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Triangles in the colored Euclidean plane

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Abstract

In this thesis we consider triangles in the colored Euclidean plane. We call a triangle monochromatic if all its vertices have the same color. First we study, how many colors are needed, such that for every triangle we can color the Euclidean plane in such a way, that there does not exist a monochromatic rotated copy of the triangle or a monochromatic translated copy of the triangle. Furthermore we show that for every triangle every coloring of the Euclidean plane in finitly many colors contains a monochromatic triangle, which is similar to the given triangle. Then we study the problem, for which triangles there exists a 6-coloring, such that the triangle is nonmonochromatic in this 6-coloring. Later we show, that for every triangle, there exists a 2-coloring of the rational plane, such that the triangle is nonmonochromatic. Finally we give a 5-coloring of a strip with height 1, such that there do not exist two points with distance 1, which have the same color.

Kurzzusammenfassung

In dieser Arbeit betrachten wir Dreiecke in der gefärbten Euklidischen Ebene. Wir sagen, ein Dreieck ist monochromatisch, wenn alle Eckpunkte die gleiche Farbe haben. Zuerst untersuchen wir, mit wie vielen Farben wir die Ebene färben müssen, sodass das Dreieck keine monochromatische Kopie hat, die nur durch Translation beziehungsweise nur durch Rotation entstanden ist. Dann beweisen wir, dass es für jedes Dreieck in jeder endliche Färbung der Euklidischen Ebene ein monochromatisches Dreieck gibt, das ähnlich zum gegebenen Dreieck ist. Wir geben dann an, für welche Dreiecke es 6-Färbungen gibt, sodass diese Dreiecke nonmonochromatisch sind. Dann betrachten wir Dreiecke in der rationalen Ebene und beweisen, dass es für jedes Dreieck eine 2-Färbung der rationalen Ebene gibt, sodass das Dreieck nonmonochromatisch ist. Abschließend zeigen wir eine 5-Färbung eines Streifens mit Breite 1, mit der Eigenschaft, dass es keine zwei Punkte mit der gleichen Farbe gibt, die Abstand 1 haben.

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1 Introduction

We study the following question.

Problem 1. *[10]*
What is the smallest number c, so that for any triangle T, there exists a coloring of the Euclidean plane with c colors, such that it is not possible, that all vertices of T have the same color if we embed it in the colored plane?

The following problem is very similar to the one above, but is more specific in the sense, that the number of colors depends on the triangle.

Problem 2. *Given a triangle T. What is the smallest number c, so that we can color the Euclidean plane with $c(T)$ colors, such that it is not possible, that all vertices of T have the same color if we embed it in the colored Euclidean plane?*

Problem 1 is a problem in Euclidean Ramsey theory. Euclidean Ramsey theory was introduced by Erdős and others in a set of papers [5, 6, 7] in 1973. As its name suggests, Euclidean Ramsey theory combines ideas from geometry and combinatorics, but also set theory and measure theory are important.

An important problem in Euclidean Ramsey theory and the motivation for Problem 1 is the so called chromatic number of the Euclidean plane.

1.1 Chromatic number of the plane

In this chapter we mean with plane the Euclidean plane if no specific norm is mentioned.

Problem 3 (Chromatic number of the plane)**.** *How many colors are at least needed, such that there exists a coloring of the Euclidean plane, so that no two points with distance* 1 *have the same color?*

This number is the so called chromatic number of the plane and is denoted by $\chi(\mathbb{E}^2)$. Here $\mathbb{E}^2$ can also be seen as an infinite graph, where the set of vertice is the set of all points of the plane and two vertices are joined by an edge, if they are distance 1 apart.

This problem can also be generalised, in the sense, that we take the Euclidean space of dimension d instead of the plane. This number is $\chi(\mathbb{E}^d)$. Another variant is, if we only consider colorings, where all sets of points with the same color are Lebesgue measurable. We denote this number as $\chi_m(\mathbb{E}^d)$ or $\chi_m(G)$ if we only look at the chromatic number of a graph G in such a coloring.

The chromatic number of the plane was introduced by Edward Nelson in 1950.

Even five years earlier Hugo Hadwiger had worked on a similar problem and had shown that every coloring of the Euclidean plane with five closed congruent sets contains a pair of points with distance 1. Therefore the problem of the chromatic number of the plane is also called Nelson-Hadwiger problem.

For the chromatic number of the plane lower and upper bounds are known. These bounds have not been improved since the problem was first mentioned.

Theorem 1. *[14] The chromatic number of the plane is at least* 4.

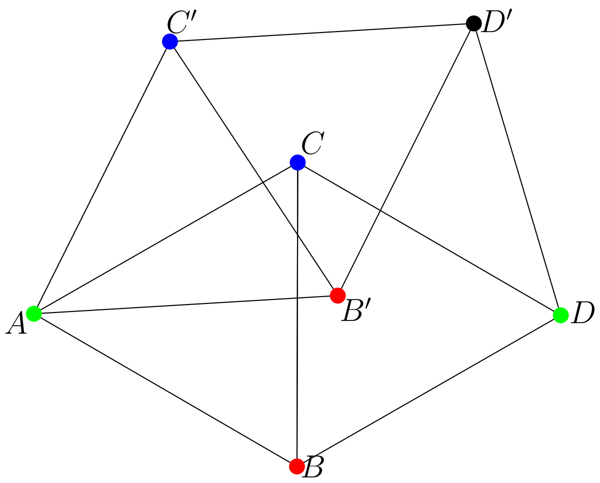


Figure 1: Moser spindle: a unit-distance graph with chromatic number 4

Proof. We look at the unit distance graph in Figure 1, the so called Moser spindle. We try to color the Moser spindle with 3 colors, say red, green and blue. Without loss of generality, let A be green. Then B, B', C and C' have to be red or blue, but B and C cannot have the same color, therefore let B be red and C be blue. Furthermore we color B' red and C' blue, because B' and C' cannot have the same color. Then D is green, but D' has a green neighbor D, a red neighbor B' and a blue neighbor C', so we cannot 3-color the Moser spindle. This means that the Moser spindle has chromatic number 4 and therefore the chromatic number of the plane is at least 4. □

There are some other examples of unit distance graphs with chromatic

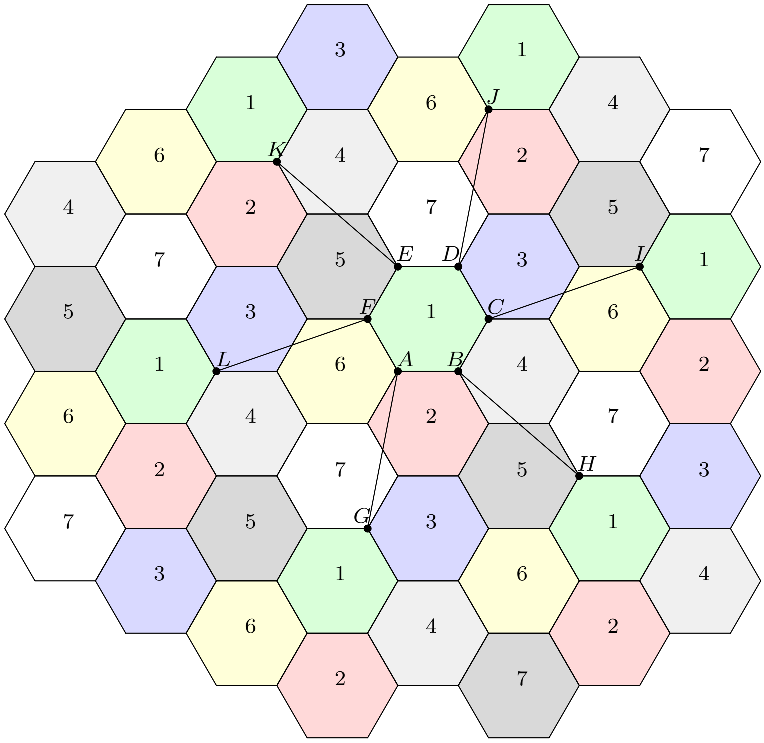


Figure 2: 7-coloring of the plane with regular hexagons with diameter $1-\epsilon$, such that no points with distance 1 have the same color

number 4. It has been proved by O'Donnel, that there exist unit distance graphs with chromatic number 4 and girth k for an arbitrary integer k [18].

Theorem 2. *[12] The chromatic number of the plane is at most* 7*.*

Proof. Let $\epsilon > 0$ be small. We study the coloring in Figure 2. The hexagons in Figure 2 have diameter $1-\epsilon$. Therefore, the distance between two points in the same hexagon is less than 1. The line segments AG, BH, CI, DJ, EK and FL are the shortest line segments between the hexagon $ABCDEF$ and other hexagons with the same color, in our case color 1. Due to rotational symmetry we have

$$\overline{AG} = \overline{BH} = \overline{CI} = \overline{DJ} = \overline{EK} = \overline{FL}. \tag{1}$$

For the distance $\overline{AG}$ we have

$$\overline{AG}^2 = \left(\frac{\sqrt{3}}{4}(1-\epsilon)\right)^2 + \left(\left(\frac{1}{2}+\frac{3}{4}\right)(1-\epsilon)\right)^2 \tag{2}$$

$$= (1-\epsilon)^2\left(\frac{3}{16}+\frac{25}{16}\right) \tag{3}$$

$$= (1-\epsilon)^2\frac{7}{4} \tag{4}$$

So we get

$$\Leftrightarrow \overline{AG} = (1-\epsilon)\frac{\sqrt{7}}{2} > 1 \tag{5}$$

Since $\overline{AG} > 1$, the distance between two points with the same color in different hexagons is more than 1. Therefore there are no two points in Figure 2 with the same color, which have distance 1. □

Erdős and de Bruijn [3] proved, that there exists a finite unit distance graph, whose chromatic number is the chromatic number of the plane. This means if the chromatic number of the plane is 7, then there exists also a finite unit distance graph which has chromatic number 7. Pritikin [16] showed, that all unit distance graphs with at most 6197 vertices are 6-colorable. So the number 6 is only an upper bound for the chromatic number of graphs with 6197 vertices. In his proof, Pritikin used a coloring of the plane with 7 colors, such that the area of one of these seven colors, say color c is very small and the vertices can be embeded in the colored plane, such that no vertex lies in the area with color c.

Townsend [19] proved, that for map colorings, we need at least 6 colors, such that no two points with distance 1 have the same color. We will see a definition for map colorings in Section 3.

In 1981 Falconer [8] showed, that $\chi_m(\mathbb{E}^n) \geq n+3$. So for $n = 2$ this gives us $\chi_m(\mathbb{E}^2) \geq 5$.

In recent years, it was studied if the chromatic number of the plane depends on the axioms we use. Payne [15] constructed some unit distance graphs G where $\chi_m(G) \neq \chi(G)$.

Recently de Grey [4] found a unit-distance graph with chromatic number 5. This gives us $\chi(\mathbb{E}^2) \geq 5$.

Before we look at some known results for Problem 1, we need some definitions.

1.2 Definitions

Definition 3. *A coloring is a partitioning of the Euclidean space, such that every point of the space has one color. An r-coloring is a coloring with r colors.*

Definition 4.

- *Two finite point sets X and Y are congruent if $|X| = |Y|$ and Y can be covered by rotation and translation of X.*
- *A finite point set X is monochromatic if every point of X has the same color.*
- *A finite point set X exists monochromatically in a coloring if there exists a congruent copy of X which is monochromatic.*
- *A finite point set X is nonmonochromatic in a coloring if it does not exist monochromatically.*

In this following chapters we will study point sets of size 3 in the plane, which we can see as vertices of a triangle.

1.3 Notation

Graham introduced the following notation for problems in Euclidean Ramsey theory [9].

Definition 5. *Let d be the dimension of the Euclidean space and let r be a positive integer. Let X be a finite point set and $Cong(X)$ the family of all congruent copies of X. We write $\mathbb{E}^d \xrightarrow{r} Cong(X)$ if X exists monochromatically in all r-colorings of $\mathbb{E}^d$.*

If it is obvious that we are looking for congruent copies of X, we can write $\mathbb{E}^d \xrightarrow{r} X$.

Definition 6. *Let A and B be any two distinct points in the Euclidean plane. We write AB for the line segment between A and B and we write $\overline{AB}$ for the length of the line segment AB.*

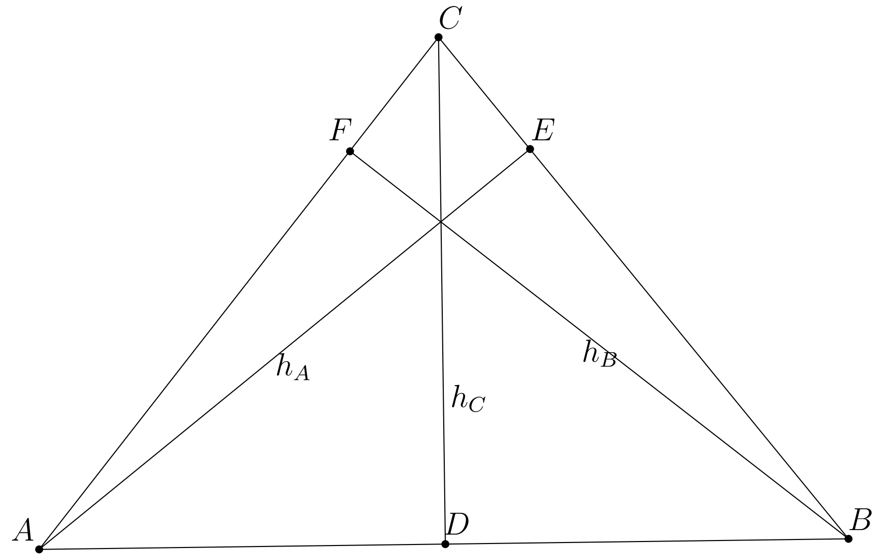


Figure 3: Normed triangle with notation

1.4 Normed triangle

Let T be a triangle with vertices A, B and C. Without loss of generality we can assume that AB is the longest side of T.

Definition 7. *Let us denote the heights of a given triangle as in Figure 3. We call a triangle a normed triangle, if the longest side is* AB *and has length* 1, *and* $\overline{BC} \leq \overline{AC}$ *holds. That means, that* h_C *is the shortest height of the triangle.*

Theorem 8. *If a triangle* T *with side lengths* a, b *and* c *does not exist monochromatically in an* r*-coloring* F, *then there exists an* r*-coloring, so that the triangle* T_a *with side lengths* 1, $\frac{b}{a}$ *and* $\frac{c}{a}$ *does not exist monochromatically.*

Proof. We construct an r-coloring F_a, so that T_a does not exist monochromatically. Consider the Euclidean plane as a vectorspace $\mathbb{R}^2$ with origin O. Every point P is colored in F_a in the same color as Q is colored in F, where $\overrightarrow{OQ} = a\,\overrightarrow{OP}$.

If T_a exists monochromatically in F_a, then there is a monochromatic triangle congruent to T_a with vertices D, E and F. Let X, Y and Z be points which satisfy

$$\begin{aligned} a\,\overrightarrow{OD} &= \overrightarrow{OX}, \\ a\,\overrightarrow{OE} &= \overrightarrow{OY}, \\ a\,\overrightarrow{OF} &= \overrightarrow{OZ}. \end{aligned}$$

Then X, Y and Z have the same color in the coloring F. Without loss of generality let $|\overrightarrow{DE}| = 1$. Then

$$\begin{aligned} |\overrightarrow{XY}| &= |\overrightarrow{OX} - \overrightarrow{OY}| \\ &= |a\overrightarrow{OE} - a\,\overrightarrow{OD}| \\ &= a|\overrightarrow{OE} - \overrightarrow{OD}| \\ &= a|\overrightarrow{DE}| = a \end{aligned}$$

holds. In the same way we get $|\overrightarrow{YZ}| = b$ and $|\overrightarrow{XZ}| = c$. So we have in F a monochromatic triangle congruent to T which is a contradiction. Therefore T_a does not exist monochromatically in F_a. □

So we only have to study normed triangles, because Theorem 8 implies if a triangle T with longest side length 1 is nonmonochromatic in a c-coloring F, then there exists a c-coloring F' such that every triangle T' with the same angles as T is nonmonochromatic in F'.

1.5 Triangles in the colored Euclidean plane

Problem 1 is interesting, because it seems to be easier than the chromatic number of the plane and we hope, if we know the solution for Problem 1, that we can get better bounds for the chromatic number of the plane.

The chromatic number of the plane is at most 7, which means that in Problem 1 the number of colors is also at most 7. On the other side, we can not use the lower bound of the chromatic number of the plane for Problem 1. There has been a lot of work done for triangles in 2-colorings of the plane.

Due to Theorem 8, we can assume that a given triangle is a normed triangle.

Theorem 9. *[5] The equilateral triangle T with sides of length 1 is nonmon-monochromatic in a coloring with parallel strips of width $\frac{\sqrt{3}}{2}$.*

Proof. We take parallel halfopen strips of height $\frac{\sqrt{3}}{2}$ as in Figure 4. Since the height of the T is the same as the height of the halfopen strips, all vertices

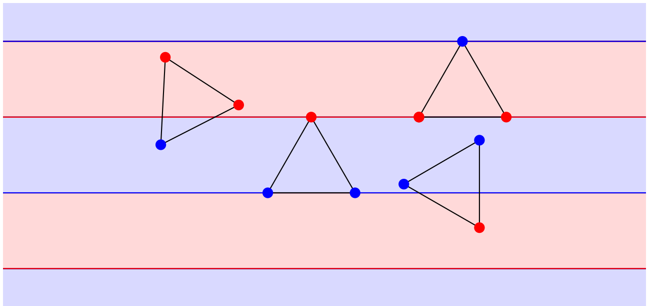


Figure 4: 2-coloring with halfopen strips for equilateral triangles

cannot lie in the same stip. It is impossible that all vertices are in different strips with the same color because $3\frac{\sqrt{3}}{2} > 1$. If two vertices are in the same strip, then the line segment between those vertices is also in the same strip due to convexity of the strips. But the height of the T is exactly the height of the strips, so the third vertex cannot be in a strip with the same color as the other two vertices. $\square$

It was conjectured, that such a coloring with strips and some freedom on the boundaries would be the only coloring with two colors, such that the equilateral triangle is nonmonochromatic in this coloring. In 2006 four czech students disproved this conjecture [13]. The counterexample is a so called zebra-like coloring, where the boundary between the colors consists of continuous curves with some properties instead of straight lines.

So there exist 2-colorings, for which equilateral triangles are nonmonochromatic, but there are also triangles which exist monochromatically in every 2-coloring.

Theorem 10. *[5] The triangle with angles* $30°$, $60°$ *and* $90°$ *exists monochromatically in every* 2*-colored Euclidean plane.*

Proof. We color the plane red and blue. We look at Figure 5. If we 2-color the vertices of the equilateral triangle with side length d, then two vertices have the same color. So we can say, that in every 2-coloring of the plane, there exist two points with the same color and distance d apart, for every positive d. Therefore we can assume that A and D have the same color, without loss of generality say red. If one of the points B, C, E and F is red, then we have a red triangle with angles $30°$, $60°$ and $90°$. Otherwise, if B, C, E and F are blue, then we have a blue triangle with angles $30°$, $60°$ and $90°$. $\square$

So we have in Problem 1 the lower bound 3. In fact, there are more examples of triangles, that exist monochromatically in every 2-coloring. For

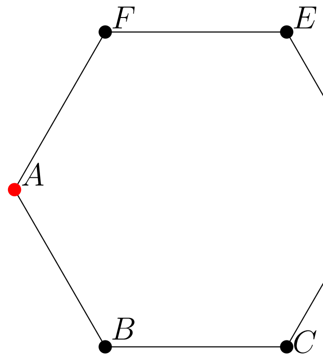


Figure 5: Regular hexagon with red colored points A and D, used in the proof of Theorem 10

example, Leslie Shader [17] proved, that every triangle with a 90° angle exists monochromatically in every 2-coloring. We will see another example of such a triangle also in a later chapter. There is a nice theorem which helps finding such triangles.

Theorem 11. *[7] Let T be a triangle with sides of lengths a, b and c Let T_a be an equilateral triangle with sides of length a, T_b be an equilateral triangle with sides of length b and T_c be an equilateral triangle with sides of length c. T exists monochromatically in a 2-coloring if and only if one of T_a, T_b and T_c exists monochromatically.*

Proof. We look at Figure 6. The triangles ABD, HBC, EDC, EFH, DFG and AHG are congruent triangles with sides of length a, b and c. These triangles are also congruent to T. The triangles ABH, EFD, BCD, FGH, ADG and CEH are equilateral triangles. We assume that the two colors are red and blue.

First we assume ABH is a red triangle. We want to show that T exists monochromatically. So if one of the triangles ABD, HBC, EDC, EFH, DFG and AHG is monochromatic, we are done. We color the points so that we avoid such a monochromatic triangle as long as possible. Therefore we color C, D and G blue, because of the triangles HBC, ABD and AHG. Then E and F are red, because of the triangles EDC and DFG. But now the triangle EFH is red, so if ABH is monochromatic then T exists monochromatically.

We can prove this for the other equilateral triangles in the same way. Now to the other direction, where T exists monochromatically.

Without loss of generality assume that A, B and D is a red triangle. We assume that ABH, EFD, BCD, FGH, ADG and CEH are not monochro-

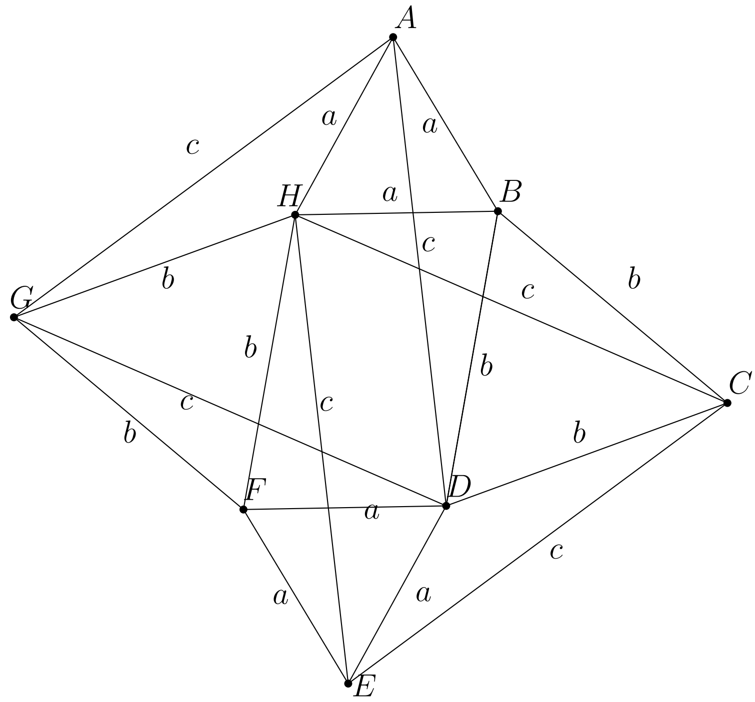


Figure 6: Illustration for the proof of Theorem 11

matic. So C, H and G are blue, because of the triangles HBC, ABD and AHG. Furthermore, E and F are red, because of the triangles CHE and GHF. So we have a red triangle EFD, which is a contradiction to our assumption. □

In [7] this theorem was used to find other triangles, that exist monochromatically in every 2-coloring of the Euclidean plane. It is conjectured, that every triangle aside from the equilateral triangle exists monochromatically in every 2-coloring of the plane.

Since 3 is the lower bound for Problem 1, it is also interesting, which triangles are nonmonmonochromatic in a 3-coloring of the plane. Graham and Tressler found some triangles, that are nonmonmonochromatic in a 3-coloring of the plane [11]. They showed, that the normed triangle with sidelengths $\frac{1}{2}$, $\frac{1}{2}$ and 1 is nonmonmonochromatic in a 3-coloring of the plane with regular hexagons. Furthermore, they also mentioned, that some triangles are nonmonmonochromatic in a 3-coloring with halfopen strips. It is also conjectured by them, that 3 is the solution for Problem 1.

2 Variants

In this chapter we look at variants of Problem 1. First we like to know, if we restrict the movement of the triangle, how much colors are needed, so that the triangle does not exist monochromaticaly. That way we get lower bounds for Problem 1.

Later in this chapter we look, what happens if we examine the class of similar triangles. Two triangles are similar if their angles are the same. The family of congruent triangles is a subfamily of the family of similar triangles and therefore we get an upper bound for the original problem.

2.1 Translation

Let T be a triangle and let $Trans(T)$ be the family of triangles which we get by translation of T. We show, that for any triangle T_1, there exists a 2-coloring of the Euclidean plane, such that every translated copy of T_1 is not monochromatic.

Theorem 12. *For all triangles T $\mathbb{E}^2 \overset{2}{\not\to} Trans(T)$ holds.*

Proof. Let l be a line that is parallel to side BC of the fixed triangle T. We project B and C onto l and call the resulting points B_l and C_l. The distance between B_l and C_l is c. We color the plane with half open strips of

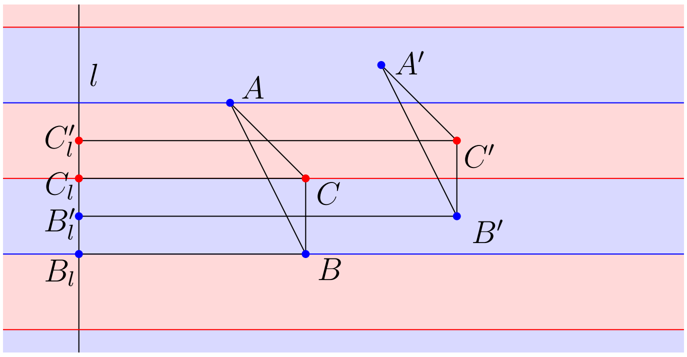


Figure 7: 2-coloring for translation

width c which are normal to l like in Figure 7. Furthermore if we project B' and C' of $T' \in Trans(T)$ onto l then the distance between these two points is again c, because with translation $B'C'$ and l are parallel for all triangles $T' \in Trans(T)$. Therefore the points B' and C' of $T' \in Trans(T)$ lie in different colored, adjacent strips in this coloring. $\square$

2.2 Rotation

Let T be a triangle and let $Rot_P(T)$ be the family of copies of T, which are rotated around a fixed point P. If P is not the circumcenter, then there are at least two vertices of the triangle, which are not on the same circle. We can color these two circles with two diffent colors as in Figure 8 and then every triangle $T' \in Rot_P(T)$ is nonmonochromatic.

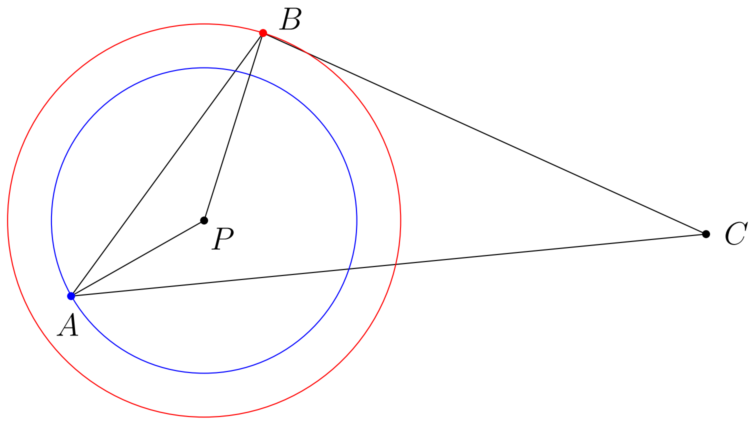


Figure 8: 2-coloring for rotation, if P is not the circumcenter

Theorem 13. *Let P be a fixed point. There exists a triangle T with $\mathbb{E}^2 \xrightarrow{2} Rot_P(T)$.*

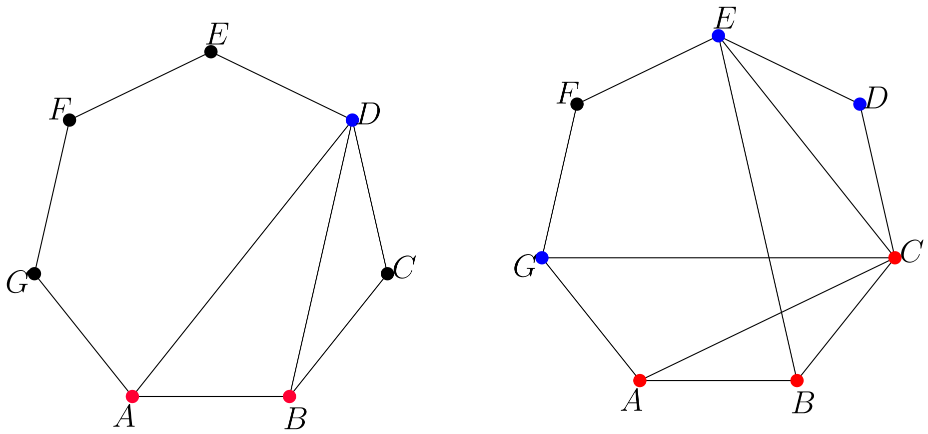
Proof. Let $ABCDEFG$ be a regular 7-gon with circumcenter P. We want to show that one of the triangles of $Rot_P(ABD)$ is monochromatic in every 2-coloring of the vertices. We try to color the 7-gon in such a way, that we avoid such a monochromatic trianlge whenever it is possible. By the pidgeonhole principle there are at least two vertices which are neighbours and have the same color. Without loss of generality we say A and B are red and then, because we want to avoid monochromatic triangles in $Rot_P(ABD)$, D is blue, as shown in Figure 9(a).

Then we have two cases:

Case 1: C is red. Then we can see in Figure 9(b), that E and G have to be blue.

But then we have the monochromatic triangle DEG which is in $Rot_P(ABD)$. This case also shows, that if there a are three or more consecutive vertices with the same color, then we have a monochromatic triangle which is in $Rot_P(ABD)$.

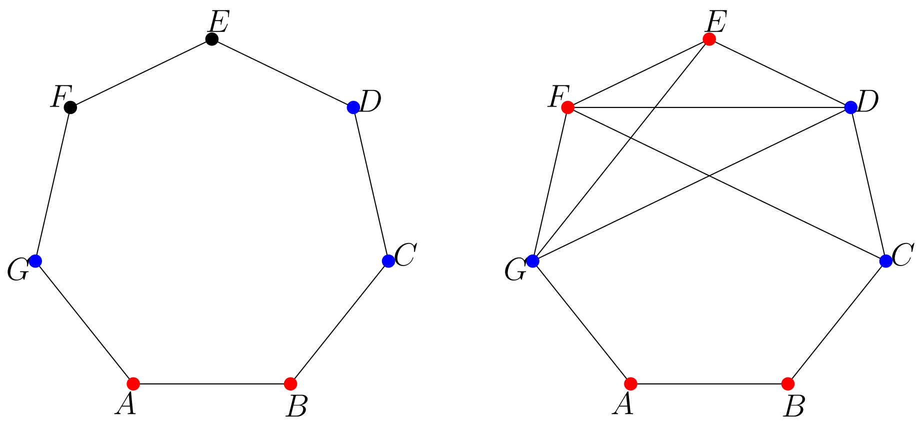
Case 2: C is blue. If G is red, then we have the three consecutive vertices with the same color and we are done. So G is blue (Figure 10(a)).



(a) Colored regular 7-gon

(b) Colored 7-gon with monochromatic triangle DEG

Figure 9: Illustration of the proof of Theorem 13 for Case 1



(a) Colored 7-gon where C and G are blue

(b) Colored 7-gon with monochromatic triangle EFA

Figure 10: Illustration for the proof of Theorem 13 for Case 2.

Due to the triangles CDF and DEG, F and E are red as shown in Figure 10(b). Now we have a monochromatic triangle EFA which is in $Rot_P(ABD)$.

So we have found a triangle T with $\mathbb{E}^2 \xrightarrow{2} Rot_P(T)$. $\square$

Theorem 14. *For all triangles T $\mathbb{E}^2 \overset{3}{\not\rightarrow} Rot_P(T)$ holds.*

Proof. If P is not the circumcenter this is true due to the remark at the beginning of Section 2.2. We will prove that we can color a circle in such a way, that every two points with angular distance $d < \pi$, do not have the same color, for d constant.

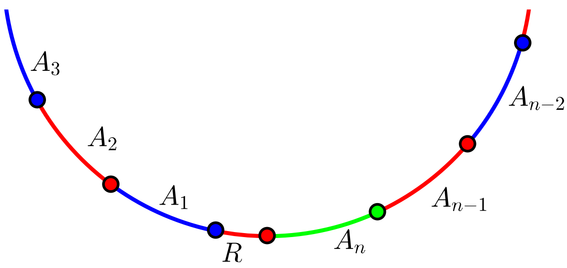


Figure 11: 3-coloring of a circle

We partition the circle into halfopen arcs $A_1, \ldots, A_k$ with angular distance d and into one arc R, with angular distance smaller than d. Without loss of generality we can say A_1 is next to R and the clockwise order of the arcs is $R, A_1, A_2, \ldots, A_n$. Let $R, A_2, A_4, \ldots, A_{2k}$ wit $2k < n$ be in color class X, let $A_1, A_3, \ldots, A_{2k-1}$ with $2k-1 < n$ be in color class Y and let A_n be color class Z like in Figure 11. Two points with angular distance a are in differents arcs of the partition. Either they are in adjacent arcs, which are different colored or one point is in A_1 and the other is in A_n which have different colors. So d does not exist monochromaticaly with this colorung.

$\square$

2.3 Similar triangles

Two triangles T_1 and T_2 are similar if we can obtain T_2 by translation, rotation, scaling or a combination of these transformations of T_1. Let T be a triangle and $Sim(T)$ the family of triangles similar to T. If two triangles are congruent, they are also similar. In this chapter we study, how many colors we need, such that every triangle in $Sim(T)$ is nonmonochromatic. This looks like Problem 1, with the difference, that we allow scaling.

Theorem 15. *For all triangles T and for all finite $r \in \mathbb{N}$, $\mathbb{E}^2 \xrightarrow{r} Sim(T)$ holds.*

Before we proof this theorem, we need some definitions.

Definition 16. *Let T be a triangle with side lengths a, b and c. We look at a triangle T_n with side lengths na, nb and nc. Let A_0, B_0 and C_0 be the vertices of T_n and $A_n = B_0$, $B_n = C_0$ and $C_n = A_0$. Further partition the side A_0A_n into n line segments with the same length by putting points A_i on A_0A_n such that $\overline{A_iA_{i+1}} = a$ and do the same for the lines B_0B_n and C_0C_n. Now add all lines A_iB_{n-i}, B_iC_{n-i} and C_iC_{n-i} and add a vertex at each intersection. We call such a graph a triangle grid of T with base n or triangle grid for short.*

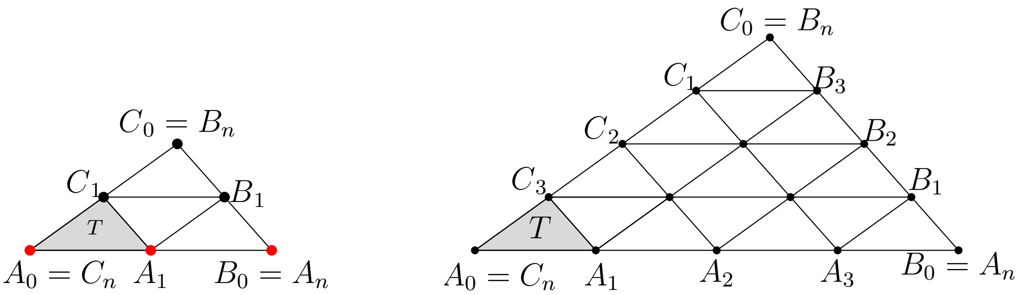
We call a triangle grid of T partial r-chromatic if in every r-coloring of the vertices of the triangle grid there is always a monochromatic triangle similar to T.

n_r is the smallest number, such that any triangle grid with base n_r is partial r-monochromatic.

Figure 12(a) and Figure 12(b) are triangle grids.

Lemma 17. *n_r is finite for all positive integers r.*

Lemma 17 implies Theorem 15.



(a) Colored triangle grid with base 2

(b) Triangle grid with base 4

Figure 12: Triangle grid examples

Before we prove Lemma 17, we look at 2-coloring of the triangle grid shown in Figure 12(a) with base 2, where A_0, A_1 and $A_2 = A_n$ are red. If one of the points B_1, C_0 and C_1 is red, then we have a monochromatic triangle similar to T. Otherwise, if none of the points B_1, C_0 and C_1 is red, then the original triangle grid without the line segments incident to A_i, $0 \leq i \leq 2$, and the vertices A_i, $0 \leq i \leq 2$, is again a triangle grid with base $n-1$ (in our case one) and is colored with $r-1$ colors, in our case one color. So we get again a monochromatic triangle similar to T.

This is the basic idea to proof Lemma 17. But how can we prove, that we always have a triangle grid, where the $A_i' \in \{A_j | 0 \leq j \leq n\}$ with $A_{i+1}' - A_i'$ is constant for $0 \leq i \leq k$, for good choices of k, have the same color.

The answer to this is to use Van der Waerdens theorem.

Theorem 18. *[20, van der Waerdens theorem] An r-coloring of the numbers from 1 to N is a partition of the numbers from 1 to N into r sets.*

Let $N = W(r,l)$ be the smallest number such that in every r-coloring of the numbers from 1 to N into r sets, there exists a monochromatic arithmetic progression of length l. If r and l are finite, then $W(r,l)$ is finte.

Proof. We will pove Lemma 17 by induction.

Induction base: $r = 1$

We choose a triangle grid with base 1, which is partial monochromatic in every 1-coloring.

Claim: n_r is finite.

We prove, that n_{r+1} is finite, if n_r is finite. Consider a triangle grid of T with base $N = W(n_r+1, r+1)$. The distance from A_0 to A_i is ia for all $i \in \{1, ..., N\}$. Now we color i with the color the point A_i has and get a coloring of the numbers from 1 to N with $r+1$ colors. With Van der Waerdens theorem we know that there is an arithmetic series $\{k, k+d, ..., k+n_r\ d\} \subset \{1, ..., N\}$ such that all A_j with $j \in S$ have the same color. Let us look at the triangle subgrid with points $A_k, ..., A_{k+n_r\ d}$. These points have the same color, say color $r+1$. If there is another point in the triangle subgrid with color $r+1$ we have a monochromatic triangle similar to T.

Otherwise if there is no other point in the subgrid, then the triangle subgrid without the points $A_k, ..., A_{k+n_r\ d}$ is colored in r colors. But the triangle subgrid without the points $A_k, ..., A_{k+n_r\ d}$ is an r-colored triangle grid of $d\,T$ with base n_r. Therefore there is a monochromatic triangle similar to T in this triangle grid of $d\,T$ with base n_r.

This means $n_{r+1} \leq W(n_r+1, r+1)$ is finite. So n_r is finite for all positive integers r. □

Theorem 15 is in fact a special case of a theorem of Gallai.

Theorem 19. *[5, 18] Let n and k be arbitrary positive integers and let A be a finite point set. In every k-coloring of $\mathbb{E}^n$ there exists a similar copy of A.*

In Theorem 15 we have $n = 2$ and $|A| = 3$.

In this chapter we have seen, that if we only allow translation or rotation of the original triangle, then we can solve Problem 1. On the other hand, if we are also allowed to scale our triangle in Problem 1, then in the r-colored plane with $r \in \mathbb{N}$, there is always a monochromatic homophetic copy of our original triangle.

3 Map like colourings of the plane

In this chapter we look at different 6-colourings of the Euclidean plane and try to find triangles, which are nonmonochromatic in such a 6-colouring.

Definition 20. *A multigraph is a graph, for which we allow multiple edges between two vertices.*

We call a multigraph, that is drawn in the plane without intersection of its edges, plane.

A multigraph is connected, if for any two vertices there is a path connecting these vertices.

An edge x of a connected multigraph G is called bridge, if $G - x$ is not connected.

A map is a plane connected multigraph without bridges. A map divides the plane into regions.

Two regions are adjacent if they share at least one edge.

A map coloring is a coloring of all regions, such that two regions with the same color do not share an edge.

A map like coloring is a coloring of all regions, edges and vertices, such that every edge between two regions with the same color has the same color as these regions and every point, whose adjacent edges have the same color, has the same color as the edges.

Since map colorings do not include zebra colorings, we need the definition of map like colorings.

We can visualize sets of triangles, because we only have to look at normed triangles.

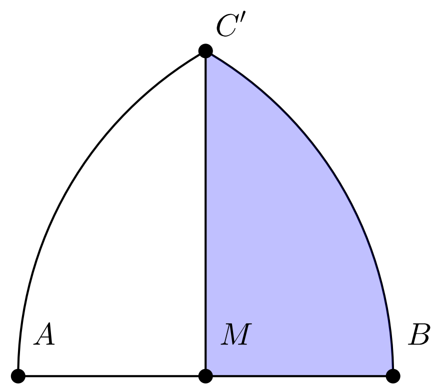


Figure 13: Visualization of the choices of C.

For normed triangles the longest side AB has length 1 and $\overline{BC} \leq \overline{AC}$. So the possible choices for C are in the area between M, B and C' in Figure 13. We will indicate choices for C in this figure and try to cover as much of this shaded area in Figure 13.

3.1 Zebra colouring

We call a coloring a zebra coloring if we color the plane with strips. In our case, the strips will be halfopen. Furthermore, we study cases, where all strips have the same height.

Given a normed triangle T. First we observe, that the height of the strips has to be at most h_C. Otherwise T could be placed in one strip and obviously would be monochromatic. Since the strips are halfopen, the height can be exactly h_C. That means, that the vertices off the triangle lie in at least two different strips. For 6 colors the distance between two points in two different strips with the same color is greater than $5h_C$, because there are five strips between two strips with the same color. So if two points have the same color and have distance at most $5h_C$, then these two points lie in the same strip. So we want h_C to be small enough, so that a given triangle cannot lie in one strip, but large enough, so that the third vertex of the triangle is not in the next strip, with the same color.

Theorem 21. *All normed triangles with $h_C \geq \frac{1}{5}$ are nonmonochromatic in a zebra coloring with 6 colors, where all strips have height h_C.*

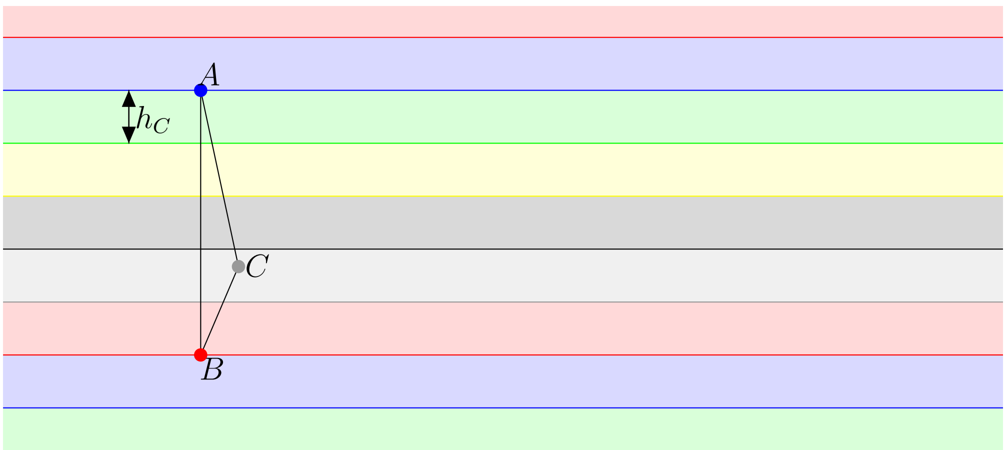


Figure 14: Zebra coloring with 6 colors with halfopen strips of height h_C

Proof. Let T be a normed triangle with $h_C \geq \frac{1}{5}$. With the above arguments we know, that the vertices do not lie in one strip. On the other hand $\overline{AB} =$

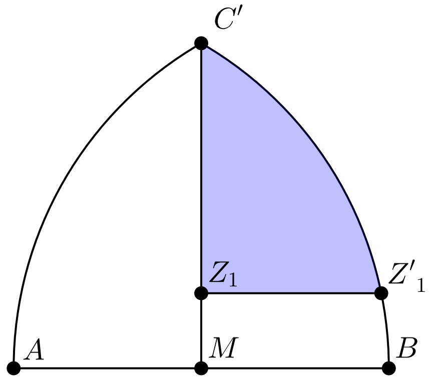


Figure 15: Possible choices for C for a nonmonochromatic triangle by Theorem 21

$1 = 5\frac{1}{5} \leq 5h_C$ and since AB is the longest side of T, it is not possible, that two points of the triangle lie in two different strips with the same color.

$\square$

Theorem 21 says, that we have a 6-coloring for every normed triangle with $h_C \geq \frac{1}{5}$. So we have a coloring for every point C in the shaded area $Z_1Z_1'C'$ in Figure 15, such that the triangle ABC is nonmonochromatic.

As we can see in Figure 14, there is room for improvement, because C seems to be in another strip too, if we try to place B in a different strip with the same color as the strip with A.

Theorem 22. *All normed triangles with $\overline{AC} \leq 5h_C$ are nonmonochromatic in a zebra coloring with* 6 *colors, where all strips have height h_C.*

Proof. Let T be a normed triangle. As seen before, A, B and C cannot lie in the same strip. Assume that T is monochromatic. Then A and C are in the same strip, since $\overline{AC} \leq 5h_C$. But, since $\overline{BC} \leq \overline{AC} \leq 5h_C$ holds, B and C lie in the same strip. So A, B and C have to be in the same strip, which is a contradiction. $\square$

This is clearly a better result than Theorem 21, but how can we visualize this result in our figure? The answer is Pythagoras theorem. Let $s = \overline{AD}$ as shown in Figure 16. We know

$$s^2 + {h_C}^2 = \overline{AC}^2 \leq (5h_C)^2 \tag{6}$$

$$\Leftrightarrow s^2 \leq 24{h_C}^2 \Leftrightarrow \frac{s}{2\sqrt{6}} \leq h_C. \tag{7}$$

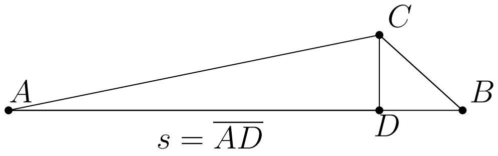
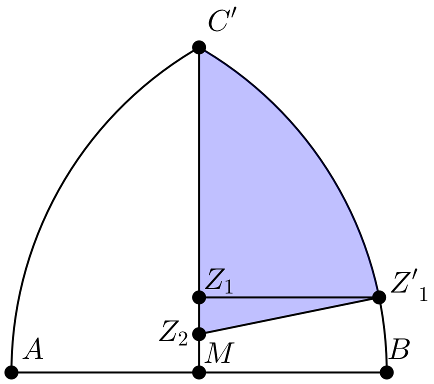
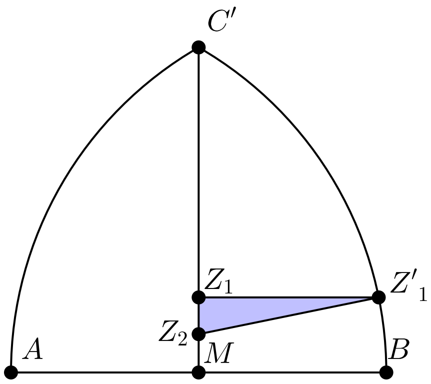


Figure 16: Triangle with $\overline{AC} = 5h_C$



(a) Possible choices for C for a nonmonochromatic triangle by Theorem 22



(b) New possible choices for C for a nonmonochromatic triangle by Theorem 22

Figure 17: Visualisation of the possible choices for C after Theorem 23

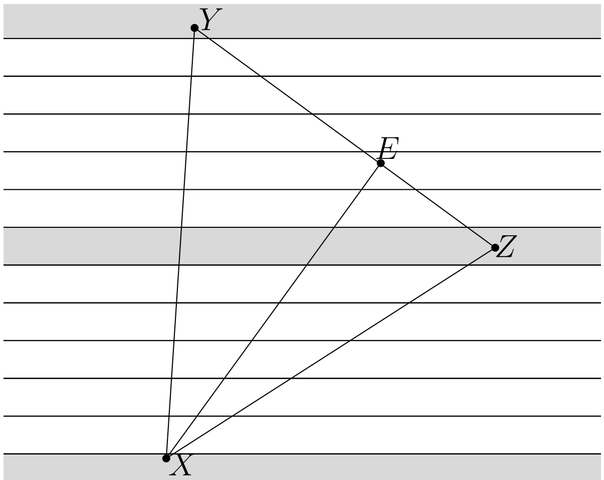


Figure 18: All vertices in different strips in Theorem 23

In Figure 17(a), if C is in the shaded area $Z_2Z_1'C'$, then it is nonmonochromatic in the zebracoloring with strips of height h_C. We can see in Figure 17(b), that we get all points in the shaded area $Z_2Z_1'Z_1$ as new possible choices for C.

The next idea is to take the largest height h_A of a triangle T, so that if B and C lie in the same strip, than A is also in the same strip. Unfortunately that only works for triangle in which every angle is at most 90°.

Theorem 23. *Every normed triangle, in which every angle is at most* 90° *and* $h_A \leq 5h_C$ *is nonmonochromatic in a zebra coloring with* 6 *colors, where all strips have height* h_C*.*

Proof. Let T be a normed triangle. Since all strips have height h_C, A, B and C cannot be in the same strip. Assume B and C are in the same strip. Let D be a point on BC, so that $\overline{AD} = h_A$. Since all angles of the triangle are at most 90°, D is between B and C. So D is in the same strip as B and C. But the distance between two strips with the same color is $5h_C \geq \overline{AD}$, so A cannot lie in a strip with the same color. The same follows, if A and B or A and C lie in the same strip, because h_A is the largest height of the triangle. So we only have to show that A, B and C cannot lie in three different strip with the same color.

We look at Figure 18 and only care about the grey strips. The triangle XYZ is congruent to our given triangle ABC. E is the point on YZ between YZ such that XE is a height of the triangle XYZ. If all vertices of XYZ are in different strips with the same color, then there is one vertex in a strip between the strips, which contain the other two vertices. In Figure 18 this is

Z. The distance between the strip containing X and the strip containing E is greater than $5h_C$, but the distance $\overline{XE}$ is at most $5h_C$, because

$$\overline{XE} \leq \overline{AD} = h_A \leq 5h_C \tag{8}$$

So this is impossible. $\square$

Note that this proof doesn't work if there is an angle greater than 90°, because then D is not necessarily between B and C.

To visualize the triangles of Theorem 23, we have to calculate some further properties. We know by calculating the area in two different ways that

$$\frac{1}{2}\overline{BC}h_A = \frac{1}{2}\overline{AB}h_C = \frac{h_C}{2} \tag{9}$$

With Theorem 23 we get

$$\frac{1}{2}\overline{BC}h_A \leq \frac{1}{2}\overline{BC}5h_C \tag{10}$$

$$\Rightarrow \frac{h_C}{2} \leq \frac{1}{2}\overline{BC}5h_C \tag{11}$$

$$\Rightarrow \frac{1}{5} \leq \overline{BC} \tag{12}$$

So we also know for a normed triangle T, if $\overline{BC} \geq \frac{1}{5}$ and all angles of T are at most 90°, then we have a 6-coloring in which T is nonmonochromatic.

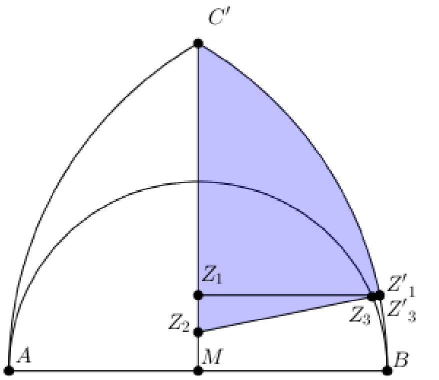
In Figure 19(a) all possible choices for C, for which we know a 6-coloring, such that the triangle ABC is nonmonochromatic, are in the shaded area. All points C, for which the triangle ABC is nonmonochromatic in a 6-zebra coloring by Theorem 23, are in the shaded area in Figure 19(b).

The new possible choices we get for C with Theorem 23 are the points in the area $Z_3Z_3'Z_1'$ in Figure 19(c). The resulting additional area is pretty small, in fact, we have to look at a figure, that is 80 times bigger than our usual figure, to see the difference.

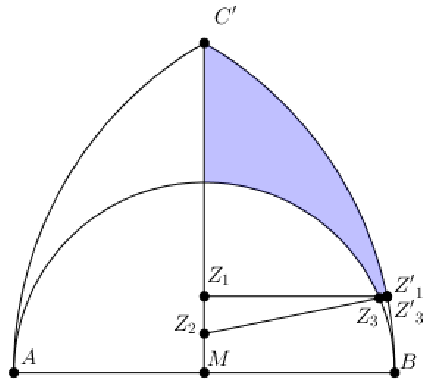
3.2 Colourings with rectangles

In this chapter we color the plane with rectangles. The diagonals of the rectangles are at most 1. If there is a diagonal longer than 1, than a normed triangle can lie in one rectangle.

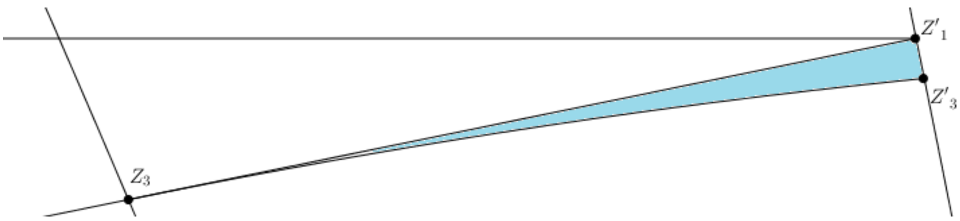
We color the plane like in Figure 20. We color the boundary of a square, so that the lowest side excluding its rightmost endpoint and the leftmost side excluding its topmost endpoint have the same color as the square. For example, we look at the square $P_1P_2P_3P_4$. The sides P_1P_2 and P_1P_4 (excluding



(a) Possible choices for C for a nonmonochromatic triangle after Theorem 23



(b) Possible choices for C for a nonmonochromatic triangle by Theorem 23



(c) New possible choices for C for a nonmonochromatic triangle after Theorem 23

Figure 19: Visualisation of the possible choices for C after Theorem 23

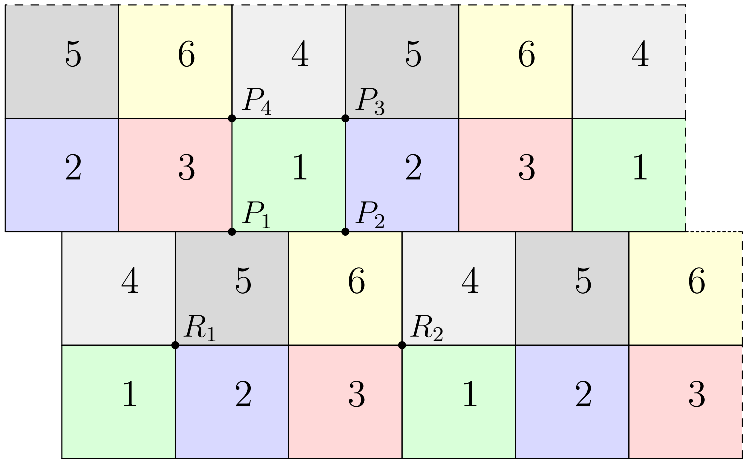


Figure 20: 6-coloring with squares

endpoints) and the point P_1 are colored with color 1, but P_4 and side P_3P_4 are colored with color 4.

The length of a side of a square in this coloring is $\frac{\sqrt{2}}{2}$, because the diagonal of every square is 1. That means that

$$\overline{R_1R_2} = 2\frac{\sqrt{2}}{2} = \sqrt{2} > 1. \tag{13}$$

Furthermore $\overline{R_1P_1} = \overline{R_2P_2}$ holds in this coloring. With Pythagoras theorem we get

$$\overline{R_1P_1}^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\frac{\sqrt{2}}{2}\right)^2 = \frac{5}{8}.$$

So $\overline{R_1P_1} = \frac{\sqrt{10}}{4}$ and therefore the shortest distance between two points in different squares with the same color is greater than $\frac{\sqrt{10}}{4}$.

Theorem 24. *All normed triangles with $\overline{AC} \leq \frac{\sqrt{10}}{4}$ are nonmonochromatic in a 6-coloring like in Figure 20.*

Proof. Let T be a normed triangle. A, B and C are not in the same square since the diagonals have length 1, but the endpoints of each diagonal are colored in distinct colors. If A is in a different square than C it cannot be in a square with the same color as the one C is in, because the shortest distance between two points in different squares with the same color is greater than $\frac{\sqrt{10}}{4}$, but $\overline{AC} \leq \frac{\sqrt{10}}{4}$. Since $\overline{BC} \leq \overline{AC} \leq \frac{\sqrt{10}}{4}$, because T is a normed triangle, B and C cannot be in different squares, if B and C have the same color. Therefore it is impossible, that A, B and C have the same color. $\square$

This result is fine, but we can get a better result if we modify our coloring.

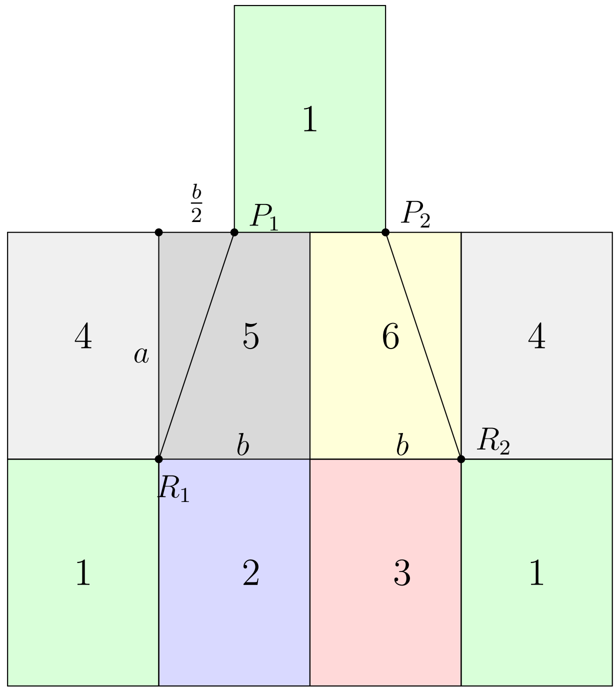


Figure 21: 6-coloring with rectangles, which diagonals have length 1

Instead of using squares we use rectangles with diagonals of length 1. For the first modification we choose our rectangles so that

$$\overline{R_1R_2} = \overline{R_1P_1} = \overline{R_2P_2}, \tag{14}$$

where R_1, R_2, P_1 and P_2 are chosen like in Figure 21. Now we calculate the lengths of the sides a and b of each rectangle. Since the diagonals have length 1, we know $a^2+b^2 = 1$ and therefore $a^2 = 1-b^2$. The other condition is

$$2b = \overline{R_1R_2} = \overline{R_1P_1} = \sqrt{a^2 + \left(\frac{b}{2}\right)^2}. \tag{15}$$

So we can calculate a and b.

$$\begin{aligned} 4b^2 = a^2 + \frac{b^2}{4} &= 1 - b^2 + \frac{b^2}{4} \\ 5b^2 - \frac{b^2}{4} = \frac{19b^2}{4} &= 1 \\ b &= \frac{2}{\sqrt{19}} \\ a &= \frac{\sqrt{15}}{\sqrt{19}} \end{aligned}$$

Theorem 25. *All normed triangles with $\overline{AC} \leq \frac{\sqrt{15}}{\sqrt{19}}$ are nonmonochromatic in a 6-coloring with rectangles with $a = \frac{\sqrt{15}}{\sqrt{19}}$ and $b = \frac{2}{\sqrt{19}}$ as shown in Figure 21.*

The proof for this theorem is essentially the same as the proof for Theorem 24, just with the values of the 6-coloring with rectangles as shown in Figure 21 instead of the values of the 6-coloring with squares, which we can see in Figure 20. The 6-coloring in Figure 21 also has the intended properties as we have calculated which meet our conditions as we have calculated.

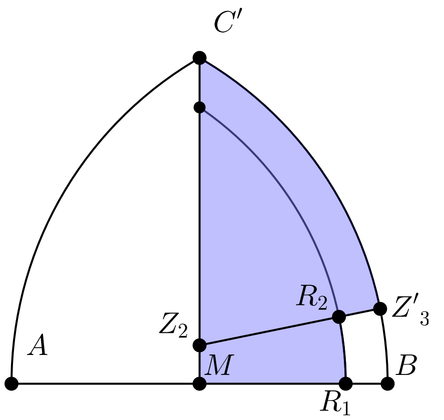
In Figure 22(a), we have for all points C, which lie in the shaded area, a 6-coloring, such that the triangle ABC is nonmonochromatic. Theorem 25 covers the shaded area in Figure 22(b). The shaded area in Figure 22(c) shows the new possible choices for C.

With this theorem we get again an upper bound for $\overline{AC}$. Can we also find a lower bound for BC? We can use a similar coloring to the one in Figure 20, but instead of squares we use again rectangles. The points R_1, R_2, P_1 and P_2 are chosen like in Figure 21. These rectangles with sides of lengths a and b should have the following properties:

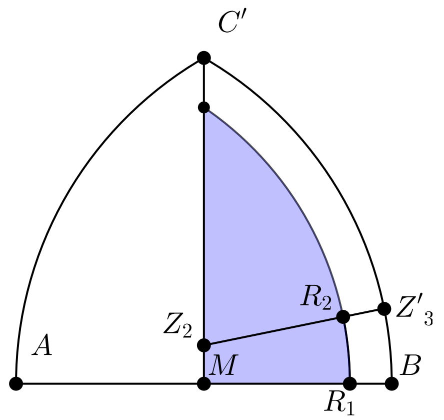
- $\overline{R_1R_2} = 2b \leq 1 \Rightarrow b \leq \frac{1}{2}$
- $a^2 + b^2 = 1 \Rightarrow a \leq \frac{\sqrt{3}}{2}$

We need the first condition, because then it is impossible that two points are in two different rectangles with the same color in the same row and have distance at most 1. The second property says that the diagonals of a rectangle are at most 1, so no normed triangle can lie in one rectangle.

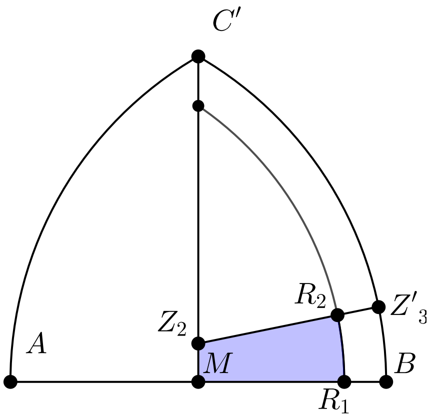
We look at Figure 23. The circle with center P_1 and points Q_1 and Q_2 and radius 1 intersects the rectangle with color 1 only inside the rectangle $R = SQ_2R_1Q_1$. Therefore, if the normed triangle T is monochromatic, then



(a) All possible choices for C for a nonmonochromatic triangle after Theorem 25



(b) Possible choices for C for a nonmonochromatic triangle by Theorem 25



(c) New possible choices for C for a nonmonochromatic triangle by Theorem 25

Figure 22: Visualisation of the possible choices for C after Theorem 23

C has to be in R. As we can see in Figure 23, if $\overline{BC} \geq \overline{Q_1Q_2}$ holds, then B and C can only be in the rectangle R if B is Q_1 and C is Q_2 or if B is Q_2 and C is Q_1. But the points Q_1 and Q_2 have different colors. On the other hand if A, B and C are in three different rectangles, then two of these rectangles have to be in the same row, because $3a = \frac{3\sqrt{3}}{2} > 1$. The distance of two points with the same color in different rectangles in the same row is greater than 1, because of our choice to color the boundaries of the rectangles. So T is nonmonochromatic if $\overline{Q_1Q_2} \leq \overline{BC}$.

Theorem 26. *All normed triangles T with $\overline{BC} \geq \frac{1}{2}\sqrt{7 - 3\sqrt{5}}$ are nonmonochromatic in a 6-coloring with rectangles with $a = \frac{\sqrt{3}}{4}$ and $b = \frac{1}{2}$.*

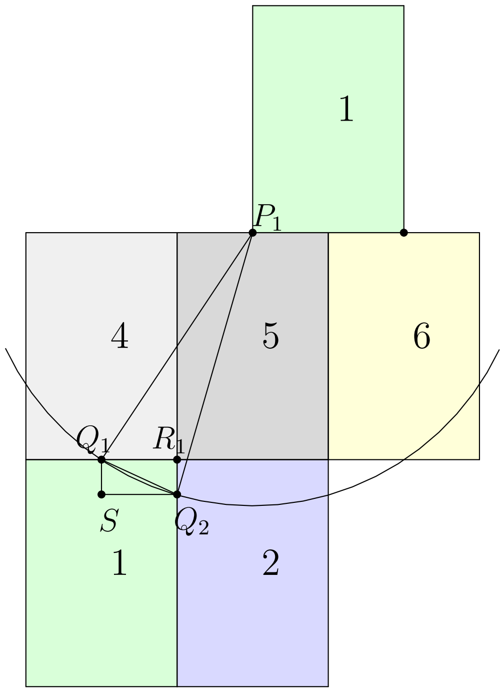


Figure 23: Lower bound for $\overline{BC}$

Proof. We have already proved before, that a normed triangle T is non-monochromatic if $\overline{Q_1Q_2} \leq \overline{BC}$. To calculate $\overline{Q_1Q_2}$, we embed our coloring in a Cartesian coordinate system. We assume $P_1 = (0,0)$ and P_1P_2 is on the x-axis of the Cartesian coordinate system. Then Q_1 and Q_2 lie on the circle $x^2 + y^2 = 1$. Furthermore the x-coordinate of Q_2 is $-\frac{1}{4}$ and the y-coordinate of Q_1 is $-\frac{\sqrt{3}}{2}$. Q_1 and Q_2 are solutions of $x^2 + y^2 = 1$ and we get $Q_1 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and $Q_2 = (-\frac{1}{4}, -\frac{\sqrt{15}}{4})$.

$$\begin{aligned}
\overline{Q_1Q_2}^2 &= \left(-\frac{1}{2} - \left(-\frac{1}{4}\right)\right)^2 + \left(-\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{15}}{4}\right)\right)^2 \\
&= \frac{1}{16} + \frac{3}{16}(2 - \sqrt{5})^2 \\
&= \frac{1}{16}(1 + 3(9 - 4\sqrt{5})) = \frac{1}{4}(7 - 3\sqrt{5}) \\
\overline{Q_1Q_2} &= \frac{1}{2}\sqrt{7 - 3\sqrt{5}}
\end{aligned}$$

$\square$

Note that $\frac{1}{2}\sqrt{7-3\sqrt{5}} \approx 0.270 > \frac{1}{5}$. This means that this bound is worse than the bound, we have for triangles, where all angles are at most $90°$. Thus this bound does not contribute to our possible choices.

3.3 Colourings with pentagons

In this chapter we will use pentagons to color the plane. So all regions have the shape of a pentagon.

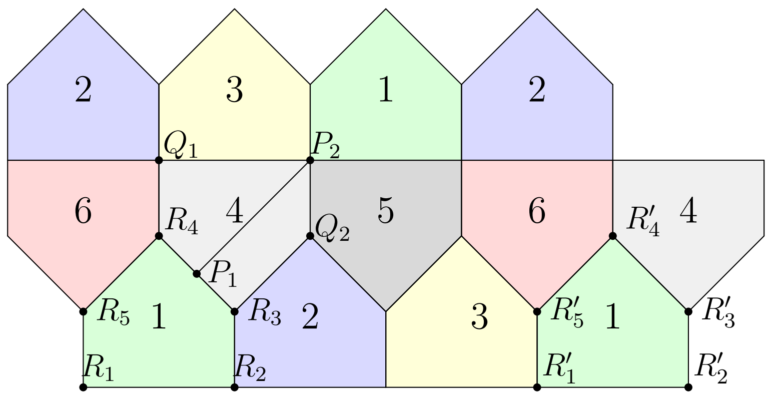


Figure 24: 6-coloring of the plane with pentagons

We look at the coloring in Figure 24. We color the boundary of the pentagon $\mathcal{P}_1 = R_1R_2R_3R_4R_5$ in the following way,

- The sides R_1R_2 and R_1R_5 and the vertex R_1 are colored with the color of the pentagon $\mathcal{P}_1$. In Figure 24 the color is 1.
- The side R_2R_3 and the vertex R_2 have the same color as the pentagon, that shares the side R_2R_3 with the pentagon $\mathcal{P}_1$. In Figure 24 the color is 2.
- The side R_3R_4 and the vertex R_3 have the same color as the pentagon, that shares the side R_3R_4 with the pentagon $\mathcal{P}_1$. In Figure 24 the color is 4.
- The side R_4R_5 and the vertices R_4 and R_5 have the same color as the pentagon, that shares the side R_4R_5 with the pentagon $\mathcal{P}_1$. In Figure 24 the color is 6.

We color all pentagons, which we are congruent to $\mathcal{P}_1$ by translation, this way. For example we get the coloring of the pentagon $R'_1R'_2R'_3R'_4R'_5$, but we do not get a coloring for the boundary of the pentagon $\mathcal{P}_2 = R_4R_3Q_2P_2Q_1$, because we cannot get $\mathcal{P}_2$ by translation of $\mathcal{P}_1$. The sides R_4R_3, R_3Q_2 and Q_1P_2 and all of its vertices are already colored by the coloring defined for $\mathcal{P}_1$. In Figure 24 R_4R_3, R_3Q_2, R_3 and Q_2 have color 4, Q_2 has color 5, P_2 has color 1 and P_2Q_1 and Q_1 have color 3. So we only have to choose colors for the sides Q_1R_4 and Q_2P_2. We color Q_1R_4 with the same color as the pentagon, which shares the side Q_1R_4 with $\mathcal{P}_2$. In Figure 24 the color of Q_1R_4 is 6. We color the side Q_2P_2 with the color of $\mathcal{P}_2$.

Furthermore, we require, that the pentagons have the following properties.

- $\overline{R_1R_3} = \overline{R_1R_4} = \overline{R_2R_5} = \overline{R_2R_4} = 1$
- $\overline{P_1P_2} = \overline{R_2R_1'} = 2\overline{R_1R_2}$
- $\overline{R_1R_2} = \overline{R_3R_5} < 1$

If a pentagon has the first property and is colored as mentioned above, then a monochromatic normed triangle, cannot be in this pentagon, because R_1 has a different color than R_3 and R_4 and R_2 has a different color than R_4 and R_5. So no monochromatic pair of points in this pentagon have distance 1. With the second property, we try to maximize $\overline{AC}$ of a normed triangle.

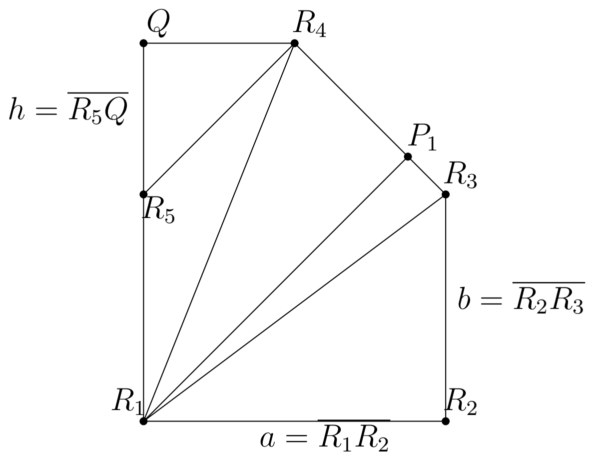


Figure 25: Notation for a pentagon to calculate $\overline{AG}$

In Figure 25 we want to calculate $\overline{R_1P_1}$, because this is also the smallest distance between two differnet pentagons with the same color. Due to Pythagoras theorem, we get

$$\overline{R_1R_3} = 1 = a^2 + b^2 \Rightarrow a^2 = 1 - b^2 \tag{16}$$

and

$$\overline{R_1R_4} = 1 = \left(\frac{a}{2}\right)^2 + (b+h)^2 \tag{17}$$
$$= \frac{1-b^2}{4} + b^2 + 2bh + h^2 \leftrightarrow \tag{18}$$
$$0 = 3b^2 + 8bh + 4h^2 - 3 \tag{19}$$
$$\leftrightarrow h = \frac{-8b + \sqrt{64b^2 - 4 \cdot 4(3h^2 - 3}}{8} = \frac{-2b + \sqrt{b^2+3}}{2} \tag{20}$$

We only have nonnegative solutions for b, since we cannot have negative length. When we look at the area $\mathcal{A}$ of the pentagon $ABCDE$, we can calculate it in two different ways.

$$\mathcal{A} = \frac{a}{2}(2b+h) \tag{21}$$
$$\mathcal{A} = \frac{1}{2}ab + \frac{1}{2}\frac{a}{2}b + \frac{1}{2}\overline{AGDC} \tag{22}$$
$$= \frac{1}{2}ab + \frac{1}{2}\frac{a}{2}b + \frac{1}{2}\overline{AG}\sqrt{\frac{a^2}{4} + h^2} \tag{23}$$
$$\tag{24}$$

With this observation and the property $\overline{R_1P_1} = 2a$, we get a third equation.

$$\frac{a}{2}(2b+h) = \frac{1}{2}ab + \frac{1}{2}\frac{a}{2}b + \frac{1}{2}\overline{AG}\sqrt{\frac{a^2}{4} + h^2} \tag{25}$$
$$= \frac{1}{2}ab + \frac{1}{2}\frac{a}{2}b + \frac{1}{2}2a\sqrt{\frac{1-b^2}{4} + h^2} \tag{26}$$
$$\Leftrightarrow \frac{b+2h}{4} = \sqrt{\frac{1-b^2}{4} + h^2} \tag{27}$$
$$\Leftrightarrow \frac{b^2 + 4bh + 4h^2}{16} = \frac{1-b^2}{4}h^2 \tag{28}$$
$$\Leftrightarrow 0 = 5b^2 + 4bh - 12h^2 - 4 \tag{29}$$

We have $h = \frac{-8b+\sqrt{64b^2-4\cdot4(3h^2-3}}{8} = \frac{-2b+\sqrt{b^2+3}}{2}$ so we get

$$\begin{align}
0 &= 5b^2 + 4b\frac{-2b+\sqrt{b^2+3}}{2} - 12\left(\frac{-2b+\sqrt{b^2+3}}{2}\right)^2 - 4 \tag{30}\\
&= 5b^2 - 4b^2 + 2b\sqrt{b^2+3} - 3(4b^2 + b^2 + 3 - 4b\sqrt{b^2+3}) - 4 \tag{31}\\
&= -14b^2 + 14b\sqrt{b^2+3} - 13 \tag{32}\\
\Leftrightarrow & 14b\sqrt{b^2+3} = 14b^2 + 13 \tag{33}\\
\Rightarrow & 196b^2(b^2+3) = 196b^4 + 364b^2 + 169 \tag{34}\\
\Leftrightarrow & 224b^2 = 169 \tag{35}\\
\Leftrightarrow & b = \frac{13}{4\sqrt{14}} \tag{36}
\end{align}$$

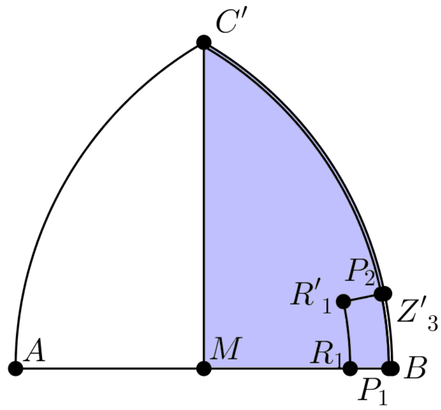
So we get

$$\begin{align}
a^2 = 1 - b^2 \Rightarrow a = \sqrt{1-\frac{169}{224}} = \sqrt{\frac{55}{224}} \tag{37}\\
\overline{AG} = 2a = 2\sqrt{\frac{55}{224}} = \sqrt{\frac{55}{56}} \approx 0.991 \tag{38}\\
h = \frac{-2b+\sqrt{b^2+3}}{2} = \frac{-2\frac{13}{4\sqrt{14}} + \sqrt{\frac{55}{224}+3}}{2} \tag{39}\\
= \frac{-26+\sqrt{727}}{8\sqrt{14}} \tag{40}
\end{align}$$

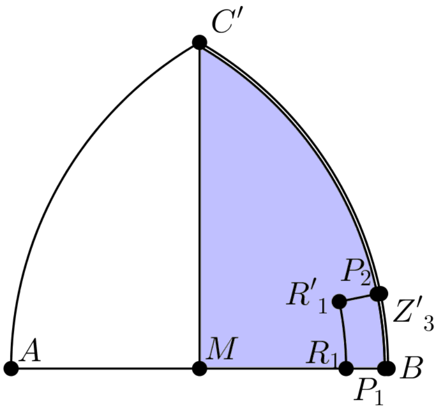
So we have proved the following theorem.

Theorem 27. *All normed triangles with $\overline{AC} \leq \sqrt{\frac{55}{56}}$ are nonmonochromatic in the coloring used in Figure 24 with $a = \sqrt{\frac{55}{224}}$, $b = \frac{13}{4\sqrt{14}}$ and $h = \frac{-26+\sqrt{727}}{8\sqrt{14}}$, where a, b and c denote the same sides as in Figure 25.*

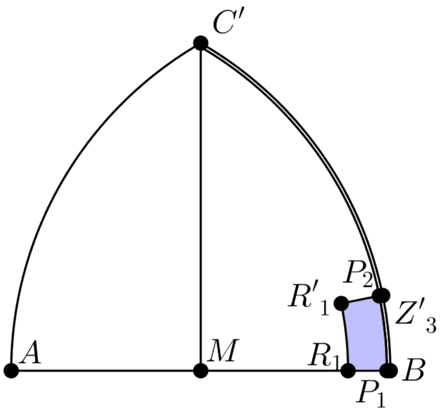
As we can see in Figure 26(a), this leaves only a small area $P_1BZ_3'P_2$, for which choices of C we do not have a coloring, such that ABC is nonmonochromatic in this coloring. It also seems, that it is much easier to maximize $\overline{AC}$ than minimizing $\overline{BC}$. The shaded area in Figure 26(b) shows all possible choices for C, for which the triangle ABC is nonmonochromatic by Theorem 27. Furthermore we get the shaded area in Figure 26(c) as new possible choices for C.



(a) All possible choices for C for a nonmonochromatic triangle after Theorem 27



(b) Possible choices for C for a nonmonochromatic triangle by Theorem 27



(c) New possible choices for C for a nonmonochromatic triangle by Theorem 27

Figure 26: Visualisation of the possible choices for C after Theorem 27

3.4 Colourings with hexagons

In this subchapter we color the plane with hexagons. If the diagonal of a hexagon is longer than 1, then AB can be in one hexagon, which we want to avoid. Since 4-colorings are also 6-colorings, we look at first at the following 4-coloring.

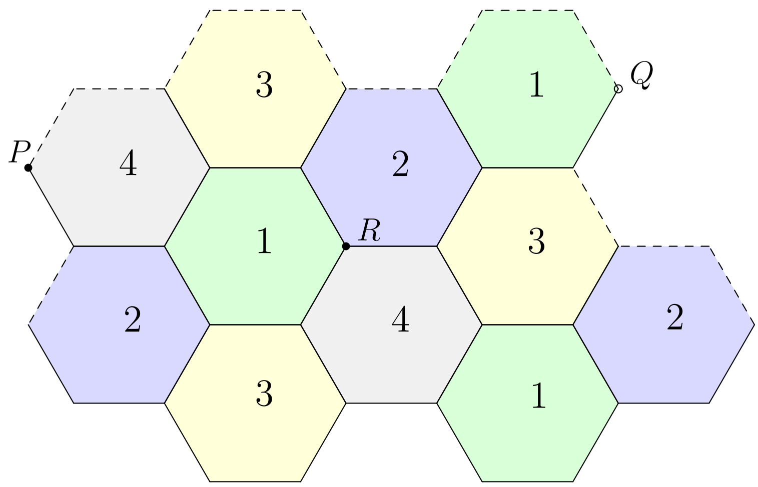


Figure 27: 4-coloring with regular hexagons of diameter 1

We take the coloring in Figure 27, where the diagonal of the hexagon is 1. The numbers denote the color and the sides below, the vertex below and the leftmost vertex of the hexagon have the same color as the hexagon. It is obvious, that this coloring can be extended to a coloring of the plane. In Figure 27 the dashed lines are not colored. Furthermore, P has color 4, R has color 2, because R is on a side, which is below the hexagon colored with 2 and Q does not have a color because it is the rightmost vertex of the hexagon.

Theorem 28. *All normed triangles with* $\overline{AC} \leq \frac{\sqrt{3}}{2}$ *are nonmonochromatic in a* 4*-coloring like in Figure 27.*

Proof. In the Figure 27 we can see that on the boundary of a hexagon, there are not two points with distance 1, that have the same color. So it is impossible, that the triangle is in one hexagon, if it is monochromatic. That means one vertex of the triangle is in a different hexagon than the other two vertices. The distance between two points in two different hexagons with the same color is more than $\frac{\sqrt{3}}{2}$. If A is in a different hexagon than C, then A and C cannot lie in hexagons with the same color, because the distance between two points in two different hexagons with the same color is more than $\overline{AC}$. The same result follows, if B and C lie in different hexagons, because $\overline{BC} \le \overline{AC}$. □

We again can improve our result with some modifications.

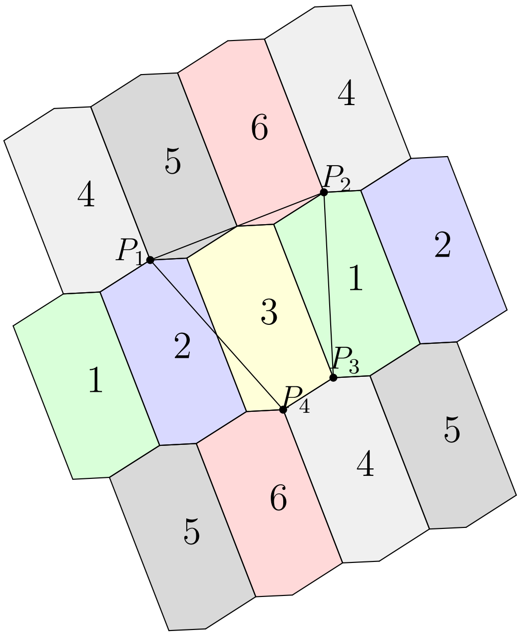


Figure 28: 6-coloring with hexagons, where all diagonals have length 1 and opposing sides are parallel

We take the 6-coloring in Figure 28. The diagonals between two opposing vertices have length 1 and all vertices of the hexagon lie on a circle with radius $\frac{1}{2}$. Furthermore opposing sides are parallel. We color the two lowest vertices of a hexagon and the three sides, which are incident to these vertices, with the same color as the hexagon. For example in Figure 28 the vertex P_4 and side P_3P_4 have color 3, but P_3 has color 1.

We want to maximize the smallest one of the lengths $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_1P_4}$, because these lengths are also the best possible lower bounds for the distance of two points in different hexagons with the same color.

We name the points of our hexagon as in Figure 29. The lengths a, h_a,

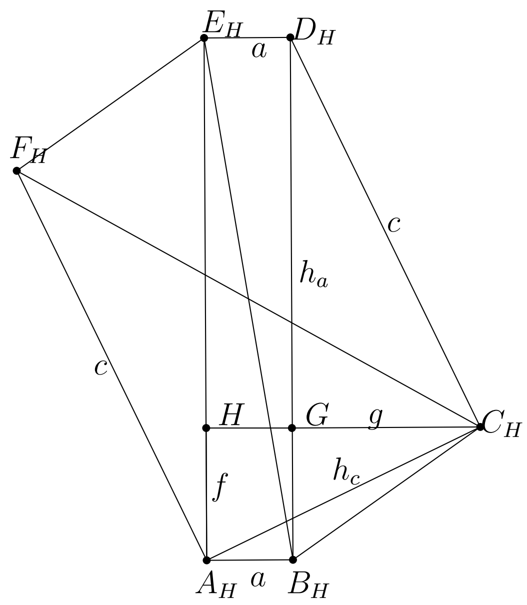


Figure 29: Notation of hexagon

c, h_c, g and f are defined as

$$a = \overline{A_H B_H} = \overline{D_H E_H} \tag{41}$$
$$h_a = \overline{A_H E_H} = \overline{B_H D_H} \tag{42}$$
$$c = \overline{C_H D_H} = \overline{A_H F_H} \tag{43}$$
$$h_c = \overline{A_H C_H} = \overline{D_H F_H} \tag{44}$$
$$f = \overline{A_H H} \tag{45}$$
$$g = \overline{C_H G} \tag{46}$$

and

$$\overline{A_H D_H} = \overline{B_H E_H} = \overline{C_H F_H} = 1 \tag{47}$$

Since all vertices of the hexagon lie on a circle, the equilateral $A_H B_H D_H E_H$ has a circumcircle, so

$$\angle A_H B_H D_H + \angle A_H E_H D_H = 180° \tag{48}$$
$$\angle B_H D_H E_H + \angle B_H A_H E_H = 180° \tag{49}$$

The sides $A_H B_H$ and $D_H E_H$ are parallel, so $A_H B_H D_H E_H$ is a rectangle. With the same arguments, we get, that $\angle A_H C_H D_H = 90°$. So we get the following equalities.

$$a^2 + {h_a}^2 = 1 \tag{50}$$
$$c^2 + {h_c}^2 = 1 \tag{51}$$
$$(a+g)^2 + f^2 = {h_c}^2 \tag{52}$$
$$(a+2g)^2 + (h_a - 2f)^2 = 1 \tag{53}$$
$$g^2 + (h_a - f)^2 = c^2 \tag{54}$$
$$h_a = \overline{P_2P_3} \tag{55}$$
$$2h_c = \overline{P_1P_2} \tag{56}$$
$$(2a+g)^2 + (h_a - f)^2 = \overline{P_1P_4}^2 \tag{57}$$

We can choose $\overline{P_1P_2} = \overline{P_2P_3} = \overline{P_1P_4}$ and get

$$h_a = 2h_c \tag{58}$$
$$h_a^2 = (2a+g)^2 + (h_a - f)^2 \tag{59}$$
$$\Leftrightarrow 2h_a f = (2a+g)^2 + f^2 = 4a^2 + 4ag + g^2 + f^2 \tag{60}$$

We eliminate c and h_c from our equation system and get new equations

$$(a+g)^2 + f^2 = {h_c}^2 = \frac{{h_a}^2}{4} \tag{61}$$
$$g^2 + (h_a - f)^2 = c^2 = 1 - {h_c}^2 = 1 - \frac{{h_a}^2}{4} \tag{62}$$
$$\Leftrightarrow g^2 + {h_a}^2 - 2h_a f + f^2 = 1 - \frac{{h_a}^2}{4} \tag{63}$$

Now we replace $2h_a f$ in Equation (63) by using the Equations (60) and (50).

$$1 - \frac{{h_a}^2}{4} = 1 - \frac{1-a^2}{4} \tag{64}$$
$$= g^2 + {h_a}^2 - 2h_a f + f^2 \tag{65}$$
$$= g^2 + 1 - a^2 - (4a^2 + 4ag + g^2 + f^2) + f^2 \tag{66}$$
$$= 1 - 5a^2 - 4ag \tag{67}$$
$$\Leftrightarrow 4ag = \frac{1-21a^2}{4} \tag{68}$$
$$\Leftrightarrow g = \frac{1-21a^2}{16a} \tag{69}$$

Furthermore we get a result for f^2 depending only on the variable a in Equation (61) by using the Equations (69) and (50).

$$\frac{1-a^2}{4} = \frac{{h_a}^2}{4} = (a+g)^2 + f^2 = \left(a + \frac{1-21a^2}{16a}\right) + f^2 \tag{70}$$

$$\Leftrightarrow f^2 = \frac{1-a^2}{4} - \left(\frac{1-5a^2}{16a}\right)^2 \tag{71}$$

$$= \frac{-1+74a^2-89a^4}{256a^2} \tag{72}$$

Now we take the square of both sides of Equation (60) and replace ${h_a}^2$, g and f^2 by the result we got in (50), (69) and (72).

$$(2h_a f)^2 = ((2a+g)^2 + f^2)^2 \tag{73}$$

$$\Leftrightarrow 4{h_a}^2 f^2 = \left(\left(2a + \frac{1-21a^2}{16a}\right)^2 + \frac{-1+74a^2-89a^4}{256a^2}\right)^2 \tag{74}$$

$$= \left(\left(\frac{1+11a^2}{16a}\right)^2 + \frac{-1+74a^2-89a^4}{256a^2}\right) \tag{75}$$

$$= \left(\frac{96a^2+32a^4}{256a^2}\right)^2 \tag{76}$$

$$= \frac{(a^2+3)^2}{64} \tag{77}$$

$$\Leftrightarrow \frac{(a^2+3)^2}{64} = 4{h_a}^2 f^2 = 4(1-a^2)\frac{-1+74a^2-89a^4}{256a^2} \tag{78}$$

$$= \frac{-1+75a^2-163a^4+89a^6}{64a^2} \tag{79}$$

$$\Leftrightarrow 0 = 88a^6 - 169a^4 + 66a^2 - 1 \tag{80}$$

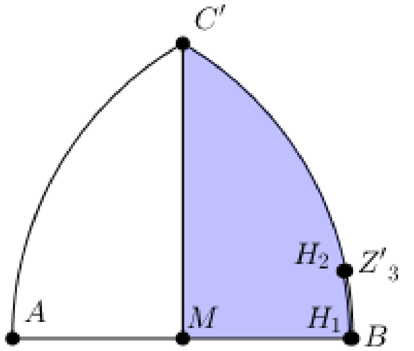
If we substitute a^2 with x we get an equation of degree 3. Since we cannot guess a solution, we take the numerical solutions of the equation. We get the solutions

$$x_1 \approx 0.015784 \Rightarrow a \approx 0.125635 \tag{81}$$

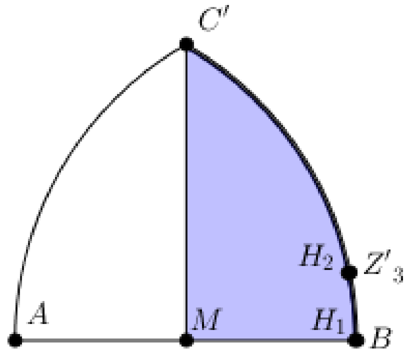
$$x_2 \approx 0.519893 \Rightarrow a \approx 0.721036 \tag{82}$$

$$x_3 \approx 1.384777 \Rightarrow a \approx 1.176766 \tag{83}$$

$$\tag{84}$$



(a) All possible choices for C for a nonmonochromatic triangle after Theorem 29



(b) Possible choices for C for a nonmonochromatic triangle by Theorem 29

Figure 30: Visualisation of the possible choices for C after Theorem 29

Since we want to maximize $h_a = \sqrt{(1-a^2)}$, we choose the solution with the smallest value. So we get

$$a \approx 0.125635 \tag{85}$$
$$h_a \approx 0.992077 \tag{86}$$
$$c \approx 0.868301 \tag{87}$$
$$h_c \approx 0.496038 \tag{88}$$
$$f \approx 0.189992 \tag{89}$$
$$g \approx 0.332575 \tag{90}$$

Note that this values were calculated by computer, without using the above value for x_1.

Theorem 29. *All normed triangles T with $\overline{AC} \leq 0.992076$ are nonmonochromatic in the hexagoncoloring in Figure 28, with the lengths as calculated above.*

So we get a slightly better result than in Theorem 27.

So we have for every C in the shaded area $MH_1H_2Z_3'C'$ in Figure 30(a) a 6-coloring, such that the triangle ABC is monochromatic. In Figure 30(b) we see all possible choices for C, such that by Theorem 29 the triangle ABC is nonmonochromatic. We also see that the area of new possible choices is small, because by Theorem 27 we get $\overline{AC} < \sqrt{\frac{55}{56}} \approx 0.991$, which is nearly 0.992076, which is the upper bound for $\overline{AC}$ by Theorem 29.

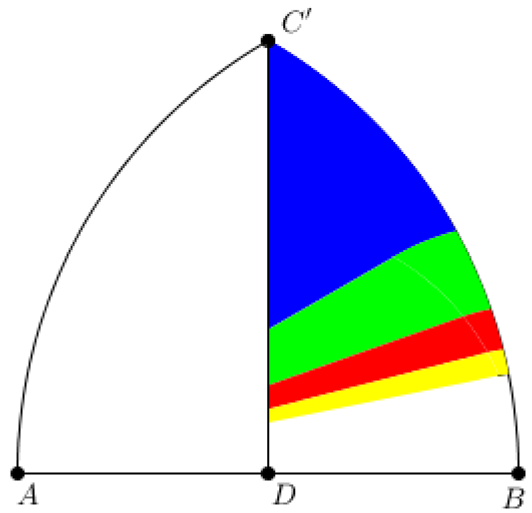


Figure 31: Triangles in zebra colorings with different number of colors: blue: 3 colors, green: 4, red: 5, yellow: 6

There has been hardly any research about nonmonochromatic triangles in 6-colorings before. So the only results, that were known, come from results, where less color were used like that the equilateral triangle is nonmonochromatic in a 2-coloring. Some results about nonmonochromatic triangles in the 3-colored plane can be found in [11].

We have proved that for every normed triangle T with $(\overline{AC} \leq 0.992076)$ or $(\overline{BC} \geq \frac{1}{5})$ or $(\overline{AC} \leq 5h_C)$, there exists a 6-coloring, such that T is nonmonochromatic in this 6-coloring.

Since we could not show, that for all triangles T there exists a 6-coloring, such that T is nonmonochromatic in this 6-coloring, the upper bound for Problem 1 is still at 7. It also seems, if we restrict Problem 1 to maplike colorings, that the answer of Problem 1 is the chromatic number of the Euclidean plane for maplike colorings.

3.5 Using theorems with less colors

In the previous chapters we have looked for triangles, which are nonmonochromatic in a 6-coloring. Now we consider the following question: Can we generalise one of the previous theorems, that we can use it for less colors? It is easy to see, that we can have zebracolorings with less than 6 colors and get similar results. So we can modify Theorem 22 and Theorem 23 and get Figure 31.

In Figure 31 the blue shaded area shows the possible choices for the third

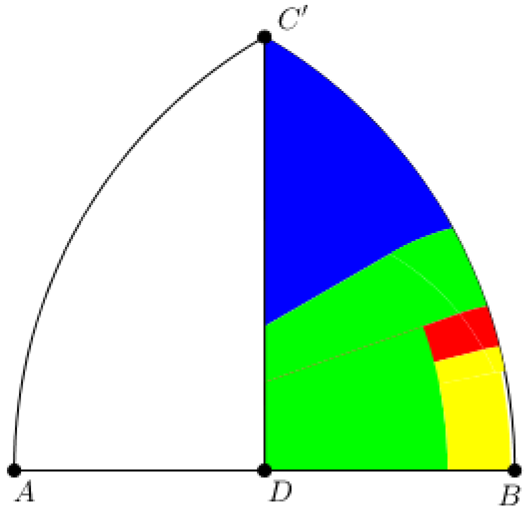


Figure 32: Nonmonochromatic triangles in colorings with different number of colors: blue: 3 colors, green: 4, red: 5, yellow: 6

vertex C of a normed triangle T, such that there exists a 3-coloring, so that T is nonmonochromatic. The green shaded area shows the possible choices in 4-colorings, the red shaded area shows the same for 5-colorings and the yellow shaded area the possible choices in a 6-coloring. We see, that the new area gets smaller, the more colors we have.

We have proved another result for less than 6 colors in Theorem 28. With this theorem we get a better bound for 4 colors. In Figure 32 we can see the areas, for which triangle we have a coloring, that the triangle is nonmonochromatic. We use the same color coding as in Figure 31.

4 Rational Colorings

We know that $\mathbb{R}^n$ with the Euclidean norm is an Euclidean space $\mathbb{E}^n$. So we can see $\mathbb{E}^2$ as $\mathbb{R}^2$ using Cartesian coordinates.

In this chapter we study colorings, where all vertices of a given triangle have rational valued coordinates. In other words we study colorings of $\mathbb{Q}^2$ instead of $\mathbb{R}^2$. We call $\mathbb{Q}^2$ the rational plane.

We will answer the question, if there exist for every triangle in $\mathbb{Q}^2$ a c-coloring of $\mathbb{Q}^2$, such that the triangle does not exist monochromatically in this coloring. Furthermore, we will determine the smallest possible value for c.

We cannot use Theorem 8, which says, that we can use normed triangles in $\mathbb{E}^n$, for this problem, if the longest side has irrational length l. Let P be a point in the rational plane. Then $\overrightarrow{OP_l} = \frac{1}{l}\overrightarrow{OP}$, but P_l is not a point in the rational plane. So we cannot assume that the longest side of the triangle has length 1.

In 1973 Woodall showed the following theorem [21].

Theorem 30 (Woodall)**.** *The chromatic number of* $\mathbb{Q}^2$ *is* 2*. In other words* $\chi(\mathbb{Q}^2) = 2$*.*

The first question is, which distances occur between two points of the rational plane. If the given triangle has a side with length l_1 and there are not two points in the rational plane with distance l_1, then the triangle is nonmonochromatic in a 1-coloring of the rational plane.

Lemma 31. *The distance between any two points in* $\mathbb{Q}^2$ *is of the form* $\sqrt{\frac{n}{m}}$ *where* $n, m \in \mathbb{N}$*.*

Proof. Let $A = (\frac{a_1}{c_1}, \frac{a_2}{c_2})$ and $B = (\frac{b_1}{d_1}, \frac{b_2}{d_2})$ points in $\mathbb{Q}^2$ with $a_1, c_1, a_2, c_2, b_1, d_1, b_2, d_2 \in \mathbb{N}$. Then

$$|AB| = \sqrt{\left(\frac{b_1}{d_1} - \frac{a_1}{c_1}\right)^2 + \left(\frac{b_2}{d_2} - \frac{a_2}{c_2}\right)^2} \tag{91}$$

$$= \sqrt{\left(\frac{b_1c_1 - a_1d_1}{c_1d_1}\right)^2 + \left(\frac{b_2c_2 - a_2d_2}{c_2d_2}\right)^2} \tag{92}$$

$$= \sqrt{\left(\frac{(b_1c_1 - a_1d_1)^2(c_2d_2)^2 + (b_2c_2 - a_2d_2)^2(c_1d_1)^2}{(c_1d_1c_2d_2)^2}\right)} \tag{93}$$

$$= \sqrt{\frac{n}{m}} \tag{94}$$

where

$$\begin{aligned} m &= (b_1c_1 - a_1d_1)^2(c_2d_2)^2 + (b_2c_2 - a_2d_2)^2(c_1d_1)^2 \in \mathbb{N} \\ n &= (c_1d_1c_2d_2)^2 \in \mathbb{N}. \end{aligned}$$

□

This means that we only need to consider triangles where all sides have length of the form $\sqrt{\frac{n}{m}}$.

Theorem 32. *Let m and n be odd integers. Then there exists a 2-coloring of the rational plane, such that the distance between two points with the same color is not $\sqrt{\frac{n}{m}}$.*

Proof. Let $(r_1, r_2), (q_1, q_2) \in \mathbb{Q}^2$.

Furthermore denote $r_1 - q_1 = \frac{w}{x}$ and $r_2 - q_2 = \frac{y}{z}$, where w, x, y and z are integers and w and x are coprime and y and z are coprime. If the distance between (r_1, r_2) and (q_1, q_2) is $\sqrt{\frac{n}{m}}$, then we get

$$(r_1 - q_1)^2 + (r_2 - q_2)^2 = \left(\frac{w}{x}\right)^2 + \left(\frac{y}{z}\right)^2 = \frac{n}{m} \tag{95}$$
$$\Leftrightarrow m(w^2z^2 + x^2y^2) = nx^2z^2 \tag{96}$$

We examine the multiplicity of 2 of both sides. If x or z is even, we get

$$v_2(m(w^2z^2 + x^2y^2)) \leq max(2v_2(x), 2v_2(z)) + 1 \tag{97}$$
$$< max(4v_2(x), 4v_2(z)) \tag{98}$$
$$\leq 4(v_2(x) + v_2(z)) = v_2(nx^2z^2) \tag{99}$$

So neither x nor z can be even, if they fullfill Equation (96). So if the denominator of $r_1 - q_1$ or $r_2 - q_2$ is even, then the distance between (r_1, r_2) and (q_1, q_2) is not 1. That means, that the right hand side of Equation (96) is odd.

If w and y are both odd or w and y are both even, then the left hand side of Equation (96) is even, but since the right hand side is odd, there is no tupel which is a solution of Equation 96. So we want to color $\mathbb{Q}^2$ with two colors, so that two points (r_1, r_2) and (q_1, q_2) have the same color only if the numerators of $r_1 - q_1$ and $r_2 - q_2$ are both even or both odd.

We partition $\mathbb{Q}^2$ into sets S_i, $i \in \mathbb{N}$, where (r_1, r_2) and (q_1, q_2) are in the same set, if the denominators of $r_1 - q_1$ and $r_2 - q_2$ are both odd. If we take two points (r_1, r_2) and (q_1, q_2) of $\mathbb{Q}^2$, which are in different sets, then the distance getween these two points is not $\sqrt{\frac{n}{m}}$, because then either $r_1 - q_1$ or $r_2 - q_2$ has an even denominator.

We have to prove, that every point is in exactly one of these set. Assume that one point (r_1, r_2) is in sets S_j and S_k. Let (q_1, q_2) be a point in S_j and (s_1, s_2) be a point in S_k. Then we know

$$r_1 - q_1 = \frac{a_1}{b_1}, r_2 - q_2 = \frac{c_1}{d_1} \tag{100}$$

$$r_1 - s_1 = \frac{a_2}{b_2}, r_2 - s_2 = \frac{c_2}{d_2} \tag{101}$$

$$q_1 - s_1 = (r_1 - s_1) - (r_1 - q_1) = \frac{a_2}{b_2} - \frac{a_1}{b_1} = \frac{a_2 b_1 - a_1 b_2}{b_1 b_2} \tag{102}$$

$$q_2 - s_2 = (r_2 - s_2) - (r_2 - q_2) = \frac{c_2}{d_2} - \frac{c_1}{d_1} = \frac{c_2 d_1 - c_1 d_2}{d_1 d_2} \tag{103}$$

Since b_1, b_2, d_1 and d_2 are odd, $b_1 b_2$ and $d_1 d_2$ are odd too. So (q_1, q_2) and (s_1, s_2) are in the same set, therefore the sets S_j and S_k are the same set. This means, that every point (r_1, r_2) cannot be in two different sets. Every point (r_1, r_2) is also in one set S_i, because

$$r_1 - r_1 = 0 = \frac{0}{1} \tag{104}$$

$$r_2 - r_2 = 0 = \frac{0}{1} \tag{105}$$

and the denominators are both odd.

Let o_i be odd integers and e_i be even integers for $i \in \mathbb{N}$. Let u and v be integers. All points of the type $\left(\frac{u}{o_2}, \frac{v}{o_4}\right)$ are in the same set. Let us call this set S_1. We look at the two points $(r_1, r_2) = \left(\frac{o_1}{o_2}, \frac{o_3}{o_4}\right)$ and $(q_1, q_2) = \left(\frac{o_5}{o_6}, \frac{o_7}{o_8}\right)$. Then

$$r_1 - q_1 = \frac{o_1}{o_2} - \frac{o_5}{o_6} = \frac{o_1 o_6 - o_2 o_5}{o_2 o_6} \tag{106}$$

$$r_2 - q_2 = \frac{o_3}{o_4} - \frac{o_7}{o_8} = \frac{o_3 o_8 - o_4 o_7}{o_4 o_8} \tag{107}$$

so the numerator of $r_1 - q_1$ and the numerator of $r_2 - q_2$ are even, which means that (r_1, r_2) and (q_1, q_2) do not lie distance $\sqrt{\frac{n}{m}}$ apart. Therefore we color all points of the type $\left(\frac{o_1}{o_2}, \frac{o_3}{o_4}\right)$ blue.

If we take a point $(q_3, q_4) = \left(\frac{e_1}{o_5}, \frac{e_2}{o_6}\right)$, then we get

$$r_1 - q_3 = \frac{o_1}{o_2} - \frac{e_1}{o_5} = \frac{o_1 o_5 - o_2 e_1}{o_1 o_5} \tag{108}$$

$$r_2 - q_4 = \frac{o_3}{o_4} - \frac{e_2}{o_6} = \frac{o_3 o_6 - o_4 e_2}{o_4 o_6} \tag{109}$$

therefore the distance between (r_1, r_2) and (q_3, q_4) is never exactly $\sqrt{\frac{n}{m}}$, since the nominator of $r_1 - q_3$ and the nominator of $r_2 - q_4$ are even. So we color all points $\left(\frac{o_1}{o_2}, \frac{o_3}{o_4}\right)$ blue.

The uncolored points in S_1 are of the type $\left(\frac{e_1}{o_1}, \frac{o_2}{o_3}\right)$ or $\left(\frac{o_4}{o_5}, \frac{e_2}{o_6}\right)$. It follows after repeating the same computations as before, that there are not two points, which are distance 1 apart, that have these coordinates. So we can color all points $\left(\frac{e_1}{o_1}, \frac{o_2}{o_3}\right)$ or $\left(\frac{o_4}{o_5}, \frac{e_2}{o_6}\right)$ red.

Let O be the point $(0,0)$. We take a point $P = (p_1, p_2)$ of the set S_k with $k \neq 1$. Let T_P be the resulting set of the translation of S_k by the vector $-\overrightarrow{OP}$. Let $P_2 = (p_3, p_4)$ be another point of S_k. Then the corresponding point in T_P is $(p_3 - p_1, p_4 - p_2)$. By the definition of S_k the denominators of $p_3 - p_1$ and $p_4 - p_2$ are odd. So T_P is a subset of S_1 and since we can 2-color S_1, we can 2-color T_P. Since the distance between two points does not change if we translate these points by the same vector, we can 2-color S_k by coloring $P_k = (p_5, p_6)$ with the same color as we color $(p_5 - p_1, p_6 - p_2)$, where p_5, p_6 are arbitrary rational numbers.

By doing this for all sets S_i, we get a 2-coloring for the rational plane, such that no pair of points has distance exactly $\sqrt{\frac{n}{m}}$. $\square$

Since the coloring does not depend on n or m, we have proven, that there is a 2-coloring such that the distance between two points with the same color is not $\sqrt{\frac{n}{m}}$, for every odd n and m.

The missing cases are, when n or m are even. First we study the case, that $n = 2n_1$, where n_1 is an odd integer.

Theorem 33. *There is a 2-coloring of the rational plane, such that, there is no pair of points with distance $\sqrt{\frac{2n}{m}}$, where n and m are odd integers, that have the same color.*

Proof. Let $R = (r_1, r_2)$ and $Q = (q_1, q_2)$ be two points in the rational plane. Let the distance between these two points be $\sqrt{\frac{2n}{m}}$. Furthermore let $r_1 - q_1 = \frac{w}{x}$ and $r_2 - q_2 = \frac{y}{z}$, where w, x, y and z are integers, w and x are coprime and y and z are coprime. Then

$$(r_1 - q_1)^2 + (r_2 - q_2)^2 = \frac{2n}{m} \tag{110}$$

$$\Leftrightarrow \left(\frac{w}{x}\right)^2 + \left(\frac{y}{z}\right)^2 = \frac{2n}{m} \tag{111}$$

$$\Leftrightarrow m((wz)^2 + (xy)^2) = 2n(xz)^2 \tag{112}$$

As in the proof of Theorem 32, we get that x and z have to be odd, if (w, x, y, z) is a solution of this equation.

If w and y are even, then $m((wz)^2+(xy)^2)$ is divisible by 4, while $2n(xz)^2$ is not divisible by 4, since x and z are odd.

If w is even and y is odd, or w is odd and y is even, then $m((wz)^2+(xy)^2)$ is not divisible by 2, while $2n(xz)^2$ is. Therefore, in all solutions of the Equation (112) w, x, y and z are odd. So the distance between R and Q is not $\sqrt{\frac{2n}{m}}$ if the nominator of $r_1 - q_1$ or the nominator of $r_2 - q_2$ is even.

As in the proof of Theorem 32, we partition $\mathbb{Q}^2$ into sets S_i, $i \in \mathbb{N}$, where (r_1, r_2) and (q_1, q_2) are in the same set, if the denominators of $r_1 - q_1$ and $r_2 - q_2$ are both odd. If we take two points (r_1, r_2) and (q_1, q_2) of $\mathbb{Q}^2$, which are in different sets, then the distance getween these two points is not $\sqrt{\frac{2n}{m}}$, because then either $r_1 - q_1$ or $r_2 - q_2$ has an even denominator.

Let o and o_i be odd integers and e and e_i be even integers for $i \in \mathbb{N}$. Let x,y and y_i be integers for $i \in \mathbb{N}$. All points of the type $\left(\frac{x}{o_1}, \frac{y}{o_2}\right)$ are in the same set of the partition. Let us call this set S_1. We color all points $\left(\frac{e}{o_1}, \frac{y}{o_2}\right)$ blue and color all points $\left(\frac{o}{o_1}, \frac{y}{o_2}\right)$ red. Let $B_1 = \left(\frac{e_1}{o_1}, \frac{y_1}{o_2}\right)$ and $B_2 = \left(\frac{e_2}{o_3}, \frac{y}{o_4}\right)$ be blue points. The nominator of

$$\frac{e_1}{o_1} - \frac{e_2}{o_3} = \frac{e_1 o_3 - e_2 o_1}{o_1 o_3} \tag{113}$$

is even, so the distance between B_1 and B_2 is not $\sqrt{\frac{2n}{m}}$.

Let $C_1 = \left(\frac{o_1}{o_2}, \frac{y_1}{o_3}\right)$ and $C_2 = \left(\frac{o_4}{o_5}, \frac{y}{o_6}\right)$ be red points. The nominator of

$$\frac{o_1}{o_2} - \frac{o_4}{o_5} = \frac{o_1 o_5 - o_2 o_4}{o_2 o_5} \tag{114}$$

is even, so the distance between C_1 and C_2 is not $\sqrt{\frac{2n}{m}}$.

As in the proof of Theorem 32 we take for every set S_i, $i > 1$, a point P_i of some S_i and translate the set by the vector $-\overrightarrow{OP_i}$ and get the set T_i. T_i is a subset of S_i. So we can 2-color T_i and since the distance between two points in S_i is the same as the distance between these two translated points in T_i, we get a 2-coloring for S_i. We do this for all $i > 1 \in \mathbb{N}$ and get a 2-coloring of the rational plane, sucht that the distance between two points with the same color is not $\sqrt{\frac{2n}{m}}$. $\square$

In the next step we prove a lemma similar to Lemma 8.

Lemma 34. *Let p and q be a positiv rational numbers. If the rational plane has a 2-coloring F, such that the distance between two points with the same color is not $\sqrt{q}$, then there exists a 2-coloring F_p, such that the distance between two points with the same color is not $p\sqrt{q}$.*

Proof. Let A and B be points in $\mathbb{Q}^2$, such that $\overline{AB} = \sqrt{q}$. We define A_p as the point such that $\overrightarrow{OA_p} = p\overrightarrow{OA}$ and B_p as the point such that $\overrightarrow{OB_p} = p\overrightarrow{OB}$. A_p and B_p are points in $\mathbb{Q}^2$ since p is a rational number. Furthermore, we color A_p with the same color as A and B with the same color as B. We do this for every point in the rational plane and get a 2-coloring F_p. The distance between A_p and B_p is

$$\overline{A_pB_p} = |\overrightarrow{A_pB_p}| \tag{115}$$
$$= |\overrightarrow{OB_p} - \overrightarrow{OA_p}| \tag{116}$$
$$= |p\overrightarrow{OB} - p\overrightarrow{OA}| \tag{117}$$
$$= p|\overrightarrow{AB}| = p\sqrt{q} \tag{118}$$

Since the distance between two points with the same color in the 2-coloring F is not $\sqrt{q}$, the the distance between two points with the same color in the 2-coloring F_p is not $p\sqrt{q}$. □

With this we can prove the following theorem.

Theorem 35. *For every positive real number d, there exists a 2-coloring of the rational plane, such that the distance between two points with the same color is not d.*

Proof. If $d \neq \sqrt{\frac{n}{m}}$, then the distance between any two points in the rational plane is not d.

If $d = \sqrt{\frac{n}{m}}$, where n and m are odd, then we use Theorem 32.

If $d = \sqrt{\frac{2^k n}{m}}$, where k is a positive integer and n and m are positive odd integers, we have two cases. If $k = 2k_1$, then $d = \sqrt{\frac{2^{2k_1} n}{m}} = 2^{k_1}\sqrt{\frac{n}{m}}$. By Theorem 32 we know there exists a 2-coloring, such that the distance between two points with the same color is not $\sqrt{\frac{n}{m}}$ and by using Lemma 34 with $p = 2^{k_1}$ and $q = \frac{n}{m}$ we know there exists a 2-coloring, such that the distance between two points with the same color is not $2^{k_1}\sqrt{\frac{n}{m}}$.

If $k = 2k_1 + 1$, then $d = \sqrt{\frac{2^{(2k_1+1)} n}{m}} = 2^{k_1}\sqrt{\frac{2n}{m}}$. By Theorem 33 we know there exists a 2-coloring, such that the distance between two points with the same color is not $\sqrt{\frac{2n}{m}}$ and by using Lemma 34 with $p = 2^{k_1}$ and $q = \frac{2n}{m}$

we know there exists a 2-coloring, such that the distance between two points with the same color is not $2^{k_1}\sqrt{\frac{2n}{m}}$.

The only missing case is $d = \sqrt{\frac{n}{2^k m}}$, where k is a positive integer and n and m are positive odd integers. But then $d = \sqrt{\frac{n}{2^k m}} = \frac{1}{2^k}\sqrt{\frac{2^k n}{m}}$ and we already proved, that there exists a 2-coloring, such that the distance between two points with the same color is not $\sqrt{\frac{2^k n}{m}}$. Using Lemma 34 with $p = \frac{1}{2^k}$ and $q = \frac{2^k n}{m}$ finishes this prove. $\square$

So we can prove the following theorem about triangles in the colored rational plane.

Theorem 36. *For every triangle T, there exist a 2-coloring of the rational plane, such that T is nonmonochromatic.*

Proof. Let s be the length of a side of T. Then Theorem 35 says, there exists a 2-coloring, such that there are not two points, which lie distance s apart, that have the same color. Therefore T is nonmonochromatic. $\square$

In this chapter we proved, that for any triangle T, there exists a 2-coloring of the rational plane, such that T is nonmonochromatic. We showed an even stronger result by proving, that for every positive real number d, there exists a 2-coloring of the rational plane, such that the distance between two points with the same color is not d. Another interesting result is, that in the coloring described in Theorem 32 all $d_i = \sqrt{\frac{p_i}{q_i}}$, $p_i, q_i \in \mathbb{Q}^+$ and p_i and q_i are odd for all i, are not distances between two points with the same color. Similarly, all distances $d_i = \sqrt{\frac{2p_i}{q_i}}$, $p_i, q_i \in \mathbb{Q}^+$ and p_i and q_i are odd for all i, are not distances between two points with the same color in the coloring described in Theorem 33.

5 Coloring strips

In this chapter we study another variant of the chromatic number of the plane.

Problem 4. *Let $c \in \mathbb{N}$. What is the largest height $h(c)$ of a strip (depending on c), such that there exists a c-coloring, so that there does not exist a pair of points of the strip with distance 1, that have the same color.*

If c is at least the chromatic number of the euclidean plane, then the strip would have infinite height. Therefore the interesting cases are, where we have at most 6 colors.

For $c = 3$, Bauslaugh [2] proved that $h(3) = \frac{\sqrt{3}}{2}$. Axenovich et al [1] got the following lower bounds.

$$h(3) \geq \frac{\sqrt{3}}{2} \geq 0.866 \tag{119}$$

$$h(4) \geq \frac{2\sqrt{2}}{3} \geq 0.94 \tag{120}$$

$$h(5) \geq \frac{\sqrt{15}}{4} \geq 0.968 \tag{121}$$

$$h(6) \geq \frac{\sqrt{15}}{2} + \sqrt{3} \geq 3.668 \tag{122}$$

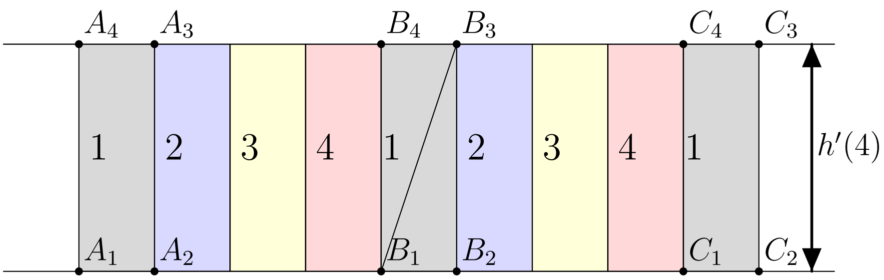


Figure 33: 4-colored strip, such that two points with the same color do not lie distance 1 apart.

Constructions for these lower bounds are the same for 3, 4 and 5 colors and are similar to zebra colorings. In Figure 33 we have 4 colors.

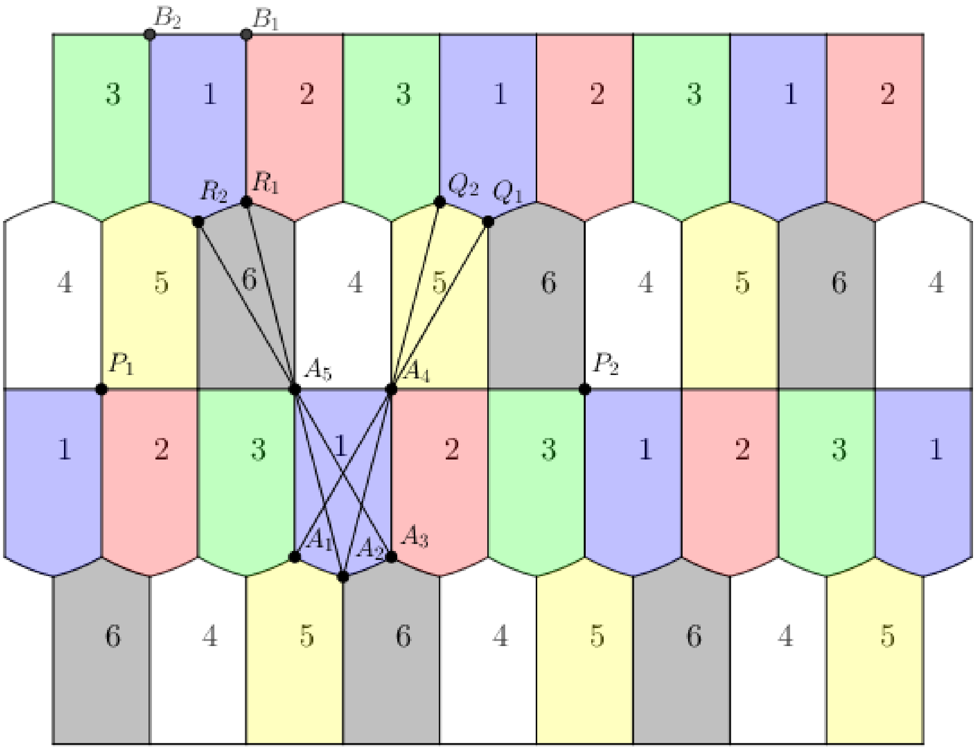


Figure 34: 6-coloring of a strip with height $\frac{\sqrt{15}}{2} + \sqrt{3}$, such that all points, which have the same color, do not have distance 1.

We take rectangles of size $h'(c) \times \frac{1}{c-1}$, where c is the number of colors and $h'(c)$ is the height of the strip. These rectangles are embeded into the strip like in Figure 33. The diagonals of the rectangles have length not greater than 1. We color the left edge with the color of the rectangle. That means that in Figure 33 the edge A_2A_3 has color 2 and the edge A_1A_4 has color 1. With Pythagoras theorem we get

$$h'(c)^2 = 1 - \left(\frac{1}{c-1}\right)^2 \tag{123}$$

$$h'(c) = \frac{\sqrt{c^2 - 2c}}{c - 1} \tag{124}$$

So we have a lower bound $h'(c)$ for $h(c)$ and for $c = 3, 4, 5$ this gives the results found by Axenovich et al.

For $c = 6$, Axenovich et al used the construction in Figure 34, which is similar to Pritikins [16] construction. In Figure 34, every tile of the coloring has width $\frac{1}{2}$. This means, that

$$\overline{A_4A_5} = \overline{B_1B_2} \qquad = \frac{1}{2} \tag{125}$$

$$\overline{P_1A_5} = \overline{P_2A_4} \qquad = 1. \tag{126}$$

The arcs R_1R_2 and A_2A_3 are on the circle with center A_5 and radius 1. The arcs A_1A_2 and Q_1Q_2 are on the circle with center A_4 and radius 1. The distances B_1R_2 and B_2R_2 are 1.

We color the leftmost edge and the the leftmost arc without its rightmost point of a tile with the same color as the tile. Furthermore, we color the line segments, which are parallel to the strip, with the same color as the tile containing it.

So the line segment A_1A_5, the arc A_1A_2 and the points A_1 and A_2 have color 1. The arc A_2A_3 and the point A_2 have color 6 and the line segment A_3A_4 and the points A_3 and A_4 have color 2. The line segment B_1B_2 has color 1. We color the line segment A_4A_5 with the colors 1 and 4.

This way, two points with distance 1 and the same color cannot be in the same tile, since the arc A_1A_2 has a different color than A_4, the arc A_2A_3 has a different color than A_5 and A_2 has not the same color as A_4 or A_5. Furthermore, R_2 has not the same color as B_1 or B_2.

The distance between two point with the same color in two different tiles is greater than 1, because the two tiles with the same color have distance 1. Each point is on the boundary of one tiles and we color the boundary in such a way, that this does not happen.

Let D be a point like in Figure 34, so that the angle $\angle A_5DB_1 = 90°$. Now we have to calculate the height of this strip, for which we want to calculate

$\overline{B_1 D}$ first. With the help of Pythagoras theorem we get

$$\overline{B_1 R_1}^2 = \overline{B_2 R_1}^2 - \overline{B_1 B_2}^2 \tag{127}$$

$$= 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} \tag{128}$$

$$\overline{B_1 R_1} = \frac{\sqrt{3}}{2} \tag{129}$$

$$\overline{D R_1}^2 = \overline{A_5 R_1}^2 - \overline{A_5 D}^2 \tag{130}$$

$$= 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16} \tag{131}$$

$$\overline{D R_1} = \frac{\sqrt{15}}{4} \tag{132}$$

$$\overline{B_1 D} = \frac{1}{2}\left(\sqrt{3} + \frac{\sqrt{15}}{2}\right) \tag{133}$$

By taking $\overline{B_1 D}$ twice we get

$$2\overline{B_1 D} = \sqrt{3} + \frac{\sqrt{15}}{2} \leq h(c) \tag{134}$$

which is the best currently known lower bound for $h(6)$.

We improve the lower bound for $h(5)$.

Theorem 37. *There exists a* 5*-coloring of a strip with height* 1*, such that there does not exist two points with the same color, that are distance* 1 *apart. In other words, this means*

$$h(5) \geq 1 \tag{135}$$

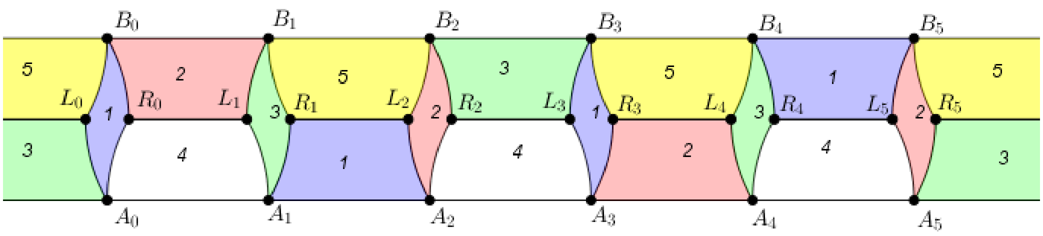


Figure 35: 5-coloring of a strip with height 1, such that all points, which have the same color, do not have distance 1.

Proof. We look at Figure 35. The strip has heigth 1 and we denote the points as in Figure 35. The boundary of this strip are two parallel lines l_A and l_B. We construct this 5-coloring with the following steps.

- Let A_0 be a point on l_A.
- Let B_0 be the point on l_B, such that the line segment A_0B_0 is orthogonal to the lines l_a and l_b.
- Let A_i be different points on l_A, such that $\overline{A_iA_{i+1}} = 1$, for all $i \in \mathbb{Z}$.
- Let B_i be different points on l_B, so that $\overline{B_iB_{i+1}} = 1$ and $\overline{A_iB_i} = 1$, for all $i \in \mathbb{Z}$.
- Let l_R be a line between l_A and l_B and let l_R be parallel to l_A. The distance between l_A and l_R is $\frac{1}{2}$.
- Let L_{i+1} and R_{i-1} be the points on l_R, which lie on the circle with center A_i and radius 1, so that R_{i-1} is left of L_{i+1}.

Before we continue, we have to mention some properties of these points. Since $\overline{A_iA_{i+1}} = 1$, A_{i-1} and A_{i+1} are also on the circle with center A_i and radius 1.

The point L_{i+1} can also be seen as a translation of the point L_i by the vector $\overrightarrow{A_iA_{i+1}}$. So $\overline{L_iL_{i+1}} = 1$. With the same arguments we get $\overline{R_iR_{i+1}} = 1$. Therefore, the points A_{i-1}, L_{i-1} and B_{i-1} lie on the circle with center L_i and radius 1. Furthermore, the points A_{i+1}, R_{i+1} and B_{i+1} lie on the circle with center R_i and radius 1.

We can now finish our construction of the coloring with these observations.

- We add all line segments R_iL_{i+1}, for all $i \in \mathbb{Z}$.
- We add all circular arcs $A_{2i}L_{2i}$ centered at L_{2i+1}, for all $i \in \mathbb{Z}$.
- We add all circular arcs $A_{2i+1}L_{2i+1}$ centered at A_{2i}, for all $i \in \mathbb{Z}$.
- We add all circular arcs $A_{2i}R_{2i}$ centered at A_{2i+1}, for all $i \in \mathbb{Z}$.
- We add all circular arcs $A_{2i+1}R_{2i+1}$ centered at R_{2i}, for all $i \in \mathbb{Z}$.
- We add all circular arcs $B_{2i}L_{2i}$ centered at B_{2i-1}, for all $i \in \mathbb{Z}$.
- We add all circular arcs $B_{2i-1}L_{2i-1}$ centered at L_{2i}, for all $i \in \mathbb{Z}$.
- We add all circular arcs $B_{2i}R_{2i}$ centered at R_{2i-1}, for all $i \in \mathbb{Z}$.
- We add all circular arcs $B_{2i-1}R_{2i-1}$ centered at B_{2i}, for all $i \in \mathbb{Z}$.

We 5-color these tiles in the following way

- We color all tiles $A_{2i}A_{2i+1}L_{2i+1}R_{2i}$ with color 4, for $i \in \mathbb{Z}$.
- We color all tiles $B_{2i-1}B_{2i}L_{2i}R_{2i-1}$ with color 5, for $i \in \mathbb{Z}$.
- We color all tiles $A_{3i}R_{3i}B_{3i}L_{3i}$ with color 1, for $i \in \mathbb{Z}$.
- We color all tiles $A_{3i+1}R_{3i+1}B_{3i+1}L_{3i+1}$ with color 3, for $i \in \mathbb{Z}$.
- We color all tiles $A_{3i+2}R_{3i+2}B_{3i+2}L_{3i+2}$ with color 2, for $i \in \mathbb{Z}$.
- We color all tiles $A_{6i+1}A_{6i+2}L_{6i+2}R_{6i+1}$ and $B_{6i+4}B_{6i+5}L_{6i+5}R_{6i+4}$ with color 1, for $i \in \mathbb{Z}$.
- We color all tiles $A_{6i+3}A_{6i+4}L_{6i+4}R_{6i+3}$ and $B_{6i}B_{6i+1}L_{6i+1}R_{6i}$ with color 2, for $i \in \mathbb{Z}$.
- We color all tiles $A_{6i-1}A_{6i}L_{6i}R_{6i-1}$ and $B_{6i+2}B_{6i+3}L_{6i+3}R_{6i+2}$ with color 3, for $i \in \mathbb{Z}$.

The boundary of all tiles are colored in the following way

- The arcs A_iL_i and B_iL_i and the point L_i are colored with the same color as the tile $A_iR_iB_iL_i$.
- The arc A_iR_i, the line segments A_iA_{i+1} and R_iL_{i+1} and the points A_i and R_i are colored with the same color as the tile $A_iA_{i+1}L_{i+1}R_i$.
- The arc B_iR_i, the line segment B_iB_{i+1} and the point B_i are colored with the same color as the tile $B_iB_{i+1}L_{i+1}R_i$.

Now we have to prove that two points with distance 1 do not have the same color. First we show, that two points with distance 1 and the same color, cannot lie in the same tile.

Case 1: Two points lie in a tile $A_{2i}A_{2i+1}L_{2i+1}R_{2i}$ or $B_{2i-1}B_{2i}L_{2i}R_{2i-1}$. These two tiles are congruent. Since these tiles are are part of a Reuleaux triangle with diameter 1, we only have to look at the boundary of these tiles. The point A_{2i} has not the same color as the arc $A_{2i+1}L_{2i+1}$ or the points A_{2i+1} and L_{2i+1}. On the other side, the point A_{2i+1} has not the same color as the arc $A_{2i}R_{2i}$ or the points A_{2i} and R_{2i}. So we do not have two points with the same color with distance 1 in a tile $A_{2i}A_{2i+1}L_{2i+1}R_{2i}$. Analogously, we do not have two points with the same color with distance 1 in a tile $B_{2i-1}B_{2i}L_{2i}R_{2i-1}$.

Case 2: Two points lie in a tile $A_{2i-1}A_{2i}L_{2i}R_{2i-1}$ or $B_{2i}B_{2i+1}L_{2i+1}R_{2i}$. These tiles are subtiles of the tiles in Case 1. The boundary is colored in the same way as in the tiles in Case 1. Therefore there are not two points with the same color with distance 1 in a tile $A_{2i-1}A_{2i}L_{2i}R_{2i-1}$ or $B_{2i}B_{2i+1}L_{2i+1}R_{2i}$.

Case 3: Two points lie in a tile $A_iR_iB_iL_i$. Each one of these tiles can be covered with a disk with diameter 1, where A_i and B_i are on the boundary of the disk. Since A_i and B_i do not have the same color, there are not two points with distance 1 and the same color in these tiles.

Therefore there is not a tile, in which there exist two points with distance 1 and the same color. Next we prove that there are also not two points with distance 1 and the same color in different tiles.

Case 4: We look at tiles $A_{2i}A_{2i+1}L_{2i+1}R_{2i}$. If a point has color 4, then it is in the tile $A_{2i}A_{2i+1}L_{2i+1}R_{2i}$, The distance between two tiles $A_{2i}A_{2i+1}L_{2i+1}R_{2i}$ and $A_{2i+2}A_{2i+3}L_{2i+3}R_{2i+2}$ is 1. This distance is only realised by the points A_{2i+1} and A_{2i+3}, but these two points have different colors. So there do not exist two points with distance 1 and color 4.

Analogously, there do not exist two points with distance 1 and color 5.

Case 5: We assume, that there exist two points with distance 1 and both points have the same color. This color is 1, 2 or 3. Without loss of generality, we assume both points have color 1. Between every tiles of the form $A_{6i+1}A_{6i+2}L_{6i+2}R_{6i+1}$ and $B_{6i+4}B_{6i+5}L_{6i+5}R_{6i+4}$ there is a tile of the form $A_{3i}R_{3i}B_{3i}L_{3i}$ with color 1.

Next we look at the distance between the tiles $A_{6i}R_{6i}B_{6i}L_{6i}$ and $A_{6i+1}A_{6i+2}L_{6i+2}R_{6i+1}$. The distance between the point A_{6i+1} and the arc $A_{6i}R_{6i}$ is 1 and the distance between the point R_{6i} and the arc $A_{6i+1}R_{6i+1}$ is 1.

We show, that for every point on the arc $A_{6i+1}R_{6i+1}$ except for A_{6i+1}, the nearest point in tile $A_{6i}R_{6i}B_{6i}L_{6i}$ is R_{6i}. Let X_{6i+1} be a point on the arc $A_{6i+1}R_{6i+1}$. We draw a circle with radius 1 and centerpoint X_{6i+1} and a cicle with radius 1 and centerpoint A_{6i+1}. These two circles have at most two intersections. One intersection is at R_{6i}. We get the other intersection by mirroring R_{6i} on the line $A_{6i+1}X_{6i+1}$. This point is not on the arc $A_{6i}R_{6i}$. Since the distance between A_{6i} and X_{6i+1} is greater than 1, we get, that R_{6i} is the nearest point to X_{6i+1} on the arc $A_{6i}R_{6i}$.

Since the point A_{6i+1} and the arc $A_{6i}R_{6i}$ and the point R_{6i} and the arc $A_{6i+1}R_{6i+1}$ have different colors, the distance between a points with color 1 in tile $A_{6i}R_{6i}B_{6i}L_{6i}$ and a point with color 1 in tile $A_{6i+1}A_{6i+2}L_{6i+2}R_{6i+1}$ is greater than 1.

The other cases follow analogously. So there are not two points with the same color with distance 1. $\square$

So we have proven, that for a strip of height 1, there exists a 5-coloring, such that there do not exist two points with distance 1 and the same color.

Previously it was proven that $h(5) \geq \frac{\sqrt{15}}{4}$ holds. We improved this by showing $h(5) \geq 1$. So far only lower bounds for $h(4)$, $h(5)$ and $h(6)$ have been proven. It thus remains an open problem, to determine these values.

6 Conclusion

In this thesis we considered the following problem:

Problem 1. *What is the smallest number c, so that for any triangle T, there exists a coloring of the Euclidean plane with c colors, such that it is not possible, that all vertices of T have the same color if we embed it in the colored plane?*

In Chapter 2 we considered only translation and rotation of a given triangle. We showed in Theorem 12, that if we only consider translation of a given triangle T, then there exists a 2-coloring of the Euclidean plane, so that every translated copy of T is nonmonochromatic. In Theorem 13 we showed, that there exists a triangle T, which has a monochromatic rotated copy of T in every 2-coloring of the Euclidean plane. On the other side, we showed in Theorem 14, that for every triangle T, there exists a 3-coloring of the Euclidean plane, so that every rotated copy of T is nonmonochromatic.

Furthermore we showed, that there does not exist a c-coloring of the Euclidean plane, such that no monochromatic triangle exists, which is similar to our given triangle. i. e., if we consider translation, rotation and scaling.

In Chapter 3 we looked at the following problem, which is similar to Problem 1:

Problem 2. *Given a triangle T. What is the smallest number c, so that we can color the Euclidean plane with $c(T)$ colors, such that it is not possible, that all vertices of T have the same color if we embed it in the colored Euclidean plane?*

In this thesis we defined a normed triangle as a triangle with side lengths $\overline{BC} \leq \overline{AC} \leq \overline{AB} \leq 1$. We proved, if a normed triangle T has at least one of the following properties,

- ($\overline{AC} \leq 0.992076$) by Theorem 29
- ($\overline{BC} \geq \frac{1}{5}$ and every angle of the triangle is at most $90°$) by Theorem 23
- ($\overline{AC} \leq 5h_C$) by Theorem 22

then there exists a 6-coloring, such that T is nonmonochromatic in this 6-coloring. The points in the shaded area in Figure 36 have one of these properties. It remains open, if for every triangle T, there is a 6-coloring, such that T is nonmonochromatic in this 6-coloring. If we could prove that, then we would get 6 as a new upper bound for Problem 1.

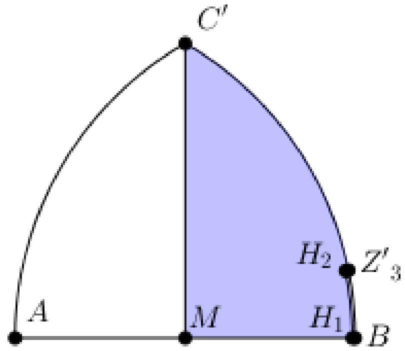


Figure 36: All found possible choices for C for a nonmonochromatic triangle in a 6-coloring

On the other side it is interesting, for which c there exists a triangle, which exists monochromatically in every c-coloring of the Euclidean plane. We know, that there exist triangles, which exist monochromatically in every 2-coloring of the Euclidean plane. But it is unknown, if there exists a triangle, which exist monochromatically in every 3-coloring of the Euclidean plane.

We showed in Chapter 4, that for every triangle T, we can 2-color the rational plane, such that T is nonmonochromatic (Theorem 36). It is interesting, how many colors are needed for other countable subsets of points of the Euclidean plane. In particular, it is interesting to know how many colors we need to color the points with coordinates in $\mathbb{Q}[\sqrt{3}]^2$, because we need at least 3 colors to avoid, that any two points with distance 1 have the same color.

Finally in Chapter 5 we proved in Theorem 37, that in the 5-coloring of a strip with height 1 in Figure 35, any two points with distance 1 do not have the same color. It is an interesting open question if there exists a 5-coloring of a strip with height greater than 1, so that any two points with distance 1 do not have the same color. In particular, it is interesting, whether the strip can have infinite height, so that any two points with distance 1 do not have the same color. If the height can be infinite, then the chromatic number of the plane is 5. Otherwise if the height is finite, then the chromatic number of the plane is at least 6. The same questions are interesting for 6-colorings.

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Definitions

Definition 3. *A coloring is a partitioning of the Euclidean space, such that every point of the space has one color. An r-coloring is a coloring with r colors.*

Definition 4.

- *Two finite point sets X and Y are congruent if $|X| = |Y|$ and Y can be covered by rotation and translation of X.*

- *A finite point set X is monochromatic if every point of X has the same color.*

- *A finite point set X exists monochromatically in a coloring if there exists a congruent copy of X which is monochromatic.*

- *A finite point set X is nonmonochromatic in a coloring if it does not exist monochromatically.*

Definition 5. *Let d be the dimension of the Euclidean space and let r be a positive integer. Let X be a finite point set and $Cong(X)$ the family of all congruent copies of X. We write $\mathbb{E}^d \xrightarrow{r} Cong(X)$ if X exists monochromatically in all r-colorings of $\mathbb{E}^d$.*

Definition 6. *Let A and B be any two distinct points in the Euclidean plane. We write AB for the line segment between A and B and we write $\overline{AB}$ for the length of the line segment AB.*

Definition 7. *Let us denote the heights of a given triangle as in Figure 3. We call a triangle a normed triangle, if the longest side is AB and has length 1, and $\overline{BC} \leq \overline{AC}$ holds. That means, that h_C is the shortest height of the triangle.*

Definition 16. *Let T be a triangle with side lengths a, b and c. We look at a triangle T_n with side lengths na, nb and nc. Let A_0, B_0 and C_0 be the vertices of T_n and $A_n = B_0$, $B_n = C_0$ and $C_n = A_0$. Further partition the side A_0A_n into n line segments with the same length by putting points A_i on A_0A_n such that $\overline{A_iA_{i+1}} = a$ and do the same for the lines B_0B_n and C_0C_n. Now add all lines A_iB_{n-i}, B_iC_{n-i} and C_iC_{n-i} and add a vertex at each intersection. We call such a graph a triangle grid of T with base n or triangle grid for short.*

We call a triangle grid of T partial r-chromatic if in every r-coloring of the vertices of the triangle grid there is always a monochromatic triangle similar to T.

n_r is the smallest number, such that any triangle grid with base n_r is partial r-monochromatic.

Definition 20. *A multigraph is a graph, for which we allow multiple edges between two vertices.*

We call a multigraph, that is drawn in the plane without intersection of its edges, plane.

A multigraph is connected, if for any two vertices there is a path connecting these vertices.

An edge x of a connected multigraph G is called bridge, if $G - x$ is not connected.

A map is a plane connected multigraph without bridges. A map divides the plane into regions.

Two regions are adjacent if they share at least one edge.

A map coloring is a coloring of all regions, such that two regions with the same color do not share an edge.

A map like coloring is a coloring of all regions, edges and vertices, such that every edge between two regions with the same color has the same color as these regions and every point, whose adjacent edges have the same color, has the same color as the edges.

Theorems

Theorem 12. *For all triangles T $\mathbb{E}^2 \stackrel{2}{\nrightarrow} Trans(T)$ holds.*

Theorem 13. *Let P be a fixed point. There exists a triangle T with $\mathbb{E}^2 \stackrel{2}{\rightarrow} Rot_P(T)$.*

Theorem 14. *For all triangles T $\mathbb{E}^2 \stackrel{3}{\nrightarrow} Rot_P(T)$ holds.*

Theorem 15. *For all triangles T and for all finite $r \in \mathbb{N}$, $\mathbb{E}^2 \stackrel{r}{\rightarrow} Sim(T)$ holds.*

Theorem 21. *All normed triangles with $h_C \geq \frac{1}{5}$ are nonmonochromatic in a zebra coloring with 6 colors, where all strips have height h_C.*

Theorem 22. *All normed triangles with $\overline{AC} \leq 5h_C$ are nonmonochromatic in a zebra coloring with 6 colors, where all strips have height h_C.*

Theorem 23. *Every normed triangle, in which every angle is at most $90°$ and $h_A \leq 5h_C$ is nonmonochromatic in a zebra coloring with 6 colors, where all strips have height h_C.*

Theorem 24. *All normed triangles with $\overline{AC} \leq \frac{\sqrt{10}}{4}$ are nonmonochromatic in a 6-coloring like in Figure 20.*

Theorem 25. *All normed triangles with $\overline{AC} \leq \frac{\sqrt{15}}{\sqrt{19}}$ are nonmonochromatic in a 6-coloring with rectangles with $a = \frac{\sqrt{15}}{\sqrt{19}}$ and $b = \frac{2}{\sqrt{19}}$ as shown in Figure 21.*

Theorem 26. *All normed triangles T with $\overline{BC} \geq \frac{1}{2}\sqrt{7 - 3\sqrt{5}}$ are nonmonochromatic in a 6-coloring with rectangles with $a = \frac{\sqrt{3}}{4}$ and $b = \frac{1}{2}$.*

Theorem 27. *All normed triangles with $\overline{AC} \leq \sqrt{\frac{55}{56}}$ are nonmonochromatic in the coloring used in Figure 24 with $a = \sqrt{\frac{55}{224}}$, $b = \frac{13}{4\sqrt{14}}$ and $h = \frac{-26+\sqrt{727}}{8\sqrt{14}}$, where a, b and c denote the same sides as in Figure 25.*

Theorem 28. *All normed triangles with $\overline{AC} \leq \frac{\sqrt{3}}{2}$ are nonmonochromatic in a 4-coloring like in Figure 27.*

Theorem 29. *All normed triangles T with $\overline{AC} \leq 0.992076$ are nonmonochromatic in the hexagoncoloring in Figure 28, with the lengths as calculated above.*

Theorem 32. *Let m and n be odd integers. Then there exists a 2-coloring of the rational plane, such that the distance between two points with the same color is not $\sqrt{\frac{n}{m}}$.*

Theorem 33. *There is a 2-coloring of the rational plane, such that, there is no pair of points with distance $\sqrt{\frac{2n}{m}}$, where n and m are odd integers, that have the same color.*

Theorem 35. *For every positive real number d, there exists a 2-coloring of the rational plane, such that the distance between two points with the same color is not d.*

Theorem 36. *For every triangle T, there exist a 2-coloring of the rational plane, such that T is nonmonochromatic.*

Theorem 37. *There exists a 5-coloring of a strip with height 1, such that there does not exist two points with the same color, that are distance 1 apart.*

Problems

Problem 1. *What is the smallest number c, so that for any triangle T, there exists a coloring of the Euclidean plane with c colors, such that it is not possible, that all vertices of T have the same color if we embed it in the colored plane?*

Problem 2. *Given a triangle T. What is the smallest number c, so that we can color the Euclidean plane with $c(T)$ colors, such that it is not possible, that all vertices of T have the same color if we embed it in the colored Euclidean plane?*

Problem 3. *How many colors are at least needed, such that there exists a coloring of the Euclidean plane, so that no two points with distance 1 have the same color?*

Problem 4. *Let $c \in \mathbb{N}$. What is the largest height $h(c)$ of a strip (depending on c), such that there exists a c-coloring, so that there does not exist a pair of points of the strip with distance 1, that have the same color.*