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Risk-minimizing Hedging Strategies in Insurance

MASTER'S THESIS

to achieve the university degree of

Diplom-Ingenieurin

Master's degree programme: Mathematics

submitted to

Graz University of Technology

Supervisor

Priv.-Doz. Dipl.-Ing. Dr.techn. Stefan Michael Thonhauser

Institut für Statistik (5060)

Graz, März 2018

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Abstract

Møller (1998, 2001) extended Föllmer and Sondermann's (1986) theory of risk-minimization in incomplete markets to determine the optimal financial portfolio to hedge the insurers risk when selling unit-linked life insurance contracts. In the first part the unique risk-minimizing financial strategy for a so called contingent T-claim, arising from pure endowment and term insurance contracts, will be specified. We will see that the insurer is able to eliminate the intrinsic risk completely, when the financial model is extended by a reinsurance possibility. In the second part, the simple world of T-claims is enlarged to general payment streams and finally some numeric results are presented.

Contents

Chapter 1

Introduction

At the beginning of a traditional life insurance contract an appointed benefit payment stream is settled between the insurer and the policy holder. An example would be a pure endowment insurance, which pays at a fixed time (the end of the contract) a previous specified amount of money, given the insured person[1] is still alive at the time of maturity. Other examples would be a term insurance, that pays a predefined amount of money at the time of death of the insured person, if the person dies before maturity of the contract, or an endowment insurance, which is a combination of both types mentioned before.

As one can see, the only stochastic part in traditional life insurance contracts is given by the mortality of the insured person. Usually the probabilities of different ages to die within one year is assumed to be given by a life table, meaning that the mortality risk is considered as diversifiable and expected.

This fact changes if one considers unit-linked or index-linked life insurance contracts. These types of contracts do not pay a fixed insurance benefit at a predefined (maybe stochastic) point in time, but consider the benefits in dependence of the price process of a specified stock, fund or index. Benefits typically are defined as the price of the stock or the maximum of the price of the stock and some guarantee. As for traditional life insurance contracts guaranteed returns are very low at the moment, unit-linked contracts are getting more popular recently.

Aase and Persson (1994) analysed unit-linked life insurance contracts

[1] In general an insurance contract can include three different persons: The one that concludes the contract, another one, whose lifetime is insured and a third person, that will receive the benefit. For reasons of simplicity in this thesis it is assumed that an insurance contract concerns only one person, i.e., the policy holder equals the insured person and equals the benefited.

under the pricing principle of equivalence implicitly assuming the insured person as risk neutral with respect to mortality. This means that the death-probabilities of the policy holder behave like the ones given in a life table. Therefore, these kinds of contracts can be priced and perfectly hedged by traditional no-arbitrage theory of financial mathematics. But when taking the the pathwise considered stochastics of mortality into account, the market no longer can be considered as complete. As a result the insurer is not able to hedge his risk perfectly by trading assets on the market and is left with some minimum obtainable risk, the so called intrinsic risk process.

Based on two articles by Møller (1998, 2001), this thesis deals with the problem of finding risk-minimizing hedging strategies for general insurance payment streams that depend on a financial asset and consider the insured event as stochastic. The theory of risk-minimization in incomplete markets introduced by Föllmer and Sondermann (1986) will be used to find those risk-minimizing hedges. The remaining risk of the insurer is going to be measured by the expected value with respect to the risk neutral martingale measure of the square of the difference of the payable benefits and the investment gains.

The first part of the current thesis will be based on Møller (1998) and will discuss simple claims that arise from unit-linked life insurance contracts where the life length of the insured person is assumed to be driven by some stochastic process and the benefits are payable at predefined points in time only. Furthermore we will assume that premiums are paid as single premiums. In this part of the paper risk-minimizing trading strategies will be obtained for pure endowment insurance contracts as well as for term insurance contracts. In both situations there will be an intrinsic risk process left for the insurer, but when considering freely traded reinsurance contracts, the direct insurer will be able to eliminate his risk completely.

After understanding the theory of risk-minimization of simple unit-linked life insurance contracts, we will be able to broadly generalize the insurance claims processes in the second part of the recent paper, based on Møller (2001). These generalized payment processes include insurance liabilities where the insurer has to face more than one possible claim in the duration of the contract. Since one can not directly use the concept of risk-minimization of Föllmer and Sondermann (1986) we need to extend their theory. After deducing the necessary results we will derive risk-minimizing hedging strategies for general unit-linked life insurance contracts and furthermore a non-life insurance contract will be considered where the claim size distribution depends on the development of the price process of a specified traded asset.

The financial market will in both cases consist of two financial products only, one bond, which will be assumed to be (locally) risk-free and used for discounting, and one stock, that is the risky asset. The discounted price process of the asset will be given by a (locally) square-integrable local martingale. There will be neither transaction costs nor liquidity constraints.

The present thesis is organized as follows: In Chapter 2 the financial model as well as the simple life insurance model are defined and briefly some required results from financial mathematics are reviewed. The theory of risk-minimization is introduced in Chapter 3 and afterwards risk-minimizing hedging strategies for simple unit-linked pure endowment and term insurance contracts are obtained in Chapter 4. In Chapter 5 we define generalized payment streams and derive the required extension of the traditional concept of risk-minimization. With this result we are able to obtain risk-minimizing hedging strategies for general unit-linked life insurance contracts. And finally in Chapter 6 some numerical results are presented.

Chapter 2

The financial market

In this chapter the financial model, that defines the environment of the two considered financial products and the desired hedging strategy, is introduced. In contrast to the insurance model, that will be generalized in the course of the present thesis, the financial model will stay mostly the same (it will just be slightly expanded).

Additionally, we will review several basic definitions and results from financial mathematics, that will be necessary to construct the forthcoming theory.[2]

2.1 The financial assets

We consider a fixed finite time horizon, that will be denoted by T and a given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, that satisfies the so called **usual conditions**.

Definition 2.1.1. *[9] Let $\mathcal{F}$ be a filtration.*

(i.) $\mathcal{F}$ is called ***right continuous*** *if $\mathcal{F} = \mathcal{F}^+$.*[3]

(ii.) $\mathcal{F}$ is called ***$\mathbb{P}$-complete*** *if $\mathcal{F}_0$ (and hence all $\mathcal{F}_t$) contain the sets of probability zero, i.e., $\mathcal{N} \subseteq \mathcal{F}_0$, where $\mathcal{N} := \{A \in \mathcal{A} | \mathbb{P}(A) = 0\}$.*

(iii.) $\mathcal{F}$ satisfies the ***usual conditions*** *if it is right continuous and $\mathbb{P}$-complete.*

[2]Most of these basic definitions, technical results and some interpretations are based on Karatzas and Shreve (1998) [5], Øksendal (2013) [11] and the lecture notes from "Stochastic Analysis" [9] and "Advanced Financial Mathematics" by Wolfgang Müller [10] lectured at TU Graz.

[3]$\mathcal{F}_t^+ := \bigcap_{s>t} \mathcal{F}_s$, it describes the information at time $t + dt$ with infinitesimal dt.

The market consists of two freely traded assets only: a (risk-free) bond with price process $B = (B_t)_{0\leq t\leq T}$ and a (risky) stock with price process $S = (S_t)_{0\leq t\leq T}$, i.e., at any time $t \in [0, T]$ both assets can be traded by prices B_t and S_t respectively. Both price processes are defined on the given probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and are driven by the following dynamics

$$dS_t = \alpha(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \tag{2.1.1}$$

$$dB_t = r(t, S_t)B_t dt, \tag{2.1.2}$$

where $(W_t)_{0\leq t\leq T}$ denotes a standard Brownian motion on the time interval $[0, T]$, the coefficients are defined as functions in time and the asset price, i.e., $\alpha, \sigma, r : [0, T] \times \mathbb{R} \to \mathbb{R}$ and the initial values are given by $B_0 = 1$ and $S_0 > 0$.

Furthermore, assume B_t to be almost surely positive at every point in time, i.e., $B_t > 0$ $\mathbb{P}$-a.s. This allows us to define the **discounted stock price** process as follows:

Definition 2.1.2. *The **discounted price** of the stock at time t is given by*

$$S_t^* := \frac{S_t}{B_t}.$$

Let us introduce the new augmented filtration $\mathcal{G} = (\mathcal{G}_t)_{0\leq t\leq T}$, which represents the economic information on $[0, T]$, i.e., $\mathcal{G}_t$ is contained in $\mathcal{F}_t$ and generated by the economy and all nullsets and therefore given by

$$\mathcal{G}_t = \sigma\{\{(S_u, B_u) \mid u \leq t\}, \mathcal{N}\} = \sigma\{S_u \mid u \leq t\}$$

where the last equality holds since B_u depends on the price process of the stock, S.

Next we want to find explicit expressions for the prices of the assets. Obviously the solution of the differential equation (2.1.2) defining the price process of the bond B is given by

$$B_t = e^{\int_0^t r_u du}, \tag{2.1.3}$$

whereat $\int_0^T r_u du$ is assumed to be finite.[4] To ensure an existing and up to indistinguishably unique solution of the stochastic differential equation (2.1.1) that defines the stock price, one has to make some regularity assumptions for the diffusion coefficient σ and the drift α. For detailed information see Appendix A and Øksendal (2013). We assume these regularity conditions to be

[4] In the following we will use the shorter form r_t instead of writing both arguments $r(t, S_t)$. The same holds for the other coefficient functions σ and α.

fulfilled in the rest of the thesis and, therefore, are able to give a reasonable integral[5] expression for the stock price itself and its discounted price process

$$S_t = S_0 e^{\int_0^t \left(\alpha_u - \frac{\sigma_u^2}{2}\right) du + \int_0^t \sigma_u dW_u}, \tag{2.1.4}$$

$$S_t^* = S_0 e^{\int_0^t \left(\alpha_u - r_u - \frac{\sigma_u^2}{2}\right) du + \int_0^t \sigma_u dW_u}. \tag{2.1.5}$$

The drift α can be considered as the *mean return* of the stock S and the diffusion coefficient σ represents the *standard deviation* of the rate of return. A very intuitive interpretation can be given for the coefficient function r, that is called the *short rate of interest*, since it represents the risk-free interest rate an investor receives for holding the bond over time. Let us define the process $\nu_t = \frac{\alpha_t - r_t}{\sigma_t}$ which denotes the *market price of risk*. "It can be interpreted as an annual risk premium (in units of the volatility σ) for the holder of the risky stock in comparison to the holder of the risk-less bond."[10].

Example 2.1.3. To give an example of a financial model we consider the coefficients α, σ and r to be constant. Then one obtains the famous *Black Scholes model*, where the two financial assets are given by

$$S_t = S_0 e^{(\alpha - \frac{\sigma^2}{2})t + \sigma W_t},$$
$$B_t = e^{rt}.$$

Here the stock price is modelled by a geometric Brownian motion.

2.2 The equivalent martingale measure

To investigate a meaningful financial model it is indispensable to define an equivalent martingale measure. We denote this new probability measure by $\mathbb{P}^*$ and want it to fulfil two conditions: First it should be equivalent to the given measure $\mathbb{P}$ and secondly the discounted stock price should be a martingale under $\mathbb{P}^*$. In the following we obtain the required new measure in two steps:

1. An equivalent measure is obtained directly by definition and an application of **Girsanov's Transformation Theorem**.

2. The martingale property is shown with the help of the **Novikov condition**.

Before starting with the definition of an equivalent measure we remember the Girsanov Transformation Theorem.

[5]Stochastic integration is meant to be in the sense of Itô.

Theorem 2.2.1. *[9] Let W be a Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P})$ and Y an W-integrable process. Further assume that*

$$U_s := e^{\int_0^s Y_u dW_u - \frac{1}{2}\int_0^s Y_u^2 du}, \quad 0 \leq s \leq t \tag{2.2.1}$$

is a $\mathbb{P}$-martingale. Then the density process $\frac{d\mathbb{P}_t^*}{d\mathbb{P}} = U_t$ defines a probability measure on $\mathcal{G}_t$, $\mathbb{P}_t^*$, which is equivalent to $\mathbb{P}$. The process $W^* = (W_s^*)_{s\leq t}$ defined by

$$W_s^* := W_s - \int_0^s Y_u du, \quad s \leq t$$

is a standard Brownian motion with respect to $\mathbb{P}_t^*$.

By setting $Y_u := -\frac{\alpha_u - r_u}{\sigma_u}$ we get the proper equivalent measure $\mathbb{P}^*$

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} := U_T := e^{\int_0^T \frac{\alpha_u - r_u}{\sigma_u} dW_u - \frac{1}{2}\int_0^T \left(\frac{\alpha_u - r_u}{\sigma_u}\right)^2 du}, \tag{2.2.2}$$

such that the discounted price process S^* given by Definition 2.1.2 is a $\mathbb{P}^*$-martingale. The martingale property can be checked by using the Novikov condition [9], which states that if $(Y_s)_{s\leq t}$ fulfils

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^t Y_u^2 du}\right] < \infty, \tag{2.2.3}$$

then $(U_s)_{s\leq t}$ is a martingale. In the current setting this would mean that we need to request the parameter functions to satisfy that

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^t \left(-\frac{\alpha_u - r_u}{\sigma}\right)^2 du}\right] < \infty.$$

So the new measure $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ and fulfils the martingale property. Therefore $\mathbb{P}^*$ given by (2.2.2) is the required equivalent martingale measure, which will be used for measuring the intrinsic risk process and finding the optimal hedging strategy in the rest of the thesis.

2.3 Further definitions

Finally the financial model is completed by defining the concept of a trading strategy in the financial model, its value and cost processes. Furthermore, we consider the risk process that depends on such a trading strategy and a claim. This financial model was considered by Møller (1998, 2001), who based this concept on the theory introduced by Föllmer and Sondermann (1986).

In the following we will work with the space of all square-integrable $\mathcal{F}$-predictable processes given by[6]

$$\mathcal{L}^2(\mathbb{P}_S^*) = \left\{ \xi | \xi \text{ is } \mathcal{F}\text{-predictable and satisfies } \mathbb{E}^* \left[\int_0^T \xi_u^2 d[S]_u \right] < \infty \right\}. \tag{2.3.1}$$

For a precise definition of **predictability** see Müller [9] Definition 2.11. A basic example for a predictable process would be any adapted process with left-continuous paths.

Next we define the concept of trading strategies and its value processes.

Definition 2.3.1. *[7] The **value process** $\hat{V}$ of a pair $\varphi = (\xi, \eta)$ is given by*

$$\hat{V}_t^\varphi := \xi_t S_t + \eta_t B_t. \tag{2.3.2}$$

*The **deflated value process** V of a pair $\varphi = (\xi, \eta)$ is given by*

$$V_t^\varphi := \hat{V}_t^\varphi B_t^{-1} = \xi_t S_t^* + \eta_t. \tag{2.3.3}$$

With the above definition of the value process we are able to define a reasonable concept of trading strategies.

Definition 2.3.2. *[7] A **trading strategy** or **portfolio strategy** is any process $\varphi = (\xi, \eta)$ with $\xi \in \mathcal{L}^2(\mathbb{P}_S^*)$ and η $\mathcal{F}$-adapted, such that the (deflated) value process V^φ is càdlàg[7] and $V_t^\varphi \in \mathcal{L}^2(\mathbb{P}^*)$ for all t.*

At any time t the portfolio (or trading strategy) $\varphi = (\xi, \eta)$ respectively ξ and η can be interpreted as number of stocks and number of bonds held at time t. The value of the portfolio at time t is given by the value process $\hat{V}_t^\varphi$ and its deflated value is obtained by discounting with the price of the risk-free asset at time t.

A special type of strategies are the so called self-financing strategies. Changes in the value of self-financing strategies are generated by changes in the underlying price processes only. Formally this property is defined as follows:

Definition 2.3.3. *[7] A trading strategy $\varphi = (\xi, \eta)$ is said to be **self-financing** if*

$$\hat{V}_t^\varphi = \hat{V}_0^\varphi + \int_0^t \xi_u dS_u + \int_0^t \eta_u dB_u, \quad \text{for all } 0 \leq t \leq T. \tag{2.3.4}$$

[6]Throughout this thesis the square brackets $[]_t$ will denote the quadratic variation of a random process at time t.

[7]A càdlàg function is a right-continuous function with existing left limits.

Obviously an equivalent definition can be given if one multiplies Equation (2.3.4) with the discounting factor B_t^{-1} and obtains

$$V_t^{\varphi} = V_0^{\varphi} + \int_0^t \xi_u dS_u^*. \tag{2.3.5}$$

The liabilities for the insurer that arise from insurance contracts are described by claims:

Definition 2.3.4. *[7] A* ***contingent claim*** *with maturity T is a random variable X that is $\mathcal{G}_T$-measurable and $\mathbb{P}^*$-square integrable. In particular*

(i) X is called ***simple claim*** *whenever $X = g(S_T)$, for some function $g : \mathbb{R}_+ \to \mathbb{R}$.*

(ii) X is called ***attainable****, if there exists a self-financing strategy φ such that $\hat{V}_T^{\varphi} = X$ $\mathbb{P}$-almost surely (a.s.), i.e., X can be perfectly hedged or perfectly duplicated.*

If all contingent claims are attainable the market is referred to as ***complete****, otherwise it is said to be* ***incomplete****.*

If $\varphi = (\xi, \eta)$ is a self-financing X-duplicating strategy, then one gets a new representation for the contingent claim X by (2.3.4) combined with (2.3.2) considered at the final point in time T

$$X = \xi_0 S_0 + \eta_0 + \int_0^T \xi_u dS_u + \int_0^T \eta_u dB_u. \tag{2.3.6}$$

Our next aim is to guarantee a "fair" market and therefore the concept of arbitrage is introduced. In an arbitrage-free market it is not possible to find a trading strategy that leads to profit without risk.

Definition 2.3.5. *[7] A self-financing strategy φ is an* ***arbitrage*** *if*

(a.i) $\hat{V}_0^{\varphi} < 0$ and

(a.ii) $\hat{V}_T^{\varphi} \geq 0$ $\mathbb{P}$-a.s. and.

Or equivalently

(b.i) $\hat{V}_0^{\varphi} \leq 0$ and

(b.ii) $\hat{V}_T^{\varphi} \geq 0$ $\mathbb{P}$-a.s. and

(b.iii) $\mathbb{P}\left(\hat{V}_T^{\varphi} > 0\right) > 0$.

The present financial model given by the assets (2.1.1) and (2.1.2) and the economic filtration $\mathcal{G}$ is arbitrage-free and complete. For detailed information about the theory of financial models see Karatzas and Shreve (1998).

In consistency with the common theory of financial mathematics the fair price at time t of a simple claim is defined as the discounted conditional expectation of the payout under the martingale measure $\mathbb{P}^*$ with respect to the economic information $\mathcal{G}_t$.

Definition 2.3.6. *The* ***arbitrage-free price process*** $(F(t, S_t))_{0\leq t\leq T}$ *of a simple claim that specifies the payment* $g(S_T)$ *at time* T *is given by*

$$F(t, S_t) = \mathbb{E}^* \left[e^{-\int_t^T r_u du} g(S_T) | \mathcal{G}_t \right]. \tag{2.3.7}$$

Øksendal (2013) (Theorem 7.1.2) shows that the processes S_t and B_t satisfy the Markov property for Itô diffusions, since they are given by the differential equations (2.1.1) - (2.1.2). And because of the Markov property of the prices S_t and B_t one sees that

$$F(t, S_t) = \mathbb{E}^* \left[e^{-\int_t^T r_u du} g(S_T) | S_t \right].$$

Remark 2.3.7. Furthermore it can be shown that the arbitrage-free price process $(F(t, S_t))_{0\leq t\leq T}$ of the simple claim $g(S_T)$ fulfils a generalization of the Black-Scholes differential equation given by the following partial differential equation with boundary condition

$$\begin{aligned} &- r(t,s)F(t,s) + F_t(t,s) + r(t,s)sF_s(t,s) + \frac{1}{2}\sigma(t,s)^2 s^2 F_{ss}(t,s) = 0, \\ &F(T,s) = g(s), \end{aligned}$$

where F_t and F_s denote the partial derivatives of F with respect to t respectively s and F_{ss} stands for the second derivative of F with respect to s.

Chapter 3

The theory of risk-minimization in incomplete markets

After the construction of a complete and arbitrage-free financial market we want to combine this model with an insurance payment stream. This leads to incompleteness in our model and to the problem of hedging general claims, since in an incomplete market it can no longer be guaranteed that a perfect duplicating self-financing trading strategy is found for every claim. Therefore we want to use the theory of risk-minimization in incomplete markets introduced by Föllmer and Sondermann (1986).

3.1 Föllmer and Sondermann's mean squared error minimization

Föllmer and Sondermann considered elementary claims of the form

$$A_t = -\kappa + I_{t \geq T} H, \quad 0 \leq t \leq T, \tag{3.1.1}$$

for some constant $\kappa \in \mathbb{R}$ and $H \in \mathcal{L}^2(\mathbb{P}^*)$, i.e., payments take place only at two points in time. From the insurer's point of view it can be interpreted as a premium income κ at time 0 and an insurance claim of amount H payable at time T.

In the first part of the present thesis we will work with basic claims in the form of (3.1.1) and hence the theory of Föllmer and Sondermann (1986) can be applied directly. Therefore their concept of risk-minimization due to squared errors is reviewed in this chapter, before we discuss the extension of their theory by Møller (2001) in the second part of the thesis.

We start with defining two new process: The cost process of a trading strategy, which is the discounted difference between the value of the portfolio and the accumulated income from the stock, and the risk process, that is given by the conditional expectation of the squared future costs.

Definition 3.1.1. *[7] The **cost process** C^φ associated with the trading strategy φ is defined by*

$$C_t^\varphi = V_t^\varphi - \int_0^t \xi_u dS_u^*, \quad 0 \leq t \leq T. \tag{3.1.2}$$

Definition 3.1.2. *[7] The **risk process** R^φ associated with the trading strategy φ is defined by*

$$R_t^\varphi = \mathbb{E}^* \left[(C_T^\varphi - C_t^\varphi)^2 | \mathcal{F}_t \right], \quad 0 \leq t \leq T. \tag{3.1.3}$$

Combining the definition of self-financing strategies (2.3.5) and Definition 3.1.1 we get the following equivalence:

$$\begin{aligned} \varphi \text{ is self-financing } &\Longleftrightarrow V_t^\varphi = V_0^\varphi + \int_0^t \xi_u dS_u^* \\ &\Longleftrightarrow C_t^\varphi + \int_0^t \xi_u dS_u^* = V_0^\varphi + \int_0^t \xi_u dS_u^* \\ &\Longleftrightarrow C_t^\varphi = V_0^\varphi = C_0^\varphi, \end{aligned}$$

meaning that a trading strategy is self-financing if and only if the corresponding cost process has constant paths.

Föllmer and Sondermann (2001) weakened the concept of self-financing strategies:

Definition 3.1.3. *[4] A strategy $\varphi = (\xi, \eta)$ is called **mean-self-financing** if the corresponding cost process is a martingale with respect to $\mathcal{F}$ and $\mathbb{P}^*$.*

Obviously every self-financing strategy is also mean-self-financing (see equivalence relation above). In contrast to the extension of the definition of self-financing strategies we need to restrict the concept of general trading strategies to those fulfilling a special property.

Definition 3.1.4. *[4] Let $H \in \mathcal{L}^2(\mathbb{P}^*)$ be a claim. A strategy is called **admissible** with respect to H if its value process has terminal value $V_T = H$ $\mathbb{P}^*$-a.s.*

Thus, an admissible strategy hedges the claim, but the hedger is left with costs defined by C_T^φ, which are not necessarily equal to $C_0^\varphi = V_0^\varphi$.

In the following a result is obtained, which specifies optimal admissible strategies, that minimize the mean squared error R_0^φ. Therefore we need to define a new process, the so called intrinsic value process.

Definition 3.1.5. *[7] The* ***intrinsic value process*** V_t^* *is defined by*

$$V_t^* := \mathbb{E}^* \left[H | \mathcal{F}_t \right].$$

For constructing the optimal trading strategies we need to apply the **Galtchouk-Kunita-Watanabe** decomposition of martingales.

Theorem 3.1.6. *[2] Let V^* be a real local martingale and S^* a local martingale with values in $\mathbb{R}^d$. Then there exists a process ξ^H with $\mathbb{P}^*\Big(\int_0^t \xi_u^H dS_u^* < \infty \Big)$ for all $0 \leq t \leq T$ and a local martingale L^H, which is orthogonal to S^* and has zero mean, such that*

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + L_t^H, \quad 0 \leq t \leq T. \tag{3.1.4}$$

This decomposition is unique.

Remark 3.1.7. Two square-integrable martingales M_1 and M_2 are called **orthogonal** if their product $M_1 M_2$ is again a martingale.

Under the martingale measure $\mathbb{P}^*$ the intrinsic value process V^* is a martingale by definition and the discounted stock price S^* is a martingale due to the construction of $\mathbb{P}^*$. Therefore, Theorem 3.1.6 can be used to find a decomposition of the intrinsic value process V^* as given in (3.1.4). With help of the process ξ^H Föllmer and Sondermann (1986, Theorem 1) proved the following:

Theorem 3.1.8. *[4] An admissible strategy $\varphi = (\xi, \eta)$ has minimal variance*

$$\mathbb{E}^* \left[(C_T^\varphi - \mathbb{E}^* \left[C_T^\varphi \right])^2 \right] = \mathbb{E}^* \left[(L_T^H)^2 \right]$$

if and only if $\xi = \xi^H$.

Proofing Theorem 3.1.8 we need the well-known **Itô** isometry

$$\mathbb{E}^* \left[\left(\int_0^t Y_s dS_s^* \right)^2 \right] = \mathbb{E}^* \left[\int_0^t Y_s^2 d[S^*]_s \right], \tag{3.1.5}$$

where $Y \in \mathcal{L}^2(B)$ and B is a Brownian motion with respect to $\mathbb{P}^*$.

Proof. Let φ be an admissible strategy for the claim H. Since $\int_0^t \xi_u dS_u^*$ has expectation 0 we get for the cost process

$$\mathbb{E}^* \left[C_T^\varphi \right] = \mathbb{E}^* \left[V_T^\varphi \right] = \mathbb{E}^* \left[H \right]. \tag{3.1.6}$$

Furthermore it holds that

$$C_T^\varphi = V_T^\varphi - \int_0^T \xi_u dS_u^* \overset{\text{Def.(3.1.4)}}{=} H - \int_0^T \xi_u dS_u^* \tag{3.1.7}$$

and since H is $\mathcal{F}_T$-measurable by definition

$$H = \mathbb{E}^*\left[H|\mathcal{F}_T\right] = V_T^*. \tag{3.1.8}$$

Using (3.1.4) together with (3.1.6), (3.1.7) and (3.1.8) we obtain

$$\begin{aligned}
\mathbb{E}^*\left[(C_T^\varphi - \mathbb{E}^*\left[C_T^\varphi\right])^2\right] &= \mathbb{E}^*\left[(C_T^\varphi - \mathbb{E}^*\left[H\right])^2\right] = \\
&= \mathbb{E}^*\left[\left(H - \int_0^T \xi_u dS_u^* - \mathbb{E}^*\left[H\right]\right)^2\right] = \\
&= \mathbb{E}^*\left[\left(V_T^* - \int_0^T \xi_u dS_u^* - \mathbb{E}^*\left[H\right]\right)^2\right] = \\
&= \mathbb{E}^*\left[\left(\mathbb{E}^*\left[H\right] + \int_0^T (\xi_u^H - \xi_u) dS_u^* + L_T^H - \mathbb{E}^*\left[H\right]\right)^2\right] = \\
&= \mathbb{E}^*\left[\left(\int_0^T (\xi_u^H - \xi_u) dS_u^*\right)^2\right] + \mathbb{E}^*\left[(L_T^H)^2\right] = \\
&= \mathbb{E}^*\left[\int_0^T (\xi_u^H - \xi_u)^2 d[S^*]_u\right] + \mathbb{E}^*\left[(L_T^H)^2\right].
\end{aligned}$$

In the last two steps we used the orthogonality of L^H and S^* and the Itô isometry (3.1.5). Thus the minimum $\mathbb{E}^*\left[(L_T^H)^2\right]$ is obtained if and only if $\xi = \xi^H$. □

Remark 3.1.9. According to Møller (1998), [7], "if the number of bonds held at time 0 is determined such that the initial value of the portfolio equals $\mathbb{E}\left[H\right]$, i.e.,

$$\eta_0 = \mathbb{E}^*\left[H\right] - \xi_0 S_0^{*}\text{''},$$

we get

$$\begin{aligned}
R_0^\varphi = \mathbb{E}^*\left[(C_T^\varphi - C_0^\varphi)^2\right] &= \mathbb{E}^*\left[(C_T^\varphi - V_0^\varphi)^2\right] = \mathbb{E}^*\left[(C_T^\varphi - \xi_0 S_0^* - \eta_0)^2\right] = \\
= \mathbb{E}^*\left[(C_T^\varphi - \xi_0 S_0^* - \mathbb{E}^*\left[H\right] + \xi_0 S_0^*)^2\right] &\overset{(3.1.6)}{=} \mathbb{E}^*\left[(C_T^\varphi - \mathbb{E}^*\left[C_T^\varphi\right])^2\right].
\end{aligned}$$

Hence minimizing the variance $\mathbb{E}^*\left[(C_T^\varphi - \mathbb{E}^*\left[(C_T^\varphi\right])^2\right]$ is equal to minimizing R_0^φ and the variance can be interpreted as the minimal obtainable risk.

Remark 3.1.10. Theorem 3.1.8 does not characterize a unique risk-minimizing trading strategy for a special claim H, but it gives an entire class of optimal portfolios, which all differ in the number of bonds held at time t for $0 \leq t < T$. This consequence follows from the definition of the mean squared error R_0^{φ}, that depends only on the cost process at time T. Therefore, "we can draw no conclusion concerning the process $\eta = (\eta_t)_{0\leq t\leq T}$ except that it must make the strategy admissible, i.e.,

$$\eta_T = H - \xi_T S_T^*."$$

see Föllmer and Sondermann (1986), [4].

Example 3.1.11. To illustrate the previous result Föllmer and Sondermann (1986, Example 1) discuss the most natural example when considering a trading strategy that is self-financing during $(0, T)$ and rebalance the portfolio only at time of maturity T to obtain an admissible variance-minimizing strategy. This strategy is given by

$$\begin{aligned}
\xi_t &= \xi_t^H, \quad 0 \leq t \leq T, \\
\eta_t &= \mathbb{E}^*[H] + \int_0^t \xi_u dS_u^* - \xi_t S_t^*, \quad 0 \leq t < T, \\
\eta_T &= H - \xi_T S_T^*.
\end{aligned}$$

Hence, this trading strategy is self-financing , i.e., no extra investment has to be made up to but excluding time of maturity T. To determine the payment stream (either an investment of the insurer or financial gain) at time T, which has to be made in order to obtain an admissible strategy, consider on the one hand

$$\eta_T = H - \xi_T S_T^* = V_T^* - \xi_T S_T^* = \mathbb{E}^*[H] + \int_0^T \xi_u dS_u^* + L_T^H - \xi_T S_T^*$$

and on the other hand we obtain from the definition of η_t

$$\eta_{T^-} = \mathbb{E}^*[H] + \int_0^{T^-} \xi_u dS_u^* - \xi_{T^-} S_{T^-}^* = \mathbb{E}^*[H] + \int_0^T \xi_u dS_u^* - \xi_T S_T^*.$$

In the last step we used the fact that η_t is càdlàg as a trading strategy (see Definition 2.3.2).

Combining these expressions we see for the payment stream at time T

$$\eta_T - \eta_{T^-} = L_T^H.$$

This means, that at time of maturity T the insurance company has to face the gain/loss L_T^H, that represents the difference between the value of the financial portfolio V_{T-}^{φ} and the claim H.

This simple trading strategy indeed minimizes the initial intrinsic risk, but at any time t within the insurance period the value of the insurance claim V_t^* will in general not be equal to the value of the financial portfolio V_t^{φ}, as Møller (1998) pointed out the disadvantage of this strategy. "Since this difference may be substantial due to adverse development within the insurance portfolio, one should at least require that the value of the portfolio equals V_t^* in order to enhance the solvency of the insurer", see [7]

Theorem 3.1.8 does not give a precise characterization of the optimal trading strategy, when minimizing the mean squared error. Therefore, as a next result Föllmer and Sondermann determined strategies, which minimize the risk in a sequential sense, i.e., at every point in time $0 \leq t \leq T$. As we will see it will be sufficient to guarantee that the value of the trading strategy equals the value of the claim V_t^* at any point in time t, as suggested in the previous example.

For this purpose let us define a class of trading strategies related to a given portfolio φ.

Definition 3.1.12. *[4] $\tilde{\varphi}$ is called an* ***admissible continuation*** *of φ at time t if $\tilde{\varphi}$ coincides with φ in the interval $[0, t)$ and has the same terminal value like φ, i.e., $V_T^{\tilde{\varphi}} = V_T^{\varphi}$.*

Definition 3.1.13. *[4] A strategy φ is called* ***risk-minimizing*** *if φ at any time minimizes the remaining risk, i.e., for any $0 \leq t < T$, we have*

$$R_t^{\varphi} \leq R_t^{\tilde{\varphi}}, \quad \mathbb{P}^* - a.s.,$$

for every admissible continuation $\tilde{\varphi}$ of φ at time t.

The next property of admissible risk-minimizing strategies was essential for Föllmer and Sondermann when proofing the final result of their theory of risk minimization.

Lemma 3.1.14. *[4] An admissible risk-minimizing strategy is mean-self-financing.*

Proof. Let $\varphi = (\xi, \eta)$ be an admissible trading strategy. For fixed time $0 \leq t_0 \leq T$ define a new strategy $\tilde{\varphi} = (\xi, \tilde{\eta})$ by

$$\tilde{\eta}_t := \begin{cases} \eta_t & \text{if } t < t_0 \\ \tilde{C}_t + \int_0^t \xi_u dS_u^* - \xi_t S_t^* & \text{if } t_0 \leq t \leq T, \end{cases}$$

where $\tilde{C}_t := \mathbb{E}^*\left[C_T^\varphi|\mathcal{F}_t\right]$ for $0 \leq t \leq T$. Then $\tilde{\varphi} = (\xi, \tilde{\eta})$ is an admissible continuation of φ at time t_0 by construction. Note that $\tilde{C}_T = \mathbb{E}^*\left[C_T^\varphi|\mathcal{F}_T\right] = C_T^\varphi$, hence for the risk process of φ at time t_0 we get

$$\begin{aligned}\mathbb{E}^*\left[\left(C_T^\varphi - C_{t_0}^\varphi\right)^2|\mathcal{F}_{t_0}\right] &= \mathbb{E}^*\left[\left(\tilde{C}_T - C_{t_0}^\varphi - \tilde{C}_{t_0} + \tilde{C}_{t_0}\right)^2|\mathcal{F}_{t_0}\right] = \\ &= \mathbb{E}^*\left[\left(\tilde{C}_T - \tilde{C}_{t_0}\right)^2|\mathcal{F}_{t_0}\right] + \left(\tilde{C}_{t_0} - C_t^\varphi\right)^2.\end{aligned}$$

Consequently, φ can be risk-minimizing if and only if $C_{t_0}^\varphi = \tilde{C}_{t_0}$ $\mathbb{P}^*$-a.s. for any $t_0 \leq T$. This means, that φ is mean-self-financing. □

Now we are able to formulate and proof Föllmer and Sondermann's (1986, Theorem 2) final result.

Theorem 3.1.15. *[4] There exists an unique admissible risk-minimizing strategy $\varphi = (\xi, \eta)$ given by*

$$(\xi_t, \eta_t) = (\xi_t^H, V_t^* - \xi_t^H S_t^*), \quad 0 \leq t \leq T. \tag{3.1.9}$$

For this strategy, the remaining risk at any time $t \leq T$ is given by

$$R_t^\varphi = \mathbb{E}^*\left[\left(L_T^H - L_t^H\right)^2|\mathcal{F}_t\right]. \tag{3.1.10}$$

Proof. The statement is shown in three steps:

(i) Admissibility:
The trading strategy φ fulfils the admissibility property, since for the value process at time T we obtain

$$V_T^\varphi = \xi_t S_T^* + \eta_T = \xi_T^H S_T^* + V_T^* - \xi_T^H S_T^* = V_T^* = H.$$

(ii) Risk-minimization:
Rewrite the difference of the cost process of the trading strategy φ in

the following way

$$
\begin{aligned}
C_T^\varphi - C_t^\varphi \overset{(3.1.7)}{=} H - \int_0^T \xi_u dS_u^* - V_t^\varphi + \int_0^t \xi_u dS_u^* \overset{(3.1.8)}{=}\\
= V_T^* - \int_0^T \xi_u dS_u^* - V_t^\varphi + \int_0^t \xi_u dS_u^* \overset{(3.1.4)}{=}\\
= \mathbb{E}^*[H] + \int_0^T \xi_u^H dS_u^* + L_T^H - \int_0^T \xi_u dS_u^* - V_t^\varphi + \int_0^t \xi_u dS_u^* =\\
= \int_t^T \xi_u^H dS_u^* + L_T^H - \int_t^T \xi_u dS_u^* - V_t^\varphi +\\
+ \mathbb{E}^*[H] + \int_0^t \xi_u^H dS_u^* + L_t^H - L_t^H \overset{(3.1.4)}{=}\\
= \int_t^T (\xi_u^H - \xi_u) dS_u^* + L_T^H - L_t^H - V_t^\varphi + V_t^*.
\end{aligned}
$$

Using this expression for the cost process together with the orthogonality of L^H and S^* we obtain for the remaining risk process

$$
\begin{aligned}
R_t^\varphi &= \mathbb{E}\left[(C_T^\varphi - C_t^\varphi)^2 \,|\mathcal{F}_t\right] =\\
&= \mathbb{E}\left[\left(\int_t^T (\xi_u^H - \xi_u) dS_u^*\right)^2 |\mathcal{F}_t\right] + \mathbb{E}\left[\left(Ł_T^H - L_t^H\right)^2 |\mathcal{F}_t\right] + (V_t^* - V_t^\varphi)^2 =\\
&= \mathbb{E}\left[\int_t^T (\xi_u^H - \xi_u)^2 d[S^*]_u |\mathcal{F}_t\right] + \mathbb{E}\left[\left(Ł_T^H - L_t^H\right)^2 |\mathcal{F}_t\right] + (V_t^* - V_t^\varphi)^2
\end{aligned}
$$

This shows that the strategy φ defined in (3.1.9) minimizes the remaining risk process and $R_t^\varphi = \mathbb{E}\left[\left(Ł_T^H - L_t^H\right)^2 |\mathcal{F}_t\right]$.

(iii) <u>Uniqueness:</u>

Consider any admissible risk-minimizing portfolio $\tilde{\varphi} = (\tilde{\xi}, \tilde{\eta})$. Theorem 3.1.8 together with (3.1.10) at time 0 implies $\tilde{\xi} = \xi^H$.

Furthermore by Lemma (3.1.14), $\tilde{\varphi}$ is mean-self-financing and therefore its value process $V^{\tilde{\varphi}}$ is a martingale, i.e.,

$$\mathbb{E}^*\left[V_T^{\tilde{\varphi}} | \mathcal{F}_t\right] = V_t^{\tilde{\varphi}}. \tag{3.1.11}$$

On the other hand, $\tilde{\varphi}$ is admissible by assumption and so

$$\mathbb{E}^*\left[V_T^{\tilde{\varphi}} | \mathcal{F}_t\right] = \mathbb{E}^*\left[H | \mathcal{F}_t\right]. \tag{3.1.12}$$

Combining (3.1.11) with (3.1.12) we get

$$V_t^{\tilde{\varphi}} = \mathbb{E}^* \left[H | \mathcal{F}_t \right].$$

Hence, $\tilde{\eta}_t = V_t^* - \xi_t^H S_t^*$. Thus, the risk-minimizing admissible strategy given in (3.1.9) is unique.

□

We call the risk process associated with the risk-minimizing strategy, that is determined in Theorem 3.1.15, the **intrinsic risk process**.

Chapter 4

First application: Unit linked life insurance contracts with single premium

After defining the financial model in Chapter 2 and discussing concepts of risk-minimization in incomplete markets by Föllmer and Sondermann (1986) in the last chapter, we will now apply the obtained results. First we need to define an insurance model, which corresponds to the model introduced by Møller (1998). This model will be rather simple in this first part of applications, before we go on to more advanced general insurance payment streams in chapter 5.

Møller (1998) considered two different types of unit-linked life insurance contracts, the pure endowment and the term insurance, both with single premiums and contingent claims payable at a pre-specified stochastic time only. Our aim will be to hedge these claims with the risk-minimizing trading strategies by Föllmer and Sondermann (1986) obtained in the last chapter.

4.1 The insurance model with single premium and payment at one point in time only

We consider a group of x-year old persons and denote the **number of persons** in this group by l_x. The **remaining lifetime** of the x-year old individuals is given by a sequence of non-negative random variables $T_1, T_2, ..., T_{l_x}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity we assume that the lifetimes of persons of the same age are independently and identically distributed with absolutely continuous distribution function F, i.e.,

$$T_i \overset{i.i.d.}{\sim} F \quad \text{for all } i \in \{1, 2, ..., l_x\}.$$

Denote the **survival probability** by

$$_tp_x = \mathbb{P}(T_i > t),$$

for some $i \in \{1, 2, ..., l_x\}$. More precisely, $_tp_x$ gives the probability of an x-year old person to die in the next t years. Let μ_{x+t} be the **hazard rate function** , which can be interpreted as the probability of an $(x+t)$-year old person to die in the infinitesimal time interval $[x+t, x+t+dt]$ and is given by

$$\mu_{x+t} := -\frac{d}{dt}\ln(_tp_x) \tag{4.1.1}$$

which is obviously equivalent to

$$_tp_x = e^{-\int_0^t \mu_{x+\tau}d\tau} \tag{4.1.2}$$

and to

$$\mathbb{P}(t < T_i < t + dt) = {_tp_x}\mu_{x+t}dt. \tag{4.1.3}$$

Next we define the stochastic process $N = (N_t)_{0\leq t\leq T}$, which takes values in $\mathbb{N}_0$ and dependens on the remaining lifetimes T_i, by

$$N_t := \sum_{i=1}^{l_x} I(T_i \leq t), \tag{4.1.4}$$

where $I(T_i \leq t)$ denotes the indicator function that gives 1 if death happens before time t, $T_i \leq t$, and 0 if the complementary event occurs. The process N can be interpreted as **counting process**, where N_t denotes the number of deaths in the group up to and inclusive time t. Note that N_t is càdlàg by definition.

Let us denote the **natural filtration** generated by N with $\mathcal{H} = (\mathcal{H}_t)_{0\leq t\leq T}$. So $\mathcal{H}_t$ can be interpreted as the information corresponding to the remaining lifetimes of the insured group available at time t. Since T_i are i.i.d., it follows that N is an $\mathcal{H}$-Markov process. Next consider the **intensity process** $\lambda = (\lambda_t)_{0\leq t\leq T}$ of the counting process N, which is given by

$$\mathbb{E}\left[dN_t|\mathcal{H}_{t^-}\right] = (l_x - N_{t^-})\mu_{x+t}dt = \lambda_t dt$$

the number of persons alive just before time t multiplied with the hazard rate function. Finally define the **compensated counting process** $M = (M_t)_{0 \leq t \leq T}$ by

$$M_t := N_t - \int_0^t \lambda_u du, \tag{4.1.5}$$

that is an $\mathcal{H}$-martingale.

In the next step we combine the insurance model with the financial market. For this purpose, we give the following list of conditions, that will be assumed to be true for the period of this chapter.

1. The available information at a specific point in time is described by the filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, which is given as the smallest filtration containing both, the economic information $\mathcal{G}$ as well as the information generated by the mortality of the insurance holders $\mathcal{H}$

$$\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t.$$

2. $\mathcal{G}$ and $\mathcal{H}$ are independent.

3. At time 0 every policyholder out of the l_x individuals subscribes to an insurance contracts, which specify a single premium and benefit payments depending on the remaining life time of the person and the development of the price of the stock S. Each contract owns maturity T and pays single premium at time 0, which is denoted by π_1. Therefore, the present value of all premiums paid equals $\pi := \pi_1 l_x$.

4. "During the period $[0, T]$ the company is allowed to trade the assets B and S freely (without transaction costs, taxes and short sales restrictions) based on the complete information $\mathcal{F}$." [7]

5. In order to find a risk-minimizing hedging strategy we allow continuous rebalancing of the amounts of stocks and bonds held in the financial portfolio.[8]

We continue with presenting the two basic forms of insurance contracts that will be considered in this section.

[8]In the real world this may cause problems since a trader has to face transactions costs for shifting the financial portfolio as well as the impossibility of continuous rebalancing itself.

4.1.1 The simple pure endowment insurance

A pure endowment insurance contract specifies a single benefit payment (the insured sum) payable at the term of the contract, if the policyholder is still alive then. Since in this thesis we are interested in unit-linked contracts, we specify the insured sum as a stipulated function of the price of the stock at time T, given by $g(S_T)$, where g is assumed to be continuous. After obtaining general results we will consider specific choices of g, such as the **pure unit-linked contract** given by $g(s) = s$ and the **unit-linked contract with guarantee** with $g(s) = \max(s, K)$, see Aase and Persson (1994).

The **present value** of the insurance contract held by individual i for $i \in \{1, ..., l_x\}$ gives the obligation of the insurance company for the single contract and is defined by the discounted insured sum conditioned on the lifetime of the policyholder

$$H_i := I(T_i > T) g(S_T) B_T^{-1} = I(T_i > T) g(S_T) e^{-\int_0^T r_u du}. \tag{4.1.6}$$

Therefore the complete claim arising by the entire insurance portfolio consisting of l_x contracts is given by the discounted insured sum times the number of survivors at time T

$$H := g(S_T) B_T^{-1} \sum_{i=1}^{l_x} (T_i > T) = g(S_T) B_T^{-1} (l_x - N_T). \tag{4.1.7}$$

Note that the present value is an $\mathcal{F}_T$-measurable random variable and will be considered as contingent claim, see Definition 2.3.4. In particular the undiscounted claim HB_T depends only on S_T and N_T. We call insurance claims payable at time T and depending on S_T and N_T only **simple T-claims** (see Definition 2.3.4 point (i)), whereas all other insurance claims payable at time T are called **general T-claims**.

4.1.2 The simple term insurance

A term insurance contract pays a benefit immediately after death of the insured person if the event of death occurs before maturity T. In contrast to Section 4.1.1 we now have to use a benefit payment function $g_t = g(t, S_t)$ which depends on the price of the stock as well as on the time, since by the nature of term insurances claims can arise at any point in time $[0, T]$. As this type of claim is no longer payable at time T only (hence no T-claim), we need to set up special assumptions to guarantee T-claims. The most simple way to transform the obligations of term insurance contracts into T-claims is to assume that the benefit payment of each contract is deferred to time

T and accumulated with the interest rate r. Therefore, insurance contract i pays the amount

$$g_t(t, S_t) B_T B_t^{-1} \tag{4.1.8}$$

at time T.[9] So we can write the present value of the insurer‘s liabilities coming from an insurance portfolio of l_x term insurance contracts with benefit payments deferred and accumulated to time T as the discounted sum over all payments of the form (4.1.8) conditioned on death before maturity

$$H_T := B_T^{-1} \sum_{i=1}^{l_x} g(T_i, S_{T_i}) B_{T_i}^{-1} B_T I(T_i \leq T) = \tag{4.1.9}$$

$$= \sum_{i=1}^{l_x} \int_0^T g(u, S_u) B_u^{-1} dI(T_i \leq u). \tag{4.1.10}$$

By interchanging sum and integral in (4.1.9) and using definition (4.1.4) we can rewrite the present value as follows

$$H_T = \int_0^T g(u, S_u) B_u^{-1} dN_u. \tag{4.1.11}$$

4.1.3 Remarks

(1) By combining the pure endowment insurance with the term insurance a multitude of different types of insurance contracts can be obtained. The most popular one is the so called **endowment insurance** which pays the insured sum either at time of death if death occurs before maturity or at maturity if the insured person is still alive then. The present value of the insurer‘s obligations arising from endowment insurance contracts is simply given by the sum of (4.1.7) and (4.1.11).

(2) As we already pointed out before in Section 2 the financial model defined by the assets S and B together with the economic filtration $\mathcal{G}$ is complete. But when we consider the market (S, B) with the overall filtration $\mathcal{F}$ we obtain an incomplete model, since there are contingent claims that can not be represented by integrals over S and B as in (2.3.6). As examples for these types of contingent claims consider

[9] This way "of modifying the contracts by deferring the benefits might seem most reasonable for contracts with short time horizons, say one year. Although time horizons associate with traditional life insurance contracts are typically much longer, we will assume that benefits are deferred to the end of the insurance period." [7]

both insurance claims introduced above, the pure endowment insurance (4.1.7) and the term insurance (4.1.9). Obviously these two claims depend not only on the price processes S and B alone but also on the uncertainty arising from the mortality risk of the insured persons (described by T_i for $i \in \{1, ..., l_x\}$).

4.1.4 Family of equivalent martingale measures in the combined model

In Section 2.2 we derived the unique equivalent martingale measure in the financial model defined by (S, B) and the economic filtration $\mathcal{G}$. Since we will not work with the smaller filtration $\mathcal{G}$ but with the complete information $\mathcal{F}$ given by the economic filtration combined with the mortality filtration, we will again have to find an equivalent martingale measure for this new situation. As Møller (1998) showed, there will not be a unique equivalent martingale measure but a whole family of measures satisfying this property. This result corresponds to the fact, that the model is not complete any more (see Remark (2) in the previous subsection).

Let $h = (h_t)_{0 \le t \le T}$ be an $\mathcal{H}$-predictable process, such that $h > -1$ and $\mathbb{E}[L_T] = 1$[10] and define a **likelihood process**, see [7], L by

$$dL_t := L_{t^-} h_t dM_t$$

with initial condition $L_0 := 1$. Now we will define the new probability measure $\widehat{\mathbb{P}}$ using Girsanov's Transformation Theorem and the density process U_T given in (2.2.2) by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = U_T L_T. \tag{4.1.12}$$

Now we want to show that the new measure $\widehat{\mathbb{P}}$ is an equivalent martingale measure. Hence, we have to show that $\widehat{\mathbb{P}}$ is equivalent to $\mathbb{P}$, which is true with Girsanov's Transformation Theorem and the definition of $\widehat{\mathbb{P}}$ (4.1.12). As a second property the discounted price process S^* needs to be a martingale with respect to $\widehat{\mathbb{P}}$. For obtaining this result, let us consider **Bayes' rule** for conditional expectation, that helps us to perform a change of measure under expectations. More precisely this theorem states

[10] As before the expectation with respect to the probability measure $\mathbb{P}$ will just be denoted by $\mathbb{E}$ whereas expectations with respect to other probability measure can be matched with the corresponding measures by additional notation like * and $\hat{}$.

Theorem 4.1.1. *[9] Let $\mathbb{P} \sim \mathbb{Q}$ be equivalent with density $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$. If X is a $\mathbb{Q}$-integrable random variable and $\mathcal{F}$ a sub σ-algebra of $\mathcal{A}$, then*

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}] = \frac{\mathbb{E}^{\mathbb{P}}[XZ|\mathcal{F}]}{\mathbb{E}^{\mathbb{P}}[Z|\mathcal{F}]}, \quad \mathbb{Q}\text{-}a.s. \text{ and } \mathbb{P}\text{-}a.s..$$

We use Bayes' rule with $\widehat{\mathbb{P}}$ instead of $\mathbb{Q}$ and setting $U_T L_T$ for the density process Z and obtain for $s \leq t$

$$\widehat{\mathbb{E}}[S_t^*|\mathcal{F}_s] = \frac{\mathbb{E}[S_t^* U_T L_T|\mathcal{F}_s]}{\mathbb{E}[U_T L_T|\mathcal{F}_s]} = \frac{\mathbb{E}[S_t^* U_T|\mathcal{F}_s]}{\mathbb{E}[U_T|\mathcal{F}_s]} \frac{\mathbb{E}[L_T|\mathcal{F}_s]}{\mathbb{E}[L_T|\mathcal{F}_s]} = \mathbb{E}[S_t^*|\mathcal{F}_s] = S_s^*,$$

where we used the independence between N and (B, S) and the martingale property of S^* with respect to $\mathbb{P}^*$. Therefore, $\widehat{\mathbb{P}}$ defines an equivalent martingale measure and since the process h was not specified, we obtain a whole family of martingale measures for the financial market.

Analogously to the compensating counting process M given by (4.1.5), we define the process $M^h = (M_t^h)_{0 \leq t \leq T}$ for admissible h by

$$M_t^h := N_t - \int_0^t \lambda_u (1 + h_u) du, \tag{4.1.13}$$

that is an $(\mathcal{F}, \widehat{\mathbb{P}})$-martingale by Girsanov's Transformation Theorem and the independence between N and (B, S).

Note the following facts:

(1) Obviously the measure $\mathbb{P}^*$ can be obtained from (4.1.12) by setting $h \equiv 0$, i.e., $\mathbb{P}^*$ is one element of the family $\widehat{\mathbb{P}}$.

(2) It is important to notice that the change of measure from $\mathbb{P}$ to $\mathbb{P}^*$ does not alter the distribution of the counting process N, since the mortality distribution is independent of the financial model.

(3) Due to the fact that we derived non-uniqueness of the equivalent martingale measure, we can define more then one fair price of contingent claims, i.e., each price given by the expectation of the claim with respect to one of the equivalent martingale measures

$$\pi(\widehat{\mathbb{P}}) := \widehat{\mathbb{E}}[H]$$

gives an arbitrage-free price.

(4) The overall filtration $\mathcal{F}$ can equivalently be generated by the $\mathbb{P}^*$-martingales S^* and M with

$$\mathcal{G}_t = \sigma\left\{S_u^*, 0 \le u \le t\right\},$$
$$\mathcal{H}_t = \sigma\left\{M_u, 0 \le u \le t\right\}.$$

As Møller (1998) justifies the choice of the specific equivalent martingale measure $\mathbb{P}^*$ defined by (2.2.2) with references to papers from Schweizer (1991, 1995) and Aase and Persson (1994), we will follow his suggestion and will use the so called **minimal martingale measure** $\mathbb{P}^*$ throughout this thesis.

4.2 Risk-minimizing strategy for pure endowment insurance

After the introduction presented in the chapters before, in this section we will eventually derive the first results by using Föllmer and Sondermann's (1986) theory to determine the risk-minimizing strategy for the unit-linked pure endowment insurance defined in Section 4.1.1. First we will construct the Galtchouk-Kunita-Watanabe decomposition (see (3.1.4)) of the claim (4.1.7) and then we will be able to apply Theorems 3.1.8 and 3.1.15 to obtain risk-minimizing strategies and the associated intrinsic risk process.

As Møller (1998) pointed out in his paper we know from classical actuarial theory "that in case of fixed premiums and sum insured, the 'relative risk' associated with the portfolio decreases as the size l_x of the portfolio increases. More precisely, this means that the ratio between the standard deviation of the present value of all payments and the size of the portfolio l_x will congerge to 0 as l_x is increased," [7]. In the model of the present thesis we can not expect such results, since claims arising from different insurance contracts all depend on the same risky stock price process S and therefore are no longer stochastically independent. Anyway, to obtain a result about the behaviour of the risk arising from the portfolio if the number of insured persons grows very large, we will use the initial intrinsic risk $R_0 = \mathbb{E}^*\left[(L_T^H - L_0^H)^2\right]$, which measures the expected squared error of the non-hedge-able part of the claims and work with the ratio $\frac{\sqrt{R_0}}{l_x}$, as Møller (1998) suggested.

4.2.1 Theory

Our first result will be to determine the Galtchouk-Kunita-Watanabe decomposition of the present value of the claim H from (4.1.7),

$$H = g(S_T)B_T^{-1}(l_x - N_T). \tag{4.2.1}$$

The following decomposition of the claim arising from the portfolio of pure endowment insurance is shown by Møller (1998, Lemma 4.1).

Theorem 4.2.1. *[7] For the contingent claim H in (4.2.1) the process V^* defined by $V_t^* := \mathbb{E}^*\left[H|\mathcal{F}_t\right]$ has the decomposition*

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \nu_u^H dM_u,$$

where (ξ^H, ν^H) are given by

$$\xi_t^H := (l_x - N_{t^-})_{T-t}p_{x+t}F_s^g(t, S_t), \tag{4.2.2}$$

$$\nu_t^H := -B_t^{-1}F^g(t, S_t)_{T-t}p_{x+t}, \tag{4.2.3}$$

for $0 \leq t \leq T$ and $F^g(t, S_t) := \mathbb{E}\left[g(S_T)B_tB_T^{-1}|\mathcal{G}_t\right]$ with first order derivative $F_s^g(t, S_t)$ with respect to the second argument S_t.

Before proving Møller's (1998) decomposition theorem, we briefly recall the well-known Itô formula for semi-martingales.

Theorem 4.2.2. *[9] Let $X = (X^1, ..., X^d)$ be a d-dimensional semi-martingale which takes almost surely values in an open set $U \subset \mathbb{R}^d$. Furthermore, let $f : U \to \mathbb{R}$ be two times differentiable. Then $f(X)$ is a semi-martingale and*

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(X_{s^-})dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s^-})d[X^i, X^j]_s$$
$$+ \sum_{s \leq t}\left\{\Delta f(X_s) - \sum_i^d \frac{\partial f}{\partial x_i}(X_{s^-})\Delta X_s^i - \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{s^-})\Delta[X^i, X^j]_s\right\}.$$

Proof of Theorem 4.2.1: For the intrinsic value process of the claim H we get

$$V_t^* = \mathbb{E}^*\left[g(S_T)B_tB_T^{-1}|\mathcal{F}_t\right]\mathbb{E}^*\left[(l_x - N_T)|\mathcal{F}_t\right]B_t^{-1} \tag{4.2.4}$$

by using the stochastic independence of N and (B, S). With the definition of the risk-free asset, the first factor in (4.2.4) equals

$$\begin{aligned}\mathbb{E}^*\left[g(S_T)B_tB_T^{-1}|\mathcal{F}_t\right] &= \mathbb{E}^*\left[e^{-\int_t^T r_u du}g(S_T)|\mathcal{F}_t\right] = \\ &= \mathbb{E}^*\left[e^{-\int_t^T r_u du}g(S_T)|\mathcal{G}_t\right] = \\ &=: F^g(t, S_t),\end{aligned}$$

Note that we used the stochastic independence of N and (B, S), which ensures that the conditional distribution of (B, S) with respect to $\mathcal{F}_t$ does not depend on information coming from the insured lives $\mathcal{H}_t$ and therefore the overall filtration $\mathcal{F}_t$ can be replaced by the economic one, denoted by $\mathcal{G}_t$. Furthermore, the function $F^g(t, S_t)$ corresponds to the unique arbitrage-free price process of a simple claim in the complete model (see Definition 2.3.6. Moreover, the generalized Black-Scholes differential equation

$$-r(t,s)F^g(t,s) + F^g_t(t,s) + r(t,s)sF^g_s(t,s) + \frac{1}{2}\sigma(t,s)^2 s^2 F^g_{ss}(t,s) = 0 \tag{4.2.5}$$

holds true for $F^g(t, S_t)$ in the incomplete case.

The second factor in (4.2.4) can be rewritten as

$$\begin{aligned}\mathbb{E}^*\left[(l_x - N_T)|\mathcal{F}_t\right] &= \mathbb{E}^*\left[\sum_{i=1}^{l_x} I(T_i > T)|\mathcal{F}_t\right] = \sum_{i=1}^{l_x} \mathbb{E}^*\left[I(T_i > T)|T_i > t\right] = \\ &= \sum_{i=1}^{l_x} {}_{T_i-t}p_{x+t} = (l_x - N_t)_{T-t}p_{x+t}.\end{aligned}$$

Therefore, at every point in time t the expected number of persons alive at maturity T equals the number of persons alive at time t weighted with the probability to survive up to time T conditioned that the person still lives at time t.

Eventually, the intrinsic value process in (4.2.4) equals

$$V^*_t = (l_x - N_t)_{T-t}p_{x+t}B^{-1}_t F^g(t, S_t). \tag{4.2.6}$$

Now we apply Itô's formula to (4.2.6). Note that the second order derivatives and the ΔX term drop out and we get

$$\begin{aligned}V^*_t =& V^*_0 + \int_0^t (l_x - N_{u-})B^{-1}_u F^g(u, S_u)d({}_{T-u}p_{x+u}) + \\ &+ \int_0^t (l_x - N_{u-})_{T-u}p_{x+u}d(B^{-1}_u F^g(u, S_u)) + \sum_{u\le t} \Delta V^*_u = \\ =& V^*_0 + \int_0^t (l_x - N_{u-})B^{-1}_u F^g(u, S_u)_{T-u}p_{x+u}\mu_{x+u}du + \\ &+ \int_0^t (l_x - N_{u-})_{T-u}p_{x+u}d(B^{-1}_u F^g(u, S_u)) + \sum_{u\le t}(V^*_u - V^*_u).\end{aligned}$$

Next we determine the integral with respect to $d(B^{-1}_u F^g(u, S_u))$. For this purpose note the following points:

(i) By applying the product rule we rewrite the differential of the stock price with the help of the deflated stock price:

$$dS_t = d(S_t^* B_t) = S_t^* dB_t + B_t dS_t^* = S_t^* r_t B_t dt + B_t dS_t^* = S_t r_t dt + B_t dS_t^*$$

(ii) Apply Itô's formula to $dF^g(t, S_t)$ and get

$$\begin{aligned} dF^g(t,S_t) =& F_t^g(t,S_t)dt + F_s^g(t,S_t)dS_t + \frac{1}{2}F_{ss}^g(t,S_t)d[S_t] = \\ =& F_t^g(t,S_t)dt + F_s^g(t,S_t)dS_t \\ &+ \frac{1}{2}F_s^g(t,S_t)d[\alpha(t,S_t)S_t dt + \sigma(t,S_t)S_t dW_t] = \\ =& F_t^g(t,S_t)dt + F_s^g(t,S_t)dS_t + \frac{1}{2}F_{ss}^g(t,S_t)\sigma^2(t,S_t)S_t^2 dt. \end{aligned}$$

Finally we obtain

$$\begin{aligned} d(B_t^{-1}F^g(t,S_t)) =& F^g(t,S_t)dB_t^{-1} + B_t^{-1}dF^g(t,S_t) \overset{(ii)}{=} \\ =& - F^g(t,S_t)B_t^{-1}r_t dt + B_t^{-1}\Bigg(F_t^g(t,S_t)dt + \\ & + F_s^g(t,S_t)dS_t + \frac{1}{2}F_{ss}^g(t,S_t)\sigma(t,S_t)^2 S_t^2 dt \Bigg) \overset{(i)}{=} \\ =& - F^g(t,S_t)B_t^{-1}r_t dt + B_t^{-1}\Bigg(F_t^g(t,S_t)dt + \\ & + F_s^g(t,S_t)S_t r_t dt + F_s^g(t,S_t)B_t dS_t^* + \frac{1}{2}F_{ss}^g(t,S_t)\sigma(t,S_t)^2 S_t^2 dt \Bigg) \overset{(4.2.5)}{=} \\ =& - F^g(t,S_t)B_t^{-1}r_t dt + F_s^g(t,S_t)dS_t^* + F^g(t,S_t)B_t^{-1}r_t dt = \\ =& F_s^g(t,S_t)dS_t^*. \end{aligned}$$

For the variation of the intrinsic value process we get

$$\sum_{u \le t}(V_u^* - V_{u^-}^*) = -\int_0^t B_u^{-1}F^g(u,S_u)_{T-u}p_{x+u}dN_u.$$

Summarizing all results we see

$$
\begin{aligned}
V_t^* =& V_0^* + \int_0^t (l_x - N_{u-}) B_u^{-1} F^g(u,S_u)_{T-u}p_{x+u}\mu_{x+u} du + \\
&+ \int_0^t (l_x - N_{u-})_{T-u}p_{x+u} d(B_u^{-1}F^g(u,S_u)) + \sum_{u\le t}(V_u^* - V_u^*) = \\
=& V_0^* + \int_0^t B_u^{-1}F^g(u,S_u)_{T-u}p_{x+u}\lambda_u du + \\
&+ \int_0^t (l_x - N_{u-})_{T-u}p_{x+u}F_s^g(u,S_u)dS_u^* - \int_0^t B_u^{-1}F^g(u,S_u)_{T-u}p_{x+u}dN_u = \\
=& V_0^* + \int_0^t (l_x - N_{u-})_{T-u}p_{x+u}F_s^g(u,S_u)dS_u^* - \int_0^t B_u^{-1}F^g(u,S_u)_{T-u}p_{x+u}dM_u = \\
=& V_0^* + \int_0^t \xi_u^H dS_t^* + \int_0^t \nu_u^H dM_u.
\end{aligned}
$$

□

Remark 4.2.3. The intrinsic risk process $V_t^* = \mathbb{E}^*[H|\mathcal{F}_t]$ represents the expectation of the future value of the claim conditioned on all information available at time t. Therefore, it can be interpreted as the fair price of the portfolio consisting of pure endowment contracts. More precisely, the initial value $V_0^* = l_{xT}p_x F^g(0,S_0)$ is the natural choice of the single premium for the complete portfolio and can be described as the benefit payment times the expected number of persons alive at maturity T. This way of pricing the portfolio would be in accordance to the traditional equivalence principle, which states that all future expected and discounted premiums should equal all future expected and discounted benefits.

Next we will combine the decomposition of V^* from Theorem 4.2.1 with Föllmer and Sondermann's Theorem 3.1.8 to obtain a family of variance-minimizing strategies, where the amount of stocks is precisely specified at any point in time t, but the amount of bonds in the portfolio varies except for maturity T.

Theorem 4.2.4. *[7] Consider the pure endowment given by the contingent claim H in (4.2.1). An admissible strategy φ^* minimizing the variance $\mathbb{E}^*[(C_T^\varphi - \mathbb{E}^*[C_T^\varphi])^2]$ is given by*

$$
\begin{aligned}
\xi_t^* &= (l_x - N_{t^-})_{T-t}p_{x+t}F_s^g(t,S_t), \quad 0 \le t \le T, \\
\eta_T^* &= H - \xi_T^* S_T^*.
\end{aligned}
$$

The minimal variance is given by

$$\mathbb{E}^*\left[(C_T^\varphi - \mathbb{E}^*[C_T^\varphi])^2\right] = l_{xT}p_x \int_0^T \mathbb{E}^*\left[(B_u^{-1}F^g(u,S_u))^2\right] {}_{T-u}p_{x+u}\mu_{x+u}du$$

Proof. Using the decomposition of the intrinsic value process V^* of Theorem 4.2.1

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \nu_u^H dM_u,$$

with

$$\xi_t^H := (l_x - N_{t^-})_{T-t}p_{x+t}F_s^g(t,S_t), \tag{4.2.7}$$

$$\nu_t^H := -B_t^{-1}F^g(t,S_t)_{T-t}p_{x+t} \tag{4.2.8}$$

and Föllmer and Sondermann's result (3.1.8) we easily see that for the amount of stocks in the variance-minimizing portfolio we have

$$\xi_t^* = \xi_t^H = (l_x - N_{t^-})_{T-t}p_{x+t}F_s^g(t,S_t).$$

To determine the amount of bonds held at maturity T, note that

$$V_T^* = \mathbb{E}^*\left[H|\mathcal{F}_T\right] = H, \tag{4.2.9}$$

since H is $\mathcal{F}_T$-measurable as contingent claim with maturity T. Furthermore,

$$H = V_T^* = \xi_T S_T^* + \eta_T \Longrightarrow \eta_T = H - \xi_T S_T^*.$$

To complete the proof, we will show the representation of the minimal variance:

$$\mathbb{E}^*\left[(C_T^\varphi - \mathbb{E}^*[C_T^\varphi])^2\right] = \mathbb{E}^*\left[(L_T^H)^2\right] = \mathbb{E}^*\left[\left(\int_0^T \nu_u^H dM_u\right)^2\right] \overset{\textit{Itô isometry (3.1.5)}}{=}$$

$$\mathbb{E}^*\left[\int_0^T (\nu_u^H)^2 d[M]_u\right] =$$

$$\mathbb{E}^*\left[\int_0^T (B_u^{-1}F^g(u,S_u)_{T-u}p_{x+u})^2\lambda_u du\right] \overset{\textit{Fubini Theorem}}{=}$$

$$\int_0^T \mathbb{E}^*\left[(B_u^{-1}F^g(u,S_u)_{T-u}p_{x+u})^2\lambda_u\right] du \overset{\textit{Independence of } N \textit{ and } (B,S)}{=}$$

$$\int_0^T \mathbb{E}^*\left[(B_u^{-1}F^g(u,S_u))^2\right]{}_{T-u}p_{x+u}^2\mathbb{E}^*[\lambda_u]du \overset{\textit{Definition of } \lambda}{=}$$

$$\int_0^T \mathbb{E}^*\left[(B_u^{-1}F^g(u,S_u))^2\right]{}_{T-u}p_{x+u}^2\mathbb{E}^*[(l_x - N_{u^-})\mu_{x+u}]du =$$

$$\int_0^T \mathbb{E}^*\left[(B_u^{-1}F^g(u,S_u))^2\right]{}_{T-u}p_{x+u}^2 l_{x\,u}p_x\mu_{x+u}du \quad \overset{{}_{T-u}p_{x+u}\cdot{}_up_x={}_Tp_x}{=}$$

$$l_{x\,T}p_x\int_0^T \mathbb{E}^*\left[(B_u^{-1}F^g(u,S_u))^2\right]{}_{T-u}p_{x+u}\mu_{x+u}du.$$

□

Remark 4.2.5. The remaining risk the insurance company has to face, when using an optimal financial strategy to hedge the claim arising from the portfolio of pure endowment contracts, corresponds to the initial intrinsic risk $R_0^\varphi = \mathbb{E}^*\left[(L_T^H)^2\right]$. Due to proportionality of R_0^φ and l_x, we see that the ratio $\frac{\sqrt{R_0^\varphi}}{l_x}$ converges to 0 if the number of insured persons l_x goes to infinity, i.e., the non-hedge-able part of the risk decreases when the group of policy holders increases.

The last step concerning simple pure endowment insurance contracts is now to determine the optimal trading strategy among the family of strategies obtained in the previous theorem. An optimal strategy should minimize the remaining risk at any point in time and therefore it should fulfil the definition of risk-minimization from 3.1.13.

Theorem 4.2.6. *[7] For the pure endowment contract given by the contingent claim (4.2.1) the unique admissible risk-minimizing strategy is given by*

$$\xi_t^* = (l_x - N_{t^-})_{T-t}p_{x+t}F_s^g(t,S_t),$$
$$\eta_t^* = (l_x - N_t)_{T-t}p_{x+t}B_t^{-1}F^g(t,S_t) - \xi_t^*S_t^*, \quad 0 \le t \le T.$$

The intrinsic risk process R^{φ^*} is given by

$$R^{\varphi^*} = (l_x - N_t) \int_t^T \mathbb{E}^* \left[(\nu_u^H)^2 | \mathcal{F}_t \right] {}_{u-t}p_{x+t} \mu_{x+u} du. \tag{4.2.10}$$

Proof. The risk-minimizing trading strategy follows directly from Föllmer and Sondermann's Theorem 3.1.15 combined with the representation for V_t^* in (4.2.6).

To see the equivalence in (4.2.10) we use the Itô isometry (3.1.5), Fubini and $\mathbb{E}\left[(l_x - N_T)|\mathcal{F}_t\right] = (l_x - N_t)_{T-t}p_{x+t}$, which is shown in the proof of Theorem 4.2.1.

$$\begin{aligned} R_t^{\varphi^*} =& \mathbb{E}^* \left[(L_T^H - L_t^H)^2 | \mathcal{F}_t \right] = \mathbb{E}\Big[\Big(\int_t^T \nu_u^H dM_u \Big)^2 | \mathcal{F}_t \Big] = \mathbb{E}^* \Big[\int_t^T (\nu_u^H)^2 d[M]_u | \mathcal{F}_t \Big] = \\ =& \mathbb{E}^* \Big[\int_t^T (\nu_u^H)^2 \lambda_u du | \mathcal{F}_t \Big] = \int_t^T \mathbb{E}^* \left[(\nu_u^H)^2 | \mathcal{F}_t \right] \mathbb{E}^* \left[\lambda_u du | \mathcal{F}_t \right] = \\ =& \int_t^T \mathbb{E}^* \left[(\nu_u^H)^2 | \mathcal{F}_t \right] \mathbb{E}^* \left[(l_x - N_u) \mu_{x+u} | \mathcal{F}_t \right] du = \\ =& (l_x - N_t) \int_t^T \mathbb{E}^* \left[(\nu_u^H)^2 | \mathcal{F}_t \right] {}_{u-t}p_{x+t} \mu_{x+u} du. \end{aligned}$$

□

As Møller (1998) points out in his paper, this model does not properly represent the real world, since the insurance company is allowed to rebalance the hedging portfolio continuously. Thus, all contingent claims can be hedged by using the two considered financial assets and the remaining uncertainty arises from the mortality risk of the insured persons only. This random quantity is described by the martingale M, which drives the insurer's loss, given by L^H

$$dL_t^H = \nu_t^H dM_t = -B_t^{-1} F^g(t, S_t)_{T-t}p_{x+t}(dN_t - \lambda_t dt). \tag{4.2.11}$$

This means, that at any time-point t the insurer shifts his financial portfolio according to the expected number of policy holders surviving the insurance period. "During the infinitesimal time interval $[t, t + dt]$ the insurer will experience the gain dM_t multiplied by the term $B_t^{-1} F^g(t, S_t)_{T-t}p_{x+t}$, the latter denoting the price at time t of one security with payment $g(S_T)$ at time T contingent on the survival of some individual. That is, a death will produce an immediate gain for the insurer due to the downwards adjustment of the expected number of survivors, whereas no deaths will cause a small loss", see [7].

4.2.2 Examples

In the following we will work through some concrete examples given by Møller (1998) to illustrate the previous results. We will specify the benefit function g and for the sake of simplicity assume constant deterministic interest rate r, drift term α and volatility σ. Therefore, the financial model simplifies to the Black-Scholes model, see Example (2.1.3).

(1) First let us consider the contract function, that describes a pure unit-linked insurance $g(s) = s$, i.e., the insured persons gets the present value of the stock S_T at maturity T. The expected value of the claim at maturity $g(S_T)$ conditioned on the information at time t is given by the process $(F^g(t,S_t))_{0\leq t\leq T}$ with

$$F^g(t,S_t) = \mathbb{E}^*\left[e^{-\int_t^T r_u du} g(S_T)|\mathcal{F}_t\right] = \mathbb{E}^*\left[e^{-r(T-t)} S_T|\mathcal{F}_t\right] = \mathbb{E}^*\left[S_T^*|\mathcal{F}_t\right] = S_t,$$

where we used the martingale property of S_t^* with respect to $\mathbb{P}^*$. Furthermore, the intrinsic value process is

$$\begin{aligned} V_t^* &= (l_x - N_t)_{T-t}p_{x+t}B_t^{-1}F^g(t,S_t) = (l_x - N_t)_{T-t}p_{x+t}e^{-rt}S_t = \\ &= (l_x - N_t)_{T-t}p_{x+t}S_t^*, \\ &\Longrightarrow V_0^* = l_{xT}p_xS_0^*. \end{aligned}$$

So we can use Theorem 4.2.6 to determine the risk-minimizing admissible strategy

$$\begin{aligned} (\xi_t,\eta_t) &= \Big((l_x - N_{t-})_{T-t}p_{x+t}F_s^g(t,S_t), (l_x - N_t)_{T-t}p_{x+t}B_t^{-1}F^g(t,S_t) - \xi_t^*S_t^*\Big) = \\ &= \Big((l_x - N_{t-})_{T-t}p_{x+t}, (l_x - N_t)_{T-t}p_{x+t}S_t^* - (l_x - N_{t-})_{T-t}p_{x+t}S_t^*\Big) = \\ &= \Big((l_x - N_{t-})_{T-t}p_{x+t}, \Delta N_{tT-t}p_{x+t}S_t^*\Big) \end{aligned}$$

with $\Delta N_t = N_t - N_{t^-}$. This means, that the number of stocks equals the expected number of individuals left in the insurance portfolio at time t. The amount of bonds is rebalanced at any time a death occures to make sure that the value of the financial strategy V_t^φ equals the value of the claim V_t^* for all t.

Furthermore, the final aggregated loss is given by

$$L_t^H = \int_0^T \nu_u dM_u = -\int_0^T B_u^{-1}F^g(u,S_u)_{T-u}p_{x+u}dM_u = -\int_0^T S_{uT-u}^*p_{x+u}dM_u$$

and the intrinsic risk process can be written as

$$\begin{aligned} R_t^{\varphi^*} &= (l_x - N_t) \int_t^T \mathbb{E}^* \Big[(\nu_u^H)^2 | \mathcal{F}_t \Big]_{u-t} p_{x+t} \mu_{x+u} du = \\ &= (l_x - N_t) \int_t^T \mathbb{E}^* \Big[(S_u^*)^2{}_{T-u} p_{x+u}^2 | \mathcal{F}_t \Big]_{u-t} p_{x+t} \mu_{x+u} du = \\ &= (l_x - N_t)_{T-t} p_{x+t} S_t^* \int_t^T {}_{T-u} p_{x+u} \mu_{x+u} du. \end{aligned}$$

(2) Now we consider a unit-linked contract with a guarantee $g(s) = \max(s, K)$, where K is a non-negative constant describing the level of guarantee. Since we can write $g(s) = K + (s - K)^+$, the price-process $(F^g(t, S_t))_{0 \leq t \leq T}$ can be determined by using Black-Scholes' formula for European call options, that states

Theorem 4.2.7. *[10] The fair price* $C^{BS}(t, S_t) = \mathbb{E}^* \Big[e^{-r(T-t)} (S_T - K)^+ | \mathcal{F}_t \Big]$ *of a call option with strike* K *at time* t *is*

$$C^{BS}(t, S_t) = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-),$$

where

$$d_\pm = \frac{\log(\frac{S_t}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma \sqrt{T-t}}$$

and Φ *denotes the distribution function of a standard-normal distributed random variable.*

Therefore, we get

$$\begin{aligned} F^g(t, S_t) &= \mathbb{E}^* \Big[e^{-\int_t^T r_u du} g(S_T) | \mathcal{F}_t \Big] = e^{-r(T-t)} K + \mathbb{E}^* \Big[e^{-r(T-t)} (S_T - K)^+ | \mathcal{F}_t \Big] = \\ &= e^{-r(T-t)} K + \mathbb{E}^* \Big[e^{-r(T-t)} (S_T - K)^+ | \mathcal{F}_t \Big] = \\ &= e^{-r(T-t)} K + S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-) = \\ &= K e^{-r(T-t)} \Phi(-d_-) K + S_t \Phi(d_+). \end{aligned}$$

Hence the first order derivative with respect to s is $F_s^g(t, S_t) = \Phi(d_+)$

and using Theorem 4.2.6 we obtain the risk-minimizing strategy

$$\begin{aligned}
\xi_t =& (l_x - N_{t^-})_{T-t}p_{x+t}\Phi(d_+),\\
\eta_t =& (l_x - N_t)_{T-t}p_{x+t}e^{-rt)}F^g(t, S_t) - (l_x - N_{t^-})_{T-t}p_{x+t}\Phi(d_+)S_t^* =\\
=& (l_x - N_t)_{T-t}p_{x+t}\Big(e^{-rt}Ke^{-r(T-t)}\Phi(-d_-) + e^{-rt}S_t\Phi(d_+)\Big) -\\
&- (l_x - N_t)_{T-t}p_{x+t}\Phi(d_+)S_t^* =\\
&= (l_x - N_t)_{T-t}p_{x+t}Ke^{-rT}\Phi(-d_-) - \Delta N_{tT-t}p_{x+t}\Phi(d_+)S_t^*
\end{aligned}$$

and the intrinsic risk process

$$\begin{aligned}
R_t^{\varphi^*} &= (l_x - N_t)\int_t^T \mathbb{E}^*\Big[(\nu_u^H)^2|\mathcal{F}_t\Big]_{u-t}p_{x+t}\mu_{x+u}du =\\
&= (l_x - N_t)\int_t^T \mathbb{E}^*\Big[\Big(-e^{-ru}F^g(u, S_u)_{T-u}p_{x+u}\Big)^2|\mathcal{F}_t\Big]_{u-t}p_{x+t}\mu_{x+u}du =\\
&= (l_x - N_t)_{T-t}p_{x+t}\int_t^T \mathbb{E}^*\Big[\Big(-e^{-ru}F^g(u, S_u)\Big)^2|\mathcal{F}_t\Big]_{T-u}p_{x+u}\mu_{x+u}du.
\end{aligned}$$

(3) The last contract function we will consider defines the benefit as a deterministic payment K, i.e., $g(s) = K$, for a non-negative constant K. Then we have for the price process

$$\begin{aligned}
F^g(t, S_t) &= \mathbb{E}^*\Big[e^{-\int_t^T r_u du}g(S_T)|\mathcal{F}_t\Big] = \mathbb{E}^*\Big[e^{-r(T-t)}K|\mathcal{F}_t\Big] =\\
&= e^{-r(T-t)}K.
\end{aligned}$$

Hence the risk-minimizing trading strategy is

$$(\xi_t, \eta_t) = \Big(0, (l_x - N_t)_{T-u}p_{x+u}e^{-rT}K\Big).$$

Since the insurance payment is deterministic now, the insurer does not need to held risky stocks in his financial portfolio to hedge the insurance claim. The only uncertainty is coming from the life length of the policy holders, which is hedged by rebalancing the amount of bonds according to the expected value of persons still alive.

The intrinsic risk process can be written as

$$\begin{aligned}
R_t^{\varphi^*} &= (l_x - N_t)\int_t^T \mathbb{E}^*\Big[(\nu_u^H)^2|\mathcal{F}_t\Big]{}_{u-t}p_{x+t}\mu_{x+u}du = \\
&= (l_x - N_t)\int_t^T \mathbb{E}^*\Big[\Big(-e^{-ru}F^g(u,S_u)_{T-u}p_{x+u}\Big)^2|\mathcal{F}_t\Big]{}_{u-t}p_{x+t}\mu_{x+u}du = \\
&= (l_x - N_t)\int_t^T \mathbb{E}^*\Big[\Big(-e^{-ru}e^{-r(T-u)}K_{T-u}p_{x+u}\Big)^2|\mathcal{F}_t\Big]{}_{u-t}p_{x+t}\mu_{x+u}du = \\
&= (l_x - N_t)\int_t^T e^{-2rT}K^2{}_{T-u}p_{x+u}^2{}_{u-t}p_{x+t}\mu_{x+u}du = \\
&= (l_x - N_t)e^{-2rT}K^2{}_{T-t}p_{x+t}(1 -_{T-t} p_{x+t}).
\end{aligned}$$

(4) As a last example let us consider the unit-linked contract function with guarantee $g(s) = \max(s, K)$ from example (4.2.2.2) with one person insured only. Therefore, specify the number of policy holders equal to one, $l_x = 1$, and obtain the following risk-minimizing strategy

$$\begin{aligned}
\xi_t =&(l_x - N_{t^-})_{T-t}p_{x+t}\Phi(d_+) = (1 - I(T_1 \le t))_{T-t}p_{x+t}\Phi(d_+) = \\
&= I(T_1 > t)_{T-t}p_{x+t}\Phi(d_+), \\
\eta_t =&(l_x - N_t)_{T-t}p_{x+t}Ke^{-rT}\Phi(-d_-) - \Delta N_{tT-t}p_{x+t}\Phi(d_+)S_t^* = \\
&= (1 - I(T_1 \le t))_{T-t}p_{x+t}Ke^{-rT}\Phi(-d_-) - I(T_1 = t)_{T-t}p_{x+t}\Phi(d_+)S_t^* \\
&= I(T_1 > t)_{T-t}p_{x+t}Ke^{-rT}\Phi(-d_-) - I(T_1 = t)_{T-t}p_{x+t}\Phi(d_+)S_t^*
\end{aligned}$$

where we used the definition of the counting process N_t (see (4.1.4)). The intrinsic value process simplifies to

$$\begin{aligned}
V_t^* &= (l_x - N_t)_{T-t}p_{x+t}B_t^{-1}F^g(t, S_t) = \\
&= (1 - I(T_1 \le t))_{T-t}p_{x+t}\Big[e^{-rT}K\Phi(-d_-) + e^{-rt}S_t\Phi(d_+)\Big] = \\
&= I(T_1 > t)_{T-t}p_{x+t}\Big[e^{-rT}K\Phi(-d_-) + S_t^*\Phi(d_+)\Big].
\end{aligned}$$

This equals the expactiation of the price of a European call option at time t in the Black-Scholes model, conditioned on the event of being still alive at time t. This term is equivalent to the **prospective reserve** from traditional life insurance mathematics, which is defined as the discounted expected difference between future benefits and future premiums.

As we can see in all previous examples, the number of stocks and bonds held in the financial portfolio heavily depend on the number of survivors in each time period. "Thus, the risk-minimizing strategies reflect the actual development in the insurance portfolio, and bring to the surface the uncertainty associated with the insured lives." [7]

4.3 Risk-minimizing strategy for term insurance

As a second example, we use Föllmer and Sondermann's (1986) theory to determine the risk-minimizing strategy for the simple term insurance, which was introduced in Section 4.1.2. As we saw in the previous section, the first step will be to find the Galtchouk-Kunita-Watanabe decomposition of the martingale V_t^* and secondly to directly apply Föllmer and Sondermann's unique risk-minimization result.

Recall, that the claim corresponding to the term insurance was given by

$$H_T = \int_0^T g(u, S_u) B_u^{-1} dN_u \tag{4.3.1}$$

and let us prove the following requested decomposition of the intrinsic value process V_t^* of the claim H_T.

Theorem 4.3.1. *[7] For the claim H_T in (4.3.1) the process V^* defined by $V_t^* = \mathbb{E}[H_T|\mathcal{F}_t]$ has the decomposition*

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \nu_u^H dM_u,$$

where (ξ^H, ν^H) are given by

$$\xi_t^H = (l_x - N_{t^-}) \int_t^T {}_{u-t}p_{x+t}\mu_{x+u} F_s^{g_u}(t, S_t) du, \tag{4.3.2}$$

$$\nu_t^H = g(t, S_t) B_t^{-1} - \int_t^T F^{g_u}(t, S_t) B_t^{-1} {}_{u-t}p_{x+t}\mu_{x+u} du. \tag{4.3.3}$$

Where we define $F^{g_u}(t, S_t)$ as the fair (arbitrage-free) price of the payout function $g(u, S_u)$ at time t and $u \geq t$

$$F^{g_u}(t, S_t) = \mathbb{E}^* \left[e^{-\int_t^u r_\tau d\tau} g(u, S_u) | \mathcal{G}_t \right] \tag{4.3.4}$$

and $F_s^{g_u}(t, S_t)$ stands for the first derivative of the fair price with respect to the price of the asset.

Proof. Note the following expression for the intrinsic value process V_t^*, where we used Fubini's Theorem twice

$$
\begin{aligned}
V_t^* =& \mathbb{E}^*[H_T|\mathcal{F}_t] = \mathbb{E}^*\Big[\int_0^T g(u,S_u)B_u^{-1}dN_u|\mathcal{F}_t\Big] \overset{N \text{ and } (B,S) \text{ are } \mathcal{F}-\text{adapted}}{=} \\
=& \int_0^t g(u,S_u)B_u^{-1}dN_u + \mathbb{E}^*\Big[\int_t^T g(u,S_u)e^{-\int_t^u r_\tau d\tau}B_t^{-1}dN_u|\mathcal{F}_t\Big] = \\
=& \int_0^t g(u,S_u)B_u^{-1}dN_u + \int_t^T F^{g_u}(t,S_t)B_t^{-1}(l_x - N_t)_{u-t}p_{x+t}\mu_{x+u}du
\end{aligned}
$$

Analogously to the proof of Theorem 4.2.1 in this chapter, we can show

$$
d(B_t^{-1}F^{g_u}(t,S_t)) = F_s^{g_u}(t,S_t)dS_t^*. \tag{4.3.5}
$$

Finally let us apply Itô's formula to the expression of V_t^*, that we derived above,

$$
\begin{aligned}
V_t^* =& V_0^* + \int_0^t -B_\tau^{-1}F^{g_\tau}(\tau,S_\tau)(l_x - N_{\tau^-})\mu_{x+\tau}d\tau + \int_0^t g(\tau,S_\tau)B_\tau^{-1}dN_\tau + \\
&+ \int_0^t \left(\int_\tau^T B_\tau^{-1}F^{g_u}(\tau,S_\tau)_{u-\tau}p_{x+\tau}\mu_{x+u}du\right)d(l_x - N_\tau) + \\
&+ \int_0^t \left(\int_\tau^T B_\tau^{-1}F^{g_u}(\tau,S_\tau)(l_x - N_{\tau^-})\mu_{x+u}d({}_{u-\tau}p_{x+\tau})\right)d\tau + \\
&+ \int_0^t \left(\int_\tau^T (l_x - N_{\tau^-})_{u-\tau}p_{x+\tau}\mu_{x+u}du\right)d\Big(B_\tau^{-1}F^{g_u}(\tau,S_\tau)\Big).
\end{aligned}
$$

By using $F^{g_t}(t,S_t) = g(t,S_t)$, $(l_x - N_{t^-})\mu_{x+t}dt = \lambda_t dt$ and (4.3.5) we simplify the integrals to

$$
\begin{aligned}
V_t^* =& V_0^* + \int_0^t -B_\tau^{-1}g(\tau,S_\tau)\lambda_\tau d\tau + \int_0^t g(\tau,S_\tau)B_\tau^{-1}dN_\tau + \\
&- \int_0^t \left(\int_\tau^T B_\tau^{-1}F^{g_u}(\tau,S_\tau)_{u-\tau}p_{x+\tau}\mu_{x+u}du\right)dN_\tau + \\
&+ \int_0^t \left(\int_\tau^T B_\tau^{-1}F^{g_u}(\tau,S_\tau)_{u-\tau}p_{x+\tau}\mu_{x+u}du\right)\mu_{x+\tau}(l_x - N_{\tau^-})d\tau + \\
&+ \int_0^t \left(\int_\tau^T F_s^{g_u}(\tau,S_\tau)_{u-\tau}p_{x+\tau}\mu_{x+u}du\right)(l_x - N_{\tau^-})dS_t^*.
\end{aligned}
$$

Note that $dM_u = dN_u - \lambda_u du$ and obtain the desired result

$$\begin{aligned}
V_t^* =& V_0^* + \int_0^t \left(\int_\tau^T F_s^{g_u}(\tau, S_\tau)_{u-\tau}p_{x+\tau}\mu_{x+u}du \right)(l_x - N_{\tau^-})dS_t^* + \\
&+ \int_0^t -B_\tau^{-1} g(\tau, S_\tau)\lambda_\tau d\tau + \int_0^t g(\tau, S_\tau)B_\tau^{-1}dN_\tau + \\
&- \int_0^t \left(\int_\tau^T B_\tau^{-1}F^{g_u}(\tau, S_\tau)_{u-\tau}p_{x+\tau}\mu_{x+u}du \right) dN_\tau + \\
&+ \int_0^t \left(\int_\tau^T B_\tau^{-1}F^{g_u}(\tau, S_\tau)_{u-\tau}p_{x+\tau}\mu_{x+u}du \right) \mu_{x+\tau}(l_x - N_{\tau^-})d\tau = \\
=& V_0^* + \int_0^t \xi_t^H dS_u^* + \int_0^t \nu_u^H dM_u.
\end{aligned}$$

□

Remark 4.3.2. As Møller (1998) points out, the term ν_t^H can be interpreted as the immediate loss of the insurer, if death of one of the policy holders occurs at time t. On the one hand the insurer has to pay $g(t, S_t)$ and on the other hand the company readjusts its expectations of future developments of the insurance portfolio, which leads to a decrease in its reserves by the amount $\int_t^T F^{g_u}(t, S_t)B_t^{-1}{}_{u-t}p_{x+t}\mu_{x+u}du$.

We now use Föllmer and Sondermann's Theorem 3.1.15 to directly determine the unique risk-minimizing trading strategy for hedging the insurer's risk process. The proof is analogously to the proof of Theorem 4.2.6 in the present thesis.

Theorem 4.3.3. *[7] For the term insurance given by the contingent claim (4.3.1) the unique admissible risk-minimizing strategy is given by*

$$\begin{aligned}
\xi_t^* =& (l_x - N_{t^-}) \int_t^T {}_{u-t}p_{x+t}\mu_{x+u}F_s^{g_u}(t, S_t)du, \\
\eta_t^* =& \int_0^t g(u, S_u)B_u^{-1}dN_u + (l_x - N_t) \int_t^T F^{g_u}(t, S_t)B_t^{-1}{}_{u-t}p_{x+t}\mu_{x+u}du + \\
&+ \xi_t^* S_t^*, \quad 0 \le t \le T.
\end{aligned}$$

The intrinsic risk process R^{φ^*} is given by

$$R_t^{\varphi^*} = (l_x - N_t) \int_t^T \mathbb{E}^* \left[(\nu_u^H)^2 | \mathcal{F}_t \right]_{u-t}p_{x+t}\mu_{x+u}du,$$

where ν^H is taken from (4.3.3).

Remark 4.3.4. Note that the final portfolio $\varphi^* = (\xi^*, \eta^*)$ is determined such that the value of this strategy at time t is given by

$$V_t^* = \int_0^t g(u, S_u) B_u^{-1} dN_u + \mathbb{E}^* \Big[\int_t^T g(u, S_u) B_u^{-1} dN_u | \mathcal{F}_t \Big].$$

"Thus, $V_t^{\varphi^*}$ is determined as the sum of the benefits set aside to deaths already occurred and the expected discounted value of payments associated with future deaths." [7]

Example 4.3.5. To illustrate the obtained results for term insurance contracts let us consider a simple example in the standard Black-Scholes financial world and a contract function with guarantee that is adjusted by a constant term of inflation δ, $g(u, s) = \max(s, Ke^{\delta u})$. First we obtain the function $F^{g_u}(t, S_t)$ with the Black-Scholes pricing formula for European options as we already did in subsection (4.2.2) Example (2)

$$F^{g_u}(t, S_t) = Ke^{\delta u} e^{-r(u-t)} \Phi(-d_-) + S_t \Phi(d_+),$$

with

$$d_\pm = \frac{\log(\frac{S_t}{Ke^{\delta u}}) + (r \pm \frac{1}{2}\sigma^2)(u-t)}{\sigma\sqrt{u-t}}$$

Applying Theorem 4.3.3 we obtain the optimal risk-minimizing financial portfolio with

$$\begin{aligned}
\xi_t^* =& (l_x - N_{t^-}) \int_t^T {}_{u-t}p_{x+t} \mu_{x+u} \Phi(d_+) du, \\
\eta_t^* =& \int_0^t g(u, S_u) B_u^{-1} dN_u + (l_x - N_t) \int_t^T {}_{u-t}p_{x+t} \mu_{x+u} B_t^{-1} K e^{\delta u} e^{-r(u-t)} \Phi(d_-) du + \\
& + (l_x - N_t) \int_t^T {}_{u-t}p_{x+t} \mu_{x+u} B_t^{-1} S_t \Phi(d_+) du - \\
& - S_t^* (l_x - N_{t^-}) \int_t^T {}_{u-t}p_{x+t} \mu_{x+u} \Phi(d_+) du = \\
=& \int_0^t g(u, S_u) B_u^{-1} dN_u + (l_x - N_t) \int_t^T {}_{u-t}p_{x+t} \mu_{x+u} K e^{-(r-\delta)u} \Phi(d_-) du - \\
& - \Delta N_t \int_t^T {}_{u-t}p_{x+t} \mu_{x+u} \Phi(d_+) du.
\end{aligned}$$

4.4 Extending the financial market by reinsurance possibilities

In the previous two sections we tried to hedge the risk for the insurer arising from different insurance contracts, where the financial model consists of two assets only, a risk-free bond and a risky stock with price process B respectively S. When combining this financial model with the uncertainty coming from the remaining lifetimes of the insured persons, T_i, we derived an incomplete model and therefore, not every claim is perfectly hedge-able. This means, that the insurer is left with some intrinsic risk, when selling a pure endowment or a term insurance contract, see Sections 4.2 and 4.3.

Møller (1998) suggested to extend the financial model by an asset corresponding to the uncertainty coming from mortality to hedge the insurer's remaining risk completely. In this section we will use Møller's additional asset to derive the perfect hedging portfolio for the pure endowment insurance. For reasons of simplicity we will consider the risk-free interest r to be constant over time.

Møller (1998) defined the price process of the new asset related to the survival probability of the insured persons by $Z = (Z_t)_{0 \leq t \leq T}$, where

$$Z_t := (l_x - N_t)_{T-t}p_{x+t}e^{-r(T-t)}, \tag{4.4.1}$$

for all $t \in [0, T]$.

The initial value $Z_0 = l_{xT}p_x e^{-rT}$ represents the discounted expected number of persons still alive at time of maturity T. Furthermore, this amount is equal to the price of l_x pure endowment insurance contracts with sum insured 1 at time 0. When we assume that premiums are paid as single premiums at time 0, Z_t equals the traditional prospective reserve at time t, since it equals the discounted expected benefits arising from l_x pure endowment insurance contracts each on with sum insured 1 at time t.

Therefore, this new financial asset given by the price process Z can be interpreted as a trade-able reinsurance possibility on the market. Even though "trading on the reinsurance markets will typically be controlled by certain restrictions such as short-selling constraints and upper limits for the amount reinsured" [7], we will not restrict trading on the new financial market (B, S, Z). Also note, that the reinsurance asset Z evolves independently from the other financial products (B, S).

Recall the definition of the claim arising from selling l_x pure endowment contracts given in (4.1.7)

$$H = (l_x - N_T)_T p_x B_T^{-1} g(S_T). \tag{4.4.2}$$

The claim H is affected by the uncertainty coming from the risky stock S as well as by the stochastic mortality of the insured persons.

Analogously to the deflated price process S^* let us define the deflated price of the reinsurance asset $Z^* = (Z_t^*)_{0\leq t\leq T}$ by

$$Z_t^* = \frac{Z_t}{B_t} = (l_x - N_t)_{T-t}p_{x+t}e^{-rT}, \tag{4.4.3}$$

for all $t \in [0,T]$.

In the new financial setup given by (B,S,Z) at any time $t \in [0,T]$ a trading strategy φ is given by a sufficiently integrable process $\varphi_t = (\xi_t, \eta_t, \vartheta_t)$, where ξ_t, η_t and ϑ_t define the amounts of stocks, bonds and reinsurance contracts, respectively, held at time t. Furthermore, ξ and ϑ needs to be $\mathcal{F}$-predictable and η is $\mathcal{F}$-adapted. Now the discounted value V_t^φ at time t of the trading strategy φ is given by

$$V_t^\varphi := \xi_t S_t^* + \vartheta_t Z_t^* + \eta_t. \tag{4.4.4}$$

Next we want to show, that the deflated price process Z^* is an $(\mathcal{F}, \mathbb{P}^*)$-martingale. This follows directly by using

$$(l_x - N_t)_{T-t}p_{x+t} = \mathbb{E}^*[(l_x - N_T)|\mathcal{F}_t],$$

which was shown in the proof of Theorem 4.2.1. For $s \leq t$ we get

$$\begin{aligned}\mathbb{E}^*[Z_t^*|\mathcal{F}_s] = \mathbb{E}^*[(l_x - N_t)_{T-t}p_{x+t}e^{-rT}|\mathcal{F}_s] = e^{-rT}\mathbb{E}^*\Big[\mathbb{E}^*[(l_x - N_T)|\mathcal{F}_t]|\mathcal{F}_s\Big] = \\ = e^{-rT}\mathbb{E}^*[(l_x - N_T)|\mathcal{F}_s] = (l_x - N_s)_{T-s}p_{x+s}e^{-rT} = Z_s^*.\end{aligned}$$

Hence, (S^*, Z^*) are $(\mathcal{F}, \mathbb{P}^*)$-martingales and we can again use $\mathbb{P}^*$ as an equivalent martingale measure to show the next statement.

Theorem 4.4.1. *[7] Consider the pure endowment with present value (4.4.2) and assume that standard pure endowment contracts with sum insured* 1 *are traded freely on a financial market with constant short rate of interest. A self-financing admissible (risk-minimizing) strategy φ^* is given by*

$$\begin{aligned}\xi_t^* &= (l_x - N_{t-})_{T-t}p_{x+t}F_s^g(t, S_t),\\ \vartheta_t^* &= e^{r(T-t)}F^g(t, S_t),\\ \eta_t^* &= V_t^* - \xi_t^* S_t^* - \vartheta_t^* Z_t^*\end{aligned}$$

for all $0 \leq t \leq T$. Furthermore, the intrinsic risk process R^{φ^*} is identically 0*.*

Proof. To show the statement we need to find a decomposition for the intrinsic value process V_t^* with respect to S^* and Z^*. From the proof of Theorem 4.2.1 we already know

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \nu_u^H dM_u, \tag{4.4.5}$$

where (ξ^H, ν^H) are given by

$$\begin{aligned} \xi_t^H &:= (l_x - N_{t-})_{T-t}p_{x+t}F_s^g(t, S_t), \\ \nu_t^H &:= -B_t^{-1}F^g(t, S_t)_{T-t}p_{x+t}, \end{aligned}$$

for $0 \leq t \leq T$.

Next let us use a slightly adapted version of the proof of Theorem 4.2.1: We set $B_t \equiv 1$, $g(S_t) \equiv 1$ and consider the claim $\widetilde{H} := B_T^{-1}(l_x - N_T) = Z_T^*$. Then using the martingale property of Z_t^* we get for the intrinsic value process of the claim $\widetilde{H}$

$$\widetilde{V}_t^* = \mathbb{E}^*[\widetilde{H}|\mathcal{F}_t] = \mathbb{E}^*[Z_T^*|\mathcal{F}_t] = Z_t^*.$$

Therefore, by inserting the special choice of $g(S_T)$ into Theorem 4.2.1, we obtain for the price of the reinsurance contract

$$\begin{aligned} Z_t^* &= Z_0^* - \int_0^t \widetilde{F}^g(u, S_u)B_u^{-1}{}_{T-u}p_{x+u}dM_u = \\ &= Z_0^* - \int_0^t B_u B_T^{-1} B_u^{-1}{}_{T-u}p_{x+u}dM_u = \\ &= Z_0^* - B_T^{-1} \int_0^t {}_{T-u}p_{x+u}dM_u. \end{aligned}$$

Here we used the fact that the first derivative of $\widetilde{F}^g$ with respect to the asset price equals zero, i.e. $\widetilde{F}_s^g(t, S_t) = 0$, since we set $g(S_t) \equiv 1$.

With this expression for Z_t^* we can write

$$dZ_u^* = -\Big(e^{-rT}{}_{T-u}p_{x+u}dM_u\Big). \tag{4.4.6}$$

Combining (4.4.5) with (4.4.6) we get the required decomposition of V_t^*

$$\begin{aligned}V_t^* &= V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \nu_u^H dM_u = \\ &= V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t -B_u^{-1} F^g(u, S_u)_{T-u}p_{x+u} dM_u = \\ &= V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t e^{r(T-u)} F^g(u, S_u) dZ_u^* = \\ &= V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t \vartheta_u^H dZ_u^*.\end{aligned}$$

We showed that the intrinsic value process can be represented as sum of two integrals with respect to the price process of the two risky financial products. Hence, the intrinsic risk process $R_t^\varphi = \mathbb{E}^*\Big[(R_T^H - R_t^H)^2|\mathcal{F}_t\Big]$ is identically 0 for all $t \in [0, T]$ in the extended financial market model. □

We showed, that the insurer is able to eliminate the risk completely by continuously rebalancing the amount of stocks, bonds and reinsurance contracts in the financial portfolio given in Theorem 4.4.1.

Using expression (4.2.6) for the intrinsic value process we obtain

$$V_t^* = (l_x - N_t)_{T-t}p_{x+t} B_t^{-1} F^g(t, S_t) = \vartheta_t^* Z_t^*. \tag{4.4.7}$$

This means, that the intrinsic value of the optimal risk-eliminating portfolio equals the value of ϑ_t^* standard pure endowment insurance contracts at any time $0 \leq t \leq T$, i.e., the insurer passes the risk arising from the insurance portfolio completely to the reinsurance company.

Furthermore, note that inserting (4.4.7) into the expression for the amount of bonds in the optimal financial portfolio η^* derived in Theorem 4.4.1 we get

$$\eta_t^* = V_t^* - \xi_t^* S_t^* - \vartheta_t^* Z_t^* = -\xi_t^* S_t^*.$$

Møller (1998) summarized this result with stating that the amount of stocks held in the financial portfolio, which equals the amount of stocks if we use the incomplete market (B, S) see Theorem 4.2.6, "is financed by an equivalent short position η_t^* in the risk-free asset"[7].

Chapter 5

Second application: Generalized insurance payment streams

This chapter deals with the problem of finding a risk-minimizing trading strategy to hedge an insurance portfolio consisting of contracts with general payment streams. The theory and results presented in this chapter are based on a paper of Møller (2001).

First we will have to point out the differences of general payment streams compared to the simple world of the previous chapters and give some additional definitions. Then, we will have to work out a generalization of Föllmer and Sondermann's (1986) theory of risk-minimization in incomplete markets. Finally, we will be able to apply the obtained results to an insurance portfolio consisting of general unit-linked life insurance contracts.

5.1 Additional definitions

Basically, all assumptions, definitions and statements presented in the previous chapters still hold true in the extended theory of this chapter. Again we work with the financial model consisting of the two financial products B and S defined in (2.1.1)-(2.1.2), the equivalent martingale measure $\mathbb{P}^*$ and all other known definitions. In the advanced set-up the only difference compared to the simple world is coming from the generalization of the insurance payment streams. Therefore, note that all assumptions, definitions and statements coming from prior chapters still hold true except those which are explicitly pointed out in this section.

In Møller's (1998) simple model of insurance payment streams all benefit payments arising from an unit-linked life insurance contract were described by a single claim H payable at some fixed time T (T-claims). Now we change the setting to Møller's (2001) advanced model of **general payment streams**. These payment streams are specified in the following definition.

Definition 5.1.1. *[8] A payment stream $A = (A_t)_{0\le t\le T}$ is an $\mathcal{F}$-adapted, square-integrable, càdlàg process.*

For some $0 \le s \le t \le T$ we will interpret $A_t - A_s$ as the total discounted outgoing cash-flows (benefit payments) minus total discounted incoming cash-flows (premium payment stream) in the time interval $(s, t]$ from the insurer's point of view.

Since we generalized the definition of insurance payment streams, we additionally have to adapt the idea of the **cost process** C associated with a trading strategy φ and a payment process A.

Definition 5.1.2. *[8] The cost process of the strategy φ and the payment process A is given by*

$$C_t^\varphi = V_t^\varphi - \int_0^t \xi_u dS_u^* + A_t, \tag{5.1.1}$$

for all $0 \le t \le T$, where ξ_t refers to the amount of stocks held at time t.

To motivate this definition we have a look at the costs associated with a trading strategy φ and a payment stream A. During the infinitesimal small time interval $(t, t+dt]$ the costs are given by the sum of changes in the stock and the bond position of the financial portfolio and the insurer's cash-flows arising from the insurance contracts. Mathematically, the change of the cost process can be written as

$$\begin{aligned} dC_t^\varphi &= C_{t+dt}^\varphi - C_t^\varphi = \\ &= (\xi_{t+dt} - \xi_t) S_{t+dt}^* + (\eta_{t+dt} - \eta_t) + A_{t+dt} - A_t, \end{aligned}$$

where the trading strategy is given by $\varphi = (\xi, \eta)$ with ξ_t and η_t the amount of stocks and bonds held at time t, respectively. Using now the definition of the deflated value process (2.3.3) we can rewrite the above equation to

$$\begin{aligned} dC_t^\varphi &= V_{t+dt}^\varphi - V_t^\varphi - \xi_t (S_{t+dt}^* - S_t^*) + A_{t+dt} - A_t = \\ &= dV_t^\varphi - \xi_t dS_t^* + dA_t \end{aligned}$$

and the formal definition of the cost process (5.1.1) follows directly.

Note that the cost process C is an $\mathcal{F}$-adapted, square-integrable process, because of the assumptions we set up for the payment process A and the trading strategy φ.

Furthermore, note that the cost process C_t^φ should be interpreted as the insurers cumulative costs during the time interval $[0, t]$. Since the payment process A owns càdlàg and hence right-continuous paths, the cost process at time t, C_t, includes all payments A_t. Therefore, the value process V_t should be interpreted as the value of the financial portfolio φ after the payments A_t. As Møller (2001) pointed out, the terminal value of the portfolio V_T is the value of the portfolio after all liabilities and premiums are settled and therefore it is a natural condition to restrict the scope of portfolios to those with terminal value 0

$$V_T^\varphi = 0,$$

the so called **0-admissibility**.

We go on by defining self-financing trading strategies and attainability for payment streams in the extended set-up.

Definition 5.1.3. *[8]*

*(i) A strategy φ is called **self-financing** for a payment process A if $C_t^\varphi = C_0^\varphi$ $\mathbb{P}^*$-a.s. for all $0 \leq t \leq T$.*

*(ii) A payment process A is said to be **attainable** if there exists a self-financing, 0-admissible strategy φ, i.e. $V_T^\varphi = 0$ $\mathbb{P}$-a.s.*

In the previous chapter we already showed that a trading strategy is self-financing (in the sense of the elementary model, see Definition 2.3.3) if and only if the condition of Definition 5.1.3 (i) holds true, i.e., if and only if $C_t^\varphi = C_0^\varphi$ $\mathbb{P}^*$-a.s. for all $0 \leq t \leq T$. Furthermore, the following Lemma states the equivalence of the concepts of attainability in both models.

Lemma 5.1.4. *[8] The payment process A is attainable (in the sense of Definition 5.1.3 (ii)) if and only if the T-claim $H = A_T$ is classically attainable.*

The proof of Lemma 5.1.4 follows directly by both definitions of attainability.

Finally, we need to slightly adapt the definition of the **intrinsic value process**, which was given by the conditional expected value of the claim H given all information up to a specific point in time.

Definition 5.1.5. *[8] The intrinsic value process $V^* = (V_t^*)_{0\leq t\leq T}$ is defined by*

$$V_t^* := \mathbb{E}^*[A_T|\mathcal{F}_t],$$

for all $0 \leq t \leq T$.

We end this section by comparing Møller's (1998) simple model with the extended model of Møller (2001):

Remark 5.1.6. *Comparison of the simple model and the generalized one*[11]*:* Definition 3.1.1 gives the cost process $\overline{C_t}^{\overline{\varphi}}$ in the elementary model with claims H in one point in time only

$$\overline{C_t}^{\overline{\varphi}} = V_t^{\varphi} - \int_0^t \overline{\xi_u} dS_u^*. \tag{5.1.2}$$

Remember that in the simple model we restricted claims H to be contingent claims with maturity T only, i.e., $V_T^{\overline{\varphi}} = H$. As one can see, this definition is independent of the claim H, because trading on the financial market and paying the insurance benefits H were clearly separated in time. In this simple model, no trading took place after paying the benefits. In the extension of the model, we exchangee T-claims H by general payment streams A, with possible payments at any point in time during the observed time horizon $[0, T]$. Therefore, the cost process can no longer be defined independently from the payment stream A.

We quit this comparison by showing that the extended framework reduces to the elementary world by setting

$$A_t = -\kappa + I_{t\geq T} H, \quad 0 \leq t \leq T,$$

for some constant $\kappa \in \mathbb{R}$ and $H \in \mathcal{L}^2(\mathbb{P}_S^*)$. Additional set $\kappa = 0$ and restrict to 0-admissible strategies φ, then the total costs at maturity T are given by

$$C_T^{\varphi} = V_T^{\varphi} - \int_0^T \xi_u dS_u^* + A_T = -\int_0^T \xi_u dS_u^* + H. \tag{5.1.3}$$

On the other hand, define a simple, elementary trading strategy $\overline{\varphi} = (\overline{\xi}, \overline{\eta})$ by setting $\overline{\xi_t} = \xi_t$ for all $0 \leq t \leq T$ and $\overline{\eta_t} = \eta_t$ for all $0 \leq t < T$ with $\eta_T = H - \overline{\xi_T} S_T^*$. Obviously, the strategies φ and $\overline{\varphi}$ differ by their terminal amount of cash only. Therefore, both cost processes are equal, i.e., $C_t^{\varphi} = \overline{C_t}^{\overline{\varphi}}$, for all $0 \leq t < T$. Furthermore, the cost process at maturity are also equivalent. This follows by (5.1.2) together with $V_T^{\overline{\varphi}} = H$ and equation (5.1.3).

[11] In this remark we equip all object corresponding to the simple model with a bar $\bar{}$.

5.2 Generalization of risk-minimization in incomplete markets

Before we are able to find risk-minimizing trading strategies hedging general unit-linked life insurance contracts, we need to extend Föllmer and Sondermann's (1986) theory of risk-minimization in incomplete markets.

As in the elementary framework, the risk-minimizing trading strategy is found by applying the Galtchouk-Kunita-Watanabe decomposition of martingales (see Theorem 3.1.6) for the martingale V^*. Therefore, let us first recall the result of decomposing the intrinsic value process in accordance to the Galtchouk-Kunita-Watanabe decomposition.

Theorem 5.2.1. *[8] The intrinsic value process V^* can be uniquely decomposed by use of the Galtchouk-Kunita-Watanabe decomposition as*

$$V_t^* = V_0^* + \int_0^t \xi_u^A dS_u^* + L_t^A, \tag{5.2.1}$$

for all $0 \leq t \leq T$, where L^A is a zero-mean martingale which is orthogonal to S^*, i.e., the process $S^* L^A$ is a martingale, and ξ^A is a predictable process satisfying the integrability condition $\xi^A \in \mathcal{L}^2(\mathbb{P}_S^*)$.

With the help of this expression for the intrinsic value process V^*, we are able to formulate the extension of Föllmer and Sondermann's (1986) mean squared error minimization of the remaining risk of the insurer.

Theorem 5.2.2. *There exists a unique 0-admissible risk-minimizing strategy $\varphi = (\xi, \eta)$ for the payment stream A given by*

$$(\xi_t, \eta_t) = (\xi_t^A, V_t^* - A_t - \xi_t^A S_t^*), \tag{5.2.2}$$

for all $0 \leq t \leq T$. The associated risk process is given by $R_t^\varphi = \mathbb{E}^*[(L_T^A - L_t^A)^2 | \mathcal{F}_t]$.

The proof is similar to the proof of the corresponding theorem in the simple set-up (see proof of Theorem 3.1.15).

Proof. The statement is shown in three steps:

(i) <u>0-Admissibility:</u>
The trading strategy φ defined in (5.2.2) fulfils the 0-admissibility property, since for the value process at time T we get

$$V_T^\varphi = \xi_T S_T^* + \eta_T = \xi_T^A S_T^* + V_T^* - A_T - \xi_T^A S_T^* = \mathbb{E}^*[A_T | \mathcal{F}_T] - A_T = 0.$$

(ii) Risk-minimization:
We begin this part of the proof by showing the following expression for the payments at maturity, A_T,

$$\begin{aligned}A_T =& \mathbb{E}^*[A_T|\mathcal{F}_T] = V_T^* = V_0^* + \int_0^T \xi_u^A dS_u^* + L_T^A = \\ =& V_t^* - \int_0^t \xi_u^A dS_u^* - L_t^A + \int_0^T \xi_u^A dS_u^* + L_T^A = \\ =& V_t^* + \int_t^T \xi_u^A dS_u^* + L_T^A - L_t^A.\end{aligned}$$

where we used (5.2.1) to rewrite the intrinsic value process at time 0 and T.

Let us introduce an arbitrary 0-admissible trading strategy by $\widetilde{\varphi} = (\widetilde{\xi}, \widetilde{\eta})$. Using $V_T^{\widetilde{\varphi}} = 0$ together with the above expression for A_T we can rewrite the difference of the costs of the strategy $\widetilde{\varphi}$ at time t and maturity T

$$\begin{aligned}C_T^{\widetilde{\varphi}} - C_t^{\widetilde{\varphi}} =& V_T^{\widetilde{\varphi}} - \int_0^T \widetilde{\xi}_u dS_u^* + A_T - \Big(V_t^{\widetilde{\varphi}} - \int_0^t \widetilde{\xi}_u dS_u^* + A_t\Big) = \\ =& - \int_0^T \widetilde{\xi}_u dS_u^* + \Big(V_t^* + \int_t^T \xi_u^A dS_u^* + L_T^A - L_t^A\Big) - \\ & - \Big(V_t^{\widetilde{\varphi}} - \int_0^t \widetilde{\xi}_u dS_u^* + A_t\Big) = \\ =& \Big(V_t^* - A_t - V_t^{\widetilde{\varphi}}\Big) + \Big(L_T^A - L_t^A\Big) + \int_t^T \big(\xi_u^A - \widetilde{\xi}_u\big) dS_u^*.\end{aligned}$$

Finally, with the orthogonality of L^A and S^* and the $\mathcal{F}_t$-measurability of $\Big(V_t^* - A_t - V_t^{\widetilde{\varphi}}\Big)$ the remaining risk process of $\widetilde{\varphi}$ can be written as

$$\begin{aligned}R_t^{\widetilde{\varphi}} =& \mathbb{E}^*\Big[\big(C_T^{\widetilde{\varphi}} - C_t^{\widetilde{\varphi}}\big)^2|\mathcal{F}_t\Big] = \\ =& \mathbb{E}^*\Big[\big(L_T^A - L_t^A\big)^2|\mathcal{F}_t\Big] + \Big(V_t^* - A_t - V_t^{\widetilde{\varphi}}\Big)^2 + \\ & + \mathbb{E}^*\Big[\int_t^T \big(\xi_u^A - \widetilde{\xi}_u\big)^2 d[S^*]_u|\mathcal{F}_t\Big].\end{aligned}$$

Now we can see, that the remaining risk can be minimized by first choosing $\widetilde{\xi} = \xi^A$ and then setting $\widetilde{\eta}$ such that $V_t^{\widetilde{\varphi}} = V_t^* - A_t$ for all

$0 \leq t \leq T$. With this choice, the remaining risk process reduces to

$$R_t^{\varphi} = \mathbb{E}^* \left[\left(L_T^A - L_t^A \right)^2 | \mathcal{F}_t \right].$$

(iii) Uniqueness:
Analogously to step (iii) in the proof of Theorem 3.1.15.

□

The result of Theorem 5.2.2 is consistent with the corresponding theorem in the elementary framework, which declared the following risk-minimizing trading strategy (see Theorem 3.1.15):

$$(\xi_t, \eta_t) = (\xi_t^H, V_t^* - \xi_t^H S_t^*), \quad 0 \leq t \leq T.$$

Setting $A_t = -\kappa + I_{t \geq T} H, 0 \leq t \leq T$ with $\kappa = 0$ the generalized result reduces to the simple trading strategy, hedging the single T-claim H.

Furthermore, notice that in the generalized set-up the amount of cash on the savings account, η, is reduced by the total outgoing cash-flow the insurer has to face at time t, i.e., A_t. This additional adjustment term ensures that the value of the financial portfolio equals the conditional expected amount of all cumulated future outgoing cash-flows from the insurer's point of view given all information up to the present point in time. Mathematically, this means:

$$\begin{aligned} V_t^{\varphi} =& \xi_t S_t^* + \eta_t = \xi_t^A S_t^* - V_t^* - A_t - \xi_t^A S_t^* = \mathbb{E}^*[A_T | \mathcal{F}_t] - A_t \overset{A_t \text{ is } \mathcal{F}_t\text{-m.a.}}{=} \\ =& \mathbb{E}^*[A_T - A_t | \mathcal{F}_t]. \end{aligned}$$

5.3 General unit-linked life insurance contracts

In this section we will find the optimal financial portfolio, which minimizes the remaining risk, the insurer has to face when selling general unit-linked life insurance contracts. These type of insurance contracts were analyzed by Møller (2001) and are specified by outgoing benefit and incoming premium payments (from the insurer's point of view), both contingent on the life-length of the policy holder. The premium payments are predefined in the insurance contract and are paid until either the maturity of the contract (T) or the time of death of the insurance holder take place. In return, the policy holder receives some benefit payments, which depend on the life length of the insured person, e.g. a fixed payment when the insurance holder survives up to time of maturity or a payment immediately upon some contractually predefined event. Both cash-flows are allowed to depend on the price process of the underlying stock, S.

We will proceed analogously to the simple set-up, meaning that we first find the Galtchouk-Kunita-Watanabe decomposition of the intrinsic value process and afterwards apply the theory of risk-minimization in incomplete markets. To complete the theory in the advanced framework, we will give some explicit examples at the end of this section.

5.3.1 The generalized insurance model

Before starting with finding the required decomposition of the intrinsic value process, we need to specify the underlying insurance model, introduced by Møller (2001). There are two big differences now, compared to the insurance model in the simple elementary set-up:

- In contrast to the previous chapters we do not analyze a whole portfolio of insurance contracts, but rather we will concentrate on one single unit-linked life insurance contract.
- The simple insurance model was based on a two-state Markov model, since with the counting process N we counted the amount of deaths up to a specific time. In this chapter we work with a multi-state Markov model and the contractual cash-flows will depend on the current state of the policy.

The insured lives

Let us define this finite set of possible policy states by a set $\mathcal{J} = \{0, 1, ..., J\}$, where 0 is the initial state. The Markov process describing the states of the policy over time is given by $Z = (Z_t)_{0 \leq t \leq T}$, which is $\mathcal{F}$-adapted and right-continuous with values in $\mathcal{J}$. Since we defined 0 to be the initial state, the initial distribution of Z is given by $(1, 0, ..., 0)$.

As in the previous chapters we again make some simplifying assumptions. First we assume $\mathcal{H} = (\mathcal{H}_t)_{0 \leq t \leq T}$ to be the $\mathbb{P}^*$-augmented natural filtration of the Markov process Z. Furthermore, let the processes Z and (B, S) be independent of each other and finally assume $\mathcal{F}$ to be the $\mathbb{P}^*$-augmented natural filtration of Z and (B, S).

Next let us define the multivariate **counting process** $N = (N^{jk})_{j \neq k}$ for all states $j, k \in \mathcal{J}$ by

$$N_t^{jk} = \#\{s | s \in (0, t], Z_{s^-} = j, Z_s = k\}, \tag{5.3.1}$$

for all $0 < t \leq T$. N_t^{jk} counts the number of transitions from state j to state k during the time interval $[0, t]$.

Further, introduce the indicator processes $I = (I^j)_{j \in \mathcal{J}}$ by

$$I_t^j = I_{\{Z_t = j\}} = \begin{cases} 1 & \text{if } Z_t = j \\ 0 & \text{otherwise} \end{cases}, \tag{5.3.2}$$

for all $0 \leq t \leq T$.

Analogously to the previous chapters we assume that there exist **transition rates** λ^{jk} and continuous deterministic **hazard rates** μ^{jk} for the Markov chain Z given by

$$\lambda_t^{jk} = I_{t-}^j \mu_t^{jk}, \quad \text{for all } 0 \leq t \leq T, \tag{5.3.3}$$

i.e., the transition rate to move from state j to state k corresponds to the hazard rate of the appropriate states at time t, if the policy stays in state j before time t. Otherwise, if the policy is in any other state at time t the transition rate equals 0.

Next define the **compensated counting processes** $M = (M^{jk})_{j \neq k}$ by

$$M_t^{jk} = N_t^{jk} - \int_0^t \lambda_u^{jk} du, \quad \text{for all } 0 \leq t \leq T. \tag{5.3.4}$$

Since we assumed μ^{jk} being continuous, deterministic functions, it follows that the compensated counting process M^{jk} are $\mathcal{H}$-martingales and the counting processes N^{jk} possess intensities. Furthermore, the martingales M^{jk} are orthogonal to each other and their quadratic variation is given by

$$[M^{jk}]_t = \int_0^t \lambda_u^{jk} du = \int_0^t I_u^j \mu_u^{jk} du, \quad \text{for all } 0 \leq t \leq T, \tag{5.3.5}$$

due to the fact that the processes N^{jk} do not have any simultaneous jumps.

Note also that the compensated counting processes M^{jk} and the discounted stock prices process S^* are independent, since we assumed Z to be independent of the economic informations given via the filtration $\mathcal{G}$. Therefore, they can be separated taking the expectation

$$\mathbb{E}^*\big[S_t^* M_t^{jk} | \mathcal{F}_s\big] = \mathbb{E}^*\big[S_t^* | \mathcal{F}_s\big] \mathbb{E}^*\big[M_t^{jk} | \mathcal{F}_s\big],$$

for $0 \leq s \leq t \leq T$. Together with Remark 3.1.7 it follows that S^* and M^{jk} are orthogonal.

Finally, we end the description of the insured lives with defining the **transition probabilities** of the Markov process Z by

$$p_{jk}(t, u) = \mathbb{P}(Z_u = k | Z_t = j), \tag{5.3.6}$$

for all $0 \leq t \leq u \leq T$ and for all $j, k \in \mathcal{J}$. This means, that $p_{jk}(t, u)$ gives the probability of being in state k at time u, if the process sojourned in state j at time t. The transition probabilities can be determined by **Kolmogorov's backward differential equations**:

$$\frac{d}{dt} p_{jk}(t, u) = \sum_{l, l \neq j} \mu_t^{jl} \big(p_{jk}(t, u) - p_{lk}(t, u)\big), \quad \text{for all } 0 \leq t \leq u, \tag{5.3.7}$$

$$p_{jk}(t, t) = 1_{\{j=k\}}. \tag{5.3.8}$$

For a detailed proof of this statement see Koller (2010) Theorem 2.3.4. Basically, Koller (2010) gives a firm and well-written overview of Markov process in insurance mathematics in Chapters 2.2-2.4.

The insurance contracts

As Møller (2001) we analyze unit-linked insurance contracts consisting of two different basic **benefit payments**:

- General life insurances:
  Immediately upon transition from state j to state k at time t the insurer has to pay the amount $g_t^{jk} = g^{jk}(t, S_t)$.

- State-wise life annuities:
  If the policy sojourns in state j at time t, the annuities are described by continuous payments with rate $g_t^j = g^j(t, S_t)$.

For all $j, k \in \mathcal{J}$, the functions $(t, s) \to g^{jk}(t, s)$ and $(t, s) \to g^j(t, s)$ are functions of the current stock price S_t only and are assumed to be measurable and fulfil

$$\mathbb{E}^* \left[(B_t^{-1} g^{jk}(t, S_t))^2 \right] < \infty, \tag{5.3.9}$$

$$\mathbb{E}^* \left[(B_t^{-1} g^{j}(t, S_t))^2 \right] < \infty, \tag{5.3.10}$$

which ensures that the processes $\int B^{-1} g^{jk} dM^{jk}$ to be square-integrable martingales. All these assumption guarantee, that the single benefit payments described by $g^{jk}(t, S_t)$ and $g^j(t, S_t)$ are just simple t-claims in the sense of Definition 2.3.4 (i). Together with the completeness of the financial market (B, S), these claims are attainable (see Definition 2.3.4 (ii)) and can uniquely be priced arbitrage-free on the financial market.

Let the **prices of the single claims** $g^{jk}(u, S_u)$ and $g^j(u, S_u)$ at time t be denoted by

$$F^{jk}(t, S_t, u) = \mathbb{E}^*\Big[B_t B_u^{-1} g^{jk}(u, S_u)|\mathcal{G}_t\Big], \tag{5.3.11}$$

$$F^{j}(t, S_t, u) = \mathbb{E}^*\Big[B_t B_u^{-1} g^{j}(u, S_u)|\mathcal{G}_t\Big], \tag{5.3.12}$$

for all $0 \le t \le u \le T$. These prices reduce to the arbitrage-free price process in the elementary model (see Definition 2.3.6) if we only allow T-claims instead of general u-claims. We assume the price processes $(t, s, u) \to F^{jk}$ and $(t, s, u) \to F^j$ to be measurable functions, that are continuously differentiable w.r.t t and two times differentiable w.r.t. s. Furthermore, we assume the first partial derivatives w.r.t. s (as in the prior chapters denoted by F_s^{jk} and F_s^j) to be uniformly bounded, i.e.$\exists K < \infty$ such that for all t, s, u and all F^{jk} and F^j it holds that

$$|F_s^{jk}(t, s, u)| \le K < \infty, \tag{5.3.13}$$

$$|F_s^{j}(t, s, u)| \le K < \infty. \tag{5.3.14}$$

Note that the prices processes F^{jk} and F^j are martingales by definition and that

$$F^{jk}(t, S_t, t) = g^{jk}(t, S_t) = g_t^{jk}, \tag{5.3.15}$$

$$F^{j}(t, S_t, t) = g^{j}(t, S_t) = g_t^{j} \tag{5.3.16}$$

holds true. Analogously, to the elementary set-up (see proof of Theorem 4.2.1), the following expressions for the differentials of the discounted price processes under the measure $\mathbb{P}^*$ can be shown

$$d(B_t^{-1} F^{jk}(t, S_t, u)) = F_s^{jk}(t, S_t, u) dS_t^*, \tag{5.3.17}$$

$$d(B_t^{-1} F^{j}(t, S_t, u)) = F_s^{j}(t, S_t, u) dS_t^*. \tag{5.3.18}$$

Next let us specify the **payment process**, $\widehat{A} = (\widehat{A}_t)_{0\le t\le T}$, arising from selling one unit-linked life insurance contract consisting of general life insurance claims, g^{jk}, and state-wise life annuities, g^j,

$$d\widehat{A}_t = \sum_{j\in\mathcal{J}} I_t^j g_t^j dt + \sum_{j,k,j\neq k} g_t^{jk} dN_t^{jk} = \sum_{j\in\mathcal{J}} \Big(I_t^j g_t^j dt + \sum_{k,k\neq j} g_t^{jk} dN_t^{jk} \Big). \tag{5.3.19}$$

From this, the following expression for the **discounted value of payments** can be easily derived:

$$A_t = A_0 + \int_0^t B_u^{-1} d\widehat{A}_u = A_0 + \int_0^t B_u^{-1} \sum_{j\in\mathcal{J}} \Big(I_{u-}^j g_u^j du + \sum_{k,k\neq j} g_u^{jk} dN_u^{jk} \Big). \tag{5.3.20}$$

Møller (2001) summarized the underlying insurance model by stating that the process A "specifies payments that are contingent on the development of the policy (as described by Z) and are linked to the development on the financial market in that the amounts g_u^j and g_u^{jk} are time-dependent functions of the stock price" [8].

Next let us consider the intrinsic value process V^* given in Definition 5.1.5 when choosing the payment process A in accordance with (5.3.20). In the following we use $\mathbb{E}^*[dN^{jk}|\mathcal{H}_{t-}] = \lambda_t^{jk}dt = I_{t-}^j\mu_t^{jk}dt$, the martingale property of $\int B^{-1}g^{jk}dM^{jk}$ and the independence between Z and (B,S) to derive

$$\begin{aligned}
V_t^* =&\mathbb{E}^*\Big[A_T \mid \mathcal{F}_t\Big] = A_t + \mathbb{E}^*\Big[(A_T - A_t) \mid \mathcal{F}_t\Big] = \\
=&A_t + \mathbb{E}^*\left[\int_t^T B_u^{-1}\sum_{j\in\mathcal{J}}\Big(I_{u-}^j g_u^j du + \sum_{k,k\neq j} g_u^{jk} dN_u^{jk}\Big) \mid \mathcal{F}_t\right] = \\
=&A_t + \mathbb{E}^*\left[\int_t^T B_u^{-1}\sum_{j\in\mathcal{J}}\Big(I_{u-}^j g_u^j du + \sum_{k,k\neq j} g_u^{jk} I_{u-}^j\mu_u^{jk} du\Big) \mid \mathcal{F}_t\right] = \\
=&A_t + B_t^{-1}\sum_{j\in\mathcal{J}}\int_t^T p_{Z_t j}(t,u)\left(F^j(t,S_t,u) + \sum_{k,k\neq j}\mu_u^{jk}F^{jk}(t,S_t,u)\right)du.
\end{aligned}$$

Instead of writing $p_{Z_t j}$ we introduce **auxiliary processes** V^i for $i \in \mathcal{J}$, that give the state-dependent expected value of total cumulated future benefits less premiums and are mathematically expressed by

$$V^i(t,S_t) = \sum_{j\in\mathcal{J}}\int_t^T p_{ij}(t,u)\left(F^j(t,S_t,u) + \sum_{k,k\neq j}\mu_u^{jk}F^{jk}(t,S_t,u)\right)du,$$

for $0 \leq t \leq T$. Equivalently, these auxiliary processes can formally be defined with the help of conditional expectation

$$V^i(t,s) = \mathbb{E}^*\Big[B_t\int_t^T B_u^{-1}d\widehat{A}_u \mid Z_t = i, S_t = s\Big]. \tag{5.3.21}$$

$V^i(t,S_t)$ can be interpreted as the current market price of the insurance contract with payment stream A at time t conditioned on the fact that the policy sojourns in state i at time t and on the stock price at time t, S_t.

Conditioned on being in state i at time t (expressed by multiplication with I_t^i) and summed up over the finite set of all possible states of the policy, $\mathcal{J}$,the intrinsic value process V^* can be written as

$$V_t^* = A_t + \sum_{i\in\mathcal{J}} I_t^i V^i(t,S_t)B_t^{-1}. \tag{5.3.22}$$

5.3.2 The risk-minimizing trading strategy

In this section we will determine the risk-minimizing trading strategy, hedging the insurer's risk arising from selling a general unit-linked life insurance contract with payment stream A defined in (5.3.20). Therefore, we will first find the Galtchouk-Kunita-Watanabe decomposition of the martingale V^* (see Møller (2001) Lemma 3.2) and then use this representation to apply Theorem 5.2.2 to obtain the insurer's optimal trading strategy (see Møller (2001) Theorem 3.4).

Theorem 5.3.1. *[8] The Galtchouk-Kunita-Watanabe decomposition of V^* is given by*

$$V_t^* = V_0^* + \int_0^t \left(\sum_{j \in \mathcal{J}} I_{u-}^i \xi_u^i \right) dS_u^* + \sum_{j,k,j \neq k} \int_0^t \nu_u^{jk} dM_u^{jk}, \tag{5.3.23}$$

where

$$\xi_t^i = \sum_{j \in \mathcal{J}} \int_t^T p_{ij}(t,u) \left(F_s^j(t,S_t,u) + \sum_{k,k \neq j} \mu_u^{jk} F_s^{jk}(t,S_t,u) \right) du, \tag{5.3.24}$$

$$\nu_t^{jk} = B_t^{-1} \big(g_t^{jk} + V^k(t,S_t) - V^j(t,S_t) \big). \tag{5.3.25}$$

Proof. The proof is structured in several steps:

(i) First we show that the discounted auxiliary processes $V^i(t,S_t)$ fulfil

$$B_t^{-1} V^i(t,S_t) = V^i(0,S_0) + \int_0^t \xi_\tau^i dS_\tau^* - \int_0^t \left(B_\tau^{-1} g_\tau^i + \sum_{k,k \neq i} \mu_\tau^{ik} \nu_\tau^{ik} \right) d\tau.$$

Therefore, we introduce the following notation for all $i \in \mathcal{J}$ and $0 \leq t \leq u \leq T$

$$Y_t^{i,u} = \sum_{j \in \mathcal{J}} p_{ij}(t,u) B_t^{-1} \left(F^j(t,S_t,u) + \sum_{k,k \neq j} \mu_u^{jk} F^{jk}(t,S_t,u) \right),$$

such that $B_t^{-1} V^i(t,S_t) = \int_t^T Y_t^{i,u} du$. Next we use the product rule, equations (5.3.17)-(5.3.18) and Kolmogorov's backward differential equation 5.3.7 for p_{ij}

$$dp_{ij}(t,u) = \Big(\sum_{l,l \neq i} \mu_t^{il} (p_{ij}(t,u) - p_{lj}(t,u) \Big) dt \tag{5.3.26}$$

to obtain

$$
\begin{aligned}
dY_t^{i,u} =& \sum_{j\in\mathcal{J}} dp_{ij}(t,u) B_t^{-1}\Bigg(F^j(t,S_t,u) + \sum_{k,k\neq j} \mu_u^{jk} F^{jk}(t,S_t,u)\Bigg) \\
&+ \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Bigg(d\Big(B_t^{-1}F^j(t,S_t,u)\Big) \\
&+ \sum_{k,k\neq j} \mu_u^{jk} d\Big(B_t^{-1}F^{jk}(t,S_t,u)\Big)\Bigg) \overset{(5.3.26)\text{ and }(5.3.17)-(5.3.18)}{=} \\
=& \sum_{j\in\mathcal{J}}\sum_{l,l\neq i} \mu_t^{il} p_{ij}(t,u) B_t^{-1}\Bigg(F^j(t,S_t,u) + \sum_{k,k\neq j} \mu_u^{jk} F^{jk}(t,S_t,u)\Bigg) dt \\
&- \sum_{j\in\mathcal{J}}\sum_{l,l\neq i} \mu_t^{il} p_{lj}(t,u) B_t^{-1}\Bigg(F^j(t,S_t,u) + \sum_{k,k\neq j} \mu_u^{jk} F^{jk}(t,S_t,u)\Bigg) dt \\
&+ \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Bigg(F_s^j(t,S_t,u)dS_t^* + \sum_{k,k\neq j} \mu_u^{jk} F_s^{jk}(t,S_t,u)dS_t^*\Bigg) = \\
=& \sum_{l,l\neq i} \mu_t^{il}\Big(Y_t^{i,u} - Y_t^{l,u}\Big) dt \\
&+ \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Bigg(F_s^j(t,S_t,u) + \sum_{k,k\neq j} \mu_u^{jk} F_s^{jk}(t,S_t,u)\Bigg) dS_t^*.
\end{aligned}
$$

For keeping notation simple we set

$$dY_t^{i,u} = \alpha_t^{i,u} dt + \beta_t^{i,u} dS_t^*, \tag{5.3.27}$$

with

$$
\begin{aligned}
\alpha_t^{i,u} &= \sum_{l,l\neq i} \mu_t^{il}\Big(Y_t^{i,u} - Y_t^{l,u}\Big), \\
\beta_t^{i,u} &= \sum_{j\in\mathcal{J}} p_{ij}(t,u)\Bigg(F_s^j(t,S_t,u) + \sum_{k,k\neq j} \mu_u^{jk} F_s^{jk}(t,S_t,u)\Bigg).
\end{aligned}
$$

Furthermore, using the integral form of (5.3.27)

$$Y_t^{i,u} = Y_0^{i,u} + \int_0^t dY_\tau^{i,u} = Y_0^{i,u} + \int_0^t \alpha_\tau^{i,u} d\tau + \int_0^t \beta_\tau^{i,u} dS_\tau^*$$

together with (5.3.27) for $t \leq u$ we obtain

$$\begin{aligned}
B_t^{-1}V^i(t,S_t) &= \int_t^T Y_t^{i,u}du = \int_t^T \left(Y_0^{i,u} + \int_0^t \alpha_\tau^{i,u}d\tau + \int_0^t \beta_\tau^{i,u}dS_\tau^*\right)du = \\
&= \int_0^T \left(Y_0^{i,u} + \int_0^t I_{\{\tau\leq u\}}\alpha_\tau^{i,u}d\tau + \int_0^t I_{\{\tau\leq u\}}\beta_\tau^{i,u}dS_\tau^*\right)du \\
&\quad - \int_0^t \left(Y_0^{i,u} + \int_0^u \alpha_\tau^{i,u}d\tau + \int_0^u \beta_\tau^{i,u}dS_\tau^*\right)du = \\
&= \int_0^T Y_0^{i,u}du - \int_0^t Y_u^{i,u}du + \int_0^T\int_0^t I_{\{\tau\leq u\}}\alpha_\tau^{i,u}d\tau du \\
&\quad + \int_0^T\int_0^t I_{\{\tau\leq u\}}\beta_\tau^{i,u}dS_\tau^* du.
\end{aligned}$$

Now we rewrite all four integrals involved in the expression above to derive the required result:

By definition of $Y^{i,u}$ the first integral reduces to

$$\int_0^T Y_0^{i,u}du = V^i(0,S_0)$$

and the second integral can be written as

$$\begin{aligned}
\int_0^t Y_u^{i,u}du &= \int_0^t \left[\sum_{j\in\mathcal{J}} p_{ij}(u,u)B_u^{-1}\Big(F^j(u,S_u,u) + \sum_{k,k\neq j}\mu_u^{jk}F^{jk}(u,S_u,u)\Big)\right]du = \\
&= \int_0^t \left[B_u^{-1}\Big(F^i(u,S_u,u) + \sum_{k,k\neq i}\mu_u^{ik}F^{ik}(u,S_u,u)\Big)\right]du \overset{(5.3.15)-(5.3.16)}{=} \\
&= \int_0^t \left[B_u^{-1}\Big(g_u^i + \sum_{k,k\neq i}\mu_u^{ik}g_u^{ik}\Big)\right]du,
\end{aligned}$$

since the policy sojourns in state i at time u.

Note that for all $i \in \mathcal{J}$ the functions $(\omega,t,u) \to \alpha_t^{i,u}(\omega)$ are measurable and

$$\int_0^T\int_0^t I_{\{\tau\leq u\}}|\alpha_\tau^{i,u}|d\tau du < \infty \quad \mathbb{P}-\text{a.s.}$$

Hence, using the standard Fubini theorem we can rewrite the first double integral by interchanging the order of the integrals and get

$$\begin{aligned}
&\int_0^T \int_0^t I_{\{\tau \le u\}} \alpha_\tau^{i,u} d\tau du = \int_0^t \int_\tau^T \alpha_\tau^{i,u} du \; d\tau = \\
&= \int_0^t \int_\tau^T \sum_{l, l \neq i} \mu_\tau^{il} \Big(Y_\tau^{i,u} - Y_\tau^{l,u} \Big) du \; d\tau = \\
&= \int_0^t \left(\sum_{l, l \neq i} \mu_\tau^{il} \int_\tau^T \Big(Y_\tau^{i,u} - Y_\tau^{l,u} \Big) du \right) d\tau = \\
&= \int_0^t \left(\sum_{l, l \neq i} \mu_\tau^{il} B_\tau^{-1} \Big(V^i(\tau, S_\tau) - V^l(\tau, S_\tau) \Big) \right) d\tau.
\end{aligned}$$

Since there is a stochastic integral involved in the second double integral, Fubini's standard theorem is not applicable, though the Fubini theorem for stochastic integral does the job, see [3]. Therefore, again note that for all $i \in \mathcal{J}$ the functions $(\omega, t, u) \to \beta_t^{i,u}(\omega)$ are measurable and due to assumptions (5.3.13)-(5.3.14) uniformly bounded for each $i \in \mathcal{J}$. Using Fubini's theorem we obtain

$$\begin{aligned}
&\int_0^T \int_0^t I_{\{\tau \le u\}} \beta_\tau^{i,u} dS_\tau^* du = \int_0^t \int_\tau^T \beta_\tau^{i,u} du \; dS_\tau^* = \\
&= \int_0^t \int_\tau^T \sum_{j \in \mathcal{J}} p_{ij}(\tau, u) \left(F_s^j(\tau, S_\tau, u) + \sum_{k, k \neq j} \mu_u^{jk} F_s^{jk}(\tau, S_\tau, u) \right) du \; dS_\tau^* = \\
&= \int_0^t \xi_\tau^i dS_\tau^*.
\end{aligned}$$

All four rewritten integrals put together show step (i):

$$\begin{aligned}
B_t^{-1} V^i(t, S_t) =& V^i(0, S_0) - \int_0^t \left[B_\tau^{-1} \Big(g_\tau^i + \sum_{k, k \neq i} \mu_\tau^{ik} g_\tau^{ik} \Big) \right] d\tau + \int_0^t \xi_\tau^i dS_\tau^* \\
&+ \int_0^t \left(\sum_{k, k \neq i} \mu_\tau^{ik} B_\tau^{-1} \Big(V^i(\tau, S_\tau) - V^k(\tau, S_\tau) \Big) \right) d\tau = \\
=& V^i(0, S_0) + \int_0^t \xi_\tau^i dS_\tau^* - \int_0^t \left(B_\tau^{-1} g_\tau^i + \sum_{k, k \neq i} \mu_\tau^{ik} \nu_\tau^{ik} \right) d\tau,
\end{aligned}$$

where we set $u = \tau$ in the second integral and $l = k$ in the fourth one.

(ii) Finally with the help of the result of point (i), we proof the decomposition of V^*.

Using

$$dI_t^i = \sum_{k,k\neq i} \big(dN_t^{ki} - dN_t^{ik}\big), \tag{5.3.28}$$

and the partial integration formula for $V_t^* = A_t + \sum_{i\in\mathcal{J}} I_t^i V^i(t,S_t)B_t^{-1}$ from (5.3.22), we obtain

$$\begin{aligned}
dV_t^* =& dA_t + \sum_{i\in\mathcal{J}} I_{t^-}^i d\big(V^i(t,S_t)B_t^{-1}\big) + \sum_{i\in\mathcal{J}} V^i(t^-,S_{t^-})B_{t^-}^{-1} dI_t^i \overset{(5.3.20),(5.3.28)\text{ and (i)}}{=} \\
=& B_t^{-1}\sum_{i\in\mathcal{J}}\left(I_t^i g_t^i dt + \sum_{k,k\neq i} g_t^{ik} dN_t^{ik}\right) - \sum_{i\in\mathcal{J}} I_{t^-}^i \left(B_t^{-1} g_t^i + \sum_{k,k\neq i}\mu_t^{ik}\nu_t^{ik}\right)dt \\
&+ \sum_{i\in\mathcal{J}} I_{t^-}^i \xi_t^i dS_t^* + \sum_{i,k,i\neq k} B_t^{-1}\Big(V^k(t,S_t) - V^i(t,S_t)\Big)dN_t^{ik}.
\end{aligned}$$

Rearranging the terms we see that the $B_t^{-1}\sum_{i\in\mathcal{J}} I_t^i g_t^i dt$ term cancels out and the expression reduces to

$$\begin{aligned}
dV_t^* =& \sum_{i,k,i\neq k} B_t^{-1}\Big(g_t^{ik} + V^k(t,S_t) - V^i(t,S_t)\Big)dN_t^{ik} \\
&- \sum_{i,k,i\neq k} I_{t^-}^i \mu_t^{ik}\nu_t^{ik} dt + \sum_{i\in\mathcal{J}} I_{t^-}^i \xi_t^i dS_t^* = \\
=& \sum_{i,k,i\neq k} B_t^{-1}\nu_t^{ik} dM_t^{ik} + \sum_{i\in\mathcal{J}} I_{t^-}^i \xi_t^i dS_t^*,
\end{aligned}$$

and that shows decomposition (5.3.24) with setting index $j = i$.

(iii) The last step is to show that the terms included in the decomposition fulfil all requirements listed in Theorem 5.2.1.

First note that assumptions (5.3.9)-(5.3.10) ensure that g^{jk}, V^k and V^{jk} are square-integrable and that the integrals w.r.t. the compensated counting process M^{jk} are square-integrable, zero-mean martingales.

Furthermore, the boundedness of F_s^{jk} and F_s^j (assumptions (5.3.13)-(5.3.14)) guarantee that the integral w.r.t. the discounted stock price S^* is a square-integrable martingale and ξ^A a predictable process.

Finally with the orthogonality of S^* and M^{jk} all conditions are fulfilled.

□

This proof is based on the proof by Møller (2001). Additionally, he presents the idea of a proof of Theorem 5.3.1 when assuming V^i to be continuously differentiable w.r.t. t and twice continuously differentiable w.r.t. s, i.e. $V^i \in \mathcal{C}^{1,2}$. With this assumption the proof is shorter than the general one we stated above, but as Møller (2001) points out "proving that $V^i \in \mathcal{C}^{1,2}$ turns out to be rather laborious" [8]. Hence this paper presents the general proof only and for detailed information about proving statement 5.3.1 with the special assumption for V^i see Møller (2001).

Before obtaining the risk-minimizing trading strategy, the Galtchouk-Kunita-Watanabe decomposition will be interpreted in the following.

Remark 5.3.2. In (5.3.23) the integral w.r.t. the compensated counting process M^{jk}

$$\sum_{j,k,j\neq k} \int_0^t \nu_u^{jk} dM_u^{jk} = \sum_{j,k,j\neq k} \int_0^t B_u^{-1}\Big(g_u^{jk} + V^k(u,S_u) - V^j(u,S_u)\Big) dM_u^{jk}$$

can be seen as the non-hedgeable part of the insurer's payment stream A. Compared with the theory of traditional life insurance the ν_u^{jk} represents the **sum at risk** of the reserve of the insurance company. In particular, $B_u^{-1} g_u^{jk}$ gives the discounted benefit payment when the policy transitions from state j to state k at time u. Furthermore, the difference $B_u^{-1}\Big(V^k(u,S_u) - V^j(u,S_u)\Big)$ represents the loss or gain of the actuarial reserve if the state of the policy changes from state j to state k at time u.

Finally we will apply Theorem 5.2.2 to determine the risk-minimizing trading strategy and its intrinsic risk process.

Theorem 5.3.3. *[8] For the payment process (5.3.20), the unique* 0*-admissible risk-minimizing hedging strategy is given by*

$$\varphi_t = (\xi_t, \eta_t) = \left(\sum_{i\in\mathcal{J}} I_{t-}^i \xi_t^i, \sum_{i\in\mathcal{J}} I_t^i B_t^{-1} V^i(t,S_t) - S_t^* \sum_{i\in\mathcal{J}} I_{t-}^i \xi_t^i \right), \tag{5.3.29}$$

with the intrinsic risk process R^φ *given by*

$$R_t^\varphi = \sum_{i\in\mathcal{J}} I_t^i \int_t^T \sum_{j,k,j\neq k} \mathbb{E}^*\left[\left(\nu_u^{jk}\right)^2 \mid \mathcal{F}_t \right] p_{ij}(t,u) \mu_u^{jk} du, \tag{5.3.30}$$

where ξ^i *and* ν^i *were determined in Theorem 5.3.1 and are given by*

$$\xi_t^i = \sum_{j\in\mathcal{J}} \int_t^T p_{ij}(t,u) \left(F_s^j(t,S_t,u) + \sum_{k,k\neq j} \mu_u^{jk} F_s^{jk}(t,S_t,u) \right) du,$$
$$\nu_t^{jk} = B_t^{-1}\big(g_t^{jk} + V^k(t,S_t) - V^j(t,S_t)\big).$$

Proof. The proof is a direct application of Theorem 5.2.2 together with Theorem 5.3.1.

For the amount of stocks held in the portfolio at time t, ξ_t, it follows that

$$\xi_t \overset{\text{Theorem 5.2.2}}{=} \xi_t^A \overset{\text{Theorem 5.3.1}}{=} \sum_{i\in\mathcal{J}} I_{t-}^i \xi_t^i,$$

and for the amount of cash at the bank account at time t, η_t, we get

$$\begin{aligned}\eta_t \overset{\text{Theorem 5.2.2}}{=} & V_t^* - A_t - \xi_t^A S_t^* \overset{(5.3.22)}{=} \\ = & A_t + \sum_{i\in\mathcal{J}} I_t^i V^i(t, S_t) B_t^{-1} - A_t - \xi_t^A S_t^* = \\ = & \sum_{i\in\mathcal{J}} I_t^i V^i(t, S_t) B_t^{-1} - S_t^* \sum_{i\in\mathcal{J}} I_{t-}^i \xi_t^i.\end{aligned}$$

To get the required expression for the intrinsic risk process use Theorem 5.2.2, the orthogonality of the M^{jk} and the independence between Z and (B, S)

$$\begin{aligned}R_t^\varphi = & \mathbb{E}^*\left[\left(L_T^A - L_t^A\right)^2 \mid \mathcal{F}_t\right] = \mathbb{E}^*\left[\left(\int_t^T \sum_{j,k,j\neq k} \nu_u^{jk} dM_u^{jk}\right)^2 \mid \mathcal{F}_t\right] = \\ = & \mathbb{E}^*\left[\int_t^T \sum_{j,k,j\neq k} \left(\nu_u^{jk}\right)^2 d[M^{jk}]_u \mid \mathcal{F}_t\right] \overset{(5.3.5)}{=} \\ = & \mathbb{E}^*\left[\int_t^T \sum_{j,k,j\neq k} \left(\nu_u^{jk}\right)^2 \lambda_u^{jk} du \mid \mathcal{F}_t\right] = \\ = & \int_t^T \sum_{j,k,j\neq k} \mathbb{E}* \left[\left(\nu_u^{jk}\right)^2 \mid \mathcal{F}_t\right] \mathbb{E}^*\left[\lambda_u^{jk} \mid \mathcal{H}_t\right] du \overset{(5.3.3)}{=} \\ = & \sum_{i\in\mathcal{J}} I_t^i \int_t^T \sum_{j,k,j\neq k} \mathbb{E}^*\left[\left(\nu_u^{jk}\right)^2 \mid \mathcal{F}_t\right] p_{ij}(t,u) \mu_u^{jk} du.\end{aligned}$$

□

The discounted value process V^φ corresponding to the risk-minimizing trading strategy specified in (5.3.29) is given by

$$V_t^\varphi = \xi_t S_t^* + \eta_t = \sum_{j\in\mathcal{J}} I_t^i B_t^{-1} V^i(t, S_t),$$

for all $0 \le t \le T$. Since by (5.3.21) the V^i equals the **prospective reserve** in state $i \in \mathcal{J}$ of classical life insurance theory, which is defined as the discounted expected value of future benefits less premiums conditioned on the policy's state. Therefore, the optimal hedging strategy φ is defined such that its value V^φ equals the prospective reserve at any time t, which makes the concept of risk-minimization very natural.

5.3.3 Examples

We conclude this chapter by discussing two examples presented by Møller (2001) (Examples 3.7-3.8). Analogously to Section 4.2.2 we work with the Black-Scholes model (see Example 2.1.3), where we assume constant deterministic interest rate r, drift term α and volatility σ.

(1) First we consider a **single term insurance with single premium**. This means that we analyze one insurance contract that pays its benefit immediately upon death of the insured person, if this happens before time of maturity T. We already discussed a term insurance in Example 4.3.5 and we will see that with the following assumptions the extended model will reduce to the simple one and we will derive an optimal trading strategy in accordance to the one in Example 4.3.5. The only difference will be the time of payments, since in the simple set-up we restricted claims to be H-claims only, in the generalized model we allow benefit payments occurring at any time $t \in [0, T]$.

We define a two-state space by $\mathcal{J} = \{0, 1\}$, where state 0 represents the initial state *insured person alive* and state 1 means *insured person dead*. Further, let x be the age of the insured person at time 0 and T_x the time of death of the insured person or the person's remaining lifetime after time 0. Since state 1 (death) is absorbing, the multivariate counting processes N^{jk} reduces to

$$N_t^{01} = I_{\{T_x \leq t\}},$$

and the intensity is given by the deterministic function $\mu^{01} = \mu$. There are two transition probabilities. The survival probability

$$p_{00}(t, u) = {}_{u-t}p_{x+t} = e^{-\int_t^u \mu_\tau d\tau} \quad \text{for } 0 \leq t \leq u \leq T,$$

which gives the probability that the insured person survives up to time u conditioned that she is still alive at time t, and the probability of death, which equals the complementary probability of p_{00}, i.e. $p_{01} = {}_{u-t}q_{x+t} = 1 - p_{00}$. The payment stream is defined by the benefit payment function

$$g^{01}(t, S_t) = \max(S_t, Ke^{\delta t}),$$

which pays at least a guarantee $Ke^{\delta t}$ at time of death, and the premium stream

$$\Delta G_0^0 = -\kappa,$$

where K, δ and κ are some constants. For the expected value of future benefit payments, the Black-Scholes pricing formula for European options (see (4.2.2) Example (2) Theorem 4.2.7) gives

$$\begin{aligned} F^{01}(t, S_t, u) =& \mathbb{E}^* \Big[e^{-r(u-t)} \big(Ke^{\delta u} + (S_u - Ke^{\delta u})^+ \big) \mid \mathcal{F}_t \Big] = \\ =& Ke^{\delta u} e^{-r(u-t)} + S_t \Phi(d_+^{(u,t)}) - Ke^{\delta u} e^{-r(u-t)} \Phi(d_-^{(u,t)}) = \\ =& Ke^{\delta u} e^{-r(u-t)} \Phi(-d_-^{(u,t)}) + S_t \Phi(d_+^{(u,t)}), \end{aligned}$$

where

$$d_\pm^{(u,t)} = \frac{\log(\frac{S_t}{Ke^{\delta u}}) + (r \pm \frac{1}{2}\sigma^2)(u-t)}{\sigma\sqrt{u-t}},$$

and Φ denotes the distribution function of a standard-normal distributed random variable. Applying Theorem 5.3.3 we obtain the risk-minimizing trading strategy

$$\begin{aligned} \xi_t =& \sum_{i \in \mathcal{J}} I_{t-}^i \xi_t^i = \\ =& \sum_{i \in \mathcal{J}} I_{t-}^i \sum_{j \in \mathcal{J}} \int_t^T p_{ij}(t,u) \left(F_s^j(t, S_t, u) + \sum_{k, k \neq j} \mu_u^{jk} F_s^{jk}(t, S_t, u) \right) du = \\ =& I_{\{T_x \geq t\}} \int_t^T p_{00}(t,u) \mu_u \Phi(d_+^{(u,t)}) du, \end{aligned}$$

$$\begin{aligned} \eta_t =& \sum_{i \in \mathcal{J}} I_t^i B_t^{-1} V^i(t, S_t) - S_t^* \sum_{i \in \mathcal{J}} I_{t-}^i \xi_t^i = \\ =& \sum_{i \in \mathcal{J}} I_t^i B_t^{-1} \left[\sum_{j \in \mathcal{J}} \int_t^T p_{ij}(t,u) \left(F^j(t, S_t, u) + \sum_{k, k \neq j} \mu_u^{jk} F^{jk}(t, S_t, u) \right) du \right] \\ &- S_t^* I_{\{T_x \geq t\}} \int_t^T p_{00}(t,u) \mu_u \Phi(d_+^{(u,t)}) du = \end{aligned}$$

$$
\begin{aligned}
=&I_{\{T_x>t\}}\int_t^T p_{00}(t,u)\mu_u\Big(Ke^{-(r-\delta)u}\Phi(-d_-^{(u,t)})+S_t^*\Phi(d_+^{(u,t)})\Big)du\\
&-S_t^*I_{\{T_x\geq t\}}\int_t^T p_{00}(t,u)\mu_u\Phi(d_+^{(u,t)})du=\\
=&I_{\{T_x>t\}}\int_t^T p_{00}(t,u)\mu_u Ke^{-(r-\delta)u}\Phi(-d_-^{(u,t)})du\\
&-S_t^*I_{\{T_x=t\}}\int_t^T p_{00}(t,u)\mu_u\Phi(d_+^{(u,t)})du.
\end{aligned}
$$

Even though the optimal strategy φ does not depend on the amount of the single premium κ, the initial cost is influenced by κ:

$$
\begin{aligned}
C_0^\varphi =&V_0^\varphi-\kappa=\xi_0S_0^*+\eta_0-\kappa=\\
=&\int_0^T p_{00}(0,u)\mu_u\Phi(d_+^{(u,0)})du\\
&+\int_0^T p_{00}(0,u)\mu_u Ke^{-(r-\delta)u}\Phi(-d_-^{(u,0)})du-\kappa=\\
=&\int_0^T p_{00}(0,u)\mu_u F^{01}(0,S_0,u)du-\kappa.
\end{aligned}
$$

Therefore, we obtain a fair market price for the term insurance contract given by the single premium

$$
\kappa=\int_0^T p_{00}(0,u)\mu_u F^{01}(0,S_0,u)du.
$$

(2) We extend Example (1) by discussing an insurance portfolio of n term insurance contracts with the same contractual terms as in (1). Furthermore, we assume the remaining lifetimes T_i to be i.i.d. and use a common hazard rate function μ. To describe a portfolio of n contracts with our present model, we need to define the state space as $\mathcal{J}=\{0,1...,n\}$, where state $i\in\mathcal{J}$ represents the state of the portfolio if exactly i insured persons having died. Then the transition rates can be written as

$$
\lambda_t^{ik}=I_t^iI_{\{k=i+1\}}(n-i)\mu_t,
$$

for $i,k\in\{0,1,...,n-1\}$ and the probability for switching from one state to another between time points u and t are given by

$$
p_{ii}(t,u)=e^{-\int_t^u(n-i)\mu_\tau d\tau},
$$

$p_{ik} = 0$ for $i > k \in \{0, 1, ..., n-1\}$ and p_{ik} for $i < k \in \{0, 1, ..., n-1\}$ determined by Kolmogorov's backward differential equation (5.3.7). We can again use F^{01} from (1) and obtain for the risk-minimizing strategy for hedging the whole insurance portfolio

$$\begin{aligned}
\xi_t =& \sum_{i \in \mathcal{J}} I^i_{t-} \xi^i_t = \\
=& \sum_{i \in \mathcal{J}} I^i_{t-} \sum_{j \in \mathcal{J}} \int_t^T p_{ij}(t,u) \left(F^j_s(t, S_t, u) + \sum_{k, k \neq j} \mu^{jk}_u F^{jk}_s(t, S_t, u) \right) du = \\
=& \int_t^T \sum_{j=Z_{t-}}^{n} p_{Z_{t-}j}(t,u)(n-j)\mu_u \Phi(d_+^{(u,t)}) du,
\end{aligned}$$

$$\begin{aligned}
\eta_t =& \sum_{i \in \mathcal{J}} I^i_t B_t^{-1} V^i(t, S_t) - S^*_t \sum_{i \in \mathcal{J}} I^i_{t-} \xi^i_t = \\
=& \sum_{i \in \mathcal{J}} I^i_t B_t^{-1} \left[\sum_{j \in \mathcal{J}} \int_t^T p_{ij}(t,u) \left(F^j(t, S_t, u) + \sum_{k, k \neq j} \mu^{jk}_u F^{jk}(t, S_t, u) \right) du \right] \\
& - S^*_t \int_t^T \sum_{j=Z_{t-}}^{n} p_{Z_{t-}j}(t,u)(n-j)\mu_u \Phi(d_+^{(u,t)}) du = \\
=& \int_t^T \sum_{j=Z_{t-}}^{n} p_{Z_{t-}j}(t,u)(n-j)\mu_u e^{-rt} \left(K e^{\delta u} e^{-r(u-t)} \Phi(-d_-^{(u,t)}) + S_t \Phi(d_+^{(u,t)}) \right) du \\
& - S^*_t \int_t^T \sum_{j=Z_{t-}}^{n} p_{Z_{t-}j}(t,u)(n-j)\mu_u \Phi(d_+^{(u,t)}) du.
\end{aligned}$$

To reduce the terms in the strategy, note the following property of the Markov process Z: Since we assumed the remaining lifetimes of the insured persons to be i.i.d. with common hazard rate function μ, the conditional distribution of $(n-Z_u)$ given Z_t is $Bin\left((n-Z_t), e^{-\int_t^u \mu_\tau d\tau}\right)$ distributed, for $0 \le t \le u \le T$. This implies

$$\sum_{j=Z_{t-}}^{n} p_{Z_{t-}j}(t,u)(n-j) = \mathbb{E}^*\left[(n-Z_u) \mid Z_t\right] = (n-Z_t)e^{-\int_t^u \mu_\tau d\tau},$$

and with this φ reduces to

$$\begin{aligned}\xi_t =&(n - Z_{t^-})e^{-\int_t^u \mu_\tau d\tau} \int_t^T \mu_u \Phi(d_+^{(u,t)})du,\\ \eta_t =&(n - Z_t) \int_t^T e^{-\int_t^u \mu_\tau d\tau} \mu_u K e^{-(r-\delta)u} \Phi(-d_-^{(u,t)})du-\\ &- \Delta Z_t S_t^* \int_t^T e^{-\int_t^u \mu_\tau d\tau} \mu_u \Phi(d_+^{(u,t)})du.\end{aligned}$$

Chapter 6

Numeric example

This thesis will be completed by the presentation of some numeric results according to the simple model of Chapter 4. We will analyze the portfolio consisting of pure endowment insurance contracts.

Financial model:
The financial world is simulated by a Black-Scholes model with risk-free interest rate $r = 0.03$ and initial stock and bond prices, $S_0 = 1$ and $B_0 = 1$, respectively. We specify a time horizon with starting point 0 and maturity T and furthermore define a number of intervals n, that divide the observed period $[0, T]$ into n parts of the same length $\Delta t = \frac{T}{n}$. Then the vector of bond prices B is given by

$$B_i = e^{r * i\Delta t} \quad \text{for } i \in \{0, 1, ..., n\}.$$

For simulating the stock prices at every point of the mesh of the partition, we need to simulate n i.i.d. standard normal distributed random variables, i.e., $z_i \overset{i.i.d}{\sim} \mathcal{N}(0, 1)$ for $i \in \{0, 1, ..., n\}$ and obtain one path of the stock price under the equivalent martingale measure $\mathbb{P}^*$ as

$$S_i = \exp\left((r - \frac{\sigma^2}{2}) i\Delta t + \sum_{k=1}^{i} z_i \sqrt{\sigma^2 \Delta t} \right), \quad \text{for } i \in \{0, 1, ..., n\}.$$

Since we want to apply Monte Carlo simulation, we need to simulate not only one path of stock prices but many different possibilities. The number of paths is specified by m. Hence we get a matrix of stock prices $S = (S_i^j)_{i \in \{0,1,...,n\}, j \in \{0,1,...,m\}}$.

Mortality model:
We model the mortality of the insured persons by the Gompertz-Makeham

law [13], which is specified by the hazard rate function

$$\mu_{x+i\Delta t} = A + Bc^{x+i\Delta t}, \quad \text{for } i \in \{0, 1, ..., n\}.$$

Because of (4.1.1) the $i\Delta t$-year survival probability of an x-year old is given by

$$1 - F(i\Delta t) := {}_{i\Delta t}p_x = \exp\left(- At - \frac{Bc^{x+i\Delta t}}{\ln(Bc)} + \frac{Bc^x}{\ln(Bc)} \right), \quad \text{for } i \in \{0, 1, ..., n\},$$

where the last term in the sum guarantees that the 0-year survival probability equals 1. Furthermore, the counting process $N_{i\Delta t}$ which gives the number of deaths up to time $i\Delta t$ is modelled by

$$N_{x+i\Delta t} = \sum_{k=1}^{l_x} I_{\{T_k \leq i\Delta t\}}, \quad \text{for } i \in \{0, 1, ..., n\},$$

with $T_k \overset{i.i.d}{\sim} F$. The random variables T_k give the time of death of the x-year old insured persons and are sampled with the help of the inversion method, i.e., for every path $j \in \{1, ..., m\}$ we sample l_x independent standard uniformly distributed random variables ($U_k \overset{i.i.d}{\sim} \mathcal{U}(0,1)$ for $k \in \{1, ..., l_x\}$) and use the well known fact that

$$T_k = F^{-1}(U_k) \Rightarrow T_k \sim F, \quad \text{for } k \in \{0, 1, ..., l_x\}.$$

Pure endowment insurance:

We want to obtain the optimal risk-minimizing hedging strategy for an insurer selling unit-linked endowment contracts with a guarantee K. The payment stream for this sort of contract is specified by $g(s) = \max(s, K)$, where K gives the non-negative constant guarantee. In Example (2) in Section 4.2.2 we used the fact that the benefit payment function g equals the payment stream of an European Call Option with strike price K and so with the help of the Black-Scholes' pricing formula, (see Theorem 4.2.7) we obtain

$$F^g(i\Delta t, S_{i\Delta t}) = Ke^{-r(T-i\Delta t)}\Phi(-d_-) + S_{i\Delta t}\Phi(d_+), \quad \text{for } i \in \{0, 1, ..., n\}.$$

Furthermore, using Monte Carlo simulations we can directly calculate the number of stocks, ξ, and bonds, η, in the insurer's risk-minimizing portfolio at all points of the grid, which splits the time horizon,

$$\begin{aligned}
\xi_{i\Delta t} =& (l_x - N_{(i\Delta t)^-})_{T-i\Delta t}p_{x+i\Delta t}\Phi(d_+), \\
\eta_{i\Delta t} =& (l_x - N_{i\Delta t})_{T-i\Delta t}p_{x+i\Delta t}Ke^{-rT}\Phi(-d_-) - \Delta N_{i\Delta t T-i\Delta t}p_{x+i\Delta t}\Phi(d_+)S^*_{i\Delta t}
\end{aligned}$$

for $i \in \{0,1,...,n\}$.

To calculate the intrinsic value process we again use Monte Carlo simulation and

$$V_{i\Delta t} = (l_x - N_{i\Delta t})_{T-i\Delta t}p_{x+i\Delta t}B_t^{-1}F^g(i\Delta t, S_{i\Delta t}), \quad \text{for } i \in \{0,1,...,n\}, \tag{6.0.1}$$

Finally, we can give explicit numeric results for the initial intrinsic risk by using

$$\begin{aligned} R_0^\varphi &= l_{xT}p_x \int_0^T \mathbb{E}^*\Big[\big(-e^{-ru}F^g(u,S_u)\big)^2\Big]_{T-u}p_{x+u}\mu_{x+u}du = \\ &= l_{xT}p_x \sum_{i=0}^{n-1} \mathbb{E}^*\Big[\big(-e^{-r*i\Delta t}F^g(i\Delta t,S_{i\Delta t})\big)^2\Big]_{T-i\Delta t}p_{x+i\Delta t}\mu_{x+i\Delta t}\Delta t, \end{aligned}$$

for $i \in \{0,1,...,n\}$, where we apply numeric integration and Monte Carlo simulation.

In the last part, some numeric results are presented. If not otherwise mentioned, all of them are based on the following parameters:

- time horizon $T = 20$,
- number of time intervals $n = 100$ and number of simulated paths $m = 1000$,
- initial age of the policy holders $x = 30$,
- starting number of policy holders $l_x = 100$,
- volatility $\sigma = 0.25$,
- guarantee $K = 1.1$ and
- mortality parameters $A = 0.05$, $B = 0.0009$ and $c = 1.01904$.

Note Table 6.1, which gives the legend for all following figures.

– Number of bonds
– Number of stocks

Table 6.1: Legend

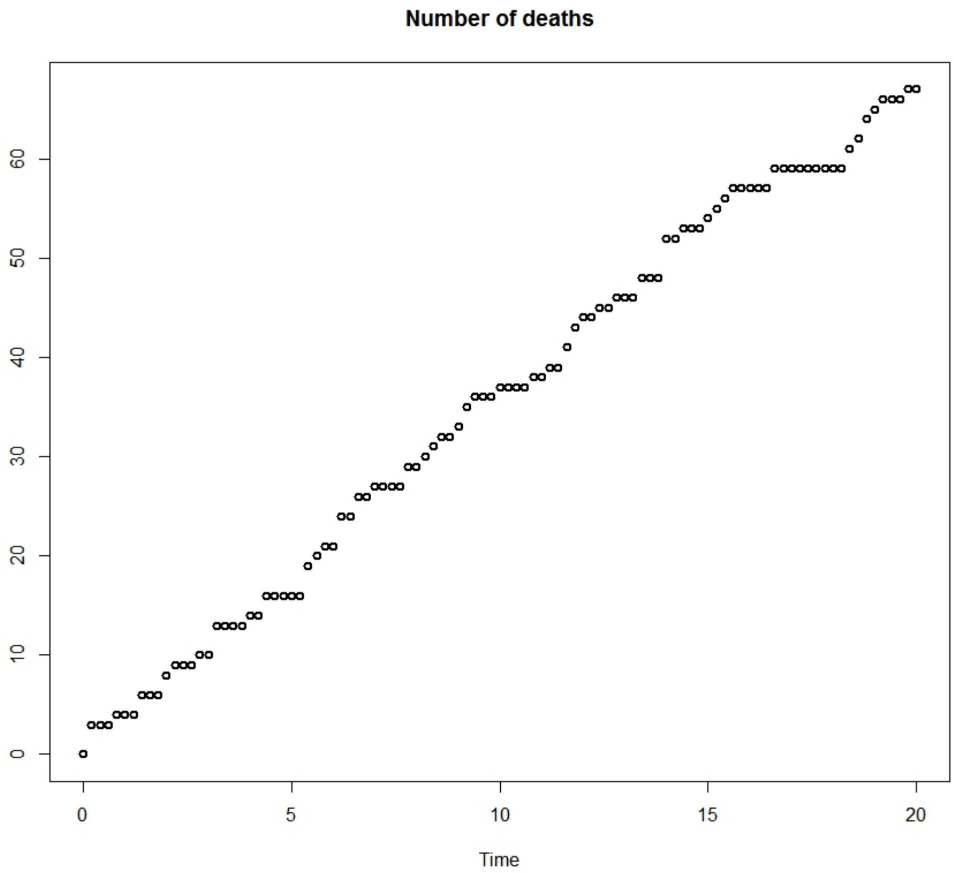

Figure 6.1: Number of deaths, one path

To give a first idea about the step function N and the number of stocks and bonds held in the portfolio $\varphi = (\xi, \eta)$, we plot one path of the $m \times l_x$-dimensional matrix N and one path of the portfolio φ, see Figure 6.1 and Figure 6.2.

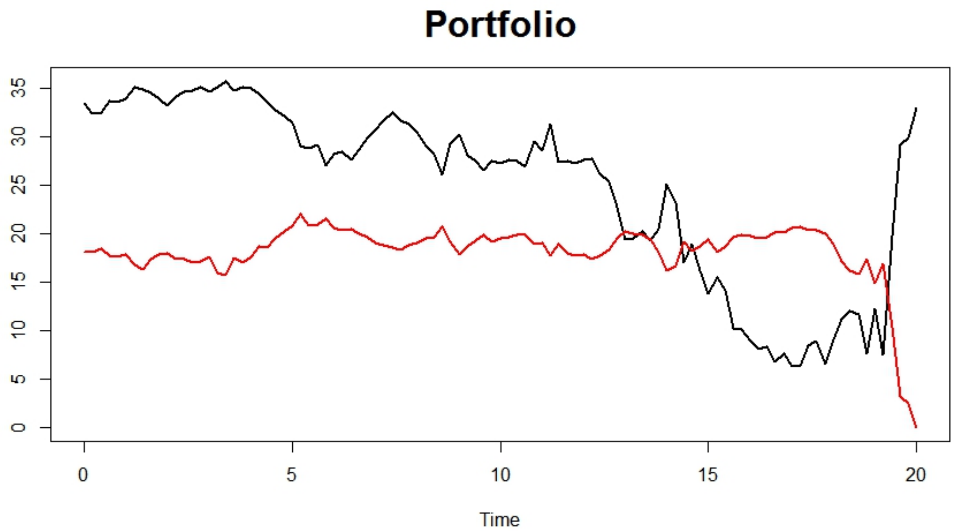

Figure 6.2: Portfolio, one path

Now we want to interpret the results by varying the parameters of the model. Therefore, we analyze the variation of single parameters only, in order to understand their influence on the results.

6.1 Variation of the number of paths m

Is 1000 a reasonable number of paths to obtain stable results from Monte Carlo simulation? To answer this question consider Figure (6.3). The average (mean) financial portfolio, consisting of ξ number of stocks (black line) and η number of bonds (red line) behaves nearly the same in all three considered situations. One can only observe a slight smoothing of the lines when the number of simulations is increased. Generally, we see that the average number of bonds in the optimal financial portfolio stays almost constant over time and the average number of stocks decreases slowly.

Furthermore, in Table 6.2 the influence of the number of paths on the initial intrinsic value, V_0^*, and the initial intrinsic risk, R_0^φ is analyzed. We can see, that the initial fair price of the insurance portfolio of pure endowment contracts, V_0^*, does not depend on the number of paths, which corresponds with expression (6.0.1) and the fact that $F^g(0, S_0)$ is independent of the future stock price. In contrast to the value of the portfolio, the initial intrinsic risk, R_0^φ, which the insurer has to face when selling pure endowment insurance contracts, varies significantly with changing the number of simulations.

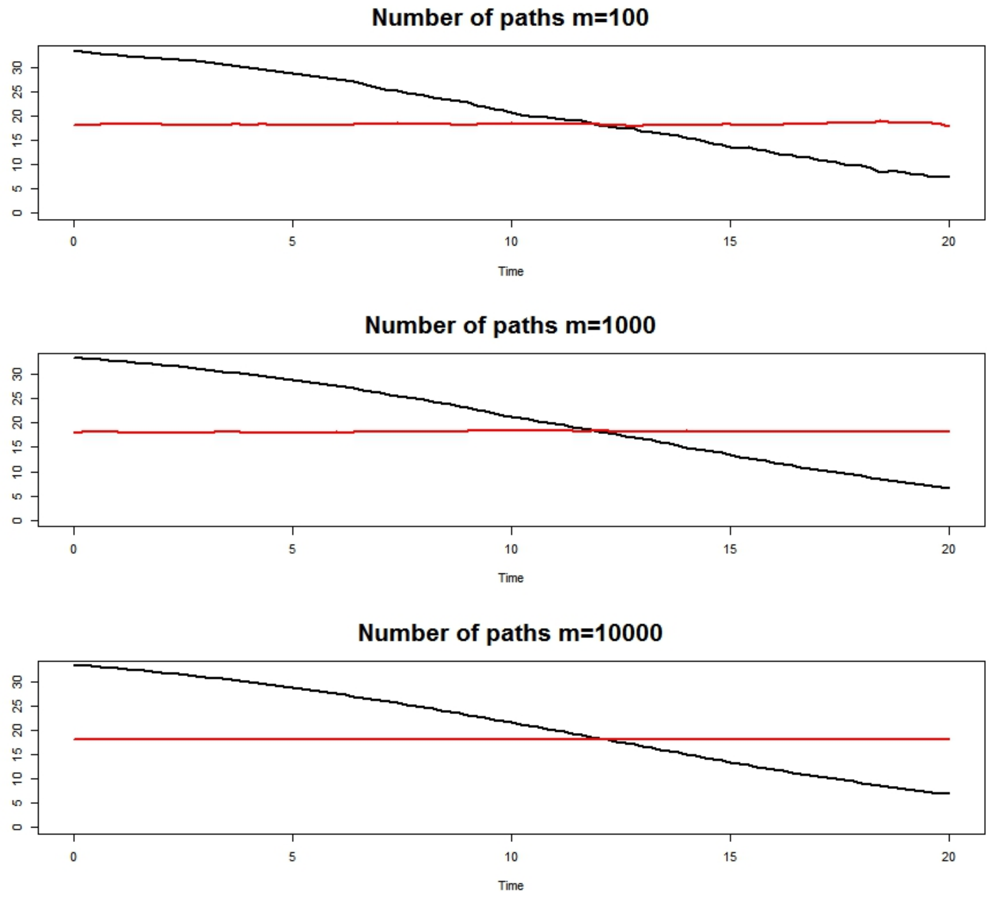

Figure 6.3: Variation of the number of paths m; Portfolio

$m = 100$	$V_0^* = 51.58$	$R_0^\varphi = 59.18$
$m = 1000$	$V_0^* = 51.58$	$R_0^\varphi = 116.98$
$m = 10000$	$V_0^* = 51.58$	$R_0^\varphi = 94.70$

Table 6.2: Variation of the number of paths m; Value and risk

6.2 Variation of the volatility σ

Next we discuss the behaviour of the financial portfolio, when varying the volatility of the stock price process in the Black-Scholes model. Figure 6.4 shows, that for all considered volatilities the optimal financial portfolios behave very similar, meaning that the average number of bonds stays almost constant and the average number of stocks decreases over time. One can observe that the optimal number of bonds increases, when the volatility of the stock increases, i.e., if the stock gets more risky the insurer should have more bonds in the portfolio to minimize the risk. Additionally, the number of stocks decreases when the volatility of the risky asset increases.

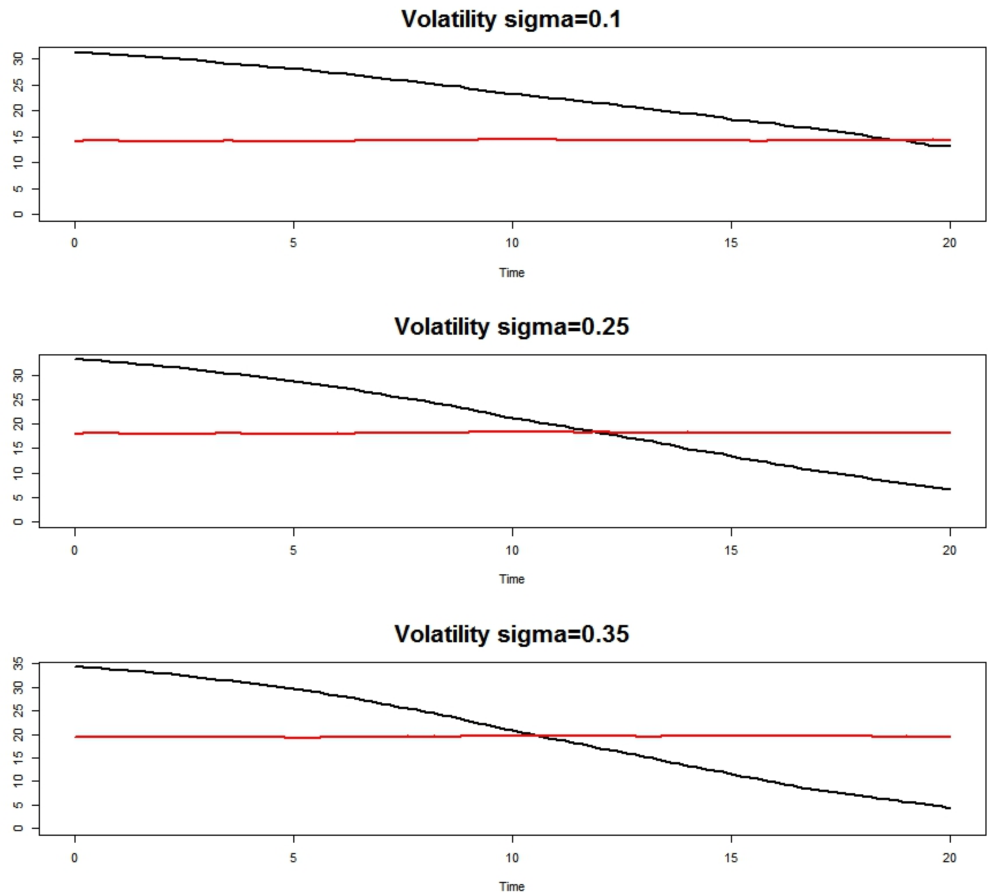

Figure 6.4: Variation of the volatility σ; Portfolio

Table 6.3 shows the influence of the variation of the financial market's volatility on the initial intrinsic value and the initial intrinsic risk. We can see that the insurer's risk at time $t = 0$ depends heavily on the uncertainty of the financial market. If the volatility of the stock increases the initial risk increases disproportionately high. The market price of the insurance portfolio also increases with growing volatility, but is less affected by the variation of the volatility then the initial risk.

$\sigma = 0.1$	$V_0^* = 45.65$	$R_0^\varphi = 75.17$
$\sigma = 0.25$	$V_0^* = 51.58$	$R_0^\varphi = 116.98$
$\sigma = 0.35$	$V_0^* = 53.75$	$R_0^\varphi = 162.48$

Table 6.3: Variation of the volatility σ; Value and risk

6.3 Variation of the mortality parameter c

Let us analyze the behaviour of the optimal trading strategy when the mortality parameter c changes. Figure 6.5 shows the number of stocks and bonds held at time t in accordance with the risk-minimizing trading strategy. The three situations give no significant difference. But when observing the initial intrinsic value and the initial intrinsic risk, see Table 6.4, we see that the initial intrinsic risk, R_0^φ, as well as the initial fair price of the insurance portfolio, V_0^*, increase when the hazard rate function increases. Note, that the risk reacts more sensitive in changing the mortality than the intrinsic value.

$c = 1$	$V_0^* = 51.57$	$R_0^\varphi = 114.27$
$c = 1.01904$	$V_0^* = 51.58$	$R_0^\varphi = 116.98$
$c = 1.2$	$V_0^* = 55.59$	$R_0^\varphi = 721.54$

Table 6.4: Variation of the mortality c; Value and risk

6.4 Variation of the time of maturity T

Finally we will see that the time of maturity has a big influence on the optimal financial portfolio as well as on the intrinsic value and the intrinsic risk. As Figure 6.6 shows, the average number of bonds and stocks in the risk-minimizing financial portfolio decreases when the time of maturity increases. This means, that if the insured persons need to survive a longer period of time to obtain their sum insured (T increases), the insurer is able to hold a small financial portfolio. In contrast to the situation when the time of maturity is very small, i.e., the insured persons will survive the end of the contract with a high probability. Then the insurer has to hold a big number of stocks and bonds in the optimal financial portfolio to ensure that all policy holders get their sum insured at the end of the contract.

Furthermore, the intrinsic value and the intrinsic risk also depend heavily on the time of maturity of the contracts. If T is small the insurer has to invest a lot of money in the financial portfolio to hedge the risk arising from the insurance contracts. Therefore, the initial fair price of the insurance portfolio

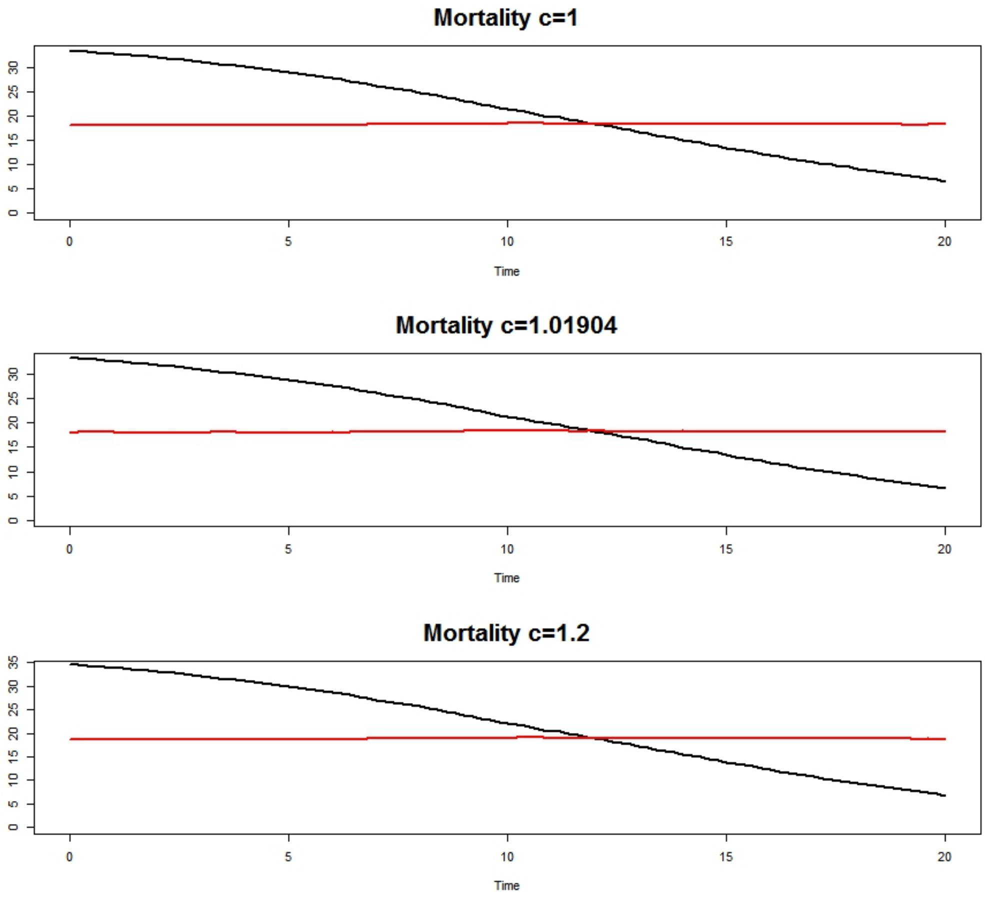

Figure 6.5: Variation of the mortality c; Portfolio

is higher than the price of the same portfolio when the time of maturity is larger. Finally, as one would expect, the intrinsic risk of the insurer increases with increasing time of maturity.

$T = 5$	$V_0^* = 107.98$	$R_0^\varphi = 38.71$
$T = 20$	$V_0^* = 51.58$	$R_0^\varphi = 116.98$
$T = 40$	$V_0^* = 17.17$	$R_0^\varphi = 385.68$

Table 6.5: Variation of the time of maturity T; Value and risk

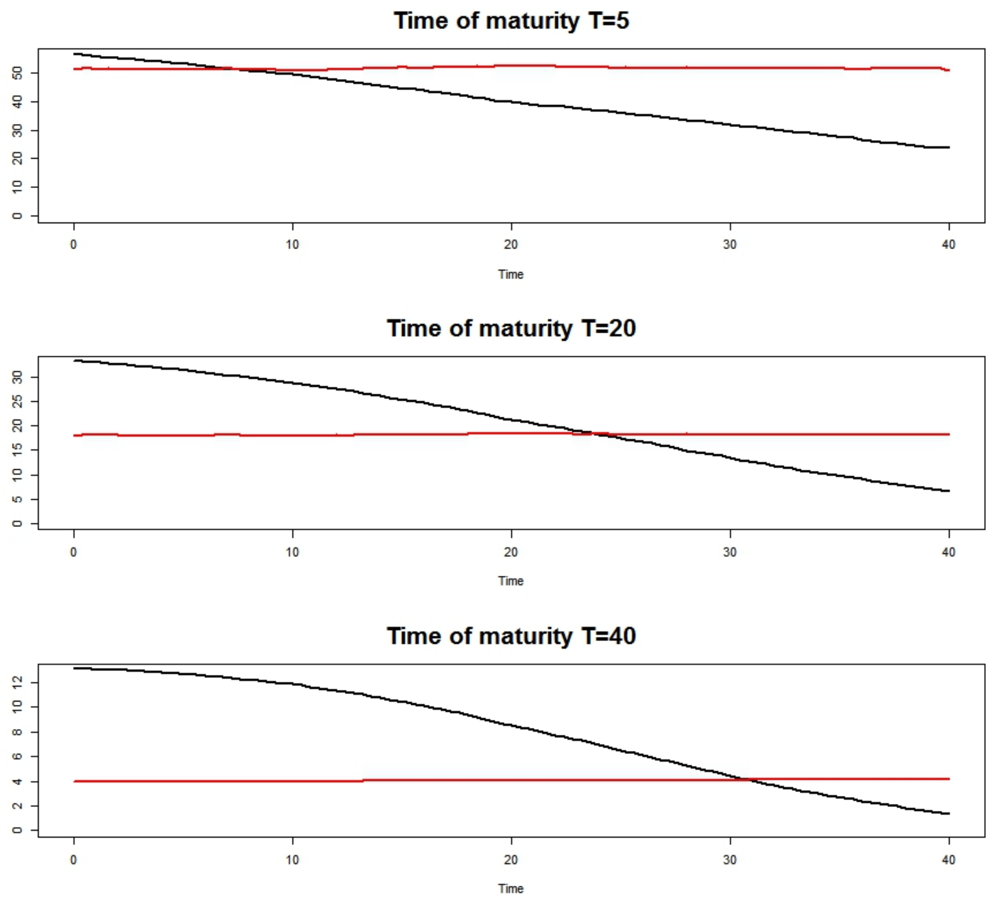

Figure 6.6: Variation of the time of maturity T; Portfolio

Chapter 7

Conclusion

The aim of this thesis was to find the risk-minimizing trading strategy to hedge the insurer's risk, that arises from selling unit-linked life insurance contracts. Therefore, we used Föllmer and Sondermann's (1986) theory of risk-minimization in incomplete markets, which is to minimize the conditional expected value of the squared difference of the cost process. Based on Møller (1998) we found the risk-minimizing hedging strategies for unit-linked insurance contracts with claims at maturity by directly using Föllmer and Sondermann's (1986) results. For this simple claims we showed that the risk could even be eliminated by extending the financial market with a reinsurance possibility. Some numeric results round up the theory of the simple model where we discussed the influence of single model parameters on the initial risk, the initial value and the financial portfolio. The second part of the thesis was based on a paper by Møller (2001). Föllmer and Sondermann's (1986) theory of risk-minimization was extended to claims payable at any time in the contract's duration. Using this generalization we were able to find risk-minimizing trading strategies for life insurance contracts specified by arbitrary payment streams.

In the course of the thesis, there were a lot of rather strict assumptions made. Hence, possible extensions of the obtained results could be generalizations of the financial model, e.g. stochastic interest rates or transaction costs. Another disadvantage is that we worked with continuous trading possibilities, but in real life a trading strategy can not be shifted infinitely often.

Appendix A

Solution of SDE for the stockprice

In Chapter 2 the financial market consisting of two assets, a stock and a bond, is introduced, where the price of the stock S_t is driven by the dynamics

$$dS_t = \alpha(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t \tag{A.0.1}$$

with $S_0 > 0$. Øksendal (2013) proved that a solution for (A.0.1) exists if one assumes a Lipschitz and a linear boundary condition for the functions σ and α. He considers the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{A.0.2}$$

with $b(t, x) \in \mathbb{R}$ and $\sigma(t, x) \in \mathbb{R}$ and a Brownian motion B and proved the following existence and uniqueness theorem

Theorem A.0.1. *[11] Let $T > 0$ and $b(.,.) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, \sigma(.,.) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be measurable functions satisfying*

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|), \quad x \in \mathbb{R}^n, t \in [0, T]$$

for some constant C, (where $|\sigma|^2 = \sum |\sigma_{ij}^2|$) and such that

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|, \quad x, y \in \mathbb{R}^n, t \in [0, T]$$

for some constant D. Let Z be a random variable which is independent of the $\sigma-$algebra $\mathcal{F}_\infty^{(m)}$ generated by $B_s(.), s \geq 0$ and such that

$$\mathbb{E}\left[|Z|^2\right] < \infty.$$

Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T, X_0 = Z$$

has the unique t-continuous solution $X_t(\omega)$ *with the property that* $X_t(\omega)$ *is adapted to the filtration* $\mathbb{F}_t^Z$ *generated by* Z *and* $B_s(.)$ *for* $s \leq t$ *and*

$$\mathbb{E}\left[\int_0^t |X_t|^2 dt\right] < \infty.$$

Obviously our original problem (A.0.1) can be written in the form of (A.0.2) and therefore owns a unique, path-wise continuous solution with existing variance. For the proof of the above theorem and more information on this topic see Øksendal (2013, Chapter 5).

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