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Properties of Random Partitions of Large Integers

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Affidavit

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly indicated all material which has been quoted either literally or by content from the sources used. The text document uploaded to TUGRAZonline is identical to the present master‘s thesis.

Date

Signature

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1 Introduction

The word *partition* has several meanings in Mathematics and most likely it will pop up any time one deals with division of some object into sub-objects. The theory of partitions has an interesting history. Some problems date back to the Middle Ages.

Leibniz was among the first mathematicians who paid attention to the developing stages in this area of mathematics. In his correspondence with Bernoulli, he asked about the number of "divulsions" of integers, in other words about the number of partitions of integers. He observed that the number 3 has three partitions, the number 4 has five partitions, the number 5 has seven partitions and the number 6 has eleven partitions. Having noticed this trend, he assumed that the number of partitions of any integer n might always be a prime number, but his assumption was quickly disproved once he computed that the number 7 has fifteen partitions. His tentative exploration of partitions though raised a lot of interesting questions.

The first discoveries of any depth were made in the eighteenth century when Euler proved several partition theorems that laid the foundations of the theory of partitions. Over the centuries a great number of mathematicians had devoted their time in a search for new identities in partition theory. The goal of Chapter 2 is to provide a brief overview of the basics regarding foundations and partition theory identities.

Let $p(n)$ denote the number of partitions of the integer n. For many years one of the most intriguing and difficult questions about partitions was to determine the asymptotic properties of $p(n)$ as n gets large. This question was finally answered by Hardy and Ramanujan in 1917, and later on by Rademacher in 1936. The details of the work of the latter are explained in Chapter 3 of this text. However, many other interesting problems in the theory of partitions remain unsolved today. One example is to find a simple criterion for deciding whether $p(n)$ is even or odd despite the good deal of effort having been expended on it. Though values of $p(n)$ have been computed for n into the billions, no pattern has been discovered to date.

For each nonnegative integer n, we can consider the uniform probability distribution on the set of partitions of n, namely to assign to each partition of n a probability of $1/p(n)$. Using this uniform distribution, one can also talk about random variables on the set of partitions and their distributions. We will merely focus on those whose values are nonnegative integers and simply call them partition statistics. We introduce in Chapter 4 some of partition statistics and important discoveries related to them.

There are several examples of pairs of partitions statistics which are identically distributed. To name a few, consider the number of even part sizes and the number of repeated part sizes; the number of consecutive even part sizes and the number of consecutive repeated part sizes; the number of part sizes that are perfect squares and the number of part sizes i whose multiplicity is greater or equal than i.

In Chapter 5, which presents the main results of the thesis, we find one more interesting example of an identically distributed pair of partition statistics. Let $X_{o,n}$ and $X_{e,n}$ denote the sums of odd and even indexed parts of a partition of n, respectively. On the other hand, let $Z_{o,n}$ and $Z_{e,n}$ denote the number of odd and even indexed parts, respectively. We prove that $X_{o,n} - X_{e,n}$ and $Z_{o,n}$ are identically distributed. We use this fact to find asymptotic expressions for $\mathbb{E}(Z_{o,n})$ and $\mathbb{E}(Z_{e,n})$. An open problem to continue the work of this thesis would be to find the limiting distributions for these statistics.

2 Introduction to Partition Theory

2.1 Basics

For the purpose of this thesis, we will be using the following definition.

Definition 1. *A partition λ of the positive integer n is a way of writing it as a sum of positive integers without regard to order. Without loss of generality, one can assume that the summands are arranged in non-increasing order. We have*

$$\lambda : \quad n = \lambda_1 + \lambda_2 + ... + \lambda_k, \qquad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k. \tag{2.1}$$

Definition 2. *The summands λ_j in (2.1) are called parts of the partition λ.*

Example 1. We will bring as an example 15 partitions of number 7 as this was the counterexample of the first conjecture of Leibniz about partitions.

$$\begin{aligned} 7 &= 7 \\ &= 6+1 = 5+2 = 4+3 \\ &= 5+1+1 = 4+2+1 = 3+3+1 = 3+2+2 \\ &= 4+1+1+1 = 3+2+1+1 = 2+2+2+1 \\ &= 3+1+1+1+1 = 2+2+1+1+1 \\ &= 2+1+1+1+1+1 \\ &= 1+1+1+1+1+1+1. \end{aligned}$$

Example 2. One way of partitioning $n = 50$ is

$$\lambda : \quad 50 = 9+6+6+5+5+5+3+3+3+2+1+1+1. \tag{2.2}$$

So, the parts of this partition are

$$\lambda_1 = 9, \lambda_2 = \lambda_3 = 6, \lambda_4 = \lambda_5 = \lambda_6 = 5, \lambda_7 = \lambda_8 = \lambda_9 = 3, \lambda_{10} = 2, \lambda_{11} = \lambda_{12} = \lambda_{13} = 1,$$

and the number of the parts of this partition is $k = 13$.

Definition 3. *The partition* λ' *such that*

$$\lambda' : \quad n = \lambda_1' + \lambda_2' + ... + \lambda_\ell' \tag{2.3}$$

is the conjugate *partition of* λ *if* λ_j' *equals the number of parts that are* $\geq j$ *in* λ.

Example 3. If λ is the partition of $n = 50$ in (2.2), the conjugate of λ would be

$$\lambda' : \quad 50 = 13 + 10 + 9 + 6 + 6 + 3 + 1 + 1 + 1. \tag{2.4}$$

The parts of λ' are

$$\begin{aligned} \lambda_1' &= 13, \\ \lambda_2' &= 10, \\ \lambda_3' &= 9, \\ \lambda_4' &= \lambda_5' = 6, \\ \lambda_6' &= 3, \\ \lambda_7' &= \lambda_8' = \lambda_9' = 1, \end{aligned}$$

and the number of parts in the conjugate partition is $\ell = 9$.

Each partition λ has a unique graphical representation called its *Ferrers Diagram*. It illustrates λ by a two-dimensional array of squares composed by λ_i squares in the $i-$th row for each $1 \leq i \leq k$. The partition of $n = 50$ in (2.2) is represented in the Figure 2.1.

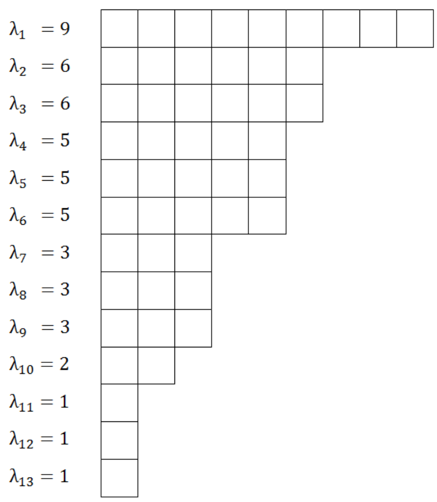


Figure 2.1: Ferrers Diagram of the partition for $n = 50$ (Example 2)

Suppose that we fix a positive integer n and λ a partition of n. It is easy to see that the Ferrers Diagram of the conjugate partition λ' of λ is obtained by reflecting λ with respect to its main diagonal. Thus conjugation defines a a one-to-one correspondence from the set of partitions of n to itself. An illustration of the previous example is given in Figure 2.2

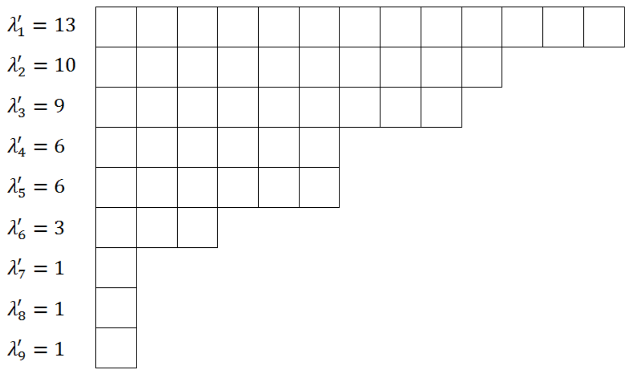


Figure 2.2: Ferrers Diagram of the conjugate partition for $n = 50$ (Example 3)

Another way to view the definition of a partition λ is through the identity

$$\lambda : \quad n = \sum_{j=1}^{n} j m_j, \tag{2.5}$$

where the integer $m_j \geqslant 0$ is called multiplicity of part j.

In Example 2 given above, we have

$$m_j = 0 \text{ for } 10 \leqslant j \leqslant 50,$$

$$m_9 = 1,\ m_8 = m_7 = 0,\ m_6 = 2,\ m_5 = 3,$$

$$m_4 = 0,\ m_3 = 3,\ m_2 = 1,\ m_1 = 3.$$

Similarly, for $\lambda^{'}$ in the Example 3, we write

$$\lambda^{'} : \quad n = \sum_{j=1}^{n} j m_j^{'}, \qquad m_j^{'} \geqslant 0. \tag{2.6}$$

In the same manner,

$$m_j^{'} = 0 \text{ for } 14 \leqslant j \leqslant 50,$$

$$m_{13}^{'} = 1,\ m_{12}^{'} = m_{11}^{'} = 0,\ m_{10}^{'} = m_9^{'} = 1,$$

$$m_8^{'} = m_7^{'} = 0,\ m_6^{'} = 2,\ m_5^{'} = m_4^{'} = 0,$$

$$m_3^{'} = 1,\ m_2^{'} = 0,\ m_1^{'} = 3.$$

Definition 4. *Let $\mathcal{P}_n$ be the set of all partitions for $n \geq 1$. Then*

$$p(n) := |\mathcal{P}_n|$$

Remark. By convention $p(0) = 1$ and $p(n) = 0$ for n negative.

In other words, the partition function $p(n)$ counts the number of unique partitions of the positive integer n. Recall that there were 7 unique partitions of 5. Thus $p(5) = 7$. Values of $p(n)$ may be found in *The Online Encyclopedia of Integer Sequences: Confessions of a Sequence Addict* by Neil J.A. Sloane [18].

n	p(n)
0	1
10	42
20	627
30	5604
40	37338
50	204226
60	966467
70	4087968
80	15796476
90	56634173
100	190569292
110	607163746
120	1844349560
130	5371315400
140	15065878135
150	40853235313
160	107437159466
170	274768617130
180	684957390936
190	1667727404093
200	3972999029388

Table 2.1: Growth of the number of partitions $p(n)$ for $n = 1, \ldots, 200$

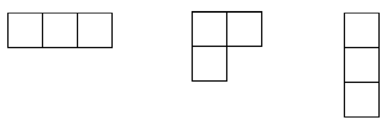
Lemma 1. *The partition function $p(n)$ is strictly increasing.*

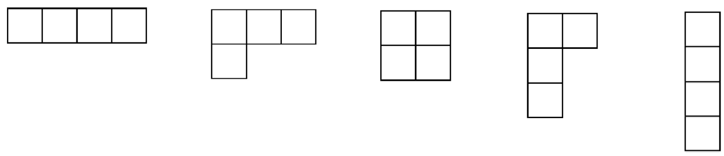
Proof. We want to show that the following holds

$$p(n) > p(n-1) \qquad \text{for all } n \geq 2.$$

We notice that for every partition of $n-1$, one can obtain a partition of n by adding a single square in a new bottom row. Conversely, every partition of n with a single square in the bottom row gives a partition of $n-1$ after we remove that square. These two procedures revert each other.

As an illustration of the above statement, we compare the partitions of the number 3 with the partitions of the number 4:





Therefore, if $q(n)$ denotes the number of partitions of n whose last part equals to 1, then

$$p(n-1) = q(n)$$

But clearly $q(n) < p(n)$ for $n \geq 2$, therefore the conclusion follows. $\square$

2.2 Generating Functions Associated to Partitions

We know that $p(n)$ counts the number of ways n can be partitioned, but how can we actually find $p(n)$? A naive way would be to simply list all the partitions, but as we will see in Chapter 3, $p(n)$ has a subexponential, but nevertheless a superpolynomial growth, making this method ineffective.

Instead, let $p(n, m)$ denote the number of partitions of n into m parts. Then clearly

$$p(n) = \sum_{m=0}^{n} p(n, m).$$

Therefore, we can compute the numbers $p(n, m)$ instead. We distinguish two different kinds of partitions of n into m parts: partitions whose last part λ_m equals to 1 and partitions whose last part λ_m does not equal to 1.

In the first case, removing the last part from the partition, we get a partition of $n-1$ in $m-1$ parts. It is easy to see that this gives a bijection between the set of partitions of n into m parts whose last part equals to 1 and the set of partitions of $n-1$ into $m-1$ parts (the inverse map being adding one square in a new last row of its Ferrers diagram if we had started with a partition of $n-1$ into $m-1$ parts).

In the second case, removing a square from each row of the partition, we obtain a partition of $n-m$ into m parts. Again, this gives a bijection between the set of partitions of n into m parts whose last part does not equal to 1 and the set of partitions of $n-m$ into m parts (the inverse map being adding one square in each row if we had started with a partition of $n-m$ into m parts). Therefore, we have that

$$p(n, m) = p(n-1, m-1) + p(n-m, m)$$

This identity allows for recursive computation of the numbers $p(n,m)$ and thus of $p(n)$ as well.

Euler also introduced generating functions, that provides with a more efficient way of computing $p(n)$. In the later chapters, they will also help us compute the partition function asymptotically.

The generating function of the partition function is the series

$$F(x) = \sum_{n=0}^{\infty} p(n)x^n,$$

where for now we simply view $F(x)$ as an element of the power series ring $\mathbb{C}[[x]]$. Then we have the following identity:

Proposition 1. *In the formal power series ring* $\mathbb{C}[[x]]$*, the following equality*

$$F(x) = \prod_{j=1}^{\infty} \frac{1}{1-x^j} \tag{2.7}$$

holds.

Proof. First, we should show that the right hand side actually makes sense as an element of $\mathbb{C}[[x]]$. Notice that at least individually, each factor of the right hand side makes sense, as $1-x^j$ is a unit in $\mathbb{C}[[x]]$ with inverse equal to $\sum_{k=0}^{\infty} x^{jk}$.

In other words, we need to show that the infinite product

$$\prod_{j=1}^{\infty} \left(\sum_{k=0}^{\infty} x^{jk} \right)$$

makes sense as an element of $\mathbb{C}[[x]]$. If we consider the partial products

$$P_n := \prod_{j=1}^{n} \left(\sum_{k=0}^{\infty} x^{jk} \right)$$

we notice that for each $n \in \mathbb{N}$, $P_{n_1} - P_{n_2}$ is a multiple of x^n for all $n_1, n_2 \geq n$. Therefore $(P_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}[[x]]$ with respect to the x-adic topology on $\mathbb{C}[[x]]$. Since $\mathbb{C}[[x]]$ is x-adically complete, i.e.

$$\mathbb{C}[[x]] \cong \varprojlim \mathbb{C}[[x]]/x^r\mathbb{C}[[x]],$$

we obtain that the infinite product is a well defined element of $\mathbb{C}[[x]]$ as well.

To show that the desired equality holds, we simply need to show that the coefficients of both sides agree. For that notice that the coefficient of x^n in right hand side agrees with the coefficient of x^n in P_n. Computing

$$\begin{aligned} P_n &= \prod_{j=1}^{n}(1 + x^j + x^{2j} + x^{3j} + \dots) \\ &= \prod_{j=1}^{n}(1 + x^j + x^{j+j} + x^{j+j+j} + \dots) \end{aligned}$$

the multiplication yields a sum of monomials of the form $x^{\sum jm_j}$. The repetitions of the monomial x^n is equal to the number of solutions of the equation

$$\sum_{j\geqslant 1} jm_j = n.$$

By (2.5), this number equals $p(n)$, as desired. □

It will also be useful in the later chapters to know about the convergence properties of $F(x)$ when regarded as a function on the complex numbers.

Proposition 2. *For every $z \in D = \{a \in \mathbb{C} : |a| < 1\}$, $F(z)$ converges absolutely and the following equality*

$$F(z) = \prod_{j=1}^{\infty} \frac{1}{1 - z^j} \tag{2.8}$$

holds.

Proof. First we need to show that both sides of the asserted equation converge. We start with the right hand side.

For that we use a fact from complex analysis which states that if $(a_n)_{n\geq 1}$ is a sequence of complex numbers such that $\sum_{n=1}^{\infty} |a_n|^2$ converges, then $\prod_{n=1}^{\infty}(1 + a_n)$ converges to a non-zero limit when $\sum_{n=1}^{\infty} a_n$ does (see [16]).

Applying the fact for $a_n = z^n$, since $\sum_{n=0}^{\infty} z^{2n}$ converges absolutely for $|z| < 1$, it follows that $\prod_{j=1}^{\infty}(1 - z^j)$ converges to a non-zero value, hence $\prod_{j=1}^{\infty} \frac{1}{1 - z^j}$ converges when $|z| < 1$.

For the left hand side, notice that for $z < |1|$ we have that

$$\sum_{i=0}^{n} p(i)|z|^i \leq \prod_{i=1}^{n} \frac{1}{1-|z|^i}$$

because if we compare the sides just formally, the left one covers all the partitions of numbers less or equal than n, while the right one covers all the partitions with parts less or equal than n. Since the right hand side converges by the above, so will the left hand side.

It remains to show that the two sides are equal. But since both of them are absolutely convergent, that means when can choose any arrangement of terms in the right hand side. Since by Proposition 1 the two sides are equal if z is replaced by a formal variable, the conclusion follows.

□

Note that in this section we use x to denote a formal variable, while z is a complex number. In the subsequent sections we do not use a distinguished notation anymore and the meaning will be clear from the context.

We are also ready to prove the following theorem of Euler, which allows us to compute recursively the values of $p(n)$ directly from smaller values of the partition function.

Theorem 1 (Euler Pentagonal Theorem)**.** *In the power series ring* $\mathbb{C}[[x]]$

$$\prod_{j=1}^{\infty}(1-x^j) = \sum_{k\in\mathbb{Z}}^{\infty}(-1)^k x^{\frac{k(3k-1)}{2}}.$$

Proof. One can use a similar argument as in the beginning of the proof of Proposition 1 to show that the left hand side of the asserted identity is well defined.

Consider the generating function in two variables

$$Q(x,y) = \prod_{j=1}^{\infty}(1+x^j y) = \sum_{n,k} q_{n,k} x^n y^k,$$

where the variable x tracks the size of a part, while the variable y tracks the number of parts. Therefore, overall $Q(x,y)$ is the 2-variable generating function tracking the partitions into distinct parts and $q_{n,k}$ is the number of partitions of n into k distinct parts.

Note that in fact not only $Q(x,y) \in \mathbb{C}[[x,y]]$, but in fact $Q(x,y) \in (\mathbb{C}[y])[[x]]$. Recall from basic commutative algebra that if $\varphi : R \to S$ is a ring homomorphism between commutative rings R and S, then φ extends to a ring homomorphism

$$\tilde{\varphi} : R[[x]] \longrightarrow S[[x]]$$
$$\sum_{n=0}^{\infty} r_n x^n \longmapsto \sum_{n=0}^{\infty} \varphi(r_n) x^n$$

of formal power series rings. Applying this construction for $R = \mathbb{C}[y], S = \mathbb{C}$ and $\varphi = \mathrm{ev}_{-1}$, where $\mathrm{ev}_{-1} : \mathbb{C}[y] \to \mathbb{C}$ is the evaluation at -1, we obtain a well defined ring homomorphism (which we also call ev_{-1})

$$\mathrm{ev}_{-1} : \mathbb{C}[y][[x]] \longrightarrow \mathbb{C}[[x]]$$
$$f(x,y) \longmapsto f(x,-1)$$

In particular, it is well defined to evaluate $Q(x,y)$ at -1. Let now $\mathcal{E}_n$ be the set of partitions of n into an even number of distinct parts, $\mathcal{O}_n$ be the set of partitions of n into an odd number of distinct parts. We then obtain that

$$\prod_{j=1}^{\infty}(1-x^j) = Q(x,-1) = \sum_{n,k\geq 0} q_{n,k} x^n (-1)^k = \sum_{n=0}^{\infty} \left(\sum_{k\text{-even}}^{\infty} q_{n,k} - \sum_{k\text{-odd}}^{\infty} q_{n,k} \right) x^n$$
$$= \sum_{n=0}^{\infty} \left(|\mathcal{E}_n| - |\mathcal{O}_n|\right) x^n.$$

We are left to show that if $t_n := |\mathcal{E}_n| - |\mathcal{O}_n|$, then

$$t_n = \begin{cases} (-1)^k, \text{ if } n = \dfrac{k(3k-1)}{2} \text{ for } k \in \mathbb{Z} \\ 0, \text{ otherwise.} \end{cases}$$

We will do so by a combinatorial bijection method.

Let $\lambda = \lambda_1 + \ldots \lambda_{2t}$ be a partition in $\mathcal{E}_n$. Then we have that $\lambda_1 > \lambda_2 > \ldots > \lambda_{2t}$. Let s be the largest integer such that $\lambda_s = \lambda_1 - s + 1$. If $\lambda_{2t} \leq s$, remove the last part λ_{2t} and add a unit to each $\lambda_1, \ldots, \lambda_s$. If $\lambda_{2t} > s$, then remove 1 unit from each $\lambda_1, \ldots, \lambda_s$ and add a part with s units after λ_{2t}. Perform an analogous process on $\mathcal{O}_n$. This yields a bijection between $\mathcal{E}_n$ and $\mathcal{O}_n$, except for the following cases:

1. $\lambda_{2t} = s+1$ and $s = 2t$. Then Then $|\mathcal{E}_n| - |\mathcal{O}_n| = 1$.

$$n = \sum_{i=1}^{2t}(2t+i) = 6t^2 + t.$$

Then $n = \dfrac{k(3k-1)}{2}$ for $k = -2t$ (the other possibility for k, namely $\dfrac{6t-1}{3}$ is not an integer). Indeed $|\mathcal{E}_n| - |\mathcal{O}_n| = 1 = (-1)^{-2t} = (-1)^k$ in this case. An analogous case occurs on the $\mathcal{O}_n$ side.

2. $\lambda_{2t} = s$ and $s = 2t$. Then $|\mathcal{E}_n| - |\mathcal{O}_n| = 1$ and

$$n = \sum_{i=0}^{2t-1}(2t+i) = 6t^2 - t.$$

Hence $n = \dfrac{k(3k-1)}{2}$ for $k = 2t$. and indeed $|\mathcal{E}_n| - |\mathcal{O}_n| = 1 = (-1)^{2t} = (-1)^k$ in this case. An analogous case occurs on the $\mathcal{O}_n$ side again.

□

Corollary 1. *If $n \geq 1$, and defining $p(n)$ to be zero for $n \leq 0$, then*

$$p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^{k+1} p\left(n - \frac{k(3k-1)}{2}\right). \tag{2.9}$$

Proof. The method of proof in Proposition 1 also shows that in the power series ring $\mathbb{C}[[x]]$ it does not actually matter with respect to which order we are taking the factors in the product $\prod_{j=1}^{\infty}(1-x^j)$ Therefore, indeed

$$F(x)\prod_{j=1}^{\infty}(1-x^j) = 1.$$

Expanding this, we obtain

$$\begin{aligned} 1 &= \left(\sum_{n=0}^{\infty} p(n)x^n\right)\left(\sum_{k\in\mathbb{Z}}(-1)^k x^{\frac{k(3k-1)}{2}}\right) \\ &= \sum_{k\in\mathbb{Z}}\left(\sum_{n\geq 0}(-1)^k x^{\frac{k(3k-1)}{2}} p(n)x^n\right) \\ &= \sum_{k\in\mathbb{Z},n\geq 0} (-1)^k p(n) x^{n+\frac{k(3k-1)}{2}} \end{aligned}$$

Let $n' = n + \dfrac{k(3k-1)}{2}$. Then

$$\begin{aligned}1 &= \sum_{k\in\mathbb{Z},n\geq 0} (-1)^k p(n) x^{n'} \\ &= \sum_{n'\geq 0}\left(\sum_{k\in\mathbb{Z}}(-1)^k p\left(n' - \frac{k(3k-1)}{2}\right)\right) x^{n'}.\end{aligned}$$

Compare the coefficients of x^n, $n \geq 1$.

$$\sum_{k\in\mathbb{Z}}(-1)^k p\left(n - \frac{k(3k-1)}{2}\right) = 0.$$

Then

$$p(n) + \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^k p\left(n - \frac{k(3k-1)}{2}\right) = 0.$$

In other words,

$$p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^k p\left(n - \frac{k(3k-1)}{2}\right),$$

as desired. □

2.3 Generating Functions Associated to Combinatorial Classes

In this section we define generating in the broader sense of combinatorial classes. We will explore some basic constructions which help us compute more complicate generating functions from the knowledge of simpler ones. Generating functions can allow us to easily prove identities involving partition numbers which are difficult to prove if one uses the combinatorial bijection method from before.

Definition 5. *A* combinatorial class *is a pair* $(\mathcal{C}, |\cdot|_{\mathcal{C}})$*, where* $\mathcal{C}$ *is a finite or countably infinite set and* $|\cdot|_{\mathcal{C}}$ *is a function from* $\mathcal{C}$ *into the nonnegative integers, called a* size function*, such that for every* $n \in \mathbb{N}_{\geq 0}$*, we have that* $|\cdot|_{\mathcal{C}}^{-1}(n)$ *is finite or in other words, the number of elements of any given size is finite.*

Throughout the section, we will denote a combinatorial class $(\mathcal{C}, |\cdot|_{\mathcal{C}})$ simply by $\mathcal{C}$ and we also let $\mathcal{C}_n$ denote the set of elements in $\mathcal{C}$ that are of size n.

Definition 6. *The* counting sequence *of a combinatorial class* $\mathcal{C}$ *is the sequence of integers* $(c_n)_{n\geq 0}$ *where for every* $n \in \mathbb{N}_{\geq 0}$*,* c_n *is the cardinality of* $\mathcal{C}_n$ *or in other words, the number of objects in the class* $\mathcal{C}$ *that have size* n*.*

Definition 7. *An isomorphism between two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ is a bijection f from $\mathcal{A}$ to $\mathcal{B}$ such that*

$$|a|_{\mathcal{A}} = |f(a)|_{\mathcal{B}}$$

for every $a \in \mathcal{A}$. In this case, one says that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic and write $\mathcal{A} \cong \mathcal{B}$.

It is easy then to show that two combinatorial classes are isomorphic precisely when the two classes have identical counting sequences.

Definition 8. *The generating function of a sequence (c_n) is the formal power series*

$$C(x) = \sum_{n=0}^{\infty} c_n x^n.$$

The generating function of a combinatorial class $\mathcal{C}$ is the generating function of its counting sequence.

Equivalently, one can think of the generating function of the class $\mathcal{C}$ as the sum

$$C(x) = \sum_{\alpha \in \mathcal{C}} x^{|\alpha|_{\mathcal{C}}}.$$

We then see that the variable x keeps track of the size function of the respective combinatorial class.

Following the notations and terminology of [7], here are some basic constructions generating functions associated to combinatorial classes:

- Let $\mathcal{E}$ denote the **neutral class**. The underlying set of this class consists of a single element which we define to have size 0. The generating function of $\mathcal{E}$ is simply $E(x) = 1$.
- Let $\mathcal{Z}$ denote the **atomic class**. The underlying set of this class consists of a single element whose size we define to be 1. The generating function associated to $\mathcal{Z}$ is $Z(x) = x$.
- Let $\mathcal{A}$ and $\mathcal{B}$ be two combinatorial classes whose underlying sets are assumed to be disjoint. Their **combinatorial sum** $\mathcal{A} + \mathcal{B}$ is defined to be the combinatorial class $\mathcal{C}$ whose underlying set is $\mathcal{A} \amalg \mathcal{B}$ and whose size function $|\cdot|_{\mathcal{C}}$ is defined in the following way. For $c \in \mathcal{A} \amalg \mathcal{B}$

$$|c|_{\mathcal{C}} = \begin{cases} |c|_{\mathcal{A}}, & \text{if } c \in \mathcal{A} \\ |c|_{\mathcal{B}}, & \text{if } c \in \mathcal{B}. \end{cases}$$

  As the notation would suggest, the generating function of $\mathcal{A} + \mathcal{B}$ indeed equals the sum of the generating functions of $\mathcal{A}$ and $\mathcal{B}$:

$$C(x) = \sum_{c \in \mathcal{A} \amalg \mathcal{B}} x^{|c|} = \sum_{\alpha \in \mathcal{A}} x^{|\alpha|} + \sum_{\beta \in \mathcal{B}} x^{|\beta|} = A(x) + B(x).$$

- For arbitrary combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ we denote by $\mathcal{A} \times \mathcal{B}$ their **Cartesian product**. Its underlying set is the set theoretic Cartesian product

$$\mathcal{A} \times \mathcal{B} = \{(\alpha, \beta) | \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}.$$

  We define the size of a pair (α, β) to be $|\alpha|_{\mathcal{A}} + |\beta|_{\mathcal{B}}$. Again, as the name would suggest, the generating function of $\mathcal{A} \times \mathcal{B}$ indeed equals the product of the generating functions of $\mathcal{A}$ and $\mathcal{B}$ since

$$\sum_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}} x^{|(\alpha,\beta)|} = \sum_{(\alpha,\beta) \in \mathcal{A} \times \mathcal{B}} x^{|\alpha|+|\beta|} = \left(\sum_{\alpha \in \mathcal{A}} x^{|\alpha|} \right) \left(\sum_{\beta \in \mathcal{B}} x^{|\beta|} \right) = A(x)B(x).$$

- For a combinatorial class $\mathcal{A}$ with no elements of size 0, we let $\mathrm{SEQ}(\mathcal{A})$ denote the combinatorial class whose underlying set is the set of all finite sequences with elements from $\mathcal{A}$. The size of such a finite sequence is defined to be the sum of the sizes of the elements of the sequence. Using our previous constructions, we see that

$$\mathrm{SEQ}(\mathcal{A}) = \mathcal{E} + \mathcal{A} + (\mathcal{A} \times \mathcal{A}) + (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) + \dots.$$

  Therefore its generating function equals

$$C(x) = 1 + A(x) + A^2(x) + A^3(x) + \dots = \frac{1}{1 - A(x)}.$$

- For a combinatorial class $\mathcal{A}$ with no elements of size 0, consider instead the multiset version $\mathrm{MSET}(\mathcal{A})$, as opposed to the previous construction $\mathrm{SEQ}(\mathcal{A})$. In other words, the underlying set will be instead the collection of all multisets with finitely many elements from $\mathcal{A}$.

  Given any finite multiset from $\mathcal{A}$, we can list its elements in a way such that it is a concatenation of finitely many sequences in each of which one element is repeated an arbitrary finite number of times. Therefore, we have the following equality in terms of combinatorial classes:

$$\mathrm{MSET}(\mathcal{A}) = \prod_{\alpha \in \mathcal{A}} \mathrm{SEQ}(\{\alpha\}).$$

  Therefore, the generating function of $\mathrm{MSET}(\mathcal{A})$ is equal to

$$C(x) = \prod_{\alpha \in \mathcal{A}} \frac{1}{1 - x^{|\alpha|}} = \prod_{n \geq 1} \left(\frac{1}{1 - x^n} \right)^{a_n},$$

  since a_0 was assumed to be 0.

Example 4 (Integer partitions)**.** Let $\mathcal{I}$ denote the combinatorial class whose underlying set is the of all positive integers and the size function $|n|_{\mathcal{I}} = n$ for all $n \geq 1$. The generating function of $\mathcal{I}$ is then

$$I(x) = \sum_{n\geq 1} x^n = \frac{x}{1-x}.$$

Comparing with Definition 1, we note that integer partitions can be regarded as as multisets of nonnegative integers, which means that

$$\mathcal{P} = \mathrm{MSET}(\mathcal{I}).$$

Its generating function then equals

$$F(x) = \prod_{n=1}^{\infty} (1-x^n)^{-I_n}$$

$I_n = 1$ for $n \geq 1$ because is exactly on object in $\mathcal{I}$ for each size $n \geq 1$ (namely, n itself), we have

$$F(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n}.$$

which agrees with the product expansion from before.

3 Rademacher's Series

Using the generating function $F(x)$ of the partition numbers defined in Chapter 2, Hardy and Ramanujan [10] also managed to find an asymptotic formula for $p(n)$. It is of the form

$$p(n) = \sum_{k<\alpha\sqrt{n}} P_k(n) + O(n^{-1/4}).$$

for a constant α. The dominant term of the sum is $P_1(n)$, for which

$$P_1(n) \sim \frac{e^{\sqrt{\frac{2n}{3}}\pi}}{4n\sqrt{3}}, \qquad \text{as } n \to \infty$$

holds. The other terms $P_i(n)$ are asymptotic to similar expressions, but whose constants in the exponential are smaller. Nevertheless, this represents just an approximation for the partition numbers and is not an exact formula because of the error term $O(n^{-1/4})$.

Instead, Hans Rademacher was able to obtain an exact formula for $p(n)$ at the expense of allowing infinitely many terms. He obtained an expansion

$$p(n) = \sum_{k=1}^{\infty} R_k(n).$$

which is also known under the name of *Rademacher's series*. Since Rademacher's approach was an adjustment of the approach of Hardy and Ramanujan, it is of no surprise that the terms $R_k(n)$ are only slightly different from the terms $P_k(n)$. The main goal of this chapter is to prove Rademacher's exact formula, to compute the exact forms of these terms $R_k(n)$ (which are also called *Rademacher terms*) and to find what are they asymptotically equivalent to.

3.1 Outline of the Proof

In this section, we present an outline of our steps. Using the notations of Proposition 1, we know that

$$\frac{F(x)}{x^{n+1}} = \sum_{k=0}^{\infty} \frac{p(k)x^k}{x^{n+1}},$$

for each $n \geq 0$ and every x in the punctured open unit disk in the complex plane. Therefore, as a function on the complex plane $\frac{F(x)}{x^{n+1}}$ is meromorphic and has a pole of order $n+1$ at the origin with residue $p(n)$. Using Cauchy's residue theorem we obtain that

$$p(n) = \frac{1}{2\pi i} \int_C \frac{F(x)}{x^{n+1}} dx$$

for any positively oriented simple closed curve C lying inside the open unit disk enclosing 0. The main idea, which goes under the name of the *circle method* in literature, is to choose a special contour C which lies near the roots of unity. These roots of unity are singularities of $F(x)$, since we showed in Chapter 2 that the latter is the infinite product of the reciprocals of the polynomials $x^i - 1$, for all $i \geq 1$.

It turns that the contours C_N we are interested in are obtained as a byproduct of the theory of Farey fractions and Ford circles. For a fixed $N \in \mathbb{N}$, the contour C_N is a glueing of circular arcs $C_{h,k}$, one for each pair (h,k) such that $0 \leq h < k \leq N$ and h, k are coprime. This arc $C_{h,k}$ lies near the point $e^{\frac{2\pi i h}{k}}$, which is a root of unity. We explore the behaviour of the integral over this contour as $N \to \infty$. Also, clearly the integral over C_N can be decomposed as a sum of integrals over the $C_{h,k}$'s:

$$\int_C = \sum_{k=1}^{N} \sum_{\substack{0 \leq h < k \\ (h,k)=1}}^{k-1} \int_{C_{h,k}}.$$

Using the Dedekind eta function and its relation to $F(x)$ we show that for each such pair (h,k) there exists an elementary function $\Psi_{h,k}(x)$ which behaves in the same way as the integrand around the singularity $e^{\frac{2\pi i h}{k}}$. The sum of the integrals of $\Psi_{h,k}$ where k is fixed and h varies equals the desired Rademacher term $R_k(n)$.

3.2 The Modular Group

As we already mentioned in our outline, we require some basics of the theory of modular forms and modular functions for obtaining our asymptotic formula for $p(n)$. In the following we explore the basic properties of Möbius transformations and define the modular group.

Consider a transformation f on the complex plane which has the form

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are complex numbers. We want to encode the set of these transformations as an algebraic object. Note that if $w \neq z \in \mathbb{C}$, then

$$f(w) - f(z) = \frac{(ad-bc)(w-z)}{(cw+d)(cz+d)},$$

therefore f is a constant function precisely when $ad - bc = 0$. We are not interested in such transformations, so from now on we assume that $ad \neq bc$. We call these functions *Möbius transformations.* Clearly then in this case, when $c \neq 0$ f, is a meromorphic function on the complex plane with a simple pole at $z = -\frac{d}{c}$, otherwise it is holomorphic on $\mathbb{C}$.

A natural way to encode these transformations is by means of matrices, where the matrix associated to f in this case would be

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

However, note that different matrices may describe the same transformation. For example if $a, b, c, d \in \mathbb{R}$ with $ad - bc = k \neq 0$, we can also write our f as

$$f(z) = \frac{\frac{a}{\sqrt{|k|}}z + \frac{b}{\sqrt{|k|}}}{\frac{c}{\sqrt{|k|}}z + \frac{d}{\sqrt{|k|}}}$$

and if $a' = \frac{a}{\sqrt{|k|}}, b' = \frac{b}{\sqrt{|k|}}, c' = \frac{v}{\sqrt{|k|}}$ and $d' = \frac{d}{\sqrt{|k|}}$, then a', b', c' and d' satisfy $a'd' - b'c' = 1$.

We will require from now that $ad - bc = 1$ and that our coefficients a, b, c and d are integers. We will call the set of corresponding Möbius transformations the *modular group.* We will also denote the modular group by Γ.

To see that the set of such Möbius transformations do actually form a group, note again that each transformation can be written as a 2×2 matrix with integer entries of determinant 1.

Then two different matrices of this kind describe the same transformation on the complex plane if and only one matrix is the negative of the other. In other words, on the level of sets

$$\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})/\sim$$

where $A \sim B$ in $\mathrm{SL}_2(\mathbb{Z})$ iff $A = -B$.

It is clear that the matrices in $\mathrm{SL}_2(\mathbb{Z})$ do form a group and $H := \{I_2, -I_2\}$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Therefore

$$\Gamma \simeq \mathrm{SL}_2(\mathbb{Z})/H$$

is also a group.

3.3 Farey Fractions

As we mentioned in the outline there are two main ingredients that were used by Rademacher in order to define the desired the desired contour of integration. One of them is the theory of Farey fractions. In the following, by a *reduced* fraction we mean a fraction in which the numerator and denominator are coprime.

Definition 9. *Let $n \in \mathbb{N}$. A fraction $\frac{h}{k}$ is called a Farey fraction of order n if $\frac{h}{k}$ is reduced and*

$$0 \leq \frac{h}{k} \leq 1, \ 0 \leq h \leq k \leq n.$$

We will denote the set of Farey fractions of order n by F_n. In other words

$$F_n = \left\{ \frac{h}{k} : 0 \leq h \leq k \leq n; \ (h,k) = 1 \right\}.$$

Furthermore, we will always assume that the elements of F_n are listed in an increasing order, which allows to talk about consecutive Farey fractions of a given order for example.

Example 5. The following are the sets of Farey fractions of order n for $1 \leq n \leq 5$:

$$F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$$
$$F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$$
$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$$
$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$
$$F_5 = \left\{ \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1} \right\}.$$

It is easy to see then that by definition, we have that $F_n \subset F_{n+1}$, for all $n \in \mathbb{N}$.

Throughout the section we will always assume that a, b, c, d are nonnegative integers.

Definition 10. *If $\frac{a}{b}$ and $\frac{c}{d}$ are two reduced fractions, then their mediant is defined to be the fraction $\frac{a+c}{b+d}$.*

However, note that the definitions that we encounter in this section depend on the specific choice of the fraction representation. In other words, these terms are not well-defined, if we would consider them as definitions on $\mathbb{Q}$. Also, note that the mediant of two reduced fractions need not be reduced as well. We will later see though that in some specific case it is.

Lemma 2. *If $\frac{a}{b}$ and $\frac{c}{d}$ are two distinct reduced fractions, then their mediant $\frac{a+c}{b+d}$ lies between them.*

Proof. Assume without loss of generality that

$$\frac{a}{b} < \frac{c}{d}$$

Then we have that $ad < bc$. By direct computation

$$\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc - ad}{b(b+d)} > 0$$

since $bc - ad > 0$. Also

$$\frac{c}{d} - \frac{a+c}{b+d} = \frac{bc - ad}{d(b+d)} > 0$$

and we are done □

Lemma 3. *Suppose that $\frac{a}{b}$ and $\frac{c}{d}$ are fractions such that $0 \leq \frac{a}{b} < \frac{c}{d} \leq 1$ and let $n \in \mathbb{N}$. Suppose that $bc - ad = 1$ and that*

$$\max(b, d) \leq n \leq b + d - 1.$$

Then $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive elements in F_n.

Proof. Since $bc - ad = 1$, the fractions $\frac{a}{b}$ and $\frac{c}{d}$ are reduced.

Since $\max(b, d) \leq n$, it means that both of them are frac in F_n. We are left to prove that they are consecutive in F_n. Assume by absurd that there is another reduced fraction $\frac{h}{k} \in F_n$ between them. In other words assume

$$\frac{a}{b} < \frac{h}{k} < \frac{c}{d}.$$

where $(h, k) = 1$ and $0 \le h \le k \le n$. We have that $bc - ad = 1$, therefore

$$\begin{aligned} k = (bc - ad)k &= b(ck) - d(ak) + (d(bh) - b(dh)) \\ &= b(ck - dh) + d(bh - ak). \end{aligned} \tag{3.1}$$

Since $\frac{a}{b} < \frac{h}{k} < \frac{c}{d}$, we obtain that $ck - dh \ge 1$ and $bh - ak \ge 1$, which shows that $k \ge b + d$.

In other words, any fraction $\frac{h}{k}$ between $\frac{a}{b}$ and $\frac{c}{d}$ has denominator at least equal to $b + d$.

Since in our case, $n \le b + d - 1$, it follows that $k \ge n + 1$, which contradicts the assumption that $\frac{h}{k} \in F_n$. □

What is remarkable is that the converse of this lemma holds as well.

Theorem 2. *Suppose that $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive terms of $\mathcal{F}_n$, with*

$$\frac{a}{b} < \frac{c}{d}.$$

Then

(i) $bc - ad = 1$

(ii) $n \le b + d - 1$.

Proof. (i) We induct on n. The theorem clearly holds true for $n = 1$, so we are left to prove the induction step. We assume it is true for $\mathcal{F}_n$ and prove it for $\mathcal{F}_{n+1}$. Suppose that $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive in $\mathcal{F}_n$. If they are still consecutive in $\mathcal{F}_{n+1}$, then there is nothing to show about them because they satisfy $ad - bc = 1$ from the induction hypothesis.

Suppose that they are not consecutive anymore in $\mathcal{F}_{n+1}$ and assume $\frac{e}{f}$ is a fraction such that

$$\frac{a}{b} < \frac{e}{f} < \frac{c}{d}$$

in $\mathcal{F}_{n+1}$, if such a fraction even exists at all. Then denote the differences of cross multiplications by $eb - fa = s > 0$ and $fc - ed = t > 0$. By the induction hypothesis we know that $bc - ad = 1$, which allows us to solve the above system in integers for the unknowns e and f. By basic linear algebra, we obtain:

$$e = at + cs \quad \text{and} \quad f = bt + ds.$$

Also then s and t must be coprime since e and f are.

Consider the set

$$S := \left\{ \frac{\mu a + \lambda c}{\mu b + \lambda d} : \lambda, \mu \ge 1, \gcd(\lambda, \mu) = 1 \right\}.$$

So far we showed that every reduced fraction between $\frac{a}{b}$ and $\frac{c}{d}$ must come from S. Moreover, every fraction in S is reduced, because a common divisor of $\mu a + \lambda c$ and $\mu b + \lambda d$ would be a common divisor of

$$-d(\mu a + \lambda c) + c(\mu b + \lambda d) = (bc - ad)\mu = \mu$$

and

$$b(\mu a + \lambda c) - a(\mu b + \lambda d) = (bc - ad)\lambda = \lambda$$

which is 1. Since every fraction in S is reduced and strictly between $\frac{a}{c}$ and $\frac{b}{d}$, and since none of them was in $\mathcal{F}_n$ by the consecutivity assumption, it means every denominator of a fraction in $\mathcal{F}_n$ is strictly greater than n.

Since we also proved that one of these fractions does lie in $\mathcal{F}_{n+1}$, it follows that one of these fractions must have denominator exactly $n+1$. Moreover, this must then be true for the fraction with the smallest denominator possible in S, since all of them exceed n anyway.

But the fraction with the smallest possible denominator is clearly the one for which $\mu = \lambda = 1$, in other words

$$\frac{a+c}{b+d}.$$

But then since $b + d = n + 1$ it follows that no other fractions in S can lie in $\mathcal{F}_{n+1}$. But then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

must be consecutive in $\mathcal{F}_{n+1}$ and clearly for these the successive cross differences are 1, because

$$(a + c)b - a(b + d) = bc - ad = 1 \text{ and } c(b + d) - d(a + c) = bc - ad = 1$$

and we are done.

(ii) Assume the contrary and suppose that $n \geq b + d$. We claim then that the mediant $\frac{a+c}{b+d}$ is also a fraction in F_n. Since by Lemma 2 this mediant lies between $\frac{a}{b}$ and $\frac{c}{d}$, this would contradict the consecutivity assumption of these two fractions.

We already assumed that $n \geq b+d$. So it suffices to check that the fraction $\frac{a+c}{b+d}$ is reduced. By part (i) we know that

$$c(b + d) - (a + c)d = bc - ad = 1$$

Therefore $(a + c, b + d) = 1$ and the fraction is reduced. □

3.4 Ford Circles

The other ingredient necessary for defining our desired contours is the theory of Ford circles, which in fact was missing from the initial approach of Rademacher, but rather appeared in a later refinement of it. There is an element interplay between Ford circles and the theory of Farey, as we will see later in the chapter.

Definition 11. *Let $\frac{h}{k}$ be a reduced fraction. The Ford circle of $\frac{h}{k}$ is the circle in the complex plane whose center is $\frac{h}{k} + i\frac{1}{2k^2}$ and radius $\frac{1}{2k^2}$. We denote this Ford circle by $C(h,k)$.*

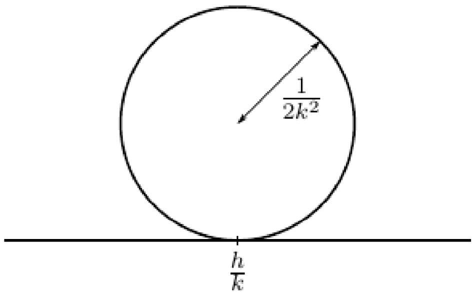


Figure 3.1: The Ford circle $C(h,k)$

Lemma 4. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be two distinct reduced fractions. The Ford circles $C(a,b)$ and $C(c,d)$ are either tangent to each other or they do not intersect at all. Moreover, $C(a,b)$ and $C(c,d)$ are tangent if and only if $bc - ad = \pm 1$.*

Proof. Let D denote the distance between the centers of $C(a,b)$ and $C(c,d)$. Then using the coordinate expressions of these centers we find that

$$D^2 = \left(\frac{a}{b} - \frac{c}{d}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2.$$

On the other hand, the square of the sum of their radii is

$$\left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2.$$

Subtracting this out of D^2, we obtain that

$$D^2 - \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2 = \left(\frac{ad-bc}{bd}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2 - \left(\frac{1}{2b^2} + \frac{1}{2d^2}\right)^2$$
$$= \frac{(ad-bc)^2 - 1}{b^2 d^2}.$$

Because $\frac{a}{b}$ and $\frac{c}{d}$ are distince, $ad - bc \neq 0$, thus the above difference is nonnegative. Furthermore, it vanishes if and only if $|ad - bc| = 1$. These is equivalent to our geometric formulation of the lemma. □

Corollary 3. *Let $\frac{h}{k}$ and $\frac{h'}{k'}$ be consecutive Farey fractions in F_N for a fixed $N \in \mathbb{N}$. Then their Ford circles $C(h,k)$ and $C(h',k')$ are tangent.*

Proof. This follows immediately from Theorem 2 and Lemma 4. □

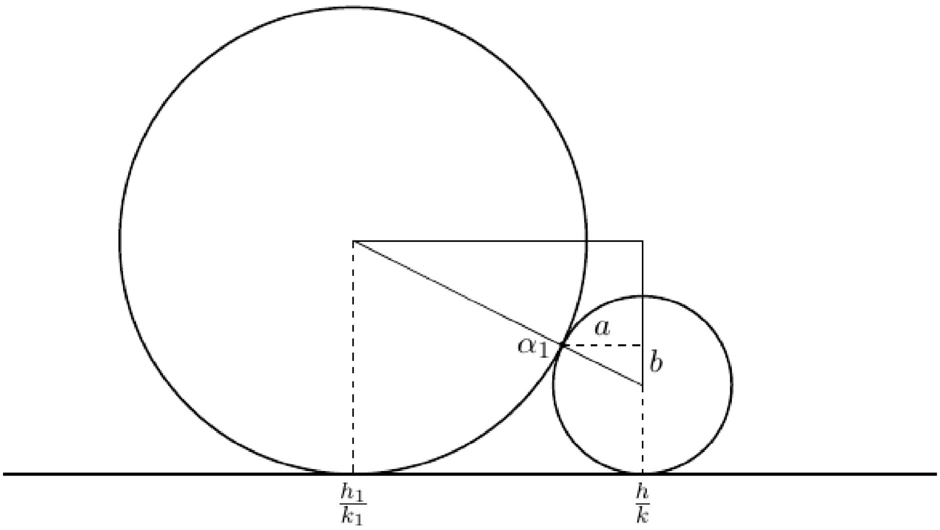


Figure 3.2: Ford Circles $C(h,k)$ and $C(h_1,k_1)$ tangent at α_1

Now we further compute the coordinate expressions of these points of tangency for Ford circles associated to Farey fractions.

Theorem 4. *Let $\dfrac{h_1}{k_1} < \dfrac{h}{k} < \dfrac{h_2}{k_2}$ be consecutive Farey fractions in F_N for a fixed $N \in \mathbb{N}$. The points of tangency of $C(h,k)$ with $C(h_1,k_1)$ and $C(h_2,k_2)$ are*

$$\alpha_1 = \frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)} + \frac{i}{k^2+k_1^2}$$

and

$$\alpha_2 = \frac{h}{k} - \frac{k_2}{k(k^2+k_2^2)} + \frac{i}{k^2+k_2^2},$$

respectively. In fact, α_1 lies on the circle whose diameter is the segment joining $\dfrac{h_1}{k_1}$ and $\dfrac{h}{k}$ in the complex plane.

Proof. We will focus on finding the coordinate expression for the point of intersection between $C(h,k)$ with $C(h_1,k_1)$, the other one being entirely similar. For the proof we will refer to the picture in Figure 3.2, for which we will assume without loss of generality that $k_1 \geq k$, the other case being completely analogous.

We will make use of similarity of two triangles from the figure. The first one is the large right triangle whose hypotenuse is the segment joining the two centers. The other one is the smaller right triangle whose hypotenuse is the segment joining the center of $C(h,k)$ with α_1.

In the large triangle all the side lengths are known. The hypotenuse has length $\frac{1}{2k_1^2} + \frac{1}{2k^2}$, the vertical side length is $\frac{1}{2k_1^2} - \frac{1}{2k^2}$, while the horizontal side length is $\frac{h}{k} - \frac{h_1}{k_1}$.

For the smaller triangle, we only know so far that the hypotenuse has length $\frac{1}{2k^2}$. We assume that the horizontal and vertical side lengths of it are a and b, respectively.

Using their similarity, we find that

$$\frac{a}{\frac{h}{k} - \frac{h_1}{k_1}} = \frac{\frac{1}{2k^2}}{\frac{1}{2k_1^2} + \frac{1}{2k^2}} = \frac{1}{2k^2} \cdot \frac{2k_1^2k^2}{k^2 + k_1^2} = \frac{k_1^2}{k^2 + k_1^2}$$

By Theorem 2 we know that $hk_1 - h_1k = 1$. This allows us to finally compute that

$$\begin{aligned} a &= \frac{k_1^2}{k^2 + k_1^2} \cdot \frac{hk_1 - h_1k}{kk_1} \\ &= \frac{k_1}{k(k^2 + k_1^2)} \end{aligned}$$

For the vertical side length b, from the triangle similarity we know that

$$\frac{b}{\frac{1}{2k_1^2} - \frac{1}{2k^2}} = \frac{k_1^2}{k^2 + k_1^2},$$

Therefore

$$b = \frac{k_1^2}{k^2 + k_1^2} \cdot \frac{k^2 - k_1^2}{2k^2k_1^2} = \frac{k^2 - k_1^2}{2k^2(k^2 + k_1^2)}.$$

It is immediate from Figure 3.2 that

$$\alpha_1 = \left(\frac{h}{k} - a\right) + i\left(\frac{1}{2k^2} + b\right) \tag{3.2}$$

Using our computations for a and b, we then find that

$$\begin{aligned}\alpha_1 &= \left(\frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)}\right) + i\left(\frac{1}{2k^2} + \frac{k^2-k_1^2}{2k^2(k^2+k_1^2)}\right)\\ &= \frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)} + i\frac{2k^2}{2k^2(k^2+k_1^2)}\\ &= \frac{h}{k} - \frac{k_1}{k(k^2+k_1^2)} + \frac{i}{k^2+k_1^2}.\end{aligned}$$

For the last statement, it simply to show that the angle formed by $\frac{h}{k}$, α_1 and $\frac{h_1}{k_1}$ is a right angle. This follows from a simple angle chase. □

3.5 Rademacher's Series

Recall from Chapter 2 that we showed that the generating function of $p(n)$ satisfied the identity

$$F(x) = \prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n)x^n \tag{3.3}$$

Moreover, we showed that for each x in the open unit disk in the complex plane, $F(x)$ converges absolutely and that the above equation holds not only in the power series ring $\mathbb{C}[[x]]$, but also whenever we plug in any complex number in the open unit disk.

As a matter of fact, the series $F(x)$ has radius of convergence 1. In other words, we also claim that $F(x)$ diverges for $x \in \mathbb{C}$ with $|x| > 1$. This holds true because by Lemma 1 of Chapter 2, $p(n)$ is unbounded and thus $p(n)|x|^n$ is unbounded for $|x| > 1$.

Recall from the outlined strategy, when dividing $F(x)$ by x^{n+1} we obtain

$$\frac{F(x)}{x^{n+1}} = \sum_{k=0}^{\infty} \frac{p(k)x^k}{x^{n+1}}, \qquad \text{for } n \geq 0.$$

which is a Laurent power series expansion at the origin. The origin is pole of order $n+1$ and the residue of this expansion or in other words, the coefficient of x^{-1} is $p(n)$. Therefore, applying Cauchy's residue theorem , one finds that

$$p(n) = \frac{1}{2\pi i}\int_C \frac{F(x)}{x^{n+1}}dx. \tag{3.4}$$

for any contour C enclosing the origin. In the following, we will explore an intelligent choice for a contour C, derived from the theory of Farey fractions and Ford circles. Of fundamental importance, is also the behaviour of $F(x)$ near the complex roots of unity. These roots of unity are singularities of $F(x)$ by the equation 3.3.

3.5.1 The Rademacher path $P(N)$

Fix $N \in \mathbb{N}$. We start by defining a path $P(N)$ in the complex plane rather than the unit disk. Later after applying an appropriate transformation this will become a closed path in the open unit disk which will enclose the origin.

Consider F_N, the set of Farey fractions of order N. For each of these fractions, consider their respective Ford circles. Recall that these circles are tangent, whenever they correspond to consecutive fractions. Note that all of these Ford circles, with the exception of $C(0,1)$ and $C(1,1)$ lie within the vertical strip

$$\mathcal{V} = \{\tau : 0 \leq \mathrm{Re}(\tau) < 1, \mathrm{Im}(\tau) > 0\}$$

inside the upper half plane $\mathcal{H} = \{\tau : \ \mathrm{Im}(\tau) > 0\}$. For $C(0,1)$ and $C(1,1)$, consider only their portions which lie inside $\mathcal{V}$.

For a Farey fraction $\frac{h}{k}$ in F_N, different from $\frac{0}{1}$ and $\frac{1}{1}$, assume that its neighbours in F_N are $\frac{h_1}{k_1}$ and $\frac{h_2}{k_2}$, in other words that

$$\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$$

are consecutive Farey fractions in F_N. As before, let α_1 denote the intersection of $C(h,k)$ with $C(h_1,k_1)$ and α_2 denote the intersection of $C(h,k)$ with $C(h_2,k_2)$. Since α_1 and α_2 are specific to the circle $C(h,k)$, we will also denote them by $\alpha_1(h,k)$ and $\alpha_2(h,k)$, respectively. We let $\gamma(h,k)$ denote the arc on $C(h,k)$ between α_1 and α_2 which does not touch the real axis, in other words, choose the upper arc.

We let then $P(N)$ denote the path obtain by joining all these arcs together with the arcs obtained from following $C(0,1)$ starting from i inside $\mathcal{V}$ until the arc of the Ford circle of the first non-zero Farey fraction, while we end with arc starting at the second endpoint of the arc of the penultimate Farey fraction following the arc of $C(1,1)$ until the point $1+i$. We will call this contour the *Rademacher path* $P(N)$. For a visualization of $P(N)$, one can check the right hand side of the Figure 3.3.

However, to apply this in the equation 3.4, we rather need a path in the open unit disk enclosing 0 and not a path in $\mathcal{V}$. For that we need a map which relates the upper half plane with the open unit disk. The needed transformation will be

$$x = e^{2\pi i\tau}$$

which maps an element $\tau \in \mathcal{V}$ into an element x inside the punctured unit disk. Indeed, if we consider a point $\tau = a + bi$ in $\mathcal{H}$, for $a, b \in \mathbb{R}$ and $b > 0$, then

$$x = e^{2\pi i(a+bi)} = e^{2a\pi i - 2b\pi}$$

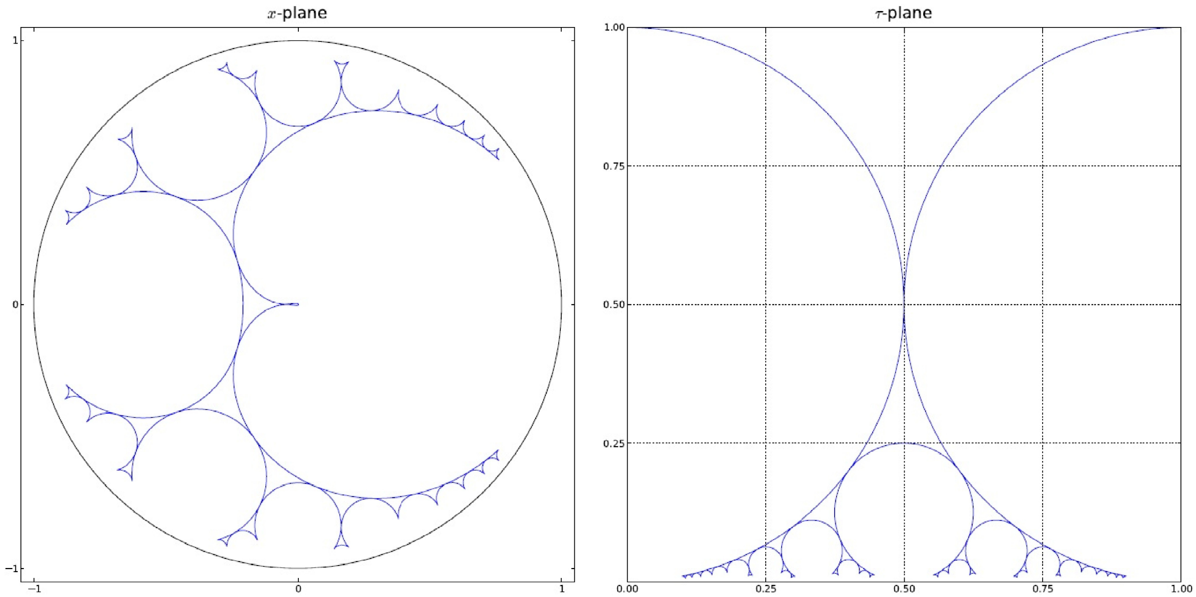


Figure 3.3: The Rademacher path $P(N)$ for the case when $N = 10$.

whose absolute value is $e^{-2b\pi}$.

We let $\tilde{P}(N)$ denote the image of the path $P(N)$ in the $x-$plane. In the following, at times we will refer to both of them as $P(N)$ with the meaning being clear from the context. In Figure 3.4 we display our transformation, while in Figure 3.3 on both sides, we see the Rademacher path $P(N)$ in both planes.

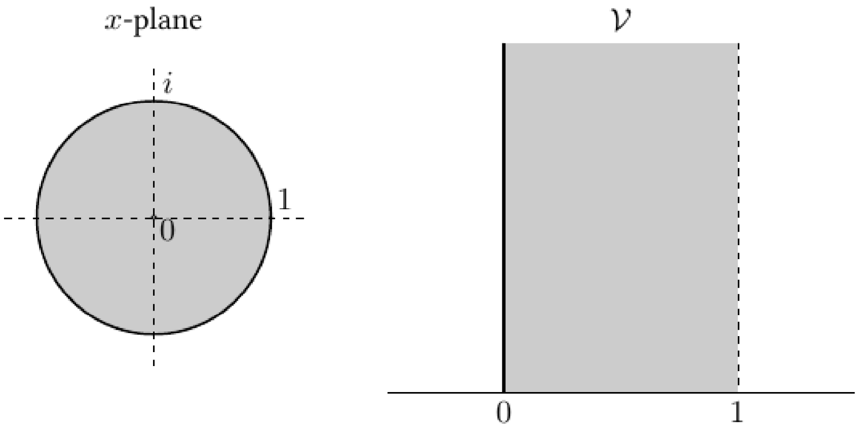


Figure 3.4: The transformation between the x-plane and the τ-plane

It is clear by definition that since $P(N)$ does not touch the real axis and avoids the points $\frac{h}{k}$ for $(h, k) = 1, 0 \leq h \leq k$, then $\widetilde{P(N)}$ avoids any $k-$th complex root of unity, whenever $k < N$. From now on, let n be a fixed number. In determining the value of $p(n)$, we will let N approach infinity.

By the above, we know that

$$p(n) = \frac{1}{2\pi i} \int_{\tilde{P}(N)} \frac{F(x)}{x^{n+1}} d\tau$$

Applying our change of variables $x = e^{2\pi i \tau}$, since

$$dx = 2\pi i e^{2\pi i \tau} d\tau,$$

our identity becomes

$$\begin{aligned} p(n) &= \frac{1}{2\pi i} \int_{P(N)} F(e^{2\pi i \tau}) e^{-2\pi i (n+1)\tau} 2\pi i e^{2\pi i \tau} d\tau \\ &= \int_{P(N)} F(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau. \end{aligned}$$

We now split $P(N)$ into the aforementioned arcs $\gamma(h,k)$, we get that

$$p(n) = \sum_{\substack{0 \le h \le k \le N \\ (h,k)=1}} \int_{\gamma(h,k)} F(e^{2\pi i \tau}) e^{-2\pi i n \tau} d\tau$$

For each such pair (h,k), we apply one more transformation which will be useful from a computational point of view later. Let

$$z = -ik^2 \left(\tau - \frac{h}{k} \right),$$

be our transformation. Reverting the transformation, it follows that

$$\tau = \frac{h}{k} + \frac{iz}{k^2}.$$

This transformation is linear, so it can be thought of as a composition of a translation, scaling and a rotation. To see what effect it has on the arc $\gamma(h,k)$, let us see first what it does on the Ford circle $C(h,k)$.

Subtracting $\frac{h}{k}$ is a shift by $\frac{h}{k}$ to left, therefore the image so far is a circle whose center is the origin. Multiplying by k^2, makes the radius of the new circle equal to

$$k^2 \cdot \frac{1}{2k^2} = \frac{1}{2}.$$

Finally, the multiplication by $-i$ rotates the figure counterclockwise by an angle $\frac{\pi}{2}$. Therefore, the image of $C(h,k)$ in the τ-plane is a circle K in the z-plane whose center is $\frac{1}{2}$ and radius $\frac{1}{2}$.

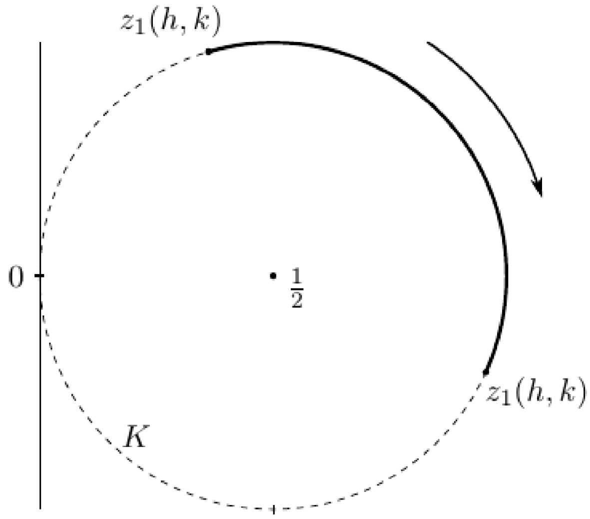


Figure 3.5: The image circle K in the z-plane

To find the image of $\alpha_1(h,k)$ (call it $z_1(h,k)$) , we just compute directly

$$\begin{aligned} z_1(h,k) &= -ik^2\left(\alpha_1(h,k) - \frac{h}{k}\right) \\ &= -ik^2\left(\frac{i}{k^2+k_1^2} - \frac{k_1}{k(k^2+k_{!}^2)}\right) \\ &= \frac{k^2}{k^2+k_1^2} + i\frac{kk_1}{k^2+k_1^2} \end{aligned} \tag{3.5}$$

while the image of $\alpha_2(h,k)$ can be similarly computed to be

$$z_2(h,k) = \frac{k^2}{k^2+k_2^2} - i\frac{kk_2}{k^2+k_2^2}. \tag{3.6}$$

Therefore the image of $\gamma(h,k)$ must one of the two arcs between $z_1(h,k)$ and $z_2(h,k)$ on the circle K. Since for $\gamma(h,k)$ we chose the arc strictly the real axis in the τ-plane, in the z-plane we must choose the arc from $z_1(h,k)$ to $z_2(h,k)$ which does not touch the imaginary axis. Let $\tilde{\gamma}(h,k)$ denote this arc. Note that for this change of variables, we have that

$$d\tau = \frac{i}{k^2}dz,$$

therefore we obtain that:

$$\begin{aligned} p(n) &= \sum_{\substack{0\le h\le k\le N\\ (h,k)=1}} \int_{\gamma(h,k)} F(e^{2\pi i\tau})e^{-2\pi i n\tau}d\tau \\ &= \sum_{\substack{0\le h\le k\le N\\ (h,k)=1}} \int_{\tilde\gamma(h,k)} F\left(\exp\left(\frac{2\pi i h}{k}-\frac{2\pi z}{k^2}\right)\right)\frac{i}{k^2}e^{-2\pi i nh/k}e^{2\pi nz/k^2}dz \\ &= \sum_{\substack{0\le h\le k\le N\\ (h,k)=1}} \frac{i}{k^2}e^{-2\pi i nh/k}\int_{\tilde\gamma(h,k)} e^{2\pi nz/k^2}F\left(\exp\left(\frac{2\pi i h}{k}-\frac{2\pi z}{k^2}\right)\right)dz. \end{aligned}$$

Now we are left to compute these integrals.

3.5.2 Dedekind's Functional Equation for F

The function F also possesses modularity properties, which we will try to explain and exploit in order to deduce certain functional equations satisfied by F, which are useful in the computation of the above integrals. We start with a

Definition 12. *The eta function η is a complex valued function defined on the upper half-plane $\mathcal{H}=\{\tau\in\mathbb{C}:\ Im(\tau)>0\}$ whose values are*

$$\eta(\tau)=e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1-e^{2\pi i m\tau}). \tag{3.7}$$

The importance of this function, which was introduced by Dedekind in the 19th century is that it satisfies the following important functional equation whose proof we omit and let the reader refer to ([2], page 52)

Theorem 5 (Dedekind's functional equation)**.** *Let $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ be one of the two possible possible representatives of an element η in the modular group Γ such that $c>0$, or in other words an element of $SL_2(\mathbb{Z})$ with $c>0$. If $\tau\in\mathcal{H}$, then*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\varepsilon(a,b,c,d)[-i(c\tau+d)]^{\frac{1}{2}}\eta(\tau), \tag{3.8}$$

where

$$\varepsilon(a,b,c,d)=\exp\left(\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right)$$

and

$$s(h,k)=\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\left\lfloor\frac{hr}{k}\right\rfloor-\frac{1}{2}\right).$$

In order to see how to apply this theorem for F, we note that in fact F and eta share a link by means of the following identities (where we use that $x = e^{2\pi i\tau}$, as well as (3.3) and (3.7))

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau}) = \frac{e^{\pi i\tau/12}}{F(e^{2\pi i\tau})}.$$

The functional equation (3.8) from the previous theorem can now be transformed into a functional equation for F.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau' = \frac{a\tau+b}{c\tau+d}$, with $c > 0$, then by (3.8) we have that

$$\eta(\tau') = \eta(\tau)[-i(c\tau+d)]^{\frac{1}{2}}\exp\left[\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right]$$

Plug in the equalities

$$\eta(\tau) = \frac{e^{\pi i\tau/12}}{F(e^{2\pi i\tau})} \text{ and } \eta(\tau') = \frac{e^{\pi i\tau'/12}}{F(e^{2\pi i\tau'})}$$

to obtain that

$$\frac{e^{\pi i\tau'/12}}{F(e^{2\pi i\tau'})} = \frac{e^{\pi i\tau/12}}{F(e^{2\pi i\tau})}[-i(c\tau+d)]^{\frac{1}{2}}\exp\left[\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right]$$

Therefore, in the τ plane F satisfies the following functional equation

$$F(e^{2\pi i\tau}) = F(e^{2\pi i\tau'})\exp\left(\frac{\pi i(\tau-\tau')}{12}\right)[-i(c\tau+d)]^{\frac{1}{2}}\exp\left[\pi i\left(\frac{a+d}{12c} + s(-d,c)\right)\right]. \quad (3.9)$$

Recall now the transformation rule between the variables x, τ and z, for a fixed coprime pair (h,k):

$$x = e^{2\pi i\tau} = \exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right).$$

Then we have that

$$x' = e^{2\pi i\tau'} = \exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right),$$

where H is any integer such that $hH \equiv -1 \pmod{k}$.

Turns out, one can always choose an appropriate matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for which (3.9) becomes instead

$$F\left(\exp\left(\frac{2\pi ih}{k} - \frac{2\pi z}{k^2}\right)\right) = \omega(h,k)\left(\frac{z}{k}\right)^{\frac{1}{2}}F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right)\exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)$$

where $\omega(h,k) = e^{\pi i s(h,k)}$. By definition $s(h,k) \in \mathbb{Q}$, therefore $\omega(h,k)$ is a complex root of unity. Note the latter expression is expressed completely in terms of z and furthermore, no more reference to the integers a, b, c, d is made. The details of the choice of the matrix are discussed in the Appendix A.

Using again the conversion between x and z, we can also express this functional equation in the following compact form

$$F(x) = \omega(h,k)\left(\frac{z}{k}\right)^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi}{12k^2}\right) F(x') \tag{3.10}$$

Now we are able to make some quantitative observations of the behaviour of $F(x)$ when is x is approaching the unit circle, more precisely a root of unity. Since

$$x = \exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right).$$

we have that x approaches $e^{2\pi i h/k}$ when $|z|$ tends to zero. But when z tends to zero, the absolute value of $\frac{2\pi}{z}$ goes to infinity, and therefore

$$x' = \exp\left(\frac{2\pi i H}{k} - \frac{2\pi}{z}\right)$$

goes to zero. Therefore, when x approaches a root of unity, $F(x')$ tends to $F(0) = 1$.

In regards to (3.10), this means that deleting the last factor in right hand side gives a good approximation to $F(x)$.

In other words

$$\omega(h,k)\left(\frac{z}{k}\right)^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)$$

should be regarded as a good approximation of $F(x)$. This differs by a constant from the expression which we call

$$\Psi_k(z) = z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right)$$

Applying Dedekind's functional equation for F into our identity for $p(n)$,

$$p(n) = \sum_{\substack{0 \le h \le k \le N \\ (h,k)=1}} i k^{-2} e^{-2\pi i n h/k} \int_{\tilde{\gamma}(h,k)} e^{2\pi n z/k^2} F\left(\exp\left(\frac{2\pi i h}{k} - \frac{2\pi z}{k^2}\right)\right) dz,$$

we obtain that

$$p(n) = \sum_{\substack{0 \le h \le k \le N \\ (h,k)=1}} i k^{-5/2} \omega(h,k) e^{-2\pi i n h/k} \int_{\tilde{\gamma}(h,k)} e^{2\pi n z/k^2} z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) F(x') dz.$$

Our quantitative observations about approximating $F(x)$ motivates writing $F(x')$ as $1 + (F(x') - 1)$ in the sum

$$p(n) = \sum_{\substack{0 \le h \le k \le N \\ (h,k)=1}} ik^{-5/2}\omega(h,k)e^{-2\pi i nh/k} \int_{\tilde{\gamma}(h,k)} e^{2\pi nz/k^2}\Psi_k(z)[1 + (F(x') - 1)]dz,$$

and splitting our integral in two terms

$$I_1(h,k) = \int_{\tilde{\gamma}(h,k)} \Psi_k(z)e^{2\pi nz/k^2}dz \tag{3.11}$$

and

$$I_2(h,k) = \int_{\tilde{\gamma}(h,k)} \Psi_k(z)e^{2\pi nz/k^2}\left[F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right) - 1\right]dz. \tag{3.12}$$

In short, our new expression for $p(n)$ can be written as

$$p(n) = \sum_{\substack{0 \le h \le k \le N \\ (h,k)=1}} ik^{-5/2}\omega(h,k)e^{-2\pi i nh/k}(I_1(h,k) + I_2(h,k)), \tag{3.13}$$

where $I_1(h,k)$ should be regarded as leading term and $I_2(h,k)$ should be regarded as the error term. In what follows we will study both of these terms separately.

3.5.3 The Error Term $I_2(h,k)$

We start with an analysis of the error term, with our goal being to show that $I_2(h,k)$ is $O(N^{-1/2})$. Therefore, the term is small and we will be able to neglect it later when will let N go to infinity and allow infinite sums. Note that the integrand is indeed holomorphic in a small neighbourhood of the disk centered at $\frac{1}{2}$ of radius $\frac{1}{2}$ punctured at the origin. Therefore we are also able to compute $I_2(h,k)$ by integrating along a different path, namely the chord between $z_1(h,k)$ and $z_2(h,k)$. We denote the integral along this cord by $\int_{z_1(h,k)}^{z_2(h,k)}$. Therefore, we have that

$$I_2(h,k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k(z)e^{2\pi nz/k^2}\left[F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right) - 1\right]dz.$$

Let us bound this integral having in mind that z varies along the aforementioned chord. We start by bounding the integrand. We use that for every complex number z we have the identity

$$|e^z| = \left|e^{\mathrm{Re}(z)}\right|.$$

Using this and the power series expansion of F, and denoting the integrand by $\Phi(z)$ we obtain

$$\begin{aligned}
|\Phi(z)| &= \left|\Psi_k(z)e^{2\pi nz/k^2}\left[F\left(\exp\left(\frac{2\pi iH}{k}-\frac{2\pi}{z}\right)\right)-1\right]\right| \\
&= \left|z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{\pi z}{12k^2}\right)\exp\left(\frac{2\pi nz}{k^2}\right)\left[\sum_{m=0}^{\infty}p(m)\exp\left(\frac{2\pi imH}{k}-\frac{2\pi m}{z}\right)-1\right]\right| \\
&= \left|z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{\pi z}{12k^2}\right)\exp\left(\frac{2\pi nz}{k^2}\right)\left[\sum_{m=1}^{\infty}p(m)\exp\left(\frac{2\pi imH}{k}-\frac{2\pi m}{z}\right)\right]\right| \\
&\leq |z|^{\frac{1}{2}}\exp\left(\frac{\pi\mathrm{Re}(1/z)}{12}-\frac{\pi\mathrm{Re}(z)}{12k^2}\right)\exp\left(\frac{2\pi n\mathrm{Re}(z)}{k^2}\right)\left(\sum_{m=1}^{\infty}p(m)\left|e^{2\pi iHm/k}\right|e^{-2\pi m\mathrm{Re}(1/z)}\right). \\
&= |z|^{\frac{1}{2}}\exp\left(\frac{\pi\mathrm{Re}(1/z)}{12}-\frac{\pi\mathrm{Re}(z)}{12k^2}\right)\exp\left(\frac{2\pi n\mathrm{Re}(z)}{k^2}\right)\left(\sum_{m=1}^{\infty}p(m)e^{-2\pi m\mathrm{Re}(1/z)}\right).
\end{aligned}$$

where in the last equality we used that $\left|e^{2\pi iHm/k}\right| = 1$. We will bound every factor in the last product besides the first one. Because z varies for values in the chord, which in particular lies inside the disk whose boundary is K, we have that $0 \leq \mathrm{Re}(z) \leq 1$. Therefore

$$\exp\left(\frac{\pi\mathrm{Re}(z)}{12k^2}\right) \leq 1 \tag{3.14}$$

and

$$\exp\left(\frac{2\pi n\mathrm{Re}(z)}{k^2}\right) \leq \exp\left(\frac{2\pi n}{k^2}\right) \leq \exp(2\pi n) \tag{3.15}$$

For the other factors, let us switch our focus on $1/z$ instead. Assume that $z = a + bi$ is inside the closed disk whose boundary is K. Since z is within distance $\frac{1}{2}$ from the center of K, we obtain

$$\left(a-\frac{1}{2}\right)^2 + b^2 \leq \frac{1}{4}$$

which rewritten means that $a^2 + b^2 \leq a$. Therefore, from the equality

$$\frac{1}{z} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

we conclude that

$$\mathrm{Re}\left(\frac{1}{z}\right) = \frac{a}{a^2+b^2} \geq 1.$$

It is also useful to keep in mind that if z lies on the circle K, then the equality case $\mathrm{Re}(1/z) = 1$ occurs. Using (3.14) and (3.15) as well, this gives

$$\begin{aligned}
|\Phi(z| &\leq |z|^{\frac{1}{2}} \exp\left(\frac{\pi \mathrm{Re}(1/z)}{12}\right) e^{2\pi n} \left[\sum_{m=1}^{\infty} p(m) \exp\left(-2\pi m \mathrm{Re}\left(\frac{1}{z}\right)\right)\right] \\
&= |z|^{\frac{1}{2}} e^{2\pi n} \left[\sum_{m=1}^{\infty} p(m) \exp\left(-2\pi \left(m - \frac{1}{24}\right) \mathrm{Re}\left(\frac{1}{z}\right)\right)\right] \\
&\leq |z|^{\frac{1}{2}} e^{2\pi n} \left[\sum_{m=1}^{\infty} p(m) \exp\left(-2\pi \left(m - \frac{1}{24}\right)\right)\right] \\
&< |z|^{\frac{1}{2}} e^{2\pi n} \left(\sum_{m=1}^{\infty} p(24m-1) e^{-\pi(24m-1)/12}\right). \\
&< |z|^{\frac{1}{2}} e^{2\pi n} \left(\sum_{j=0}^{\infty} p(j) e^{-\pi j/12}\right) \\
&= c|z|^{\frac{1}{2}}
\end{aligned}$$

where $c = e^{2\pi n} F\left(\exp\left(-\frac{\pi}{12}\right)\right)$, the latter factor being well defined because

$$0 < \exp\left(-\frac{\pi}{12}\right) < 1.$$

Note that c is a constant whose expression whose expression depends on our fixed n and is independent of N and z, which means it can be applied universally.

To bound the integrand, now it suffices to bound $|z|^{\frac{1}{2}}$ and the length of the chord between $z_1(h,k)$ and $z_2(h,k)$.

Recalling our formulas for the coordinate expressions for $z_1(h,k)$ and $z_2(h,k)$, we obtain

$$\begin{aligned}
|z_1(h,k)|^2 &= \left|\frac{k^2}{k^2 + k_1^2}\right|^2 + \left|\frac{kk_1}{k^2 + k_1^2}\right|^2 \\
&= \frac{k^4 + k^2 k_1^2}{(k^2 + k_1^2)^2} \\
&= \frac{k^2}{k^2 + k_1^2}.
\end{aligned}$$

After performing an analogous computation for $z_2(h,k)$, our results are summarized by the equalities

$$|z_1(h,k)| = \frac{k}{\sqrt{k^2+k_1^2}}$$
$$|z_2(h,k)| = \frac{k}{\sqrt{k^2+k_2^2}}$$

Using the second part of Theorem 2 of Chapter 2, we know that $k+k_1 \geq N+1$, because $\frac{h}{k}$ and $\frac{h_1}{k_1}$ are consecutive Farey fractions. Combining this with the basic inequalities

$$0 \leq (k-k_1)^2 = 2(k^2+k_1^2) - (k+k_1)^2$$

we obtain that:

$$|z_1(h,k)| = \frac{k}{\sqrt{k^2+k_1^2}} \leq \frac{\sqrt{2}k}{k+k_1} \leq \frac{\sqrt{2}k}{N+1} \leq \frac{\sqrt{2}k}{N} \tag{3.16}$$

In a similar way we also obtain that

$$|z_2(h,k)| < \frac{\sqrt{2}k}{N} \tag{3.17}$$

As z varies through the chord between these 2 points, using the unit interval parametrization

$$\begin{aligned} |z| &= |tz_1(h,k) + (1-t)z_2(h,k)| \leq t|z_1(h,k)| + (1-t)|z_2(h,k)| \\ &\leq \max(|z_1(h,k)|, |z_2(h,k)|) < \frac{\sqrt{2}k}{N}. \end{aligned}$$

where t varies in the unit interval. Thus

$$c|z|^{\frac{1}{2}} \leq c2^{\frac{1}{4}} \left(\frac{k}{N}\right)^{\frac{1}{2}}.$$

Also, using the triangle inequality, the length of the chord is bounded by

$$|z_1(h,k)| + |z_2(h,k)| = \frac{2\sqrt{2}k}{N}.$$

Now that every component of the integral is bounded, we can finally conclude that

$$|I_2(h,k)| < C\left(\frac{k}{N}\right)^{\frac{3}{2}}$$

where C is a constant that depends solely on n. Therefore, the sum of the error terms is bounded by

$$\begin{aligned}
\left|\sum_{\substack{0\le h\le k\le N\\(h,k)=1}} \left(ik^{-5/2}\omega(h,k)e^{-2\pi i n h/k}I_2(h,k)\right)\right| &= \left|\sum_{\substack{0\le h\le k\le N\\(h,k)=1}} \left(k^{-5/2}I_2(h,k)\right)\right| \\
&\le \sum_{\substack{0\le h\le k\le N\\(h,k)=1}} \left(k^{-5/2}|I_2(h,k)|\right) \\
&< \sum_{k=1}^{N}\left(\sum_{\substack{0\le h<k\\(h,k)=1}} \frac{C}{kN^{3/2}}\right) \\
&= \frac{C}{N^{3/2}}\left(\sum_{k=1}^{N}\left(\frac{1}{k}\sum_{\substack{0\le h<k\\(h,k)=1}} 1\right)\right) \\
&\le \frac{C}{N^{3/2}}\left(\sum_{k=1}^{N}\frac{1}{k}k\right) \\
&= \frac{C}{\sqrt{N}}.
\end{aligned}$$

where in the first equality we used that i, $\omega(h,k)$ and $e^{-2\pi i n h/k}$ have absolute value 1. Therefore the equation (3.13) for $p(n)$ becomes

$$p(n) = \sum_{\substack{0\le h\le k\le N\\(h,k)=1}} ik^{-5/2}\omega(h,k)e^{-2\pi i n h/k}I_1(h,k) + O(N^{-1/2}) \tag{3.18}$$

3.5.4 The Term $I_1(h,k)$

Turns out that the term $I_1(h,k)$ can be modified in a way that the difference of $I_1(h,k)$ to its modification is of small order. In fact, just like in the case of the error $I_2(h,k)$, when considered in the sum, this order of the difference is $O(1/\sqrt{N})$. This change consists in modifying the path of integration.

Having in mind the path independence discussed in the previous subsection, we can omit mentioning $\tilde{\gamma}(h,k)$ and simply write

$$I_1(h,k) = \int_{z_1(h,k)}^{z_2(h,k)} \Psi_k(z)e^{2\pi n z/k^2}dz.$$

If K^- denotes the path which goes along the circle K in clockwise direction. Since our previous choice of the arc meant it was also clockwise, we have that

$$I_1(h,k) = \int_{K^-} - \int_{z_2(h,k)}^{0} - \int_{0}^{z_1(h,k)} .$$

the reason for splitting the integral at the origin being the singularity of the integrand there. Let

$$J_1 := J_1(h,k) = \int_{z_2(h,k)}^{0} \Psi_k(z) e^{2\pi n z/k^2} \text{ and } J_2 := J_2(h,k) = \int_{0}^{z_1(h,k)} \Psi_k(z) e^{2\pi n z/k^2}$$

We will that both the error term created by summing up over all J_1 and J_2 is of order $O(N^{-1/2})$.

Refer from now to the figure 3.6. Let us bound the length of the arc on K from the origin to $z_1(h,k)$. From the figure it is clear that this length is smaller than the length of the arc between the same points, in the situation when these two points determine the diameter of the circle.

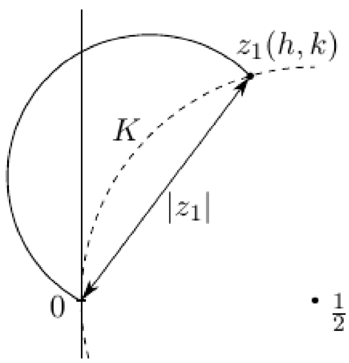


Figure 3.6: The arc on K from 0 to z_1

But for the latter arc, we know that its length is half of the circumference, namely $\pi|z_1(h,k)|/2$. Using (3.16) we know thus the length our arc on K from 0 $z_1(h,k)$ is bounded by $\sqrt{2}\pi k/N$.

Similarly, the length of the arc from the origin to $z_2(h,k)$ is bounded by the same constant.

It is also clear by geometric considerations, that the absolute value of any z on these two arcs is bounded by $\max(|z_1|,|z_2|)$, thus $|z| < \sqrt{2}k/N$ there by (3.16) and (3.17).

Recall that the transformation $\mathrm{Re}(1/z) = 1$ for points z on the circle K, We are ready to bound the integrand:

$$\begin{aligned}\left|\Psi_k(z)e^{2\pi nz/k^2}\right| &= \left|z^{\frac{1}{2}}\exp\left(\frac{\pi}{12z}-\frac{\pi z}{12k^2}\right)e^{2\pi nz/k^2}\right| \\ &= |z|^{\frac{1}{2}}\exp\left(\frac{\pi\mathrm{Re}(1/z)}{12}-\frac{\pi\mathrm{Re}(z)}{12k^2}\right)e^{2\pi n\mathrm{Re}(z)/k^2} \\ &\leq |z|^{\frac{1}{2}}e^{\pi/12}e^{2\pi n} \\ &< c_1\left(\frac{k}{N}\right)^{\frac{1}{2}},\end{aligned}$$

where c_1 which depends only on our fixed n. In the first inequality we used that $\mathrm{Re}(1/z) = 1$ together with (3.14) and (3.15), while in the second one we used the bound for $|z|$ above. Combine this with the bound on the length of the arcs from before, we find that both

$$|J_1| < C_1\left(\frac{k}{N}\right)^{\frac{3}{2}} \qquad \text{and} \qquad |J_2| < C_1\left(\frac{k}{N}\right)^{\frac{3}{2}},$$

where C_1 is constant.

Therefore the total error when using the curve K^- instead of the arcs is bounded by

$$\begin{aligned}\left|\sum_{h,k} ik^{-5/2}\omega(h,k)e^{-2\pi inh/k}(J_1+J_2)\right| &\leq \sum_{h,k} k^{-5/2}|J_1+J_2| \\ &< \sum_{k=1}^{N}\sum_{\substack{0\leq h<k \\ (h,k)=1}} \frac{2C_1}{kN^{-3/2}} \\ &= \frac{2C_1}{N^{-3/2}}\sum_{k=1}^{N}\frac{1}{k}\sum_{\substack{0\leq h<k \\ (h,k)=1}} 1 \\ &\leq \frac{2C_1}{\sqrt{N}}.\end{aligned}$$

which is indeed of the claimed order. Applying to our expansion of $p(n)$, we obtain

$$p(n) = \sum_{h,k} ik^{-5/2}\omega(h,k)e^{-2\pi inh/k}\int_{K^-}\Psi_k(z)e^{2\pi nz/k^2}dz + O(N^{-1/2})$$

3.5.5 The Rademacher Terms

Since n was fixed and N could vary, we allow now $N \to \infty$ and our our equation for $p(n)$ becomes

$$p(n) = \sum_{k=1}^{\infty} \sum_{\substack{0 \le h < k \\ (h,k)=1}} ik^{-5/2} \omega(h,k) e^{-2\pi inh/k} \int_{K^-} \Psi_k(z) e^{2\pi nz/k^2} dz$$

Let $A_k(n) = \sum_{\substack{0 \le h < k \\ (h,k)=1}} \omega(h,k) e^{-2\pi inh/k}$. Then our expression for $p(n)$ becomes

$$\begin{aligned} p(n) &= \sum_{k=1}^{\infty} ik^{-5/2} A_k(n) \int_{K^-} z^{\frac{1}{2}} \exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2}\right) e^{2\pi nz/k^2} dz \\ &= \sum_{k=1}^{\infty} ik^{-5/2} A_k(n) \int_{K^-} z^{\frac{1}{2}} \exp\left\{\frac{\pi}{12z} - \frac{2\pi z}{k^2}\left(n - \frac{1}{24}\right)\right\} dz, \end{aligned}$$

We perform one more change of variables

$$z = \frac{\pi}{12t} \quad \text{with change of differentials} \quad dz = -\frac{\pi}{12t^2} dt,$$

This is just a scaled version of the reverse transform $\frac{1}{z}$. Using our analysis of the latter, we conclude thus that the circle K is onto the line $\mathrm{Re}(z) = \dfrac{\pi}{12}$.

Our sum expansion of $p(n)$ can be rewritten as

$$\begin{aligned} p(n) &= \sum_{k=1}^{\infty} -ik^{-5/2} A_k(n) \int_{\pi/12-\infty i}^{\pi/12+\infty i} \left(\frac{\pi}{12t}\right)^{\frac{1}{2}} \exp\left\{t + \frac{\pi^2}{6k^2 t}\left(n - \frac{1}{24}\right)\right\} \frac{\pi}{12t^2} dt \\ &= \sum_{k=1}^{\infty} -ik^{-5/2} A_k(n) \left(\frac{\pi}{12}\right)^{\frac{3}{2}} \int_{\pi/12-\infty i}^{\pi/12+\infty i} t^{-5/2} \exp\left\{t + \frac{\pi^2}{6k^2 t}\left(n - \frac{1}{24}\right)\right\} dt \\ &= 2\pi \left(\frac{\pi}{12}\right)^{\frac{3}{2}} \sum_{k=1}^{\infty} k^{-5/2} A_k(n) \int_{\pi/12-\infty i}^{\pi/12+\infty i} t^{-5/2} \exp\left\{t + \frac{\pi^2}{6k^2 t}\left(n - \frac{1}{24}\right)\right\} dt. \end{aligned}$$

This is an integral which can be evaluated by means of the theory of Bessel functions, about which we provide details in the Appendix B. Using (B.1), we know that

$$\frac{\left(\frac{1}{2}z\right)^{\nu}}{2\pi i} \int_{\pi/12-\infty i}^{\pi/12+\infty i} t^{-\nu-1} \exp\left(t + \frac{z^2}{4t}\right) dt = I_{\nu}(z),$$

and if we plug in the values

$$z = \frac{1}{2}\left[\frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\right]^{\frac{1}{2}} \quad \text{and} \quad \nu = \frac{3}{2},$$

we find that the integration term can be rewritten as

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\pi/12-\infty i}^{\pi/12+\infty i} t^{-5/2}\exp\left\{t + \frac{\pi^2}{6k^2 t}\left(n - \frac{1}{24}\right)\right\}dt \\ &= \left\{\frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\right\}^{-\frac{3}{4}} I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right).\end{aligned}$$

This gives us the following form of the equation of $p(n)$

$$p(n) = \frac{2\pi(n - \frac{1}{24})^{-\frac{3}{4}}}{(24)^{\frac{3}{4}}}\sum_{k=1}^{\infty} k^{-1}A_k(n) I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)$$

$I_{\frac{3}{2}}(z)$ is a Bessel function whose order is a half integer, for which we computed in (B.3) that

$$I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right) = \frac{6^{\frac{3}{4}}k^{\frac{3}{2}}}{\pi^2}\left(n - \frac{1}{24}\right)^{\frac{3}{4}}\frac{d}{dn}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right\}}{\sqrt{n - \frac{1}{24}}}\right).$$

Once again simplifying our equation for $p(n)$, we arrive at our infinite series representation:

$$\begin{aligned} p(n) &= \frac{2\pi\left(n - \frac{1}{24}\right)^{-\frac{3}{4}}}{24^{\frac{3}{4}}}\frac{6^{\frac{3}{4}}\left(n - \frac{1}{24}\right)^{\frac{3}{4}}}{\pi^2}\sum_{k=1}^{\infty} k^{-1}k^{\frac{3}{2}}A_k(n)\frac{d}{dn}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right\}}{\sqrt{\left(n - \frac{1}{24}\right)}}\right) \\ &= \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} k^{\frac{1}{2}}A_k(n)\frac{d}{dn}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right\}}{\sqrt{\left(n - \frac{1}{24}\right)}}\right). \qquad (3.19)\end{aligned}$$

This realizes our main goal for this chapter, which was a convergent series expansion for $p(n)$. We also denote by

$$R_k(n) = k^{\frac{1}{2}}A_k(n)\frac{d}{dn}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right\}}{\sqrt{n - \frac{1}{24}}}\right),$$

and call it the kth *Rademacher term*.

3.6 Asymptotics of $p(n)$

We can now also deduce the asymptotic formula for $p(n)$ found by Hardy and Ramanujan. For this, turns out, it is enough to consider only the first term of the Rademacher series.

We will show that this term dominates the remaining sum and that it has the desired asymptotics. Let us compute the Rademacher term for the case when $k = 1$.

$$A_1(n) = \sum_{\substack{0 \le h < k \\ (h,k)=1}} e^{\pi i s(h,k)} e^{-2\pi i n h/k} = e^{\pi i \cdot 0} e^{-2\pi n \cdot 0} = 1,$$

Therefore, differentiating, we obtain

$$\begin{aligned} R_1(n) &= \frac{d}{dn}\left(\frac{\sinh\left(\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)}{\sqrt{n-\frac{1}{24}}}\right) \\ &= \frac{1}{4}\left(e^{\pi\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}} + e^{-\pi\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}}\right)\pi\sqrt{\frac{2}{3}}\left(n-\frac{1}{24}\right)^{-1} \\ &\quad - \frac{1}{4}\left(e^{\pi\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}} - e^{-\pi\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}}\right)\left(n-\frac{1}{24}\right)^{-\frac{3}{2}} \end{aligned}$$

Using the binomial series for the exponent $\frac{1}{2}$, we compute

$$\begin{aligned} \sqrt{n-\frac{1}{24}} = \sqrt{n}\left(1-\frac{1}{24n}\right)^{\frac{1}{2}} &= \sqrt{n}\sum_{m=0}^{\infty}\binom{\frac{1}{2}}{m}(-1)^k\left(\frac{1}{24n}\right)^m \\ &= \sqrt{n}\left(1-\frac{1}{2\cdot 24n}-\frac{1}{8\cdot(24n)^2}-\dots\right) \\ &= \sqrt{n}\left(1+O\left(\frac{1}{n}\right)\right) \\ &= \sqrt{n}+O\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

and using in turn the exponential series, we obtain

$$e^{\pi\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}} = e^{\pi\sqrt{\frac{2n}{3}}}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right).$$

Since the other terms will not affect this exponential term, we obtain an asymptotic expression of the first Rademacher term.

$$R_1(n) = \frac{\pi}{4n}\sqrt{\frac{2}{3}}e^{\pi\sqrt{\frac{2n}{3}}}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right).$$

We are left to explore the size of the other terms in the Rademacher series. Consider the change of variables

$$t = \frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}, \text{ so that } n - \frac{1}{24} = \frac{3}{2}\left(\frac{kt}{\pi}\right)^2,$$

we find that

$$\begin{aligned}
\frac{d}{dn}\left(\frac{\sinh t}{\sqrt{n-\frac{1}{24}}}\right) &= \frac{\pi}{2k}\sqrt{\frac{2}{3}}\frac{\cosh t}{\left(n-\frac{1}{24}\right)} - \frac{1}{2}\frac{\sinh t}{\left(n-\frac{1}{24}\right)^{\frac{3}{2}}} \\
&= \frac{\pi}{2k}\left(\frac{2}{3}\right)^{\frac{3}{2}}\left(\frac{\pi}{kt}\right)^2 \cosh t - \frac{1}{2}\left(\frac{2}{3}\right)^{\frac{3}{2}}\left(\frac{\pi}{kt}\right)^3 \sinh t \\
&= \frac{2}{3\sqrt{6}}\frac{\pi^3}{k^3}\frac{\cosh t}{t^2} - \frac{\sqrt{2}}{3\sqrt{3}}\frac{\pi^3}{k^3}\frac{\sinh t}{t^3} \\
&= \frac{2}{3\sqrt{6}}\frac{\pi^3}{k^3}\left(\frac{\cosh t}{t^2} - \frac{\sinh t}{t^3}\right).
\end{aligned} \tag{3.20}$$

Let

$$f(t) = \left(\frac{\cosh t}{t^2} - \frac{\sinh t}{t^3}\right)$$

Its Taylor series equals to

$$\begin{aligned}
f(t) &= \left(\frac{1}{t^2} + \frac{t^2}{2!t^2} + \frac{t^4}{4!t^2} + \dots\right) - \left(\frac{t}{t^3} + \frac{t^3}{3!t^3} + \frac{t^5}{5!t^3} + \dots\right) \\
&= \left(\frac{1}{2!} - \frac{1}{3!}\right) + t^2\left(\frac{1}{4!} - \frac{1}{5!}\right) + t^4\left(\frac{1}{6!} - \frac{1}{7!}\right) + \dots,
\end{aligned}$$

therefore it is increasing and its limit at $t = 0$ exists and is finite.

Now we can bound the remaining Rademacher terms. Note $A_k(n)$ is a sum of roots of unity with at most k summands. Thus $|A_k(n)| \leq k$. It follows that, for a constant C:

$$\begin{aligned}
\left|\sum_{k\geq 2} k^{\frac{1}{2}} A_k(n) \frac{2}{3\sqrt{6}}\frac{\pi^3}{k^3} f\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)\right| &\leq C\sum_{k\geq 2} k^{-\frac{3}{2}} f\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right) \\
&\leq Cf\left(\frac{\pi}{2}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right)\left(\sum_{k\geq 2} k^{-\frac{3}{2}}\right),
\end{aligned}$$

because as k increases, t then decreases and f then also must decrease. Furthermore, the argument of f approaches ∞ for which we showed the limit is finite. The last factor converges sum is a p-series with $p = 3/2 > 1$. In summary

$$\begin{aligned}\left|\sum_{k\geq 2} R_k(n)\right| &= O\left\{ f\left(\frac{\pi}{2}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)\right\} \\ &= O(R_2(n)).\end{aligned}$$

Performing now an analogous computation for $R_2(n)$ as in the case of $R_1(n)$ we will find that its dominating term is again an exponential function, with a similar but nevertheless smaller constant part. Therefore $R_2(n) = O(R_1(n))$ and

$$\begin{aligned}p(n) &\sim \frac{1}{\pi\sqrt{2}} R_1(n) \\ &= \frac{1}{\pi\sqrt{2}}\frac{\pi}{4n}\sqrt{\frac{2}{3}} e^{\pi\sqrt{\frac{2n}{3}}}\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \\ &\sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}} \qquad \text{as } n \to \infty.\end{aligned} \tag{3.21}$$

4 Statistics of a Random Partition

Assume that $n \geq 1$ is a fixed number. We can assign to the set of partitions $\mathcal{P}_n$ a probability measure in the simplest way, by means of the uniform probability measure. This means that we assume that every individual outcome, in other words any partition in $\mathcal{P}_n$, has a probability of $1/p(n)$.

This allows us to talk about random variables on the sets of partitions. More precisely, we will interested in those which are integer valued and simply call them them partition statistics. We analyze a few partition statistics and results known about them.

4.1 Partition Length and the Number of Parts

Two of the most basic partition statistics are the partition length and the number of parts. As we will see these two are closely related.

Definition 13. *The length of a partition $\lambda \in \mathcal{P}_n$, denoted by $L_n(\lambda)$, is the size of the largest part of λ. Let L_n denote the corresponding partition statistic on $\mathcal{P}_n$.*

Example 6. Consider the following partition for $n = 34$:

$$\mu : \quad 34 = 15 + 8 + 6 + 3 + 1 + 1.$$

The size of the largest part of this partition is 15, therefore $L_{34}(\mu) = 15$.

Definition 14. *Let $Z_n(\lambda)$ denote the number of parts in $\lambda \in \mathcal{P}_n$. Let Z_n be the corresponding partition statistic on $\mathcal{P}_n$.*

Example 7. On the above partition μ of 34, $Z_{34}(\mu) = 6$.

The two statistics are related by the following lemma.

Lemma 5. *If $\mathbb{P}$ is the uniform probability measure on $\mathcal{P}_n$, then*

$$\mathbb{P}(L_n = k) = \mathbb{P}(Z_n = k), \quad k = 1, 2, \ldots, n. \tag{4.1}$$

Proof. For each partition $\lambda \in \mathcal{P}_n$ such that $L_n(\lambda) = k$ consider its conjugate partition λ'. Then $Z_n(\lambda') = k$.

Since conjugation is a set bijection from $\mathcal{P}_n$ onto itself, with reverse the conjugation, it follows that there are equally many partitions of order n whose length is k and partitions of order n whose number of parts is k. Since the probability measure is uniform, we are done. □

Corollary 6. *For $n \geq 1$, we have that $\mathbb{E}(L_n) = \mathbb{E}(Z_n)$.*

If we allow n to vary, recall that one could generating functions to encode quantitative information about partitions of several orders in one stroke.

In the one variable case, as we considered before, we could usually keep track of only one size measure, but if one needs more, one can add extra variables.

If we want a generating function which keeps track of size of the partitions and their lengths one can consider the following two variable generating function

$$P(x,y) = \prod_{j=1}^{\infty}(1 - x^j y)^{-1},$$

where the variable x keep track of the order of the partition and y keeps track of the length. This can be proven either directly, or by the above lemma.

This generating functions allows in theory the computation of the expected value $\mathbb{E}(L_n) = \mathbb{E}(Z_n)$. In order to obtain a generating function whose nth coefficient describes the total length summed over all partitions of order n, one simply computes the nth coefficient in

$$\frac{\partial}{\partial u} P(x,y)\bigg|_{y=1} = \prod_{j=1}^{\infty}(1 - x^j)^{-1} \cdot \sum_{j=1}^{\infty} \frac{x^j}{1 - x^j}.$$

If we divide the coefficient of x^n by $p(n)$ of partitions of n, we will have computed the expected value $\mathbb{E}(L_n)$. In Chapter 5, we will see what $\mathbb{E}(L_n) = \mathbb{E}(Z_n)$ equal to. The main goal there will be compute expectations for their refined counterparts after a finding a more sophisticated, but easily proven analog of Lemma 5.

4.2 Number of Distinct Parts

Definition 15. *Let $D_n(\lambda)$ be the number of distinct parts of the given partition $\lambda \in \mathcal{P}_n$ and let D_n denote the corresponding partition statistic.*

For the computing the expect value, one can repeat the method in the case of case finding the average length. One can instead compute the total value when we consider all the partitions by means of the following two variable generating function:

$$\prod_{j=1}^{\infty}\left(1+\frac{x^j y}{1-x^j y}\right).$$

where x keeps track of the order of the partition and y keeps track of the number of distinct parts. Differentiating with respect to y and plugging in $y=1$ yields

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\frac{x}{1-x}.$$

Dividing the nth coefficient by $p(n)$ yields the desired expected value. For the distribution, the research by Goh and Schmutz [8] shows that D_n follows a normal distribution in the limit.

One can introduce several generalisations of this statistic too. One of them is $D_{n,m}$, the sum of the m-th powers of all distinct parts of a partition of order n. The two variable generating function corresponding to it is

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}j^m x^j.$$

A study of this statistic can be found in [12].

4.3 Moments of a Partition

Definition 16. *Let $n\geq 1$ and $\lambda\in\mathcal{P}_n$. Suppose $\lambda=\lambda_1+\lambda_2+\ldots$, then the k-th moment of λ is defined to be*

$$M_{k,n}(\lambda)=\lambda_1^k+\lambda_2^k+\ldots$$

Clearly for $k=0$, we have that $M_{0,n}=L_n$ recovers the length of a partition. Also $M_{1,n}$ equals the constant function n. Its corresponding two variable generating function is

$$\prod_{j=1}^{\infty}(1-x^j y^{j^k})^{-1}$$

for which if we do the above trick by differentiating at y and setting $y=1$ we obtain the one variable generating function

$$\prod_{j=1}^{\infty}(1-x^j)^{-1}\cdot\sum_{j=1}^{\infty}\frac{j^k x^j}{1-x^j}.$$

which allows for direct computation of the mean $\mathbb{E}(M_{k,n})$ for example.

4.4 Number of Parts of a Given Size

Consider the statistic which counts the number of parts of size d in partitions of order n. Its multivariable generating function is given by

$$\frac{1-x^d}{1-x^d y} \cdot \prod_{j=1}^{\infty}(1-x^j)^{-1}.$$

For the studying its mean, which counts the average number of occurences of the number d as a part in a partition of order n. one repeats the procedure for the other statistics and computes the one variable counterpart

$$\frac{x^d}{1-x^d} \cdot \prod_{j=1}^{\infty}(1-x^j)^{-1}.$$

4.5 Number of Parts with Given Multiplicity

A natural counterpart of the statistic in the previous section is the statistic counting the number of a given multiplicity d. Again if we allow a second variable y to keep track of these of the given multiplicity, we analyze the bivariable generating function

$$\prod_{j=1}^{\infty}\left(\frac{1}{1-x^j} + (y-1)x^{dj}\right),$$

Differentiating y and evaluating at 1, we obtain the generating function

$$\prod_{j=1}^{\infty}(1-x^j)^{-1} \cdot \sum_{j=1}^{\infty} x^{dj}(1-x^j) = \prod_{j=1}^{\infty}(1-x^j)^{-1} \cdot \frac{(1-x)x^d}{(1-x^d)(1-x^{d+1})}.$$

whose nth coefficient is the total number of parts of the given multiplicity d.

The study in [5] shows that asymptotically the value of the average number of parts of multiplicity d is equivalent to $\frac{6n}{\pi d(d+1)}$.

It turns out that the above statistic is closely related to the number of d-successions, which counts the number of times when the difference between two subsequent parts is d. The two statistic have almost the exact same distribution. A recent paper studying successions is [14]. For related research see also [3] and [15].

4.6 The Largest Repeated Part and the Longest Run

Consider the statistic with counts what is the size of the largest repeated part in a partition, which we define to be zero in the case when there is no repeated part size. Following the account in [9], it turns that the one variable generating function whose coefficients sum up the largest repeated part sizes accounting for all partitions of n equals

$$\prod_{j=1}^{\infty}(1-x^j)^{-1} \cdot \sum_{j=1}^{\infty} \frac{x^{2j}}{1-x^{2j}},$$

This can be generalized further if instead we are considering the statistic that instead counts the largest part that is repeated d or more times. Again, if no part is repeated enough times, we assign a value of 0 to out statistic. Analogously, the generating function for sum of this statistic over all partitions equals to

$$\prod_{j=1}^{\infty}(1-x^j)^{-1} \cdot \sum_{j=1}^{\infty} \frac{x^{dj}}{1-x^{dj}}.$$

For the case when $d = 1$ we recover the statistic L_n. In [20] one can find further results concerning the distribution of these statistics.

We call the *longest run* of a partition the maximum multiplicity of a number in the partition.

It can be easily shown that the generating function whose nth coefficient is the number of partitions of order n whose longest run is less than k equals

$$\prod_{j=1}^{\infty} \frac{1-x^{jk}}{1-x^j} = P(x)P(x^k)^{-1},$$

Therefore generating function whose nth coefficient is the number of partitions of order n whose longest run is exactly k equals

$$P(x)(P(x^{k+1})^{-1} - P(x^k)^{-1}).$$

Thus if we want to study, we are interested in the generating function

$$P(x) \cdot \sum_{k=1}^{\infty} k(P(x^{k+1})^{-1} - P(x^k)^{-1}).$$

Two useful references on the topic are [17] and [19], according to which the limit of this statistic is rather unusual. The expected value of this statistic was found to be asymptotically equivalent to $\left(4\sqrt{2} - \frac{6\sqrt{6}}{\pi}\right)\sqrt{n}$.

5 Main Results

We arrive at the main part of this thesis, in which we compute expectations of generalized versions of some of the statistics considered in the previous chapters. Let us define the statistics of interest to us.

Definition 17. *The sums of odd and even indexed parts are defined by*

$$X_{o,n}(\lambda) = X_{o,n} = \lambda_1 + \lambda_3 + \dots,$$

and

$$X_{e,n}(\lambda) = X_{e,n} = \lambda_2 + \lambda_4 + \dots,$$

respectively.

Obviously, for each $\lambda \in \mathcal{P}_n$,

$$X_{o,n} + X_{e,n} = n \qquad \text{and} \qquad X_{o,n} \geqslant X_{e,n}.$$

The last inequality is easily established since in the sum

$$(\lambda_1 - \lambda_2) + (\lambda_3 - \lambda_4) + \dots = X_{o,n} - X_{e,n}$$

each summand is non-negative.

Canfield, Savage, and Wilf [4] showed that $X_{e,n}$ might not contribute very much less than $n/2$ and asymptotically, as $n \to \infty$, the expected values $\mathbb{E}(X_{o,n})$ and $\mathbb{E}(X_{e,n})$ have the same leading term $n/2$. The difference in the second leading terms is, however, of order $c \times \sqrt{n}$.

Canfield et al. [4] consider the more general statistics $X_{m,i}(\lambda)$, the sum of those parts in partition λ whose index j is congruent to i mod m:

$$X_{m,i} = X_{m,i}(\lambda) = \sum_{j \equiv i \pmod m} \lambda_j, \qquad m \geqslant 1, \quad 1 \leqslant i \leqslant m.$$

They proved the following result.

Theorem 7. *For fixed integers $m \geq 1$ and i, there exists a constant $c_{m,i}$ such that*

$$\mathbb{E}(X_{m,i}) - \frac{n}{m} = \frac{m+1-2i}{2Cm}\sqrt{n}\log n + c_{m,i}\sqrt{n} + O(\log n). \tag{5.1}$$

The constants $c_{m,i}$ are given by

$$c_{m,i} = \frac{(\gamma + \log(2/C))(m+1-2i)}{Cm} + \frac{2}{Cm}\sum_{\ell=1}^{m-1} \frac{\omega^{-\ell(i-1)}}{1-\omega^{\ell}} \log(1-\omega^{\ell}),$$

where $C = \pi\sqrt{2/3}$, $\omega = e^{2\pi\sqrt{-1}/m}$, and γ is the Euler constant.

We can use this result to describe the asymptotics of the expectations of $X_{o,n}$ and $X_{e,n}$.

Lemma 6. *For large enough n, we have:*

$$\mathbb{E}(X_{o,n}) = \frac{n}{2} + \frac{\sqrt{6n}}{8\pi}\log n + \frac{\sqrt{6n}\log 2}{4\pi} + O(\log n),$$

$$\mathbb{E}(X_{e,n}) = \frac{n}{2} - \frac{\sqrt{6n}}{8\pi}\log n - \frac{\sqrt{6n}\log 2}{4\pi} + O(\log n).$$

Proof. To prove this Lemma, we apply the formula (5.1) in the Theorem 7 with $m = 2$ and $i = 1$. We obtain

$$\mathbb{E}(X_{o,n}) = \frac{n}{2} + \frac{3-2}{2^2\pi\sqrt{\frac{2}{3}}}\sqrt{n}\log n + \frac{1}{\pi\sqrt{\frac{2}{3}}}\frac{\omega^0}{1-\omega}(\log(1-\omega))\sqrt{n} + O(\log n).$$

For $m = 2$ we have $\omega = e^{2\pi i/2} = e^{\pi i} = \cos\pi = -1$. Substituting this in the above result we get

$$\begin{aligned}\mathbb{E}(X_{o,n}) &= \frac{n}{2} + \frac{3-2}{4\pi\sqrt{\frac{2}{3}}}\sqrt{n}\log n + \frac{\sqrt{3}\log 2}{2\sqrt{2}\pi}\sqrt{n} + O(\log n) \\ &= \frac{n}{2} + \frac{\sqrt{6n}}{8\pi}\log n + \frac{\sqrt{6n}\log 2}{4\pi} + O(\log n).\end{aligned}$$

We proceed in the same way with $\mathbb{E}(X_{e,n})$. In this case we set $m = 2$ and $i = 2$. So, we have

$$\mathbb{E}(X_{e,n}) = \frac{n}{2} + \frac{3-4}{2^2\pi\sqrt{\frac{2}{3}}}\sqrt{n}\log n + \frac{1}{\pi\sqrt{\frac{2}{3}}}\frac{\omega^{-1}}{1-\omega}(\log(1-\omega))\sqrt{n} + O(\log n).$$

For $m = 2$ and $i = 2$, we have the same value of $\omega = -1$ and

$$\mathbb{E}(X_{e,n}) = \frac{n}{2} - \frac{1}{4\pi\sqrt{\frac{2}{3}}}\sqrt{n}\log n - \frac{\sqrt{3}\log 2}{2\sqrt{2}\pi}\sqrt{n} + O(\log n)$$
$$= \frac{n}{2} - \frac{\sqrt{6n}}{8\pi}\log n - \frac{\sqrt{6n}\log 2}{4\pi} + O(\log n).$$

□

The novel results of this work focus on the study of other generalized versions of previously statistics considered, namely the number of odd and even indexed parts.

Definition 18. *Let*

$$Z_{o,n}(\lambda) = Z_{o,n} = m_1 + m_3 + \ldots,$$

and

$$Z_{e,n}(\lambda) = Z_{e,n} = m_2 + m_4 + \ldots,$$

denote the number of odd and even indexed parts of the partition $\lambda \in \mathcal{P}_n$, respectively.

Recall that m_j is the multiplicity of part j, see (2.5). Obviously,

$$Z_{o,n} + Z_{e,n} = Z_n.$$

For the simpler statistic Z_n, the expected value $\mathbb{E}(Z_n)$ was first studied in terms of statistical physics by Husimi [11], and later on by Kessler and Livingston [13]. We present these results in the following lemma:

Lemma 7. *For large enough n, the following holds:*

$$\mathbb{E}(Z_n) = \mathbb{E}(L_n) = \frac{\sqrt{6n}}{2\pi}\left(\log n + 2\gamma - \log\frac{\pi^2}{6}\right) + O(\log n),$$

where $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right) \approx 0.5772156$ *is the Euler constant.*

Erdös and Lehner were apparently the first who studied limiting distributions of random partition statistics. In particular, they showed in [6] that $\frac{\pi Z_n}{\sqrt{6n}} - \log\sqrt{\frac{6n}{\pi}}$ converges in distribution (weakly), as $n \to \infty$, to a random variable whose distribution function is $e^{-e^{-x}}$, $-\infty < x < \infty$.

The main *goal* of this work is to find asymptotic expressions for $\mathbb{E}(Z_{o,n})$ and $\mathbb{E}(Z_{e,n})$ as $n \to \infty$.

In order to do that, we use the following clever trick, which will bring us in the position to apply the results of Canfield et al. [4] in order to compute the expectations of $Z_{o,n}$ and $Z_{e,n}$.

Lemma 8. *The random variables $X_{o,n} - X_{e,n}$ and $Z_{o,n}$ have coinciding probability distributions with respect to probability measure $\mathbb{P}$.*

Proof. Consider

$$\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathcal{P}_n$$

and its conjugate partition

$$\lambda' = (\lambda'_1, \ldots, \lambda'_l) \in \mathcal{P}_n.$$

Recall that λ'_j in fact equals to the number of parts in λ that are greater or equal than j.

Because λ is also the conjugate of λ' this also implies that λ_j equals the number of parts in λ' that are greater or equal than j, and λ_{j+1} is the number of parts in λ' that are greater or equal than $j+1$.

Hence, $\lambda_j - \lambda_{j+1}$ is the number of parts in λ' that are equal to j, or in other words m'_j, where m'_j is the multiplicity of part j in λ'. Conversely, from the point of view of λ' we have that

$$\lambda'_j - \lambda'_{j+1} = m_j$$

for all $j \geq 1$. It then follows that

$$\begin{aligned} X_{o,n}(\lambda') - X_{e,n}(\lambda') &= (\lambda'_1 - \lambda'_2) + (\lambda'_3 - \lambda'_4) + \ldots \\ &= m_1 + m_3 + \ldots \\ &= \sum_{i=0}^{\infty} m_{2i+1} \\ &= Z_{o,n}(\lambda) \end{aligned}$$

Since conjugation is a bijection on the set of partitions of n onto itself, then it follows that for each $s \in \mathbb{N}$, there are equally many partitions $\lambda \in \mathcal{P}_n$ whose statistics $X_{o,n} - X_{e,n}$ and $Z_{o,n}$ equal to s simultaneously. □

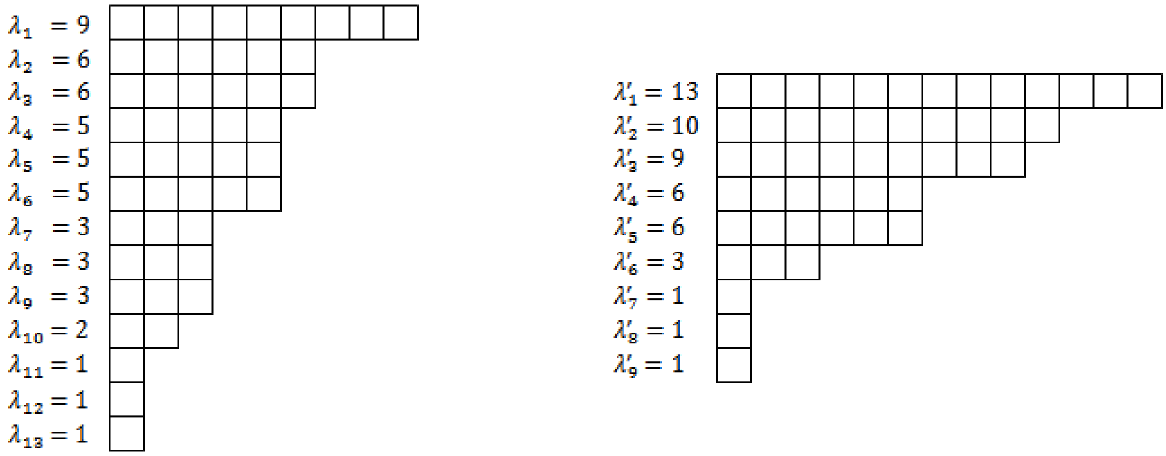


Figure 5.1: Ferrers Diagram of the partition and the conjugate partition for $n = 50$ (Examples 2 and 3)

Example 8. Going back to the partition of $n = 50$ in Example 2 and 3, we can see that $\lambda_3^{'} = 9$, which means that there are 9 parts in λ that are $\geqslant 3$. This is also confirmed by Figure 5.1. This implies that there are $\lambda_3 = 6$ parts in $\lambda^{'}$ that are $\geqslant 3$ and there are $\lambda_4 = 5$ parts in $\lambda^{'}$ that are $\geqslant 4$. When we take the difference $\lambda_3 - \lambda_4$, we have just 1 part in $\lambda^{'}$ that is 3. This also yields the multiplicity $m_3^{'} = 1$.

Theorem 8. *If $n \to \infty$, then*

$$\mathbb{E}(Z_{o,n}) = c_0\sqrt{n}\log n + c_1\sqrt{n} + O(\log n)$$

and

$$\mathbb{E}(Z_{e,n}) = c_0\sqrt{n}\log n - c_2\sqrt{n} + O(\log n),$$

where

$$c_0 = \frac{\sqrt{6}}{4\pi} = 0.1949242\ldots$$
$$c_1 = \frac{\sqrt{6}}{2\pi}\log 2 = 0.27022319\ldots$$
$$c_2 = \frac{\sqrt{6}}{2\pi}\left(2\gamma - \log\frac{\pi^2}{3}\right) = 0.460467898\ldots$$

Proof. From Lemma 8, which attributes to the random variables $X_{o,n} - X_{e,n}$ and $Z_{o,n}$ coinciding probability distributions it follows that

$$\mathbb{E}(Z_{o,n}) = \mathbb{E}(X_{o,n}) - \mathbb{E}(X_{e,n}).$$

Replacing $\mathbb{E}(X_{o,n})$ and $\mathbb{E}(X_{e,n})$ by the asymptotic expressions given in Lemma 6, we obtain

$$\begin{aligned}\mathbb{E}(Z_{o,n}) &= \left(\frac{n}{2} + \frac{\sqrt{6n}}{8\pi}\log n + \frac{\sqrt{6n}\log 2}{4\pi} + O(\log n)\right) \\ &\quad - \left(\frac{n}{2} - \frac{\sqrt{6n}}{8\pi}\log n - \frac{\sqrt{6n}\log 2}{4\pi} + O(\log n)\right) \\ &= \frac{\sqrt{6n}}{4\pi}\log n + \frac{\sqrt{6n}}{2\pi}\log 2 + O(\log n).\end{aligned}$$

In the same way, using the result of Lemmas 6 and 7, we have

$$\begin{aligned}&\mathbb{E}(Z_{e,n}) = \mathbb{E}(X_{e,n} - X_{o,n} + L_n) = -\mathbb{E}(X_{o,n} - X_{e,n}) + \mathbb{E}(L_n) = -\mathbb{E}(Z_{o,n}) + \mathbb{E}(L_n) \\ &= -\left(\frac{\sqrt{6n}}{4\pi}\log n + \frac{\sqrt{6n}}{2\pi}\log 2 + O(\log n)\right) + \frac{\sqrt{6n}}{2\pi}\left(\log n + 2\gamma - \log\frac{\pi^2}{6}\right) + O(\log n) \\ &= \frac{\sqrt{6n}}{4\pi}\log n - \frac{\sqrt{6n}}{2\pi}\left(2\gamma - \log\frac{\pi^2}{3}\right) + O(\log n).\end{aligned}$$

□

Intuitively, it seems that $Z_{o,n}$ and $Z_{e,n}$ should not essentially differ. This is true for the first term asymptotics, however, the coefficients of $\sqrt{n}$ in the second terms are essentially different as $n \to \infty$.

A Dedekind's Functional Equation in Terms of F

We will assume the notations of Chapter 3 throughout this appendix. The purpose of this chapter is to ptove the following

Claim 1. *The function F satisfies the following functional equation*

$$F\left(\exp\left(\frac{2\pi ih}{k}-\frac{2\pi z}{k^2}\right)\right)=\omega(h,k)\left(\frac{z}{k}\right)^{\frac{1}{2}}F\left(\exp\left(\frac{2\pi iH}{k}-\frac{2\pi}{z}\right)\right)\exp\left(\frac{\pi}{12z}-\frac{\pi z}{12k^2}\right)$$

where $\omega(h,k)=e^{\pi is(h,k)}$.

We recall that from the functional equation satisfied by the Dedekind eta and its relation F we deduced that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$, $c>0$ and if we denoted $\tau'=\frac{a\tau+b}{c\tau+d}$, then

$$F(e^{2\pi i\tau})=F(e^{2\pi i\tau'})\exp\left(\frac{\pi i(\tau-\tau')}{12}\right)[-i(c\tau+d)]^{\frac{1}{2}}\exp\left[\pi i\left(\frac{a+d}{12c}+s(-d,c)\right)\right].$$

It suffices to make the appropriate choice of the matrix. We will show that $a=H$, $b=-\dfrac{hH+1}{k}$ $c=k$ and $d=-h$ works. Note that the determinant of the matrix with these entries is indeed 1. Also, because of the choice of H, it follows that $b\in\mathbb{Z}$ and clearly $c>0$ is satisfied as well.

Note that automatically we have then that $s(-d,c)=s(h,k)$. using that $\tau=\dfrac{iz+hk}{k^2}$, we compute

$$\begin{aligned}a\tau+b&=\frac{Hiz+hHk}{k^2}-\frac{hHk+k}{k^2}=\frac{Hiz-k}{k^2},\\ c\tau+d&=\frac{iz+hk}{k}-h=\frac{iz}{k},\end{aligned}$$

hence their ratio equals

$$\tau' = \frac{Hiz - k}{kiz} = \frac{iz^{-1}k + H}{k}.$$

Finally, set this into our functional equation and compute

$$F\left(\exp\left(\frac{2\pi ihk}{k^2} - \frac{2\pi z}{k^2}\right)\right) = F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi k}{zk}\right)\right)\left[-i\left(\frac{iz}{k}\right)\right]^{\frac{1}{2}}$$
$$\times \exp\left(\frac{\pi i}{12}\left(\frac{iz - iz^{-1}k^2 + hk - Hk}{k^2}\right)\right)\exp\left[\pi i\left(\frac{H-h}{12k} + s(h,k)\right)\right]$$
$$= F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right)\left(\frac{z}{k}\right)^{\frac{1}{2}}\exp\left(\frac{\pi i}{12}\left(\frac{h-H}{k} + \frac{H-h}{k} + \frac{iz}{k^2} - \frac{i}{z}\right)\right)\exp(\pi is(h,k))$$
$$= F\left(\exp\left(\frac{2\pi iH}{k} - \frac{2\pi}{z}\right)\right)\left(\frac{z}{k}\right)^{\frac{1}{2}}\exp\left(\frac{\pi}{12z} - \frac{\pi z}{12k^2} + \pi is(h,k)\right),$$

which is the desired identity.

B Bessel Functions and Their Evaluation

The goal of this chapter is to provide a brief of the properties of Bessel functions that are needed in obtaining an expression for the Rademacher series. For a more thorough resource on Bessel functions the reader is invited to refer to [21].

Bessel functions arrive as solutions of differential equations, which are also named after Bessel, and their are defined by means of a power series

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}.$$

In the following we will though work a modified version of the Bessel functions which we define as

$$\begin{aligned} I_\nu(z) &= e^{-\frac{1}{2}\nu\pi i} J_\nu(iz) \\ &= \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \end{aligned}$$

One of our points of interest is to describe this by means of an integral equation. The first is to use an integral equation of the Gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} t^{-z} e^t dt,$$

to obtain that

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{2\pi i} \sum_{m=0}^{\infty} \int_{c-\infty i}^{c+\infty i} \frac{\left(\frac{z}{2}\right)^{2m}}{m!} t^{-\nu-m-1} e^t dt.$$

Using that summation and integration commute, we refine this to

$$\begin{aligned} I_\nu(z) &= \frac{\left(\frac{z}{2}\right)^\nu}{2\pi i} \int_{c-\infty i}^{c+\infty i} \sum_{m=0}^{\infty} \frac{\left(\left(\frac{z}{2}\right)^2 t^{-1}\right)^m}{m!} t^{-\nu-1} e^t dt \\ &= \frac{\left(\frac{z}{2}\right)^\nu}{2\pi i} \int_{c-\infty i}^{c+\infty i} \exp\left\{\left(\frac{z}{2}\right)^2 \frac{1}{t}\right\} t^{-\nu-1} e^t dt. \end{aligned}$$

Summarizing, the above, we obtain the desired integral equation for $I_\nu(z)$

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{2\pi i} \int_{c-\infty i}^{c+\infty i} t^{-\nu-1} \exp\left(t + \frac{z^2}{4t}\right) dt. \tag{B.1}$$

assuming that $c > 0$ and $\mathrm{Re}(\nu) > 0$.

Of this special interest to us are the Bessel functions whose orders are half integers, in other words for which $\nu = n + \frac{1}{2}$, where $n \in \mathbb{Z}$. Turns, these admit expressions in terms of certain trigonometric and algebraic functions on z. We only need the case $n = 1$, for which we have that

Lemma 9. *For* $\nu = \frac{3}{2}$

$$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz}\left(\frac{\sinh z}{z}\right). \tag{B.2}$$

Proof. We will show that both sides equal a certain expression. Start with the right hand side

$$\begin{aligned} \sqrt{\frac{2z}{\pi}} \frac{d}{dz}\left(\frac{\sinh z}{z}\right) &= \sqrt{\frac{2z}{\pi}} \left(\frac{\cosh z}{z} - \frac{\sinh z}{z}\right) \\ &= \sqrt{\frac{2z}{\pi}} \left\{\left(\frac{1}{2!} - \frac{1}{3!}\right) z + \left(\frac{1}{4!} - \frac{1}{5!}\right) z^3 + \left(\frac{1}{6!} - \frac{1}{7!}\right) z^5 + \dots\right\} \\ &= \sqrt{\frac{2z}{\pi}} \sum_{m=0}^{\infty} \left\{\frac{1}{(2m+2)!} - \frac{1}{(2m+3)!}\right\} z^{2m+1} \\ &= \sqrt{\frac{2z}{\pi}} \sum_{m=0}^{\infty} \left\{\frac{2m+2}{(2m+3)!}\right\} z^{2m+1} \\ &= \sqrt{\frac{2z}{\pi}} \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m)!(2m+1)(2m+3)}. \end{aligned}$$

For the left hand side, we will make use of the following identities involving the Gamma function

$$\Gamma(z+1) = z\Gamma(z) \quad \text{and} \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi},$$

and the following elementary identity on factorials

$$2^m m!(2m-1)!! = (2m)!!(2m-1)!! = (2m)!.$$

Therefore, we compute

$$\begin{aligned}
I_{\frac{3}{2}}(z) &= \left(\frac{z}{2}\right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\Gamma\left(m+\frac{5}{2}\right)} \\
&= \left(\frac{z}{2}\right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\frac{1}{2}\right)} \\
&= \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\frac{3}{2}} \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m} 2^m}{m!(2m-1)!!\left(m+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)} \\
&= \frac{1}{4}\frac{2z}{\pi} \sum_{m=0}^{\infty} \frac{z^{2m+1}}{2^m m!(2m-1)!!\left(m+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)} \\
&= \frac{2z}{\pi} \sum_{m=0}^{\infty} \frac{z^{2m+1}}{(2m)!(2m+1)(2m+3)},
\end{aligned}$$

and obtain the desired result. □

B.1 Evaluation of Bessel Functions of Half Order

Here we compute the evaluation of $I_{\frac{3}{2}}(z)$ at the point

$$\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}$$

Plug-in this value in equation (B.2), we obtain

$$I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right) = \frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\frac{\pi^{\frac{1}{2}}}{k^{\frac{1}{2}}}\frac{2^{\frac{1}{4}}}{3^{\frac{1}{4}}}\left(n-\frac{1}{24}\right)^{\frac{1}{4}}\frac{d}{dz}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right\}}{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}}\right).$$

Making the conversion between dz and dn, we obtain

$$dz = \frac{\pi}{2k}\left(\frac{2}{3}\left(n-\frac{1}{24}\right)\right)^{-\frac{1}{2}}\frac{2}{3}dn,$$

therefore

$$I_{\frac{3}{2}}\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right) = \frac{2^{\frac{1}{2}}}{k^{\frac{1}{2}}}\frac{2^{\frac{1}{4}}}{3^{\frac{1}{4}}}\left(n-\frac{1}{24}\right)^{\frac{1}{4}}\frac{2k}{\pi}\frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}}\left(n-\frac{1}{24}\right)^{\frac{1}{2}}\frac{3}{2}\frac{k}{\pi}\frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}}\frac{d}{dn}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right\}}{\sqrt{\left(n-\frac{1}{24}\right)}}\right)$$

$$= \frac{6^{\frac{3}{4}}k^{\frac{3}{2}}}{\pi^2}\left(n-\frac{1}{24}\right)^{\frac{3}{4}}\frac{d}{dn}\left(\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n-\frac{1}{24}\right)}\right\}}{\sqrt{\left(n-\frac{1}{24}\right)}}\right) \quad \text{(B.3)}$$

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