^1(Q)}$, which confirms the error estimate (4.59).

For the singular solution u_2, the related results are given in Table 4.13, where reduced orders of convergence in $\|\cdot\|_{L^2(Q)}$ and in $|\cdot|_{H^1(Q)}$ are observed. These convergence rates correspond to the reduced Sobolev regularity $u_2 \in H^{5/4-\varepsilon}(Q)$, $\varepsilon > 0$.

Remark 4.2.33. *The Galerkin-Petrov finite element method* (4.55) *seems to fulfil a kind of conservation of the total energy*

$$E(t) := \frac{1}{2}\|\partial_t u(\cdot,t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla_x u(\cdot,t)\|_{L^2(\Omega)}^2, \quad t \in [0,T].$$

As illustration, consider a solution of the homogeneous wave equation, i.e.

$$u_3(x,t) = (\cos(\pi t) + \sin(\pi t))\sin(\pi x) \quad \textit{for } (x,t) \in Q = (0,1) \times (0,10)$$

with the total energy

$$E(t) = \frac{\pi^2}{2} \quad \textit{for } t \in [0,10].$$

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_1 - \widetilde{u}_{1,h}\|_{L^2(Q)}$	eoc	$\|u_1 - \widetilde{u}_{1,h}\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	3.579e+00	-	1.289e+01	-
132	0.18750	0.03125	1.87500	0.31250	1.975e+00	0.86	9.849e+00	0.39
552	0.09375	0.01562	0.93750	0.15625	9.213e-01	1.10	6.534e+00	0.59
2256	0.04688	0.00781	0.46875	0.07812	6.829e-01	0.43	5.210e+00	0.33
9120	0.02344	0.00391	0.23438	0.03906	2.466e-01	1.47	2.848e+00	0.87
36672	0.01172	0.00195	0.11719	0.01953	7.029e-02	1.81	1.435e+00	0.99
147072	0.00586	0.00098	0.05859	0.00977	1.819e-02	1.95	7.159e-01	1.00
589056	0.00293	0.00049	0.02930	0.00488	4.588e-03	1.99	3.576e-01	1.00
2357760	0.00146	0.00024	0.01465	0.00244	1.149e-03	2.00	1.788e-01	1.00
9434112	0.00073	0.00012	0.00732	0.00122	2.875e-04	2.00	8.938e-02	1.00
37742592	0.00037	0.00006	0.00366	0.00061	7.189e-05	2.00	4.469e-02	1.00

Table 4.12: Numerical results of the Galerkin-Petrov finite element discretisation (4.55) with nonuniform meshes for $Q = (0,1) \times (0,10)$ and for the function u_1.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_2 - \widetilde{u}_{2,h}\|_{L^2(Q)}$	eoc	$\|u_2 - \widetilde{u}_{2,h}\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	7.836e+01	-	3.173e+02	-
132	0.18750	0.03125	1.87500	0.31250	2.166e+01	1.86	1.191e+02	1.41
552	0.09375	0.01562	0.93750	0.15625	5.487e+00	1.98	5.225e+01	1.19
2256	0.04688	0.00781	0.46875	0.07812	1.777e+00	1.63	2.696e+01	0.95
9120	0.02344	0.00391	0.23438	0.03906	6.476e-01	1.46	1.593e+01	0.76
36672	0.01172	0.00195	0.11719	0.01953	3.001e-01	1.11	1.076e+01	0.57
147072	0.00586	0.00098	0.05859	0.00977	1.393e-01	1.11	8.077e+00	0.41
589056	0.00293	0.00049	0.02930	0.00488	6.156e-02	1.18	6.452e+00	0.32
2357760	0.00146	0.00024	0.01465	0.00244	2.650e-02	1.22	5.308e+00	0.28
9434112	0.00073	0.00012	0.00732	0.00122	1.126e-02	1.23	4.423e+00	0.26
37742592	0.00037	0.00006	0.00366	0.00061	4.758e-03	1.24	3.704e+00	0.26

Table 4.13: Numerical results of the Galerkin-Petrov finite element discretisation (4.55) with nonuniform meshes for $Q = (0,1) \times (0,10)$ and for the function u_2.

Here, the initial condition

$$u_3(x,0) = u_0(x) = \sin(\pi x), \quad x \in \Omega,$$

is treated via homogenisation, while the initial condition

$$\partial_t u_3(x,0) = v_0(x) = \pi \sin(\pi x), \quad x \in \Omega,$$

is incorporated in a weak sense. For the solution u_3*, the discrete total energy*

$$E_h(t) := \frac{1}{2}\|\partial_t \widetilde{u}_h(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla_x \widetilde{u}_h(\cdot,t)\|^2_{L^2(\Omega)}, \quad t \in [0,T],$$

is computed. In Figure 4.3 the difference

$$E_h(t) - E(t) = E_h(t) - \frac{\pi^2}{2} \quad \text{for } t \in [0,10]$$

is plotted pointwise for the refinement level with uniform mesh sizes

$$h_t = \frac{10}{6 \cdot 2^{10}} \approx 0.00162760 \quad \text{and} \quad h_x = \frac{1}{6 \cdot 2^{10}} \approx 0.00016276.$$

Note that $\partial_t \widetilde{u}_h$ *is piecewise constant in time. Probably due to the used space-time approximation, some oscillations within the finite element accuracy are observed, but no energy loss occurs, see also Figure 4.2.*

For a comparison with the Newmark Galerkin method (4.5), (4.6) *of Section 4.1, the errors in the space-time norms* $\|\cdot\|_{L^2(Q)}$ *and* $|\cdot|_{H^1(Q)}$ *are given in Table 4.14, where the convergence rates are as expected, when the nonuniform meshes as in Figure 4.1 are used.*

Remark 4.2.34. *A comparison of Table 4.12 with Table 4.1 and Table 4.14 with Table 4.2 shows that the Newmark Galerkin method* (4.5), (4.6) *of Section 4.1 and the Galerkin-Petrov finite element discretisation give similar results, provided the right-hand side* $f \in L^2(Q)$ *has no singularity with respect to time. Note that the Newmark Galerkin method* (4.5), (4.6) *is not applicable to the solution* u_2*.*

Remark 4.2.35. *The to* (4.55) *related inf-sup constant*

$$\inf_{0 \neq u_h \in Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)} \sup_{0 \neq w_h \in Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)} \frac{a_h(u_h, w_h)}{|u_h|_{H^1(Q)}\,|w_h|_{H^1(Q)}} =: c_S(h) \tag{4.60}$$

seems to depend linearly on the mesh size h. The inf-sup constant $c_S(h)$ *is given as*

$$c_S(h) = \sqrt{\lambda_{\min}},$$

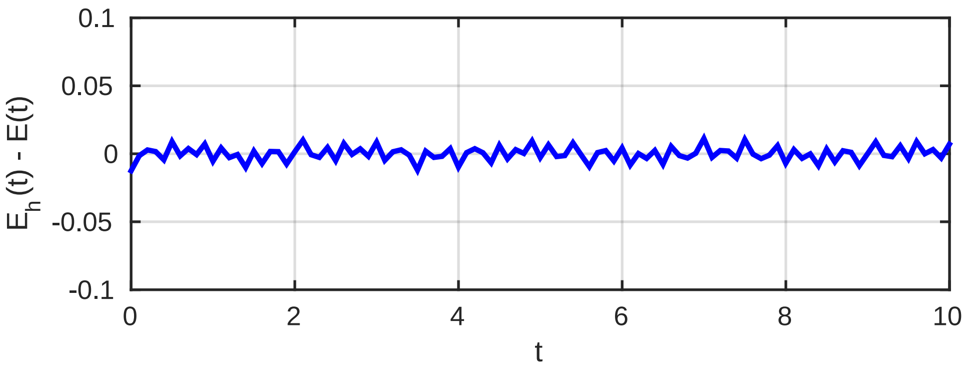

Figure 4.3: Difference of the total energy $E(t) = \frac{\pi^2}{2}$ and $E_h(t)$ of the Galerkin-Petrov finite element discretisation (4.55) with a uniform mesh for $Q = (0,1) \times (0,10)$ and for the function u_3.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_3 - \widetilde{u}_{3,h}\|_{L^2(Q)}$	eoc	$\|u_3 - \widetilde{u}_{3,h}\|_{H^1(Q)}$	eoc
30	0.37500	0.06250	3.75000	0.62500	2.503e+00	-	1.051e+01	-
132	0.18750	0.03125	1.87500	0.31250	2.496e+00	0.00	9.798e+00	0.09
552	0.09375	0.01562	0.93750	0.15625	2.241e+00	0.15	9.774e+00	0.00
2256	0.04688	0.00781	0.46875	0.07812	2.580e+00	-0.20	1.152e+01	-0.23
9120	0.02344	0.00391	0.23438	0.03906	1.082e+00	1.24	4.846e+00	1.24
36672	0.01172	0.00195	0.11719	0.01953	3.013e-01	1.84	1.445e+00	1.74
147072	0.00586	0.00098	0.05859	0.00977	7.697e-02	1.97	4.606e-01	1.65
589056	0.00293	0.00049	0.02930	0.00488	1.934e-02	1.99	1.804e-01	1.35
2357760	0.00146	0.00024	0.01465	0.00244	4.841e-03	2.00	8.268e-02	1.12
9434112	0.00073	0.00012	0.00732	0.00122	1.211e-03	2.00	4.034e-02	1.03
37742592	0.00037	0.00006	0.00366	0.00061	3.027e-04	2.00	2.005e-02	1.01

Table 4.14: Numerical results of the Galerkin-Petrov finite element discretisation (4.55) with nonuniform meshes for $Q = (0,1) \times (0,10)$ and for the function u_3.

where $\lambda_{\min}$ is the minimal eigenvalue of the generalised eigenvalue problem [84, Subsection 3.6.6, page 124]

$$\widetilde{K}_h^\top A_{h,,0}^{-1} \widetilde{K}_h \underline{u} = \lambda A_{h,0,} \underline{u}$$

with the matrices

$$\begin{aligned}
\widetilde{K}_h[k,i] &= a_h(\chi_i, \eta_k), \\
A_{h,0,}[k,i] &= \langle \partial_t \chi_i, \partial_t \chi_k \rangle_{L^2(Q)} + \langle \nabla_x \chi_i, \nabla_x \chi_k \rangle_{L^2(Q)}, \\
A_{h,,0}[k,i] &= \langle \partial_t \eta_i, \partial_t \eta_k \rangle_{L^2(Q)} + \langle \nabla_x \eta_i, \nabla_x \eta_k \rangle_{L^2(Q)}
\end{aligned}$$

for $i,k = 1,\dots,\mathrm{dof}$, *where χ_i are the nodal basis functions of $Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q)$ and η_k are the nodal basis functions of $Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q)$, i.e.*

$$\begin{aligned}
Q_h^1(Q) \cap H^{1,1}_{0;0,}(Q) &= \operatorname{span}\{\chi_i\}_{i=1}^{\mathrm{dof}}, \\
Q_h^1(Q) \cap H^{1,1}_{0;,0}(Q) &= \operatorname{span}\{\eta_k\}_{k=1}^{\mathrm{dof}}.
\end{aligned}$$

As illustration, consider the rectangular space-time domain

$$Q = \Omega \times (0,T) = (0,1) \times (0,2)$$

with uniform discretisations with mesh sizes $h_x = h_{x,\max} = h_{x,\min}$ and $h_t = h_{t,\max} = h_{t,\min}$, where a uniform refinement strategy is applied. The inf-sup constant $c_S(h)$ of (4.60) *is given in Table 4.15, where a linear dependency on the mesh size h is observed.*

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$c_S(h)$
30	0.16667	0.16667	0.33333	0.33333	0.13867820
132	0.08333	0.08333	0.16667	0.16667	0.07504415
552	0.04167	0.04167	0.08333	0.08333	0.03971295
2256	0.02083	0.02083	0.04167	0.04167	0.02028705
9120	0.01042	0.01042	0.02083	0.02083	0.01012171
36672	0.00521	0.00521	0.01042	0.01042	0.00510211

Table 4.15: Optimal discrete inf-sup constant $c_S(h)$ of (4.60) for the perturbed bilinear form (4.54) with a uniform temporal mesh size h_t and a uniform spatial mesh size h_x for the space-time cylinder $Q = (0,1) \times (0,2)$.

In the remainder of this section, the two-dimensional spatial L-shaped domain

$$\Omega := (-1,1)^2 \setminus ([0,1] \times [-1,0]) \subset \mathbb{R}^2 \tag{4.61}$$

and the finite time $T = \frac{1}{4}$ are considered for the solutions

$$u_4(x_1,x_2,t) = \sin(\pi x_1)\sin(\pi x_2)\sin^2\left(\frac{5}{4}\pi t\right), \quad (x_1,x_2,t) \in Q,$$
$$u_5(x_1,x_2,t) = r(x_1,x_2)^{2/3} \cdot \sin\left(\frac{2}{3}\arg(x_1,x_2)\right) \cdot \sin(\pi t), \quad (x_1,x_2,t) \in Q,$$

where $(r(x_1,x_2), \arg(x_1,x_2)) \subset [0,\infty) \times [0,2\pi)$ are polar coordinates located in $\underline{0} \in \mathbb{R}^2$ with the radial coordinate $r(x_1,x_2)$ and the angular coordinate $\arg(x_1,x_2)$. For the solution u_5, the inhomogeneous Dirichlet boundary condition

$$u_5(x_1,x_2,t) = g(x_1,x_2,t), \quad (x_1,x_2,t) \in \Sigma,$$

is treated via homogenisation, and the second initial condition

$$\partial_t u_5(x_1,x_2,0) = v_0(x_1,x_2), \quad (x_1,x_2) \in \Omega,$$

is incorporated in a weak sense. The spatial domain Ω is decomposed into uniform triangles with uniform mesh size $h_x = h_{x,\max} = h_{x,\min}$ as given in Figure 4.4 for level 0. The temporal domain $(0,1/4)$ is decomposed into nonuniform elements with the nodes

$$t_0 = 0.0, \quad t_1 = 0.125, \quad t_2 = 0.1875, \quad t_3 = 0.25 = T.$$

The appearing integrals to compute the related right-hand side in (4.55) and the vector, related to the second initial condition $\partial_t u_5(\cdot,\cdot,0) = v_0$, are calculated by using high-order quadrature rules. The numerical results for the smooth solution u_4, when a uniform refinement strategy is applied as in Figure 4.4, are given in Table 4.16, where unconditional stability and quadratic convergence in $\|\cdot\|_{L^2(Q)}$, as predicted by the error estimate (4.58), are observed. Moreover, linear convergence is seen, when measuring the error in $|\cdot|_{H^1(Q)}$, which confirms the error estimate (4.59). For the singular solution u_5, the related results are given in Table 4.17, where a reduced order of convergence in $\|\cdot\|_{L^2(Q)}$ and in $|\cdot|_{H^1(Q)}$ are observed. However, for a fixed uniform time mesh with $h_t = \frac{1}{4000}$, an adaptive meshing for the spatial domain Ω is considered with respect to the time element $\tau_{\hat{\ell}} = (t_{\hat{\ell}-1}, t_{\hat{\ell}})$ with

$$\hat{\ell} = \min\left(\underset{\ell=1,\dots,N_t}{\arg\max} \|u_5 - \widetilde{u}_{5,h}\|_{L^2(\Omega\times\tau_\ell)}\right),$$

i.e. the spatial decomposition $\overline{\Omega} = \bigcup_{i=1}^{N_x} \overline{\omega}_i \subset \mathbb{R}^2$ is adaptively refined with respect to the local error indicator

$$\|u_5 - \widetilde{u}_{5,h}\|_{L^2(\omega_i\times\tau_{\hat{\ell}})}, \quad i = 1,\dots,N_x,$$

where Dörfler marking [41] with parameter $\theta = 0.2$ is used.

This adaptive scheme seems to lead to optimal convergence rates in $\|\cdot\|_{L^2(Q)}$ and $|\cdot|_{H^1(Q)}$ with respect to the spatial variable, see Figure 4.5, Figure 4.6, and see Figure 4.7 for the meshes produced by the adaptive scheme. Since the stabilised method (4.55) is unconditionally stable, the usage of spatially adaptive refinement strategies is possible, which is confirmed by this example. Note that without the stabilisation such spatial refinement, as in Figure 4.7, is only possible for a sufficiently small temporal mesh size h_t due to the CFL condition (4.51). However, adaptive refinement strategies are left for future work. See also [115], for the approach of spatially graded meshes.

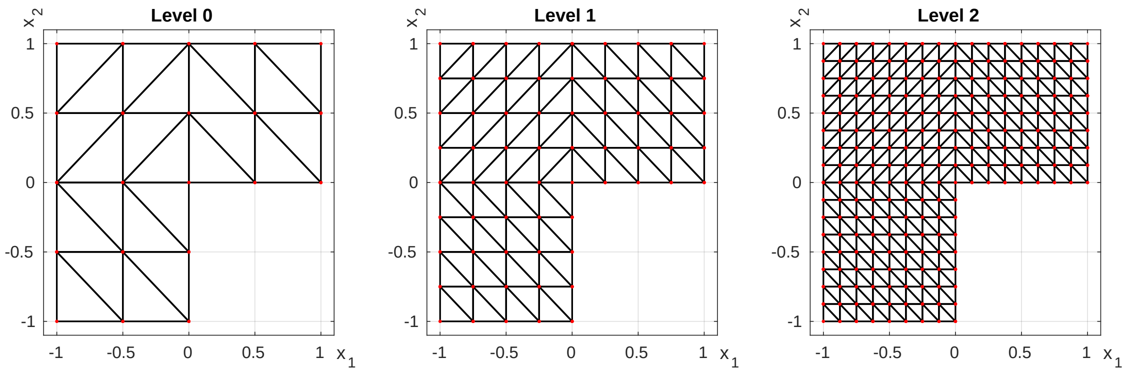

Figure 4.4: Uniform refinement strategy: Starting mesh, the meshes after one and two uniform refinement steps.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_4 - \widetilde{u}_{4,h}\|_{L^2(Q)}$	eoc	$\|u_4 - \widetilde{u}_{4,h}\|_{H^1(Q)}$	eoc
15	0.35355	0.35355	0.12500	0.06250	5.400e-02	-	7.381e-01	-
198	0.17678	0.17678	0.06250	0.03125	1.160e-02	1.79	3.280e-01	0.94
1932	0.08839	0.08839	0.03125	0.01562	2.627e-03	1.96	1.565e-01	0.97
16920	0.04419	0.04419	0.01562	0.00781	6.379e-04	1.96	7.719e-02	0.98
141360	0.02210	0.02210	0.00781	0.00391	1.582e-04	1.97	3.846e-02	0.98
1155168	0.01105	0.01105	0.00391	0.00195	3.948e-05	1.98	1.921e-02	0.99
9339072	0.00552	0.00552	0.00195	0.00098	9.865e-06	1.99	9.604e-03	1.00

Table 4.16: Numerical results of the Galerkin-Petrov finite element discretisation (4.55) for the L-shape (4.61) and $T = \frac{1}{4}$ for the function u_4 for a uniform refinement strategy.

dof	$h_{x,\max}$	$h_{x,\min}$	$h_{t,\max}$	$h_{t,\min}$	$\|u_5 - \widetilde{u}_{5,h}\|_{L^2(Q)}$	eoc	$\|u_5 - \widetilde{u}_{5,h}\|_{H^1(Q)}$	eoc
15	0.35355	0.35355	0.12500	0.06250	4.897e-03	-	8.597e-02	-
198	0.17678	0.17678	0.06250	0.03125	1.729e-03	1.21	5.026e-02	0.62
1932	0.08839	0.08839	0.03125	0.01562	6.675e-04	1.25	3.016e-02	0.67
16920	0.04419	0.04419	0.01562	0.00781	2.737e-04	1.23	1.844e-02	0.68
141360	0.02210	0.02210	0.00781	0.00391	1.159e-04	1.21	1.140e-02	0.68
1155168	0.01105	0.01105	0.00391	0.00195	4.888e-05	1.23	7.106e-03	0.68
9339072	0.00552	0.00552	0.00195	0.00098	2.034e-05	1.26	4.448e-03	0.67

Table 4.17: Numerical results of the Galerkin-Petrov finite element discretisation (4.55) for the L-shape (4.61) and $T = \frac{1}{4}$ for the function u_5 for a uniform refinement strategy.

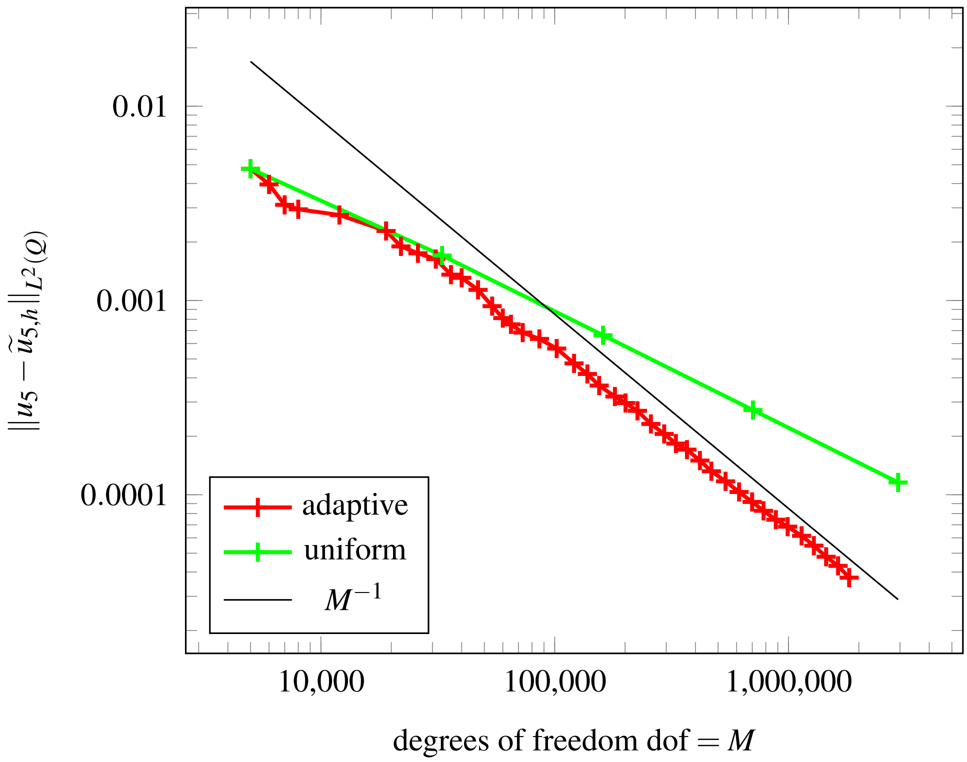

Figure 4.5: $L^2(Q)$ error of u_5 for (4.55) for the L-shape (4.61) and $T = \frac{1}{4}$ for $h_t = \frac{1}{4000}$ and for a spatially adaptive refinement strategy with the meshes of Figure 4.7.

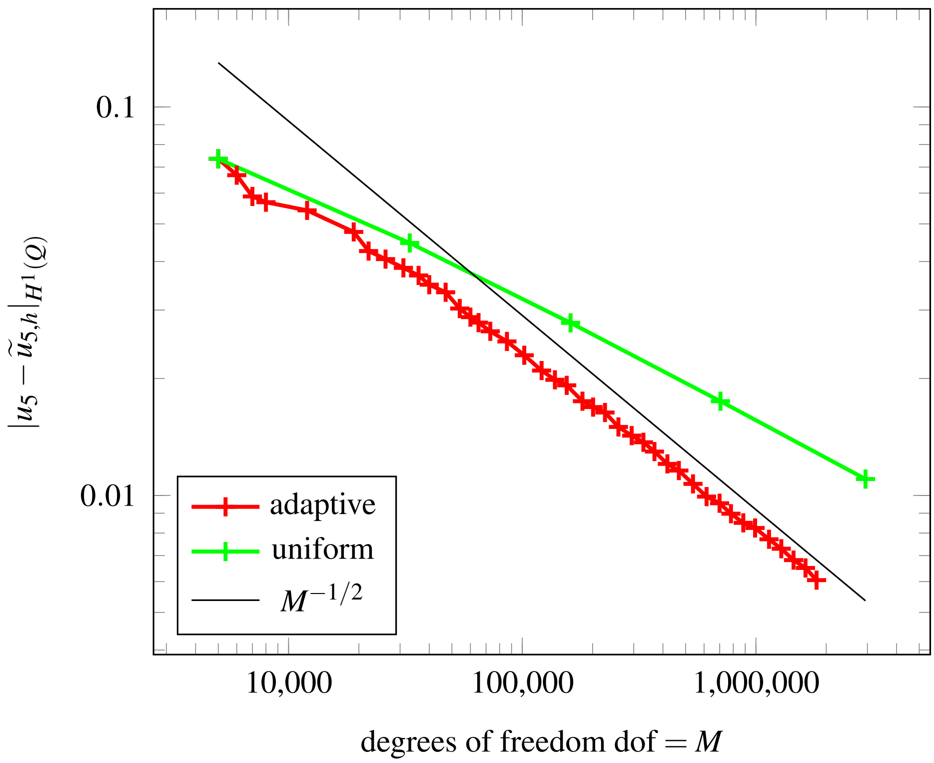

Figure 4.6: $H^1(Q)$ error of u_5 for (4.55) for the L-shape (4.61) and $T = \frac{1}{4}$ for $h_t = \frac{1}{4000}$ and for a spatially adaptive refinement strategy with the meshes of Figure 4.7.

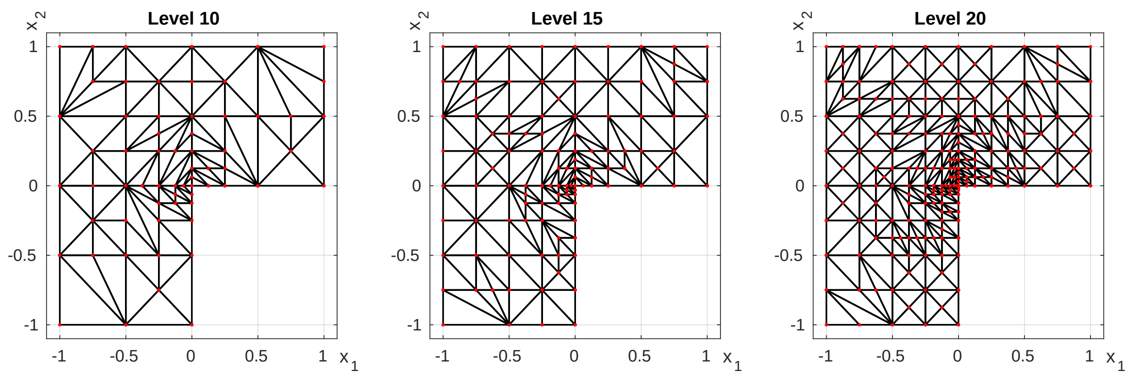

Figure 4.7: Spatially adaptive refinement strategy for the function u_5.

4.3 Space-Time Variational Formulation in a Strong Sense

In this section, the wave equation (4.10) is considered in $L^2(Q)$. Therefore, with the notations of Section 2.1, define

$$H^1(Q;\Box) := \{v \in H^1(Q) \colon \ \Box_Q v \in L^2(Q)\}$$

with the inner product

$$\langle u,v\rangle_{H^1(Q;\Box)} := \langle u,v\rangle_{L^2(Q)} + \langle \partial_t u, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} + \langle \Box_Q u, \Box_Q v\rangle_{L^2(Q)}$$

and the induced norm

$$\|u\|_{H^1(Q;\Box)} := \sqrt{\langle u,u\rangle_{H^1(Q;\Box)}} = \sqrt{\|u\|^2_{H^1(Q)} + \|\Box_Q u\|^2_{L^2(Q)}}.$$

For a function $v \in H^1(Q;\Box)$, the condition $\Box_Q v \in L^2(Q)$ involves that there exists a function $f_v \in L^2(Q)$ with

$$\Box_Q T_v(\varphi) = \int_Q f_v(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t \quad \text{ for all } \varphi \in \mathcal{D}(Q),$$

where $T_v \colon \mathcal{D}(Q) \to \mathbb{R}$, with

$$T_v(\varphi) = \int_Q v(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t \quad \text{ for all } \varphi \in \mathcal{D}(Q),$$

is the distribution related to $v \in L^2(Q)$. Note that the function $f_v \in L^2(Q)$ is unique, because $C_0^\infty(Q)$ is dense in $L^2(Q)$.

Clearly, $(H^1(Q;\Box), \langle\cdot,\cdot\rangle_{H^1(Q;\Box)})$ is an inner product space.

Lemma 4.3.1. *The inner product space* $(H^1(Q;\Box), \langle\cdot,\cdot\rangle_{H^1(Q;\Box)})$ *is a Hilbert space.*

Proof. Consider a Cauchy sequence $(v_n)_{n\in\mathbb{N}} \subset H^1(Q;\Box)$. Hence, $(v_n)_{n\in\mathbb{N}} \subset H^1(Q)$ is also a Cauchy sequence in $H^1(Q)$ and $(\Box_Q v_n)_{n\in\mathbb{N}} \subset L^2(Q)$ is also a Cauchy sequence in $L^2(Q)$. So, there exist $v \in H^1(Q)$ with $\|v_n - v\|_{H^1(Q)} \to 0$ as $n \to \infty$ and $f \in L^2(Q)$ with $\|\Box_Q v_n - f\|_{L^2(Q)} \to 0$ as $n \to \infty$. Let $T_v \colon \mathcal{D}(Q) \to \mathbb{R}$, $T_{v_n} \colon \mathcal{D}(Q) \to \mathbb{R}$, with

$$T_v(\varphi) = \int_Q v(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t, \quad T_{v_n}(\varphi) = \int_Q v_n(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t \quad \forall \varphi \in \mathcal{D}(Q),$$

be the distributions related to the limit $v \in H^1(Q)$ and to $v_n \in H^1(Q;\Box)$ for every $n \in \mathbb{N}$. There follow for all $\varphi \in \mathcal{D}(Q)$

$$\begin{aligned}\Box_Q T_v(\varphi) = T_v(\Box\varphi) &= \int_Q v(x,t)\Box\varphi(x,t)\mathrm{d}x\mathrm{d}t = \lim_{n\to\infty}\int_Q v_n(x,t)\Box\varphi(x,t)\mathrm{d}x\mathrm{d}t \\ &= \lim_{n\to\infty} T_{v_n}(\Box\varphi) = \lim_{n\to\infty}\Box_Q T_{v_n}(\varphi) = \lim_{n\to\infty}\int_Q \Box_Q v_n(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t \\ &= \int_Q f(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t\end{aligned}$$

and so, it holds $\Box_Q v = f \in L^2(Q)$. Hence, $v \in H^1(Q;\Box)$. □

Set

$$H^2_{0,}(0,T;L^2(\Omega)) := \{v \in H^2(0,T;L^2(\Omega))\colon\ v(\cdot,0) = \partial_t v(\cdot,0) = 0 \text{ in } L^2(\Omega)\}$$

and

$$H^1_0(\Omega;\Delta) := \{w \in H^1_0(\Omega)\colon\ \Delta_x w \in L^2(\Omega)\},$$

where Δ_x is the distributional Laplace operator for distributions $\mathcal{D}'(\Omega)$. Furthermore, define the subspace

$$\tilde{H}^2_{0,}(Q;\Box) := L^2(0,T;H^1_0(\Omega;\Delta)) \cap H^1_{0,}(0,T;H^1_0(\Omega)) \cap H^2_{0,}(0,T;L^2(\Omega)) \subset H^1(Q;\Box)$$

and by completion the Hilbert space

$$\tilde{H}^1_{0,}(Q;\Box) := \overline{\tilde{H}^2_{0,}(Q;\Box)}^{\|\cdot\|_{H^1(Q;\Box)}} \subset H^1(Q;\Box)$$

endowed with the inner product $\langle\cdot,\cdot\rangle_{H^1(Q;\Box)}$. That means,

$$\tilde{H}^1_{0,}(Q;\Box) = \{v \in H^1(Q;\Box)\colon\ \exists (v_n)_{n\in\mathbb{N}} \subset \tilde{H}^2_{0,}(Q;\Box) \text{ with } \|v_n - v\|_{H^1(Q;\Box)} \to 0\}.$$

Lemma 4.3.2. *It holds*

$$H^1_{0,}(Q;\Box) \subset H^{1,1}_{0;0,}(Q),$$

i.e. for $u \in \tilde{H}^1_{0,}(Q;\Box)$, there follow

$$\left\|\gamma_{0,x}^{\mathrm{int}} v\right\|_{L^2(\Sigma)} = \|v(\cdot,0)\|_{L^2(\Omega)} = 0,$$

where $\gamma_{0,x}^{\mathrm{int}}\colon\ L^2(0,T;H^1(\Omega)) \to L^2(\Sigma)$ is the extended trace operator (2.15).

Proof. Let $u \in \tilde{H}^1_{0,}(Q;\square)$ be fixed. Because of the completion, there exists an approximating sequence $(u_n)_{n\in\mathbb{N}} \subset \tilde{H}^2_{0,}(Q;\square)$ with $\|u-u_n\|_{H^1(Q;\square)} \to 0$ as $n \to \infty$. It holds with the constant $C_{\mathrm{Tr}} > 0$ from the extended trace operator, see (2.15), that

$$\left\|\gamma_{0,x}^{\mathrm{int}}(u-u_n)\right\|_{L^2(\Sigma)} \le C_{\mathrm{Tr}}\|u-u_n\|_{L^2(0,T;H^1(\Omega))} \le C_{\mathrm{Tr}}\|u-u_n\|_{H^1(Q;\square)} \to 0 \quad \text{as } n \to \infty$$

and hence,

$$\left\|\gamma_{0,x}^{\mathrm{int}}u\right\|_{L^2(\Sigma)} = \lim_{n\to\infty} \underbrace{\left\|\gamma_{0,x}^{\mathrm{int}}u_n\right\|_{L^2(\Sigma)}}_{=0} = 0.$$

With the continuous embedding (2.12), there follow

$$\|u(\cdot,0)-u_n(\cdot,0)\|_{L^2(\Omega)} \le C\|u-u_n\|_{H^1(Q)} \le C\|u-u_n\|_{H^1(Q;\square)} \to 0 \quad \text{as } n \to \infty$$

and so,

$$\|u(\cdot,0)\|_{L^2(\Omega)} = \lim_{n\to\infty} \underbrace{\|u_n(\cdot,0)\|_{L^2(\Omega)}}_{=0} = 0.$$

Hence, the assertion is proven. □

Lemma 4.3.3. *For $u \in \tilde{H}^1_{0,}(Q;\square)$, there holds*

$$\|u\|_{H^1(Q)} \le \sqrt{1+\frac{4T^2}{\pi^2}}\,|u|_{H^1(Q)}\,.$$

Proof. Let $u \in \tilde{H}^1_{0,}(Q;\square)$ be fixed. Because of the completion, there exists an approximating sequence $(u_n)_{n\in\mathbb{N}} \subset \tilde{H}^2_{0,}(Q;\square)$ with $\|u-u_n\|_{H^1(Q;\square)} \to 0$ as $n \to \infty$. Hence, it holds $u_n(x,\cdot) \in H^1_{0,}(0,T)$ for almost all $x \in \Omega$. By using the Poincaré inequality with respect to time, see Lemma 3.4.5, there follow

$$\|u_n\|^2_{L^2(Q)} = \int_\Omega \int_0^T u_n(x,t)^2 \mathrm{d}t\mathrm{d}x \le \frac{4T^2}{\pi^2} \int_\Omega \int_0^T \partial_t u_n(x,t)^2 \mathrm{d}t\mathrm{d}x = \frac{4T^2}{\pi^2}\|\partial_t u_n\|^2_{L^2(Q)}$$

and with

$$\begin{aligned}\|u_n\|^2_{H^1(Q)} &= \|u_n\|^2_{L^2(Q)} + \|\partial_t u_n\|^2_{L^2(Q)} + \|\nabla_x u_n\|^2_{L^2(Q)} \\ &\le \left(1+\frac{4T^2}{\pi^2}\right)\|\partial_t u_n\|^2_{L^2(Q)} + \|\nabla_x u_n\|^2_{L^2(Q)},\end{aligned}$$

the assertion by completion. □

Lemma 4.3.4. *For* $u \in \tilde{H}^1_{0,}(Q;\Box)$*, there hold*

$$|u|^2_{H^1(Q)} = 2\int_0^T\int_0^t \langle \partial_t u(\cdot,s), \Box_Q u(\cdot,s)\rangle_{L^2(\Omega)} \mathrm{d}s\mathrm{d}t \tag{4.62}$$

and hence,

$$|u|_{H^1(Q)} \le 2T\|\Box_Q u\|_{L^2(Q)}.$$

Proof. Let $u \in \tilde{H}^1_{0,}(Q;\Box)$ be fixed. Because of the completion, there exists an approximating sequence $(u_n)_{n\in\mathbb{N}} \subset \tilde{H}^2_{0,}(Q;\Box)$ with $\|u-u_n\|_{H^1(Q;\Box)} \to 0$ as $n \to \infty$. It holds with integration by parts for almost all $s \in (0,T)$

$$\begin{aligned}
\left\langle \begin{pmatrix} \nabla_x u_n(\cdot,s) \\ \partial_t u_n(\cdot,s) \end{pmatrix}, \begin{pmatrix} \nabla_x \partial_t u_n(\cdot,s) \\ \Delta_x u_n(\cdot,s) \end{pmatrix} \right\rangle_{L^2(\Omega)} &= \langle \nabla_x u_n(\cdot,s), \nabla_x \partial_t u_n(\cdot,s)\rangle_{L^2(\Omega)} \\
&\quad + \langle \partial_t u_n(\cdot,s), \Delta_x u_n(\cdot,s)\rangle_{L^2(\Omega)} \\
&= \langle \nabla_x u_n(\cdot,s), \nabla_x \partial_t u_n(\cdot,s)\rangle_{L^2(\Omega)} \\
&\quad - \langle \nabla_x \partial_t u_n(\cdot,s), \nabla_x u_n(\cdot,s)\rangle_{L^2(\Omega)} \\
&\quad + \langle \underbrace{\gamma_0^{\mathrm{int}} \partial_t u_n(\cdot,s)}_{=0}, \partial_n u_n(\cdot,s)\rangle_{L^2(\partial\Omega)} \\
&= 0,
\end{aligned}$$

where γ_0^{int} is the trace operator and ∂_n is the normal derivative, see Section 2.5.

The function u_n satisfies $\partial_t u_n(\cdot,0) = 0$ in $L^2(\Omega)$ and $\nabla_x u_n(\cdot,0) = \underline{0}$ in $[L^2(\Omega)]^d$ and so, there follow

$$\begin{aligned}
|u_n|^2_{H^1(Q)} &= \|\partial_t u_n\|^2_{L^2(Q)} + \|\nabla_x u_n\|^2_{L^2(Q)} \\
&= \int_0^T\int_0^t \partial_s \left[\langle \partial_t u_n(\cdot,s), \partial_t u_n(\cdot,s)\rangle_{L^2(\Omega)} + \sum_{i=1}^d \langle \partial_{x_i} u_n(\cdot,s), \partial_{x_i} u_n(\cdot,s)\rangle_{L^2(\Omega)} \right] \mathrm{d}s\mathrm{d}t \\
&= 2\int_0^T\int_0^t \left[\langle \partial_t u_n(\cdot,s), \partial_{tt} u_n(\cdot,s)\rangle_{L^2(\Omega)} + \sum_{i=1}^d \langle \partial_{x_i} u_n(\cdot,s), \partial_{x_i t} u_n(\cdot,s)\rangle_{L^2(\Omega)} \right] \mathrm{d}s\mathrm{d}t
\end{aligned}$$

and together with the help of the last relation,

$$
\begin{aligned}
|u_n|^2_{H^1(Q)} &= 2\int_0^T\int_0^t \left\langle \begin{pmatrix} \nabla_x u_n(\cdot,s) \\ \partial_t u_n(\cdot,s) \end{pmatrix}, \begin{pmatrix} \nabla_x \partial_t u_n(\cdot,s) \\ \partial_{tt} u_n(\cdot,s) \end{pmatrix} \right\rangle_{L^2(\Omega)} \mathrm{d}s\mathrm{d}t \\
&\quad - 2\int_0^T\int_0^t \underbrace{\left\langle \begin{pmatrix} \nabla_x u_n(\cdot,s) \\ \partial_t u_n(\cdot,s) \end{pmatrix}, \begin{pmatrix} \nabla_x \partial_t u_n(\cdot,s) \\ \Delta_x u_n(\cdot,s) \end{pmatrix} \right\rangle_{L^2(\Omega)}}_{=0} \mathrm{d}s\mathrm{d}t \\
&= 2\int_0^T\int_0^t \left\langle \begin{pmatrix} \nabla_x u_n(\cdot,s) \\ \partial_t u_n(\cdot,s) \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_{tt} u_n(\cdot,s) - \Delta_x u_n(\cdot,s) \end{pmatrix} \right\rangle_{L^2(\Omega)} \mathrm{d}s\mathrm{d}t. \\
&= 2\int_0^T\int_0^t \langle \partial_t u_n(\cdot,s), \partial_{tt} u_n(\cdot,s) - \Delta_x u_n(\cdot,s)\rangle_{L^2(\Omega)} \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

The completion procedure gives the equality (4.62). The Cauchy-Schwarz inequality yields the asserted estimate. □

Corollary 4.3.5. *The inner product space* $(\tilde{H}^1_{0,}(Q;\Box), \langle \Box_Q(\cdot), \Box_Q(\cdot)\rangle_{L^2(Q)})$ *is a Hilbert space.*

Proof. The assertion follows immediately from Lemma 4.3.3 and Lemma 4.3.4. □

In the following, $\tilde{H}^1_{0,}(Q;\Box)$ is endowed with the inner product $\langle \Box_Q(\cdot), \Box_Q(\cdot)\rangle_{L^2(Q)}$ and hence, with the norm $\|\Box_Q(\cdot)\|_{L^2(Q)}$. The strong variational formulation of the wave equation (4.10) for given $f \in L^2(Q)$ is as follows:

Find $u \in \tilde{H}^1_{0,}(Q;\Box)$ such that

$$a_S(u,v) = \langle f, v\rangle_{L^2(Q)} \tag{4.63}$$

for all $v \in L^2(Q)$, where the bilinear form $a_S(\cdot,\cdot)\colon \tilde{H}^1_{0,}(Q;\Box) \times L^2(Q) \to \mathbb{R}$ is defined by

$$a_S(u,v) := \langle \Box_Q u, v\rangle_{L^2(Q)} \quad u \in \tilde{H}^1_{0,}(Q;\Box),\, v \in L^2(Q).$$

Next, properties of the bilinear form $a_S(\cdot,\cdot)\colon \tilde{H}^1_{0,}(Q;\Box) \times L^2(Q) \to \mathbb{R}$ are shown and finally, unique solvability of the strong variational formulation (4.63) is proven.

Lemma 4.3.6. *The bilinear form* $a_S(\cdot,\cdot)\colon \tilde{H}^1_{0,}(Q;\Box) \times L^2(Q) \to \mathbb{R}$ *is bounded, i.e.*

$$|a_S(u,v)| \le \|\Box_Q u\|_{L^2(Q)} \|v\|_{L^2(Q)}$$

for all $u \in \tilde{H}^1_{0,}(Q;\Box)$, $v \in L^2(Q)$.

Proof. The assertion follows immediately from the Cauchy-Schwarz inequality. □

Lemma 4.3.7. *For the bilinear form* $a_S(\cdot,\cdot)\colon \tilde{H}^1_{0,}(Q;\Box) \times L^2(Q) \to \mathbb{R}$, *there holds the inf-sup condition*

$$\sup_{0\neq v\in L^2(Q)} \frac{|a_S(u,v)|}{\|v\|_{L^2(Q)}} = \|\Box_Q u\|_{L^2(Q)} \quad \text{for all } u \in \tilde{H}^1_{0,}(Q;\Box).$$

Proof. The inf-sup condition follows by the representation of the norm $\|\cdot\|_{L^2(Q)}$. □

Lemma 4.3.8. *For the bilinear form* $a_S(\cdot,\cdot)\colon \tilde{H}^1_{0,}(Q;\Box) \times L^2(Q) \to \mathbb{R}$, *there holds the surjectivity condition:*

For each function $0 \neq v \in L^2(Q)$, *there exists an element* $u \in \tilde{H}^1_{0,}(Q;\Box)$ *with* $a_S(u,v) \neq 0$.

Proof. Let $v \in L^2(Q)$ be fixed. There exists an approximating sequence $(\hat{v}_n)_{n\in\mathbb{N}} \subset C^\infty_0(Q)$ such that $\|\hat{v}_n - v\|_{L^2(Q)} \to 0$ as $n \to \infty$. Consider the eigenfunctions $\phi_i \in H^1_0(\Omega)$ and eigenvalues $\mu_i > 0$ of the Dirichlet eigenvalue problem of the Laplacian, see (2.4). Write v and $\hat{v}_n$ for $n \in \mathbb{N}$ as Fourier series

$$v(x,t) = \sum_{i=1}^\infty v_i(t)\phi_i(x)$$

and

$$\hat{v}_n(x,t) = \sum_{i=1}^\infty \hat{v}_{n,i}(t)\phi_i(x)$$

for $(x,t) \in Q$ with coefficients $v_i \in L^2(0,T)$, $\hat{v}_{n,i} \in L^2(0,T)$, see (3.65). It holds

$$\infty > \|v\|^2_{L^2(Q)} = \sum_{i=1}^\infty \|v_i\|^2_{L^2(0,T)}$$

and for $n \in \mathbb{N}$

$$\infty > \|\hat{v}_n\|^2_{L^2(Q)} = \sum_{i=1}^\infty \|\hat{v}_{n,i}\|^2_{L^2(0,T)}, \qquad \infty > \|\nabla_x \hat{v}_n\|^2_{L^2(Q)} = \sum_{i=1}^\infty \mu_i \|\hat{v}_{n,i}\|^2_{L^2(0,T)}.$$

Fix an index $n \in \mathbb{N}$. Define for $M \in \mathbb{N}$

$$u_{n,M}(x,t) = \sum_{i=1}^M \frac{1}{\sqrt{\mu_i}} \int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x), \quad (x,t) \in Q,$$

and

$$u_n(x,t) = \sum_{i=1}^\infty \frac{1}{\sqrt{\mu_i}} \int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x), \quad (x,t) \in Q.$$

Since

$$\begin{aligned}
\|u_n\|^2_{L^2(Q)} &= \sum_{i=1}^{\infty} \frac{1}{\mu_i} \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \right]^2 \mathrm{d}t \\
&\le \sum_{i=1}^{\infty} \frac{1}{\mu_i} \int_0^T \int_0^t \hat{v}_{n,i}(s)^2 \mathrm{d}s \int_0^t \sin^2\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s\mathrm{d}t \\
&\le \frac{T^2}{2} \sum_{i=1}^{\infty} \frac{1}{\mu_i} \|\hat{v}_{n,i}\|^2_{L^2(0,T)} \le \frac{1}{\mu_1} \frac{T^2}{2} \|\hat{v}_n\|^2_{L^2(Q)} < \infty,
\end{aligned}$$

it holds $u_n \in L^2(Q)$ and $u_{n,M} \to u_n$ in $L^2(Q)$ as $M \to \infty$.

To compute the time derivative ∂_t of u_n, set for $M \in \mathbb{N}$

$$w_{n,M}(x,t) = \sum_{i=1}^{M} \int_0^t \hat{v}_{n,i}(s) \cos\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x), \quad (x,t) \in Q,$$

$$w_n(x,t) = \sum_{i=1}^{\infty} \int_0^t \hat{v}_{n,i}(s) \cos\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x), \quad (x,t) \in Q,$$

and compute

$$\begin{aligned}
\|w_n\|^2_{L^2(Q)} &= \sum_{i=1}^{\infty} \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \cos\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \right]^2 \mathrm{d}t \\
&\le \frac{T^2}{2} \sum_{i=1}^{\infty} \|\hat{v}_{n,i}\|^2_{L^2(0,T)} = \frac{T^2}{2} \|\hat{v}_n\|^2_{L^2(Q)} < \infty,
\end{aligned}$$

i.e. $w_n \in L^2(Q)$ and $w_{n,M} \to w_n$ in $L^2(Q)$ as $M \to \infty$. It follows

$$\begin{aligned}
\int_0^T \int_\Omega u_n(x,t) \partial_t \varphi(x,t) \mathrm{d}x\mathrm{d}t &= \lim_{M\to\infty} \int_0^T \int_\Omega u_{n,M}(x,t) \partial_t \varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&= -\lim_{M\to\infty} \int_0^T \int_\Omega \partial_t u_{n,M}(x,t) \varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&= -\lim_{M\to\infty} \int_0^T \int_\Omega w_{n,M}(x,t) \varphi(x,t) \mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \int_\Omega w_n(x,t) \varphi(x,t) \mathrm{d}x\mathrm{d}t
\end{aligned}$$

for all $\varphi \in C_0^\infty(Q)$, i.e. $\partial_t u_n = w_n \in L^2(Q)$. Analogously, the weak derivatives $\partial_{tt} u_n$, $\nabla_x u_n$, $\nabla_x \partial_t u_n$ and $\Delta_x u_n$ are derived, since one formally computes for $(x,t) \in Q$

$$\partial_{tt} u_n(x,t) = -\sum_{i=1}^{\infty} \sqrt{\mu_i} \int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x) + \sum_{i=1}^{\infty} \hat{v}_{n,i}(t) \phi_i(x),$$

$$\nabla_x u_n(x,t) = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\mu_i}} \int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \nabla_x \phi_i(x),$$

$$\nabla_x \partial_t u_n(x,t) = \sum_{i=1}^{\infty} \int_0^t \hat{v}_{n,i}(s) \cos\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \nabla_x \phi_i(x),$$

$$\Delta_x u_n(x,t) = -\sum_{i=1}^{\infty} \sqrt{\mu_i} \int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x),$$

where the term-by-term differentiation is allowed because of the estimates

$$\begin{aligned}
\|\partial_{tt} u_n\|_{L^2(Q)}^2 &\le 2 \sum_{i=1}^{\infty} \mu_i \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \right]^2 \mathrm{d}t + 2 \|\hat{v}_n\|_{L^2(Q)}^2 \\
&\le T^2 \sum_{i=1}^{\infty} \mu_i \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 + 2 \|\hat{v}_n\|_{L^2(Q)}^2 \\
&= T^2 \|\nabla_x \hat{v}_n\|_{L^2(Q)}^2 + 2 \|\hat{v}_n\|_{L^2(Q)}^2 < \infty,
\end{aligned}$$

$$\begin{aligned}
\|\nabla_x u_n\|_{L^2(Q)}^2 &= \sum_{i=1}^{\infty} \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \right]^2 \mathrm{d}t \\
&\le \frac{T^2}{2} \sum_{i=1}^{\infty} \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 = \frac{T^2}{2} \|\hat{v}_n\|_{L^2(Q)}^2 < \infty,
\end{aligned}$$

$$\begin{aligned}
\|\nabla_x \partial_t u_n\|_{L^2(Q)}^2 &= \sum_{i=1}^{\infty} \mu_i \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \cos\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \right]^2 \mathrm{d}t \\
&\le \frac{T^2}{2} \sum_{i=1}^{\infty} \mu_i \|\hat{v}_{n,i}\|_{L^2(0,T)}^2 = \frac{T^2}{2} \|\nabla_x \hat{v}_n\|_{L^2(Q)}^2 < \infty
\end{aligned}$$

and

$$\|\Delta_x u_n\|^2_{L^2(Q)} = \sum_{i=1}^{\infty} \mu_i \int_0^T \left[\int_0^t \hat{v}_{n,i}(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \right]^2 \mathrm{d}t$$
$$\leq \frac{T^2}{2} \sum_{i=1}^{\infty} \mu_i \|\hat{v}_{n,i}\|^2_{L^2(0,T)} = \frac{T^2}{2} \|\nabla_x \hat{v}_n\|^2_{L^2(Q)} < \infty.$$

Therefore, $u_n \in \tilde{H}^2_{0,}(Q;\Box)$.

Analogously, define for $(x,t) \in Q$

$$u(x,t) = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\mu_i}} \int_0^t v_i(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x)$$

and partial sums

$$u_M(x,t) = \sum_{i=1}^{M} \frac{1}{\sqrt{\mu_i}} \int_0^t v_i(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x)$$

for $M \in \mathbb{N}$. With the same arguments as above, there hold

$$\lim_{M\to\infty} u_M = u \quad \text{in } L^2(Q),$$
$$\lim_{M\to\infty} \partial_t u_M = \partial_t u \quad \text{in } L^2(Q),$$
$$\lim_{M\to\infty} \nabla_x u_M = \nabla_x u \quad \text{in } [L^2(Q)]^d.$$

For each $M \in \mathbb{N}$, one computes for $(x,t) \in Q$

$$\Box u_M(x,t) = \Box \left[\sum_{i=1}^{M} \frac{1}{\sqrt{\mu_i}} \int_0^t v_i(s) \sin\left(\sqrt{\mu_i}(t-s)\right) \mathrm{d}s \cdot \phi_i(x) \right] = \sum_{i=1}^{M} v_i(t)\phi_i(x)$$

and hence, it follows

$$\langle u, \Box\varphi \rangle_{L^2(Q)} = \int_0^T \int_\Omega u(x,t)\Box\varphi(x,t)\mathrm{d}x\mathrm{d}t = \lim_{M\to\infty} \int_0^T \int_\Omega u_M(x,t)\Box\varphi(x,t)\mathrm{d}x\mathrm{d}t$$
$$= \lim_{M\to\infty} \int_0^T \int_\Omega \Box u_M(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t = \lim_{M\to\infty} \int_0^T \int_\Omega \sum_{i=1}^{M} v_i(t)\phi_i(x)\varphi(x,t)\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega v(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t = \langle v, \varphi \rangle_{L^2(Q)}$$

for all $\varphi \in C_0^\infty(Q)$, i.e. $\Box_Q u = v \in L^2(Q)$. The estimates

$$|u_n - u|_{H^1(Q)} \le C\|\hat{v}_n - v\|_{L^2(Q)} \quad \text{and} \quad \|\Box_Q(u_n - u)\|_{L^2(Q)} \le \|\hat{v}_n - v\|_{L^2(Q)}$$

yields $\|u_n - u\|_{H^1(Q;\Box)} \to 0$ as $n \to \infty$ and hence, $u \in H^1(Q;\Box)$.

To summarise, the sequence $(u_n)_{n\in\mathbb{N}} \subset \tilde{H}^2_{0,}(Q;\Box)$ converges to $u \in H^1(Q;\Box)$ in $\|\cdot\|_{H^1(Q;\Box)}$ and hence, $u \in \tilde{H}^1_{0,}(Q;\Box)$ by the completion procedure. With $\Box_Q u = v \in L^2(Q)$ and therefore,

$$a_S(u,v) \ge \|v\|^2_{L^2(Q)} > 0,$$

the assertion follows. □

Theorem 4.3.9. *For each given $f \in L^2(Q)$, there exists a unique solution $u \in \tilde{H}^1_{0,}(Q;\Box)$ of the variational formulation* (4.63). *Furthermore,*

$$\mathcal{L}_S\colon\ L^2(Q) \to \tilde{H}^1_{0,}(Q;\Box), \qquad \mathcal{L}_S f := u,$$

is an isomorphism satisfying

$$\|\Box_Q u\|_{L^2(Q)} = \|\Box_Q \mathcal{L}_S f\|_{L^2(Q)} = \|f\|_{L^2(Q)}.$$

Proof. With the help of the Nečas Theorem 2.9.1, the results in Lemma 4.3.6, Lemma 4.3.7 and Lemma 4.3.8 yield the existence and uniqueness of a solution $u \in \tilde{H}^1_{0,}(Q;\Box)$. In addition, it holds with the variational formulation (4.63) the equality

$$\|f\|_{L^2(Q)} = \sup_{0\ne v\in L^2(Q)} \frac{|\langle f,v\rangle_{L^2(Q)}|}{\|v\|_{L^2(Q)}} = \sup_{0\ne v\in L^2(Q)} \frac{|\langle \Box_Q u,v\rangle_{L^2(Q)}|}{\|v\|_{L^2(Q)}} = \|\Box_Q u\|_{L^2(Q)}$$

and therefore, the assertion. □

Remark 4.3.10. *The following functions are given to get a first impression of the solution space $\tilde{H}^1_{0,}(Q;\Box)$.*

1. *Functions $u \in C^2(\overline{Q})$ with $u_{|\Sigma_0} = 0$, $\partial_t u_{|\Sigma_0} = 0$ and $\nabla_x u_{|\Omega_0} = \underline{0}$ are contained in $\tilde{H}^2_{0,}(Q;\Box) \subset \tilde{H}^1_{0,}(Q;\Box)$.*
2. *A function $u \in H^1(Q;\Box)$ with*
$$\left\|\gamma^{\text{int}}_{0,x} v\right\|_{L^2(\Sigma)} = \|v(\cdot,0)\|_{L^2(\Omega)} = 0 \quad \text{and} \quad \Box_Q u = 0 \quad \text{in } Q$$
is in $\tilde{H}^1_{0,}(Q;\Box)$ if and only if $u = 0$ in Q. This follows immediately from the representation (4.62).
3. *Consider the smooth function*
$$u(x,t) = \sin(\pi x)\sin(\pi t) \quad \text{for } (x,t) \in (0,1)\times(0,1) = Q,$$
satisfying $\left\|\gamma^{\text{int}}_{0,x} v\right\|_{L^2(\Sigma)} = \|v(\cdot,0)\|_{L^2(\Omega)} = 0$ and $\Box_Q u = 0$ in Q. The representation (4.62) *yields that $u \notin \tilde{H}^1_{0,}(Q;\Box)$.*

4.4 Space-Time Variational Formulation in a Weak Sense

In this section, the wave equation (4.10) is considered in a weaker sense than $L^2(Q)$. Therefore, with the notations of Section 2.1 and Section 2.5, define

$$\mathcal{H}(Q) := \{v_{|Q} \colon\ v \in L^2(Q_-),\, v_{|\Omega\times(-\infty,0)} = 0,\, \Box_{Q_-} v \in [H^{1,1}_{0;,0}(Q)]'\}$$

with the norm

$$\|v\|_{\mathcal{H}(Q)} := \sqrt{\|v\|^2_{L^2(Q)} + \left\|\Box_{Q_-} v\right\|^2_{[H^{1,1}_{0;,0}(Q)]'}},$$

where

$$Q_- = \Omega \times (-\infty, T) \subset \mathbb{R}^{d+1}$$

is the unbounded domain with respect to time. For a function $v \in \mathcal{H}(Q)$, the condition $\Box_{Q_-} v \in [H^{1,1}_{0;,0}(Q)]'$ involves that there exists an element $f_v \in [H^{1,1}_{0;,0}(Q)]'$ with

$$\Box_{Q_-} T_v(\varphi) = \left\langle f_v, \varphi_{|Q} \right\rangle_Q \quad \text{for all } \varphi \in \mathcal{D}(Q_-),$$

where $T_v \colon \mathcal{D}(Q_-) \to \mathbb{R}$, with

$$T_v(\varphi) = \int\limits_{Q_-} v(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t = \int\limits_{Q} v(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t \quad \text{for all } \varphi \in \mathcal{D}(Q_-),$$

is the distribution related to $v \in L^2(Q_-)$ and as in Section 2.5, $\langle \cdot,\cdot \rangle_Q$ denotes the duality pairing in $[H^{1,1}_{0;,0}(Q)]' \times H^{1,1}_{0;,0}(Q)$ as extension of the $L^2(Q)$ inner product. Note that it holds $\varphi_{|Q} \in H^{1,1}_{0;,0}(Q)$ for $\varphi \in \mathcal{D}(Q_-)$ and that $C_0^\infty(Q_-)_{|Q}$ is dense in $H^{1,1}_{0;,0}(Q)$. Hence, the element $f_v \in [H^{1,1}_{0;,0}(Q)]'$ is unique.

Clearly, $(\mathcal{H}(Q), \|\cdot\|_{\mathcal{H}(Q)})$ is a normed vector space and it is even a Banach space.

Lemma 4.4.1. *The normed vector space* $(\mathcal{H}(Q), \|\cdot\|_{\mathcal{H}(Q)})$ *is a Banach space.*

Proof. Consider a Cauchy sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{H}(Q)$. Hence, $(v_n)_{n\in\mathbb{N}} \subset L^2(Q)$ is also a Cauchy sequence in $L^2(Q)$ and $(\Box_{Q_-} v_n)_{n\in\mathbb{N}} \subset [H^{1,1}_{0;,0}(Q)]'$ is also a Cauchy sequence in $[H^{1,1}_{0;,0}(Q)]'$. So, there exist $v \in L^2(Q)$ with $\|v_n - v\|_{L^2(Q)} \to 0$ as $n \to \infty$ and $f \in [H^{1,1}_{0;,0}(Q)]'$ with

$$\left\|\Box_{Q_-} v_n - f\right\|_{[H^{1,1}_{0;,0}(Q)]'} = \sup_{0\neq w \in H^{1,1}_{0;,0}(Q)} \frac{\left|\left\langle \Box_{Q_-} v_n - f, w \right\rangle_Q\right|}{|w|_{H^1(Q)}} \to 0 \quad \text{as } n \to \infty.$$

Let $T_v\colon \mathcal{D}(Q_-)\to\mathbb{R}$, $T_{v_n}\colon \mathcal{D}(Q_-)\to\mathbb{R}$, with

$$T_v(\varphi)=\int\limits_Q v(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t,\quad T_{v_n}(\varphi)=\int\limits_Q v_n(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t\qquad\text{for all }\varphi\in\mathcal{D}(Q_-),$$

be the distributions related to the limit $v\in L^2(Q)$ and to $v_n\in L^2(Q)$ for every $n\in\mathbb{N}$. It follows for all $\varphi\in\mathcal{D}(Q_-)$

$$\begin{aligned}\Box_{Q_-}T_v(\varphi)=T_v(\Box\varphi)&=\int\limits_Q v(x,t)\Box\varphi(x,t)\mathrm{d}x\mathrm{d}t=\lim_{n\to\infty}\int\limits_Q v_n(x,t)\Box\varphi(x,t)\mathrm{d}x\mathrm{d}t\\&=\lim_{n\to\infty}T_{v_n}(\Box\varphi)=\lim_{n\to\infty}\Box_{Q_-}T_{v_n}(\varphi)=\lim_{n\to\infty}\left\langle\Box_{Q_-}v_n,\varphi_{|Q}\right\rangle_Q=\left\langle f,\varphi_{|Q}\right\rangle_Q\end{aligned}$$

because $\varphi_{|Q}\in H^{1,1}_{0;,0}(Q)$. So, it holds $\Box_{Q_-}v=f\in[H^{1,1}_{0;,0}(Q)]'$. Hence, $v\in\mathcal{H}(Q)$. □

Since the norm $\|\cdot\|_{\mathcal{H}(Q)}$ is a Hilbertian norm, see Section 2.5, $\mathcal{H}(Q)$ is even a Hilbert space with respect to an abstract inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}(Q)}$, which induces the norm $\|\cdot\|_{\mathcal{H}(Q)}$.

Lemma 4.4.2. *It holds $H^{1,1}_{0;0,}(Q)\subset\mathcal{H}(Q)$. Furthermore, each function $u\in H^{1,1}_{0;0,}(Q)$ fulfils*

$$\left\langle\Box_{Q_-}u,w\right\rangle_Q=a_{H^1}(u,w):=-\langle\partial_t u,\partial_t w\rangle_{L^2(Q)}+\langle\nabla_x u,\nabla_x w\rangle_{L^2(Q)}\tag{4.64}$$

for all $w\in H^{1,1}_{0;,0}(Q)$.

Proof. Let $u\in H^{1,1}_{0;0,}(Q)$ be fixed. Set $v:=u$ in Q and $v:=0$ in $Q_-\setminus Q$. Clearly, it holds $v\in L^2(Q_-)$ and $v_{|\Omega\times(-\infty,0)}=0$. It remains to prove that $\Box_{Q_-}v\in[H^{1,1}_{0;,0}(Q)]'$. Therefore, let $T_v\colon\mathcal{D}(Q_-)\to\mathbb{R}$, with

$$T_v(\varphi)=\int\limits_{Q_-}v(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t=\int\limits_Q u(x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t\qquad\text{for all }\varphi\in\mathcal{D}(Q_-),$$

be the distribution related to $v\in L^2(Q_-)$ and define $f_u\in[H^{1,1}_{0;,0}(Q)]'$ by

$$\langle f_u,w\rangle_Q:=a_{H^1}(u,w)=-\langle\partial_t u,\partial_t w\rangle_{L^2(Q)}+\langle\nabla_x u,\nabla_x w\rangle_{L^2(Q)}\qquad\text{for all }w\in H^{1,1}_{0;,0}(Q).$$

Note that $f_u\in[H^{1,1}_{0;,0}(Q)]'$ is bounded by the Cauchy-Schwarz inequality, satisfying the estimate $\|f_u\|_{[H^{1,1}_{0;,0}(Q)]'}\le|u|_{H^1(Q)}$, see Lemma 4.2.21. It follows for all $\varphi\in\mathcal{D}(Q_-)$ with integration by parts with respect to time and space that

$$\begin{aligned}\Box_{Q_-}T_v(\varphi)=T_v(\Box\varphi)&=\int\limits_Q u(x,t)\Box\varphi(x,t)\mathrm{d}x\mathrm{d}t=\int\limits_Q u(x,t)(\partial_{tt}-\Delta_x)\varphi(x,t)\mathrm{d}x\mathrm{d}t\\&=-\left\langle\partial_t u,\partial_t\varphi_{|Q}\right\rangle_{L^2(Q)}+\left\langle\nabla_x u,\nabla_x\varphi_{|Q}\right\rangle_{L^2(Q)}=\left\langle f_u,\varphi_{|Q}\right\rangle_Q.\end{aligned}$$

Hence, it holds $\Box_{Q_-}v=f_u\in[H^{1,1}_{0;,0}(Q)]'$ and so, $u\in\mathcal{H}(Q)$. The equality (4.64) follows also from the last relation, because $C_0^\infty(Q_-)_{|Q}$ is dense in $H^{1,1}_{0;,0}(Q)$. □

Define by completion the Hilbert space

$$\mathcal{H}_{0,}(Q) := \overline{H^{1,1}_{0;0,}(Q)}^{\|\cdot\|_{\mathcal{H}(Q)}} \subset \mathcal{H}(Q)$$

endowed with the Hilbertian norm $\|\cdot\|_{\mathcal{H}(Q)}$. That means,

$$\mathcal{H}_{0,}(Q) = \{v \in \mathcal{H}(Q) \colon \exists (v_n)_{n\in\mathbb{N}} \subset H^{1,1}_{0;0,}(Q) \text{ with } \|v_n - v\|_{\mathcal{H}(Q)} \to 0\}.$$

Lemma 4.4.3. *For $u \in \mathcal{H}_{0,}(Q)$, it holds*

$$\left\|\Box_{Q_-} u\right\|_{[H^{1,1}_{0;,0}(Q)]'} \geq \frac{\sqrt{2}}{T}\|u\|_{L^2(Q)}.$$

Proof. Let $0 \neq u \in \mathcal{H}(Q)$ be fixed. Because of the completion, there exists an approximating sequence $(u_n)_{n\in\mathbb{N}} \subset H^{1,1}_{0;0,}(Q)$ with $\|u - u_n\|_{\mathcal{H}(Q)} \to 0$ as $n \to \infty$. Because of Theorem 4.2.22, there exists for each $n \in \mathbb{N}$ a unique solution $\tilde{w}_n \in H^{1,1}_{0;,0}(Q)$ of

$$a_{H^1}(v, \tilde{w}) = -\langle \partial_t \tilde{w}_n, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x \tilde{w}_n, \nabla_x v\rangle_{L^2(Q)} \stackrel{!}{=} \langle u_n, v\rangle_{L^2(Q)} \quad \forall v \in H^{1,1}_{0;0,}(Q), \quad (4.65)$$

satisfying $|\tilde{w}_n|_{H^1(Q)} \leq \frac{1}{\sqrt{2}}T\|u_n\|_{L^2(Q)}$. With equality (4.64), with $v = u_n \in H^{1,1}_{0;0,}(Q)$ in the variational formulation (4.65) and with the stability estimate $|\tilde{w}_n|_{H^1(Q)} \leq \frac{1}{\sqrt{2}}T\|u_n\|_{L^2(Q)}$, there follow

$$\begin{aligned}\left\|\Box_{Q_-} u_n\right\|_{[H^{1,1}_{0;,0}(Q)]'} &= \sup_{0\neq w \in H^{1,1}_{0;,0}(Q)} \frac{|\langle \Box_{Q_-} u_n, w\rangle_Q|}{|w|_{H^1(Q)}} \geq \frac{|\langle \Box_{Q_-} u_n, \tilde{w}_n\rangle_Q|}{|\tilde{w}_n|_{H^1(Q)}} = \frac{|a_{H^1}(u_n, \tilde{w}_n)|}{|\tilde{w}_n|_{H^1(Q)}} \\ &\geq \frac{\sqrt{2}\|u_n\|^2_{L^2(Q)}}{T\|u_n\|_{L^2(Q)}} = \frac{\sqrt{2}}{T}\|u_n\|_{L^2(Q)}\end{aligned}$$

and hence, the assertion by completion. □

Corollary 4.4.4. *The inner product space $\left(\mathcal{H}_{0,}(Q), \langle \Box_{Q_-}(\cdot), \Box_{Q_-}(\cdot)\rangle_{[H^{1,1}_{0;,0}(Q)]'}\right)$ is complete, i.e. a Hilbert space.*

Proof. The assertion follows immediately from Lemma 4.4.3. □

In the following, $\mathcal{H}_{0,}(Q)$ is endowed with the Hilbertian norm $\left\|\Box_{Q_-}(\cdot)\right\|_{[H^{1,1}_{0;,0}(Q)]'}$. The weak variational formulation for given $f \in [H^{1,1}_{0;,0}(Q)]'$ is as follows:

Find $u \in \mathcal{H}_{0,}(Q)$ such that

$$a_W(u, w) = \langle f, w\rangle_Q \quad (4.66)$$

for all $w \in H^{1,1}_{0;,0}(Q)$, where the bilinear form $a_W(\cdot,\cdot)\colon \mathcal{H}_{0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}$ is defined by

$$a_W(u,w) := \left\langle \Box_{Q\_} u, w \right\rangle_Q \quad u \in \mathcal{H}_{0,}(Q), w \in H^{1,1}_{0;,0}(Q).$$

Next, properties of the bilinear form $a_W(\cdot,\cdot)\colon \mathcal{H}_{0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}$ are shown and finally, unique solvability of the weak variational formulation (4.66) is proven.

Lemma 4.4.5. *The bilinear form $a_W(\cdot,\cdot)\colon \mathcal{H}_{0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}$ is bounded, i.e.*

$$|a_W(u,w)| \le \left\|\Box_{Q\_} u\right\|_{[H^{1,1}_{0;,0}(Q)]'} |w|_{H^1(Q)} \qquad \textit{for all } u \in \mathcal{H}_{0,}(Q), w \in H^{1,1}_{0;,0}(Q).$$

Proof. The assertion follows immediately by the definition of the space $\mathcal{H}_{0,}(Q)$. □

Lemma 4.4.6. *For the bilinear form $a_W(\cdot,\cdot)\colon \mathcal{H}_{0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}$, there holds the inf-sup condition*

$$\sup_{0 \neq w \in H^{1,1}_{0;,0}(Q)} \frac{|a_W(u,w)|}{|w|_{H^1(Q)}} = \left\|\Box_{Q\_} u\right\|_{[H^{1,1}_{0;,0}(Q)]'} \quad \textit{for all } u \in \mathcal{H}_{0,}(Q).$$

Proof. The inf-sup condition follows by the definition of the norm $\left\|\Box_{Q\_}(\cdot)\right\|_{[H^{1,1}_{0;,0}(Q)]'}$. □

Lemma 4.4.7. *For the bilinear form $a_W(\cdot,\cdot)\colon \mathcal{H}_{0,}(Q) \times H^{1,1}_{0;,0}(Q) \to \mathbb{R}$, there holds the surjectivity condition:*

For each $0 \neq w \in H^{1,1}_{0;,0}(Q)$, there exists an element $u \in \mathcal{H}_{0,}(Q)$ with $a_W(u,w) \neq 0$.

Proof. Let $0 \neq w \in H^{1,1}_{0;,0}(Q)$ be a given function. By Theorem 4.2.22 exists a unique solution $\tilde{u} \in H^{1,1}_{0;0,}(Q)$ of

$$a_{H^1}(\tilde{u},z) = -\langle \partial_t \tilde{u}, \partial_t z\rangle_{L^2(Q)} + \langle \nabla_x \tilde{u}, \nabla_x z\rangle_{L^2(Q)} \stackrel{!}{=} \langle w, z\rangle_{L^2(Q)} \qquad \forall z \in H^{1,1}_{0;,0}(Q) \quad (4.67)$$

satisfying $\frac{\sqrt{2}}{T} |\tilde{u}|_{H^1(Q)} \le \|w\|_{L^2(Q)}$. With the help of representation (4.64) and the variational formulation (4.67) for $z = w \in H^{1,1}_{0;,0}(Q)$, there follow

$$\begin{aligned} a_W(\tilde{u},w) &= \left\langle \Box_{Q\_} \tilde{u}, w \right\rangle_Q \\ &= a_{H^1}(\tilde{u},w) = -\langle \partial_t \tilde{u}, \partial_t w\rangle_{L^2(Q)} + \langle \nabla_x \tilde{u}, \nabla_x w\rangle_{L^2(Q)} = \langle w, w\rangle_{L^2(Q)} > 0 \end{aligned}$$

and hence, the assertion. □

Theorem 4.4.8. *For each given $f \in [H^{1,1}_{0;,0}(Q)]'$, there exists a unique solution $u \in \mathcal{H}_{0,}(Q)$ of the variational formulation* (4.66). *Furthermore,*

$$\mathcal{L}_W \colon\ [H^{1,1}_{0;,0}(Q)]' \to \mathcal{H}_{0,}(Q), \qquad \mathcal{L}_W f := u,$$

is an isomorphism satisfying

$$\left\|\Box_{Q_-} u\right\|_{[H^{1,1}_{0;,0}(Q)]'} = \left\|\Box_{Q_-} \mathcal{L}_W f\right\|_{[H^{1,1}_{0;,0}(Q)]'} = \|f\|_{[H^{1,1}_{0;,0}(Q)]'}.$$

Proof. With the help of the Nečas Theorem 2.9.1, the results in Lemma 4.4.5, Lemma 4.4.6 and Lemma 4.4.7 yield the existence and uniqueness of a solution $u \in \mathcal{H}_{0,}(Q)$. In addition, it holds with the variational formulation (4.66) the equality

$$\|f\|_{[H^{1,1}_{0;,0}(Q)]'} = \sup_{0 \neq w \in H^{1,1}_{0;,0}(Q)} \frac{|\langle f, w\rangle_Q|}{|w|_{H^1(Q)}} = \sup_{0 \neq w \in H^{1,1}_{0;,0}(Q)} \frac{|a_W(u,w)|}{|w|_{H^1(Q)}} = \left\|\Box_{Q_-} u\right\|_{[H^{1,1}_{0;,0}(Q)]'}$$

and therefore, the assertion. □

Remark 4.4.9. *The following functions are given to get a first impression of the solution space $\mathcal{H}_{0,}(Q)$.*

1. *Functions $u \in C^2(\overline{Q})$ with $u_{|\Sigma_0} = 0$ are contained in $H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0,}(Q)$.*
2. *Consider the smooth function*

$$u(x,t) = \sin(\pi x)\sin(\pi t) \quad \textit{for } (x,t) \in (0,1) \times (0,1) = Q$$

satisfying $\gamma_0^{\mathrm{int}} u = 0$ in Σ_0 and $\Box_Q u = 0$ in Q. But there is

$$\Box_{Q_-} u \neq 0$$

because the related distribution

$$T_u(\varphi) = \int_Q u(x,t)\varphi(x,t)\,\mathrm{d}x\mathrm{d}t \quad \textit{for all } \varphi \in \mathcal{D}(Q_-)$$

fulfils with integration by parts for all $\varphi \in \mathcal{D}(Q_-)$

$$\Box_{Q_-} T_u(\varphi) = T_u(\Box\varphi) = \int_Q u(x,t)\Box\varphi(x,t)\,\mathrm{d}x\mathrm{d}t = \int_Q u(x,t)(\partial_{tt} - \partial_{xx})\varphi(x,t)\,\mathrm{d}x\mathrm{d}t$$

$$= -\langle \partial_t u, \partial_t \varphi\rangle_{L^2(Q)} + \langle \partial_x u, \partial_x \varphi\rangle_{L^2(Q)} = \pi \int_0^1 \sin(\pi x)\varphi(x,0)\,\mathrm{d}x.$$

On the other hand, it holds $u \in H^{1,1}_{0;0,}(Q) \subset \mathcal{H}_{0,}(Q)$. In other words, the second initial condition $\partial_t u(\cdot,0) = 0$ in Ω is not incorporated in the ansatz space $\mathcal{H}_{0,}(Q)$.

5 CONCLUSIONS AND OUTLOOK

In this work, space-time variational formulations and their discretisations with conforming, piecewise polynomial functions for the heat and wave equation are considered in a bounded space-time cylinder Q with a finite time T.

The main result for the heat equation is an unconditionally stable finite element method of Galerkin-Bubnov type with piecewise linear, continuous functions, which is based on a variational formulation in a subspace of the anisotropic Sobolev space $H^{1,1/2}(Q)$. This space-time variational formulation is analysed with the help of Fourier series, and a kind of Hilbert transform $\mathcal{H}_T$ is introduced. This leads to a symmetric and elliptic variational formulation and hence, to a symmetric Galerkin discretisation of the first-order time derivative ∂_t. For the heat equation, unconditional stability for unstructured space-time meshes, error estimates in $L^2(Q)$, in $H^1(Q)$ and in the anisotropic Sobolev space $H^{1/2}_{0,}(0,T;L^2(\Omega))$ for a tensor-product approach are proven. Furthermore, numerical examples, which confirm the theoretical results, are presented. The main advantage of this formulation is the possibility of a combination with the standard boundary element method for the heat equation, i.e. a FEM-BEM coupling, see [39]. An investigation in such directions is a possible future work. In addition, proving a discrete inf-sup condition and error estimates for unstructured space-time meshes and a fast realisation of the kind of Hilbert transform $\mathcal{H}_T$ given in (3.67) are also of interest in future.

For the wave equation, a space-time variational formulation in a subspace of the Sobolev space $H^1(Q)$, which is not inf-sup stable, is used for a conforming space-time finite element method, which leads to a conditionally stable method, i.e. a CFL condition is required. For a tensor-product approach, an unconditionally stable method with piecewise linear, continuous functions is investigated. An extension to a space-time approximation with unstructured space-time meshes remains open for the future. A first possibility is the use of locally refined meshes with hanging nodes, including related constraints to satisfy the continuity requirements of the ansatz space. A second possibility is to transfer the stabilisation to unstructured but admissible space-time meshes. In both cases, one constant source of difficulties is the situation of different initial and end conditions of the ansatz and test spaces $H^{1,1}_{0;0,}(Q)$, $H^{1,1}_{0;,0}(Q)$, which may lead to a nonsquare system of linear equations for discretisations based on unstructured space-time meshes. Using the transformation $\overline{\mathcal{H}}_T$ given in (4.44), i.e. a Galerkin-Bubnov scheme, is a possible way out.

Moreover, existence and uniqueness results for the wave equation as a partial differential equation in $L^2(Q)$ and in a weaker sense than $L^2(Q)$ are proven, including isomorphic solution operators and corresponding inf-sup conditions. These inf-sup stable space-time variational formulations in the strong or weak sense might be useful not only for other discretisation methods, e.g., wavelets, but also for the analysis of the related boundary integral equations.

For both equations, i.e. for the heat equation and for the wave equation, moving boundaries, space-time adaptive schemes, space-time parallelisations and especially fast solvers and preconditioning, which are based on space-time variational formulations given in this thesis, are left for future considerations. Finally, any extensions to more involved equations, e.g., Stokes equations or Maxwell's equations, are of interest in future.

REFERENCES

[1] ABBOUD, T., JOLY, P., RODRIǴUEZ, J., AND TERRASSE, I. Coupling discontinuous Galerkin methods and retarded potentials for transient wave propagation on unbounded domains. *J. Comput. Phys. 230*, 15 (2011), 5877–5907.

[2] AIMI, A., DESIDERIO, L., DILIGENTI, M., AND GUARDASONI, C. A numerical study of energetic BEM-FEM applied to wave propagation in 2D multidomains. *Publ. Inst. Math., Nouv. Sér. 96* (2014), 5–22.

[3] AIMI, A., DILIGENTI, M., FRANGI, A., AND GUARDASONI, C. A stable 3D energetic Galerkin BEM approach for wave propagation interior problems. *Eng. Anal. Bound. Elem. 36*, 12 (2012), 1756–1765.

[4] AIMI, A., DILIGENTI, M., FRANGI, A., AND GUARDASONI, C. Neumann exterior wave propagation problems: computational aspects of 3D energetic Galerkin BEM. *Comput. Mech. 51*, 4 (2013), 475–493.

[5] AIMI, A., DILIGENTI, M., FRANGI, A., AND GUARDASONI, C. Energetic BEM-FEM coupling for wave propagation in 3D multidomains. *Int. J. Numer. Methods Eng. 97*, 5 (2014), 377–394.

[6] AIMI, A., DILIGENTI, M., GUARDASONI, C., MAZZIERI, I., AND PANIZZI, S. An energy approach to space-time Galerkin BEM for wave propagation problems. *Int. J. Numer. Methods Eng. 80*, 9 (2009), 1196–1240.

[7] AIMI, A., DILIGENTI, M., GUARDASONI, C., AND PANIZZI, S. A space-time energetic formulation for wave propagation analysis by BEMs. *Riv. Mat. Univ. Parma (7) 8* (2008), 171–207.

[8] AIMI, A., DILIGENTI, M., AND PANIZZI, S. Energetic Galerkin BEM for wave propagation Neumann exterior problems. *CMES, Comput. Model. Eng. Sci. 58*, 2 (2010), 185–219.

[9] AKOPJAN, J. R., AND OGANESJAN, L. A. A variational-difference method for the solution of two-dimensional linear parabolic equations. *Ž. Vyčisl. Mat. i Mat. Fiz. 17*, 1 (1977), 109–118, 284.

[10] ANDREEV, R. *Stability of space-time Petrov-Galerkin discretizations for parabolic evolution equations*. PhD thesis, ETH Zürich, 2012.

[11] ANDREEV, R. Stability of sparse space-time finite element discretizations of linear parabolic evolution equations. *IMA J. Numer. Anal. 33*, 1 (2013), 242–260.

[12] ARENDT, W., AND KREUTER, M. Mapping theorems for Sobolev spaces of vector-valued functions. *Stud. Math. 240*, 3 (2018), 275–299.

[13] ARENDT, W., AND URBAN, K. *Partielle Differenzialgleichungen. Eine Einführung in analytische und numerische Methoden.* Heidelberg: Spektrum Akademischer Verlag, 2010.

[14] ARNOLD, D. N., AND NOON, P. J. Coercivity of the single layer heat potential. *J. Comput. Math. 7*, 2 (1989), 100–104. China-US Seminar on Boundary Integral and Boundary Element Methods in Physics and Engineering (Xi'an, 1987–88).

[15] AUBIN, J.-P. *Applied functional analysis*, 2nd ed. ed. New York, NY: Wiley, 2000.

[16] AZIZ, A. K., AND MONK, P. Continuous finite elements in space and time for the heat equation. *Math. Comp. 52*, 186 (1989), 255–274.

[17] BAIOCCHI, C., AND BREZZI, F. Optimal error estimates for linear parabolic problems under minimal regularity assumptions. *Calcolo 20*, 2 (1983), 143–176 (1984).

[18] BAKER, G. A. Error estimation for finite element methods for second order hyperbolic equations. *SIAM J. Numer. Anal. 13* (1976), 564–576.

[19] BALES, L., AND LASIECKA, I. Continuous finite elements in space and time for the nonhomogeneous wave equation. *Comput. Math. Appl. 27*, 3 (1994), 91–102.

[20] BAMBERGER, A., AND HA DUONG, T. Formulation variationnelle espace-temps pour le calcul par potentiel retardé de la diffraction d'une onde acoustique. I. *Math. Methods Appl. Sci. 8* (1986), 405–435.

[21] BAMBERGER, A., AND HA DUONG, T. Formulation variationnelle pour le calcul de la diffraction d'une onde acoustique par une surface rigide. *Math. Methods Appl. Sci. 8*, 4 (1986), 598–608.

[22] BANGERTH, W., GEIGER, M., AND RANNACHER, R. Adaptive Galerkin finite element methods for the wave equation. *Comput. Methods Appl. Math. 10*, 1 (2010), 3–48.

[23] BANJAI, L., AND SCHANZ, M. Wave propagation problems treated with convolution quadrature and BEM. In *Fast boundary element methods in engineering and industrial applications*, vol. 63 of *Lect. Notes Appl. Comput. Mech.* Springer, Heidelberg, 2012, pp. 145–184.

[24] BAUSE, M., KÖCHER, U., RADU, F. A., AND SCHIEWECK, F. Post-processed Galerkin approximation of improved order for wave equations. *arXiv e-prints* (Mar. 2018).

[25] BAUSE, M., RADU, F. A., AND KÖCHER, U. Error analysis for discretizations of parabolic problems using continuous finite elements in time and mixed finite elements in space. *Numer. Math. 137*, 4 (2017), 773–818.

[26] BERGH, J., AND LÖFSTRÖM, J. *Interpolation spaces. An introduction.*, vol. 223. Springer, Berlin, 1976.

[27] BRADJI, A. Note on a new high order piecewise linear finite element approximation for the wave equation in one dimensional space. In *Numerical analysis and its applications. 6th international conference, NAA 2016, Lozenetz, Bulgaria, June 15–22, 2016. Revised selected papers*. Cham: Springer, 2017, pp. 235–242.

[28] BRADJI, A., AND FUHRMANN, J. Some new error estimates for finite element methods for second order hyperbolic equations using the Newmark method. *Math. Bohem. 139*, 2 (2014), 125–136.

[29] BRAESS, D. *Finite Elemente. Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie.*, 5th revised ed. ed. Berlin: Springer Spektrum, 2013.

[30] BRENNER, S. C., AND SCOTT, L. R. *The mathematical theory of finite element methods*, third ed., vol. 15 of *Texts in Applied Mathematics*. Springer, New York, 2008.

[31] CHAZARAIN, J., AND PIRIOU, A. *Introduction à la théorie des équations aux dérivées partielles linéaires*. Gauthier-Villars, Paris, 1981.

[32] COHEN, G. C. *Higher-order numerical methods for transient wave equations*. Scientific Computation. Springer-Verlag, Berlin, 2002.

[33] COHEN, G. C., AND PERNET, S. *Finite element and discontinuous Galerkin methods for transient wave equations*. Scientific Computation. Springer, Dordrecht, 2017.

[34] COSTABEL, M. Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal. 19*, 3 (1988), 613–626.

[35] COSTABEL, M. Boundary integral operators for the heat equation. *Integral Equations and Operator Theory 13*, 4 (1990), 498–552.

[36] DAUTRAY, R., AND LIONS, J.-L. *Mathematical analysis and numerical methods for science and technology. Vol. 5*. Springer-Verlag, Berlin, 1992. Evolution problems. I, With the collaboration of Michel Artola, Michel Cessenat and Hélène Lanchon, Translated from the French by Alan Craig.

[37] DEVAUD, D. *hp-approximation of linear parabolic evolution problems in* $H^{1/2}$. PhD thesis, ETH Zürich, 2017.

[38] DEVAUD, D., AND SCHWAB, C. Space-time *hp*-approximation of parabolic equations. *Calcolo 55*, 3 (2018), 23.

[39] DOHR, S. *Space-time boundary element methods for the heat equation*. PhD thesis, Technische Universität Graz, to be submitted, 2019.

[40] DOHR, S., NIINO, K., AND STEINBACH, O. Space-time boundary element methods for the heat equation. *Submitted.* (2018).

[41] DÖRFLER, W. A convergent adaptive algorithm for Poisson's equation. *SIAM J. Numer. Anal. 33*, 3 (1996), 1106–1124.

[42] DÖRFLER, W., FINDEISEN, S., AND WIENERS, C. Space-time discontinuous Galerkin discretizations for linear first-order hyperbolic evolution systems. *Comput. Methods Appl. Math. 16*, 3 (2016), 409–428.

[43] DOUGLAS, JR., J., AND DUPONT, T. Galerkin methods for parabolic equations. *SIAM J. Numer. Anal. 7* (1970), 575–626.

[44] DUPONT, T. L^2-estimates for Galerkin methods for second order hyperbolic equations. *SIAM J. Numer. Anal. 10* (1973), 880–889.

[45] EGGER, H., AND RADU, B. A mass-lumped mixed finite element method for acoustic wave propagation. *arXiv e-prints* (Mar. 2018).

[46] EGGER, H., AND RADU, B. Super-convergence and post-processing for mixed finite element approximations of the wave equation. *Numer. Math. 140*, 2 (2018), 427–447.

[47] ELSTRODT, J. *Maß-und Integrationstheorie*. Springer, Berlin, Germany, 2011.

[48] ERIKSSON, K., AND JOHNSON, C. Adaptive finite element methods for parabolic problems. I: A linear model problem. *SIAM J. Numer. Anal. 28*, 1 (1991), 43–77.

[49] ERIKSSON, K., AND JOHNSON, C. Adaptive finite element methods for parabolic problems. II: Optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$. *SIAM J. Numer. Anal. 32*, 3 (1995), 706–740.

[50] ERIKSSON, K., JOHNSON, C., AND THOMÉE, V. Time discretization of parabolic problems by the discontinuous Galerkin method. *RAIRO, Modélisation Math. Anal. Numér. 19* (1985), 611–643.

[51] ERN, A., AND GUERMOND, J.-L. *Theory and practice of finite elements*, vol. 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[52] FONTES, M. *Parabolic equations with low regularity*. PhD thesis, Lunds universitet, 1996.

[53] FRENCH, D. A. A space-time finite element method for the wave equation. *Comput. Methods Appl. Mech. Eng. 107*, 1-2 (1993), 145–157.

[54] FRENCH, D. A., AND KING, J. T. Analysis of a robust finite element approximation for a parabolic equation with rough boundary data. *Math. Comput. 60*, 201 (1993), 79–104.

[55] FRENCH, D. A., AND PETERSON, T. E. A continuous space-time finite element method for the wave equation. *Math. Comp. 65*, 214 (1996), 491–506.

[56] GANDER, M. J., AND NEUMÜLLER, M. Analysis of a new space-time parallel multigrid algorithm for parabolic problems. *SIAM J. Sci. Comput. 38*, 4 (2016), a2173–a2208.

[57] GIMPERLEIN, H., MEYER, F., ÖZDEMIR, C., STARK, D., AND STEPHAN, E. P. Boundary elements with mesh refinements for the wave equation. *Numer. Math. 139*, 4 (2018), 867–912.

[58] GIMPERLEIN, H., NEZHI, Z., AND STEPHAN, E. P. A priori error estimates for a time-dependent boundary element method for the acoustic wave equation in a half-space. *Math. Methods Appl. Sci. 40*, 2 (2017), 448–462.

[59] GIMPERLEIN, H., OEZDEMIR, C., STARK, D., AND STEPHAN, E. P. hp-version time domain boundary elements for the wave equation on quasi-uniform meshes. *arXiv e-prints* (Nov. 2018).

[60] GIMPERLEIN, H., OZDEMIR, C., AND STEPHAN, E. P. Time domain boundary element methods for the Neumann problem: error estimates and acoustic problems. *J. Comput. Math. 36*, 1 (2018), 70–89.

[61] GIMPERLEIN, H., AND STARK, D. On a preconditioner for time domain boundary element methods. *Eng. Anal. Bound. Elem. 96* (2018), 109–114.

[62] GLÄFKE, M. *Adaptive Methods for Time Domain Boundary Integral Equations*. PhD thesis, Brunel University London, 2012.

[63] GOPALAKRISHNAN, J., SCHÖBERL, J., AND WINTERSTEIGER, C. Mapped tent pitching schemes for hyperbolic systems. *SIAM J. Sci. Comput. 39*, 6 (2017), b1043–b1063.

[64] GRISVARD, P. *Elliptic problems in nonsmooth domains*, vol. 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[65] GROSSMANN, C., ROOS, H.-G., AND STYNES, M. *Numerical treatment of partial differential equations.* Berlin: Springer, 2007.

[66] GROTE, M. J., MEHLIN, M., AND SAUTER, S. A. Convergence analysis of energy conserving explicit local time-stepping methods for the wave equation. *SIAM J. Numer. Anal. 56*, 2 (2018), 994–1021.

[67] GROTE, M. J., SCHNEEBELI, A., AND SCHÖTZAU, D. Discontinuous Galerkin finite element method for the wave equation. *SIAM J. Numer. Anal. 44*, 6 (2006), 2408–2431.

[68] GRUBB, G. *Distributions and operators.*, vol. 252. New York, NY: Springer, 2009.

[69] GUARDASONI, C. *Wave Propagation Analysis with Boundary Element Method.* PhD thesis, Università degli Studi di Milano, 2010.

[70] GWINNER, J., AND STEPHAN, E. P. *Advanced boundary element methods. Treatment of boundary value, transmission and contact problems.*, vol. 52. Cham: Springer, 2018.

[71] HA-DUONG, T. *Equations intégrales pour la résolution numérique de problèmes de diffraction d'ondes acoustiques dans* $\mathbb{R}^3$. PhD thesis, Université Pierre-et-Marie-Curie, 1987.

[72] HA-DUONG, T. On the transient acoustic scattering by a flat object. *Japan J. Appl. Math. 7*, 3 (1990), 489–513.

[73] HA-DUONG, T. On retarded potential boundary integral equations and their discretisation. In *Topics in computational wave propagation*, vol. 31 of *Lect. Notes Comput. Sci. Eng.* Springer, Berlin, 2003, pp. 301–336.

[74] HACKBUSCH, W. Optimal $H^{p,p/2}$ error estimates for a parabolic Galerkin method. *SIAM J. Numer. Anal. 18*, 4 (1981), 681–692.

[75] HAIRER, E., AND WANNER, G. *Solving ordinary differential equations. II: Stiff and differential-algebraic problems. Reprint of the 1996 2nd revised ed.*, reprint of the 1996 2nd revised ed. ed., vol. 14. Berlin: Springer, 2010.

[76] HARBRECHT, H., AND TAUSCH, J. A fast sparse grid based space-time boundary element method for the nonstationary heat equation. *Numer. Math. 140*, 1 (2018), 239–264.

[77] HASSELL, M. E., QIU, T., SÁNCHEZ-VIZUET, T., AND SAYAS, F.-J. A new and improved analysis of the time domain boundary integral operators for the acoustic wave equation. *J. Integral Equations Appl. 29*, 1 (2017), 107–136.

[78] HOFER, C., LANGER, U., NEUMÜLLER, M., AND TOULOPOULOS, I. Time-multipatch discontinuous Galerkin space-time isogeometric analysis of parabolic evolution problems. *ETNA, Electron. Trans. Numer. Anal. 49* (2018), 126–150.

[79] HOU, L. S., AND ZHU, W. Error estimates under minimal regularity for single step finite element approximations of parabolic partial differential equations. *Int. J. Numer. Anal. Model. 3*, 4 (2006), 504–524.

[80] HUGHES, T. J. R., AND HULBERT, G. M. Space-time finite element methods for elastodynamics: Formulations and error estimates. *Comput. Methods Appl. Mech. Eng. 66*, 3 (1988), 339–363.

[81] HULBERT, G. M., AND HUGHES, T. J. R. Space-time finite element methods for second-order hyperbolic equations. *Comput. Methods Appl. Mech. Eng. 84*, 3 (1990), 327–348.

[82] HYTÖNEN, T., VAN NEERVEN, J., VERAAR, M., AND WEIS, L. *Analysis in Banach spaces. Volume I. Martingales and Littlewood-Paley theory.*, vol. 63. Cham: Springer, 2016.

[83] JENKINS, E. W., RIVIÈRE, B., AND WHEELER, M. F. A priori error estimates for mixed finite element approximations of the acoustic wave equation. *SIAM J. Numer. Anal. 40*, 5 (2002), 1698–1715.

[84] JOHN, V. *Finite element methods for incompressible flow problems.*, vol. 51. Cham: Springer, 2016.

[85] JOHNSON, C. *Numerical solution of partial differential equations by the finite element method.* Cambridge: Cambridge University Press, 1987.

[86] JOHNSON, C. Discontinuous Galerkin finite element methods for second order hyperbolic problems. *Comput. Methods Appl. Mech. Eng. 107*, 1-2 (1993), 117–129.

[87] JOLY, P., AND RODRÍGUEZ, J. Mathematical aspects of variational boundary integral equations for time dependent wave propagation. *J. Integral Equations Appl. 29*, 1 (2017), 137–187.

[88] KARAA, S. Finite element θ-schemes for the acoustic wave equation. *Adv. Appl. Math. Mech. 3*, 2 (2011), 181–203.

[89] KARAA, S. Stability and convergence of fully discrete finite element schemes for the acoustic wave equation. *J. Appl. Math. Comput. 40*, 1-2 (2012), 659–682.

[90] KATO, T. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[91] KESAVAN, S. *Topics in functional analysis and applications*. John Wiley & Sons, Inc., New York, 1989.

[92] KING, F. W. *Hilbert transforms. Volume 1.*, vol. 124. Cambridge: Cambridge University Press, 2009.

[93] KNABNER, P., AND ANGERMANN, L. *Numerical methods for elliptic and parabolic partial differential equations*, vol. 44 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 2003.

[94] KÖCHER, U., AND BAUSE, M. Variational space-time methods for the wave equation. *J. Sci. Comput. 61*, 2 (2014), 424–453.

[95] KRENK, S. Energy conservation in Newmark based time integration algorithms. *Comput. Methods Appl. Mech. Engrg. 195*, 44-47 (2006), 6110–6124.

[96] KRESS, R. *Linear integral equations*, third ed., vol. 82 of *Applied Mathematical Sciences*. Springer, New York, 2014.

[97] LADYZHENSKAYA, O. A. *The boundary value problems of mathematical physics*, vol. 49 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1985.

[98] LADYZHENSKAYA, O. A., SOLONNIKOV, V. A., AND URAL'TSEVA, N. N. *Linear and quasi-linear equations of parabolic type.*, vol. 23. American Mathematical Society (AMS), Providence, RI, 1968.

[99] LANGER, U., MOORE, S. E., AND NEUMÜLLER, M. Space-time isogeometric analysis of parabolic evolution problems. *Comput. Methods Appl. Mech. Engrg. 306* (2016), 342–363.

[100] LARSSON, S., AND MOLTENI, M. Numerical solution of parabolic problems based on a weak space-time formulation. *Comput. Methods Appl. Math. 17*, 1 (2017), 65–84.

[101] LARSSON, S., AND SCHWAB, C. Compressive Space-Time Galerkin Discretizations of Parabolic Partial Differential Equations. *arXiv e-prints* (Jan. 2015).

[102] LIONS, J.-L., AND MAGENES, E. *Problèmes aux limites non homogènes et applications. Vol. 1.* Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.

[103] LIONS, J.-L., AND MAGENES, E. *Problèmes aux limites non homogènes et applications. Vol. 2.* Travaux et Recherches Mathématiques, No. 18. Dunod, Paris, 1968.

[104] LOPEZ-FERNANDEZ, M., AND SAUTER, S. Generalized convolution quadrature with variable time stepping. *IMA J. Numer. Anal. 33*, 4 (2013), 1156–1175.

[105] LUBICH, C. Convolution quadrature and discretized operational calculus. I. *Numer. Math. 52*, 2 (1988), 129–145.

[106] LUBICH, C. On the multistep time discretization of linear initial-boundary value problems and their boundary integral equations. *Numer. Math. 67*, 3 (1994), 365–389.

[107] LUBICH, C., AND SCHNEIDER, R. Time discretization of parabolic boundary integral equations. *Numer. Math. 63*, 4 (1992), 455–481.

[108] MEIDNER, D. *Adaptive Space-Time Finite Element Methods for Optimization Problems Governed by Nonlinear Parabolic Systems.* PhD thesis, Ruprecht-Karls-Universität Heidelberg, 2008.

[109] MESSNER, M. *A fast multipole Galerkin boundary element method for the transient heat equation*. PhD thesis, Technische Universität Graz, 2013.

[110] MIGÓRSKI, S. A., OCHAL, A., AND SOFONEA, M. *Nonlinear inclusions and hemivariational inequalities*, vol. 26 of *Advances in Mechanics and Mathematics*. Springer, New York, 2013. Models and analysis of contact problems.

[111] MOIOLA, A., AND PERUGIA, I. A space-time Trefftz discontinuous Galerkin method for the acoustic wave equation in first-order formulation. *Numer. Math. 138*, 2 (2018), 389–435.

[112] MOLLET, C. Stability of Petrov-Galerkin discretizations: application to the space-time weak formulation for parabolic evolution problems. *Comput. Methods Appl. Math. 14*, 2 (2014), 231–255.

[113] MÜLLER, F. *Numerical analysis of Finite Element methods for second order wave equations in polygons*. PhD thesis, ETH Zürich, 2017.

[114] MÜLLER, F., SCHÖTZAU, D., AND SCHWAB, C. Discontinuous Galerkin Methods for Acoustic Wave Propagation in Polygons. *J. Sci. Comput. 77*, 3 (2018), 1909–1935.

[115] MÜLLER, F. L., AND SCHWAB, C. Finite elements with mesh refinement for wave equations in polygons. *J. Comput. Appl. Math. 283* (2015), 163–181.

[116] NEUMÜLLER, M. *Space-time methods: fast solvers and applications*. PhD thesis, Technische Universität Graz, 2013.

[117] NEUMÜLLER, M., AND STEINBACH, O. Refinement of flexible space-time finite element meshes and discontinuous Galerkin methods. *Comput. Vis. Sci. 14*, 5 (2011), 189–205.

[118] NEČAS, J. Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle. *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser. 16* (1962), 305–326.

[119] NEČAS, J. *Les méthodes directes en théorie des équations elliptiques*. Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague, 1967.

[120] NOON, P. J. *The single layer heat potential and Galerkin boundary element methods for the heat equation*. PhD thesis, University of Maryland, College Park, 1988.

[121] OSTERMANN, E. *Numerical Methods for Space-Time Variational Formulations of Retarded Potential Boundary Integral Equations*. PhD thesis, Gottfried Wilhelm Leibniz Universität Hannover, 2009.

[122] PANDEY, J. N. *The Hilbert transform of Schwartz distributions and applications.* New York, NY: John Wiley & Sons, 1996.

[123] PAZY, A. *Semigroups of linear operators and applications to partial differential equations*, vol. 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[124] PETERSEIM, D., AND SCHEDENSACK, M. Relaxing the CFL condition for the wave equation on adaptive meshes. *J. Sci. Comput. 72*, 3 (2017), 1196–1213.

[125] QIU, T., RIEDER, A., SAYAS, F.-J., AND ZHANG, S. Time-domain boundary integral equation modeling of heat transmission problems. *arXiv e-prints* (Nov. 2017).

[126] QUARTERONI, A., AND VALLI, A. *Numerical approximation of partial differential equations.*, vol. 23. Berlin: Springer, 1994.

[127] RAVIART, P.-A., AND THOMAS, J. M. *Introduction à l'analyse numérique des équations aux dérivées partielles.* 1983.

[128] REED, M., AND SIMON, B. *Methods of modern mathematical physics. I*, second ed. Academic Press, New York, 1980.

[129] REINARZ, A. *Sparse space-time boundary element methods for the heat equation*. PhD thesis, University of Reading, 2015.

[130] RICHTER, G. R. An explicit finite element method for the wave equation. *Appl. Numer. Math. 16*, 1-2 (1994), 65–80.

[131] SAUTER, S. A., AND SCHWAB, C. *Boundary element methods.* Berlin: Springer, 2011.

[132] SAYAS, F.-J. *Retarded potentials and time domain boundary integral equations. A road map.* Cham: Springer, 2016.

[133] SCHMÜDGEN, K. *Unbounded self-adjoint operators on Hilbert space.*, vol. 265. Dordrecht: Springer, 2012.

[134] SCHÖTZAU, D., AND SCHWAB, C. Time discretization of parabolic problems by the *hp*-version of the discontinuous Galerkin finite element method. *SIAM J. Numer. Anal. 38*, 3 (2000), 837–875.

[135] SCHÖTZAU, D., AND SCHWAB, C. *hp*-discontinuous Galerkin time-stepping for parabolic problems. *C. R. Acad. Sci. Paris Sér. I Math. 333*, 12 (2001), 1121–1126.

[136] SCHULZ, H., AND STEINBACH, O. A new a posteriori error estimator in adaptive direct boundary element methods: the Dirichlet problem. *Calcolo 37*, 2 (2000), 79–96.

[137] SCHWAB, C., AND STEVENSON, R. Space-time adaptive wavelet methods for parabolic evolution problems. *Math. Comp. 78*, 267 (2009), 1293–1318.

[138] SCHWARTZ, L. *Théorie des distributions*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966.

[139] SCHWEIZER, B. *Partielle Differentialgleichungen. Eine anwendungsorientierte Einführung.* Heidelberg: Springer Spektrum, 2018.

[140] SEPÙLVEDA SALAS, P. E. *Spacetime Numerical Techniques for the Wave and Schrödinger Equations*. PhD thesis, Portland State University, 2018.

[141] STEINBACH, O. *Numerical approximation methods for elliptic boundary value problems. Finite and boundary elements.* New York, NY: Springer, 2008.

[142] STEINBACH, O. Space-time finite element methods for parabolic problems. *Comput. Methods Appl. Math. 15*, 4 (2015), 551–566.

[143] STEINBACH, O., AND YANG, H. An algebraic multigrid method for an adaptive space–time finite element discretization. *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics) 10665 LNCS* (2018), 66–73. cited By 0.

[144] STEINBACH, O., AND YANG, H. Comparison of algebraic multigrid methods for an adaptive space-time finite-element discretization of the heat equation in 3D and 4D. *Numer. Linear Algebra Appl. 25*, 3 (2018), e2143, 17.

[145] STEINBACH, O., AND ZANK, M. Coercive space-time finite element methods for initial boundary value problems. *Submitted.* (2018).

[146] STEINBACH, O., AND ZANK, M. A stabilized space–time finite element method for the wave equation. In *Advanced Finite Element Methods with Applications. Selected papers from the 30th Chemnitz FEM Symposium 2017, (T. Apel, U. Langer, A. Meyer, O. Steinbach eds.), Lecture Notes in Computational Science and Engineering*. Springer, 2019, pp. 315–342.

[147] STRIKWERDA, J. C. *Finite difference schemes and partial differential equations. 2nd ed.*, 2nd ed. ed. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2004.

[148] TANTARDINI, F. *Quasi-optimality in the backward Euler-Galerkin method for linear parabolic problems*. PhD thesis, Università degli Studi di Milano, 2014.

[149] TAUSCH, J. Fast Nyström methods for parabolic boundary integral equations. In *Fast boundary element methods in engineering and industrial applications.* Berlin: Springer, 2012, pp. 185–219.

[150] THOMÉE, V. *Galerkin finite element methods for parabolic problems*, second ed., vol. 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2006.

[151] TOMARELLI, F. Regularity theorems and optimal error estimates for linear parabolic Cauchy problems. *Numer. Math. 45*, 1 (1984), 23–50.

[152] TRÈVES, F. *Topological vector spaces, distributions and kernels*. Academic Press, New York-London, 1967.

[153] TRIEBEL, H. *Interpolation theory, function spaces, differential operators. 2nd rev. a. enl. ed.*, 2nd rev. a. enl. ed. ed. Leipzig: Barth, 1995.

[154] URBAN, K., AND PATERA, A. T. An improved error bound for reduced basis approximation of linear parabolic problems. *Math. Comp. 83*, 288 (2014), 1599–1615.

[155] VEIT, A. *Numerical Methods for Time-Domain Boundary Integral Equations*. PhD thesis, Universität Zürich, 2011.

[156] VEIT, A., MERTA, M., ZAPLETAL, J., AND LUKÁŠ, D. Efficient solution of time-domain boundary integral equations arising in sound-hard scattering. *Internat. J. Numer. Methods Engrg. 107*, 5 (2016), 430–449.

[157] WEIDMANN, J. *Lineare Operatoren in Hilberträumen. Teil I: Grundlagen.* Wiesbaden: B. G. Teubner, 2000.

[158] WERNER, D. *Funktionalanalysis.*, 8th revised edition ed. Berlin: Springer Spektrum, 2018.

[159] WHEELER, M. F. A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations. *SIAM J. Numer. Anal. 10* (1973), 723–759.

[160] WLOKA, J. *Partielle Differentialgleichungen*. B. G. Teubner, Stuttgart, 1982.

[161] ZANK, M., AND STEINBACH, O. Adaptive space-time boundary element methods for the wave equation. *PAMM 16*, 1 (2016), 777–778.

[162] ZEIDLER, E. *Nonlinear functional analysis and its applications. II/A*. Springer-Verlag, New York, 1990.

[163] ZHAO, Z., AND LI, H. Convergence of a space-time continuous Galerkin method for the wave equation. *J. Inequal. Appl.* (2016), Paper No. 280, 18.

[164] ZLOTNIK, A. A. Convergence rate estimates of finite-element methods for second-order hyperbolic equations. In *Numerical methods and applications*. CRC, Boca Raton, FL, 1994, pp. 155–220.