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Reflected and stopped random walks and the distinguishing number of graphs

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AFFIDAVIT

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Chapter 1

An introduction based on a short story

In this introduction I want to give a short overview over my research of the last years within some non-mathematical and easily understandable words in a short story. The motivation to do it in this way was given in a very useful soft skill and presentation seminar that Mathematic PhD-Students have to absolve during their studies.

Let us start with a quotation by Shizuo Kakutani who said that: "A drunk man will find his way home, but a drunk bird may get lost forever." I could imagine you wonder about the meaning of this citation and where the connection to mathematics occurs. Let me explain.

We make it simple and imagine that the drunk man can move in four directions. To the left, to the right, to the front and to the rear, in other words he moves on a two-dimensional grid. Now assume that the drunk man is so drunk that he does not know where his home is. With each new step he throws a tetrahedron ("a four sided dice") that gives him the direction with the same probability of 1/4 where he can go. Moreover, after each step he has forgotten where he went before.

Now one may ask the question: "Will he ever find his home with probability one?"

Since the drunk man describes a two dimensional random walk it is known that the answer is "yes".

But what is about the drunk bird? For the bird we assume that it can move additionally up and down and that it takes every step independently from the one before, analog to the drunk man. It chooses the direction with a probability of 1/6 by for example throwing a dice. That is not really realistic for a bird but it is only to give you a pictorial imagination of a three dimensional random walk. For the drunk bird the answer on the question is "no" as the statement of Kakutani already conveyed.

The corresponding mathematical theorem goes back to Pólya and handles the question of how the return probability of a symmetric random walk to its starting point changes if the dimension of the room grows up. As I told you through the example such a one-step symmetric random walk is recurrent (returns with probability 1) in dimension one or two and it is transient (returns with probability smaller than 1) if it moves in a three or higher dimensional room.

Now I want to modify this situation and add some walls or barriers in the world of the man and the bird. First imagine that these barriers are something like mirrors so that in case they would "want to go through this barriers" they are reflected. The question now is how the mirrors influence the behavior of the return probability of the man/bird? These and some more questions will be answered in Chapter 5, of course in a more mathematical way.

A further question is, what will happen if the barriers are built in a way that the man and the bird are really stopped there instead of reflected? The solving of this problem can also be motivated by a different model. Hence, the story goes on and we imagine that the man stops by a supermarket on his way to buy some chocolate for his children. If he comes to the checkstand, there is a queue and he has to wait. The waiting time of each customer can be also modeled by a one dimensional random walk that is stopped at the barriers from before and one can ask the following questions. "What is the relation between the waiting time of the man and the waiting time of the customer in front of him?" "How fast does the checkout clerk have to be in order for it to be possible to handle all customers and, furthermore, when a second checkout is needed?" as well as " How will the waiting time of the man be changed if there are two or three checkouts?" In the theory of queuing processes one can find answers to these questions with a lot of further extensions and applications. We stick to the random walk behind this process which is known as Lindley process in the one dimensional case and extend it to higher dimensions. Results regarding this process are specified in Chapter 5.

As I mentioned before the probability that the drunk man will come back to his home is one. But if he arrives at home, he has to face another problem. The man carries a bunch of keys where every key has exactly the same (symmetric) shape, meaning they are indistinguishable. It is not really a problem because the wife of the man was so clever and colored some of his keys in a way that he can identify each key and of course now also will find the right key to get into his house. But the interesting question is: "How many colors did his wife need if she tried to use as few as possible colors?" This question was raised by Albertson and Collins in 1996. The maybe unexpected answer is that one needs three colors if one has three, four or five keys and in all other cases one only needs two colors. In Figure 1.1 you can see some examples. The idea is to distinguish the keys not only by their own color but also by their sequence.

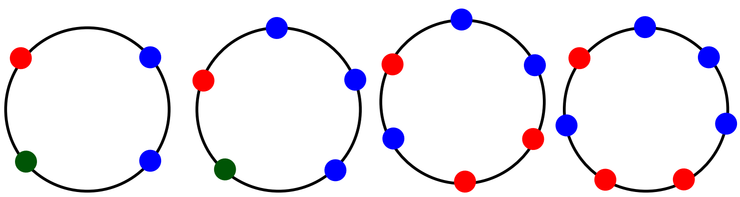

Figure 1.1: Distinguished colored cycle with $k = 4, \ldots, 7$ vertices.

The ring of keys represents a cycle graph. But the similar question one could ask for other graphs for example trees (= a connected graph without cycles) or subcubic graphs (= every vertex of the graph has at most three neighbors). That is what is discussed in Chapter 6.

In the end I want to mention that I am sorry about the allocation of the role of the man and the woman, but to stick to the quotation of Kakutani it was only possible to present this short story in that way.

Acknowledgement

I am very happy that I had the opportunity to do my PhD-Studies at Graz University of Technology. During this time I learned a lot of things. First, of course I enhanced my mathematical skills and got a feeling for the width of this science. Second, I build up a lot of autonomy that I am sure will help me also in situations outside mathematics. And third, I enhanced my abilities in cooperation with other people due to common projects and the membership in the DK-College.

I believe every PhD-Student or Scientist knows the situation where one works for a long time for example on a proof of a theorem and one finally is very happy to get it. But after thinking about all steps made again one realizes that there was a small mistake in the beginning and one has to start up again. Although such common situations of a researcher can be very hard I see that they strengthen my power to handle problems a lot.

For all of that I want to thank my advisor Wolfgang Woess who gave me the possibility to learn so many things, who supported me all the time and who accepted my strengths and weaknesses.

I also want to thank my co-authors that have contributed to my work, especially Wojciech Cygan who spend a long period in Graz and has helped me a lot to improve my mode of practice and style of writing as well as Wilfried Imrich who gave me the opportunity of working on a project outside of my main research topic. Through this I extended my mind and came in contact to further mathematicians. Moreover, the discussions and support by my colleagues was every time very fruitful. In particular I want to thank Marc Peigné, Johannes Cuno and Sebastian Müller.

Last but not least I want to thank my family and my boyfriend who have always supported and encouraged me on my way.

Chapter 2

Summary

In this thesis we start with an introductory chapter where we give some general definitions and a short overview about known facts regarding the topics of the subsequent chapters. First, we define Markov chains and random walks and consider their limiting behavior regarding recurrence and transience. After that the thesis covers the definitions of graphs and their properties as transitivity, their coloring and distinguishing number.

The rest of the PHD-Thesis is divided into three parts. In the first part we present the research results concerning the reflected random walk. After introducing the reflected random walk in the one-dimensional case we consider the reflected random walk in higher dimensions with one or more axes of reflection and provide conditions for its recurrence behavior. The used methods are based on the works about the corresponding one-dimensional reflected random walk. The model was first considered by von Schelling who pointed out the application of this process to telephone networks. The reflected random walk is defined by $X_0 = x_0$ and $X_{n+1} = |X_n - Y_{n+1}|$, $n \geq 0$, where $Y_1, Y_2, \ldots$ is a sequence of independent and identically distributed real valued random variables. It is a special example of a random dynamical system and can also be seen as a Markov chain. We give a summary of some interesting results about this stochastic process, which was described and studied among others by Feller, Spitzer, Peigné and Woess, Boudiba and Rabeherimanana. After this we focus on the recurrence behavior of the multi-dimensional reflected random walk. We consider processes with reflections in each coordinate and processes where in some coordinates there are reflections and in all other coordinates there are ordinary random walks.

In the second part we give results concerning the multidimensional Lindley process. Recall the one-dimensional Lindley process $(W_n)_{n\geq 1}$, which is defined by setting $W_0 = 0$ and $W_n = \max\{0, W_{n-1} - Y_n\}$, $n \geq 1$, where $Y_1, Y_2, \ldots$ is a sequence of independent and identically distributed discrete random variables. We summarize the known results about this process and introduce the multi-dimensional case which is build upon the process $(W_n)_{n\geq 1}$. We study the recurrence behavior and present our approach which is based on the technique of discrete subordination and the theory of regular variation.

The third part contains a more graph theoretical topic. The direct connection to the other two topics is quite small apart from the introductory story. But since there was a chance to collaborate with other DK-Students and to get an insight to an other interesting field in mathematics I was not reluctant to work on the questions that were stated by Wilfried

Imrich in an fruitful seminar given by him and Wolfgang Woess. The upraising question is how many colors do we need to color a graph such that the only color preserving automorphism is the identity? The smallest such quantity of colors is called the distinguishing number. Since its introduction by Albertson and Collins [1] more than 20 years ago, an extensive literature on this topic has developed. Collins and Trenk [26] and, independently, Klavžar, Wong and Zhu [57] proved that for any finite graph G of maximum valence $\Delta(G) = d$, $D(G) \leq d + 1$. For infinite graphs $D(G) \leq d$, see Imrich, Klavžar and Trofimov [48]. We want to improve this bounds for subcubic graphs ($d = 3$) and give a complete classification regarding its distinguishing number. As one of the consequences we get that all infinite connected graphs with $\Delta(G) = 3$ are 2-distinguishable. In part four we are interested in coloring the vertices of a finite or infinite tree T of bounded finite valence k by c colors ($2 \leq c \leq k$), such that every color preserving automorphism fixes as many vertices as possible. In this sense we show that there always exists a c coloring for which all vertices whose distance from the next leaf is at least $\lceil \log_c k \rceil$ are fixed by any color preserving automorphism, and that one can do much better in many cases.

Chapter 3

Preliminaries

In this chapter we want to give a short collection of some definitions and facts that build a basis for the upcoming three parts. It contains some properties of Markov chains, random walks as well as some graph theoretical aspects. A lot of things are comparable to the books of Woess [85], [86].

Remember the man and the bird from the introductory story. I called them "random walker". The term "random walk" is often used instead of Markov chain since there is a strong connectivity.

A *Markov chain* (X_k) is build upon a state space $\mathcal{X}$ that consists of finite or countably infinite elements. In the example of the drunk man $\mathcal{X}$ was the space of all points in a two-dimensional grid. Moreover, we need a start distribution $p_0(x)$ that describes the probability that the Markov chain starts in x. The so called transition probabilities $p(x, y)$ that can be collect in the transition matrix P represent the probability that the Markov chain jumps from x to y. If we multiply the transition matrix n times by itself, we get a matrix P^n whose entries are the probabilities to come from one point to another in exactly n steps. We write

$$p(x,y) = \mathbb{P}(X_n = y | X_{n-1} = x_{n-1}),$$

$$p^{(n)}(x,y) = \mathbb{P}(X_1 = x_1 | X_0 = x)\mathbb{P}(X_2 = x_2 | X_1 = x_1) \dots \mathbb{P}(X_n = y | X_{n-1} = x_{n-1}).$$

For example the one step transition probabilities of the man are

$$p((i,j),(i \pm 1, j)) = p((i,j),(i, j \pm 1)) = \frac{1}{4}$$

and zero else where (i, j) describes a point in $\mathbb{Z}^2$. At last for every Markov chain the Markov property 3.1 and the property of time homogeneity 3.2 holds

$$\mathbb{P}(X_{k+1} = x_{k+1} | X_k = x_k, \dots, X_1 = x_1) = \mathbb{P}(X_{k+1} = x_{k+1} | X_k = x_k), \tag{3.1}$$

$$\mathbb{P}(X_{j+1} = x | X_j = y,) = \mathbb{P}(X_{k+1} = x | X_k = y). \tag{3.2}$$

The first one means that the future state of the process depends only on the present state and not on the past. The second one describes that the transition probabilities are independent from the time or index.

For each Markov chain one has an underlying graph.

Definition 3.0.1 (Graph)**.** A graph G is a pair (V, E) where V is the set of vertices and E is the set of edges which is build by subsets of two elements of V. For an undirected graph there is at most one edge between two vertices so that (x, y) is the same edge as (y, x). In the case of a directed graph the edges are ordered pairs. Thus, (x, y) is an edge that is directed from x to y while (y, x) is directed from y to x. If a vertex is connected by an edge to itself, we call it a loop.

Regarding a Markov chain we have a directed graph with loops where the vertices are build by the state space $\mathcal{X}$. There is a directed edge between two vertices x and y if and only if $p(x, y) > 0$. It is possible to give $p(x, y)$ as weight to the edge (x, y) for every edge of the graph and obtain a weighted graph in that way. A Markov chain is irreducible if from every state in the transition graph it is possible to reach every other state with positive probability. Considering the drunk man in our main example the underlying graph of this Markov chain is the two-dimensional grid where every edge is a double edge with weight 1/4 in each direction. As we have seen one can imagine that the Markov chain describes a random walk on that graph. Thus, one often speaks (also in this thesis) of a random walk instead of a Markov chain especially if the Markov chain is adjusted to the underlying state space. In general a random walk is defined as follows.

Definition 3.0.2 (Random Walk)**.** Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables in $\mathbb{R}^d$. The stochastic process $(X_n)_{n\in\mathbb{N}_0}$ given by

$$X_n = X_0 + \sum_{j=1}^{n} Y_j, \quad n \in \mathbb{N}_0$$

is called a d-dimensional random walk in $\mathbb{R}^d$. Often one takes $X_0 = 0 \in \mathbb{R}^d$. A random walk is a discrete process with independent and stationary increments, compare [33].

A typical example is the d-dimensional simple random walk like the drunk man or the drunk bird from the introduction. An often used and studied property of a random walk or of a Markov chain is the return behavior to the starting point that can be described through recurrence and transience.

Definition 3.0.3 (Recurrence)**.** We say that a state x of a stochastic process (Markov chain, random walk) is recurrent if

$$\mathbb{P}_x(X_n = x \text{ for infinitely many } n) = 1.$$

A very helpful tool to study the recurrence is the Green function.

Definition 3.0.4 (Green function)**.** The Green function is defined by

$$G(x, y|z) = \sum p^{(n)}(x, y) z^n,$$
$$G(x, y|1) = \mathbb{E}(\text{number of visits of } y \text{ if starting in } x).$$

If

$$G(x, x) = \infty$$

then x is recurrent (and vice versa). For a random walk $G(x, y) = \infty$ means that if the random walk starts in x the expected number of visits to y is infinity. In addition recurrence is equivalent to each of the following conditions

$$\mathbb{P}_x(\exists n > 0 : X_n = x) = 1,$$
$$\mathbb{P}(\tau_x < \infty | X_0 = x) = 1,$$

where

$$\tau_x = \inf\{n \geq 1 : X_n = x\}$$

is the time of the first return to the starting point x of the process.

Definition 3.0.5 (Transience)**.** A stochastic process is transient if it is not recurrent. It means that

$$\mathbb{P}_x(X_n = x \text{ for infinitly many } n) = 0.$$

Transience is equivalent to each of the following conditions

$$\mathbb{P}_x(\exists n > 0 : X_n = x) < 1,$$
$$G(x, x) = \sum_n p^{(n)}(x, x) < \infty,$$
$$\mathbb{P}(\tau_x < \infty | X_0 = x) < 1.$$

For a irreducible Markov chain either all states are recurrent or transient A well known criteria for recurrence and transience of a random walk is the following where we take $Y \sim Y_1$.

Theorem 3.0.6. *(Chung-Fuchs, 1951) Consider a d-dimensional random walk X_n. Then the following holds.*

(i) For $d = 1$ and if $X_n/n \to 0$ in probability, then X_n is recurrent. This happens if $\mathbb{E}Y = 0$.

(ii) For $d = 2$ and if $X_n/\sqrt{n}$ converges in distribution to a centered normal distribution, then X_n is recurrent. This happens if $\mathbb{E}Y = 0$ and $\mathbb{E}Y^2 < \infty$.

(iii) For $d = 3$ X_n is transient.

Remark 3.0.7. If $\mathbb{E}Y = \mu$, then by the strong law of large numbers $X_n/n \to \mu$ almost surely. Therefore, if $\mathbb{E}Y$ exists and is non-zero, it is obvious that the random walk is transient.

For recurrent random walks or Markov chains we can further differentiate between null and positive recurrence. A recurrent state x is called positive recurrent if $\mathbb{E}\tau_x < \infty$ and null recurrent else. If all states of a Markov chain are positive recurrent, then we say that the Markov chain is positive recurrent.

Definition 3.0.8 (Stationary/invariant measure)**.** A measure $\nu = (\nu_1, \nu_2, \dots)$ that satisfies

$$\nu P = \nu$$

is called a stationary measure. If $0 \leq \nu_i \leq 1$ for all i and $\sum_i \nu_i = 1$, we call it a stationary probability measure or stationary distribution. One often also call it invariant measure.

For a Markov chain the existence of a stationary probability measure is equivalent to positive recurrence.

In Definition 3.0.1 we have specified what we mean by a graph and described the underlying graph of a Markov chain. If there is no connection to a Markov chain, we will consider undirected graphs without multi-edges in this thesis. Some classical examples are k-regular graphs where each vertex has k neighbored vertices, the complete graph having an edge between every pair of vertices or more general a connected graph.

Definition 3.0.9 (Path, Connected graph, Locally finite graph)**.** A path between two vertices v_1 and v_n is a sequence of vertices $v_1, \dots v_n$ with $v_i \sim v_{i+1}$ for $i = 1, \dots n-1$ where $u \sim v$ means that there is an edge between u and v.

We say that a graph is connected if there exists a path between u and v for all vertices $u, v \in V$. If the number of neighbored vertices is finite for each vertex, the graph is locally finite.

Definition 3.0.10 (Isomorphism, Automorphism of a graph)**.** Two graphs G_1 and G_2 are isomorph if there is a bijection $\gamma : V(G_1) \to V(G_2)$, such that x is adjacent to y in G_1 if and only if $\gamma(x)$ is adjacent to $\gamma(y)$ in $V(G_2)$.

An automorphism of a graph G is an isomorphism from G to G. That means it is a bijective map $\gamma : V(G) \to V(G)$ such that

$$x \sim y \Leftrightarrow \gamma(x) \sim \gamma(y) \quad \forall x, y \in V(G).$$

It is a permutation of the vertices where the edges in both graphs are the same. The identical mapping is an automorphism which we can take as neutral element. If we have two automorphisms, then the product is also an automorphism. Moreover, for every automorphism we can find a second automorphism such that the product is the identical mapping. So the set of all automorphisms of a graph G build a group, the automorphism group which we will write as $\mathrm{Aut}(G)$.

If one collect all automorphisms that leave a vertex v invariant, then we call this the stabilizer of v

$$\mathrm{Aut}(G)_v = \{\alpha \in \mathrm{Aut}(G) | \alpha(v) = v\}.$$

It is a subgroup of the automorphism group $\mathrm{Aut}(G)$.

In Chapter 6 we want to color the vertices of a graph such that each vertex can be uniquely identified through the structure of the graph and the coloring of the vertices. We want to determine how many colors we need to color trees and subcubic graphs. That number of colors is defined as distinguishing number.

Definition 3.0.11 (Coloring, Distinguishing number). A map $c : V \to \{1, \ldots, d\}$ is called d-coloring of a graph. The distinguishing number $D(G)$ of a graph G is the the smallest d such that there exists a d-coloring c of G that is only preserved by the identity automorphism.

Example 3.0.12. The distinguishing number of the $K_{2,3}$ and the cube is 2 while it is 3 for the $K_{3,3}$ and the K_4.

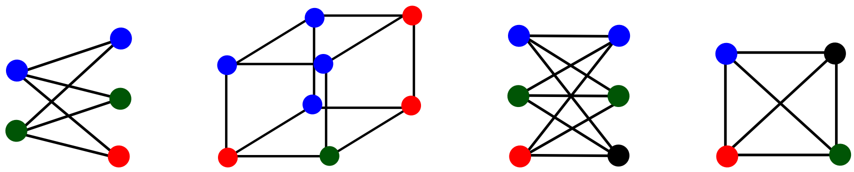

Figure 3.1: Distinguishing colorings for the $K_{2,3}$, the cube, the $K_{3,3}$ and K_4.

Sometimes the distinguishing number is given for graphs where each automorphism moves at least a certain number of vertices. For this we use the term motion of a graph.

Definition 3.0.13 (Motion)**.** The motion $m(\phi)$ of a permutation ϕ is defined as the number of vertices that are moved by ϕ, indeed

$$m(\phi) = |\{v|\phi(v) \neq v\}|.$$

The motion $m(G)$ of a graph G is the minimal motion of all its automorphisms (excluding the identity). That means

$$m(G) = \min\{m(\phi)|\phi \in \mathrm{Aut}(G)\backslash\{id\}\}.$$

Thus, if we consider a graph with for example motion 3, it means that every automorphism of the graph moves at least three vertices and there is no automorphism moving only two vertices and fixing the rest.

Definition 3.0.14 (degree/valence)**.** We say that a vertex v of a graph has degree or valence k if there exists exactly k vertices v_i, $i = 1, \ldots, k$ such that there is an edge between v and v_i for each i.

Definition 3.0.15 (edge and vertex transitivity)**.** A graph G is called edge transitive if for every (u, v), $(u', v') \in E$ there exists an automorphism γ such that $(\gamma(u), \gamma(v)) = (u', v')$. We say that G is vertex transitive if for every $u, v \in V$ there exists an automorphism ϕ such that $\phi(u) = v$.

A vertex transitive graph is always regular. That means every vertex has the same degree. A typical example of a vertex transitive graph is the peterson graph. Complete graphs are vertex as well as edge transitive.

Chapter 4

Random Walks with Reflection

The following part is based on the submitted joint paper with Wolfgang Woess *Multidimensional random walk with reflections.*

4.1 Introduction

Let $(Y_n)_{n\geq 0}$ be a sequence of i.i.d. real valued random variables, and let $S_n = Y_1 + \ldots + Y_n$ be the classical associated random walk. Reflected random walk (RRW) is the process $(X_n^x)_{n\geq 0}$ given by

$$X_0^x = x \geq 0\,, \qquad X_n^x = |X_{n-1}^x - Y_n|.$$

It was first considered by von Schelling [83] in the context of telephone networks. A rigorous examination appeared in Feller [38], and was then developed further by Knight [59], Boudiba [18], [19] and Leguesdron [63]. The PhD Thesis of Benda [10] and his unpublished papers [11], [12] contain important contributions that will also play a role here.

Our main interest is in recurrence of this process. Positive recurrence is settled in the above references via exhibiting a unique stationary probability measure for the process; proving uniqueness is a non-trivial task. Criteria for null recurrence were given by Smirnov [75] and Rabeherimanana [71], and also by Peigné and Woess [67], [68].

In the present paper, we are interested in the multidimensional variant, where we have a random walk which is reflected in the first coordinate(s) and remains an ordinary random walk in the other coordinate(s). Thus, we have a probability measure μ on $\mathbb{R}^{r+s}$ and the state space $\mathbb{R}_+^r \times \mathbb{R}^s$, whose elements we write as (x, w) or just xw, where $x \in \mathbb{R}_+^r$ and $w \in \mathbb{R}^s$. For $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$, we write

$$|x| = \big(|x_1|, \ldots, |x_r|\big) \quad \text{and} \quad \|x\| = \sqrt{x_1^2 + \cdots + x_r^2}\,.$$

We consider a sequence $(Y_n\,, V_n)$ of i.i.d. μ-distributed random vectors with $Y_n \in \mathbb{R}^r$ and $V_n \in \mathbb{R}^s$. Then our process starting at (x, v) is given by

$$(X_n^x\,, v + Z_n)\,, \quad \text{where} \quad X_0^x = x,\; X_n^x = |X_{n-1}^x - Y_n|\,, \quad \text{and} \quad Z_n = V_1 + \cdots + V_n\,. \tag{4.1}$$

We shall usually start with $w = 0$. For studying transience / recurrence, only the cases $s \in \{0, 1, 2\}$ are of interest, since otherwise already (Z_n) is transient.

We remark immediately that the process (4.1) factorises in each coordinate.

- If $i \le r$, then the i-th coordinate of $(X_n^x, v + Z_n)$ is the reflected random walk on $\mathbb{R}_+$ which starts at x_i and is driven by the i-th marginal μ_i of μ.
- If $r+1 \le i \le r+s$ then the i-th coordinate is the random walk (sum of i.i.d. random variables) which starts at v_i and whose law is the i-th marginal of μ.
- In particular, (X_n^x) is the reflected random walk on $\mathbb{R}_+^r$ driven by $\mu_{\lfloor r \rfloor}$, the overall marginal of μ on the first r coordinates, and $(v + Z_n)$ is the (ordinary) random walk on $\mathbb{R}^s$ whose law $\mu_{\lceil s \rceil}$ is the overall marginal of μ on the last s coordinates.

The given process is comparable to other reflected processes. Some examples can be found in Fayolle, Iasnogorodski and Malyshev [37] and Jonckheere and Shneer [52]. There are some similarities to the well known queuing process $X_n = \max\{0, X_{n-1} - Y_n\}$ that was studied by Cygan and Kloas [28] in the multidimensional case. We mention that one could suppose to see connections to the large field on Brownian motion approximations as in Bramson, Dai and Harrison [22] and Hobson and Rogers [45] but we emphasize that their are big differences especially since in our case increments with arbitrary distributions are considered.

As usual, we shall distinguish between the *lattice* and the *non-lattice* cases in each coordinate. The lattice case arises when there is $\kappa > 0$ such that $\operatorname{supp}(\mu_i) \subset \kappa \cdot \mathbb{Z}$. In this case, we can and will always assume without loss of generality that

$$\operatorname{supp}(\mu_i) \subset \mathbb{Z} \quad \text{and} \quad \gcd \operatorname{supp}(\mu) = 1. \tag{4.2}$$

The marginal μ_i is non-lattice if no κ as above exists.

Thus, we shall assume that $r = r_1 + r_2$ and $s = s_1 + s_2$ such that the marginals μ_i satisfy (4.3.3) for $i = 1, \dots, r_1$ and $i = r+1, \dots, r+s_1$, while they are non-lattice in the other coordinates. Consequently, it is natural that we restrict our state space to

$$\mathcal{X} = \mathbb{N}_0^{r_1} \times \mathbb{R}_+^{r_2} \times \mathbb{Z}^{s_1} \times \mathbb{R}^{s_2}\,. \tag{4.3}$$

In the non-discrete situation, our study of recurrence and stationary probability distributions focuses on topological recurrence.

One of our basic tools is *local contractivity,* a property of stochastic dynamical systems that was introduced by Babillot, Bougerol and Elie [8] and studied in detail by Benda [10]. We summarise the basic facts in the short §2. In §3, we review one-dimensional reflected random walk and display the clever method of [10] in the lattice case to induce a locally contractive process on the even integers (Proposition 4.2.4). We also display an example of a transient reflected random walk where the non-reflected walk is recurrent.

In §4, we consider the multidimensional case with reflection in all coordinates. The main result is Theorem 4.4.2, characterising positive recurrence. While the case where all marginals are non-lattice is covered by Peigné [66], the presence of lattice marginals leads to considerable additional difficulties which we elaborate in detail. Subsequently, we provide several partial results and examples regarding the null-recurrent situation, where however a complete characterisation remains a challenging open problem.

In the last §5, we consider the general situation where some coordinates are reflected and others (at most 2) are "free" (non-reflected). Our second main result is Theorem 4.5.1, where we assume that the reflected part is (topologically) recurrent and the non-reflected coordinates are centred and satisfy the natural moment conditions. While this is easy when the reflected part is discrete (lattice), additional tools from Ergodic Theory are needed in general, invoking results on recurrence of stationary random walks which are due to Atkinson [5] and Schmidt [74]. This leads to recurrence of the process. Again, it is a challenging open problem to handle the case when the reflected part is only null-recurrent.

4.2 A summary on local contractivity

We recall a few facts that were explained in [68], plus additional features. Unless otherwise stated, the facts displayed in this section can be found in [68], resp. the remarkable PhD thesis [10].

In general, we consider a proper metric space $(\mathcal{X}, d)$ and the monoid $\mathfrak{C}(\mathcal{X})$ of all continuous mappings $\mathcal{X} \to \mathcal{X}$. It carries the topology of uniform convergence on compact sets. Now let $\widetilde{\mu}$ be a Borel probability measure on $\mathfrak{C}(\mathcal{X})$, and let $(F_n)_{n\geq 1}$ be a sequence of i.i.d. $\mathfrak{C}(\mathcal{X})$-valued random variables (functions) with common distribution $\widetilde{\mu}$, defined on a suitable probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The associated *stochastic dynamical system (SDS)* $\omega \mapsto X_n^x(\omega)$ is given by

$$X_0^x = x \in \mathcal{X}\,, \quad \text{and} \quad X_n^x = F_n \circ F_{n-1} \circ \cdots \circ F_1(x)\,, \quad n \geq 1\,. \tag{4.4}$$

In case of reflected random walk on $\mathcal{X} = \mathbb{N}_0^{r_1} \times \mathbb{R}_+^{r_2}$ (as in (4.3) with $s_1 = s_2 = 0$), we have $F_n(x) = |x - Y_n|$, and these mappings are contractions, whence we may replace $\mathfrak{C}(\mathcal{X})$ by the closed sub-monoid $\mathfrak{Lip}_1(\mathcal{X})$ of all Lipschitz mappings with Lipschitz constant ≤ 1. If μ is the distribution on $\mathbb{R}^d$ of the increments Y_n, then $\widetilde{\mu}$ is the image of μ under the mapping $\mathbb{R} \to \mathfrak{Lip}_1(\mathcal{X})$, $y \mapsto f_y$, where $f_y(x) = |x - y|$.

Definition 4.2.1. The SDS is called *locally contractive,* if for every $x \in \mathcal{X}$ and every compact $K \subset \mathcal{X}$,

$$\mathbb{P}[d(X_n^x, X_n^y) \cdot \mathbb{1}_K(X_n^x) \to 0 \quad \text{for all } y \in \mathcal{X}] = 1\,.$$

It is called *strongly contractive,* if for every $x \in \mathcal{X}$,

$$\mathbb{P}[d(X_n^x, X_n^y) \to 0 \quad \text{for all } y \in \mathcal{X}] = 1\,.$$

Proposition 4.2.2. *A locally contractive SDS is either* transient,

$$\mathbb{P}[d(X_n^x, x) \to \infty] = 1 \quad \textit{for every } x \in \mathcal{X}$$

or (topologically) recurrent *in the sense that there is a maximal non-empty closed subset $\mathcal{L} \subset \mathcal{X}$ with the property that for every open set U that intersects $\mathcal{L}$,*

$$\mathbb{P}[X_n^x \in U \textit{ infinitely often}] = 1 \quad \textit{for every } x \in \mathcal{X}.$$

In the recurrent case, $\mathcal{L}$ coincides almost surely with the set of accumulation points of any trajectory $\big(X_n^x(\omega)\big)$, i.e. every neighbourhood of any point in $\mathcal{L}$ is visited infinitely often a.s.

$\mathcal{L}$ is also characterised as the smallest non-empty closed subset of $\mathcal{X}$ with the property that $f(\mathcal{L}) \subset \mathcal{L}$ for every $f \in \operatorname{supp}(\widetilde{\mu}) \subset \mathfrak{C}(\mathcal{X})$.

Note that the last characterisation does not rely on recurrence; it depends only on $\operatorname{supp}(\widetilde{\mu})$. In the recurrent case, the set $\mathcal{L}$ is called the *attractor,* and the SDS is strongly contractive.

An *invariant measure* for an SDS is a Radon measure ν on $\mathcal{X}$ such that for any Borel set $B \subset \mathcal{X}$,

$$\int \mathbb{1}_B(X_1^x)\, d\nu(x) = \nu(B).$$

Part (a) of the following is obvious; for (b) see [67].

Proposition 4.2.3. (a) *A locally contractive SDS which has an invariant probability measure is recurrent.*
(b) *A locally contractive SDS which is recurrent has an invariant measure ν which is unique up to multiplication by constants. In this case, the following holds.*

- $\operatorname{supp}(\nu) = \mathcal{L}$.
- $\nu(\mathcal{L}) < \infty$ *if and only if the SDS is positive recurrent*
  (the return time to any open set which intersects $\mathcal{L}$ has finite expectation).

For an SDS of contractions, let $\mathfrak{S}(\widetilde{\mu})$ be the sub-semigroup of $\mathfrak{Lip}_1(\mathcal{X})$ generated by $\operatorname{supp}(\widetilde{\mu})$ and $\overline{\mathfrak{S}}(\widetilde{\mu})$ its closure.

Proposition 4.2.4. *A non-transient SDS of contractions is locally contractive if and only if $\overline{\mathfrak{S}}(\widetilde{\mu})$ contains a constant function. In this case, it is recurrent as well as strongly contractive, so that it is absorbed by the attractor: for any starting point x,*

$$d(X_n^x, \mathcal{L}) \to 0 \quad \textit{almost surely.}$$

See [63], [66], [10] and [68, Theorem 4.2]. An important tool is going to be the following.

Proposition 4.2.5. *Suppose that our SDS of contractions is locally contractive and has an invariant probability measure ν. Then there is an $\mathcal{X}$-valued random variable Z such that for any starting point $x \in X$,*

$$\widehat{X}_n^x = F_1 \circ \cdots \circ F_n(x) \to Z \quad \textit{almost surely.}$$

The distribution of Z is ν.

Note that in general, $(\widehat{X}_n^x)_{n\geq 0}$ is not Markovian. The proposition is proved in [63] and [66] under the assumption that $\mathcal{X} = \mathbb{R}^r$. In [66], it concerns more general SDS of contractions which are not necessarily compositions of i.i.d. mappings, but driven by a positive recurrent Markov chain. It readily extends to any proper metric space $\mathcal{X}$ in place of $\mathbb{R}^r$.

4.3 A review of one-dimensional reflected random walk

Here, the Y_n are real random variables with common distribution μ. We always assume that

$$\mu\big((0\,,\,\infty)\big) > 0\,. \tag{4.5}$$

The state space is $\mathcal{X} = \mathbb{R}_+$ in the non-lattice case, and $\mathcal{X} = \mathbb{N}_0 = \{0, 1, \dots\}$ in the lattice case (4.3.3).

A. Irreducibility and local contractivity

We set

$$\begin{aligned} &N = \sup \operatorname{supp}(\mu)\,, \quad \text{if} \quad \operatorname{supp}(\mu) \subset \mathbb{R}_+\,, \quad \text{resp.} \quad N = \infty\,, \quad \text{otherwise, and}\\ &\mathcal{L} = [0\,,\,N] \cap \mathcal{X}\,, \quad \text{if} \quad N < \infty\,, \quad \text{resp.} \quad \mathcal{L} = \mathbb{R}_+ \cap \mathcal{X}\,, \quad \text{if} \quad N = \infty\,. \end{aligned} \tag{4.6}$$

Then (X_n^x) is (topologically) irreducible on $\mathcal{L}$, see [63], [19], [71], [67], [68]. Regarding local contractivity, the following is known; compare with [63], [10], [11], [67] and [68].

Proposition 4.3.1. *Assume that μ is non-lattice and satisfies* (4.5)*. Then the reflected random walk induced by μ is locally contractive.*

In the lattice case, we cannot have local contractivity. Indeed, if $x, y \in \mathbb{N}_0$ then $X_n^x - X_n^y$ always has the same parity as $x - y$. However, the PhD thesis [10] contains a smart observation & method which we now explain. For the remainder of this sub-section, we assume that μ satisfies (4.3.3).

For $x \in \mathbb{Z}$, let $\pi(x) = 0$ if x is even, and $\pi(x) = 1$ if x is odd. Then the following is obvious.

Lemma 4.3.2. *The process $\big(\pi(X_n^x)\big)_{n\ge 0}$ is a Markov chain on $\{0,1\}$ with transition probabilities $p(i,j) = \mathbb{P}\left(\pi(X_n^x) = j | \pi(X_{n-1}^x\right)$ where*

$$p(0,0) = p(1,1) = \mu(2\cdot\mathbb{Z}) \quad \textit{and} \quad p(0,1) = p(1,0) = \mu(2\cdot\mathbb{Z}+1)\,.$$

In particular, it depends only on the parity of the starting point x, and by (4.3.3) it is irreducible. It is therefore positive recurrent, the return times to each of the two states coincide, their distribution is easily computed, and the expected value is 2. We can consider the induced process on $2\cdot\mathbb{N}_0$, resp. on $2\cdot\mathbb{N}_0+1$. That is, we consider the a.s. finite stopping times

$$\begin{aligned} &\mathbf{t}(0) = 0\,, \quad \text{and, setting} \quad S_k = Y_1 + \dots + Y_k\,,\\ &\mathbf{t}(n) = \inf\{k > \mathbf{t}(n-1) : \pi(X_k^x) = \pi(x)\} = \inf\{k > \mathbf{t}(n-1) : S_k \text{ is even}\}\,. \end{aligned} \tag{4.7}$$

No matter whether the starting point of (X_n^x) is even or odd, the induced process $(X_{\mathbf{t}(n)}^x)$ on the respective class $2\cdot\mathbb{N}_0$ or $2\cdot\mathbb{N}_0+1$ is again an SDS generated by i.i.d. contractions:

$$X_{\mathbf{t}(n)}^x = \bar F_n \circ \bar F_{n-1} \circ \dots \circ \bar F_1(x) \quad \text{with} \quad \bar F_n = f_{Y_{\mathbf{t}(n)}} \circ f_{Y_{\mathbf{t}(n)-1}} \circ \dots \circ f_{Y_{\mathbf{t}(n-1)+1}}\,. \tag{4.8}$$

Let $\widetilde{\mu}_{\mathbf{t}}$ be the distribution of $\bar F_1$ on $\mathfrak{Lip}_1(\mathbb{N}_0)$. Since the proof of the following is not easily accessible [10], we present it here.

Proposition 4.3.3. *If μ satisfies*

$$\operatorname{supp}(\mu_i) \subset \mathbb{Z} \quad \textit{and} \quad \gcd \operatorname{supp}(\mu) = 1.$$

then $\mathbb{1}_{2\cdot\mathbb{N}_0} \in \overline{\mathfrak{S}}(\widetilde{\mu}_{\mathbf{t}})$.

Proof Recall the notation $f_y(x) = |x - y|$.
Step 1. There are elements $y_0 \,, \dots, y_m \in \mathbb{N}$ such that

$$0 < y_0 < \cdots < y_m \,, \quad \gcd\{y_0 \,, \dots, y_m\} = 1 \,, \quad \text{and} \quad f_{y_k} \in \mathfrak{S}(\widetilde{\mu}).$$

Indeed, there is $b \in \operatorname{supp}(\mu)$ with $b \geq 1$, and if $a < 0$ then $a' = a + (\lfloor -a/b \rfloor + 1)b \geq 1$, and we check easily that

$$f_{a'} = f_b^{\lfloor -a/b \rfloor + 1} \circ f_a \,, \quad \text{where } f_b^n = \underbrace{f_b \circ \cdots \circ f_b}_{n \text{ times}}$$

whence $f_{a'} \in \mathfrak{S}(\widetilde{\mu})$ whenever $a \in \operatorname{supp}(\mu)$. Now, there are $a_1 \,, \dots, a_n \in \operatorname{supp}(\mu) \setminus \{0\}$ with greatest common divisor 1. We replace each $a_k < 0$ by a'_k and add b to the updated collection of elements. Then we order them and eliminate possibly redundant ones to get $y_0 \,, \dots, y_m$.
Step 2. We now set $d_k = \gcd\{y_0 \,, \dots, y_k\}$, so that $y_0 = d_0 > d_1 > \cdots > d_m = 1$. We construct recursively elements $g_0 \,, \dots, g_m \in \mathfrak{S}(\widetilde{\mu})$ such that

$$g_k(n) = f_{d_k}(n) \quad \text{for all } \ n \in \{0, 1, \dots, d_k\} \,.$$

We start with $g_0 = f_{y_0}$. If we already have g_{k-1} then we follow the steps of the Euclidean algorithm getting d_k as the greatest common divisor of y_k and d_{k-1}. We pose $r_0 = y_k$, $r_1 = d_{k-1} < y_k$ and applying repeated integer division $r_{i-1} = q_i r_i + r_{i+1}$ with $0 \leq r_{i+1} < r_i$. If j is the first index for which $r_{j+1} = 0$ then $r_j = \gcd\{y_k \,, d_{k-1}\} = d_k$. We let

$$h_0 = f_{y_k} \,, \; h_1 = g_{k-1} \,, \quad \text{and} \quad h_i = h_{i-1}^{q_{i-1}} \circ h_{i-2} \,, \; i = 2, \dots, j \,.$$

Then we set $g_k = h_j$. (The h_i as well as j depend on k.) One checks easily that also g_k has the proposed properties.
Step 3. We now have $g_m(n) = f_1(n)$ for $n \in \{0, 1\}$. Because $f_1(x) - f_1(y)$ always has the same parity as $x - y$, g_m sends even numbers to odd ones and vice versa. Since it is a contraction, this implies that $|g_m(n+1) - g_m(n)| = 1$ for all n. From this we deduce inductively that for all $n \in \mathbb{N}$,

$$g_m(2n-1) \in \{0, 2, \dots, 2n-2\} \quad \text{and} \quad g_m(2n) \in \{1, 3, \dots, 2n-1\}.$$

Therefore $h = g_m^2 \in \mathfrak{S}(\widetilde{\mu})$ preserves the parity of any $n \in \mathbb{Z}$. But this just means that $h \in \mathfrak{S}(\widetilde{\mu}_{\mathbf{t}})$. The above yields that

$$h^k(2n-1) = 1 \quad \text{and} \quad h^k(2n-2) = 0 \quad \text{for} \quad n = 1, \dots, k.$$

As $k \to \infty$, we see that $h^k \to \mathbb{1}_{2\cdot\mathbb{N}_0+1}$ pointwise, so that $\mathbb{1}_{2\cdot\mathbb{N}_0+1} \in \overline{\mathfrak{S}}(\widetilde{\mu}_{\mathbf{t}})$.

Next Proposition 4.2.4 implies the following.

Corollary 4.3.4. *The induced process* $(X^x_{\mathbf{t}(n)})$ *is locally contractive on each of the classes* $2{\cdot}\mathbb{N}_0$ *and* $2{\cdot}\mathbb{N}_0{+}1$. *The respective limit sets are* $\mathcal{L}_0 = \mathcal{L}\cap(2{\cdot}\mathbb{N}_0)$, *resp.* $\mathcal{L}_1 = \mathcal{L}\cap(2{\cdot}\mathbb{N}_0{+}1)$, *where* $\mathcal{L}$ *is as in* (4.6).

If the original reflected random walk (X^x_n) *is positive, resp. null recurrent, then so is the induced process on each of the two classes, and* $X^x_n - X^y_n \to 0$ *a.s. whenever* $x-y$ *is even.*

The statement on recurrence is clear from the fact that the return time to the starting point of the induced process is bounded by the return time of the original process. We remark that [10] has general results in the same spirit, where the SDS has a finite, irreducible factor chain.

B. Non-negative Y_n

We first consider the situation when $Y_n \ge 0$ (of course excluding the trivial case $Y_n \equiv 0$), so that the increments of (X^x_n) are non-positive except possibly at the moments of reflection. In this case, Feller [38] and Knight [59] have computed an invariant measure for the process when the Y_n are non-lattice random variables, while first Kemperman [54] and later Boudiba [18], [19] have provided such a measure when the Y_n are lattice variables. Compare also with Foss and Rogozin [39] and Borovkov [17].

Lemma 4.3.5. *Suppose that* $\operatorname{supp}\mu \subset [0\,,\infty)$.

(a) *If* μ *is non-lattice then an invariant measure is given by*

$$\nu(dx) = \mu\big((x\,,\,\infty)\big)\,dx\,.$$

(b) *If* μ *is lattice, then an invariant measure is*

$$\nu(0) = \frac{1-\mu(0)}{2} \quad \textit{and} \quad \nu(x) = \frac{\mu(x)}{2} + \mu\big((x\,,\,\infty)\big)\,, \quad \textit{if } x \in \mathbb{N}\,.$$

In both cases, $\nu\big([0\,,\infty)\big) = \mathbb{E}(Y_1)$. This leads to the following well-known property.

Corollary 4.3.6. *The reflected random walk is positive recurrent on* $\mathcal{L}$ *if and only if* $\mathbb{E}(Y_1) < \infty$.

The next question is when we have null-recurrence. The following sufficient conditions are due to [75], [67] and [71] (in this order for (i)-(iii)). We want to remark that [54] also give conditions for recurrence of reflected lattice random walks.

Proposition 4.3.7. *Suppose that* $\operatorname{supp}(\mu) \subset \mathbb{R}_+$. *Then each of the following conditions implies the next one (but not vice versa) and is sufficient for recurrence of the reflected random walk on* $\mathcal{L}$.

$$\mathbb{E}\big(\sqrt{Y_1}\,\big) < \infty \tag{i}$$

$$\int_{\mathbb{R}^+} \mu\big((x\,,\,\infty)\big)^2\,dx < \infty \tag{ii}$$

$$\lim_{y\to\infty} \mu\big((y\,,\,\infty)\big) \int_0^y \mu\big((x\,,\,y]\big)\,dx = 0 \tag{iii}$$

(In the lattice case, the integrals reduce to sums and dx *is the counting measure on* $\mathbb{N}_0$ *.)*

C. Two-sided increments

We now drop the assumption that $Y_n \geq 0$. Of course, we require that μ is such that we do not have $S_n = Y_1 + \cdots + Y_n \to -\infty$ with positive probability (= probability 1 by Kolmogorov's 0-1 law), because in this case there are only finitely many reflections, and $X_n^x \to \infty$ almost surely.

Let $Y_n^+ = \max\{Y_n, 0\}$ and $Y_n^- = \max\{-Y_n, 0\}$. If *(a)* $\mathbb{E}(Y_1^-) < \mathbb{E}(Y_1^+) \leq \infty$, or if *(b)* $0 < \mathbb{E}(Y_1^-) = \mathbb{E}(Y_1^+) < \infty$, then $\limsup S_n = \infty$ almost surely, so that there are infinitely many reflections.

We now assume that $\limsup S_n = \infty$ almost surely. Then the (non-strictly) ascending *ladder epochs*

$$\boldsymbol{\ell}(0) = 0\,, \quad \boldsymbol{\ell}(k+1) = \inf\{n > \boldsymbol{\ell}(k) : S_n \geq S_{\boldsymbol{\ell}(k)}\}$$

are all almost surely finite, and the random variables $\boldsymbol{\ell}(k+1) - \boldsymbol{\ell}(k)$ are i.i.d. We can consider the *embedded random walk* $S_{\boldsymbol{\ell}(k)}$, $k \geq 0$, which tends to ∞ almost surely. Its increments $\overline{Y}_k = S_{\boldsymbol{\ell}(k)} - S_{\boldsymbol{\ell}(k-1)}$, $k \geq 1$, are i.i.d. non-negative random variables with distribution denoted $\overline{\mu}$. Furthermore, if $(\overline{X}_k^x)$ denotes the reflected random walk associated with the sequence $(\overline{Y}_k)$, while X_n^x is our original reflected random walk associated with (Y_n), then

$$\overline{X}_k^x = X_{\boldsymbol{\ell}(k)}^x\,,$$

since no reflection can occur between times $\boldsymbol{\ell}(k)$ and $\boldsymbol{\ell}(k+1)$. It is easy to see that the embedded reflected random walk $(\overline{X}_k^x)$ is recurrent if and only the original reflected random walk is recurrent. This leads to the following sufficient recurrence criteria [68].

Proposition 4.3.8. *Reflected random walk (X_n^x) is (topologically) recurrent on $\mathcal{L}$, if*

(a) $\mathbb{E}(Y_1^-) < \mathbb{E}(Y_1^+) \leq \infty$ *and* $\mathbb{E}\big(\sqrt{Y_1^+}\big) < \infty$, *or if*

(b) $0 < \mathbb{E}(Y_1^-) = \mathbb{E}(Y_1^+)$ *and* $\mathbb{E}\Big(\sqrt{Y_1^+}^{\,3}\Big) < \infty$.

In case (a), one has positive recurrence if and only if $\mathbb{E}(Y_1^+) < \infty$, and in case (b), one has null recurrence.

In the positive recurrent case of (a), we also explain how to get the invariant probability measure from the one for the embedded process. Write ν for the latter. It is computed from $\overline{\mu}$ according to Lemma 4.3.5. For any Borel set $B \subset \mathbb{R}$,

$$\nu(B) = \int_{\mathcal{L}} \mathbb{E}\left(\sum_{k=0}^{\boldsymbol{\ell}(1)-1} \mathbb{1}_B(X_k^x)\right) d\nu(x)\,, \tag{4.9}$$

and it is finite because $\boldsymbol{\ell}(1)$ has finite expectation. (Note that for $k < \boldsymbol{\ell}(1)$ we have $X_k^x = x - S_k$.) Among the observations from [10] and [68], we also recall the following.

Lemma 4.3.9. *If μ is symmetric on $\mathbb{R}$ (resp. $\mathbb{Z}$), then reflected random walk is (topologically) recurrent if and only if the random walk (S_n) is recurrent.*

In particular, if μ is symmetric and has finite first moment, then the associated reflected random walk is recurrent.

The last statement follows from the classical result that when $\mathbb{E}(|Y_1|) < \infty$ and $\mathbb{E}(Y_1) = 0$ then S_n is recurrent; see Chung and Fuchs [25].

At this point we can ask whether also in the non-symmetric case, recurrence of the ordinary random walk (S_n) always implies recurrence of the associated reflected random walk. The answer is "no", as the following example shows.

Example 4.3.10. Let the Y_n be i.i.d. with centred distribution μ supported by $\{k \in \mathbb{Z} : k \geq -1\}$, and $\overline{\mu}$ the distribution of the $\overline{Y}_k$. By Wiener-Hopf-factorisation as in [38] (see [68] in the present context),

$$\mu = \overline{\mu} + \delta_{-1} - \overline{\mu} * \delta_{-1} ,$$

because δ_{-1} is the first strictly descending ladder distribution associated with μ. Thus, we have

$$\mu(-1) = 1 - \overline{\mu}(0) \quad \text{and} \quad \mu(x) = \overline{\mu}(x) - \overline{\mu}(x+1) \quad \text{for } x \in \mathbb{N}_0 .$$

If we *start* with a probability measure $\overline{\mu}$ on $\mathbb{N}_0$ which satisfies $\overline{\mu}(x) \geq \overline{\mu}(x+1)$ for all x then we can *construct* μ in this way, whence μ has finite first moment and is centred. By the uniqueness of the Wiener-Hopf decomposition, $\overline{\mu}$ is indeed the first ascending ladder distribution of μ. Now define $\overline{\mu}(x) = c \log(x+2)/(x+2)^{3/2}$, $x \in \mathbb{N}_0$. Then the random walk (S_n) with law μ is recurrent. But by [68, Ex. 5.11], resp. its discrete variant in [67], the embedded reflected random walk is transient, and so is the reflected random walk induced by μ.

4.4 Reflection in all coordinates

In this section, we study the multidimensional case (4.1) with $r = r_1 + r_2 \geq 2$ and $s = 0$. Our state space is $\mathcal{X} = \mathbb{N}_0^{r_1} \times \mathbb{R}_+^{r_2}$. We suppose that all one-dimensional marginals of the probability measure μ satisfy (4.5). Suppose initially that $r_1 \geq 1$. For $x = (x_1, \dots, x_r) \in \mathcal{X}$, write

$$X_n^x = (X_{n,1}^{x_1}, \dots, X_{n,r}^{x_r}) ,$$

so that $(X_{n,i}^{x_i})_{n \geq 0}$ is the reflected random walk induced by μ_i. When the latter is recurrent on its unique essential class, we know from propositions 4.3.1 and 4.2.4 that $X_{n,i}^{x_i} - X_{n,i}^{y_i} \to 0$ almost surely, when $i > r_1$ and $x_i, y_i \in \mathbb{R}_+$ are arbitrary. On the other hand, when $i \leq r_1$, by Corollary 4.3.4 the same holds as long as $x_i, y_i \in \mathbb{N}_0$ have the same parity. Recall the mapping $\pi(k) = \mathbb{1}_{2\cdot\mathbb{Z}+1}(k)$ and define

$$\boldsymbol{\pi} : \mathcal{X} \to \{0,1\}^{r_1} , \quad \boldsymbol{\pi}(x_1, \dots, x_r) = \big(\pi(x_1), \dots, \pi(x_{r_1})\big).$$

Then recurrence of the marginal processes implies

$$\|X_n^x - X_n^y\| \to 0 \quad \text{almost surely, whenever} \quad \boldsymbol{\pi}(x) = \boldsymbol{\pi}(y) . \tag{4.10}$$

For an element $\boldsymbol{\varepsilon}$ of the hypercube $\mathbb{Z}_2^{r_1} = \{0,1\}^{r_1}$, let

$$\mathcal{X}_{\boldsymbol{\varepsilon}} = \{x \in \mathcal{X} : \boldsymbol{\pi}(x) = \boldsymbol{\varepsilon}\} .$$

We note that $\boldsymbol{\pi}(X_n^x) = \boldsymbol{\pi}(x + S_n)$, where again $S_n = Y_1 + \cdots + Y_n$. The process $\big(\boldsymbol{\pi}(X_n^x)\big)_{n \geq 0}$ is a random walk on the hypercube which is translation invariant with respect to addition mod 2 and automatically symmetric. It is driven by the probability measure $\boldsymbol{\pi}\mu(\boldsymbol{\varepsilon}) = \mu\big((2\cdot\mathbb{Z})^{r_1} + \boldsymbol{\varepsilon}\big)$. Since by assumption (4.3.3), each $\mathrm{supp}(\mu_i)$, $i \le r_1$, contains odd elements, $\boldsymbol{\pi}\mu$ charges elements different from $\mathbf{0} = (0,\dots,0)$. The random walk is not necessarily irreducible; the group $\mathbb{Z}_2^{r_1}$ decomposes into a subgroup Γ (consisting of $\mathbf{0}$ and the elements that can be reached from $\mathbf{0}$) and its cosets, on each of which that random walk is irreducible. This leads us to the following.

Observation 4.4.1. Let $\Gamma^{(j)}$, $j = 1,\dots,2^d$, be the cosets of Γ in $\mathbb{Z}_2^{r_1}$. Then $1 \le d < r_1$, and our state space decomposes into the classes

$$\mathcal{X}^{(j)} = \bigcup_{\boldsymbol{\varepsilon}\in\Gamma^{(j)}} \mathcal{X}_{\boldsymbol{\varepsilon}}\,,$$

so that reflected random walk started in some $x \in \mathcal{X}^{(j)}$ never exits from that class.

Thus, even though all marginal one-dimensional reflected walks are (topologically) irreducible on the respective sets $\mathcal{L}_i$ $(i = 1,\dots,r)$, the multidimensional reflected random walk may have a decomposition into non-interacting parts. We shall see an example further below; in particular, the structure of the essential class(es) is not as simple as in the one-dimensional case (4.6). Of course, in the non-lattice case $r_1 = 0$, we will not have more than one class; in that case, we set $d = 0$ and $\mathcal{X}^{(1)} = \mathcal{X}$.

Theorem 4.4.2. *Let μ be a probability measure on $\mathbb{Z}^{r_1}\times\mathbb{R}^{r_2}$ whose lattice marginals μ_i $(i = 1,\dots,r_1)$ satisfy* (4.3.3)*, while for $i > r_1$, the marginals are non-lattice.*

Suppose that for each $i \in \{1,\dots,r\}$, the one-dimensional reflected random walk induced by μ_i is positive recurrent on the respective set $\mathcal{L}_i$ according to (4.6)*.*

Then each class $\mathcal{X}^{(j)}$ of (4.4.1) *carries a unique invariant probability measure $\nu^{(j)}$ for the r-dimensional reflected random walk induced by μ. Reflected random walk started in any point of $\mathcal{X}^{(j)}$ is a.s. absorbed by $\mathcal{L}^{(j)} = \mathrm{supp}(\nu^{(j)})$, and it is positive recurrent on $\mathcal{L}^{(j)}$.*

Proof If $r_1 = 0$ then the proof simplifies, as we shall clarify at the end. So assume $r_1 \ge 1$. As in (4.7), we consider the a.s. finite stopping times

$$\begin{aligned}\boldsymbol{\tau}(0) = 0 \quad\text{and}\quad \boldsymbol{\tau}(n) &= \inf\{k > \boldsymbol{\tau}(n-1) : \boldsymbol{\pi}(X_k^x) = \boldsymbol{\pi}(x)\}\\ &= \inf\{k > \boldsymbol{\tau}(n-1) : \boldsymbol{\pi}(S_k) = \mathbf{0}\}\,,\end{aligned}\tag{4.11}$$

where again $S_k = Y_1 + \dots + Y_k \in \mathbb{Z}^{r_1}\times\mathbb{R}^{r_2}$. Once more, the increments $\boldsymbol{\tau}(n) - \boldsymbol{\tau}(n-1)$, $n \ge 1$, are i.i.d. The stationary probability distribution of $\big(\boldsymbol{\pi}(X_n^x)\big)$ on $\Gamma^{(j)}$ is uniform, whence $\mathbb{E}\big(\boldsymbol{\tau}(1)\big) = |\Gamma|$. We look at the induced process $(X^x_{\boldsymbol{\tau}(n)})_{n\ge0}$ on each set $\mathcal{X}_{\boldsymbol{\varepsilon}}$, where $\boldsymbol{\varepsilon} \in \{0,1\}^{r_1}$. As in (4.8), it is an SDS induced by the i.i.d. multidimensional contractions

$$\begin{aligned}F_n &= f_{Y_{\boldsymbol{\tau}(n)}} \circ f_{Y_{\boldsymbol{\tau}(n)-1}} \circ \dots \circ f_{Y_{\boldsymbol{\tau}(n-1)+1}}\,, \quad\text{with}\\ F_n(x_1,\dots,x_r) &= \Big(F_{n,1}(x_1),\dots,F_{n,r}(x_r)\Big)\,, \quad\text{where}\\ F_{n,i} &= f_{Y_{\boldsymbol{\tau}(n)-1,i}} \circ \dots \circ f_{Y_{\boldsymbol{\tau}(n-1)+1,i}}\,.\end{aligned}\tag{4.12}$$

Here, $Y_{k,i}$ is of course the i-th coordinate of the random vector Y_k, and as above $f_b(x_i) = |x_i - b|$ for $b, x_i \in \mathbb{R}$. Note that the random mappings F_n do not depend on the point x or the class $\mathcal{X}_{\varepsilon}$ where the process starts. By (4.10), the SDS $(X^x_{\boldsymbol{\tau}(n)})$ is strongly contractive on each $\mathcal{X}_{\varepsilon}$. We write $\mathcal{L}_{\varepsilon}$ for its attractor. Hence, each of its marginal processes is also strongly contractive; for any starting point, it is absorbed by its attractor, which is the respective projection of $\mathcal{L}_{\varepsilon}$. (Here, "absorbed" means in the lattice case that with probability 1 it belongs to the attractor from some time onwards, while in the non-lattice case, the distance to the attractor tends to 0.)

Claim. Each marginal process $(X^{x_i}_{\boldsymbol{\tau}(n),i})_{n\geq 0}$ is positive recurrent on its attractor.

In spite of being "obvious", this needs justification.

We start by considering the first marginal of (X^x_n), which is driven by the lattice distribution μ_1. We can apply the reasoning of Lemma 4.3.2 and the subsequent lines to $(X^{x_1}_{n,1})$. Define

$$\boldsymbol{\pi}' : \mathcal{X} \to \mathbb{N}_0 \times \{0,1\}^{r_1-1}\,, \quad \boldsymbol{\pi}'(x_1,\dots,x_r) = \big(x_1, \pi(x_2),\dots,\pi(x_{r_1})\big).$$

The process $\big(\boldsymbol{\pi}'(X^x_n)\big)_{n\geq 0}$ is "reflected random walk on $\mathbb{N}_0$ with internal degrees of freedom". Its transition probabilities are

$$p'\big((x_1,\boldsymbol{\varepsilon}'),(y_1,\bar{\boldsymbol{\varepsilon}}')\big) = \mathbb{P}\big[\,|x_1 - Y_{1,1}| = y_1\,,\ \big(\pi(Y_{1,2}),\dots,\pi(Y_{1,r_1})\big) = \bar{\boldsymbol{\varepsilon}}' - \boldsymbol{\varepsilon}'\,\big]\,, \qquad (4.13)$$

where of course $\bar{\boldsymbol{\varepsilon}}' - \boldsymbol{\varepsilon}'$ is taken mod 2. Observation 4.4.1 applies to $\big(\boldsymbol{\pi}'(X^x_n)\big)$ if one replaces $\mathcal{X}^{(j)}$ with

$$\boldsymbol{\pi}'(\mathcal{X}^{(j)}) = \big\{(x_1,\boldsymbol{\varepsilon}') : \big(\pi(x_1),\boldsymbol{\varepsilon}'\big) \in \Gamma^{(j)}\big\}.$$

Since the transition probabilities (4.13) are additive mod 2 in the $\boldsymbol{\varepsilon}'$-coordinates, an invariant measure with finite total mass for $\big(\boldsymbol{\pi}'(X^x_n)\big)$ is given by

$$\nu_1'(x_1,\boldsymbol{\varepsilon}') = \nu_1(x_1)\,,$$

where ν_1 is the invariant probability distribution for the first marginal process driven by μ_1. We let $\nu_1^{(j)}$ be the probability measure obtained by restricting ν_1' to $\pi'(\mathcal{X}^{(j)})$ and normalising it. We shall see that $\mathrm{supp}(\nu_1^{(j)})$ is the only essential class of $\big(\boldsymbol{\pi}'(X^x_n)\big)$ within $\pi'(\mathcal{X}^{(j)})$.

In any case, $\big(\boldsymbol{\pi}'(X^x_n)\big)$ is positive recurrent in the irreducible (whence essential) class of each point $(x_1,\boldsymbol{\varepsilon}')$ with $x_1 \in \mathrm{supp}(\nu_1)$. We have $\boldsymbol{\pi}(x) = \boldsymbol{\varepsilon} = (\varepsilon_1,\boldsymbol{\varepsilon}')$, where $\boldsymbol{\pi}'(x) = (x_1\,,\boldsymbol{\varepsilon}')$ and $\varepsilon_1 = \pi(x_1)$. The stopping times $\boldsymbol{\tau}(n)$ are the successive instants when $\big(\boldsymbol{\pi}'(X^x_n)\big)$ visits the subset $(2\cdot\mathbb{N}_0 + \varepsilon_1)\times\{\boldsymbol{\varepsilon}'\}$. Thus, if x is such that $x_1 \in \mathrm{supp}(\nu_1)$, then the return time of $\big(\boldsymbol{\pi}'(X^x_n)\big)$ to $(x_1,\boldsymbol{\varepsilon}')$ has finite expectation. At that return time, also $\big(\boldsymbol{\pi}'(X^x_{\boldsymbol{\tau}(n)})\big)$ is back at $(x_1,\boldsymbol{\varepsilon}')$, whence also the return time of $\big(\boldsymbol{\pi}'(X^x_{\boldsymbol{\tau}(n)})\big)$ has finite expectation. But the first marginal of $\big(\boldsymbol{\pi}'(X^x_{\boldsymbol{\tau}(n)})\big)$ is just the first marginal of $(X^x_{\boldsymbol{\tau}(n)})$, so that the return time of the first marginal process also has finite expectation.

This argument shows that all the lattice marginal processes $(X^{x_i}_{\boldsymbol{\tau}(n),i})$, $i = 1,\dots,r_1$, are positive recurrent on their respective attractors (as we know that they are strongly contractive, whence the respective attractor – depending on $\boldsymbol{\pi}(x)$ – is the unique essential class).

Now suppose that there are also non-lattice marginals, i.e., $r > r_1$. Then we consider the last marginal of (X_n^x), which is driven by the non-lattice distribution μ_r. We know from propositions 4.3.1 and 4.2.3 that this marginal SDS is strongly contractive with invariant probability measure ν_r. Its attractor is $\mathrm{supp}(\nu_r)$.

For any $x \in \mathcal{X}_\varepsilon$, the r^{th} marginal process $(X^{x_r}_{\boldsymbol{\tau}(n),r})$ is a strongly contractive sub-SDS of $(X^{x_r}_{n,r})$. This time we define

$$\boldsymbol{\pi}'' : \mathcal{X} \to \{0,1\}^{r_1} \times \mathbb{R}_+ \,, \quad \boldsymbol{\pi}''(x) = \big(\boldsymbol{\pi}(x), x_r\big).$$

The transition probabilities of the process $\big(\boldsymbol{\pi}''(X_n^x)\big)_{n\geq 0}$ are

$$p''\big((\boldsymbol{\varepsilon}, x_1), \{\overline{\boldsymbol{\varepsilon}}\} \times B\big) = \mathbb{P}\big[\, |x_r - Y_{1,r}| \in B \,,\ \big(\pi(Y_{1,1}), \dots, \pi(Y_{1,r_1})\big) = \overline{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon} \,\big] \,, \tag{4.14}$$

again taking $\overline{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}$ mod 2, where $B \subset \mathbb{R}_+$ is a Borel set. Again, Observation 4.4.1 applies to $\big(\boldsymbol{\pi}''(X_n^x)\big)$ if one replaces $\mathcal{X}^{(j)}$ with

$$\boldsymbol{\pi}''(\mathcal{X}^{(j)}) = \Gamma^{(j)} \times \mathbb{R}_+ \,.$$

Once more, since the transition probabilities (4.14) are additive mod 2 in the $\boldsymbol{\varepsilon}$-coordinates, an invariant measure with finite total mass for $\big(\boldsymbol{\pi}''(X_n^x)\big)$ is given by

$$\nu_r''(\{\boldsymbol{\varepsilon}\} \times B) = \nu_r(B) \,,$$

where ν_r is the invariant probability distribution for the r^{th} marginal process driven by μ_r. That marginal process is strongly contractive, and its attractor is $\mathrm{supp}(\nu_r)$.

The projected random walk $\big(\boldsymbol{\pi}(X_n^x)\big)$ is positive recurrent on each of its irreducible classes $\Gamma^{(j)}$. If $\boldsymbol{\varepsilon} \in \Gamma^{(j)}$ and $x \in \mathcal{X}_{\boldsymbol{\varepsilon}}$ then $\pi''(Xx_{\boldsymbol{\varepsilon}}) = \{\boldsymbol{\varepsilon}\} \times \mathbb{R}_+$ is a recurrent set for $\big(\boldsymbol{\pi}''(X_n^x)\big)$. It is a straightforward and well-known consequence that the restriction of ν_r'' to $\{\boldsymbol{\varepsilon}\} \times \mathbb{R}_+$ is an invariant measure for the induced process on that recurrent set; see e.g. the proof of [68, Lemma 2.6] (which at first yields excessivity of the restriction, while invariance follows from the fact that the restricted measure has finite total mass). Now, that induced process is nothing but $\big(\boldsymbol{\varepsilon}, X^x_{\boldsymbol{\tau}(n),r}\big)$. Therefore ν_r is the unique invariant probability measure of $\big(X^x_{\boldsymbol{\tau}(n),r}\big)$. Since the latter process is strongly contractive, $\mathrm{supp}(\nu_r)$ is its attractor, and the process is positive recurrent on that set.

Again, this argument applies to all non-lattice marginals of our SDS, and the claim is proved.

We know (via Proposition 4.2.3) that for every starting point $x \in \mathcal{X}$, each marginal SDS $(X^{x_i}_{\boldsymbol{\tau}(n),i})$ has a unique invariant probability measure $\nu_{i,\boldsymbol{\varepsilon}}$ on its attractor, which depends on $\boldsymbol{\varepsilon} = \boldsymbol{\pi}(x)$. By Proposition 4.2.5, where we introduced the right product (Furstenberg's Principle) there is a non-negative integer, resp. real random variable $Z_{i,\boldsymbol{\varepsilon}}$ such that for the reversed process, we have

$$\widehat{X}^{x_i}_{\boldsymbol{\tau}(n),i} = F_{1,i} \circ F_{2,i} \circ \cdots \circ F_{n,i}(x_i) \to Z_{i,\boldsymbol{\varepsilon}} \quad \text{almost surely}$$

for each $x = (x_1, \dots, x_r) \in \mathcal{X}_{\boldsymbol{\varepsilon}}$, with the $F_{k,i}$ given in (4.12) . But then we get that

$$\widehat{X}^{x}_{\boldsymbol{\tau}(n)} = F_1 \circ F_2 \circ \cdots \circ F_n(x) \to Z_{\boldsymbol{\varepsilon}} = (Z_{1,\boldsymbol{\varepsilon}} \,, \dots, Z_{r,\boldsymbol{\varepsilon}}) \quad \text{almost surely}$$

for each $x \in \mathcal{X}_{\boldsymbol{\varepsilon}}$. Since the limit random variable $Z_{\boldsymbol{\varepsilon}}$ does not depend on the starting point, its distribution $\nu_{\boldsymbol{\varepsilon}}$ is an invariant probability measure for $(X^x_{\boldsymbol{\tau}(n)})$, and $\mathcal{L}_{\boldsymbol{\varepsilon}} = \text{supp}(\nu_{\boldsymbol{\varepsilon}})$. We note that the marginals of $\nu_{\boldsymbol{\varepsilon}}$ are the above measures $\nu_{i,\boldsymbol{\varepsilon}}$. (Recall here that for $r_1 < i \le r$, we have $\nu_{i,\boldsymbol{\varepsilon}} = \nu_i$, the invariant probability measure for the reflected random walk driven by the marginal μ_i .)

Now suppose that the starting point x lies in $\mathcal{X}^{(j)}$. The projected random walk $\big(\boldsymbol{\pi}(X^x_n)\big)$ is positive recurrent on $\Gamma^{(j)}$. Therefore (X^x_n) visits each $\mathcal{X}_{\boldsymbol{\varepsilon}} \subset \mathcal{X}^{(j)}$ infinitely often with probability 1. Since the $\boldsymbol{\tau}(n)$ are the times of the successive return visits to each of those $\mathcal{X}_{\boldsymbol{\varepsilon}}$, we see that the set of accumulation points of (X^x_n) coincides almost surely with

$$\mathcal{L}^{(j)} = \bigcup_{\boldsymbol{\varepsilon} \in \Gamma^{(j)}} \mathcal{L}_{\boldsymbol{\varepsilon}} \,. \tag{4.15}$$

We choose $\boldsymbol{\varepsilon} \in \Gamma^{(j)}$ and use $\nu_{\boldsymbol{\varepsilon}}$ to construct a probability measure on $\mathcal{X}^{(j)}$ by

$$\begin{aligned}
\nu^{(j)}(B) &= \frac{1}{\mathbb{E}\big(\boldsymbol{\tau}(1)\big)} \int_{\mathcal{L}_{\boldsymbol{\varepsilon}}} \mathbb{E}\left(\sum_{n=0}^{\boldsymbol{\tau}(1)-1} \mathbb{1}_B(X^x_n) \right) d\nu_{\boldsymbol{\varepsilon}}(x) \\
&= \frac{1}{|\Gamma|} \sum_{n=0}^{\infty} \int_{\mathcal{L}_{\boldsymbol{\varepsilon}}} \mathbb{P}[X^x_n \in B\,,\ \boldsymbol{\tau}(1) \ge n+1]\, d\nu_{\boldsymbol{\varepsilon}}(x)\,,
\end{aligned}$$

where $B \subset \mathcal{X}^{(j)}$ is a Borel set. It is well known and easy to verify that this is an invariant probability measure for (X^x_n).

Suppose that ν is an arbitrary invariant probability measure for (X^x_n) on $\mathcal{X}^{(j)}$. Every point in $\mathcal{X}^{(j)} \setminus \mathcal{L}^{(j)}$, not being an accumulation point of (X^x_n), is transient (has a neighbourhood which is visited only finitely often). Thus, we must have $\text{supp}(\nu) \subset \mathcal{L}^{(j)}$. On the other hand, invariance of ν implies that $X^x_1 \in \text{supp}(\nu)$ a.s. for any $x \in \text{supp}(\nu^{(j)})$, and iterating, the entire trajectory of (X^x_n) is in $\text{supp}(\nu)$. We see that $\text{supp}(\nu) = \mathcal{L}^{(j)}$.

The projected probability measure $\boldsymbol{\pi}(\nu)$ must be invariant for the factor chain $\big(\boldsymbol{\pi}(X^x_n)\big)$ in $\Gamma^{(j)}$. Therefore $\nu(\mathcal{X}_{\boldsymbol{\varepsilon}}) = 1/|\Gamma|$ for every $\boldsymbol{\varepsilon} \in \Gamma^{(j)}$. It is again a well-known fact that the normalised restriction of ν to $\mathcal{X}_{\boldsymbol{\varepsilon}}$ must be the (as we know, unique) invariant probability measure for the induced process $(X^x_{\boldsymbol{\tau}(n)})$ on that set. Thus, $\nu = \nu^{(j)}$ is unique,

$$\nu^{(j)} = \frac{1}{|\Gamma|} \sum_{\boldsymbol{\varepsilon} \in \Gamma^{(j)}} \nu_{\boldsymbol{\varepsilon}}\,,$$

where $\nu_{\boldsymbol{\varepsilon}}$ is viewed as a measure on the whole of $\mathcal{X}^{(j)}$. This concludes the proof in the presence of lattice marginals.

In the purely non-lattice case when $r_1 = 0$, we do not need to pass to an induced subsystem which becomes contractive: the reversal argument applies directly to the original reflected random walk. Indeed, this is the case treated by [66, Th. 24].

We know that the one-dimensional marginals of each of the invariant probability measures $\nu^{(j)}$ on the different parts $\mathcal{X}^{(j)}$ of the state space are the invariant measures ν_i of the marginal processes, which are supported by the intervals $[0\,, N_i] \cap \mathbb{R}_+$, resp. $[0\,, N_i] \cap \mathbb{N}_0$. In the higher-dimensional case, the essential classes $\mathcal{X}^{(j)}$ where the reflected random walk takes place – the respective support of the $\nu^{(j)}$ - are not easily determined. We illustrate this by the following simple examples.

Example 4.4.3. We let $\mathcal{X} = \mathbb{N}_0^2$.
(a) Let $\mu = \frac{1}{2}(\delta_{(2,3)}+\delta_{(3,2)})$. Then $N_1 = N_2 = 3$ and the reflected random walk is absorbed by (a subset of) $\{0,1,2,3\}^2$. We have $\Gamma = \{0,1\}^2$, and there is only one essential class. Indeed, there are the three irreducible classes

$$\{(0,0),(2,3),(3,2)\}\,, \quad \{(3,3)\} \quad \text{and} \quad \{0,1,2,3\}^2 \setminus \{(0,0),(2,3),(3,2),(3,3)\}.$$

The latter is the essential one.
(b) Let $\mu = \frac{1}{2}(\delta_{(-1,2)} + \delta_{(2,-1)})$. Then $N_1 = N_2 = \infty$ and $\Gamma = \{0,1\}^2$. Again, there is only one essential class, and one finds that this is $\mathbb{N}_0^2 \setminus \{(0,0)\}$.
(c) Let $\mu = \frac{1}{2}(\delta_{(-1,3)} + \delta_{(3,-1)})$. Again, $N_1 = N_2 = \infty$ but $\Gamma = \{(0,0),(1,1)\}$. Reflected random walk evolves on the two separated parts

$$\mathcal{X}^{(1)} = \{(k,l) \in \mathbb{N}_0^2 : k+l \text{ is odd}\} \quad \text{and} \quad \mathcal{X}^{(2)} = \{(k,l) \in \mathbb{N}_0^2 : k+l \text{ is even}\}.$$

While the whole of $\mathcal{X}^{(1)}$ is an essential class and thus equal to $\mathcal{L}^{(1)}$, the essential class within $\mathcal{X}^{(2)}$ is $\mathcal{L}^{(2)} = \mathcal{X}^{(2)} \setminus \{(0,0)\}$.
One can also find examples as in (b) or (c) where a bigger region around the origin is not part of the attractor. □

Remark 4.4.4. (a) In view of Proposition 4.2.2, the sets $\mathcal{L}_{\boldsymbol{\varepsilon}}$ only depend on $\text{supp}(\mu)$, and thus also the set $\mathcal{L}^{(j)}$ of (4.15) does not depend on recurrence, but just on $\text{supp}(\mu)$. And as long as all marginals satisfy $\mu_i\big((0\,,\,\infty)\big) > 0$, we can modify μ to obtain another probability measure with the same support that induces a reflected random walk which is positive recurrent on each $\mathcal{X}^{(j)}$ (or, more precisely, $\mathcal{L}^{(j)}$).

(b) There is a very simple argument, communicated to us by Nina Gantert, which shows at least in the discrete case ($r_2 = 0$) that positive recurrence of each of the marginal processes implies that RRW starting from any point in $\mathbb{N}_0^r$ must be absorbed by a positive recurrent essential class. We display that argument here, for simplicity taking only $r = 2$. There must be finite sets $A_1\,, A_2 \subset \mathbb{N}_0$ such that $\nu_1(A_1)+\nu_2(A_2) > 1$, where the ν_i are the respective stationary probability measures. Then for $x \in \mathbb{N}_0^2$, by the convergence theorem,

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{P}[X_k^x \in A_1 \times A_2] \ge \frac{1}{n}\sum_{k=0}^{n-1}\Big(\mathbb{P}[X_{k,1}^{x_1} \in A_1] + \mathbb{P}[X_{k,2}^{x_2} \in A_2] - 1\Big)$$
$$\to \nu_1(A_1) + \nu_2(A_2) - 1 > 0\,, \quad \text{as } n \to \infty\,.$$

Thus, one would think that the first issue is to use purely algebraic arguments involving only $\text{supp}(\mu)$ which should lead to a description of the essential classes of RRW, showing that there is precisely one within each $\mathcal{X}^{(j)}$. However, to the authors it is by no means obvious how to achieve this without involving the local contractivity arguments used above. Indeed, already in the one-dimensional case, without use of local contractivity (which works via the algebraic Proposition 4.3.3), the corresponding reasoning is amazingly hard: quoting [19, p. 100], "d'une surprenante difficulté" – even though in dimension 1 the stationary distribution is known explicitly. □

The next question is whether one can get a more general recurrence result regarding null recurrence, that is, when some of the marginal distributions give rise to null recurrent reflected random walks; compare with propositions 4.3.7 and 4.3.8. This appears to be a hard task. We next show that in general, for recurrence one cannot have more than two marginals which are only null recurrent.

Consider μ on $\mathbb{R}^r$. We take a sequence $(\mathbf{e}_{n,i})_{n\geq 0, 1\leq i\leq r}$ of i.i.d. random variables which are equidistributed on $\{\pm 1\}$ and independent of $(Y_n)_{n\geq 1}$. For each one-dimensional marginal μ_i and the associated coordinates $Y_{n,i}$ we consider the associated process

$$W_{0,i}^{x_i} = x_i\,, \quad \text{and} \quad W_{n+1,i}^{x_i} = W_{n,i}^{x_i} + E_{n,i}^{x_i} Y_{n+1,i}\,, \quad \text{where} \quad E_{n,i}^{x_i} = \begin{cases} -1\,, & \text{if } W_{n,i}^{x_i} > 0\,, \\ \mathbf{e}_{n,i}\,, & \text{if } W_{n,i}^{x_i} = 0\,, \\ 1\,, & \text{if } W_{n,i}^{x_i} < 0\,. \end{cases}$$

Then we have

$$|W_n^x| = X_n^{|x|}\,,$$

where (recall) absolute values are taken coordinate-wise. The following is a straightforward exercise.

Lemma 4.4.5. *If μ is* fully symmetric, *that is, invariant under all coordinate reflections $x_i \mapsto -x_i$ $(i = 1, \ldots, d)$, then the r-dimensional increments*

$$\widetilde{Y}_n = E_{n-1}^x \cdot Y_n = \left(E_{n-1,1}^{x_1} Y_{n,1}\,, \ldots, E_{n-1,d}^{x_d} Y_{n,d}\right)$$

are i.i.d. μ-distributed. In particular, for any $x \in \mathcal{X}$ and Borel set $B \in \mathbb{R}_+^r$,

$$\begin{aligned} \mathbb{P}[X_n^x \in B] &= \mathbb{P}[x + S_n \in B^*]\,, \quad \textit{where} \\ B^* &= \{(\pm y_1\,, \ldots, \pm y_r) : (y_1\,, \ldots, y_r) \in B\}. \end{aligned} \tag{4.16}$$

We observe that when $\mathrm{supp}(\mu)$ is a fully symmetric set, then the induced reflected random walk is such that $\mathcal{L}^{(j)} = \mathcal{X}^{(j)}$ for the essential classes given by (4.15), resp. the respective partition (4.4.1) of the state space $\mathcal{X}$.

Corollary 4.4.6. *Suppose that μ is fully symmetric. Then reflected random walk induced by μ is transient whenever the dimension is $r \geq 3$. When $r \in \{1, 2\}$, a sufficient condition for recurrence is that μ has finite moment of order r.*

We shall deduce from Theorem 4.5.1 below that this has the following generalisation.

Corollary 4.4.7. *Let μ be a probability measure on $\mathbb{R}^{r+s}$ whose lattice marginals satisfy (4.3.3). Write $\mu_{\lfloor r\rfloor}$ for the r-dimensional marginal of μ in the first r coordinates and $\mu_{\lceil s\rceil}$ for the s-dimensional marginal of μ in the last s coordinates, where $s \in \{1, 2\}$. Suppose that the reflected random walk induced by $\mu_{\lfloor r\rfloor}$ is positive recurrent on each of its essential classes.*

If $\mu_{\lceil s\rceil}$ is fully symmetric and has finite moment of order s, then the reflected random walk induced by μ is (topologically) null recurrent on its essential classes.

The following, regarding the joint observation of independent parts, is obvious.

Lemma 4.4.8. *Suppose that the probability measure μ on $\mathbb{R}^{r_1+r_2}$ is such that all lattice marginals satisfy* (4.3.3) *and*

$$\mu = \mu_{\lfloor r_1 \rfloor} \otimes \mu_{\lceil r_2 \rceil} .$$

If RRW driven by $\mu_{\lfloor r_1 \rfloor}$ is positive recurrent and RRW driven by $\mu_{\lceil r_2 \rceil}$ is null recurrent, then RRW driven by μ is null recurrent (on the respective essential classes).

This holds in particular, when $r_2 = 1$ and one of the conditions of for null recurrence of §3 is satisfied.

The following provides a class of examples regarding null recurrence in dimension 2.

Lemma 4.4.9. *Let μ_1 and μ_2 be probability measures on $\mathbb{Z}$ which satisfy* (4.3.3)*. Suppose they have exponential moments of all orders and are centred. Then RRW on $\mathbb{N}_0^2$ induced by $\mu_1 \otimes \mu_2$ is null recurrent on its essential classes.*

Proof Under the above assumptions, it was shown by Essifi and Peigné [35] that for all $x, y \in \mathbb{N}_0$

$$\mathbb{P}[X_{n,i}^x = y] \sim C_y^{(i)}\, n^{-1/2} \quad \text{as} \quad n \to \infty,$$

where $C_y^{(i)} > 0$, for $i = 1, 2$. The statement follows.

With weaker moment conditions, one can well have two independent RRWs, each of which is null recurrent, while the resulting two-dimensional RRW is transient.

Example 4.4.10. On $\mathbb{Z}$, let (Y_n) be equidistributed on $\{\pm 1\}$, so that $S_n = Y_1 + \cdots + Y_n$ is simple random walk. Let $\big(\tau(n)\big)_{n \geq 0}$ be a sequence of random times which is independent of (Y_n) and such that $\tau(0) = 0$ and $\tau(n) - \tau(n-1)$ are i.i.d. $\mathbb{N}$-valued. The associated *subordinated random walk* is

$$S_{\tau(n)} = \widetilde{Y}_1 + \cdots + \widetilde{Y}_n\,, \quad \text{where} \quad \widetilde{Y}_k = Y_{\tau(k-1)+1} + \cdots + Y_{\tau(k)}\,.$$

Now let $0 < \alpha < 1$ and consider $\tau(n) = \tau_\alpha(n)$, where

$$\mathbb{P}[\tau_\alpha(n) - \tau_\alpha(n-1) = k] = \frac{\alpha\, \Gamma(k-\alpha)}{k!\, \Gamma(1-\alpha)} \sim \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{k^{1+\alpha}} \quad (1 \leq k \to \infty).$$

By Bendikov and Saloff-Coste [15, Thm.3.4],

$$\mathbb{P}[S_{\tau(2n)} = 0] \simeq n^{-\frac{1}{2\alpha}}\,,$$

where $\simeq$ means asymptotic equivalence of sequences. Let μ_α be the distribution of $\widetilde{Y}_1$. We see that $(S_{\tau(n)})$, the symmetric random walk on $\mathbb{Z}$ with law μ_α, as well as the associated RRW on $\mathbb{N}_0$ are recurrent if and only if $\alpha \geq 1/2$.

Now consider $\mu = \mu_\alpha \otimes \mu_\alpha$ on $\mathbb{Z}^2$. It is fully symmetric, and we get that for any $\alpha \in (0\,,\,1)$, the random walk induced by μ with reflection in none, one or both coordinates is transient.
□

4.5 Reflected plus non-reflected coordinates

We now consider the situation of (4.1) in dimension $r+s$ with $s \in \{1,2\}$, and state space as in (4.3). As before, we write $\mu_{\lfloor r \rfloor}$ and $\mu_{\lceil s \rceil}$ for the overall marginal distributions of μ in the first r and last s variables, respectively.

Theorem 4.5.1. *Suppose that $\mu_{\lfloor r \rfloor}$ satisfies the assumptions of Theorem 4.4.2, so that the associated reflected random walk (X_n^x) on $\mathbb{N}_0^{r_1} \times \mathbb{R}_+^{r_2}$ is positive recurrent.*

If $\mu_{\lceil s \rceil}$ has finite moment of order s, then the process (X_n^x, Z_n) of (4.1) is (topologically) recurrent if and only if $\mu_{\lceil s \rceil}$ is centred.

Here, we mean that when $\mathcal{L}^{(j)}$ is one of the essential classes (4.15) of (X_n^x) according to Theorem 4.4.2, then for each $x \in \mathcal{L}^{(j)}$ and $v \in \mathbb{R}^s$, the process $(X_n^x, v + Z_n)$ returns to any neighbourhood of (x,v) infinitely often with probability 1. Of course, when $\mu_{\lceil s \rceil}$ is lattice, there are infinitely many returns to (x,v) itself. Note that we may assume w.l.o.g. that $v = 0$. We also remark here that (X_n^x, Z_n) is a typical case of a *Markov random walk* or *random walk with internal degrees of freedom* with positive recurrent *driving Markov chain* (X_n^x). There is an ample literature on processes of this type, see e.g. Jacod [51], Krámli and Szász [60], Babillot [7] or Uchiyama [80] and the references in those papers.

Proof Because this is considerably simpler, we first consider the case when $\mu_{\lfloor r \rfloor}$ is purely lattice, that is, $r_2 = 0$. Let $x \in \mathcal{L}^{(j)}$, and let $\mathbf{t}(n)$ be the successive return times of (X_n^x) to x, with $\mathbf{t}(0) = 0$. They have i.i.d. increments with finite expectation by positive recurrence. Then

$$Z_{\mathbf{t}(n)} = \widetilde{V}_1 + \cdots + \widetilde{V}_n\,, \quad \text{where} \quad \widetilde{V}_k = Z_{\mathbf{t}(k)} - Z_{\mathbf{t}(k-1)}\,,$$

and the $\widetilde{V}_k$ are i.i.d. By Wald's identity,

$$\mathbb{E}(\widetilde{V}_1) = \mathbb{E}\big(\mathbf{t}(1)\big)\,\mathbb{E}(V_1)\,,$$

and if V_1 has finite second moment then so does $\widetilde{V}_1$. The result follows.

The situation is more complicated when the reflected part is not purely lattice. In this case, we start with a compact neighbourhood U of some point $x \in \mathcal{L}^{(j)}$. We know that for any $y \in U$, the chain (X_n^y) returns to U almost surely. Thus, we can consider the induced process $(X_{\mathbf{t}(n)}^y)$ on U, where $\mathbf{t}(n)$ are the times of the successive visits to U. Note that they depend on the starting point y and do not have i.i.d. increments. In any case, it is a well known fact that the normalised restriction $\nu_U = \nu_U^{(j)}$ of the invariant probability measure $\nu^{(j)}$ to U is an invariant probability for the induced process.

We shall use a method of [8, Thm. 4.1]. For any probability distribution ν supported in $\mathcal{L}^{(j)}$, we let $\mathbb{P}_\nu = \mathbb{P}_{(\nu,0)}$ be the probability on the trajectory space of $(X_n^x, v + Z_n)$, where (X_n^x) has starting distribution ν – so that we might as well use the notation (X_n^ν) – while (S_n) starts at $v = 0$. In other words,

$$\mathbb{P}_\nu = \int \mathbb{P}_{(x,0)}\, d\nu(x)\,,$$

where in general $\mathbb{P}_{(x,v)}$ refers to starting the process at the deterministic point (x,v). Since $\nu^{(j)}$ is the unique invariant probability for the original process on $\mathcal{X}^{(j)}$, resp. $\mathcal{L}^{(j)}$, also ν_U

is the unique invariant probability for the induced process. Therefore the induced process on $\mathcal{L}^{(j)}$ with initial distribution ν_U – which we denote by $(X^{\nu_U}_{\mathbf{t}(n)})$ – is not only stationary, but ergodic under $\mathbb{P}_{\nu_U}$ – see e.g. Hernández-Lerma and Lasserre [44, Prop. 2.4.3]

Claim. The sequences of random variables $(\widetilde{V}_n)$ and $\big(\mathbf{t}(n)-\mathbf{t}(n-1)\big)$ are stationary ergodic under $\mathbb{P}_{\nu_U}$.

Proof of the claim. Stationarity is straightforward, and contained in the first part of the following. Let $(\mathcal{F}_n)$ be the filtration of the σ-algebra on the trajectory space generated by (X^x_n, Z_n). Take a measurable function $\phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^+$. Then, since the distribution of $X^{\nu_U}_{\mathbf{t}(n)}$ is ν_U and the transitions of (Z_n) are translation invariant,

$$\begin{aligned}\mathbb{E}_{\nu_U}\big(\phi(\widetilde{V}_{n+1}, \widetilde{V}_{n+2}, \dots) \mid \mathcal{F}_{\mathbf{t}(n)}\big) &= \mathbb{E}_{(X^{\nu_U}_{\mathbf{t}(n)}, Z_{\mathbf{t}(n)})}\big(\phi(\widetilde{V}_1, \widetilde{V}_2, \dots)\big)\\ &= \mathbb{E}_{(X^{\nu_U}_{\mathbf{t}(n)}, 0)}\big(\phi(\widetilde{V}_1, \widetilde{V}_2, \dots)\big) = \mathbb{E}_{\nu_U}\big(\phi(\widetilde{V}_1, \widetilde{V}_2, \dots)\big).\end{aligned}$$

Now suppose in addition that $W = \phi(\widetilde{V}_1, \widetilde{V}_2, \dots)$ is measurable with respect to the invariant σ-algebra of $(\widetilde{V}_1, \widetilde{V}_2, \dots)$, so that $W = \phi(\widetilde{V}_n, \widetilde{V}_{n+1}, \dots)$ for each n. Then by martingale convergence and the above,

$$W = \lim_{n\to\infty} \mathbb{E}_{\nu_U}(W \mid \mathcal{F}_{\mathbf{t}(n)}) = \lim_{n\to\infty} \mathbb{E}_{(X^{\nu_U}_{\mathbf{t}(n)}, 0)}(W).$$

Therefore W is also an invariant function of $(X^{\nu_U}_{\mathbf{t}(n)})$, which is ergodic, so that W is $\mathbb{P}_{\nu_U}$-almost surely constant. This shows ergodicity of $(\widetilde{V}_n)$. The proof for the increments $\big(\mathbf{t}(n) - \mathbf{t}(n-1)\big)$ is analogous.

Having proved the Claim, we recall that as in the lattice case $\mathbb{E}\big(\mathbf{t}(1)\big) < \infty$, and by Wald's identity $\mathbb{E}(\widetilde{V}_1) = 0$ if and only $\mathbb{E}(V_1) = 0$.

If $s = 1$, then we see that under $\mathbb{P}_{\nu_U}$, the random walk $(Z_{\mathbf{t}(n)})$ on $\mathbb{R}$ arises from the sums of the stationary ergodic sequence of the random variables $\widetilde{V}_n$, which have finite expectation. By a theorem of [5], $(Z_{\mathbf{t}(n)})$ is recurrent (= returns infinitely often to any neighbourhood of 0 with probability 1) if and only if V_1 is centred. This proves that $(X^x_n, v + Z_n)$ is recurrent (where RRW is considered one one of its attractors $\mathcal{L}^{(j)}$) if and only if $\mathbb{E}(V_1) = 0$.

If $s = 2$, then our assumption is that $\mathbb{E}(V_1^2) < \infty$, so that Z_n satisfies the Central Limit Theorem. If $\mathbb{E}(V_1) \neq 0$ then we have of course transience. So suppose that $\mathbb{E}(V_1) = 0$. Then $Z_n/\sqrt{n}$ converges in law to a non-degenerate 2-dimensional centred normal distribution. By Birkhoff's Ergodic Theorem, $\mathbf{t}(n)/n \to \mathbb{E}\big(\mathbf{t}(1)\big)$ almost surely under $\mathbb{P}_{\nu_U}$. Then, by an old theorem of Rényi [72] (going back to Anscombe [3]), also $Z_{\mathbf{t}(n)}/\sqrt{\mathbf{t}(n)}$ is asymptotically normal with the same limit distribution. Now we can apply the theorem of [74] to deduce that $(Z_{\mathbf{t}(n)})$ is recurrent. This concludes the proof.

Proof [Proof of Corollary 4.4.7] Let (X^x_n) be RRW induced by $\mu_{\lfloor r \rfloor}$ and $(v + Z_n)$ be the ordinary random walk induced by $\mu_{\lceil s \rceil}$. By Theorem 4.5.1, the process $(X^x_n, v + Z_n)$ is recurrent on its essential classes. A straightforward adaptation of Lemma 4.4.5 yields that we also have recurrence when there is reflection in the last two coordinates.

Note that the last phrase of the proof remains true also when $s = 2$ and there only is reflection in one of the last two coordinates, while the other coordinate remains non-reflected.

This observation together with Corollary 4.4.7 and Theorem 4.5.1 clarifies that there can not be a general result on recurrence with more than two null-recurrent coordinates, be they reflecting or “free”.

We conclude with an open problem. Suppose that $r = s = 1$, so that we have reflection in the first coordinate only, and no reflection in the second one. Also suppose that the second marginal gives rise to a recurrent (ordinary) random walk (e.g., having finite first moment and being centred.) Provide general recurrence criteria, when the reflected process in the first coordinate is null-recurrent.

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Chapter 5

The multidimensional Lindley process

The following part is based on the submitted joint paper with Wojciech Cygan *On recurrence of the multidimensional Lindley process.*

5.1 Introduction

Let $(Y_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with common distribution μ and let $S_n = Y_1 + \cdots + Y_n$ be the classical random walk. A Lindley process (LP) is a discrete time stochastic process $(W_n)_{n\geq 0}$ defined recursively by

$$W_0 = w_0 \geq 0 \quad \text{and} \quad W_n = \max\{W_{n-1} - Y_n, 0\}, \quad \text{for } n \geq 1. \tag{5.1}$$

The process (W_n) is a Markov chain on the state space $[0, \infty)$ with one-step transition probabilities given by

$$p(w_0, [0, w]) = \mathbb{P}(W_1 \leq w \,|\, W_0 = w_0) = \mu\left([w_0 - w, \infty)\right), \quad \text{for } w \geq 0$$

Relation (5.1) reveals that the LP which starts at 0 obeys the same transition rules as the underlying random walk (S_n), except the times when (S_n) crosses its successive maximal levels, since at these moments (W_n) stays at 0. In other words, the return times to 0, denoted by $T_W(k)$, $k \geq 0$, for the process (W_n) started at 0 coincide with the ascending ladder epochs of the random walk (S_n). Let us recall that the (non-strict) ascending ladder epochs are defined as

$$\bar{\tau}(0) = 0, \quad \bar{\tau}(k+1) = \inf\{n > \bar{\tau}(k) : S_n \geq S_{\bar{\tau}(k)}\}, \quad \text{for } k \geq 0,$$

where $S_0 = 0$ and we use the convention that $\inf \emptyset = \infty$. It is straightforward to check that $T_W(k) = \bar{\tau}(k)$. There are also more connections like this and one of the most significant is that, given $W_0 = 0$, the random variable W_n has the same distribution as $M_n = \max\{0, S_1, \dots, S_n\}$. All the mentioned facts bear a lot of fruitful consequences and we exploit them repeatedly in our paper.

We briefly state the well-known facts about recurrence of the LP in the one-dimensional case. Recall that an essential class for a Markov chain is a subset of the state space which is irreducible and absorbing. Given $\mathbb{P}(Y_1 > 0) > 0$ there is only one essential class for (W_n) and it contains all the states that can be visited after the process reached 0. Thus to study its recurrence it suffices to concentrate on the behaviour at the origin.

We recall from Feller [38, Ch. XII, Sec. 2, Theorem 1] that there are three types of random walks:

- (S_n) is either oscillating, then $\liminf_{n\to\infty} S_n = -\infty$ and $\limsup_{n\to\infty} S_n = \infty$;
- or it has a positive drift with $\lim_{n\to\infty} S_n = \infty$;
- or it has a negative drift meaning that $\lim_{n\to\infty} S_n = -\infty$.

In the first two cases we have $\mathbb{P}(\bar{\tau}(1) < \infty) = 1$, whereas in the negative drift case $\mathbb{P}(\bar{\tau}(1) < \infty) < 1$. By the correspondence between the ladder epochs of (S_n) and the return times of (W_n), we conclude that (W_n) is recurrent if and only if $\mathbb{P}(\bar{\tau}(1) < \infty) = 1$. Therefore (W_n) is recurrent if and only if (S_n) is oscillating or if it has a positive drift and the following dichotomy holds true:

1) The process (W_n) is null recurrent if and only if (S_n) is oscillating. Then $\bar{\tau} = \bar{\tau}(1)$ has infinite first moment, cf. Gut [43, Theorem 9.1]. It happens if $\mathbb{E}Y_1 = 0$ or if μ is symmetric.

2) The process (W_n) is positive recurrent if and only if (S_n) has a positive drift. In this case $\mathbb{E}\bar{\tau}$ is finite and W_n converges weakly to the random variable $M_\infty = \sup\{S_0, S_1, \ldots\}$ which is finite a.s. This holds in particular if $\mathbb{E}|Y_1| < \infty$ and $\mathbb{E}Y_1 > 0$.

We observe that for a general distribution μ on $\mathbb{R}$ and the associated LP with an arbitrary initial random variable $W_0 \geq 0$ which is independent of (Y_n) we have equality in law $W_n = \max\{M_{n-1}, W_0 + S_n\}$, for $n \geq 1$. This in turn implies that, given $\mathbb{E}(Y_1) > 0$, $W_n \to M_\infty$ in law and thus the distribution of M_∞ is the unique stationary measure for (W_n), cf. also Diaconis and Freedman [29, Theorem 4.1].

The LP comes up naturally in the framework of single server queues and thus it was extensively studied over the past decades, see e.g. the seminal paper Kendall [55] with references therein and cf. also the books by Feller [38], Borovkov [17] and Asmussen [4]. Lindley [65] was the first who investigated the limit behaviour of (W_n) and discovered its connections with the Wiener-Hopf integral equations. More recently, asymptotics of the return probabilities of (W_n) were computed by Essifi, Peigné and Raschel [36].

The LP may be also viewed as a random walk with a certain barrier at zero and in this spirit we mention the reflected random walk (RRW), denoted by $(X_n^x)_{n\geq 0}$, which is defined analogously to (W_n) but instead of the maximum function in (5.1) one sets $X_0^x = x \geq 0$ and $X_n^x = |X_{n-1}^x - Y_n|$, for $n \geq 1$. There is an obvious and striking resemblance between the two processes and in this note we take advantage of this aspect. In particular, the question of recurrence of RRW received much attention in the literature, see Peigné and Woess [67] with references therein and Kloas and Woess [58] respective chapter 4 for a treatment of the multidimensional case.

The multidimensional counterpart of the LP arises from the studies on many server queueing models which were initiated by Kiefer and Wolfowitz [56]. In this note we aim at finding sufficient conditions for recurrence of the multidimensional LP as well as for a process of which some coordinates are LP and the other are ordinary random walks. We focus mainly on the two-dimensional lattice case but we also present a satisfactory result for higher dimensions. More precisely, the paper is organized as follows: Section 5.2 is devoted to the study of the asymptotic behaviour of a given random walk on integers which is evaluated at some random stopping times that are ladder epochs of a second independent random walk. Further, we take advantage of the result and construct a pair of examples of random walks with infinite second moment and investigate their recurrence. In section 5.3 we treat the two-dimensional LP in the lattice quadrant and investigate its recurrence under various assumptions on the tail behaviour of the underlying random walk. Among other methods, we apply the asymptotics obtained in Section 5.2. In the last paragraph we use a technique of local contractivity, which is related to stochastic dynamical systems, to study the LP in higher dimensions.

5.2 Subordination tools for random walks

Let $S_n = Y_1 + \ldots + Y_n$ be an oscillating random walk such that $S_0 = 0$. We always assume that the distribution μ of the increment Y is supported by $\mathbb{Z}$. Since (S_n) is oscillating, the first strict ascending ladder epoch $\tau = \tau(1) = \min\{n \geq 1 : S_n > 0\}$ is well-defined. Following Vatutin and Wachtel [81], for $\alpha, \beta \in \mathbb{R}$ we consider the set

$$\mathcal{A} = \{0 < \alpha < 1;\ |\beta| < 1\} \cup \{1 < \alpha < 2;\ |\beta| \leq 1\} \cup \{\alpha = 1, 2;\ \beta = 0\}. \tag{5.2}$$

For $(\alpha, \beta) \in \mathcal{A}$ we write $Y \in D(\alpha, \beta)$ if the distribution of Y belongs to the domain of attraction of the stable law with characteristic function

$$\Phi(\xi) = \exp\left\{-c|\xi|^\alpha \left(1 - i\beta \frac{\xi}{|\xi|} \tan \frac{\alpha\pi}{2}\right)\right\},$$

for $c > 0$. If $1 < \alpha \leq 2$ we assume that $\mathbb{E}(Y) = 0$. It is known Doney [32] that if $Y \in D(\alpha, \beta)$ then

$$\mathbb{P}(S_n > 0) \to \rho \in (0, 1), \quad n \to \infty, \tag{5.3}$$

where the parameter ρ is given by

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan\left(\beta \tan \frac{\pi\alpha}{2}\right). \tag{5.4}$$

Moreover, condition (5.3) is equivalent to the existence of a slowly varying (at infinity) function ℓ such that

$$\mathbb{P}(\tau > n) \sim \frac{1}{\Gamma(\rho)\Gamma(1-\rho) n^\rho \ell(n)}, \quad n \to \infty. \tag{5.5}$$

Recall that a function f is regularly varying of index γ at infinity if $\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^\gamma$, for all $\lambda > 0$, and f is called slowly varying if $\gamma = 0$. Equation (5.5) means that τ belongs to the domain of attraction of the one-sided stable law of index ρ. According to [81, Theorem 3] and [82, Theorem 10] for $\alpha = 1$ or $\beta = -1$ we also have the following local result

$$\mathbb{P}(\tau = n) \sim \frac{\rho}{\Gamma(\rho)\Gamma(1-\rho) n^{\rho+1} \ell(n)}, \quad n \to \infty, \tag{5.6}$$

with the same slowly varying function ℓ as in (5.5).

We study the local asymptotic behaviour of a random walk which is evaluated at ladder epochs of the random walk (S_n). More precisely, we consider a finite range and centered random walk (Z_n) on $\mathbb{Z}$ (i.e. the support of the law of Z_1 is bounded and $\mathbb{E}Z_1 = 0$) and we look more closely at the tail decay of the random variable Z_τ, where τ is the first strictly ascending ladder epoch of (S_n). The proof of the theorem below is based on the similar result obtained in Bendikov and Cygan [13] for the Green function of the subordinated random walk in $\mathbb{Z}^d$ but it requires numerous improvements and adjustments to the present setting. To our best knowledge, this is the first result of this type in the centred but not necessarily symmetric case.

We emphasise that the scope of the theorem is wider than it is stated. One can consider an arbitrary increasing random walk (η_n) on non-negative integers and then a new subordinated random walk (Z_{η_n}). The result is applicable given that the increments of (η_n) behave locally as in (5.6). We obtain the local behaviour of the subordinated random walk without any assumption on the structure of the distribution of η_1, cf. Bendikov, Cygan and Trojan [14] for the detailed discussion on the asymptotic behaviour of subordinated random walks under the assumption that the Laplace transform of η_1 is governed by a Bernstein function.

Theorem 5.2.1. *Suppose that (S_n) is an oscillating random walk such that its increment $Y \in D(\alpha, \beta)$. Let τ be the first strict ascending ladder epoch of (S_n). Assume that (S_n) is independent of (Z_n), then*

$$\mathbb{P}\left(Z_\tau = x\right) \sim \frac{C(\rho)}{|x|^{2\rho+1}\ell(|x|^2)}, \quad \text{as } |x| \to \infty, \tag{5.7}$$

where ℓ is the slowly varying function from (5.6) and

$$C(\rho) = \frac{\rho(2\sigma^2)^{\rho-1}\Gamma\left(\rho + \frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\rho)\Gamma(1-\rho)}, \quad \text{with } \sigma^2 = \mathrm{Var}(Z_1). \tag{5.8}$$

Proof We set $p_n(x) = \mathbb{P}(Z_n = x)$ and write

$$\mathbb{P}(Z_\tau = x) = \sum_{n=1}^{\left[|x|^{5/3}\right]} p_n(x)\mathbb{P}(\tau = n) + \sum_{n>\left[|x|^{5/3}\right]} p_n(x)\mathbb{P}(\tau = n) = I_1(x) + I_2(x).$$

Let $\overline{p}_n(x) = (\sqrt{2\pi n}\sigma)^{-1}e^{-|x|^2/(2\sigma^2 n)}$ and $E(n,x) = p_n(x) - \overline{p}_n(x)$. Applying Lawler and Limic [62, Theorem 2.1.1] (see the discussion following Proposition 2.1.2), we get that for a centered irreducible and aperiodic random walk in $\mathbb{Z}^d$ with finite third moment there is some $C > 0$ such that

$$|E(n,x)| \le Cn^{-\frac{d+1}{2}}, \quad n \ge 1. \tag{5.9}$$

We decompose $I_2(x)$ into two parts

$$I_2(x) = \sum_{n>\left[|x|^{5/3}\right]} \overline{p}_n(x)\mathbb{P}(\tau = n) + \sum_{n>\left[|x|^{5/3}\right]} E(n,x)\mathbb{P}(\tau = n) = I_{21}(x) + I_{22}(x),$$

and first we establish that $I_{22}(x) = o\left(|x|^{-2\rho-1}/\ell(|x|^2)\right)$. Our assumption followed by (5.6) and (5.9) with $d = 1$ implies that for some $C > 0$

$$I_{22}(x) \leq C \sum_{n > \left[|x|^{5/3}\right]} \frac{1}{n^{\rho+2}\ell(n)} \sim C \int_{|x|^{5/3}}^{\infty} \frac{1}{t^{\rho+2}\ell(t)} \mathrm{d}t, \quad \text{as } |x| \to \infty.$$

By Bingham, Goldie and Teugels [16, Proposition 1.5.10], we have

$$\int_{|x|^{5/3}}^{\infty} \frac{1}{t^{\rho+2}\ell(t)} \mathrm{d}t \sim \frac{1}{(\rho+1)|x|^{5(\rho+1)/3}\ell\left(|x|^{5/3}\right)}, \quad \text{as } |x| \to \infty$$

and whence, for $|x|$ large enough,

$$I_{22}(x)|x|^{2\rho+1}\ell(|x|^2) \leq C \frac{1}{(\rho+1)|x|^{(2-\rho)/3}} \frac{\ell(|x|^2)}{\ell\left(|x|^{5/3}\right)}.$$

Potter bounds [16, Theorem 1.5.6] yield $\ell(|x|^2) \leq 2|x|^{1/3}\ell\left(|x|^{5/3}\right)$, for $|x|$ large, and therefore

$$I_{22}(x)|x|^{2\rho+1}\ell(|x|^2) \leq C \frac{1}{(\rho+1)|x|^{(1-\rho)/3}} \to 0, \quad \text{as } |x| \to \infty,$$

as desired. Next, with $I_{21}(x)$ we proceed as follows. For $|x| \to \infty$,

$$I_{21}(x) \sim C_1 \sum_{n > \left[|x|^{5/3}\right]} e^{-\frac{|x|^2}{2\sigma^2 n}} \frac{1}{n^{\rho+3/2}\ell(n)} \sim C_1 \int_{|x|^{5/3}}^{\infty} e^{-\frac{|x|^2}{2\sigma^2 t}} \frac{1}{t^{\rho+3/2}\ell(t)} \mathrm{d}t,$$

where $C_1 = \rho(\sigma\sqrt{2\pi}\Gamma(\rho)\Gamma(1-\rho))^{-1}$. By a suitable change of variables we have

$$\int_{|x|^{5/3}}^{\infty} e^{-\frac{|x|^2}{2\sigma^2 t}} \frac{1}{t^{\rho+3/2}\ell(t)} \mathrm{d}t = \frac{(2\sigma^2)^{\rho+\frac{1}{2}}}{|x|^{2\rho+1}\ell(|x|^2)} \int_0^{|x|^{1/3}/(2\sigma^2)} e^{-s} s^{\rho-\frac{1}{2}} \frac{\ell(|x|^2)}{\ell\left(|x^2|/(2\sigma^2 s)\right)} \mathrm{d}s.$$

We choose an arbitrary $\varepsilon \in (0, (2\rho+1)/2)$. By Potter bounds we get that for $|x|$ big enough,

$$\ell(|x|^2) \leq 2\max\left\{(2\sigma^2 s)^{-\varepsilon}, (2\sigma^2 s)^{\varepsilon}\right\} \ell\left(|x|^2/(2\sigma^2 s)\right).$$

Hence, we are allowed to apply the dominated convergence theorem to the above integral which implies

$$I_{21}(x) \sim \frac{\rho(2\sigma^2)^{\rho-1}\Gamma\left(\rho+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(\rho)\Gamma(1-\rho)} \frac{1}{|x|^{2\rho+1}\ell(|x|^2)}, \quad \text{as } |x| \to \infty.$$

To finish the proof we are left to show that $I_1(x) = o\left(|x|^{-2\rho-1}/\ell(|x|^2)\right)$. Here we use our assumption that the random walk (Z_n) has finite range. The Gaussian upper bound of Alexopoulus [2, Theorem 1.8] yields

$$I_1(x) \leq C_2 e^{-C_3|x|^{1/3}} \sum_{n=1}^{\left[|x|^{5/3}\right]} n^{-1/2} e^{-\frac{C_3 x^2}{n}} \mathbb{P}(\tau = n) \leq C_2 e^{-C_3|x|^{1/3}}.$$

Since $x^{-\nu}\ell(|x|^2)$ tends to 0 for any $\nu > 0$, we get that

$$|x|^{2q+1}\ell(|x|^2)I_1(x) \leq C_2|x|^{2\rho+1+\nu}e^{-C_3|x|^{1/3}},$$

where the last quantity tends to 0 as $|x| \to \infty$.

Corollary 5.2.2. *Under the assumptions of Theorem 5.2.1, $Z_\tau \in D(2\rho, 0)$.*

Proof Let $F(x) = \mathbb{P}(Z_\tau \le x)$. By Theorem 5.2.1, as $x \to \infty$,

$$1 - F(x) = \sum_{k>x} P(Z_\tau = k) \sim C(\rho) \sum_{k>x} \frac{1}{k^{2\rho+1}\ell(k^2)} \sim C(\rho) \int_x^\infty \frac{1}{t^{2\rho+1}\ell(t^2)} \mathrm{d}t.$$

Hence, by [16, Prop. 1.5.10], $1 - F(x) \sim C(\rho)/(2\rho\, x^{2\rho}\ell(x^2))$ at infinity. Asymptotics (5.7) are symmetric in x and whence one easily shows that $F(-x)/(1 - F(x))$ tends to 1 as x goes to infinity. We conclude that $1 - F(x) + F(-x) \sim C(\rho)/(\rho\, x^{2\rho}\ell(x))$ at infinity. The conditions of Gnedenko and Kolmogorov [42, §35, Thm. 2] are fulfilled and we obtain that Z_τ belongs to the domain of attraction of the stable law of index 2ρ. Since $F(-x)/(1 - F(x) + F(-x))$ tends to $1/2$ as x goes to infinity, the skewness parameter β equals 0.

We next present a variety of examples of random walks on $\mathbb{Z}$ which are constructed according to the discussed procedure of the random change of time. As we proved that $Z_\tau \in D(2\rho, 0)$, we get that $\mathbb{E}(|Z_\tau|^\gamma) < \infty$, for all $\gamma < 2\rho$. First we handle the case $\rho \ne 1/2$.

Proposition 5.2.3. *If (Z_n) is symmetric then under the conditions of Theorem 5.2.1, the random walk $(Z_{\tau(n)})$ is transient if $0 < \rho < \frac{1}{2}$ and recurrent if $\frac{1}{2} < \rho < 1$.*

Proof If $\frac{1}{2} < \rho < 1$ then $\mathbb{E}(|Z_\tau|) < \infty$ and by symmetry we have $E(Z_\tau) = 0$ which yields recurrence. If $0 < \rho < \frac{1}{2}$ we set $F(x) = \mathbb{P}(Z_\tau \le x)$ and let $H(x) = 1 - F(x) + F(-x)$ be the tail function. Then by symmetry and Theorem 5.2.1, for some $C > 0$,

$$\frac{H(x)}{1 - F(x)} = \frac{2(1 - F(x)) + P(Z_\tau = x)}{1 - F(x)} \sim 2 + \frac{C(\rho) 2\rho\, x^{2\rho}\ell(x^2)}{C x^{2\rho+1}\ell(x^2)} \to 2, \quad \text{as } x \to \infty.$$

Thus $H(x) \sim 2(1 - F(x)) \sim C(\rho)/(\rho\, x^{2\rho}\ell(x^2))$ at infinity.

Let $\phi(t) = \mathbb{E}(e^{itZ_\tau})$ be the characteristic function of Z_τ. By symmetry it is a real and even function. The result by Pitman [70, Theorem 1] implies that, as $t \to 0$,

$$1 - \phi(t) \sim \frac{\pi\, H(t^{-1})}{2\Gamma(2\rho)\sin(\rho\pi)} \sim C_1(\rho) t^{2\rho}\ell(t^{-2}), \text{ with } C_1(\rho) = C(\rho)\frac{\pi}{4\rho\Gamma(2\rho)\sin(\rho\pi)}.$$

To prove transience we apply the Chung and Fuchs criterion [25], see also Spitzer [77, Ch. 2, Sec. 8, T2]. Since the random walk $(Z_{\tau(n)})$ is aperiodic (according to [77, Ch. 1, Sec. 2, Def. D2]), $\phi(\theta) = 1$ if and only if $\theta = 2k\pi$, $k \in \mathbb{Z}$, and whence it suffices to prove that $\int_0^\epsilon (1 - \phi(t))^{-1}\mathrm{d}t$ is finite for small $\epsilon > 0$ which in front of the previous formula is equivalent to the convergence of $\int_0^\epsilon (t^{2\rho}\ell(t^{-2}))^{-1}\mathrm{d}t$. For any $\nu > 0$ we have $\ell(t^{-2}) > t^{2\nu}$, for $t > 0$ small enough. Choosing ν such that $2(\rho + \nu) < 1$ the considered integral converges.

In the (critical) case $\rho = \frac{1}{2}$ we give an example of a recurrent random walk $(Z_{\tau(n)})$ with increments that have no finite first moment. First we recall an important notion of α-conjugate pairs from the theory of regular variation which we extract from Doney [31].

For a given slowly varying function ℓ set $f(x) = x^\alpha\ell(x)$, with some $\alpha > 0$. By [16, Theorem 1.5.12], there is a regularly varying function g of index $1/\alpha$ and such that $g(f(x)) \sim x$

at infinity. Since g varies regularly, $g(x) = x^{1/\alpha}\ell^*_\alpha(x)$, for some slowly varying ℓ^*_α. By definition, ℓ^*_α satisfies

$$(\ell(x))^{1/\alpha}\ell^*_\alpha(x^\alpha\ell(x)) \to 1, \text{ equivalently } (\ell^*_\alpha(x))^\alpha\ell(x^{1/\alpha}\ell^*_\alpha(x)) \to 1, \quad \text{as } x \to \infty. \tag{5.10}$$

The function ℓ^*_α is called the α-conjugate of the function ℓ. The way to remember the meaning of ℓ^*_α is that $y \sim x^\alpha\ell(x)$, when x goes to infinity, if and only if $x \sim y^{1/\alpha}\ell^*_\alpha(y)$, as y goes to infinity. One easily checks that if

$$\lim_{x\to\infty}\frac{\ell(x)}{\ell(x^\alpha\ell(x))} = C(\alpha) > 0 \quad \text{then} \quad \ell^*_\alpha \sim (C(\alpha)\ell(x))^{-\frac{1}{\alpha}}, \quad \text{as } x \to \infty. \tag{5.11}$$

This holds for many slowly varying functions, for example $\log x$, $\log\log x$ etc.

Example 5.2.4. Let (S_n) be an oscillating random walk such that its increment Y belongs to the domain of attraction of the normal distribution. It is known from Feller [38, Ch. XVII, Sec. 5, Thm. 1a] that it holds if and only if the truncated variance of Y is slowly varying, that is

$$\mathbb{E}\left(Y^2\mathbf{1}_{(|Y|\le y)}\right) \sim \frac{2}{\ell(y)}, \qquad \text{as } y \to \infty, \tag{5.12}$$

for some slowly varying function ℓ. We additionally assume that $\mathbb{E}(Y^2) = \infty$ and $\mathbb{E}(S_\tau) < \infty$. Then the result by Uchiyama [80, Thm. 1.2 and Rem. 2] implies that

$$\mathbb{P}(\tau > n) \sim \frac{1}{\sqrt{\pi}\mathbb{E}(S_\tau)n^{1/2}\ell^*(n)}, \quad \text{as } n \to \infty,$$

where $\ell^* = \ell^*_2$ is the 2-conjugate of ℓ as defined in (5.10). By [81] we obtain that

$$\mathbb{P}(\tau = n) \sim \frac{1}{2\sqrt{\pi}\mathbb{E}(S_\tau)n^{3/2}\ell^*(n)}, \quad \text{as } n \to \infty.$$

Next, if we take a symmetric random walk (Z_n) then Theorem 5.2.1 gives us

$$\mathbb{P}(Z_\tau = x) \sim \frac{1}{\sqrt{2}\pi\mathbb{E}(S_\tau)x^2\ell^*(x^2)}, \quad \text{as } x \to \infty.$$

As in the proof of Proposition 5.2.3,

$$H(x) = 1 - F(x) + F(-x) \sim \frac{1}{\sqrt{2}\pi\mathbb{E}(S_\tau)x\ell^*(x^2)}, \quad \text{as } x \to \infty,$$

and

$$1 - \phi(t) \sim \frac{\pi}{2}H(t^{-1}) \sim \frac{t}{2^{3/2}\mathbb{E}(S_\tau)\ell^*(t^{-2})}, \quad \text{as } t \to 0.$$

Thus, to study recurrence of $(Z_{\tau(n)})$ we investigate convergence of the integral $\int_0^\epsilon \frac{1}{1-\phi(t)}\mathrm{d}t$ around zero. To simplify the calculations we restrict our attention to the specific choice of the slowly varying function in (5.12) and for that reason we take $\ell(x) = \log^\eta x$, for $\eta \in \mathbb{R}$. We immediately get by (5.11) that $\ell^*(x) \sim 2^{\frac{\eta}{2}}\log^{-\frac{\eta}{2}} x$ at infinity. Finally we are left with the integral $\int_0^\epsilon t^{-1}\log^{-\frac{\eta}{2}}(t^{-2})\mathrm{d}t$ which diverges for $\eta \le 2$ (and we get recurrence), whereas for $\eta > 2$ it converges and implies transience. Moreover, one easily verifies that $\mathbb{E}(|Z_\tau|) = \infty$ for $\eta \ge -2$ and $\mathbb{E}(|Z_\tau|) < \infty$ otherwise. Thus we have the following possibilities

- for $\eta < -2$ the random walk $(Z_{\tau(n)})$ is recurrent with finite first absolute moment,
- for $-2 \leq \eta \leq 2$ the random walk $(Z_{\tau(n)})$ is recurrent and $\mathbb{E}(|Z_\tau|) = \infty$,
- for $2 < \eta$ the random walk $(Z_{\tau(n)})$ is transient.

We end this section with a result concerning Theorem 5.2.1 when the increments of the random walk (S_n) have finite second moment.

Proposition 5.2.5. *Let (S_n) be an oscillating random walk with the increment Y having finite second moment and let (Z_n) be a centred and finite range random walk on $\mathbb{Z}$ independent of (S_n). Then there is some $C > 0$ such that*

$$\lim_{y\to\infty} y^2 \mathbb{P}\left(Z_\tau = y\right) = C, \tag{5.13}$$

and in this case $\mathbb{E}(|Z_\tau|) = \infty$. Equation (5.13) holds also when Y is symmetric and has a density.

Proof The proof is similar to that of Theorem 5.2.1, but in place of formula (5.6) one uses the result by ÉPPEL [34],

$$\mathbb{P}(\tau = n) \sim cn^{-3/2}, \quad c > 0,\ n \to \infty.$$

We prove that $\mathbb{E}(|Z_\tau|) = \infty$. We set $F(x) = \mathbb{P}(Z_\tau \leq x)$ and by (5.13) we get that, for some $C_1 > 0$,

$$1 - F(n) \sim \frac{C_1}{n}, \quad n \to \infty, \tag{5.14}$$

which means that $1-F(n)$ is regularly varying at infinity of index -1. In view of symmetry this implies that Z_τ is in the domain of attraction of the Cauchy law. Therefore, if $\mathcal{C}$ is the distribution function of the Cauchy law, then there are sequences $b_n > 0$ and $a_n > 0$ such that $F^{*n}(b_n x + b_n a_n) \to \mathcal{C}(x)$, for all x as n goes to infinity. We find the asymptotic behaviour of the normalizing sequence (b_n). It is known [42] that (b_n) satisfies $1 - F(b_n) \sim \frac{C_2}{n}$ at infinity and, by (5.14), we obtain that $b_n \sim C_3 n$ at infinity. Finally, by TUCKER [78], the integral $\int |x| \mathrm{d}F(x)$ is finite if and only if $\sum_{n\geq 1} n^{-2} b_n < \infty$ and the proof is finished.

5.3 Multidimensional Lindley Process

We proceed to study recurrence of the LP in higher dimensions. We start by discussing the two-dimensional case and finally we display a result and various ideas for higher dimensions.

Two-dimensional LP

Let (W_n^i), $i = 1, 2$, be two Lindley processes as defined in (5.1) with the underlying random walks (S_n^i) with increments Y^i which have distributions μ^i supported in $\mathbb{Z}$. We consider a process (W_n^1, W_n^2) in the lattice quadrant $\mathbb{N}_0 \times \mathbb{N}_0$ and we assume that

$\mathbb{P}\left(Y^1 > 0, Y^2 > 0\right) > 0$. Then the probability to reach $(0,0)$ from an arbitrary state after finitely many steps is positive. Thus, the origin and all the states that can be reached from it build a unique essential class. Without our assumption it may happen that some states will never be reached by the process even when $\gcd(\operatorname{supp}\mu^i) = 1$, see the following example. We also emphasize that a precise description of essential classes in a general case is a very hard task.

Example 5.3.1. Set $\mu = \frac{1}{4}\left(\delta_{(-4,1)} + \delta_{(-3,2)} + \delta_{(1,-4)} + \delta_{(2,-3)}\right)$. Then $\gcd(\operatorname{supp}\mu^i) = 1$ and clearly the two coordinates of (W_n^1, W_n^2) are transient and whence also the two dimensional process is transient. In this case every point in $\mathbb{N}_0 \times \mathbb{N}_0$ will be visited at most one time a.s.

On the other hand, setting $\mu = \frac{1}{4}\left(\delta_{(-1,1)} + \delta_{(-1,2)} + \delta_{(1,-1)} + \delta_{(2,-1)}\right)$ we also have that $\gcd(\operatorname{supp}\mu^i) = 1$ with positive recurrent coordinates and the two-dimensional Lindley process (W_n^1, W_n^2) will never reach $(0,0)$ in this case. We notice however that (W_n^1, W_n^2) is positive recurrent in its essential class, cf. Theorem 5.3.7.

We begin our discussion on recurrence with a very simple but fruitful lemma.

Lemma 5.3.2. *Let $\bar{\tau}^1(n)$ be the n-th non-strict ascending ladder epoch of (S_n^1). Assume that the first coordinate process (W_n^1) and the projected process $(0, W^2_{\bar{\tau}^1(n)})$ are recurrent then the two-dimensional process (W_n^1, W_n^2) is recurrent. If W_n^1 and $W^2_{\bar{\tau}^1(n)}$ are positive recurrent then (W_n^1, W_n^2) is positive recurrent.*

Proof Let T and $\widetilde{T}$ be the first return times to the point $(0,0)$ of (W_n^1, W_n^2) and $(0, W^2_{\bar{\tau}^1(n)})$ respectively. By the assumption, $\widetilde{T}$ is a.s. finite. We claim that $T = \bar{\tau}^1(\widetilde{T})$. Indeed, we have

$$\begin{aligned} T &= \inf\{n \geq 1 : (W_n^1, W_n^2) = (0,0)\} \\ &= \inf\{\bar{\tau}^1(n) \geq 1 : (W^1_{\bar{\tau}^1(n)}, W^2_{\bar{\tau}^1(n)}) = (0,0)\} \\ &= \bar{\tau}^1\left(\inf\{n \geq 1 : (0, W^2_{\bar{\tau}^1(n)}) = (0,0)\}\right) = \bar{\tau}^1(\widetilde{T}), \end{aligned}$$

where we used the fact that (W_n^1, W_n^2) attains the value $(0,0)$ only if $n \in \{\bar{\tau}^1(k) : k \geq 0\}$. This in turn implies that T is a.s. finite and we get the first part of the result.

For the positive recurrent case, we consider a random walk $\bar{\tau}^1(n) = \xi_1 + \ldots + \xi_n$ with the independent increments $\xi_i = \bar{\tau}^1(i) - \bar{\tau}^1(i-1)$ which have the same law as $\bar{\tau}^1(1)$. We build a new filtration $\{\mathcal{F}_n\}_{n\geq 1}$ given by

$$\mathcal{F}_n = \sigma\left(\bar{\tau}^1(1), \ldots, \bar{\tau}^1(n), \left(Y_1^1, Y_1^2\right), \ldots, \left(Y^1_{\bar{\tau}^1(n)}, Y^2_{\bar{\tau}^1(n)}\right)\right)$$

and notice that, since $\{\widetilde{T} \leq n\} \in \mathcal{F}_n$, $\widetilde{T}$ is a stopping time with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$. Moreover, the increments of the random walk $(\bar{\tau}^1(n))$ have the form

$$\xi_i = \inf_{k\geq 0}\left\{Y^1_{\bar{\tau}^1(i-1)+1} + \ldots + Y^1_{\bar{\tau}^1(i-1)+k}\right\}$$

and therefore ξ_n is independent of $\mathcal{F}_{n-1}$. This allows us to apply the Wald's identity in the form $\mathbb{E}\bar{\tau}^1(\widetilde{T}) = \mathbb{E}\bar{\tau}^1(1)\mathbb{E}\widetilde{T} < \infty$ which implies positive recurrence of (W_n^1, W_n^2).

In the next proposition we apply Lemma 5.3.2 and combine it with an argument which comes from renewal theory.

Proposition 5.3.3. *Suppose that the random walks (S_n^i), $i = 1, 2$, are independent and oscillating with increments $Y^i \in D(\alpha, \beta)$ satisfying $\rho_1 + \rho_2 > 1$, where ρ_i are defined in (5.4). Then the process (W_n^1, W_n^2) is null recurrent.*

Proof Since (S_n^1) is oscillating, (W_n^1) is null recurrent. Let $\tau^1(n)$ denote the n-th strict ladder epoch of (S_n^1). We show that the Green function $G(0,0)$ of the process $(0, W^2_{\tau^1(n)})$ is infinite and whence it is a recurrent Markov chain. Evidently, this implies recurrence of the process $(0, W^2_{\bar{\tau}^1(n)})$ which in front of Lemma 5.3.2 forces the result. An independence-based argument allows us to compute

$$
\begin{aligned}
G(0,0) &= \sum_{n=0}^{\infty} \mathbb{P}(W^2_{\tau^1(n)} = 0) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(W_k^2 = 0)\mathbb{P}(\tau^1(n) = k) \\
&= \sum_{k=0}^{\infty} \mathbb{P}(W_k^2 = 0) \sum_{n=0}^{\infty} \mathbb{P}(\tau^1(n) = k) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} \mathbb{P}(\tau^2(m) = k) \sum_{n=0}^{k} \mathbb{P}(\tau^1(n) = k) \\
&= \sum_{k=0}^{\infty} u_k^1 u_k^2, \quad \text{where } u_k^i = \sum_{n=0}^{k} \mathbb{P}(\tau^i(n) = k).
\end{aligned}
$$

The sequence (u_k^1) is a renewal sequence, that is it satisfies the recursive equation

$$u_0^1 = 1, \quad u_k^1 = \sum_{n=1}^{k} \mathbb{P}(\tau^1(1) = n) u^1_{k-n}.$$

Since $\mathbb{P}(S_1^1 > 0) > 0$, we have $\gcd\{\mathrm{k} : \mathbb{P}(\tau^1(1) = \mathrm{k}) > 0\} = 1$. Moreover, our assumption implies that (5.5) holds with some slowly varying function ℓ. Therefore, applying the celebrated renewal theorem by Garsia and Lamperti [40, Theorem 1.1] we obtain that

$$\liminf_{k\to\infty} \frac{u_k^1}{k^{\rho_1 - 1}\ell(k)} = \pi^{-1}\Gamma(\rho_1)\Gamma(1-\rho_1)\sin(\rho_1\pi) = C.$$

Thus, for some $\epsilon > 0$, $k_0 > 1$, and for all $k \ge k_0$, we have

$$u_k^1 \ge (C - \epsilon) k^{\rho_1 - 1}\ell(k) \ge (C-\epsilon) k^{\rho_1 - 1 - \nu}, \quad \text{for any} \quad \nu > 0.$$

Clearly, all the same holds for the sequence (u_k^2) and whence for $C_1 > 0$ we get that $G(0,0) \ge C_1 \sum_{k > k_0} k^{-2\nu + \rho_1 + \rho_2 - 2}$. Choosing ν such that $0 < 2\nu \le \rho_1 + \rho_2 - 1$ we conclude the claim.

We also present a positive result in the case when $\rho_1 = \rho_2 = 1/2$.

Proposition 5.3.4. *If the random walks (S_n^i), $i = 1, 2$, are independent, centered and with finite second moment then (W_n^1, W_n^2) is null recurrent.*

Proof The proof is similar as that of Proposition 5.3.3 but instead of asymptotics (5.5) we use the result by Éppel [34, Theorem 1], that is

$$\mathbb{P}(\tau^1(1) = n) \sim K n^{-\frac{3}{2}}, \quad \text{for } K > 0, \text{ as } n \to \infty.$$

This allows us to show that $G(0,0)$ is infinite and we again get the result.

Our next result concerns recurrence of the two-dimensional process (W_n, Z_n), where in the first coordinate (W_n) is a LP with the underlying random walk $S_n = Y_1 + \ldots + Y_n$ and the second coordinate (Z_n) is a random walk on $\mathbb{Z}$ with increments $V_1, V_2, \ldots$.

Theorem 5.3.5. *The two-dimensional process (W_n, Z_n) is recurrent in each of the following cases.*
1. If (W_n) is positive recurrent and (Z_n) is a centered random walk.
2. If (S_n) is oscillating with the increment $Y \in D(\alpha, \beta)$ such that $1/2 < \rho < 1$, and (Z_n) is a symmetric finite range random walk independent of (S_n).
3. If (W_n) is null recurrent with $\mathbb{E}(|Y|^2) < \infty$ and independent of (Z_n) which we assume to be a symmetric random walk of finite range.

Proof Recall that the k-th return time of the first coordinate (W_n) to 0 is equal to the k-th non-strict ascending ladder epoch $\bar{\tau}(k)$ of the underlying random walk (S_n). Thus, the return times to the origin of (W_n, Z_n) are the same as for the induced random walk $(Z_{\bar{\tau}(n)})$. Moreover, the process $(Z_{\bar{\tau}(n)})$ is recurrent if the random walk $(Z_{\tau(n)})$ is recurrent, where $\tau(n)$ is the n-th strict ascending ladder epoch of (S_n).

To prove the first assertion we notice that as W_n is positive recurrent we known that $\mathbb{E}\tau < \infty$, where $\tau = \tau(1)$. We also notice that τ is a stopping time for the two-dimensional random walk (S_n, Z_n) which implies that the event $\{\tau \leq n\}$ is independent of V_{n+1} and whence we are allowed to apply the Wald's identity in the form $\mathbb{E}Z_\tau = \mathbb{E}Z_1\mathbb{E}\tau = 0$. Therefore $(Z_{\tau(n)})$ is recurrent.

The second claim is a direct consequence of Proposition 5.2.3. In the last case we have $\mathbb{E}(Y) = 0$ and, as follows by Proposition 5.2.5, the following asymptotic relation holds

$$y^2\mathbb{P}(Z_\tau = y) \to C > 0, \quad \text{as } y \to \infty.$$

Since $(Z_{\tau(n)})$ is a symmetric random walk, we conclude, for instance by Spitzer [77, Sec. 8, E2], that it is recurrent.

LP in higher dimensions

We consider the multidimensional Lindley process $(W_n^1, \ldots, W_n^d)$ in $\mathbb{N}_0 \times \cdots \times \mathbb{N}_0$. The underlying random walks (S_n^i) are governed by distributions μ^i which are supported by $\mathbb{Z}$ and such that $\mathbb{P}(Y^i > 0) > 0$.

The following theorem treats positive recurrence of the multidimensional LP. Its proof uses elements from the theory of stochastic dynamical systems and thus we briefly present necessary definitions and facts, see Peigné and Woess [68] and [69] for more detailed description where in particular the substantial PhD work of Benda [10] is outlined.

Let (X, d) be a proper metric space and denote by $\mathcal{C} = \mathcal{C}(X)$ the monoid of all continuous functions from X to X equipped with the topology of uniform convergence on compact sets. Fix a probability space $(\Omega, \mathbb{P})$ and consider a sequence $(F_n)_{n\geq 1}$ of independent and identically distributed $\mathcal{C}$-valued random functions with a common distribution $\tilde{\mu}$. The corresponding stochastic dynamical system $\omega \mapsto X_n^x(\omega)$ is given by

$$X_0^x = x, \quad X_n^x = F_n \circ \ldots \circ F_1(x), \quad n \geq 1.$$

For a LP on $\mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ we have $F_n(x) = \max\{x - Y_n, 0\}$ and these mappings are contractions so that we can restrict our attention to the set $\mathcal{L} \subset \mathcal{C}$ of all Lipschitz mappings with Lipschitz constants ≤ 1. Notice that if μ is the distribution of Y_n then $\tilde{\mu}$ is the image of μ under $y \mapsto f_y$, $f_y(x) = \max\{x - y, 0\}$. A stochastic dynamical system is called conservative if

$$\mathbb{P}\left(\liminf_{n\to\infty} d(X_n^x, x) < \infty\right) = 1, \quad \text{for every } x \in X,$$

and it is locally contractive if for every $x \in X$ and every compact set $K \subset X$,

$$\mathbb{P}\left(d(X_n^x, X_n^y)\mathbf{1}_K(X_n^x) \to 0, \text{ for all } y \in X\right) = 1.$$

For $\omega \in \Omega$ we consider the set $L^x(\omega)$ of all accumulation points of the sequence $(X_n^x(\omega))_{n\geq 0}$ in X. The following lemma allows us to show that there is only one essential class for the multidimensional LP, see [68, Lemma (2.5)].

Lemma 5.3.6. *For a conservative and locally contractive stochastic dynamical system, there exists a set $L \subset X$ such that*

$$\mathbb{P}\left(L^x(\cdot) = L, \text{ for all } x \in X\right) = 1.$$

We now prove the main theorem of this section.

Theorem 5.3.7. *Suppose that each of the Lindley processes (W_n^i) is positive recurrent. Then there exists a unique invariant probability measure of the process $(W_n^1, \ldots, W_n^d)$. The stationary measure is the distribution of a random variable U which is the limit of the backward process*

$$F_1 \circ \cdots \circ F_n(x) \xrightarrow{a.s} U, \quad \text{for every } x \in \mathbb{R}^d, \tag{5.15}$$

where $F_n(x) = \max\{x - Y_n, 0\}$. In particular, the multidimensional process $(W_n^1, \ldots, W_n^d)$ is positive recurrent in its unique essential class.

Proof We start by showing that the LP (W_n^i) is locally contractive in each coordinate. We set $f_y(x) = \max\{x - y, 0\}$ and consider random contractions $F_n^i = f_{Y_n^i}$ with law $\tilde{\mu}^i$ which is the image of μ^i under the mapping f_y. Let $\mathfrak{S}^i$ be the closed sub-semigroup of $\mathcal{L}$ generated by $\operatorname{supp}(\tilde{\mu}^i)$. Our aim is to show that there is a constant function in $\mathfrak{S}^i$ and this, in view of [68, Corollary 4.4], will force local contractivity. The claim follows by the assumption $\mathbb{P}(Y^i > 0) > 0$. Indeed, there is $y > 0$ such that for any $x \in \mathbb{R}$ there is $N_x > 1$ such that for all $n \geq N_x$ we obtain that the n-fold composition $f_y \circ \cdots \circ f_y(x) \equiv 0$, and thus the null-function lies in $\mathfrak{S}$ as desired.

Next, by positive recurrence, each (W_n^i) has a unique invariant probability measure, say ν^i. This together with local contractivity imply that for any starting point x^i we have the a.s.-convergence of the backward process $F_1^i \circ \cdots \circ F_n^i(x^i) \to U^i$, where U^i is a random variable with distribution ν^i. This goes back to LEGUESDRON [63], compare with [58, Prop. 2.6]. But this means that (5.15) holds. Applying Furstenberg's contraction principle, see [68, Prop. 1.3], we conclude that the distribution ν of the random vector $(U_1, U_2, \ldots, U_d)$ is the unique invariant probability measure for $(W_n^1, \ldots, W_n^d)$ which is equivalent to the positive recurrence.

It is left to show that there is only one essential class. Indeed, by the very definition, the local contractivity of the coordinates implies that the multidimensional process $(W_n^1, \dots, W_n^d)$ is locally contractive as well. Since we have proved it is recurrent, it must be conservative. In our case, the Lindley process lives on the grid and thus the deterministic set $L \subset \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ from Lemma 5.3.6 is such that, independently of the starting point,

$$\mathbb{P}\left((W_n^1, \dots, W_n^d) = l \text{ for infinitely many } n\right) = 1, \quad \text{for every } l \in L.$$

We clearly conclude that L is the unique essential class of $(W_n^1, \dots, W_n^d)$.

Remark 5.3.8. To prove the existence of positive recurrence states of $(W_n^1, \dots, W_n^d)$ there is a simple argument which was presented to us by Nina Gantert and mentioned already in the context of RRW in [58, Remark (4.10)]. However, this argument yields no understanding of the number of essential classes and their absorption properties. The use of local contractivity leads to an answer and additional insight, namely a.s. convergence of the backward process.

Chapter 6

Distinguishing graphs of maximum valence three

This chapter is based on the submitted paper *The distinguishing number of locally finite trees* which is joint work with Svenja Hüning, Wilfried Imrich, Hannah Schreiber and Thomas Tucker.

6.1 Introduction

The distinguishing number of a group A acting faithfully on a set Ω is the least number of colors needed to color the elements of Ω such that the only color-preserving element of A is the one that fixes all elements of Ω. If A is the automorphism group of a graph G, then the distinguishing number $D(G)$ of G is the distinguishing number of the action of A on the vertex set of G. Since its introduction by Albertson and Collins [1] more than 20 years ago, there has developed an extensive literature on this topic.

Actually Babai [6] showed already 1977 that a tree has a distinguishing coloring with two colors if all vertices have the same valence $\alpha \geq 2$, where α can be an arbitrary finite or infinite cardinal[1], but the subject lay dormant until the seminal paper of Albertson and Collins [1].

The concept also has had an independent separate history in the theory of permutation groups [24], unknown to graph theorists until recently [9].

The first motivation for this paper is a bound by Collins and Trenk [26] and, independently, Klavžar, Wong and Zhu [57]. They proved that for any finite graph G of maximum valence $\Delta(G) = d$, $D(G) \leq d+1$ with equality only if G is the complete graph K_{d+1}, the complete bipartite graph $K_{d,d}$, or the C_5. For infinite graphs the bound is the supremum of the valences, see Imrich, Klavžar and Trofimov [48]. Hence, for infinite graphs $D(G) \leq d$ if G has bounded valence d. If one wishes to improve this bound, it is reasonable to begin with $d = 3$.

The second, equally important motivation, is the Infinite Motion Conjecture of Tucker [79],

[1] If α is smaller than the first uncountable inaccessible cardinal, then there also exists a coloring with a finite number of colors that is only preserved by the identity endomorphism.

who conjectured that each connected, locally finite infinite graph is 2-distinguishable if every automorphism that is not the identity moves infinitely many vertices. The conjecture is still open, although it has been shown to be true for many classes of graphs [27, 50, 76], in particular for graphs of subexponential growth [64], and thus for all graphs of polynomial growth. For a long time it was not clear whether it holds for graphs of maximal valence 3, and whether infinite motion was really needed. This was first solved under the additional condition of vertex transitivity [49]. It turns out that all finite or infinite connected, vertex transitive graphs are 2-distinguishable unless they are one of four exceptional graphs.

Here the result is extended to a complete classification of all finite or infinite connected graphs of maximal valence 3 that are not 2-distinguishable.

We begin with a general observation about graphs of bounded valence.

For any graph G with $\Delta(G) = d$ and $D(G) = d - 1$, one can subdivide an edge with a vertex v and add an edge between v and a vertex of a disjoint copy of K_d to get a graph G' with $\Delta(G') = d$ and $D(G') = d-1$ (if G is d-valent, then G' can be as well, simply by attaching $d-2$ copies of K_d). Thus, the only cases where one might expect a classification of graphs with a given distinguishing number are $D(G) = d$.

There are infinitely many graphs with $\Delta(G) = d$ and $D(G) = d$. Let $T(n, d)$ be the tree where all vertices have valence 1 or d and every vertex of valence 1 has the same distance n from a root vertex v. Clearly, $D(T(n, d)) \geq d - 1$ and $D(T(n, d)) = d - 1$ if and only if $D(T(n + 1, d)) = d - 1$. But $T(1, d) = K_{1,d}$ so $D(T(1, d)) = d$ and hence $D(T(n, d)) = d$.

From now on we assume that the maximum valence is 3, unless otherwise stated.

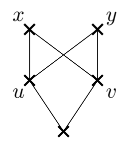
We call a vertex of valence 1 a *leaf.*

If u and v have a common neighbor, we say they are *siblings* or a *sibling pair*. Vertex v is an *only child* of a vertex u if v is the only neighbor of u with $valence(v) = 1$. For $d = 3$, we abbreviate $T_n = T(n, d)$.

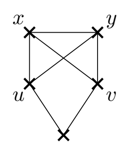
We give some variations of the trees T_n which also have distinguishing number 3. The most obvious one is simply to join each sibling pair of leaves by an edge. Denote this graph S_n. We do this only for $n > 1$ since T_1 has three sibling pairs and adding all such edges gives K_4. Note that we can also think of S_n as obtained from T_n by attaching a triangle to each leaf of T_{n-1}.

The other three variation are obtained by replacing the edges between sibling pairs in S_n by three other "gadgets". In each case, the sibling pair vertices are labeled u, v.

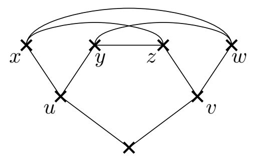
- *Gadget of type 1*: A 4-cycle $uxvy$ with x, y valence 2.



- *Gadget of type 2*: The same as the type 1 gadget but with an edge xy.



- *Gadget of type 3*: A hexagon $uxzvyw$ with edges xy, zw (this can be viewed as $K_{2,2}$ with u joined to one part and v to the other part).



Now, we define three graphs, R_n^1, R_n^2 and R_n^3, by adding the respective gadgets between each sibling pair of leaves in the tree T_n. See Figure 6.1 for examples.

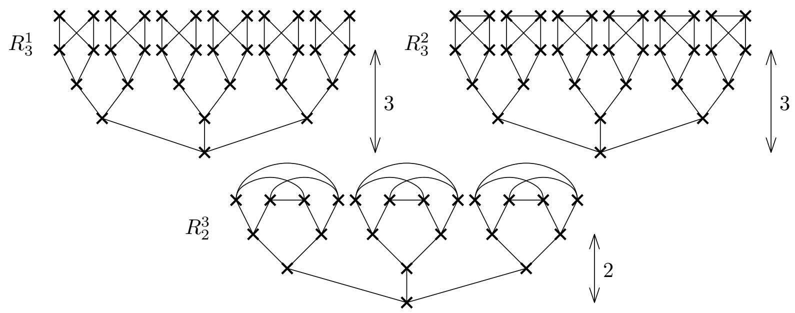


Figure 6.1: R_3^1 (top left), R_3^2 (top right) and R_2^3 (bottom).

Since $\mathrm{Aut}(S_n)$ acts on its vertices the same way as $\mathrm{Aut}(T_n)$, we have $D(S_n) = 3$. If there existed a distinguishing 2-coloring for one of the graphs R_n^1, R_n^2 or R_n^3, then this would induce a distinguishing 2-coloring of the associated T_n. Therefore $D(G) = 3$ for $G = R_n^1, R_n^2, R_n^3$.

Our classification of graphs with $\Delta(G) = 3$ and $D(G) = 3$ is the following:

Theorem 6.1.1 (Main Theorem)**.** *Let G be a finite or infinite connected graph with $\Delta(G) = 3$. Then $D(G) = 3$ if and only if G is either $K_{1,3}$, $K_{2,3}$, the cube Q, the Petersen graph P, or a member of one of the five families T_n, S_n, R_n^1, R_n^2, R_n^3 for $n > 1$.*

We note for the four exceptions, clearly $D(K_{1,3}) = D(K_{2,3}) = 3$, and it is an exercise to verify that $D(Q) \neq 2$ (or see [79]). It is slightly more work to show $D(P) \neq 2$; we will sketch a proof in Section 2.

The proof that these are the only graphs G with maximum valence $d = 3$ and $D(G) = 3$ occupies most of the rest of this paper. In Section 2, we first give some Corollaries that may shed some light on the general problem when $d > 3$. In Section 3 we introduce a 2-coloring which either is distinguishing or leads to restrictions on the local structure of G. This coloring is then used throughout the rest of the paper. In Section 4, we show that if G has any leaves, then $D(G) = 2$ unless $G = T_n$. The goal of Section 5 is to reduce to the case of edge transitive graphs by analyzing how the stabilizer of a vertex v acts on the neighbors of v. In Section 6, we show that any cubic graph G with girth at least 6 has $D(G) = 2$; this proof does not use edge transitivity. This completes the proof of the Main Theorem, since the five edge transitive cubic graphs of girth less than 6 are easily analyzed. In Section 7 we pose a variety of questions.

6.2 Corollaries

We give some corollaries of the Main Theorem, mostly just observations about our list of graphs with $D(G) = 3$. Each gives some insight into the relationship between distinguishing number and graph structure. Each suggests ways one might generalize the case of maximum valence 3 to graphs of higher valence.

Corollary 6.2.1. *Every infinite connected graph of maximal valence 3 is 2-distinguishable.*

Corollary 6.2.2. *Every connected, vertex transitive graph of maximal valence 3 is 2-distinguishable, except for $K_{3,3}$, K_4, Q and P.*

Corollary 6.2.3. *Every connected, edge transitive graph of maximal valence 3 that is not vertex transitive is 2-distinguishable, except for $K_{1,3}$ and $K_{2,3}$.*

For more direct proofs of Corollary 6.2.2 and 6.2.3 see [49].

Corollary 6.2.4. *Every 2-connected graph of maximal valence 3 is 2-distinguishable, except for $K_{2,3}$, $K_{3,3}$, K_4, Q, and P.*

The length of the shortest cycle in G is its *girth*. The following result is in fact one of the steps in the proof of the Main Theorem.

Corollary 6.2.5. *If G has girth at least 6, then $D(G) = 2$.*

The *motion* of a group A acting on a set Ω is the smallest integer m such that some element of A moves exactly m points. The motion of a graph G, which we denote $m(G)$, is the motion of $\mathrm{Aut}(G)$ acting on the vertex set. The Motion Lemma [27, 73] states that if $m > 2\log_2(|A|)$, then the action has distinguishing number 2; the proof is elementary and short. Thus large enough motion gives 2-distinguishability. For graphs of maximum valence 3, large enough means 3 or more, except for Q and P, since it is easily checked that all other G in our Main Theorem have motion 2.

Corollary 6.2.6. *If $m(G) > 2$, then $D(G) = 2$ with the exception of Q and P.*

In fact, when $D(G) = 3$ and G is not Q or P, we can isolate an automorphism of motion 2 using a 2-coloring of G. We say a coloring *fixes* a set of vertices if any color-preserving automorphism fixes all vertices in that set.

Corollary 6.2.7. *If $D(G) = 3$ and G is not Q or P, then G admits a 2-coloring that fixes all vertices except two siblings.*

Proof Clearly, such a 2-coloring exists for $K_{1,3}$ and $K_{2,3}$. All other graphs that satisfy the assumptions of the lemma have a root vertex, say v_0, corresponding to the root of T_n. The T_n are the only graphs in the class with leaves and they all come in sibling pairs. We first construct the desired coloring for T_n.

We begin with a 2-coloring of T_1: we color v_0 black, two of its neighbors white and one black. Clearly this coloring fixes all vertices except for one pair of interchangeable siblings. To color T_n we color its subgraph T_1 as before, and continue inductively by assigning different colors to any two vertices of distance $k > 1$ from v_0 if they have a common neighbor of distance $k-1$ from v_0. When $k=n$ we make an exception for a single sibling pair of vertices whose shortest paths to v_0 contain a white neighbor of v_0. Both vertices in that pair are colored white. It is easy to see that this coloring fixes all vertices not in this pair.

For S_n, R_n^1, R_n^2 we proceed analogously, and let the vertices x, y in the gadgets play the role of the sibling pairs in T_n. For R_n^3 we assign different colors to all gadget vertices z, w, but treat the pairs x, y as before. Again, our coloring fixes all vertices except the ones in the white (x, y)-pair. □

It is easily verified that $m(Q) = 4$.

Proposition 6.2.8. *For the Petersen graph, $m(P) = 6$ and $D(P) = 3$.*

Proof We follow a remark of Lehner[2]. He observed that P is the complement of the line graph of K_5. Thus any edge coloring of K_5 is a vertex coloring of the complement $\overline{P}$ of P, and thus also of P. Because every 2-edge coloring of K_5 corresponds to a subgraph of K_5 and its complement, every subgraphs of K_5, together with its complement, yields correspond a 2-colorings of P. Furthermore, given a subgraph, say H, of K_5 the group induced by $\mathrm{Aut}(H)$ on $E(H)$ is the same as the group that preserves the coloring of the vertices of P induced by H.

As the smallest graph with trivial automorphism group has a least six vertices, every subgraph H of K_5 has a non-trivial automorphism a. We consider the cycle decomposition of a. If it has a five-cycle or a four-cycle, then $\mathrm{Aut}(H)$ moves all ten pairs of distinct vertices of K_5. If it is a three-cycle it moves at least nine pairs. If a has only 2-cycles or fixed points it moves at least six pairs (but can fix four).

For a 3-distinguishing coloring of P we let H be a path of length 4 in K_5 and choose an end-edge e of H. We color e red, the other edges of H black and the edges of the complement of H white. This yields a 3-coloring of P. As the group of H that preserves its edge-coloring is trivial, this is a distinguishing 3-coloring of P. □

6.3 Canonical 2-colorings rooted at a subgraph

Let G be a cubic or subcubic graph and K be a vertex-induced, connected subgraph with at least one *internal* vertex, that is, a vertex all of whose neighbors are in K. Define $S_n(K)$ as the set of vertices of distance n from K; one might call it the sphere of radius n about K. Thus $S_0(K) = K$ and $S_1(K)$ is the set of vertices not in K but adjacent to some vertex in K. Let $B_n(K)$ denote all vertices of distance at most n from K (the ball of radius n about K). For a vertex v in $S_n(K)$, we call its neighbors in $S_{n+1}(K)$, $S_n(K)$, $S_{n-1}(K)$, respectively, its *up, cross, down* neighbors. Notice that all vertices which are not in K have at least one down neighbor and that not every vertex has to have an up neighbor.

The idea of constructing a 2-distinguishing coloring of the vertices of G is to color all vertices of K black and then to extend the coloring inductively from one $S_n(K)$ to the

[2]Private communication.

next. Our objective is to obtain a 2-coloring of G such that the only color-preserving automorphism fixing the vertices of K is the identity. Thus at stage n we have this:

Goal *Assume $B_n(K)$, $n > 0$, has been 2-colored so that any automorphism of G fixing the vertices of K and preserving the coloring of $B_n(K)$ is the identity on $B_n(K)$. Then extend this to a 2-coloring of $S_{n+1}(K)$ that has the same property on $B_{n+1}(K)$.*

The plan for extending the coloring to $S_{n+1}(K)$ is simple enough: if a vertex v in $S_n(K)$ has a single up neighbor, color it white, and if it has two up neighbors that can be switched by an automorphism of G that fixes $S_n(K)$, color one white and one black. The problem is that the up vertices of v may have already been colored when we colored the up neighbors of a different vertex. In the following three paragraphs we will make this procedure more precise.

Assume we have colored the graph up to the sphere $S(n, v)$. Let V be the set of vertices of $S(n, v)$ and U be the set of vertices of $S(n + 1, v)$. Moreover, let H be the subgraph of G determined by all edges between vertices of V and U. In fact, these are the up-edges from V. Note that for the moment we do not care about possible cross-edges in U or V. We are interested in coloring the vertices of U such that any automorphism of G which fixes the vertices of V and preserves the coloring of U also fixes the vertices of U.

Suppose that U has a vertex x of valence 3 in H. If there is another vertex $y \in U$ adjacent to the same vertices in V as x, color x black and y white. If there is no such vertex y, color x white.

Now, consider the subgraph H' of H obtained by removing all valence 3 vertices of U in H, i.e. we remove all colored vertices of U. The remaining subgraph H' contains only vertices which are of valence 1 or 2, so it is a union of paths and/or cycles such that the vertices in each component alternate between U and V. See Figure 6.2 for some examples. By assumption the vertices of V are fixed. Therefore, any component of H' is fixed except for two configurations. One is a 2-path, which consists of three vertices such that the middle vertex is in V, and the other one is quadrilateral, see Figure 6.2. For all other possible components of H' there is no automorphism of G fixing V but acting non trivial on the considered component of U. Thus, we color all vertices which are neither in a 2-path nor in a quadrilateral white. For the remaining pairs $x, y \in V$ in a 2-path or quadrilateral, choose a coloring using black the fewest times such that any automorphism of G fixing U also fixes these remaining pairs. Note that we know there are such colorings because we could simply color each such pair black-white.

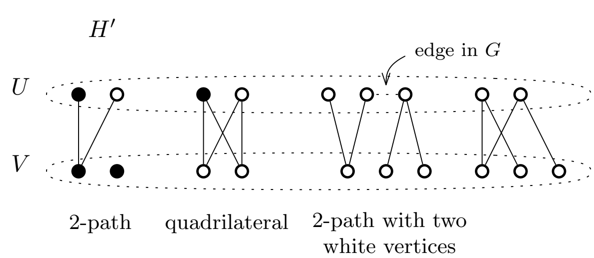


Figure 6.2: Possible configurations in H' used in the description of the canonical coloring.

We call this a *canonical 2-coloring* of G *rooted at* K.

In what follows, we use subscripts to denote which sphere a vertex is in: so $u_n, v_n, x_n \dots$

are vertices in $S_n(K)$. We also extend the notion of siblings as we have defined it in Section 6.1 by calling two (distinct) vertices u_n, v_n *siblings* if they have a common down neighbor.

We make the following observations about the resulting coloring:

Proposition 6.3.1 (White Up). *If v_n has an up neighbor, it has a white up neighbor.*

Proposition 6.3.2 (Black-white Siblings). *If v_n, $n > 0$, is black, it has a white sibling u_n and there is an automorphism of G interchanging v_n and u_n, but fixing all other vertices of $B_n(K)$.*

Proof Suppose there was no such automorphism. Then we could color v_n white, contradicting the minimality in the use of black. □

Proposition 6.3.3 (Black Cross). *Suppose $n > 0$. If u_n and v_n are both black and adjacent, then there is a quadrilateral $u_n v_n x_n y_n$, where x_n is a white sibling of u_n and y_n a white sibling of v_n.*

Proof Since v_n is black, it has a white sibling y_n with an automorphism interchanging v_n and y_n but fixing u_n, forcing an edge $u_n y_n$. Similarly, since u_n is also black there is a white vertex x_n and edge $v_n x_n$. Since the interchange of v_n and y_n also leaves x_n fixed, which is adjacent to v_n, we have y_n adjacent to x_n. □

Proposition 6.3.4 (All Black). *There is no vertex v_n for $n > 1$ of valence 2 or 3 that is black with all neighbors black.*

Proof By Propositions White Up (6.3.1) and Black Cross (6.3.3), all neighbors of v_n are down neighbors. Since v_n is black, it has a white sibling u_n (with the same valence) and an automorphism interchanging u_n and v_n fixing $S_{n-1}(K)$. Since all the neighbors of v_n are down neighbors, u_n has the same neighbors as v_n. Let x_{n-1} be one of the common neighbors. Since it is black, there is an automorphism ϕ interchanging x_{n-1} with the white vertex z_{n-1} and leaving fixed all other vertices of $S_{n-1}(K)$. Since the other down neighbors of u_n, v_n are all black, ϕ fixes all of them. Since each of these other possible down neighbors has u_n, v_n as up neighbors, ϕ either fixes u_n, v_n or interchanges them. This forces another edge from either u_n or v_n to the white vertex z_{n-1}, a contradiction, see Figure 6.3.

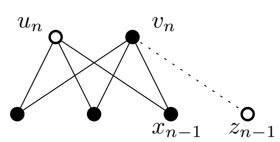


Figure 6.3: The vertex v_n has already three neighbors. Therefore there can not be an edge between v_n and z_{n-1}.

□

Call a vertex of a canonical coloring *kiwi* if it is black surrounded by black. Proposition All Black (6.3.4) says the only kiwi vertices are in $K \cup S_1(K)$.

Proposition 6.3.5 (Internal)**.** *The only non-identity color-preserving automorphisms of a canonical coloring rooted at the subgraph K are those taking an internal vertex of K to either an internal vertex of K or a kiwi vertex of $S_1(K)$.*

Note that the neighbors of a kiwi vertex in $S_1(K)$ must be in K by Propositions White Up (6.3.1) and Black Cross (6.3.3).

6.4 Leaves

We first show that the only graphs G having a leaf with $D(G) = 3$ are the trees T_n.

Theorem 6.4.1. *If G has a leaf, then $D(G) = 2$ or $G = T_n$ for some n.*

Proof The smallest two subcubic graphs with a leaf are the T_1 and a triangle where one of the three vertices has a further neighbor. For these two graphs the theorem holds. Consider now a graph G with more than 4 vertices. Suppose some vertex v in G has valence 1 and is an only child of u. The canonical coloring where K is the edge uv breaks all automorphisms in $\mathrm{Aut}_v(G)$ since any automorphism fixing v fixes u as well. Since all only children other than v are colored white in a canonical coloring, there is no automorphism moving v, so the canonical coloring is distinguishing.

Now suppose that all leaves of G come in sibling pairs and assume first that G is finite. Prune all such sibling pairs. Since $\Delta(G) = 3$, there is at least one leaf in the new graph. We have two cases. The new graph has again sibling pairs. Then prune again all sibling pairs and continue like this inductively until you get T_1 or a graph with an only child. If we ended with T_1 we know that $G = T_n$ for some n. Else we get at some point a graph G' with an only child. Give G' a distinguishing 2-coloring and color all sibling pairs, one black and one white. Any automorphism ϕ of G takes G' to G', so if ϕ is color-preserving, $\phi|_{G'}$ is the identity. But then ϕ is the identity on G since all successive removed sibling pairs are colored black-white.

Now assume that G is infinite. As before, if G has an only child leaf, then $D(G) = 2$. Suppose instead all leaves come in sibling pairs. We want to prune all such pairs to form a graph G'. First assume that G has a cycle C. Now we can induct on the distance s from C to the closest leaf. If $s = 1$, the closest leaf is an only child, since the parent has valence 2 on the cycle C, so $D(G) = 2$. Assume $D(G) = 2$ for all infinite graphs with a leaf of distance $s = n$ from a cycle. Then for $s = n + 1$, either G has an only child leaf, or pruning all sibling pairs gives a graph G' with $s = n$. In either case, $D(G) = 2$.

The same argument applies if we replace cycles by 2-way infinite paths. Thus there remain only trees without 2-way infinite paths, but 1-way infinite paths. In this case there must be a maximal infinite path whose origin is an only child leaf. To see this, let R be a 1-way infinite path. The set of all 1-way infinite paths that contain R is partially ordered by inclusion and every totally ordered subset has a maximal element, namely its union. By Zorn's Lemma there must be a maximal element. It cannot be a 2-way infinite path. Hence the maximal element is a 1-way infinite path. Because it is maximal its origin must be an only child leaf. □

Lemma 6.4.2. *If G has adjacent sibling vertices of valence 2, then $D(G) = 2$ or $G = S_n$. If G has a gadget of type 1, 2, or 3, respectively, then $D(G) = 2$ or $G = R_n^1, R_n^2, R_n^3$, respectively.*

Proof Suppose that G has adjacent siblings of valence 2. Thus we know that G cannot be T_n. If G has a vertex of valence 1, then $D(G) = 2$ by Theorem 6.4.1. Let G' be the graph obtained by removing all edges between sibling pairs of valence 2. We note that $\mathrm{Aut}(G)$ is a subgroup of $\mathrm{Aut}(G')$. Thus if $D(G') = 2$, we have $D(G) = 2$. Suppose instead that $D(G') = 3$.

Since G' has vertices of valence 1, $G' = T_n$ by Theorem 6.4.1. Since G has no vertex of valence 1, every vertex of valence 1 in G' comes from the removal of an edge between a sibling pair in G, making $G = S_n$.

The proof for gadgets of type 1, 2, 3 is the same, where G' is obtained by removing all gadgets of one type, creating vertices of valence 1. Any distinguishing 2-coloring of G' extends to one of G by coloring u_n, v_n and x_n, y_n black white, and for gadgets 3, z_n, w_n black-white. Otherwise, $G = R_n^1, R_n^2, R_n^3$. □

6.5 Vertex types

The general plan is to understand distinguishability of cubic or subcubic graphs by looking at the way $\mathrm{Aut}_v(G)$ acts on the edges incident to a vertex v of valence 3. If that action is trivial, call v *type 1*. If the action leaves one edge fixed but interchanges the other two edges, call it *type 2*. If it fixes no edge, but is not S_3, call it *type 3* and *type 6* otherwise. We note that $\mathrm{Aut}_v(G)$ defines a permutation group A on the neighbors of v and the type is the same as the order of A. See Figure 6.4 for some examples.

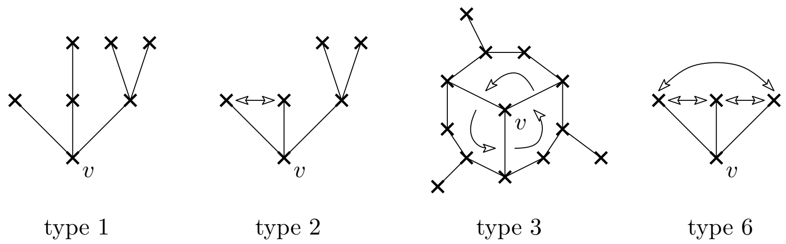


Figure 6.4: Examples of the different types of vertices.

We can define similarly type 1 and type 2 for vertices of valence 2. All vertices of valence 1 are type 1.

Observation 1 If every valence 3 vertex of G is type 3 or 6 and every valence 2 vertex is type 2, then G is edge transitive.

Observation 2 In each of the five families $T_n, S_n, R_n^1, R_n^2, R_n^3$ with $D(G) = 3$, the root vertex is type 6 and all other vertices of valence 3 are type 2. For S_n, the valence 2 vertices are type 1. For R_n^1, the valence 2 vertices are type 2.

6.5.1 Vertices of type 1

Theorem 6.5.1. *Suppose that G has a valence 2 vertex v of type 1. Then either $D(G) = 2$ or $G = S_n$ for some n.*

Proof Take a canonical coloring with K the graph spanned by v and its two neighbors u and w. By Proposition Internal (6.3.5), any non-identity, color-preserving automorphism must move v to another vertex x, which is either internal to K or a kiwi vertex of $S_1(K)$.

If x is internal to K, then K must be a triangle with two vertices of valence 2. Let G' be obtained by removing all such triangles. If $D(G') = 2$, then we can extend any distinguishing coloring of G to G' by coloring the two valence 2 vertices in each such triangle, one white and one black. Therefore $D(G') = 3$, forcing $G' = T_n$, so G is S_n.

Suppose instead that x is a kiwi vertex in $S_1(K)$. Then by Proposition Internal (6.3.5), the neighbors of x are u and w. But x would only be black if there is an automorphism fixing u, w and interchanging x with some other z. Then z has valence 2 as well and is adjacent to u, w, forcing $G = K_{2,3}$. But then v is not a type 1 vertex. □

In every figure accompanying the definition of a gadget there are vertices u and v. If u and v are not siblings, call the corresponding gadget a *non-sibling gadget.* See Figure 6.5 for an example.

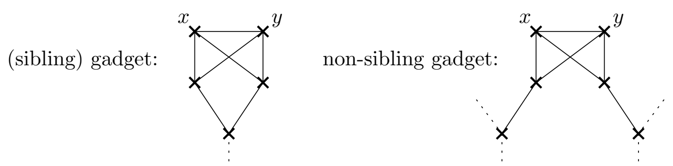


Figure 6.5: Example of a (sibling) gadget and a non-sibling gadget.

Corollary 6.5.2. *If G contains a non-sibling gadget, then $D(G) = 2$.*

Proof Suppose first that u, v are adjacent valence 2 vertices that are not siblings. The last vertex in the path of valence 2 vertices containing u, v (in either direction) is a valence 2 vertex of type 1. Since S_n does not contain a pair of adjacent vertices of valence 2 that are not siblings, $G \neq S_n$. Thus $D(G) = 2$ by Theorem 6.5.1.

For each of the gadgets, replace all appearances of a non-sibling gadget by an edge to create a graph G', with adjacent vertices of valence 2 that are not siblings. Then, by the above, $D(G') = 2$ and one easily extends a distinguishing 2-coloring from G' to G, coloring x, y black-white for R_n^1, R_n^2 and also z, w black-white for R_n^3. □

Theorem 6.5.3. *If G has a type 1 vertex v of valence 3, then $D(G) = 2$.*

Proof Choose a canonical 2-coloring with K spanned by v and its three neighbors. This breaks all automorphisms in $\mathrm{Aut}_v(G)$. Thus the only color-preserving automorphisms left must move v to a kiwi vertex u. By Proposition All Black (6.3.4), u is internal to $K \cup S_1(K)$.

Suppose that $u \neq v$ is internal to K. This forces K to be a sibling or non-sibling type 2 gadget, making $D(G) = 2$ or $G = R_n^2$ by Corollary 6.5.2 and Lemma 6.4.2. Suppose instead that $n = 1$ and x_1 is kiwi, which forces its down neighbors to be u_0, w_0, z_0. But x_1 would only be black if there was an automorphism fixing K and interchanging x_1 with another y_1. This forces $G = K_{3,3}$, contradicting that v has type 1. □

6.5.2 Type 2 vertices of valence 2

Theorem 6.5.4. *If G has a valence 2 vertex of type 2 but none of type 1, then $G = K_{2,3}, R_n^1$ or $D(G) = 2$.*

Proof Let G' be the cubic graph obtained by smoothing over all valence 2 vertices. Thus G is obtained from G' by inserting valence 2 vertices in some edges. Note that we cannot have more than one such vertex in any edge of G', since otherwise along this edge there will be a type 1 vertex of valence 2 in G.

A multiple edge in G', comes from a gadget of type 1 or a non-sibling gadget of type 1 or from $G = K_{2,3}$. Thus by Corollary 6.5.2 either $D(G) = 2$ or $G = R_n^1$. We therefore assume that G' has no multiple edges. If $D(G') = 2$, then $D(G) = 2$. Otherwise, either $G = R_n^2$ or $G = R_n^3$, since G' is cubic. Since all edges of G' except those in the gadgets have no automorphism interchanging the endpoints, the inserted vertices being type 2 must be in the gadget edges.

Color one gadget vertex black and the rest of the valence 2 vertices white. In effect, this fixes one leaf w of T_n. Now canonically color T_n rooted at the center v so that the neighbor of v in the branch containing w is colored white and the other two neighbors are colored black-white. This fixes the neighbors of v and hence breaks all automorphisms. We conclude that $D(G') = 2$ and so $D(G) = 2$. □

At this point, our classification is complete for subcubic graphs.

6.5.3 Type 2 vertices of valence 3

Lemma 6.5.5. *If the cubic graph G contains $K_{2,3}$ as a subgraph, then $G = K_{3,3}, R_n^3$ or $D(G) = 2$.*

Proof We note that the subgraphs of the gadget of type 3 spanned by $\{x, w, u, y, z\}$ and $\{y, z, v, x, w\}$ are both isomorphic to $K_{2,3}$. We claim this the only way two copies of $K_{2,3}$ in G can overlap. We view their union H as obtained from two copies of $K_{2,3}$ with vertices identified in pairs. We consider three different cases.

- *Case 1:* Identifying a pair of valence 3 vertices also identifies in pairs their 3 neighbors (so the resulting vertex has valence 3); this yields $K_{3,3}$. See Figure 6.6 for illustration.

- *Case 2:* Identifying two vertices of valence 2 forces the identification of a pair of their neighbors of valence 3, which was just considered (see Case 1).

- *Case 3:* Any identification of a valence 2 with a valence 3 vertex forces the identification of two other neighboring pairs of neighboring vertices, which in turn forces further identification. Thus H has at most 6 vertices and must be obtained by adding a single vertex of valence 2 to $K_{2,3}$, yielding gadget 3.

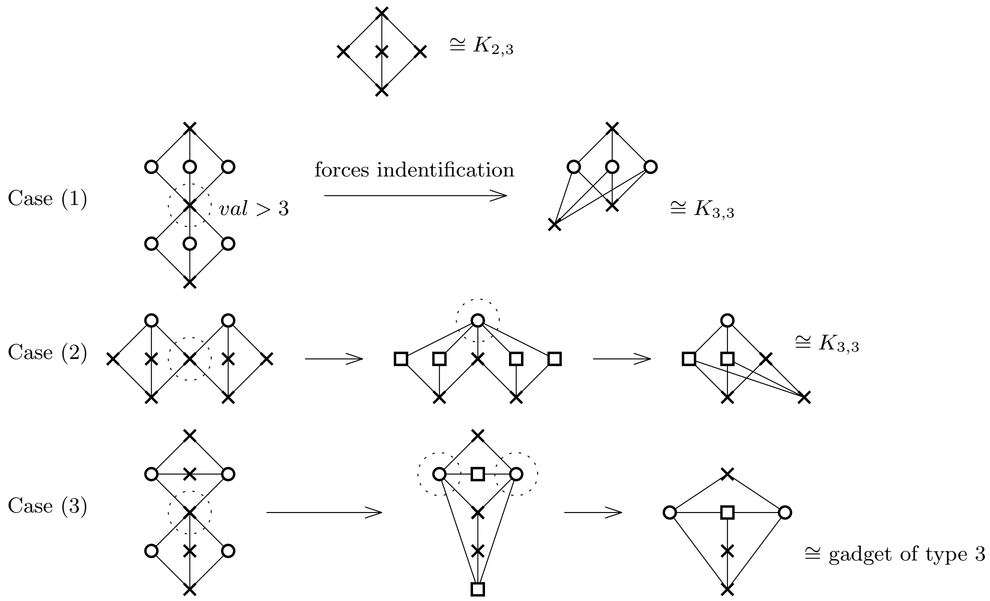


Figure 6.6: Identification of vertices.

If G contains a gadget of type 3, then $D(G) = 2$ or $G = R_n^3$. Therefore we assume that all copies in G of $K_{2,3}$ are disjoint. Let G' be the graph obtained by identifying in each copy of $K_{2,3}$ the two valence 3 vertices. The resulting graph has vertices of valence 2 with some vertices of valence 3 surrounded by vertices of valence 2. Thus G' is not S_n or R_n^1, so $D(G') = 2$. Any distinguishing coloring of G' can be extended to one of G by coloring the two valence 3 vertices of each $K_{2,3}$ black and white. $\square$

Theorem 6.5.6. *If the cubic graph G has a type 2 vertex of valence 3, then $D(G) = 2$ or G is R_n^2 or R_n^3. If G has a type 3 vertex, then $D(G) = 2$.*

Proof Let v be a type 2 vertex of valence 3 and let u_1, v_1 be its neighbors which can be interchanged by $\mathrm{Aut}_v(G)$. Color v black and all its neighbors black as in a canonical coloring rooted at the graph spanned by v and its neighbors. Suppose that u_1 and v_1 are adjacent. If their up neighbors are different, color one white and one black. This will also fix u_1, v_1 and can be continued to a canonical coloring that breaks all automorphisms in $\mathrm{Aut}_v(G)$. If instead u_1, v_1 have a common up neighbor, we have a gadget 2 or a non-sibling gadget 2. Then either $D(G) = 2$ or $G = R_n^2$ by Lemma 6.4.2 and Corollary 6.5.2.

We assume therefore that u_1 and v_1 are not adjacent. If they have one common up neighbor but not two, color that neighbor white and the other two up neighbors black and white. This distinguishes u_1, v_1. Suppose u_1, v_1 have two common up neighbors. Then G contains a $K_{2,3}$ so by Lemma 6.5.5 we infer that $D(G) = 2$ or $G = R_n^3$.

Therefore, assume that the up neighbors of u_1 and v_1 are distinct. Color the up neighbors of u_1 black-white and the up neighbors u_2, v_2 of v_1 both white. This distinguishes u_1, v_1 but allows an interchange of u_2, v_2. Repeat this process on u_2, v_2. Either we distinguish u_2, v_2 or we find u_3, v_3 that can be interchanged. Continue the process. If G is finite, the process must end either with $G = R_n^2, R_n^3$ or with a 2-coloring which breaks all automorphisms in $\mathrm{Aut}_v(G)$ with v and its three neighbors colored black. If G is infinite, we continue the process as long as u_n, v_n have distinct up neighbors, yielding a 2-coloring that breaks any automorphism fixing v see Figure 6.7.

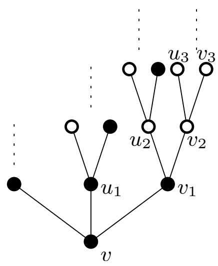


Figure 6.7: Example of the coloring to fix the vertices u_1 and v_1 by any automorphism that fixes v.

We then can proceed as in the proof of Proposition All Black (6.3.4) to show that there is no other black vertex surrounded by three black vertices for $S_n(K)$, $n > 1$, and by construction there is none in $S_1(K)$. We conclude that no color-preserving automorphism moves v, so the coloring is distinguishing.

Suppose instead that v is a type 3 vertex. We proceed exactly as before except now we are breaking any automorphism taking u_1 to v_1 or v_1 to u_1 (but not interchanging them). We still break all non-identity elements of $\mathrm{Aut}_v(G)$. Since neither R_n^2 or R_n^3 have a type 3 vertex, we must have $D(G) = 2$. □

We have now completed the classification for graphs G with a vertex of type 1,2,3. There remains only the case where all vertices have type 6. Then G must be cubic and, as is easily seen, edge transitive. We will treat the distinguishability of edge transitive graphs in the next section.

6.6 Girth

Our analysis of edge transitive cubic graphs G uses the girth of G. One easily verifies that the only edge transitive cubic graph of girth 3 is K_4, and that $K_{3,3}$ and the cube are the only edge transitive graphs of girth 4. We know $D(K_4) = D(K_{3,3}) = 4$ and $D(Q) = 3$.

For girth 5 we observe that edge transitive graphs that are not vertex transitive must be bipartite. Hence all edge transitive graphs of odd girth are also vertex transitive. But there are only two vertex transitive cubic graphs of girth 5, the dodecahedron H and the Petersen graph [41]. In Lemma 6.2.8 we have shown that $D(P) = 3$. However, $D(H) = 2$. To find a 2-distinguishing coloring color black a vertex v, its three neighbors x, y, z, and a vertex w adjacent to x.

For girth $s > 5$ the situation changes drastically. Although there are only five cubic edge transitive graphs of girth at most 5, and no infinite ones, there are infinitely many such graphs already for girth 6, and an infinite one is the honeycomb lattice, which is also edge transitive.

Thus it remains for us to show that edge transitive cubic graphs of girth $s > 5$ are 2-distinguishable. In fact, our proof does not use edge transitivity at all.

Lemma 6.6.1. *If G is a cubic graph with girth $s > 6$, then $D(G) = 2$.*

Proof Let C be a cycle of length s. Since $s > 6$, each vertex in $S_1(C)$ is adjacent to only one vertex in C. Moreover, if two vertices in $S_1(C)$ are adjacent, then they can be used to form a path of length three between two vertices in C of distance at most $s/2$, contradicting $s > 6$. Let the vertices of C be denoted by $1, 2, 3, \ldots, s$. Let K be C together with the whiskers at vertices 1 and 4 as well as $6, \ldots, s$, see Figure 6.8.

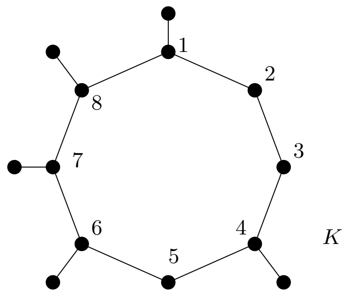


Figure 6.8: The set K for $s = 8$. The vertices 1 and 4 together with 6, 7 and 8 are the kiwi vertices.

Choose a canonical 2-coloring rooted at K. By Proposition All Black (6.3.4), there is no kiwi vertex in $S_n(K)$ for $n > 1$. There is none in $S_1(K)$ either, since any such vertex would be adjacent to three vertices in K, forcing a $K_{2,3}$ in G, contradicting $s > 6$. Thus any color-preserving automorphism ϕ must leave invariant the kiwi vertices $1, 4, 6, \ldots, s$. The graph spanned by these vertices consists of an isolated vertex 4 and a path $6, \ldots, s, 1$. Thus ϕ fixes 4, and either leaves the path fixed or reverses it (interchanging vertices $1, 6$). In the first case, ϕ fixes 5, since $s > 4$, and ϕ fixes 2, 3 since $1, \phi(2), \phi(3), 4$ provides another path of length 3 between 1 and 4, contradicting $s > 6$. In the second case, $4, \phi(5), \phi(6) = 1$ provides a path of length 2 from 4 to 1, contradicting $s > 5$. We conclude that ϕ fixes all vertices of C and hence all vertices of K, so ϕ is the identity. □

For girth $s = 6$, we also have $D(G) = 2$ but the argument is slightly more complicated.

Lemma 6.6.2. *Let G be a cubic graph with girth $s = 6$. Then $D(G) = 2$.*

Proof We note that the proof of Lemma 6.6.1 for girth $s > 6$ only uses $s \neq 6$ to insure that for a cycle C of length s, the graph K obtained from C with whiskers to vertices $1', 4', 6', \ldots, s'$ has no edges between the whiskers. The rest of the proof only requires $s > 5$. In particular, for girth $s = 6$, if G contains a cycle 123456 with no edge between $1'$ and $4'$, then $D(G) = 2$ (as there can be no edge between $4'$ and $6'$ as otherwise $s < 6$). By cyclically permuting $1, 2, 3, 4, 5, 6$, we conclude that for every 6-cycle C, we must have edges $1'4', 2'5', 3'6'$ in $S_1(C)$ or else $D(G) = 2$. Applying this to the cycle $1'12344'$, we have that $1'$ and $3'$ must have a common neighbor. Since the choice of which vertex is labeled 1 does not matter, $3'$ and $5'$ have a common neighbor, as do $5'$ and $1'$. Since all vertices have valence 3, it must be that $1', 3', 5'$ have one common neighbor 7. Similarly, $2', 4', 6'$ have one common neighbor 8.

At this point all 14 of the vertices have valence 3 so we have the entire graph, see Figure 6.9. This is the Heawood graph (the dual of the triangulation of the torus with underlying graph K_7).

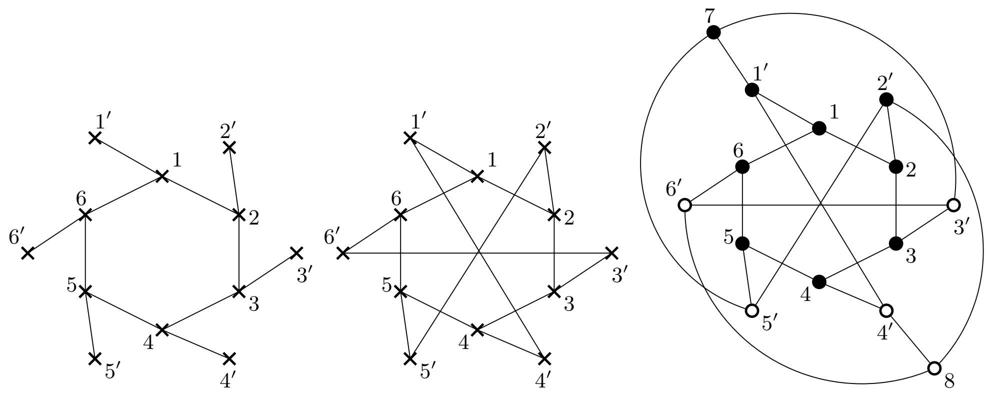


Figure 6.9: Construction of the graph described in the proof of Lemma 6.6.2.

Now consider the following 2-coloring of the graph. Let $1, 2, 3, 4, 5, 6, 1', 2', 7$ be black and the remaining vertices white. In the graph H spanned by the black vertices, 7 is the only vertex of valence 1 adjacent to a vertex of valence 2 (namely $1'$). Thus any color-preserving automorphism ϕ fixes 7. Thus ϕ also fixes $1'$ and hence 1. Since $2'$ is the only vertex of valence one, ϕ also fixes 2. Thus ϕ fixes the remaining vertices of the cycle C, so ϕ fixes all black vertices. But then ϕ also fixes the white vertices adjacent to $3, 4, 5, 6$. That leaves only 8 so it must be fixed as well, making ϕ the identity (compare the right graph in Figure 6.9 for the coloring). □

6.7 Questions

There are a number of questions worth further study,

Question 1 (Higher Valence) Can we classify graphs G with $\Delta(G) = d = D(G)$?

As we have observed, if $G = T(n, d)$, then $\Delta(G) = d = D(G)$. We could add edges within each sibling family of size $d - 1$ to form a graph $S(n, d)$ analogous to S_n (the vertices of a sibling family then have valence $d - 1$). We can also attach $d - 1$ independent vertices to a sibling family using $K_{d-1,d-1}$ to obtain a graph analogous to R_n^1. There do not appear to be analogues for R_n^2 and R_n^3.

We can define a canonical $d - 1$ coloring rooted at a graph K such that the only color-preserving automorphism of G fixing the vertices of K is the identity. Then we have to identify properties of such colorings that restrict the structure of K and $S_1(K)$. Note that a variation of the canonical coloring using $d+1$ colors, with color $d+1$ for a vertex v , colors $1, 2, \ldots, d$ for the neighbors of v, and colors $1, 2, \ldots, d-1$ is how one gets $D(G) \leq d+1$. And to show $D(G) = d+1$ only for $G = K_{d+1}$ or $G = K_{d,d}$, one uses canonical d-colorings rooted at an asymmetric vertex-induced subgraph K with color d used only on K.

Question 2 (Highly Symmetric Graphs) If $\Delta(G) = d$ and G is vertex transitive, must $D(G) = 2$ for all but finitely many G?

Question 3 (Connectivity) What is the relationship between vertex or edge connectivity, valence, and distinguishing number?

The examples with $D(G) = d$ are not 2-connected. What happens if we require, say, 3-connectivity? For example, we can get $D(G) = d-3$ with G 3-connected by attaching K_{d-1} at three vertices of valence 2. As the connectivity goes up, the distinguishing number seems to go down, with finitely many exceptions like K_{d+1}.

Question 4 (Infinite Graphs) What happens for infinite G with $\Delta(G) = d > 3$?

For infinite graphs, we expect that if $\Delta(G) = d$, then $D(G) < d$, just as for $d = 3$. But there are also interesting questions just for such G with $D(G) = 2$. As we observed before Corollary 2.6, for finite graphs, large enough motion implies $D(G) = 2$. The Infinite Motion Conjecture [79] is that if G is locally finite and $m(G) = \infty$, then $D(G) = 2$. On the other hand, for the case $d = 3$, we see there is no need for the hypothesis of infinite motion to get $D(G) = 2$, and there are other classes of graphs with $D(G) = 2$ that do not depend intrinsically on infinite motion [76]. As we observed, however, it is easy to construct an infinite d-valent graph G with $D(G) = d-1$, so for $d > 2$, we expect infinite motion to be involved.

Question 5 (Motion). For cubic graphs, if the motion $m(G) > 2$, then $D(G) = 2$ with the exception of Q and P. For $d > 3$, is it the case that if $m(G) > d$, then $D(G) = 2$ with finitely many exceptions?

Perhaps, even $m(G) > 2$ gives $D(G) = 2$ with finitely many exceptions.

Question 6 (Chromatic Distinguishing Number) Suppose all colorings are required to be proper (adjacent vertices get different colors). What happens when $\Delta(G) = 3$?

Collins and Trenk [26] define the chromatics distinguishing number $\chi_D(G)$ to be the least k such that G has a proper k-coloring whose only color-preserving automorphism is the identity. They prove that $\chi_D(G) \leq 2d$ with equality only for $K_{d,d}$ and C_6. For $d = 3$, there is the possibility of classifying graphs with $D(G) = 5$, especially in the case that G is bipartite.

In [47] the chromatic distinguishing number of infinite graphs is investigated. For connected graphs of bounded valence d it is shown that $\chi_D(G) \leq 2d-1$, and for infinite subcubic graphs of infinite motion this improves to $\chi_D(G) \leq 4$.

Question 7 (Edge Distinguishing) One can also define [53] the *distinguishing index* (or edge distinguishing number) $D'(G)$ as the least k such that some k-coloring of the edges of G is preserved only by the identity. In [53] it is shown that $D'(G) \leq \Delta(G)$ for finite graphs. For infinite graphs $\Delta(G)$ has to be replaced by the supremum of the valences [23].

What happens with $D'(G)$ when $\Delta(G) = 3$?

Question 8 (Cost) When $D(G) = 2$, the cost [20, 21] is the least number of times the color black is used. When $\Delta(G) = D(G) = 2$, what can we say about the cost? For cubic graphs this is treated in [47].

The canonical coloring tends to use black as few times as possible for $S_n(K)$, $n > 0$. How close does this number come to the cost?

Chapter 7

Distinguishing locally finite trees

This chapter is based on the submitted paper *Distinguishing locally finite trees* which is joint work with Svenja Hüning, Wilfried Imrich, Hannah Schreiber and Thomas Tucker.

7.1 Introduction

This paper is concerned with automorphism breaking of finite and infinite trees of bounded valence by vertex colorings. 1977 Babai [6] showed that the vertices of every k-regular tree, where $k \geq 2$ is an arbitrary cardinal, can be colored with two colors such that only the identity automorphism preserves the coloring. For trees that are not regular such a 2-coloring need not be possible. This raises the question of how many colors are needed to break all automorphisms of a given tree T, that is, of finding the smallest cardinal d to which there exists a d-coloring of the vertices of T that is only preserved by the identity automorphism.

Another question is to find, for a given $c \geq 2$, c-colorings of the vertices of a given tree T such that the color-preserving automorphisms fix subtrees of T that are maximal in some sense. This is the problem, which we consider here.

Our note is related to [46], where both questions were answered for connected graphs of maximum valence 3. That paper, in turn, was motivated by the general problem of determining the distinguishing number of graphs and the Infinite Motion Conjecture of Tucker.

The *distinguishing number* $\mathrm{D}(G)$ of a graph G is the smallest cardinal number d such that there exists a d-coloring of the vertices of G which is only preserved by the identity automorphism. We also say that such a coloring *breaks* $\mathrm{Aut}(G)$ and that G is *d-distinguishable*. These concepts were introduced 1996 by Albertson and Collins [1] and have spawned a series of related papers. In two of them, by Collins and Trenk [26] and Klavžar, Wong and Zhu [57], it is shown that the distinguishing number $\mathrm{D}(G)$ of any finite connected graphs G of maximal valence k is at most k, unless G is K_k, $K_{k,k}$ or C_5. Then $\mathrm{D}(G) = k + 1$. In [48] this was extended to infinite graphs. In that case $\mathrm{D}(G)$ is bounded by the supremum of the valences of the vertices.

Despite the fact that the distinguishing number can be arbitrarily large, it is 1 for asymmetric graphs. As almost all finite graphs are asymmetric, this means that almost all graphs have distinguishing number 1. Furthermore, almost all finite graphs that are not asymmetric have just one non-identity automorphisms, which is an involution[1]. One can break it by coloring one selected vertex black and all others white. Clearly such graphs are 2-distinguishable.

For infinite graphs we have the Infinite Motion Conjecture [79]. It says that all connected, locally finite, infinite graphs are 2-distinguishable if every non-identity automorphism moves infinitely many vertices. Despite the fact that it is true for many classes of graphs, for example for graphs of subexponential growth, see Lehner [64], the conjecture is still open. Until recently it was not even clear whether it holds for graphs of maximum valence 3, but in [46] it was shown that all connected infinite graphs of maximal valence 3 are 2-distinguishable and that no motion assumption is needed.

In the case of connected finite graphs G of maximum valence 3 it is even enough to require that every automorphism moves at least three vertices to ensure 2-distinguishability, unless G is the cube or the Petersen graph. In fact, with the exception of K_4, $K_{3,3}$, the cube and the Petersen graph, all connected finite graphs G of maximum valence 3 admit a 2-coloring where every automorphism that preserves the coloring fixes all vertices, with the exception of at most one pair of interchangeable vertices. We say the 2-coloring fixes all but this pair of vertices. For example, consider three copies of $K_{1,3}$, select a vertex of valence 1 in each copy and identify them. The resulting tree has distinguishing number 3, and it is easy to find a 2-coloring that fixes all but two vertices.

In this note we generalize this to trees T of maximal valence k and an arbitrary number of colors c. We wish to find c-colorings that *fix as many vertices as possible*, that is, we wish to maximize the sets of vertices that are fixed by each color preserving automorphism. As this is hard to control, we look for the smallest number $r(c, k)$ such that there exists a c-coloring of T that fixes in the worst case at least all vertices of T whose distance from the next leaf is at least $r(c, k) = \lceil \log_c k \rceil$. This is made more precise in Section 7.5, where one can see that in a lot of cases this number will be significantly smaller.

7.2 Preliminaries

7.2.1 Graph representation

Let $G = (V(G), E(G))$ be a connected graph, $V(G)$ its set of vertices and $E(G)$ its set of edges. To simplify the notation we write V and E if the graph is clear by the context. If the vertices of G have maximal valence 3, then G is called *subcubic*. As usual we call the number of edges on a shortest path between two vertices u and v be the *distance* $d(u, v)$ between u and v.

Definition 7.2.1. Let $v \in V$. The *ball* of radius n with center v is defined as the set $B(n, v) = \{u \in V \mid d(u, w) \leq n\}$. The sphere of radius n around v is the set $S(n, v) = \{u \in V \mid d(u, v) = n\}$, that means it is the set of all vertices of distance n from v, see Figure 7.1.

In this paper, we mostly represent a graph G by an arrangement of spheres with a common center $v \in V(G)$, and say that G is *rooted* in v. In that context, a *down-neighbor*, a *cross-*

[1] Florian Lehner, private communication.

neighbor and an *up-neighbor* of $\in S(n,v)$ is a neighbor w of u that it is in $S(n-1,v)$, $S(n,v)$ or $S(n+1,v)$ respectively; see Figure 7.1. The corresponding edges are called *down-edges*, *cross-edges* and *up-edges*.

Definition 7.2.2. Two vertices z, z' are *siblings* if they have the same down-neighbor. We call a vertex w an *only child* of a vertex u if w is the only neighbor of u of valence 1.

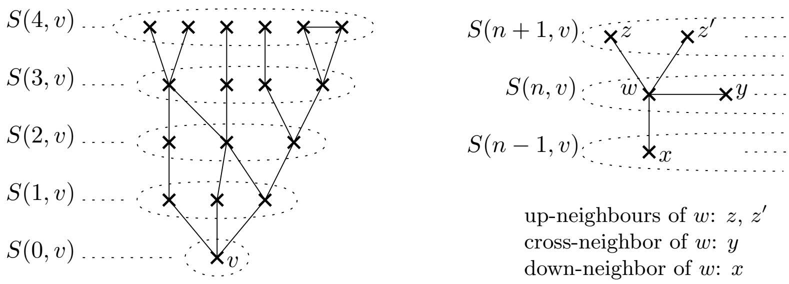


Figure 7.1: Decomposition of a graph into spheres centered at a vertex v. Each vertex in $S(i,v)$ has distance i from v.

7.2.2 Ends and rays

A *subgraph* of a graph $G=(V(G),E(G))$ is a graph $H=(V(H),E(H))$ such that $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$. If $E(H)$ contains all edges between vertices of $V(H)$ that are also in $E(G)$, we say H is an *induced subgraph* of G and denote it by $\langle V(H)\rangle$. If $S\subset V(G)$, then $G\setminus S$ is the subgraph of G induced by the vertices of the set $V(G)\setminus S$.

Definition 7.2.3. A *ray* is an infinite graph $R=(V,E)$, with $V=\{v_0,v_1,v_2,...\}$ and $E=\{v_0v_1,v_1v_2,v_2v_3,...\}$ where the v_i are pairwise different. If a ray $R_1=(V_1,E_1)$ is a subgraph of a ray $R_2=(V_2,E_2)$, then R_1 is called a *tail* of R_2.

Two rays R_1 and R_2 in a graph $G=(V(G),E(G))$ are *equivalent*, in symbols $R_1\sim R_2$, if for every finite set $S\subset V(G)$ there exists a connected component of $G\setminus S$ containing a tail of both R_1 and R_2. One can show that $\sim$ is an equivalence relation. The equivalence classes of $\sim$ are called *ends* of G.

If $T=(V,E)$ is an infinite tree, then the set of vertices in V such that $T-v_i=\langle V\setminus v_i\rangle$ contains at least two infinite components is denoted by $V^C=\{v_1,v_2,\dots\}$. We call the induced subgraph $\langle V^C\rangle$ the *trunk* of T and denote it by T^C.

7.2.3 Automorphisms and Colorings

An *isomorphism* $\varphi:V(G)\to V(H)$ between two graphs G and H is a bijection such that $uv\in E(G)$ if and only if $\varphi(u)\varphi(v)\in E(H)$. The isomorphisms of G onto itself are called *automorphisms*. They form a group $A(G)$. The *stabilizer* of a vertex $v\in V(G)$ is the set $A(G)_v=\{\alpha\in A(G)\mid\alpha(v)=v\}$.

Definition 7.2.4. A *d-coloring* $\mathfrak{c}:V\to\{1,...,d\}$ of a graph G is a map that gives each vertex of G a color $i\in\{1,...,d\}$. We say that an automorphism *preserves* a coloring $\mathfrak{c}$, if $\mathfrak{c}(v)=\mathfrak{c}(\varphi(v))$ for all $v\in V$. Otherwise, we say that the coloring *breaks* the automorphism.

The set $A(G)_{\mathfrak{c}}$ of all automorphisms preserving $\mathfrak{c}$ forms a group. If $v \in V(G)$ and $A(G)_{\mathfrak{c}} \subseteq A(G)_v$, we say that $\mathfrak{c}$ *fixes* v. As mentioned in the introduction our aim is to construct graph colorings that fix as many vertices as possile. The minimal number of colors needed to fix all vertices is called the distinguishing number.

Definition 7.2.5. The *distinguishing number* $\mathrm{D}(G)$ of a graph G is the the smallest d for which there exists a d-coloring $\mathfrak{c}$ of G such that the only automorphism preserving $\mathfrak{c}$ is the identity.

Definition 7.2.6. The *center* of a finite graph is a vertex for which the greatest distance from v to any other vertex of the graph is minimal.

In a finite tree T the center is either a single vertex or an edge. If the center of T is a vertex w, then $A(T)_w = A(T)$. If the center is an edge uv, any automorphism $\alpha \in A(T)$ satisfies either $\alpha(v) = v$ and $\alpha(u) = u$ or $\alpha(v) = u$ and $\alpha(u) = v$. We define a subtree T_w of the tree T rooted in v, with $w \in S(n, v)$, as the tree induced by w and all the vertices in $S(m, v)$, $m > n$, for which there exists a path to w in T not containing v.

7.3 Coloring locally finite infinite trees

In this section we prove that every locally finite infinite tree with maximal valence k has distinguishing number $k - 1$. For the proof we need the following two lemmas.

Lemma 7.3.1. *Let T be a tree with maximal valence k. If T has a vertex v of valence $1 \leq val(v) \leq k - 1$ then $\mathrm{D}(A(T)_v) \leq k - 1$. In other words, there is a $(k-1)$-coloring $\mathfrak{c}$ of T that breaks all automorphisms of $A(T)_v$.*

Proof Let v be a vertex of valence $1 \leq val(v) \leq k - 1$. We color v with an arbitrary color and all its neighbors with different colors. This coloring fixes $B(1, v)$ in $A(T)_v$. We prove that for all $i > 0$ there is a $k - 1$-coloring of $B(i, v)$ that breaks all automorphisms of $A(B(i, v))_v$. To show this, it suffices to extend a given coloring $\mathfrak{c}(i)$ of $B(i, v)$ with $k-1$ colors to $B(i + 1, v)$. We do this as follows. Every vertex in $S(i, v)$ has at most $k - 1$ up-neighbors. Each of them can be colored with a different color. If we continue to color the tree in this way, we obtain a coloring that breaks all automorphisms of $A(T)_v$ but the identity. □

The second result that we need is Königs Lemma, see e.g. [30].

Lemma 7.3.2 (König's Lemma)**.** *Let $V_0, V_1, V_2, \ldots$ be an infinite sequence of non-empty, disjoint sets. Let $V = \cup_{i\geq 0}$ denote their union. If $G = (V, E)$ is a graph such that for all $v \in V_n$, $n \geq 1$, there exits a vertex $f(v) \in V_{n-1}$ adjacent to v, then there exists an infinite path $v_0v_1 \ldots$ in G with $v_n \in V_n$ for all $n \geq 0$.*

Note that parts of the next result (namely the finite case) can be extracted from a recent classification result from [46]. Here we provide a direct and constructive proof which also generalizes to trees with maximal valence k.

Theorem 7.3.3. *Every locally finite, infinite tree with maximal valence k has distinguishing number $D(G) \leq k - 1$.*

Proof Let T be a locally finite infinite tree. It is well known that such a tree without vertices of valence 1 is 2-distinguishable, see for example [84]. Hence, we can assume that T has at least one vertex of valence 1.

By König's Lemma, T has at least one end. We first treat the case, where T has exactly one end, say e. Let $R \in e$ be a ray with origin v_0, where v_0 is a vertex of valence 1.

We color all vertices of R with color 1. Thus every automorphism that stabilizes R also fixes all vertices of R. Consider a vertex z in R and its j neighbors that are not in R. Since $j \leq k-2$ we can color these vertices with different colors without using color 1. We proceed by doing this for all neighbored vertices of R. Let v be one neighbor of z not in R and let T_v be the component of $T - z$ that contains v. Now, color T_v as in Lemma 7.3.1. Continue by coloring the neighbors of all vertices in R in the same way. If w is another vertex of T of valence 1, then the unique ray R_w in e with origin w contains at least one vertex which is not colored with 1. Hence, independently of how the coloring of T will be finished, any color preserving automorphism α of T will satisfy $\alpha(v_0) \neq w$. Because this holds for all vertices of valence 1 that are different from v_0 we infer that $\alpha(v_0) = v_0$ and that α fixes every vertex of R. Because the vertices in R are fixed also the neighbors and finally all finite trees connected to R are fixed.

We now treat the case when T has 2 or more ends. Consider the trunk T^C of T which is a locally finite, infinite tree without leaves. Since it is unique any automorphism of T leaves T^C invariant. By [84] we know that T^C is 2-distinguishable. Any vertex z of T^C has at most $k-2$ neighbors not in T^C that can be fixed by coloring them with different colors. Again, for each of these neighbors v_i of z we consider the underlying tree T_{v_i} of $T - z$ that contains v_i as before and apply Lemma 7.3.1. □

The following example shows that Theorem 7.3.3 cannot be generalized to finite trees.

Example 7.3.4. Consider a finite tree with center v where every vertex is of valence 3 or 1 and every leaf has the same distance from v. The distinguishing number of such trees is 3. For any 2-coloring of such a tree, there remains a pair of leaves that is indistinguishable, see Figure 7.2. An other easy example is the $K_{1,3}$.

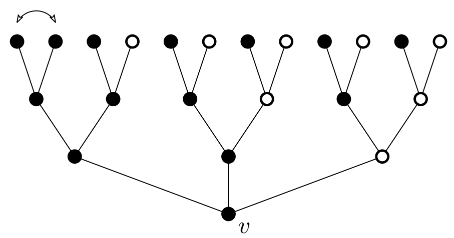


Figure 7.2: Example of a finite tree T with $\mathrm{D}(T) = 3$. With a 2-coloring, two leaves are interchangeable.

7.4 Finite trees

We begin with the analog of Theorem 7.3.3 for finite trees.

Lemma 7.4.1. *Let T be a finite tree with maximal valence k. Then there is a $k-1$-coloring, which fixes all vertices, with the possible exception of two sibling leaves.*

Proof If the center of T consists of a single vertex v, assign v an arbitrary color and color its neighbors $v_1, v_2, \ldots$ with as many different colors as possible. Only in the case where v has valence k we have to use one color twice. As before let T_{v_i} be the component of $T - v$ containing v_i. We extend the coloring of T_{v_i} as in Lemma 7.3.1 for all i. In the case where v has k neighbors it is possible that there is an automorphism that interchanges the two subtrees, say T_{v_m} and T_{v_n} (with v_m and v_n of the same color). We take an endpoint a of T_{v_m} and change its color so that T_{v_m} and T_{v_n} are indistinguishable.

However, now there might be a color preserving automorphism that moves a. This is only possible if there is another vertex of valence 1, say b, that has a common neighbor with a. This is the only pair of that kind in T.

We still have to consider the case where the center of T is an edge, say uv. In that case we color u and v with different colors, and consider the components T_u and T_v of $T - uv$ that contain u or v, respectively. We now obtain a distinguishing $k-1$-coloring by applying Lemma 7.3.1 to T_u and T_v. □

It is known from Babai [6] that each (infinite) homogeneous tree of valence $k > 2$ is 2-distinguishable. This result cannot be adapted to the finite case, but note that we can fix all vertices except the leaves by a 2-coloring.

Lemma 7.4.2. *Let T be a finite tree where every vertex has valence 1 or k. Then there exist a 2-coloring of T that fixes all vertices except (some of) the leaves.*

Proof Suppose the center of T consists of the single vertex v. This vertex is then automatically fixed by any automorphism. Let v be a white vertex and color all k neighbors of v black.

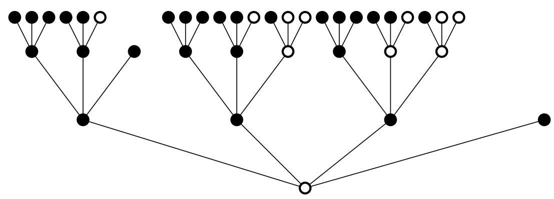


Figure 7.3: Example of the coloring algorithm given in the proof of Lemma 7.4.2 with $k = 4$.

To avoid that they can be changed we assign different colorings to the next $k-1$ up-neighbors. Since the number of different 2-colorings of $k-1$ indistinguishable vertices is k, each of the black vertices can be fixed. We proceed with this coloring process until we reach the leaves. Then, all vertices except the leaves are fixed. See Figure 7.3 for an example.

If the center of T is an edge, say uv, we color u white, v black and continue with a coloring for the subtrees T_u respectively T_v as before. □

7.5 Main Theorem

We now construct a c-coloring of a tree with maximal valence k which only leaves certain vertices of the tree indistinguishable. In the next theorem, we consider a tree rooted in a vertex v. Here, that is, in the finite case, we define T_u as the vertex u together with the components of $T - u$ that do not contain v. In the infinite case T_u consists of the vertex u together with all finite subtrees of $T - u$, see Figure 7.4.

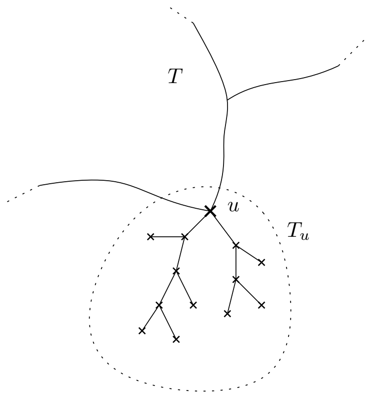


Figure 7.4: After removal of u, all remaining finite trees are attached to u in T_u.

Main Theorem. Let T be a finite or infinite tree of maximal valence $k < \infty$. We assume T to be rooted in a fixed vertex v (the center of the tree in the finite case) and choose a number $c \geq 2$. Then, for every pair c, k, there exists a number $r(c,k)$ and a c-coloring of T that fixes all vertices $u \in V(T)$, for which there exists a leaf w in T_u such that $d(u,w) \geq r(c,k)$. If $k \in \{0,1,2\}$ or $c \geq k$, then $r(c,k) = 0$, which means that the entire graph is fixed. If $k \geq 3$ and $c = k-1$, then $r(c,k) = 1$. Otherwise

$$r(c,k) := \begin{cases} \log_2(\max\{3, k-2\}) + 1 & \text{for } k \geq 4 \text{ and } c = 2. \\ \log_c\left(\max\left\{3, \left\lceil \frac{k-2}{c-1} \right\rceil\right\}\right) & \text{for } k \geq 4 \text{ and } 2 < c \leq k-2. \end{cases} \tag{7.1}$$

The strategy to prove the theorem is the following. First we introduce an explicit coloring algorithm and then we show two of its properties (see Lemmas 7.5.1 and 7.5.2) which lead to the result.

We start by considering a finite tree T with maximal valence $4 \leq k < \infty$. We assume that we have $2 \leq c \leq k-2$ colors. For simplicity we write $\{0, 1, 2, 3, ..., c-1\}$ for the c different colors (in the following figures, 0 is white, 1 black and 2 gray).

Furthermore, a c-coloring of a set of siblings *optimal* if the maximal number of vertices with the same color is minimal. Moreover, a vertex u is said to satisfy the *distance condition* if there exists a leaf w in T_u of distance $d(u,w) \geq r(c,k)$.

7.5.1 Coloring Algorithm

We first assume that the center of T consists of a single vertex v. We root T in v and color v with color 0. Let $n' \in \mathbb{N}$ be the maximal radius such that $S(n', v) \neq \emptyset$. Now consider the following steps, each of which maybe processed several times..

Step 0: If there exist indices ℓ in $\{1,\dots,n'\}$ such that there still exist uncolored vertices in $S(\ell,v)$, then let n be the minimum of these indices and continue with Step 1.

Otherwise stop the coloring algorithm[2].

Step 1: If there exists an uncolored vertex in $S(n,v)$ call it u and go to Step 2. Otherwise go back to Step 0.

Step 2: If the vertex u does not fulfill the distance condition, color u with 0 and go back to Step 1. Otherwise continue with Step 3.

Step 3: Note that the vertex u satisfies the distance condition. Consider u and all of its uncolored siblings $\{v_1,\dots v_r\}$ which fulfill the distance condition. If this set is not empty, then color them optimally. If there are no indistinguishable vertices within $\{u,v_1,\dots v_r\}$ with the same color, go back to Step 1. Otherwise continue the coloring in the following way (which is still part of Step 3):

Main Line Coloring: Let V_j be the set of vertices among $\{u,v_1,\dots v_r\}$ with color j, where at least two vertices have color j, i.e. $|V_j|>1$ (vertices with a color that appears only once are clearly fixed).

We can assume that for all $v\in V_j$, and for all j, there exists a leaf w in T_v of distance $d(v,w)\geq r(c,k)$. Let $V_j=\{v_1^j,v_2^j,\dots,v_{m_j}^j\}$ for each of the colors $j=1,\dots,\ell\leq c$ and do the following:

Consider (one of) the longest path(s) R_i from v_i^j to a leaf in $T_{v_i^j}$ for each of these vertices v_i^j, $i=1,...,m_j$. We call the chosen paths R_i *main lines*. To distinguish $\{v_1^j,v_2^j,\dots,v_{m_j}^j\}$ we color the main lines R_i with pairwise different finite sequences of colorings.

For example, if $c=2$, we color the paths with different "reverse binary colorings". That means R_1 will be colored with 00000..., R_2 with 10000..., R_3 with 01000..., R_4 with 11000... and so on. If $c=3$, we use "reverse ternary colorings", meaning that R_1 will be colored with 00000..., R_2 with 10000..., R_3 with 20000..., R_4 with 01000... and so on, see Figure 7.5.

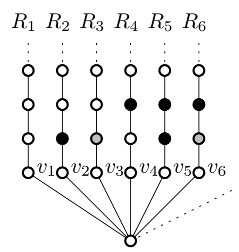


Figure 7.5: Coloring of main lines for $c=3$.

In general we use a reversed coloring based on the number system with base c. Having done this for all $j\in\{1,\dots\ell\}$ continue with Step 4.

Step 4: We consider all vertices $\{w_1,\dots w_r\}$ produced in Step 3 that are part of one of the main lines and have valence ≥ 3. That means we consider vertices that have at least one second up-neighbor, which is not in the main line. Consider a vertex w_i. For simplicity

[2] In this step we always look after the lowest sphere where there are uncolored vertices, such that in the end we can guarantee that we colored all vertices.

we denote its up-neighbors (it has at least two) by v_1, v_2, …, v_ℓ. By construction only one vertex is in a main line, let it be v_1. Therefore, v_1 is already colored, whereas v_2, …, v_ℓ are uncolored. Let a be the color of v_1. We consider two cases:

Case 1: $\ell = 2$.

Then color v_2 with $(a+1) \mod c$, see Figure 7.6 for an example, where $c = 3$.

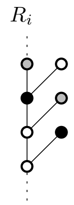


Figure 7.6: How to color unique siblings of vertices in a main line for $c = 3$.

If $c > 2$ continue with Step 4 for the next w_i. Otherwise do the following:

If the subtree T_{v_2} contains only vertices of valence one or two (we say if there is no branching), color all vertices of $T_{v_2} - v_2$ with 1. Now, consider the case in which there is a branching in T_{v_2}. Color all vertices up to the branching in $T_{v_2} - v_2$ with 1. If there are 2 or 3 sibling vertices in that branching, color them all with 1. If the branching consists of 4 or more vertices, then assign them an optimal coloring, see Figure 7.7. Restart Step 4 for the next w_i.

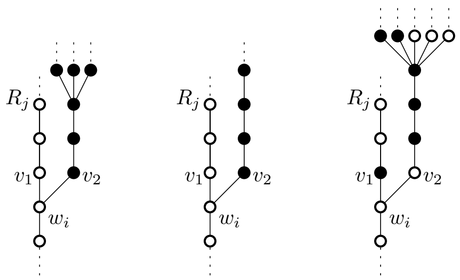


Figure 7.7: Coloring of secondary lines for $c = 2$.

Case 2: $3 \leq \ell \leq k - 1$.

Color $v_2, v_3, \ldots, v_\ell$ with an optimal coloring considering two additional restrictions:

First, we do not use the color a. Second, if there exists $j \in \{2, \ldots, \ell\}$ such that v_j has the color $b \in \{0, \ldots, c-1\} \setminus \{a\}$, then there exists $j' \in \{2, \ldots, \ell\}$, $j \neq j'$, such that $v_{j'}$ also has color b. This means that a color is always used at least twice, see Figure 7.8. Note that v_1 is the only vertex among the siblings whose color appears just once.

Having done this for all $w_1, ..., w_k$ check whether there are indistinguishable vertices of the same color satisfying the distance condition (as an example consider the case that there is a branching in the subtree T_{v_2} when $c = 2$ or within $\{v_2, v_3, \ldots, v_\ell\}$), then repeat the Main Line Coloring method and apply Step 4 to these vertices. If there are no colored indistinguishable vertices, continue with Step 1.

This completes the algorithm for the case that the center of T is a single vertex.

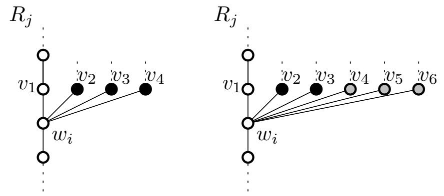


Figure 7.8: Examples of how to color the four (left) or the six (right) up-neighbors of a vertex in some main line, with $c \geq 3$.

Now, assume that the center of T is not a vertex but a an edge uv. Let T_u be the tree containing u in $T - v$ and T_v be the tree containing v in $T - u$. Color u with 1, v with 0, and proceed with the coloring of T_u and T_v as explained above for trees whose center consists of a single vertex.

7.5.2 Proof of the Main Theorem

Lemma 7.5.1. *Let V be a set of t vertices. Assume they are colored with an optimal coloring consisting of j colors, with the additional restriction that every color appears at least twice. Then, the maximal number p of vertices with the same color in V is* $\max\left\{3, \left\lceil \frac{t}{j} \right\rceil\right\}$.

Proof If t is even and $j \geq \frac{t}{2}$, then there are enough colors such that the vertices are colored pairwise differently, meaning that $p = 2$. If t is even but $j < \frac{t}{2}$, we are forced to use every possible color, and every color has to be used at least twice. Thus, the restriction is automatically fulfilled and $p = \left\lceil \frac{t}{j} \right\rceil$.

Now, consider the case when t is odd. We first ignore one vertex and proceed as in the even case for the remaining $t - 1$ vertices. The ignored vertex then has to get the same color as one of the remaining vertices due to the additional restriction. That means

$$\text{if } j \geq \frac{t-1}{2}, \text{ then } p = 2 + 1 = 3, \text{ and}$$
$$\text{if } j < \frac{t-1}{2}, \text{ then } p = \left\lceil \frac{t-1}{j} \right\rceil + 1.$$

The $+1$ arises from the vertex that was first ignored. It is needed in the case where each color is used equally often. It is easy to verify that $\left\lceil \frac{t}{j} \right\rceil \geq \left\lceil \frac{t-1}{j} \right\rceil + 1$ for every possible pair of t and j. Therefore an upper bound for p is always $\max\left\{3, \left\lceil \frac{t}{j} \right\rceil\right\}$. □

Lemma 7.5.2. *Let T be a finite tree with maximal valence $0 < k < \infty$. If T is colored as described in the Coloring Algorithm where $c \geq 2$, then the largest number of sibling vertices fulfilling the distance condition and having the same color that can appear is* $\max\left\{3, \left\lceil \frac{k-2}{c-1} \right\rceil\right\}$.

Proof Within our Coloring Algorithm there are three situations in which sibling vertices with the same color fulfilling the distance condition might appear.

Situation 1: If we apply an optimal coloring to m vertices, we obtain a maximum of $\lceil \frac{m}{c} \rceil$ vertices with the same color. For a tree with maximal valence k it is bounded by $s_1 = \lceil \frac{k}{c} \rceil$.

Situation 2: In Step 4 case 2 of the Coloring Algorithm, we use an optimal coloring with $c-1$ colors such that each color appears at least twice. By applying Lemma 7.5.1, we obtain a maximum number $s_2 = \max\left\{3, \left\lceil \frac{k-2}{c-1} \right\rceil\right\}$ of sibling vertices with the same color in that case.

Situation 3: Consider the case $c = 2$ in Step 4 Case 1 of the algorithm where we are in the branching situation. There are at most $s_3 = \max\{3, \lceil \frac{k-1}{2} \rceil\}$ vertices with the same color. We see that $s_2 \geq s_3$ for $c = 2$ and $k \geq 4$.

It remains to compute the maximum of s_1 and s_2. Straight forward calculations show that

$$k < 2c \quad \Longleftrightarrow \quad \frac{k-2}{c-1} < \frac{k}{c} < 2.$$

Therefore, for $k < 2c$ the maximum is 3, while for $k \geq 2c$ it is $\frac{k-2}{c-1}$. Thus, the maximum of s_1 and s_2 is $\max\left\{3, \left\lceil \frac{k-2}{c-1} \right\rceil\right\}$. $\square$

Proof of the Main Theorem The cases $k \in \{0, 1, 2\}$ and $c \geq k$ are trivial. For $c = k-1$, we refer to Corollary 7.4.1 in Section 7.4. So, assume that we have $2 \leq c \leq k-2$ and $k \geq 4$.

First, we consider a finite tree T and apply the Coloring Algorithm. Assume that the center of T consists of a single vertex v. This vertex is fixed by each automorphism of T. Thus, it remains to show that for all $n \in \mathbb{N}$ the vertices with the same color in $S(n, v)$ which satisfy the distance condition are indistinguishable due to our algorithm.

The distance $r(c, k)$ is built upon the maximal number of indistinguishable vertices with the same color that can appear in the same sphere and the given Main Line Coloring in the algorithm. The aim of these main lines is to fix indistinguishable siblings that have the same color and the vertices on these main lines themselves.

Let $n' \in \mathbb{N}$ be the smallest index such that there exist indistinguishable sibling vertices with the same color $v_1, \ldots, v_m$, $m \geq 2$, in $S(n', v)$ which fulfill the distance condition.

Consider $v_1, \ldots, v_m$ and their main lines R_i given by Step 3 of the algorithm. We see that if each v_i has distance at least $\log_c m$ to a leaf in T_{v_i}, then the main lines R_i are colored pairwise differently. Due to Lemma 7.5.2 this distance is bounded by $\log_c \max\left\{3, \left\lceil \frac{k-2}{c-1} \right\rceil\right\}$. Since this coincides with our distance condition regarding the given $r(c, k)$, for any color-preserving automorphism α of T with $\alpha(v_i) = v_j$ and $\alpha(R_i) = R_j$ for some $i, j \in \{1, \ldots, m\}$, we see that $i = j$. This means that the main lines are not interchangeable.

Next we argue why a main line R_i cannot be mapped to *any* other string of some T_{v_j}. By a string we mean a path, that has at most one vertex of each sphere. Therefore we show that there is no automorphism α such that $\alpha(T_{v_i}) = T_{v_j}$ for any $i, j \in \{1, \ldots, m\}$.

Without loss of generality we consider T_{v_i} and T_{v_j}, $i \neq j$ and assume that there exists at least one vertex in R_i resp. R_j with at least two up-neighbors (otherwise T_{v_i} and T_{v_j} are stabilized by the Main Line Coloring method).

Case 1: There exists a vertex w_0 in R_i with one up-neighbor w_1 in R_i and at least two more up-neighbors, called $w_2, \ldots, w_\ell$, $\ell \geq 3$. Let us assume that there exists a color-preserving automorphism α of T that maps T_{v_j} to T_{v_i}. By the Main Line Coloring method we know that $\alpha(R_i) \neq \alpha(R_j)$. Thus, there exists a vertex $\tilde{w} \in R_j$ (in the same sphere as $w_1, \ldots, w_\ell$) and $k \in \{2, \ldots, \ell\}$ such that $\alpha(\tilde{w}) = w_k$. See Figure 7.9.

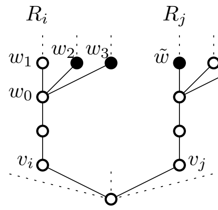


Figure 7.9: Example of w_0 in R_i with three up-neighbors. The vertex $\tilde{w}$ has the same color as w_2 and w_3.

Due to Step 4, Case 2, in the algorithm, $\tilde{w}$ does not have a sibling vertex with the same color, whereas w_k for sure has. So, α cannot map $\tilde{w}$ to w_k, and we have a contradiction. We conclude that α does not exists.

Case 2: There exists a vertex w_0 in R_i with one up-neighbor w_1 in R_j and exactly one additional up-neighbor, called w_2.

Let us again assume that there exists a color-preserving automorphism α of T that maps T_{v_j} to T_{v_i}. By main line coloring we know that $\alpha(w_1) \notin R_j$. Thus there exits a vertex $\tilde{w} \in R_j$ such that $\alpha(w_2) = \tilde{w}$.

Case 2.1: Let $c \geq 3$. If α swaps $\tilde{w}$ and w_2, then α also swaps w_1 and the unique sibling of $\tilde{w}$. But if $\tilde{w}$ and w_2 have the same color, their siblings do not, because of the shifting of the colors modulo c in Case 1 of Step 4 of the algorithm. Thus, such an α does not exist. Note, here it is important that we assume that $c \geq 3$. For $c = 2$, we only have the color pairs $(0, 1)$ and $(1, 0)$ which cannot be distinguished.

Case 2.2: Let $c = 2$. If the subtree T_{w_2} contains only vertices of valence one or two (no branching), all vertices of T_{w_2} (except maybe w_2) are colored with 1. In contrast to this, at least the last vertex of R_j (the leaf) is colored with 0. This is due to the given $r(2, k)$, see Figure 7.8 (here we need the +1 in the case where $r(c, k) = \log_2(\max\{3, k - 2\}) + 1$). Thus, such an α does not exist.

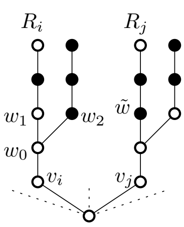


Figure 7.10: The distance $r(2, k)$ guarantees that the last vertex of a main line is always white if $c = 2$.

Now, assume there is a branching in T_{w_2}. By our algorithm, we have avoided that there exist single vertices of a certain color in such a branching (see the left and the right picture in Figure 7.7) while in every branching in the main line of R_j there exist a vertex (the vertex in the main line itself) with color d that is the only vertex with this color in that branching[3].

All in all, we showed that $v_1, \ldots, v_m$ are fixed. Now, we can use a final inductive argument. Assume that all vertices with the same color up to the sphere $S(n, v)$ are fixed. Consider indistinguishable sibling vertices that fulfill the distance condition in the sphere $S(n+1, v)$. At this point, we can apply the same argument as above to ensure that they have to be fixed. Note that here it is important to assume that everything below these vertices is already fixed.

If the center of T consists of an edge uv, color u with 0 and v with 1. Then u and v are fixed by the properties of a tree and we can apply the Coloring Algorithm to the remaining vertices in the two subtrees containing u and v after removing the edge uv rooted in, respectively, u and v. Then, we use the same reasoning as above. This completes the proof for the finite case.

Now, let T be an infinite, locally finite tree. We first assume that T has at least two ends. In that case consider the trunk T^C of T which is a locally finite, infinite tree without leaves. Since it is unique any automorphism of T leaves T^C invariant. By [84] we know that T^C is 2-distinguishable. Now, for every vertex u in the trunk we consider the subtree T_u that contains, as explained above, the vertex u together with all finite subtrees of $T-u$. We apply the Coloring Algorithm to each of these subtrees and fix all vertices which fulfill the distance condition.

If T has only one end, then there exists a vertex v with valence 1. Let R be the ray with root v. We color all vertices of R with 0. For each vertex in R we color the maximal $k-2$ neighbors not in R with an optimal coloring with the $c-1$ colors different from 0. Thus, there are at most $\lceil \frac{k-2}{c-1} \rceil$ vertices with the same color. If there are indistinguishable vertices we apply the Coloring Algorithm from above to fix all vertices which fulfill the distance condition. □

We end with an example that shows that the given constant $r(c, k)$ in (7.1) is tight in some special cases.

Example 7.5.3. Consider a tree as in Figure 7.11 with maximal valence $k = 10$, which we would like to color with $c = 3$ colors. Without a coloring the vertices in the first sphere are indistinguishable. Using an optimal coloring for these vertices there are $\lceil \frac{10}{3} \rceil = 4$ vertices $v_1, \ldots, v_4$ with the same color. Thus in our algorithm they are starting points of main lines of length two and we can distinguish them in that way, see Figure 7.11. Clearly, it is not possible to distinguish $v_1, \ldots, v_4$ by using only 3-colors in the next sphere, but we can fix them by coloring all vertices of the next two spheres. That is what the upper bound $r(3, 10) = \log_3 \max\{3, \frac{10-2}{2}\} \approx 1.26$ yields, which means we need at least distance 2.

One easily deduces that the number of vertices which can be distinguished by the given coloring in Theorem 7.5 depends on the structure of the graph. Especially it depends on the number and the distribution of the leaves. Of course there are enough cases where it is possible to distinguish considerably more vertices through a coloring than provided by our theorem (e.g. if a lot of vertices are fixed by the structure of the tree). But for

[3]Coloring the neighbors of a vertex in R_j we excluded the use of the color of this particular vertex.

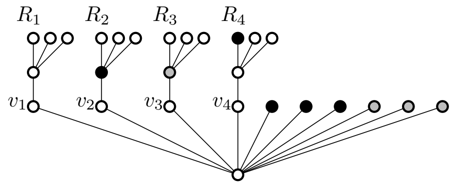


Figure 7.11: Example of a 3-colored tree with maximal valence $k = 10$.

a general result one has to consider the critical cases as in Example 7.5.3 where we have seen that the bound is tight.

We always considered the center of a finite tree as a fixed starting vertex in our Coloring Algorithm. But one also could take another vertex of the graph, that is fixed by all automorphisms. In some cases this may lead to colorings, that fix more vertices of the graph than in the case where we start the coloring algorithm in the center.

Bibliography

[1] M. O. Albertsons and K. L. Collins. Symmetry breaking in graphs. *Electronic Journal Combinatorics*, 1996.

[2] G. K. Alexopoulos. Random walks on discrete groups of polynomial volume growth. *Ann. Probab.*, 30(2):723–801, 2002.

[3] F. J. Anscombe. Large-sample theory of sequential estimation. *Proc. Cambridge Philos. Soc.*, 48:600–607, 1952.

[4] S. Asmussen. *Applied Probability and Queues.* Springer, 2000.

[5] G. Atkinson. Recurrence of co-cycles and random walks. *J. London Math. Soc.*, 13:486–488, 1976.

[6] L. Babai. Asymmetric trees with two prescribed valences. *Acta Math. Acad. Sci. Hungar.*, 29:193–200, 1977.

[7] M. Babillot. Théorie du renouvellement pour des chaînes semi-markoviennes transientes. *Ann. Inst. H. Poincaré (4)*, 24:507–569, 1988.

[8] M. Babillot, Ph. Bougerol, and L. Elie. The random difference equation $x_n = a_n x_{n-1} + b_n$ in the critical case. *Ann. Probab.*, 25:478–493, 1997.

[9] R. F. Bailey and P. J. Cameron. Base size, metric dimension and other iinvariant of groups and graphs. *Bull. Lond. Math. Soc.*, 43(2):209–242, 2011.

[10] M. Benda. *Schwach kontraktive dynamische Systeme.* PhD thesis, Ludwig-Maximilians-Universität München, 1998.

[11] M. Benda. Contractive stochastic dynamical systems. 1999. unpublished.

[12] M. Benda. A reflected random walk on the half line. 1999. unpublished.

[13] A. Bendikov and W. Cygan. Alpha-stable random walk has massive thorns. *Colloq. Math.*, 138:105–129, 2015.

[14] A. Bendikov, W. Cygan, and B. Trojan. Limit theorems for random walks. 2017. to appear in *Stochastic Process. Appl.*, arXiv:1504.01759.

[15] A. Bendikov and L. Saloff-Coste. Random walks on groups and discrete subordination. *Math. Nachr.*, 285:580–605, 2012.

[16] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular Variation*. Cambridge University Press, 1987.

[17] A. A. Borovkov. *Stochastic Processes in Queueing Theory*. Springer-Verlag, New York-Berlin, 1976.

[18] M. A. Boudiba. La chaîne de feller $x_{n+1} = |x_n - y_{n+1}|$ où les $(y_n)_{n\geq 1}$ sont indépendantes et identiquement distribuées. *C. R. Acad. Sc. Paris Sér. I Math.*, 301:517–519, 1985.

[19] M. A. Boudiba. La chaîne de Feller $X_{n+1} = |X_n - Y_{n+1}|$ et les chaînes associées. *Ann. Sci. Univ. Clermont-Ferrand II Probab. Appl.*, 5:91–132, 1986.

[20] D. Boutin. The cost of 2-distinguishing cartesian powers. *Electron. J. Combin.*, 20, 2013.

[21] D. Boutin and W. Imrich. The cost of distinguishing graphs. *London Math. Society, Lecture Notes Series 436, Cambridge University Press*, 2017.

[22] M. Bramson, J. G. Dai, and J. M. Harrison. Positive recurrence of reflecting brownian motion in three dimensions. *Ann. Appl. Probab*, 20:753–783, 2010.

[23] I. Broere and M. Pilśniak. The distinguishing index of infinite graphs. *Electron. J. Combin.*, 22, 2015.

[24] P. J. Cameron, P. M. Neumann, and J. Saxl. On groups with no regular orbits on the set of subsets. *Arch. Math. (Basel)*, 43(4):295–296, 1984.

[25] K. L. Chung and W. H. J. Fuchs. On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc.*, No. 6:12, 1951.

[26] K. L. Collins and A. N. Trenk. The distinguishing chromatic number. *Electron. J. Combin.*, 13, 2006.

[27] M. Conder and T. Tucker. The motion lemma and distinguishability two. *Ars Mathematica Contemporanea*, 4:63–72, 2011.

[28] W. Cygan and J. Kloas. On recurrence of the multidimensional lindley process. 2017. submitted, arxiv.org/pdf/1707.01714.

[29] P. Diaconis and D. Freedman. Iterated random functions. *SIAM Rev.*, 41(1):45–76, 1999.

[30] R. Diestel. *Graph Theor.* Springer, 3 edition, 2006.

[31] R. A. Doney. On the exact asymptotic behaviour of the distribution of ladder epochs. *Stochastic Process. Appl.*, 12(2):203–214, 1982.

[32] R. A. Doney. Spitzer's condition and ladder variables in random walks. *Probab. Theory Related Fields*, 101(4):577–580, 1995.

[33] R. Durrett. *Probability: Theory and Examples*, volume 3. Curt Hinrichs, 2005.

[34] M. S. Éppel'. A local limit theorem for the first passage time. *Sib. Math. J.*, 20(1):181–191, 1979.

[35] R. Essifi and M. Peigné. Return probabilities for the reflected random walk on $\mathbb{N}_0$. *J. Theoret. Probab.*, 28(1):231–258, 2015.

[36] R. Essifi, M. Peigné, and K. Raschel. Some aspects of fluctuations of random walks on $\mathbb{R}$ and applications to random walks on $\mathbb{R}^+$ with non-elastic reflection at 0. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10(2):591–607, 2013.

[37] R. Fayolle, G. Iasnogorodski and V. Malyshev. Random walks in the quarter plane. algebraic methods, boundary value problems and applications. *Springer*, 1999.

[38] W. Feller. *An Introduction to Probability Theory and its Applications*, volume II. Wiley, New York, 1971.

[39] S. Foss and B. Rogozin. The recurrence of oscillating random walk. *Theory of Probability and its Applications.*, 23, 1978.

[40] A. Garsia and J. Lamperti. A discrete renewal theorem with infinite mean. *Comment. Math. Helv.*, 37:221–234, 1962/1963.

[41] H. H. Glover and D. Marušičl. Hamiltonicity of cubic cayley graph. *J. Eur. Math. Soc*, 9:775–787, 2007.

[42] B. V. Gnedenko and A. N. Kolmogorov. *Limit Distributions for Sums of Random Variables.* Addison-Wesley publishing company, 1968.

[43] A. Gut. *Stopped Random Walks.* Springer, 1988.

[44] O. Hernández-Lerma and J. B. Lasserre. Markov chains and invariant probabilities. *Progress in Mathematics*, 211, 2003.

[45] D. G. Hobson and L. Rogers. Recurrence and transience of reflected brownian motion in the quadrant. *Matg. Proc. Cambridge Phil. Soc.*, 113:387–399, 1994.

[46] W. Imrich, S. Hüning, J. Kloas, H. Schreiber, and T.Tucker. Distinguishing graphs of maximum valence 3. 2017. submitted, https://arxiv.org/abs/1709.05797.

[47] W. Imrich, M. Kalinowski, R. andPilśniak, and M.H. Shekarriz. Bounds for distinguishing invariants of infinite graphs. *Electron. J. Combin.*, 24, 2017.

[48] W. Imrich, S. Klavžar, and V. Trofimov. Distinguishing infinite graphs. *Electron. J. Combin.*, 14, 2007.

[49] W. Imrich, T. Lachmann, T. Tucker, and G. Wiegel. The cost of distinguishing graphs of maximum valence 3. *Electron. J. Combin.*, 2017. in preparation.

[50] W. Imrich, S. M. Smith, T. Tucker, and M. E. Watkins. Infinite motion and 2-distinguishability of graphs and groups. *J. Algebraic Combin.*, 41:109–122, 2015.

[51] J. Jacod. Théorème de renouvellement et classification pour les chaînes semi-markoviennes. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 7:201–209, 1974.

[52] M. Jonckheere and S. Shneer. Stability of multi-dimensional birth-and-death processes with state-dependent 0-homogeneous jumps. *Adv. Appl. Probab*, 46:59–75, 2014.

[53] R. Kalinowski and M. Pilśniak. Distinguishing graphs by edge-colourings. *European J. Combin.*, 45:124–131, 2015.

[54] J. H. B. Kemperman. The oscillating random walk. *Stoch. Processes and their Applications*, 2:1–29, 1974.

[55] D. G. Kendall. Some problems in the theory of queues. *J. Roy. Statist. Soc. Ser. B.*, 13:151–173; discussion: 173–185, 1951.

[56] J. Kiefer and J. Wolfowitz. On the theory of queues with many servers. *Trans. Amer. Math. Soc.*, 78:1–18, 1955.

[57] S. Klavžar, T.-L. Wong, and X. Zhu. Distinguishing labelings of group action on vector spaces and graphs. *J. Algebra*, 303:626–641, 2006.

[58] J. Kloas and W. Woess. Multidimensional random walk with reflections. 2016. preprint, arXiv:1704.06055v1.

[59] F. B. Knight. On the absolute difference chains. *Z. Wahrsch. Verw. Gebiete*, 43(1):57–63, 1978.

[60] A. Krámli and D. Szász. Random walks with internal degrees of freedom. *Z. Wahrsch. verw. Gebiete*, 63:85–95, 1983.

[61] S. Lalley. Lecture notes - One Dimensional Random Walks. *The University of Chicago*, 2010.

[62] G. F. Lawler and V. Limic. *Random Walk: A Modern Introduction.* Cambridge Studies in Advanced Mathematics, 2010.

[63] J. P. Leguesdron. Marche aléatoire sur le semi-groupe des contractions de $\mathbb{R}^d$. Cas de la marche aléatoire sur $\mathbb{R}_+$ avec choc élastique en zéro. *Ann. Inst. H. Poincaré Probab. Statist.*, 25(4):483–502, 1989.

[64] F. Lehner. Distinguishing graphs with intermediate growth. *Combinatorica*, 36:333–347, 2016.

[65] D. V. Lindley. The theory of queues with a single server. *Proc. Cambridge Philos Soc.*, 48:277–289, 1952.

[66] M. Peigné. Marches de Markov sur le semi-groupe des contractions de $\mathbf{R}^d$. Cas de la marche aléatoire à pas markoviens sur $(\mathbf{R}^+)^d$ avec chocs élastiques sur les axes. *Ann. Inst. H. Poincaré Probab. Statist.*, 28(1):63–94, 1992.

[67] M. Peigné and W. Woess. On Recurrence of Reflected Random Walk on the Half-line. 2008. unpublished manuscript, arXiv:0612306v1.

[68] M. Peigné and W. Woess. Stochastic dynamical systems with weak contractivity properties I. Strong and local contractivity. *Colloq. Math.*, 125(1):31–54, 2011.

[69] M. Peigné and W. Woess. Stochastic dynamical systems with weak contractivity properties II. Iteration of Lipschitz mappings. *Colloq. Math.*, 125(1):55–81, 2011.

[70] E. J. G. Pitman. On the behavior of the characteristic function of a probability distribution in the neighborhood of the origin. *J. Austral. Math. Soc.*, 8:423–443, 1968.

[71] T. J. Rabeherimanana. Résultats sur l'irréductibilité et la conservativité de la chaîne de différences absolues. *Ann. Fac. Sci. Toulouse Math. (6)*, 11(2):177–199, 2002.

[72] A. Rényi. On the asymptotic distribution of the sum of a random number of independent random variables. *Acta Math. Acad. Sci. Hungar.*, 8:193–199, 1957.

[73] A. Russell and R. Sundaram. A note on the asymptotics and computational complexity of graph distinguishability. *Electron. J. Combin.*, 5, 1998.

[74] K. Schmidt. On joint recurrence. *C. R. Acad. Sci. Paris Sér. I Math.*, 327:837–842, 1998.

[75] S. N. Smirnov. The recurrence of an absolute difference chain. *Mat. Zametki*, 42(2):336–342, 347, 1987.

[76] S. M. Smith, T. Tucker, and M. E. Watkins. Distinguishing maps. *Electron. J. Combin.*, 19, 2012.

[77] F. Spitzer. *Principles of Random Walk*. Springer, 1976.

[78] H. G. Tucker. On moments of distribution functions attracted to stable laws. *Houston J. Math.*, 1(1):149–152, 1975.

[79] T. Tucker. Distinguishing maps. *Electron. J. Combin.*, 18, 2011.

[80] K. Uchiyama. A note on summability of ladder heights and the distributions of ladder epochs for random walks. *Stochastic Process. Appl.*, 121(9):1938–1961, 2011.

[81] V. A. Vatutin and V. Wachtel. Local limit theorem for ladder epochs. 2007. unpublished manuscript, arXiv:0701914.

[82] V. A. Vatutin and V. Wachtel. Local probabilities for random walks conditioned to stay positive. *Probab. Theory Relat. Fields*, 143:177–217, 2009.

[83] H. von Schelling. Über die Verteilung der Kopplungswerte in gekreuzten Fernmeldekabeln großer Länge. *Eltektrische Nachrichten-Technik 20*, 1943.

[84] M. E. Watkins and X. Zhou. Distinguishability of locally finite trees. *Electron. J. Combin.*, (14), 2007.

[85] W. Woess. *Random Walks on Infinitv Graphs and Groups.* Cambridge University Press, 2000.

[86] W. Woess. *Denumerable Markov Chains.* European Mathematical Society, 2009.