ASYMPTOTICS AND BLENDING IN THE MODELING OF WELDING

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ABSTRACT

Important welding questions are often easy to ask and difficult to answer. For example, the question "what is the width of the weld?" is essential for understanding the strength of a weld, but it is currently answered through trial and error, or through sophisticated numerical modeling. In this work, it is proposed that there is a third approach based on a deep understanding of physics, and a basic command of mathematics. From the point of view of the practitioner, the answer can be approximated using formula, tables, and graphs of great generality. In this approach, the aspect of interest of the weld is reduced to its minimal representation, neglecting all secondary physical phenomenon. Mathematically, this corresponds to an asymptotic regime. In contrast with other asymptotic techniques such as perturbation analysis, in the proposed methodology, blending techniques are applied. The advantage of these blending techniques is that they approach the exact solutions (typically within a few percentage points) but involve only a few constants that are suitable to be transmitted in print. Much of the existing work on heat transfer outside welding is summarized in this form, but the approach has not been applied to welding yet. Some welding problems are outside the range of standard blending techniques, and an extension of the techniques will be discussed. The application of this approach will also be discussed using the width of the weld and other related problems.

Keywords: Heat transfer, Asymptotics, Blending, Scaling, Modeling

INTRODUCTION

The complexity of an engineering problem can be decomposed into the complexity of its physics, which can be assessed by the number of parameters related to the physics of the problem, and the complexity of its geometry, given by the number of parameters necessary to capture the shape of the problem [1]. In the field of welding, the geometry of welding processes (e.g. bead geometry and joint preparation) is generally not complex; however, the large number of parameters relative to relevant physics such as heat and mass transfer, fluid dynamics, electromagnetism, thermodynamics, as well as their tight coupling make welding very difficult to understand at intuitive level and more complex than most other engineering fields.

Typically, there are three approaches to analyze welding problems: trial and error (making a prototype), numerical simulation and design rules. Trial and error is the most common and reliable way in welding procedure development as it is capable of producing the reality with complex geometries and provides direct measurements from experiments. However, case-by-case results of specific processes are difficult to be synthesized for the

next trial or extended to new operating parameters. The horizontal axis in Fig. 1 is the cost of making a prototype and the vertical axis represents the total cost.

Sophisticated computer simulations are cost effective in the fields such as aerospace and nuclear industry where the prototype is time-consuming and expensive to make. In welding, running a weld may cost on the order of one to thousand dollars and within this range of cost, making a prototype or trial and error usually costs much less money than developing or implementing a comprehensive simulation. Numerical models excel at dealing with the complexity of geometries, but implementing multicoupled physics is often challenging.

Design rules (the red horizontal line in Fig. 1) can significantly lower costs, and enable a design approach to welding procedures which are usually done by trial and error. Typically, the design rules can be expressed in the form of simple formulae and correction factors. The simple formulae are the asymptotic solutions to ideal case and their correction factors are developed to capture the departures from the ideal cases.

Instead of considering every point in the space-time domain, the target of the design rules is to predict the characteristic values such as a maximum velocity, or maximum temperature. The concept of characteristic values was discussed in detail in [2] to normalize the governing equations into proper magnitudes and it is typically chosen as the maximum absolute value of the function of interest. Asymptotic analysis and blending do not present convergence problems and they are well suited for multiphysics problems.

The use of design rules can provide guidelines for follow-up tests and significantly reduce the money and effort put in the trial and error stage. There are very few of these design rules in the field of welding; for example, the practical question "what is the resulting width of a weld bead by a given set of parameters" is difficult to be answered quickly within the existing knowledge.
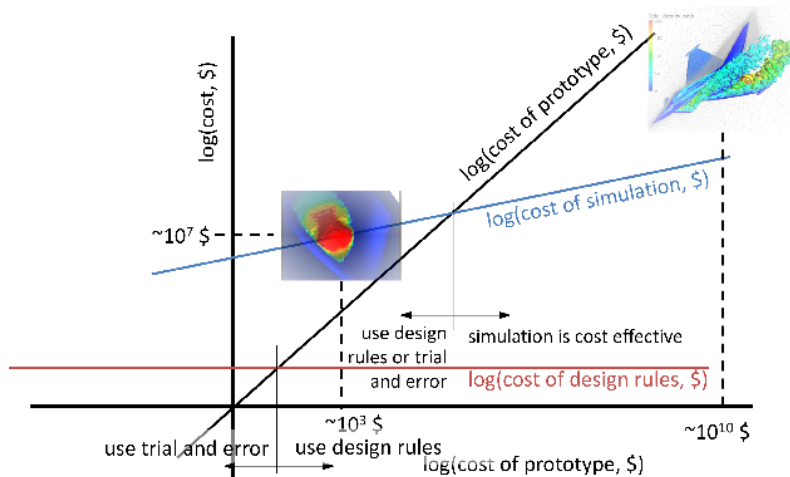

Fig. 1 Costs of simulation, prototypes, and design rules. For the aerospace industry, a prototype is much more expensive than simulations. For welding, a prototype is often cheaper than simulations, resulting in trial-and-error approaches. Design rules are much less expensive than simulations, and should enable a design approach where it is seldom done currently.

Expressions in explicit form which are general, accurate, easy-to-calculate are convenient for transmission and amenable to be used by practitioners in industry. This

paper proposes a systematic methodology based on asymptotics and blending techniques to develop a set of formulae in standardized form (asymptotics and correction factors) that can predict (or estimate reasonably) important weld properties valid for general welding procedures with different alloys, process conditions and operating parameters. The predictions sought would target the size of the weld pool, heating and cooling rates, melting efficiencies, etc., such that they could be coupled with metallurgical and performance models.

The systematic methodology advocated here to obtain engineering formulas of interest can be roughly summarized as a six-step procedure call the Minimal Representation and Correction Factors (MRCF) [3]. MRCF considers the dominant phenomena and the deviations from the ideal case using the exact analytical solutions, experimental measurements or results from numerical models. Once a complex multicoupled problem is minimally represented by its dominant mechanisms, explicit, closed-form expressions can be developed.

Blending techniques will be applied to extend the asymptotic solutions to the whole domain of dimensionless groups capturing the characteristic values of interest. Blending has been widely applied in many disciplines such as heat transfer [4, 5], including moving heat sources [6], mass transfer [7], and fluid dynamics [8]. With blending, correction factors can be rigorously derived to capture the deviation from asymptotic behavior and define the validity of the asymptotics. Blending techniques and their extension are the main focus of this paper.

MRCF: MINIMAL REPRESENTATION AND CORRECTION FACTOR

The approach presented here for asymptotic analysis of engineering problems is the Minimal Representation and Correction Factor (MRCF). It was first proposed in [3] and has been successfully implemented to construct general expressions for thermal characteristics of moving heat sources in [6, 9]. The MRCF approach consists of the following six steps:

- List all physics considered relevant
- Identify dominant factors
- Solve approximate problem considering only dominant factors (Minimal Representation)
- Check for self-consistency
- Compare predictions to "reality"
- Create correction factors

The minimal representation of a problem corresponds to the formulation of the problem with parameters such that all secondary phenomena become negligible. There are various techniques for obtaining the minimal representation. They can be roughly divided into manual and computational techniques. Manual Asymptotics include "Informed" Dimensional Analysis: generate dimensionless groups based on knowledge about system [10-15], Inspectional Analysis: dimensionless groups from normalized equations[16-19],

Order of Magnitude/Balancing Techniques [20-22]. Computational Asymptotics include Statistical analysis/data mining[23-25], and analysis of the Governing equations[2, 3, 26].

The MRCF is an iterative approach of identifying dominant phenomena and solve the simple solutions of this minimal represented problem by considering only dominant factors and characteristic values of the simplified problem can be addressed by asymptotics to capture the behavior of the system for combinations of parameters that result in the same dominant phenomena (which is called a Regime). Thus, a multicoupled complex problem will be decomposed into its minimal representation of the dominant mechanisms. The dominant factors are typically unknown and need ingenuity and expertise to make a reasonable postulation. The advantage of the MRCF is that this iterative process transverses all possible choice of dominant factors and self-consistency is checked for the obtained results to assure the neglected phenomenon are secondary. Departure from the reality caused by the neglect of secondary phenomenon will be accounted by the correction factors, which capture the deviation from the ideal solutions. Correction factors can be calibrated with experimental numerical results to account for secondary phenomena not considered explicitly.

<div align="center">IMPORTANT CONCEPTS AND NOTATION IN MRCF</div>

Because of the complexity of the concepts involved in MRCF, careful notation is essential. We will call here $\{U\}$ the set of dependent variables with elements $u(\{X\}, \{P\})$, where $\{X\}$ is the set of independent variables, and $\{P\}$ is the set of problem parameters.

When the parameters of an engineering problem change, the resulting characteristic values will be influenced even if the independent variables remain unchanged. Thus, the set of problem parameters $\{P\}$ is also included as arguments of the solution functions. Accordingly, u_c represents the characteristic value of $u(\{X\}, \{P\})$.

For the sake of greater generality, normalization will be applied to transform relevant variables into dimensionless form and thus, results in different units or from different operating parameters can be generalized and compared within the same scale. The magnitude $\hat{u}_{c,i}$ is the asymptotic behavior of $u_c(\{P\})$ in Regime i (e.g. Regime I, Regime II) where the value of the dimensionless groups based on parameters is asymptotically large or small. Very often, the asymptotic behavior has power-law dependence on the dimensionless groups.

In normalized form, $u^*(\{X^*\}, \{\Pi\})$ is the dimensionless dependent variable where $\{X^*\}$ is the set of normalized independent variables, and $\{\Pi\}$ is a set of independent dimensionless groups based only on the problem parameters. $u_c^*(\{\Pi\})$ is the characteristic value of $u^*(\{X^*\}, \{\Pi\})$. Similarly, $\widehat{u^*}_{c,i}(\{\Pi\})$ is the asymptotic behavior of $u_c^*(\{\Pi\})$ in Regime i (often in the form of a power law).

Dimensional analysis indicates that a problem can be captured by a set of n dimensionless groups, where $n = m - k$, m is the number of independent physical magnitudes involved in the problem, and k is the set of independent reference units of the problem [27]. Typically, the functional dependence between the characteristic value targeted (u_c) and the dimensionless groups is monotonic, thus the number of regimes is two per dimensionless group (close to zero, and close to infinity, then the number of regimes n_r is:

$$n_r = 2^n \tag{1}$$

FOUNDATIONS OF BLENDING TECHNIQUES

Blending is the general description of a family of techniques of interpolation between two asymptotic solutions. In this work, the focus is on techniques that use a very small number of parameters (typically one or two). The advantage is that they are very easy to communicate and implement. Surprisingly, in many practical cases these techniques have a very small error (typically few percentage points) against the exact solution they are interpolating.

In this work we will use the notation $\mathcal{B}$ to represent the blending of asymptotics $\widehat{u^*}_{c,i}(\{\Pi\})$, corresponding to regime i (Regime I, Regime II, etc.) using the set of blending parameters $\{B\}$, thus:

$$u_c^*(\{\Pi\}) \approx \widehat{u_c^*}^+(\{\Pi\}) = \mathcal{B}(\widehat{u^*}_{c,I}(\{\Pi\}), \widehat{u^*}_{c,II}(\{\Pi\}), \widehat{u^*}_{c,III}(\{\Pi\}) \dots \{B\}) \tag{2}$$

where the symbol $\widehat{}$ indicates that the magnitude is an asymptotic approximation and the $^+$ superscript represents the improvement over the asymptotics after blending.

There are several approaches to determine the set of blending constants $\{B\}$ and the minimax optimum procedure [6] applied in this paper is used to determine the optimum set of blending constants which minimizes the maximum error defined as:

$$\text{Maxerror} = \max_{\Pi}\left[\ln\frac{\widehat{u_c^*}^+(\{\Pi\})}{u_c^*(\{\Pi\})}\right] \tag{3}$$

where $u_c^*(\{\Pi\})$ is the target dimensionless characteristic value, which can be an exact analytical solution, experimental measurements or numerical simulation results. The maximum value is explored over the whole domain of $\{\Pi\}$, and the minimum is explored over the blending parameters $\{B\}$.

Eqn. (3) has the advantage of revealing the comparable magnitudes for large errors and it is equivalent to the definition of relative error when the error is small. The set of blending constants $\{B\}$ needs be determined only once for each blending function.

One essential property of the blending function $\mathcal{B}$ is that it shares the asymptotic behavior of the original function:

$$u_c^*(\{\Pi\}) \to \widehat{u^*}_{c,i}(\{\Pi\}) \quad \text{when}\{\Pi\} \to \{\Pi\}_i \quad (\text{Regime i})$$

DEVELOPMENT OF CORRECTION FACTORS

Based on the blending of Eqn. (1), correction factors can be established for each asymptotic regime to capture the departure of the blended approximation from asymptotic behavior. In dimensionless form:

$$u_c^*(\{\Pi\}) \approx \widehat{u_{c,i}^*}^+(\{\Pi\}) = \widehat{u^*}_{c,i}(\{\Pi\})\mathcal{B}\left(\frac{\widehat{u^*}_{c,I}(\{\Pi\})}{\widehat{u^*}_{c,i}(\{\Pi\})}, \frac{\widehat{u^*}_{c,II}(\{\Pi\})}{\widehat{u^*}_{c,i}(\{\Pi\})}, \frac{\widehat{u^*}_{c,III}(\{\Pi\})}{\widehat{u^*}_{c,i}(\{\Pi\})} \dots \{B\}\right)$$
$$= \widehat{u^*}_{c,i}\, f_{u_{c,i}}(\Pi) \quad \text{for Regime i} \tag{4}$$

and its dimensional counterpart:

$$u_c(\{\Pi\}) \approx \widehat{u_{c,i}}^+(\{\Pi\}) = \hat{u}_{c,i}(\{\Pi\})\mathcal{B}\left(\frac{\hat{u}_{c,I}(\{\Pi\})}{\hat{u}_{c,i}(\{\Pi\})}, \frac{\hat{u}_{c,II}(\{\Pi\})}{\hat{u}_{c,i}(\{\Pi\})}, \frac{\hat{u}_{c,III}(\{\Pi\})}{\hat{u}_{c,i}(\{\Pi\})} \dots \{B\}\right)$$
$$= \hat{u}_{c,i}\, f_{u_{c,i}}(\Pi) \quad \text{for Regime i} \tag{5}$$

where the ith term inside the correction factor is equal to 1 and the set of blending constants $\{B\}$ is the same as in Eqn. (2) and $f_{u_{c,i}}(\Pi)$ is a shorthand notation for the blending function that leads to correction factors. In the shorthand notation, the set of blending parameters is not stated explicitly, but it is present.

As the blending function $\mathcal{B}$ maintains the asymptotic behavior of the original function, the value of the correction factor $f_{u_{c,i}}(\Pi)$ tends to the value of 1 or at the same order as 1 in its corresponding Regime i:

$$f_{u_{c,i}}(\Pi) \to 1 \quad \text{when}\{\Pi\} \to \{\Pi\}_i \quad (\text{Regime i})$$

Eqn. (4) and Eqn. (5) capture the deviation of the blended approximation from the asymptotic behavior. These equations can also be used to determine the range of validity of the asymptotic equations for a given acceptable error, and can also be used to account for systematic errors from the mathematical treatment of the asymptotics, or random errors from the physics caused by neglected secondary phenomenon. The correction factors developed are explicit and can typically be calculated with a handheld calculator or a spreadsheet.

1D BLENDING

When the target function depends on only one dimensionless group, the system has two asymptotic regimes (Eqn. (1)). Thus, according to Eqn. (4):

$$u_c^*(\{\Pi\}) \approx \widehat{u_c^*}^+(\{\Pi\}) = \mathcal{B}(\widehat{u^*}_{c,I}(\{\Pi\}), \widehat{u^*}_{c,II}(\{\Pi\}), \{B\}) \tag{6}$$

Blending in 1D (when blending depends on only one dimensionless group) is the simplest case and the basis of the extension of blending to more complex scenarios. Standard 1D blending was treated in [28-31]; however, these studies could not account for

the case when the asymptotes did not cross. Also, the standard blending technique can result in high errors when the functions involved are not power laws.

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STANDARD 1D BLENDING

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In the case where the two asymptotic solutions are known and intersect with each other only once, the two asymptotes can be blended with only one blending constant (in this case $\{B\} = \{n\}$):

$$u_c^*(\{\Pi\}) \approx \widehat{u_c^*}^+(\{\Pi\}) = [\widehat{u^*}_{c,I}(\{\Pi\})^n + \widehat{u^*}_{c,II}(\{\Pi\})^n]^{\frac{1}{n}} \tag{7}$$

where the optimal value of n needs to be determined only once for each blending function. A sufficient (more restrictive than necessary) condition to permit the use of Eqn. (7) is that the target function is monotonous on its dependence on the dimensionless group. A vast number of engineering problems have an asymptotic behavior in the form of power laws. Eqn. (7) can also be used to create the correction factors $f_{u_{c,I}}(\Pi)$ (based on asymptotic in Regime I) and $f_{u_{c,II}}(\Pi)$ (based on asymptotic in Regime II) as follows:

$$f_{u_{c,I}}(\Pi) = \mathcal{B}\left(1, \frac{\widehat{u^*}_{c,II}(\{\Pi\})}{\widehat{u^*}_{c,I}(\{\Pi\})}, n\right) = \left\{1 + \left[\frac{\widehat{u^*}_{c,II}(\{\Pi\})}{\widehat{u^*}_{c,I}(\{\Pi\})}\right]^n\right\}^{\frac{1}{n}} \tag{8}$$

$$f_{u_{c,II}}(\Pi) = \mathcal{B}\left(\frac{\widehat{u^*}_{c,I}(\{\Pi\})}{\widehat{u^*}_{c,II}(\{\Pi\})}, 1, n\right) = \left\{1 + \left[\frac{\widehat{u^*}_{c,II}(\{\Pi\})}{\widehat{u^*}_{c,I}(\{\Pi\})}\right]^{-n}\right\}^{\frac{1}{n}} \tag{9}$$

Eqn. (8) and Eqn. (9) are exactly equivalent and they capture the difference between the blended approximation and different starting asymptotic expressions. They tend to the exact value of 1 in their corresponding limits for all finite values of n. The value of n for Eqn. (8) and Eqn. (9) are the same as for Eqn. (7).

Standard 1D blending provides a new paradigm to obtain a general solution over the whole domain in terms of simple, known, limiting solutions with minimal degree of explicit empiricism, which is typically caused by the additional introduction of the blending constants $\{B\}$. Eqn. (7) is a canonical expression for the formulation of correlating equations and has the advantage of simplicity, generality, inherent accuracy, and convergence to theoretical solutions in the limits. Nevertheless, it must be applied with an understanding of its restrictions. It can not represent processes with irregular transition in the non-asymptotic part and Eqn. (7) only applies for the situation where asymptotes in the limiting regimes have a single intersection. What's more, the asymptotic solutions must be free of singularities because the existence of singularity will persist and disrupt the prediction even though the singularity occurs outside asserted range of the asymptote [30, 31].

1D Blending of Non-Crossing Asymptotics

Eqn. (7) is not applicable for non-crossing asymptotics. For any finite positive value of n, the blended result will always be the asymptotic of larger magnitude and the smaller asymptotic would be chosen as the approximation for any negative value n.

A modified blending technique is proposed here to extend the applicability of Eqn. (7), by applying a factor $\exp(a\Pi^b)$ to either of the asymptotes to force the two asymptotics to have a single intersection. This approach is useful for asymptotic behavior weaker than exponential; for example, it can not be used in the case such as a Γ function. The 1D blending function for non-crossing asymptotics has the following expression:

$$u_c^*(\{\Pi\}) \approx \widehat{u_c^*}^+(\{\Pi\}) = \mathcal{B}(\widehat{u^*}_{c,I}(\{\Pi\}), \widehat{u^*}_{c,II}(\{\Pi\}), \{n, a, b\})$$
$$= \{\widehat{u^*}_{c,I}(\{\Pi\})^n + [\widehat{u^*}_{c,II}(\{\Pi\}) \exp(a\Pi^b)]^n\}^{\frac{1}{n}} \tag{10}$$

where in this case, $\{B\} = \{n, a, b\}$.

The reason of choosing exponential function as the form of modification factor is due to its simplicity and that its dependence on the dimensionless group is stronger than any power law or other common functional relationships in engineering design. The sign of the coefficient a determines the type of the modification factor: when $a > 0$, the modification factor is always larger than 1 while for $a < 0$, the modification factor $\exp(a\Pi^b)$ varies within the interval of $(0,1]$. Both the sign and the magnitude of b matter as the modification factor has increasing dependence on the independent variable for $ab > 0$ while for the case of $ab < 0$, it decreases as the independent variable increases. An appropriate magnitude of b should be chosen to ensure the modified asymptote intersects the other. With a suitable selection of the blending constants $\{n, a, b\}$, the modified asymptote still maintains its asymptotic behavior, while the exponential term changes its intermediate behavior such that the two asymptotes intersect. A similar approach was made in [7] to obtain a general correlation of three functions, with the applied modification factor using $b = 1$.

1D Blending of Constant Asymptotics

Many phenomena demonstrate the asymptotic behavior of two limiting solutions of different constant values, for example, in the field of welding, the maximum hardness of the heat affected zone [32, 33] and in heat transfer, the temperature distribution of a slab heated on both surfaces at different constant temperatures. Although the treatment of non-crossing asymptotes could apply to constant asymptotics, a specific treatment of this particular problem results in convenient solutions of smaller complexity.

The following three blending functions are explored:

$$u_c^*(\{\Pi_1\}) \approx \widehat{u_c^*}^+(\{\Pi_1\}) = \mathcal{B}(\widehat{u^*}_{c,I}, \widehat{u^*}_{c,II}, \{a_1, b_1\})$$
$$= \widehat{u^*}_{c,I} + \frac{\widehat{u^*}_{c,II} - \widehat{u^*}_{c,I}}{2}\left(1 + \frac{1 - a_1\Pi_1^{b_1}}{1 + a_1\Pi_1^{b_1}}\right) \tag{11}$$

$$u_c^*(\{\Pi_2\}) \approx \widehat{u_c^*}^+(\{\Pi_2\}) = \mathcal{B}(\widehat{u^*}_{c,I}, \widehat{u^*}_{c,II}, \{a_2, b_2\})$$

$$= \widehat{u^*}_{c,I} + \frac{\widehat{u^*}_{c,II} - \widehat{u^*}_{c,I}}{2}[1 + \tanh(a_2\Pi_2 + b_2)] \qquad (12)$$

$$u_c^*(\{\Pi_3\}) \approx \widehat{u_c^*}^+(\{\Pi_3\}) = \mathcal{B}(\widehat{u^*}_{c,I}, \widehat{u^*}_{c,II}, \{a_3, b_3\})$$

$$= \widehat{u^*}_{c,I} + \frac{\widehat{u^*}_{c,II} - \widehat{u^*}_{c,I}}{2}\left[1 + \frac{2}{\pi}\arctan(a_3\Pi_3 + b_3)\right] \qquad (13)$$

where Π_1, Π_2, Π_3 represents three different blending expressions, not implying multi-variable dependence, and $\{a_1, b_1\}$, $\{a_2, b_2\}$, $\{a_3, b_3\}$ are the corresponding sets of blending constants. Eqn. (11)-(13) converge to exact asymptotics in both regimes for all finite values of $\{a, b\}$. Eqn. (11) is consistent with the correlation method proposed by Churchill [31] when only limiting values are known by replacing the constant asymptotic behavior of either regime with a postulated functional dependence (typically power functions) and then blending the constructed function with constant asymptotic in the other regime with a blending exponent n. Eqn. (11) is the special case with $n = 1$. Eqn. (11) and Eqn. (12) are essentially equivalent with the transformation of variables: $\Pi_2 = -\frac{1}{2}\ln(\Pi_1)$, $a_2 = -\frac{1}{2}b_1$ and $b_2 = -\frac{1}{2}\ln(a_1)$.

The two blending constants $\{a, b\}$ capture the two degrees-of-freedom of the functional behavior after blending: the location of the midpoint $u_c^*(\Pi) = (\widehat{u^*}_{c,I} + \widehat{u^*}_{c,II})/2$ and the slope of tangent at the midpoint, $\frac{d\widehat{u_c^*}^+(\{\Pi\})}{d\Pi}$, which represents the steepness of the transformation between the two asymptotics.

The differences in the form of the three blending expressions make them applicable for different scenarios. The main difference between Eqn. (11) and Eqn. (12) is the domain, which is $(0, \infty)$ of Eqn. (11), and $(-\infty, \infty)$ of Eqn. (12). Eqn. (12) and Eqn. (13) share the same domain of $(-\infty, \infty)$ and but the curvature is different, with the same slope at the value of 1 and the same location of midpoint, Eqn. (13) is less sharp than Eqn. (12) as shown in Fig. 2.
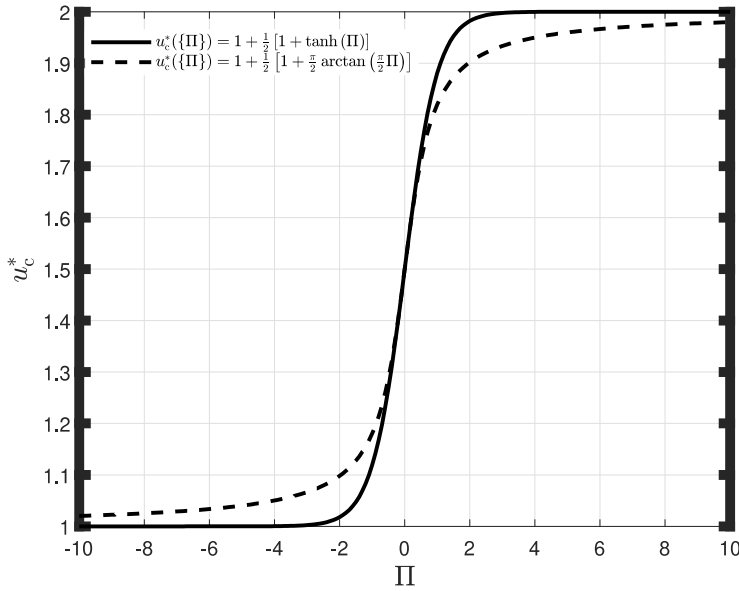
Fig. 2 Comparison between Eqn. (12) and Eqn. (13). The solid line is $u_c^*(\{\Pi\}) = 1 + \frac{1}{2}[1 + \tanh(\Pi)]$, and the dash line represents $u_c^*(\{\Pi\}) = 1 + \frac{1}{2}\left[1 + \frac{\pi}{2}\arctan\left(\frac{\pi}{2}\Pi\right)\right]$. Two constant asymptotics are 1 and 2. The center point $\Pi = 0$ and its slope is the same for both functions but the dash line is less steep than the solid line.

ASYMPTOTES WITH CHANGES IN THE SIGN

The blending techniques proposed require that all asymptotes are positive over the whole domain; however, in some cases the asymptotic function might be negative outside its range of asymptotic validity. For example, when the asymptotic behavior towards infinity is logarithmic (e.g. $\widehat{u_c^*}(\{\Pi\}) = \ln\Pi$ as $\Pi \to \infty$), the asymptotic tends to minus infinity as Π tends to zero, which is invalid for the techniques proposed. One possible solution in this case is replacing the asymptote $\ln\Pi$ with $\ln(1 + \Pi)$, such that asymptotic behavior keeps unaffected as $\ln(1 + \Pi) \approx \ln(\Pi)$, when $\Pi \to \infty$ and the modified expression is always positive over the whole domain of $(0, \infty)$.

ADDITION OF INTERMEDIATE TERMS

The blending function Eqn. (7) guarantees the limiting solutions in asymptotic regions and estimates values of the intermediate region by optimization of the blending constant n. For power-law functions, this approach is simple and accurate; however, when the functions to be blended are not power-laws, such as logarithmic or exponential functions, Eqn. (7) converges very slowly towards small errors. When convergence is slow, the blending error

is small at unrealistically large or small orders of magnitude. An alternative approach is proposed here to deal with these functions while preserving simplicity and accuracy. To improve the accuracy of estimation in the intermediate regime, an additive term $G(\Pi)$ can be introduced as:

$$u_c^*(\{\Pi\}) \approx \widehat{u_c^*}^+(\{\Pi\}) = [\widehat{u^*}_{c,I}(\{\Pi\})^n + \widehat{u^*}_{c,II}(\{\Pi\})^n + G(\Pi)^n]^{\frac{1}{n}} \qquad (14)$$

where the choice of $G(\Pi)$ is flexible as long as it does not change the asymptotic behavior, i.e., for the case of $n > 0$, $G(\Pi) \ll \widehat{u_{c,I}^*}(\{\Pi\})$ as $\Pi \to \infty$, and $G(\Pi) \ll \widehat{u_{c,II}^*}(\{\Pi\})$ as $\Pi \to 0$. Although the blended result could be more accurate with a sophisticated $G(\Pi)$, the format of $G(\Pi)$ should be as simple as possible to minimize the number of involved blending constants.

The added term can be interpreted as the departure between normal blending equation and the exact value of the intermediate regime, or the ``asymptote'' of the intermediate regime.

As the absolute value of blending constant n is typically of the order of magnitude of 1, it is reasonable to set the blending parameter $n = \pm 1$ to reduce the number of optimization variables to two. These variables can be optimized by the proposed minimax optimum procedure of minimizing the maximum error defined in Eqn. (3). The sign of n is chosen based on the asymptotic solutions. When both asymptotes are the lower bounds in their corresponding regimes, $n = 1$ while the dependent variable, $u_c^*(\{\Pi\})$, has a decreasing power of Π, $n = -1$.

For the case of two power asymptotes, the additive term can be chosen in the form of power law as $G(\Pi) = a\Pi^b$,, with blending constants $\{B\} = \{a, b, \pm 1\}$. Denoting the two power asymptotes as $\widehat{u_{c,I}^*}(\{\Pi\}) = c_I \Pi^{d_I}$ and $\widehat{u_{c,II}^*}(\{\Pi\}) = c_{II} \Pi^{d_{II}}$, the exponent of the additive term b has to be set between d_I and d_{II} to make sure $G(\Pi)$ is negligible in the asymptotic regions. It is not clear in what cases this approach is more accurate than Eqn. (7) . One obvious difference is that this approach needs to optimize two variables instead of one.

For the case of logarithmic asymptotes (e.g. $\widehat{u_{c,II}^*}(\{\Pi\}) = \ln \Pi$), $G(\Pi) = a\Pi^b$ fails in maintaining asymptotic behavior, because $\ln \Pi$ is always smaller than power laws when Π tends to infinity, no matter the value of exponent. In this case, additive term $G(\Pi)$ can be expressed in the fractional form as $G(\Pi) = a\Pi/(b + \Pi)$ instead with blending constants $\{B\} = \{a, b, \pm 1\}$, such that the additive term can be smaller than the logarithmic asymptote in the limiting regime when $\Pi \to \infty$.

AN EXTENSION OF 1D BLENDING: PARAMETRIC 1D BLENDING

Some problems involve multiple dimensionless groups, so they are not 1D blending problems. However, when the some dimensionless groups have a reduced range of variation, they can be considered as another parameter in the blending problem, as opposed to another dimensionless group to be blended. For example, solid mechanics problems involve the dimensionless parameter ν (Poisson's ratio), which is seldom far from 0.3 for

most engineering materials. In this case, it is convenient to consider v as a constant instead of performing blending over unrealistic values.

This approach is especially valuable when the dimensionless groups that can be considered as parameters is such that blending can be reduced to 1D.

CHALLENGES BEYOND 1D BLENDING

The 1D blending describes many engineering problems concerning to one independent variable with much rigor and elegance, however, in some cases, the secondary factors play a significant role and are not negligible indeed, for example, welding on a plate of medium thickness[34, 35], or where the size of a Gaussian distributed heat source can not be neglected [36]. It is necessary to develop a series of practical and systematic 2D blending methods on the basis of current 1D blending theory.

In normalized form, 2D blending problem can be defined as $u_c^*(\Pi_1, \Pi_2)$, where u_c^* is the target characteristic value dependent on two dimensionless variables $\{\Pi\} = \{\Pi_1, \Pi_2\}$. The 2D blending problem can also be addressed with an equivalent expression in implicit form: $F(u_c^*, \{\Pi\}) = F(u_c^*, \Pi_1, \Pi_2) = 0$, which could be one equation or a group of equations associated with problem parameters. The general strategy to construct 2D blending is to decompose the 2D blending problem into several 1D blending branches, where 1D blending theory is well-developed, and then combine the solved 1D blending equation of each branch to assemble the solution to 2D problem.

Asymptotes in 2D problems are typically derived with limitation theorems, but the limitation theorems of two variable or multi-variable functions are much more complex than functions of single dependence. The complexity includes but not restrict to the division of limiting regimes, different types of asymptotics, relationship between double limits and iterated limits, and so on. The decomposition of 2D blending into 1D blending problems and the assembly of 2D blending solution based on 1D blending functions have high demand on mathematical skills and are never trivial. The simplest case of 2D blending is that the target function $u_c^*(\Pi_1, \Pi_2)$ is assumed variable separable over its entire domain, which means $u_c^*(\Pi_1, \Pi_2) = v(\Pi_1) \cdot w(\Pi_2)$, and the 2D blending could be directly split into two 1D blending functions of $v(\Pi_1)$ and $w(\Pi_2)$. Thus the assembled 2D blending is in form of the product of both 1D blending equation $\widehat{u_c^*}^+(\Pi_1, \Pi_2) = \hat{v}^+(\Pi_1) \cdot \hat{w}^+(\Pi_2)$, where $\hat{v}^+(\Pi_1)$ $\hat{w}^+(\Pi_2)$ are relatively simple and easily tractable with in 1D blending theory proposed in this paper. Systematic 2D blending methods to obtain general estimation of the target characteristic value in terms of asymptotics and correction factors with high accuracy are the focus of current research.

CASE STUDIES

Four case studies are be presented to demonstrate the application of MRCF approach. Target characteristic values of interest are expressed in terms of asymptotics (simple case solutions in the extreme cases) and the type of the blending approach for each case will be identified and applied to generate correction factors in explicit form.

CASE A: ESTIMATION OF MAXIMUM BEAD WIDTH BASED ON ROSENTHAL 3D MODEL

Step 1: List all physics considered relevant
Relevant physics involved in the shape of the weld pool include heat transfer, heat dissipation on the surface to the environment, convective flow, the effects of fluid dynamics and so on. It is impossible to list all relevant mechanisms but the dominant phenomenon and the best practice is to list typical and essentially relevant approximations, which can be suggested by published papers and experimental observations.

Step 2: Identify dominant factors
For the case of the maximum bead width, the dominant mechanism is postulated as heat transfer via conduction and the classic Rosenthal 3D model [37, 38] was utilized to calculate thermal characteristics which assumes the heat source as a point moving with constant velocity in a straight line on the surface of the semi-infinite base materials with constant thermal properties. Heat dissipation on the mental surface, convective flow, and the effects of fluid dynamics are neglected as secondary phenomenon. This Rosenthal 3D model assumes the heat source as a infinitely small point (which is actually not), and considers everything as solid by neglecting the effect of fluid dynamics and phase changes. What' more, it only focuses on the conduction within the solid base metal and the advection due to the relative motion between heat source and the substrate without considering the effects of convective heat transfer in the molten metal and heat dissipation to the external environment (convection and radiation). Due to the huge simplifications, this idealized model with the assumption of identifying pure conduction as the most dominant phenomena and the obtained estimations must check self-consistency which will be discussed in step 4.

Step 3: Solve approximate problem considering only dominant factors (Minimal Representation)
Starting from the Fourier's law and the boundary conditions of the moving system, the simplified problem has been solved in [6] and the obtained analytical solution of the temperature field has the following expression:

$$T(x, y, z) = T_0 + \frac{q}{2\pi kr} \exp\left[-\frac{U}{2\alpha}(r + x)\right] \qquad (15)$$

where q is the net thermal power absorbed by the base material, k is the thermal conductivity of the base material, and T_0 is the temperature of the substrate far from the heat source or the preheat temperature. In welding, the power q is estimated as the product of nominal power of the heat source and its thermal efficiency.

Eqn. (15) provides the value of temperature for each point in the domain with a singularity at $r = 0$, which is the location of the point heat source and the temperature value there is infinite. It is the theoretical solution of the temperature field as a function of position to the simplified welding problem by considering conduction as the dominant mechanism. However, in practical applications, temperatures of interest always appear as known conditions, for example, melting temperature and the A $_1$, A $_3$ where phase transformations typical occur are readily known beforehand. Characteristic values which capture the essence of thermal history such as maximum bead width and the location where it occurs are unknown and difficult to obtain.

In this idealized model, dependent variable is the temperature $\{U\} = \{T\}$, and there are three spatial independent variables $\{X\} = \{x, y, z\}$, and five parameters $\{P\} = \{T_0, q, k, U, \alpha\}$. The characteristic value of interest is the maximum isotherm width: $u_c(\{P\}) = y_{\max}$. After normalization and dimensional analysis of Eqn. (15), the conclusion is that if we denote the temperature value of the isotherm under consideration as $T = T_c$, the dimensionless characteristic values associated with $T = T_c$ depend only on one dimensionless group (more details can be found in [6]): The Rykalin number (Ry), first proposed by Fuerschbach [39] and has the following expression in terms of parameters:

$$\mathrm{Ry} = \frac{qU}{4\pi k \alpha (T_c - T_0)} \tag{16}$$

where the constant of $1/4\pi$ is added to simplify the final expressions detailed below.

All dimensionless characteristic values associated with an isotherm $T = T_c$ can be captured with functions depending only on Ry, thus the set of independent dimensionless groups based only on parameters turns into $\{\Pi\} = \{\mathrm{Ry}\}$ and the dimensionless characteristic value $u_c^*(\{\Pi\}) = y_{\max}^*(\{\Pi\})$ can be expressed with sole dependence on Ry as $y_{\max}^*(\mathrm{Ry})$. As Ry is the only one dimensionless group necessary to capture all characteristic values, the number of asymptotic regimes is given by $2^1 = 2$ and they are Regime I, corresponding to large values of Ry ("fast heat source" where advection dominates over conduction), and Regime II, corresponding to small values of Ry ("slow heat source" with heat transfer dominated by conduction) and the two asymptotics can be represented as $\widehat{u^*}_{c,\mathrm{I}}(\{\Pi\}) = \widehat{y^*}_{\max_\mathrm{I}}(\mathrm{Ry})$ and $\widehat{u^*}_{c,\mathrm{II}}(\{\Pi\}) = \widehat{y^*}_{\max_\mathrm{II}}(\mathrm{Ry})$, respectively. These asymptotic regimes yield simple expressions for the characteristic values, usually in the form of power laws. Asymptotic analysis of Eqn. (15) yields the following two asymptotics of the dimensionless maximum isotherm width dependent only on Ry [6]:

$$\widehat{y^*}_{\max_\mathrm{I}}(\mathrm{Ry}) = \sqrt{\frac{2\mathrm{Ry}}{e}} \quad \text{for Regime I (fast)} \tag{17}$$

$$\widehat{y^*}_{\max_\mathrm{II}}(\mathrm{Ry}) = \mathrm{Ry} \quad \text{for Regime II (slow)} \tag{18}$$

Step 4: Check for self-consistency

As Eqn. (15) comes directly from the postulated model, the validity of calculations based on Eqn. (15) (Eqn. (17) and Eqn. (18)) will remain in the limit of the validity of this model itself. The self-consistency of this idealized model has been accomplished by comparison with experiments as it is impossible to prove the accuracy of the Rosenthal 3D model from the mathematical point of view. Christensen in the 1960s has done a series of experiments to test the validity of this Rosenthal 3D model [16] of different operating parameters and materials. It has shown that despite of the great simplifications, the model by considering conduction and advection as the dominant phenomena can still generate reasonable results for points far away from the heat source at temperatures below the melting temperature.

Step 5: Compare predictions to "reality"

Eqn. (17) and Eqn. (18) are less accurate for intermediate values (Ry=O(1)). As the obtained two asymptotics are in the form of power laws, Standard 1D Blending method is

applicable for this case. For the non-asymptotic region, simple and accurate expression can be obtained with Eqn. (7).

Substitute Eqn. (17) and Eqn. (18) into Eqn. (7), the blended function of the maximum isotherm width in dimensionless form is obtained:

$$y^*_{max}(Ry) \approx \widehat{y^*}_{max}^{+}(Ry) = \left[\widehat{y^*}_{max_I}(Ry)^n + \widehat{y^*}_{max_{II}}(Ry)^n\right]^{\frac{1}{n}} \qquad (19)$$

The optimal blending constant n was determined using a minimax approach with two nested optimizations as detailed in [6]. With Rosenthal 3D model and the resulting temperature field, the "reality" used here to calibrate the deviations from the idealized behavior is the maximum value of the isotherm width calculated from Eqn. (15) numerically. When n is equal to -1.7312, the maximum error over the whole domain of Ry is less than 0.7236%.

Step 6: Create correction factors

Substituting the two asymptotics $\widehat{u^*}_{c,I}(\{\Pi\}) = \widehat{y^*}_{max_I}(Ry)$ and $\widehat{u^*}_{c,II}(\{\Pi\}) = \widehat{y^*}_{max_{II}}(Ry)$ into Eqn. (8) and Eqn. (9), the following expression for the correction factors are obtained:

$$f_{y_{max}}(Ry) = \left[1 + \left(\sqrt{\frac{eRy}{2}}\right)^{\pm n}\right]^{\frac{1}{n}} \quad \begin{array}{l} +\text{for Regime I (fast)} \\ \\ -\text{for Regime II (slow)} \end{array} \qquad (20)$$

where the exponent $+n$ corresponds to Regime I, and $-n$ corresponds to Regime II. The minimax error is always smaller than 0.7236% at the optimal value of n=-1.7312.

Correction factors based on both asymptotics are plotted in Fig. 3 and the reflected symmetry are consistent with Eqn. (20) as expected. They can also be used to check the validity of both asymptotics. The intersection of the correction factors, Ry $_c$ = 0.7359, can be considered as the rough divider of the two asymptotic regions: Regime I, Regime II and the intermediate region (the vicinity of Ry $_c$) where the typically the maximum departure occurs.
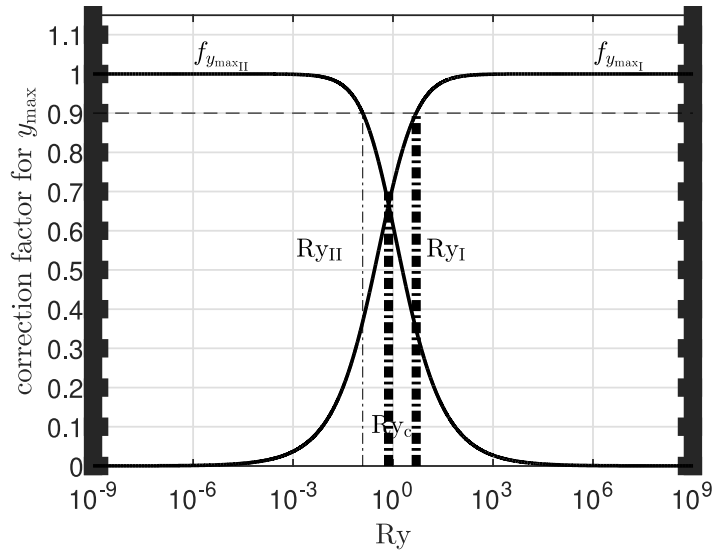
Fig. 3 Correction factors for maximum isotherm width y_{max} [6]

CASE B: PREDICTIONS OF THE MAXIMUM HARDNESS OF THE HAZ

One of the major challenges in welding industry is predicting the maximum hardness of the heat affected zone that depends the chemical composition and the welding parameters. During the 1970s and the 1980s, many empirical models (based on statistical regression of experimental data) was proposed to address this problem [33, 40]. Although each proposed model is different, they all treat the problematic with a similar nature using asymptotic values to obtain a hardness value as a function of the cooling rate.

Basically, the hardness of a specific point on heat affected zone depends on the present microstructures and their volume fraction. At the same time, the type and amount of each microstructure is a function of the thermal history and chemical composition. In the case, the characteristic value of interest is the the maximum hardness of the HAZ, many variables can be discarded as the assumptions are made. For example, austenitization peak temperature, which plays an important role on the kinetics of austenite decomposition, can be assumed to be equal for all cases by considering that the maximum hardness will be found adjacent to the fusion line where the peak temperature is maximum, very close to the fusion temperature (i.e in the coarse grain heat affected zone). By making this simplification, hardness can be written down as a function of the cooling rate at 700 °C (or cooling time, $t_{8/5}$), and the nominal chemical composition of the welded plate.
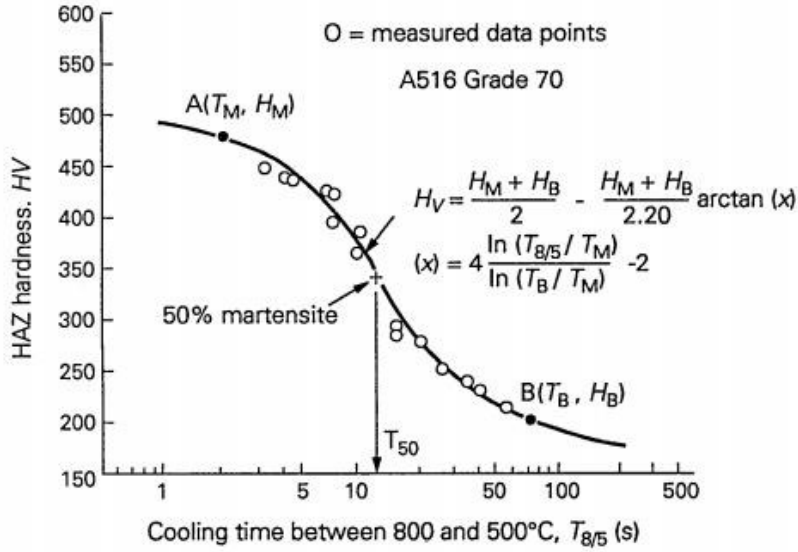
Fig. 4 Characteristic hardness in HAZ as a function of cooling time between 800 °C and 500 °C [33]

One clear example of using asymptotes to address this problem is given by the Yurioka's model [33] and a proposed Continuous Cooling Transformation Structure Hardness (CCTSH) curve is represented in Fig. 4. In this case, the maximum hardness in HAZ depends only on the cooling time between 800 °C and 500 °C. With only one dependence on the cooling time, there are in total two regimes and the characteristic points given by the hardness upper (H_M) and lower limits (H_B) and their characteristic times are used to develop a equation able to predict the hardness value for any time between τ_M and τ_B.

Upper and lower hardness asymptotes are easy to picture when this problem is addressed, if the different possible microstructures are consider. The upper limit, is directly related with the hardness of the martensite, which means $\hat{u}_{c,\mathrm{II}} = H_M$. No other microstructure is harder than the fresh martensite obtained from cooling from the austenite region with cooling rates higher than a critical value τ_M. On the other hand, the lower limit it can be quite more complicated to picture, since different criteria used depends on the author. However, the main idea resides on using a lower limit, $\hat{u}_{c,\mathrm{I}} = H_B$, that represents a microstructure with no martensite, τ_B. After the formulation of the two limiting constant hardness values empirically, a blended expression in the form of Equation 13 was proposed:

$$HV = \frac{H_M + H_B}{2} - \frac{H_M - H_B}{2.2}\arctan x \qquad (21)$$

where x is defined as:

$$x(\mathrm{rad}) = 4\frac{\log\dfrac{t_{8/5}}{\tau_M}}{\log\dfrac{t_B}{\tau_M}} - 2$$

17

It is important to remark that, when blending two constant asymptotics (point A and B in Fig. 4), Eqn. (13) must be applied with a understanding of its applicable scope as for cooling times that are not between τ_M and τ_B, Eqn. (21) is not applicable. As it was mentioned before, Eqn. (11) and Eqn. (12) can also be applied for this case and there are many other types of functions proposed by different authors.

CASE C: THERMAL STRESS FIELD IN THE VICINITY OF A MOVING POINT HEAT SOURCE

Step 1: List all physics considered relevant

The thermal stress field produced by non-uniform temperature field surrounding a moving point heat source is of significant interest in welding research due to the implications for the size and magnitude of the residual stress field. This example will focus on estimation of the thermal stresses produced by a moving heat source in a thin-plate (i.e. 2D material). The characteristic values (u_c) which are of primary interest in the analysis of thermal stresses are the longitudinal stress ($\sigma_{x,c}$) and transverse stress ($\sigma_{y,c}$) produced by some critical temperature change (ΔT_c). The dimensionless form ($u_c^* = \{\sigma_x^*, \sigma_y^*\}$) of the characteristic values is obtained by normalizing with respect to the product of the elastic modulus (E), coefficient of thermal expansion (a), and critical temperature change (ΔT_c):

$$\{u_c^*\} = \{\sigma_{x,c}^*, \sigma_{y,c}^*\} = \{\sigma_{x,c}/(Ea\Delta T_c), \sigma_{y,c}/(Ea\Delta T_c)\}$$

The mathematical formulation for the thermal stress field in a 2D plane composed of a homogeneous, isotropic, linear elastic material involves a total of six dependent variables, three independent variables and six fixed parameters. The dependent variables are:

$\{U\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \sigma_x, \sigma_y, \tau_{xy}\}$

which represent the x, y, and shear components of the strain and stress fields; the independent variables are:

$\{X\} = \{\Delta T, x, y\}$

where ΔT is the temperature field at a given location (x,y) in a 2D cartesian plane; and the parameters are:

$\{P\} = \{\Delta T_c, x_c, y_c, E, G, a\}$

where ΔT_c is some characteristic temperature change, x_c is the maximum length of the characteristic temperature isotherm, y_c is the maximum width of the characteristic temperature isotherm, G is the shear modulus of the material, E is the elastic modulus of the material, and a is the linear coefficient of thermal expansion.

Step 2: Identify dominant factors

Applying the Buckingham Π theory, it is found that there can only be exactly two independent dimensionless groups in this analysis. Any 2 dimensionless groups that form an independent set may be selected. In this example, the dimensionless parameters are chosen as:

$\{\Pi\} = \{E/G, x_c/y_c\}$

---

by convention stresses/strains oriented parallel to the heat source movement (i.e. x direction) are termed longitudinal, and stresses/strains oriented perpendicular to heat source movement (i.e. y direction) are referred to as transverse

where E/G is the ratio of the elastic modulus to shear modulus which can be equivalently expressed in terms of the Poisson's ratio (ν) as $E/G = 2(1 + \nu)$ and $x_c/y_c = A_R$ defined as the aspect ratio of the critical isotherm ΔT_c.

Note that in this example, since there are two dimensionless groups (G/E and A_R), there will be four asymptotic regimes:

- Regime I: $A_R \rightarrow \infty, E/G \rightarrow 0$
- Regime II: $A_R \rightarrow 1, E/G \rightarrow 0$
- Regime III: $A_R \rightarrow \infty, E/G \rightarrow \infty$
- Regime IV: $A_R \rightarrow 1, E/G \rightarrow \infty$

Note that the lower asymptotic value of the aspect ratio A_R is 1, which corresponds to a perfectly circular temperature isotherm (i.e. an infinitely slow or stationary heat source).

For all isotropic materials it is necessary that $-1 \leq \nu \leq 0.5$[41], which constrains the allowable range of the modulus ratio to $0 \leq E/G \leq 3$. For all commonly welded metallic materials, Poisson's ratio is approximately $\nu = 0.3$ and the modulus ratio is approximately $E/G = 2.6$. Since for all practical problems, the modulus ratio is relatively small, the Regime I, and Regime II are of practical interest.

Step 3: Solve approximate problem considering only dominant factors (Minimal Representation)

The asymptotic solutions for the dimensionless stresses in Regime I are [42]:

$$\widehat{\sigma^*}_{x,c,I} = 1$$
$$\widehat{\sigma^*}_{y,c,I} = 1/A_R{}^2$$

and the corresponding dimensional stress values are:

$$\hat{\sigma}_{x,c,I} = aE\Delta T_c$$
$$\hat{\sigma}_{y,c,I} = aE\Delta T_c/A_R{}^2$$

Analytical expressions for the asymptotic solutions of the dimensionless stresses in Regime II is the subject of an ongoing study, but these expressions will have the general form:

$$\widehat{\sigma^*}_{x,c,II} = \widehat{\sigma^*}_{x,c,II}(\{A_R, \nu\})$$
$$\widehat{\sigma^*}_{y,c,II} = \widehat{\sigma^*}_{y,c,II}(\{A_R, \nu\})$$

Step 4: Check for self-consistency

For single-pass full-penetration welding of plain carbon and low-alloy steel, the aspect ratio is expected to vary between 2-20 depending on the welding process. Although the asymptotic equations may provide reasonable estimates in some of these cases, it is clearly an issue of practical interest necessary to obtain estimates of the characteristic stresses at intermediate values of the aspect ratio. The asymptotic blending technique provides a powerful and convenient method to develop a closed form equation for estimating the stresses at these intermediate values.

Step 5 and 6: Compare predictions to "reality" and create correction factors

In this case, the standard approach for 1D asymptotic blending may be applied with only one minor modification. Although the asymptotic solutions in Regime I depend on only A_R, this will not necessarily be the case in Regime II. If the asymptotic expressions in Regime II are found to depend on the value of Poisson's ratio, it will be necessary to

optimize the value of the set of blending constants $\{B\}$ over a range of typical values of v (ex. 0.25-0.35) using Parametric 1D blending. In this case, the value of v is treated as a parameter and the optimization for the 1D blending coefficients is evaluated in the 2D space defined by $1 < A_R < \infty$ and v is around the value of 0.3. The 1D parametric blending technique provides a simple and powerful alternative to full 2D blending in 2-parameter systems where the range of one parameter is relatively restricted.

Although significant challenges are associated with obtaining experimental measurements of thermal stresses, the "reality" for the critical stress at a selection of intermediate values used in the optimization may be readily obtained from finite-element analysis (FEA). A limited number of experimental measurements may then be performed to validate the final solution and aid in the development of additional correction factors for secondary effects.

CASE D: MAXIMUM TEMPERATURE IN FRICTION STIR WELDING (FSW)

Step 1: List all physics considered relevant
Relevant physics involved in FSW are the plastic flow near the pin and the generated heat, heat conduction into the base material, heat loss to the environment, inertial factors, kinematics and forces related in the deformation and so on.

Step 2: Identify dominant factors

For the case of FSW, the dominant factors are identified as the heat diffusion in a localized soft layer in [1, 43]. Four groups of competing phenomena are considered: heat diffusion vs. heat advection, kinematics of rotational flow vs. translational flow, thickness of the soft shear layer vs. the radius of the pin and the peak temperature jump caused by shoulder vs. the contribution of plastic area around the pin. Among the four groups, dominant factors are identified as the heat diffusion, rotational flow, thickness of the shear layer and the dominance of the pin on peak temperature. Self consistency will be check for the secondary factors in step 4 by comparing the obtained expression for maximum temperature against experiments published in literature.

Step 3: Solve approximate problem considering only dominant factors (Minimal Representation)

The problem is greatly simplified to its minimal representation by considering only dominant factors listed above. Target characteristic value is the maximum temperature developed during process can be expressed as $u_c = T_s$ and asymptotics obtained have the following form[43]:

$$\widehat{T_s} = T_0 + \Delta T_{\mathrm{m}} \left[\frac{3}{2} \frac{\eta_s}{AB\Delta T_{\mathrm{m}}} \left(\frac{\eta K_0}{\Delta T_0} \right)^{n-1} \left(\frac{a^2 \tau_{\mathrm{R}}}{k} \right)^n \omega^{n+1} \right]^{\frac{1}{2}} \tag{22}$$

where T_s is the maximum temperature achieved at the pin-shear layer interface, T_0 is the temperature at the interface between shear layer and the base plate, $\Delta T_{\mathrm{m}} = T_m - T_0$ and T_m is the melting temperature of the substrate, η_s is the efficiency which considers mechanical energy converted into heat, excluding the mechanical energy accumulated in the form of dislocations, A is the constant of constitutive behavior of the alloy and η is the total

efficiency of the process, accounts for energy stored in the form of dislocations in the shear layer and heat lost to the tool, K_0 is the Modified Bessel function of second kind and 0 order, $\Delta T_0 = T_0 - T_\infty$, a is the pin radius, τ_R is the reference shear stress of the alloy, k is the thermal conductivity of the alloy at T_0 and ω is the angular velocity of rotation of the pin.

Step 4: Check for self-consistency

All predictions based on the minimally represented problem must be checked for self-consistency. For the case of FSW, Eqn. (22) evaluated by comparing the effect of assumed secondary factors on the accuracy of the estimation. The relative magnitude of selected secondary factors can be represented by its value to that of the dominant factors. In this way, a value of 1 indicates that the secondary factor is of similar magnitude than the dominant factor. Very small values on the horizontal axis correspond to the self-consistent regime, while high values correspond to inconsistent cases in which the factors considered secondary are actually larger than the dominant factors. There are four independent dimensionless groups that characterize the secondary phenomena $\{\Pi\} = \{\text{Pe}, V/\omega\hat{\delta}, \hat{\delta}/a, (T_p - T_\infty)/(T_s^+ - T_\infty)\}$

Step 5: Compare predictions to "reality"

Eqn. (22) must be validated through comparisons with reality. In summary, three factors considered secondary (advective heat transfer, pin translation on maximum temperature and shoulder heat input on the peak temperature in FSW) have been proved indeed secondary for the vast majority of cases in [1, 43], and that for those valid cases, Eqn. (22) predicts the proper order of magnitude and dependence on process parameters. However, the approximation by considering the shear layer thickness on peak temperature as secondary phenomena is not always valid and correction factor for the estimation of maximum temperature are necessary. The 4D blending problem can be simplified into Parametric Blending of one dependence: $\{\Pi\} = \{\text{Pe}, V/\omega\hat{\delta}, \hat{\delta}/a, (T_p - T_\infty/T_s^+ - T_\infty)\} \approx \{\hat{\delta}/a\}$

Step 6: Create correction factors

In this FSW case, systematic errors in the math and physics can be accounted for by the correction factor in Standard 1D blending form with blending constants $\{B\} = \{C_1, C_2, C_3\}$:

$$f_T^+ = C_1 \left(1 + C_2 \frac{\hat{\delta}}{a}\right)^{C_3}$$

Blending constant C_1 takes care of the systematic error in the mathematics while C_2 and C_3 are used to account the secondary phenomena for best accuracy. When C_1=0.764, C_2=0.259, and C_3=-0.857, optimal match between Eqn. (22) and the published experimental values has been achieved with 12% standard deviation.

CONCLUSIONS

Design rules in the form of a simple formula (asymptotic behavior of the simplified problem by considering only dominant mechanisms) and correction factors which capture the deviation from the ideal cases are proposed. This approach is of much help to solve complex multiphysics problems such as welding and is based on a six-step framework we termed Minimal Representation and Correction Factor (MRCF). The formulation of the design

rules makes it convenient to couple them to represent multi-coupled phenomena and embed them in numerical models and control systems. It is shown through examples that the technique of blending has wide applicability for correlating experimental or numerical data or theoretical values for processes with known asymptotic solutions.

The proposed blending theory extends the applicability of standard blending approaches to obtain general, accurate and explicit expressions in terms of known asymptotic solutions and blending constants determined with a systematic optimization procedure. The blending expressions are also used to develop correction factors that capture the deviations from ideal cases.

Blending is also useful in summarizing isolated experimental results and numerical data from simulation. The general and systematic formulation can be extended to other multi-coupled phenomena besides welding.

APPENDIX: NOTATION

Table 1 Notation

Symbols	Description
Varialbes	
a	Blending constant
b	Blending constant
f	Correction factors
k	Number of independent reference units
m	Number of independent physical magnitudes
n	Blending exponent
n_r	Number of regimes
u	Dependent variable and related function
v	Variable separated function of u
w	Variable separated function of u
B	Blending constants
F	Implicit Function
P	Problem parameter
X	Independent variable
U	Dependent variable
Π	Independent dimensionless groups
$\mathcal{B}$	Blending functions
Superscripts	
$*$	Normalized value
$\hat{}$	Asymptotic behavior
$+$	Blended approximation
Subscripts	
I	Corresponding to Regime I
II	Corresponding to Regime II
III	Corresponding to Regime III
i	Regime i
c	Characteristic values
Others	
$\{\cdots\}$	Set

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