Diploma Thesis

MODELLING USER ATTENTION WITH COMPETITION MODELS

Aleksandar Gagovic, Bsc.

7070 Knowledge Technologies Institute
Faculty of Computer Science and Biomedical Engineering
Graz University of Technology

Supervisor: Assoc.Prof. Dipl.-Ing. Dr. techn. Denis Helic

Submitted to **Graz University of Technology**

To achieve the university degree of Master of Science
Master's degree programme: Information and Computer Engineering
Graz, in August 2018

Masterarbeit

MODELLING USER ATTENTION WITH COMPETITION MODELS

Aleksandar Gagovic, Bsc.

7070 Institut für Wissenstechnologien
Fakultät für Informatik und Biomedizinische Technik
Technische Universität Graz

Betreuer: Assoc.Prof. Dipl.-Ing. Dr. techn. Denis Helic

Graz University of Technology

eingereicht an der **Techinsche Universität Graz**

zur Erlangung des akademischen Grades Master of Science
Masterstudium: Telematik

Graz, im August 2018

Abstract

This master's thesis is dealing with the distribution of the user attention that is occurring when multiple agents are competing for it. There is a broad use of competition models based on one or another form of competition in modern science. It is equally represented in natural as well in social science. Competition models have been proven to be very helpful when applied to solve complex relations between participants in a particular system. Since the non-linear nature of such a dynamic system is very hard to predict on its own, modern technologies aid in recreating those systems for experimental purposes. They help us save so much time and resources otherwise needed for the collection and processing of data. Having this modern technology at our disposal, it is only matter of choosing the right competition model as the tool for carving out the system dynamics of that particular competition. The aim of this thesis is to illustrate a competition model that includes participants who are competing for same resource within a closed system. It also tries to demonstrate how different input parameters affect the end results due to the system's non-linearity. By experimenting with different system setups this thesis also attempts to establish some general rules that are universally applicable by predicting end results of the competition within the system. The tool used in this thesis for creating a particular competition model with various competition setups is the Lotka-Volterra generalized competition model. The implementation has been done by using a Python-based framework specifically developed for this paper. Finally, the attained data and end results are being discussed.

Keywords: Competitive models, Evolutionary Game Theory, Lotka-Volterra equations,Predator prey model, non-linear system dynamics

Kurzfassung

In dieser Masterarbeit geht es um die Verteilung der Aufmerksamkeit des Benutzers, wenn mehrere Agenten darüber konkurrieren. Es gibt eine breite Verwendung von Modellen auf der Grundlage einiger Wettbewerbsformen in der modernen Wissenschaft. Es ist gleichermaßen in der Natur wie auch in den Sozialwissenschaften vertreten. Wettbewerbsmodelle haben sich als sehr hilfreich erwiesen, wenn komplexe Beziehungen zwischen den Teilnehmern eines Systems gelöst werden müssen. Da die Nichtlinearität eines solchen dynamischen Systems sehr schwer vorhersagbar ist, kann die Verwendung moderner Technologien zur Wiederherstellung eines solchen Systems sehr hilfreich sein. Sie könnten viel Zeit und Ressourcen für das Sammeln benötigter Daten sparen. Es geht dann eher darum, das richtige Wettbewerbsmodell als Werkzeug für die Systemdynamik eines solchen Wettbewerbs zu wählen. Das Ziel dieser Arbeit ist es, das Wettbewerbsmodell zwischen Mitgliedern zu demonstrieren, die im System über dieselbe Ressource konkurrieren. Es wird versucht zu zeigen, wie verschiedene Eingangsparameter aufgrund der Nichtlinearität des Systems signifikante Auswirkungen auf das Endergebnis haben können. Durch verschiedene Setups des Systems wird versucht zu lösen, ob es Postulate gibt, die allgemeingültig für die Vorhersage von Endergebnissen der Konkurrenz im System sind. Das Werkzeug der Auswahl, das für die Erstellung eines bestimmten Abschlussmodells mit verschiedenen Wettbewerbsaufbauten in dieser Diplomarbeit verwendet wird, ist das allgemeine Wettbewerbsmodell von Lotka-Volterra. Die Implementierung erfolgt in einem speziellen Python-basierten Framework, das speziell für dieses Papier entwickelt wurde. Abschließend werden die erzielten Daten und Ergebnisse diskutiert.

Keywords: Competitive models, Evolutionary Game Theory, Lotka-Volterra equations,Predator prey model, non-linear system dynamics

Statutory Declaration

I declare that I have authored this thesis independently, that I have not used other than the declared sources / resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.The text document uploaded to TUGRAZonline is identical to the present master's thesis dissertation.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst, andere als die angegebenen Quellen/Hilfsmittel nicht benutzt und die den benutzten Quellen wörtlich und inhaltlich entnommene Stellen als solche kenntlich gemacht habe. Das in TUGRAZonline hochgeladene Textdokument ist mit der vorliegenden Masterarbeit identisch.

Ort	Datum	Unterschrift

Acknowledgement

This master thesis was written in year 2018 at the Institute for Informations technologies at the Technical University of Graz.

This project would not have been possible without the support of many people. At the beginning i would like to thank my mentor, Assoc-Prof Dr.tech Denis Helic, who read my numerous revisions and helped make some sense of the confusion. Also thanks to my committee members, Ao.Univ.-Prof. Dipl.-Ing. Dr.techn. Erich Leitgeb and Ao.Univ.-Prof. Dipl.-Ing. Dr.techn. Elisabeth Lex , who offered guidance and support.

Thanks to the Technical University of Graz, providing me with means to make this thesis as well as much needed help and support throughout my entire studies. I would like to thank my family: my parents and to my sister for supporting me spiritually and materially throughout writing this thesis, my studies and my life in general.

And finally, thanks to my fiancée, who during the work on this thesis became also my spouse. She helped me with putting this thesis into cohesive structure and more readable work by correcting my many grammatical and syntactical errors, enduring this long process with me, always offering support and love.

Graz, in December 2018 Aleksandar Gagovic

Danksagung

Diese Masterarbeit wurde im Jahr 2018 am Institut für Wissenstechnologien an der Technischen Universität Graz durchgeführt.

An dieser Stelle möchte ich mich bei all denjenigen bedanken, die mich während der Anfertigung dieser Masterarbeit unterstützt und motiviert haben.

Zuerst gebührt mein Dank Herr Assoc-Prof Dr.tech Denis Helic, die meine Masterarbeit betreut und begutachtet hat. Für die hilfreichen Anregungen und die konstruktive Kritik bei der Erstellung dieser Arbeit möchte ich mich herzlich bedanken.

Ebenfalls möchte ich mich bei meinen Kommilitonen Ao.Univ.-Prof. Dipl.-Ing. Dr.techn. Erich Leitgeb und Ao.Univ.-Prof. Dipl.-Ing. Dr.techn. Elisabeth Lex bedanken, die mir mit viel Geduld, Interesse und Hilfsbereitschaft zur Seite standen. Bedanken möchte ich mich für die zahlreichen interessanten Debatten und Ideen, die maßgeblich dazu beigetragen haben, dass diese Masterarbeit in dieser Form vorliegt.

Danke an die Technische Universität Graz, die mir die nötigen Mittel zur Verfügung gestellt hat, um diese These sowie die dringend benötigte Hilfe und Unterstützung während meines gesamten Studiums zu erstellen. Ich möchte meiner Familie danken: meinen Eltern und meiner Schwester, dass sie mich geistig und materiell während des Schreibens dieser These, meines gesamten Studiums und meinem alltäglichen Leben unterstützt haben.

Abschließend möchte ich mich bei meiner Verlobten bedanken, die während der Arbeit an dieser These auch mein Ehepartner wurde. Sie half mir dabei, diese These in eine lesbarere Arbeit zu verwandeln, indem sie meine vielen grammatikalischen und syntaktischen Fehler korrigierte, diesen langen Prozess mit mir durchlebte und immer Rückhalt und Liebe bot

Graz, im Dezember 2018 Aleksandar Gagovic

Contents

List of Figures

List of Tables

1 Introduction

In the modern times, the use of different simulations is wide spread. To simulate a particular process (i.e public traffic) instead of doing it in the real world conditions usually means achieving the goal with much less effort, time and resources. It is by no means a new concept and it has been done in different forms for centuries. With the rise of modern technologies, especially computers and their processing power, the use and need for the simulations is even more obvious. From the basic mathematics to the modelling of the nuclear explosions on the stars and evolution of the Solar system, there are practically limitless possibilities and approaches where computers and their potential can be used to help humanity.

A very interesting field where different types of simulations can be applied is the closed system competition. In the physical science the term closed system represents the system that does not allow specific types of relocation in or out of the system[RN91][p.78]. This means that there are no foreign (outer) influences on the system values, thus it can be considered as a ”hermetically closed”. The entire dynamics of the system depends strictly of the inner parameters.

Competition in such system can be then regarded as a simplified model of a particular open system competition. By effectively excluding outer factors that can influence the system, it is easier to simulate and predict competition progress. Participants of the system can only compete among each other. Every participant has some features that can only be changed in the interaction with other participants, or according to some predefined parameters of the system itself. They don’t have to take into consideration some outer factors that can occur in the system. By knowing this, they can then adjust their competition strategies accordingly.

1.1 Motivation

There is no need to make some theoretical use case when there is more than enough of good examples from the everyday life. Modern social networks’ and search engines’ competing for user attention is one of such examples. As with most businesses, different social network platforms are competing for the market share. In this case, users are the resource they are competing for. When more users are spending more of their free time on a particular social media platform, instead of on one of their competitors’, that particular network is establishing "supremacy" over other networks. As with every competition, they all want to win by acquiring majority of the resources. All of them want that an average user spends more time in their ”ecosystem” than in the competitor’s system. Let’s say that an average user spends three hours a day surfing the web and browsing various social networks. This means that if this user spends two hours a day on Facebook (e.g.), he has only one hour left for all other networks. It is also very likely that user's attention given to a particular social network will gradually decrease over the time, and eventually cease to exist. In theory, an average user may spend less than three hours on the Internet, but will never exceed this upper limit. This means that only the attention given to the individual social network can vary, but not the total time spend on all of the social networks.

There are various competition models trough which we can illustrate this issue. Those competition models were researched by many different scientists, who provided diverse representations of the competition, as well as solutions and tools to determine possible results. This thesis will not only cover some of them, but also implement one specific tool in order to see if some general rules can be establish within the competition in closed systems. The competition itself can be regarded as a "fight" between participants in a particular system over some kind of a resource. Our example in the previous paragraph serves as a model to showcase some basics of such a competition model.

There is, however, a problem that can often occur with such an illustration as the one above. A typical representation of N participants (hereinafter agents) in the competition doesn't take into consideration the fact that the number of users is a definite number that all of the agents have to share. Instead, a typical competition model only shows the user attention flow between agents (social networks in this case). If we measure all users on all networks at some random point in time, it is very likely that the total number of users will be greater than the real number of all users, which should not be possible. This happens because majority of such studies concentrate on the user as a resource, and not on the attention and time that the user can invest in a particular social network. A simple example would be an user who uses 3 of the 6 most popular social networks. In a typical competition model it is usually supposed that the user is spending $\frac{1}{3}$ of his time on each network. But such an assumption doesn't take into consideration the fact that the user may spend two hours per day on the first, and just half an hour on the second and third social network. Consequently, the first social network actually takes 66% of the user's attention.

In this thesis we are changing the approach and implementing the fact that all agents in the system are sharing mutual resources, i.e. users attention as shown in the example above. If such a system is going to be modelled with these conditions, that would mean that, at some random point in time, neither of agents should have more than three hours of user's attention. Concurrently the entire sum of attentions at that point in time can be i.g. 5 hours, since this system does not have an upper value sum restriction, but just a time restriction for a single agent.

A very important segment where these simulations can be applied is called game theory (GT), and it has become a very popular concept in the last century. The concept, history and components of the game theory will be covered in the next chapter. The most significant branch of the GT is Evolutionary Game Theory (EGT), which will be used as a specific tool for showcasing the possibilities of the competition models later on in the thesis.

1.2 Objective

Why is this tool so important? Here is a simple example to showcase its significance. Imagine there was a national or presidential election. The usual way to predict the outcome of the elections is to take a random sample of people and ask them which option they are going to vote for. This method is based on a simple statistic. If the sample is broader, taken from various parts of the country, with mixed age-, educational-, and gender structure, the prediction will be more reliable and the results will probably be more accurate. Acquiring this amount of data is neither easy nor cheap. If an adequate type of competition model was used instead, with some predefined parameters, a certain prediction with similar results could be made with far less effort and resources. Of course, it is not always possible to determine the correct value, but with enough parameters it is possible to construct an adequate model with a lower margin of error. Such models are not being applied only to social sciences, but also to natural sciences. Its name is self-explanatory, since the first real usage of such models was in biology, in order to determine population growth of particular fish species[J.L25]. About 100 years ago, a corner stone for modern evolution and competition model among the species was laid down. More about its history and development will be covered in next chapter.

The motivation behind this thesis was to construct a competition model of different agents that are competing for the same resource. This is very important because most of other competition models do not have that joint parameter. Instead, agents within them are competing for their own supremacy. The general problem here is that there can be a situation when there are two species (agents) and one can eat up another (predator and prey). This is the most common system in population dynamics, a predator-prey model, which will also be covered in more detail later on. But what if those two species are competing for the same resource, without being able to harm each other, at least not directly? How to construct a system if that is the case? This thesis will try to answer these questions by giving a better illustration of such a system.

1.3 Disclaimer

The base for this thesis is the book *"Evolutionary Games and Population Dynamics"*, which was written by Josef Hofbauer and Karl Sigmund[Hof88].

As with most simulations, the provided results are just an approximation and not the exact values. The systems that are going to be described further in the thesis are very dynamic in nature, and as such, they are also very unpredictable. There are far too many parameters that have to be taken into account in order to make the right assumption. Because of all of this, it is not easy to quantify the exact value of the simulation results precision (percentage-wise). On the other hand, they can give the better comprehension of the system dynamics development within such models, as well as predict possible outcomes. We just have to keep in mind the fact that the biggest limitation of such models is the model itself, or to be more accurate, the parameters that are describing them.

1.4 Testing Methods

An obvious solution for this type of problem would be to use a processing power of modern computers in order to speed up the calculation. There are already many mathematical competition models that can be constructed by using programming languages. This method is beneficial on both fronts. Firstly, it is much easier to generate data by applying an appropriate competition model, than to obtain it by physical measuring of any kind. And secondly, once gathered, it is much easier to utilize the computers to do the "heavy lifting" instead of manually calculating and processing that data. On top of that, it is also much more simple and faster to visually present those results in form of diagrams and graphs.

The programming language that was here used to model simulation and to compute its values is Python. There are many reasons why exactly this programming language was chosen. It is fast and often used for large mathematical computations. It has quite large open community support with different libraries and native integrations across various platforms. Since it is not very verbose, it is easily understandable for people that did not write the code, as well for those without programming background. Moreover, it is one of those natural programming languages, which makes it very easy to learn. Although I personally have extensive knowledge of several object-oriented languages, my Python background was on the beginner's level when I first started working on this thesis. Even with those basic skills, the entire practical part of this thesis was written relatively easy. There were far less lines of code needed than it would be the case with Java or C-like languages. In short, the less is more in this case.

As previously stated, the code written in Python is going to be used to make the calculations and present the results in form of graphs. The input parameters of every species are chosen randomly, with no specific reason why exactly these numbers are being used and not some others. However, since each agent has several parameters, the practical part of the thesis will show how different combinations of different parameters change the competition behaviour for that single agent. The purpose of testing is also to explore which parameters have a greater influence upon the system of equations. In this way we will be able to establish some rules and constraints within the system, as well as to determine how to set parameters in order to achieve different outcomes.

Input values will be given in form of a .csv file, and output values are going to be plotted to graphs and saved to another .csv file. Python is responsible for the entire data processing. More about the implementation and the code itself will be covered in Chapter 4.

1.5 Chapters overview

In the next chapter, Chapter 2 the theoretical background and related work that was used as the base for this thesis will be covered. This chapter also covers important terminology and definitions concerning the competition models and game theory in general. Starting with some historical background of the game theory, the chapter examines elementary terms important for basic understanding of this area of system dynamics, as well as some significant terms and concepts that will be applied later on in the thesis. Second part of the chapter focuses on the evolutionary game theory, its definitions and basics principles. Beside the basics, this chapter also covers concepts of logistic growth and Lotka-Volterra's non-linear dynamic systems, which are the foundation of the practical part of this thesis.

Chapter 3 presents the problem of the thesis and shows how current models of evolutionary game theory can be adjusted in order to achieve a desired output. This chapter also provides a solution to the problem based on the mathematical analysis of the given competition model.

Chapter 4 represents the experimental setup of the thesis. The simulation framework used for computing the competition model will be shown in the introductory part of the chapter, as well as a brief explanation of the used libraries and input files. Main part of this chapter covers the experiments that are being executed in order to establish potential rules that could be applied to competition models. All input data, mathematical representations, as well as output plot with end results are also presented in this chapter.

Chapter 5 evaluates the results of the simulation framework. The purpose of this chapter is to process all the data gathered from the executed experiments and to present the results in a cohesive form. Only generally valid findings for a specific model setup are covered. Additionally, this chapter also covers the limitations of the implemented model.

Finally, the last chapter, Chapter 6,draws the final conclusions about the entire modelling process. It also indicates possible improvements and suggest interesting topics that some additional/further/ research could cover.

2 Theoretical Background and Related Work

One of the most important sources for this thesis was the book ”Theory of Games and Economic Behaviour” [NM47, p.1] written by the founder of modern game theory, John Von Neumann and Oscar Morgenstern. Another significant source of information as well as inspiration was a book called ”Contributions to the Theory of Games” written by Alfred W. Tucker and Harold W Kuhn, which has had four different editions [KT50] [HWK53] [MD57] [HWK59].

Although these two books represent knowledge base and the foundation of this thesis, there is a plenty of other scientific papers that have been consulted in order to better understand and present the theory behind it.

Since the game theory in general is already extensively researched, finding adequate literature to be referenced in this thesis was not that difficult. In order to avoid plagiarism, all the ideas, statements, quotations, explanations, etc., have been referenced and cited properly. Since the papers that were consulted are mostly quite short and concise, there is no need to explain them in much detail. Consequently, there are usually just couple of sentences (and not the entire content) taken from a particular paper to back up some specific statements I was trying to make.

As a foundation block for the theoretical part of the evolutionary game theory were used three books. The first one was the ”Evolutionary Game Theory”[Wei97] written by Joergen W. Weibull. The second one is a book named ”Evolution and the Theory of Games”[Smi82] by John Maynard Smith. Last important book that was used as a basis for this thesis was the book ”Evolutionary Games and Population Dynamics”[Hof88, p.1] by Josef Hofbauer. The theoretical background of evolutionary game theory as itself will be covered in the last part of this chapter.

Same as with the general GT, the evolutionary game theory is broadly explored as well, especially over the last twenty years. Therefore, finding adequate literature that would make for the backbone of the theoretical part of the thesis was not particularly complicated. Majority of the references were used to provide an introduction to both theories and to cover some basic ideas and terminology in order to enable better and easier practical implementation of the latter.

The last major literature section covers the main implementation tool itself - the Lotka-Volterra competition model[J.L25]. This competition model was used in the practical part of this thesis as a tool for modelling the competition for user attention. The underlying theory behind this model is 90 years old, which means that it has already been considerably researched, with enough resources to create a solid basis for the further development of that chapter in this thesis. In this case, there was no need to use books as a starting point, since the required theoretical background is less complex. For that reason, several papers and web links were more than sufficient. Those have been also referenced appropriately whenever some segments from those papers were incorporated into this thesis.

One last thing worth noting here is the fact that this thesis is not and cannot be considered as a direct extension of any specific paper or book. This thesis simply gives an overview of most relevant literature on a given topic and in its practical part showcases the theoretical implementation on a particular example. Should anybody insist on imposing some kind of a correlation between this thesis and those previously mentioned works, then it is safe to say that this paper compresses and summarizes several scientific papers on competition models in the evolutionary game theory. More about all of this later on in this chapter..

2.1 Modelling Competition

The term *competition*, as well as the entire concept behind it, is not anything new. Since the beginning of the time there were always various types of competitions among all kind of live beings. Biology as a science is based upon diverse competitions, and such, it has laid a corner stone for the evolution in general. Of course, the notion of competition is not only present in biology, but also in many other natural and social sciences. In fact, one could say that the competition, in one way or another, is omnipresent in every being's life, and as such, it seems almost impossible to avoid.

Competition can be defined as an interaction between diverse organisms, whereas the existence of one can endanger or decrease the fitness of the other.[LD07][p.324] Fitness is a key concept of the evolutionary theories and can have various meaning. Apart from the most commonly known physical fitness which portrays a person's general stamina or conditioning, there is also the Darwinian fitness[LD07][p.323]. This type of fitness implies the ability of some species to reproduce, or to be more precise, their reproductive ability to add their genes to the gene pool. The better the fitness, the more likely it is for a particular species to have offspring and to further evolve in the future. If the fitness is lower, there is more chance for that species to become extinct.

There are many models that can describe different kinds of competitions. But in the end, depending on the type of the competitor, they can all be divided into two main categories:

1. Intra-specific
2. Inter-specific

An intra-specific competition occurs among the entities of the same species. If the competition occurs among different species, than we are talking about the inter-specific competition. Even though these terms are mostly used in biology, they do not reflect only the biological species. The competition can be a football match between two teams, whereby each team is a ”species”. Or among car manufactures competing for the biggest market share. The term ”species” is here used to generalize the occurrence of such competition in order to make it more translatable and applicable in different fields. It does not even need to be a plurality of the same entity in order to represent species. One party can also depict it within a particular competition model. Apart from these two main competition types (and their subcategories), there are also many other different types of competition that are not really that relevant for this thesis.

What actually is relevant for this thesis is a particular system within which the agents are competing for the user attention. In this context agents can be referred to as species, and the amount of the user's attention is considered to be a resource the agents (species) are competing for. A competition model can be relatively easily simplified and represented in form of a mathematical model. This is especially important when there is a need to test system dynamics of such model and its various parameters (such as species behaviour over time, importance of different input values on the final outcome, etc.). By doing so, the models can be represented as mathematical equations, and then translated into the software frameworks, in order to test exponentially more complexed settings.

There is a particular branch of mathematical science that explores different system dynamics of the non-linear systems (and most of the typical competition models fall under this category). This branch is called game theory and it can be a very helpful tool for creation of a distinct competition model.

Before proceeding with development of one specific competition model that is required in this thesis, it is necessary to first get more familiar with the available tools for creating it, as well as with the game theory in general.

2.2 Introduction to Game Theory

In short, game theory is a branch of applied mathematics which research and develops a strategic situations, respectively a situation of conflict and cooperation in which success of the player or participant depends from other players and participants. As such, there can be many examples for the applied use of the game theory in different fields of science and engineering in general. Although game theory as such is not a new idea, the main focus and its development was made in 20^{th} century.

Probably the most important reading which defined the game theory was the book of John von Neumann "Theory of Games and Economic Behaviour"[NM47]. He wrote this book together with economist Oscar Morgenstern. In the second edition of the book, the authors made assumptions of why the game theory is so useful and gave the examples that are important for statistics and economics. Those examples shown how to determine actors (players) decisions under certain conditions.

The importance of game theory was acknowledge with the Nobel Prize. Until today eleven game-theorists won the Nobel Prize, the last one, Jean Tirole in 2014 in the field of economics[AB14]. This number should not surprise anyone since the game theory first found its applied use directly in economics. The reason behind it was simple. Find and recognize certain situation in which some entity or organization is in conflict with some other entity or organization. In such cases, right judgement and execution can be crucial for survival of that entity. Game theory is obviously not only bound for economics, but also for politics, computer science, law, sports and more or less everywhere where there is a for need decision making of some kind. Even some popular games such as chess and Monopoly are based on some game theory model which stems from real life conflicts.

Game theory is very broad spectrum of different cases and it is not very easy to systematically derive that spectrum is general categories. With that being said, by looking the number of active players for different game theory examples, it can be clearly seen that usually there are either single or multi player systems. This difference is going to be useful to divide game theory into two main categories.

First sub group are the one player games. In this type of games participant player usually plays against the surrounding(nature). This sort of game theory is called decision theory. For such games the probability theory is very important, since it is useful to know the chances of different possible outcomes in the game. That is why is this theory broadly used in the analysis of decision, in order to showcase best solutions given by the provided informations.

The second sub group are multi player theories. They can be also divided in even more subgroups. The most important difference from the one player games is that the players are now competing against each other and not against their surrounding. For the game theorist these type of games are generally much more interesting than the first sub group. This thesis main goal will be to explain and demonstrate one specific example of game theory and what is it important. Before beginning with specifics, some formalities have to be covered, and thus answer to some crucial questions, in order to better understand the multi-player systems.

2.2.1 History of Game Theory

As already mentioned in the introduction, John von Neunmann, a mathematician and pioneer of modern computer science, is considered to be the father of game theory. Even though there were some papers already published by other authors, von Neumann set the mathematical basis of game theory in his articles from the year 1928 and 1937. In his book ”Theory of Games and Economic Behavior”[NM47, p.1], he showcases the implementation of mathematical theories into strategic games, which was developed by him and his co-author, Morgenstern, in 1928 and between 1940-1941. The book also discusses various economic issues, however, due to World War II happening at that time, much more attention was paid to translating the game theory to modulation of war situation, whereby the main premise was the conflict between two sides (players).

Another also very important concept was established by John Forbes Nash, who wrote his doctoral thesis under the title ”Non-Cooperative Games”, and introduced the term ***equilibrium*** for the first time. This term is today called the Nash’s Equilibrium and Nash won the Nobel Prize for his discovery[Tho84, p.20] and also one very good movie[1]. In years that followed, the game theory was further developed to find its usage in different fields, such as politics and negotiations, just to name a few. The fourth edition of the book ”Theory of Games”Kuhn and Tucker provides an useful overview on how game theory could be applied to relevant socio-economic issues of that particular time (1950.-1959.).[KT50, p.1]

Next important discovery for the game theory happened in 1984, when the book ”The Evolution of Cooperation”[Axe84, p.1] by Robert Alexrod was published. He studied the game theory from a different point of view than authors before him. His main premise was, what if there was no need for conflict, but for cooperation instead. His book proved that mutual and continuous collaboration between agents within a particular system is not only desirable, but also mutually beneficial for all parties. He even organized a tournament to play the famous iterative game "Prisoners Dilemma" only to demonstrate that it is definitely well worth while to work together in the context of long-term interactions. More about this topic and its findings is covered later in this chapter.

2.2.2 Types of Games

There are many criteria based on which games can be categorized. The most obvious one is the number of players, in which case they can be divided into following categories:

1. games with one player, which are not very interesting to the game theory
2. games with two players, most commonly studied type of game
3. games with more players.

Even by categorizing the game solely based on the number of players, there are still some other distinctive features that enable further classification. One of those features is interest of the player. In that case, games can be divided into 3 additional subcategories[KS06][Mar71]:

[1] More information about the movie "A Beautiful Mind": *http://www.imdb.com/title/tt0268978*

- cooperative games
- non-cooperative games
- games with combined motives.

The first type would be the games in which all players have the common interest. They will make coalitions (alliances) which help them to meet their needs and, consequently, achieve their goals. A simple practical example of such situation can be found in traffic. Drivers in cars, driving in opposite directions, whereby each driver (player) makes decisions independently regarding the direction he is taking (turn left, turn right, stop, go straightforward, etc.), but they all have to cooperate in order to avoid accidents and traffic jam. This implies that drivers have to signal their decisions to other players, i.e. drivers.

Most of the games, however, fall under the second category. Goals of the players and their interest are completely opposite, and there is no cooperation of any kind between them. A good example of such game are most of the board games, card games, chess, etc.

The third type of games are games that have both of those elements at the same time, the so called games with combined motives. In these games, players have to cooperate to some extent. A practical example from real life would be the labour union and top management within a company, with both parties having different agendas and goals. For example, company executives will usually try to get as much as possible from the workers at the lowest cost possible, while the labour union members, on the other hand, will always try to make better working conditions for themselves, such as higher wages, less work, more vacation days, etc. However, if they want the company to be successful, they have to find a common ground. Only in that way is the company able to make profit. That is why there is a need to find a balance between opposite sides in order to achieve success in games with combined motives.

Taking into account the overall gain in the game, the games can be further divided into two categories: ones with null sum and ones without null sum.[Lac84, p.11-26] Former are games where the gain of winner equals the loss of loser (just opposite sign), but the sum of these two values is null. Simply put, the winner gains everything that the loser has lost. The latter game type does not have this feature, and the overall sum in the end depends on many factors.

Games can also be classified by the amount of information and rules a game has, which leads us to systems with complete and incomplete information and rules. First type implies that all players are familiar with all the rules, strategies and have same knowledge about a particular game system. Chess would be example of such system because both players know all the moves and rules they are playing by. The second type in this subcategory are games that do not have strictly defined rules. Example of such system would be auctions, because players do not know how much other players are willing to bid for an item.

An finally, the last type of games that we want to mention here are iterative games. Basically, those are games where adversaries play same game multiple times and then they adjust their strategies according to output parameters.[Lac84, p.11-26]

2.2.3 Game Theory Terminology

In game theory there are different terms that describe the theory itself and they are used in the modelling of games.[Tho84, p.17] The term **game** in the game theory is the model of conflict situation between two or more agents. It is already said in the definition, the game tries to summarize key problems in the conflict of interests. Next important term is **players**. The players are described as game participants and they are the models of real entities which participate in some type of conflict interaction. Throughout this thesis we use the term **agents**in this context.

Another important term is **move**. Every game consists of sequence of moves. The move is defined as an event in the game which is the consequence of a player's decision or of a random event. For example, in the game of poker the first move is always random (dealing cards to the players).[Tho84, p.18]

Gain represents the value (sum) at the end of the game and an aftermath of all played moves. It is usually represented in a form of numerical value. This can be also simplified by saying that a player gets +1 point if he wins, 0 points if the final result/outcome is even, and -1 if he loses.[Tho84, p.18]

Strategy is very important term in game theory. It represents description of all decisions which player can make in any given situation in the game. The strategy can be illustrated as a decision tree in which every possible situation within the specific game is defined . One thing worth noting here is that this representation is relevant with some simple games. But coming up with a decision tree for the game of chess, for instance, would be practically impossible, since there are 20 possible starting moves, and with each move the number of total moves grows exponentially.

2.2.4 Game Analysis

After the basic terms and form of the game, next thing that will follow in game theory is its analysis. Game theory does not offer a question of how to play a particular game in order to win, it just offers a possible strategies and offers a solution to the game.[Gib82, p.115]

The question here is how can be decided in some particular game which output is better (in games with two or more players). For an external viewer it is difficult to decide if the gain of one player is more important (greater) than the other. Therefore it is necessary to have a correct approach for comparing different outcomes of the game. The simple example of such game would be a game where the gain of a player is his pay in his currency. If there are two players with two different currencies and the course between them is unknown, it is not possible to determine which player had gain more.

It is not possible to determine the gains in every case, but some assumptions are relatively easy to make in order to better understand the gains. The course is maybe unknown, but if the first player has 10 units of currency X and 5 units of currency Y, and the second has 10 units of currency X and 3 units of currency Y, it is clear that the first player had maid greater gains. From such conclusion stems the definition of ***Pereto dominance.***[SLB08, p.61]

Definition of Pereto dominance: Strategic profile with Pereto dominance dominates over strategic profile s' if for every $i \in N$ $u_j(s) \geqslant u_j(s')$ and there is $u_j(s) \geqq u_j(s')$

Under strategic profile is considered some of possible combinations of strategies that are available to the players. In other words, in Pereto-Dominated profile, some player can have more gains (profit) without diminishing gains of another player. From this statement a term of Pereto optimality can be defined.

Definition of Pereto optimality: Strategic profile s is Pereto optimal (efficient) if there is no other strategic profile $s' \in S$ which dominates over s.

Therefore Pereto optimal profile is the one where profit of one player cannot be increased without diminishing the profit of some other.

2.2.5 Dominant Strategy

Some assumptions have to be made when talking about game theory rules. One of the most important ones is that game is played by rational players. Under the term rational is meant that a rational player would never play a strategy which is dominant. Definition of the dominant strategy is the following[Gib82, p.5]:

In normal form of game $G = (S_1, \ldots, S_N; u_1, \ldots, u_n)$, s_i' and s_i'' are the allowed strategies for the player i. Strategy s_i' is hard dominated from strategy s_i'' if for every strategies combination of the competitor player, gain of the Player I is less if he plays strategy s_i' instead s_i''.

On the top of the next example, such strategy can be further investigated. On the table 2.1 is given the gain matrix of some abstract game[Gib82, p.6]:

		Player 2		
		left	middle	right
Player 1	up	1,0	1,2	0,1
	down	0,3	0,1	2,0

Table 2.1: Example of dominant strategy gain matrix

From this example it is possible to eliminate dominant strategy. Player 1 has two strategies: Up and Down, and Player 2 has three: Left, Middle and Right. For Player 2, Right is the dominated from the strategy Middle because the gain of the strategy Middle is always greater than the gain of the strategy Right indifferent of what strategy Player 1 chooses ($2 > 1$ and $1 > 0$)

		Player 2	
		left	middle
Player 1	up	1,0	1,2
	down	0,3	0,1

Table 2.2: Table to test captions and labels

After eliminating strategy Right, there is another dominated strategy in the table, this time for the Player 1 and that is the strategy Down. The players know dominated strategies and Player 1 is left with only one strategy, up. Player 2 has two strategies left, but it can be easily seen from the table 2.2 that the best against strategy Up, strategy Middle. By playing strategy Middle in this abstract system, it can be considered as a solution for the given game.[Gib82, p.6]

In order to make such iterative elimination of possible strategies in the system, there is one pre-requirement that has to be fulfilled. All players have to know all strategies. The problem is that most of the games cannot come up to solution by applying iterative elimination of the dominated strategies.

2.2.6 Nash Equilibrium (Balance)

The term of Nash's equilibrium is one of the most important and integral parts of the game theory in general, and it is going to be of big use in the Chapters 3 and 5 when the implementation of the evolutionary game theory comes to place.

Even through by applying the iterative elimination of the dominated strategies it is possible to find the solution of the abstract game, much stronger (and better) way to this is by applying the Nash's equilibrium. If strategies that are going to be played in the game can be assumed, the rationale players will always play the best strategy that is feasible against the opponent. If two strategies of the opponents are at the same time the best response from one to another then they are in the state of Nash's equilibrium. The following definitions if given by[Gib82, p.8]:

In normal form of game with n players $G\{S_1, \ldots, S_n; u_1, \ldots, u_n\}$, strategies $\{s_1^*, \ldots, s_n^*\}$ are in Nash's equilibrium if for every player I s_i^* (least tied) best answer of player I onto strategies of the rest n-1 players $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$:

$$U_1(s_1^*, \ldots, s_{i-1}, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geqq u_1(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \tag{2.1}$$

For every possible strategy s_i in S_i , so that s_i^* is solving:

$$Maxu_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \tag{2.2}$$

This formal definition means that the strategies are in Nash's equilibrium if there are no other combinations of strategies which can give better profit for every strategy when they deal directly. Following simple example is here to provide better understanding of how the procedure of finding the Nash's equilibrium is implemented. This example can be seen on the table 2.3.

The process of finding the Nash equilibrium in this example is consisting of finding the best response (strategy) on every strategy. Best response to the strategy T is the strategy L, because it gives the biggest profit to the Player 2 (4). By observing strategy C of the second

		Player 2		
		left	middle	right
Player 1	T	3	2	0
	M	2	3	1
	B	2	3	1

Table 2.3: Example of Nash Equilibrium

player, the best response in this case is the strategy T from the first player. By comparing every strategy of Player 1 with every strategy of the Player 2, it can be seen that the only relation that satisfies the Nash equilibrium is the one between the strategy B of the Player 1 and the strategy R of the Player 2, because they are the best responses one to another. Even through the Nash equilibrium is fundamental part of game theory and it is considered the best way to find a solution in some game, the problem remains still that not every game has a strategies that can correlate with Nash equilibrium.

2.2.7 Evolutionary Game Theory

The name evolutionary game theory was derive from the mathematical game theory which was applied to the biological conditions. It was original developed as a concept and a theory by R. A. Fisher in 1930[Fis30] In the context of evolution it was realized that its fitness was the strategic part of the evolution. Over the course of years, the appliances of this model was not only used in biology, but it found its use in the fields of economy, anthropology and sociology, and social science in general (for example in politics in order to make vote predictions, by implementing competition behaviour).

There are three main reasons why is this model of such interest for non-biological fields. The first is that the term of "evolution" used in the evolutionary game theory does not have to be strictly considered as biological evolution. It can also be consider as a cultural for example.[SP73] The example of this type of evolution would be religious (changing the sort of religion in one area over given time). The second reason would be the premises behind this sort of game theory are very useful for creating some social systems, then the ones behind the standard game theory. And thirdly, the evolutionary game theory is dynamic theory, and as such gives the decisive aspect that is missed from standard game theory.

The difference from the standard game theory is that evolutionary can be applied on systems that are not necessary defined with explicit rules and decisions. Furthermore even the agents of the system do not have to be completely reasonable as in normal game theory. Instead of this, the analysis will be utilized in such context, that every agent of the system can show various behavioural states (thus including ones that are not totally reasonable). Now, by analysing the behaviours, it should be possible to determine which one of them are making the influence of particular agents to grow (or persist), and which one are doing completely opposite (they are driven out from the closed system by other agent).

This type of approach as the name itself suggests is usually used in the area of evolutionary biology. This term was firstly used by John Maynard Smith and G.R Price. Evolutionary biology is based on the idea that an organisms genes largely determine its observable characteristics, and hence its fitness in a given environment[Smi82] . The species (organisms) that are more capable will probably give more descendants, which will have positive effect to the genes by boosting their fitness to again give more descendants, thus raising the population of the specific specie. What this means is that the genes that are more fitted will win eventually just because of the specie ability to reproduce faster than others.

The key of evolutionary game theory is the behaviour of every specie. The success will largely depend from the interaction between them in a mutual system. Because the competition is the key, and there has to be an interaction among species this means that if one specie is isolated, its fitness cannot be measured. This is only possible when that particular specie is competing with others. And here the relations to the standard game theory can be made. The parameters of the specie (genetic characteristics) are the same as the strategy in game theory, and the fitness is the same as the pay-off. The pay-off is built upon strategies (or parameters if evolutionary terminology is used) of the specie with which it interact. This is off course a little generalized representation, by the ideas from standard game theories do indeed have their counterparts in the evolutionary model. Even some ideas like equilibrium are used to make predictions of output values in the populations evolutions, and these states will be shown latter on in the practical part of this thesis, when the main focus would be the prediction of the possible end results.

2.2.8 Difference Between Evolutionary Game Theory and Conventional Game Theory

It was already mentioned that the EGT was derived as a part of more general CGT. As such, EGT inherits some of the CGT features, and then adds a layer of its own on the top. Most important mutual feature of both theories is its purpose. Both are used to make interactive decision making. Core aspect of them is the Nash's Equilibrium [WAD68][p.540]. But even though both theories are trying to achieve the same goal, the agents interacting in those games are having different starting factors. Omnipresent feature of every model based on the CGT is that the agents participating in it will try to do "rationale choice". This means that in the case of CGT agents will always be perfectly rational, trying to make most of the gain matrix. This feature makes them very predictable, since if the pay-off matrix of the agent is known, it is possible to determine its next move (since it will always try to make the biggest gain). This feature is point of departure between CGT and EGT. The later theory assumes that the agents are having "lack of rationality" [DE10][p.209] , based on some other components of the agent (biological, sociological, cultural) and as such strategy that emerges is the product of trial and error (it does not have to be perfectly rational all the time). This component makes the game more realistic.

Evolutionary stable strategy that was already mentioned in the previous section is also an important difference. To put it simple, ESS is a combination of Nash's equilibrium and stability. This means that a part of the ESS is still coupled with the stability strategy of the CGT (which is the Nash Equilibrium) but there is also the stability parameter which is an exclusive evolutionary parameter. Strategies that have larger Darwinian fitness will be preferable and more stable. Selection will than lead to an ESS. As a pure biological term this is called ”survival of the fittest”, where only the most equipped species will survive and the ones less capable will go extinct. This feature can be seen even better in modern times in the industry, since it usually takes much less time to see big changes in some systems, that in would take in some biological system. Obvious example would be smart-phone market. In the last 10 years so much things have changed that the predictions for the next 10 would be in the realm of blind guess. The Figure2.1 shows the smart-phone sales per quarter per operating system. It is very easily seen how the situation dramatically changed in just couple of years. Symbian, who was the dominant operating system at the beginning of the 21st century lost practically all of its intakes, were the Google's Android sky-rocketed obtaining over 70% of today's smart-phone market share. Such systems are perfect examples to showcase potential usage of EGT.

On the other hand, in order for some biological specie to go extinct (or take dominance over system) it would take years and centuries (and probably even more if there are no outer influences and the system is closed)

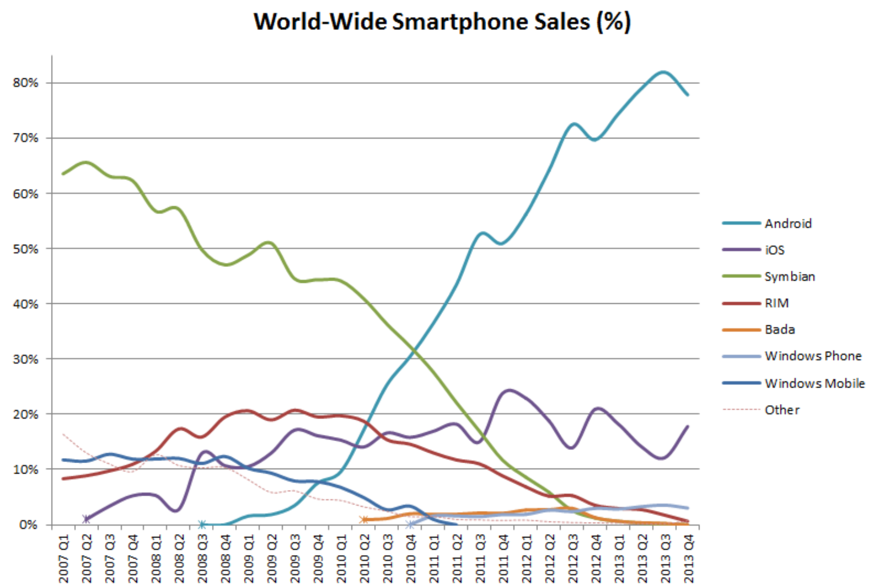

Figure 2.1: Smartphone sales given by the quarter [2]

[2] This picture was created based on the World-Wide Smartphone Sales based on Gartner actuals from the following web addresse: *http://www.statista.com/statistics/263437/global-smartphone-sales-to-end-users-since-2007/*

2.2.9 Application of Evolutionary Game Theory

Foundation of evolutionary game theory were made on top of traditional game theory. There are specific aspects of different science disciplines were the CGT is not very suited. Some problems can be explained with less effort using EGT. In the heart of the system the model is still based on the equilibrium. Rationalization of agents is what sets these two models apart. Knowing that the agent will not always make ideal move gives a certain form of realism which lacks in a CGT. Through the trial and error process, agent in the system will try to make optimal strategy. Such concept is very similar to typical human behaviour, since most of human learnings is some sort of trial and error. Different human behaviours were also explained with the use of EGT. Some of the most notable are: altruism[FZ07], behaviour in public goods game[CR06], human culture[EG07], moral behaviour[HS08], social learning[KN03].

EGT is not only relevant in natural sciences, but in social as well. It modern times it is very common in solving different economic issues[Fri98]. This is due the face that such models are able to produce richer prediction than the ones using CGT. EGT was found by economic theorist to be very effective in exploring the base of CGT solution approaches[Fri98].

Social dilemmas can also be a good starting point for application of a EGT. A reputation of one individual which it has in some society can be a good example of social dilemma. Reputation forms the system of the evolution of corporation in larger societies where people may interact frequently with people that they may not know personally but because of various reputation systems they're able to identify those who are cooperative and enter into mutually beneficial reciprocal relationships. The more sophisticated and secure these reputation systems the greater the capacity for cooperative organizations. We can create large systems where we know who to cooperate with, potentially creating a successful community. But of course, as the society gets bigger, more complex institutions for enabling functional reputation systems must be created. In such a way we have gone from small communities where local gossip was suffice to know everyone's capacity per corporation to large modern societies where centralized organizations vouched for people's reputation to today's burgeoning global reputation system based on information technology and mediated through the Internet. Research shows the co-operators create better opportunities from themselves than non-co-operators[HS08]. They are selectively preferred as collaborative partners, romantic partners and group leaders. This only occurs however when people social dilemma choices are seen and recorded by others in some way. However this kind of indirect reciprocity is cognitively complex. No other creature has mastered it to even a fraction of what humans have.

Potential fields of use don't even need to have humans as main focus of inquiry. The usage is much wider and can range anywhere from law and politics to the industrial organization. To put it short, whenever there is a mechanics in which agent interact in a system with each other strategically (which means the agents outcome could depend from its but also other agents influences), EGT is worth considering as a possible tool for solving such problem. This is the main reason why the author of this thesis has chosen EGT as medium for helping in modelling competition model for user attention.

2.2.10 Two Approaches of Evolutionary Game Theory

Upon closer check, it can been noticed that there are typically two main approaches to the evolutionary game theory. These approaches can be simply called static and dynamic one.

First type of approach is explained in the work of Maynard Smith and Price[Smi82]. They have employed here the idea of the solid strategy as the main engine of the analysis. It can be thus considered as static since the definitions do not refer to the underlying process which can change the population by implementing different strategies (behaviours).

The second approach doesn't try to determine the evolutionary stability assumption. Here a model of the process have to be defined, who can change the strategy frequency in the population, and then examine and analyses the parameters of the dynamics of the evolution in that particular model. This means when the dynamic is defined, all given stability approaches in the dynamical systems analysis can be used.

First Approach, Definitions of Evolutionary Stability

In the book *"The Logic of Animal Conflict"* by Maynard Smith and Price, the first approach is explained in more detail. They have used the Hawk-Dove game in order to make the analysis. The concept behind this game is the following: there are two species (agents) that are competing for the same resource which has some fixed value V (as a biological term, value V would be the resource that represents the increase of the Darwinian fitness of the specie that gathered the resource). Each of those two specie have only one strategies at its disposal, and they are explained below.

HAWK: they have aggressive behaviour, in which case they cannot be stopped until they are injured or the other specie retreats.

DOVE: they have defensive behaviour, which means, they will retreat as soon other agents commence aggressive behaviour.

If this is put into a matrix representation, there are four different outcomes, because every specie can interact with itself and the other). (1) Two hawks interact with each other, they both initiate aggressive behaviour, thus resulting as probable outcome equal probability for both agents to be injured. The cost of this conflict will be the reduced size of every unit fitness by some constant value C. If the Hawk meets the Dove, the Dove will retreat at once, thus leaving all the resources to the Hawk (2) and remaining without any (3). If the two Doves are collided, they will just share them equally (4) between each other. This pay-off matrix can be seen in the picture below:

As it was already stated, there can be only one strategy as it is shown in the pay-off matrix. This means that every single actor of both species will have to use this rules in the game. This will result for the strategy to be evolutionary stable. If this is implemented, no mutant (this is an agent of system that utilizes some different, innovative strategy) can successfully "infiltrate" into the system.

This system can be then described as followed. The discrete value $\Delta F(a_1,a_2)$ stands for the variance (change) of the fitness for the agent that follows his strategy a_1 against another agent that follows strategy a_2. F(s) would stand for the total fitness of the agent that follows strategy s, and there is also an initial state (value of fitness) that is going to be noted as F_0. Then there are 3 more parameters:

- σ - Evolutionary stable strategy,
- μ - Mutual attempt to invade the population
- p - proportion of the population that follows the μ

The first two parameters can be then described in form of derivative functions:

$$F(\sigma) = F_0 + (1 - p)\Delta F(\sigma, \sigma) + p\Delta F(\sigma, \mu) \tag{2.3}$$

$$F(\mu) = F_0 + (1 - p)\Delta F(\mu, \sigma) + p\Delta F(\mu, \mu) \tag{2.4}$$

There are also some rules that need to be followed in order to achieve the system stability. Since the parameter σ is evolutionary stable, fitness of the agent that follows it must be larger than the one that follows μ. If this is not the case, that the mutant agent is going to be capable to invade this system. Translated to the mathematical expression these means that F(σ) < F(μ).

Second rule in concern about proportion p. This value should drive to zero and thus requiring one of following two statements (this is how evolutionary stable system by Maynard Smith and Price was defined):

- ΔF(σ, σ) > ΔF(μ,σ)
- ΔF(σ, σ) = ΔF(μ,σ) and ΔF(σ, μ) > ΔF(μ,μ)

Explaining it in simple words what does this all means is the following. Evolutionary stable strategy σ is fulfilled if one of two requirements are met:

1. Parameter σ does better playing against itself, then any mutant agent against σ
2. If there is some mutant agent that plays just as good against σ as the σ itself, then the σ plays better against that mutant then the mutant agent itself.

The outcome which stems from such type of strategy in the Hawk-Dove game is following: Dove strategy is not evolutionary stable. Considering that the population can be overrun by other factor in the system, in this case a Hawk mutant agent. When the cost of injury C is lower than the value of V (acquiring new resource) then the evolutionary strategy of the Hawk is stable. This means that it is worth risking getting injured in order to acquire new resource. In the opposite situation, evolutionary strategy is not stable if it used only the pure strategy. But if the players fuse different strategies, than it is possible to gain stability in the evolutionary strategy [Sel83].

	Cooperate	Defect
Cooperate	(R,R')	(S,T')
Defect	(T,S')	(P,P')

Table 2.4: Prisoners dilemma payoff matrix

After the original work of Maynard Smith and Price, there were some other solution and concepts that were submitted. Two most prominent ideas were the evolutionary stable set[Tho84], and the idea of a "limit ESS"[Sel83][Sel88]

The first work provided s generalization of the ESS concept and the second protracted this concept to the specific type of game, an extensive game form with two players.

2.3 Second Approach, Specifying Dynamics for the Population

Most well-known example for the second approach of the evolutionary game theory is the Prisoners Dilemma game. Here, every player can have two types of strategies, and they are usually called *"Cooperate"* and *"Defect"*. The pay-off matrix of this game can be seen in a Table 2.4 above

Pay-offs are listed as (row, column). When it is represented in this form, they do not have to be symmetric in the matrix. Only requirement is that the pay-offs for each player have to be ordered properly.

Obvious question here is, how a group of individuals can evolve by constantly playing the game such as Prisoner's Dilemma. In order to answer on that question, couple of assumptions had to be made. First assumption is that the particular group has to be large in numbers. Population state can be then portrayed by tracking relation of how many individuals are using every strategy. In this case there are only two of them, Cooperate and Deflect, which greatly simplifies things. Percentages of those two strategies can be labelled as P_c and P_d. Fitness of both parties can be then denoted as W_c and W_d, and the median value of entire group (population) can be represented with W. From here there are three equations that can depict those values using pay-off values and proportions:

$$W_C = F_0 + p_c \Delta F(C, C') + p_d \Delta F(C, D) \tag{2.5}$$

$$W_D = F_0 + p_c \Delta F(D, C') + p_d \Delta F(D, D) \tag{2.6}$$

$$\bar{W} = p_c W_c + p_d W_d \tag{2.7}$$

Second assumption is the "inheritance" of the strategy from their off springs. For the next generation, both strategies can be related to the proportion of the entire group which follows either Cooperate or Deflect strategy in the current generation. Those relations can be represented with following equations.

$$p_c' = \frac{p_c W_C}{\bar{W}} \tag{2.8}$$

$$p_d' = \frac{p_d W_D}{\bar{W}} \tag{2.9}$$

These equations can be also written in that manner that they represent difference between current and next generation:

$$p_c' = \frac{p_c(W_C - \bar{W})}{\bar{W}} \tag{2.10}$$

$$p_d' = \frac{p_d(W_D - \bar{W})}{\bar{W}} \tag{2.11}$$

Finally, it can be assumed that the difference can be substitute with approximation of the differential equations. These equations were first defined by Taylor and Jonker in 1978[TJ78], to provide continual evolution of the particular dynamics needed for the evolutionary game theory. They called it the *Replicator Dynamics*:

$$\frac{dp_c}{dt} = \frac{p_c(W_C - \bar{W})}{\bar{W}} \tag{2.12}$$

$$\frac{dp_d}{dt} = \frac{p_d(W_D - \bar{W})}{\bar{W}} \tag{2.13}$$

There is a distinct difference between replicator dynamics and evolutionary stable strategy. While the former can be formed to use either pure or mixed strategy, the latter presumes that the members of such group can use only pure strategies. For the replicators these means that they can copy there "features" to the off springs without an error. So as the state of the population modifies, so does the fitness of that group as well as the pay-off of the used (pure) strategy.

The replicator dynamics may be used to model a population of individuals playing the Prisoner's Dilemma. For the Prisoner's Dilemma, the expected fitness of Cooperating and Defecting are:

$$W_C = F_0 + p_c \Delta F(C, C') + (p_c)\Delta F(C, D) = F_0 + p_c R + p_d S \tag{2.14}$$

$$W_D = F_0 + p_c \Delta F(D, C) + (p_d)\Delta F(D, D) = F_0 + p_c T + p_d P \tag{2.15}$$

Because $T > R$ and $P > S$, it follows as well that $W_D > W_C$ and hence $W_D > \overline{W} > W_C$. This means that $\frac{W_D - \overline{W}}{\overline{W}} > 0$ and $\frac{W_C - \overline{W}}{\overline{W}} < 0$

Since the strategy frequencies for Defect and Cooperate in the next generation are given by the following equations:

$$p'_d = p_d * \frac{W_D - \overline{W}}{\overline{W}} > 0 \tag{2.16}$$

$$p'_c = p_c * \frac{W_C - \overline{W}}{\overline{W}} < 0 \tag{2.17}$$

respectively, we see that over time the proportion of the population choosing the strategy Cooperate eventually becomes extinct. Figure 2.2 illustrates one way of representing the replicator dynamical model of the prisoner's dilemma, known as a state-space diagram.

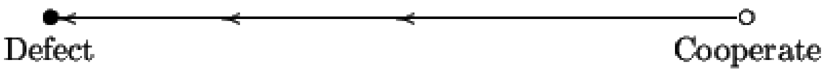

Figure 2.2: The Replicator Dynamical Model of the Prisoner's Dilemma [3]

This graph can be interrupted as following: On the left, the point shows the case where all of the agents in the system are deflecting by default. The far right point is the state when all of the agents are cooperating. Everything what is in between is the percentage-wise relationship between these two states. Evolutionary trajectory is represented with the arrows that go through the line. This line is also then followed with the amount of agents over time. Open circle on the right implies that at this point (where all of the agents are cooperating) are unstable equilibrium. The reason for this is simply the fact that a small group of the mutant agents that would play the other strategy (deflect) would draw the rest of the agents away from the equilibrium (since the pay-off of this strategy would be higher). This is also the reason for the solid circle on the right. It represents the stable equilibrium, because in this case, the mutant strategy is cooperate and some small percentage of agents will play it, while the majority remains at the deflect, it will return to the equilibrium state due to the evolutionary dynamics.

It can be seen here, that there is not a lot of difference these two approaches of the EGT. In the particular game it is clear that only ESS is the deflect strategy. Under the replicator dynamics this is the only equilibrium which is stable in nature. Both approaches then fit together rather elegant. The only SE happens when all of the agents in the system are following the given ESS. But it should be also clear, that here this is only the case, since the game is pretty simple. Normally, such games are much more complex, usually involving more than two players. This subsequently is also multiplexing the complexity in a relationship between the stable states of the replicator dynamics and the evolutionary stable strategy.

What also can be noticed as the interested point of such approach, it is that solution is not always obvious. This can be seen even on the simple games such as the Prisoners

Dilemma. The political scientists Robert Axelrod in the late 70s performed a number of highly influential computer experiments asking what is a good strategy for playing a repeated prisoner's dilemma[Axe84]. He asked the various researches the submit the algorithm that can play game, and to see, which kind of strategy would win. Computer models of the evolution of cooperation showed that indiscriminate co-operators almost always end up losing against defectors, who accept helpful acts from others but do not reciprocate them.

However, this is where the things are getting more interesting. This game clearly showed one distinct peculiarity that is not very different from the real life. People who are cooperative and helpful indiscriminately all of the time will end up getting taken advantage of by others, just like in the game. But, if all agents in the system are playing the pure defective strategy, then they will also lose out on the possible rewards of cooperation that would give them all higher pay-offs.. Many strategies have been tested and the best competitive strategies are generally cooperative with the reserved retaliatory response, if necessary. The most famous and one of the most successful of these is tit for tat. Tit for tat is a very simple algorithm of just three rules.

- I start with cooperation
- If you cooperate then I'll cooperate
- If you defect then I'll defect follows the

Computer tournaments in which different strategies were pitted against each other showed tit the tat to be the most successful strategy in social dilemmas. Tit for tat is a common strategy in real-world social dilemmas because it is nice but firm it makes cooperation a possibility but is also quick to reprimand the defectors. It is a strategy that can be found naturally in everything from international trade politics to people borrowing and lending money. In repeated iterations cooperation can emerge when people adopt a tit-for-tat strategy.Evolutionary selection can then start to favour them and they do not get exploited by all the defectors because they immediately switched defect in retaliation.

But the tit for tat attack strategy did not last long in this setting as a new solution came to emerge given this new context. This strategy was a mutant of tit for tat it was more forgiving called generous tit for tat. Generous tit for tat is an algorithm that starts with cooperation and then will reciprocate cooperation from others but if the other defects it will defect with some probability. Thus it uses probability to enable the quality of what we might call forgiveness. It cooperates when others do but when they defects there is still a probability that it will continue to cooperate. This is a random decision by the algorithm so it is not possible for others to predict when it will continue to cooperate which is an important part of this strategy. It turns out that this forgiveness strategy is optimal in environments when there is some degree of noise in communications as is characteristic of real-world environments. In a world of errors in action and perception such a strategy it can be a Nash equilibrium and evolutionary stable. The more beneficial the cooperation is the more forgiving generous tit for tat can be, while still resisting invasion by defectors.

The extraordinary thing that now happens is the ones everyone has moved towards playing generous tit for tat. Cooperation becomes a much stronger tractor and at this stage players can now play an unconditional cooperative strategy without having any disadvantages. In a world of generous tip for tat there is no longer a need for any other actions and thus unconditional co-operators survive.

As it was already explained in the previous paragraph, in order for a strategy to be evolutionary stable it must have the property that if almost every agent of the system follows unanimous strategy, no mutant can successfully invade. A mutant is an agent who adopts a novel strategy different to the default one. In many situations cooperation is favoured and it even benefits an individual to forgive an occasional defection. The cooperative societies are always unstable because mutants inclined to defect can upset any balance and this is the downfall of the cooperative strategy. What happens next is somewhat predictable. In a system, where everyone is cooperating, unconditional defection becomes an optimal strategy once it takes hold. Thus it is obvious that in such games a dynamic cyclical process as higher forms of cooperation arise and then collapse. In many ways then this reflects what can be seen in the real worlds. Economies and empires rising and falling as institutional structures for cooperation of forms mature and eventually decline.

The basic problem of the evolution of cooperation is that the nice guys get taken advantage of and thus there must be some form of supporting structure to enable cooperation. More than any other primate species humans have overcome this problem through a variety of mechanisms such as reciprocating cooperative acts forming reputations of others and the self and caring about those reputations. We create pro-social norms about good behaviour that everyone in the group will enforce on others through disapproval, if not punishment. And will enforce on themselves through feelings of guilt and shame. All of which form the fabric of our socio-cultural institutions that enable advanced forms of cooperation

Both approaches are dynamic in nature and therefore can be represented in a form of the system of ordinary differential equations. This aspect can greatly help into representing the needed competition model. It is far simpler to represent such model in a form of mathematical equations, as well as to translate it to the programming language in order to make the calculations much faster. Exactly that characteristic will be used and explained in the next section of this thesis. But before the illustration of the envisaged mathematical system used to model the competition model, same basics of the dynamic competition model, which is going to be used latter, have to be covered.

2.4 Logistic Growth

Logistic growth is the most basic model for (generally) representation of the biological population in a closed system. By term system is meant that specie's environment which has plenty of food, enough space for growth and multiplying and no threats from predators. If there is no upper limit for the quantity for specific specie, this number could potentially rise to infinity. If it is represented in a form of function, and then logistic growth as a discrete function is considered, in every next iteration, the value will be a little higher than in the last. This means that the variance will always be positive, and will also increase with every next step. In the real world, this type of system is usually non-existent, since it is impossible to grow continuously without slowing down, or without some upper limit.

This is why it should be consider a value K which represents a carrying capacity of the system. Carrying capacity is maximum number of some units that can occupy the system. In the real world scenarios, when all the same settings are made, and the single change is the existence of the carrying capacity, the logistic growth will then rise faster until some point, and as it approaches the carrying capacity, its rise will be slowed down, and as the time t increases (number of repetitions) it will always strive to the carrying capacity of the system, but never exceed that value, thus making the growth virtually non-existent.

By comparing those two models, it is easily noticeable that for the lower values of P (population of the specie) both functions are almost identical, but after some time, with the rise of population, they start to differ, more and more as the time progresses. This model is widely known as logistic growth model or the Verhulst model that is given in honour of the famous Belgian mathematical P.F. Verhulst. He was famous for predicting the US population 100 years in future with the mistake that was less than 1 percent (the prediction was made by him in 1840).[Ver38] Even though he made this as an approximation, he was remarkably accurate, and the model was more and less were accurate until 60' of the 20th century.

This model is not only useful for predicting biological population rates, but in various different sciences, natural or social. Logistic growth is used today to describe different types of behaviours, in medicine, statistics, politics, linguistics, economics and many others.

2.4.1 Logistic Growth Terminology and Mathematical Representation

The following formula is the most used representation of the logistic growth and it was modelled by P.F Verhulst in 1838:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) \tag{2.18}$$

The parameters are given in form of a differential equation, are the following:

- K - Carrying capacity of the system (maximum value of units/agents in the system)
- r - Growth rate of the specie
- P - Current population of the specie in the system
- $\frac{dP}{dt}$ - Change of the population between two discrete time slices (it must be positive)

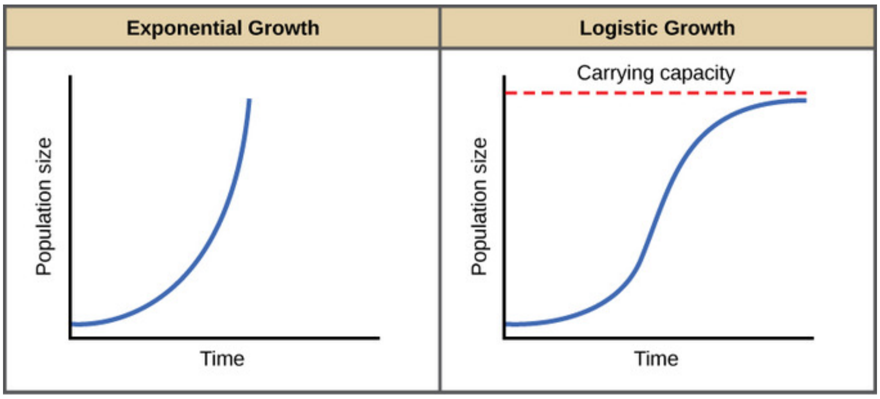

Figure 2.3: Exponential and Logistic growth functions[4]

On the Figure 2.3 the differences between the exponential and logistic growth of the same function can be seen. As the time goes by, first one will start to grow even faster, while on the other hand, the logistic growth will decrease its speed as it approaches to the carrying capacity limit.

Both of these function are not very useful for modelling competition systems. This is due the fact that they cannot be used to represent multiple agents competing in same system. On the other hand, there is a tool than can enable exactly that feature, making possible to implement effects that agents are having on each other. Such model is called Lotka-Volterra model.

[4] This picture was taken from the following web addresse: *https://figures.boundless.com/21346/large/figure-45-03-01.jpe*

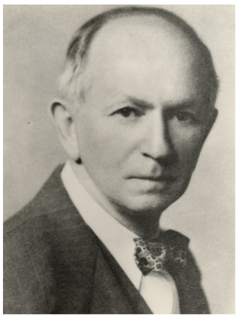

Figure 2.4: Alfred J. Lotka

Figure 2.5: Vito Volterra

2.5 Lotka-Volterra Model

2.5.1 Introduction to the Lotka-Volterra Equations

The Lotka-Volterra predator prey equation was independently developed from Alfred J. Lotka, US mathematician physical chemist and statistician in 1925[J.L25] and Vito Voltera, an Italian mathematician and physicist in 1926. These equations have given rise to a vast literature. The key examinations that triggered both mathematicians to develop that model, was the result of the predatory fish population in the Adriatic see after the World War I. The results have shown that the number of such fish species were much greater than before the war.

Vito Volterra[5]was an Italian mathematician and physicist. In 1931 he was one of a small minority of an Italian professors who refused to sign the Oath of Legions to the Mussolini and he was thus forced to resign his university position and go abroad. He was stripped of all his honours and privileges at the university, but he was later returned to Rome just before his death in 1940.

Alfred Lotka[6]on the other hand was not scientist in usual sense, and he was not associated to any university or scientific institution. He was a supervisor of a statistical office of a Metropolitan life insurance company of New York, so he did have a very strong grasp about population statistics.

Because the Lotka-Volterra model is not very realistic for use in real cases, further development of this equations occurred over the years. Expending the model by implementing the functional response method in two separate papers, C.S Holing made in 1959 anther step forward in creating more realistic representation of the predator prey model. In his case, it was implemented the concept of intake rate of the consumer (either prey or predator). This rate depended from the food density function[Hol59a]. The predator prey model was based on the moose and wolf population in the Isle Royale National Park in the state Michigan, US.

[5] The Figure 2.3 was taken from the following web addresse: *http://www3.varesenews.it/immaginiarticoli/200711/volterra.jpg*

[6] The Figure 2.4 was taken from the following web address: *https://wikisofia.cz/images/4/4c/Lotka.jpg*

This relation is one of the most relevant and most studied, since more than four dozens of different papers were published with this thematic.[Hol59b]

Next big step happened in the 80' when the alternative representation of the Lotka-Volterra model was presented. This system of differential equations is named Ariditi-Ginzburg equation. In contrast to the original model which only depends from the birth and death rates of both species, in this case one more constant has to be considered and that is the predator to prey ratio. This means that a number of preys hunted by a single predator will decrease if the number of predators is increased (more competition for the same resource).

The mathematical representation of this model will be shown in the next section.

2.5.2 Lotka-Volterra Mathematical Representation

After given short introduction to the basics of the predator prey model there are some assumption that had to be taken into consideration before proceeding with mathematical representation. It was already stated that the generalization of this model is relatively simple and that it is not very accurate. The constraints are these[Bai10]

- Population of prey specie can always find enough food (thus not dying from starvation). The same doesn't apply for predator specie
- The only source of the food for the predator specie is the prey population
- As it can be seen from the model, the variation rate of the population is in direct correlation with its size
- There are no outer changes that can affect system, for example the environment cannot change in that manner that one specie will profit more than the other
- All members of predator specie have unaffected appetite, this means they will eat constantly the same amount of food without any changes.

Lotka-Volterra model is primarily used to describe dynamical biological systems, but its use is far wider than this. Economics is another branch where it can be very useful to implement these equation, and here the use of such model has long history. Richard Goodwin is consider as a pioneer of implementing this model in the economics between 1965 and 1967. Economics and industry are also a vast dynamical systems just as biological between species. Many branches are interconnected, and the model that was introduced to connect various sector in form of a single dynamical system was based primarily on the trophic functions.

2.5.3 Lotka-Volterra Predator Prey Model

Lotka-Volterra equations[Hyd11, p.2]:

$$\frac{dV}{dt} = bV - aVP \tag{2.19}$$

$$\frac{dP}{dt} = caVP - dP \tag{2.20}$$

As it can be seen from the equations above , the system of differential equations used in the Lotka-Volterra model has two equations, first one is representing the prey (2.19) and second one the predator (2.20). There are several important parameters in this equation:

- **V** - Population density of the prey specie
- **P** - Population density of the predator specie
- **b** - Intrinsic rate of prey specie population increase
- **a** - Coefficient of predation rate
- **c** - Reproduction rate of predators per one eaten prey
- **d** - Mortality rate of the predators

Predator prey model based with Lokta-Volterra equations have some assumptions that have to be fulfilled in order to be implemented.

Prey specie V has a density independent exponential growth, if there is no presence of the predatory specie in the system, and the rate of the growth is denoted with ”b”.

Rate of predation (hunting) has to be constant. It is denoted with ”a” and it represent amount of prey population consumed by single predator. This value is not variable, and there is no slowing down for example if the population of prey decreases (or the ratio between predator and prey specie changes). There is also no interference between predators.

Mortality rate of predators, which is denoted with ”d” is also a constant value, as well as the reproduction rate of predators per eaten prey.

Predator prey model is the basic model of interaction between two species in one closed system in which one of the species is considered to be the predator which hunts other specie which is the prey.[Hyd11, p.3] This representation can be represented with the system of non-linear differential equations of first order.[Hyd11, p.4] Since the system is closed this means that the predator can only consume the prey specie, and if this specie goes extinct so will the predator.

There are off course many shortcomings for such system. The most obvious one is that in case of absence of predator, the prey specie would grow indefinitely. Such behaviour is rather unrealistic, and in order to make the system a bit more real the exponential term in the first equation should be replaced with the two term logistic growth expression

$$x' = (ax - rx^2) - axy \qquad y' = -cy + yxy \tag{2.21}$$

When implemented as shown on the equations above, if there is a vacancy for the predator specie, the first term is going to be transformed to the logistic equation. In such case, the population of the prey specie would stabilize at carrying capacity of the system.

Typical output of such relation between predator and prey is given on the picture below(Figure 3.4[7]). It is clear, that as long neither of agents goes extinct the growing rates in both populations will be of continues nature with different phases (if they are observed as pure mathematically represented signals). By increasing the number of predators, the population of prey specie will decline, which will result in decline of predator specie after some time (since there will be not enough prey to hunt), which will then result with increase of prey specie. This cycle will just repeat itself.

The second problem is that this system only applies for two agents of one system. In most situation this is not the case, since there is usually more agents. The system that can explain such type of competition between species is called *Lotka-Volterra competition model* and is often considered as a generalized representation of the Lotka-Volterra equations. This model is much more interesting for observation, because of its non-linearisation is much harder to predict, and even more since there can be N number of agents competing for the same resource.

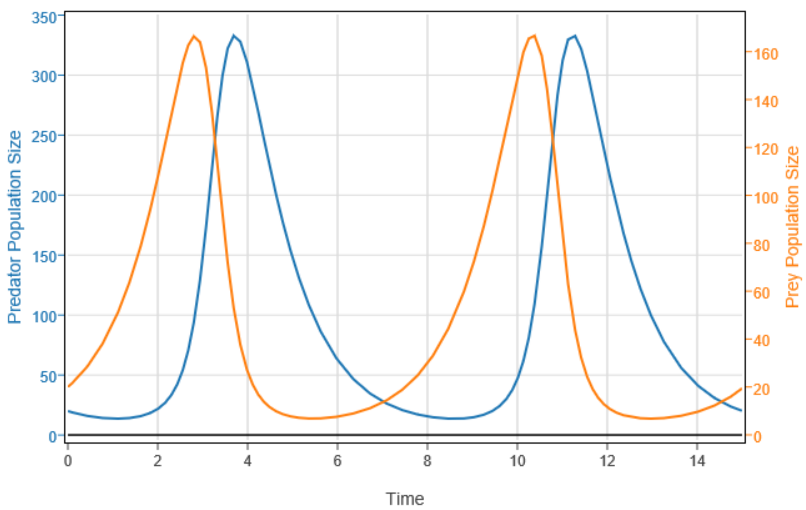

Figure 2.6: Lotka-Volterra predator prey model

[7] The Figure 3.4 was taken from the following web addresse: *https://plot.ly/ chris/1635/lotka_volterra_predator_prey_model/*

Stability:

From the preceding expressions for the equilibria it can be seen that the zero-growth isocline are straight and perfectly horizontal (for prey) or vertical (for predators). This reflects the complete absence of self-density dependence in both species' dynamics[Bai10, p.16].

* Picture of zero growth isocline for predator and prey *

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial V} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial V} & \frac{\partial f_2}{\partial P} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial V}(b-ap)V & \frac{\partial}{\partial P}(b-ap)V \\ \frac{\partial}{\partial V}(kaV-d)P & \frac{\partial}{\partial P}(kaV-d)P \end{bmatrix} = \begin{bmatrix} b-aP^* & -aV^* \\ caP & caV^*-d \end{bmatrix}$$

$$= \begin{bmatrix} b-a\frac{b}{a} & -a\frac{d}{ka} \\ ca\frac{b}{a} & ca\frac{d}{ka}-d \end{bmatrix} = \begin{bmatrix} 0 & -\frac{d}{c} \\ cb & 0 \end{bmatrix}$$

The trace of J therefore is exactly equal to 0, not meeting the stability condition that the trace is negative, so the equilibrium is not stable.[MF99, p.56]

In fact (as shown on the bottom of the page), the eigenvalues are purely imaginary: their real parts are exactly 0. This creates a kind of dynamics right on the cusp between stability and instability, called neutral stability: cycling with no trend either in towards the equilibrium or out away from it

Equilibrium (assuming V > 0 and P > 0):

$$\frac{dV}{dt} = 0 \Rightarrow (b-aP)V = 0 \Rightarrow b-aP = 0 \Rightarrow \boxed{P^* = \frac{b}{a}} \tag{2.22}$$

$$\frac{dP}{dt} = 0 \Rightarrow (caV-d)P = 0 \Rightarrow caV-d = 0 \Rightarrow \boxed{V^* = \frac{d}{ca}} \tag{2.23}$$

$$\det A = \begin{vmatrix} 0-\lambda & -\frac{d}{c} \\ cb & 0-\lambda \end{vmatrix} \Rightarrow (0-\lambda^2) - (-\frac{d}{c}cb) = \lambda^2 + db = 0$$

$$\lambda^2 - db = 0 \Rightarrow \lambda = \sqrt{-db} \Rightarrow \boxed{\lambda = \pm idb}$$

2.5.4 Lotka-Volterra Competition Model

Competition models are of significant importance in the different science fields. Strictly speaking in terms of biology it can be said that the competition model describes the changes in number of individuals from each specie, when more than one specie is competing for the same resource. This term is not only in correlation with biology, it can also be implemented in other natural and social science branches.

Lotka-Volterra competition model is one of the most used for such representation. After examining the logistic growth model which is primarily the intra-specific completion model of single species, by extending it, the competition model between two species can be built. If such representation is used, the competition can be divided in two groups

- ***Intra-specific*** - among individuals of the same species
- ***Inter-specific*** - between different species

If that is the case, it can be regarded that each species has its own population, denoted with N_1 and N_2. Furthermore each specie has its own growth rate r_1 and r_2 respectively as well as carrying capacity for individual specie (K_1 and K_2). The last parameter which is important is the competition coefficient between species. More about this coefficient attributes will be said later in this thesis. It represents the value of inter-specific interaction between two species. Worth nothing here is that this value in not the same for 2 species (α_{21} is not the same as α_{12}, they are not commutative in nature). This means that one species can be more competitive towards other and therefore will have greater competition coefficient. In order the have the competition this value is usually considered to be negative [Bai10, p.7]. If this value is zero, this means that there is no competition between species.

Beside sign, there is also the matter of inter-specific coefficient magnitude. This can play a huge role in the final outcome of the system. By default, it is assumed that the value of α_{ii} coefficient is 1(since it is the same agent type, it can not have some diminishing factor on itself).[Bai10, p.7]. The values of all IEC can be completely random.The IEC measure the per capita effect of one agent on the population growth of the other, measured relative to the effect of intra-specific competition. That is why when $\alpha_{ii} = 1$, then per capita intra-specific effects will be inter-specific effects. But if the values are not 1, then there are 2 possible outcomes:

- $\alpha_{ij} < 1$, then intra-specific effects are more deleterious to agent itself then inter-specific effects
- $\alpha_{ij} > 1$, then inter-specific effects are more deleterious then to the agent

There is also one more assumption which is important for the model, logistic assumption. It means that the growth rate of agent declines with the amount of competition[Bai10, p.11]. When all these setting are in place the equations for both agents can be written:

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N_1 - \alpha_{21} N_2}{K_1}) \tag{2.24}$$

$$\frac{dN_2}{dt} = r_2 N_2 (1 - \frac{N_2 - \alpha_{12} N_1}{K_2}) \tag{2.25}$$

$\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ represent the change of number of individuals per period of time. If the value is positive this means that the number of that particular specie is growing, otherwise it is declining. In special case when this value is zero, than this means that the specie is in equilibrium state (which will also be covered in more detail later). If the coefficients are zero (the species don't interact in any way), this model can be reduced to separate logistic growth.

Lotka-Volterra model is very general in its nature. By further inspecting of competition coefficients, it is noticeable that divergent types of relationships can be created if the coefficients have different signs.[Bai10, p.9]. On the table 2.5 the appropriate relationships depending from the sign can be seen:

		sign of interspecies coefficient	
		α_{12}	α_{21}
Type of relationship	Competition	-	-
	Predator prey	-	+
	Amensalism	-	0
	Commensalism	+	0
	Mutualism	+	+

Table 2.5: Table of interspecific coefficients

The strength of the interaction depends from the absolute value of the coefficient.

By taking into consideration just first possibility, that all given values of inter-specific coefficients are negative, four distinct outcomes for two specie competition can be constructed, depending strictly from carrying capacity and inter specific coefficients. For simplifying this statement it can also be assumed that the carrying capacities of the both species are the same. This assumption can help later on in testing part, since it is assumed that the agent will compete over same resource (and this value will be represented with the carrying capacity). By observing figure 2.7 below it can be clearly seen that if the coefficient values are negative and carrying capacity of both species is the same, than the specie which has smaller magnitude of the inter-specific competition will always win.

This hypotheses can be easily determine over equilibrium isocline. This lines represent the equilibrium points for the specie at every given time. If the population of the specie is under the equilibrium isocline, it will grow, if it above it will decline.

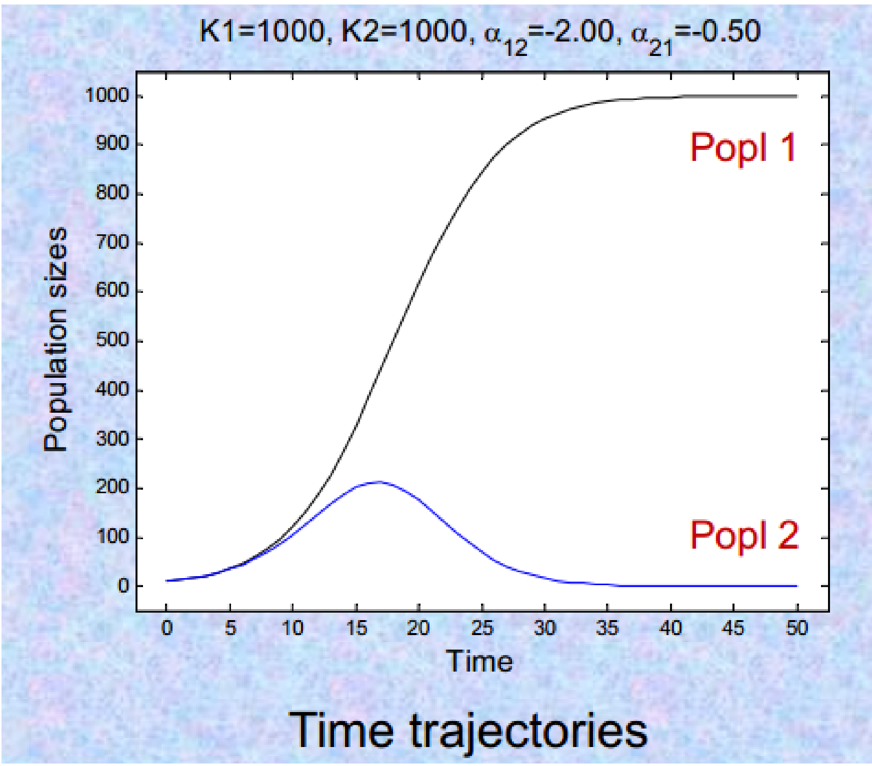

Figure 2.7: Lotka-Volterra model with two species competition plot

Take into consideration first specie. This first specie has some carrying capacity K_1 and inter-specific coefficient α_{12} with second specie. Second specie has another carrying capacity K_2 (and as it was already mentioned, its value is the same as K_1 carrying capacity), as well as inter-specific coefficient α_{21} (its coefficient with specie 1).

It is presumed that the alpha values are negative, because a competitions behaviour is wanted. In this case, four possible different outcomes can occur, and they are represented on the following pictures:

2.6 Summarized Notes

Classical game theory was developed during the mid-20th century primarily for application in economics and political science but in the 1970s a number of biologists started to recognize how similar the games being studied were to the interaction between animals within ecosystems. Game theory then quickly became a hot topic in biology as they started to find it relevant to all sorts of animal and microbial interactions.

Originally evolutionary game theory was simply the application of game theory to evolving populations in biology asking how cooperative systems could have evolved over time from various strategies the biological creatures might have adopted. However the development of evolutionary game theory has produced a theory which holds great promise as a general theory of games.

The general question of interest in evolutionary game theory is in how do patterns of cooperation evolve and what are the optimal strategies to use in a game that evolves over time. Just as equilibrium is the central idea within static non-cooperative games the central idea in dynamic games is that of evolutionary stable strategies, as those that will endure over time.

Representing and consecutively solving the dynamic system based on the evolutionary game theory can be executed via system of differential equations. In that regard Lotka Volterra competition model depicts a prime example of a such system. Primarily used in the population dynamics, this model can easily be adjusted for virtually any type of closed system competition. The two agent game was demonstrated in this chapter, but the most interesting appliance of this approach would be with extended number of agents.

Dynamic approach for the problem of multiple agents in the system will be covered and analysed in depth in the next chapter.

3 Methodology

Before it is possible to begin with the practical implementation of the competition model, some key aspects of the Lokta-Volterra competition model should be covered. In order to develop such model properly, there have to be laid some foundation of how this model can be presented analytically. That is why it can be helpful to know some basic mathematical representations that can be used to unburden the problems of solving system of non linear equations.

The scope of the methodology chapter is to make the round up of all important theoretical aspects of this thesis before proceeding with the practical implementation and experimentation. Main goal here is to help reader better understand how the competitive models function, what are the constraints and the assumption that are expected from such model. And by setting all requirements correctly, to help predict, to some extent, the outcomes of the competitions in such systems, even without the help of simulation frameworks.

This chapter is divided into three main subsections. The first section is covering some basics in order to get more familiar with the limitations of the current competition models. It will be shown why it is so important to change it before implementing it into simulation framework. Second and third sections are connected more tightly. The second will demonstrate to proposed solution that is going to be used in this thesis. It will display what changes had to be made, as well as what effects should be expected as final results. Finally, the third section is trying to make the mathematical representation of the proposed solution. The idea is to try to mathematically prove (or at least assume) the output result based on the input values in the system.

The intent of this methodology and its mathematical analysis is not to give the precise value of the end result for every single setup. Such prediction would be nearly impossible to expect. That is the main reason why there are computer simulations will be done latter in the first place. The primary objective here is just to make a relatively accurate approximation of the agent's intakes in the system. In the Chapter 5 the results from the simulation framework will be then compared with the ones from this chapter in order to see how accurate can those assumption really be.

3.1 Presented Problem

The problem that usually happens with Lotka-Volterra models was already shortly mentioned in the introductory chapter. Lotka-Volterra equations are given for two types of relationship, either as predator prey model or as competition model. The first model is narrowed to two species (agents) in the system, one being the prey and the other being the predator. Such model is mostly used for assumptions in the context of biology, and does not have vast potential outside that area, even though the predations (as type of relation) is not only relevant in biology.

That is why the second model of LV equations has more potential for this simulations. There is unfortunately some other difficulties that arise from such representation. By default, this model is also used for two species model, but in can be generalized (as it was already shown in previous chapter) and as such can derive to make an N-species model. The problem here is that by default, every specie (agent) has its own carrying capacity. This means, when that is translated to the equations and the needed calculations are made, the output value will be completely independent. It would look like that they are competing in the same realm, but over different resources, and when the sum of those resource in some random point is made, it would largely overshadow the maximum value of a single resource.

Lastly, there is also a problem of the connections between agents itself. This means that the coefficients will always remain the same, no matter what happens it the system. This is because the agent in the system of LV equations are playing pure strategy, and not the mixed one. The system would make a bit more sense if the coefficients can somehow vary trough out the execution of the simulation. This would however imply that the Lotka-Volterra model is implemented differently, thus using entirely different population model.

The most eminent model that further derives the basic LVCM is the Lesli-Gower predator prey model [AAM03], which uses the differential equations of second grade instead of first grade, as with the LVPPM. Since the idea is to be in the realm of relatively simple implementation, this model can be neglected and eventually left for some further work. By making the system more generalized large amount of the input parameters can be computed really quickly, even if the results are not the most accurate one. Such output can done be ”fine-tuned” with some other (more precise) competition model in order to estimate the deviations from the original output.

This would also mean that there is a need to find the balance between precision and speed. It is obvious that the more generalized systems are less prone to errors, but they can compute the results multiple times faster. Since the goal of this thesis was to find the coherences that exist between agents in such systems, simpler representation of the inter specific coefficients would also make this possible, even if the precision of the LVCM is not that good as some other models.

3.2 Envisaged Solution

It is now pretty evident that there are two parts of the equation that need to be coupled in order to solve the problem. On one side, there are the agents that are competing over user's attention in some closed system. On the other, there is a proposed tool that should be by some mean adapt in order to be implemented into the competition model.

As it was said in the last section, major obstacle of the LVCM is the representation of the joint resource over which the agents in the system should compete. Agents are still interacting with each other, but they only have influence on the resource that is strictly connected to one agent. This means that the agents-resource ratio of such LVCM is N:N. For the given system, on the other hand, ratio 1:N is needed.

This is why some adjustments to existent LVCM had to be made. By setting the value of carrying capacity the same for all agents, it is effectively "forced" onto them to compete over the same resource. In this model, as it will be shown later, only relative values in the system will be calculated.

One simple example can give better insight over the proposed solution. It is already clear that the main objective of this thesis is construction of competition model in which the participants are competing over user attention. How can they compete over user attention? Most people are using the search engines and meta-sites in order to find some specific piece of information. There are more than enough different sources where those information can be found, to name only the few: Google, Bing, Yahoo, Wikipedia or Facebook.

Let's say that some random person has to write a master thesis. This person wants to find some useful information sources, and can spend three hours per day doing that "data mining". Majority of people would probably start by using more of some search engines (the ones that they are accustomed to), less of some other. If the data is proven to be more relevant on some engine, user will start to use it more, neglecting others. After a couple of weeks, user is already familiar with matter and sources, which means that he will most certainly only going to use two or three sources, and the rest is not going to get any more of his attention, which will result that they go "extinct" (which means these search engines will not be more used from user for particular research).

It is very important to note that this setting is by no means the only one in which following system can be implemented. The reason why this particular depiction was used, is to make the reader more comfortable with understanding the idea behind such model. As it was already mentioned in the introduction, this model can be used for determining intakes of political parties in the election, where the resource is the voters. The practical appliance of such model is limitless. As long as there is one single resource over which different agents are competing for, it can be adjusted for the specific use case and later implemented. That is why the user attention model mentioned here is only given more as a particular use case that should be examined, rather than a more generalized depiction.

Lastly, as it was said in the precious section, the speed of the implemented framework was also important for the final conclusion. That was one of the most important design decision, and why the LVCM was taken in the first place (as well as Python and its frameworks for the practical part of this thesis). Some of the experiments that were conducted latter in the next chapter had big amount of data to compute. This was made in order to be certain that the output results are not going to be changed over the time (when no equilibrium in the system was achieved). By iterating some experiments more then million times, implemented equations can make a huge time difference, no matter how fast the computer used for those calculations is. And the repetition of every single experiment multiple times for different set of input parameters can just add more to that contrast.

Following sections will try to explain the competition model from the mathematical standpoint. First, it will be covered a basic competition model with only two agents competing over the same resource in the closed system. It will be then continued with the expanding of the system to a random N number of users which is greater than two. To goal of that part will be to make the solution as much as scalable and generalized as possible.

3.2.1 Basic Two Agent Model Calculation Depiction

In the previous chapter the default representation of the LVCM equations for the system of the agents have be given in the following equations 2.24 and 2.25). The symbols representing the following

- K - Carrying capacity of the system
- N_i - Value of the agent i influence
- $\frac{dN_i}{dt}$ - Change of the influence of the agent per time slice
- r_i - Growth rate of the agent i
- α, β - inter-specific coefficients

Now the stationary points of the system can be calculated, in which the equilibrium can be achieved.

$$0 = r_1 N_1 (\frac{K - N_1 - \alpha N_2}{K}) \Rightarrow K - N_1 - \alpha N_2 = 0 \Rightarrow \boxed{N_1 = K - \alpha N_2} \tag{3.1}$$

$$0 = r_2 N_2 (\frac{K - N_2 - \beta N_1}{K}) \Rightarrow K - N_2 - \beta N_1 = 0 \Rightarrow \boxed{N_2 = K - \beta N_1} \tag{3.2}$$

From here you can get a steady states that occur on following points

$$N_1 = K - \alpha(K - \beta N_1) \Rightarrow \boxed{N_1 = \frac{1-\alpha}{1-\alpha\beta} K} \tag{3.3}$$

$$N_2 = K - \beta(K - \alpha N_2) \Rightarrow \boxed{N_2 = \frac{1-\beta}{1-\alpha\beta} K} \tag{3.4}$$

This depiction is giving four stable states for the two agent competition model, giving the equilibrium points at following value pairs : (0,0) , (1,0) , (0,1) and (N_1, N_2)

Stationary points of the system:

$\frac{1-\alpha}{1-\alpha\beta} < 1$ since N_1 can not be larger than K. By dismantling this equation we are getting that: $\alpha > \alpha\beta \Rightarrow \boxed{\beta < 1}$

Same derivation will be done to the second equation (in this case N_2 has to be lower than K) and the output result will be $\boxed{\alpha < 1}$

From here is obvious how the final distribution will look like

Case 1: $\alpha, \beta > 1 \Rightarrow \alpha > \beta \Rightarrow N_1 > N_2$ Case 2: $\alpha, \beta < 1 \Rightarrow \alpha > \beta \Rightarrow N_1 < N_2$

This was expected, since graphical solution showed the same end result. Only way to have a proper competition is to have both of the IEC values lower than their respective IACs. In any other case, when agent will eliminate the other, and there equilibrium in a form of coexistence will not be possible.

3.2.2 Theoretical Analysis

By considering two separate equations for two hypothetical agents their logistic growths can be represented. The depiction of the agents isocline is given in the next section. First agent is represented in blue and the second in red color. Variable notations have not changed since before, thus K still represents the logistic growth, N_1 resource allocation of the first agent, N_2 resource allocation of the second agent, and r_1 and r_2 their growth rates, respectively. In order to account the effects of competition the competition coefficient factor α is added to the equations, thus making them interconnected. Looking the things from the general perspective, in the first equation coefficient alpha represent the effect of single member of N_2 group of agents to a single agent from the N_1 group of agents. Because all of the agents from the N_2 group will effect on the N_1 resources, this coefficient is going to be multiplied by the number of group N_2 members. In the particular case of this thesis, this can be depicted as an influence of N_1 agent on the user attention allocated by the N_2 agent. The depiction N_1 will than represent the user attention allocated from the first agent, instead of the number of agents(as in the general representation). The exact same thing is done happening to the resources of the second agent. In this case there is an completion coefficient beta, and it is applied as before, just that it now represents the effect an individual agent N_1 can have on the user attention of the second agent.

This equations can provide the user with very powerful tool to determine the outcome of the competition between the agents. If the change of every agent is already known, as well as the competitive effect that they have at each other, then it is possible to predict if the agents are going to be able to split the resources and coexist, or one agent will "destroy" the other (effectively taking the entire attention of the user for himself).

First derivative of *Specie 1*, $\frac{dN_1}{dt}$, represent the change of size in each instance of time. If this is a positive value, then the allocated user attention will increase, if it is not, then it will decrease. In case that this value is zero, there is a state of equilibrium, attention stays constant and doesn't change. From the equations it can be seen that it is a simple product of three values. In order to get the equilibrium of the system one of this values (or more) have to be zero. Since it is clear that if the attention rate or its growth rate are zero, will simply state the obvious, the product will be zero, and the attention will not grow (or decline). There is only one more possibility to create the equilibrium state. What is inside the brackets can also reproduce the equilibrium. That cannot be applied to carrying capacity, since it is obvious that this value has to positive (from mathematical and logical standpoint). Thus remains the numerator, which has to be zero in order to have the zero growth. By transforming the equations both species can have two zero conditions, thus creating t he isocline starting and end points of the zero population growth, which can be then drawn onto diagram in order to represent the equilibrium.

This also makes logical conclusion. If the system has two agents, in order to have stable distribution of resources, one agent can allocate entire carrying capacity, which will lead to the extinction of the second agent (and thus reproducing the logistic growth function from before), or there will be some stable state in which case the first agent will have less acquired resources in comparison when it is alone in the system, and that difference will be acquired by the second agent. The system can't sustain the resource allocation of one agent which is as big as carrying capacity of that system due to the presence interspecies competitor which will always prevent this from happing (until at least intakes of one agent goes extinct).

3.2.3 Mathematical Representation of the Proposed Solution with Two Agents in Competition

It is relatively straight forward chain of events, when only two agents are beetling among themselves. When they are represented with the two differential equations, it is easy to determine the probable outcome of the interaction. Not every interaction between two agents is considered a competition, even if the requirements are fulfilled. There are four possible outcomes. Beside competition, where it is not immediately clear who is going to win (it depends from starting parameters of the equations), there is also extinction of one of two agents, as well as coexistence. In that particular case, both agents can claim some amount of resources, which is not as big as it would be without competition (they are settling for less) but still better solution than to lose all of the resources. In case of two agents, by simple observing the equations, it can be deduced which outcome is going to happen. There is standard way of determining isocline. Following section can graphically explain possible dealing of such situations.

- One of the magnitudes larger and other is smaller than 1

  Having magnitude of the inter-specific coefficient that is higher than one means that you the agent is affecting more on itself than on its opponent. If the other magnitude is smaller than one, there is no actual competition, since the agent with smaller magnitude will always win. This is very clear from two figures below. Such state is called competitive exclusion. [Har60] (of *Agent 2* by *Agent 1*). There can never be an equilibrium point for both agents. The Figure 3.1 shows direction in which the change of the user attention will split. One agent will dominate the other and take all of the attention for himself. Second figure (given on the Figure 3.2) shows the same situation when the values are opposite. Now the second agent will dominate over the first one and the entire carrying capacity of the system will be allocated by N_2 agent.

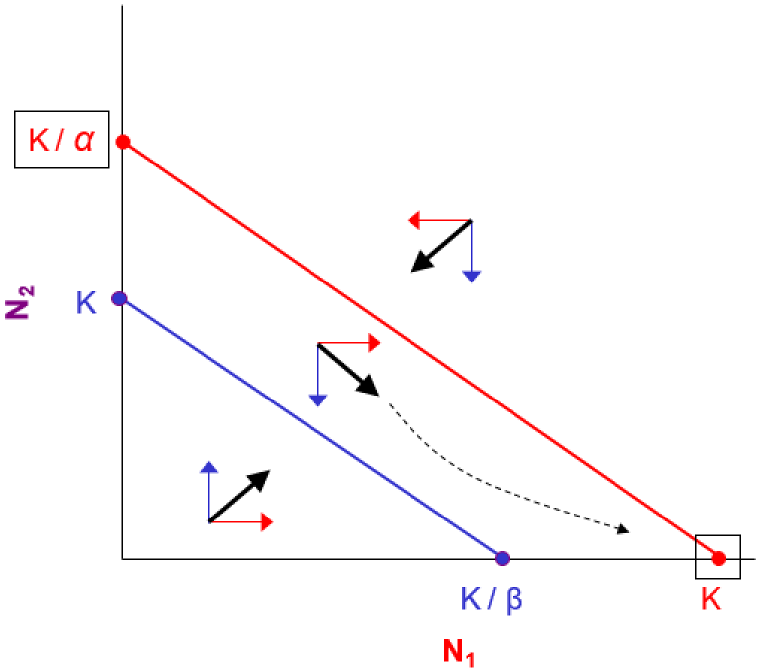

Figure 3.1: Magnitude of $\alpha_{21} > 1$ and $\alpha_{12} < 1$

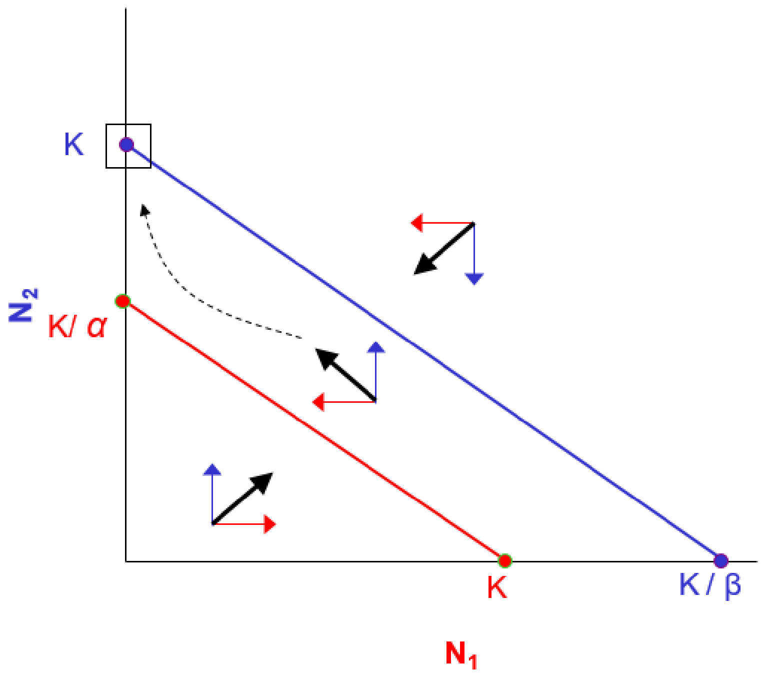

Figure 3.2: Magnitude of $\alpha_{21} > 1$ and $\alpha_{12} < 1$

- Both IEC magnitudes are higher than 1

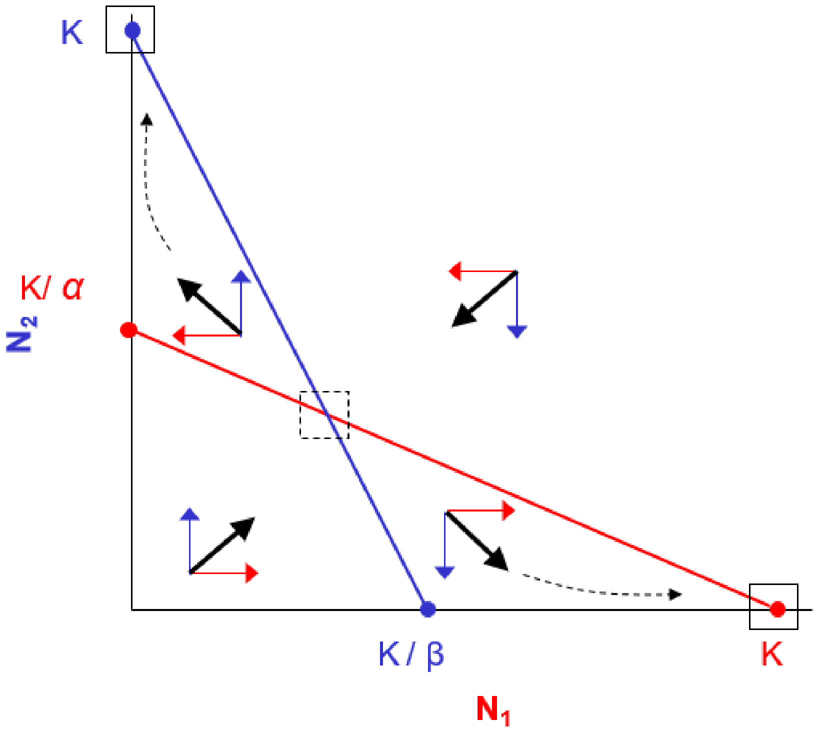

Figure 3.3: Magnitude of $\alpha_{21} > 1$ and $\alpha_{12} > 1$

On the Figure 3.3 above, the special case when both magnitudes of inter-specific coefficients are greater than one can be seen. Such state makes the agents influencing themselves more than the other one. This means that the agent that is influence .

Vector help better understanding what happens if the gathered attention is not on the equilibrium isocline. If below, it will increase for both agents, and when above it will decrease. If it is the upper left corner, above the isocline of the *Agent 1* and under the one of the *Agent 2*, the direction of the vector will go toward the carrying capacity which we completely belong in the end to the *Agent 2*, while the other agents acquired resources will be non-existent. Same will happen if their values are in the lower right corner, in which case the *Agent 1* will have all of the users' attention. Both isocline do cross, which means that there is a fourth equilibrium point beside (0,0), (K,0) and (0,K). But this point is non-stable one, since the system will to go toward other points as soon as the values are changed. This can be compared with the ball on the top of the hill. It will be in the steady state, but as soon as the ball is pushed to the left or right, it will go done the hill and it will never be able to return to that position on itself.

Same will happen with two agents. The end result will depend from two factors, starting position of both values and their IEC. If both values of the agents are known, it will be easy to predict the final outcome of the competition.

- Both IEC magnitudes are smaller then 1

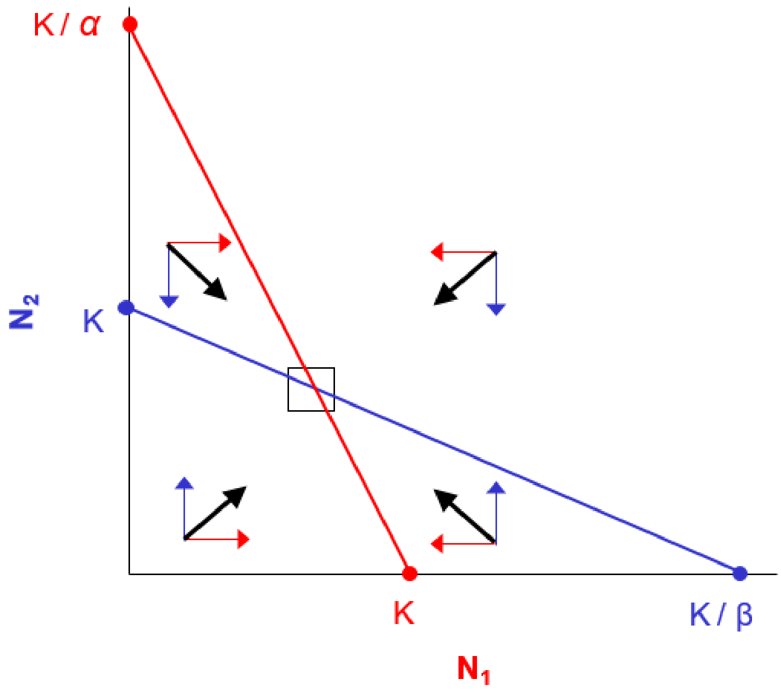

Figure 3.4: Magnitude of $\alpha_{21} < 1$ and $\alpha_{12} < 1$

The last possible outcome is the only one with the real competition. Both of the IEC magnitudes are smaller than IAC, which means that the agents will inflicting more "damage" to the opponent than to itself (and not self-destruct as in some previous depictions), as it can be deduced on the Figure 3.4 above.

The provided flow vectors are given again to make the understanding of the resource distribution clearer. In this case there is a stable, attracting equilibrium in the intersection points, and the agents will lean toward that value. It also should be noted, that even though it is represented as theoretical possible, the values of the agents will always be just under the intersection, since they cannot go over the carrying capacity of the system. Such system will also have same three equilibrium points as the previous one. But the fourth EP will be the opposite of the last one, since the agents will both try to achieve it. It can be called the "compromised point", since neither of the agents can fulfil its total potential, but both of the agents can coexist in the system.

Such system is the only solution for the coexistence between the agents. They will presumably not gather enough user attention as they would like, but it is still far better outcome than not having any attention at all. As long as their IEC smaller than the IAC, they will survive. In this case, only parameter needed to determine the final outcome is the IEC. The agent that has smaller IEC will have larger share in the user attention, independent of the starting position.

3.3 Further Expansion for the N Species Lotka-Volterra Model

In comparison with a two species Lotka-Volterra competition model, by making it generalized for N agents in the system it will make the things a little bit more complicated. First of all, it has to be assumed that such system has an interaction matrix in which all given values have positive sign (intra and inter specific coefficients have to be: $\alpha_{ij} > 0$). Only then when this requirement is checked, it can be assured that such system is competitive in nature.

Furthermore, if the competition among different agents is neglected, their shares in the system will rise, until the carrying capacity of the system is reached. In order to simulate this type of behaviour it has to be assumed that the growth rates of all agents are positive as well, which translates to : $r_i > 0$. This also means that every agent's share (of the user attention) will be in range between 0 and 1 at all times ($0 < x_i < 1$). This basically means that all share values will always be positive, and in range between 0 and 100% of the entire system capacity.

It is assumed that such system has more than 2 agents, but it is safe to say that the N should be larger than 5. If this is the case Smale[Sma76] already proved that when all of the mentioned conditions are fulfilled system can go into any of the asymptotic behaviours (such state can include limit cycles, n-torus, attracting equilibrium or fix point). Simply said, in such circumstances, the output of the system will always try to achieve equilibrium state.

Another important finding was made by Hirsch[Hir85][Hir88] who showed that attractor dynamics can happen on a set of N-1 dimensions. This means that the dimensions of the attractor itself have to be smaller than the N-1 limit. And this is crucial because such finite cycle can be present only if there are more than one dimension. Chaos needs at least 3 dimensions to occur and n-torus at least n dimensions. Having said that, Hirsch showcased that a Lotka-Volterra competitive system have to have at least 3 dimensions to have a finite cycle output, or one more to be chaotic other torus. This matches with the statement from Smale[Sma76] that in order to have dynamics it has to be $N \geq 5$ dimensions.

Hirsch also proved[Hir90] that there is and (N-1) dimensional simplex that is a irrelevant set C

$$\Delta_{N-1} = \left\{ x_i : x_i \geq 0, \sum x_i = 1 \right\} \tag{3.5}$$

and is a global attractor of every point excluding the origin. This carrying simplex contains all of the asymptotic dynamics of the system.

As it was already stated, all intra and inter specific values of the system have to positive in order for him to be stable. This can be considered as a Jacobean matrix with eigenvalues. Konodoh[Kon03] showed that if the system is large in scale in will be either unstable or with depressed connectivity among agents, ant that if the system wants to be stable it has to evolve (eigenvalues of the system have to change) in accordance with the natural selection.

3.3.1 Generalized Representation for the System with N Competitors

Having a two agent model is good foundation for the extension of the system. In most cases, such non-linear dynamics will have more than two agents that are competing for the same resource. The problem that arises with such model, is its complexity. As it will be seen in the following section, even the three agent system has some of the limitations which are making harder to predict the outcome of the competition. Having even more agents, just makes the thing exponentially more complicated.

As a good stating point for the further extension of the Lotka Volterra model, additional agent should be incorporated into the system, having than three in total. If it is possible to make some conclusions from such system, further increase of the agents in the system should make some predictions more clearer.

The generalized Lotka Volterra equation for N agents is the following:

$$\frac{dN_i}{dt} = r_i N_i - (\sum_{j=1}^{n} a_{ij} N_j) N_i \quad i, j = 1...n \tag{3.6}$$

From here, the system with three agents would look the like:

$$\begin{aligned}
\frac{dN_1}{dt} &= r_1 N_1 - a_{11} {N_1}^2 - a_{12} N_1 N_2 - a_{13} N_1 N_3 \\
\frac{dN_2}{dt} &= r_2 N_2 - a_{21} N_1 N_2 - a_{22} {N_2}^2 - a_{23} N_2 N_3 \\
\frac{dN_3}{dt} &= r_3 N_3 - a_{31} N_1 N_3 - a_{32} N_2 N_3 - a_{33} {N_3}^2
\end{aligned} \tag{3.7}$$

This system of equation can also be represented with dimensionless parameters in order to simplify the numerical analysis of such system.

$$\begin{aligned}
\dot{x_1} &= x_1(1 - x_1 - \alpha x_2 - \beta x_3) \\
\dot{x_2} &= x_2(1 - \beta x_1 - x_2 - \alpha x_3) \\
\dot{x_3} &= x_3(1 - \alpha x_1 - \beta x_2 - x_3)
\end{aligned} \tag{3.8}$$

The parameters α and β are in this case both have positive magnitudes in order to make the system competitive (but as it will be shown in the experimental part, this does not have to be the case). By looking in the previous chapter, same conclusion can be made here. Such system has eight equilibriums, where some of them are stable, and some of them are unstable (saddle points). Aside from trivial equilibrium in the point (0,0,0), there are also three equilibriums where only one agent takes dominance over the entire system. These are the following equilibrium points: (1,0,0), (0,1,0) and (0,0,1).

There are also three equilibrium points for the two agent coexistence system. These are the following points:

$$\frac{1}{1-\alpha\beta}\begin{pmatrix}1-\alpha\\1-\beta\\0\end{pmatrix}\quad\frac{1}{1-\alpha\beta}\begin{pmatrix}0\\1-\alpha\\1-\beta\end{pmatrix}\quad\frac{1}{1-\alpha\beta}\begin{pmatrix}1-\beta\\0\\1-\alpha\end{pmatrix}\tag{3.9}$$

And finally there is also an equilibrium point in which there coexistence between all agents of the system.

$$\frac{1}{1+\alpha+\beta}\begin{pmatrix}1\\1\\1\end{pmatrix}\tag{3.10}$$

Dynamics of the system, as well as the stability itself are in the case depends from the magnitudes of both α and β parameters. It is obvious that the most engaging equilibrium in this case is the one from previous equation, with all three agents coexisting with their influences. From here can bee seen that the Jacobian matrix $\frac{\delta f_i}{\delta x_j}$ is given by following:

$$-(\frac{\delta f_i}{\delta x_j}) = \begin{pmatrix}1 & \alpha & \beta\\ \beta & 1 & \alpha\\ \alpha & \beta & 1\end{pmatrix}\tag{3.11}$$

From the matrix J three eigenvalues can be detriment:

$$\lambda_0 = 1+\alpha+\beta\tag{3.12}$$

$$\lambda_{\pm} = 1-\frac{1}{2}(\alpha+\beta)\pm\frac{i}{2}\sqrt{3}(\alpha+\beta)\tag{3.13}$$

From these two equations it is clear that the stable equilibrium is in case $\alpha+\beta<2$. For $\alpha+\beta>2$ and both $\alpha,\beta\geq 1$ the stable equilibrium would be the one with only one agent surviving. Which agent would survive in this case depends entirely from the starting parameters of the system. Depending from such conditions the stability is going to gravitate toward one of these points.

With such parameter setup, it is then also possible to represent a dependency of the three agent system in two dimensional plane. Figure 3.5 showcase how the relation between inter-specific parameters governs the stability of the system. It can be seen that only the (I) area will present a stable equilibrium in which it is possible for more than one agent to coexist. In the area (II) there can be only one winner of the competition, while in the area (III) no parameter setup can provide a stable equilibrium

There are some interesting edge cases that can appear in such system. If there is an assumption that $0<\alpha<1<\beta$ where $\alpha+\beta\geq 2$ Figure 3.5 can showcase such dynamics in more detail. What can be seen from such special parameter setup, is that the eigenvalues of the equilibrium

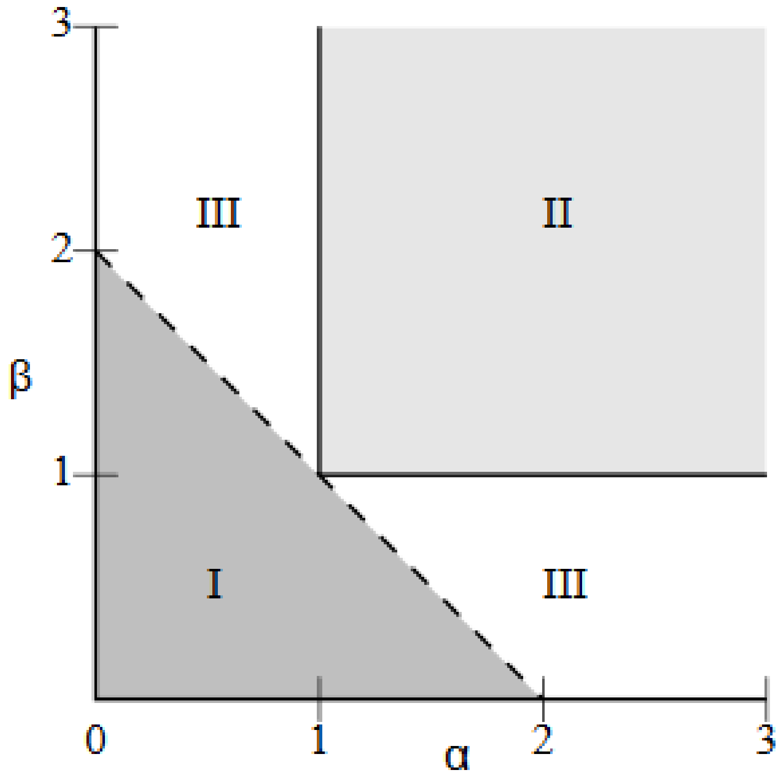

Figure 3.5: Three different system dynamic possibilities (I): $\alpha+\beta<1$ three agent coexistence, system is in stable equilibrium (II): $\alpha+\beta>2$ and $\alpha,\beta\geq 1$, single agent equilibriums are stable; (III): $\alpha+\beta\geq 2$, and α or $\beta\leq 1$ there is no possible stable equilibrium point, only a periodic orbit

have only imaginary value(both of them).In other words, equilibrium does not have hyperbolic motion. But even without such feature, periodic orbit still must remain.

This can be proven in the following manner. In order to have a asymptotic solution, there has to be intersection of two surfaces which are described by the agents parameter. The first surface can be shaped with the sum of all agent values into value N, as provided by following formula:

$$N(t) \equiv x_1(t) + x_2(t) + x_3(t) \tag{3.14}$$

Sum of the equation 3.8 creates the following equation for N

$$\dot{N} = N(1 - N) \tag{3.15}$$

Generalized solution of this equation can be represented as following:

$$N(t) = \frac{N(0)}{N(0) + (1 - N(0))e^{-t}} \tag{3.16}$$

Asymptotic limit of this sum is then:

$$\lim_{t \to \infty} N(t) = 1 \tag{3.17}$$

and from equations 3.8 solution of the first surface rests in the plane $N = x_1 + x_2 + x_3 = 1$

For the second surface , the product of same three agent values can be used:

$$P(t) \equiv x_1(t)x_2(t)x_3(t) \tag{3.18}$$

In order to convert product to sum, logarithmic function has to be used, as following:

$$\begin{aligned} \frac{d \log x_1}{dt} &= 1 - x_1 - \alpha x_2 - \beta x_3 \\ \frac{d \log x_2}{dt} &= 1 - \beta x_1 - x_2 - \alpha x_3 \\ \frac{d \log x_3}{dt} &= 1 - \alpha x_1 - \beta x_2 - x_3 \end{aligned} \tag{3.19}$$

Sum of these three equation, alongside with know condition $\alpha + \beta = 2$ gives then the following:

$$\frac{d}{dt} \log(x_1 x_2 x_3) = 3 - 3N \tag{3.20}$$

From both 3.15 and 3.20 comes then:

$$\frac{d \log P(t)}{dt} = 3\frac{d \log N(t)}{dt} \rightarrow \frac{P(t)}{P(0)} = (\frac{N(t)}{N(0)})^3 \tag{3.21}$$

Asymptotic limit can be then found:

$$\lim_{t \to \infty} P(t) = C \equiv \frac{P(0)}{(N(0))^3} \tag{3.22}$$

Solution of 3.8 then lies on the following hyperboloid $P = x_1 x_2 x_3 = C$ in R^3. Constant C depends from the starting conditions of the dynamic system.

Combination of outcomes for N and P (when precondition is Int(R^3_+) gives the following conclusion. All possible solutions gravitate toward intersection of plane N = 1 and hyperboloid P = C. This hyperboloid is mere periodic orbit (which again lies also in the plane $N = 1$). The representation can be seen on the Figure 3.6

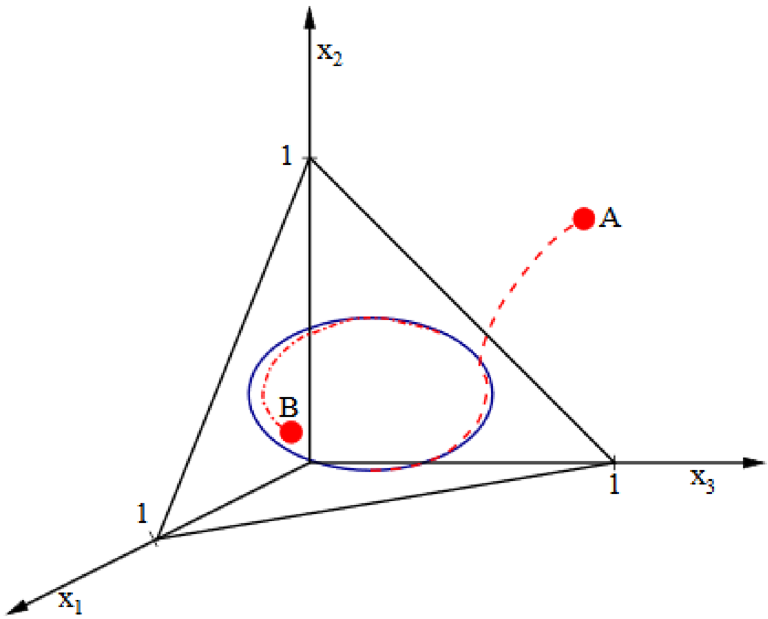

Figure 3.6: Graphical representation for the system 3.8 for $\alpha + \beta = 2$

From the Figure 3.6 it can be also seen that all orbits of the non trivial solution are lying in the plane $x_1 + x_2 + x_3 = 1$

It is very oblivious that even for only three agents things are getting quite a bit complicated, and the predictions are getting even harder. For more than three agents is practically not possible to represent the solution in any graphical way, only analyticity. There is also existing a connection between the number of agents in the system, and the possible equilibrium points (both stable and unstable). This value is given in form 2^n, where the n depicts the number of agents. Also, if the matrix equations are considered, if there exist more than 1 solution, then it can exist also the infinite number of non-unique solutions. Accordingly, finding the stability of such equilibrium is even harder, even if all possible simplifications are implemented.

That is why the solution of the non linear dynamic system with more then three agents should be restricted to the computational simulation. It will increase the solving of such system immensely. The mathematical depiction that was used in this section was mere a proof of concept, that such outcome can be found be using strictly analytical tools. But when the things are getting more complex, it is simple better to just call the cavalry.

4 Experimental Setup

This chapter going to cover two important things, and they will be presented in two separate sections.

- Simulation framework
- Experimental and results

First section is dedicated to the used Python framework that was applied to model different examples of competition models. This section will briefly cover used libraries as well as structure of the input files for the framework. In order to give a better understanding of the implemented program, an UML diagram of the used framework will also be provided.

Second section is significantly extensive. It has to cover all executed experiments, their real life depictions as well as mathematical counterparts. Every covered experiment will be represented in the same manner. First the potential real life situation where such model can be implemented. After it, the mathematical representation of the input parameters for the given experiment. And at the end, the output plot that represents the final (potential) result of the competition. If there are some conclusions that can be found in the setup, they will also be presented in that part.

In it also worth noting that usually more experiment with different input parameters per type of depiction will be executed. This is needed in order to make the results more universally valid. Since the system is logarithmic in nature, the adjustments of the input parameters (usually) have to cover more different basis (10^1, 10^2 , ...) in order the prove the correctness of the presented hypothesis. The simplest solution for this problem is the just iterate basis of the input coefficients in the same experiment setup and compare the output results. If the output are staying the same, or with small deviations, it is safe to say that they are universally valid.

4.1 Simulation Framework

In the introductory chapter of this thesis, it was already been stated that the Python programming language will be used as a tool to model the competition models. There are some key advantages of Python in comparison with some other object-oriented languages. To name just the few of them:

- Easy to use for beginners
- Light to read and understand for other people

- Rich open source libraries, no need for derivation of function for special purpose
- Very fast, optimized for matrix multiplication and mathematical calculation in general
- In few lines of code can be written much more than with some other languages

Since the programming requirements of this thesis were just to make the necessary calculations, and to present them in a form of plots, without some specialized GUI for the framework, Python was an obvious choice.

4.1.1 Input File

All requested parameters of the system that are needed for the experiments are going to be provided in the form of .csv file. Because of that, it is worth noting to shortly explain the .csv file used in the calculations and its key features, as an input file for the Python program.

CSV acronym means comma separated file, and such types of files are almost always used to store the tabular data (that can be either typical text or numbers) in a form of plain text[1]. Every value of such file is separated with comma sign. Every row represents a single array of values. This means that .csv file structure can be considered as a matrix. Many different software can use such type of files, from Microsoft Excel, Open Office, Calc and Apple Numbers to the regular text manipulating software such as Notepad++. The zip file which is provided along with diploma thesis has a specific folder with the .csv files that are used to represent different behaviours of competitive model in evolutionary game theory. But its structure is the same for all files, the only difference is the number of columns and rows, and the values of same fields in order to simulate different types of outputs. From the Figure 4.1 below, it can be seen how the typical structure of the .csv file used in the conducted experiments looks like.

```
#name    ,carring_capacity ,growth_rate ,starting_value ,x1   ,x2   ,x3   ,x4   ,x5   ,x6
Player_1 ,100              ,0.005       ,20             ,1.00 ,0.02 ,0.22 ,0.03 ,0.03 ,0.04
Player_2 ,100              ,0.004       ,10             ,0.02 ,1.00 ,0.23 ,0.03 ,0.03 ,0.03
Player_3 ,100              ,0.004       ,10             ,0.02 ,0.03 ,1.00 ,0.04 ,1.04 ,0.04
Player_4 ,100              ,0.003       ,25             ,0.02 ,0.23 ,1.04 ,1.00 ,0.05 ,0.05
Player_5 ,100              ,0.003       ,20             ,0.02 ,0.03 ,0.04 ,0.05 ,1.00 ,0.06
Player_6 ,100              ,0.005       ,5              ,0.02 ,0.03 ,0.04 ,0.05 ,0.06 ,1.00
```

Figure 4.1: Typical csv file structure

If one of those .csv file is opened, it can be seen that the first row has a # sign before text. This means that first row is just a comment and will not be processed in the software. The first row is given just as title, in order to help user understand what does exactly which value means. Without it, it would be very hard to guess what is what, since only first value in every row is the text (name of the agent) and the rest are just plain float or integer values.

Upon inspecting the file, it can be spotted that after the name of the agent, the second value (first numerical) is the carrying capacity of that agent, which means the maximum number of

[1] This statement was taken from the following web page: *http://www.computerhope.com/jargon/c/csv.htm*

that particular ”specie” units. It was already explained in previous chapter, that all agents have to have the same carrying capacity. That is the main premise if joint resources is wanted for which all agents are competing for. If smaller number are provided here to one or more agents, that would effectively ”crippled” them, since then it wouldn’t be the fair competition(some agents would achieve greater gains even though others have more optimized parameters, but the value of carrying capacity is not allowing them to fulfil 100% of their potential).

After carrying capacity, next parameter is the growth rate. The parameter is not constants for all species, and it can vary in different values. The only rule that it has to abide is that it has to be a positive value. If that should not be the case, the growth of the agent’s attention would be negative, and no matter what happens, that agent would always ”die” (loses all gathered attention) at the end.

Next value is named starting value, and it represents, just as the name says, the starting value of the agent. Since the mathematical representation of Lotka-Volterra model is based on iterative calculation (in order to calculate next iteration, the last one has to be known), this value can be considered as a x[0] value of the array. Only when the starting value is given, it is possible to move along with the calculations. There is one pre-requirement for this variable, and that is that it has to be positive value. In case of growth rate, where that value should be positive, here is a must. And this is because the values represented on graph are relative (depicted in the form of percentages that they acquired in the competition with a other agents. Therefore, they cannot be non-positive.

After starting value there are a number of values that are only labelled as x_1, x_2, and so on. This values represent the influences which specific agent has on the other agents of the system. The number of these parameters depends from the number of agents, thus if there are six agents like on the Figure 4.1, there will be six ”x” parameters per row. Standard pre requirement for such system is that a self-interacting ”x” parameter is set to 1 (it can be seen that in the first player row, parameter x_1 is 1, the x_2 parameter of second player is 1, and so on). This is usually done in this manner in order to simplify the system. As for the values that can be assigned, it is pretty much allowed complete randomness. That means, that the values can be positive and negative, greater than 1 or smaller than -1, and almost all possibilities per agent are allowed. By adjusting different values, it is possible to achieve different states of the system, which is also going to be shown in the next chapter.

These are all important parameters of the typical .csv file used in the further calculations. As already denoted, the basic structure of them all is the same. The only difference is the number of players in the given system and their values.

4.1.2 Used Libraries

In this section, used libraries for the practical part of the thesis will be explained. There are 3 main libraries that are used in the project:

- csv
- NumPy
- myplotlib

csv module

Entire Python code was structured using these three core libraries. Python's csv module is used for the manipulation of the csv file (read, write options). In the code it was primarily used to parse the input .csv file into the program and serialize it for further calculations. The only thing that the user should consider when using this module is that the format of the .csv file should be either UTF-8 or ASCII code to be safe. The Unicode format should be avoided since it is not supported.

NumPy module

Python's NumPy module is the part of much bigger SciPy package which is used for almost all mathematical and scientific work in general that is performed with Python. NumPy module is one of its fundamental packages. Its main purpose is efficient array-processing of the large multi-dimensional arrays and matrices. By doing so the NumPy also does not sacrifice a lot of its speed for some smaller multi-dimensional arrays. It can also operate on such type of structures with implemented high-level mathematical functions. Beside this abilities, it is very useful for Fourier transform, linear algebra, and it also has special tools for integration with C/C++ code as well as Fortran. NumPy is distributed and licensed under the BSD license

In the code, NumPy array structures are often used. Actually, all of the arrays used in calculations are some type of NumPy array. As such, they are much more optimized for matrix multiplications used in the code instead of the regular arrays. The time gains using these arrays are not that big, relatively speaking since the input objects are not that big (there are never more than 10 agents in the .csv file) and the number of repeats does not exceed the five thousand iterations. But, the code was made scalable from the start, which means, if the user want to load much bigger .csv file, with 50 agents for example, and to make tens of thousands iterations, then the entire NumPy structure would make significant gains in comparison with traditional Python array.

matplotlib module

The last major module that was used in the Python project is the matplotlib. This is a plotting library which is used for creating all of the graphs that are used and presented in this paper. Matplotlib is also part of the SciPy package, and it is used together with the NumPy module as the object oriented API for incorporating plotting features into programming language. It is very powerful tool that is used in different scientific areas for diverse researches. It has a programming interface semantics which is very similar to MATLAB's (which can be seen as an advantage or disadvantage) and deep integration with the Python itself. Most notable disadvantages of the matplotlib are that it depends greatly from other packages (in most cases, the NumPy itself) and that is almost impossible to use in other programming languages (works only for Python). In the simulation framework is implemented in such way that after all calculations with the NumPy are done, the results are saved to specific arrays in order to be plotted with the help of matplotlib.

4.1.3 UML Diagram of the Simulation Framework

The following Figure showcase an UML diagramm of the framework that was used in the practical part of the thesis. It is provided to help better understanding Python code. The code itself is provided alongside the master thesis.

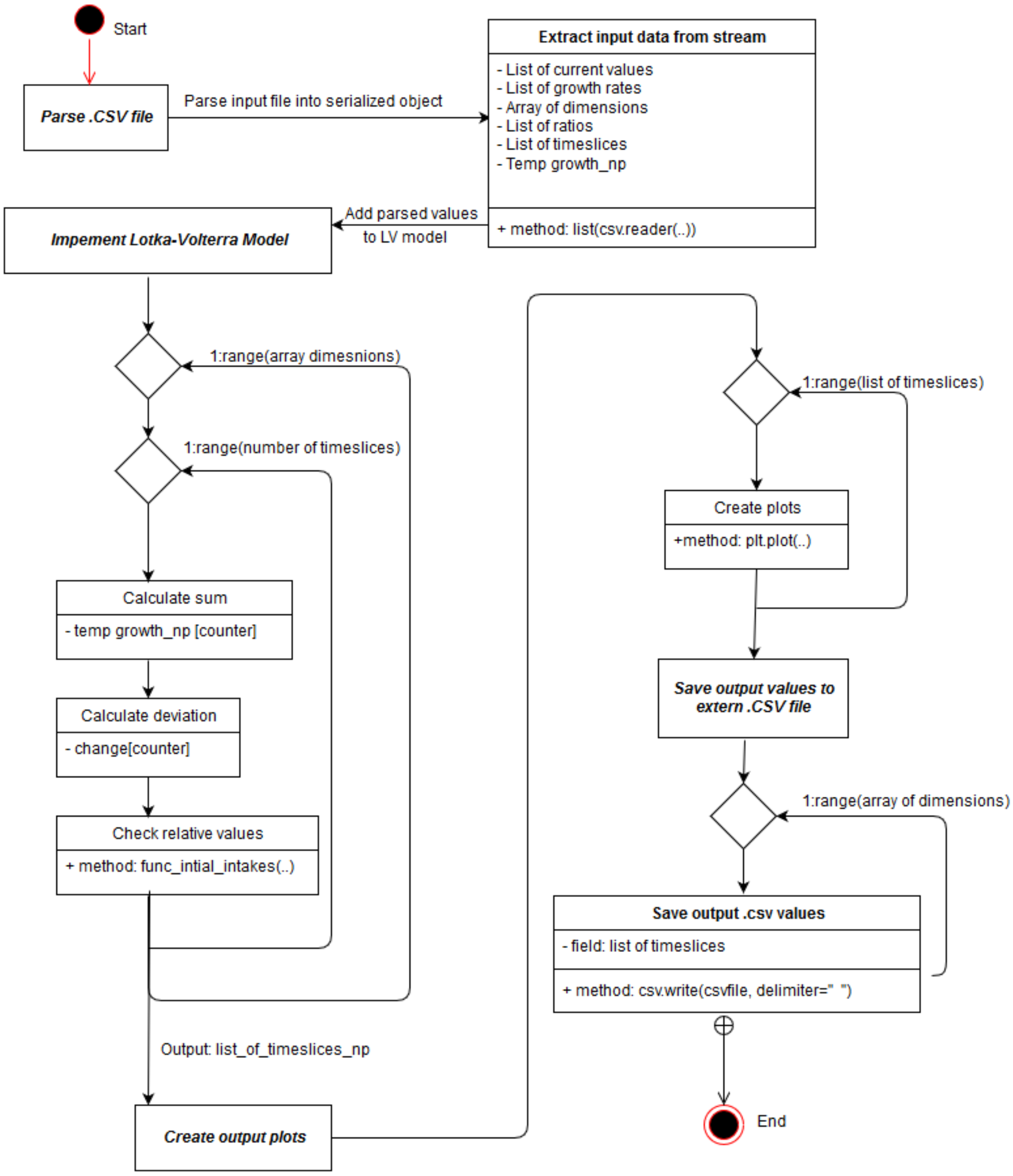

Figure 4.2: UML diagram representation of the used Python framework

4.2 Experiments and Results

This chapter is the most extensive part of the entire thesis and it covers all the experiments that were performed. The main objective of this chapter is to translate a real life competition model scenario into mathematical representation and to execute it in the specified framework to see the potential output results. By setting up a specific environment with distinctive parameters, the goal is to determine how to recreate different templates of the system. Those include scenarios such as when there is a dominant agent in the system, coexistence, extinction, symbiosis or any other type of competition that can occur when multiple participants are competing for same resources. Every type of experiment encompasses various number of experiments that were made in order to get an argument that is most universally valid as possible. At the end, in cases when this was possible, the mathematical representation was translated back into real life scenario in order to demonstrate how particular values affect final results.

It is important to point out that the values in the table are only partially random. This means that the values do have to meet certain criteria for the specific experiment, but they were randomly chosen from the range of values that are compatible with the experiment requirements. The real number of experiments that were made in order to come up with various competition patterns represented in this section exceeds by far the number of experiments that are covered in this thesis. Presenting every single experiment that was done during the writing of this thesis would simply make no sense because in some cases the outcomes were similar (or even identical), and in other cases there was no clear conclusion. That is why there are two kinds of experiments. First one is a kind of experiment that requires understanding of what happens with the system under particular predefined conditions (e.g., all IEC are competitive in nature and have minor effects on other agents). Other type of experiment is the one where a certain competition pattern is observed (e.g., continual non equilibrium state) and it needs to be established whether there are some pre-requirements among IEC in order to achieve this state. For this second type a lot of different experiments with really "fine tuning" needed to be done, in order to establish just one setup of input parameters that will determine rules applicable for that particular state.

The experiments that were carried out cover just one part of the spectrum, since there is practically unlimited amount of different setups that can be examined. The idea was to cover most of the basic setups that are easily translated into the real life. Such case, for example, would be a system in which all agents have competitive IEC whose magnitudes are less than one. Afterwards, some specific cases were also covered, in order to find if particular general rules can be applied to different multi agent systems.

The chapter is divided into several subsections, with each covering one particular experimental setup, but in some cases there are more than just one experiment per setup. The reason for that is the fact that sometimes a minor adjustment of the input parameters can result in completely opposite output values.

4.2.1 Type 1 - Pure Competition Model

First type of competition models are the ones that are only competitive in nature, in other words, there is no agent with the predatory characteristics in the system. This means that the agents are only competing for same resource and do not attack each other. The agents are only indirectly affecting each other, since obtaining more resource by one agent will lead to the decrease of the gathered resources of other agents. There is no direct correlation between agents, therefore, the system implies a pure competition.

Weak relations between agents mean that their own effects on themselves are more prominent than the effects of all other agents onto that specific agent. Most of the modern industries can be observed in this manner. Car, smartphone or computer manufacturers are competing for the same resource (users, buyers) and the relations between them can be weaker or stronger. For this specific type of experiment it is assumed that all relations between agents in the closed competition system are weak.

Experiment 1.1 - Pure Competition Model with Exclusively Weak Relations Between Agents

This is the default experiment for the given setup. Having multiple participants in the system which is a pure competition model. All of the agents are interacting with each other, but effects they are having on the adversaries are looser than the ones of them selves. These effects are also of similar strength, since the idea is to undermine their effect on the final outcome. All of the experiments conducted in this section will be formed with that premise in mind.

Mathematical representation of Experiment 1.1

The given system setup can be transformed with relative ease into an mathematical model. Since it is established that the system is pure competition model, translated into mathematical environment it means that all of the coefficients will be positive. Experiment 1.1 will analyse such setup when the relations between all agents are weak. In order to simulate such behaviour all of the IEC values will be lower than the IAC values.This setup is shown in the Table 4.1.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	0.002	0.003	0.004	0.005	0.005	1.000

Table 4.1: (Exp 1.1) Pure competition model with exclusively weak relations, $0 < \alpha_{ij} < 1$

Results for Experiment 1.1

First executed experiment was performed in the case that all IEC of the competing agents are positive, which means that the system is purely competitive in nature, and that the magnitudes of all values are less than one. This implies that the intra-specific coefficient of every agent is lager than the IEC, and thus makes bigger influence on the outcome of the competition than IEC. The output plot for such setting is given on the Figure 4.3.

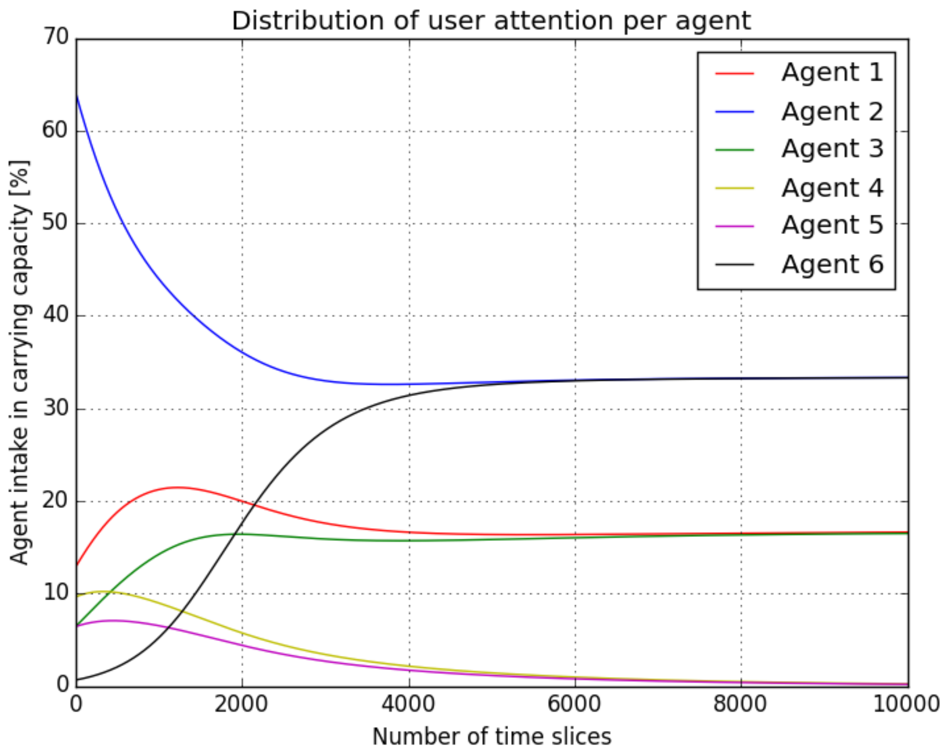

Figure 4.3: Output plot for Experiment 1.1

Comparing the input values with the output plot, it is very clear, that in such setting the stable equilibrium of the system will occur (in this case after some 5000 repeats). And it will afterwards not deviate from those values. It is also clear that the agents with the same growth rate will have the same intake of the available resources. The agents with highest growth rate will have largest intake, while the one with the smallest will go extinct after some time, losing the resources constantly as the time goes by.

In order to compare influences of inter-specific coefficients on the competitions outcome there is two more test that have to be executed. The growth rates and starting percentages are left on the same level, and the only parameter that is going to be alerted is the inter-specific coefficient of the agent with the biggest growth rate in order to see if that makes any changes in the outcome.

Experiment 1.2 - Pure Competition Model with Exclusively Weak Relations Between Agents, where the Observed Relation is even Weaker

Second experiment is notably similar to the last one, just in this case the idea is to observe agent which relations with other participants in the system are even weaker. Goal behind this concept is to see, if the exclusion of the agent from the influences of other agents can have any sort of effect on the agent values and thus intake in the users attention.

Mathematical representation of Experiment 1.2

As it can be seen on the table below Table 4.2 the only difference between the default experiment and this one are the IECs of the dominant agent that are decreased 10 times. The question is, can the change of this values increase/decrease dominant position of the agent in the system

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	0.0002	0.0003	0.0004	0.0005	0.0005	1.000

Table 4.2: (Exp 1.2) Pure competition model with very weak relations of the observed agent

Results for Experiment 1.2

If this plot is compared to the previous it is immediately clear that they look almost exactly the same, and further decrease of the IEC does not change the output (as long the IEC is a positive value). This was already established analytically in the Chapter 3 when it was spoked about the equilibrium isocline and how the output should look like in case of two agent system. This is why it is safe to say that, with the given settings, **the agent with the highest growth rate will take the biggest percentage of the entire capacity.** The dominant agent with remain intact, as long the parameters remain in the range defined by the experiment setup.

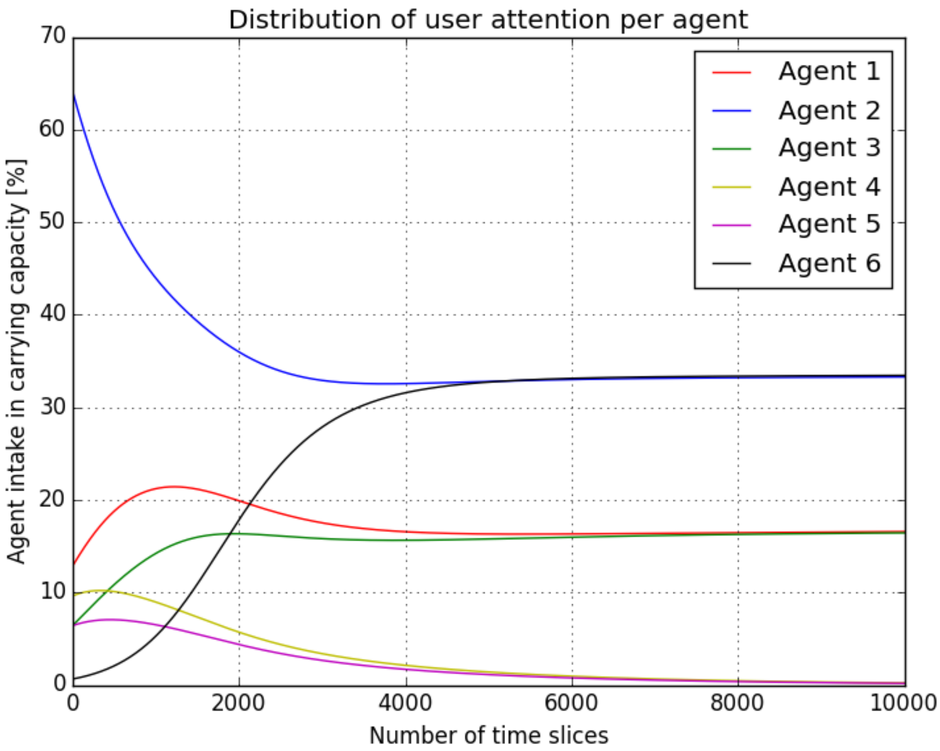

Figure 4.4: Output plot for Experiment 1.2

Experiment 1.3 - Pure Competition Model with Exclusively Weak Relations Between Agents, where the Observed Relation is Stronger than All Others

Experiment 1.3 is staying in the same realm as the last two experiments. The question here is if having stronger relations than the rest of the system (but still weaker than the effect that agent is having on itself) can produce some noticeable differences than the last couple of experiments.

Mathematical representation of Experiment 1.3

For the following experiment, the values of the observed IEC will be increased by the power of 10 (Table 4.3). This can be seen on the table below. Since the system consist of equations with logarithmic basis, this change should provide distinct difference between outputs of Experiment 1.1 and Experiment 1.3

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.05	0.02	0.03	0.04	0.05	0.05	1.000

Table 4.3: Competition model values for $0 < \alpha_{ij} < 1$ (Experiment 1.3)

Results for Experiment 1.3

Here the first embark of deviation can be seen on the Figure 4.5. Both *Agent 2* and *Agent 6* have the same growth rates, but the IEC value of the *Agent 6* are on average ten times lower than of *Agent 6*. The rest of agents will have the same values as before, and the equilibrium state for the entire system will be established as before. The only real difference will be the intake of tested agent, which will decrease slightly.

Experiment 1.4 - Pure Cmpetition Model with Exclusively Weak Relations Between Agents, where the Observed Relation is much Stronger then All Others

Last experiment in this section in only derivation of the last one. Since it is finally noticeable some change of output values on the last plot, the goal is to see how far can they really change.

Mathematical representation of Experiment 1.4

There is one more experiment that should be conducted with this values. What happens with the system if there are ICs of the observed agent increased by the power of 100 ? The values will still be lower than intra-specific coefficient, but still very close(Table 4.4). Can such setup lead to the extinction of the agent's intakes or will he only partially lose his share

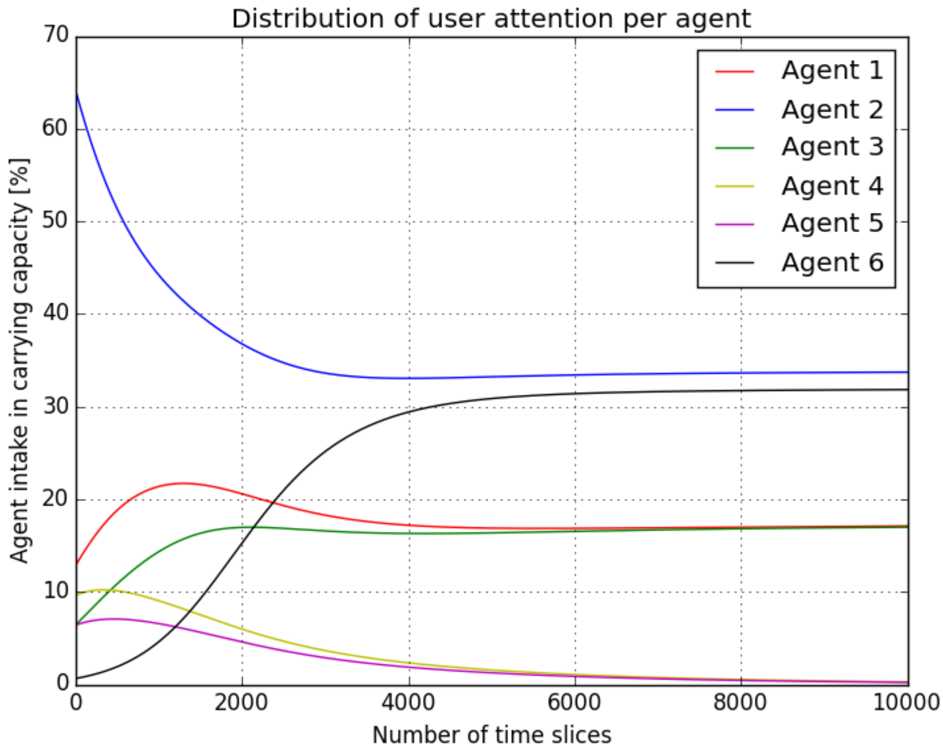

Figure 4.5: Output plot for Experiment 1.3

of available resources. The output plots and findings from given set-ups will be provided and further examined in the next chapter.

Results for Experiment 1.4

If the goal is to stay in the same realm of boundaries, but still to provide different output, the IEC has to be increased once again by factor of 10 (100 in total). Figure 4.6 is showing what will happen with the influence distribution if such parameters are applied:

As it can be seen on the plot, the changes in this case are much drastic than before. For starters there is only one dominant agent (with the largest rate, there is no change to this condition). Agent 2 that was one of the dominant agents has lost his position falling under two more agents. It is also noticeable that none of the agents went extinct as with earlier examples. The test was made with ten thousands of repeats, but still no equilibrium had been achieved (but the plot clearly shows that the values are stabilized, and it would achieve this state if more repeats are made).

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
Agent 1	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
Agent 2	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
Agent 3	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
Agent 4	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
Agent 5	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
Agent 6	1	0.5	0.2	0.3	0.4	0.5	0.5	1.000

Table 4.4: Competition model values for $0 < \alpha_{ij} < 1$ (Experiment 1.4)

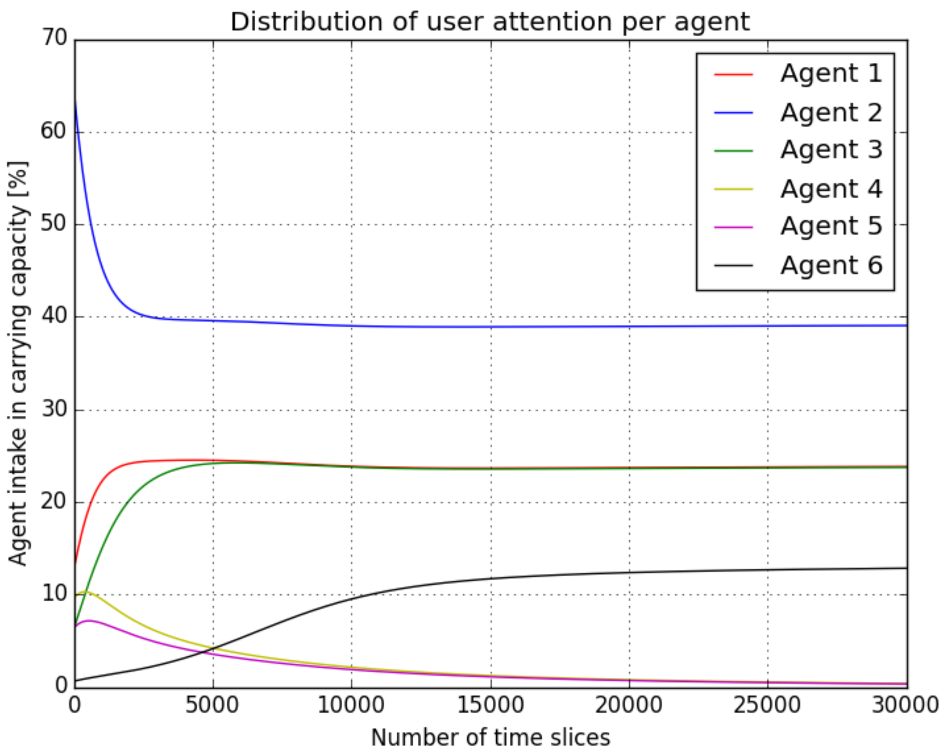

Figure 4.6: Output plot Experiment 1.4

4.2.2 Type 2 – Pure Competition Model with Only One Strong Relation Between Specific Two Agents

Second group of experiments is continuing on the same path as the Type 1, with one major difference. Until now, all of the relations between agents were weak, and the system was loosely coupled. The agents did not interfere with each other a lot, and the most dominant parameter was the effect they had on them selves. The results that emerged from such setup clearly show was it needed to gather most of the users attention. The premise of this group of experiments is to "inject" an agent in the system, which will have more effective influence on one or more agents, than on itself. The goal is to see if such change can be noticeable, when comparing with the last group.

Experiment 2.1 - Pure competition model where the single strong relation is just above the rest of the relation

Starting experiment for these group would be just to check system. Inject one agent that will have more deleterious influence on some specific agent than on itself and see what happens. The question here is, can such small change to the entire system make a greater impact to the final outcome.

Mathematical representation of Experiment 2.1

Experiments from the Type 2 will cover similar behaviour as Type 1 (subsection 4.2.1), since it is still only a competition relation among agents, but in this case, the dominant agent will have inter-specific coefficient which is higher than intra-specific coefficient. Only change of the values will be the for the one of the agent's IEC, to be larger than 1 in order to demonstrate how much influence on the entire system such change can make. This experiment will be conducted with the parameters shown on the table 4.5

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	1.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	0.002	0.003	0.004	0.005	0.005	1.000

Table 4.5: Competition model values for $0 < \alpha_{ij} < 1$ (Experiment 2.1)

Results of experiment 2.1

There is an easy way to really demonstrate how much power a single IEC can have on the entire observed system. Experiment 1.1 is again taken into consideration. As it can been seen on the Figure 5.2, *Agent 2* is the dominant player in the game (alongside *Agent 6*). All IEC values are of the same basis (10^{-3}). Now, a subtle change happens. By taking a random IEC parameter a changing its value to 1 (which is now increase by 3 decades). With the help of simple deduction and previous examples, it is clear that Agent 2 will not be any more the dominant player in the game (as it was already shown In the Figure 5.4), but it is still somewhere in the middle.This state is represented on the Figure 5.5. It can be clearly seen, that the intakes of Agent 2 are rapidly decreasing to some point, after which he enters the equilibrium state.

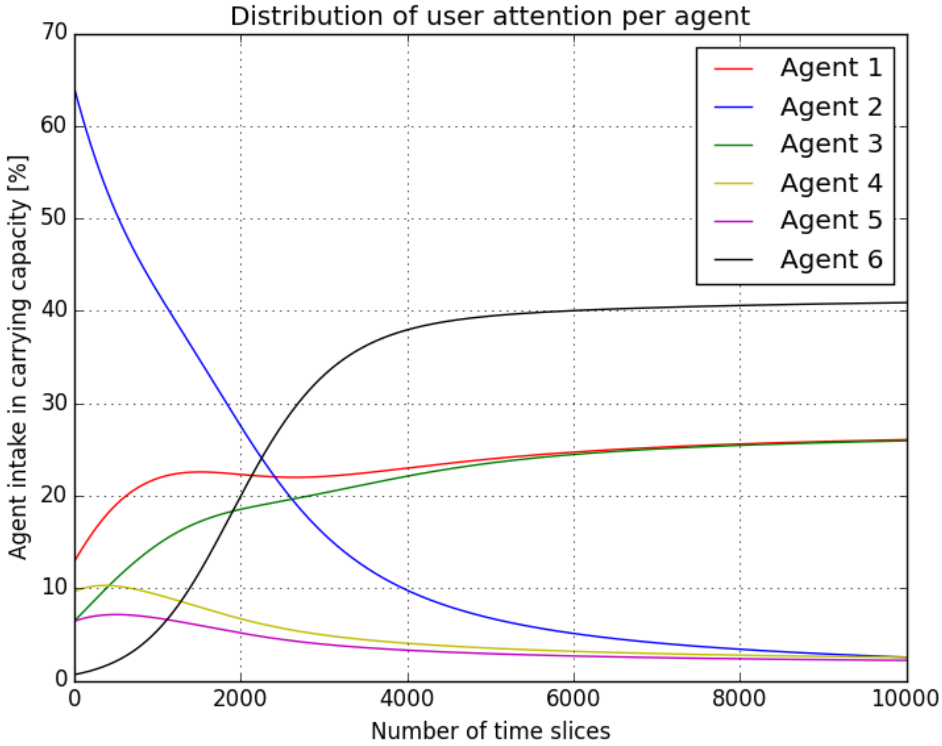

Figure 4.7: Output plot for Experiment 2.1

Experiment 2.2 - Pure competition model where the single strong relation is decisively above the rest of the relation

From the last experiment was obvious that the user attention of the targeted agent will decline if its competition factor is higher than the one influencing him. The outcome that will follow will most certainly always lead to the loss of entire user attention, even for the smallest of changes. Because of that feature, purpose of the following experiment is to exam, how fast this loss of intake can really be.

Mathematical representation of Experiment 2.2

One more similar experiment will be executed, staying in the same basis for the changed IEC, just doubling its value. These input values are shown on the table 4.6. The last experiment barely crossed the mark of IAC, and increased it only tenth of the percentage over that border.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	2.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	0.002	0.003	0.004	0.005	0.005	1.000

Table 4.6: Competition model values for $0 < \alpha_{ij} < 1$ (Experiment 2.2)

Still, it managed to cut off all intakes in some 40000 iterations. This experiment will try to establish how much faster can that happen, if the parameter is continued to be increased.

Results for Experiment 2.2

By increasing same parameter at setting the value to two, the presence of the agent in the competition model is effectively killed . This example is shown on the Figure 4.8. The rapid decline of *Agent 6* is even more obvious than in a previous example. After some 6000 time slices this agent's intake will be completely extinct. This only shows how much sensitive is the system to small changes of one single parameter.

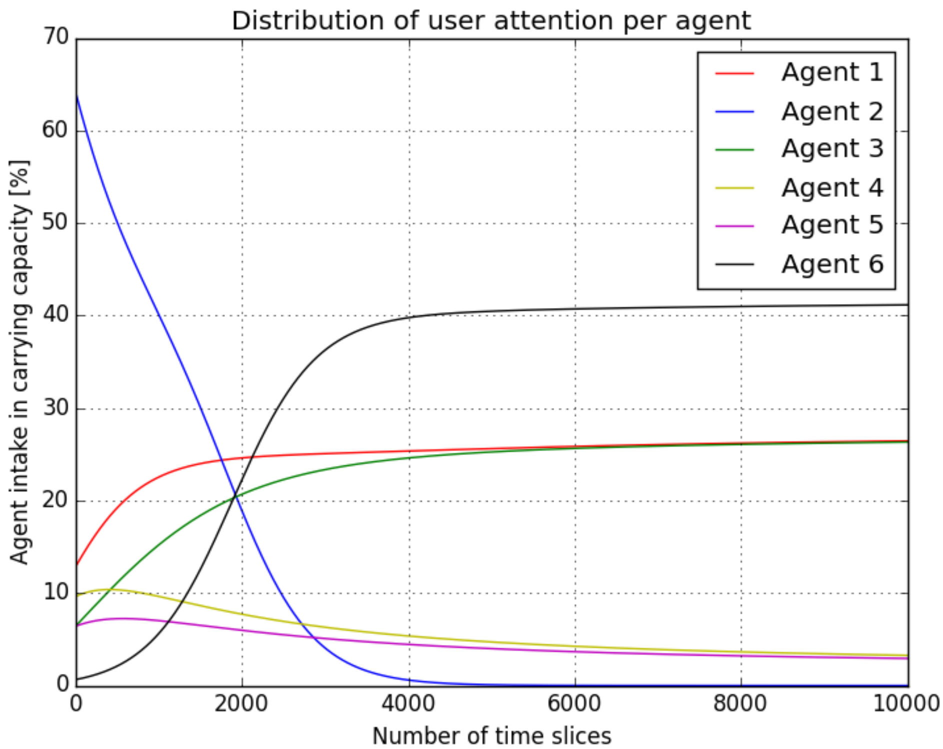

Figure 4.8: Output plot for Experiment 2.2

With this experiment generalization of the competition-only type of relation can be concluded. In real-life scenarios the agents will not always have only such sort of mentality, but they can inter alia also have more diversified sorts of relations between then (which were already covered to some degree in the methodology chapter. Next section with experiments will handle more with predatory type of relations between agents.

This is the reason why the things starting to get really complicated when the number of competitors is increased. It is very hard to predict a possible output without the use of same software that can simulate potential outcomes in the competition.

4.2.3 Type 3 – Mixed Model with Only One Weak Predatory Relation Between Agents

First two groups of experiments focused only on the pure competition models, mostly with weak relations between the participants in the system. This group however introduces the term of predatory relation among agents. Usually, such relation is common for the biological systems, but it can be found in many other surroundings. With pure competition models, every relation is harmful for both agents that interact. Predation effect is , on other hand, deleterious only to one side. Predator agent takes the resources by effectively ”killing off” other agent influence, and does not looses its own in the process. Following experiments will try to unveil how much effect such relation can really have on the entire system.

Experiment 3.1 - Competition model with only one weak predatory relation between agents

Mathematical representation of Experiment 3.1

For the Experiment 3.1, different sort of relation between agents will be implemented. In the table 4.7, there is only one agent with predatory characteristics, and all other remain the same as with former experiments. There is one important question that rises from this setup. Does the predatory relation has the same effect on the agent’s intakes as the competitive, or some changes can occur? This is especially important when the IAC is greater then IEC, because in former examples the IAC was always dominant when its magnitude was greater than the one of the IEC.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	-0.002	0.003	0.004	0.005	0.005	1.000

Table 4.7: Competition model with one predatory value of observed agent (Experiment 3.1)

Next step that needs to be investigated is what happens with the system when one or more IECs is negative, but its magnitude remains smaller than the one of the IAC. Logical deduction would be, that in that case such agent would become predominant in the system because it would not compete for the resources with given specie, but directly taking from it. The results of the following experiments will give better understanding of such behaviour.

Results for Experiment 3.1

In order to observe the changes that can happen with negative IEC, same type of simulation will be conducted as for the previous examples, which means, absolute values remains the same, only the sign of a particular IEC will change. The one other difference is going to be, that now *Agent 6* values will be changed (since his starting value is smallest), and the question here is to see, if the negative coefficient can improve the capacity intake in the system.

As it can be seen from the Figure 4.9, the results look identical to plot which has same input values without negative value (plot 5.2). This is clear, since the IEC value is less than 1, thus the intra-specific coefficient still has more influence on the capacity intake of the agent.

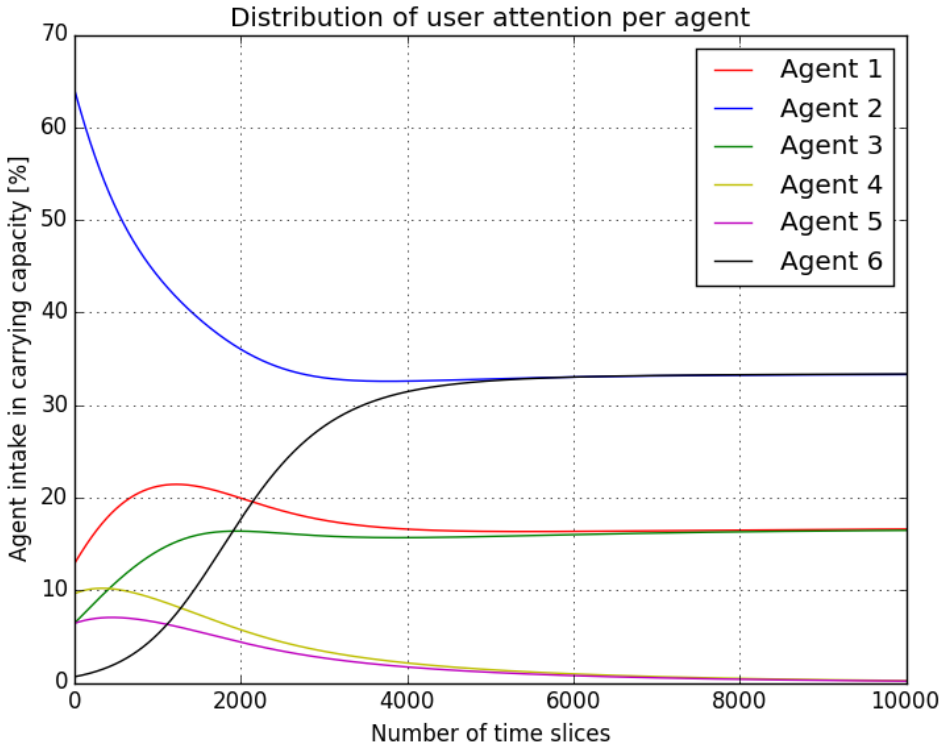

Figure 4.9: Output plot for the Experiment 3.1

When basis of the IEC is changed to the power of 10^2 (10 times higher coefficient, there is still no changes to the output plot). In order for changes to be visible, the basis of the IEC has to be increased to the power of 10^1, as it can been seen on the Figure 4.9. Only when this value is 100 times higher than any other coefficient, deviation on the output plot can be seen. As long as the IEC with negative sign is smaller by magnitude than the intra-specific coefficient, this changes will not be drastic.

This experiment was done for multiple parameters of the observed agent. Targeting particular agent with predatory relation did not make any significant change to the final outcome, that is why is safe to say that the IAC will always have greater influence on the end intake of the agent, even if some of his coefficients are predatory in nature.

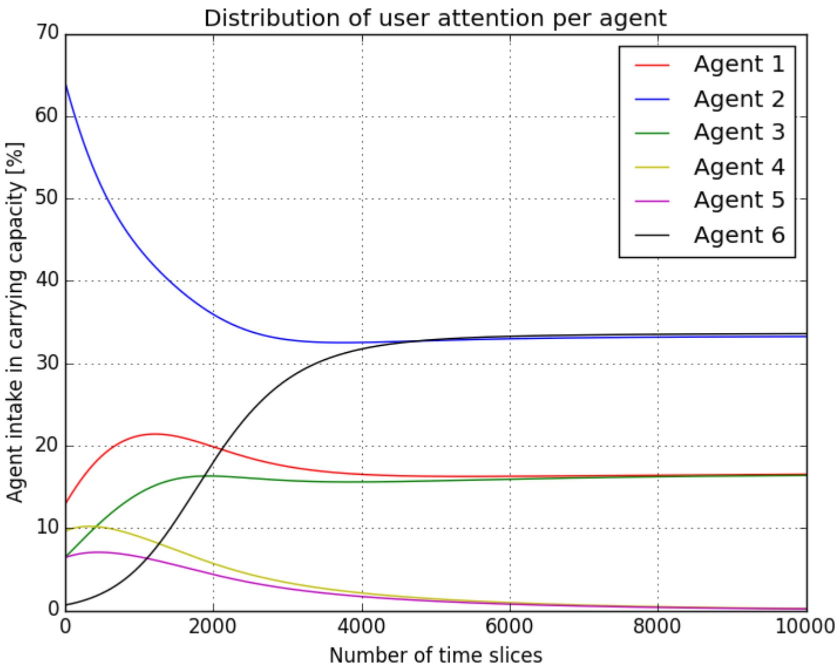

Figure 4.10: Output plot for Experiment 3.1 (increased predatory IEC 10 times)

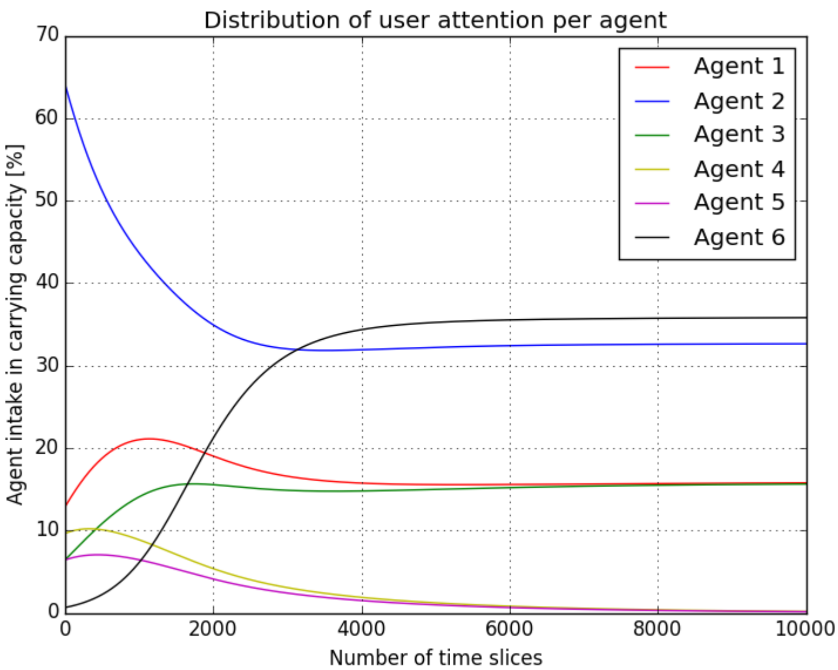

Figure 4.11: Output plot for Experiment 3.1 (increased predatory IEC 100 times)

Experiment 3.2 - Mixed model with predator agent how has weak predatory relations towards all other agents

The difference between this and the last experiment is that the observed agent will now become purely predatory in nature, not having any of competition effects, as in the last one. By diminishing any possible damaging factors to the agent, it would be safe to predict that such agent would dominate the system. The question is, if its predatory characteristics are weak, is it enough to still achieve its goal

Mathematical representation of Experiment 3.2

Special type of the Experiment 3.1 will be conducted afterwards. This experiment will determine, if only one predatory parameter can't make change, can the change to the agents' intakes occur if all parameters of one agent are switched up to be predatory. The table 4.8 shows the input values for the given test.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
Agent 1	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
Agent 2	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
Agent 3	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
Agent 4	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
Agent 5	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
Agent 6	1	0.005	-0.002	-0.003	-0.004	-0.005	-0.005	1.000

Table 4.8: Competition model with all predatory values of observed agent (Experiment 3.2)

Results for Experiment 3.2

This is the case when all of the IECs of the observed agent are having predatory nature.

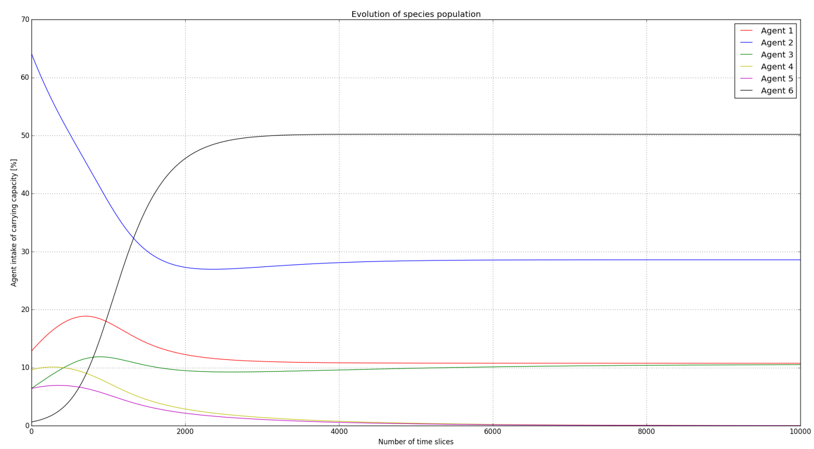

Figure 4.12: Output plot for the Experiment 3.2

Experiment 3.3 - Mixed model with predator agent how has weak predatory relations towards all other agents, but with magnitudes that are stronger than the ones of the competition agents

Following experiment will be based on the last one. One predatory agent will remain in the system, but its predatory effects will be steadily increased in order to show how such feature influences on the outcome.

Mathematical representation of Experiment 3.3

This is similar experiment as the last two, except now the basis of the predatory IEC will be increased. It will still be smaller than the intra-specific coefficient, but 10^1 or 10^2 times higher than the other parameters. The premise here is to show if such setup can lead to dominance of agent with predatory parameters, or will its intake still be limited with its growth rate. The table 4.9 below gives the input parameters for such representation.

There are few experiments that have to be conducted here in order to make some assumptions. What needs to be examined is, if increase of only one parameter can lead to intakes changes. Then, it is also important to show if targeting specific agent with predatory intra-specific coefficient can lead to different outcome (does it matter if the current dominant agent is "attacked" from observed agent, or will the result be the same if the same thing is done to some agent that will go extinct). Does further increase of IEC basis can accelerate changes? And finally, is it possible to make some predictions of how the final agents share will look like if more than one parameter of the observed agent is changed to be of predatory type.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	-0.02	-0.03	-0.04	-0.05	-0.05	1.000

Table 4.9: Competition model with all predatory values of observed agent(increased 10 times) (Experiment 3.3)

Results for Experiment 3.3

This is similar experiment as before, only this time the predatory coefficients magnitudes will be increased 10 times.

This experiment was also conducted for even more increase of the IEC, 100 times.

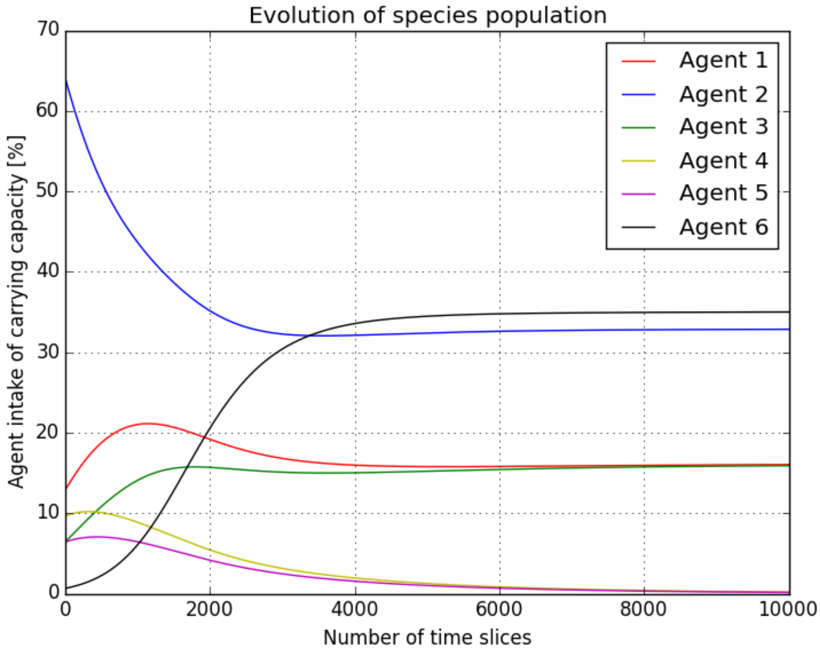

Figure 4.13: Output plot for the Experiment 3.3, 10 time increase for IECs

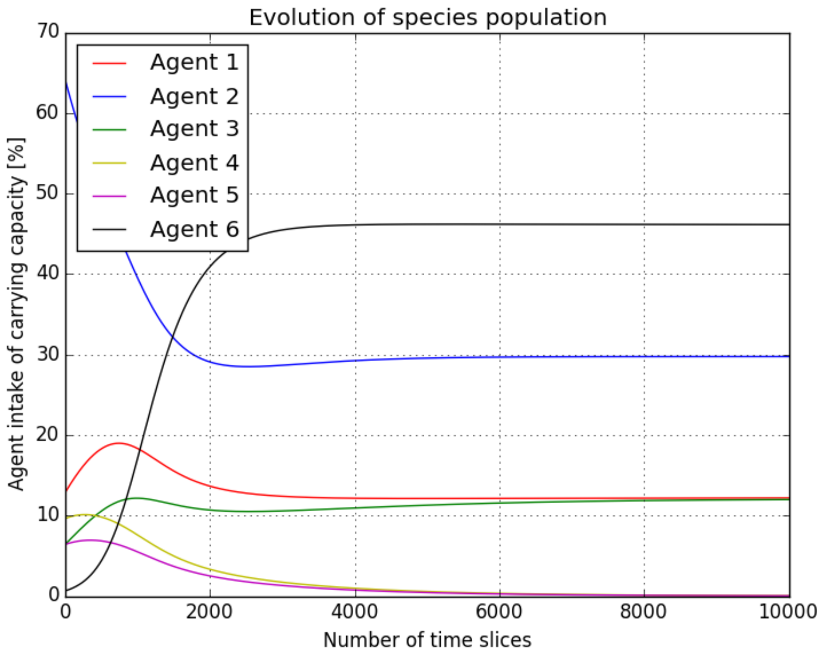

Figure 4.14: Output plot for the Experiment 3.3, 100 time increase for IECs

4.2.4 Type 4 – Mixed Competition Model With Only One Strong Predatory Relation Between Agents

Last group of experiments has showed the mixed competition model that has agent with weak predatory interaction toward other agents. Even with such features it was more than enough for the particular agent to take control over user attention in the system. From logical stand point that aspect alone is more than sufficient to support assertion that if the predatory interaction is strong enough, such agent will take utter control over system's resources. Such statement is obvious, and does not need to be proven. The premise of experiments in this group is to show if targeting particular agent with strong predatory interaction can lead to acquiring more resources then bunch of them. Or if targeting just one particular agent can be sufficient to achieve total dominance in the system.

Experiment 4.1 - Mixed Model Where the Only Strong Predatory Relation Between the Agents is Just Above the Rest of the Relation

First experiment that is going to executed will try to showcase what happens with the system when one agent's predatory effect is greater than all others. And does this feature alone is enough for taking the dominance of the users' attention.

Mathematical representation of Experiment 4.1

This sample will be more concentrated on the inter-specific magnitudes that are larger than IAC. This type of setting is very important, because in theory (covered in Chapter 3) it was obvious that such scenario can easily bring big changes to the entire system. This setup should be the one used when main goal of the experiment is to have a dominant agent, but that the entire system remains in equilibrium.

Default settings for such system would be to have all values same as in the Experiment 1.1 (Table 4.1), and then just change value of some random IEC of the observed agent to have magnitude which is larger than its intra-specific coefficient (but with negative sign). This experiment will be conducted for all parameters and the results will be shown afterwards. On the Table 4.10 are given the values used in the experiment.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	-1.002	0.003	0.004	0.005	0.005	1.000

Table 4.10: (Exp. 4.1) Mixed model with only one strong predatory relationship in the system

Results for Experiment 4.1

Just as it was assumed from theoretical background, such setting produced drastic changes. The magnitude of the IEC is greater than 1, thus making it the dominant factor in the matrix. Because such value will then have larger effect than the deleteriousness of the agent's IAC, final outcome was relatively easy to predict. As it can be seen on the Figure 4.15, even though the *Agent 6* had the lowest starting value, he obtained in relatively short period of them more than 50% of entire capacity. And by doing that, he became the dominant agent in the system.

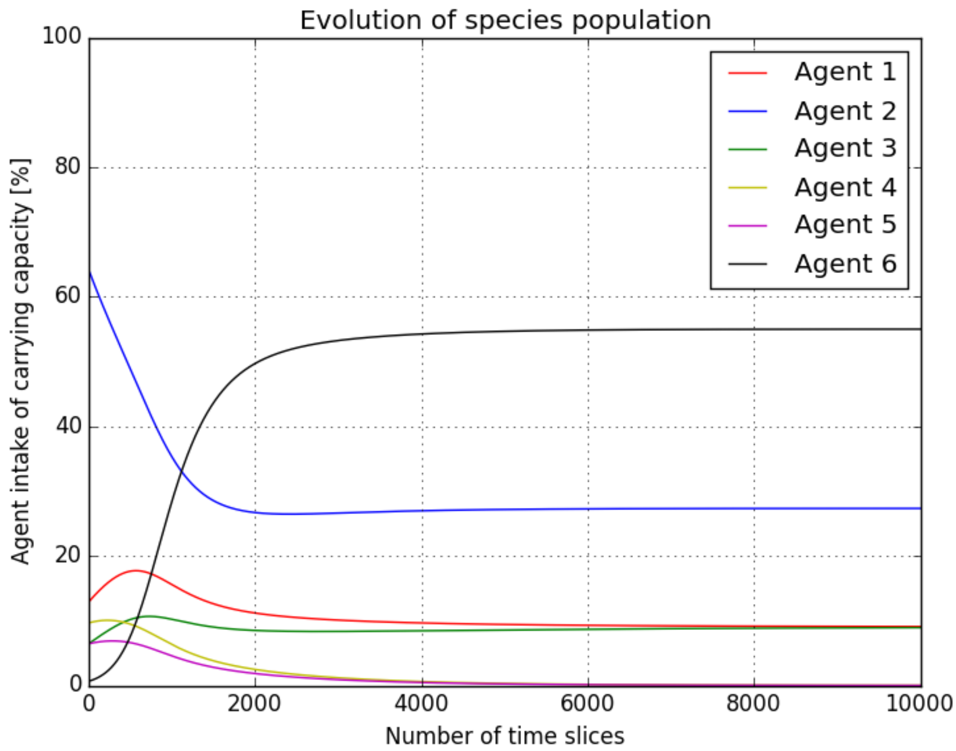

Figure 4.15: Output plot for Experiment 4.1

Experiment 4.2 - Mixed model where the peculiar targeting of one agent

Last experiment was predictable, since it was expected from the system to behave in that manner. Next experiment will focus more on the specific targeting of different agent in order to establish some connections between strength of predatory interaction and strength of the targeted agent's features.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
Agent 1	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
Agent 2	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
Agent 3	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
Agent 4	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
Agent 5	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
Agent 6	1	0.005	-2.002	0.003	0.004	0.005	0.005	1.000

Table 4.11: Competition model values with one predatory value in range $\alpha_{ij} > 2$ (Experiment 4.2)

Peculiar targeting of the agents is very common tactics in many different branches of life. It is typical political game, part of the political warfare[2]. Observed agent is laying low against most (all but one) adversaries. And then focuses all of its efforts upon one target in order to discredit in mass media, among people, domestic and foreign representatives. Mathematical model of such agent would be smaller values of competitive IEC values (positive) and one as big as possible predatory relation towards targeting entity. Such tactics can give significant gains, if implemented correctly. Of course, it is easier said then done.

Mathematical representation of Experiment 4.2

This experiment will not be so much different from the former, because in this case the premise is to examine if small increases of predatory parameters can develop massive influence to the agents' intakes. The Table 4.11 with the input values for this experiment shows that only the predatory parameter will be double. This is interesting for experimentation because of the nature of the Lotka-Volterra equations. They are logarithmic, and the applied changes should be made for the base of the parameter in order to stimulate changes, but as it will be seen in the next chapter, in this case, there is no need for such increase. As with last example, this experiment will be conducted to different IECs of the observed agent in order to see if some pattern may appear.

Results for Experiment 4.2

This experiment is to some degree similar as the last one. There is however one key difference. The idea here was to target not just the random agent with predatory behaviour as it can be seen on the Figure 4.16., but also to target specific the agent with biggest intake in the system Figure 4.17. Just comparing this two plots with the one from the previous example, it can easily be determined that targeting the agent with biggest intake will produce better results for predatory agent for almost 20 %, which is significant difference. What is also interesting to notice is that such targeting will take a toll onto other agents. The will also loose their gains in the process. That means that if peculiar targeting implemented adequately it is possible not just to overthrow the dominant agent in the system, but to diminish strengths of other agents as well.

[2] James Bovard: A Brief History of IRS Political Targeting, online article: *http://www.wsj.com/articles/SB10001424127887324715704578482823301630836*

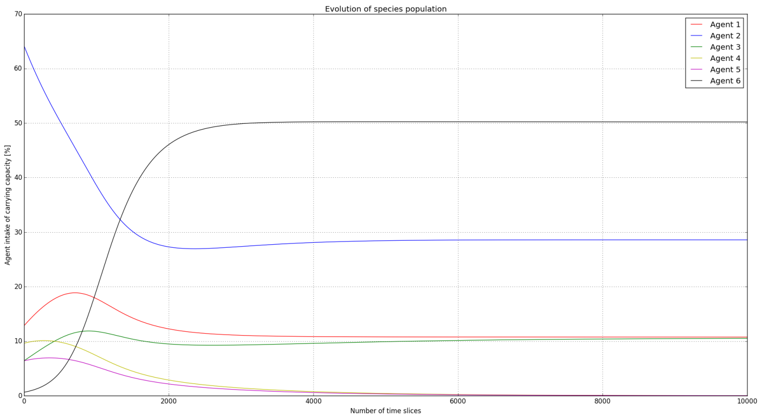

Figure 4.16: Output plot for Experiment 4.2a, target random agent

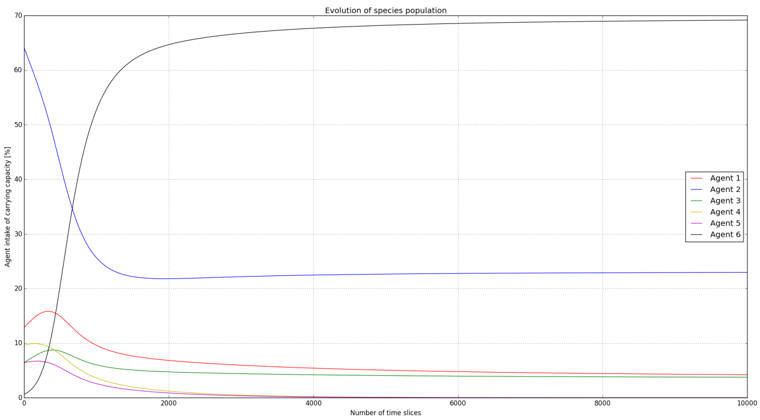

Figure 4.17: Output plot for Experiment 4.2b, target dominant agent

4.2.5 Type 5 – Mixed Competition Model With One Very Strong Predatory Relation Between Agents

This is very special case that can occur. Usually in the real life scenarios, there will be no such big discrepancy between parameters, but if such experiment would happen, how will the agents interact with each other. Can they coexist, or will just the one with largest values have the total dominance over entire system? If we are that agent, is there a successful strategy that can be done in order to prevent (or minimize) losses that will most certainly happen when there are such extremes in the system.

Experiment 5.1 - Mixed model with one very strong peculiar target among agents

This experiment's focus is to examine how can a single intense relation affect a stability of the system. Even though such system is not very plausible, there is always a change for it to mutate into such state. The question here is, can "over saturation" of relation lead to counter effect, that the dominant agents somehow self-destroys its gains.

Mathematical representation of Experiment 5.1

Having just one extreme in the entire system can be prove to be very challenging. Table 4.12 provides the look on the input values for such experiment. Only one parameter is predatory in nature, and this parameter is 10^4 times larger than all other parameters. For a logarithmic equations system, such value can already be considered as extreme. As with previous examples, the goal is to examine, if targeting specific agent with predatory parameter can produce different end values and intakes in the system. This experiment will also be conducted when the base of the extreme parameter is even greater, but all of this settings, as well as results, will be covered in depth in the next chapter.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	-10.002	0.003	0.004	0.005	0.005	1.000

Table 4.12: Competition model values with one predatory value of observed agent in range $\alpha_{ij} > 10$ (Experiment 5.1)

Results for Experiment 5.1

By further increase of the parameters interesting pattern appears. Since it is the logarithmic function, by increasing the value by the power of 10^2 the output clearly shows, that even though the predatory IEC is more than 10^5 times higher than all other IECs, it is still not enough to completely overtake system resources. ***Agent with higher growth rates than predator will not go extinct*** and the rest will simply lose their intakes. As it can been seen from the Table 4.12 input parameters, *Agent 6* has same growth rate as *Agent 1* and lower than *Agent 2* and *Agent 3.* Two later agents are still present in capacity distribution even after 10000 cycles, the others are not there any more. Predatory agent will have its peak after some 1000 repeats, with steady decrease in percentage until 6000 repeats, where it will reach its equilibrium point, and will not change his values afterwards.

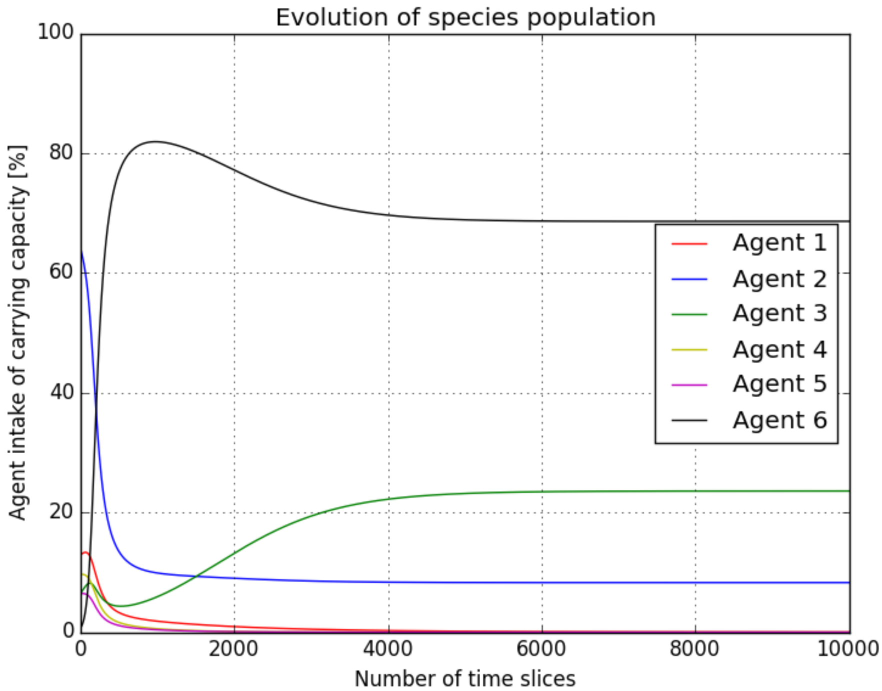

Figure 4.18: Output plot for Experiment 5.1 (With predatory IEC > 10)

Experiment 5.2 - Mixed model with one extremely stronger predatory relation between agents

Mathematical representation of Experiment 5.2

This is the last experiment conducted for the systems which will produce some sort of attracting equilibrium state. In this experiment examines what will happen with the system when there is an invading agent, which has IEC values that exceeds by far average values of the system. And these values do not have to be all of the predatory or competition type, it can be a mix of both. The idea is to find some proof that can be helpful in forecasting possible ending result of such system. Can targeting different agents with different relation bring any difference, or does competition parameter have larger influence on the system than predatory if they are of same basis. Starting input values for such system are provided on the table 4.13, and experiment's results will be covered in the next chapter.

Results for Experiment 5.2

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
`Agent 2`	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
`Agent 3`	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
`Agent 4`	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
`Agent 5`	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
`Agent 6`	1	0.005	-100.02	0.003	0.004	0.005	0.005	1.000

Table 4.13: Competition model with one predatory value of observed agent with extreme magnitude $\alpha_{ij} > 100$ (Experiment 5.2)

Then there is the question of total dominance. What does a predatory agent type has to the in order to ensure complete capacity of the system only for itself. The answer to this question is rather simple, and it was already showcased in the Experiment 4.2. Just ***target the non-predatory agent with biggest growth rate.***

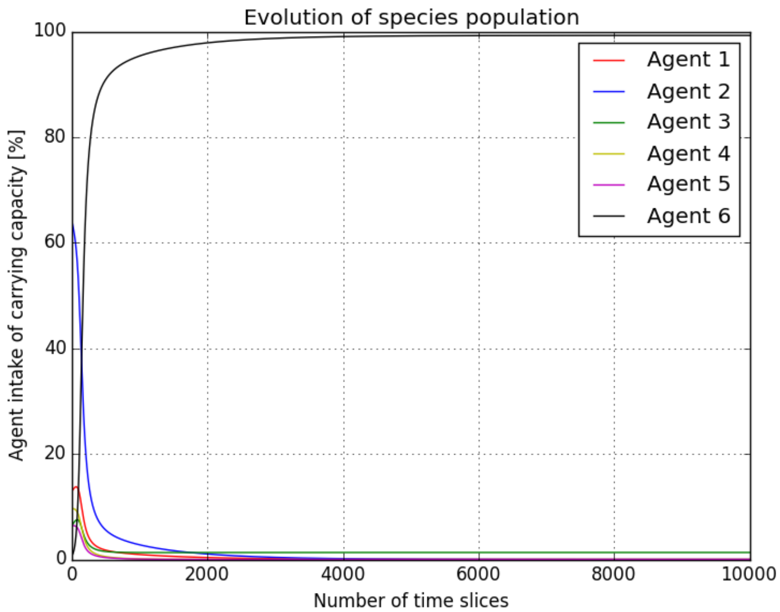

Figure 4.19: Output plot for Experiment 5.2 (Total dominance)

By *increasing just one IEC value of the predatory agent*,it is possible to establish complete dominance in the system after relatively short period of time. What is even more interesting, if there is one more parameter of the same magnitude as predatory IEC value (but with positive sign, settings are shown on the Table 4.14), it will still not change anything as it can be seen on the picture below (Figure 4.20)

The predatory agent would simply be in some form of dormitory state until some period of time, and then after a while would just ”explode” and overtake capacity afterwards lining

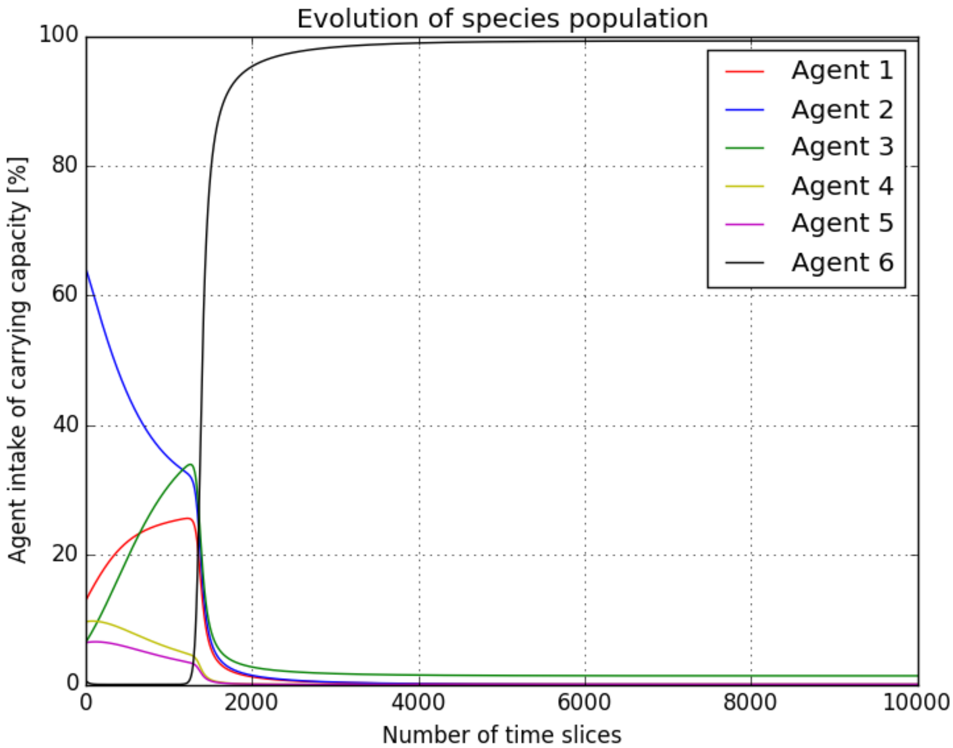

Figure 4.20: Output plot Experiment 5.2b (competition IEC with same magnitude as predatory)

rest of the agents extinct. This behaviour is very important, since the positive IEC of that magnitude would ”destroy” the agent in normal case, and he would lose its capacity intake very fast. Instead, it look like that in this case in only delays the inevitable, which is the total dominance of the predator agent.

In order to clarify this behaviour even more, the signs of the two IEC with biggest magnitude can just swap places. Now, in this case, *Agent 6* has predatory relationship with *Agent 1*, and extremely disadvantageous relationship with *Agent 3*. *Agent 3* has the highest growth rate and *Agent 1* has the second largest. The predatory agent has the third largest GR. The results are shown on the table below.

Now, even though *Agent 6* will have at some point over 70% of entire capacity, after reaching its amplitude, he will swiftly start to lose its position in the system, effectively ”dying” some 400 time slices after reaching its maximum. Agents with higher growth rates than predatory agent will survive, and the one with highest GR will have also the highest percentage of total user attention. Two agents with second largest GR (which is in this case the same value, just 0.001 points lower than the top growth rate) will have some 10% less of the share than the *Agent 3*. All surviving agents will reach point of equilibrium after some time, in this case after 4000 repeats.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
Agent 1	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
Agent 2	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
Agent 3	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
Agent 4	15	0.003	0.004	0.004	0.004	1.000	0.005	0.005
Agent 5	10	0.003	0.005	0.005	0.005	0.005	1.000	0.006
Agent 6	1	0.005	110.02	0.003	-110.02	0.005	0.005	1.000

Table 4.14: Competition model with one predatory value of observed agent with extreme magnitude $\alpha_{ij} > 100$ (Experiment 5.2c)

This example clearly confirms the previous statement. In order to have the largest share (or complete dominance in the system) as an agent you have to target the agent with highest growth rate. Even the slightest mistake of targeting the agent with basis of 10^{-3} lower growth rate can prove disastrous for the predatory agent as this tiny line can make the difference between total dominance and total oblivion.

There is one more important notice in this use case. Again, it has to be mentioned that the function are logarithmic in nature and that the basis of the IEC values plays the most important role. This means, if there is one agent with two IEC values with same basis (which are the higher than all other parameters) but they have opposite signs, then the output explained in above use case will occur. But, if there is more than two IECs with same basis, what happens in such situation? In that case, there is one more rule that can predict the output. If there are more than one $\alpha_{ij} > 1$ of same agent with the biggest basis in the system, that agent will always go extinct, indifferent of the other parameters values.

This is very important statement, since there is no need to observe the growth rates of the agents, because they are not playing an important role in the final output. It can be completely random, as soon as the previous condition is fulfilled, the same output for the observed agent is going to occur. This also means that the number of predator-prey relations with other agents (IEC < 0) is also irrelevant for final outcome (as long the basis of such parameter is not greater than the one of the competitions type of parameters)

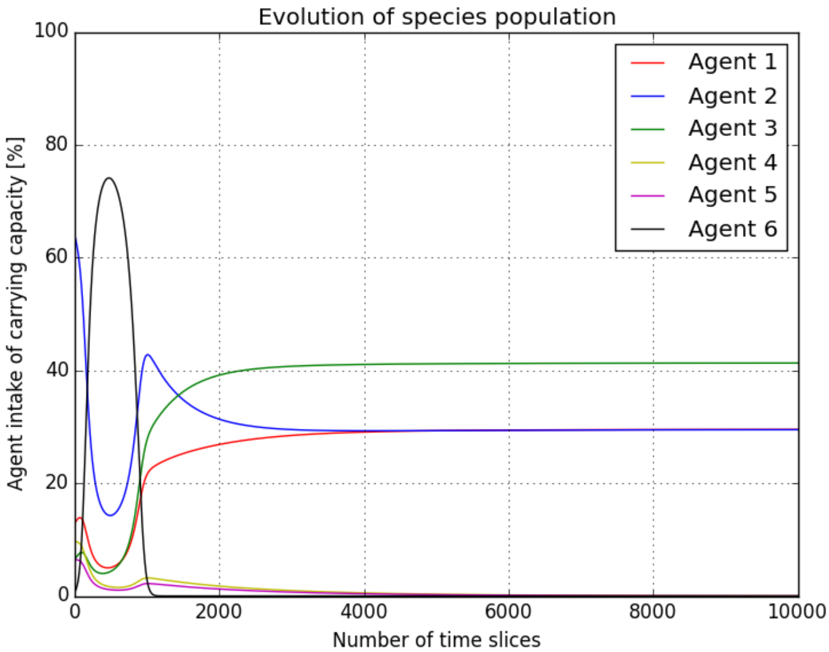

Figure 4.21: Output plot for Experiment 5.2c - Case 7b

4.2.6 Type 6 – Pure Competition Model With All Strong Relations Between Agents

Next group of experiments have objectively more actual purpose, since in most of the competition models, other agent will commonly have greater influence on target agent values, than that agent itself. What is very important to underline for these experiments is that all of the IECs values have to have magnitudes that are larger than the IAC values.

Mathematical representation of Experiment 6.1a

There is one experiment that should be executed first in order to showcase why its important for the system to have all of its IEC as strong connections. The following table (Table 4.15) provides input values for such special case, in which just one relation is weak, and therefore its mathematical representation is that the magnitude of that value is lower than the IAC values.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	1.002	1.002	1.002	1.002	1.002
`Agent 2`	100	0.005	1.002	1.000	1.003	1.003	1.003	1.003
`Agent 3`	10	0.004	0.003	1.003	1.000	1.004	1.004	1.004
`Agent 4`	15	0.003	1.004	1.004	1.004	1.000	1.005	1.005
`Agent 5`	10	0.003	1.005	1.005	1.005	1.005	1.000	1.006
`Agent 6`	1	0.005	1.002	1.003	1.004	1.005	1.005	1.000

Table 4.15: Competition model with all IEC $\alpha_{ij} > 1$ except one value (Experiment 6.1a)

Results for Experiment 6.1a

It is immediately noticeable that the only agent with the weaker relation will take prevail over user attention, taking some 65% of the entire intake. This statement is generally applicable to all setups. As long there is only one relation is weak, that agent will dominate the system, no matter of input values. Because of such property, this experiment is not very interesting. It is very limited special case, which does not have a particular real life scenario to which it can be parsed. It is only demonstrated in order to show the importance of every single relation among agents on the final outcome of the competition.

The Sample 6 experiments are focused on the system in which all of the IECs are competitive by nature, and higher than IAC value. First plot, shown on the Figure 4.22, shows the situation in which all IEC values are higher except one. No matter which value of the system is lower than IAC, if such state happens, that Agent will inevitably overtake dominion in the system, aggregating most of the resources. Such case is because of that feature were easy to predict.

First "real" example for this subset of experiments will be when the desired demands are achieved.

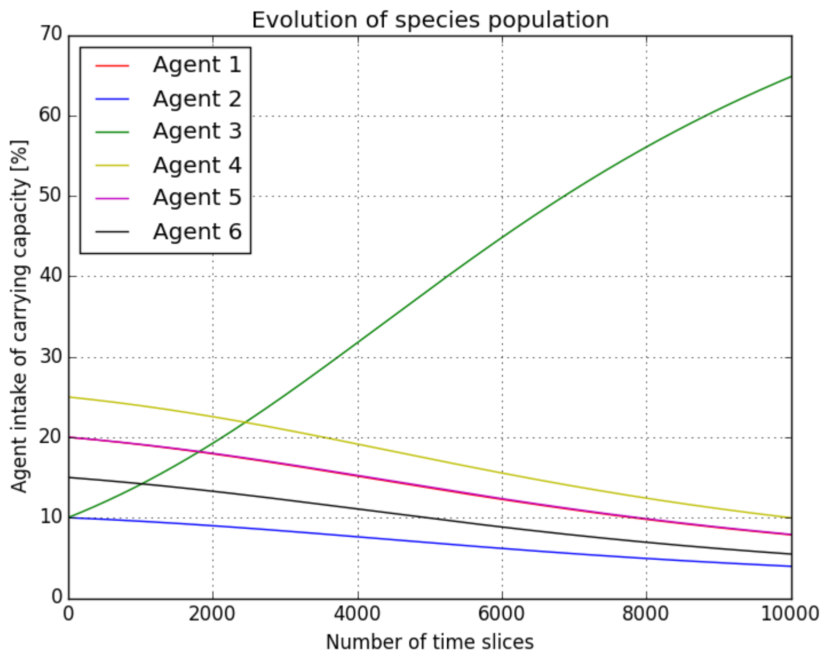

Figure 4.22: Output plot for Experiment 6.1a (One weak relation)

Experiment 6.1b - Pure competition model with only strong relations among the agents of similar strength

The following case should demonstrate what happens with the system, if all agents are similar in strength. Any by "strength" it is meant the relations that are existing between the participants in the system. Furthermore, all of the relations should have strong characteristic. If we compare it with the previous example in can be regarded as a opposite variant of the first group of experiments. Instead that all of the competition IEC values are weaker than IAC, now they are all going to be stronger. The logical question here is : Is the growth rates of the agents still going to play the crucial role, just as they did in first group ?

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	1.002	1.002	1.002	1.002	1.002
`Agent 2`	100	0.005	1.002	1.000	1.003	1.003	1.003	1.003
`Agent 3`	10	0.004	1.003	1.003	1.000	1.004	1.004	1.004
`Agent 4`	15	0.003	1.004	1.004	1.004	1.000	1.005	1.005
`Agent 5`	10	0.003	1.005	1.005	1.005	1.005	1.000	1.006
`Agent 6`	1	0.005	1.002	1.003	1.004	1.005	1.005	1.000

Table 4.16: Competition model with all IEC $\alpha_{ij} > 1$ without exception(Experiment 6.1b)

Mathematical representation of Experiment 6.1b

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	1.002	1.002	1.002	1.002	1.002
`Agent 2`	100	0.005	1.002	1.000	1.003	1.003	1.003	1.003
`Agent 3`	10	0.004	1.203	1.003	1.000	1.004	1.004	1.004
`Agent 4`	15	0.003	1.004	1.004	1.004	1.000	1.005	1.005
`Agent 5`	10	0.003	1.005	1.005	1.005	1.005	1.000	1.006
`Agent 6`	1	0.005	1.002	1.003	1.004	1.005	1.005	1.000

Table 4.17: Competition model with all IEC $\alpha_{ij} > 1$, 20% increase (Experiment 6.1c)

As it can be seen on the table above, all of the IECs are positive and their magnitudes are larger than the ones of the IACs. The default case at begging of the experiments section had the similar values. The only difference is that every IEC value is +1 increased. All values should be in the same range, and not just their basis, but the values should be in some $\pm$ 10% interval from median value of all IECs.

Results for Experiment 6.1b

Default case, which output plot is given on the Figure 4.23 below shows somewhat different behaviour. From the experiments in Type 1 is clear that when all of agents IEC values are lower than IAC, the agent with biggest GR will have the largest intake. However, if the settings are opposite (all IEC values are higher than IAC), than **the agent with the biggest starting value in the system is going to win**. What is event more interesting, if these values are in some similar range of values, all agents will remain with some intakes in the system, and not one will go extinct. This means, that when the *coexistence of the all agents* is the desired feature of the system, this is the way to go (no other state of the system can reproduce that requirement).

Results for Experiment 6.1c

This is however by no means a define example for above use case. The problem with magnitudes that are greater than IAC is that they are much harder to anticipate, and to predict how the output result will look like. To showcase this statement another similar experiment as the last one will be executed, but in that case change of the IAC value will be approximately 20%, as it can be seen on the table 4.17 and the differences between outputs of this two input settings can be seen on the next page, when comparing plots on the Figure 4.23 and Figure 4.24.

Next plot shows unfortunately how sensitive is the system, and that even the smallest changes to the system can neglect previous statement. By increasing just one IEC of the random agent for 20%, that agent's intake will decrease and eventually disappear. This also means that prediction of the non linear dynamic system is still very hard, no matter in which direction the relation parameters are going. Analytical approach for solving such problems would definitely not be the appropriated solution (and in most cases not even feasible), which means that only computational approach with some simulation framework remains.

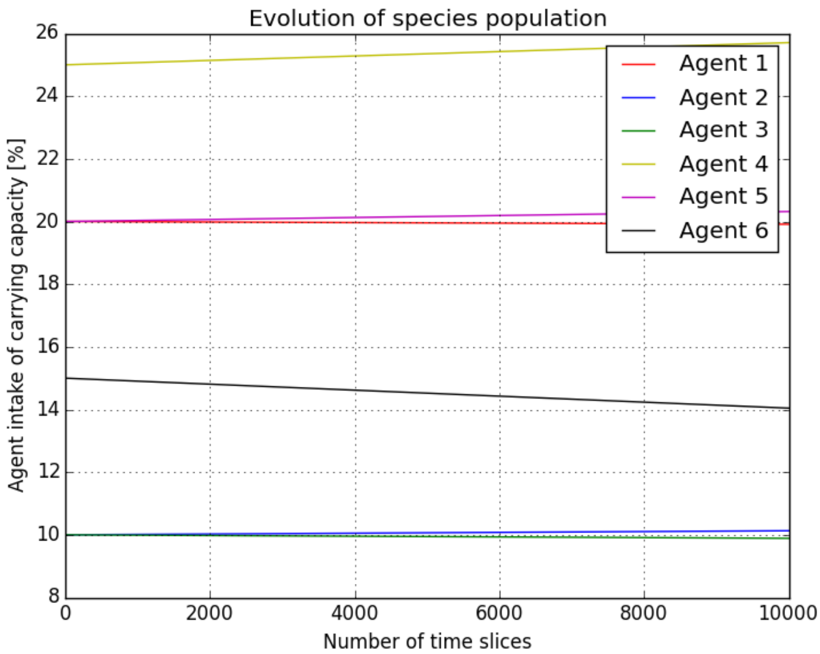

Figure 4.23: Output plot for Experiment 6.1b (Default case)

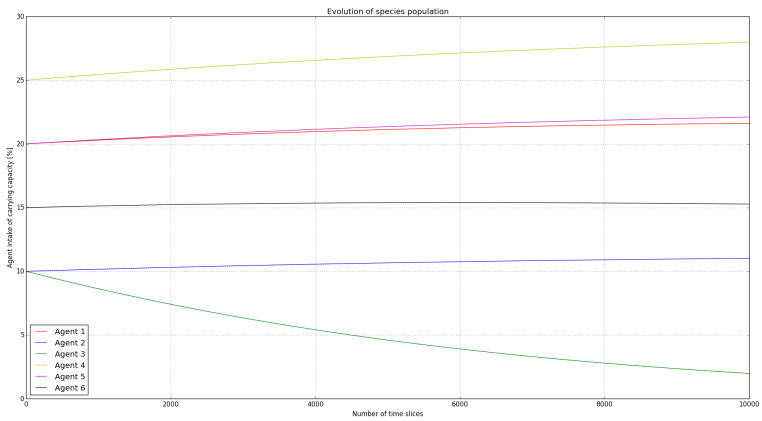

Figure 4.24: Output plot for Experiment 6.1c (20% increase of one IEC)

4.2.7 Type 7 - Mixed Competition Model That Does Not Achieve Equilibrium State

The purpose of this experiment is to showcase the sensitivity of the competition systems. The border between equilibrium and non-equilibrium state is very thin. By mixing weak and strong relation, and making them predatory or competitive in nature it is plausible to achieve completely different states of the system. Good news here is that if there is a need to recreate a specific circumstance in the competition model, than this would be the road to take. Unfortunately, the bad news is the sensitivity of such system, where only one parameter can play an important role. Furthermore, predicament of the output by using conventional means (i.e. mathematical analysis) would not be possible. Only safe way to solve such systems is by using computing power of modern computers.

Until now, all of the conducted experiments have achieved an equilibrium state. Since the Smale[Sma76] already proved that the system with only competition relations between agents will always have end up in the asymptotic state and reach equilibrium. When there are more than five agents in the system, remains the question, what happens if there are also predatory relations in the mix.

Mathematical representation of Experiment 7.1

Table below shows some random parameters that are provided in order to achieve continuous non-equilibrium state. This means that the equilibrium state will never be achieved and the participants in the system are uniformly changing their intakes of the available resources. It was already proven by [Sma76] that such state in the system cannot be achieved only with competition values (only positive values of the IEC) but there has to be a predatory sort of bond among some of the agents. This requirement is fulfilled in the following experiment, and as it can be seen from the Table 4.18, the parameters don't even have to have magnitudes that are larger than one.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.005	1.000	-1.002	1.002	3.002	-4.003	-3.004
`Agent 2`	100	0.004	3.002	1.000	5.003	-1.003	-3.003	0.003
`Agent 3`	10	0.004	1.002	-5.003	1.000	1.004	-1.004	1.004
`Agent 4`	25	0.003	-1.004	5.003	1.004	1.000	-3.005	1.005
`Agent 5`	20	0.003	-1.005	5.005	-2.005	-1.005	1.000	4.006
`Agent 6`	5	0.005	5.002	-3.003	1.004	-1.005	-2.006	1.000

Table 4.18: Mixed model that goes into perpetual state (Experiment 7.1a)

The output of this experiment is providing one interesting observation for this experiment. The fragility of the perpetual state, that can be completely changed with adjustment of just one IEC value. By doing that, the system will go into stable state.

Results for Experiment 7.1

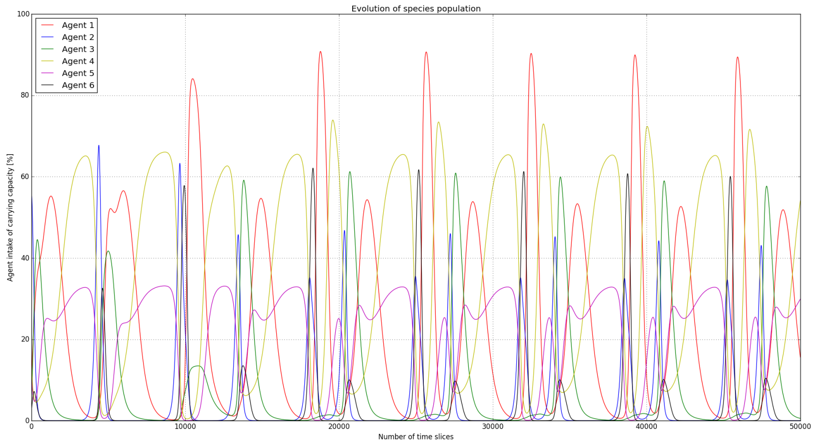

Figure 4.25: Output plot for Experiment 7.1a (Non equilibrium state)

The following plot shows the continuous output, non of the agents in the system will ever reach equilibrium state. This test for done multiple times for different values and number of repeats. As it can be seen from the plot, in this case, there is one million repeats, and there is no sign of potential equilibrium. Usually the test would achieve this state in less than $\frac{1}{100}$ of the time given in this test.

What is even more astonishing is the sensitivity of the system on the different values. Comparing this plot with the next, it is very clear that the system will achieve equilibrium state for all of its agents in a little over of $\frac{1}{50}$ provided time. The input matrix that gives the second plot differs from the first input matrix in just one value (instead of -1, α_{56} is set to -4). This value cleanly demonstrates the power of every single IEC value in the system. By directly influencing just one agent, it can indirectly change the outcomes of all other agents in the system, independently from their size.

That feature arises unfortunately some other problems. Because of the system's sensitivity and its stability, it it practically impossible to determine the outcome of the closed system by just observing the values. Even when all parameters of the system are known, without proper experiment it would not be feasible to determine end result. Author of this thesis did not found any evidences on the Internet that would support different opinion. That is the blessing and the curse of the non linear dynamic systems, it is not conceivable to predict them (at least if not some pre-requirements are not fulfilled).

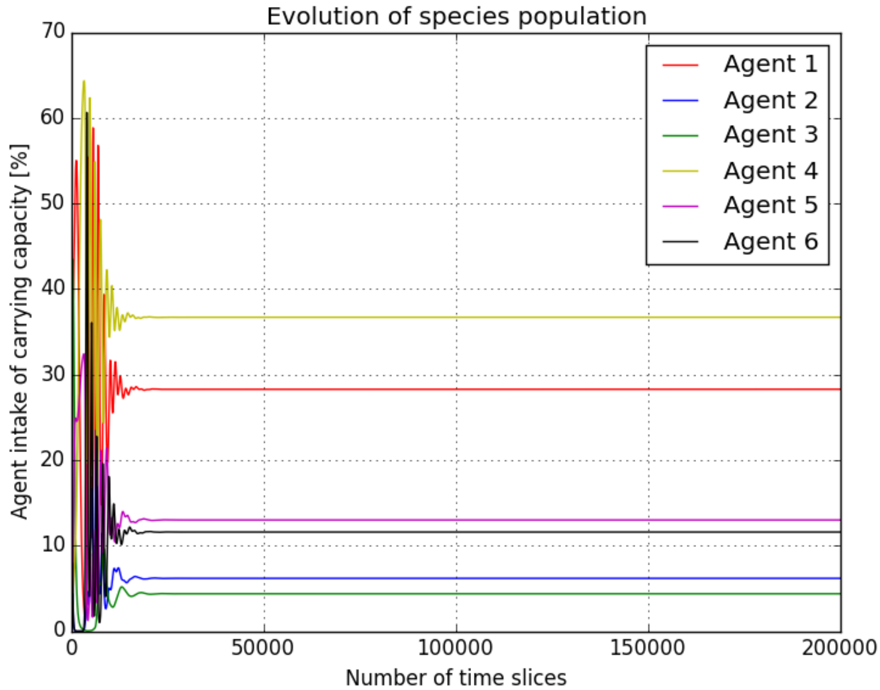

Figure 4.26: Output plot for Experiment 7.1b (Equilibrium state)

4.2.8 Type 8 – Mixed Competition Model Where Two Agents are in Symbiosis in Order to Ensure Survival in the System

Last type of experiments is going to concentrate on a following situation. There are two random agents in the system, and they collaborate on the common interest. The idea behind this interest is to make both agents dominant in the system, and by doing so, decrease the intakes of all other agents. The question is, how should the IEC should be distributed in order to predict possible cooperation among two agents, and their supremacy over others. And if there is some general rule that the agents can follow in order to achieve this state in the competition model.

Experiment 8.1 - Mutualism between two agents in the pure competition system

Last group of experiments will be concentrating on the principle of symbiosis. If two agents are trying to join their efforts, such relation could be prove to be beneficial for both sides. This relation can be especially helpful if there is already a dominant agent in the system, and none of other agents can overtake his dominance alone. This is very common behaviour in modern economics, where different companies in the same (or similar branch) can join their endeavours in order to increase market share. It is just a more subtle way of saying old idiom: If you cannot beat them, than join them. Mutualism should be beneficial for all parties that are participating in it. Sometimes it's only needed to survive in the system (such as the cooperation between Fiat and Chrysler)[3], and sometimes its goal is to establish a monopoly over some system resources (the merger between NXP and Freescale is good example of such symbiosis)[4]. Following experiments will try to cover some of the most common relations that can happen during mutualism.

Mathematical representation of Experiment 8.1a

First executed experiment is going to be based upon weak system of pure competition model. Both competitive and altruistic parameters are weak, which means that there values are lower than intra-specific values of the agents. Mathematical representation of mutualism doesn't differ much from the predatory relation between two agents. Only noticeable difference is that both parameters α_{ij} and α_{ji} have to be with negative sign. In order to make the symbiosis fair, both parameters will also have the same value.

For this experiment, Agent 4 and Agent 5 will try to achieve greater gains trough symbiosis. The setup of the system will be the same as with the Experiment 1.1, where the only noticeable difference will be the symbiotic relation between two agents. These two agents were chosen particularly because of their lower values, since they didn't manage to obtain any user attention on their own, before the system has reached a equilibrium state.

[3] More informations about this merger: *http://www.detroitnews.com/story/business/autos/chrysler/2014/10/07/fiat-chrysler-automobiles-merger-cleared/16851393/*

[4] More informations on this merger: *http://www.wsj.com/articles/nxp-freescale-agree-to-merger- 1425245923?tesla=y*

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
Agent 1	20	0.004	1.000	0.002	0.002	0.002	0.002	0.002
Agent 2	100	0.005	0.002	1.000	0.003	0.003	0.003	0.003
Agent 3	10	0.004	0.003	0.003	1.000	0.004	0.004	0.004
Agent 4	15	0.003	0.004	0.004	0.004	1.000	-0.005	0.005
Agent 5	10	0.003	0.005	0.005	0.005	-0.005	1.000	0.006
Agent 6	1	0.005	0.002	0.003	0.004	0.005	0.005	1.000

Table 4.19: (Exp 8.1a) Mutualism of two agents in the pure competition model with weak relations

- Results for Experiment 8.1a - Establishing a dominance over system by symbiosis among two random agents)

As it can be see from the Figure4.27 below, not much has changed in the system, when comparing to the default graph of the Experiment 1.1 (Figure 4.3). Both agents still lost all of the users attention. These experiments were done multiple times for many different values of the α_{45} and α_{54}, and as long those coefficients are smaller than IAC, the outcome would be the same. After making the value peaking over IAC value, the behaviour remain the same.

The values had to be increase 500 times in order for system to finally start showing some changes. Only after such large increase, both agent did overtake the system resources from the rest of competition. This is unfortunately not very realistic and such representation doesn't look very plausible for the real life system. In the end, this means that **if the competition system has mostly weak relations in the system, mutualism between two agents would not be beneficial at all in almost every case.**

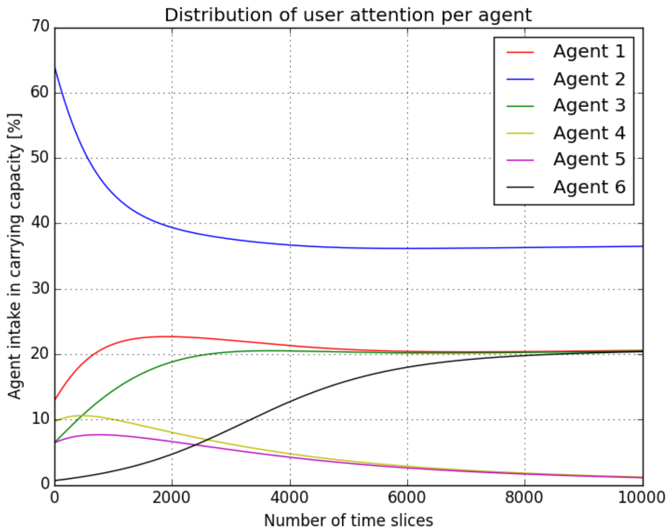

Figure 4.27: Output plot for Experiment 8.1a (Mutualism with weak relations)

Second variant of mutualism in such system would be to check what happens with the agents if the relations in the system are mostly strong.

Mathematical representation of Experiment 8.1b

The setup of this experiment is taken from the Experiment 6.1b. In this case since, strong relations between are simulated. This is done by setting the values of the IEC to be greater of the IAC. Only change, as with the last experiment are the parameters α_{45} and α_{54} which are both negative, but also greater than IAC values. On the Table4.20 the setup for the following experiment can be seen.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	1.002	1.002	1.002	1.002	1.002
`Agent 2`	100	0.005	1.002	1.000	1.003	1.003	1.003	1.003
`Agent 3`	10	0.004	0.003	1.003	1.000	1.004	1.004	1.004
`Agent 4`	15	0.003	1.004	1.004	1.004	1.000	-1.005	1.005
`Agent 5`	10	0.003	1.005	1.005	1.005	-1.005	1.000	1.006
`Agent 6`	1	0.005	1.002	1.003	1.004	1.005	1.005	1.000

Table 4.20: (Exp 8.1) Mutualism of two agents in the pure competition model with strong relations

- Results for Experiment 8.1b - Establishing a dominance over system by symbiosis among two random agents)

As it can be seen from the Figure 4.28 both agents will take over all of the system resources, splitting them in half. All other agents will lose their intakes and eventually be "expelled" from the system. This means that the influence of mutualism is much more beneficial for both agents if the relations between agents are predominately strong. This experiment was then executed also for much lower values of symbioses parameters, and the results were almost the same. On the Figure 4.29 is given the setup when does parameters are 500 times weaker than the rest of IEC values.

Such outputs are complete opposite to the previous experiment. In this case, parameters for mutualism are having much greater influence to the system in general, no matter how weak they are. From such standpoint it is safe to say that the **influences of the symbiosis parameters are conversely proportionate to the strength of relations in the system.**

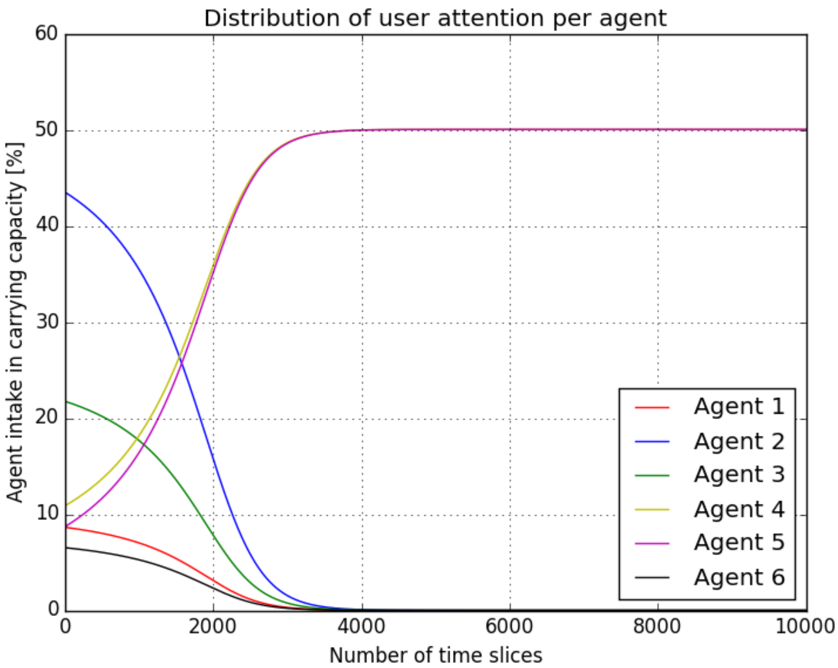

Figure 4.28: Output plot for Experiment 8.1b (Mutualism with strong relations)

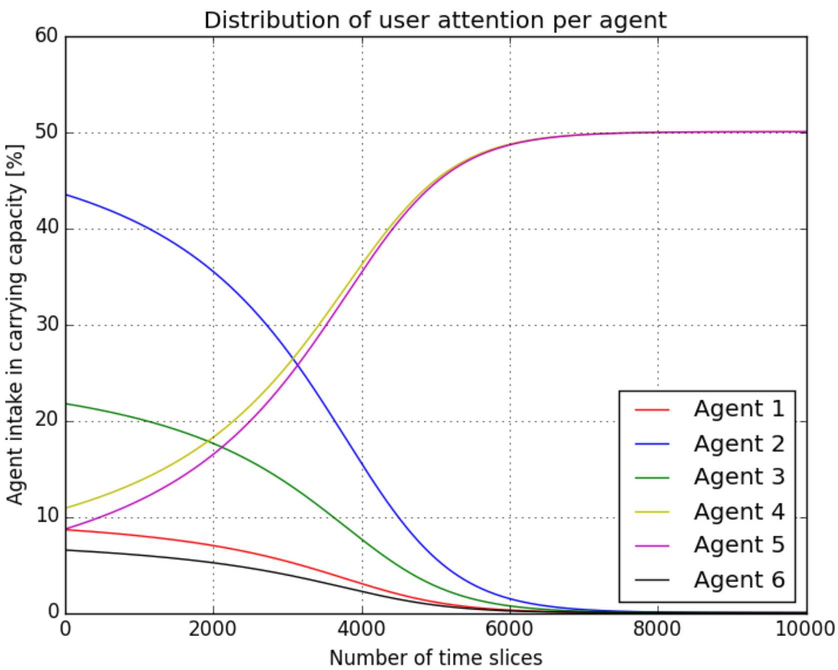

Figure 4.29: Output plot for Experiment 8.1c (Mutualism with strong relations)

Experiment 8.2 - Peculiar targeting of symbiotic agents in the pure competition system

The last experiment in this entire chapter will be concentrating on the use case that the agent is aware that in the competition system two or more agents are having mutually beneficial relationship. If that agent know that such relation can lead to the lose of resources for all other agents, how can he adjust his tactics in order to better accommodate new conditions, and prevent lose of his intakes.

Enter the peculiar targeting tactic. It is assumed that the one agent in the system is already informed about the symbiotic bound between two agents in the system. As such he knows that regular tactics are insufficient for him to remain at his level, so he changes his strategy to be more aggressive towards (predatory) one of the symbiotic agents. Specific targeting of just one agent with different relation than all other one in the system is called peculiar targeting.

This is a very realistic example, since it is not very uncommon that in the some sort of competition two or more participants are secretly working together in order to achieve better results for them both. If some third party can find out about their plans, it could prepare its strategy accordingly in other to preserve its resources. This experiment will try to put an emphasise on how precisely one should set defensive strategy for different input values.

Last two experiments are having a action-reaction sort-of bond, since this setting only makes sense if the agents are already in symbiosis.

Mathematical representation of Experiment 8.2

Mathematical representation of such model doesn't differ in great extent from the previous experiments. There are still weak relation among agents, and mutual cooperation between Agent 4 and Agent 5. Agent 3 is aware of this situation, and he knows that neither his starting values nor his growth rate is sufficient enough for him to preserve his intakes. By changing some of the parameters he will try to turn the tides into his favour.

Name	St.Value	Gr.Rate	IC P1	IC P2	IC P4	IC P4	IC P5	IC P6
`Agent 1`	20	0.004	1.000	1.002	1.002	1.002	1.002	1.002
`Agent 2`	100	0.005	1.002	1.000	1.003	1.003	1.003	1.003
`Agent 3`	10	0.004	0.003	1.003	1.000	1.004	1.004	1.004
`Agent 4`	15	0.003	1.004	1.004	1.004	1.000	-1.005	1.005
`Agent 5`	10	0.003	1.005	1.005	1.005	-1.005	1.000	1.006
`Agent 6`	1	0.005	1.002	1.003	1.004	1.005	1.005	1.000

Table 4.21: (Exp 8.2) Peculiar targeting of symbiotic agents in the pure competition system

Results for Experiment 8.2 - Peculiar targeting of symbiotic agents in the pure competition system

There are two key findings made that are deriving from this experiment. The only difference between two setups needed to demonstrate those findings is the growth rate of the agent that makes the peculiar targeting. In the first case, its growth rate is the higher that any of the two agents in the symbiosis. If that is the case, as it can be seen on the Figure 4.30, *Agent 6* will overtake the **majority of the user's attention by having only one predatory relation with one of the agents in the symbiosis.** The other possibility to have the same outcome is to have more than one predatory relations with other agents that are not in symbiosis.

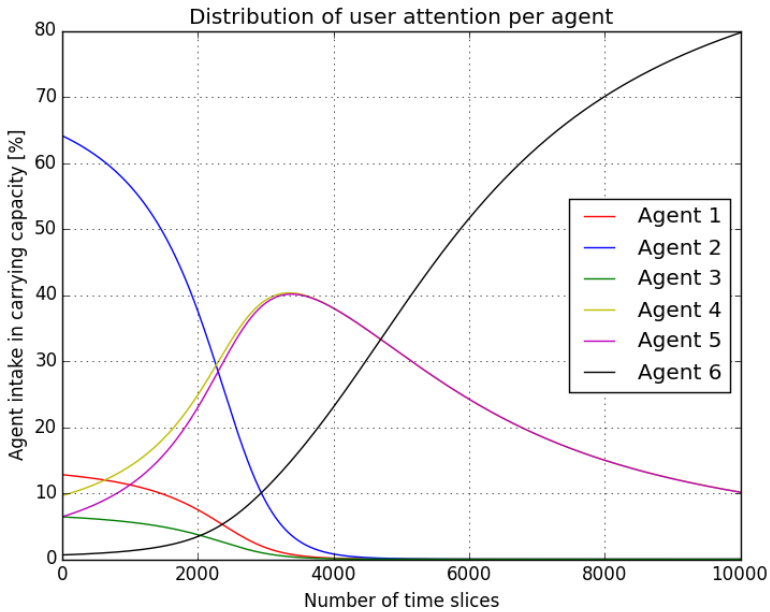

Figure 4.30: Output plot for Experiment 8.2a (Targeting of the non-symbiotic agents)

Second case is for the setup when the agent is having growth rate that is smaller or same as the one of the agents in symbiosis. In this case, he will have to target both agents with predatory relation in order to have the largest intake of the systems resources. Also, by targeting other agents with predatory relation, it will not result with success in the long term, since he will lose all of its intakes, as it can be seen on Figure 4.32. This means than the only solution for survival of an agent which starting values and growth rates are smaller than the one of symbiotic agents(or any other agent for that matter) is to target both with predatory relation. That case is depicted on the Figure 4.31, where it can be clearly seen , that *Agent 6* will overtake most of the resource, while all agent that are not in symbiotic bound will loose their intakes. Starting values do not have important influence on the outcome, growth rate is the dominant factor in such experiment.

With this last experiment is this chapter concluded. This are by no means all possible setups that can be created to demonstrate the competition among agents in the closed system. The idea was to show some of the possible scenarios that can happen in real life situations, and to determine if it is possible to find some universally valid statements and from them deduce the outcomes, by just knowing the input parameters. There are many more interesting setups, but the space needed to describe them all largely exceeds the given frame of this thesis.

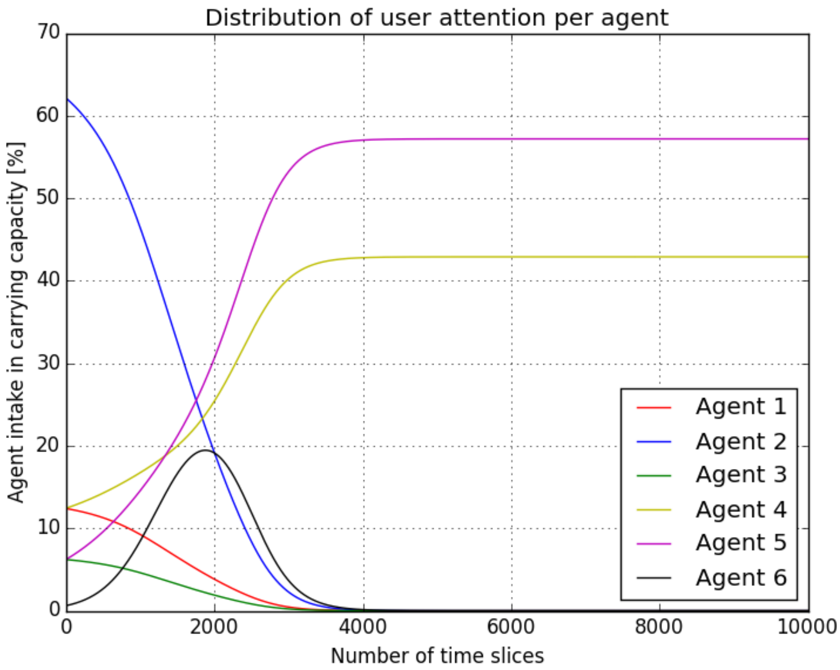

Figure 4.31: Output plot for Experiment 8.2b (Targeting of the non-symbiotic agents)

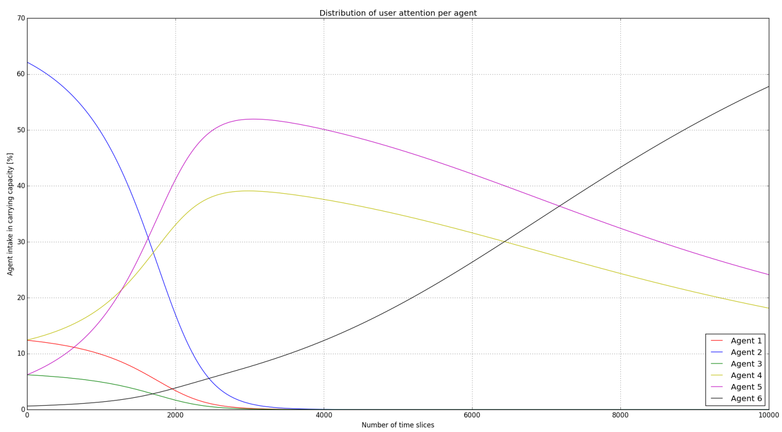

Figure 4.32: Output plot for Experiment 8.2b (Targeting of the agents in the symbiosis)

5 Discussion

The aim of this chapter is to summarize all important findings made through the executed experiments. Beside presenting them in a coherent way, there is also a short discussion about their impacts on the competition system, as well as about advantages and disadvantages of these findings. They are also showcasing which types of outcomes could be predicted and which not.

This summary is only covering findings that were made within the experimental framework. Additional assumptions or conclusions can be deduced from current findings, but I wanted to stay in realm of only use cases that were covered in this thesis.

The second part of this chapter is presenting some of the limitations of the implemented competition model, possible shortcomings, as well as potential area for improvement.

5.1 Key Findings of the Experiments

- *If the system is a pure competition model, with all weak relations between agents, the agent with the highest growth rate will win most of users' attention*

This finding is supported by first type of experiments and it was already discussed within the Jacobian matrix described in the Methodology chapter. Starting values neither affect the end value percentages, nor the relations between the agents. If there is more than one agent with the same highest growth rate, all of them will have same end results, they will be on top and achieve equilibrium, regardless of starting point values. The final outcome will always be the same, and looking from mathematical standpoint, this type of setup is the easiest one to predict.

- *If the system is a pure competition model, with all weak relations between agents, the agent with the lowest growth rate will lose all of its resources*

On the opposite side of the spectrum there is the agent that will go "extinct" because of its growth rate. The same principle applies to the weakest link. If there is more than one agent with lowest growth rate, they all will go extinct, and other (stronger) agents will share all resources. Same as with the previous finding, it is just as easy to predict outcomes in this kind of systems.

- *If the system is a pure competition model, the agent who has the highest competition coefficient will lose all of its resources*

This statement can also be translated to the real life situations. The more you fight, the greater the probability that you will eventually lose due to fatigue and constant damage being done to you by other agents. In the mathematical sense, it is inevitable. Such system will reach its equilibrium, and agent's resources in such system will be lost.

- *If the system is a pure competition model with all strong relations between agents, the one with the highest starting position will always win*

This situation is a completely opposite case from the first one. Strength of the relations, i.e. inter specific coefficients will directly affect the most dominant agent in the system. If these relations can be characterized as strong (which mathematically speaking means that IEC > IAC), then the agent that has the highest user attention rate at the beginning of experiment will always have it highest at the end. This automatically implies that the agent with the lowest starting value will lose all of its resources eventually.

There is just one precondition that has to be taken into account in order to be able to make this statement generally valid: all inter-specific coefficients have to be at the same level of strength. Following finding explains this further.

- *If the system is a pure competition model, with strong relations between agents, quantity is more important than quality*

If a particular agent has more than one strong competition relation toward other agents, these relations will have a worse impact on him than having only one very strong relation. From the Type 6 experiments it was made clear that having more strong IECs will inevitably lead to loss of the attention. An agent can have one very strong competition relation and still dominate the system, but it can also have more less stronger relations and go extinct (even when the sum of all those relations is higher than the value of that one very strong relation). Logarithmic nature of the LVM makes predictions for such systems relatively complicated and inaccurate.

- *If the system is a mixed competition model, the agent with a higher growth rate than the predator will not go extinct*

This finding could be called the "law of preservation". When there is a pure competition system with weak relations, and a predatory agent is introduced to it, as long as one or more agents are having growth rate that is higher than the one of the predator, they will not lose their resources. No matter how higher the predation rate is in comparison to competitive effects, it is not possible to establish a total dominance in the system. Except in the case that follows.

- *By targeting the non-predatory agent with the highest growth rate it is possible to gain dominance in the system (Focused targeting)*

There is only one possible way for a predatory agent to obtain all resources of the system (complete dominance). It has to target a non-predatory agent with the highest growth rate. This property is very well known in many different branches of social and natural sciences. Better result will be achieved by predatory targeting a specific agent, than by selecting it randomly (any other agent apart from the dominant one). If the targeted agent has the highest growth rate in the system, and the predatory relation is strong enough, it can be enough to establish complete dominance in the system.

- *Mutualism of two agents is enough to establish dominance in the system for both agents*

This statement can be called ”strength through unity”. Best bet for two weaker agents in the system is to cooperate if they are to survive. But mutualism does not only enable them to survive, but in some cases also to overtake the majority of system’s resources. An important conclusion drawn from this experiment is the predatory targeting of such agents. If a predatory agent targets one of both agents it will easily overtake their resources, and if strong enough, he will completely eradicate them from the system. This interdependence shows just how sensitive the relations between the agents actually are. Even the slightest change of the system features can lead to complete different outcomes.

5.2 Limitations of the Implemented Competition Model

It was already mentioned more than once that an estimation of a competition model by using Lotka-Volterra model has some constraints. It is not always possible to project the system outcomes for every single combination of input parameters.

The most sizeable problem of every non-linear dynamic system is prediction of its outcome. If the values are adequately mixed, a purely mathematical approach would be not enough. Therefore, it is necessary to conduct experiments in order to determine output values.

By just observing the input values it is not possible to instantly determine end result for every single input setup. Reason for this is, obviously, the nature of a non-linear system. It is its biggest flaw. A mathematical analysis is another tool, but it cannot also be always performed since it can get very complex to solve the system of differential equations. This fact was already discussed in the methodology chapter. By having more than two agents in the systems, it gets exponentially harder to predict an end result. This leads us to the simulation framework as the last tool for solving the problem. It is always possible to implement a simulation framework in order to find the end result.

As far as a mathematical representation is concerned, it is not always easy to do the representation of such system, neither graphically nor analytically. This was discussed in the Section 2.3. The complexity of the equilibrium points and thus the stability of the system is increasing exponentially with the number of agents that are competing in the closed system.

In the end, we also have to mention limitations from the game theory standpoint. This model basically incorporates a pure strategy for all agents in the competition, which does not changes over time. Such a behaviour is not common in a real life scenario, since people are usually adapting their strategies to a given situation. On the other hand, implementing a mixed strategy on top of the non linear dynamic system would make the end results exponentially more complicated to predict. Since the current task is already challenging on its own, adding this layer of complexity would exceed the scope of this thesis. But it would for sure be a good idea for some future experiments.

6 Conclusion

This thesis was set to explore the concept of a competition model among numerous agents in the closed system. The results from the Chapter 4 showcased the importance of different starting parameters and strategies and their impact on the final outcome in the system. The use cases in this thesis were purely theoretical in their nature, and although they can be found or applied in real life situations, it does not mean that the end results will match. This is for certain the major weakness of most simulations, because making predictions is only possible to a certain extent. The outcome can be estimated by their dependencies, but to give a definite answer on how the outcome will look like, is unfortunately not entirely possible. But this was not the purpose of this thesis anyway. The key objective was to set the theoretical model which will represent the competition behaviour involving the same resource. The mathematical model and Python code model provided in the thesis are methodological tools used to support these hypothesis.

Even with all those limitations it is still possible to determine outcomes for the given situation. The provided experiments helped a lot by providing the closer insight of how the various parameters affect the final outcome. If the real-life scenario can be simplified to an extent where it can be represented by using Lotka-Volterra competition model, then the deviation of such system will not be as significant as with the actual experiment. And this is perfectly feasible. Even the Einstein himself said that *everything should be made as simple as possible, but no simpler*[1]. This brings us back to the beginning of the thesis. The more precisely defined the simulation framework is, the more accurate the end results will the simulation provide. In order to be able to comprehend this better, we have to bear in mind that the interaction between various competitors usually cannot be represented by a single parameter. It is more likely to be a polynomial equation of some sort, and even when that is the case, it still can not guarantee the precise output value. This divergence will decrease over the time, as it is the case with all simulations that progress in their complexity.

The conclusions made throughout this thesis were pretty clear about the restraints of the theoretical model. If a practical example would really be implemented, and the results deviate from the model significantly, the entire model would probably have to be revisited and adjusted in order to output more accurate results. Every system has several coefficients that can have an impact on the final outcomes. Predicting such outcomes is even more difficult when there are non-linear dynamic systems. The problem with such systems is that the outcome cannot be predicted by simply taking into account input values, since they can neither follow the superposition principle nor they can be directly proportional.

[1] 1

[1] More information about this and some other famous Einstein quotes on the following website: *https://championingscience.com/2013/11/10/everything-should-be-made-as-simple-as-possible-but-no-simpler/*

6.1 Achievements and Impact

Last section of this chapter covers various findings that this thesis provides to the topic based on the conducted experiments. Due to the above mentioned limitations that can occur with a given model, it is not possible to just make a general conclusion which confirms the initial hypotheses. The general summary of the experimental illustrates precisely what happens when a random competition model with some random parameters is implemented. From a mathematical standpoint this is already a significant progress, since the prediction of a non-linear equations is not an easy task. On the other hand, if some theoretical model needs to be transformed into a particular real life scenario, it is not realistic to confidently claim that an exact sequence of events will occur. This is why the final conclusion is given more in a form of what this thesis covers and what is beyond the scope of research. As with all theories, hypotheses and experiments, there is always room for improvement of the existing and forming of new ones. But those thoughts will be left for some other future works.

6.2 Future Work

Next logical step in a study like this would be to implement the theoretical findings we've come up with into some practical example. Such a system would have to be closed, with finite number of competitors, given intra- and inter-specific parameters and resources that they are competing for. By observing the output values of the system and comparing them with the theoretical model provided here, it would be possible to determine what is the magnitude of deviation from the real life situation. Such results could prove the accuracy of the implemented model, but that is already material for a completely new thesis.

Beside that, further development of the competition model would be a good place to start some future works with. As it has been already mentioned, the Lotka-Volterra model that was used here is relatively simple, with only one inter-specific parameter that connects two random agents. That kind of a relation is not very common in the real world, even within the closed systems, without any external influence. Since this feature was a major limitation for the provided models, that is the first thing that needs adjustment. If it is possible to create more complex relations between various agents (such as that those relations are dependable on more than just one parameter), then the end results would be more accurate. The "problem" with such change is that then it is not the Lotka-Volterra competition model any more, but an entirely new approach to research and experimentation.

And finally, one last thought for the end of this conclusion. Should one try developing this thesis further, s/he should not concentrate on just one aspect of the competition model. An inter-specific coefficient does not have to be the deciding factor in the model, even though it is very important. The entire simulation can be considered as a chain of many links (features). And since the chain is only as strong as it is its weakest link, more effort should be put into making the individual parts of the simulation chain and its links as strong as possible. Such way of thinking will most certainly produce better end results and more legit simulation in general. I personally have no doubts that this is going to be achieved in the forthcoming period, and can only hope that this thesis will contribute to carrying that task out.

List of Symbols

s	strategic profile for Pereto dominance
$u_s(i)$	gain of strategic profile
G	normal form of game
X	min max criterion
$p_i j$	probability of agents strategy
$v_1(p1, p2)$	expected gain of the strategy
J, K	number of pure strategies in S1 and S2 respectivly
σ	Evolutionary stable strategy
μ	Mutual attempt to invade the population
p	proportion of the population that follows the μ
K	Carrying capacity
P	Current population of the specie
r	Growth rate
$\frac{dP}{dt}$	Change of the population between two discrete time slices
H	Population density of the prey specie
V	Population density of the predator specie
R	Intrinsic rate of prey specie population increase
a	Coefficient of predation rate
b	Reproduction rate of predators per one eaten prey
M	Mortality rate of the predators
K_i	Carrying capacity of individual specie
$\alpha_i j$	interspecific coefficient
$\alpha_i j$	intraspecific coefficient
dN_1/dt	Individual change of specie per time

List of Abbreviations

NLSD	Non-linear system dynamics
GT	Game Theory
CM	Competition Model
EGT	Evolutionary Game Theory
PD	Population Dynamics
PPM	Predator Prey Model
SN	Social Networks
UA	User Attention
CSV	Comma separated values
SD	System Dynamics
LV	Lotka-Volterra
COG	Cooperative Games
NCOG	Non-cooperative Games
GWCM	Games with combined motives
PD	Pereto Dominance
NE	Nash equilibrium
MGT	Mathematical Game Theory
ES	Evolutionary Stability
LGR	Logistic Growth
LVE	Lotka-Volterra Equations
LVRM	Lotka-Volterra Mathematical Representation
LVM	Lotka-Volterra Model
LVPPM	Lotka-Volterra Predator-Prey Model
IAC	Intra-specific Coefficient
IEC	Inter-specific Coefficient

EQI	Equilibrium Isocline
CC	Carrying Capacity
SIF	Simulation framework
OOL	Object Oriented Languages
MAS	Multi Agent Systems
NAS	N-agent systems
UC	Use Case
RLS	Real-life Scenarios
PDC	Predatory Characteristic
COC	Competitive Characteristic

Bibliography

[AAM03] Daher OM. Aziz-Alaoui MA. Boundedness and global stability for a predator–prey model with modified leslie–gower and holling-type ii schemes. *Appl Math Lett*, page 1069–1075, 2003.

[AB14] Nobel Media AB. Jean tirole - facts, 2014.

[Axe84] Robert Axelrod. *The evolution of cooperation*. Basic Books, United States of America, 1984.

[Bai10] Steve Baigent. Lotka-volterra dynamics - an introduction. *UCL*, pages 1–58, March 2010.

[CR06] Christiane Clemens and Thomas Riechmann. Evolutionary dynamics in public goods games. *Computational Economics*, 28:399–420, 2006.

[DE10] Jon Kleinberg. David Easley. *Networks, Crowds, and Markets: Reasoning about a Highly Connected World*. ambridge University Press, 2010.

[EG07] Magnus Enquist and Stefano Ghirlanda. Evolution of social learning does not explain the origin of human cumulative culture. *Journal of Theoretical Biology*, 246:129–135, 2007.

[Fis30] Ronald Fisher. *The Genetic Theory of Natural Selection*. The Clarendon Press, Oxford, England, UK, 1930.

[Fri98] Daniel Friedman. On economic applications of evolutionary game theory. *J Evol Econ*, 8:15–43, 1998.

[FZ07] Jeffrey A. Fletcher and Martin Zwick. The evolution of altruism: Game theory in multilevel selection and inclusive fitness. *Journal of Theoretical Biology*, 245:26–36, 2007.

[Gib82] Robert Gibson. *A Primer in Game Theory*. Prentice Hall, United States of America, 1982.

[Har60] G Hardin. The competitive exclusion principle. *Science*, 131:1292–1297, 1960.

[Hir85] M. Hirsch. Systems of differential equations that are competitive or cooperative, ii: convergence almost everywhere. *SIAM J. Math. Anal.*, 423:16, 1985.

[Hir88] M. Hirsch. Systems of differential equations which are competitive or cooperative, iii: competing species. *Nonlinearity*, 1:51–71, 1988.

[Hir90] M. Hirsch. Systems of differential equations that are competitive or cooperative, iv: structural stability in three-dimensional systems. *SIAM J. Math. Anal.*, 1225:21, 1990.

[Hof88] Josef Hofbauer. *Evolutionary Games and Population Dynamics.* Cambridge University Press, England, UK, 1988.

[Hol59a] C. S. Holling. The components of predation as revealed by a study of small-mammal predation of the european pine sawfly. *The Canadian Entomologist*, 91.5:293–320, 1959.

[Hol59b] C. S. Holling. Some characteristics of simple types of predation and parasitism. *The Canadian Entomologist*, 91.7:385–398, 1959.

[HS08] William Harms and Brian Skyrms. Evolution of moral norms. *Oxford Handbook of Philosophy of Biology*, 2008.

[HWK53] A. W. Tucker H. W. Kuhn. *Contributions to the Theory of Games (AM-28), Volume II.* Princeton University Press, 1953.

[HWK59] A. W. Tucker H. W. Kuhn. *Contributions to the Theory of Games (AM-40), Volume IV.* Princeton University Press, 1959.

[Hyd11] David Hyde. Predator prey modeling with the lotka-volterra equations. *OMC, Lecture 27*, pages 1–18, July 2011.

[J.L25] Alfred J.Lotka. *Elements of Physical Biology.* Williams and Wilkins Company, USA, 1925.

[KN03] Tatsuya Kameda and Daisuke Nakanishi. Does social/cultural learning increase human adaptability? rogers's question revisited. *Evolution and Human Behavior*, 24:242–260, 2003.

[Kon03] Michio Kondoh. Foraging adaptation and the relationship between food-web complexity and stability. *Science*, 28:1388–1391, 2003.

[KS06] Kovacevic M. Krcevinac S., Cangalovic M. *Operaciona istraživanja 2.* Fakultet organizacionih nauka - FON, Belgrade, Serbia, 2006.

[KT50] H. W. Kuhn and A. W. Tucker. *Contirbution to the theory of games, volume II.* Annals of Mathematical Studies, United States of America, 1950.

[Lac84] Pavel Lacra. *Game Theory for control of optical networks.* Beerhauser Basel, Switzwerland, 1984.

[LD07] Martin Ziehe Lloyd Demetrius. Darwinian fitness. *Theoretical Population Biology*, 72:323–345, July 2007.

[Mar71] Lj. Martic. *Primjena matematickih metoda u ekonomskoj analizi(Zbirka zadataka).* Novinsko – izdavacki, stamparski i birotehnicki zavod, Zagreb, Croatia, 1971.

[MD57] P. Wolfe M. Dresher, A. W. Tucker. *Contributions to the Theory of Games (AM-39), Volume III.* Princeton University Press, 1957.

[MF99] Diqing Jiang Meng Fan, Ke Wang. Exsistence and global attractivity of positive periodic solutions of periodic n-species lotka-volterra competition systems with several deviating arguments. *Mathematical Biosciences*, 160:47–61, March 1999.

[NM47] Johna Von Neumann and Oscar Morgenstern. *Theory of games and Economic Behaviour*. Princeton University Press, United States of America, 1947.

[RN91] P.S: Joaq Rana N.C. *Classical Mechanics*. Tata McGrow Hill, New Delhi, 1991.

[Sel83] Reinhard Selten. Evolutionary stability in extensive two-person games. *Mathematical Social Sciences*, 5:269–363, 1983.

[Sel88] Reinhard Selten. Evolutionary stability in extensive two-person games – correction and further development. *Mathematical Social Sciences*, 16:223–266, 1988.

[SLB08] Yoav Shoham and Kevin Leyton-Brown. *Multiagent systems*. Cambridge University Press; 1 edition, England, UK, 2008.

[Sma76] S. Smale. On the differential equations of species in competition. *Journal of Mathematical Biology*, 3:5–7, 1976.

[Smi82] John Maynard Smith. *Evolution and the Theory of Games*. Cambridge University Press, England, UK, 1982.

[SP73] J.Maynard Smith and G. R. Price. The logic of animal conflict. *Nature*, 5:269–363, Novembar 1973.

[Tho84] L.C. Thomas. *Games, Theory and Applications*. Dover Publications INC., Mineola, New york, 1984.

[TJ78] Peter D. Taylor and Leo B. Jonker. Evolutionary stable strategies and game dynamics. *Mathematical Biosciences*, 40:145–156, 1978.

[Ver38] Pierre-Francois Verhulst. Notice sur la loi que la population suit dans son accroissement. *Correspondance mathématique et physique*, 10:113–121, 1838.

[WAD68] Jr. William A. Darity. *International Encyclopedia of the Social Sciences; 2nd edition*. Gale, 1968.

[Wei97] Jorgen W. Weibull. *Evolutionary Game Theory*. MIT Press, Cambridge, MA, USA, 1997.