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On stochastic optimization problems and an application in finance

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Abstract

In this thesis a special stochastic optimal control problem is investigated. The problem under study arose from a dynamic cash management model in finance, where decisions about the dividend and the financing policy of a firm have to be made. The control problem regarding this model contains, in addition to the ordinary stochastic control part stemming from the financing opportunity, a singular stochastic control part caused by the dividend control. Furthermore, due to the fact that the firm can be liquidated by its management, a stopping problem is included. The main discourse is about the theoretical solution of stochastic optimal control problems, using the dynamic programming approach. For that purpose and due to the features of stochastic optimal control problems, the theoretical background with a focus on Markov processes is provided. For the considered model the Hamilton-Jacobi-Bellman equation is formally derived and the verification step is carefully examined. Finally the problem is solved numerically using the policy iteration algorithm.

Zusammenfassung

In dieser Arbeit wird ein spezielles stochastisches Kontrollproblem untersucht. Dieses Problem entstand durch ein dynamisches Gelddispositionsmodell in der Finanzwissenschaft, in welchem Entscheidungen über die Dividenden- und die Finanzierungspolitik einer Firma getroffen werden müssen. Das Kontrollproblem bezüglich dieses Modells enthält, über den gewöhnlichen stochastischen Kontrollteil, stammend von der Finanzierungsmöglichkeit, hinaus, einen singulären Kontrollteil, veranlasst durch die Kontrolle der Dividenden. Da die Firma liquidiert werden kann ist auch ein Stoppproblem beinhaltet. Der größte Teil der Arbeit behandelt die theoretische Lösung des stochastischen optimalen Kontrollproblems unter der Verwendung der Methode des dynamischen Programmierens. Zu diesem Zweck und durch die Eigenschaften des Kontrollproblems wird der theoretische Hintergrund, speziell Markov Prozesse, betrachtet. Die Hamilton-Jacobi-Bellman Gleichung wird für das herangezogene Modell formal hergeleitet, außerdem wird die Verifikation untersucht und ausgearbeitet. Schließlich wird das Problem unter der Verwendung des Policy-Iteration Algorithmus numerisch gelöst.

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Contents

Chapter 1

Preliminaries

1.1 Introduction

This thesis is primarily concerned with a special stochastic optimal control problem based on the paper "Capital supply uncertainty, cash holdings, and investment" authored by Julien Hugonnier, Semyon Malamud and Erwan Morellec [6]. In particular for the considered problem, we want to provide the underlying theory about stochastic optimal control, extend and discuss the existing material in the mentioned paper about the theoretical solution and finally solve it numerically.

This first chapter contains the basic mathematical theory with which questions and problems in finance and insurance can be handled. In the following chapters we will specify the conditions of the considered problems in order to use the tools which will be provided here. The starting point builds some introductory theory about Markov processes, especially we are looking at Markov diffusion processes, which play an important role in the context of stochastic optimal control problems. To prepare the theory about these problems is the main goal we want to achieve in this first chapter. The following is mainly based on two books namely on the book by Rolski et al. [14] and on the book by Fleming and Soner [5]. In particular on the one hand, regarding Markov processes, we refer to [14, p. 269 - 270 and p. 437 - 443] and [5, p. 125 - 136]. On the other hand concerning stochastic optimal control theory it is referred to [5, p. 136 - 151], which covers this topic in a more general view and to [5, p. 157 - 177], which provides the theory about controlled Markov diffusions in $\mathbb{R}^n$.

1.2 Markov processes

Starting with Markov processes in discrete time will make it easier to understand how these processes will behave in continuous time. The probability space we are working on is $(\Omega, \mathcal{F}, P)$. We consider a finite state space Σ such that $\Sigma = \{1, 2, \ldots, l\}$ and a so called initial distribution $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \in [0, 1]^l$. α_i is the probability that the Markov process in discrete time $(X_n)_{n\in\mathbb{N}_0}$, which is called Markov chain, starts in state i at time 0:

$$P(X_0 = i) = \alpha_i.$$

Next it is common that this Markov chain has a possibility to evolve from the initial state to another state and so we set $p_{ij} \in [0, 1]$ the probability such that X_n moves in one time step from state i to state j:

$$P(X_n = j | X_{n-1} = i) = p_{ij}.$$

Note that p_{ij} does not depend on n, therefore the considered Markov chain is called homogeneous, in contrast to an inhomogeneous one, where the transition probabilities p_{ij} depend on n. Moreover in this context we call a matrix $\boldsymbol{P} = (p_{ij})_{i,j\in\Sigma}$ a stochastic matrix, if it fulfills

$$p_{ij} \geq 0 \ \ \forall \ i, j \in \Sigma \text{ and } \sum_{j=1}^{l} p_{ij} = 1, \forall \ \ i \in \Sigma.$$

A Markov chain has the special property that its evolution at a time step only depends on the last state which the process has achieved and not on the whole history. The next definition makes this argument precise.

Definition 1. *A homogeneous Markov chain is a sequence of random variables $(X_n)_{n\in\mathbb{N}_0}$ with values in $\Sigma = \{1, 2, \ldots, l\}$, for which there exists a vector of probabilities $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \in [0, 1]^l$ with $\sum_{i=1}^{l} \alpha_i = 1$, which is called initial distribution and a stochastic matrix $\boldsymbol{P} = (p_{ij})_{i,j\in\Sigma}$, which is called one step transition matrix, such that $\forall \ n \in \mathbb{N}_0$ and $i_0, i_1, \ldots, i_n \in \Sigma$*

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \alpha_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}.$$

For our purposes it is necessary to consider Markov processes in continuous time and with continuous state space. Therefore we introduce some general definitions and results about stochastic processes in continuous time, which are taken from the lecture notes “Stochastic analysis” by Müller [9]:

Definition 2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ be a filtration and Σ be a finite dimensional state space.*

- *A stochastic process $X = (X_t)_{t\geq 0}$ can be viewed as a function of both variables $X : [0,\infty)\times\Omega \to \Sigma$ such that $(t,\omega) \mapsto X_t(\omega)$, where $[0,\infty)\times\Omega$ is equipped with the σ-algebra $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$.*

- *$(X_t)_{t\geq 0}$ is called adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$ measurable $\forall\ t \geq 0$ and it is always adapted to its canonical filtration $(\mathcal{F}^X_t)_{t\geq 0}$, where $\mathcal{F}^X_t = \sigma(X_s : s \leq t)$.*

- *$(X_t)_{t\geq 0}$ is called càdlàg, if its paths are right-continuous with existing left-hand limits.*

- *$(X_t)_{t\geq 0}$ is called measurable if $(t,\omega) \mapsto X_t(\omega)$ is measurable, i.e. $\{(t,\omega) \in [0,\infty) \times \Omega : X_t(\omega) \in B\} \in \mathcal{B}([0,\infty)) \otimes \mathcal{F} \quad \forall\ B \in \mathcal{B}(\Sigma)$.*

- *$(X_t)_{t\geq 0}$ is called progressive if for every $t \geq 0$ the map*

  $$\Psi_t : [0,t] \times \Omega \to \Sigma, \qquad (s,\omega) \mapsto X_s(\omega)$$

  is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$ measurable.

- *Denote by $\mathcal{P}$ the sub σ-algebra of $\mathcal{B}([0,\infty)) \otimes \mathcal{F}$ generated by all left continuous adapted processes. $\mathcal{P}$ is called the σ-algebra of predictable events. A process X is called predictable if it is $\mathcal{P}$-measurable.*

- *Given a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ is said to fulfill the usual conditions if on the one hand it is right-continuous i.e. $\mathcal{F}_{t+} = \mathcal{F}_t\ \forall\ t \geq 0$. Moreover on the other hand it is P complete, which means that $\mathcal{F}_0$ (and hence all $\mathcal{F}_t$) contains all sets of measure 0, i.e. $\mathcal{N} \subseteq \mathcal{F}_0$, where $\mathcal{N} := \{A \in \mathcal{F} | P(A) = 0\}$, in other words $\mathcal{F}_t$ must be equal to $\sigma(\mathcal{F}_t \cup \mathcal{N})$ for all $t \geq 0$.*

Lemma 1. *[12, p. 182]*
With the definitions above it holds that

$$\textit{predictable} \;\Rightarrow\; \textit{progressive} \;\Rightarrow\; \textit{adapted and measurable.}$$

The proof is left out. Note that in [8, p. 10] it is mentioned that due to the definition of predictable processes every left-continuous and adapted process is predictable. Subsequently there are listed some notations and assumptions, which will be necessary for the further proceeding:

- we consider the probability space denoted by $(\Omega, \mathcal{F}, P)$ with filtration $(\mathcal{F}_t)_{t\geq 0}$,
- we restrict ourselves to the case $\Sigma = \mathbb{R}^d$ (or disconnected components in $\mathbb{R}^d$), where $d \geq 1$, in a more general setting it would be enough to allow Σ to be any complete separable metric space,
- the stochastic process $(X_t)_{t\geq 0}$ is assumed to be càdlàg,
- let $\mathcal{B}(\Sigma)$ be the σ-algebra of Borel sets in Σ,
  $M(\Sigma) = \{f : \Sigma \to \mathbb{R} | f \text{ measurable } \}$,
  $M_b(\Sigma) = \{f : \Sigma \to \mathbb{R} | f \text{ measurable and bounded } \}$ such that the supremum norm $\forall g \in M_b(\Sigma)$ is defined as $\|g\| = \sup_{x\in\Sigma} |g(x)|$,
- the following notation considers conditional expectations, for $s \leq t$: $E(X_t|X_s = x) = E_{sx}(X_t)$ and $E(X_t|X_0 = x) = E_x(X_t)$.

The next definition is the analogue to the previous one, the transition kernel takes the part of the transition probability in the continuous time and continuous space situation.

Definition 3. *Let* $\hat{P} : \mathbb{R}_+ \times \Sigma \times \mathcal{B}(\Sigma) \to [0, 1]$ *be a function which satisfies* $\forall\ h,\ h_1,\ h_2, \geq 0, x \in \Sigma, B \in \mathcal{B}(\Sigma)$ *:*

$$\hat{P}(h, x, \cdot) \text{ is a probability measure on } (\Sigma, \mathcal{B}(\Sigma)), \tag{1.2.1}$$

$$\hat{P}(0, x, \{x\}) = 1, \tag{1.2.2}$$

$$\hat{P}(\cdot, \cdot, B) \in M(\mathbb{R}_+ \times \Sigma) \text{ and} \tag{1.2.3}$$

$$\hat{P}(h_1 + h_2, x, B) = \int_\Sigma \hat{P}(h_2, y, B)\hat{P}(h_1, x, dy). \tag{1.2.4}$$

Then $\hat{P}$ *is called transition kernel.*

The property (1.2.4) can be seen as the analogue to the so called Chapman-Kolmogorov equation for more details especially in the more restrictive cases see [14, p. 271 and p. 309].

Definition 4. *Let $(X_t)_{t\geq 0}$ be a stochastic process with values in Σ and let $\hat{P}$ be its transition kernel and α be a probability measure on $(\Sigma, \mathcal{B}(\Sigma))$, which we call initial distribution, such that $\forall\ n \in \mathbb{N}_0, B_0, B_1, \ldots, B_n \in \mathcal{B}(\Sigma)$, $t_0 = 0 \leq t_1 \leq \cdots \leq t_n$:*

$$\begin{aligned} &P(X_0 \in B_0, X_{t_1} \in B_1, X_{t_n} \in B_n) \\ &= \int_{B_0} \int_{B_1} \cdots \int_{B_n} \hat{P}(t_n - t_{n-1}, x_{n-1}, dx_n) \ldots \hat{P}(t_1, x_0, dx_1)\alpha(dx_0), \end{aligned} \tag{1.2.5}$$

then $(X_t)_{t\geq 0}$ is called a (homogeneous) Markov process.

The interpretation of the transition kernel is the following the probability that X_t goes from state x to a state which is contained in B in time h is meant to be $\hat{P}(h, x, B)$. Just as a remark it is worth to mention that it is possible to show if $\Sigma = \mathbb{R}^d$ (or more general if Σ is a complete separable metric space) and for given $(\alpha, \hat{P})$ there exists a Markov process $(X_t)_{t\geq 0}$ with initial distribution α and transition kernel $\hat{P}$, see [14, p. 438].

The next Theorem states the better known conditional independence property about Markov processes.

Theorem 1 ([14, p. 438 - 439])**.** *A stochastic process $(X_t)_{t\geq 0}$ with values in Σ is a (time homogeneous) Markov process if and only if $\exists\ \hat{P} = (\hat{P}(h, x, B))$ a transition kernel such that $\forall\ t, h \geq 0, B \in \mathcal{B}(\Sigma)$*

$$P(X_{t+h} \in B | \mathcal{F}_t^X) = \hat{P}(h, X_t, B) \tag{1.2.6}$$

or equivalent to that $\forall\ t, h \geq 0, g \in M_b(\Sigma)$

$$E(g(X_{t+h}) | \mathcal{F}_t^X) = \int_{\Sigma} g(y)\hat{P}(h, X_t, dy). \tag{1.2.7}$$

Furthermore a process $(X_t)_{t\geq 0}$ is said to be a strong Markov process with respect to its history $(\mathcal{F}_t^X)_{t\geq 0}$ if the time t in (1.2.6) can be replaced by any $(\mathcal{F}_t^X)_{t\geq 0}$-stopping time τ

$$P(X_{\tau+h} \in B | \mathcal{F}_\tau^X) = \hat{P}(h, X_\tau, B) \quad P - a.s. \tag{1.2.8}$$

on $\{\tau < \infty\}$ and $\forall\ h \geq 0, B \in \mathcal{B}(\Sigma)$.

In the next step we want to go further and introduce some interesting facts about these class of stochastic processes, which made them so important in the theory of stochastic optimal control. In particular we will find some requirements such that a somehow modified Markov process becomes a martingale. This leads us to the consideration of the infinitesimal generator of the transition kernel and to a generalization of the so called Dynkin formula.

Definition 5. *We call $(T(h))_{h\geq 0}$ a contraction semigroup on $M_b(\Sigma)$ if it is a family of mappings $T(h) : M_b(\Sigma) \to M_b(\Sigma)\ \forall\ h \geq 0$ such that $\forall h, h_1, h_2 \geq 0$ and $g \in M_b(\Sigma)$ it holds that*

$$T(0) = I, \tag{1.2.9}$$

$$T(h_1 + h_2) = T(h_1)T(h_2) \text{ and} \tag{1.2.10}$$

$$\|T(h)g\| \leq \|g\|, \tag{1.2.11}$$

where I is the identity mapping.

Lemma 2. *For the Markov process $(X_t)_{t\geq 0}$ with values in Σ, a transition kernel $\hat{P} = (\hat{P}(h, x, B))$ and a function $g \in M_b(\Sigma)$ it holds that*

$$T(h)g(x) := \int_\Sigma g(y)\hat{P}(h, x, dy) = E(g(X_h)|X_0 = x) \tag{1.2.12}$$

is a contraction semigroup on $M_b(\Sigma)$.

Proof. Let $g \in M_b(\Sigma)$, $x \in \Sigma$ and $h_1, h_2 \geq 0$ then the first equation (1.2.9) holds because with (1.2.2) we get

$$T(0)g(x) = E(g(X_0)|X_0 = x) = g(x) \Rightarrow T(0) = I.$$

The property (1.2.4) gives (1.2.10) as follows

$$\begin{aligned} T(h_1 + h_2)g(x) &= \int_\Sigma g(y)\hat{P}(h_1 + h_2, x, dy) \\ &= \int_\Sigma g(y) \int_\Sigma \hat{P}(h_2, z, dy)\hat{P}(h_1, x, dz) \\ &= \int_\Sigma \left(\int_\Sigma g(y)\hat{P}(h_2, z, dy)\right) \hat{P}(h_1, x, dz) \\ &= \int_\Sigma T(h_2)g(z)\hat{P}(h_1, x, dz) = T(h_1)T(h_2)g(x). \end{aligned}$$

Finally with (1.2.1) we have $\forall\ x \in \Sigma$ that

$$|T(h)g(x)| = \left|\int_\Sigma g(y)\hat{P}(h,x,dy)\right| \leq \int_\Sigma \|g\|\hat{P}(h,x,dy) = \|g\|$$

which yields (1.2.11), $\|T(h)g\| \leq \|g\|$. □

Definition 6. *Let $\mathcal{D}(A) \subseteq M_b(\Sigma)$ and $(T(h))_{h\geq 0}$ a contraction semigroup, then the mapping $A : \mathcal{D}(A) \to M_b(\Sigma)$ is defined for every $g \in M_b(\Sigma)$ such that*

$$Ag := \lim_{h\searrow 0} \frac{1}{h}(T(h)g - g), \tag{1.2.13}$$

exists. A is called the infinitesimal generator of $(T(h))_{h\geq 0}$ and $\mathcal{D}(A) = \{g \in M_b(\Sigma) |$the limit (1.2.13) exists in the supremum norm and $Ag \in M_b(\Sigma)\}$ its domain.

Furthermore it is referred to [5, p. 129], where it is stated that the *Hille-Yosida Theorem* provides sufficient conditions such that $\mathcal{D}(A)$ contains enough functions and that the transition distributions $\hat{P}$ are determined by A in the case of (1.2.12).

Moreover if we consider functions of the form $F : (a,b) \subseteq \mathbb{R} \to M_b(\Sigma)$, then the derivative and the Riemann integral are defined in the common way $\left[\frac{\partial T(t)g}{\partial t}\Big|_{t+=0}\right.$ respectively $\left.\int_0^t T(s)gds\right]$, where the convergence in the sense of the supremum norm is used. It is stated that such integrals exist for right-continuous semigroups. For $g \in M_b(\Sigma)$ such that $\lim_{h\searrow 0} T(h)g = g$ with the special contraction semigroup (1.2.12) the mapping $h \mapsto T(h)g$ is right-continuous and the Riemann integral $\int_0^t T(s+h)g\ ds$ exists $\forall\ t, h \geq 0$. [Note that $\sum_{i=1}^n (t_i^n - t_{i-1}^n)T(t_{i-1}^n)g \longrightarrow \int_0^t T(s+h)g\ ds$ and $T(s+h)g(x) = E(g(X_{s+h})|X_0 = x)\ \forall\ x \in \Sigma$.] With that we can formulate the next theorem about contraction semigroups.

Theorem 2. *The statements (a), (b) and (c) below hold for a contraction semigroup $(T(h))_{h\geq 0}$ and its infinitesimal generator A:*

(a) Let $g \in M_b(\Sigma)$ such that the mapping $h \mapsto T(h)g$ is right continuous at $h = 0$, this yields that $\forall\ t \geq 0$ $\int_0^t T(v)gdv \in \mathcal{D}(A)$ and

$$T(t)g - g = A\int_0^t T(v)g\ dv. \tag{1.2.14}$$

(b) Let $g \in \mathcal{D}(A)$ and $t \geq 0$, then $T(t)g \in \mathcal{D}(A)$ and

$$\frac{d^+}{dt} T(t)g = AT(t)g = T(t)Ag, \tag{1.2.15}$$

note that $\frac{d^+}{dt}$ equals the derivative from the right.

(c) Let $g \in \mathcal{D}(A)$ and $t \geq 0$, then $\int_0^t T(v)g dv \in \mathcal{D}(A)$ and

$$T(t)g - g = A \int_0^t T(v)g \; dv = \int_0^t A \; T(v)g \; dv = \int_0^t T(v) \; Ag \; dv. \tag{1.2.16}$$

Proof. (a) The mapping $v \mapsto T(v)g$ is right continuous $\forall \; v \geq 0$ and hence the Riemann integral $\int_0^t T(v+h)g \; dv$ exists $\forall \; t, h \geq 0$, because we have assumed that $(T(h))_{h \geq 0}$ is a contraction semigroup. Further for $t_i^n = t\frac{i}{n}$ we have

$$\lim_{n \to \infty} \frac{t}{n} \sum_{i=1}^n T(t_i^n)g = \int_0^t T(v)g \; dv.$$

Moreover it holds that

$$T(h) \int_0^t T(v)g \; dv \stackrel{(*)}{=} \int_0^t T(h)T(v)g \; dv \stackrel{(**)}{=} \int_0^t T(v+h)g \; dv,$$

where $(**)$ is true because of (1.2.10) and $(*)$ since

$$T(h) \int_0^t T(v)g \; dv$$

$$= T(h) \left(\int_0^t T(v)g \; dv - \frac{t}{n} \sum_{i=1}^n T(t_i^n)g \right) + \frac{t}{n} \sum_{i=1}^n T(h)T(t_i^n)g$$

and by property (1.2.11)

$$\left\| T(h) \left(\int_0^t T(v)g \; dv - \frac{t}{n} \sum_{i=1}^n T(t_i^n)g \right) \right\| \leq$$

$$\left\| \int_0^t T(v)g \; dv - \frac{t}{n} \sum_{i=1}^n T(t_i^n)g \right\| \to 0.$$

Hence we obtain that

$$\frac{1}{h}(T(h)-I)\int_0^t T(v)g\ dv = \frac{1}{h}\int_0^t (T(v+h)g - T(v)g)\ dv$$
$$= \underbrace{\int_t^{t+h} T(v)g\ dv - T(v)g)\ dv - \frac{1}{h}\int_0^h T(v)g\ dv.}_{\to T(t)g-g \text{ for } h\searrow 0, \text{ because } v\mapsto T(v)g \text{ is right continuous.}}$$

Considering the limit $h \searrow 0$ yields the equation (1.2.14).

(b) The following term

$$\frac{1}{h}(T(h)T(t)g - T(t)g) = \frac{1}{h}(T(h+t)g - T(t)g)$$

$$= \frac{1}{h}(T(t+h)g - T(t)g) = \frac{1}{h}(T(t)T(h)g - T(t)g) = T(t)\frac{1}{h}(T(h)g-g)$$

together with (1.2.11) yields that $T(t)g \in \mathcal{D}(A)$ and that $AT(t)g = T(t)Ag$. And the statement concerning the derivative from the right is obtained by evaluation of the limit $h \searrow 0$ in the equation

$$\underbrace{\frac{1}{h}(T(h+t)g - T(t)g)}_{\to \frac{d^+}{dt}T(t)g} = \underbrace{\frac{1}{h}(T(h)-I)T(t)g}_{\to AT(t)g}.$$

(c) For $g \in \mathcal{D}(A)$ we get that $T(h)g \to g$ for $h \searrow 0$ and hence for $t \geq 0$ $\int_0^t T(v)gdv \in \mathcal{D}(A)$, because of (a). With part (b) and the equation (fundamental theorem of calculus)

$$\int_0^t \underbrace{\frac{d^+}{dv}T(v)g}_{=AT(v)g=T(v)Ag}\quad dv = T(t)g - T(0)g,$$

we get the statement of part (c). □

In order to construct martingales out of Markov process we exploit the connection between martingales and infinitesimal generators. Subsequently one has to show the following property called Dynkin‘s formula for the considered process in order to solve stochastic optimal control problems.

Theorem 3. *Let $(X_t)_{t\geq 0}$ be a Markov process with values in Σ and transition kernel $\hat{P} = (\hat{P}(h,x,B))$ and let $(T(h))_{h\geq 0}$ be the corresponding contraction semigroup* (1.2.12) *and A its generator. Then it holds for every $g \in \mathcal{D}(A)$ that the stochastic process $(M_t)_{t\geq 0}$ with*

$$M_t = g(X_t) - g(X_0) - \int_0^t Ag(X_v)dv \tag{1.2.17}$$

is a $(\mathcal{F}_t^X)_{t\geq 0}$ martingale.

Proof. First of all we get that $Ag \in M_b(\Sigma)$ since $g \in \mathcal{D}(A)$ and so Ag is measurable and bounded. Further $(X_t)_{t\geq 0}$ is cadlag and hence $Ag(X_t(\omega))$ is also measurable. Moreover knowing Ag is bounded the integral $\int_0^t Ag(X_s(\omega))ds$ is (pathwise) well-defined as Lebesgue integral for every $\omega \in \Omega$. With $t, h \geq 0$ one obtains

$$\begin{aligned}
&E[M_{t+h}|\mathcal{F}_t^X] + g(X_0)\\
&= E\left[g(X_{t+h}) - \int_t^{t+h} Ag(X_s)ds \middle| \mathcal{F}_t^X\right] - \int_0^t Ag(X_s)ds\\
&= \int_\Sigma g(y)\hat{P}(h,X_t,dy) - \int_t^{t+h}\int_\Sigma Ag(y)\hat{P}(s-t,X_t,dy)ds - \int_0^t Ag(X_s)ds\\
&= T(h)g(X_t) - \int_0^h T(s)Ag(X_t)ds - \int_0^t Ag(X_s)ds \overset{(\star)}{=} g(X_t) - \int_0^t Ag(X_s)ds\\
&= M_t + g(X_0).
\end{aligned}$$

At $(\star)$ the equation (1.2.16) of Theorem 2 part (c) was used. □

Note that the martingale $(M_t)_{t\geq 0}$ from above has expectation zero since $M_0 = 0$ and $E(M_t) = E(M_0) = 0$. Further we consider some special types of Markov processes and some useful results, which we exploit in the subsequent chapter.

1.2.1 Diffusion processes

First of all we consider a quite simple example namely a one dimensional diffusion process, here Dynkin's formula can be directly verified. For that reason let $(B_t)_{t\geq 0}$ be a Brownian motion and $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous. Then the equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x$$

defines a homogeneous diffusion process fulfilling the strong Markov property as we will see few lines below. Further let $f \in C_b^2$ (denoting the twice continuously differentiable and bounded functions) with compact support then

$$\begin{aligned} df(X_t) &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X]_t \\ &= \underbrace{(\mu(X_t)f'(X_t) + \frac{1}{2}\sigma^2(X_t)f''(X_t))}_{=Af(X_t)}dt + f'(X_t)\sigma(X_t)dB_t \end{aligned}$$

and we obtain that

$$f(X_t) - f(X_0) - \int_0^t Af(X_s)ds = \int_0^t f'(X_s)\sigma(X_s)dB_s.$$

Because the term on the right-hand side is a local martingale and if we assume that $E_x\left[\int_0^\infty ||f'(X_s)\sigma(X_s)||^2 ds\right] < \infty$ it follows that this process is a martingale and hence Dynkin's formula holds by taking expectations on both sides. This will be also true if t is replaced by an almost surely finite stopping time τ. As a conclusion we can assume

$$\mathcal{D}(\mathcal{A}) = \{f \in C_b^2 | f \text{ has compact support and } E_x\left[\int_0^\infty ||f'(X_s)\sigma(X_s)||^2 ds\right] < \infty\}.$$

Now the above example is generalized by showing that Dynkin's formula holds for general diffusion processes and a further interesting result is proven, the so called Feynman-Kac formula. The following considerations are based on [5, p. 126 - 129, p. 134 - 135 and p. 397 - 401] and on [15, p. 9 - 11]. The starting point will be the d-dimensional stochastic differential equation for $t_0 \leq t < t_1$:

$$\begin{aligned} dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, \quad t \leq s \leq t_1, \\ \text{and } X_t = x \in \mathbb{R}^d, \end{aligned} \tag{1.2.18}$$

where $(W_s) \quad t \leq s \leq t_1$, is a n-dimensional standard Brownian motion and

$$b : [t, t_1] \times \mathbb{R}^d \to \mathbb{R}^d,$$

$$\sigma : [t, t_1] \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$$

are functions fulfilling the requirements of the Theorem for SDE's ensuring existence, uniqueness and the strong Markov property of the continuous solution X_s of (1.2.18) and for $m = 1, 2, \ldots$ there exists a constant B_m, which depends on m and $t_1 - t$, such that

$$E_{tx}(|X_s|^m) \leq B_m(1 + |x|^m), \tag{1.2.19}$$

in particular the component functions have to be Lipschitz continuous and satisfy a linear growth condition.

In fact we are working here with time-inhomogeneous Markov processes and therefore one has to extend the definition of the infinitesimal generator. Everything considered above can be reviewed just where the transition distribution is able to change in time. The following is a quick overview on the important facts about time-inhomogeneous Markov processes.

Definition 7. *The transition distribution of a Markov process $(X_r)_{r \geq 0}$ has the following form for $t < s$, where $s, t \in [t_0, t_1)$ and $x \in \Sigma$:*

$$\hat{P}(t, x, s, B) = P(X_s \in B | X_t = x), \quad \forall B \in \mathcal{B}(\Sigma), \tag{1.2.20}$$

where

- *for s, t, B fixed $\hat{P}(t, \cdot, s, B)$ is $\mathcal{B}(\Sigma)$ measurable,*
- *for s, t, x fixed $\hat{P}(t, x, s, \cdot)$ is a probability measure and*
- *for $t < r < s$ and $t, r, s \in [t_0, t_1)$ the Chapman-Kolmogorov equation holds true*

$$\hat{P}(t, x, s, B) = \int_\Sigma \hat{P}(r, y, s, B) \hat{P}(t, x, r, dy). \tag{1.2.21}$$

With that definition one can write down the usual Markov property in an equivalent form as follows. For $\mathcal{F}_r^X = \mathcal{F}(X_\theta, \theta \leq r)$ the smallest σ-algebra such that for all $\theta \leq r$ $(\theta, r \in I$, where I is a subinterval of $[t_0, t_1])$, the random variables X_θ with vales in Σ are measurable. It holds that for $r < s$ $r, s \in I$ and $B \in \mathcal{B}(\Sigma)$

$$P(X_s \in B | \mathcal{F}_r^X) = \hat{P}(r, X_r, s, B). \tag{1.2.22}$$

The linear operators are analogously defined as in the above case just with respect to the changed transition distribution $\hat{P}$. In this setting we are able to define the analogue of the infinitesimal generator namely the backward evolution operator $\mathcal{A}$. In general the definition is analogous to the previous one but in order to work with this operator in the framework of the theory of stochastic optimal control one assumes that the domain of the operator $\mathcal{D}(\mathcal{A})$ is quite restricted as it is done in [5, p. 127, 128]. In fact some properties are demanded in addition to the natural assumption that the respective limit exists.

Definition 8. *Let $\Phi : [t_0, t_1] \times \Sigma \to \mathbb{R}$ be a function, then the linear operator $\mathcal{A}$ is defined by*

$$\mathcal{A}\Phi(t, x) = \lim_{t \searrow 0} \frac{1}{h} [E_{tx} (\Phi(t + h, X_{t+h})) - \Phi(t, x)], \tag{1.2.23}$$

given that the limit exists for all $x \in \Sigma$ and all $t \in [t_0, t_1)$ where the domain $\mathcal{D}(\mathcal{A})$ contains all functions Φ such that $\mathcal{A}\Phi$ exists.

Nevertheless it is additionally assumed that for all $\Phi \in \mathcal{D}(\mathcal{A})$ it holds that

- Φ, $\frac{\partial \Phi}{\partial t}$ and $\mathcal{A}\Phi$ are continuous on $[t_0, t_1] \times \Sigma$,
- for $t < s$ $(s, t \in [t_0, t_1])$ it holds that $E_{tx}(|\Phi(s, X_s)|) < \infty$ and $E_{tx}(\int_t^s |\mathcal{A}\Phi(r, X_r)|)dr < \infty$ and
- Dynkin's formula holds for $t < s$

$$E_{tx}(\Phi(s, X_s)) - \Phi(t, x) = E_{tx} \left(\int_t^s \mathcal{A}\Phi(r, X_r) dr. \right) \tag{1.2.24}$$

In order to obtain Dynkin's formula one can assume that

$$\Phi(s, X_s) - \Phi(t, x) - \int_t^s \mathcal{A}\Phi(r, X_r) dr \tag{1.2.25}$$

is a martingale with respect to $\{\mathcal{F}_s, P\}$, where the filtration $\{\mathcal{F}_s\}$ is such that X_s is adapted to it, i.e. X_s is $\mathcal{F}_s$ measurable.

Now we reveal the connection between the above considered operators. For time - homogeneous Markov processes we consider the time interval $[0,\infty)$ and obtain that the transition distribution written in terms of Definition 7 satisfies for $0 \le t \le s, B \in \mathcal{B}(\Sigma)$ and $x \in \Sigma$

$$\hat{P}(t,x,s,B) = \hat{P}(0,x,s-t,B),$$

which means that

$$P(X_s \in B | X_t = x) = P(X_{s-t} \in B | X_0 = x).$$

That means that the transition distribution does not depend on the actual time point of the process but only on the state of the process and the time lapse which goes by. Because of that reason time - homogeneous Markov processes are always considered starting in time point 0, which is in fact just a time shift of the process. Furthermore this justifies the definition of the transition kernel above taking only three arguments. On top of this we ought to think in a formal framework that for the backward evolution operator $\mathcal{A}$ and the infinitesimal generator A the relation

$$\mathcal{A}\Phi = \frac{\partial \Phi}{\partial t} + A\Phi(t,\cdot) \tag{1.2.26}$$

holds, whereby A operates in the second variable x.
Next we are able to write down the so called backward evolution equation. Let $l(t,x)$ and $\psi(x)$ be continuous functions on $[t_0,t_1]\times\Sigma$ respectively on Σ, then the linear and inhomogeneous equation

$$\begin{aligned} \mathcal{A}\Phi + l(t,x) = 0, \quad t_0 \le t \le t_1, \\ \Phi(t_1,x) = \psi(x) \end{aligned} \tag{1.2.27}$$

is called a backward evolution equation. Let $\Phi \in \mathcal{D}(\mathcal{A})$ be the solution of this backward evolution equation provided that it exists, then we obtain by Dynkin's formula (for that reason one has to assume that Dynkin holds, but in this framework this was an additional property of the domain $\mathcal{D}(\mathcal{A})$)

$$\Phi(t,x) = E_{tx}\left(\int_{t_0}^{t_1} l(s,X_s)ds + \psi(X_{t_1})\right). \tag{1.2.28}$$

This expression can be considered as the total expected gain respectively cost over the time period $[t, t_1]$, whereby l denotes the running gain respectively cost function and ψ the terminal gain respectively cost function. This is a common designation in stochastic optimal control theory.

For the next theorem we need the following definition.

Definition 9. *[10, p. 110]*
A (time-homogeneous) Itô diffusion is a stochastic process $X_t(\omega) = X(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}^d$ *satisfying a stochastic differential equation of the form*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad s \leq t, X_s = x$$

where W_t *is a* n*-dimensional Brownian motion and* $b : \mathbb{R}^d \to \mathbb{R}^d, \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times n}$ *satisfy the conditions in the first part of the next theorem, which in this case simplify to:*

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq D|x - y|; \qquad x, y \in \mathbb{R}^d,$$

where $|\sigma|^2 = \sum |\sigma_{ij}|^2$.

The following theorem is split up into two main parts. For another version of the first part of the theorem we refer to [7, p. 289].

Theorem 4. *1. Existence and uniqueness theorem for stochastic differential equations [10, p. 66]:*
Let $T > 0$ *and*

$$b(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d,$$
$$\sigma(\cdot, \cdot) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$$

be measurable functions satisfying

$$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|); \qquad x \in \mathbb{R}^d, t \in [0, T]$$

for some constant C, *(where* $|\sigma|^2 = \sum |\sigma_{ij}|^2$*) and such that*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|; \qquad x, y \in \mathbb{R}^d, t \in [0, T]$$

for some constant D. *Let* Z *be a random variable which is independent of the* σ*-algebra* $\mathcal{F}_\infty^{(n)}$ *generated by* $W_s, s \geq 0$ *and such that*

$$E\left[|Z|^2\right] < \infty.$$

Then the stochastic differential equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad 0 \le t \le T, X_0 = Z$$

has a unique t-continuous solution $X_t(\omega)$ with the property that $X_t(\omega)$ is adapted to the filtration $\mathcal{F}_t^Z$ generated by Z and W_s, $s \le t$ and

$$E\left[\int_0^T |X_t|^2 dt\right] < \infty.$$

2. The strong Markov property for Itô diffusions [10, p. 113]:
Let f be a bounded Borel function on $\mathbb{R}^d$, τ a stopping time w.r.t. $\mathcal{F}_t^{(n)}$, $\tau < \infty$ a.s.. Then

$$E^x\left[f(X_{\tau+h})|\mathcal{F}_\tau^{(n)}\right] = E^{X_\tau}\left[f(X_h)\right] \quad \forall h \ge 0.$$

Next we assume that $\mathcal{D}(A) = C_p^{1,2}([t_0, t_1] \times \mathbb{R}^d)$, where the subscript p states that this is the space of all $\Phi \in C^{1,2}([t_0, t_1] \times \mathbb{R}^d)$ where $\Phi, \Phi_t, \Phi_{x_i}, \Phi_{x_i x_j}$ for $i, j = 1, \ldots, d$ fulfill the following polynomial growth condition:

$$\exists\ K, m \text{ such that } \forall\ (t,x) \in [t_0, t_1] \times \mathbb{R}^d : \\ |\Phi(t,x)| \le K(1 + |x|^m). \tag{1.2.29}$$

With that we can show Dynkin's formula directly in the case of time-inhomogeneous processes for a function $\Phi \in \mathcal{D}(A)$ and the solution X_s of (1.2.18). Itô's formula gives

$$d\Phi(s, X_s) =$$

$$\Phi_t ds + \sum_{i=1}^d \Phi_{x_i} dX_s^i + \frac{1}{2}\left[\sum_{i=1}^d \Phi_{tx_i} d[t, X^i]_s + \Phi_{tt} d[t,t] + \sum_{i=1}^d \sum_{j=1}^d \Phi_{x_i x_j} d[X^i, X^j]_s\right],$$

where

$$dX_s^i = b^i(s, X_s)ds + \sum_{l=1}^n \sigma_{il}(s, X_s)dW_s^l,$$

since $d[t, X^i] = 0$, $\forall\ i = 1, \ldots, d$, $\quad d[t,t] = 0$ and

$$d[X^i, X^j]_s = (\sum_{l=1}^n \sigma_{il}(s, X_s)dW_s^l)(\sum_{k=1}^n \sigma_{jk}(s, X_s)dW_s^k) = \sum_{l=1}^n \sigma_{il}(s, X_s)\sigma_{jl}(s, X_s)ds.$$

Moreover one obtains with $\sum_{l=1}^{n}\sigma_{il}(s,X_s)\sigma_{jl}(s,X_s)=a_{ij}(s,X_s)$ for $i,j=1,\dots d$:

$$\begin{aligned}d\Phi(s,X_s)=&\Phi_t ds+\sum_{i=1}^{d}\Phi_{x_i}\left(b^i(s,X_s)ds+\sum_{l=1}^{n}\sigma_{il}(s,X_s)dW_s^l\right)\\&+\frac{1}{2}\left[\sum_{i=1}^{d}\sum_{j=1}^{d}\Phi_{x_ix_j}\left(\sum_{l=1}^{n}\sigma_{il}(s,X_s)\sigma_{jl}(s,X_s)\right)ds\right]\\=&\left(\Phi_t+\sum_{i=1}^{d}\Phi_{x_i}b^i(s,X_s)+\frac{1}{2}\left[\sum_{i=1}^{d}\sum_{j=1}^{d}\Phi_{x_ix_j}a_{ij}(s,X_s)\right]\right)ds\\&+\sum_{i=1}^{d}\Phi_{x_i}\sum_{l=1}^{n}\sigma_{il}(s,X_s)dW_s^l.\end{aligned}\tag{1.2.30}$$

Which has the following form changing the notation to vector valued functions and processes

$$\begin{aligned}&d\Phi(s,X_s)\\&=\left(\Phi_t+\langle D_x\Phi,b(s,X_s)\rangle+\frac{1}{2}\left[\sum_{i,j=1}^{d}\Phi_{x_ix_j}a_{ij}(s,X_s)\right]\right)ds+((D_x\Phi)^T\sigma)^T(s,X_s)dW_s\\&=A\Phi(s,X_s)ds+((D_x\Phi)^T\sigma)^T(s,X_s)dW_s,\end{aligned}\tag{1.2.31}$$

where with $(a_{ij}(s,X_s))_{1\leq i,j\leq d}=a(s,X_s)=\sigma(s,X_s)\sigma(s,X_s)^T$ we have

$$\begin{aligned}A\Phi(s,X_s)&=\Phi_t+\sum_{i=1}^{d}\Phi_{x_i}b^i(s,X_s)+\frac{1}{2}\left[\sum_{i=1}^{d}\sum_{j=1}^{d}\Phi_{x_ix_j}a_{ij}(s,X_s)\right]\\&=\Phi_t+(D_x\Phi)^T\ b(s,X_s)+\frac{1}{2}tr((D_x^2\Phi)a(s,X_s)).\end{aligned}\tag{1.2.32}$$

Moreover using the integral representation

$$\Phi(s,X_s)-\Phi(t,x)-\int_t^s A\Phi(r,X_r)dr=\int_t^s((D_x\Phi)^T\sigma)^T(r,X_r)dW_r,\tag{1.2.33}$$

we obtain Dynkin's formula by taking expectations on both sides

$$E_{tx}\left(\Phi(s,X_s)\right)-\Phi(t,x)=E_{tx}\left(\int_t^s A\Phi(r,X_r)dr\right), \tag{1.2.34}$$

because the right-hand side in (1.2.33) is a local martingale and because σ is assumed to grow linearly and hence $(D_x\Phi)^T\sigma$ has polynomial growth we get with (1.2.29) that the right-hand side is a true martingale. In fact one has to ensure that $E\left[\int_t^\infty ||((D_x\Phi)^T\sigma)^T(r,X_r)||^2dr\right]<\infty$ to verify that the local martingale is a true martingale. Note that the partial derivatives e.g. $\Phi_t=\Phi_t(s,X_s)$ depend on (s,X_s), but this was left out in order to simplify the notation. Furthermore note that the infinitesimal generator A for diffusion processes (as solutions of (1.2.18)) is in fact a partial differential operator as already seen in (1.2.32), provided that for the integrals the following hold (e.g. assuming that Φ has compact support)

$$E_{t,x}\left[\int_t^s |A\Phi(r,X_r)|dr\right]<\infty \text{ and } E_{t,x}\left[\int_t^\infty ||((D_x\Phi)^T\sigma)^T(r,X_r)||^2dr\right]<\infty.$$

Finally we state and prove a version of the well-known Feynman-Kac formula [5, p. 400 - 401], which reads as follows.

Theorem 5. *Let $V\in C^{1,2}(\overline{Q})$, where $Q=[t_0,t_1)\times O$, O an open and bounded set such that $O\subset\mathbb{R}^d$ and (X_s) be the solution of (1.2.18). Furthermore let*

$$\Gamma_s=\exp\left(\int_t^s c_r dr\right),$$

where c is $\mathcal{F}_s$ - progressive and $\exists\ M$ sucht that $c_r\le M<\infty$ and let τ be the exit time of (s,X_s) from Q and θ be a $\{\mathcal{F}_s\}$-stopping time with $t\le\theta\le\tau$. Then it holds that

$$V(t,x)=E\left[-\int_t^\theta \Gamma_s(AV(s,X_s)+c_sV(s,X_s))ds+\Gamma_\theta V(\theta,X_\theta)\right], \tag{1.2.35}$$

where we have as in (1.2.32)

$$AV(s,X_s)=V_t(s,X_s)+(D_xV(s,X_s))^T\, b(s,X_s)+\frac{1}{2}tr((D_x^2V(s,X_s))a(s,X_s)).$$

Proof. First of all the partial integration formula respectively also called product rule is applied to $\Gamma_s V(s, X_s)$:

$$d(\Gamma_s V(s, X_s)) = V(s-, X_{s-}) d\Gamma_s + \Gamma_{s-} dV(s, X_s) + d[\Gamma, V(\cdot, X)]_s$$

and since the sample paths of Γ are of bounded variation and all protagonists are at least left-continuous we obtain

$$d(\Gamma_s V(s, X_s)) = V(s, X_s) d\Gamma_s + \Gamma_s dV(s, X_s).$$

Furthermore we get for the single differentials

$$d\Gamma_s = c_s \Gamma_s ds$$

and analogously to (1.2.22)

$$dV(s, X_s) = AV(s, X_s) ds + ((D_x V)^T \sigma)^T (s, X_s) dW_s.$$

In its entirety we have

$$\begin{aligned} &d(\Gamma_s V(s, X_s)) \\ &= V(s, X_s) c_s \Gamma_s ds + \Gamma_s \left(AV(s, X_s) ds + ((D_x V)^T \sigma)^T (s, X_s) dW_s \right) \\ &= \Gamma_s \left(AV(s, X_s) + V(s, X_s) c_s \right) ds + \Gamma_s ((D_x V)^T \sigma)^T (s, X_s) dW_s. \end{aligned}$$

Now integrating from t to θ and using the fact that $\Gamma_t = 1$ yields

$$\begin{aligned} &\Gamma_\theta V(\theta, X_\theta) - V(t, x) = \\ &\int_t^\theta \Gamma_s (AV(s, X_s) + c_s V(s, X_s)) ds + \int_t^\theta \Gamma_s ((D_x V)^T \sigma)^T (s, X_s) dW_s. \end{aligned}$$

Taking expectations on both sides and using that the expectation of the last term is zero yields the statement. □

The Theorem 5 is extending the backward evolution equation (1.2.27), which follows if V solves $AV + cV + l = 0$ and gives (1.2.28), where $l(t, x) = \Gamma_t l(x)$.

1.2.2 Compound Poisson processes

Up next we consider $(X_t)_{t\geq 0}$, which is a compound Poisson process with drift $c > 0$, $(N_t)_{t\geq 0}$ the corresponding Poisson process with intensity $\lambda > 0$ and $(Y_i)_{i\in\mathbb{N}} \overset{iid}{\sim} F_Y$ ($F_Y(0) = 0$). $(X_t)_{t\geq 0}$ has independent and stationary increments and is therefore a Markov process, analogously as argued in [14, p. 442]. For $h > 0$ one has that the first jump-time is exponentially distributed with $E(T_1) = \frac{1}{\lambda}$. Thus, the law of total probability gives

$$E_x(f(X_{h\wedge T_1})) = f(x+ch)e^{-\lambda h} + \int_0^h \int_0^\infty f(x+ch-y)dF_Y(y)\lambda e^{-\lambda t}dt,$$

and

$$\frac{1}{h}E_x(f(X_{h\wedge T_1}) - f(x))$$
$$= e^{-\lambda h}\frac{f(x+ch)-f(x)}{h} + \frac{e^{-\lambda h}-1}{h}f(x) + \frac{1}{h}\int_0^h \int_0^\infty f(x+ch-y)dF_Y(y)\lambda e^{-\lambda t}dt.$$

Now we let $h \searrow 0$ and obtain

$$Af(x) = cf'(x) - \lambda f(x) + \lambda \int_0^\infty f(x-y)dF_Y(y).$$

Hence we get that the following is a martingale with expectation zero

$$f(X_t) - f(X_0) - \int_0^t \left[cf'(X_s) + \lambda \int_0^\infty f(X_s - y) - f(X_s)dF_Y(y)\right] ds,$$

where in the most general case f' here denotes the right-hand derivative of f. Finally, following the lines of [14, p. 442 - 443, 449] and considering the requirements which were needed in the above treatment, we set
$\mathcal{D}(A) = \{f \in M_b(\mathbb{R}) | f \text{ is differentiable with } f' \in M_b(\mathbb{R})\}$.

1.3 Optimal control of Markov processes

As we have seen in the last section Markov processes have some useful properties which, will be applicable in the theory of stochastic optimal control. In fact Markov processes are of special interest because they are very often the natural modeling choice, but note that this does not has to be always the case. Hence only controlled Markov processes are considered and later on we will restrict ourselves to some classes of Markov processes namely diffusion processes. The following is based on the discourse in [5, p. 136 - 151] about controlled Markov processes. The subsequent section is very general and for explicit problems one has to think about the real mathematical requirements in greater detail.

At the beginning some additional assumptions and notations will be introduced:

- $(X_t)_{t\geq 0}$ will denote a (time-homogeneous) Markov process as before with state space Σ, but now the behaviour of the distribution of $(X_t)_{t\geq 0}$ depends on a stochastic process $(u_t)_{t\geq 0}$ and is therefore called the controlled process or state process,
- $(u_t)_{t\geq 0}$ is called the control process or briefly the control and assumes values in the control space U, which will be a complete separable metric space,
- if it is essential, we will express the dependence of $(X_t)_{t\geq 0}$ on the control $(u_t)_{t\geq 0}$ as $X = X^u$ as it is done in [4, p. 27],
- the information which is available to choose the control u_s will contain all states X_r for $r \leq s$ and will be modeled by the canonical filtration of X in particular $(\mathcal{F}_t^X)_{t\geq 0}$, where $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$.

In general it is assumed that for a given constant control $v \in U$ the corresponding controlled process X^v is a Markov process with infinitesimal generator A^v, with the domain $D(A^v)$ depending on v. One has to assume that there exists $D \subset D(A^v)\ \forall\ v \in U$ with D sufficiently large.

Because X is considered to be a Markov process and the information flow is modeled via $(\mathcal{F}_t^X)_{t\geq 0}$ one would expect that the control policies are of the form

$$u_s = \underline{u}(s, X_s), \tag{1.3.1}$$

such that they depend only on the last state X_s at time s. These controls are called Markov control policies. Moreover in such cases $X_s^{\underline{u}}$ is given by a Markov process and its infinitesimal generator, $\forall\ \Phi \in D$

$$A^{\underline{u}}\Phi(t, x) = A^{\underline{u}(t,x)}\Phi(t, x). \tag{1.3.2}$$

In general the assumptions which have to be made depend on the considered problem, in order to ensure that $A^{\underline{u}}$ really defines a Markov process. In subsequent considerations the notion of Markov control policies will be generalized via admissible control systems, which depend on the probabilistic ingredients in a more complex way than as in (1.3.1), in general they depend additionally on the underlying probability space.

Now we define the general two objectives which are considered in the context of this theory. In the first case the problem is considered on the finite time interval $t \leq s \leq t_1$ and one has the objective:

$$J(t, x, u) = E_{tx}\left[\int_t^{t_1} L(s, X_s, u_s)ds + \psi(X_{t_1})\right]. \tag{1.3.3}$$

So in this setting the problem is called the finite time horizon problem. J is called either gain function if one wants to maximize one's yield or cost function if one wants to minimize the costs. The function $L(s, x, v)$ is called running gain/cost function and $\psi(x)$ terminal gain/cost function. Additionally, if $L(s, x, v) \equiv 0$ the problem is said to be in Mayer form and if $\psi(x) \equiv 0$ it is said to be in Lagrange form. Of course the technical assumptions on L and ψ have to be specified such that J is well defined. L and ψ are generally assumed to be continuous and such that the integral (1.3.3) exists. Note that these kind of problems can also be considered in the case, where the Markov process is time-inhomogeneous, see [5].

In the alternative case the problem is called infinite time horizon problem or more precisely the infinite horizon discounted gain/cost problem. It is considered on the time interval $[0, \infty)$ and the objective reads as follows

$$J(x, u) = E_x\left[\int_0^{\infty} e^{-\beta s} L(X_s, u_s)ds\right], \tag{1.3.4}$$

where the function L is analogously as above and $\beta > 0$ is the preference or interest rate.

1.3.1 Dynamic programming

Here the formal dynamic programming principle (DPP) is stated and the dynamic programming equation (DPE) is derivated. This approach leaves out the proof of the DPP, because this has to be done separately for every specific task. In a subsequent step a so called Verification Theorem is stated and proven, for which one has to assume that the dynamic programming equation has a sufficiently smooth classical solution, which suffices the former heuristic derivation. If there is no such solution then solutions have to be considered in some weaker sense, one possibly relies on so called viscosity solutions. In order to consider that topic it is referred to [5, p. 53 - 123 and p. 213 - 251]. An alternative to the verification approach above is mentioned in the paper by Albrecher and Thonhauser [2, p. 307 - 308], namely one also can prove that the "[...] obtained solution of the HJB equation actually dominates the values of all other possible strategies (usually by martingale arguments)."

Finite time horizon problem:

We start with the finite time horizon problem formulated above with objective (1.3.3). So we start at time t and know the initial state $X_t = x \in \Sigma$, with that we can define the value function, which is a function of the initial data:

$$V(t,x) = \sup_{u \in C} J(t,x;u), \tag{1.3.5}$$

where J is supposed to be the gain function and the supremum is taken over the set C of admissible controls. Again C has to be specified in every specific problem, hence this formulation is a heuristic one. The dynamic programming principle or Bellman's principle of dynamic programming is then stated as follows for $t \leq t+h \leq t_1$ it holds that

$$V(t,x) = \sup_{C} E_{tx}\left[\int_t^{t+h} L(s, X_s^u, u_s)ds + V(t+h, X_{t+h}^u)\right]. \tag{1.3.6}$$

This means to be optimal one has to optimize the running gain on the interval $[t, t+h]$ and continue in an optimal way from $t+h$ to the endpoint t_1 with initial data $(t+h, X_{t+h})$. As a next step we state the dynamic programming equation (1.3.7) and show heuristically how it is obtained from the DPP:

$$0 = \max_{v \in U}\left[A^v V(t,x) + L(t,x,v)\right], \tag{1.3.7}$$

considered in $[t_0, t_1] \times \Sigma$ with terminal (Cauchy) data

$$V(t_1, x) = \psi(x), \quad x \in \Sigma. \tag{1.3.8}$$

Considering a constant control $u_s = v$ on the time interval $s \in [t, t+h]$ one gets from (1.3.6)

$$V(t,x) \geq E_{tx}\left[\int_t^{t+h} L(s, X_s^v, v)ds\right] + E_{tx}\left[V(t+h, X_{t+h}^v)\right]. \tag{1.3.9}$$

The inequality is equivalent to

$$0 \geq \frac{1}{h}E_{tx}\left[\int_t^{t+h} L(s, X_s^v, v)ds\right] + \frac{1}{h}E_{tx}\left[V(t+h, X_{t+h}^v) - V(t,x)\right].$$

Then consider the limit $h \to 0$ from above and we obtain

$$\lim_{h\to 0+} \frac{1}{h}E_{tx}\left[\int_t^{t+h} L(s, X_s^v, v)ds\right] = L(t,x,v)$$

and

$$\begin{aligned} &\lim_{h\to 0+} \frac{1}{h}E_{tx}\left[V(t+h, X_{t+h}^v) - V(t,x)\right] \\ \overset{(*)}{=} &\lim_{h\to 0+} \frac{1}{h}E_{tx}\left[\int_t^{t+h} A^v V(s, X_s^v)ds\right] = A^v V(t,x). \end{aligned}$$

Note that on has to think about the requirements needed such that all these steps are verified, in particular one has to assume something like $V \in D$ in order to ensure using Dynkin's formula at $(*)$. Note that in this context it is more restrictive to say that for a function f the term $\lim_{h\searrow 0} \frac{1}{h}E_{tx}[f(X_{t+h}) - f(x)]$ is considered with respect to the supremum norm than to assume that f fulfills Dynkin's formula and hence it is required that Dynkin holds. This will give us for all $v \in U$ the inequality

$$0 \geq A^v V(t,x) + L(t,x,v). \tag{1.3.10}$$

Assuming existence of an optimal Markov control policy $\underline{u}^*$ and the Markov process X^* generated by $A^{\underline{u}^*}$ we obtain equality in (1.3.9)

$$V(t,x) = E_{tx}\left[\int_t^{t+h} L(s, X_s^*, \underline{u}(s, X_s^*))ds\right] + E_{tx}\left[V(t+h, X_{t+h}^*)\right].$$

Analogously, as above one obtains

$$0 = A^{\underline{u}^*} V(t,x) + L(t,x,\underline{u}^*(t,x)), \tag{1.3.11}$$

assuming that all considered steps must be accompanied by requirements on the used mathematical objects, e.g. continuity of $\underline{u}^*$ at (t, x). And finally we see that inequality (1.3.10) and equality (1.3.11) are equivalent to the dynamic programming equation (1.3.7). Moreover the above argumentation yields that for an optimal Markov control policy it should hold that

$$\underline{u}^*(t, x) \in \arg\max[A^v V(t, x) + L(t, x, v)], \tag{1.3.12}$$

whereby

$$\arg\max g(v) = \{u^* \in U : g(u^*) \geq g(v) \ \forall v \in U\}.$$

Infinite horizon discounted gain problem:

In this case the considered objective is (1.3.4):

$$J(x, u) = E_x \left[\int_0^\infty e^{-\beta s} L(X_s, u_s) ds \right], \quad \text{for some } \beta > 0$$

and the formal value function has the form

$$V(x) = \sup_{u \in C_1} J(x; u). \tag{1.3.13}$$

Similar as above C_1 denotes the set of admissible controls and the data is given by the initial state $X_0 = x$. The proceeding here is analogous to the previous case and one obtains for the infinite time interval $[0, \infty)$ the dynamic programming equation

$$\beta V(x) = \max_{v \in U}[A^v V(x) + L(x, v)]. \tag{1.3.14}$$

In this framework it is natural to consider so called stationary Markov control policies $\underline{u}(x)$. As above the formal approach yields that for an optimal stationary Markov control policy $\underline{u}^*$ it should hold that

$$\underline{u}^*(x) \in \arg\max[A^v V(x) + L(x, v)]. \tag{1.3.15}$$

1.3.2 Verification step

In the former derivation one gets that the value function should fulfill the dynamic programming equation, under the restriction that the dynamic programming principle holds. So if one wants to solve stochastic optimal control problems the starting point in order to obtain information about the value function will be the corresponding dynamic programming equation, provided that the DPP is fulfilled. Therefore solving the DPE generates a candidate for the value function, but in fact one has to verify that this really is the value function. This is done in the so called verification step, the corresponding statement about the solution is called verification theorem. This theorem claims that if a classical solution of the DPE has been found, than this is the minimum, respectively the maximum, of the considered expected value among the set of all admissible control systems, which are defined below. Nevertheless the assumptions of the theorem are very strong, it is not always possible that a solution in the classical sense does exist. Hence also solutions in a weaker sense are maybe considered (e.g. viscosity solutions). For this purpose it is referred to [5, p. 53 - 123 and p. 213 - 251] as mentioned above. For example in the case of controlled Markov diffusion processes one needs that $W \in C^{1,2}$ (one has to apply Itô's formula) and W and $A^v W$ need to fulfill a polynomial growth condition.

Finite time horizon problem:

We are now in the first setting with the finite time interval $[t_0, t_1]$. $W(t, x)$ is called a classical solution of the DPE (1.3.7) with data (1.3.8) if

$$W \in D, \tag{1.3.16}$$

where D was defined above,

$$0 = \max_{v \in U} \left[A^v W(t, x) + L(t, x, v) \right], \tag{1.3.17}$$

for $(t, x) \in [t_0, t_1] \times \Sigma$ and

$$W(t_1, x) = \psi(x), \quad \forall x \in \Sigma. \tag{1.3.18}$$

Furthermore for given initial data (t, x), let

$$\pi = (\Omega, (\mathcal{F}_s), P, X, u).$$

π is called an admissible control system if

- $(\Omega, \mathcal{F}_{t_1}, P)$ is a probability space,
- $(\mathcal{F}_s)$ for $t \leq s \leq t_1$ is an increasing family of σ-algebras,
- $X = (X_s)$ and $u = (u_s)$ are stochastic processes on $[t, t_1]$ such that:
  - (i) $X_s \in \Sigma$ for $t \leq s \leq t_1$ and $X_t = x$,
    X is càdlàg and adapted w.r.t. $(\mathcal{F}_s)$;
  - (ii) $u_s \in U$ for $t \leq s \leq t_1$,
    u is adapted and as a process measurable w.r.t. $(\mathcal{F}_s)$;
  - (iii) the Dynkin formula

$$E_{tx}(\Phi(t_1, X_{t_1})) - \Phi(t, x) = E_{tx}\left(\int_t^{t_1} A^{u_s}\Phi(s, X_s)ds\right) \quad (1.3.19)$$

holds for all $\Phi \in D$ such that the properties

$$E_{tx}(|\Phi(t_1, X_{t_1})|) < \infty \text{ and } E_{tx}\left(\int_t^{t_1} |A^{u_s}\Phi(s, X_s)|ds\right) < \infty$$

are fulfilled.

It is worth to mention that regarding π these technical points have to be verified, when tackling a specific problem. In this setting we get that

$$J(t, x; \pi) = E_{tx}\left(\int_t^{t_1} L(s, X_s, u_s)ds + \psi(X_{t_1})\right) \quad (1.3.20)$$

is the total expected gain (respectively cost) associated to π.

Theorem 6. *If* (1.3.17)-(1.3.18) *has a classical solution W with $W \in D$, then $\forall$ $(t, x) \in [t_0, t_1] \times \Sigma$ it holds that*

(a) $W(t, x) \geq J(t, x; \pi)$ for every admissible control system π.

(b) If $\exists$ $\pi^* = (\Omega^*, (\mathcal{F}_s^*), P^*, X^*, u^*)$ an admissible control system such that

$$u^*(s) \in \arg\max[A^v W(s, X_s^*) + L(s, X_s^*, v)]$$

for $\lambda \otimes P^*$ - almost all $(s, \omega) \in [t_0, t_1] \times \Omega^*$, then

$$W(t, x) = J(t, x; \pi^*),$$

where λ denotes the Lebesgue-measure.

Proof. Consider an admissible control system π, because of the assumptions we have that $u_s \in U$ and therefor

$$A^{u_s}W(s, X_s) + L(s, X_s, u_s) \leq 0.$$

Moreover we get

$$\begin{aligned} W(t,x) &= E_{tx}\left(\int_t^{t_1} -A^{u_s}W(s,X_s)ds\right) + E_{tx}(W(t_1, X_{t_1})) \\ &= E_{tx}\left(\int_t^{t_1} -A^{u_s}W(s,X_s)ds + \psi(X_{t_1})\right) \\ &\geq E_{tx}\left(\int_t^{t_1} L(s,X_s,u_s)ds + \psi(X_{t_1})\right) = J(t,x;\pi), \end{aligned}$$

using first Dynkin's formula (1.3.19) and the terminal data (1.3.18) and then the above inequality.

And on the other hand if, we take π^* instead of an arbitrary control system π one obtains equality instead of inequality and hence (b) holds too. □

Analogously, as above, let C be the class of all admissible control systems, then we consider

$$V_{AS}(t,x) = \sup_C J(t,x;\pi).$$

For W such that the assumptions of the Verification Theorem are fulfilled, we obtain

$$V_{AS} = W.$$

The requirements in this Theorem are very restrictive, but a procedure to obtain an optimal Markov control policy would be the following one. We have to choose a Markov control policy $\underline{u}^*$ with the property that for each $(t,x) \in [t_0, t_1] \times \Sigma$ it holds that

$$\underline{u}^*(t,x) \in \arg\max[A^v W(t,x) + L(t,x,v)].$$

Additionally, if for any initial data (t,x) the process $\underline{u}^*$ determines a Markov process X_s^* with infinitesimal generator $A^{\underline{u}^*}$, one is able to take

$$u_s^* = \underline{u}^*(s, X_s^*). \tag{1.3.21}$$

If the associated control system π^* is admissible, then π^* is optimal. u^* is said to be an optimal Markov control policy, provided the method is working, see [5, p. 142].

Infinite time horizon problem:

Now we consider the above type of problem in another framework. The objective is of the form (1.3.4):

$$J(x,u) = E_x\left[\int_0^\infty e^{-\beta s} L(X_s, u_s)ds\right], \quad \beta > 0.$$

One has to assume for the generator A^v for $v \in U$ of a time-homogeneous Markov process, that $D \subset D(A^v)$ for all v, such that D contains sufficiently many functions.
For an initial state x

$$\pi = (\Omega, (\mathcal{F}_s), P, X, u)$$

is called an admissible control system if

- $(\Omega, \mathcal{F}_\infty, P)$ is a probability space,
- $(\mathcal{F}_s)$ for $s \geq 0$ is an increasing family of σ-algebras, $\mathcal{F}_s \subset \mathcal{F}$
- $X = (X_s)$ and $u = (u_s)$ are stochastic processes on $[0, \infty)$ such that:
  (a) $X_s \in \Sigma$ for $s \geq 0$ and $X_t = x$,
  X is càdlàg and adapted w.r.t. $(\mathcal{F}_s)$;
  $u_s \in U$ for $s \geq 0$,
  u is adapted and as a process measurable w.r.t. $(\mathcal{F}_s)$;

  (b) the Dynkin formula

$$E_{tx}(\Phi(t_1, X^u_{t_1})) - \Phi(t,x) = E_{tx}\left(\int_t^{t_1} A^{u_s}\Phi(s, X^u_s)ds\right) \quad (1.3.22)$$

  holds for $\Phi(t,x) = e^{-\beta t}\phi(x)$ for all $\phi \in D$ such that the properties

$$E_{tx}(|\Phi(t_1, X^u_{t_1})|) < \infty \text{ and } E_{tx}\left(\int_t^{t_1} |A^{u_s}\Phi(s, X^u_s)|ds\right) < \infty$$

  for $0 < t_1 < \infty$ are fulfilled and

  (c)

$$E_x\left(\int_0^\infty e^{-\beta s}|L(X^u_s, u_s)|ds\right) < \infty.$$

For $\Phi(t,x) = e^{-\beta t}\phi(x)$ the Dynkin formula has the following form

$$e^{-\beta t_1} E_x(\phi(X^u_{t_1})) - \phi(x) = E_x\left(\int_0^{t_1} e^{-\beta s}[A^{u_s}\phi - \beta\phi](X^u_s)ds\right). \quad (1.3.23)$$

Further, the DPE reads as follows compare (1.3.14):

$$\beta W(x) = \max_{v\in U}[A^v W(x) + L(x,v)]. \quad (1.3.24)$$

If W fulfills the DPE (1.3.24) for all $x \in \Sigma$ and $W \in D$, then it is called a classical solution.

Lemma 3. *For $W \in D$ a classical solution of* (1.3.24) *it holds that:*

(a) $W(x) \geq J(x;\pi)$ for those admissible π for which

$$\limsup_{t_1\to\infty} e^{-\beta t_1} E_x(W(X_{t_1})) \geq 0. \quad (1.3.25)$$

(b) If an admissible system π^* exists with

$$u^*_s \in \arg\max[A^v W(X^*_s) + L(X^*_s, v)],$$

for $\lambda \otimes P^*$ - almost all $(s,\omega) \in [0,\infty) \times \Omega^*$, and

$$\liminf_{t_1\to\infty} e^{-\beta t_1} E_x(W(X^*_{t_1})) \leq 0, \quad (1.3.26)$$

one obtains that

$$W(x) \leq J(x;\pi^*),$$

where λ denotes the Lebesgue-measure.

Proof. For an admissible control system π we know by assumption that $u_s \in U$ and therefore

$$A^{u_s} W(X_s) - \beta W(X_s) + L(X_s, u_s) \leq 0.$$

Now Dynkin's formula is used with $\Phi = e^{-\beta t}W$ and with formula (1.3.23) one obtains

$$\begin{aligned} W(x) &= e^{-\beta t_1} E_x(W(X_{t_1})) + E_x\left(\int_0^{t_1} e^{-\beta s}[-A^{u_s}W + \beta W](X_s)ds\right) \\ &\geq E_x\left(\int_0^{t_1} e^{-\beta s} L(X_s, u_s)ds\right) + e^{-\beta t_1} E_x(W(X_{t_1})). \end{aligned} \tag{1.3.27}$$

Now consider a subsequence such that $t_1 \to \infty$ and $e^{-\beta t_1}E_x(W(X_{t_1}))$ tends to a limit larger or equal than 0, this can be done according to (1.3.25). Hence (a) is proven.

For (b) the only difference to the proof of (a) is that the inequality in (1.3.27) becomes an equality and one has to consider a subsequence such that $t_1 \to \infty$ and that $e^{-\beta t_1}E_x(W(X_{t_1}))$ tends to a limit smaller or equal than 0, which is possible due to (1.3.26). □

Similarly to the first case we call C_1 the class of admissible controls systems π which additionally fulfill (1.3.25) and set

$$V_{AS} = \sup_{C_1} J(x; \pi). \tag{1.3.28}$$

Theorem 7. *For a classical solution W of* (1.3.24) *with $W \in D$ it holds that*

$$W(x) \geq V_{AS}(x).$$

Furthermore, we have if $\exists\ \pi^* \in C_1$ such that (1.3.26) *is fulfilled and*

$$u_s^* \in \arg\max[A^v W(X_s^*) + L(X_s^*, v)]$$

for $\lambda \otimes P^*$ - almost all $(s,\omega) \in [0,\infty) \times \Omega^*$, then

$$W(x) = V_{AS}(x) = J(x; \pi^*),$$

where λ denotes the Lebesgue-measure.

The theorem follows by applying Lemma 3. Note that π^* is the optimal control system in the class of admissible control systems C_1. Analogously, to the former case the main task is to identify an optimal stationary Markov control policy $\underline{u}^*$ fulfilling

$$\underline{u}^*(x) \in \arg\max[A^v W(x) + L(x, v)], \tag{1.3.29}$$

$$u_s^* = \underline{u}^*(X_s^*) \text{ and}$$

$(X_s^*)_{s\geq 0}$ is a Markov process, with initial state $X_0^* = x$ and generator $A^{\underline{u}^*}$.

Chapter 2

A singular stochastic control problem

2.1 Introduction

In this chapter the main focus lies on investigating and reproducing the paper "Capital supply uncertainty, cash holdings, and, investment" by Hugonnier et al. [6]. This paper models a financial environment, where the considered firm has to optimize its capital policy under some assumptions about availability of outside financing and investment opportunities. This optimization is done due to control of the dividends paid to the shareholders and of the financing strategy respectively the investment strategy. They start with a discussion about the economic impact on the posed problem and how such models are considered in different economic environments, meaning to allow several assumptions, which somehow rely on observed scenarios in the real world, as for example a financial crisis, to be incorporated. We will lay the focus on the mathematical aspects of the problem and verify those steps, where the paper refers to the literature. In the paper the descriptive part and the rigorous mathematical part, where the proofs are carried out, are separated. In fact all mathematical verifications are gathered in various parts of an appendix. Some statements and additional assumptions are justified and written down in the so called "Supplementary Appendix", see e.g. [6, p. 8, footnote 5]. We have to comment that this "Supplementary Appendix" was not considered in this framework, because at this point it was not available. In our treatment there will be no segregation of these two parts, because of the better understanding of the mathematical implications and proofs. Furthermore we will evaluate the problem, which is in fact a singular stochastic control problem, mainly under the use of the corresponding chapter "Singular Stochastic Control" in [5, p. 315 - 362] as it is referred to at [6, p. 38]. Nevertheless also other chapters of the book [5] were used, because the singular stochastic control theory is obviously strongly related to the already considered aspects of the optimal stochastic control theory. Furthermore we use the paper [16] by Shreve et al. in order to investigate the considered model. The way of proceeding will be to implement the theory in coincidence with the provided problem.

2.2 Model assumptions

Lets begin with some general assumptions about modeling uncertainty and the financial background, which is based on [6, p. 8 - 12].

- The time is assumed to be continuous.
- The probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where the filtration is $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, with $\mathcal{F}_\infty \subset \mathcal{F}$ and fulfills the usual conditions.
- The involved agents are acting risk neutral.
- The discount rate is constant and denoted by $\rho > 0$.

Concerning the firm we have the following conditions. First of all note that the management of the firm represents interest of the shareholders and works therefore with steering mechanisms in order to manipulate the investment, payout, liquidation and financing strategy. How this is done can be easily seen by considering the cash reserves of the firm (2.2.4).

We have that the assets in place of the firm generate a continuous cash flow stream $X = (X_t)_{t\geq 0}$, that satisfies

$$dX_t = \mu_0 dt + \sigma dB_t, \tag{2.2.1}$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$, μ_0 is a constant, denoting the mean of the cash flow, and σ is also a constant, denoting the volatility of the cash flow.

Moreover the firm has the opportunity to invest, which has a positive impact on the mean income of the firm, namely the cash flow then satisfies

$$dX_t = \mu_0 dt + \sigma dB_t + (\mu_1 - \mu_0)dt,$$

where $\mu_1 > \mu_0$ in order to have an increase. The firm can decide when it uses this growth option, but at time of investment, denoted by the stopping time T, it has to pay a lump sum cost $K > 0$.

The firms payout policy is managed by a nondecreasing, adapted and left-continuous process $D = (D_t)_{t\geq 0}$ with $D_{0-} = 0$, which denotes the cumulative dividends paid out to the shareholders up to time t. Note that in the paper it is assumed that $(D_t)_{t\geq 0}$ is only adapted and nondecreasing, but to ensure that it is progressive in order to be in line with the theory about singular stochastic control, we assume that it is left-continuous. Since it is assumed that it is adapted and left-continuous we make sure that the dividend process is even predictable. Hence with Lemma 1 we get that $(D_t)_{t\geq 0}$ is therefore progressive.

The financing in this model stands in contrast to other model approaches of similar type. In fact the provision of fresh capital is not unrestrictedly possible. The capital supply uncertainty is modeled as follows. If the firm is willing to accept outside financing, it has to search for investors. This searching procedure follows a Poisson process $(N_t)_{t\geq 0}$ with intensity λ. Meaning that conditional on searching, the jump times of the Poisson process are the occurrence times of investors and hence these are the times where financing happens. The hight of the funding at time $t \geq 0$ is denoted by a nonnegative predictable process $f = (f_t)_{t\geq 0}$.
The intensity λ of the Poisson process represents the arrival rate of investors. Considering extremal values for λ leads to different capital market conditions. On the one hand if $\lambda = 0$, then there will not be one single investor available at any time and hence the firm has to finance losses or investment with capital of the aggregated cash holdings. On the other hand if we let $\lambda \to \infty$, then investors are always on-hand and hence the firm can use outside funding all the time. This situation implies that cash reserves are redundant and issuing new equity is able to cover losses and costs of investment.
In this environment where availability of capital respectively investors is not always given one has that at these times of refinancing, the new investors can

acquire part of the arising surplus. This is done in a bargaining procedure over the conditions of the corresponding new issue in order to set the "[...] cost of capital or, equivalently, the proceeds from the stock issue", see [6, p. 11]. The resulting distribution of the surplus between new investors and the established shareholders is determined via, so-called *Nash bargaining*. The corresponding protagonists are the following.

- Let $\eta \in [0,1]$ be the bargaining power of the new investors, further η is set to depend on the availability of investors and hence we assume $\eta = \frac{a}{a+\lambda}$ with $a > 0$,
- $V(c)$ be the value of the firm given as a function of the cash reserves c,
- $\mathcal{S}_f V(c)$ be the financing surplus, which satisfies

$$\mathcal{S}_f V(c) := V(c+f) - f - V(c) \tag{2.2.2}$$

  and

- π^* be the amount new investors obtain, if funding in the height of $f \geq 0$ is raised, so we have

$$\pi^* = \arg\max_{\pi \geq 0} \left[\pi^\eta (\mathcal{S}_f V(c) - \pi)^{1-\eta}\right] = \eta \mathcal{S}_f V(c). \tag{2.2.3}$$

Note that the bargaining power and hence the corresponding surplus of the investors respectively the cost of capital declines if the supply of capital respectively λ increases.

The firm's rulers are able to retain gains, which therefore increases the cash reserves. These cash holdings generate interest at a constant rate r such that $r < \rho$. The corresponding carry cost of cash is therefore denoted by δ and satisfies $\delta = \rho - r > 0$. The corresponding cash reserves are denoted by C_t at time $t \geq 0$, which evolves as follows

$$dC_t = \left(rC_{t-} + \mu_0 + \mathbb{1}_{\{T \leq t\}}(\mu_1 - \mu_0)\right) dt + \sigma dB_t + f_t dN_t - dD_t - \mathbb{1}_{\{T=t\}} K. \tag{2.2.4}$$

So we have that on the one hand cash reserves increase due to gaining interest, earnings of the assets in place and outside funding and on the other hand decrease due to paying dividends to the shareholders and due to the lump sum cost of the investment opportunity. In the paper it is stated that in this model "[...]T, D, and f are endogenously determined", see [6, p. 10].

Finally, it is possible that the firm is liquidated. This is done if the cash holdings hit the value zero. That situation can have two potential reasons. Primarily, they can reach zero due to a negative evolution of the cash flows and this form of liquidation is in some sense unintended or in other words not directly intended. On the contrary the management has the force to pay a so called liquidating dividend, which means that all cash reserves are paid out to the shareholders. The liquidation value of a firm's assets l_i related to the current corresponding mean cash flow rate μ_i satisfies

$$l_i = \frac{\varphi \mu_i}{\rho} = \varphi \int_0^\infty e^{-\rho t} \mu_i dt, \tag{2.2.5}$$

where $\varphi < 1$ denotes the tangibility of assets, that is the fraction of the assets which can be converted into cash in case of liquidation. Therefore $1 - \varphi$ denotes the haircut which stands for the part of the investment that is not returned. These are economically based assumptions for the liquidation value, which seem to be quite reasonable, but there is also some mathematical point of view in this context. Let $a(x)$ denote the drift of the stochastic differential equation for the state process, which yields in our case $a(x) = rx + \mu_i$. In fact, if one compares this model to the absorption problem in the paper by Shreve et al. [16], then we notice that the condition at [16, p. 65] namely

$$a'(x) = r \leq \rho$$

is fulfilled. And moreover we realize that the drift at time zero $a(0)$ in the considered SDE for the state process in our model fulfills

$$a(0) = \mu_i > \varphi \mu_i = \rho l_i$$

too, which gives in the the absorption problem a condition such that the optimal strategy is of barrier type. On the other hand if this condition is not fulfilled, then the corresponding theorem for the absorption problem at [16, p. 67] states that the optimal strategy would be an initial jump to zero with reward in our notation $l_i + c$, if we start with the cash reserves c. Note that in the absorption problem there is no outside financing considered.

The stochastic liquidation time of the firm is called τ_0, where

$$\tau_0 = \inf\{t \geq 0 | C_t = 0\}.$$

Because of capital supply uncertainty and the omnipresent possibility of paying a liquidating dividend, liquidation automatically occurs if cash holdings hit the value zero. That means we can either stop, if a liquidating dividend is payed out or the cash reserve process hits the value zero. That is why we obtain $C_{t-} - \Delta D_t \geq 0$, which stands in contrast to the general dividend problem in the respective literature, where it must hold that $C_{t-} - \Delta D_t > 0$.

Hence the firm's management faces the following task. It controls the payout $(D_t)_{t\geq 0}$, the funding $(f_t)_{t\geq 0}$ and the time of investment (T) in order to maximize the present value of future dividends. The problem the management has to solve is precisely described by the following terms:

$$\begin{aligned} V(c) = \sup_{(f,D,T)} E_c\bigg[&\int_0^{\tau_0} e^{-\rho t}\left(dD_t - (f_t + \eta \mathcal{S}_{f_t} V(C_{t-}))dN_t\right) \\ &+ e^{-\rho\tau_0}\left(l_0 + \mathbb{1}_{\{\tau_0 > T\}}(l_1 - l_0)\right)\bigg]. \end{aligned} \tag{2.2.6}$$

Note that this equation is implicit in V, but in the further proceeding we are switching to an auxiliary problem where $\eta = 0$ and therefore the implicit representation disappears. Further note that the value in zero of V depends whether the investment opportunity was exploited or not. The description of the above term is as follows. The dD_t - term stands for the present value of the paid amount to current shareholders up to the liquidation time τ_0, the dN_t-term is representing the "[...] net of the claim of new (outside) investors on future cash flows", see [6, p. 12]. Moreover the last term stands for the discounted liquidation value, which depends on the condition if investment takes place before liquidation or not.

Note that the firm is not allowed to fund the investment with debt. This restriction is described in the paper to have no effect on the obtained results in the environment, where the supply of credits is uncertain.

In the end we discuss the crucial fact that the above supremum is taken of an expression, which contains itself, the value function namely, in the term of the surplus in case of a funding. Due to this fact in the paper it is stated that the optimization problem is similar to a so called rational expectations problem. Furthermore it is proclaimed that in the Supplementary Appendix, which is, as mentioned above, not available, the authors of the paper showed that "[...] introducing bargaining in the model and solving the corresponding rational expectations equilibrium is equivalent to reducing the arrival rate of investors from λ to $\lambda^* \equiv \lambda(1-\eta)$ in an otherwise similar model where outside investors have no bargaining power", see [6, p. 12]. So a new model is considered where some parameters are changed. The new parameters are labeled with a star. So we have the new bargaining power $\eta^* = 0$ and the intensity of the Poisson process changes to $\lambda^* \equiv \lambda(1-\eta)$ and hence one obtains for the objective:

$$V(c) = \sup_{(f,D,T)} E_c \left[\int_0^{\tau_0} e^{-\rho t} \left(dD_t - f_t dN_t \right) + e^{-\rho \tau_0} \left(l_0 + \mathbb{1}_{\{\tau_0 > T\}} (l_1 - l_0) \right) \right]. \tag{2.2.7}$$

Note that we only consider the model where the firm has no growth option, because otherwise this would go beyond the scope of this thesis. As a remark we just want to mention that the resulting switching problem, where the firm is able to change the drift of the underlying cash flow process as described above, is considered in the respective paper by Hugonnier et al. [6].

2.3 Singular stochastic control theory

This section serves as introduction to singular stochastic control theory. The main purpose is that we want to provide some theoretical background in order to evaluate the above problem in the corresponding context.

We start as before with a formal and heuristic derivation of some important properties. The considered problem will be a restricted version of the infinite horizon problem of a Markov diffusion process in $\mathbb{R}^d$. That is why we are beginning with a general description of this type of problem, which is based on [5, p. 171 - 172] and then make some further assumptions in order to start a formal discussion of the singular stochastic control case, which is based on [5, p. 315 - 321]. The starting point will be the following stochastic differential equation in the time - homogeneous case similar to (1.2.18) for the state process and a objective function similar to (1.3.4).

2.3.1 The infinite time horizon problem for Markov diffusions in $\mathbb{R}^d$

Let $(X_s)_{s\geq 0}$ be a d-dimensional stochastic process, namely the state process, satisfying

$$\begin{aligned} dX_s = b(X_s, u_s)ds + \sigma(X_s, u_s)dW_s, \quad s \geq 0, \\ \text{and with initial data } X_0 = x \in \mathbb{R}^d. \end{aligned} \tag{2.3.1}$$

It is assumed that $(W_s)_{s\geq 0}$ is a n-dimensional standard Brownian motion, $u_s \in U$ is the control at time s and

$$b : \mathbb{R}^d \times U \to \mathbb{R}^d,$$

$$\sigma : \mathbb{R}^d \times U \to \mathbb{R}^{d\times n}$$

are continuous functions, where $U \subset \mathbb{R}^m$ for some m is closed. Moreover $b(\cdot, v)$ and $\sigma(\cdot, v)$ are $C^1(\mathbb{R}^d)$ and for some constant C it holds that

$$\begin{aligned} |b_x| \leq C, \quad |\sigma_x| \leq C, \\ |b(x, v)| \leq C(1 + |x| + |v|) \text{ and} \\ |\sigma(x, v)| \leq C(1 + |x| + |v|). \end{aligned} \tag{2.3.2}$$

Note that b_x denotes the gradient with respect to x and σ_x denotes the differential $D_x\sigma$ because σ is a matrix valued function. With $|\sigma|$ we consider the operator norm. Furthermore let $O \subseteq \mathbb{R}^d$ be open and we have that either $O = \mathbb{R}^d$ or ∂O is a compact $(d-1)$-dimensional C^3 manifold. The state process is controlled up to the first time it leaves the open set O. The exit time of X_s from O is denoted by τ and if $X_s \in O \ \forall \ s \geq 0$ we have $\tau = \infty$, in particular $\tau = \inf\{s \geq 0 : X_s \notin O\}$ is a stopping time. Additionally let

$$L : \mathbb{R}^d \times U \to \mathbb{R},$$

$$g : \mathbb{R}^d \to \mathbb{R}$$

continuous functions such that L satisfies the following polynomial growth condition

$$|L(x, v)| \leq \tilde{C}(1 + |x|^k + |v|^k) \tag{2.3.3}$$

for appropriate constants $\tilde{C} \geq 0$ and $k \geq 0$. Note that regarding the definition in [5] a function $f : \Sigma \mapsto \mathbb{R}$, where Σ is a metric space, is called polynomial growing if for constants $M, l \geq 0$ it holds that

$$|f(x)| \leq M(1 + |x|^l), \forall \ x \in \Sigma.$$

Now with $\beta \geq 0$ a discount factor, the objective function reads as follows

$$J(x;u) = E_x \left[\int_0^\tau e^{-\beta s} L(X_s, u_s) ds + \mathbb{1}_{\{\tau<\infty\}} e^{-\beta\tau} g(X_\tau) \right]. \tag{2.3.4}$$

Now we want to establish the considered optimal stochastic control problem accurately. Therefore the notion of a reference probability system is introduced similar to the admissible control system in the more general view.

Definition 10. *A 5-tupel*

$$\nu = (\Omega, \mathcal{F}, \{\mathcal{F}_s\}, P, W)$$

is called a reference probability system, if

- $(\Omega, \mathcal{F}, P)$ *is a probability space,*
- $\{\mathcal{F}_s\}$ *is a filtration such that* $\{\mathcal{F}_s\} \subset \mathcal{F}$ *and*
- $W = (W_s)_{s\geq 0}$ *is a* $\mathcal{F}_s$*-adapted Brownian motion on* $[0,\infty)$.

In order to restrict the set of controls to some extent that all considered objects are defined and make sense we consider the following.

Definition 11. *The set* $\mathfrak{A}_\nu$ *contains all* $\mathcal{F}_s$ *- progressively measurable processes* $(u_s)_{s\geq 0}$ *with values in* U *such that*

$$E \left[\int_0^{t_1} |u_s|^p ds \right] < \infty \tag{2.3.5}$$

for $t_1 < \infty$ *and* $p = 1, 2, 3, \ldots$ *and it holds that*

$$E_x \left[\int_0^\tau e^{-\beta s} |L(X_s, u_s)| ds \right] < \infty. \tag{2.3.6}$$

Now we distinguish between the supremum over all controls in $\mathfrak{A}_\nu$:

$$V_\nu(x) = \sup_{\mathfrak{A}_\nu} J(x;u) \tag{2.3.7}$$

and the supremum over all reference probability systems:

$$V_{PM}(x) = \sup_\nu V_\nu(x). \tag{2.3.8}$$

The terminology reads as follows, if $V_\nu(x) = J(x; u^*)$ for $u^* \in \mathfrak{A}_\nu$, then u^* is called $\nu-$ *optimal*. On top of this if $V_{PM}(x) = J(x; u^*)$, where $u^* \in \mathfrak{A}_{\nu^*}$ for some reference probability system $\nu^* = (\Omega^*, \{\mathcal{F}^*{}_s\}, P^*, W^*)$, then u^* is called an optimal admissible progressively measurable control process. In this context it is referred to [5, p. 160], where it is moreover stated that $V_\nu = V_{PM}$ if specific conditions are satisfied.

In the following we use previously stated results especially from Section 1.2.1 and Section 1.3 in order to obtain the dynamic programming equation also called the Hamilton-Jacobi-Bellman equation of this type of infinite time horizon problem. The starting point is the DPE (1.3.14), (1.3.24):

$$\beta W(x) = \max_{v \in U}[A^v W(x) + L(x, v)].$$

This equation is transformed, using for the operator A^v the analogous operator as in the uncontrolled case (1.2.32). We obtain the following (2.3.9) just by using (2.3.1) instead of (1.2.18) in the derivation of (1.2.32). So now the coefficient functions do not depend on the time t, which means that we are in the time-homogeneous setup. Additionally, the coefficient functions depend now on the control u, but following the lines in the derivation of (1.2.32) with the changed setting should lead us to the analogous results. This gives us that Dynkin's formula holds as above and the operator has the form

$$\begin{aligned} A^v W(x) &= \sum_{i=1}^{d} W_{x_i} b^i(x, v) + \frac{1}{2}\left[\sum_{i=1}^{d}\sum_{j=1}^{d} W_{x_i x_j} a_{ij}(x, v)\right] \\ &= (D_x W)^T\ b(x, v) + \frac{1}{2} tr((D_x^2 W) a(x, v)), \end{aligned} \tag{2.3.9}$$

where $a(x, v) = \sigma(x, v)\sigma(x, v)^T$. For further details see [5, p. 160 - 161].

Up next with these results we obtain the Hamilton-Jacobi-Bellman equation, which is here a second-order nonlinear partial differential equation

$$-\beta V + \sup_{v \in U}\left[(DV)^T\ b(x, v) + \frac{1}{2} tr((D^2 V) a(x, v)) + L(x, v)\right] = 0,\ x \in O \tag{2.3.10}$$

and the boundary condition

$$V(x) = g(x),\ x \in \partial O. \tag{2.3.11}$$

Further the HJB equation is uniformly elliptic if $\exists\ c > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x,v)\xi_i\xi_j \geq c|\xi|^2 \qquad \forall\ \xi \in \mathbb{R}^d \text{ and } v \in U. \tag{2.3.12}$$

In this case, regarding to the literature, the HJB equation (2.3.10) together with the boundary data (2.3.11) is expected to have a smooth solution that is even unique if one additionally assumes that O is bounded. In the opposite if the condition (2.3.12) is not fulfilled the HJB equation is degenerate elliptic. Here it is not so easy to derive the desired conclusions. But it is possible with other restrictions to consider the task in some weaker sense and one is able to verify the existence of a viscosity solution.

2.3.2 The singular stochastic control case

Now we have to make additional assumptions on the considered objects of the above infinite time horizon setting in order to admit that the variation of the state through the control is not continuous. Which stands in opposition to the classical view of control problems, where this variation is differentiable in time. The following is, as already mentioned at the beginning of this section, based on [5, p. 315 - 321].

The considered setting is the one from above and we make on the back of this, the following assumptions. Consider the reference probability system $\nu = (\Omega, \mathcal{F}, \{\mathcal{F}_s\}, P, W)$, but now additionally to the requirements in the above definition the filtration $\{\mathcal{F}_s\}$ is assumed to be right continuous. Further let as above $O \subset \mathbb{R}^d$ and let $U \subset \mathbb{R}^d$ such that

$$\forall v \in U, \lambda \geq 0 \text{ it holds that } \lambda v \in U, \tag{2.3.13}$$

which means that U is a closed cone in $\mathbb{R}^d$. Moreover we set

$$b(x,v) = \hat{b}(x) + v, \tag{2.3.14}$$

$$\sigma(x,v) = \hat{\sigma}(x), \tag{2.3.15}$$

$$L(x,v) = \hat{L}(x) + \hat{c}(v), \tag{2.3.16}$$

and

$$\hat{c}(\lambda v) = \lambda \hat{c}(v), \quad \forall\ \lambda \geq 0, \tag{2.3.17}$$

where $x \in \mathbb{R}^d$ and $v \in U$. Further let $\hat{b}, \hat{\sigma} \in C^1(\mathbb{R}^d)$, where their first order partial derivatives are bounded and let $\hat{L}, \hat{c} \in C(\mathbb{R}^d)$. Note that the set U of controls is not bounded anymore.

Now we consider the related HJB equation, we obtain from (2.3.10) plugging in the previous terms:

$$\begin{aligned} -\beta V(x) + \sup_{v \in U} \Big[(DV)^T(x)\hat{b}(x) + (DV)^T(x)v \\ + \frac{1}{2} tr((D^2V)(x)\hat{a}(x)) + \hat{L}(x) + \hat{c}(v) \Big] = 0, \quad (2.3.18) \end{aligned}$$

where $x \in O$, $\hat{a}(x) = \hat{\sigma}(x)\hat{\sigma}(x)^T$ and the boundary condition

$$V(x) = g(x), \; x \in \partial O. \tag{2.3.19}$$

So we can rewrite (2.3.18) such that

$$\begin{aligned} -\beta V(x) + (DV)^T(x)\hat{b}(x) + \frac{1}{2} tr((D^2V)(x)\hat{a}(x)) \\ + \hat{L}(x) + \sup_{v \in U} \left[(DV)^T(x)v + \hat{c}(v) \right] = 0, \quad (2.3.20) \end{aligned}$$

where $x \in O$. We look at the term with the supremum more closely by setting

$$\mathcal{H}(DV(x)) := \sup_{v \in U} \left[(DV)^T(x)v + \hat{c}(v) \right]. \tag{2.3.21}$$

Then we have, if there exists a $v \in U$ such that $(DV)^T v + \hat{c}(v) > 0$, by the property (2.3.13) of U and by (2.3.17) of $\hat{c}$ that $\mathcal{H}(DV(x)) = +\infty$. Considering only normalized controls in the set

$$K := \{v \in U : |v| = 1\}, \tag{2.3.22}$$

which contains, loosely speaking, the allowed directions in that the control is able to move. Further set

$$H(DV(x)) := \sup_{v \in K} \left[(DV)^T(x)v + \hat{c}(v) \right] \tag{2.3.23}$$

and finally we deduce that

$$\mathcal{H}(DV(x)) = \begin{cases} +\infty & \text{if } H(DV(x)) > 0, \\ 0 & \text{if } H(DV(x)) \leq 0. \end{cases} \tag{2.3.24}$$

With these results we see that in some cases parts of the HJB equation explode. In order to handle this fact it is formally assumed that for the value function V it holds that

$$H(DV(x)) \leq 0, x \in O. \tag{2.3.25}$$

On top of this we define the following differential operator

$$\mathcal{L}V(x) := -\beta V(x) + (DV)^T(x)\hat{b}(x) + \frac{1}{2}tr((D^2V)(x)\hat{a}(x)) \tag{2.3.26}$$

with these notations we have that the HJB equation (2.3.20) reads as follows

$$\mathcal{L}V(x) + \hat{L}(x) + \mathcal{H}(DV(x)) = 0, x \in O, \tag{2.3.27}$$

with (2.3.25) and due to the properties of the supremum, we have that V formally fulfills

$$\mathcal{L}V(x) + \hat{L}(x) \leq 0, x \in O. \tag{2.3.28}$$

Next, if we consider the case where there exists a $x \in O$ such that the inequality is strict $H(DV(x)) < 0$, then we obtain that the unique maximizer in (2.3.21) is zero in a neighborhood of x. So this yields that in a formal way around x the optimal feedback control should be zero. So it is stated that due to the relation of the uncontrolled diffusion processes to linear equations one should get

$$\mathcal{L}V(x) + \hat{L}(x) = 0, \text{ if } H(DV(x)) < 0. \tag{2.3.29}$$

So using (2.3.25) together with (2.3.28) and using (2.3.29) one can conclude that

$$\max\{\mathcal{L}V(x) + \hat{L}(x), H(DV(x))\} = 0, x \in O, \tag{2.3.30}$$

and from above we have the boundary condition

$$V(x) = g(x), x \in \partial O. \tag{2.3.31}$$

For a rigorous basis see the Verification Theorem 4.1 of the DPE (2.3.30) with boundary data (2.3.31) at [5, p. 322].

Following the lines of [5, p. 315 - 319] it is stated that considering the above quantities generally does not lead to optimal controls respectively almost optimal ones become arbitrary large. The key point is to integrate over u_s and take this as the control process. First of all we write down the stochastic differential equation (2.3.1) for the state process X with the above assumptions on the coefficient functions:

$$dX_s = \hat{b}(X_s)ds + u_s ds + \hat{\sigma}(X_s)dW_s, \quad s \geq 0, \\ \text{with initial data } X_0 = x \in \mathbb{R}^d. \tag{2.3.32}$$

Because of this special form of the SDE we consider the following, set

$$\hat{u}_s = \begin{cases} |u_s|^{-1}u_s & \text{if } u_s \neq 0, \\ 0 & \text{if } u_s = 0, \end{cases} \tag{2.3.33}$$

and

$$\xi_t = \int_0^t |u_s|ds. \tag{2.3.34}$$

As a consequence (2.3.32) transforms to

$$dX_s = \hat{b}(X_s)ds + \hat{\sigma}(X_s)dW_s + \hat{u}_s d\xi_s, \quad s \geq 0. \tag{2.3.35}$$

The considered control at time t is therefore

$$z_t = \int_{[0,t)} \hat{u}_s d\xi_s. \tag{2.3.36}$$

The class of controls has to be extended in order to admit the control z_t because this does not necessarily need to be absolutely continuous with respect to t. Further assumptions are that on every interval $[0, t]$, z_t is a function of bounded variation, which means that every component of z_t is the difference of two monotone functions of t. Further let μ be the the total variation measure of z and

$$\xi_t = \int_{[0,t)} d\mu_s. \tag{2.3.37}$$

We have:

$$\xi \text{ is nondecreasing, real-valued and} \\ \text{left continuous furthermore } \xi_0 = 0. \tag{2.3.38}$$

The process z is identified with the pair $(\xi, \hat{u})$. In fact by the Radon-Nikodym theorem one has $\exists \hat{u}_s \in \mathbb{R}^d$ such that (2.3.36) holds and further $|\hat{u}_s| \leq 1$. Moreover for z given, ξ_t is determined uniquely $\forall t \geq 0$ and $\hat{u}_s$ is also determined uniquely for μ-a.e. $s \geq 0$. Concerning the progressive-measurability we have that z_s is $\mathcal{F}_s$-progressively measurable implies that first ξ_s is $\mathcal{F}_s$-progressively measurable and second that there exists a version of $\hat{u}_s$ which is again $\mathcal{F}_s$-progressively measurable, see [5, p. 318] and [5, Appendix D, p. 397 - 402]. Hence z_s is assumed to be $\mathcal{F}_s$-progressively measurable, and we assume that the function $\hat{u}_s$ is $\mathcal{F}_s$-progressively measurable and additionally fulfills the relation (2.3.36). Moreover the following conditions are assumed

$$\hat{u}_s \in U, \text{ for } \mu\text{-a.e. } \; s \geq 0, \tag{2.3.39}$$

and

$$E\left(|z_t|^m\right) < \infty \text{ for } m \in \mathbb{N}. \tag{2.3.40}$$

Definition 12. *Let $\hat{\mathfrak{A}}_\nu$ be the set of all $z = (\xi, \hat{u})$, which are progressively measurable and fulfill* (2.3.38)*,* (2.3.39)*and* (2.3.40)*.*

For the proof of existence of $(X_s)_{s\geq 0}$ we have that, given an initial state $x \in O$ the Picard iteration yields $X_t^{(m)}$ for $m = 1, 2, 3, \ldots$, such that $X_t^{(m)} - z_t$ converges against $X_t - z_t$, where the convergence is almost surely (with probability 1) and uniformly for bounded t, see [5, p. 319]. Hence the equation

$$X_t = x + \int_0^t \hat{b}(X_s)ds + \int_0^t \hat{\sigma}(X_s)dW_s + z_t, \quad t \geq 0, \tag{2.3.41}$$

with

$$X_{t+} - X_t = z_{t+} - z_t,$$

(which means that jumps in the state process are jumps in the control process) has a unique and left-continuous solution X. Note that we have

$$X_{t+} = \lim_{s \searrow t} X_s = X_t + z_{t+} - z_t = X_t + \hat{u}_t(\xi_{t+} - \xi_t)$$

for every $t \geq 0$, generally X_t is not continuous. Further X is left-continuous and $\mathcal{F}_t$ is right-continuous yields that the exit time τ of X_s from $\overline{O}$ is a $\mathcal{F}_t$ - stopping time. Finally the objective as in (2.3.4) reads as follows

$$J(x; \xi, \hat{u}) = E_x\left[\int_{[0,\tau]} e^{-\beta s}(\hat{L}(X_s)ds + \hat{c}(\hat{u}_s)d\xi_s)\right] \tag{2.3.42}$$

and we have to maximize this term first over all pairs $(\xi, \hat{u}) \in \hat{\mathfrak{A}}_\nu$ and second over all reference probability systems ν. $J(x; \xi, \hat{u})$ is defined $\forall (\xi, \hat{u}) \in \hat{\mathfrak{A}}_\nu$, because $\hat{L} \geq 0$ and $\hat{c} \geq 0$, but it is possible that J becomes $+\infty$. In the end we set

$$V_\nu(x) = \sup_{\hat{\mathfrak{A}}_\nu} J(x; \xi, \hat{u}) \tag{2.3.43}$$

and

$$V(x) = V_{PM}(x) = \sup_\nu V_\nu(x). \tag{2.3.44}$$

Chapter 3

Establishing the solution

In this chapter we apply the dynamic programming approach with the corresponding HJB equation derived in the preceding parts in order to solve a special case of the above presented model. Note that the following solution approach is based on [6, p. 13 - 16 and p. 37 - 44]. In fact the model is restricted such that we are only considering firms without growth option, therefore the two main terms change. The SDE (2.2.4) becomes

$$dC_t = (rC_{t-} + \mu)\, dt + \sigma dB_t + f_t dN_t - dD_t, \text{ with } C_{0-} = c \tag{3.0.1}$$

and the value function (2.2.7) transforms to

$$\hat{V}(c) = \sup_{(f,D)\in\Theta} E_c \left[\int_0^{\tau_0} e^{-\rho t} \left(dD_t - f_t dN_t \right) + e^{-\rho \tau_0} l \right], \tag{3.0.2}$$

where τ_0 denotes again the time when the process C_t equals zero for the first time and Θ is the set of admissible strategies. In particular we have that Θ contains all pairs (f, D) of financing and dividend strategies with the properties listed in the Section 2.2 and additionally they need to fulfill

$$E_c \left[\int_0^{\tau_0} e^{-\rho s} (dD_s + f_s dN_s) \right] < \infty.$$

It is important to mention that now the optimal value of the firm is here denoted with a hat and on the other hand the value of the firm corresponding to the conjectured optimal control strategy will be denoted with V. Besides, note that one could also let V depend on the considered mean cash flow rate μ_i and therefore on l_i and denote it with V_i. In that notation one can solve the problem for different μ_i and has for all i the respective solution, but in order to improve the readability we suppress the subscript notation and point out that in the general case the problem can be obviously solved for V_0, where the firm has not yet used the growth option with rate μ_0 and for V_1 with rate μ_1, where the growth opportunity was already realized. Nevertheless, these solutions can be used to derive a solution in the general case where the possibility to invest exists. Further the differential operator (2.3.26) reads now as follows:

$$\mathcal{L}\phi(c) := -\rho\phi(c) + \phi'(c)(rc + \mu) + \frac{1}{2}\sigma^2\phi''(c). \tag{3.0.3}$$

The above presented approach can be used to derive the HJB equation for the model without the outside financing term $f_t dN_t$, where the control is the cumulated dividend process D_t. The HJB equation for the whole model is derived by splitting up the two problems into the part with the optimal dividends and the part with the optimal financing height at given points in time. Regarding the second part of the problem we define additionally the following operator

$$\mathfrak{F}\phi(c) := \max_{f\geq 0}\left[\lambda^*\left(\phi(c+f) - \phi(c) - f\right)\right], \tag{3.0.4}$$

where λ^* is the new intensity of the Poisson process $(N_t)_{t\geq 0}$ as mentioned above and the definition is reasoned in the Section 3.1. Fortunately, this separation can be done, because we can split up the conditional expectation in the value function and use special properties of the Poisson process, this is implemented in the next section. Furthermore the behavior of the problem considered with respect to a barrier strategy, which is stated to be the optimal one, is of a special kind. Following a barrier strategy means that the firm is paying dividends if the cash reserves reach a prespecified threshold b in order to keep those cash holdings at the barrier level. Beneath this level it is stated to be optimal to raise the cash reserves up to the barrier b. This is done due to the natural evolution of the process C_t driven by interest gains and earnings and additionally due to the possible outside financing. This implies that beneath b the firm is always searching for investors and given that the Poisson process N_t jumps, which means that investors are available, the firm makes use of the financing possibility and the height is determined via the current position of the cash reserves. In fact f_t will be exactly that amount such that the cash holdings reach the barrier level b.

Otherwise, if the funding would let C_t exceed the barrier, then this surplus would instantaneously be paid out in the form of dividends.

Exactly with this consideration the optimal barrier is determined, because as long as the value function is concave, the firm is searching for investors and tries to raise its cash reserves and exactly at the first point in time when the concavity does not hold anymore it is optimal for the firm to start paying dividends. Similar observations are made in the paper by Shreve et al. [16], where a general absorption problem is considered. This type of problem is more or less an analogue to our model problem, despite the fact that they only can control their state process via the in our case called dividend process analogously to the theory on singular stochastic control in Section 2.3.2. This paper gives a very good view how general problems of this type namely absorption problems can be solved. Nevertheless both, in the paper [16] by Shreve et al. and in the paper [6] by Hugonnier et al., they are choosing not to derive the Hamilton-Jacobi-Bellman equation for their problem, but they state the HJB equation and use martingale techniques in order to do the verification of their proclaimed solution. This method is explained in the four steps of solving the problem declared in Section 3.1.

Moreover note that we consider singular stochastic controls because in the above case it could be possible that $c > b$, which means that the reserves start beyond the barrier and in order to compensate this, the optimal dividend strategy will jump immediately to the barrier, hence this generates a jump in the control D at time $0+$ and as we have seen in Section 2.3.2, this yields a jump in the cash holdings at the same point in time.

3.1 Dynamic programming approach

Now the main ingredients of the problem have been introduced and we start with the step-wise solution of the problem. Using dynamic programming it is very common to declare the main line of attack before tackling the problem.

We follow the steps below in order to solve the problem:

1. First of all we start with the derivation of the related Hamilton-Jacobi-Bellman equation.
2. We must show that, if we have a smooth solution of the HJB equation, then this solution is larger or equal than the value function $\hat{V}$.

3. Next we make a special guess respectively a well-considered construction in order to have a candidate for the conjectured optimal policy and additionally we have to find the related value of the firm.

4. In the end we have to verify that this firm value of the former step solves the HJB equation and is sufficiently smooth.

At this point it is worth mentioning that the first step can be treated specifically. Of course one needs to derive the HJB equation at least heuristically in order to solve the stochastic optimal control problem, but if one has obtained the correct equation and the subsequent steps can be carried out appropriately, then it is not necessary to derive the HJB equation. Loosely speaking that means one can even let the equation show up without any information of its derivation. This is based on the approach, written down in the above steps and also implemented in [16], that first states the HJB equation, then claims that if one has a smooth enough solution, then this function dominates the value function of the control problem. Finally, due to the shape of the problem, one is able to formulate a conjectured solution of the control problem and verify that it is optimal by proving that it solves the HJB equation in the desired way.

3.1.1 First step: The Hamilton-Jacobi-Bellman equation

If we use $l(c) := l + c$ as the related value in case of liquidation of the firm with cash holdings c, we have that in the considered paper [6, Appendix B, p. 38] the HJB equation for the value function is stated to be

$$\max\{\mathcal{L}\phi(c) + \mathfrak{F}\phi(c), 1 - \phi'(c), l(c) - \phi(c)\} = 0. \tag{3.1.1}$$

As reported above, in the paper the derivation is more or less left out, nevertheless this paper refers to the theory about singular stochastic control in the book [5]. With the previously presented results on this topic we have provided a basis for some parts of this gap in the discourse of the problem. In fact the equation is obtained by considering a mixture of a singular stochastic optimal control problem and an optimal stopping problem. The first two parts under the maximum are similar to (2.3.30) obtained in Section 2.3.2. The differential operator is extended by the term $\mathfrak{F}\phi(c)$ which comes from the investment opportunity part with the Poisson process and the control f. Moreover if we compare (2.3.35) respectively (2.3.41) and (3.0.1) we get with $D_t = \xi_t$ that $\hat{u} \equiv -1$, furthermore we get with (2.3.42) and (3.0.2) that $\hat{c} \equiv 1$. Hence we obtain that $H(DV(x))$ from (2.3.23) corresponds to $1-\phi'(c)$, whereby $\hat{u} \equiv -1 \in K = \{1, -1\}$. For another point of view regarding this topic see [2, p. 309 - 310]. The third term $l(c) - \phi(c)$ comes from the stopping problem. We realize that $\phi(c) \geq l(c)$ shall hold is quite reasonable, because the management has at any point in time the opportunity to liquidate the firm and get the amount $l(c) = l + c$. Hence the firm value is at least as large as the liquidation value $l(c)$. For further information on optimal stopping in this context see [4, p. 53 - 57].

In the following we give a heuristic derivation of the HJB equation including the part, which corresponds to the financing opportunity controlled by f. The subsequent considerations are, to some extent, a detailed accomplishment of the above description. At the beginning we need some additional results on stochastic processes with jumps, especially integrals with respect to Poisson processes, for that reason see [11, Chapter 15: Stochastic Calculus for Jump Processes]. The stochastic integral with respect to a Poisson process fulfills

$$\int_0^t f_s dN_s = \sum_{k=1}^{N_t} f_{T_k}, \tag{3.1.2}$$

where $(T_1, \ldots, T_{N_t})$ are the jump times of the Poisson process $(N_s)_{s\geq 0}$ up to time t.

Theorem 8. *[11, p. 508 - 509]*
Consider a process $(f_s)_{s\geq 0}$, which is adapted to the filtration generated by the compound Poisson process $(Y_s)_{s\geq 0}$, whereby $Y_s = \sum_{k=1}^{N_t} Z_k$ and Z_k are i.i.d. random variables denoting the jump heights and λ denoting the intensity, such that

$$E\left[\int_0^t |f_s| ds\right] < \infty, \qquad t > 0.$$

Then, we obtain that the expected value of the compound Poisson compensated stochastic integral has the following representation

$$E\left[\int_0^t f_{s-} dY_s\right] = E\left[\int_0^t f_{s-} Z_s dN_s\right] = \lambda E[Z] E\left[\int_0^t f_s ds\right]. \tag{3.1.3}$$

With these results we can rewrite the equation 3.0.2 concerning the value function:

$$\begin{aligned}\hat{V}(c) &= \sup_{(f,D)\in\Theta} E_c\left[\int_0^{\tau_0} e^{-\rho t}\left(dD_t - f_t dN_t\right) + e^{-\rho\tau_0} l\right] \\ &= \sup_{(f,D)\in\Theta}\left\{E_c\left[\int_0^{\tau_0} e^{-\rho t} dD_t + e^{-\rho\tau_0} l\right] - E_c\left[\int_0^{\tau_0} e^{-\rho t}\lambda^* f_t dt\right]\right\}.\end{aligned} \tag{3.1.4}$$

Now we can see that the dD_t term corresponds to the singular part and the dN_t term respectively the dt term corresponds to the normal stochastic control part. Further we rewrite (3.0.1) and obtain

$$C_t = c + \int_0^t \left(rC_{s-} + \mu\right) ds + \sigma dB_t - dD_t + \sum_{i=1}^{N_t} f_{T_i}, \tag{3.1.5}$$

where $(T_1, \dots, T_{N_t})$ denote the jump times of the Poisson process $(N_s)_{s\geq 0}$ up to time t. For the moment and in order to derive the HJB equation we assume that D_t has a representation as $D_t = \int_0^t d_s ds$, and if we want to let D_t be singular then just consider the limit $d_s \to \infty$ at some time point $\bar{s}$ and we obtain that the control D_t has a jump at the point $\bar{s}$. So d as a density process could be unbounded, analogous to the considerations in the paper by Albrecher and Thonhauser [2, p. 309]. Moreover we assume in the subsequent part of the derivation that the controls are fixed constants denoted by $\bar{f}$ and $\bar{d}$ on a small interval $[0, h]$ for $h > 0$. This leads to the representation

$$C_h^{\bar{f},\bar{d}} = c + \int_0^h \left(rC_{s-}^{\bar{f},\bar{d}} + \mu - \bar{d}\right) ds + \sigma dB_h + \sum_{i=1}^{N_h} \bar{f}.$$

In the next step we apply Itô's formula to the function $\hat{V}(C_h^{\bar{f},\bar{d}})$ to obtain

$$\begin{aligned}\hat{V}(C_h^{\bar{f},\bar{d}}) - \hat{V}(c) = &\int_0^h \hat{V}'(C_{s-}^{\bar{f},\bar{d}}) d\left(C_s^{\bar{f},\bar{d}}\right)^{\text{cont.}} + \frac{\sigma^2}{2}\int_0^h \hat{V}''(C_{s-}^{\bar{f},\bar{d}}) ds \\ &+ \sum_{0\leq s\leq h} \Delta\hat{V}(C_s^{\bar{f},\bar{d}}).\end{aligned}$$

Applying Itô's formula to $e^{-\rho h}\hat{V}(C_h^{\bar{f},\bar{d}})$ and using the above obtained results yields

$$\begin{aligned}
&e^{-\rho h}\hat{V}(C_h^{\bar{f},\bar{d}}) - \hat{V}(c) = \\
&= \int_0^h e^{-\rho s} d\hat{V}(C_s^{\bar{f},\bar{d}}) + \int_0^h \hat{V}(C_{s-}^{\bar{f},\bar{d}})(-\rho)e^{-\rho s}ds \\
&= \int_0^h e^{-\rho s}\hat{V}'(C_{s-}^{\bar{f},\bar{d}})d\left(C_s^{\bar{f},\bar{d}}\right)^{\text{cont.}} + \int_0^h e^{-\rho s}\frac{\sigma^2}{2}\hat{V}''(C_{s-}^{\bar{f},\bar{d}})ds \\
&\quad + \sum_{0\le s\le h} e^{-\rho s}\Delta\hat{V}(C_s^{\bar{f},\bar{d}}) - \int_0^h e^{-\rho s}\rho\hat{V}(C_{s-}^{\bar{f},\bar{d}})ds \\
&= \int_0^h e^{-\rho s}\left(-\rho\hat{V}(C_{s-}^{\bar{f},\bar{d}}) + (rC_{s-}^{\bar{f},\bar{d}} + \mu - \bar{d})\hat{V}'(C_{s-}^{\bar{f},\bar{d}}) + \frac{\sigma^2}{2}\hat{V}''(C_{s-}^{\bar{f},\bar{d}})\right)ds \\
&\quad + \sum_{0\le s\le h} e^{-\rho s}\Delta\hat{V}(C_s^{\bar{f},\bar{d}}) + \int_0^h e^{-\rho s}\sigma dB_t.
\end{aligned}$$

Furthermore note that the following is true

$$\begin{aligned}
E_c\left[\sum_{0\le s\le h} e^{-\rho s}\Delta\hat{V}(C_s^{\bar{f},\bar{d}})\right] &= E_c\left[\sum_{i=1}^{N_h} e^{-\rho s}\left(\hat{V}(C_{T_i-}^{\bar{f},\bar{d}} + \bar{f}) - \hat{V}(C_{T_i-}^{\bar{f},\bar{d}})\right)\right] \\
&= E_c\left[\int_0^h e^{-\rho s}\left(\hat{V}(C_{s-}^{\bar{f},\bar{d}} + \bar{f}) - \hat{V}(C_{s-}^{\bar{f},\bar{d}})\right)dN_s\right] \\
&= E_c\left[\lambda^*\int_0^h e^{-\rho s}\left(\hat{V}(C_{s-}^{\bar{f},\bar{d}} + \bar{f}) - \hat{V}(C_{s-}^{\bar{f},\bar{d}})\right)ds\right].
\end{aligned}$$

Based on the dynamic programming principle

$$\hat{V}(c) = \sup_{(f,D)\in\Theta} E_c\left[\int_0^{h\wedge\tau_0} e^{-\rho s}(d_s - \lambda^* f_s)ds + e^{-\rho(h\wedge\tau_0)}\hat{V}(C_{h\wedge\tau_0})\right],$$

we obtain the following

$$0 = \sup_{(f,D)\in\Theta} E_c\left[\int_0^{h\wedge\tau_0} e^{-\rho s}(d_s - \lambda^* f_s)ds\right] + E_c\left[e^{-\rho(h\wedge\tau_0)}\hat{V}(C_{h\wedge\tau_0}) - \hat{V}(c)\right].$$

Now we choose constant controls as above, divide by h and consider the limit $h \searrow 0$, hence we get

$$0 \ge \lim_{h\searrow 0}\left(\frac{1}{h}E_c\left[\int_0^{h\wedge\tau_0} e^{-\rho s}(\bar{d} - \lambda^*\bar{f})ds\right] + \frac{1}{h}E_c\left[e^{-\rho(h\wedge\tau_0)}\hat{V}(C_{h\wedge\tau_0}) - \hat{V}(c)\right]\right).$$

This immediately gives

$$0 \geq \bar{d} - \lambda^* \bar{f} + \lim_{h \searrow 0} \left(\frac{1}{h} E_c \left[e^{-\rho(h \wedge \tau_0)} \hat{V}(C_{h \wedge \tau_0}) - \hat{V}(c) \right] \right).$$

Further with the results from above we obtain for the last term

$$\begin{aligned}
&\lim_{h \searrow 0} \left(\frac{1}{h} E_c \left[e^{-\rho(h \wedge \tau_0)} \hat{V}(C_{h \wedge \tau_0}) - \hat{V}(c) \right] \right) \\
&= \lim_{h \searrow 0} \left(\frac{1}{h} E_c \left[\int_0^h e^{-\rho s} \left(-\rho \hat{V}(C_{s-}^{\bar{f},\bar{d}}) + (r C_{s-}^{\bar{f},\bar{d}} + \mu - \bar{d}) \hat{V}'(C_{s-}^{\bar{f},\bar{d}}) + \frac{\sigma^2}{2} \hat{V}''(C_{s-}^{\bar{f},\bar{d}}) \right) ds \right] \right. \\
&\left. + \frac{1}{h} E_c \left[\sum_{0 \leq s \leq h} e^{-\rho s} \Delta \hat{V}(C_s^{\bar{f},\bar{d}}) + \int_0^h e^{-\rho s} \sigma dB_t \right] \right) \\
&= \lim_{h \searrow 0} \left(\frac{1}{h} E_c \left[\int_0^h e^{-\rho s} \left(\mathcal{L}(\hat{V}(C_{s-}^{\bar{f},\bar{d}})) - \bar{d}\, \hat{V}'(C_{s-}^{\bar{f},\bar{d}}) \right) ds \right] \right. \\
&\left. + \frac{1}{h} E_c \left[\lambda^* \int_0^h e^{-\rho s} \left(\hat{V}(C_{s-}^{\bar{f},\bar{d}} + \bar{f}) - \hat{V}(C_{s-}^{\bar{f},\bar{d}}) \right) ds \right] + \frac{1}{h} E_c \left[\int_0^h e^{-\rho s} \sigma dB_t \right] \right) \\
&= \mathcal{L}(\hat{V}(c)) - \bar{d}\, \hat{V}'(c) + \lambda^* (\hat{V}(c + \bar{f}) - \hat{V}(c)),
\end{aligned}$$

where the last equality holds since the expectation of the integral with respect to the Brownian motion is zero. Using this result in addition to the preceding one we get

$$0 \geq \mathcal{L}(\hat{V}(c)) + \lambda^* (\hat{V}(c + \bar{f}) - \hat{V}(c) - \bar{f}) + (1 - \hat{V}'(c)) \bar{d}.$$

Analogous to the results obtained in Section 2.3.2 for the general singular stochastic control case, and analogous to the considerations in the paper by Albrecher and Thonhauser [2, p. 309] in an actuarial framework, we have the following

$$0 = \sup_{(f,D) \in \Theta} \{ \mathcal{L}(\hat{V}(c)) + \lambda^* (\hat{V}(c + f) - \hat{V}(c) - f) + (1 - \hat{V}'(c)) d \},$$

where d and D are connected via $\int_0^t d_s ds = D_t$. Note that this expression is only meaningful up to some extent. Since, as in the above mentioned literature we have that, if $1 - \hat{V}'(c) > 0$ at some value c the locally maximizing control would be unbounded and hence also the above supremum would be unbounded. But on the contrary if $1 - \hat{V}'(c) < 0$ the locally maximizing control would be equal to zero, so

$$0 = \mathcal{L}(\hat{V}(c)) + \max_{f \geq 0}\{\lambda^*(\hat{V}(c+f) - \hat{V}(c) - f)\} = \mathcal{L}(\hat{V}(c)) + \mathfrak{F}(\hat{V}(c)).$$

As a consequence we get the following Hamilton-Jacobi-Bellman equation for this part of the problem, compare (2.3.30):

$$0 = \max\{\mathcal{L}(\hat{V}(c)) + \mathfrak{F}(\hat{V}(c)), 1 - \hat{V}'(c)\}. \tag{3.1.6}$$

Note that we have to formally assume that $1 - \hat{V}'(c) \leq 0$ for all $c \geq 0$ holds true for the value function $\hat{V}$, compare this to (2.3.25).

Finally we take into consideration the comment on the optimal stopping part of the problem made at the beginning of this section and obtain the entire Hamilton-Jacobi-Bellman equation for the whole model (3.1.1):

$$\max\{\mathcal{L}\phi(c) + \mathfrak{F}\phi(c), 1 - \phi'(c), l(c) - \phi(c)\} = 0.$$

3.1.2 Second step: HJB solutions exceed the value function

This step is quite important for further actions. In fact we want to check that, if we have constructed a control strategy which is admissible, then this control and the related firm value is the optimal one. This check can be done, if we show that every smooth enough solution to the HJB equation is larger or equal than the value function. Hence, considering an admissible control policy with a firm value that is a solution of the HJB equation gives us an admissible candidate, which then is, if it is smooth enough, in fact the optimal one and its firm value is the value function, which is the supremum over all firm values respectively gain functions. In the following theorem the wanted conclusion is proven via martingale techniques, which is very common in the framework of stochastic optimal control problems.

Theorem 9. *Let $\phi \in C^2(0,\infty)$ be a solution to the HJB equation* (3.1.1) *and further let ϕ be concave, then $\phi(c) \geq \hat{V}(c)$.*

Proof. Consider $\phi \in C^2(0,\infty)$ a solution of (3.1.1) and a given admissible control $(f, D) \in \Theta$. In this setting we want to apply the product rule respectively Itô's formula in order to investigate the process

$$Y_t := e^{-\rho t}\phi(C_t) + \int_{0+}^{t} e^{-\rho s}(dD_s - f_s dN_s). \tag{3.1.7}$$

Analogously as proceeded in [16, p. 60 - 62] we have to make sure that the process, to which we want to apply results from stochastic analysis [9], is a semimartingale and hence adapted and cadlag. But in this situation the state process C is due to the control process D not right-continuous. So we work with the process

$$\bar{C}_t := C_{t+},$$

which firstly ensures the right continuity and secondly since the underlying filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is right continuous, we get that $\bar{C} = (\bar{C}_t)_{t\geq 0}$ is adapted to this filtration. On top of this we have to consider some needed results for stochastic processes. The above dividend process D is left-continuous and satisfies $[D]_s^c = 0$, because it is assumed to be nondecreasing and therefore has finite variation. Further by the properties of the Poisson process N, if we consider the continuous part, it holds that $N_s^c = 0$ and moreover we have $[N]_s^c = N_s^c = 0$. Now we start by applying the product rule, also called partial integration formula, as follows for $0 \leq t_1 \leq t$:

$$e^{-\rho t}\phi(\bar{C}_t) = e^{-\rho t_1}\phi(\bar{C}_{t_1}) + \int_{t_1}^{t} (-\rho)e^{-\rho s}\phi(\bar{C}_{s-})ds + \int_{t_1}^{t} e^{-\rho s} d\phi(\bar{C}_s)$$

In the next step we use Itô's formula to obtain

$$d\phi(\bar{C}_s) = \phi'(\bar{C}_{s-})d\bar{C}_s + \frac{1}{2}\phi''(\bar{C}_{s-})d[\bar{C}]_s^c + \Delta\phi(\bar{C}_s) - \phi'(\bar{C}_{s-})\Delta\bar{C}_s,$$

which gives that

$$\begin{aligned} e^{-\rho t}\phi(\bar{C}_t) =& e^{-\rho t_1}\phi(\bar{C}_{t_1}) + \int_{t_1}^{t} (-\rho)e^{-\rho s}\phi(\bar{C}_{s-})ds \\ &+ \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})d\bar{C}_s + \int_{t_1}^{t} e^{-\rho s}\frac{1}{2}\phi''(\bar{C}_{s-})d[\bar{C}]_s^c \\ &+ \sum_{t_1\leq s\leq t} e^{-\rho s}\left(\Delta\phi(\bar{C}_s) - \phi'(\bar{C}_{s-})\Delta\bar{C}_s\right). \end{aligned}$$

Next we have from (3.0.1) the evolution of the state process C that

$$d\bar{C}_s = (r\bar{C}_{s-} + \mu)ds + \sigma dB_s + f_s dN_s - dD_{s+}$$

and

$$d[\bar{C}]_s^c = \sigma^2 ds + f_s^2 d[N]_s^c - d[D]_s^c = \sigma^2 ds.$$

Note that all quadratic covariation continuous part terms are zero due to the properties of the processes B, N and D, in particular we have that on the one hand B is a Brownian motion and N is a Poisson process, so $[B, N]^c = 0$ and on the other hand D is of finite variation. Using this in the above equation we obtain

$$\begin{aligned}
e^{-\rho t}\phi(\bar{C}_t) =& e^{-\rho t_1}\phi(\bar{C}_{t_1}) + \int_{t_1}^{t} (-\rho) e^{-\rho s}\phi(\bar{C}_{s-})ds \\
&+ \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})(r\bar{C}_{s-} + \mu)ds + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s \\
&+ \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-}) f_s dN_s - \int_{(t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD_{s+} \\
&+ \int_{t_1}^{t} e^{-\rho s}\frac{1}{2}\phi''(\bar{C}_{s-})\sigma^2 ds \\
&+ \sum_{t_1 \le s \le t} e^{-\rho s}\left(\Delta\phi(\bar{C}_s) - \phi'(\bar{C}_{s-})\Delta\bar{C}_s\right) \\
=& e^{-\rho t_1}\phi(\bar{C}_{t_1}) + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s \\
&+ \int_{t_1}^{t} e^{-\rho s}\left[-\rho\phi(\bar{C}_{s-}) + \phi'(\bar{C}_{s-})(r\bar{C}_{s-} + \mu) + \frac{1}{2}\phi''(\bar{C}_{s-})\sigma^2\right] ds \\
&+ \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-}) f_s dN_s - \int_{(t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD_{s+} \\
&+ \sum_{t_1 \le s \le t} e^{-\rho s}\left(\Delta\phi(\bar{C}_s) - \phi'(\bar{C}_{s-})\Delta\bar{C}_s\right).
\end{aligned}$$

Now we exploit the fact that ϕ solves the HJB equation (3.1.1), which yields that

$$\begin{aligned}
\mathcal{L}\phi(\bar{C}_{s-}) &\le -\mathfrak{F}\phi(\bar{C}_{s-}) = -\max_{f\ge 0}[\lambda^*(\phi(\bar{C}_{s-} + f) - \phi(\bar{C}_{s-}) - f) \\
&\le -\lambda^*(\phi(\bar{C}_{s-} + f_s) - \phi(\bar{C}_{s-}) - f_s).
\end{aligned}$$

Note that the Poisson process is right-continuous and therefore $N_{s+} = N_s$, together with the above inequality this yields

$$\begin{aligned}e^{-\rho t}\phi(\bar{C}_t) \leq & e^{-\rho t_1}\phi(\bar{C}_{t_1}) + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s \\ & - \int_{t_1}^{t} e^{-\rho s}\lambda^*(\phi(\bar{C}_{s-} + f_s) - \phi(\bar{C}_{s-}) - f_s)ds \\ & + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})f_s dN_s - \int_{(t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD_{s+} \\ & + \sum_{t_1\leq s\leq t} e^{-\rho s}\left(\Delta\phi(\bar{C}_s) - \phi'(\bar{C}_{s-})\Delta\bar{C}_s\right).\end{aligned}$$

Since we are working with the process C_{t+} we have to distinguish between jumps coming from the process D, where we have $D_{t+} \neq D_t$ if and only if $C_{t+} \neq C_t$ and on the other hand jumps coming from the right-continuous process N, where $N_t \neq N_{t-}$ if and only if $C_t \neq C_{t-}$. So we have that

$$C_{t+} - C_t = -(D_{t+} - D_t),$$

if D jumps and

$$C_t - C_{t-} = f_t,$$

if N jumps. So the sum in the above term can be spitted up, which reads as follows

$$\begin{aligned}e^{-\rho t}\phi(\bar{C}_t) \leq & e^{-\rho t_1}\phi(\bar{C}_{t_1}) + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s \\ & - \int_{t_1}^{t} e^{-\rho s}\lambda^*(\phi(\bar{C}_{s-} + f_s) - \phi(\bar{C}_{s-}) - f_s)ds \\ & + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})f_s dN_s - \int_{(t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD_{s+} \\ & + \sum_{t_1\leq s\leq t,\ D_{s+}\neq D_s} e^{-\rho s}\left(\Delta\phi(\bar{C}_s) - \phi'(\bar{C}_{s-})\Delta\bar{C}_s\right) \\ & + \sum_{t_1\leq s\leq t,\ N_s\neq N_{s-}} e^{-\rho s}\left(\Delta\phi(\bar{C}_s) - \phi'(\bar{C}_{s-})\Delta\bar{C}_s\right).\end{aligned}$$

If we go further and write the terms in greater detail, we observe

$$
\begin{aligned}
e^{-\rho t}\phi(\bar{C}_t) \leq & e^{-\rho t_1}\phi(\bar{C}_{t_1}) + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s \\
& - \int_{t_1}^{t} e^{-\rho s}\lambda^*(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-}) - f_s)ds \\
& + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})f_s dN_s - \int_{(t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD_{s+} \\
& + \sum_{t_1\leq s\leq t,\ D_{s+}\neq D_s} e^{-\rho s}\left(\phi(\bar{C}_{s-} - (D_{s+} - \underbrace{D_{s-}}_{=D_s})) - \phi(\bar{C}_{s-})\right) \\
& - \sum_{t_1\leq s\leq t,\ D_{s+}\neq D_s} e^{-\rho s}\left(\phi'(\bar{C}_{s-})(-(D_{s+} - \underbrace{D_{s-}}_{=D_s}))\right) \\
& + \sum_{t_1\leq s\leq t,\ N_s\neq N_{s-}} e^{-\rho s}\left(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-}) - \phi'(\bar{C}_{s-})f_s\right).
\end{aligned}
$$

Note that, in accordance to this delta notation, we have the following

$$\Delta\bar{C}_s = \bar{C}_s - \bar{C}_{s-} = C_{s+} - C_{s-}.$$

Further, analogously as in [16, p. 61], we obtain that

$$0 \leq e^{-\rho t_1}\left[(\phi(C_{t_1}) - \phi(C_{t_1+})) - \phi'(C_{t_1+})(D_{t_1+} - D_{t_1})\right],$$

since ϕ is assumed to be concave and note that

$$(D_{t_1+} - D_{t_1}) = -(C_{t_1+} - C_{t_1}).$$

We add this non-negative expression to the right hand side of the inequality, rearrange some terms and get that the lower bound of the dD integral is now appended:

$$
\begin{aligned}
e^{-\rho t}\phi(\bar{C}_t) \leq & e^{-\rho t_1}\phi(C_{t_1}) + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s \\
& + \int_{t_1}^{t} e^{-\rho s}\lambda^* f_s ds - \int_{[t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD_{s+} \\
& - \int_{t_1}^{t} e^{-\rho s}\lambda^*(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-}))ds \\
& + \sum_{t_1 \leq s \leq t,\ N_s \neq N_{s-}} e^{-\rho s}\left(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-})\right) \\
& + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})f_s dN_s - \sum_{t_1 \leq s \leq t,\ N_s \neq N_{s-}} e^{-\rho s}\phi'(\bar{C}_{s-})f_s \\
& + \sum_{t_1 \leq s \leq t,\ D_{s+} \neq D_s} e^{-\rho s}\left(\phi(\bar{C}_{s-} - (D_{s+} - D_s)) - \phi(\bar{C}_{s-})\right) \\
& - \sum_{t_1 \leq s \leq t,\ D_{s+} \neq D_s} e^{-\rho s}\phi'(\bar{C}_{s-})\left(-(D_{s+} - D_s)\right).
\end{aligned}
$$

Now we use that $\phi' \geq 1$ in order to modify the above terms and use an approach analogously as in the paper by Azcue and Muller [3, p. 19 - 20], which reads as follows

$$
\begin{aligned}
& \sum_{t_1 \leq s \leq t,\ D_{s+} \neq D_s} e^{-\rho s}\left(\phi(\bar{C}_{s-} - (D_{s+} - D_s)) - \phi(\bar{C}_{s-})\right) \\
& = \sum_{t_1 \leq s \leq t,\ D_{s+} \neq D_s} e^{-\rho s}\left(\int_0^{D_{s+}-D_s} -\phi'(\bar{C}_{s-} - u)du\right) \\
& \leq - \sum_{t_1 \leq s \leq t,\ D_{s+} \neq D_s} e^{-\rho s}\left(\int_0^{D_{s+}-D_s} du\right) \\
& = - \sum_{t_1 \leq s \leq t,\ D_{s+} \neq D_s} e^{-\rho s}(D_{s+} - D_s).
\end{aligned}
$$

Moreover note that $D_t = \int_0^t dD_s^c + \sum_{s<t,\ D_{s+} \neq D_s}(D_{s+} - D_s)$ is a separation in a continuous part and a discontinuous part, which holds since D is non-decreasing and left-continuous. Therefore we get that

$$-\int_{[t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD_{s+} + \sum_{\substack{t_1\le s\le t\\ D_{s+}\neq D_s}} e^{-\rho s}\phi'(\bar{C}_{s-})\left(D_{s+}-D_s\right)$$
$$= -\int_{[t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD^c_{s+},$$

and hence

$$\begin{aligned}
e^{-\rho t}\phi(\bar{C}_t) \le & e^{-\rho t_1}\phi(C_{t_1}) + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s \\
& + \int_{t_1}^{t} e^{-\rho s}\lambda^* f_s ds - \int_{[t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD^c_{s+} \\
& - \int_{t_1}^{t} e^{-\rho s}\lambda^*(\phi(\bar{C}_{s-}+f_s)-\phi(\bar{C}_{s-}))ds \\
& + \sum_{t_1\le s\le t,\ N_s\neq N_{s-}} e^{-\rho s}\left(\phi(\bar{C}_{s-}+f_s)-\phi(\bar{C}_{s-})\right) \\
& + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})f_s dN_s - \sum_{t_1\le s\le t,\ N_s\neq N_{s-}} e^{-\rho s}\phi'(\bar{C}_{s-})f_s \\
& - \sum_{t_1\le s\le t,\ D_{s+}\neq D_s} e^{-\rho s}(D_{s+}-D_s).
\end{aligned}$$

Again with $\phi' \geq 1$ we obtain

$$\begin{aligned}
& -\int_{[t_1,t]} e^{-\rho s}\phi'(\bar{C}_{s-})dD^c_{s+} - \sum_{t_1\le s\le t,\ D_{s+}\neq D_s} e^{-\rho s}(D_{s+}-D_s) \\
& \le -\int_{[t_1,t]} e^{-\rho s}dD^c_{s+} - \sum_{t_1\le s\le t,\ D_{s+}\neq D_s} e^{-\rho s}(D_{s+}-D_s) \\
& = -\int_{[t_1,t]} e^{-\rho s}dD_{s+}.
\end{aligned}$$

So this estimation gives

$$
\begin{aligned}
e^{-\rho t}\phi(\bar{C}_t) \leq & e^{-\rho t_1}\phi(C_{t_1}) + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s \\
& + \int_{t_1}^{t} e^{-\rho s}\lambda^* f_s ds - \int_{[t_1,t]} e^{-\rho s} dD_{s+} \\
& - \int_{t_1}^{t} e^{-\rho s}\lambda^*(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-}))ds \\
& + \sum_{t_1 \leq s \leq t,\ N_s \neq N_{s-}} e^{-\rho s}\left(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-})\right) \\
& + \int_{t_1}^{t} e^{-\rho s}\phi'(\bar{C}_{s-}) f_s dN_s - \sum_{t_1 \leq s \leq t,\ N_s \neq N_{s-}} e^{-\rho s}\phi'(\bar{C}_{s-}) f_s.
\end{aligned}
$$

Analogously as in [16, p. 61], with $0 \leq t_1 \leq t < t_2$ we consider the limit $t \nearrow t_2$ and additionally we take conditional expectations on both sides. Moreover take into consideration that we further know from [11, Chapter 15] and the properties of the jump process N:

$$
\begin{aligned}
\int_{[t_1,t_2)} e^{-\rho s}\phi'(\bar{C}_{s-}) f_s dN_s &= \sum_{t_1 \leq s < t_2,\ N_s \neq N_{s-}} e^{-\rho s}\phi'(\bar{C}_{s-}) f_s, \\
E_c\left[\int_{[t_1,t_2)} e^{-\rho s}\lambda^* f_s ds \middle| \mathcal{F}_{t_1}\right] &= E_c\left[\int_{[t_1,t_2)} e^{-\rho s} f_s dN_s \middle| \mathcal{F}_{t_1}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& E_c\left[\int_{[t_1,t_2)} e^{-\rho s}\lambda^*(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-}))ds \middle| \mathcal{F}_{t_1}\right] \\
&= E_c\left[\int_{[t_1,t_2)} e^{-\rho s}(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-}))dN_s \middle| \mathcal{F}_{t_1}\right] \\
&= E_c\left[\sum_{t_1 \leq s < t_2,\ N_s \neq N_{s-}} e^{-\rho s}(\phi(\bar{C}_{s-}+f_s) - \phi(\bar{C}_{s-})) \middle| \mathcal{F}_{t_1}\right].
\end{aligned}
$$

So in the end we remain with the following inequality

$$\begin{aligned}&E_c\left[e^{-\rho t_2}\phi(C_{t_2})\middle|\mathcal{F}_{t_1}\right]\\ \le& E_c\Bigg[e^{-\rho t_1}\phi(C_{t_1})+\int_{t_1}^{t_2}e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s\\ &+\int_{[t_1,t_2)}e^{-\rho s}f_s dN_s-\int_{[t_1,t_2)}e^{-\rho s}dD_s\Bigg|\mathcal{F}_{t_1}\Bigg].\end{aligned}$$

Adding the missing integral $E_c\left[\int_{0+}^{t_2}e^{-\rho s}(dD_s-f_s dN_s)\middle|\mathcal{F}_{t_1}\right]$ on both sides yields

$$\begin{aligned}E_c&\left[e^{-\rho t_2}\phi(C_{t_2})+\int_{0+}^{t_2}e^{-\rho s}(dD_s-f_s dN_s)\middle|\mathcal{F}_{t_1}\right]\\ &\le e^{-\rho t_1}\phi(C_{t_1})+\int_{0+}^{t_1}e^{-\rho s}(dD_s-f_s dN_s)\\ &\quad+E_c\left[\int_{t_1}^{t_2}e^{-\rho s}\phi'(\bar{C}_{s-})\sigma dB_s\middle|\mathcal{F}_{t_1}\right].\end{aligned}$$

We know that $\phi'\ge 1$ and that ϕ is concave so $\phi''\le 0$, this together implies that $1\le\phi'(c)\le\phi'(0)$. This gives us that

$$\int_{t_1}^{t_2}\left[e^{-\rho s}\phi'(\bar{C}_{s-})\sigma\right]^2ds\le(\phi'(0))^2\int_{t_1}^{t_2}e^{-\rho 2s}\sigma^2 ds<\infty.$$

Hence the local martingale under the last expectation is a true martingale with expectation zero and we obtain that Y is a supermartingale.

Next we consider the stopped process

$$Z_t:=Y_{t\wedge\tau_0}.$$

With the important result from above we obtain

$$\begin{aligned}
\phi(C_{0-}) &= \phi(C_0) - \Delta\phi(C_0) = Z_0 - \Delta\phi(C_0) \overset{(*)}{\geq} E_c\left(Z_{\tau_0}\right) - \Delta\phi(C_0) \\
&= E_c\left(e^{-\rho\tau_0}\phi(\underbrace{C_{\tau_0}}_{=0}) + \int_{0+}^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right) - \Delta\phi(C_0) \\
&\overset{(**)}{=} E_c\left(e^{-\rho\tau_0}\underbrace{\phi(0)}_{=l(0)} + \int_0^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right) - \Delta D_0 - \Delta\phi(C_0) \\
&\overset{(\star)}{=} E_c\left(e^{-\rho\tau_0}l(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right) \\
&- \Delta D_0 - \phi(C_{0-} - \Delta D_0) + \phi(C_{0-}) \\
&\overset{(\star)}{=} E_c\left(e^{-\rho\tau_0}l(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right) \\
&+ \int_{C_{0-}-\Delta D_0}^{C_{0-}} \underbrace{(\phi'(c) - 1)}_{\geq 0}\, dc \\
&\geq E_c\left(e^{-\rho\tau_0}l(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right).
\end{aligned} \tag{3.1.8}$$

In the above treatment we used at $(*)$ the optional sampling theorem for the supermartingale Z and at $(**)$ we used the fact that the Poisson process N does not jump with probability one at the time point zero. In addition to this and in coherence with the at (2.3.41) mentioned circumstance that jumps in the control are jumps in the state process and the properties of the state process (3.0.1) we have especially at time zero $\Delta C_0 = -\Delta D_0$, which is used at $(\star)$.

Finally if we take the supremum over all admissible strategies in Θ we obtain from (3.1.7)

$$\phi(C_{0-}) \geq \sup_{(f,D)\in\Theta} E_c\left(e^{-\rho\tau_0}l(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s - f_s dN_s)\right), \tag{3.1.9}$$

which means, keeping in mind that $C_{0-} = c$ and $l(0) = l$,

$$\phi(c) \geq \hat{V}(c). \tag{3.1.10}$$

□

3.1.3 Third step: The conjectured optimal policy

Due to economic considerations of the underlying model assumptions it is conjectured that there exists a threshold denoted by C^* such that above of it it is optimal to pay dividends and below it the firm is acting optimally if it is searching for investors and retain earnings in order to reach the barrier C^*. On top of this, the firm will only be liquidated if the cash reserves reach the value zero. For further economical comments see [6, p. 13 - 16]. This barrier strategy was also explained at the beginning of this section. For further types of control strategies related to dividend problems, but considered in an actuarial mathematical point of view we refer to [2, p. 302 - 304]. So the conjectured barrier strategy defines a value of the firm following this strategy, which has to be determined. For this purpose the starting point is a modified version of the HJB equation (3.1.1). In fact the value of the firm acting due to this strategy with barrier b is denoted by $v(c; b)$. Moreover, if we are below the barrier, an investor occurs and the current cash holdings are equal to c, we have for the financing strategy that $f = b - c$, which reads in terms of processes as follows

$$f_t = f(C_{t-}) = b - C_{t-},$$

so we arrive with the cash reserves at the barrier b. Further we want to explain how the dividend control process will look like in the case of this barrier strategy. In fact, using the corresponding funding control yields that the cash reserve process is reflected at the barrier b. This means that the process D, as already announced, has to compensate the cash process in order to keep it below respectively at the barrier level. For that reason D grows if C touches the barrier and stays constant otherwise. Analogously to the models considered in [16], D corresponds to the local time of the state process at the barrier level b.

Theorem 10. *(Tanaka's formula) [13, p. 96]*
Let X be a continuous semimartingale. Then there exists a continuous increasing and adapted process $l = (l_t)_{t\geq 0}$ such that

$$|X_t| - |X_0| = \int_0^t sgn(X_s)dX_s + l_t,$$

where $sgn(x) = -1$*, if* $x < 0$ *and* $sgn(x) = 1$*, if* $x > 0$ *and* $sgn(0) = -1$*. The process* l *is called the (semimartingale) local time of* X *at zero. It grows only when* $X = 0$*:*

$$\int_0^t \mathbf{1}_{\{X_s \neq 0\}} dl_s = 0. \tag{3.1.11}$$

So we have that $v(c;b)$ satisfies

$$\begin{cases} 0 = \mathcal{L}v(c;b) + \lambda^*[v(b;b) - v(c;b) - (b-c)] & \text{if } 0 < c < b, \\ 0 = v(b;b) - v(c;b) - (b-c) & \text{if } c \geq b, \\ v(0;b) = l. \end{cases} \tag{3.1.12}$$

The second equality implies that for $c > b$

$$\frac{v(c;b) - v(b;b)}{c-b} = 1$$

and therefore we want to have on the other hand considering the limit of c to b from below

$$\lim_{c \nearrow b} v'(c;b) \stackrel{!}{=} 1. \tag{3.1.13}$$

Now we focus on solving the problem (3.1.12) and due to this we start solving the homogeneous equation. For that purpose we need some results about the solution of the homogeneous differential equation, which was treated by Shreve et al. [16]. Therefore the following is based on [16, p. 56-58].

For $0 \leq a < b < \infty$ let $u, v : [a,b] \to \mathbb{R}$ be two Lipschitz continuous functions, where $v(x) > 0$, for all $x \in [a,b]$. For $i = 1, 2$ consider stochastic processes $X^i = (X^i_t)_{t \geq 0}$, which satisfy if $a < X^i_t < b$ the stochastic differential equation

$$dX^i_t = u(X^i_t)dt + v(X^i_t)dW_t, \quad X^i_0 = x_0 \in [a,b]. \tag{3.1.14}$$

Further if X^1 reaches the boundary a respectively b, then it is absorbed and if X^2 reaches the boundary, then it is absorbed at a and reflected at b. Moreover the process $W = (W_t)_{t \geq 0}$ is as usual a standard Brownian motion. Concerning X^1, if one has a given Brownian motion, then existence and pathwise uniqueness of this process can be guaranteed. On the other hand in the case of X^2 we have that there exists a Brownian motion W and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ which satisfy

- W is adapted to $\mathbb{F}$,
- $(W_s)_{s\geq t}$ is independent of $\mathcal{F}_t$,
- $\exists$ a process X^2 adapted to $\mathbb{F}$, for which it holds that

$$X_t^2 = x_0 + \int_0^t u(X_s^2)ds + \int_0^t v(X_s^2)dW_s - \zeta_t, \tag{3.1.15}$$

until the process reaches the absorbing boundary at a. Note that concerning the conditions on the process ζ it is stated at [16, p. 57] that this process is adapted to $\mathbb{F}$, nondecreasing, continuous with $\zeta_0 = 0$. Moreover for a fixed $\omega \in \Omega$ the process ζ is constant on any time interval where $X^2 < b$ holds. So even uniqueness is given, but as in the case of existence, Shreve et al. [16] refer to further literature. Such kind of problems are generally known as the Skorokhod problem. As above consider the following stopping times, which are defined as the first time points when the process reaches the respective boundary:

$$\tau_y^i := \inf\{t \geq 0 : X_t^i = y\},$$

and further we set $\tau_y^i = \infty$, if the point y is never reached by X^i.

Now we consider the following differential equation,

$$\beta f(x) = u(x)f'(x) + \frac{1}{2}v^2(x)f''(x), \tag{3.1.16}$$

which we are going to solve with functions depending on stopping times respectively local times.

Theorem 11. *For a solution f of* (3.1.16) *and $x \in [a, b]$ it holds that*

$$\begin{aligned} f(x) &= E_x\left[e^{-\beta(\tau_a^1\wedge\tau_b^1)} f(X^1_{\tau_a^1\wedge\tau_b^1})\right] \\ &= E_x\left[e^{-\beta\tau_a^1} f(X^1_{\tau_a^1}) 1_{\{\tau_a^1\leq\tau_b^1\}}\right] + E_x\left[e^{-\beta\tau_b^1} f(X^1_{\tau_b^1}) 1_{\{\tau_b^1\leq\tau_a^1\}}\right] \\ &= f(a)E_x\left[e^{-\beta\tau_a^1} 1_{\{\tau_a^1\leq\tau_b^1\}}\right] + f(b)E_x\left[e^{-\beta\tau_b^1} 1_{\{\tau_b^1\leq\tau_a^1\}}\right] \end{aligned} \tag{3.1.17}$$

and

$$f(x) = f(a)E_x\left[e^{-\beta\tau_a^2}\right] + f'(b)E_x\left[\int_0^{\tau_a^2} e^{-\beta t} d\zeta_t\right]. \tag{3.1.18}$$

Proof. We start with the proof of (3.1.17). For that purpose we apply the partial integration formula respectively Itô's formula to $e^{-\beta t} f(X_t^1)$ which reads as follows

$$\begin{aligned}
d(e^{-\beta t} f(X_t^1)) =& -\beta e^{-\beta t} f(X_t^1) dt + e^{-\beta t} df(X_t^1) \\
=& -\beta e^{-\beta t} f(X_t^1) dt + e^{-\beta t} \left(f'(X_t^1) dX_t^1 + \frac{1}{2} f''(X_t^1) d[X^1]_t \right) \\
=& -\beta e^{-\beta t} f(X_t^1) dt \\
&+ e^{-\beta t} \left(f'(X_t^1) u(X_t^1) dt + f'(X_t^1) v(X_t^1) dW_t + \frac{1}{2} f''(X_t^1) v^2(X_t^1) dt \right) \\
=& e^{-\beta t} \left(-\beta f(X_t^1) + f'(X_t^1) u(X_t^1) + \frac{1}{2} f''(X_t^1) v^2(X_t^1) \right) dt \\
&+ e^{-\beta t} f'(X_t^1) v(X_t^1) dW_t \\
=& e^{-\beta t} f'(X_t^1) v(X_t^1) dW_t.
\end{aligned}$$

Now we integrate from 0 to $\tau_a^1 \wedge \tau_b^1$, which reads as follows

$$e^{-\beta(\tau_a^1 \wedge \tau_b^1)} f(X^1_{\tau_a^1 \wedge \tau_b^1}) - f(x) = \int_0^{\tau_a^1 \wedge \tau_b^1} e^{-\beta t} f'(X_t^1) v(X_t^1) dW_t$$

and take expectations on both sides

$$E_x \left[e^{-\beta(\tau_a^1 \wedge \tau_b^1)} f(X^1_{\tau_a^1 \wedge \tau_b^1}) - f(x) \right] = E_x \left[\int_0^{\tau_a^1 \wedge \tau_b^1} e^{-\beta t} f'(X_t^1) v(X_t^1) dW_t \right].$$

Note that for $a \leq X_t^1 \leq b$ the integrand of the stochastic integral with respect to the Brownian motion W on the right hand side is bounded. Hence the stochastic integral is a martingale with expectation zero, which yields that

$$E_x \left[e^{-\beta(\tau_a^1 \wedge \tau_b^1)} f(X^1_{\tau_a^1 \wedge \tau_b^1}) - f(x) \right] = 0$$

and finally

$$f(x) = E_x \left[e^{-\beta(\tau_a^1 \wedge \tau_b^1)} f(X^1_{\tau_a^1 \wedge \tau_b^1}) \right].$$

In order to prove (3.1.18) we analogously apply the partial integration formula respectively the more general version of Itô's formula to $e^{-\beta t} f(X_t^2)$ and obtain

$$d(e^{-\beta t} f(X_t^2)) = e^{-\beta t} f'(X_t^2) \left(v(X_t^2) dW_t - d\zeta_t \right).$$

Since, (ζ_t) is a monotone function it induces a measure on $[0, \infty)$, which assigns measure zero to the set $\{t \in [0, \infty) | X_t^2 \neq b\}$ we have that

$$f'(X_t^2) d\zeta_t = f'(b) d\zeta_t.$$

Now we integrate from 0 to τ_a^2 to obtain

$$e^{-\beta \tau_a^2} f(X_{\tau_a^2}^1) - f(x) = \int_0^{\tau_a^2} e^{-\beta t} f'(X_t^2) v(X_t^2) dW_t - \int_0^{\tau_a^2} e^{-\beta t} f'(b) d\zeta_t.$$

Moreover taking expectations and exploiting the martingale property analogously as above yields that

$$E_x \left[e^{-\beta \tau_a^2} f(X_{\tau_a^2}^1) - f(x) \right] = -E_x \left[\int_0^{\tau_a^2} e^{-\beta t} f'(b) d\zeta_t \right], \tag{3.1.19}$$

which is equivalent to

$$f(x) = f(a) E_x \left[e^{-\beta \tau_a^2} \right] + f'(b) E_x \left[\int_0^{\tau_a^2} e^{-\beta t} d\zeta_t \right].$$

Finally, in order to assure that the above calculations make sense we have to check that

$$E_x \left[\int_0^{\infty} e^{-\beta t} d\zeta_t \right] < \infty.$$

For that reason we do the similar integration as above from 0 to T for $T > 0$ and take expectations, so we get an analogous result as (3.1.19). But instead of the solution f, we consider a solution g of (3.1.16), with the special properties $g(a) = 0$ and $g'(b) = 1$ to obtain

$$E_x \left[e^{-\beta T} g(X_T^1) - g(x) \right] = -E_x \left[\int_0^T e^{-\beta t} d\zeta_t \right].$$

In the end we let $T \to \infty$, which yields the wanted property

$$E_x \left[\int_0^T e^{-\beta t} d\zeta_t \right] = g(x) < \infty.$$

□

A useful corollary of this theorem is the following.

Corollary 1. *For* $x \in [a,b]$ *the functions*

$$\varphi_1(x) = E_x\left[e^{-\beta\tau_a^1}\right], \tag{3.1.20}$$

$$\psi_1(x) = E_x\left[e^{-\beta\tau_b^1}\right], \tag{3.1.21}$$

$$\varphi_2(x) = E_x\left[e^{-\beta\tau_a^2}\right], \tag{3.1.22}$$

and

$$\psi_2(x) = E_x\left[\int_0^{\tau_a^2} e^{-\beta t} d\zeta_t\right] \tag{3.1.23}$$

are solutions to the differential equation (3.1.16) *with the boundary conditions*

$$\varphi_1(a) = 1, \varphi_1(b) = 0,$$

$$\psi_1(a) = 0, \psi_1(b) = 1,$$

$$\varphi_2(a) = 1, \varphi_2'(b) = 0,$$

$$\psi_2(a) = 0, \psi_2'(b) = 1.$$

Proof. For every function assume that f is a solution to (3.1.16) which fulfills the respective boundary conditions and apply the above theorem to verify that f has indeed the shape as proclaimed in (3.1.20) - (3.1.23). □

Considering the absorption problem (AP) in the paper by Shreve et al. [16], which is an analogue to our considered model problem, as mentioned above, if $\lambda = 0$, because there is no outside financing possible. Here we also work with a barrier strategy, where the state process X^U, which is of the form (3.1.15) with control ζ^U, is reflected downward at a barrier U. Now the associated reward to this strategy has the form

$$V_U(x) := \begin{cases} E_x\left[\int_0^{\tau_0^U} e^{-\beta t} d\zeta_t^U + e^{-\beta\tau_0^U} P\right], & \text{if } 0 \le x \le U, \\ x - U + V_U(U), & \text{if } x \ge U, \end{cases} \tag{3.1.24}$$

where P is a so called penalty for absorption. If $x > U$, ζ^U is meant to jump at $t = 0$ with jump height $U - x$ and further after this initial jump

X^U is a diffusion reflected at U and absorbed at 0. This is the meaning of the definition of V_U for $x > U$. Moreover we see that for $x \in [0, U]$, V_U is a solution to the differential equation (3.1.16) with boundary conditions

$$V_U(0) = P \text{ and } V_U'(U) = 1. \tag{3.1.25}$$

A quite important fact, which can be seen here is that the first derivative $V_U'(U)$ is equal to one, but the second derivative $V_U''(U)$ could be nonexistent. Nevertheless the right-hand second derivative is zero, but the left-hand second derivative could not be equal to that. In the absorption problem the optimal barrier U^* has to satisfy

$$V_{U^*}''^{-}(U^*) = 0,$$

where $V_{U^*}''^{-}$ denotes the left-hand second derivative. Moreover it is mentioned that U^* is chosen such that V_{U^*} becomes "[...] twice continuously differentiable is a manifestation of the 'heuristic principle of smooth fit' [...]", see [16, p. 69]. On top of this in the absorption problem to fulfill the principle of smooth fit is equivalent to maximize some expression of $V_U(x)$ with $x \in [0, U]$ over all barrier levels U, which is of course exactly what is wanted in that model. For further information we refer to [16].

Now we come back to our model and therefore we use these results to define the following functions.

Definition 13. *For the uncontrolled cash reserve process denoted by* $C^0 = (C_t^0)_{t \geq 0}$*, which satisfies*

$$dC_t^0 = (rC_{t-}^0 + \mu)dt + \sigma dB_t,$$

and for $y \in \mathbb{R}$ *we set* $\tau_y := \inf\{t \geq 0 : C_t^0 = y\}$ *as above, for* $c \in [0, b]$ *we define*

$$L(c; b) := E_c \left[e^{-(\rho+\lambda^*)\tau_0} 1_{\{\tau_0 \leq \tau_b\}} \right] \tag{3.1.26}$$

and

$$H(c; b) := E_c \left[e^{-(\rho+\lambda^*)\tau_b} 1_{\{\tau_b \leq \tau_0\}} \right]. \tag{3.1.27}$$

So we know that these functions solve the homogeneous equation

$$\mathcal{L}\phi(c) = \lambda^* \phi(c), \tag{3.1.28}$$

where $u(x) = rx + \mu$, $v(x) = \sigma$ and $\beta = \rho + \lambda^*$ have to be used to apply the related theorem. In this framework (3.1.28) is the homogeneous differential equation of (3.1.12) considering $c \in [0, b]$.

Next in order to obtain a solution for the first equality in (3.1.12), we have to use the above results such that we can verify the statements at [6, p. 39 - 40]. Now we are searching for a solution of the inhomogeneous equation (3.1.12). For that reason we consider the subsequent lemma, where in the following it is assumed that the values $\phi(b)$ and $\phi(0)$ are given.

Lemma 4. *For the function*

$$\Pi(c;b) := \frac{\lambda^*}{\rho + \lambda^*}\left(\phi(b) - b + c + \frac{\mu + rc}{\rho + \lambda^* - r}\right) \tag{3.1.29}$$

it holds that

$$\mathcal{L}\Pi(c;b) - \lambda^*\Pi(c;b) + \lambda^*(\phi(b) - b + c) = 0. \tag{3.1.30}$$

Hence the function $\Pi(c;b)$ is the particular solution we are searching for. Of course Π depends on the value $\phi(b)$, but in the most cases we omit the notation Π^ϕ, which would imply that Π depends on the respective value. The proof is done just by plugging in the found equation in order to verify that this is indeed a solution. Of course if the special form of the function was not known we would have to apply special techniques in order to solve the differential equation. Combining these results culminates in the next theorem.

Theorem 12. *For fixed $b \geq 0$ the unique solution of*

$$\begin{cases} 0 = \mathcal{L}\phi(c;b) + \lambda^*[\phi(b) - \phi(c;b) - (b - c)] & \text{if } c \leq b, \\ 0 = \phi(b) - \phi(c;b) - (b - c) & \text{if } c \geq b, \end{cases} \tag{3.1.31}$$

has the following form

$$\phi(c;b) = \phi(b)H(c;b) + \phi(0)L(c;b) - \Pi(b;b)H(c;b) - \Pi(0;b)L(c;b) + \Pi(c;b) \tag{3.1.32}$$

and $\phi \in C^1$.

Proof. By the preceding results we know that H and L solve the homogeneous equation and Π solves the inhomogeneous equation, which yields that

$$\mathcal{L}\left[(\phi(b) - \Pi(b;b))H(c;b)\right] - \lambda^* \left[(\phi(b) - \Pi(b;b))H(c;b)\right] = 0,$$

$$\mathcal{L}\left[(\phi(0) - \Pi(0;b))L(c;b)\right] - \lambda^* \left[(\phi(0) - \Pi(0;b))L(c;b)\right] = 0$$

and

$$\mathcal{L}\Pi(c;b) + \lambda^*[\phi(b) - \Pi(c;b) - (b-c)] = 0.$$

All these equations together prove the claim. □

Moreover the solution has a representation via so called hypergeometric functions. The following theory on these types of functions especially used in the proof of Lemma 5 is based on the book of Abramowitz and Stegun [1, p. 504].

Definition 14. *The confluent hypergeometric function $M(a,b;z)$ is of the form*

$$\begin{aligned} M(a,b;z) &:= \sum_{k=0}^{\infty} \frac{(a-1+k)!}{(a-1)!} \frac{(b-1)!}{(b-1+k)!} \frac{z^k}{k!} \\ &= 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} + \dots, \end{aligned} \tag{3.1.33}$$

on top of this, we use the following notation for these hypergeometric functions

$$F(x) := M\left(\frac{-(\rho+\lambda^*)}{2r}, \frac{1}{2}; \frac{-(rx+\mu)^2}{\sigma^2 r}\right) \tag{3.1.34}$$

and

$$G(x) := \frac{rx+\mu}{\sigma\sqrt{r}} M\left(\frac{-(\rho+\lambda^*)}{2r} + 1, \frac{3}{2}; \frac{-(rx+\mu)^2}{\sigma^2 r}\right). \tag{3.1.35}$$

With these functions we are able to find an alternative expression for our solutions L and H. As a remark in the underlying paper [6, p. 40] there probably is in the definition of G a typing error, because in the second argument of the function M there is written $\frac{23}{2}$ instead of $\frac{3}{2}$.

Lemma 5. *For a fixed $b \geq 0$ the solutions of the homogeneous differential equation* (3.1.28) *satisfy*

$$L(c;b) = E_c\left[e^{-(\rho+\lambda^*)\tau_0} 1_{\{\tau_0 \leq \tau_b\}}\right] = \frac{G(b)F(c) - F(b)G(c)}{G(b)F(0) - F(b)G(0)} \tag{3.1.36}$$

and

$$H(c;b) := E_c\left[e^{-(\rho+\lambda^*)\tau_b} 1_{\{\tau_b \leq \tau_0\}}\right] = \frac{F(0)G(c) - G(0)F(c)}{G(b)F(0) - F(b)G(0)}. \tag{3.1.37}$$

Proof. The general solution to the equation (3.1.28) $\mathcal{L}\phi(c) = \lambda^*\phi(c)$ is given by

$$\phi(c) = \gamma_1 F(c) + \gamma_2 G(c),$$

where γ_1 and γ_2 are constants. This holds because if we perform a change of variables

$$\phi(c;b) = g\left(\frac{-(rc+\mu)^2}{\sigma^2 r}\right) = g(z),$$

where the term $z = \frac{-(rc+\mu)^2}{\sigma^2 r}$ corresponds to the parameter in the hypergeometric functions F and G, which will be immediately clear by the following calculation, in fact (3.1.28) is equivalent to

$$\begin{aligned}
0 =& \mathcal{L}g(z) - \lambda^* g(z) \\
=& -(\rho+\lambda^*)g(z) + (rc+\mu)g'(z)z' + \frac{\sigma^2}{2}\left[g''(z)(z')^2 + g'(z)z''\right] \\
=& -(\rho+\lambda^*)g(z) + (rc+\mu)g'(z)\left(\frac{-2(rc+\mu)}{\sigma^2 r}r\right) \\
&+ \frac{\sigma^2}{2}\left[g''(z)\left(\frac{4(rc+\mu)^2}{\sigma^4 r^2}r^2\right) + g'(z)\frac{-2r}{\sigma^2 r}r\right] \\
=& -(\rho+\lambda^*)g(z) + g'(z)\left(\frac{-2(rc+\mu)^2}{\sigma^2}\right) \\
&+ g''(z)\left(\frac{2(rc+\mu)^2}{\sigma^2}\right) + g'(z)(-r) \\
=& -(\rho+\lambda^*)g(z) + g'(z)(2rz - r) + g''(z)(-2r)z.
\end{aligned}$$

Next we divide the whole equation with $(-2r) < 0$ to get

$$0 = \frac{(\rho+\lambda^*)}{2r}g(z) + g'(z)\left(\frac{1}{2} - z\right) + g''(z)z.$$

This equation is exactly Kummer's differential equation, which generally looks like

$$0 = -ag(z) + (b - z)g'(z) + zg''(z), \tag{3.1.38}$$

in our case we have $a = -\frac{-(\rho+\lambda^*)}{2r}$, $b = \frac{1}{2}$ and $z = \frac{-(rc+\mu)^2}{\sigma^2 r}$. Now by applying results from the theory about differential equations of this type, we know that $M(a, b; z)$ and $z^{1-b}M(a + 1 - b, 2 - b; z)$ solve Kummer's equation (3.1.38), which are in our case exactly F and G. Hence L and H can be written as linear combinations of the functions F and G, determined via their special boundary behaviour. □

As before our main goal is still to prove that the barrier strategy is the optimal one. So far we see that for fixed $b \geq 0$ the firm value $v(c; b)$ as solution of (3.1.12) has for $c \in (0, b)$ the following form

$$v(c; b) = v(b; b)H(c; b) + lL(c; b) - \Pi(b; b)H(c; b) - \Pi(0; b)L(c; b) + \Pi(c; b), \tag{3.1.39}$$

where Π satisfies

$$\Pi(c; b) = \frac{\lambda^*}{\rho + \lambda^*}\left(v(b; b) - b + c + \frac{\mu + rc}{\rho + \lambda^* - r}\right).$$

Additionally, it fulfills (3.1.13) $\lim_{c \nearrow b} v'(c; b) = 1$. Furthermore we have to find the optimal barrier level C^*. This value can be achieved by considering the first derivative of the firm value with respect to the barrier level

$$\frac{\partial v}{\partial b}(c; b)\bigg|_{b=C^*} = 0, \quad c > 0. \tag{3.1.40}$$

Moreover it is stated that this condition is equivalent to the so called high-contact condition

$$\lim_{c \nearrow C^*} v''(c; C^*) = 0. \tag{3.1.41}$$

In order to verify this statement one has to use the second equation in (3.1.12) namely $0 = v(b; b) - v(c; b) - (b - c)$ if $c \geq b$, (3.1.13), (3.1.39) and further the representation of L and H as in (3.1.36) and (3.1.37). For additional information see [6, p. 15]. Note that this high-contact condition is in general required to make sure that the firm value v is twice continuously differentiable. Because if we consider the function $v(c; b)$ with $c \geq b$, then the right-hand second derivative in b is zero but in general the left-hand one does not necessarily has to be zero. So in order to verify the optimality we have to make sure that $v \in C^2$ in order to use Theorem 9, where we needed this degree of smoothness since we applied Itô's formula. The analogous situation was already mentioned above for the case of the general absorption problem, in this context we refer to [16, p. 69]. In conclusion we need to find a barrier level C^* for our control strategy such that the high-contact condition (3.1.41) is satisfied.

Furthermore if we have that the first equation in (3.1.12) holds in addition to (3.1.13), then the condition (3.1.41) yields that

$$\lim_{c \nearrow C^*} v(c; C^*) = \frac{\mu}{\rho} + \frac{r}{\rho} C^*. \tag{3.1.42}$$

So most parts of the work in this step have been done respectively have been well prepared. Now we need that the derived firm value with respect to the barrier strategy is concave, which is a very important property needed when we want to prove that this firm value satisfies the HJB equation (3.1.1) in order to apply Theorem 9 in step 4. For that purpose we consider the function $w(c; b)$, where $b \geq 0$ remains in this context arbitrary but fixed. Besides w is assumed to fulfill the following properties

(a) $w(c; b)$ is the unique continuously differentiable solution to (3.1.31),

(b) it satisfies the boundary conditions

  (i) $w'(b; b) = 1$ and

  (ii) $w''(b; b) = 0$,

(c) further by the above conditions, in particular these are (b)(i), (b)(ii) and the first equation in (3.1.31), we have that (3.1.42) holds. This determines the value of w at the boundary and reads as follows

$$w(b; b) = \frac{\mu}{\rho} + \frac{r}{\rho} b.$$

Note that with these assumptions on w it is mentioned in [6, p. 41] that by the theory of linear ordinary differential equations existence and uniqueness is ensured. Further, it is important to notice that our conjectured optimal firm value function $v(c; b)$ satisfies these conditions, in particular compare to the first two equations of (3.1.12), (3.1.13) in addition to (3.1.41). The next theorem, as already mentioned, deals with the special behavior of such a function w, above all, it provides the concavity. But in order to verify the statement, we have to make use of some auxiliary lemmas. Nevertheless, the following results are quite interesting on their own.

Lemma 6. *Let $\phi(c;b)$ be an arbitrary function and let f be a solution to the differential equation*

$$\mathcal{L}f(c) + \phi(c;b) = 0. \tag{3.1.43}$$

Then, if $\phi(c;b) \geq 0$ the function f has no local minima, which are negative and on the other hand, if $\phi(c;b) \leq 0$ f has no local maxima, which are positive.

Proof. Assume that $c_{\min}$ is a local minimum and $\phi(c;b) \geq 0$, then we have that $f'(c_{\min}) = 0$ and $f''(c_{\min}) \geq 0$ so this in addition to (3.1.43) implies that

$$(-\rho)f(c_{\min}) + \frac{\sigma^2}{2}f''(c_{\min}) + \phi(c;b) = 0,$$

so we obtain

$$f(c_{\min}) \geq 0.$$

The proof of the second case where $\phi(c;b) \leq 0$ is done analogously. □

Lemma 7. *Let f be a solution to* (3.1.43) *with $\phi(c;b) \leq 0$ and further for a $c_0 \geq 0$ assume that $f(c_0) \geq 0$, $f'(c_0) \leq 0$ and that $|f(c_0)| + |f'(c_0)| + |\phi(c_0;b)| > 0$. Then for all $c < c_0$ it holds that $f(c) > 0$ and $f'(c) < 0$.*

Proof. The prove is done via contradiction. Consider the function f, where we assume that there exists at least one $c \in [0, c_0)$ such that $f'(c) \geq 0$. With respect to that, let $\bar{c} \in [0, c_0)$ be the largest value where f' changes sign and hence $f'(\bar{c}) = 0$, keep in mind that $f'(c_0) \leq 0$. These conditions yield that $f(\bar{c})$ is a local maximum. Moreover we know that $f(c_0) \geq 0$ and $\bar{c} < c_0$. This yields that $f(\bar{c})$ is a positive local maximum, which is indeed a contradiction to Lemma 6 and the assertion follows. □

Lemma 8. *Let f be a solution to* (3.1.43)*, where $\phi(c;b)$ satisfies $\phi'(c;b) \leq 0$, further for a $c_0 \geq 0$ assume that $f'(c_0) \geq 0$, $f''(c_0) \leq 0$ and that $|f'(c_0)| + |f''(c_0)| + |\phi'(c_0;b)| > 0$. Then for all $c < c_0$ it holds that $f'(c) > 0$ and $f''(c) < 0$. If additionally for f holds that $f''(c_0) = 0$, then $f''(c) > 0$ for $c > c_0$ and $f'(c_0) = \min_{c \geq 0} f'(c)$.*

Proof. First of all we differentiate (3.1.43) to obtain

$$-\rho f'(c) + rf'(c) + (rc+\mu)f''(c) + \frac{\sigma^2}{2}f'''(c) + \phi'(c) = 0.$$

So we can see that the function $g = f'$ solves

$$\mathcal{L}g(c) + rg(c) + \phi'(c) = 0.$$

So we are able to apply Lemma 7 to $g = f'$ in the usual way just with ϕ' instead of ϕ and $\bar{\rho} = -\rho + r$ instead of ρ in the differential operator $\mathcal{L}$, because we still have $\bar{\rho} = -\rho + r < 0$. This verifies the case $c < c_0$ and the other case $c > c_0$ can be done in an analogous way. □

Theorem 13. *With respect to $c \geq 0$ the above considered function $w(c; b)$ is increasing and concave, and with respect to b it is strictly monotone decreasing.*

Proof. Consider the function

$$f(c) = w(c;b) - \Pi^w(c;b) = w(c;b) - \frac{\lambda^*}{\rho+\lambda^*}\left(w(b;b) - b + c + \frac{\mu + rc}{\rho+\lambda^*-r}\right),$$

then we know that this function solves the homogeneous differential equation (3.1.28), because it is the difference of two particular solutions, respectively compare f to (3.1.32). Moreover it holds that $f'(b) > 0$, because we have that

$$\begin{aligned} & f'(b) = 1 - \frac{\lambda^*}{\rho+\lambda^*-r} > 0 \\ & \Leftrightarrow 1 > \frac{\lambda^*}{\rho+\lambda^*-r} \\ & \Leftrightarrow \rho + \lambda^* - r > \lambda^* \\ & \Leftrightarrow \rho > r. \end{aligned}$$

And $\rho > r$ is assumed to be true. We also used that for the denominator $\rho + \lambda^* - r > 0$ holds. Furthermore we have that $f''(b) = 0$, hence by Lemma 8 we have that $f(c)$ and on top of this also $w(c; b)$ is increasing and concave for $c \leq b$. Where we used in Lemma 8 that ρ in the differential operator is replaced by $\rho + \lambda^*$ and ϕ in (3.1.43) is considered to be zero, which does not harm the statement respectively the proof.

For the next part, namely to prove that $w(x;b)$ is monotone decreasing with respect to b, we consider $b_1 < b_2$ and let $g(c) = w(c;b_1) - w(c;b_2)$. Then we know by the properties of w in particular by item (c) that

$$w(b;b) = \frac{\mu}{\rho} + \frac{r}{\rho}b \tag{3.1.44}$$

and therefore we see that g solves the equation

$$\mathcal{L}g(c) - \lambda^* g(c) + \lambda^* \left(\frac{r}{\rho} - 1\right)(b_1 - b_2) = 0.$$

Since due to the linearity of the operator $\mathcal{L}$ we obtain

$$\begin{aligned}\mathcal{L}w(c;b_1) - \lambda^* w(c;b_1) - [\mathcal{L}w(c;b_2) - \lambda^* w(c;b_2)]& \\ = -\lambda^*(w(c;b_1) - b_1 - (w(c;b_2) - b_2))& \\ = -\lambda^* \left(\frac{r}{\rho} - 1\right)(b_1 - b_2).&\end{aligned}$$

On top of this, g satisfies the boundary conditions $g'(b_1) = 1 - w'(b_1;b_2) < 0$ and $g''(b_1) = -w''(b_1;b_2) \geq 0$, because $w(c;b)$ is for $c \leq b$ increasing and concave. Knowing these properties and if we modify Lemma 8, we get that w is monotone decreasing, but we still have to prove that $g(b_1) > 0$. So we consider the following

$$\begin{aligned}g(b_1) =& w(b_1;b_1) - w(b_1;b_2) = w(b_1;b_1) - w(b_2;b_2) + \int_{b_1}^{b_2} w'(c;b_2)dc \\ &\geq w(b_1;b_1) - w(b_2;b_2) + b_2 - b_1 = \left(\frac{r}{\rho} - 1\right)(b_1 - b_2) > 0.\end{aligned}$$

Note that we have $w'(b_2, b_2) = 1$ and further $w''(c, b_2) \leq 0$ for $c \leq b_2$ by the above results and hence $w'(c, b_2) \geq 1$ for $c \leq b_2$. □

The next considerations will lead to the determination of the optimal barrier level C^*. In fact we know that for $v(c;b)$ (3.1.12) holds, especially $v(0;b) = l(0) = l$. So regarding $w(c;b)$, where the barrier level b was arbitrary but fixed up to now, the only thing left to do is to determine a value b such that $w(0;b) = l(0)$. The solution of this equation will be denoted by C^* and as a consequence $v(c;C^*)$ and $w(c;C^*)$ will coincide. This will be needed in order to verify that v relative to that special barrier will be a smooth solution to the HJB equation. For that purpose we consider the next theorem.

Theorem 14. *The equation*

$$w(0; C^*) = l(0) \tag{3.1.45}$$

has a unique solution C^*.

Proof. We know by Theorem 13 that $w(0; b)$ is monotone decreasing with respect to b. Moreover using (3.1.44) we get that

$$w(0; 0) = \frac{\mu}{\rho} > l(0),$$

since by (2.2.5) we have $l(0) = l = \frac{\varphi\mu}{\rho}$ with $\varphi < 1$. On the other hand a straightforward calculation gives us that $w(0, \infty) < 0$. These properties imply that (3.1.45) has a unique solution. □

3.1.4 Fourth step: Optimality of the conjectured solution

In this step we verify that choosing the control processes with respect to the barrier strategy leads to a firm value which is indeed the optimal value $\hat{V}$. For that purpose we use the obtained value function for our barrier strategy with respect to the conjectured optimal barrier threshold C^* namely $v(c; C^*)$.

Theorem 15. *Let* C^* *be the unique solution of the equation* $w(0; C^*) = l(0)$, *then the function*

$$V(c) := w(c; C^*) = v(c; C^*) \tag{3.1.46}$$

is a solution to the HJB equation (3.1.1), *which satisfies* $V(c) \in C^2(0, \infty)$.

Proof. We know by Theorem 12 that $V(c) = w(c; C^*) \in C^2(0, \infty)$ solves the first two equations of (3.1.12), in addition to (3.1.13) and (3.1.41). Further since C^* solves (3.1.45), the third condition of (3.1.12) is also fulfilled by $w(c; C^*)$ and hence $w(c; C^*) = v(c; C^*)$.

Next we have to show that $V(c)$ solves (3.1.1). We start with the prove of

$$V'(c) \geq 1 \quad \forall c \geq 0.$$

Since $V(c) = w(c; C^*)$ is concave by Theorem 13 so $w''(c; C^*) \leq 0$ for all $c \geq 0$ and the smooth pasting condition, which is another denomination of the high-contact condition (3.1.41), holds we get that $w''(C^*, C^*) = 0$ in addition to $w'(C^*, C^*) = 1$. So we further obtain that

$$\begin{cases} V'(c) > 1, & \text{if } c < C^*, \\ V'(c) = 1, & \text{if } c \geq C^*. \end{cases} \tag{3.1.47}$$

The next part is to prove

$$V(c) \geq l(c) \quad \forall c \geq 0.$$

This follows from

$$l(c) - V(c) = \int_0^c (1 - V'(x))dx \leq 0,$$

note that $V(0) = l$. The remaining part is the proof of

$$\mathcal{L}V(c) + \mathfrak{F}V(c) \leq 0 \quad \forall c \geq 0.$$

First of all we consider the case $c \leq C^*$, where by the Theorem 12 it holds that:

$$\mathcal{L}V(c) + \mathfrak{F}V(c) = -\lambda^* \left[V(C^*) - C^* + c - V(c) \right] + \mathfrak{F}V(c)$$

and if we want to find the maximizing f for

$$\mathfrak{F}V(c) = \max_{f \geq 0} \lambda^* \left[V(c + f) - f - V(c) \right],$$

then

$$\frac{\partial}{\partial f} \lambda^* \left[V(c + f) - f - V(c) \right] \stackrel{!}{=} 0.$$

which yields that

$$V'(c + f) \stackrel{!}{=} 1$$

and therefore

$$f = C^* - c \geq 0,$$

since (3.1.47) holds and further V is concave, increasing and the smooth pasting condition holds. So we have that

$$\mathfrak{F}V(c) = \lambda^* \left[V(C^*) - C^* + c - V(c) \right]$$

and therefore

$$\mathcal{L}V(c) + \mathfrak{F}V(c) = 0 \quad \forall c \leq C^*.$$

On the other hand if $c > C^*$, due to Theorem 12 we have that

$$\mathfrak{F}V(c) = \max_{f \geq 0} \lambda^* \left[V(C^*) - C^* + c + f - f - V(C^*) + C^* - c\right] = 0.$$

Using (3.1.44) and we have for the other operator

$$\begin{aligned} \mathcal{L}V(c) &= \mathcal{L}(V(C^*) - C^* + c) \\ &= -\rho\left(\frac{\mu}{\rho} + \frac{r}{\rho}C^* - C^* + c\right) + (rc + \mu) \\ &= (\rho - r)(C^* - c) < 0, \end{aligned}$$

since by assumption $\rho > r$ and $C^* < c$. As a conclusion we obtain that $V(c)$ fulfills the HJB equation (3.1.1). □

In the end almost everything is done and the last and remaining part will be to prove the following final theorem, which states the optimality of the conjectured solution of the stochastic control problem.

Theorem 16. *For the value function in* (3.0.2) *denoted by* $\hat{V}(c)$ *and the function* $V(c)$ *defined in* (3.1.46) *it holds that*

$$\hat{V}(c) = V(c) \quad \forall c \geq 0. \tag{3.1.48}$$

Proof. The proof is done via showing that both inequalities in (3.1.48) hold. The first one namely $\hat{V}(c) \leq V(c) \quad \forall c \geq 0$ holds true. Since knowing Theorem 15 is true we are able to apply Theorem 9, which gives us the correctness of the first inequality.
For the other inequality we consider the following control strategy

$$D_t^* = L_t \text{ and } f_t^* = (C^* - C_{t-})^+.$$

Further $L_t = \sup_{s \leq t} (b_t - C^*)^+$, where

$$db_t = (rb_{t-} + \mu)dt + \sigma dB_t + (C^* - b_{t-})^+ \, dN_t$$

and for C it holds that

$$dC_t = (rC_{t-} + \mu)dt + \sigma dB_t - dD_t^* + f_t^* dN_t, \quad C_{0-} = c \geq 0.$$

First of all we have to verify that (D^*, f^*) is admissible, i.e. $(D^*, f^*) \in \Theta$. Note that in the following we will need that $C_t \geq 0\ \forall t \geq 0$. For the financing control we obtain from the definition of f_t^* that

$$E_c\left[\int_0^\infty e^{-\rho t} f_t^* dN_t\right] \leq E_c\left[\int_0^\infty e^{-\rho t} C^* dN_t\right] = \frac{\lambda^* C^*}{\rho}.$$

Next we use the partial integration formula for $\bar{C}_t := C_{t+}$ to get

$$\begin{aligned}
0 \leq & e^{-\rho t}\bar{C}_t = \bar{C}_0 + \int_0^t (-\rho) e^{-\rho s}\bar{C}_{s-} ds + \int_0^t e^{-\rho s} d\bar{C}_s \\
=& \bar{C}_0 + \int_0^t (-\rho) e^{-\rho s}\bar{C}_{s-} ds + \int_0^t e^{-\rho s}(r\bar{C}_{s-} + \mu) ds \\
&+ \int_0^t e^{-\rho s}\sigma dB_s - \int_{(0,t]} e^{-\rho s} dD_{s+}^* + \int_0^t e^{-\rho s} f_s^* dN_s \\
=& \bar{C}_0 + \int_0^t e^{-\rho s}((r-\rho)\bar{C}_{s-} + \mu) ds \\
&+ \int_0^t e^{-\rho s}\sigma dB_s - \int_{(0,t]} e^{-\rho s} dD_{s+}^* + \int_0^t e^{-\rho s} f_s^* dN_s \\
\leq& \bar{C}_0 + \int_0^t e^{-\rho s}\mu ds + \int_0^t e^{-\rho s}\sigma dB_s - \int_{(0,t]} e^{-\rho s} dD_{s+}^* + \int_0^t e^{-\rho s} f_s^* dN_s.
\end{aligned}$$

Further we have that $\bar{C}_0 = C_{0+} = C_0 - (D_{0+} - D_0)$ and hence

$$0 \leq C_0 + \int_0^t e^{-\rho s}\mu ds + \int_0^t e^{-\rho s}\sigma dB_s - \int_{[0,t]} e^{-\rho s} dD_{s+}^* + \int_0^t e^{-\rho s} f_s^* dN_s.$$

Now taking expectations on both sides and rearranging the terms yields that

$$\begin{aligned}
E_c\left[\int_0^t e^{-\rho s} dD_s^*\right] \leq& C_0 + E_c\left[\int_0^t e^{-\rho s}\mu ds\right] + E_c\left[\int_0^t e^{-\rho s} f_s^* dN_s\right] \\
\leq& C_0 + E_c\left[\int_0^\infty e^{-\rho s}\mu ds\right] + E_c\left[\int_0^\infty e^{-\rho s} f_s^* dN_s\right] \\
\leq& C_0 + \frac{\mu}{\rho} + \frac{\lambda^* C^*}{\rho},
\end{aligned}$$

holds for $t < \infty$ and with Fatou's lemma we obtain that $(D^*, f^*) \in \Theta$. Similarly as in the proof of Theorem 9 we consider the process:

$$Y_t := e^{-\rho(t\wedge\tau_0)} V(C_{t\wedge\tau_0}) + \int_{0+}^{t\wedge\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s), \tag{3.1.49}$$

where τ_0 is the same as in the proof of Theorem 9, namely the first time C_t reaches the value zero. Next we apply Itô's formula for semimartingales to Y_t, since $V(c)$ is a solution to the HJB equation and due to the special choice of (D^*, f^*), we obtain that Y is a local martingale. In contrast to the former proof we can use the fact that $C_t \in [0, C^*]\ \ \forall t \geq 0$ in addition to the increase of V, which implies that for a stopping time θ

$$|Y_\theta| \leq |V(C^*)| + \int_0^\infty e^{-\rho t}(dD_s^* + f_{t-}^* dN_t).$$

Moreover we know from above that the right hand side in particular the integral is integrable, because (D^*, f^*) is admissible, so as a conclusion the process $Y = (Y_t)_{t \geq 0}$ is a uniformly integrable martingale. Here we are able to identify quite clearly the difference between this proof and the preceding one. In this case we have the martingale property, where in the former one the considered process was just a supermartingale. With this important property we can go further as follows

$$\begin{aligned} V(c) = Y_{0-} = Y_0 - \Delta Y_0 &\overset{(*)}{=} Y_0 + \Delta D_0^* \overset{(**)}{=} E_c\left(Y_{\tau_0}\right) + \Delta D_0^* \\ &= E_c\left[e^{-\rho\tau_0} V(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s}(dD_s^* - f_s^* dN_s)\right] + \Delta D_0^* \\ &= E_c\left[e^{-\rho\tau_0} l(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s^* - f_s^* dN_s)\right]. \end{aligned}$$

Note that we used at $(*)$ the special definition of V and at $(**)$ that Y is a martingale. Finally this verifies the other equality $V(c) \leq \hat{V}(c)$ and both inequalities together yield the desired result. □

3.2 Resulting consequences and outlook

In the end taking all these steps into account leads to the following conclusion, see [6, p. 16].

Proposition 1. *For a firm with mean cash flow rate* μ *and without growth option the firm value reads as follows* $V(c) = v(c; C^*)$*. The function* $v(c; b)$ *is defined by* (3.1.12) *respectively* (3.1.39) *with the properties* (3.1.13), (3.1.41) *and* (3.1.42)*, further* C^* *is the unique solution to* (3.1.41)*. This leads to the representation*

$$\begin{aligned} v(c; C^*) = \left(\frac{\mu}{\rho} + \frac{r}{\rho} C^*\right) H(c; C^*) + lL(c; C^*) - \Pi(C^*; C^*) H(c; C^*) \\ - \Pi(0; C^*) L(c; C^*) + \Pi(c; C^*). \end{aligned} \tag{3.2.1}$$

A further quite interesting fact, which we will not prove is the next lemma about the dependencies of the optimal barrier on the underlying parameters of the model.

Lemma 9. *[6, p. 16, 43-44]*
The optimal barrier level C^* *of a firm without growth option is increasing in the cash flow volatility* σ^2 *and monotone decreasing in the capital supply* λ^* *and the asset tangibility* φ*.*

As an outlook one also is able to solve the control problem if the growth option is possible. In that case one needs to solve our considered problem with respect to different mean cash flow rates, in particular with respect to the rate μ_0 before and μ_1 after the investment. Moreover it is stated that the option to invest is only profitable if the lump sum cost, which has to be paid in case of investment, is below some identified threshold and in this case either a barrier strategy or a band strategy is optimal depending on the investment cost, see [6, p. 16 - 23]. In fact if the investment cost is below the threshold of profitability and rather low, then the barrier strategy is optimal. On the other hand if the cost is larger, but still below the threshold, then optimal strategy is of band type. If it is above the threshold, then the optimal control strategy for a firm with growth option is the same as for a firm without such a possibility, because then the firm never uses its option and remains with the mean cash flow rate μ_0 and we are in the case of Proposition 1, see [6, p. 16 -17]. The optimal switching time, where we change the mean cash flow rate from μ_0 to μ_1 is $\tau_{0,1} = \inf\{t > 0 | V_1(C_t - K) > V_0(C_t)\}$, where V_i denotes the value function with respect to μ_i, $i = 0, 1$. These comments were just a small outlook on the treated topics, for further information we refer to [6].

In the end the main discourse is completed with the result of Proposition 1. We have prepared the regarding theory, considered and adapted the treatment in [6] with the assistance of the below listed literature. Especially the paper by Shreve et al. [16] and its results were of special use and therefore crucial to the success for this discourse. Hereinafter we present a iterative procedure, which can be used to solve the above problem numerically.

Appendix

Numerical complement

The stated aim in this chapter is to find an approximation respectively a numerical solution to the considered stochastic optimal control problem. For that reason we consider the so-called policy iteration algorithm, see [5, p. 363 - 388]. In the subsequent lines we explain how this iterative method is applied to our model problem. The starting point is as in many numerical solution approaches the discretization of the state space in which the cash reserves process attains its values. For that purpose we primarily operate on the interval $[0, x_{end}]$, for a given endpoint and discretize this interval such that we remain with the the grid points

$$0 = x_0 < x_1 < \cdots < x_n = x_{end},$$

where $x_i = h \cdot i$, $i \in \{0, 1, \ldots, n\}$ for a prespecified number of grid points n, an artificial upper bound x_{end} and a given step size $h = \frac{x_{end}}{n}$. On top of this we discretize the above introduced operators (3.0.3) and (3.0.4). For the first one, the differential operator, we use forward difference quotients to approximate the first derivative and the standard finite difference approximation for the second derivative. This reads as follows

$$g'(x) \approx \frac{g(x+h) - g(x)}{h}$$

and

$$g''(x) \approx \frac{g(x+h) - 2g(x) + g(x-h)}{h^2}.$$

Based on that the differential operator

$$\mathcal{L}g(x) = -\rho g(x) + g'(x)(rx + \mu) + \frac{1}{2}\sigma^2 g''(x)$$

transforms to

$$-\rho g(x) + \frac{g(x+h)-g(x)}{h}(rx+\mu) + \frac{1}{2}\sigma^2 \frac{g(x+h)-2g(x)+g(x-h)}{h^2}.$$

If we rearrange the terms an take $x = x_i$ for $i \in \{1, \ldots, n-1\}$, we will get for the equation $\mathcal{L}g = 0$ the following discrete representation

$$\begin{aligned} 0 = &\left[(rx_i+\mu)\frac{1}{h} + \frac{\sigma^2}{2}\frac{1}{h^2}\right] g(x_{i+1}) \\ &+ \left[-(rx_i+\mu)\frac{1}{h} - \frac{\sigma^2}{h^2} - \rho\right] g(x_i) + \frac{\sigma^2}{2}\frac{1}{h^2} g(x_{i-1}), \end{aligned}$$

for $i \in \{1, \ldots, n-1\}$. On top of this we define

$$a_i = \left[-(rx_i+\mu)\frac{1}{h} - \frac{\sigma^2}{h^2} - \rho\right],$$

$$b_i = \left[(rx_i+\mu)\frac{1}{h} + \frac{\sigma^2}{2}\frac{1}{h^2}\right]$$

and

$$c = \frac{\sigma^2}{2}\frac{1}{h^2}$$

for $i \in \{1, \ldots, n-1\}$. The values on the endpoints of the considered interval are determined by $g(x_0) = g(0) = l$ and on the other one by $g'(x_{end}) = 1$, which yields $g(x_n) = g(x_{n-1}) + h$ for the first iteration. The last condition is due to the considerations made on singular stochastic optimal control theory. Since defining the first derivative at the endpoint to be one corresponds to the strategy that there are no dividend payments until the endpoint of the interval. As usual we gather this system of equations in the common matrix notation and obtain

$$\begin{pmatrix} a_1 & b_1 & 0 & \ldots & 0 \\ c & a_2 & b_2 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & c & a_{n-2} & b_{n-2} \\ 0 & 0 & 0 & c & a_{n-1}+b_{n-1} \end{pmatrix} \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_{n-2}) \\ g(x_{n-1}) \end{pmatrix} = \begin{pmatrix} -cl \\ 0 \\ \vdots \\ 0 \\ -b_{n-1}h \end{pmatrix}.$$

Further we denote the above matrix with L, the vector on the left-hand side with $\bar{g}$ and the one on the right-hand side with $\bar{b}$ in order to obtain $L\bar{g} = \bar{b}$.

The discretization of the second operator (3.0.4)

$$\mathfrak{F}g(x) = \max_{v \geq 0} \left[\lambda^* \left(g(x+v) - g(x) - v \right) \right],$$

will be done as follows. First of all, if $x_0 = 0$ we assumed that outside financing is not possible, so we search for the funding of height $j \cdot h$ which maximizes

$$g(x_i + j \cdot h) - g(x_i) - j \cdot h$$

for every $i \in \{1, \dots, n-1\}$ representing the actual height of the cash reserves respectively the actual state and every $j \in \{1, \dots, n-i\}$ representing the height of the financing in state i. This can be understood as follows, if in state i the optimal financing amount equals $j \cdot h = x_j$, then the cash holdings jump to the state $i+j$, since $x_{i+j} = x_i + x_j$. The highest permitted financing in state i is thus $(n-i) \cdot h$, because with this funding amount one reaches the last state x_{end}. There would be no reason to go beyond the last state, because in that area we are paying dividends so the outside funding above this level would lead to an dividend payout of this part of the financing amount. Again we collect the terms in the more compact matrix notation:

$$Fg := \begin{pmatrix} Fg(x_1) \\ \vdots \\ Fg(x_i) \\ \vdots \\ Fg(x_{n-1}) \end{pmatrix} = \begin{pmatrix} \max_{0 \leq j \leq (n-1)} \left\{ \lambda^* (g(x_1 + j \cdot h) - g(x_1) - j \cdot h) \right\} \\ \vdots \\ \max_{0 \leq j \leq (n-i)} \left\{ \lambda^* (g(x_i + j \cdot h) - g(x_i) - j \cdot h) \right\} \\ \vdots \\ \max_{0 \leq j \leq 1} \left\{ \lambda^* (g(x_{n-1} + j \cdot h) - g(x_{n-1}) - j \cdot h) \right\} \end{pmatrix}.$$

For the first iteration step we take as control strategy doing nothing until we reach the endpoint of the interval and from that point on the dividend payments are started. Then within this first iteration we compute the approximative value function via solving $\mathcal{L}g = 0$, which has as discrete representation $Lg = \bar{b}$. Next we investigate the object g and detect the area denoted by A, where $g' > 1$ holds. For that purpose we use further on as approximation for the first derivative the forward difference quotient. Regarding the second solution iteration we shave to solve the equation $L^{(1)}g^{(1)} + F^{(1)}g = 0$ on the area A. This leads to a second approximation for the value function denoted by $g^{(1)}$. As already mentioned in the former sections we know from stochastic singular control theory that if the first derivate is strictly larger than one in a point x, then the optimal control has to be zero in a neighborhood of x, see [5, p. 317]. Due to that we are looking for areas where we either do nothing or use the control to manipulate the state process. Of course we need some termination argument, for that reason we state that we will stop the iterative procedure, if the value function does not change anymore up to a small variation.

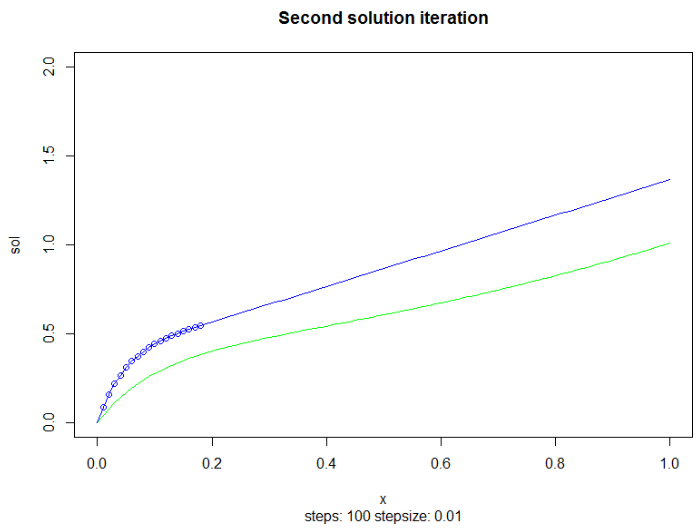

Figure 1: Second solution iteration

In the above picture we have that the green line corresponds to the first solution iteration and the blue line with the circles to the second one. The area where the equation was solved is marked by the circles on the curve and the remaining straight blue line corresponds to the linear extension of the obtained solution.

In iteration step k we have to solve the following equation

$$L^{(k)}g^{(k)} + F^{(k)}g^{(k-1)} = 0.$$

Where $g^{(k)}$ denotes the new solution of this step and $g^{(k-1)}$ denotes the solution of the preceding step. Furthermore the matrices $L^{(k)}$ denote the modified versions of the matrix L in order to solve the differential equation on the new area determined by making use of the solution $g^{(k-1)}$. Concerning the second operator we have that $F^{(k)}$ denotes the latest operator analogous to F searching for the maximal financing control on the new area. Starting the next iteration step we again investigate the last obtained approximation $g^{(k)}$ in order to determine the region where the first derivative of the obtained function is greater than one and in addition to this we consider the second derivative and search for the first point when it reaches the value zero. Using these informations we rethink the upper bound for the interval on which the equation is solved and modify it appropriately. In addition to this we select this upper bound in such a way that the sequence of approximations of the value function is non-decreasing. This is done since, due to the features of the policy iteration algorithm, exactly the control has to be chosen, which enlarges the value function, in order to have an improvement in every iterative step.

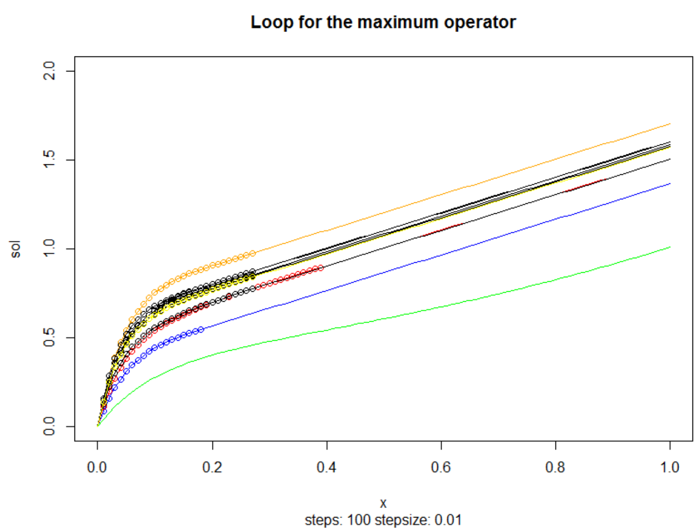

Figure 2: Auxiliary iteration

Note that as already mentioned above the F operator can only be applied to the solution from the preceding step. But in fact we want to determine the optimal funding control for the current solution approximation. Based on this consideration we do the following. We introduce an auxiliary iteration after every main iteration step, which will ensure that we work with the best approximation for the corresponding financing control of the current approximative value function. This means that we have to solve the current equation on the same region until the obtained approximation for the value function does not change anymore up to some small error. This approach is illustrated especially in the fourth iteration of our numeric example. Indeed by looking at the Figure 2 we observe that the first obtained approximative solution on the new area, which corresponds to the curve in orange, is much larger than the curve in yellow, which represents the solution obtained after twenty times solving on the same area. Moreover the solutions of the auxiliary iteration are represented by the plotted black curves. Based on this we decide to reject the initially obtained orange solution for the benefit of the yellow one in order to use the better related financing control for the current step.

Finally following this iteration procedure we arrive with the below graphically illustrated solutions:

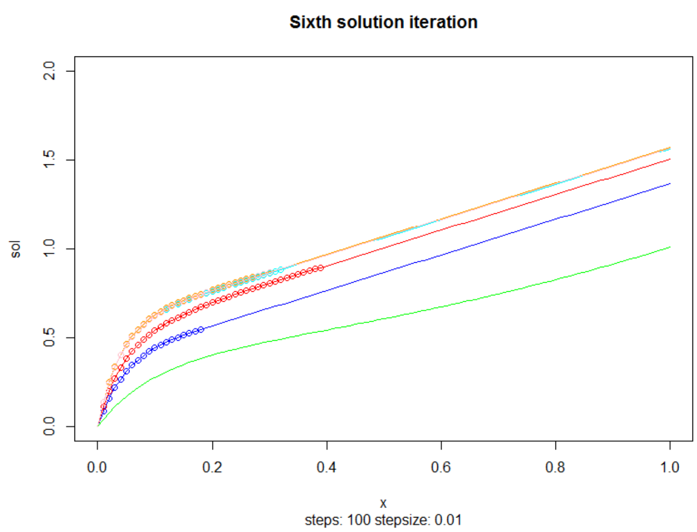

Figure 3: Convergence of the iteration

At this point we can observe that the six solutions of this iterative procedure build a non-decreasing sequence of functions, where the pink represents the last solution. The procedure stopped since, as the euclidean norm of the difference of the last two iterative solutions was below a prespecified constant, the termination criterion was fulfilled. Note that there was applied a slightly modified version of the usual policy iteration algorithm. In fact in the common algorithm every admissible control strategy has to be considered and further, in order to identify the optimal one, every associated gain function has to be evaluated. In this numerical example we tried to focus only on those control strategies which improve the respective gain function. For that purpose and in order to obtain a non-decreasing sequence of approximative value functions in accordance with the algorithm, we used the additional information about the behavior of the derivatives.

Finally we should have some benchmark function in order to review our numerical result. To this end we use the theoretical solution, in particular we have to solve a similar equation as above using the optimal dividend and financing policy. The correspond system of equations is written down subsequently.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ c & a_1-\lambda & b_1 & 0 & \dots & 0 & \lambda \\ 0 & c & a_2-\lambda & b_2 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & \lambda \\ \vdots & \vdots & & \ddots & c & a_{n-1}-\lambda & b_{n-1}+\lambda \\ 0 & 0 & & \dots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} V(x_0) \\ V(x_1) \\ V(x_2) \\ \vdots \\ V(x_{n-2}) \\ V(x_{n-1}) \\ V(x_n) \end{pmatrix} = \begin{pmatrix} l \\ -\lambda(x_1-x_n) \\ \\ \vdots \\ \\ -\lambda(x_{n-1}-x_n) \\ h \end{pmatrix}.$$

We solve this equation in matrix notation for a sequence of different barriers $b = x_{end}$. In a next step we select exactly the barrier which leads to the maximum representation of the value function. The computation of the theoretical value function is done in such a way, because the confluent hypergeometric functions, which are stated to be the solution of this problem, turned out to be numerically unstable with respect to the given set of parameters. Due to this fact the most practicable way was to choose this alternative to compute the value function. In Figure 4 below the two solutions are displayed graphically. In particular the value function is plotted in gold, whereas the numerical solution is plotted in black with its linear extension.

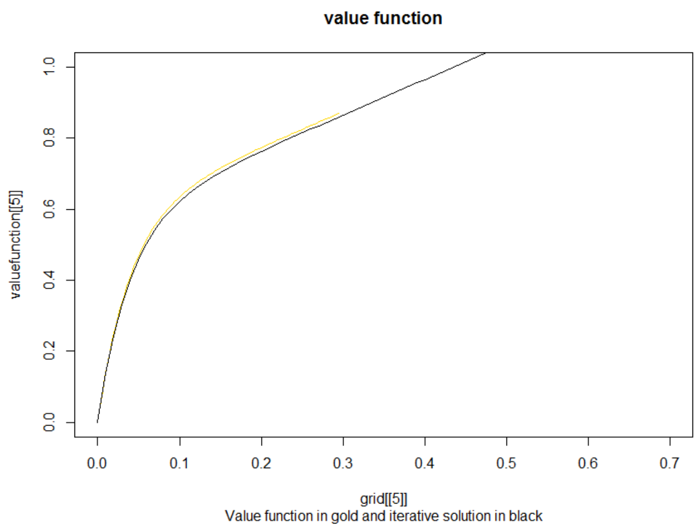

Figure 4: Comparison of the value function and the numerical solution

For the numeric example we have used the parameters as given at [6, Table 1]:

Discount rate:	$\rho = 0.06$,	Interest rate on cash:	$r = 0.0055$,
Mean cash flow rate:	$\mu = 0.05$,	Volatility of cash flow:	$\sigma = 0.1$,
Asset tangibility:	$\varphi = 0.00$,	Arrival rate of investors:	$\lambda = 2$.

Moreover we have used the following parameters for the discretization in the first step:

Number of steps	$n = 100$
End of grid interval	$x_{end} = 1$.

The above described iteration has been performed six times with these parameters and in the end we arrived with the already presented solution.

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