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Design and Analysis of Robust Homogeneous Control Systems

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Affidavit

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Abstract

This thesis proposes a new design algorithm for nonlinear state observers for linear time-invariant systems. The approach is based on the assignment of real homogeneous eigenvalues to the estimation error dynamics. The result is a design formula which can be regarded as a generalization of Ackermann's eigenvalue assignment. The choice of the homogeneous eigenvalues involves necessary conditions for asymptotic stability of the estimation error dynamics. The presented method allows an arbitrary selection of the homogeneity degree and yields conditions for the choice of the homogeneity weights. Depending on the particular choice of the homogeneity degree and the weights the obtained observer is either linear or nonlinear. The approach includes the Luenberger observer as well as a sliding mode based observer. For strongly observable systems with bounded unknown input the approach also enables the construction of a robust observer. Furthermore, an inequality condition for robustness in terms of homogeneous eigenvalues is presented. A tutorial example illustrates the algorithms applicability and numerical simulations demonstrate the effectiveness of the proposed approach.

Kurzfassung

Diese Arbeit präsentiert einen neuen Entwurfsalgorithmus für nichtlineare Zustandsbeobachter für lineare zeitinvariante Systeme. Dieser basiert auf der Vorgabe von reellen homogenen Eigenwerten der Schätzfehlerdynamik. Das Ergebnis ist eine Entwurfsformel, welche als Verallgemeinerung der Eigenwertvorgabe nach Ackermann angesehen werden kann. Die Wahl der homogenen Eigenwerte wird dabei an ein notwendiges Stabilitätskriterium für die Schätzfehlerdynamik geknüpft. Die präsentierte Methode ermöglicht eine beliebige Vorgabe des Homogenitätsgrades und beinhaltet Bedingungen zur Wahl der Homogenitätsgewichte. Abhängig vom gewählten Homogenitätsgrad und den Gewichten ergibt sich ein linearer oder ein nichtlinearer Beobachter. Inkludierte Spezialfälle sind der Luenberger Beobachter und ein Sliding-Mode basierender Beobachter. Für stark beobachtbare Systeme mit einer beschränkten Störung am Eingang kann mithilfe dieses Algorithmus auch ein robuster Beobachter entworfen werden. Der Einfluss der homogenen Eigenwerte auf die Robustheit wird dabei durch eine Ungleichung beschrieben. Ein abschließendes Beispiel zeigt die Anwendbarkeit der vorgestellten Entwurfsmethode und numerische Simulationen demonstrieren deren Funktionsweise.

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1 Introduction

Linear time-invariant systems are the simplest and best-studied class of dynamic systems. The right-hand side of those systems offers the properties of additivity and homogeneity. Due to the linearity the stability of LTI systems is very simple to analyze, the computation of explicit solutions is easy and Laplace and frequency domain considerations deliver insight into the system behavior. However, LTI systems suffer from some essential drawbacks. Asymptotically stable LTI systems only provide an exponential convergence rate and, therefore, do not enable finite-time convergence. Moreover, they are not able to compensate for arbitrary unknown bounded disturbances.

The extension to homogeneous systems skips additivity and, therefore, also involves a special class of non-linear systems. A further generalization of the standard homogeneity known from linear systems leads to the weighted homogeneity proposed by V. I. Zubov [1] in 1958. This approach assigns real positive weights to the states and introduces the so-called homogeneity degree which yields an additional scaling of time as the result of a scaling of the trajectories.

Homogeneous systems provide many desirable properties. Homogeneous systems offer finite-time stability for negative homogeneity degree, see [2]. A specific choice of the weights and the homogeneity degree leads to a discontinuous right-hand side which enables the suppression of bounded perturbations [3]. This robustness property is strongly related to many sliding-mode based control strategies. Both, the twisting algorithm and the well-known super-twisting algorithm [4] are discontinuous homogeneous systems with negative homogeneity degree. Levant's robust exact differentiator [5], [6] is a further example of a higher order homogeneous system. A lot of work deals with deterministic control of uncertain homogeneous systems, see e.g. [7], [8], [9].

L. Rosier proved the existence of homogeneous Lyapunov functions for the stability of the origin of homogeneous systems with continuous right-hand side [10]. Many approaches in homogeneous system design rely on this theory. However, construction of Lyapunov functions may be a hard and cumbersome job. H. Nakamura et al. proposed the homogeneous eigenvalue approach [11] which generalizes the idea of eigenvalues and eigenvectors knwon from the linear case. Furthermore, necessary and sufficient stability criteria for the origin of homogeneous systems were presented [12].

The publications of H. Nakakura et al. utilize the homogeneous eigenvalues as a stability analysis tool for homogeneous systems. This thesis in contrast deals with the design of homogeneous state observers for LTI systems based on homogeneous eigenvalue assignment. First of all, basic properties of homogeneous systems are reviewed. Moreover, the concept of homogeneous eigenvalues and eigenvectors is summarized and necessary and sufficient stability criteria are shown. Finally, a homogeneous observer design algorithm for arbitrary homogeneity degree is derived. This yields a generalization of Ackermann's formula. For a special choice of the homogeneity degree and the homogeneity weights a robust observer is obtained and its functionality is shown using a simulation example.

2 Basics of homogeneous systems

Homogeneity describes an important property of dynamic systems. In this Chapter the basic terms and definitions are shown. First of all, an introduction is given by means of homogeneous scalar functions. A special type of homogeneity well-known from the definition of linearity is recalled. Furthermore, a more general approach is evolved introducing weighted homogeneity of arbitrary degree proposed by Rothschild and Stein [13]. The homogeneous norm which is a fundamental homogeneous function regarding homogeneous eigenvalue analysis is introduced. In the next step the notion of homogeneity is extended to vector fields. Finally the concept of homogeneous dynamical systems is explained, its properties are analyzed and examples are given.

2.1 Homogeneous functions

The considered mappings are assumed to be real-valued functions of n real variables, i.e.

$$V : \mathbb{R}^n \to \mathbb{R}. \tag{2.1}$$

2.1.1 Linearity

A real scalar function $V(x_1, x_2, \ldots, x_n)$ is called linear if it satisfies the conditions of additivity and homogeneity.

Additivity

The function is called additive if the relation

$$V(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) = V(x_1, x_2, \ldots, x_n) + V(y_1, y_2, \ldots, y_n) \tag{2.2}$$

holds for every x_i and $y_i \in \mathbb{R}$, $i = 1 \ldots n$. So the function evaluated at the sum of two different inputs equals the sum of the function outputs evaluated separately for each input.

Homogeneity

In order to analyze homogeneity each input variable is multiplied with a scalar factor $\varepsilon \in \mathbb{R}$. If the function V satisfies the property

$$V(\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_n) = \varepsilon \cdot V(x_1, x_2, \ldots, x_n) \tag{2.3}$$

it is called homogeneous. A multiplication of every input argument with ε scales the function output with the same factor.

2.1.2 Homogeneity of arbitrary degree

The type of homogeneity appearing in the context of linear functions refers to the special case of homogeneity of degree $q = 1$. A more general definition calls a function $V(x_1, x_2, \ldots, x_n)$ homogeneous of degree q if there exists $q \in \mathbb{R}$ such that the relationship

$$V(\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_n) = \varepsilon^q \cdot V(x_1, x_2, \ldots, x_n), \qquad \varepsilon > 0 \tag{2.4}$$

is valid [14]. A scaling of the arguments by ε causes a scaling of the function output by ε^q.

Examples:

- Consider $V(x_1, x_2, \ldots, x_n)$ to be a linear combination of the input arguments

  $$V(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{i=1}^{n} a_i x_i, \qquad a_i \in \mathbb{R} \quad \forall i. \tag{2.5}$$

  A scaling of the parameters leads to

  $$V(\varepsilon x_1, \varepsilon x_2, \ldots, \varepsilon x_n) = \sum_{i=1}^{n} a_i \varepsilon x_i = \varepsilon \sum_{i=1}^{n} a_i x_i = \varepsilon^1 \cdot V(x_1, x_2, \ldots, x_n) \tag{2.6}$$

  and for this reason V is homogeneous of degree $q = 1$. Furthermore, V in this example is additive too and, therefore, V is linear.
- Consider

  $$V(x_1, x_2) = -2x_1^2 x_2 + x_2^3. \tag{2.7}$$

  A multiplication of the arguments with ε results in

  $$V(\varepsilon x_1, \varepsilon x_2) = -2(\varepsilon x_1)^2 \varepsilon x_2 + (\varepsilon x_2)^3 = \varepsilon^3(-2x_1^2 x_2 + x_2^3) = \varepsilon^3 \cdot V(x_1, x_2). \tag{2.8}$$

  V is homogeneous of degree $q = 3$.
- The function

  $$V(x_1, x_2) = \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}} \tag{2.9}$$

  is homogeneous of degree $q = 1$ because

  $$V(\varepsilon x_1, \varepsilon x_2) = \frac{\varepsilon x_1 \varepsilon x_2}{\sqrt{(\varepsilon x_1)^2 + (\varepsilon x_2)^2}} = \frac{\varepsilon^2 x_1 x_2}{\varepsilon \sqrt{x_1^2 + x_2^2}} = \varepsilon^1 \cdot V(x_1, x_2). \tag{2.10}$$

  Obviously, the function is not linear although the homogeneity degree $q = 1$.

2.1.3 Weighted positive homogeneity

A further generalization proposed by V. I. Zubov [1] is achieved by assigning a real weight $r_i > 0$ to each input argument x_i. A function $V(x_1, x_2, \ldots, x_n)$ is called weighted homogeneous of degree q w.r.t. the weights $r_1, r_2, \ldots, r_n$ if

$$V(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \ldots, \varepsilon^{r_n}x_n) = \varepsilon^q \cdot V(x_1, x_2, \ldots, x_n), \qquad q \in \mathbb{R},\, \varepsilon \in \mathbb{R},\, r_i > 0 \quad \forall i \tag{2.11}$$

holds. Many publications (see e.g. [11], [7], [3]) use the even more general approach of weighted positive homogeneity which does not enforce the symmetry for positive and negative values of ε. Therefore, ε is restricted to positive real numbers. Then, a function $V(x_1, x_2, \ldots, x_n)$ is called weighted positively homogeneous of degree q w.r.t. the weights $r_1, r_2, \ldots, r_n$ if

$$V(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2, \ldots, \varepsilon^{r_n}x_n) = \varepsilon^q \cdot V(x_1, x_2, \ldots, x_n), \qquad q \in \mathbb{R},\, \varepsilon > 0,\, r_i > 0 \quad \forall i \tag{2.12}$$

is satisfied. This definition of homogeneity is in the focus of this thesis.

Example:

- In this example the function

  $$V(x_1, x_2) = x_1^3 x_2 - 3x_2^2 \tag{2.13}$$

  is considered. Weighted scaling of the input leads to

  $$\begin{aligned} V(\varepsilon^{r_1}x_1, \varepsilon^{r_2}x_2) &= (\varepsilon^{r_1}x_1)^3\varepsilon^{r_2}x_2 - 3(\varepsilon^{r_2}x_2)^2 = \varepsilon^{3r_1+r_2}x_1^3x_2 - 3\varepsilon^{2r_2}x_2^2 \stackrel{!}{=} \\ &\stackrel{!}{=} \varepsilon^q(x_1^3x_2 - 3x_2^2) = \varepsilon^q V(x_1, x_2). \end{aligned} \tag{2.14}$$

  A comparison of the coefficients results in

  $$\begin{aligned} 3r_1 + r_2 &= q \\ 2r_2 &= q \end{aligned} \tag{2.15}$$

  which obviously is an underdetermined system of linear equations. The choice $r_1 = 1$ induces $r_2 = 3$ and $q = 6$. Therefore, $V(x_1, x_2)$ is homogeneous of degree $q = 6$ w.r.t. the weights $r_1 = 1$ and $r_2 = 3$.
  On the other hand $r_1 = 2$, $r_2 = 6$ and $q = 12$ also is a valid solution of the system of equations (2.15) and, therefore, $V(x_1, x_2)$ is homogeneous of degree $q = 12$ w.r.t. the weights $r_1 = 2$ and $r_2 = 6$ too. For this reason the choice of the weights and the homogeneity degree is not unique.

A more compact notation used for example by H. Nakamura et al. [11] and Meigoli and Nikravesh [15] is achieved by the introduction of vectors. The input vector $\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}^T$ contains the input arguments x_i and the weights r_i are summarized in the vector $\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & \ldots & r_n \end{pmatrix}^T$.

The linear mapping $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} : \mathbb{R}^n \to \mathbb{R}^n$, where

$$\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x} = \begin{pmatrix} \varepsilon^{r_1}x_1 & \varepsilon^{r_2}x_2 & \ldots & \varepsilon^{r_n}x_n \end{pmatrix}^T, \qquad \varepsilon > 0,\, r_i > 0 \quad i = 1, 2 \ldots, n \tag{2.16}$$

is called a dilation of $\boldsymbol{x}$ w.r.t. ε and the dilation coefficients $\boldsymbol{r}$. The dilation can also be expressed by a multiplication with a diagonal matrix

$$\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x} = \boldsymbol{\Gamma}_{\boldsymbol{r}}(\varepsilon)\boldsymbol{x}, \tag{2.17}$$

with

$$\boldsymbol{\Gamma}_{\boldsymbol{r}}(\varepsilon) = \begin{pmatrix} \varepsilon^{r_1} & 0 & \dots & 0 \\ 0 & \varepsilon^{r_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \varepsilon^{r_n} \end{pmatrix}. \tag{2.18}$$

With this notation the definition of weighted homogeneity given in (2.11) can be reformulated to

$$V(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) = \varepsilon^q \cdot V(\boldsymbol{x}), \qquad q \in \mathbb{R},\, \varepsilon > 0,\, r_i > 0 \quad \forall i. \tag{2.19}$$

2.1.4 Homogeneous norm

A homogeneous norm is a homogeneous function that is frequently used e.g in homogeneous eigenvalue analysis, see Chapter 3. The so-called homogeneous p-norm, see [16], $\|\boldsymbol{x}\|_{\{\boldsymbol{r},p\}} : \mathbb{R}^n \to \mathbb{R}_0^+$ w.r.t. the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix}^T$ is defined by

$$\|\boldsymbol{x}\|_{\{\boldsymbol{r},p\}} = (|x_1|^{\frac{p}{r_1}} + |x_2|^{\frac{p}{r_2}} + \dots + |x_n|^{\frac{p}{r_n}})^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |x_i|^{\frac{p}{r_i}}\right)^{\frac{1}{p}}, \qquad p \in \mathbb{N}. \tag{2.20}$$

Analyzing the homogeneous norm w.r.t. homogeneity yields for $\varepsilon > 0$ yields

$$\begin{aligned} \|\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}\|_{\{\boldsymbol{r},p\}} &= (|\varepsilon^{r_1} x_1|^{\frac{p}{r_1}} + |\varepsilon^{r_2} x_2|^{\frac{p}{r_2}} + \dots + |\varepsilon^{r_n} x_n|^{\frac{p}{r_n}})^{\frac{1}{p}} = \\ &= (\varepsilon^p |x_1|^{\frac{p}{r_1}} + \varepsilon^p |x_2|^{\frac{p}{r_2}} + \dots + \varepsilon^p |x_n|^{\frac{p}{r_n}})^{\frac{1}{p}} = \\ &= \varepsilon(|x_1|^{\frac{p}{r_1}} + |x_2|^{\frac{p}{r_2}} + \dots + |x_n|^{\frac{p}{r_n}})^{\frac{1}{p}} = \\ &= \varepsilon \|\boldsymbol{x}\|_{\{\boldsymbol{r},p\}}. \end{aligned} \tag{2.21}$$

Therefore, the homogeneous norm $\|\boldsymbol{x}\|_{\{\boldsymbol{r},p\}}$ is a homogeneous function of degree $q = 1$ w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$.

The second important property of the homogeneous norm is the positive definiteness

$$\begin{aligned} \|\boldsymbol{x}\|_{\{\boldsymbol{r},p\}} &= 0 \qquad \text{for } \boldsymbol{x} = \boldsymbol{0}, \\ \|\boldsymbol{x}\|_{\{\boldsymbol{r},p\}} &> 0 \qquad \text{for } \boldsymbol{x} \neq \boldsymbol{0}. \end{aligned} \tag{2.22}$$

The homogeneous norm is zero if $\boldsymbol{x}$ equals the zero vector. For all other $\boldsymbol{x}$ the norm is strictly positive.

Although the homogeneous norm looks very similar to the well-known p-norm it does not fulfill the triangular inequality which is an axiom to be satisfied for norms in linear algebra [17].

2.2 Homogeneous vector fields

A further important concept in control engineering is the homogeneity of vector fields. The considered vector fields $\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) & f_2(\boldsymbol{x}) & \dots & f_n(\boldsymbol{x}) \end{pmatrix}^T$ are maps from a n-dimensional real vector space onto another n-dimensional real vector space, i.e.

$$\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n. \tag{2.23}$$

The vector field $\boldsymbol{f}(\boldsymbol{x})$ is called homogeneous of degree $q \in \mathbb{R}$ w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}$ if each scalar function $f_i(\boldsymbol{x})$ is homogeneous of degree $q + r_i$, i.e.

$$f_i(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}) = \varepsilon^{q+r_i} \cdot f_i(\boldsymbol{x}), \qquad \varepsilon > 0, r_i > 0 \quad \forall i. \tag{2.24}$$

Extracting ε^q from the single elements of the vector field $f_i(\boldsymbol{x})$ leads to

$$\boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}) = \varepsilon^q \cdot \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{f}(\boldsymbol{x}). \tag{2.25}$$

A dilation of the input causes a dilation of the vector field and a multiplication with ε^q.

Ambiguity of the homogeneity degree and the dilation coefficients

The homogeneity degree q and the dilation coefficients $\boldsymbol{r}$ are not uniquely determined. Let $\boldsymbol{f}(\boldsymbol{x})$ be homogeneous of degree q w.r.t. the dilation coefficients $\boldsymbol{r}$ which is characterized by

$$\boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}) = \varepsilon^q \cdot \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{f}(\boldsymbol{x}), \qquad \varepsilon > 0. \tag{2.26}$$

Substituting

$$\varepsilon = \hat{\varepsilon}^k, \qquad \hat{\varepsilon} > 0,\ k > 0 \tag{2.27}$$

into (2.26) gives

$$\boldsymbol{f}(\boldsymbol{\Delta}_{\hat{\varepsilon}^k}^{\boldsymbol{r}}\boldsymbol{x}) = \hat{\varepsilon}^{k\cdot q} \cdot \boldsymbol{\Delta}_{\hat{\varepsilon}^k}^{\boldsymbol{r}}\boldsymbol{f}(\boldsymbol{x}). \tag{2.28}$$

The dilation operator in equation (2.28) is rewritten using the relation

$$\boldsymbol{\Delta}_{\hat{\varepsilon}^k}^{\boldsymbol{r}}\boldsymbol{x} = \begin{pmatrix} \hat{\varepsilon}^{k\cdot r_1}x_1 \\ \hat{\varepsilon}^{k\cdot r_2}x_2 \\ \vdots \\ \hat{\varepsilon}^{k\cdot r_n}x_n \end{pmatrix} = \boldsymbol{\Delta}_{\hat{\varepsilon}}^{k\cdot\boldsymbol{r}}\boldsymbol{x} \tag{2.29}$$

which yields

$$\boldsymbol{f}(\boldsymbol{\Delta}_{\hat{\varepsilon}}^{k\cdot\boldsymbol{r}}\boldsymbol{x}) = \hat{\varepsilon}^{k\cdot q} \cdot \boldsymbol{\Delta}_{\hat{\varepsilon}}^{k\cdot\boldsymbol{r}}\boldsymbol{f}(\boldsymbol{x}). \tag{2.30}$$

Substituting

$$\hat{\boldsymbol{r}} = k \cdot \boldsymbol{r}, \qquad \hat{q} = k \cdot q \tag{2.31}$$

leads to

$$\boldsymbol{f}(\boldsymbol{\Delta}_{\hat{\varepsilon}}^{\hat{\boldsymbol{r}}}\boldsymbol{x}) = \hat{\varepsilon}^{\hat{q}} \cdot \boldsymbol{\Delta}_{\hat{\varepsilon}}^{\hat{\boldsymbol{r}}}\boldsymbol{f}(\boldsymbol{x}) \tag{2.32}$$

which is again the homogeneity condition of the vector field $\boldsymbol{f}(\boldsymbol{x})$. Therefore, $\boldsymbol{f}(\boldsymbol{x})$ is also homogeneous of degree $\hat{q} = k \cdot q$ w.r.t. the dilation coefficients $\hat{\boldsymbol{r}} = k \cdot \boldsymbol{r}$ for $k > 0$. Hence, the homogeneity degree q and the dilation coefficients $\boldsymbol{r}$ are defined uniquely except for a positive real scaling factor k.

Example:

- Consider the vector field $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} 2x_1x_2 - 5x_1^3 \\ -4x_1^4 \end{pmatrix} \tag{2.33}$$

A dilation of the input yields

$$\boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}) = \begin{pmatrix} 2\varepsilon^{r_1}x_1\varepsilon^{r_2}x_2 - 5(\varepsilon^{r_1}x_1)^3 \\ -4(\varepsilon^{r_1}x_1)^4 \end{pmatrix} = \begin{pmatrix} 2\varepsilon^{r_1+r_2}x_1x_2 - 5\varepsilon^{3r_1}x_1^3 \\ -4\varepsilon^{4r_1}x_1^4 \end{pmatrix} \stackrel{!}{=}$$
$$\stackrel{!}{=} \varepsilon^q \begin{pmatrix} 2\varepsilon^{r_1}x_1x_2 - 5\varepsilon^{r_1}x_1^3 \\ -4\varepsilon^{r_2}x_1^4 \end{pmatrix} = \varepsilon^q \begin{pmatrix} \varepsilon^{r_1} f_1(\boldsymbol{x}) \\ \varepsilon^{r_2} f_2(\boldsymbol{x}) \end{pmatrix}. \tag{2.34}$$

A comparison of the coefficients leads to the system of equations

$$\begin{aligned} r_1 + r_2 &= q + r_1 \\ 3r_1 &= q + r_1 \\ 4r_1 &= q + r_2. \end{aligned} \tag{2.35}$$

The first equation in (2.35) can be simplified to

$$r_2 = q \tag{2.36}$$

and the second one to

$$r_1 = \frac{q}{2}. \tag{2.37}$$

Inserting equation (2.36) and (2.37) into the third equation of (2.35) yields

$$4\frac{q}{2} = q + q \tag{2.38}$$

which holds independent of q. Hence, the system of equations is underdetermined. One solution is

$$q = 2, r_1 = 1 \text{ and } r_2 = 2. \tag{2.39}$$

Therefore, the given vector field $\boldsymbol{f}(\boldsymbol{x})$ is homogeneous of degree $q = 2$ w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ with dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$.

The solution $q = 4$ and $\boldsymbol{r} = \begin{pmatrix} 2 & 4 \end{pmatrix}^T$ is feasible too. Again the choice is not unique.

2.3 Homogeneous systems

In control theory homogeneous systems play a prominent role. Due to the homogeneity of a system one can infer that local properties are even preserved globally. The definition of homogeneous systems is reviewed and it is shown that a scaling of the initial state causes a scaling of the trajectory. Furthermore, it is illustrated how to find a corresponding homogeneous system of degree $\tilde{q} = 0$ w.r.t. a homogeneous system of arbitrary degree q and the equivalence of the trajectories is discussed. Finally, finite-time stability and finite-time blow-up in the context of homogeneous systems are reviewed.

2.3.1 Definitions and properties of solutions of homogeneous systems

The autonomous time-invariant homogeneous system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}) \tag{2.40}$$

is considered. A system (2.40) is called homogeneous of degree q w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ if the vector field $\boldsymbol{f}(\boldsymbol{x})$ on the right-hand side is homogeneous of degree q w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$, see [11].

Let $\boldsymbol{x}(t)$ be the solution of the differential equation (2.40) with initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$. The goal is to obtain further solutions for a dilation of the initial state

$$\boldsymbol{z}_0 = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_0. \tag{2.41}$$

The solution $\boldsymbol{z}(t)$ has to fulfill the differential equation (2.40) and must coincide with $\boldsymbol{x}(t)$ if $\varepsilon = 1$. Therefore, one might try

$$\boldsymbol{z} = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}. \tag{2.42}$$

Computing the derivative w.r.t. time yields

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{x}). \tag{2.43}$$

Due to the homogeneity property

$$\boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) = \varepsilon^q \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{x}) \Leftrightarrow \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{x}) = \varepsilon^{-q} \boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) \tag{2.44}$$

differential equation (2.43) further simplifies to

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \varepsilon^{-q} \boldsymbol{f}(\underbrace{\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}}_{\boldsymbol{z}}) = \varepsilon^{-q} \boldsymbol{f}(\boldsymbol{z}). \tag{2.45}$$

Dividing differential equation (2.45) by ε^{-q} results in

$$\frac{1}{\varepsilon^{-q}} \frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}(\varepsilon^{-q} t)} = \boldsymbol{f}(\boldsymbol{z}). \tag{2.46}$$

Substituting

$$\tau = \varepsilon^{-q} t \tag{2.47}$$

yields

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}\tau} = \boldsymbol{f}(\boldsymbol{z}). \tag{2.48}$$

Obviously $\boldsymbol{z}(\tau)$ fulfills the differential equation (2.40). From one known solution $\boldsymbol{x}(t)$ with initial state $\boldsymbol{x}_0$ further solutions with initial state $\boldsymbol{z}_0 = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_0$ can be derived by scaling the coordinates with $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ and the time t with ε^{-q}

$$(\boldsymbol{x}, t) \curvearrowright (\boldsymbol{z}, \tau) = (\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}, \varepsilon^{-q} t). \tag{2.49}$$

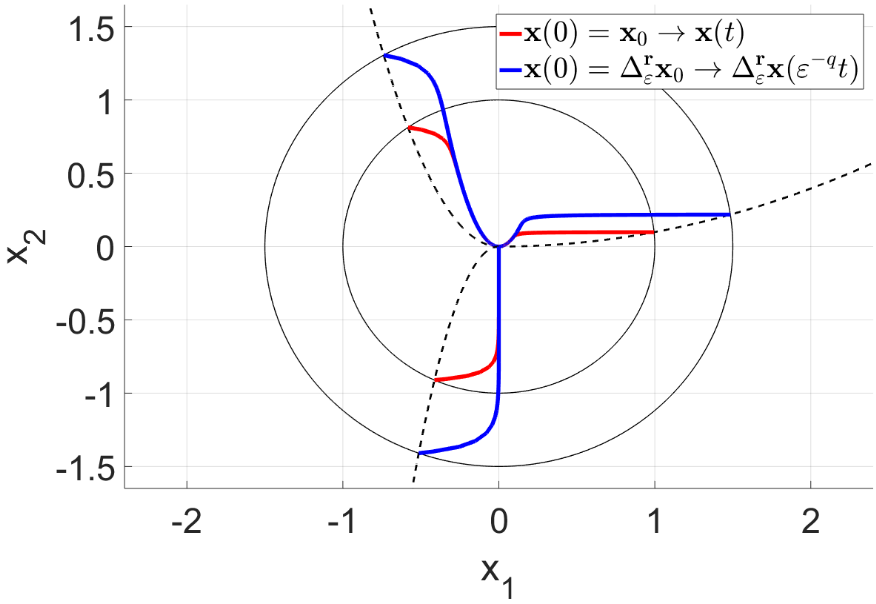

Figure 2.1: Trajectory plot of system (2.50) for initial states $\boldsymbol{x}_0$ on the unit circle and dilated initial states $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}_0$ located on the circle with radius 1.5. The dashed lines indicate all the possible dilations of the initial states.

The findings (2.49) allow to draw the conclusion that local properties of a homogeneous system are even preserved globally.

For example consider the equilibrium state $\boldsymbol{x} = \boldsymbol{0}$ of the homogeneous system (2.40) to be locally asymptotically stable. In other words, each trajectory starting sufficiently close to the origin converges to it as $t \to \infty$. The homogeneity of the system allows to create solutions with dilated initial states and arbitrary large values of $\varepsilon > 0$. All these solutions converge to the origin for $t \to \infty$ independent of the scaling variable ε. Hence, local stability implies global stability too.

Figure 2.1 shows a trajectory plot of the system

$$\begin{aligned} \dot{x}_1 &= -1.7x_1^2 \operatorname{sign}(x_1) + 0.6x_1\sqrt{|x_2|}\operatorname{sign}(x_2) \\ \dot{x}_2 &= -0.05|x_2|^{\frac{3}{2}} \operatorname{sign}(x_2) \end{aligned} \tag{2.50}$$

which is homogeneous of degree $q = 1$ w.r.t. the dilation coefficients $\boldsymbol{r} = \begin{pmatrix}1 & 2\end{pmatrix}^T$. The red trajectories are obtained for different initial states $\boldsymbol{x}_0$ distributed along the unit circle. The dashed lines indicate possible dilations of the initial states $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}_0$ for arbitrary $\varepsilon > 0$. Due to the dilation coefficients $\boldsymbol{r} = \begin{pmatrix}1 & 2\end{pmatrix}^T$ these dashed lines are parabolas in state space. The initial states of the blue trajectories are chosen in the intersection points of the dilation lines and the circle with radius 1.5. The solution curves are dilated versions of the red ones following relation (2.49). The blue trajectories converge to the origin because the red ones do so.

In nonlinear control theory it might be advantageous to design controllers such that the closed loop system is homogeneous. In this case, global stability is ensured if the origin is locally asymptotically stable.

2.3.2 Finding a corresponding homogeneous system of degree $\tilde{q} = 0$

The theory summarized in this Section was presented by H. Nakamura et al. [12]. The right-hand side of a homogeneous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{2.51}$$

of arbitrary degree q w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ can be decomposed into

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \zeta(\boldsymbol{x})\tilde{\boldsymbol{f}}(\boldsymbol{x}), \qquad \forall t \in \mathbb{R} : \boldsymbol{x}(t) \neq \boldsymbol{0}, \tag{2.52}$$

where $\zeta : \mathbb{R}^n \to \mathbb{R}$ is a scalar function and $\tilde{\boldsymbol{f}} : \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous vector field of degree $\tilde{q} = 0$ w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$.

A solution $\boldsymbol{x}(t)$ of system (2.51) is equal to a solution $\tilde{\boldsymbol{x}}(\tau)$ of the system

$$\frac{\mathrm{d}\tilde{\boldsymbol{x}}}{\mathrm{d}\tau} = \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}}) \tag{2.53}$$

if the variable $\tau = \tau(t)$ satisfies

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \zeta(\tilde{\boldsymbol{x}}). \tag{2.54}$$

This is shown by differentiating the solution $\boldsymbol{x}(t)$, i.e.

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \overset{(2.51)}{=} \boldsymbol{f}(\boldsymbol{x}) = \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\boldsymbol{x}}(\tau)) = \frac{\mathrm{d}\tilde{\boldsymbol{x}}}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} \underset{(2.54)}{\overset{(2.53)}{=}} \tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}})\zeta(\tilde{\boldsymbol{x}}). \tag{2.55}$$

Hence, the function $\zeta(\boldsymbol{x})$ may be regarded as a time scaling. In a trajectory plot the solutions of (2.51) and (2.53) for the same initial state $\boldsymbol{x}(t = 0) = \tilde{\boldsymbol{x}}(\tau = 0) = \boldsymbol{x}_0$ are identical because time is not visible in such a representation. Therefore, system (2.53) is called a corresponding system of homogeneity degree $\tilde{q} = 0$ of system (2.51) with equivalent trajectories.

Stability of the corresponding system of degree $\tilde{q} = 0$

The trajectories of the original system of homogeneity degree q and the corresponding system of homogeneity degree $\tilde{q} = 0$ coincide. For all initial states for those the solutions $\boldsymbol{x}(t)$ converge to an equilibrium point the solutions of $\tilde{\boldsymbol{x}}(t)$ converge to it too. Hence, both systems have the same stability properties.

Requirements concerning the time scaling function

First of all

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \zeta(\boldsymbol{x}) > 0, \qquad \forall \boldsymbol{x} \neq \boldsymbol{0} \tag{2.56}$$

has to hold to ensure that both time variables t and τ move into the same direction. Furthermore, $\zeta(\boldsymbol{x})$ has to be well-defined for all $\boldsymbol{x} \neq \boldsymbol{0}$ to ensure continuity of the solution $\tau(t)$.

A homogeneity analysis of the time scaling function yields

$$\boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) = \zeta(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) \tilde{\boldsymbol{f}}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}). \tag{2.57}$$

Exploitation of the homogeneity of $\boldsymbol{f}(\boldsymbol{x})$ and $\tilde{\boldsymbol{f}}(\boldsymbol{x})$ leads to

$$\varepsilon^q \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{x}) = \zeta(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \tilde{\boldsymbol{f}}(\boldsymbol{x}). \tag{2.58}$$

Again equation (2.52) can be used to decompose $\boldsymbol{f}(\boldsymbol{x})$

$$\varepsilon^q \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} (\zeta(\boldsymbol{x}) \tilde{\boldsymbol{f}}(\boldsymbol{x})) = \zeta(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \tilde{\boldsymbol{f}}(\boldsymbol{x}) \tag{2.59}$$

and $\zeta(\boldsymbol{x})$ can be pulled out of the dilation because it is scalar. A comparison of both sides of the resulting equation

$$\varepsilon^q \zeta(\boldsymbol{x}) \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \tilde{\boldsymbol{f}}(\boldsymbol{x}) = \zeta(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \tilde{\boldsymbol{f}}(\boldsymbol{x}) \tag{2.60}$$

leads to

$$\varepsilon^q \zeta(\boldsymbol{x}) = \zeta(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) \tag{2.61}$$

which exactly is a homogeneity condition for $\zeta(\boldsymbol{x})$.

Therefore, a valid time scaling function $\zeta(\boldsymbol{x})$

- has to be strictly positive $\zeta(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$ (additional zeros in $\zeta(\boldsymbol{x})$ cause some extra effects, see Section 4.1),
- must be well-defined for all $\boldsymbol{x} \neq \boldsymbol{0}$ (singularities cause some additional effects, see 2.3.5)
- and must be homogeneous of degree q w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$.

Every $\zeta(\boldsymbol{x})$ that meets these requirements is a valid time scaling function. Hence, the choice is not unique. A reasonable class of functions for $\zeta(\boldsymbol{x})$ are powers of homogeneous norms $\|\boldsymbol{x}\|_{\{\boldsymbol{r},p\}}^q$ explained in Section 2.1.4. They always fulfill the conditions.

2.3.3 Finite-time convergence and finite-time blow-up

Again a homogeneous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{2.62}$$

of homogeneity degree q w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ is examined.

If

- the origin of system (2.62) is asymptotically stable and
- the homogeneity degree is negative $q < 0$,

all solutions converge to the origin in finite time [2], [12]. Such a system is called finite-time stable.

On the other hand, if

- the origin of system (2.62) is unstable and
- the homogeneity degree is positive $q > 0$,

all unstable solutions blow up in finite time [12].

2.3.4 Example: Twisting Algorithm

Consider the twisting algorithm

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -k_1 \operatorname{sign}(x_1) - k_2 \operatorname{sign}(x_2) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{x}) \tag{2.63}$$

well-known from sliding mode based control which features a discontinuous right-hand side. Solutions of such systems are understood in the sense of Filippov [18]. The analysis of system (2.63) regarding homogeneity yields

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}) &= \begin{pmatrix} \varepsilon^{r_2} x_2 \\ -k_1 \operatorname{sign}(\varepsilon^{r_1} x_1) - k_2 \operatorname{sign}(\varepsilon^{r_2} x_2) \end{pmatrix} = \begin{pmatrix} \varepsilon^{r_2} x_2 \\ -k_1 \operatorname{sign}(x_1) - k_2 \operatorname{sign}(x_2) \end{pmatrix} \stackrel{!}{=} \\ &\stackrel{!}{=} \varepsilon^q \begin{pmatrix} \varepsilon^{r_1} x_2 \\ -k_1 \varepsilon^{r_2} \operatorname{sign}(x_1) - k_2 \varepsilon^{r_2} \operatorname{sign}(x_2) \end{pmatrix} = \varepsilon^q \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{x}) \end{aligned} \tag{2.64}$$

and results in the linear system of equations

$$\begin{aligned} r_2 &= q + r_1 \\ 0 &= q + r_2 \\ 0 &= q + r_2. \end{aligned} \tag{2.65}$$

The last two equations of (2.65) are equal and lead to

$$r_2 = -q. \tag{2.66}$$

Inserting equation (2.66) in the first equation of (2.65) results in

$$r_1 = -2q. \tag{2.67}$$

$q < 0$ has to hold because $r_i > 0$. The choice $q = -1$ leads to $\boldsymbol{r} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$. For this reason the twisting algorithm is homogeneous of degree $q = -1$ w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ with the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$.

2.3.5 Example: Super-Twisting Algorithm

Another sliding mode based control law is the super-twisting algorithm

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 - k_1 \sqrt{|x_1|}\,\mathrm{sign}(x_1) \\ -k_2\,\mathrm{sign}(x_1) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{x}). \tag{2.68}$$

Testing for homogeneity yields

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}) = \begin{pmatrix} \varepsilon^{r_2} x_2 - k_1 \sqrt{|\varepsilon^{r_1} x_1|}\,\mathrm{sign}(\varepsilon^{r_1} x_1) \\ -k_2\,\mathrm{sign}(\varepsilon^{r_1} x_1) \end{pmatrix} = \begin{pmatrix} \varepsilon^{r_2} x_2 - k_1 \varepsilon^{\frac{r_1}{2}} \sqrt{|x_1|}\,\mathrm{sign}(x_1) \\ -k_2\,\mathrm{sign}(x_1) \end{pmatrix} \overset{!}{=} \\ \overset{!}{=} \varepsilon^q \begin{pmatrix} \varepsilon^{r_1} x_2 - k_1 \varepsilon^{r_1} \sqrt{|x_1|}\,\mathrm{sign}(x_1) \\ -k_2 \varepsilon^{r_2}\,\mathrm{sign}(x_1) \end{pmatrix} = \varepsilon^q \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{x}). \end{aligned} \tag{2.69}$$

The linear system of equations

$$\begin{aligned} r_2 &= q + r_1 \\ \frac{r_1}{2} &= q + r_1 \\ 0 &= q + r_2 \end{aligned} \tag{2.70}$$

is obtained by a comparison of the coefficients. Again the third equation of (2.65) gives

$$r_2 = -q \tag{2.71}$$

and the second one solved for r_1 yields

$$r_1 = -2q. \tag{2.72}$$

Inserting equation (2.71) and (2.72) into the first equation of (2.70) leads to

$$-q = q + 2(-q) \tag{2.73}$$

which is always true. Choosing $q = -1$ produces $r_1 = 2$ and $r_2 = 1$. So the super-twisting algorithm is homogeneous of degree $q = -1$ w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ with the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$.

Corresponding system of homogeneity degree $\tilde{q} = 0$

The function

$$\zeta_1(\boldsymbol{x}) = |x_1|^{-\frac{1}{2}} \tag{2.74}$$

is homogeneous of degree $q = -1$ w.r.t. the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ and seems to be a suitable choice for the time scaling function because the system description stays simple. The corresponding system of degree $\tilde{q} = 0$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau_1} = \begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}\tau_1} \\ \frac{\mathrm{d}x_2}{\mathrm{d}\tau_1} \end{pmatrix} = \begin{pmatrix} |x_1|^{\frac{1}{2}} x_2 - k_1 x_1 \\ -k_2 |x_1|^{\frac{1}{2}}\,\mathrm{sign}(x_1) \end{pmatrix} = \tilde{\boldsymbol{f}}_1(\boldsymbol{x}). \tag{2.75}$$

possesses additional equilibrium points

$$\boldsymbol{x}_e = \begin{pmatrix} 0 \\ x_{2,e} \end{pmatrix}, \qquad x_{2,e} \in \mathbb{R} \tag{2.76}$$

located along the x_2-axis in state space. The problem is that $\zeta_1(\boldsymbol{x})$ is not defined for $x_1 = 0$. τ is the solution of

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}t} = \zeta_1(\boldsymbol{x}(t)) = |x_1(t)|^{-\frac{1}{2}}. \tag{2.77}$$

The case $x_1 = 0$ leads to discontinuities in the solution of τ even if $x_2 \neq 0$ and, therefore, time stands still. This causes the additional equilibrium points of system (2.75). Note that the equilibrium points created by the time scaling are always unstable due to the equivalence of trajectories.

It is possible to use $\zeta_1(\boldsymbol{x}) = |x_1|^{-\frac{1}{2}}$ but one has to mind the problems that may occur. A safe alternative choice instead would be

$$\zeta_2(\boldsymbol{x}) = \|\boldsymbol{x}\|^{q}_{\{\boldsymbol{r},p\}} = (|x_1|^{\frac{p}{2}} + |x_2|^{p})^{-\frac{1}{p}}, \qquad p \in \mathbb{N}. \tag{2.78}$$

The drawback is a more complicated description of the corresponding system of degree $\tilde{q} = 0$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau_2} = \begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}\tau_2} \\ \frac{\mathrm{d}x_2}{\mathrm{d}\tau_2} \end{pmatrix} = \begin{pmatrix} x_2(|x_1|^{\frac{p}{2}} + |x_2|^{p})^{\frac{1}{p}} - k_1\sqrt{|x_1|}(|x_1|^{\frac{p}{2}} + |x_2|^{p})^{\frac{1}{p}} \operatorname{sign}(x_1) \\ -k_2(|x_1|^{\frac{p}{2}} + |x_2|^{p})^{\frac{1}{p}} \operatorname{sign}(x_1) \end{pmatrix} = \boldsymbol{f}_2(\boldsymbol{x}). \tag{2.79}$$

3 Homogeneous eigenvalue analysis

In the theory of linear time-invariant systems linear eigenvalues and eigenvectors play a prominent role. Eigenvalues determine the stability of the system and are a measure for the speed of convergence or divergence, respectively. Many linear controller design methods are based on the properties of linear eigenvalues. A basic example is the linear state-feedback controller which aims to assign desired eigenvalues to the dynamic matrix of the closed-loop system. For homogeneous systems H. Nakamura et al. [11] introduced the so-called homogeneous eigenvalues and homogeneous eigenvectors. These homogeneous eigenvalues contain information about the stability of equilibrium points too. They offer the possibility of analyzing the stability properties without e.g. using Lyapunov's second method. This may be advantageous because constructing Lyapunov functions often is a very difficult job.

In this Chapter the significance of eigenvalues and eigenvectors in the context of linear systems is recapitulated. This concept is extended to the so-called point-wise eigenvalue approach. The drawback of this straightforward extension is the loss of the meaningful interpretation for the eigenvectors. Furthermore, the existing stability criteria are limited to a small class of non-linear systems and do not involve the property of homogeneity. Then, real homogeneous eigenvalues and eigenvectors are introduced and their properties are analyzed. Finally, the connections between homogeneous eigenvalues and the stability properties are summarized and examples are given.

3.1 Eigenvalues and eigenvectors of linear time-invariant systems

Consider the linear time-invariant system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{3.1}$$

with the state vector $\boldsymbol{x} \in \mathbb{R}^n$ and the constant system matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. The eigenvalues $\lambda_i \in \mathbb{C}$ and eigenvectors $\boldsymbol{v}_i \in \mathbb{C}^n$ of the matrix $\boldsymbol{A}$ are computed from

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i, \qquad i = 1, \ldots, n \tag{3.2}$$

which is called the eigenvalue equation. The multiplication of an eigenvector $\boldsymbol{v}_i$ with the system matrix $\boldsymbol{A}$ has the same effect as scaling the eigenvector by the factor λ_i. The solution for the eigenvector $\boldsymbol{v}_i$ is unique except for a real scaling factor. This means that the direction of the eigenvector is unique and every length except zero is valid. Therefore, every real eigenvector is a straight line through the origin in the state space.

3.1.1 Trajectories starting on real eigenvectors

If the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$ of a trajectory $\boldsymbol{x}(t)$ is located somewhere on a real eigenvector $\boldsymbol{v}_i$, the dynamical behavior (3.1) reduces to

$$\dot{\boldsymbol{x}} = \lambda_i \boldsymbol{x} \qquad \text{with } \lambda_i \in \mathbb{R}. \tag{3.3}$$

Hence, the direction of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are the same. As a consequence the trajectory of $\boldsymbol{x}(t)$ stays on the eigenvector $\boldsymbol{v}_i$ forever. Moreover, the solution of the reduced differential equation (3.3) system yields

$$\boldsymbol{x}(t) = \mathrm{e}^{\lambda_i t} \boldsymbol{x}_0, \qquad \forall t \geq 0. \tag{3.4}$$

Obviously, $\boldsymbol{x}(t)$ diverges along the direction of the eigenvector if $\lambda_i > 0$ and converges exponentially along the eigenvector to the origin, i.e. $\lim_{t \to \infty} \boldsymbol{x}(t) = \boldsymbol{0}$ if $\lambda_i < 0$. If $\lambda_i = 0$, then $\boldsymbol{x}(t) = \boldsymbol{x}_0$ remains constant over time. Hence, the eigenvalues λ_i are responsible for the stability of the system and also determine the speed of convergence and divergence, respectively.

3.1.2 Stability of LTI systems

For an LTI system with n linear independent real eigenvectors the solution for an arbitrary initial state $\boldsymbol{x}_0$, exploiting the superposition principle, is given by

$$\boldsymbol{x}(t) = \sum_{i=1}^{n} c_i \mathrm{e}^{\lambda_i t} \boldsymbol{v}_i. \tag{3.5}$$

The constants $c_i \in \mathbb{R}$ must satisfy

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 = \sum_{i=1}^{n} c_i \boldsymbol{v}_i \tag{3.6}$$

which results in the solution of a linear system of equations. From equation (3.5) it becomes obvious that the equilibrium point in the origin is exponentially stable if all the real eigenvalues λ_i are negative.

This concept can be easily extended to complex eigenvalues and systems with repeating eigenvalues which may result in less than n linear independent eigenvectors. An LTI system is asymptotically stable if and only if all the eigenvalues satisfy

$$\mathrm{Re}\{\lambda_i\} < 0 \qquad \forall i, \tag{3.7}$$

see [19]. Hence, the real part of every eigenvalue has to be negative.

3.2 Point-wise eigenvalues and eigenvectors

The straight-forward extension to the non-linear case are the so-called point-wise eigenvalues and eigenvectors. This concept has been analyzed by J. Medanic [20], [21] and has been applied for example by S. Koch and M. Reichhartinger [22]. The theory summarized below is taken from J. Medanic [20]. First of all the pseudo-linear system representation of non-linear systems is discussed and subsequently the point-wise eigenvalues and eigenvectors are introduced and their influence on the stability behavior is analyzed.

3.2.1 Pseudo-linear system representation

The considered system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0},\, \boldsymbol{x} \in \mathbb{R}^n \tag{3.8}$$

is non-linear and time-invariant with an equilibrium point in the origin. It is based on the structured representation also known as pseudo-linear system representation of non-linear systems which decomposes the right-hand side of the differential equation (3.8) into

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}(\boldsymbol{x})\boldsymbol{x} \tag{3.9}$$

which yields

$$\dot{\boldsymbol{x}} = \boldsymbol{M}(\boldsymbol{x})\boldsymbol{x}, \tag{3.10}$$

where $\boldsymbol{M}(\boldsymbol{x})$ denotes the $n \times n$-dimensional state-dependent system matrix. First of all it is obvious that the choice of $\boldsymbol{M}(\boldsymbol{x})$ is not unique for $n \geq 2$. Secondly it is necessary that $\boldsymbol{M}(\boldsymbol{x})$ is well-defined for all possible state vectors $\boldsymbol{x}$. For this reason there exists a class of systems for those a valid decomposition is not possible.

3.2.2 Equilibrium points in pseudo-linear system representation

An analysis of the equilibrium points in pseudo-linear system representation results in the non-linear system of equations

$$\boldsymbol{M}(\boldsymbol{x}_e)\boldsymbol{x}_e = \boldsymbol{0}. \tag{3.11}$$

Obviously the trivial solution $\boldsymbol{x}_e = \boldsymbol{0}$ yields one equilibrium point in the origin which is already assumed in differential equation (3.8). The state-dependent dynamic matrix $\boldsymbol{M}(\boldsymbol{x})$ can be represented by its column vectors $\boldsymbol{m}_i(\boldsymbol{x})$

$$\boldsymbol{M}(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{x}) & \boldsymbol{m}_2(\boldsymbol{x}) & \dots & \boldsymbol{m}_n(\boldsymbol{x}) \end{pmatrix} \qquad \boldsymbol{m}_i(\boldsymbol{x}) \in \mathbb{R}^{n\times 1},\, i = 1, \dots, n. \tag{3.12}$$

Furthermore, the state vector can be expressed by its components x_i which gives an alternative representation of the system of equations (3.11)

$$\begin{pmatrix} \boldsymbol{m}_1(\boldsymbol{x}_e) & \boldsymbol{m}_2(\boldsymbol{x}_e) & \dots & \boldsymbol{m}_n(\boldsymbol{x}_e) \end{pmatrix} \begin{pmatrix} x_{1,e} \\ x_{2,e} \\ \vdots \\ x_{n,e} \end{pmatrix} = \boldsymbol{m}_1(\boldsymbol{x}_e)x_{1,e} + \boldsymbol{m}_2(\boldsymbol{x}_e)x_{2,e} + \dots + \boldsymbol{m}_n(\boldsymbol{x}_e)x_{n,e} =$$

$$= \sum_{i=1}^{n} x_{i,e}\boldsymbol{m}_i(\boldsymbol{x}_e) = \boldsymbol{0}. \tag{3.13}$$

For additional equilibrium points $\boldsymbol{x}_e$ a linear combination of the column vectors $\boldsymbol{m}_i(\boldsymbol{x}_e)$ has to result in the zero vector. In consequence the column vectors $\boldsymbol{m}_i(\boldsymbol{x}_e)$ have to be linear dependent i.e.

$$\operatorname{rank}\{\boldsymbol{M}(\boldsymbol{x}_e)\} < n. \tag{3.14}$$

$\boldsymbol{M}(\boldsymbol{x}_e)$ is quadratic and, therefore, it has to be singular and all candidates for equilibrium points have to satisfy

$$\det(\boldsymbol{M}(\boldsymbol{x}_e)) = 0. \tag{3.15}$$

In consequence, there is only one single equilibrium point located in the origin if

$$\det(\boldsymbol{M}(\boldsymbol{x})) \neq 0, \qquad \forall \boldsymbol{x} \neq \boldsymbol{0}. \tag{3.16}$$

3.2.3 Definition and properties of point-wise eigenvalues

The simple idea is to examine the eigenvalues and eigenvectors of $\boldsymbol{M}(\boldsymbol{x})$ which yields the point-wise eigenvalue equation

$$\boldsymbol{M}(\boldsymbol{x})\boldsymbol{v}_i(\boldsymbol{x}) = \lambda_i(\boldsymbol{x})\boldsymbol{v}_i(\boldsymbol{x}). \tag{3.17}$$

Due to the state dependency of $\boldsymbol{M}(\boldsymbol{x})$ the point-wise eigenvalues $\lambda_i(\boldsymbol{x})$ and the single elements of the eigenvectors $\boldsymbol{v}_i(\boldsymbol{x})$ may be functions of the state vector too. Therefore, the computation of point-wise eigenvalues and eigenvectors means to symbolically determine the eigenvalues and eigenvectors of $\boldsymbol{M}(\boldsymbol{x})$.

From linear algebra it is known that the determinant of a matrix is the product of its eigenvalues

$$\det(\boldsymbol{M}(\boldsymbol{x})) = \prod_{i=1}^{n} \lambda_i(\boldsymbol{x}). \tag{3.18}$$

Therefore, the determinant is zero if at least one eigenvalue is zero. If all the point-wise eigenvalues are nonzero, also the determinant is nonzero. According to Section 3.2.2 there are no equilibrium points except the origin if the determinant of $\boldsymbol{M}(\boldsymbol{x})$ is not equal to zero for all possible $\boldsymbol{x} \neq \boldsymbol{0}$. In consequence there is only a single equilibrium point in the origin if the point-wise eigenvalues satisfy

$$\lambda_i(\boldsymbol{x}) \neq 0, \qquad i = 1, \ldots, n, \, \forall \boldsymbol{x} \neq \boldsymbol{0}. \tag{3.19}$$

The point-wise eigenvalues depend on the chosen pseudo-linear representation of the system. Therefore, different representations lead to different eigenvalues and eigenvectors. It is not sufficiently studied yet if there exist some invariances or meaningful relations between different representation schemes.

3.2.4 Significance regarding stability

It seems intuitive to suppose that it is sufficient for asymptotic stability for all initial states if there exists a pseudo-linear representation of the system for which the real part of all the eigenvalues is negative. A counter example by P. Tsiotras et al. [23] proves that this assumption is incorrect. Although this simple stability criterion can not be carried over from the linear case a more specific one is shown by J. Medanic [20]. It is assumed that the structured representation (3.10) of some non-linear time-invariant system leads to n different real point-wise eigenvalues $\lambda_i(\boldsymbol{x})$ and, therefore, n linear independent eigenvectors $\boldsymbol{v}_i(\boldsymbol{x})$. Furthermore, all the eigenvectors are assumed to be constant i.e. they do not depend on the state vector

$$\boldsymbol{v}_i(\boldsymbol{x}) = \boldsymbol{v}_i = const. \tag{3.20}$$

The eigenvalues are collected in the diagonal matrix

$$\boldsymbol{\Lambda}(\boldsymbol{x}) = \begin{pmatrix} \lambda_1(\boldsymbol{x}) & 0 & \dots & \dots & 0 \\ 0 & \lambda_2(\boldsymbol{x}) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda_{n-1}(\boldsymbol{x}) & 0 \\ 0 & \dots & \dots & 0 & \lambda_n(\boldsymbol{x}) \end{pmatrix} \tag{3.21}$$

and the eigenvectors are assembled in the columns of the matrix

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \end{pmatrix}. \tag{3.22}$$

Then, a similarity transformation of the state-dependent system matrix $\boldsymbol{M}(\boldsymbol{x})$ to diagonal form is

$$\boldsymbol{\Lambda}(\boldsymbol{x}) = \boldsymbol{P}^{-1}\boldsymbol{M}(\boldsymbol{x})\boldsymbol{P} \tag{3.23}$$

which rearranges to

$$\boldsymbol{M}(\boldsymbol{x}) = \boldsymbol{P}\boldsymbol{\Lambda}(\boldsymbol{x})\boldsymbol{P}^{-1}. \tag{3.24}$$

The Lyapunov function

$$V(\boldsymbol{x}) = \boldsymbol{x}^T(\boldsymbol{P}^{-1})^T\boldsymbol{P}^{-1}\boldsymbol{x} \tag{3.25}$$

can be used to analyze stability properties. Due to the transposition $V(\boldsymbol{x})$ can be modified to

$$\begin{aligned} V(\boldsymbol{x}) &= \boldsymbol{x}^T(\boldsymbol{P}^{-1})^T\boldsymbol{P}^{-1}\boldsymbol{x} = \\ &= (\boldsymbol{P}^{-1}\boldsymbol{x})^T\boldsymbol{P}^{-1}\boldsymbol{x}. \end{aligned} \tag{3.26}$$

Introducing the vector

$$\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{P}^{-1}\boldsymbol{x}, \qquad \boldsymbol{w}(\boldsymbol{x}) \in \mathbb{R}^n \tag{3.27}$$

allows to write

$$V(\boldsymbol{x}) = \boldsymbol{w}^T(\boldsymbol{x})\boldsymbol{w}(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^2(\boldsymbol{x}). \tag{3.28}$$

The matrix $\boldsymbol{P}^{-1}$ is regular because $\boldsymbol{P}$ is regular and, therefore, its column vectors are linear independent. This means that $\boldsymbol{w} = \boldsymbol{0}$ is only possible for $\boldsymbol{x} = \boldsymbol{0}$. Therefore,

$$V(\boldsymbol{x}) > 0, \quad \forall \boldsymbol{x} \neq \boldsymbol{0} \tag{3.29}$$

is satisfied i.e. $V(\boldsymbol{x})$ is globally positive definite.

The derivative w.r.t. time then computes to

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= 2\boldsymbol{x}^T(\boldsymbol{P}^{-1})^T\boldsymbol{P}^{-1}\dot{\boldsymbol{x}} = \\ &= 2\boldsymbol{x}^T(\boldsymbol{P}^{-1})^T\boldsymbol{P}^{-1}\boldsymbol{M}(\boldsymbol{x})\boldsymbol{x}. \end{aligned} \tag{3.30}$$

Substituting $\boldsymbol{M}(\boldsymbol{x})$ by (3.24) further simplifies the equation to

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= 2\boldsymbol{x}^T(\boldsymbol{P}^{-1})^T\underbrace{\boldsymbol{P}^{-1}\boldsymbol{P}}_{\boldsymbol{I}}\boldsymbol{\Lambda}(\boldsymbol{x})\boldsymbol{P}^{-1}\boldsymbol{x} = \\ &= 2\boldsymbol{x}^T(\boldsymbol{P}^{-1})^T\boldsymbol{\Lambda}(\boldsymbol{x})\boldsymbol{P}^{-1}\boldsymbol{x} = \\ &= 2\boldsymbol{x}^T(\boldsymbol{P}\boldsymbol{\Lambda}^{-1}(\boldsymbol{x})\boldsymbol{P}^T)^{-1}\boldsymbol{x}. \end{aligned} \tag{3.31}$$

The remaining task is to find conditions regarding the eigenvalues in $\boldsymbol{\Lambda}(\boldsymbol{x})$ to ensure that $(\boldsymbol{P}\boldsymbol{\Lambda}^{-1}(\boldsymbol{x})\boldsymbol{P}^T)^{-1}$ is negative definite. The inversion does not have any influence because the inverse of a negative/positive definite matrix is again negative/positive definite, see e.g. [24]. Hence, it is sufficient to show that

$$\boldsymbol{P}\boldsymbol{\Lambda}^{-1}(\boldsymbol{x})\boldsymbol{P}^T \prec 0 \tag{3.32}$$

holds. It is advantageous to rearrange the product of matrices to

$$\begin{aligned} \boldsymbol{P}\boldsymbol{\Lambda}^{-1}(\boldsymbol{x})\boldsymbol{P}^T &= \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_1(\boldsymbol{x})} & & & \\ & \frac{1}{\lambda_2(\boldsymbol{x})} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_n(\boldsymbol{x})} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1^T \\ \boldsymbol{v}_2^T \\ \vdots \\ \boldsymbol{v}_n^T \end{pmatrix} = \\ &= \frac{1}{\lambda_1(\boldsymbol{x})}\boldsymbol{v}_1\boldsymbol{v}_1^T + \frac{1}{\lambda_2(\boldsymbol{x})}\boldsymbol{v}_2\boldsymbol{v}_2^T + \dots + \frac{1}{\lambda_n(\boldsymbol{x})}\boldsymbol{v}_n\boldsymbol{v}_n^T = \\ &= \sum_{i=1}^{n} \frac{1}{\lambda_i(\boldsymbol{x})}\boldsymbol{v}_i\boldsymbol{v}_i^T. \end{aligned} \tag{3.33}$$

The quadratic form of this matrix has to be negative

$$\begin{aligned}\boldsymbol{x}^T \boldsymbol{P}\boldsymbol{\Lambda}^{-1}(\boldsymbol{x})\boldsymbol{P}^T\boldsymbol{x} &= \boldsymbol{x}^T\left(\sum_{i=1}^{n}\frac{1}{\lambda_i(\boldsymbol{x})}\boldsymbol{v}_i\boldsymbol{v}_i^T\right)\boldsymbol{x} = \\ &= \sum_{i=1}^{n}\frac{1}{\lambda_i(\boldsymbol{x})}\boldsymbol{x}^T\boldsymbol{v}_i\boldsymbol{v}_i^T\boldsymbol{x} = \\ &= \sum_{i=1}^{n}\frac{1}{\lambda_i(\boldsymbol{x})}(\boldsymbol{x}^T\boldsymbol{v}_i)^2 \overset{!}{<} 0, \qquad \forall \boldsymbol{x} \neq \boldsymbol{0}.\end{aligned} \tag{3.34}$$

The sum can never be zero for $\boldsymbol{x} \neq \boldsymbol{0}$ because all the scalar products $\boldsymbol{x}^T\boldsymbol{v}_i$ would have to be zero. $\boldsymbol{x}$ would have to be orthogonal on every single eigenvector $\boldsymbol{v}_i$ which is in fact not possible due to the assumption that there are n linear independent eigenvectors which span the whole $\mathbb{R}^n$. This means condition (3.34) is satisfied if

$$\lambda_i(\boldsymbol{x}) \overset{!}{<} 0, \qquad i = 1, \dots, n, \ \forall \boldsymbol{x} \neq \boldsymbol{0}. \tag{3.35}$$

Obviously, the sign of the point-wise eigenvalues determines whether $\dot{V}(\boldsymbol{x})$ is negative definite. If the point-wise eigenvectors are constant and the point-wise eigenvalues are negative, $V(\boldsymbol{x})$ is a Lyapunov function.

This concept can be easily extended for complex and repeated point-wise eigenvalues [20]. In summary the equilibrium point in the origin of system (3.8) is globally asymptotically stable if there exists a valid pseudo-linear representation for which

- the point-wise eigenvectors $\boldsymbol{v}_i$ are constant, i.e.

$$\boldsymbol{v}_i(\boldsymbol{x}) = \boldsymbol{v}_i = const. \tag{3.36}$$

  and
- the real part of the point-wise eigenvalues $\lambda_i(\boldsymbol{x})$ is negative for all values of $\boldsymbol{x} \neq \boldsymbol{0}$, i.e.

$$\mathrm{Re}\{\lambda_i(\boldsymbol{x})\} < 0, \qquad i = 1, \dots, n, \ \forall \boldsymbol{x} \neq \boldsymbol{0}. \tag{3.37}$$

Despite this sufficient stability criterion for non-linear systems the point-wise eigenvalue approach has some fundamental drawbacks. First of all it is restricted to constant eigenvectors which may be a difficult and limiting property when designing controllers based on point-wise eigenvalue assignment. Furthermore, the point-wise eigenvectors are no longer solution curves of the differential equation (3.8). Trajectories starting on an eigenvector do not necessarily stay on it forever. Finally the point-wise eigenvalue concept does not take advantage of the homogeneity of systems which is in the focus of this thesis.

3.3 Homogeneous eigenvalues and eigenvectors

Homogeneous eigenvalues and eigenvectors adopt and generalize many properties of the linear case for non-linear but homogeneous systems. First of all the Euler vector field and homogeneous rays are introduced. Furthermore, projections of trajectories onto a so-called Euler sphere are examined and the projection system is established. With this background the real homogeneous eigenvalues proposed by H. Nakamura et al. [11] are analyzed.

3.3.1 Euler vector field and homogeneous rays

The Euler vector field $\boldsymbol{\nu}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^n$, see [16], corresponding to the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix}^T$ is defined by the linear mapping

$$\boldsymbol{\nu}(\boldsymbol{x}) = \begin{pmatrix} r_1 x_1 \\ r_2 x_2 \\ \vdots \\ r_{n-1} x_{n-1} \\ r_n x_n \end{pmatrix}, \qquad r_i > 0, \quad \forall i. \tag{3.38}$$

An alternative representation of the Euler vector field is a multiplication of a constant diagonal matrix $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ and the state vector $\boldsymbol{x}$

$$\boldsymbol{\nu}(\boldsymbol{x}) = \begin{pmatrix} r_1 x_1 \\ r_2 x_2 \\ \vdots \\ r_{n-1} x_{n-1} \\ r_n x_n \end{pmatrix} = \underbrace{\begin{pmatrix} r_1 & 0 & \dots & \dots & 0 \\ 0 & r_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & r_{n-1} & 0 \\ 0 & \dots & \dots & 0 & r_n \end{pmatrix}}_{=\boldsymbol{B}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \boldsymbol{B}\boldsymbol{x}. \tag{3.39}$$

Homogeneous rays, see [16], are the solutions of the differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{\nu}(\boldsymbol{x}). \tag{3.40}$$

The solution of differential equation (3.40) yields

$$\boldsymbol{x}(t) = \begin{pmatrix} x_{1,0} \mathrm{e}^{r_1 t} \\ x_{2,0} \mathrm{e}^{r_2 t} \\ \vdots \\ x_{n,0} \mathrm{e}^{r_n t} \end{pmatrix} \tag{3.41}$$

for a given initial state vector $\boldsymbol{x}_0 = \begin{pmatrix} x_{1,0} & x_{2,0} & \dots & x_{n,0} \end{pmatrix}^T$. The solution (3.41) is very similar to the behavior of trajectories on real eigenvectors in case of a linear system in Section 3.1.1. The only difference is the additional weighting of the exponential functions with the dilation coefficients r_i.

It seems perspicuous to analyze a dilated solution

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}(t) = \begin{pmatrix} \varepsilon^{r_1} x_{1,0} \mathrm{e}^{r_1 t} \\ \varepsilon^{r_2} x_{2,0} \mathrm{e}^{r_2 t} \\ \vdots \\ \varepsilon^{r_n} x_{n,0} \mathrm{e}^{r_n t} \end{pmatrix}. \tag{3.42}$$

The usage of the relationship

$$\varepsilon^{r_i} = \mathrm{e}^{\ln(\varepsilon^{r_i})} = \mathrm{e}^{r_i \ln(\varepsilon)}, \qquad \varepsilon > 0 \tag{3.43}$$

allows further modifications

$$\tilde{\boldsymbol{x}}(t) = \begin{pmatrix} x_{1,0}\mathrm{e}^{r_1 \ln(\varepsilon)}\mathrm{e}^{r_1 t} \\ x_{2,0}\mathrm{e}^{r_2 \ln(\varepsilon)}\mathrm{e}^{r_2 t} \\ \vdots \\ x_{n,0}\mathrm{e}^{r_n \ln(\varepsilon)}\mathrm{e}^{r_n t} \end{pmatrix} = \begin{pmatrix} x_{1,0}\mathrm{e}^{r_1 \ln(\varepsilon)+r_1 t} \\ x_{2,0}\mathrm{e}^{r_2 \ln(\varepsilon)+r_2 t} \\ \vdots \\ x_{n,0}\mathrm{e}^{r_n \ln(\varepsilon)+r_n t} \end{pmatrix} = \begin{pmatrix} x_{1,0}\mathrm{e}^{r_1 (\ln(\varepsilon)+t)} \\ x_{2,0}\mathrm{e}^{r_2 (\ln(\varepsilon)+t)} \\ \vdots \\ x_{n,0}\mathrm{e}^{r_n (\ln(\varepsilon)+t)} \end{pmatrix} = \boldsymbol{x}(t+\ln(\varepsilon)). \tag{3.44}$$

Obviously, the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ acts like a time shift $T = \ln(\varepsilon)$. Differentiating equation (3.44) results in

$$\frac{\mathrm{d}\tilde{\boldsymbol{x}}(t)}{\mathrm{d}t} = \begin{pmatrix} r_1 x_{1,0}\mathrm{e}^{r_1 (\ln(\varepsilon)+t)} \\ r_2 x_{2,0}\mathrm{e}^{r_2 (\ln(\varepsilon)+t)} \\ \vdots \\ r_n x_{n,0}\mathrm{e}^{r_n (\ln(\varepsilon)+t)} \end{pmatrix} = \boldsymbol{\nu}(\tilde{\boldsymbol{x}}) \tag{3.45}$$

which proves that $\tilde{\boldsymbol{x}}(t)$ is again a valid solution of the differential equation (3.40).

In consequence new solutions of differential equation (3.40) can be generated by applying the dilation operator $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$ to a given solution $\boldsymbol{x}(t)$. Both solutions are located on the same homogeneous ray because the dilation only produces a time shift.

Moreover, a homogeneous ray is uniquely defined by one single point $\boldsymbol{x}_0$. Hence, the homogeneous ray are represented by the set of all possible dilations [12]

$$\Psi(\boldsymbol{x}_0) = \left\{ \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_0 \middle| \varepsilon > 0 \right\}. \tag{3.46}$$

Homogeneous rays in the plane

In the planar case homogeneous rays reduce to simple functions. Equation (3.46) states that every point of a homogeneous ray $\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$ is a dilated version of any arbitrary other point $\boldsymbol{x}_0 = \begin{pmatrix} x_{1,0} & x_{2,0} \end{pmatrix}^T$ of the homogeneous ray

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \varepsilon^{r_1} x_{1,0} \\ \varepsilon^{r_2} x_{2,0} \end{pmatrix}. \tag{3.47}$$

Assuming $x_{1,0} \neq 0$ enables to solve the first equation of (3.47) for ε which yields

$$\varepsilon = \left(\frac{x_1}{x_{1,0}} \right)^{\frac{1}{r_1}}. \tag{3.48}$$

The sign of x_1 and $x_{1,0}$ is equivalent due to $\varepsilon > 0$ and, therefore, the ratio is always positive. For this reason it is feasible to modify equation (3.48) to

$$\varepsilon = \left| \frac{x_1}{x_{1,0}} \right|^{\frac{1}{r_1}} = \frac{|x_1|^{\frac{1}{r_1}}}{|x_{1,0}|^{\frac{1}{r_1}}}. \tag{3.49}$$

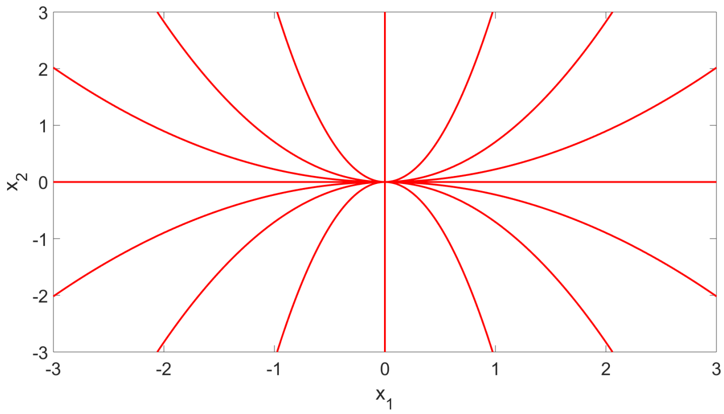

Figure 3.1: Homogeneous rays w.r.t. dilation coefficients $\boldsymbol{r} = \begin{pmatrix}1 & 2\end{pmatrix}^T$ are parabolic functions $x_2(x_1) = a_{\boldsymbol{r}}(\boldsymbol{x}_0)x_1^2$.

Insertion of equation (3.49) into the second equation of (3.47) leads to

$$x_2 = \frac{|x_1|^{\frac{r_2}{r_1}}}{|x_{1,0}|^{\frac{r_2}{r_1}}} x_{2,0} = \frac{x_{2,0}}{|x_{1,0}|^{\frac{r_2}{r_1}}} |x_1|^{\frac{r_2}{r_1}}. \tag{3.50}$$

Substituting

$$a_{\boldsymbol{r}}(\boldsymbol{x}_0) = \frac{x_{2,0}}{|x_{1,0}|^{\frac{r_2}{r_1}}} \tag{3.51}$$

yields

$$x_2(x_1) = a_{\boldsymbol{r}}(\boldsymbol{x}_0)|x_1|^{\frac{r_2}{r_1}}. \tag{3.52}$$

Equation (3.51) is homogeneous of degree 0 due to

$$\begin{aligned} a_{\boldsymbol{r}}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_0) &= \frac{\varepsilon^{r_2} x_{2,0}}{|\varepsilon^{r_1} x_{1,0}|^{\frac{r_2}{r_1}}} = \frac{\varepsilon^{r_2} x_{2,0}}{\varepsilon^{r_2} |x_{1,0}|^{\frac{r_2}{r_1}}} = \\ &= \frac{x_{2,0}}{|x_{1,0}|^{\frac{r_2}{r_1}}} = a_{\boldsymbol{r}}(\boldsymbol{x}_0) \end{aligned} \tag{3.53}$$

and, therefore, constant along a homogeneous ray.

From equation (3.52) it becomes clear that homogeneous rays in the plane are power functions (with fractional exponents for $r_1, r_2 \in \mathbb{N}$). For example homogeneous rays w.r.t. dilation coefficients $\boldsymbol{r} = \begin{pmatrix}1 & 2\end{pmatrix}^T$ are quadratic parabolas as illustrated in Figure 3.1.

3.3.2 Euler sphere, projection system and projection solution

H. Nakamura et al. [12] introduced the so-called Euler sphere to prove a sufficient stability criterion for the origin of homogeneous systems. In this Section the Euler sphere is introduced and its properties are summarized. Furthermore, the projection of a vector field and the projection of solution curves of homogeneous systems onto the Euler sphere are analyzed. The presented theory summarizes the results of H. Nakamura et al. [12].

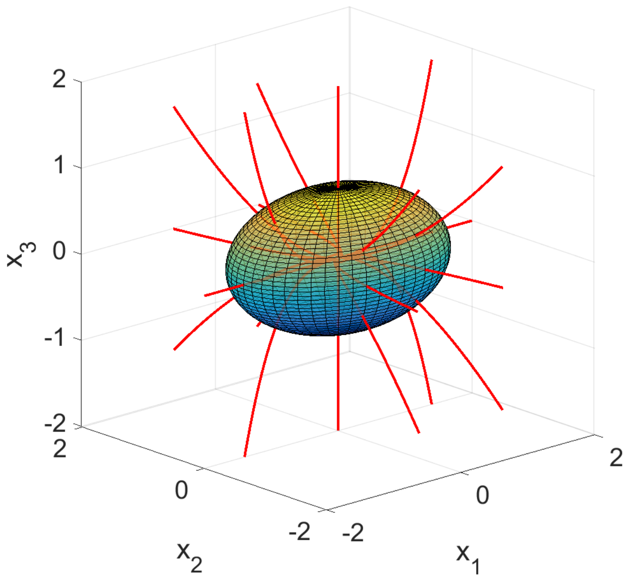

Figure 3.2: Euler sphere $E_1^{\boldsymbol{r}}$ for $n = 3$ w.r.t. dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$. The red lines are homogeneous rays which are orthogonal to the ellipsoids surface.

Euler sphere

The Euler sphere $E_1^{\boldsymbol{r}}$ w.r.t. dilation coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix}^T$ is defined by the set

$$E_1^{\boldsymbol{r}} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \frac{r_1}{2} x_1^2 + \frac{r_2}{2} x_2^2 + \dots \frac{r_n}{2} x_n^2 = 1 \right\}. \tag{3.54}$$

The Euler sphere is the surface of a hyperellipsoid in state space with its center in the origin. For $n = 2$ it reduces to an ellipse and for $n = 3$ it is an ellipsoid.

The Euler vector field $\nu(\boldsymbol{x})$ w.r.t. dilation coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix}^T$ is normal to the surface of the Euler sphere [12] and, therefore, the homogeneous rays are normal to it. Figure 3.2 shows the Euler sphere $E_1^{\boldsymbol{r}}$ for $n = 3$ w.r.t. dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$. The red lines indicate some exemplary homogeneous rays.

A dilation of the Euler sphere $E_\varepsilon^{\boldsymbol{r}} = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} E_1^{\boldsymbol{r}}$ is defined by the set of all dilated points of the Euler sphere $E_1^{\boldsymbol{r}}$

$$E_\varepsilon^{\boldsymbol{r}} = \left\{ \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_0 \middle| \boldsymbol{x}_0 \in E_1^{\boldsymbol{r}} \right\}, \qquad \varepsilon > 0. \tag{3.55}$$

Inserting the definition of the Euler sphere (3.54) into (3.55) leads to

$$E_\varepsilon^{\boldsymbol{r}} = \left\{ \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_0 \middle| \sum_{i=1}^{n} \frac{r_i}{2} x_{0,i}^2 = 1 \right\}, \qquad \varepsilon > 0. \tag{3.56}$$

Substituting $\boldsymbol{x} = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_0$ yields

$$E_\varepsilon^{\boldsymbol{r}} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \middle| \sum_{i=1}^{n} \frac{r_i x_i^2}{2\varepsilon^{2r_i}} = 1 \right\}, \qquad \varepsilon > 0 \tag{3.57}$$

which is unfortunately inconsistent in [12]. Homogeneous rays are the solution of system (3.40) which is linear and, therefore, homogeneous of degree 0 w.r.t. dilation coefficients $\begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^T$. For this reason, dilations of the Euler sphere $E_\varepsilon^{\boldsymbol{r}}$ are not orthogonal to the Euler vector field in general for $\varepsilon \neq 1$.

Projection system

Consider the homogeneous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0},\, \boldsymbol{x} \in \mathbb{R}^n \tag{3.58}$$

of arbitrary homogeneity degree q w.r.t. dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$. The projection vector field $\boldsymbol{f}_0(\boldsymbol{x})$ is the projection of the vector field $\boldsymbol{f}(\boldsymbol{x})$ onto the Euler sphere $E_1^{\boldsymbol{r}}$. The projection onto a surface is done by subtracting the projection of the vector onto the orthogonal vector of the tangential plane. In case of the Euler sphere the orthogonal vector is the Euler vector field $\boldsymbol{\nu}(\boldsymbol{x})$. Hence, the projection vector field of system (3.58) computes to

$$\boldsymbol{f}_0(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - \frac{\boldsymbol{\nu}^T(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})}{\|\boldsymbol{\nu}(\boldsymbol{x})\|^2}\boldsymbol{\nu}(\boldsymbol{x}), \tag{3.59}$$

where $\|\cdot\|$ denotes the Euclidean vector norm. The orthogonality of $\boldsymbol{\nu}(\boldsymbol{x})$ and $\boldsymbol{f}_0(\boldsymbol{x})$ can be checked by evaluation of the scalar product

$$\begin{aligned} \boldsymbol{\nu}(\boldsymbol{x})^T \boldsymbol{f}_0(\boldsymbol{x}) &= \boldsymbol{\nu}(\boldsymbol{x})^T \left(\boldsymbol{f}(\boldsymbol{x}) - \frac{\boldsymbol{\nu}^T(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})}{\|\boldsymbol{\nu}(\boldsymbol{x})\|^2}\boldsymbol{\nu}(\boldsymbol{x}) \right) = \\ &= \boldsymbol{\nu}(\boldsymbol{x})^T \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{\nu}(\boldsymbol{x})^T \boldsymbol{\nu}(\boldsymbol{x}) \frac{\boldsymbol{\nu}^T(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})}{\|\boldsymbol{\nu}(\boldsymbol{x})\|^2} = \\ &= \boldsymbol{\nu}(\boldsymbol{x})^T \boldsymbol{f}(\boldsymbol{x}) - \|\boldsymbol{\nu}(\boldsymbol{x})\|^2 \frac{\boldsymbol{\nu}^T(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})}{\|\boldsymbol{\nu}(\boldsymbol{x})\|^2} = 0. \end{aligned} \tag{3.60}$$

Figure 3.3 illustrates the projection of the vector field $\boldsymbol{f}(\boldsymbol{x})$ onto the Euler sphere for $n = 3$ and the chosen dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$. The Euler vector field $\boldsymbol{\nu}(\boldsymbol{x})$ is orthogonal to the tangential plane and, therefore, to $\boldsymbol{f}_0(\boldsymbol{x})$.

The system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_0(\boldsymbol{x}), \qquad \boldsymbol{x}(0) \in E_1^{\boldsymbol{r}} \tag{3.61}$$

is called a projection system of (3.58). The solution of the projection system (3.61) for initial state $\boldsymbol{x}(0) \in E_1^{\boldsymbol{r}}$ remains on the Euler sphere

$$\boldsymbol{x}_0 \in E_1^{\boldsymbol{r}} \Rightarrow \boldsymbol{x}(t) \in E_1^{\boldsymbol{r}} \quad \forall t \tag{3.62}$$

because of the tangential direction of $\boldsymbol{f}_0(\boldsymbol{x})$.

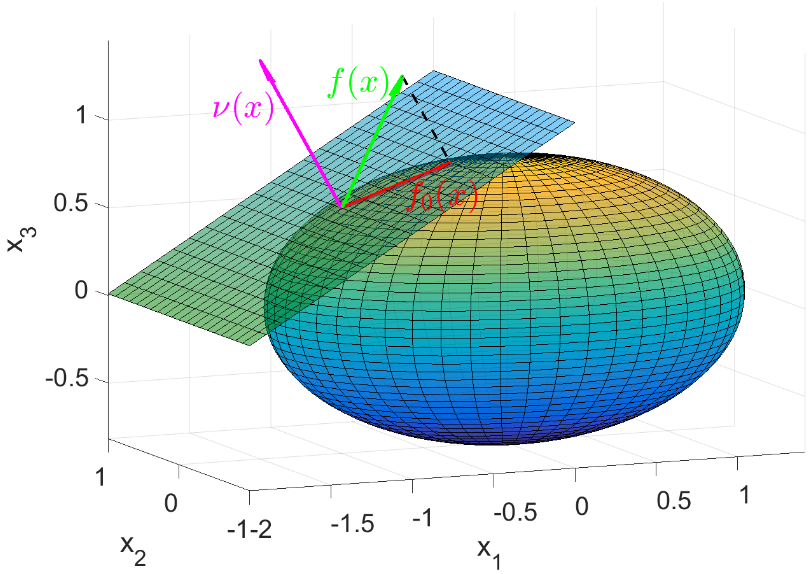

Figure 3.3: Illustration of the projection of the vector field $\boldsymbol{f}(\boldsymbol{x})$ onto the Euler sphere $E_1^{\boldsymbol{r}}$ which results in the projection vector field $\boldsymbol{f}_0(\boldsymbol{x})$. The dilation coefficients are $\boldsymbol{r} = \begin{pmatrix}1 & 2 & 3\end{pmatrix}^T$.

Projection solution

The projection of a solution $\boldsymbol{x}(t)$ of the original system (3.58) along homogeneous rays is called a projection solution $\boldsymbol{x}_E(t)$. Figure 3.4 shows the projection of a trajectory $\boldsymbol{x}(t)$ onto the Euler sphere $E_1^{\boldsymbol{r}}$ for a homogeneous system w.r.t. dilation coefficients $\boldsymbol{r} = \begin{pmatrix}1 & 2 & 3\end{pmatrix}^T$.

Consider two solutions $\boldsymbol{x}_a(t)$ and $\boldsymbol{x}_b(t)$ of system (3.58) with initial states $\boldsymbol{x}_{a,0}$ and $\boldsymbol{x}_{b,0}$ located on the same homogeneous ray, i.e. $\boldsymbol{x}_{b,0} = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_{a,0}$. In Section 2.3.1 it is shown that a dilation of the initial state results in a dilation of the trajectory and an additional scaling of time, i.e.

$$\boldsymbol{x}_{b,0} = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_{a,0} \Rightarrow \boldsymbol{x}_b(t) = \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{x}_a(\varepsilon^{-q} t). \tag{3.63}$$

Obviously, $\boldsymbol{x}_a(\varepsilon^{-q}t)$ and $\boldsymbol{x}_b(t)$ are located on the same homogeneous ray $\forall t \geq 0$ and, therefore, the projections onto the Euler sphere are identical. Hence, the trajectories of their projection solutions $\boldsymbol{x}_{a,E}(t)$ and $\boldsymbol{x}_{b,E}(t)$ coincide.

Equivalence of trajectories of projection solution and solution of the projection system

Let $\boldsymbol{x}(t) = \begin{pmatrix}x_1(t) & \dots & x_n(t)\end{pmatrix}^T$ be the solution of (3.58) for initial state $\boldsymbol{x}_0$. Projection onto the Euler sphere $E_1^{\boldsymbol{r}}$ means to find a state-depending $\varepsilon\big(\boldsymbol{x}(t)\big)$ that satisfies

$$\sum_{i=1}^{n} \frac{r_i}{2} \Big(\varepsilon\big(\boldsymbol{x}(t)\big)^{r_i} x_i\Big)^2 = 1. \tag{3.64}$$

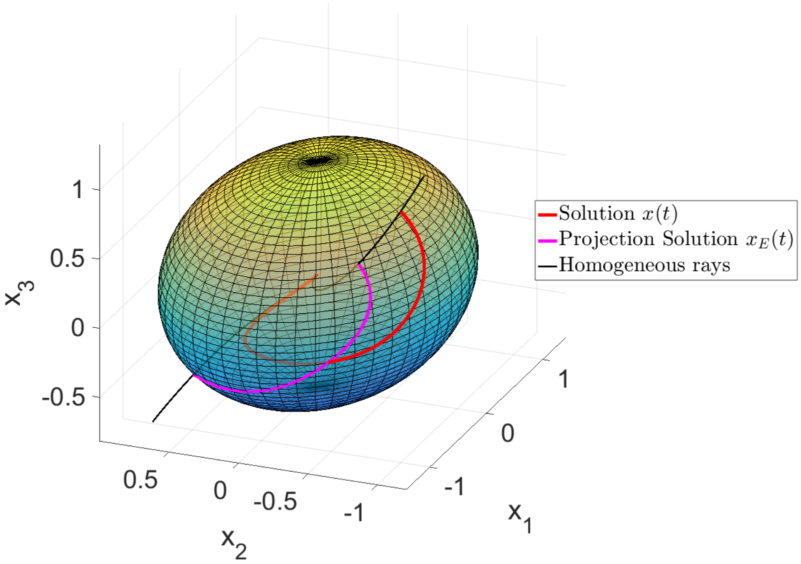

Figure 3.4: Illustration of the projection solution $\boldsymbol{x}_E(t)$ which is a point-wise projection of the solution $\boldsymbol{x}(t)$ onto the Euler sphere $E_1^{\boldsymbol{r}}$ along homogeneous rays. The dilation coefficients are chosen $\boldsymbol{r} = \begin{pmatrix}1 & 2 & 3\end{pmatrix}^T$.

Then, the projection solution can be written as a dilation of $\boldsymbol{x}(t)$

$$\boldsymbol{x}_E(t) = \begin{pmatrix} x_{E,1}(t) \\ \vdots \\ x_{E,n}(t) \end{pmatrix} = \begin{pmatrix} \varepsilon^{r_1}\big(\boldsymbol{x}(t)\big)x_1(t) \\ \vdots \\ \varepsilon^{r_n}\big(\boldsymbol{x}(t)\big)x_n(t) \end{pmatrix} = \boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon(\boldsymbol{x}(t))}\boldsymbol{x}(t). \tag{3.65}$$

Differentiation of equation (3.64) w.r.t. time yields

$$\sum_{i=1}^{n} \frac{r_i}{2} 2\varepsilon^{r_i} x_i \Big(r_i \varepsilon^{r_i - 1} \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} x_i + \varepsilon^{r_i} f_i(\boldsymbol{x}) \Big) = 0, \tag{3.66}$$

where $f_i(\boldsymbol{x})$ denotes the i^{th} component of right-hand side of system (3.58). Solving (3.66) for $\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$ results in the ordinary differential equation

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -\varepsilon \frac{\sum\limits_{i=1}^{n} r_i \varepsilon^{2r_i} x_i f_i(\boldsymbol{x})}{\sum\limits_{i=1}^{n} r_i^2 \varepsilon^{2r_i} x_i^2}. \tag{3.67}$$

The definition of homogeneous vector fields (see 2.2) allows the substitution

$$f_i(\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon \boldsymbol{x}) = \varepsilon^{q+r_i} f_i(\boldsymbol{x}) \Rightarrow \varepsilon^{r_i} f_i(\boldsymbol{x}) = \varepsilon^{-q} f_i(\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon \boldsymbol{x}) = \varepsilon^{-q} f_i(\boldsymbol{x}_E). \tag{3.68}$$

Furthermore, the i^{th} component of the Euler vector field is given by

$$\nu_i(\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon \boldsymbol{x}) = \nu_i(\boldsymbol{x}_E) = r_i \varepsilon^{r_i} x_i. \tag{3.69}$$

In consequence (3.67) further simplifies to the intermediate result

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -\varepsilon \frac{\sum\limits_{i=1}^{n} \nu_i(\boldsymbol{x}_E)\varepsilon^{-q} f_i(\boldsymbol{x}_E)}{\sum\limits_{i=1}^{n} \nu_i^2(\boldsymbol{x}_E)} = -\varepsilon \cdot \varepsilon^{-q} \frac{\boldsymbol{\nu}^T(\boldsymbol{x}_E)\boldsymbol{f}(\boldsymbol{x}_E)}{\|\boldsymbol{\nu}(\boldsymbol{x}_E)\|^2}. \tag{3.70}$$

Differentiation of the i^{th} element $x_{E,i}$ of equation (3.65) yields

$$\dot{x}_{E,i} = \varepsilon^{r_i}(\boldsymbol{x}) f_i(\boldsymbol{x}) + r_i \varepsilon^{r_i - 1}(\boldsymbol{x}) \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} x_i, \qquad i = 1 \dots n. \tag{3.71}$$

Insertion of differential equation (3.70) into (3.71) results in

$$\dot{x}_{E,i} = \varepsilon^{r_i}(\boldsymbol{x}) f_i(\boldsymbol{x}) - \varepsilon^{-q}(\boldsymbol{x}) r_i \varepsilon^{r_i}(\boldsymbol{x}) x_i \cdot \frac{\boldsymbol{\nu}^T(\boldsymbol{x}_E)\boldsymbol{f}(\boldsymbol{x}_E)}{\|\boldsymbol{\nu}(\boldsymbol{x}_E)\|^2}. \tag{3.72}$$

Equation (3.68) and (3.69) are inserted to simplify (3.72) to

$$\begin{aligned} \dot{x}_{E,i} &= \varepsilon^{-q}(\boldsymbol{x}) f_i(\boldsymbol{x}_E) - \varepsilon^{-q}(\boldsymbol{x}) \nu_i(\boldsymbol{x}_E) \frac{\boldsymbol{\nu}^T(\boldsymbol{x}_E)\boldsymbol{f}(\boldsymbol{x}_E)}{\|\boldsymbol{\nu}(\boldsymbol{x}_E)\|^2} = \\ &= \varepsilon^{-q}(\boldsymbol{x}) \Big(f_i(\boldsymbol{x}_E) - \nu_i(\boldsymbol{x}_E) \frac{\boldsymbol{\nu}^T(\boldsymbol{x}_E)\boldsymbol{f}(\boldsymbol{x}_E)}{\|\boldsymbol{\nu}(\boldsymbol{x}_E)\|^2} \Big), \qquad i = 1 \dots n. \end{aligned} \tag{3.73}$$

Rewriting (3.73) in vector notation eventually yields

$$\dot{\boldsymbol{x}}_E = \varepsilon^{-q}(\boldsymbol{x}) \Big(\boldsymbol{f}(\boldsymbol{x}_E) - \boldsymbol{\nu}(\boldsymbol{x}_E) \frac{\boldsymbol{\nu}^T(\boldsymbol{x}_E)\boldsymbol{f}(\boldsymbol{x}_E)}{\|\boldsymbol{\nu}(\boldsymbol{x}_E)\|^2} \Big) = \varepsilon^{-q}(\boldsymbol{x}) \boldsymbol{f}_0(\boldsymbol{x}_E). \tag{3.74}$$

Obviously, the direction of $\dot{\boldsymbol{x}}_E$ and the projection vector field $\boldsymbol{f}_0(\boldsymbol{x}_E)$ are equivalent. For this reason the trajectories of the projection solution $\boldsymbol{x}_E$ and the solution of the projection system (3.61) coincide if the initial values are located on the same homogeneous ray. Moreover, both solutions are identical if $q = 0$.

3.3.3 Definitions and properties of homogeneous eigenvalues and eigenvectors

Again the homogeneous system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0},\, \boldsymbol{x} \in \mathbb{R}^n \tag{3.75}$$

is considered. The homogeneous eigenvalue equation proposed by H. Nakamura et al. [11] is given by

$$\boldsymbol{f}(\boldsymbol{v}) = \lambda \|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}} \boldsymbol{\nu}(\boldsymbol{v}). \tag{3.76}$$

The vector

$$\boldsymbol{v} = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix}^T \tag{3.77}$$

is called a homogeneous eigenvector and λ denotes a homogeneous eigenvalue. In the further analysis both are assumed to be real.

In many publications, see e.g. [15], the homogeneous eigenvalues are not constant but depend on the state vector $\boldsymbol{x}$. The homogeneous eigenvalue equation then modifies to

$$\boldsymbol{f}(\boldsymbol{v}) = \lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{\nu}(\boldsymbol{v}). \tag{3.78}$$

In the subsequent analysis this notation is analyzed and it is shown that the eigenvalues evaluated on a homogeneous eigenvector are constant.

Solution of the homogeneous eigenvalue equation

In order to solve equation (3.78) for $\lambda(\boldsymbol{v})$ it is advantageous to rewrite system (3.75) in pseudo-linear system representation as shown in Section 3.2.1, i.e.

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}(\boldsymbol{x})\boldsymbol{x} \tag{3.79}$$

in which $\boldsymbol{M}(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$ denotes the state-dependent system matrix. Again the choice of $\boldsymbol{M}(\boldsymbol{x})$ is not unique for $n \geq 2$. The Euler vector field can be expressed by a multiplication of a constant diagonal matrix $\boldsymbol{B}$ containing the dilation coefficients with the state vector $\boldsymbol{x}$, as shown in Section 3.3.1. This allows to rewrite the homogeneous eigenvalue equation as

$$\boldsymbol{M}(\boldsymbol{v})\boldsymbol{v} = \lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{B}\boldsymbol{v}. \tag{3.80}$$

Multiplying with $\|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\boldsymbol{B}^{-1}$ from the left-hand side produces

$$\|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{v})\boldsymbol{v} = \lambda(\boldsymbol{v})\boldsymbol{v}. \tag{3.81}$$

The homogeneous eigenvector has to fulfill $\boldsymbol{v} \neq \boldsymbol{0}$ and for this reason the homogeneous norm is ensured to be not equal to zero. The inverse of $\boldsymbol{B}$ always exists because $\boldsymbol{B}$ is a diagonal matrix with strictly positive values in the main diagonal. Subtracting $\|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{v})\boldsymbol{v}$ and extracting $\boldsymbol{v}$ leads to

$$\left(\lambda(\boldsymbol{v})\boldsymbol{I} - \|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{v})\right)\boldsymbol{v} = \boldsymbol{0}, \tag{3.82}$$

with $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ denoting the identity matrix. In this notation the homogeneous eigenvalue problem is reduced to a classical eigenvalue problem of the matrix $\|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{v})$. This system of equations offers a non-trivial solution for $\boldsymbol{v}$ if the determinant

$$\det\left(\lambda(\boldsymbol{v})\boldsymbol{I} - \|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{v})\right) = 0. \tag{3.83}$$

Hence, $\lambda(\boldsymbol{v})$ is computed by symbolically calculating the zeros of the characteristic polynomial of $\|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{v})$. With the knowledge of the homogeneous eigenvalues $\lambda(\boldsymbol{v})$ the eigenvectors can be calculated by solving the system of equations (3.82) [25].

Properties of homogeneous eigenvectors

In the case of linear systems the solution curves of the differential equation stay on the eigenvector forever if the initial state $\boldsymbol{x}_0$ is located on the eigenvector (see Section 3.1.1). Homogeneous eigenvectors preserve this important property which is shown below.

Solving the homogeneous eigenvalue equation

$$\boldsymbol{f}(\boldsymbol{v}) = \lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{\nu}(\boldsymbol{v}) \tag{3.84}$$

can be interpreted as identifying all the points $\boldsymbol{v}$ in state space where the direction of the vector field $\boldsymbol{f}(\boldsymbol{v})$ equals the direction of the Euler vector field $\boldsymbol{\nu}(\boldsymbol{v})$. The scalar term $\lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}$ determines the sign and the speed of change but does not affect the direction. Assume that $\boldsymbol{v}$ is a valid solution of the homogeneous eigenvalue equation (3.84). A dilation of the homogeneous eigenvector yields

$$\boldsymbol{f}(\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}) = \varepsilon^q\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{f}(\boldsymbol{v}). \tag{3.85}$$

Substitution of $\boldsymbol{f}(\boldsymbol{v})$ using equation (3.84) leads to

$$\boldsymbol{f}(\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}) = \varepsilon^q\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\big(\lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{\nu}(\boldsymbol{v})\big). \tag{3.86}$$

The dilation operator does not have any influence on the scalar parts, i.e.

$$\boldsymbol{f}(\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}) = \varepsilon^q\lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{\nu}(\boldsymbol{v}). \tag{3.87}$$

The homogeneous norm is a homogeneous function of degree 1. Therefore,

$$\|\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}} = \varepsilon^q\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}} \tag{3.88}$$

holds. Both, the dilation operator and the Euler vector field act like a multiplication of the vector $\boldsymbol{v}$ with a diagonal matrix. For this reason these operations are commutative

$$\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{\nu}(\boldsymbol{v}) = \boldsymbol{\nu}(\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}). \tag{3.89}$$

Insertion of equations (3.88) and (3.89) into (3.87) yields

$$\boldsymbol{f}(\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}) = \lambda(\boldsymbol{v})\|\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{\nu}(\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}). \tag{3.90}$$

Obviously, $\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}$ is a valid solution for the eigenvector too. The homogeneous eigenvalue $\lambda(\boldsymbol{v})$ stays the same which is discussed later in Section 3.3.3. Again, the direction of the Euler vector field $\boldsymbol{\nu}(\boldsymbol{x})$ and the vector field $\boldsymbol{f}(\boldsymbol{x})$ coincide in the point $\boldsymbol{x} = \boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}$. Therefore, the trajectory of the system evolves along the homogeneous eigenvector which coincides with a homogeneous ray in a phase plot if the initial state $\boldsymbol{x}_0$ is located somewhere on the eigenvector $\boldsymbol{\Delta}^{\boldsymbol{r}}_{\varepsilon}\boldsymbol{v}$ with arbitrary $\varepsilon > 0$.

This means that the differential equation (3.75) reduces to

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \lambda(\boldsymbol{x})\|\boldsymbol{x}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{\nu}(\boldsymbol{x}) \tag{3.91}$$

for every trajectory with initial state $\boldsymbol{x}_0 = \begin{pmatrix} x_{1,0} & x_{2,0} & \dots & x_{n,0} \end{pmatrix}^T$ located on a real homogeneous eigenvector. A division by $\lambda(\boldsymbol{x})\|\boldsymbol{x}\|^q_{\{\boldsymbol{r},2\}}$ yields

$$\frac{1}{\lambda(\boldsymbol{x})\|\boldsymbol{x}\|^q_{\{\boldsymbol{r},2\}}} \cdot \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{\nu}(\boldsymbol{x}). \tag{3.92}$$

The scalar function $\lambda(\boldsymbol{x})\|\boldsymbol{x}\|^q_{\{\boldsymbol{r},2\}}$ can be treated as a scaling of time. The introduction of the scaled time variable τ, which satisfies the differential equation

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \lambda(\boldsymbol{x})\|\boldsymbol{x}\|^q_{\{\boldsymbol{r},2\}}, \tag{3.93}$$

leads to

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \boldsymbol{\nu}(\boldsymbol{x}). \tag{3.94}$$

The solution

$$\boldsymbol{x}(\tau) = \begin{pmatrix} x_{1,0}\mathrm{e}^{r_1\tau} \\ x_{2,0}\mathrm{e}^{r_2\tau} \\ \vdots \\ x_{n,0}\mathrm{e}^{r_n\tau} \end{pmatrix} \tag{3.95}$$

of differential equation (3.94) is already known from Section 3.3.1 which coincides with a homogeneous ray. For this reason, real homogeneous eigenvectors match with homogeneous rays in a trajectory plot because different time evolution is not visible.

In the linear case real eigenvectors are straight lines. The direction of the eigenvector is defined but the length is arbitrary. In contrast to that, real homogeneous eigenvectors are not necessarily straight lines but homogeneous rays. If one point of the eigenvector is known, all the other points can be found by applying the dilation operator $\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon$ with arbitrary $\varepsilon > 0$. The property that real eigenvectors are solution curves of the differential equation is consistent with the linear case.

Properties of homogeneous eigenvalues of systems with homogeneity degree $q = 0$

If the homogeneity degree of system (3.75) is zero, i.e. $q = 0$, the homogeneous eigenvalue equation (3.76) reduces to

$$\boldsymbol{f}(\boldsymbol{v}) = \lambda(\boldsymbol{v})\boldsymbol{\nu}(\boldsymbol{v}). \tag{3.96}$$

Then, dilation of $\boldsymbol{v}$ yields

$$\boldsymbol{f}(\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon \boldsymbol{v}) = \boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon \boldsymbol{f}(\boldsymbol{v}). \tag{3.97}$$

Exploiting (3.39) for representation of the Euler vector field leads to

$$\lambda(\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon \boldsymbol{v})\boldsymbol{B}\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon \boldsymbol{v} = \boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon(\lambda(\boldsymbol{v})\boldsymbol{B}\boldsymbol{v}). \tag{3.98}$$

As explained in Section 2.1.3, the dilation operator corresponds to a multiplication with a diagonal matrix $\boldsymbol{\Gamma_r}(\varepsilon)$

$$\lambda(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{v})\boldsymbol{B}\boldsymbol{\Gamma_r}(\varepsilon)\boldsymbol{v} = \boldsymbol{\Gamma_r}(\varepsilon)\lambda(\boldsymbol{v})\boldsymbol{B}\boldsymbol{v}. \tag{3.99}$$

The commutativity of the diagonal matrices is used to swap the order of $\boldsymbol{B}$ and $\boldsymbol{\Gamma_r}(\varepsilon)$ which leads to

$$\boldsymbol{B}\boldsymbol{\Gamma_r}(\varepsilon)\lambda(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{v})\boldsymbol{v} = \boldsymbol{B}\boldsymbol{\Gamma_r}(\varepsilon)\lambda(\boldsymbol{v})\boldsymbol{v}. \tag{3.100}$$

Both matrices $\boldsymbol{B}$ and $\boldsymbol{\Gamma_r}(\varepsilon)$ are invertible because they are real diagonal matrices with strictly positive numbers in the main diagonal. A multiplication with $(\boldsymbol{\Gamma_r}(\varepsilon)^{-1}\boldsymbol{B}^{-1})$ from the left-hand side results in

$$\lambda(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{v})\boldsymbol{v} = \lambda(\boldsymbol{v})\boldsymbol{v}. \tag{3.101}$$

The trivial solution $\boldsymbol{v} = \boldsymbol{0}$ for the homogeneous eigenvector is not permitted. Therefore,

$$\lambda(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{v}) = \lambda(\boldsymbol{v}) \tag{3.102}$$

has to hold i.e. $\lambda(\boldsymbol{v})$ is a homogeneous function of degree 0 w.r.t. the dilation coefficient vector $\boldsymbol{r}$. Moreover, the eigenvalue evaluated on the eigenvector is constant because the homogeneous eigenvector is a homogeneous ray. Therefore, it is feasible to write

$$\lambda(\boldsymbol{v}) = \lambda = \text{const.} \tag{3.103}$$

Note that in this regard (3.76) coincides with the notation used by H. Nakamura et al. [11].

Properties of homogeneous eigenvalues of systems with arbitrary homogeneity degree

In Section 2.3.2 it is shown that for every system of arbitrary homogeneity degree q a corresponding system of homogeneity degree $\tilde{q} = 0$ can be found. The trajectories of the original system and the corresponding system coincide and, therefore, both systems offer the same stability behavior. It is be of interest to find a relation between the homogeneous eigenvalues of the systems.

System (3.75) is assumed to be homogeneous of arbitrary degree q. As shown in Section 2.3.2 the right-hand side of the differential equation can be decomposed into a scalar time scaling function $\zeta(\boldsymbol{x})$ and the vector field $\tilde{\boldsymbol{f}}(\boldsymbol{x})$ of the corresponding system, i.e.

$$\boldsymbol{f}(\boldsymbol{x}) = \zeta(\boldsymbol{x})\tilde{\boldsymbol{f}}(\boldsymbol{x}). \tag{3.104}$$

Insertion of equation (3.104) into the homogeneous eigenvalue equation yields

$$\zeta(\boldsymbol{v})\tilde{\boldsymbol{f}}(\boldsymbol{v}) = \lambda(\boldsymbol{v})\|\boldsymbol{v}\|_{\{\boldsymbol{r},2\}}^{q}\boldsymbol{\nu}(\boldsymbol{v}). \tag{3.105}$$

The homogeneous eigenvalue equation of the corresponding system of homogeneity degree $\tilde{q} = 0$ is

$$\tilde{\boldsymbol{f}}(\boldsymbol{v}) = \tilde{\lambda}\boldsymbol{\nu}(\boldsymbol{v}). \tag{3.106}$$

Both, the original and the corresponding system produce the same trajectories and, therefore, the homogeneous eigenvectors $\boldsymbol{v}$ are equivalent. For this reason insertion of equation (3.106) into equation (3.105) is feasible which leads to

$$\zeta(\boldsymbol{v})\tilde{\lambda}\boldsymbol{\nu}(\boldsymbol{v}) = \lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{\nu}(\boldsymbol{v}). \tag{3.107}$$

Again the Euler vector field $\boldsymbol{\nu}(\boldsymbol{v})$ is rewritten as a product as shown in (3.39), which yields

$$\zeta(\boldsymbol{v})\tilde{\lambda}\boldsymbol{B}\boldsymbol{v} = \lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}}\boldsymbol{B}\boldsymbol{v}. \tag{3.108}$$

The diagonal matrix $\boldsymbol{B}$ is eliminated by multiplying with $\boldsymbol{B}^{-1}$ from the left-hand side. For non-trivial solutions for the eigenvector $\boldsymbol{v}$ the relationship

$$\zeta(\boldsymbol{v})\tilde{\lambda} = \lambda(\boldsymbol{v})\|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}} \tag{3.109}$$

has to hold. Equation (3.109) is solved for the homogeneous eigenvalue $\lambda(\boldsymbol{v})$ of the original system of arbitrary homogeneity degree q. The resulting equation

$$\lambda(\boldsymbol{v}) = \|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\zeta(\boldsymbol{v})\tilde{\lambda} \tag{3.110}$$

relates the homogeneous eigenvalues of the original system of arbitrary degree q to the corresponding system of degree $\tilde{q} = 0$. Moreover, if the time scaling function is chosen to

$$\zeta(\boldsymbol{v}) = \|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}} \tag{3.111}$$

as suggested in Section 2.3.2, the homogeneous eigenvalues of the original and the corresponding system are equivalent, i.e.

$$\zeta(\boldsymbol{v}) = \|\boldsymbol{v}\|^q_{\{\boldsymbol{r},2\}} \Rightarrow \lambda(\boldsymbol{v}) = \tilde{\lambda}. \tag{3.112}$$

Again the homogeneity properties of $\lambda(\boldsymbol{v})$ are analyzed which yields

$$\lambda(\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon\boldsymbol{v}) = \|\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\zeta(\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon\boldsymbol{v})\tilde{\lambda}. \tag{3.113}$$

In Section 2.1.4 it is proved that the homogeneous norm is a homogeneous function of degree one. A valid time scaling function has to be homogeneous of degree q which is shown in Section 2.3.2. The homogeneous eigenvalues of a system with homogeneity degree $\tilde{q} = 0$ are constant. Therefore, equation (3.113) simplifies to

$$\begin{aligned}\lambda(\boldsymbol{\Delta}^{\boldsymbol{r}}_\varepsilon\boldsymbol{v}) &= \varepsilon^{-q}\|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\varepsilon^q\zeta(\boldsymbol{v})\varepsilon^0\tilde{\lambda} = \\ &= \|\boldsymbol{v}\|^{-q}_{\{\boldsymbol{r},2\}}\zeta(\boldsymbol{v})\tilde{\lambda} = \\ &= \lambda(\boldsymbol{v}).\end{aligned} \tag{3.114}$$

This means that homogeneous eigenvalues are homogeneous of degree zero and, therefore, constant along eigenvectors because eigenvectors are homogeneous rays. For this reason the constant notation by H. Nakamura et al. is valid

$$\lambda(\boldsymbol{v}) = \lambda = \text{const.} \tag{3.115}$$

for systems of arbitrary homogeneity degree q.

In Section 3.3.4 the connection between the homogeneous eigenvalues and the stability of the origin is analyzed. Similar to the linear case the sign of the homogeneous eigenvalues is an essential element. Hence, it is beneficial to relate the sign of the homogeneous eigenvalues λ of the original system of homogeneity degree q and the homogeneous eigenvalues $\tilde{\lambda}$ of the corresponding system of homogeneity degree $\tilde{q} = 0$. Equation (3.110) provides a link between λ and $\tilde{\lambda}$. Application of the sign operator to (3.110) yields

$$\operatorname{sign}(\lambda) = \operatorname{sign}(\|\boldsymbol{v}\|_{\{\boldsymbol{r},2\}}^{-q} \zeta(\boldsymbol{v}) \tilde{\lambda}). \tag{3.116}$$

The homogeneous norm is a positive definite function, see Section 2.1.4, and the time scaling has to satisfy at least $\zeta(\boldsymbol{x}) \geq 0$, see Section 2.3.2. For this reason, they do not have any influence on the sign which simplifies equation (3.116) to

$$\operatorname{sign}(\lambda) = \operatorname{sign}(\tilde{\lambda}). \tag{3.117}$$

The signs of the homogeneous eigenvalues λ of the original system of homogeneity degree q and the homogeneous eigenvalues $\tilde{\lambda}$ of the corresponding system of homogeneity degree $\tilde{q} = 0$ are equal.

3.3.4 Homogeneous eigenvalues and stability

Homogeneous eigenvalues are an extension of the linear eigenvalue concept to non-linear, but homogeneous systems. Therefore, a link between the homogeneous eigenvalues and the stability properties of a system seems to be evident. H. Nakamura et al. presented a necessary criterion for asymptotic stability of the origin [11] and provided a sufficient criterion [26] which was simplified for planar systems and systems of order $n = 3$ [12].

Necessary criterion for asymptotic stability of the origin

The results presented in this Section are taken from H. Nakamura et al. [11]. Consider the homogeneous system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0},\ \boldsymbol{x} \in \mathbb{R}^n \tag{3.118}$$

of arbitrary homogeneity degree q w.r.t. the dilation $\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}$. The right-hand side of differential equation (3.118) is assumed to be continuous. The corresponding system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \boldsymbol{f}(\boldsymbol{x}) \|\boldsymbol{x}\|_{\{\boldsymbol{r},2\}}^{-q} = \tilde{\boldsymbol{f}}(\boldsymbol{x}) \tag{3.119}$$

of homogeneity degree $\tilde{q} = 0$ with equivalent trajectories is obtained using the time scaling $\zeta(\boldsymbol{x}) = \|\boldsymbol{x}\|_{\{\boldsymbol{r},2\}}^{q}$, see Section 2.3.2. For this choice of $\zeta(\boldsymbol{x})$ the homogeneous eigenvalue equation of the original system (3.118), i.e.

$$\boldsymbol{f}(\boldsymbol{v}) = \lambda \|\boldsymbol{v}\|_{\{\boldsymbol{r},2\}}^{q} \boldsymbol{\nu}(\boldsymbol{v}) \tag{3.120}$$

and the corresponding system (3.119), i.e.

$$\tilde{\boldsymbol{f}}(\boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{v})\|\boldsymbol{v}\|_{\{\boldsymbol{r},2\}}^{-q} = \tilde{\lambda}\boldsymbol{\nu}(\boldsymbol{v}) \tag{3.121}$$

are equivalent and, therefore, the homogeneous eigenvalues are identical

$$\lambda = \tilde{\lambda}. \tag{3.122}$$

For an initial state $\boldsymbol{x}_0 = \begin{pmatrix} x_{1,0} & x_{2,0} & \dots & x_{n,0} \end{pmatrix}^T$ on a real homogeneous eigenvector $\boldsymbol{v}$ the differential equation of the corresponding system (3.119) reduces to

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \lambda\boldsymbol{\nu}(\boldsymbol{x}) \Leftrightarrow \begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}\tau} \\ \frac{\mathrm{d}x_2}{\mathrm{d}\tau} \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}\tau} \end{pmatrix} = \begin{pmatrix} r_1\lambda x_1 \\ r_2\lambda x_2 \\ \vdots \\ r_n\lambda x_n \end{pmatrix}. \tag{3.123}$$

The solution of differential equation (3.123) yields

$$\boldsymbol{x}(\tau) = \begin{pmatrix} x_{1,0}\mathrm{e}^{r_1\lambda\tau} \\ x_{2,0}\mathrm{e}^{r_2\lambda\tau} \\ \vdots \\ x_{n,0}\mathrm{e}^{r_n\lambda\tau} \end{pmatrix}. \tag{3.124}$$

The dilation coefficients $r_i > 0\ \forall i$ are strictly positive numbers. Three cases of λ can be distinguished.

- If $\lambda < 0$, then $\lim\limits_{\tau\to\infty} \boldsymbol{x}(\tau) = \boldsymbol{0}$. Due to the equivalence of trajectories the solution of system (3.118) for arbitrary $\boldsymbol{x}_0$ located on the eigenvector $\boldsymbol{v}$ converges to the zero equilibrium state.
- $\lambda = 0$ yields the constant solution $\boldsymbol{x}(\tau) = \boldsymbol{x}_0$. This means that in an arbitrary small vicinity of the origin there exists a solution which is constant and does not converge to the zero equilibrium state. Therefore, the origin is not asymptotically stable.
- If $\lambda > 0$, the trajectory (3.124) diverges along the eigenvector for $\tau \to \infty$ even if the initial state $\boldsymbol{x}_0 \in \boldsymbol{v}$ is located arbitrarily close to the origin. For this reason the origin of the corresponding system (3.119) is unstable and due to the equivalence of trajectories the origin of system (3.118) is unstable.

In summary, if there exists a real homogeneous eigenvector $\boldsymbol{v}_i$ with $\lambda_i > 0$ / $\lambda_i = 0$, the origin of system (3.118) is unstable / not asymptotically stable. In conclusion, if the zero equilibrium state of system (3.118)

- is stable, all real homogeneous eigenvalues satisfy $\lambda_i \leq 0$, $\forall i$.
- is asymptotically stable, all real homogeneous eigenvalues satisfy $\lambda_i < 0$, $\forall i$.

Sufficient criterion for asymptotic stability of the origin

This Section summarizes the results of H. Nakamura et al. [12] which provide a sufficient condition for global asymptotic stability of the origin of the considered homogeneous system (3.118).

In Section 3.3.3 it is proved that homogeneous eigenvectors are homogeneous rays. Therefore, the projection of a homogeneous eigenvector $\boldsymbol{v}$ onto the Euler sphere $E_1^{\boldsymbol{r}}$ along homogeneous rays is the intersection point $\boldsymbol{v}_E$ of the eigenvector and the Euler sphere. Due to $\boldsymbol{v}_E \in \boldsymbol{v}$ the right-hand side of system (3.118) reduces to

$$\boldsymbol{f}(\boldsymbol{v}_E) = \lambda \|\boldsymbol{v}_E\|_{\{\boldsymbol{r},2\}}^q \boldsymbol{\nu}(\boldsymbol{v}_E) \tag{3.125}$$

in the point $\boldsymbol{x} = \boldsymbol{v}_E$. The evaluation of the projection vector field $\boldsymbol{f}_0(\boldsymbol{v}_E)$ yields

$$\begin{aligned}
\boldsymbol{f}_0(\boldsymbol{v}_E) &= \boldsymbol{f}(\boldsymbol{v}_E) - \frac{\boldsymbol{\nu}^T(\boldsymbol{v}_E)\boldsymbol{f}(\boldsymbol{v}_E)}{\|\boldsymbol{\nu}(\boldsymbol{v}_E)\|^2}\boldsymbol{\nu}(\boldsymbol{v}_E) = \\
&= \lambda \|\boldsymbol{v}_E\|_{\{\boldsymbol{r},2\}}^q \boldsymbol{\nu}(\boldsymbol{v}_E) - \frac{\boldsymbol{\nu}^T(\boldsymbol{v}_E)\lambda\|\boldsymbol{v}_E\|_{\{\boldsymbol{r},2\}}^q\boldsymbol{\nu}(\boldsymbol{v}_E)}{\|\boldsymbol{\nu}(\boldsymbol{v}_E)\|^2}\boldsymbol{\nu}(\boldsymbol{v}_E) = \\
&= \lambda \|\boldsymbol{v}_E\|_{\{\boldsymbol{r},2\}}^q \boldsymbol{\nu}(\boldsymbol{v}_E) - \lambda \|\boldsymbol{v}_E\|_{\{\boldsymbol{r},2\}}^q \frac{\|\boldsymbol{\nu}(\boldsymbol{v}_E)\|^2}{\|\boldsymbol{\nu}(\boldsymbol{v}_E)\|^2}\boldsymbol{\nu}(\boldsymbol{v}_E) = \mathbf{0}.
\end{aligned} \tag{3.126}$$

Obviously the projection system possesses an equilibrium point in $\boldsymbol{v}_E$. In fact, all the equilibrium points of the projection system correspond to homogeneous eigenvectors which is shown below. Consider one equilibrium point $\boldsymbol{x}_{proj,e} \in E_1^{\boldsymbol{r}}$ of the projection system which has to satisfy

$$\boldsymbol{f}_0(\boldsymbol{x}_{proj,e}) = \mathbf{0}. \tag{3.127}$$

The projection vector field $\boldsymbol{f}_0(\boldsymbol{x})$ describes the tangential component of the original vector field $\boldsymbol{f}(\boldsymbol{x})$. Therefore, $\boldsymbol{f}(\boldsymbol{x}_{proj,e})$ has to be orthogonal to the Euler sphere because the tangential component is zero. The Euler vector field is orthogonal to the Euler sphere. For this reason, the direction of $\boldsymbol{f}(\boldsymbol{x}_{proj,e})$ and $\boldsymbol{\nu}(\boldsymbol{x}_{proj,e})$ coincide and the relation

$$\boldsymbol{f}(\boldsymbol{x}_{proj,e}) = K\boldsymbol{\nu}(\boldsymbol{x}_{proj,e}) \tag{3.128}$$

holds, where K is a real constant. Substituting

$$K = \lambda \|\boldsymbol{x}_{proj,e}\|_{\{\boldsymbol{r},2\}}^q \tag{3.129}$$

is feasible because $\|\boldsymbol{x}_{proj,e}\|_{\{\boldsymbol{r},2\}}^q$ is constant too and λ is obtained by solving equation (3.129) which yields

$$\lambda = \frac{K}{\|\boldsymbol{x}_{proj,e}\|_{\{\boldsymbol{r},2\}}^q}. \tag{3.130}$$

Therefore, $\boldsymbol{x}_{proj,e}$ fulfills the homogeneous eigenvalue equation

$$\boldsymbol{f}(\boldsymbol{x}_{proj,e}) = \lambda \|\boldsymbol{x}_{proj,e}\|_{\{\boldsymbol{r},2\}}^q \boldsymbol{\nu}(\boldsymbol{x}_{proj,e}). \tag{3.131}$$

In consequence, every equilibrium point of the projection system corresponds to a homogeneous eigenvector and vice versa.

Consider a solution $\boldsymbol{x}_{proj}(t)$ of the projection system with initial state $\boldsymbol{x}_{proj}(0) \in E_1^{\boldsymbol{r}}$ which converges to the intersection point of the Euler sphere and a real homogeneous eigenvector $\boldsymbol{v}_i$ for $t \to \infty$. Due to the equivalence of trajectories of projection solution and solution of the projection system, every solution $\boldsymbol{x}(t)$ of the original system (3.118) converges to the eigenvector $\boldsymbol{v}_i$ for $t \to \infty$ if the projection of $\boldsymbol{x}(0)$ is located somewhere on the trajectory of $\boldsymbol{x}_{proj}(t)$. In Section 3.3.4 it is shown that the solution curve of the original system converges to the origin if the initial state is located on the homogeneous eigenvector and the homogeneous eigenvalue is negative.

Therefore, every solution $\boldsymbol{x}(t)$ converges to the origin for $t \to \infty$ if

- the projection of the initial state $\boldsymbol{x}(0)$ is located somewhere on $\boldsymbol{x}_{proj}(t)$ and
- $\lambda_i < 0$ is satisfied.

If all solution curves of the projection system converge to points for arbitrary $\boldsymbol{x}_{proj}(0) \in E_1^{\boldsymbol{r}}$, the trajectories of the original system (3.118) converge to real homogeneous eigenvectors for arbitrary $\boldsymbol{x}(0)$. So if all the homogeneous eigenvalues λ_i are negative, the trajectories converge to the origin for arbitrary initial state.

Therefore, the origin of system (3.118) is globally asymptotically stable if

- the solution curves of the projection system converge to points for arbitrary initial state $\boldsymbol{x}_{proj}(0) \in E_1^{\boldsymbol{r}}$ and
- all the homogeneous eigenvalues are negative $\lambda_i < 0 \ \forall i$.

The first condition is very abstract but can be simplified for lower dimensional systems. In the planar case the Euler sphere is an ellipse. The only possibility to violate the first condition is that the projection system produces a limit cycle along this ellipse. So if there exists at least one real homogeneous eigenvector/eigenvalue the projection system has an equilibrium point located somewhere on the ellipse and no limit cycles can occur.

For this reason, the origin of system (3.118) is globally asymptotically stable if

- the system is planar $n = 2$,
- there exists at least one real homogeneous eigenvalue and
- all the homogeneous eigenvalues are negative $\lambda_i < 0 \ \forall i$.

For $n = 3$ the Euler sphere is an ellipsoid. In this case only limit cycles on the ellipsoid's surface would violate the first condition.

Therefore, the origin of system (3.118) is globally asymptotically stable if

- $n = 3$,
- the the projection system does not produce any limit cycles for arbitrary initial state $\boldsymbol{x}_{proj}(0) \in E_1^{\boldsymbol{r}}$ and
- all the homogeneous eigenvalues are negative $\lambda_i < 0 \ \forall i$.

3.3.5 Example: LTI System

In this example the homogeneous eigenvalues and the homogeneous eigenvectors of an LTI system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}) \tag{3.132}$$

of arbitrary order n are analyzed. The right-hand side of system (3.132) is a vector field

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{1,1}x_1 + \dots + a_{1,n}x_n \\ \vdots \\ a_{n,1}x_1 + \dots + a_{n,n}x_n \end{pmatrix} \tag{3.133}$$

in which each row is a linear combination of the state variables. First of all the degree of homogeneity and the dilation coefficients have to be determined by applying the dilation operator to the state vector

$$\boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{x}) = \begin{pmatrix} a_{1,1}\varepsilon^{r_1}x_1 + \dots + a_{1,n}\varepsilon^{r_n}x_n \\ \vdots \\ a_{n,1}\varepsilon^{r_1}x_1 + \dots + a_{n,n}\varepsilon^{r_n}x_n \end{pmatrix} \stackrel{!}{=} \varepsilon^q \begin{pmatrix} a_{1,1}\varepsilon^{r_1}x_1 + \dots + a_{1,n}\varepsilon^{r_1}x_n \\ \vdots \\ a_{n,1}\varepsilon^{r_n}x_1 + \dots + a_{n,n}\varepsilon^{r_n}x_n \end{pmatrix} = \varepsilon^q \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{x}). \tag{3.134}$$

A comparison of the coefficients leads to the linear system of equations

$$r_i = q + r_j, \qquad i = 1, \dots, n, \; j = 1, \dots, n. \tag{3.135}$$

From $i = j$ one obtains

$$q = 0. \tag{3.136}$$

For $i \neq j$ one has

$$r_i = r_j. \tag{3.137}$$

The most general choice of the dilation coefficients is an arbitrary positive constant

$$r_i = \eta, \qquad \eta > 0, \; i = 1, \dots, n. \tag{3.138}$$

This means that every linear time-invariant system is homogeneous of degree $q = 0$ w.r.t. the dilation coefficients $\boldsymbol{r} = \eta \cdot \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^T$.

The corresponding homogeneous eigenvalue equation

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{\nu}(\boldsymbol{v}) \tag{3.139}$$

is further simplified using

$$\boldsymbol{\nu}(\boldsymbol{v}) = \eta \boldsymbol{v} \tag{3.140}$$

which arises from the dilation coefficient vector $\boldsymbol{r} = \eta \cdot \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^T$. The resulting relation

$$\boldsymbol{A}\boldsymbol{v} = \lambda \eta \boldsymbol{v} \tag{3.141}$$

recovers the linear eigenvalue equation with the linear eigenvalues

$$s = \lambda \eta. \tag{3.142}$$

The linear and the homogeneous eigenvalues are equal for the choice $\eta = 1$. The concept of homogeneous eigenvalues is a generalization of the special case of linear eigenvalues.

3.3.6 Example: Super-Twisting Algorithm

The super-twisting algorithm produces a second order homogeneous system of homogeneity degree $q = -1$ w.r.t. the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ as shown in Section 2.3.5. In this Section the homogeneous eigenvalues and eigenvectors of the super-twisting algorithm are calculated.

The computation of the the homogeneous eigenvalues and eigenvectors is done for the corresponding system of homogeneity degree $\tilde{q} = 0$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}\tau} \\ \frac{\mathrm{d}x_2}{\mathrm{d}\tau} \end{pmatrix} = \begin{pmatrix} |x_1|^{\frac{1}{2}} x_2 - k_1 x_1 \\ -k_2 |x_1|^{\frac{1}{2}} \operatorname{sign}(x_1) \end{pmatrix} = \tilde{\boldsymbol{f}}(\boldsymbol{x}) \tag{3.143}$$

which is derived in Section 2.3.5. Then the homogeneous eigenvalues of the original system are determined using relation (3.110).

The corresponding system (3.143) is rewritten in pseudo-linear system representation

$$\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}\tau} \\ \frac{\mathrm{d}x_2}{\mathrm{d}\tau} \end{pmatrix} = \underbrace{\begin{pmatrix} -k_1 & |x_1|^{\frac{1}{2}} \\ -k_2 |x_1|^{-\frac{1}{2}} & 0 \end{pmatrix}}_{\boldsymbol{M}(\boldsymbol{x})} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.144}$$

with the state-dependent system matrix $\boldsymbol{M}(\boldsymbol{x})$. The eigenvalues are computed using equation (3.83), i.e.

$$\det\left(\tilde{\lambda}\boldsymbol{I} - \|\boldsymbol{v}\|_{\{\boldsymbol{r},2\}}^{-\tilde{q}} \boldsymbol{B}^{-1} \boldsymbol{M}(\boldsymbol{v})\right) = 0. \tag{3.145}$$

derived in Section 3.3.3. Due to $\tilde{q} = 0$ the homogeneous norm vanishes and (3.145) simplifies to

$$\det\left(\tilde{\lambda}\boldsymbol{I} - \boldsymbol{B}^{-1} \boldsymbol{M}(\boldsymbol{v})\right) = 0. \tag{3.146}$$

The matrix $\boldsymbol{B}$ is a constant diagonal matrix with the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$ in the main diagonal, i.e.

$$\boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.147}$$

The inverse of a diagonal matrix is found by inverting each element of the main diagonal which results in

$$\boldsymbol{B}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.148}$$

Then the characteristic polynomial of $\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{v})$ computes to

$$
\begin{aligned}
\det\left(\tilde{\lambda}\boldsymbol{I} - \boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{v})\right) &= \det\left(\begin{pmatrix}\tilde{\lambda} & 0\\ 0 & \tilde{\lambda}\end{pmatrix} - \begin{pmatrix}\frac{1}{2} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-k_1 & |v_1|^{\frac{1}{2}}\\ -k_2|v_1|^{-\frac{1}{2}} & 0\end{pmatrix}\right) = \\
&= \det\left(\begin{pmatrix}\tilde{\lambda} & 0\\ 0 & \tilde{\lambda}\end{pmatrix} - \begin{pmatrix}-\frac{1}{2}k_1 & \frac{1}{2}|v_1|^{\frac{1}{2}}\\ -k_2|v_1|^{-\frac{1}{2}} & 0\end{pmatrix}\right) = \\
&= \det\begin{pmatrix}\tilde{\lambda} + \frac{1}{2}k_1 & -\frac{1}{2}|v_1|^{\frac{1}{2}}\\ k_2|v_1|^{-\frac{1}{2}} & \tilde{\lambda}\end{pmatrix} = \\
&= \tilde{\lambda}^2 + \frac{1}{2}k_1\tilde{\lambda} + \frac{1}{2}k_2.
\end{aligned} \tag{3.149}
$$

The homogeneous eigenvalues of the corresponding system are calculated by setting the characteristic polynomial (3.149) to zero and solving the quadratic equation which yields

$$
\tilde{\lambda}_{1,2} = -\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2 - 8k_2}. \tag{3.150}
$$

The homogeneous eigenvalues of the original system with homogeneity degree $q = 1$ are then found using relation (3.110) which gives

$$
\lambda_{1,2} = \underbrace{(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2 - 8k_2})}_{\tilde{\lambda}_{1,2}} \cdot \underbrace{|v_1|^{-\frac{1}{2}}}_{\zeta(\boldsymbol{v})} \cdot \underbrace{\sqrt{|v_1| + |v_2|^2}}_{\|\boldsymbol{v}\|_{\{\boldsymbol{r},2\}}^{-q}} \tag{3.151}
$$

In the further derivations it is assumed that the control parameters k_1 and k_2 are chosen such that inequality

$$
k_1^2 - 8k_2 \geq 0 \tag{3.152}
$$

holds. This ensures that the homogeneous eigenvalues are real numbers.

The homogeneous eigenvectors are computed by solving the system of equations (3.82) which yields

$$
\begin{pmatrix}\tilde{\lambda} + \frac{1}{2}k_1 & -\frac{1}{2}|v_1|^{\frac{1}{2}}\\ k_2|v_1|^{-\frac{1}{2}} & \tilde{\lambda}\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{3.153}
$$

for the corresponding system of homogeneity degree $\tilde{q} = 0$. By inserting the homogeneous eigenvalues of the corresponding system (3.150) the system of equations (3.153) is rewritten to

$$
\begin{pmatrix}\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2 - 8k_2} & -\frac{1}{2}|v_1|^{\frac{1}{2}}\\ k_2|v_1|^{-\frac{1}{2}} & -\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2 - 8k_2}\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{3.154}
$$

The evaluation of the matrix-vector product leads to

$$
\begin{aligned}
\Big(\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2 - 8k_2}\Big)v_1 + \Big(-\frac{1}{2}|v_1|^{\frac{1}{2}}\Big)v_2 &= 0,\\
\Big(k_2|v_1|^{-\frac{1}{2}}\Big)v_1 + \Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2 - 8k_2}\Big)v_2 &= 0.
\end{aligned} \tag{3.155}
$$

The variable v_2 is eliminated by multiplying the first equation of (3.155) with $2|v_1|^{-\frac{1}{2}}\Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)$ and adding the result to the second equation which yields

$$\Big(k_2|v_1|^{-\frac{1}{2}}\Big)v_1 + 2|v_1|^{-\frac{1}{2}}\Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)\Big(\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)v_1 = 0. \tag{3.156}$$

Under assumption (3.152) expansion of equation (3.156) and extraction of v_1 leads to

$$\Big(k_2|v_1|^{-\frac{1}{2}} + 2|v_1|^{-\frac{1}{2}}\big(-\frac{1}{16}k_1^2 + \frac{1}{16}(k_1^2-8k_2)\big)\Big)v_1 = 0. \tag{3.157}$$

The trivial solution for the eigenvector is not permitted. For this reason

$$k_2|v_1|^{-\frac{1}{2}} + 2|v_1|^{-\frac{1}{2}}\big(-\frac{1}{16}k_1^2 + \frac{1}{16}(k_1^2-8k_2)\big) = 0 \tag{3.158}$$

has to hold. The left-hand side of equation (3.158) further simplifies to

$$\begin{aligned}
&k_2|v_1|^{-\frac{1}{2}} + 2|v_1|^{-\frac{1}{2}}\big(-\frac{1}{16}k_1^2 + \frac{1}{16}(k_1^2-8k_2)\big) = \\
&\quad = k_2|v_1|^{-\frac{1}{2}} + 2|v_1|^{-\frac{1}{2}}\big(-\frac{1}{16}k_1^2 + \frac{1}{16}k_1^2 - \frac{1}{2}k_2\big) = \\
&\quad = k_2|v_1|^{-\frac{1}{2}} - 2|v_1|^{-\frac{1}{2}}\cdot\frac{1}{2}k_2 = \\
&\quad = 0.
\end{aligned} \tag{3.159}$$

This means that equation (3.156) is always true. In consequence the second equation of (3.155) can be omitted because both equations are linear dependent. This linear dependence is mandatory in eigenvalue problems and can be used to check the correctness of the calculations.

The reduced system of equations

$$\Big(\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)v_1 + \Big(-\frac{1}{2}|v_1|^{\frac{1}{2}}\Big)v_2 = 0 \tag{3.160}$$

is underdetermined and, therefore, every arbitrary real number η is a valid choice for v_1, i.e.

$$v_1 = \eta, \qquad \eta \in \mathbb{R}. \tag{3.161}$$

In the next step equation (3.160) is solved for v_2 which results in

$$v_{2_{1,2}} = \frac{1}{2}\Big(k_1 \pm \sqrt{k_1^2-8k_2}\Big)|\eta|^{\frac{1}{2}}\,\mathrm{sign}(\eta). \tag{3.162}$$

The eigenvectors of the super-twisting algorithm are expressed in vector notation

$$\boldsymbol{v}_{1,2} = \begin{pmatrix} \eta \\ \frac{1}{2}\Big(k_1 \pm \sqrt{k_1^2-8k_2}\Big)|\eta|^{\frac{1}{2}}\,\mathrm{sign}(\eta)\end{pmatrix}, \qquad \eta \in \mathbb{R}. \tag{3.163}$$

Alternatively, function notation of the eigenvectors yields

$$v_{2_{1,2}}(v_1) = \frac{1}{2}\Big(k_1 \pm \sqrt{k_1^2-8k_2}\Big)|v_1|^{\frac{1}{2}}\,\mathrm{sign}(v_1). \tag{3.164}$$

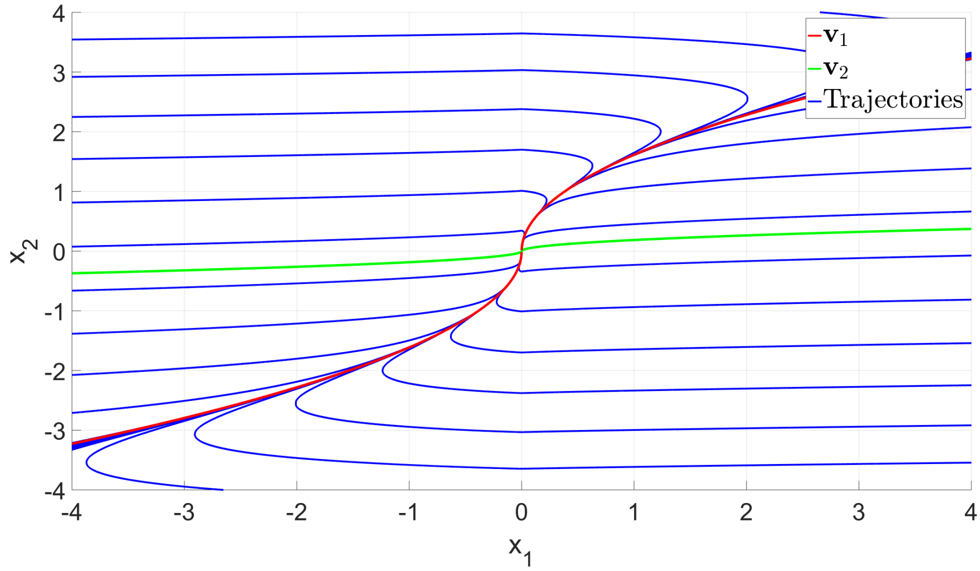

Figure 3.5: Trajectory plot and eigenvectors of the super-twisting algorithm. The control parameters are chosen $k_1 = 1.8$ and $k_2 = 0.15$.

For this reason the eigenvectors of the super-twisting algorithm are sign preserving square root functions in state space.

With the knowledge of the eigenvectors the eigenvalues in equation (3.151) can be computed. Insertion of equation (3.163) and some simplifications yield

$$\begin{aligned}\lambda_{1,2} &= \Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)\cdot|v_1|^{-\frac{1}{2}}\cdot\sqrt{|v_1|+|v_2|^2} = \\ &= \Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)\cdot|\eta|^{-\frac{1}{2}}\cdot\sqrt{|\eta|+\Big|\frac{1}{2}\Big(k_1\pm\sqrt{k_1^2-8k_2}\Big)|\eta|^{\frac{1}{2}}\operatorname{sign}(\eta)\Big|^2} = \\ &= \Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)\cdot|\eta|^{-\frac{1}{2}}\cdot\sqrt{|\eta|\Big(1+\Big|\frac{1}{2}\Big(k_1\pm\sqrt{k_1^2-8k_2}\Big)\Big|^2\Big)} = \\ &= \Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)\cdot|\eta|^{-\frac{1}{2}}\cdot|\eta|^{\frac{1}{2}}\cdot\sqrt{\Big(1+\Big|\frac{1}{2}\Big(k_1\pm\sqrt{k_1^2-8k_2}\Big)\Big|^2\Big)} = \\ &= \Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)\cdot\sqrt{\Big(1+\Big|\frac{1}{2}\Big(k_1\pm\sqrt{k_1^2-8k_2}\Big)\Big|^2\Big)}.\end{aligned} \tag{3.165}$$

Due to assumption (3.152) the absolute value function can be dropped which leads to the final result

$$\lambda_{1,2} = \Big(-\frac{1}{4}k_1 \pm \frac{1}{4}\sqrt{k_1^2-8k_2}\Big)\cdot\sqrt{\Big(1+\frac{1}{4}\big(k_1\pm\sqrt{k_1^2-8k_2}\big)^2\Big)}. \tag{3.166}$$

Figure 3.5 shows a trajectory plot of the super-twisting algorithm. The parameters are chosen

$k_1 = 1.8$ and $k_2 = 0.15$. The homogeneous eigenvalues

$$\lambda_1 \approx -0.1765, \lambda_2 \approx -0.8209 \tag{3.167}$$

are obtained using equation (3.166). The homogeneous eigenvectors are described by the functions

$$\begin{aligned} x_{2_1}(x_1) &\approx 1.6141 \cdot |x_1|^{\frac{1}{2}} \operatorname{sign}(x_1), \\ x_{2_2}(x_1) &\approx 0.1859 \cdot |x_1|^{\frac{1}{2}} \operatorname{sign}(x_1) \end{aligned} \tag{3.168}$$

using equation (3.164). All trajectories seem to converge to the first eigenvector and in consequence to the origin.

The sufficient stability criterion from Section 3.3.4 is not applicable directly because the right-hand side of the super-twisting algorithm is discontinuous. In this case the stability of the origin can be proved using the corresponding system of homogeneity degree $\tilde{q} = 0$ which, in contrast, has a continuous right-hand side. From equation (3.151) it is obvious that the homogeneous eigenvalues of the original system and the corresponding system have the same sign. For this reason, the origin of the corresponding system of homogeneity degree $\tilde{q} = 0$ is globally asymptotically stable for this choice of parameters k_1 and k_2 because the system is planar and there exist two real homogeneous eigenvalues which are both negative. Due to the equivalence of trajectories the origin of the original system of the super-twisting algorithm is globally asymptotically stable.

4 Observer design based on homogeneous eigenvalue assignment

The relation between the stability behavior of a homogeneous system and its homogeneous eigenvalues can be exploited for controller and observer design. In this Chapter an approach for the design of homogeneous observers for LTI-systems is presented. First of all, a homogeneous observer for a chain of integrators is derived which makes use of homogeneous eigenvalue assignment. This illustrative example shows the difficulties that may occur and points out some conditions on the choice of the dilation coefficients and the homogeneity degree. Subsequently, this approach is further generalized to create homogeneous observers for arbitrary LTI-systems. For this purpose a generalization of Ackermann's formula is derived. Additionally, a robust observer design for strongly observable systems is proposed. Finally, a simulation demonstrates the effectiveness of the presented design approach.

4.1 Homogeneous observer for a chain of n integrators

Consider the chain of n integrators

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= u \\
\\
y &= x_1
\end{aligned} \tag{4.1}$$

where $\boldsymbol{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix}^T$ is the state vector, u is the input and y is the output of the system. The observer

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + l_1(\sigma_1) \\
\dot{\hat{x}}_2 &= \hat{x}_3 + l_2(\sigma_1) \\
&\vdots \\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + l_{n-1}(\sigma_1) \\
\dot{\hat{x}}_n &= u + l_n(\sigma_1) \\
\\
\hat{y} &= \hat{x}_1
\end{aligned} \tag{4.2}$$

consists of a copy of the plant and non-linear injection terms $l_i(\sigma_1)$, $i = 1 \dots n$ which possibly depend on the estimation error of the output

$$\sigma_1 = y - \hat{y} = x_1 - \hat{x}_1. \tag{4.3}$$

The dynamics of the estimation errors $\sigma_i = x_i - \hat{x}_i$, $i = 1 \dots n$ result in

$$\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 - l_1(\sigma_1)\\
\dot{\sigma}_2 &= \sigma_3 - l_2(\sigma_1)\\
&\vdots\\
\dot{\sigma}_{n-1} &= \sigma_n - l_{n-1}(\sigma_1)\\
\dot{\sigma}_n &= -l_n(\sigma_1).
\end{aligned} \tag{4.4}$$

The non-linear injection terms $l_i(\sigma_1)$ are chosen such that the dynamics of the estimation error (4.4) are homogeneous of degree q w.r.t. the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & r_2 & \dots & r_n \end{pmatrix}^T$ and the homogeneous eigenvalues of the corresponding system of homogeneity degree $\tilde{q} = 0$ are $\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n$. The corresponding system of homogeneity degree $\tilde{q} = 0$ with equivalent trajectories yields

$$\begin{aligned}
\frac{\mathrm{d}\sigma_1}{\mathrm{d}\tau} &= \zeta(\boldsymbol{\sigma})^{-1}\sigma_2 - \zeta(\boldsymbol{\sigma})^{-1} l_1(\sigma_1)\\
\frac{\mathrm{d}\sigma_2}{\mathrm{d}\tau} &= \zeta(\boldsymbol{\sigma})^{-1}\sigma_3 - \zeta(\boldsymbol{\sigma})^{-1} l_2(\sigma_1)\\
&\vdots\\
\frac{\mathrm{d}\sigma_{n-1}}{\mathrm{d}\tau} &= \zeta(\boldsymbol{\sigma})^{-1}\sigma_n - \zeta(\boldsymbol{\sigma})^{-1} l_{n-1}(\sigma_1)\\
\frac{\mathrm{d}\sigma_n}{\mathrm{d}\tau} &= -\zeta(\boldsymbol{\sigma})^{-1} l_n(\sigma_1)
\end{aligned} \tag{4.5}$$

for the general time scaling $\zeta(\boldsymbol{\sigma})$ which depends on the elements of the estimation error vector $\boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix}^T$. The pseudo-linear system representation of system (4.5), given by

$$\begin{pmatrix} \frac{\mathrm{d}\sigma_1}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\sigma_2}{\mathrm{d}\tau} \\ \vdots \\ \frac{\mathrm{d}\sigma_{n-2}}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\sigma_{n-1}}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\sigma_n}{\mathrm{d}\tau} \end{pmatrix} = \underbrace{\begin{pmatrix}
-\zeta(\boldsymbol{\sigma})^{-1} l_1(\sigma_1)\sigma_1^{-1} & \zeta(\boldsymbol{\sigma})^{-1} & 0 & \dots & \dots & 0\\
-\zeta(\boldsymbol{\sigma})^{-1} l_2(\sigma_1)\sigma_1^{-1} & 0 & \zeta(\boldsymbol{\sigma})^{-1} & \ddots & & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
-\zeta(\boldsymbol{\sigma})^{-1} l_{n-2}(\sigma_1)\sigma_1^{-1} & \vdots & & \ddots & \zeta(\boldsymbol{\sigma})^{-1} & 0\\
-\zeta(\boldsymbol{\sigma})^{-1} l_{n-1}(\sigma_1)\sigma_1^{-1} & \vdots & & & \ddots & \zeta(\boldsymbol{\sigma})^{-1}\\
-\zeta(\boldsymbol{\sigma})^{-1} l_n(\sigma_1)\sigma_1^{-1} & 0 & \dots & \dots & \dots & 0
\end{pmatrix}}_{\boldsymbol{M}(\boldsymbol{\sigma})} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{n-2} \\ \sigma_{n-1} \\ \sigma_n \end{pmatrix} \tag{4.6}$$

is well-defined for all $\boldsymbol{\sigma} \in \mathbb{R}^n$ except for possibly $\sigma_1 = 0$ and zeros of the time scaling $\zeta(\boldsymbol{\sigma})$ which is considered later. It is assumed that $l_i(\sigma_1)$ does not contain any singularities $\forall i$ because the right-hand side of the error dynamics (4.4) would tend to $\pm\infty$. Therefore, this is independent of the pseudo-linear system representation but rather a problem of the system itself.

The homogeneous eigenvalues $\tilde{\lambda}_i$ of the corresponding system (4.5) are the roots of the characteristic polynomial of the matrix $\|\boldsymbol{\sigma}\|_{\{\boldsymbol{r},2\}}^{-\tilde{q}}\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{\sigma})$ which yields for $\tilde{q}=0$

$$\det\left(\tilde{\lambda}\boldsymbol{I}-\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{\sigma})\right)\stackrel{!}{=}0. \tag{4.7}$$

The matrix $\boldsymbol{B}$ is a diagonal matrix with the dilation coefficients r_i in the main diagonal. Due to $r_i>0\ \forall i$ the inverse always exists and results in an inversion of the diagonal elements

$$\boldsymbol{B}=\begin{pmatrix} r_1 & 0 & \dots & 0\\ 0 & r_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & r_n\end{pmatrix}\Leftrightarrow\boldsymbol{B}^{-1}=\begin{pmatrix} \frac{1}{r_1} & 0 & \dots & 0\\ 0 & \frac{1}{r_2} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & \frac{1}{r_n}\end{pmatrix}. \tag{4.8}$$

Insertion of the state-dependent system matrix $\boldsymbol{M}(\boldsymbol{\sigma})$ and $\boldsymbol{B}^{-1}$ allows to rewrite the left-hand side of equation (4.7) to

$$\det\left(\tilde{\lambda}\boldsymbol{I}-\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{\sigma})\right)=\begin{vmatrix} \tilde{\lambda}+\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_1}l_1(\sigma_1)\sigma_1^{-1} & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_1} & 0 & \dots & \dots & 0\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_2}l_2(\sigma_1)\sigma_1^{-1} & \tilde{\lambda} & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_2} & \ddots & & \vdots\\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-2}}l_{n-2}(\sigma_1)\sigma_1^{-1} & \vdots & \ddots & \ddots & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-2}} & 0\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-1}}l_{n-1}(\sigma_1)\sigma_1^{-1} & \vdots & & \ddots & \tilde{\lambda} & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-1}}\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_n}l_n(\sigma_1)\sigma_1^{-1} & 0 & \dots & \dots & 0 & \tilde{\lambda}\end{vmatrix}. \tag{4.9}$$

From linear algebra it is known that the determinant of a matrix does not change if a multiple of one row is added to another one. This property can be used to eliminate the entries in the upper secondary diagonal of (4.9) which yields

$$\begin{vmatrix} \tilde{\lambda}+\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_1}l_1(\sigma_1)\sigma_1^{-1} & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_1} & 0 & \dots & \dots & 0\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_2}l_2(\sigma_1)\sigma_1^{-1} & \tilde{\lambda} & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_2} & \ddots & & \vdots\\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-2}}l_{n-2}(\sigma_1)\sigma_1^{-1} & \vdots & \ddots & \ddots & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-2}} & 0\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-1}}l_{n-1}(\sigma_1)\sigma_1^{-1} & \vdots & & \ddots & \tilde{\lambda} & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-1}}\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_n}l_n(\sigma_1)\sigma_1^{-1} & 0 & \dots & \dots & 0 & \tilde{\lambda}\end{vmatrix}\begin{array}{l}\leftarrow +\\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-1}}\tilde{\lambda}^{-1}\end{array}=$$

$$
= \begin{vmatrix}
\tilde{\lambda} + \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_1} l_1(\sigma_1)\sigma_1^{-1} & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_1} & 0 & \dots & \dots & 0 \\
\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_2} l_2(\sigma_1)\sigma_1^{-1} & \tilde{\lambda} & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_2} & \ddots & & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-2}} l_{n-2}(\sigma_1)\sigma_1^{-1} & \vdots & \ddots & \ddots & -\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-2}} & 0 \\
\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-1}} l_{n-1}(\sigma_1)\sigma_1^{-1} + \frac{\zeta(\boldsymbol{\sigma})^{-2}}{r_{n-1}r_n}\tilde{\lambda}^{-1} l_n(\sigma_1)\sigma_1^{-1} & \vdots & & \ddots & \tilde{\lambda} & 0 \\
\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_n} l_n(\sigma_1)\sigma_1^{-1} & 0 & \dots & \dots & 0 & \tilde{\lambda}
\end{vmatrix}
\begin{array}{l} \\ \\ \\ \leftarrow + \\ \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-2}}\tilde{\lambda}^{-1} \\ \\ \end{array}
=
$$

$$
= \quad \dots \quad =
$$

$$
= \begin{vmatrix}
\tilde{\lambda} + \sum_{i=1}^{n} \frac{1}{\prod_{k=1}^{i} r_k} \zeta(\boldsymbol{\sigma})^{-i} \tilde{\lambda}^{-(i-1)} l_i(\sigma_1)\sigma_1^{-1} & 0 & \dots & \dots & \dots & 0 \\
\vdots & \tilde{\lambda} & \ddots & & & \vdots \\
\vdots & 0 & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-1}} l_{n-1}(\sigma_1)\sigma_1^{-1} + \frac{\zeta(\boldsymbol{\sigma})^{-2}}{r_{n-1}r_n}\tilde{\lambda}^{-1} l_n(\sigma_1)\sigma_1^{-1} & \vdots & & \ddots & \tilde{\lambda} & 0 \\
\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_n} l_n(\sigma_1)\sigma_1^{-1} & 0 & \dots & \dots & 0 & \tilde{\lambda}
\end{vmatrix} . \tag{4.10}
$$

Note that the inversion of $\tilde{\lambda}$ in the calculations above is not critical because the choice of a homogeneous eigenvalue at $\tilde{\lambda} = 0$ does not make sense anyway. The determinant of the resulting lower triangular matrix is given by the product of its diagonal elements which leads to the characteristic polynomial in $\tilde{\lambda}$

$$
\det\left(\tilde{\lambda}\boldsymbol{I} - \boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{\sigma})\right) = \begin{vmatrix}
\tilde{\lambda} + \sum_{i=1}^{n} \frac{1}{\prod_{k=1}^{i} r_k} \zeta(\boldsymbol{\sigma})^{-i} \tilde{\lambda}^{-(i-1)} l_i(\sigma_1)\sigma_1^{-1} & 0 & \dots & \dots & \dots & 0 \\
\vdots & \tilde{\lambda} & \ddots & & & \vdots \\
\vdots & 0 & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_{n-1}} l_{n-1}(\sigma_1)\sigma_1^{-1} + \frac{\zeta(\boldsymbol{\sigma})^{-2}}{r_{n-1}r_n}\tilde{\lambda}^{-1} l_n(\sigma_1)\sigma_1^{-1} & \vdots & & \ddots & \tilde{\lambda} & 0 \\
\frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_n} l_n(\sigma_1)\sigma_1^{-1} & 0 & \dots & \dots & 0 & \tilde{\lambda}
\end{vmatrix} =
$$

$$
\begin{aligned}
&=\tilde{\lambda}^n + \sum_{i=1}^{n} \frac{\zeta(\boldsymbol{\sigma})^{-i}}{\prod\limits_{k=1}^{i} r_k} l_i(\sigma_1)\sigma_1^{-1}\tilde{\lambda}^{n-i} = \\
&=\tilde{\lambda}^n + \frac{\zeta(\boldsymbol{\sigma})^{-1}}{r_1} l_1(\sigma_1)\sigma_1^{-1}\tilde{\lambda}^{n-1} + \frac{\zeta(\boldsymbol{\sigma})^{-2}}{r_1 r_2} l_2(\sigma_1)\sigma_1^{-1}\tilde{\lambda}^{n-2} + \cdots + \\
&\quad + \frac{\zeta(\boldsymbol{\sigma})^{-(n-1)}}{r_1 r_2 \cdots r_{n-1}} l_{n-1}(\sigma_1)\sigma_1^{-1}\tilde{\lambda} + \frac{\zeta(\boldsymbol{\sigma})^{-n}}{r_1 r_2 \cdots r_n} l_n(\sigma_1)\sigma_1^{-1}.
\end{aligned}
\tag{4.11}
$$

The desired roots of the characteristic polynomial are $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$. Consequently, the characteristic polynomial has to be decomposable into the linear factors $(\tilde{\lambda} - \tilde{\lambda}_i)$, $i = 1, \ldots, n$ which yields

$$
\begin{aligned}
\det\left(\tilde{\lambda}\boldsymbol{I} - \boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{\sigma})\right) &= \tilde{\lambda}^n + \sum_{i=1}^{n} \frac{\zeta(\boldsymbol{\sigma})^{-i}}{\prod\limits_{k=1}^{i} r_k} l_i(\sigma_1)\sigma_1^{-1}\tilde{\lambda}^{n-i} \stackrel{!}{=} \\
&\stackrel{!}{=} \prod_{i=1}^{n}(\tilde{\lambda} - \tilde{\lambda}_i) = \tilde{\lambda}^n + \sum_{i=1}^{n} \gamma_{n-i}\tilde{\lambda}^{n-i}
\end{aligned}
\tag{4.12}
$$

where γ_{n-i} is obtained by applying Vieta's formula [27]

$$
\gamma_{n-i} = (-1)^i \sum_{1 \le m_1 < m_2 < \cdots < m_i \le n} \tilde{\lambda}_{m_1}\tilde{\lambda}_{m_2}\cdots\tilde{\lambda}_{m_i}, \qquad i = 1, \ldots, n. \tag{4.13}
$$

A comparison of the polynomial coefficient of $\tilde{\lambda}^{n-i}$ in equation (4.12) results in

$$
\frac{\zeta(\boldsymbol{\sigma})^{-i}}{\prod\limits_{k=1}^{i} r_k} l_i(\sigma_1)\sigma_1^{-1} = \gamma_{n-i}, \qquad i = 1, \ldots, n. \tag{4.14}
$$

Hence, the non-linear injection terms obtained by solving equation (4.14) are given by

$$
l_i(\sigma_1) = \left(\prod_{k=1}^{i} r_k\right) \gamma_{n-i}\zeta(\boldsymbol{\sigma})^i \sigma_1 \qquad i = 1, \ldots, n. \tag{4.15}
$$

The left-hand side of (4.15) depends only on the error σ_1. Therefore, the right-hand side and in particular the time scaling $\zeta(\boldsymbol{\sigma})$ have to be chosen as a function of σ_1. For this reason

$$
\zeta(\boldsymbol{\sigma}) = |\sigma_1|^{\frac{q}{r_1}} \tag{4.16}
$$

seems a reasonable choice which satisfies the homogeneity condition for time scaling functions, see Section 2.3.2. Insertion of the time scaling (4.16) into equation (4.15) eventually gives

$$
\begin{aligned}
l_i(\sigma_1) &= \left(\prod_{k=1}^{i} r_k\right) \gamma_{n-i}|\sigma_1|^{\frac{i\cdot q}{r_1}}\sigma_1 \\
&= \left(\prod_{k=1}^{i} r_k\right) \gamma_{n-i}|\sigma_1|^{\frac{i\cdot q}{r_1}+1}\operatorname{sign}(\sigma_1), \qquad i = 1, \ldots, n.
\end{aligned}
\tag{4.17}
$$

Inserting the non-linear injection terms (4.17) into the corresponding system of homogeneity degree $\tilde{q} = 0$ (4.5) yields

$$\begin{aligned}
\frac{\mathrm{d}\sigma_1}{\mathrm{d}\tau} &= |\sigma_1|^{-\frac{q}{r_1}}\sigma_2 - |\sigma_1|^{-\frac{q}{r_1}} r_1\gamma_{n-1}|\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
\frac{\mathrm{d}\sigma_2}{\mathrm{d}\tau} &= |\sigma_1|^{-\frac{q}{r_1}}\sigma_3 - |\sigma_1|^{-\frac{q}{r_1}} r_1 r_2\gamma_{n-2}|\sigma_1|^{\frac{2q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
&\vdots\\
\frac{\mathrm{d}\sigma_{n-1}}{\mathrm{d}\tau} &= |\sigma_1|^{-\frac{q}{r_1}}\sigma_n - |\sigma_1|^{-\frac{q}{r_1}} r_1 r_2\cdots r_{n-1}\gamma_1|\sigma_1|^{\frac{(n-1)q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
\frac{\mathrm{d}\sigma_n}{\mathrm{d}\tau} &= -|\sigma_1|^{-\frac{q}{r_1}} r_1 r_2\cdots r_n\gamma_0|\sigma_1|^{\frac{n\cdot q}{r_1}+1}\operatorname{sign}(\sigma_1)
\end{aligned}\tag{4.18}$$

which further simplifies to

$$\begin{aligned}
\frac{\mathrm{d}\sigma_1}{\mathrm{d}\tau} &= |\sigma_1|^{-\frac{q}{r_1}}\sigma_2 - r_1\gamma_{n-1}|\sigma_1|\operatorname{sign}(\sigma_1)\\
\frac{\mathrm{d}\sigma_2}{\mathrm{d}\tau} &= |\sigma_1|^{-\frac{q}{r_1}}\sigma_3 - r_1 r_2\gamma_{n-2}|\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
&\vdots\\
\frac{\mathrm{d}\sigma_{n-1}}{\mathrm{d}\tau} &= |\sigma_1|^{-\frac{q}{r_1}}\sigma_n - r_1 r_2\cdots r_{n-1}\gamma_1|\sigma_1|^{\frac{(n-2)q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
\frac{\mathrm{d}\sigma_n}{\mathrm{d}\tau} &= -r_1 r_2\cdots r_n\gamma_0|\sigma_1|^{\frac{(n-1)q}{r_1}+1}\operatorname{sign}(\sigma_1).
\end{aligned}\tag{4.19}$$

Parts of the right-hand side of the corresponding system (4.19) tend to $\pm\infty$ for $\sigma_1 \to 0$ whenever $q > 0$ because the time scaling is not strictly positive for all $\boldsymbol{\sigma} \neq \mathbf{0}$ but zero for $\sigma_1 = 0$. Nevertheless, the right-hand side is well-defined and the trajectories of the original system and the corresponding system coincide where $\sigma_1 \neq 0$. Anyway, the corresponding system of degree $\tilde{q} = 0$ is exploited for design purposes only and the singularities are not present in the original system.

The pseudo-linear system representation of system (4.19) that is consistent with (4.6) is

$$\begin{pmatrix} \frac{\mathrm{d}\sigma_1}{\mathrm{d}\tau}\\ \frac{\mathrm{d}\sigma_2}{\mathrm{d}\tau}\\ \vdots\\ \frac{\mathrm{d}\sigma_{n-2}}{\mathrm{d}\tau}\\ \frac{\mathrm{d}\sigma_{n-1}}{\mathrm{d}\tau}\\ \frac{\mathrm{d}\sigma_n}{\mathrm{d}\tau}\end{pmatrix} = \underbrace{\begin{pmatrix}
-r_1\gamma_{n-1} & |\sigma_1|^{-\frac{q}{r_1}} & 0 & \dots & \dots & 0\\
-r_1 r_2\gamma_{n-2}|\sigma_1|^{\frac{q}{r_1}} & 0 & |\sigma_1|^{-\frac{q}{r_1}} & \ddots & & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\
-r_1 r_2\cdots r_{n-2}\gamma_2|\sigma_1|^{\frac{(n-3)q}{r_1}} & \vdots & & \ddots & |\sigma_1|^{-\frac{q}{r_1}} & 0\\
-r_1 r_2\cdots r_{n-1}\gamma_1|\sigma_1|^{\frac{(n-2)q}{r_1}} & \vdots & & & \ddots & |\sigma_1|^{-\frac{q}{r_1}}\\
-r_1 r_2\cdots r_n\gamma_0|\sigma_1|^{\frac{(n-1)q}{r_1}} & 0 & \dots & \dots & \dots & 0
\end{pmatrix}}_{\boldsymbol{M}(\boldsymbol{\sigma})} \begin{pmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_{n-2}\\ \sigma_{n-1}\\ \sigma_n\end{pmatrix}.\tag{4.20}$$

Three cases are considered to check whether the decomposition in pseudo-linear system representation results in loss of information about additional homogeneous eigenvalues and eigenvectors:

- $q = 0$ yields a linear system and, therefore, $\boldsymbol{M}(\boldsymbol{\sigma}) = \boldsymbol{M}$ is a constant matrix. No additional singularities in its entries can occur.
- $q > 0$ leads to singularities in the secondary diagonal of $\boldsymbol{M}(\boldsymbol{\sigma})$ for $\sigma_1 = 0$, but they are already part of the system (4.19). For this reason no information about additional homogeneous eigenvalues and eigenvectors is lost.
- $q < 0$ produces singularities in the first column of $\boldsymbol{M}(\boldsymbol{\sigma})$ for $\sigma_1 = 0$ which are not part of system (4.19) if

$$\frac{(n-1)q}{r_1} + 1 > 0 \Leftrightarrow r_1 > -(n-1)q \tag{4.21}$$

is satisfied. However, this inequality is always fulfilled as will be shown later on, see (4.32). In consequence of these singularities there may be additional homogeneous eigenvalues and eigenvectors for $\sigma_1 = 0$ which are not visible in the pseudo-linear representation. Inserting of $\sigma_1 = 0$ into the homogeneous eigenvalue equation of the corresponding system (4.19) yields

$$\boldsymbol{0} = \tilde{\lambda}\|v\|^0_{\{\boldsymbol{r},2\}}\boldsymbol{\nu}(\boldsymbol{v}) = \tilde{\lambda}\boldsymbol{\nu}(\boldsymbol{v}), \tag{4.22}$$

which obviously induces an additional homogeneous eigenvalue

$$\tilde{\lambda}_{n+1} = 0. \tag{4.23}$$

This homogeneous eigenvalue of the corresponding system of homogeneity degree $\tilde{q} = 0$ occurs as a result of the time scaling $\zeta(\boldsymbol{\sigma}) = |\sigma_1|^{\frac{q}{r_1}}$ which contains singularities at $\sigma_1 = 0$. This fact has to be considered in the stability analysis later on.

The original estimation error dynamics for arbitrary homogeneity degree q and dilation coefficients $\boldsymbol{r}$ are given by

$$\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 - r_1\gamma_{n-1}|\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
\dot{\sigma}_2 &= \sigma_3 - r_1 r_2\gamma_{n-2}|\sigma_1|^{\frac{2q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
&\vdots\\
\dot{\sigma}_{n-1} &= \sigma_n - r_1 r_2\cdots r_{n-1}\gamma_1|\sigma_1|^{\frac{(n-1)q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
\dot{\sigma}_n &= -r_1 r_2\cdots r_n\gamma_0|\sigma_1|^{\frac{nq}{r_1}+1}\operatorname{sign}(\sigma_1).
\end{aligned} \tag{4.24}$$

In the previous considerations the homogeneity of the resulting estimation error dynamics is postulated but not ensured. It is necessary to recheck the homogeneity condition for the right-hand side of system (4.24) which leads to

$$\begin{aligned}
\varepsilon^{r_2}\sigma_2 - r_1\gamma_{n-1}|\varepsilon^{r_1}\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\varepsilon^{r_1}\sigma_1) &\overset{!}{=} \varepsilon^{q+r_1}\sigma_2 - r_1\gamma_{n-1}\varepsilon^{q+r_1}|\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
\varepsilon^{r_3}\sigma_3 - r_1 r_2\gamma_{n-2}|\varepsilon^{r_1}\sigma_1|^{\frac{2q}{r_1}+1}\operatorname{sign}(\varepsilon^{r_1}\sigma_1) &\overset{!}{=} \varepsilon^{q+r_2}\sigma_3 - r_1 r_2\gamma_{n-2}\varepsilon^{q+r_2}|\sigma_1|^{\frac{2q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
&\vdots\\
\varepsilon^{r_n}\sigma_n - r_1\cdots r_{n-1}\gamma_1|\varepsilon^{r_1}\sigma_1|^{\frac{(n-1)q}{r_1}+1}\operatorname{sign}(\varepsilon^{r_1}\sigma_1) &\overset{!}{=} \varepsilon^{q+r_{n-1}}\sigma_n - r_1\cdots r_{n-1}\gamma_1\varepsilon^{q+r_{n-1}}|\sigma_1|^{\frac{(n-1)q}{r_1}+1}\operatorname{sign}(\sigma_1)\\
-r_1\cdots r_n\gamma_0|\varepsilon^{r_1}\sigma_1|^{\frac{nq}{r_1}+1}\operatorname{sign}(\varepsilon^{r_1}\sigma_1) &\overset{!}{=} -r_1\cdots r_n\gamma_0\varepsilon^{q+r_n}|\sigma_1|^{\frac{nq}{r_1}+1}\operatorname{sign}(\sigma_1).
\end{aligned} \tag{4.25}$$

A comparison of the coefficients yields the system of linear equations

$$
\begin{aligned}
r_{k+1} &= q + r_k \qquad && k = 1, \dots, n-1 && (4.26)\\
iq + r_1 &= q + r_i \qquad && i = 1, \dots, n. && (4.27)
\end{aligned}
$$

Equation (4.26) is a recursive formula for the dilation coefficients. For an arbitrary choice of $r_1 > 0$ the dilation coefficients compute to

$$
\begin{aligned}
r_2 &= q + r_1\\
r_3 &= q + r_2 = 2q + r_1\\
r_4 &= q + r_3 = 3q + r_1\\
&\vdots\\
r_n &= q + r_{n-1} = (n-1)q + r_1 \qquad (4.28)
\end{aligned}
$$

and, therefore,

$$
r_i = (i-1)q + r_1, \qquad i = 1, \dots, n \tag{4.29}
$$

holds. The second condition (4.27), solved for r_i, is equivalent to equation (4.29) and, consequently, to the first condition (4.26). For this reason system (4.24) is homogeneous of degree q w.r.t. the dilation coefficients

$$
\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} r_1 \\ q + r_1 \\ 2q + r_1 \\ \vdots \\ (n-1)q + r_1 \end{pmatrix}. \tag{4.30}
$$

Furthermore, the dilation coefficients are limited to positive numbers

$$
r_i > 0, \qquad \forall i \tag{4.31}
$$

which results in the additional condition

$$
r_1 > \begin{cases} 0 & \text{if } q \geq 0 \\ -(n-1)q & \text{if } q < 0. \end{cases} \tag{4.32}
$$

Finally, the observer is given by

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + r_1 \gamma_{n-1} |\sigma_1|^{\frac{q}{r_1}+1} \operatorname{sign}(\sigma_1)\\
\dot{\hat{x}}_2 &= \hat{x}_3 + r_1 r_2 \gamma_{n-2} |\sigma_1|^{\frac{2q}{r_1}+1} \operatorname{sign}(\sigma_1)\\
&\vdots\\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + r_1 r_2 \cdots r_{n-1} \gamma_1 |\sigma_1|^{\frac{(n-1)q}{r_1}+1} \operatorname{sign}(\sigma_1)\\
\dot{\hat{x}}_n &= u + r_1 r_2 \cdots r_n \gamma_0 |\sigma_1|^{\frac{nq}{r_1}+1} \operatorname{sign}(\sigma_1)\\
\\
\sigma_1 &= x_1 - \hat{x}_1. \qquad (4.33)
\end{aligned}
$$

For $q > 0$ the right-hand side of the observer differential equations tends to $\pm\infty$ as $\sigma_1 \to 0$. To avoid this undesired behavior one has to ensure that inequality

$$\frac{nq}{r_1} + 1 \geq 0 \quad \Leftrightarrow \quad r_1 \geq -nq, \qquad \text{for } q < 0 \tag{4.34}$$

holds. This inequality further restricts homogeneity condition (4.32), which finally yields the condition for the choice of the dilation coefficient r_1

$$\begin{aligned} r_1 &> 0 && \text{if } q \geq 0, \\ r_1 &\geq -nq && \text{if } q < 0. \end{aligned} \tag{4.35}$$

The proposed observer given in (4.33) provides three degrees of freedom:

- the homogeneity degree q which can be chosen arbitrarily in $\mathbb{R}$,
- the dilation coefficient r_1 which has to be chosen according to condition (4.35) and determines all the other dilation coefficients according to relation (4.30),
- the homogeneous eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$ which, of course, have to be chosen negative due to the necessary condition for stability of the origin from Section 3.3.4. The parameters $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ are the result of relation (4.13).

4.1.1 Observer of homogeneity degree $q = 0$

The choice $q = 0$ and $r_1 = \eta$ with $\eta > 0$ satisfies the condition for homogeneity (4.35). The dilation coefficients result in

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} = \eta \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{4.36}$$

according to relation (4.30). Both, the resulting observer

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \eta\gamma_{n-1}\sigma_1 \\ \dot{\hat{x}}_2 &= \hat{x}_3 + \eta^2\gamma_{n-2}\sigma_1 \\ &\vdots \\ \dot{\hat{x}}_{n-1} &= \hat{x}_n + \eta^{n-1}\gamma_1\sigma_1 \\ \dot{\hat{x}}_n &= u + \eta^n\gamma_0\sigma_1 \\ \\ \sigma_1 &= x_1 - \hat{x}_1 \end{aligned} \tag{4.37}$$

and its estimation error dynamics

$$\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 - \eta\gamma_{n-1}\sigma_1 \\
\dot{\sigma}_2 &= \sigma_3 - \eta^2\gamma_{n-2}\sigma_1 \\
&\vdots \\
\dot{\sigma}_{n-1} &= \sigma_n - \eta^{n-1}\gamma_1\sigma_1 \\
\dot{\sigma}_n &= -\eta^n\gamma_0\sigma_1
\end{aligned} \tag{4.38}$$

are linear systems. In Section 3.3.5 it is shown that the linear eigenvalue approach matches with the homogeneous eigenvalue approach in the case of linear systems. Hence, the obtained observer (4.37) is a Luenberger observer and the linear eigenvalues of the estimation error dynamics (4.38) are located at

$$s_i = \eta \cdot \tilde{\lambda}_i \qquad i = 1, 2, \ldots, n. \tag{4.39}$$

In this case $\tilde{\lambda}_i < 0 \ \forall i$ is a necessary and sufficient condition for stability of the origin if the eigenvalues are limited to real numbers.

4.1.2 Observer of homogeneity degree $q = -1$ with dilation coefficient $r_1 = n$

The choice $q = -1$ and $r_1 = n$ fulfills homogeneity condition (4.35) and yields the dilation coefficients

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_{n-1} \\ r_n \end{pmatrix} = \begin{pmatrix} n \\ n-1 \\ \vdots \\ 2 \\ 1 \end{pmatrix} \tag{4.40}$$

due to relation (4.30). The observer results in

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + n\gamma_{n-1}|\sigma_1|^{\frac{n-1}{n}} \operatorname{sign}(\sigma_1) \\
\dot{\hat{x}}_2 &= \hat{x}_3 + n(n-1)\gamma_{n-2}|\sigma_1|^{\frac{n-2}{n}} \operatorname{sign}(\sigma_1) \\
&\vdots \\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + n(n-1)\cdots 2 \cdot \gamma_1|\sigma_1|^{\frac{1}{n}} \operatorname{sign}(\sigma_1) \\
\dot{\hat{x}}_n &= u + n(n-1)\cdots 1 \cdot \gamma_0 \operatorname{sign}(\sigma_1) \\
\\
\sigma_1 &= x_1 - \hat{x}_1.
\end{aligned} \tag{4.41}$$

Merging the constants to

$$\kappa_i = \frac{n!}{(n-i)!}\gamma_{n-i} \qquad i = 1, \ldots, n \tag{4.42}$$

simplifies the differential equations of the observer (4.41) to

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + \kappa_1 |\sigma_1|^{\frac{n-1}{n}} \operatorname{sign}(\sigma_1) \\
\dot{\hat{x}}_2 &= \hat{x}_3 + \kappa_2 |\sigma_1|^{\frac{n-2}{n}} \operatorname{sign}(\sigma_1) \\
&\vdots \\
\dot{\hat{x}}_{n-1} &= \hat{x}_n + \kappa_{n-1} |\sigma_1|^{\frac{1}{n}} \operatorname{sign}(\sigma_1) \\
\dot{\hat{x}}_n &= u + \kappa_n \operatorname{sign}(\sigma_1) \\
\\
\sigma_1 &= x_1 - \hat{x}_1.
\end{aligned} \tag{4.43}
$$

which coincides with the arbitrary order robust exact differentiator proposed by A. Levant [5], [6] for a known input u. The state variables $x_i(t)$ corresponds to the $(i-1)^{\text{th}}$ derivative of the given signal $x_1(t)$. The observer parameters κ_i depend on the variables γ_{n-i} which are related to the homogeneous eigenvalues $\tilde{\lambda}_i$ according to equation (4.13). For this reason a choice $\tilde{\lambda}_i < 0 \ \forall i$ offers a reasonable starting point to find an appropriate parameter setting because the necessary stability criterion derived in Section 3.3.4 is satisfied.

Uncertainty of the input

In differentiation problems usually all the derivatives of the signal $x_1(t)$ are unknown and the input u corresponds to the unknown n^{th} derivative of the signal to be differentiated. Therefore, the n^{th} derivative is treated as an unknown disturbance $\Delta(t)$ that is assumed to be bounded

$$|\Delta(t)| \leq L, \qquad \forall t,\, L \geq 0. \tag{4.44}$$

The estimation error dynamics of the robust exact differentiator (4.43) with unknown input $u = \Delta(t)$ are given by

$$
\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 - \kappa_1 |\sigma_1|^{\frac{n-1}{n}} \operatorname{sign}(\sigma_1) \\
\dot{\sigma}_2 &= \sigma_3 - \kappa_2 |\sigma_1|^{\frac{n-2}{n}} \operatorname{sign}(\sigma_1) \\
&\vdots \\
\dot{\sigma}_{n-1} &= \sigma_n - \kappa_{n-1} |\sigma_1|^{\frac{1}{n}} \operatorname{sign}(\sigma_1) \\
\dot{\sigma}_n &= \Delta(t) - \kappa_n \operatorname{sign}(\sigma_1).
\end{aligned} \tag{4.45}
$$

To ensure stability of the origin of (4.45) it is necessary that the sign function is able to dominate the uncertainty $\Delta(t)$. Thus, the inequality

$$|\kappa_n| > L \tag{4.46}$$

has to hold, where κ_n can be expressed by the homogeneous eigenvalues using equations (4.42) and (4.13) which yields

$$\kappa_n = n! \cdot \gamma_0 = (-1)^n n! \cdot \prod_{i=1}^{n} \tilde{\lambda}_i. \tag{4.47}$$

Insertion of equation (4.47) into inequality (4.46) leads to

$$\left|(-1)^n n! \cdot \prod_{i=1}^{n} \tilde{\lambda}_i\right| > L. \tag{4.48}$$

Division by $n!$ yields

$$\left|\prod_{i=1}^{n} \tilde{\lambda}_i\right| > \frac{L}{n!} \tag{4.49}$$

which is a necessary condition for the homogeneous eigenvalues regarding stability of the origin of the perturbed system (4.45). The absolute value of the product of the homogeneous eigenvalues must be larger than $\frac{L}{n!}$ in order to compensate the disturbance $\Delta(t) \in [-L, L]$ in the zero equilibrium state.

4.1.3 Second order observer

In Section 3.3.4) a sufficient criterion for the stability of the origin of planar homogeneous systems has been discussed. In the following the results will be applied to the achieved observer. According to (4.33) the observer's differential equations result in

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + r_1\gamma_1|\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\sigma_1) \\
\dot{\hat{x}}_2 &= u + r_1 r_2\gamma_0|\sigma_1|^{\frac{2q}{r_1}+1}\operatorname{sign}(\sigma_1) \\
\\
\sigma_1 &= x_1 - \hat{x}_1.
\end{aligned} \tag{4.50}$$

The dilation coefficients are given by

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ q + r_1 \end{pmatrix} \tag{4.51}$$

due to homogeneity condition (4.30). The parameters

$$\begin{aligned}
\gamma_1 &= -(\tilde{\lambda}_1 + \tilde{\lambda}_2), \\
\gamma_0 &= \tilde{\lambda}_1\tilde{\lambda}_2
\end{aligned} \tag{4.52}$$

are related to the homogeneous eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ by equation (4.13). Insertion of equations (4.51) and (4.52) modifies the observer (4.50) to

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 - r_1(\tilde{\lambda}_1 + \tilde{\lambda}_2)|\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\sigma_1) \\
\dot{\hat{x}}_2 &= u + r_1(q + r_1)\tilde{\lambda}_1\tilde{\lambda}_2|\sigma_1|^{\frac{2q}{r_1}+1}\operatorname{sign}(\sigma_1) \\
\\
\sigma_1 &= x_1 - \hat{x}_1.
\end{aligned} \tag{4.53}$$

Stability behavior

The estimation error dynamics of system (4.53) are given by

$$
\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 + r_1(\tilde{\lambda}_1 + \tilde{\lambda}_2)|\sigma_1|^{\frac{q}{r_1}+1} \operatorname{sign}(\sigma_1) \\
\dot{\sigma}_2 &= -r_1(q + r_1)\tilde{\lambda}_1\tilde{\lambda}_2|\sigma_1|^{\frac{2q}{r_1}+1} \operatorname{sign}(\sigma_1).
\end{aligned}
\tag{4.54}
$$

The sufficient stability criterion from Section 3.3.4 is only applicable to continuous systems. The right-hand side of the estimation dynamics (4.54) is continuous for

$$
\begin{aligned}
& r_1 > 0 \quad \text{if } q \geq 0 \\
& \frac{2q}{r_1} + 1 > 0 \Leftrightarrow r_1 > -2q \quad \text{if } q < 0.
\end{aligned}
\tag{4.55}
$$

In these cases system (4.54) is globally asymptotically stable for the choice

$$
\tilde{\lambda}_1 < 0, \, \tilde{\lambda}_2 < 0 \tag{4.56}
$$

because it is planar, there exist two real homogeneous eigenvalues and both are negative due to relation (3.117).

The case

$$
\frac{2q}{r_1} + 1 = 0 \Leftrightarrow r_1 = -2q, \qquad q < 0 \tag{4.57}
$$

yields the estimation error dynamics

$$
\begin{aligned}
\dot{\sigma}_1 &= \sigma_2 + r_1(\tilde{\lambda}_1 + \tilde{\lambda}_2)|\sigma_1|^{\frac{1}{2}} \operatorname{sign}(\sigma_1) \\
\dot{\sigma}_2 &= -r_1(q + r_1)\tilde{\lambda}_1\tilde{\lambda}_2 \operatorname{sign}(\sigma_1)
\end{aligned}
\tag{4.58}
$$

which has a discontinuity in the second differential equation and coincides with the super-twisting algorithm with

$$
k_1 = -r_1(\tilde{\lambda}_1 + \tilde{\lambda}_2), \quad k_2 = r_1(q + r_1)\tilde{\lambda}_1\tilde{\lambda}_2. \tag{4.59}
$$

The origin of the super-twisting algorithm is globally asymptotically stable if the homogeneous eigenvalues are negative. The proof is already done in Section 3.3.6 using the corresponding system of homogeneity degree $\tilde{q} = 0$ which is continuous.

All the possibilities offered by condition (4.35) are covered above. For this reason the origin of the estimation error dynamics is globally asymptotically stable for any choice of q and r_1 satisfying (4.35).

4.2 Nonlinear observer design for observable LTI-systems

In this Section a design algorithm for nonlinear observers is presented. The approach is applicable to observable LTI-systems. First of all, the problems of homogeneous observer design for LTI-systems are pointed out. In consequence of this considerations a modified observability normal form of LTI-systems is proposed. The design is done for systems which offer this special structure. Furthermore, it is shown how to transform a general observable LTI-system to this modified observability normal form. The inverse transformation of the observer leads to a design formula which generalizes the approach of Ackermann's eigenvalue assignment [28].

4.2.1 Difficulties in homogeneous observer design for arbitrary observable LTI-systems

Let

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \\ y &= \boldsymbol{c}^T\boldsymbol{x} \end{aligned} \tag{4.60}$$

be an arbitrary observable LTI-system with dynamic matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \tag{4.61}$$

and output vector

$$\boldsymbol{c}^T = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}. \tag{4.62}$$

The general ansatz

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{b}u + \boldsymbol{l}(y - \hat{y}) \\ \hat{y} &= \boldsymbol{c}^T\hat{\boldsymbol{x}} \end{aligned} \tag{4.63}$$

for the observer introduces the output injection terms

$$\boldsymbol{l}(y-\hat{y}) = \begin{pmatrix} l_1(y-\hat{y}) \\ \vdots \\ l_n(y-\hat{y}) \end{pmatrix}. \tag{4.64}$$

The definition of the estimation error $\boldsymbol{\xi} = \boldsymbol{x} - \hat{\boldsymbol{x}} = \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{pmatrix}^T$ leads to the error dynamics

$$\dot{\boldsymbol{\xi}} = \boldsymbol{A}\boldsymbol{\xi} - \boldsymbol{l}(\boldsymbol{c}^T\boldsymbol{\xi}) \quad \Leftrightarrow \quad \begin{pmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_n \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^{n} a_{1,i}\xi_i - l_1\left(\sum\limits_{i=1}^{n} c_i\xi_i\right) \\ \vdots \\ \sum\limits_{i=1}^{n} a_{n,i}\xi_i - l_n\left(\sum\limits_{i=1}^{n} c_i\xi_i\right) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{\xi}). \tag{4.65}$$

The overall goal is to choose the injection terms $l_j\left(\sum_{i=1}^{n} c_i\xi_i\right)$, $j = 1,\dots,n$ such that system (4.65) is homogeneous of degree q w.r.t. the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & \dots & r_n \end{pmatrix}^T$ and some desired homogeneous eigenvalues. Hence, the right-hand side of system (4.65) has to satisfy the homogeneity condition

$$\boldsymbol{f}(\boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}}\boldsymbol{\xi}) = \varepsilon^q \boldsymbol{\Delta}_\varepsilon^{\boldsymbol{r}} \boldsymbol{f}(\boldsymbol{\xi}) \tag{4.66}$$

which yields

$$\begin{pmatrix} \sum_{i=1}^{n} a_{1,i}\varepsilon^{r_i}\xi_i - l_1\left(\sum_{i=1}^{n} c_i\varepsilon^{r_i}\xi_i\right) \\ \vdots \\ \sum_{i=1}^{n} a_{n,i}\varepsilon^{r_i}\xi_i - l_n\left(\sum_{i=1}^{n} c_i\varepsilon^{r_i}\xi_i\right) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} a_{1,i}\varepsilon^{q+r_1}\xi_i - \varepsilon^{q+r_1} l_1\left(\sum_{i=1}^{n} c_i\xi_i\right) \\ \vdots \\ \sum_{i=1}^{n} a_{n,i}\varepsilon^{q+r_n}\xi_i - \varepsilon^{q+r_n} l_n\left(\sum_{i=1}^{n} c_i\xi_i\right) \end{pmatrix}. \tag{4.67}$$

Consider e.g. $a_{1,1} \neq 0$ and $c_1 = 0$. Then a comparison of the coefficient ξ_1 in the first equation of (4.67) yields

$$a_{1,1}\varepsilon^{r_1} = a_{1,1}\varepsilon^{q+r_1} \quad \Rightarrow \quad r_1 = q + r_1. \tag{4.68}$$

Obviously this case enforces

$$q = 0. \tag{4.69}$$

For this reason an arbitrary choice of the homogeneity degree q is impossible which means a strong limitation to the observer design.

In general, if system (4.65) does not offer a very special structure, the direct design of homogeneous estimation dynamics of arbitrary homogeneity degree q is impossible. Hence, a transformation of the system to a special structure is necessary.

4.2.2 Homogeneous observer design for LTI-systems in modified observability normal form

Consider the linear, time-invariant system

$$\begin{aligned} \dot{\boldsymbol{z}} &= \bar{\boldsymbol{A}}\boldsymbol{z} + \bar{\boldsymbol{b}}u \\ y &= \bar{\boldsymbol{c}}^T\boldsymbol{z}, \end{aligned} \tag{4.70}$$

where $\boldsymbol{z} = \begin{pmatrix} z_1 & z_2 & \dots & z_n \end{pmatrix}^T$ is the state vector, the output vector

$$\bar{\boldsymbol{c}}^T = \boldsymbol{e}_1^T = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \tag{4.71}$$

is the unit vector along the first dimension of state space and the dynamic matrix $\bar{\boldsymbol{A}}$ offers the special structure

$$\bar{\boldsymbol{A}} = \begin{pmatrix} -r_1\beta_{n-1} & r_1 & 0 & \dots & 0 \\ -r_2\beta_{n-2} & 0 & r_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -r_{n-1}\beta_1 & 0 & & \ddots & r_{n-1} \\ -r_n\beta_0 & 0 & \dots & \dots & 0 \end{pmatrix}. \tag{4.72}$$

Again the coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & \dots & r_n \end{pmatrix}^T$ correspond to the homogeneity weights of the homogeneous observer to be constructed and $\beta_0, \dots, \beta_{n-1}$ are arbitrary real numbers. Although the structure of system (4.70) looks very similar to an observability normal form the coefficients $\beta_0, \dots, \beta_{n-1}$ do not match with the coefficients $\alpha_0, \dots, \alpha_{n-1}$ of the characteristic polynomial of $\bar{\boldsymbol{A}}$ in general. However, for a system in this specific structure the design of a homogeneous observer of arbitrary degree q w.r.t. dilation coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & \dots & r_n \end{pmatrix}^T$ is very simple.

The observer

$$\begin{aligned} \dot{\hat{\boldsymbol{z}}} &= \bar{\boldsymbol{A}}\hat{\boldsymbol{z}} + \bar{\boldsymbol{b}}u + \bar{\boldsymbol{l}}(y - \hat{y}) \\ \hat{y} &= \bar{\boldsymbol{c}}^T\hat{\boldsymbol{z}} \end{aligned} \tag{4.73}$$

consists of a copy of the plant and some non-linear injection terms

$$\bar{\boldsymbol{l}}(y - \hat{y}) = \begin{pmatrix} \bar{l}_1(y - \hat{y}) \\ \vdots \\ \bar{l}_n(y - \hat{y}) \end{pmatrix}. \tag{4.74}$$

The introduction of the estimation error vector

$$\boldsymbol{\sigma} = \boldsymbol{z} - \hat{\boldsymbol{z}} = \begin{pmatrix} \sigma_1 & \dots & \sigma_n \end{pmatrix}^T \tag{4.75}$$

leads to the output error

$$y - \hat{y} = \bar{\boldsymbol{c}}^T\boldsymbol{z} - \bar{\boldsymbol{c}}^T\hat{\boldsymbol{z}} = \bar{\boldsymbol{c}}^T(\boldsymbol{z} - \hat{\boldsymbol{z}}) = \bar{\boldsymbol{c}}^T\boldsymbol{\sigma} = \sigma_1 \tag{4.76}$$

and the estimation error dynamics

$$\dot{\boldsymbol{\sigma}} = \bar{\boldsymbol{A}}\boldsymbol{\sigma} - \bar{\boldsymbol{l}}(\sigma_1). \tag{4.77}$$

The basic idea for the design of the nonlinear observer is the same as applied for the chain of integrators in Section 4.1. The non-linear injection terms $\bar{\boldsymbol{l}}(\sigma_1)$ are chosen such that the estimation error dynamics (4.77) are homogeneous of degree q w.r.t. the dilation coefficients $\boldsymbol{r} = \begin{pmatrix} r_1 & \dots & r_n \end{pmatrix}^T$ and the homogeneous eigenvalues of the corresponding system of homogeneity degree $\tilde{q} = 0$ are located at $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$.

The corresponding system of homogeneity degree $\tilde{q} = 0$ for a general time scaling $\zeta(\boldsymbol{\sigma})$ yields

$$\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\tau} = \zeta(\boldsymbol{\sigma})^{-1}\Big(\bar{\boldsymbol{A}}\boldsymbol{\sigma} - \bar{\boldsymbol{l}}(\sigma_1)\Big). \tag{4.78}$$

The pseudo-linear system representation which is given by

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\tau} &= \zeta(\boldsymbol{\sigma})^{-1}\left(\bar{\boldsymbol{A}}\boldsymbol{\sigma} - \bar{\boldsymbol{l}}(\sigma_1)\frac{\bar{\boldsymbol{c}}^T\boldsymbol{\sigma}}{\bar{\boldsymbol{c}}^T\boldsymbol{\sigma}}\right) = \\ &= \underbrace{\zeta(\boldsymbol{\sigma})^{-1}\left(\bar{\boldsymbol{A}} - \bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\right)}_{\boldsymbol{M}(\boldsymbol{\sigma})}\boldsymbol{\sigma} = \\ &= \boldsymbol{M}(\boldsymbol{\sigma})\boldsymbol{\sigma}\end{aligned} \tag{4.79}$$

simplifies the assignment of the homogeneous eigenvalues. Let

$$\gamma(\tilde{\lambda}) = \tilde{\lambda}^n + \gamma_{n-1}\tilde{\lambda}^{n-1} + \cdots + \gamma_1\tilde{\lambda} + \gamma_0 = \prod_{i=1}^{n}(\tilde{\lambda} - \tilde{\lambda}_i) \tag{4.80}$$

be the desired characteristic polynomial of the matrix $\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{\sigma})$ with the coefficients $\gamma_0, \ldots, \gamma_{n-1}$ and zeros located at the desired homogeneous eigenvalues $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$. Modification of the characteristic polynomial of $\boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{\sigma})$ leads to

$$\begin{aligned}\det\left(\tilde{\lambda}\boldsymbol{I} - \boldsymbol{B}^{-1}\boldsymbol{M}(\boldsymbol{\sigma})\right) &= \det\left(\tilde{\lambda}\boldsymbol{I} - \zeta(\boldsymbol{\sigma})^{-1}\boldsymbol{B}^{-1}\left(\bar{\boldsymbol{A}} - \bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\right)\right) = \\ &= \zeta(\boldsymbol{\sigma})^{-n}\cdot\det\left(\zeta(\boldsymbol{\sigma})\tilde{\lambda}\boldsymbol{I} - \boldsymbol{B}^{-1}\left(\bar{\boldsymbol{A}} - \bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\right)\right).\end{aligned} \tag{4.81}$$

Assigning the desired homogeneous eigenvalues means setting equations (4.80) and (4.81) to be equal which yields

$$\prod_{i=1}^{n}(\tilde{\lambda} - \tilde{\lambda}_i) = \zeta(\boldsymbol{\sigma})^{-n}\cdot\det\left(\zeta(\boldsymbol{\sigma})\tilde{\lambda}\boldsymbol{I} - \boldsymbol{B}^{-1}\left(\bar{\boldsymbol{A}} - \bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\right)\right). \tag{4.82}$$

Multiplying both sides of (4.82) with $\zeta(\boldsymbol{\sigma})^n$ and substituting

$$\bar{\lambda} = \zeta(\boldsymbol{\sigma})\tilde{\lambda} \tag{4.83}$$

results in

$$\prod_{i=1}^{n}(\bar{\lambda} - \tilde{\lambda}_i\zeta(\boldsymbol{\sigma})) = \det\left(\bar{\lambda}\boldsymbol{I} - \boldsymbol{B}^{-1}\left(\bar{\boldsymbol{A}} - \bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\right)\right). \tag{4.84}$$

This means that assigning the eigenvalues $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$ to the matrix $\zeta(\boldsymbol{\sigma})^{-1}\boldsymbol{B}^{-1}\big(\bar{\boldsymbol{A}} - \bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\big)$ is equivalent to assigning the eigenvalues $\tilde{\lambda}_1\zeta(\boldsymbol{\sigma}), \ldots, \tilde{\lambda}_n\zeta(\boldsymbol{\sigma})$ to the matrix $\boldsymbol{B}^{-1}\big(\bar{\boldsymbol{A}} - \bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\big)$ which is given by

$$\boldsymbol{B}^{-1}\big(\bar{\boldsymbol{A}} - \bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\big) = \begin{pmatrix} -\beta_{n-1} - \frac{1}{r_1}\sigma_1^{-1}\bar{l}_1(\sigma_1) & 1 & 0 & \ldots & 0 \\ -\beta_{n-2} - \frac{1}{r_2}\sigma_1^{-1}\bar{l}_2(\sigma_1) & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\beta_1 - \frac{1}{r_{n-1}}\sigma_1^{-1}\bar{l}_{n-1}(\sigma_1) & \vdots & & \ddots & 1 \\ -\beta_0 - \frac{1}{r_n}\sigma_1^{-1}\bar{l}_n(\sigma_1) & 0 & \ldots & \ldots & 0 \end{pmatrix}. \tag{4.85}$$

The matrix (4.85) is in observable canonical form and, therefore, the first column contains the coefficients of its characteristic polynomial

$$\begin{aligned}\det\Big(\bar{\lambda}\boldsymbol{I}-\boldsymbol{B}^{-1}\big(\bar{\boldsymbol{A}}-\bar{\boldsymbol{l}}(\sigma_1)\boldsymbol{e}_1^T\sigma_1^{-1}\big)\Big)=&\bar{\lambda}^n+\big(\beta_{n-1}+\frac{1}{r_1}\sigma_1^{-1}\bar{l}_1(\sigma_1)\big)\bar{\lambda}^{n-1}+\big(\beta_{n-2}+\frac{1}{r_2}\sigma_1^{-1}\bar{l}_2(\sigma_1)\big)\bar{\lambda}^{n-2}+\\ &+\cdots+\big(\beta_1+\frac{1}{r_{n-1}}\sigma_1^{-1}\bar{l}_{n-1}(\sigma_1)\big)\bar{\lambda}+\big(\beta_0+\frac{1}{r_n}\sigma_1^{-1}\bar{l}_n(\sigma_1)\big)=\\ =&\bar{\lambda}^n+\sum_{i=1}^{n}\big(\beta_{n-i}+\frac{1}{r_i}\sigma_1^{-1}\bar{l}_i(\sigma_1)\big)\bar{\lambda}^{n-i}.\end{aligned}\tag{4.86}$$

The zeros of the characteristic polynomial (4.86) should be located at $\tilde{\lambda}_1\zeta(\boldsymbol{\sigma}),\ldots,\tilde{\lambda}_n\zeta(\boldsymbol{\sigma})$ which yields

$$\bar{\lambda}^n+\sum_{i=1}^{n}\big(\beta_{n-i}+\frac{1}{r_i}\sigma_1^{-1}\bar{l}_i(\sigma_1)\big)\bar{\lambda}^{n-i}\stackrel{!}{=}\prod_{i=1}^{n}\big(\bar{\lambda}-\tilde{\lambda}_i\zeta(\boldsymbol{\sigma})\big)=\bar{\lambda}^n+\sum_{i=1}^{n}\gamma_{n-i}\zeta(\boldsymbol{\sigma})^i\bar{\lambda}^{n-i}\tag{4.87}$$

where γ_{n-i} are the coefficients of the desired polynomial (4.80). A comparison of the coefficients $\bar{\lambda}^{n-i}$ results in

$$\beta_{n-i}+\frac{1}{r_i}\sigma_1^{-1}\bar{l}_i(\sigma_1)=\gamma_{n-i}\zeta(\boldsymbol{\sigma})^i,\qquad i=1,\ldots,n,\tag{4.88}$$

which is solved for the non-linear injection term

$$\bar{l}_i(\sigma_1)=r_i\sigma_1\big(\gamma_{n-i}\zeta(\boldsymbol{\sigma})^i-\beta_{n-i}\big),\qquad i=1,\ldots,n.\tag{4.89}$$

The left-hand side of equation (4.89) should only depend on σ_1 and, therefore, the right-hand also has to do so. Hence, again the choice

$$\zeta(\boldsymbol{\sigma})=|\sigma_1|^{\frac{q}{r_1}}\tag{4.90}$$

is reasonable. Insertion of (4.90) into (4.89) finally leads to

$$\bar{l}_i(\sigma_1)=r_i\sigma_1\big(\gamma_{n-i}|\sigma_1|^{\frac{i\cdot q}{r_1}}-\beta_{n-i}\big),\qquad i=1,\ldots,n.\tag{4.91}$$

Homogeneity of the estimation error dynamics

Insertion of the injection terms (4.91) into the estimation error dynamics (4.77) yields

$$\begin{pmatrix}\dot{\sigma}_1\\ \dot{\sigma}_2\\ \vdots\\ \dot{\sigma}_{n-1}\\ \dot{\sigma}_n\end{pmatrix}=\begin{pmatrix}r_1\sigma_2-r_1\sigma_1\gamma_{n-1}|\sigma_1|^{\frac{q}{r_1}}\\ r_2\sigma_3-r_2\sigma_1\gamma_{n-2}|\sigma_1|^{\frac{2q}{r_1}}\\ \vdots\\ r_{n-1}\sigma_n-r_{n-1}\sigma_1\gamma_1|\sigma_1|^{\frac{(n-1)q}{r_1}}\\ -r_n\sigma_1\gamma_0|\sigma_1|^{\frac{nq}{r_1}}\end{pmatrix}=\begin{pmatrix}r_1\sigma_2-r_1\gamma_{n-1}|\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\sigma_1)\\ r_2\sigma_3-r_2\gamma_{n-2}|\sigma_1|^{\frac{2q}{r_1}+1}\operatorname{sign}(\sigma_1)\\ \vdots\\ r_{n-1}\sigma_n-r_{n-1}\gamma_1|\sigma_1|^{\frac{(n-1)q}{r_1}+1}\operatorname{sign}(\sigma_1)\\ -r_n\gamma_0|\sigma_1|^{\frac{nq}{r_1}+1}\operatorname{sign}(\sigma_1)\end{pmatrix}.\tag{4.92}$$

The homogeneity of the error dynamics (4.92) is not ensured for an arbitrary choice of the dilation coefficients $\boldsymbol{r}$ and the homogeneity degree q. The system is equivalent to the estimation error dynamics (4.24) of the integrator chain except for some scaling constants. For this reason the same conditions for the choice of the parameters have to hold which are given by

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{pmatrix} = \begin{pmatrix} r_1 \\ q + r_1 \\ 2q + r_1 \\ \vdots \\ (n-1)q + r_1 \end{pmatrix} \tag{4.93}$$

and

$$\begin{aligned} r_1 &> 0 && \text{if } q \geq 0, \\ r_1 &\geq -nq && \text{if } q < 0. \end{aligned} \tag{4.94}$$

4.2.3 Transformation to modified observability normal form

Consider the observable LTI-system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \\ y &= \boldsymbol{c}^T \boldsymbol{x} \end{aligned} \tag{4.95}$$

of arbitrary structure and order n. The goal is to find a transformation of system (4.95) such that the transformed system is in modified normal form proposed in Section 4.2.2. This is achieved by applying a regular, linear coordinate transformation

$$\boldsymbol{z} = \boldsymbol{T}^{-1}\boldsymbol{x} \quad \Leftrightarrow \quad \boldsymbol{x} = \boldsymbol{T}\boldsymbol{z}, \qquad \boldsymbol{T} \in \mathbb{R}^{n \times n}. \tag{4.96}$$

The the transformed system computes to

$$\begin{aligned} \dot{\boldsymbol{z}} &= \boldsymbol{T}^{-1}\dot{\boldsymbol{x}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{T}^{-1}\boldsymbol{b}u = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\boldsymbol{z} + \boldsymbol{T}^{-1}\boldsymbol{b}u \\ y &= \boldsymbol{c}^T\boldsymbol{T}\boldsymbol{z}. \end{aligned} \tag{4.97}$$

The dynamic matrix, the input vector and the output vector of the transformed system

$$\begin{aligned} \dot{\boldsymbol{z}} &= \bar{\boldsymbol{A}}\boldsymbol{z} + \bar{\boldsymbol{b}}u \\ y &= \bar{\boldsymbol{c}}^T \boldsymbol{z} \end{aligned} \tag{4.98}$$

are given by

$$\bar{\boldsymbol{A}} = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} \tag{4.99}$$

$$\bar{\boldsymbol{b}} = \boldsymbol{T}^{-1}\boldsymbol{b} \tag{4.100}$$

$$\bar{\boldsymbol{c}}^T = \boldsymbol{c}^T\boldsymbol{T}. \tag{4.101}$$

Multiplication of (4.99) with the transformation matrix $\boldsymbol{T}$ from the left-hand side leads to

$$\boldsymbol{T}\bar{\boldsymbol{A}} = \boldsymbol{A}\boldsymbol{T}. \tag{4.102}$$

The transformation matrix can be expressed by its column vectors $\boldsymbol{t}_i$, i.e.

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \dots & \boldsymbol{t}_n \end{pmatrix}. \tag{4.103}$$

Insertion of the matrix $\bar{\boldsymbol{A}}$ given in (4.72) and (4.103) into equation (4.102) results in

$$\begin{pmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \dots & \boldsymbol{t}_n \end{pmatrix} \begin{pmatrix} -r_1\beta_{n-1} & r_1 & 0 & \dots & 0 \\ -r_2\beta_{n-2} & 0 & r_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -r_{n-1}\beta_1 & 0 & & \ddots & r_{n-1} \\ -r_n\beta_0 & 0 & \dots & \dots & 0 \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \dots & \boldsymbol{t}_n \end{pmatrix}. \tag{4.104}$$

Expansion of equation (4.104) yields

$$\begin{pmatrix} -r_1\beta_{n-1}\boldsymbol{t}_1 - r_2\beta_{n-2}\boldsymbol{t}_2 - \dots - r_n\beta_0\boldsymbol{t}_n & r_1\boldsymbol{t}_1 & \dots & r_{n-1}\boldsymbol{t}_{n-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}\boldsymbol{t}_1 & \boldsymbol{A}\boldsymbol{t}_2 & \dots & \boldsymbol{A}\boldsymbol{t}_n \end{pmatrix} \tag{4.105}$$

which ends up in

$$-r_1\beta_{n-1}\boldsymbol{t}_1 - r_2\beta_{n-2}\boldsymbol{t}_2 - \dots - r_n\beta_0\boldsymbol{t}_n = \boldsymbol{A}\boldsymbol{t}_1 \tag{4.106}$$

and a recursive condition for the column vectors

$$\boldsymbol{t}_k = \frac{1}{r_k}\boldsymbol{A}\boldsymbol{t}_{k+1} \qquad k = 1, \dots, n-1. \tag{4.107}$$

The recursion (4.107) is used to express $\boldsymbol{t}_1, \dots, \boldsymbol{t}_{n-1}$ as a function of the last column vector $\boldsymbol{t}_n$

$$\boldsymbol{t}_i = \left(\prod_{j=i}^{n-1} \frac{1}{r_j}\right) \boldsymbol{A}^{n-i}\boldsymbol{t}_n, \qquad i = 1, \dots, n. \tag{4.108}$$

The vector $\boldsymbol{t}_n$ is obtained from relation (4.101), where $\bar{\boldsymbol{c}}^T$ corresponds to the unit vector $\boldsymbol{e}_1^T = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}^T$, which yields

$$\begin{aligned} \boldsymbol{e}_1^T &= \boldsymbol{c}^T \left(\left(\prod_{j=1}^{n-1} \tfrac{1}{r_j}\right) \boldsymbol{A}^{n-1}\boldsymbol{t}_n \quad \left(\prod_{j=2}^{n-1} \tfrac{1}{r_j}\right) \boldsymbol{A}^{n-2}\boldsymbol{t}_n \quad \dots \quad \tfrac{1}{r_{n-1}}\boldsymbol{A}\boldsymbol{t}_n \quad \boldsymbol{t}_n \right) = \\ &= \left(\left(\prod_{j=1}^{n-1} \tfrac{1}{r_j}\right) \boldsymbol{c}^T\boldsymbol{A}^{n-1}\boldsymbol{t}_n \quad \left(\prod_{j=2}^{n-1} \tfrac{1}{r_j}\right) \boldsymbol{c}^T\boldsymbol{A}^{n-2}\boldsymbol{t}_n \quad \dots \quad \tfrac{1}{r_{n-1}}\boldsymbol{c}^T\boldsymbol{A}\boldsymbol{t}_n \quad \boldsymbol{c}^T\boldsymbol{t}_n \right). \end{aligned} \tag{4.109}$$

Transposition of equation (4.109) allows extraction of $\boldsymbol{t}_n$

$$\boldsymbol{e}_1 = \begin{pmatrix} \left(\prod_{j=1}^{n-1} \frac{1}{r_j}\right) \boldsymbol{c}^T \boldsymbol{A}^{n-1} \boldsymbol{t}_n \\ \left(\prod_{j=2}^{n-1} \frac{1}{r_j}\right) \boldsymbol{c}^T \boldsymbol{A}^{n-2} \boldsymbol{t}_n \\ \vdots \\ \frac{1}{r_{n-1}} \boldsymbol{c}^T \boldsymbol{A} \boldsymbol{t}_n \\ \boldsymbol{c}^T \boldsymbol{t}_n \end{pmatrix} = \begin{pmatrix} \left(\prod_{j=1}^{n-1} \frac{1}{r_j}\right) \boldsymbol{c}^T \boldsymbol{A}^{n-1} \\ \left(\prod_{j=2}^{n-1} \frac{1}{r_j}\right) \boldsymbol{c}^T \boldsymbol{A}^{n-2} \\ \vdots \\ \frac{1}{r_{n-1}} \boldsymbol{c}^T \boldsymbol{A} \\ \boldsymbol{c}^T \end{pmatrix} \boldsymbol{t}_n. \tag{4.110}$$

The matrix in equation (4.110) is decomposed into a diagonal matrix containing the dilation coefficients, an anti-diagonal matrix which flips the rows and the observability matrix $\boldsymbol{\mathcal{O}}_{obsv}$

$$\boldsymbol{e}_1 = \begin{pmatrix} \prod_{j=1}^{n-1} \frac{1}{r_j} & 0 & \cdots & \cdots & 0 \\ 0 & \prod_{j=2}^{n-1} \frac{1}{r_j} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \frac{1}{r_{n-1}} & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T \boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^T \boldsymbol{A}^{n-2} \\ \boldsymbol{c}^T \boldsymbol{A}^{n-1} \end{pmatrix}}_{\boldsymbol{\mathcal{O}}_{obsv}} \boldsymbol{t}_n. \tag{4.111}$$

Inversion of all the matrices is not critical. The diagonal entries of the diagonal matrix are strictly positive due to $r_i > 0$, $\forall i$, the anti-diagonal matrix is even an involutory matrix and $\boldsymbol{\mathcal{O}}_{obsv}$ is regular because the pair $(\boldsymbol{A}, \boldsymbol{c}^T)$ is observable by assumption. For this reason $\boldsymbol{t}_n$ is given by

$$\begin{aligned} \boldsymbol{t}_n &= \boldsymbol{\mathcal{O}}_{obsv}^{-1} \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} \prod_{j=1}^{n-1} r_j & 0 & \cdots & \cdots & 0 \\ 0 & \prod_{j=2}^{n-1} r_j & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & r_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \boldsymbol{e}_1 = \\ &= \boldsymbol{\mathcal{O}}_{obsv}^{-1} \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} \prod_{j=1}^{n-1} r_j \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \\ &= \left(\prod_{j=1}^{n-1} r_j\right) \boldsymbol{\mathcal{O}}_{obsv}^{-1} \boldsymbol{e}_n = \left(\prod_{j=1}^{n-1} r_j\right) \bar{\boldsymbol{t}}_n, \end{aligned} \tag{4.112}$$

where $\bar{\boldsymbol{t}}_n$ is the last column of the inverse observability matrix

$$\bar{\boldsymbol{t}}_n = \boldsymbol{\mathcal{O}}_{obsv}^{-1} \boldsymbol{e}_n. \tag{4.113}$$

Inserting the result for $\boldsymbol{t}_n$ (4.112) into (4.108) yields the column vectors of $\boldsymbol{T}$

$$\begin{aligned}\boldsymbol{t}_i &= \left(\prod_{j=i}^{n-1} \frac{1}{r_j}\right) \boldsymbol{A}^{n-i} \left(\prod_{j=1}^{n-1} r_j\right) \bar{\boldsymbol{t}}_n = \\ &= \left(\prod_{j=1}^{i-1} r_j\right) \boldsymbol{A}^{n-i} \bar{\boldsymbol{t}}_n, \qquad i = 1, \dots, n\end{aligned} \tag{4.114}$$

and finally ends up in

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \dots & \boldsymbol{t}_{n-1} & \boldsymbol{t}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}^{n-1}\bar{\boldsymbol{t}}_n & r_1 \boldsymbol{A}^{n-2} \bar{\boldsymbol{t}}_n & \dots & \left(\prod_{j=1}^{n-2} r_j\right) \boldsymbol{A}\bar{\boldsymbol{t}}_n & \left(\prod_{j=1}^{n-1} r_j\right) \bar{\boldsymbol{t}}_n \end{pmatrix}. \tag{4.115}$$

Condition (4.106) has not been considered yet. Insertion of the column vectors (4.114) yields

$$-r_1\beta_{n-1}\boldsymbol{A}^{n-1}\bar{\boldsymbol{t}}_n - r_1 r_2 \beta_{n-2} \boldsymbol{A}^{n-2}\bar{\boldsymbol{t}}_n - \dots - \left(\prod_{j=1}^{n-1} r_j\right) \beta_1 \boldsymbol{A}\bar{\boldsymbol{t}}_n - \left(\prod_{j=1}^{n} r_j\right) \beta_0 \bar{\boldsymbol{t}}_n = \boldsymbol{A}^n \bar{\boldsymbol{t}}_n. \tag{4.116}$$

Collecting all terms on one side of the equation and extracting $\bar{\boldsymbol{t}}_n$ results in

$$\left(\boldsymbol{A}^n + r_1\beta_{n-1}\boldsymbol{A}^{n-1} + r_1 r_2 \beta_{n-2}\boldsymbol{A}^{n-2} + \dots + \left(\prod_{j=1}^{n-1} r_j\right) \beta_1 \boldsymbol{A} + \left(\prod_{j=1}^{n} r_j\right) \beta_0 \boldsymbol{I}\right) \bar{\boldsymbol{t}}_n = \boldsymbol{0}. \tag{4.117}$$

The trivial solution is not permitted and, therefore,

$$\boldsymbol{A}^n + \underbrace{r_1\beta_{n-1}}_{\alpha_{n-1}} \boldsymbol{A}^{n-1} + \underbrace{r_1 r_2 \beta_{n-2}}_{\alpha_{n-2}} \boldsymbol{A}^{n-2} + \dots + \underbrace{\left(\prod_{j=1}^{n-1} r_j\right) \beta_1}_{\alpha_1} \boldsymbol{A} + \underbrace{\left(\prod_{j=1}^{n} r_j\right) \beta_0}_{\alpha_0} \boldsymbol{I} = \boldsymbol{0}_{n\times n}. \tag{4.118}$$

has to hold. Equation (4.118) is a polynomial of order n which is $\boldsymbol{0}_{n\times n}$ evaluated at $\boldsymbol{A}$. The Caley-Hamilton theorem [29] states that every square matrix fulfills its characteristic equation. For this reason the left-hand side of equation (4.118) is the characteristic polynomial of $\boldsymbol{A}$ and the coefficients β_k are the result of

$$\beta_k = \alpha_k \prod_{j=1}^{n-k} \frac{1}{r_j}, \qquad k = 0, \dots, n-1, \tag{4.119}$$

where α_k are the coefficients of the characteristic polynomial of $\boldsymbol{A}$, i.e.

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1 s + \alpha_0. \tag{4.120}$$

4.2.4 Generalization of Ackermann's formula

Again consider the observable LTI-system (4.95) and its transformation in modified observability normal form (4.98). A homogeneous observer is designed for the system in modified observability normal form like suggested in Section 4.2.2 and transformed back to the original coordinate space. The relation between the non-linear injection terms of the observer in modified normal form $\bar{\boldsymbol{l}}(y-\hat{y})$ and the injection terms of the observer for the original system $\boldsymbol{l}(y-\hat{y})$ is given by the transformation

$$\boldsymbol{l}(y-\hat{y}) = \boldsymbol{T}\cdot\bar{\boldsymbol{l}}(y-\hat{y}). \tag{4.121}$$

Insertion of the transformation matrix (4.115) and the injection terms $\bar{\boldsymbol{l}}(y-\hat{y})$ (4.91) yields

$$\begin{aligned}\boldsymbol{l}(y-\hat{y}) &= \begin{pmatrix}\boldsymbol{A}^{n-1}\bar{\boldsymbol{t}}_n & r_1\boldsymbol{A}^{n-2}\bar{\boldsymbol{t}}_n & \dots & \left(\prod\limits_{j=1}^{n-2} r_j\right)\boldsymbol{A}\bar{\boldsymbol{t}}_n & \left(\prod\limits_{j=1}^{n-1} r_j\right)\bar{\boldsymbol{t}}_n\end{pmatrix}\begin{pmatrix} r_1(y-\hat{y})\big(\gamma_{n-1}|y-\hat{y}|^{\frac{q}{r_1}}-\beta_{n-1}\big)\\ r_2(y-\hat{y})\big(\gamma_{n-2}|y-\hat{y}|^{\frac{2q}{r_1}}-\beta_{n-2}\big)\\ \vdots\\ r_{n-1}(y-\hat{y})\big(\gamma_1|y-\hat{y}|^{\frac{(n-1)q}{r_1}}-\beta_1\big)\\ r_n(y-\hat{y})\big(\gamma_0|y-\hat{y}|^{\frac{nq}{r_1}}-\beta_0\big)\end{pmatrix} =\\ &= (y-\hat{y})\cdot\sum_{i=1}^{n}\left(\prod_{j=1}^{i} r_j\right)\left(\gamma_{n-i}|y-\hat{y}|^{\frac{i\cdot q}{r_1}}-\beta_{n-i}\right)\boldsymbol{A}^{n-i}\bar{\boldsymbol{t}}_n.\end{aligned} \tag{4.122}$$

Insertion of the coefficients β_k (4.119) leads to

$$\begin{aligned}\boldsymbol{l}(y-\hat{y}) &= (y-\hat{y})\cdot\sum_{i=1}^{n}\left(\prod_{j=1}^{i} r_j\right)\left(\gamma_{n-i}|y-\hat{y}|^{\frac{i\cdot q}{r_1}}-\alpha_{n-i}\prod_{j=1}^{i}\frac{1}{r_j}\right)\boldsymbol{A}^{n-i}\bar{\boldsymbol{t}}_n =\\ &= (y-\hat{y})\cdot\left(\sum_{i=1}^{n}\left(\prod_{j=1}^{i} r_j\right)\gamma_{n-i}|y-\hat{y}|^{\frac{i\cdot q}{r_1}}\boldsymbol{A}^{n-i}-\sum_{i=1}^{n}\alpha_{n-i}\boldsymbol{A}^{n-i}\right)\bar{\boldsymbol{t}}_n.\end{aligned} \tag{4.123}$$

Addition of the zero vector $(y-\hat{y})(\boldsymbol{A}^n-\boldsymbol{A}^n)\bar{\boldsymbol{t}}_n$ modifies equation (4.123) to

$$\boldsymbol{l}(y-\hat{y}) = (y-\hat{y})\cdot\left(\boldsymbol{A}^n+\sum_{i=1}^{n}\left(\prod_{j=1}^{i} r_j\right)\gamma_{n-i}|y-\hat{y}|^{\frac{i\cdot q}{r_1}}\boldsymbol{A}^{n-i}-\underbrace{\left(\boldsymbol{A}^n+\sum_{i=1}^{n}\alpha_{n-i}\boldsymbol{A}^{n-i}\right)}_{\boldsymbol{0}}\right)\bar{\boldsymbol{t}}_n. \tag{4.124}$$

Again the Caley-Hamilton theorem can be applied because the last term is the characteristic polynomial of $\boldsymbol{A}$ evaluated at $\boldsymbol{A}$ itself and, therefore, it vanishes.

Finally, the observer's injection terms are given by

$$\boldsymbol{l}(y-\hat{y}) = (y-\hat{y})\chi(\boldsymbol{A}, y-\hat{y})\bar{\boldsymbol{t}}_n, \tag{4.125}$$

where $\bar{\boldsymbol{t}}_n$ is the last column of the inverse observability matrix (4.113) and $\chi(\boldsymbol{A}, y-\hat{y})$ denotes the polynomial

$$\chi(\boldsymbol{A}, y-\hat{y}) = \boldsymbol{A}^n + \sum_{i=1}^{n} \chi_{n-i}(y-\hat{y})\boldsymbol{A}^{n-i} \tag{4.126}$$

with the error depending polynomial coefficients

$$\chi_{n-i}(y-\hat{y}) = \left(\prod_{j=1}^{i} r_j\right)\gamma_{n-i}|y-\hat{y}|^{\frac{i\cdot q}{r_1}}. \tag{4.127}$$

γ_{n-i} are the coefficients of the desired polynomial (4.80) of the corresponding system with homogeneity degree $\tilde{q}=0$. The homogeneity degree q and the dilation coefficients $\boldsymbol{r}$ have to be chosen such that they satisfy conditions (4.93) and (4.94).

The choice $q=0$ and $r_i=1$, $\forall i$ simplifies polynomial (4.126) to

$$\chi(\boldsymbol{A}) = \boldsymbol{A}^n + \sum_{i=1}^{n} \gamma_{n-i}\boldsymbol{A}^{n-i} \tag{4.128}$$

and, therefore, yields a linear observer with the eigenvalues of the estimation error dynamics located at $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$. This special case exactly matches with Ackermann's formula.

4.3 Robust observer design for observable LTI-systems

In this Section the approach for homogeneous observer design derived in Section 4.2 is used to construct observers for LTI-systems with uncertain input. Consider the observable system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}_\Delta \Delta(t) \\ y &= \boldsymbol{c}^T\boldsymbol{x} \end{aligned} \tag{4.129}$$

with disturbance input vector $\boldsymbol{b}_\Delta$ and unknown disturbance $\Delta(t)$ whose absolute value is bounded by a constant L, i.e.

$$|\Delta(t)| \leq L, \qquad \forall t,\, L \geq 0. \tag{4.130}$$

Transformation of system (4.129) to modified observability normal form like shown in 4.2.3 results in

$$\begin{aligned} \dot{\boldsymbol{z}} &= \bar{\boldsymbol{A}}\boldsymbol{z} + \bar{\boldsymbol{b}}_\Delta \Delta(t) \\ y &= \bar{\boldsymbol{c}}^T\boldsymbol{z} \end{aligned} \tag{4.131}$$

with

$$\bar{\boldsymbol{b}}_\Delta = \begin{pmatrix}\bar{b}_{\Delta,1} & \bar{b}_{\Delta,2} & \dots & \bar{b}_{\Delta,n}\end{pmatrix}^T = \boldsymbol{T}^{-1}\boldsymbol{b}_\Delta. \tag{4.132}$$

The observer design is done in the same way as described in Section 4.2.2 which leads to the estimation error dynamics

$$\begin{pmatrix}\dot{\sigma}_1\\ \dot{\sigma}_2\\ \vdots\\ \dot{\sigma}_{n-1}\\ \dot{\sigma}_n\end{pmatrix} = \begin{pmatrix} r_1\sigma_2 - r_1\gamma_{n-1}|\sigma_1|^{\frac{q}{r_1}+1}\operatorname{sign}(\sigma_1) + \bar{b}_{\Delta,1}\Delta(t)\\ r_2\sigma_3 - r_2\gamma_{n-2}|\sigma_1|^{\frac{2q}{r_1}+1}\operatorname{sign}(\sigma_1) + \bar{b}_{\Delta,2}\Delta(t)\\ \vdots\\ r_{n-1}\sigma_n - r_{n-1}\gamma_1|\sigma_1|^{\frac{(n-1)q}{r_1}+1}\operatorname{sign}(\sigma_1) + \bar{b}_{\Delta,n-1}\Delta(t)\\ -r_n\gamma_0|\sigma_1|^{\frac{nq}{r_1}+1}\operatorname{sign}(\sigma_1) + \bar{b}_{\Delta,n}\Delta(t)\end{pmatrix}. \tag{4.133}$$

The choice $q = -1$ and $r_1 = n$ yields the dilation coefficients

$$\boldsymbol{r} = \begin{pmatrix}r_1 & r_2 & \dots & r_n\end{pmatrix}^T = \begin{pmatrix}n & n-1 & \dots & 1\end{pmatrix}^T \tag{4.134}$$

due to relation (4.93) and modifies system (4.133) to

$$\begin{pmatrix}\dot{\sigma}_1\\ \dot{\sigma}_2\\ \vdots\\ \dot{\sigma}_{n-1}\\ \dot{\sigma}_n\end{pmatrix} = \begin{pmatrix} n\sigma_2 - n\gamma_{n-1}|\sigma_1|^{\frac{n-1}{n}}\operatorname{sign}(\sigma_1) + \bar{b}_{\Delta,1}\Delta(t)\\ (n-1)\sigma_3 - (n-1)\gamma_{n-2}|\sigma_1|^{\frac{n-2}{n}}\operatorname{sign}(\sigma_1) + \bar{b}_{\Delta,2}\Delta(t)\\ \vdots\\ 2\sigma_n - 2\gamma_1|\sigma_1|^{\frac{1}{n}}\operatorname{sign}(\sigma_1) + \bar{b}_{\Delta,n-1}\Delta(t)\\ -\gamma_0\operatorname{sign}(\sigma_1) + \bar{b}_{\Delta,n}\Delta(t)\end{pmatrix} \tag{4.135}$$

which, in terms of structure resembles the robust exact differentiator [6] except for the perturbation terms $\bar{b}_{\Delta,1},\dots,\bar{b}_{\Delta,n-1}$. It is known that (4.135) is robust against unknown bounded perturbations present in the last channel. Therefore, all other elements of the disturbance input vector must vanish

$$\bar{b}_{\Delta,1} = \bar{b}_{\Delta,2} = \dots = \bar{b}_{\Delta,n-1} = 0, \tag{4.136}$$

which means that $\bar{\boldsymbol{b}}_\Delta$ is a scaled unit vector along the last dimension in state space

$$\bar{\boldsymbol{b}}_\Delta = \bar{b}_{\Delta,n}\boldsymbol{e}_n. \tag{4.137}$$

In the following it is investigated which systems satisfy restriction (4.137). The inverse transformation of the disturbance input vector (4.132) is given by

$$\boldsymbol{b}_\Delta = \boldsymbol{T}\bar{\boldsymbol{b}}_\Delta = \bar{b}_{\Delta,n}\boldsymbol{T}\boldsymbol{e}_n. \tag{4.138}$$

Insertion of the transformation matrix (4.115) into (4.138) yields

$$\begin{aligned}\boldsymbol{b}_\Delta &= \bar{b}_{\Delta,n}\begin{pmatrix}\boldsymbol{A}^{n-1}\bar{\boldsymbol{t}}_n & r_1\boldsymbol{A}^{n-2}\bar{\boldsymbol{t}}_n & \dots & \left(\prod_{j=1}^{n-2} r_j\right)\boldsymbol{A}\bar{\boldsymbol{t}}_n & \left(\prod_{j=1}^{n-1} r_j\right)\bar{\boldsymbol{t}}_n\end{pmatrix}\boldsymbol{e}_n =\\ &= \bar{b}_{\Delta,n}\left(\prod_{j=1}^{n-1} r_j\right)\bar{\boldsymbol{t}}_n,\end{aligned} \tag{4.139}$$

where $\bar{\boldsymbol{t}}_n$ is the last column of the inverse observability matrix, see equation (4.113), and the dilation coefficients are given by (4.134) which modifies equation (4.139) to

$$\boldsymbol{b}_\Delta = \bar{b}_{\Delta,n} \cdot n! \cdot \boldsymbol{\mathcal{O}}_{obsv}^{-1} \boldsymbol{e}_n. \tag{4.140}$$

Multiplication with $\boldsymbol{\mathcal{O}}_{obsv}$ from the left-hand side and replacing it by its definition leads to

$$\begin{pmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T \boldsymbol{A} \\ \vdots \\ \boldsymbol{c}^T \boldsymbol{A}^{n-2} \\ \boldsymbol{c}^T \boldsymbol{A}^{n-1} \end{pmatrix} \boldsymbol{b}_\Delta = \bar{b}_{\Delta,n} \cdot n! \cdot \boldsymbol{e}_n. \tag{4.141}$$

Execution of the multiplications results in the system of equations

$$\begin{pmatrix} \boldsymbol{c}^T \boldsymbol{b}_\Delta \\ \boldsymbol{c}^T \boldsymbol{A} \boldsymbol{b}_\Delta \\ \vdots \\ \boldsymbol{c}^T \boldsymbol{A}^{n-2} \boldsymbol{b}_\Delta \\ \boldsymbol{c}^T \boldsymbol{A}^{n-1} \boldsymbol{b}_\Delta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \bar{b}_{\Delta,n} \cdot n! \end{pmatrix}. \tag{4.142}$$

The last row contains the information about the formation of $\bar{b}_{\Delta,n}$ which is given by

$$\bar{b}_{\Delta,n} = \frac{1}{n!} \boldsymbol{c}^T \boldsymbol{A}^{n-1} \boldsymbol{b}_\Delta. \tag{4.143}$$

To find the meaning of the other rows of equation (4.142) the output y of system (4.129) and its derivatives are examined, i.e.

$$\begin{aligned}
y &= \boldsymbol{c}^T \boldsymbol{x} \\
\dot{y} &= \boldsymbol{c}^T \dot{\boldsymbol{x}} = \boldsymbol{c}^T \boldsymbol{A} \boldsymbol{x} + \underbrace{\boldsymbol{c}^T \boldsymbol{b}_\Delta}_{=0} \Delta(t) \\
\ddot{y} &= \boldsymbol{c}^T \boldsymbol{A} \dot{\boldsymbol{x}} = \boldsymbol{c}^T \boldsymbol{A}^2 \boldsymbol{x} + \underbrace{\boldsymbol{c}^T \boldsymbol{A} \boldsymbol{b}_\Delta}_{=0} \Delta(t) \\
&\vdots \\
y^{(n-1)} &= \boldsymbol{c}^T \boldsymbol{A}^{n-2} \dot{\boldsymbol{x}} = \boldsymbol{c}^T \boldsymbol{A}^{n-1} \boldsymbol{x} + \underbrace{\boldsymbol{c}^T \boldsymbol{A}^{n-2} \boldsymbol{b}_\Delta}_{=0} \Delta(t) \\
y^{(n)} &= \boldsymbol{c}^T \boldsymbol{A}^{n-1} \dot{\boldsymbol{x}} = \boldsymbol{c}^T \boldsymbol{A}^{n} \boldsymbol{x} + \underbrace{\boldsymbol{c}^T \boldsymbol{A}^{n-1} \boldsymbol{b}_\Delta}_{=\bar{b}_{\Delta,n} \cdot n! \neq 0} \Delta(t),
\end{aligned} \tag{4.144}$$

where $y^{(i)}$ denotes the i^{th} derivative of the output y. Obviously, the disturbance $\Delta(t)$ is only allowed to act directly to the n^{th} derivative of the output which means that the system output y must offer a relative degree

$$\delta = n \tag{4.145}$$

w.r.t. the unknown disturbance $\Delta(t)$. A system of the form (4.129) which is observable and satisfies condition (4.145) is called strongly observable [30], [31].

In addition to strong observability, it has to be ensured that the discontinuity in the last differential equation of (4.135) is capable to dominate the disturbance which requires

$$|\gamma_0| > \left|\bar{b}_{\Delta,n}\right| L. \tag{4.146}$$

From relation (4.80) it is clear that γ_0 is the product of the homogeneous eigenvalues $\tilde{\lambda}$ of the corresponding system with degree $\tilde{q} = 0$, i.e.

$$\gamma_0 = (-1)^n \prod_{i=1}^{n} \tilde{\lambda}_i. \tag{4.147}$$

Insertion of equations (4.147) and (4.143) into inequality (4.146) results in the necessary condition for the choice of the homogeneous eigenvalues

$$\left|\prod_{i=1}^{n} \tilde{\lambda}_i\right| > \frac{L}{n!}\left|\boldsymbol{c}^T \boldsymbol{A}^{n-1} \boldsymbol{b}_\Delta\right|. \tag{4.148}$$

The sufficient stability criterion presented in Section 3.3.4 is not suitable for the analysis of higher-order systems. Hence, the choice of the homogeneous eigenvalues provides necessary conditions for the stability of the estimation error dynamics only. Nevertheless, a parameter setting, which guarantees global asymptotic stability, always exists because the structure of the estimation error dynamics equals the robust exact differentiator [5], [32].

4.4 Example for robust homogeneous observer design

In this Section the methods for robust homogeneous observer design developed in Section 4.2 and 4.3 are applied to the perturbed third order LTI-system

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \begin{pmatrix} -5 & 2 & 0 \\ 5 & -6 & -5 \\ -15 & 11 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \Delta(t) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}_\Delta \Delta(t) \\ y &= \begin{pmatrix} \frac{1}{10} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \boldsymbol{c}^T \boldsymbol{x} \end{aligned} \tag{4.149}$$

with unknown input $\Delta(t)$. Furthermore, it is known that the amplitude of $\Delta(t)$ does not exceed $L = 23$, i.e.

$$|\Delta(t)| \leq L = 23, \qquad \forall t. \tag{4.150}$$

4.4.1 Verification of strong observability

The application of the presented method for robust homogeneous observer design is limited to strongly observable systems. For this reason system (4.149) is checked for observability and the relative degree δ of the output w.r.t. $\Delta(t)$ is determined. The observability matrix computes

$$\boldsymbol{\mathcal{O}}_{obsv} = \begin{pmatrix} \boldsymbol{c}^T \\ \boldsymbol{c}^T\boldsymbol{A} \\ \boldsymbol{c}^T\boldsymbol{A}^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{5} & 0 \\ \frac{7}{2} & -\frac{11}{5} & -1 \end{pmatrix}. \tag{4.151}$$

Due to its triangular structure it is obvious that the rows of $\boldsymbol{\mathcal{O}}_{obsv}$ are linearly independent which means that the observability matrix is regular. Therefore, the system is observable. Furthermore, the first and the second row of $\boldsymbol{\mathcal{O}}_{obsv}$ are orthogonal to the input vector $\boldsymbol{b}_\Delta$, i.e.

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{b}_\Delta &= 0, \\ \boldsymbol{c}^T\boldsymbol{A}\boldsymbol{b}_\Delta &= 0. \end{aligned} \tag{4.152}$$

In consequence of this the output y has the relative degree $\delta = n = 3$ and, therefore, system (4.149) is strongly observable.

4.4.2 Choice of the homogeneous eigenvalues

In Section 4.3 it is derived that the product of the homogeneous eigenvalues $\tilde{\lambda}_i$ has to satisfy the inequality

$$\left|\prod_{i=1}^{n}\tilde{\lambda}_i\right| > \frac{L}{n!}\left|\boldsymbol{c}^T\boldsymbol{A}^{n-1}\boldsymbol{b}_\Delta\right| \tag{4.153}$$

in order to compensate the unknown disturbance $\Delta(t)$. Insertion of the given values yields

$$\left|\prod_{i=1}^{3}\tilde{\lambda}_i\right| > \frac{23}{3!}|2| \approx 7.67. \tag{4.154}$$

Due to stability reasons, see Section 3.3.4, it is clear that all the eigenvalues $\tilde{\lambda}_i$, $i = 1, 2, 3$ have to be negative. A suitable choice of the homogeneous eigenvalues that fulfills inequality (4.154) is

$$\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = -2. \tag{4.155}$$

4.4.3 Application of the generalized formula of Ackermann

According to Section 4.2.4 the non-linear injection terms of the observer are given by

$$\boldsymbol{l}(y-\hat{y}) = (y-\hat{y})\chi(\boldsymbol{A}, y-\hat{y})\bar{\boldsymbol{t}}_n. \tag{4.156}$$

The coefficients of the polynomial

$$\chi(\boldsymbol{A}, y-\hat{y}) = \boldsymbol{A}^n + \sum_{i=1}^{n} \chi_{n-i}(y-\hat{y})\boldsymbol{A}^{n-i} \tag{4.157}$$

compute

$$\chi_{n-i}(y-\hat{y}) = \left(\prod_{j=1}^{i} r_j\right)\gamma_{n-i}|y-\hat{y}|^{\frac{i\cdot q}{r_1}} \tag{4.158}$$

which yields

$$\chi(\boldsymbol{A}, y-\hat{y}) = \boldsymbol{A}^3 + r_1\gamma_2|y-\hat{y}|^{\frac{q}{r_1}}\boldsymbol{A}^2 + r_1r_2\gamma_1|y-\hat{y}|^{\frac{2q}{r_1}}\boldsymbol{A} + r_1r_2r_3\gamma_0|y-\hat{y}|^{\frac{3q}{r_1}}\boldsymbol{I}. \tag{4.159}$$

The homogeneity degree q and the dilation coefficients $\boldsymbol{r}$ are chosen as

$$q = -1, \qquad \boldsymbol{r} = \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}^T \tag{4.160}$$

in order to achieve a robust observer, see Section 4.3. Denoting the coefficients of the desired polynomial as γ_{n-i}, the desired polynomial reads as

$$\gamma(\tilde{\lambda}) = \prod_{i=1}^{n}(\tilde{\lambda} - \tilde{\lambda}_i) = \tilde{\lambda}^n + \gamma_{n-1}\tilde{\lambda}^{n-1} + \cdots + \gamma_1\tilde{\lambda} + \gamma_0, \tag{4.161}$$

which, for the choice (4.155) yields

$$\gamma(\tilde{\lambda}) = (\tilde{\lambda} + 2)^3 = \tilde{\lambda}^3 + 6\tilde{\lambda}^2 + 12\tilde{\lambda} + 8. \tag{4.162}$$

Insertion of the coefficients of the desired polynomial (4.162), the homogeneity degree $q = -1$ and the dilation coefficients $\boldsymbol{r}$ (4.160) simplifies polynomial (4.159) to

$$\chi(\boldsymbol{A}, y-\hat{y}) = \boldsymbol{A}^3 + 18|y-\hat{y}|^{-\frac{1}{3}}\boldsymbol{A}^2 + 72|y-\hat{y}|^{-\frac{2}{3}}\boldsymbol{A} + 48|y-\hat{y}|^{-1}\boldsymbol{I}. \tag{4.163}$$

The last missing part in the generalized formula of Ackermann (4.156) is $\bar{\boldsymbol{t}}_n$ which is given by

$$\bar{\boldsymbol{t}}_n = \boldsymbol{\mathcal{O}}_{obsv}^{-1}\boldsymbol{e}_n = \begin{pmatrix} 10 & 0 & 0 \\ 25 & 5 & 0 \\ -20 & -11 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}. \tag{4.164}$$

Finally, equation (4.156) is evaluated which ends up in the non-linear injection vector

$$\boldsymbol{l}(y-\hat{y}) = \left(|y-\hat{y}| \cdot \begin{pmatrix} -50 \\ -45 \\ -36 \end{pmatrix} + |y-\hat{y}|^{\frac{2}{3}} \cdot \begin{pmatrix} 180 \\ 0 \\ 342 \end{pmatrix} + |y-\hat{y}|^{\frac{1}{3}} \cdot \begin{pmatrix} 0 \\ 360 \\ -432 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -48 \end{pmatrix}\right)\operatorname{sign}(y-\hat{y}) \tag{4.165}$$

for the robust observer

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{l}(y-\hat{y}) \\ \hat{y} &= \boldsymbol{c}^T\hat{\boldsymbol{x}}. \end{aligned} \tag{4.166}$$

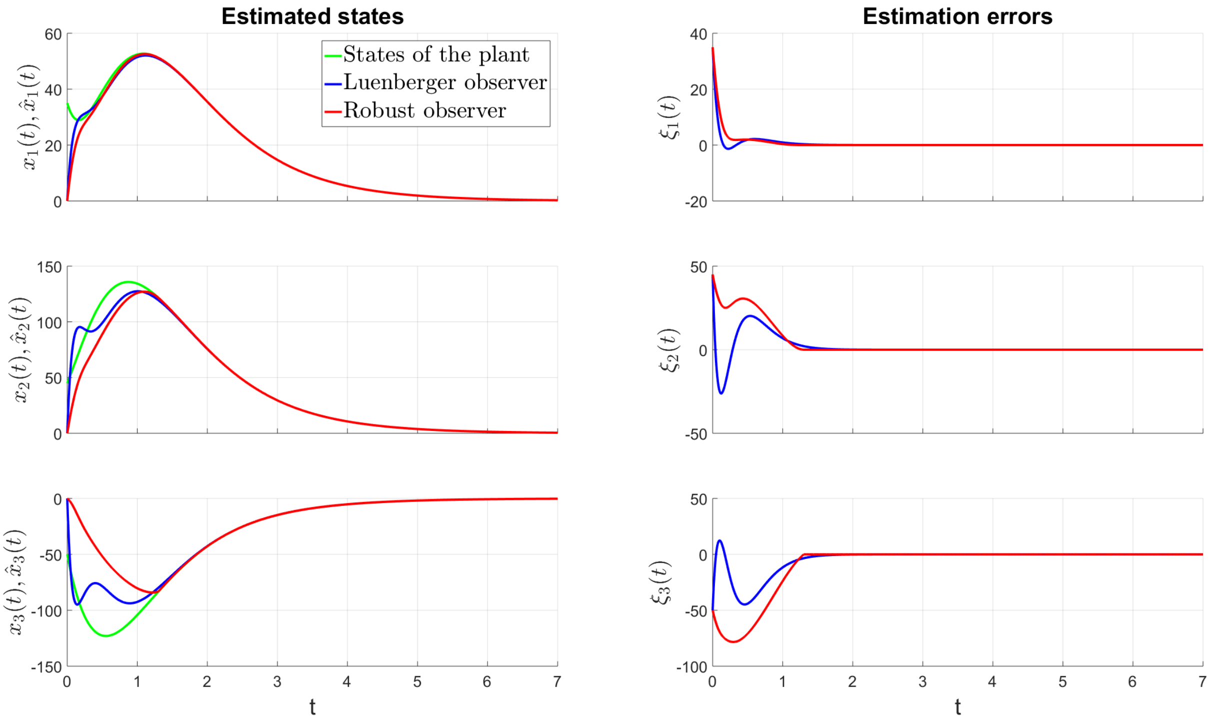

Figure 4.1: Comparison of the estimated states and estimation errors over time in the unperturbed case.

4.4.4 Simulation

A simulation in Matlab/Simulink of the designed observer (4.166) demonstrates its suitability. Additionally, the robust observer is compared to a Luenberger observer with eigenvalues $s_i = -6$, $i = 1, 2, 3$. The initial state of the system is chosen as

$$\boldsymbol{x}_0 = \begin{pmatrix} 35 & 45 & -50 \end{pmatrix}^T \tag{4.167}$$

and the initial states of both observers are selected to be the zero vector, i.e.

$$\hat{\boldsymbol{x}}_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T. \tag{4.168}$$

Unperturbed Case

First of all, the unperturbed case is considered, i.e.

$$\Delta(t) = 0, \qquad \forall t. \tag{4.169}$$

The estimation of the states and the estimation errors over time are shown in Figure 4.1. The estimation errors of the Luenberger observer decay exponentially. The robust homogeneous observer in contrast converges in finite time $T_c \approx 1.3s$.

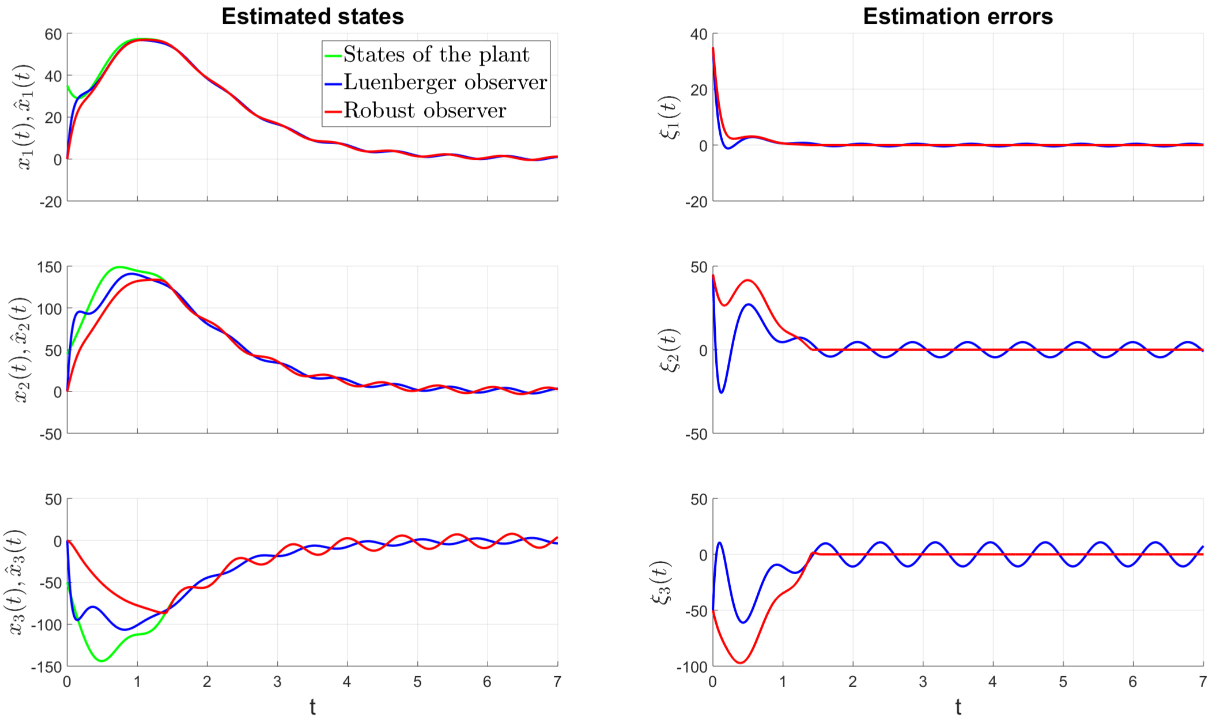

Figure 4.2: Comparison of the estimated states and estimation errors over time for $\Delta(t) = 23 \cdot \sin(8t)$.

Perturbed Case

Now the system is disturbed by the perturbation

$$\Delta(t) = 23 \cdot \sin(8t) \tag{4.170}$$

which satisfies assumption (4.150) the robust observer is designed for.

Figure 4.2 shows the simulation result. The Luenberger observer does not ensure convergence any more. Even when the transients died away, the unknown disturbance excites the system. The estimation errors stay bounded because of the BIBO property of the estimation error dynamic.

As expected the robust observer, i.e. $q = -1$, again ensures convergence within finite time. The convergence time $T_c \approx 1.5s$ takes slightly longer than in the unperturbed case. Once the observer error is driven to zero, the disturbance $\Delta(t)$ is entirely suppressed.

In Figure 4.3 the perturbation is changed to

$$\Delta(t) = 30 \cdot \sin(2t). \tag{4.171}$$

The disturbance amplitude exceeds $L = 24$ which is the maximum amplitude that can be

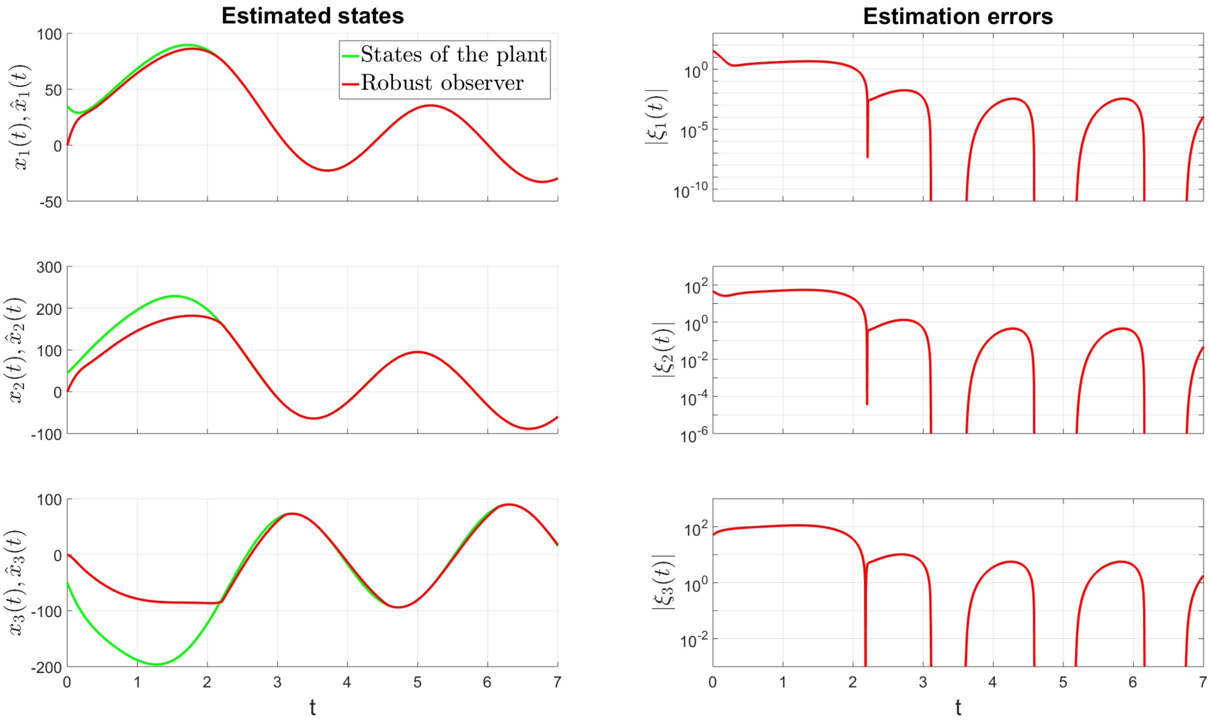

Figure 4.3: Estimated states and estimation errors over time for $\Delta(t) = 30 \cdot \sin(2t)$.

compensated by the robust observer with this choice of the gains. Obviously, the observer error can not be forced to zero or remain in zero in the time intervals when the absolute value of the sinusoidal perturbation overshoots $L = 24$. In between the disturbance amplitude is smaller than $L = 24$ and the observer again converges in finite time.

5 Conclusion

A new design algorithm for homogeneous observers for linear time-invariant systems of arbitrary homogeneity degree has been proposed. The approach is based on the assignment of homogeneous eigenvalues proposed by H. Nakamura et al. [11]. The observer's injection terms generalize Ackermann's eigenvalue assignment for homogeneous systems. Conditions regarding the choice of the dilation coefficients are derived in order to ensure that the estimation error dynamics are homogeneous.

Moreover, the approach is exploited to construct robust observers for strongly observable systems with bounded perturbations. The resulting observers introduce discontinuities on the right-hand side of the estimation error dynamics which allow the compensation of disturbances. It is shown that the product of the selected homogeneous eigenvalues affects the robustness against the perturbations. A final tutorial example demonstrates the effectiveness of the presented approach and the theoretical findings are confirmed by numerical simulations.

The established algorithm unifies many well-known methods, i.e. the Luenberger observer, the super-twisting algorithm and Levant's robust exact differentiator [6]. It is simple to apply due to its similarity to Ackermann's formula. Although the homogeneous eigenvalues in general only provide necessary conditions for the stability of the estimation error dynamics they are a reasonable starting point for the choice of the observer parameters.

5.1 Outlook

The sufficient stability criteria [12] regarding homogeneous eigenvalues are not suitable for the analysis of higher-order systems. In consequence, the choice of the homogeneous eigenvalues currently relies on necessary conditions. Simplifications of the sufficient stability criteria should be addressed in the future.

Future work should also deal with the development of proper discretization techniques for the resulting observers. This if of great interest, especially if the observer includes discontinuous terms on the right-hand side as those terms may lead to the so-called chattering phenomenon. This is necessary to utilize the presented algorithm in real world applications.

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