Analytical Solution for Level Pool Routing Equation Based on Various Inflow Hydrographs

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Abstract

Reservoir routing is one of the most important issues in reservoir management. Estimating outflow from a dam outlet, which is the target of reservoir routing, is significantly vital since unpredicted floods can bring about severe damages to infrastructures and human beings downstream of storage reservoirs. In essence, general analytical computation of storage and outflow may be impossible in practice and subsequently, application of numerical methods seems to be inevitable for reservoir routing. However, analytical relations for temporal variation of water storage in a typical reservoir for several cases of inflow hydrographs including triangular, abrupt wave, flood pulse and broad peak hydrographs are derived in this paper. . In these scenarios, an orifice was assumed for reservoir outlet. According to the literature, presented analytical solutions for the last three cases are proposed for the first time. The applicability of these analytical solutions is tested using four numerical examples while a robust numerical scheme, i.e., fourth-order Runge-Kutta, is also applied for solving these examples. The obtained results demonstrate that the recommended analytical solutions have a good agreement with the corresponding results of the utilized numerical scheme for all examples.

Keywords: Reservoir Routing, Analytical Solution, Inflow Hydrograph, Fourth-Order Runge-Kutta Scheme.

1. INTRODUCTION

Effective reservoirs which are decisive factors in water supply system in many communities usually have periodic high water levels. In order to not only maximize water usage but also prevent severe damages, hydraulic engineers are supposed to predict reservoir responses under various circumstances especially for high water levels [1]. From this point of view, reservoir routing is the process of determining the propagation corresponds to inflow hydrographs through reservoir [2]. Level pool routing is a robust tool for computation of outflow hydrograph from a single reservoir based on inflow hydrograph and the characteristics of the system [3]. In this regard, clarification of the hydraulic head equation with respect to time and detailed descriptions of hydrological characteristic related to the system play a key role in designing efficient reservoirs [4].

Various routing approaches can be classified into two main categories including (1) hydrologic routing and (2) hydraulic routing methods. The former employs continuity equation exclusively while the latter utilizes the continuity equation together with equation of motion [5-9]. This paper focuses on the hydrologic routing method which uses the nonlinear first-order ordinary differential equation.

The governing equation for level pool routing which is named as continuity equation is shown in Eq. 1:

$$\frac{dS}{dt} = I - Q \tag{1}$$

where S , I and Q are storage, inflow and outflow at time t , respectively.

Although there are numerous numerical solutions for the above mentioned equation [10-11], this equation can be analytically solved with some assumptions such as the hypothesis of power function for the storage-outflow relationship [12]. Furthermore, Paik [13] studied routing for a triangular inflow hydrograph in 2008. However, no recommendation is available for treating other types of inflow hydrographs in literature. In this study, it is tried to determine the analytical solution of outflow hydrograph for four common functions of inflow including triangular, abrupt wave, flood pulse and broad peak hydrographs.

The remainder of this paper is structured as follows: The methodology and governing equations of reservoir routing for various types of inflow hydrograph are developed in section 2. Afterwards, a comparison between obtained analytical solutions and numerical solutions of Runge-Kutta method is conducted for four numerical examples in section 3. Conclusion is given in the final section.

2. METHODOLOGY

The storage of a typical reservoir with constant area A may be computed as illustrated in Eq. 2.

$$S = Ah \tag{2}$$

where h is the hydraulic head of water.

The outflow from an orifice outlet can be computed using Eq. 3:

$$Q = \lambda Ca\sqrt{2gh} \tag{3}$$

where λ is the orifice formula error correction factor, C is the orifice coefficient, a is the total cross-sectional area of orifice outlet, and g is the gravitational acceleration.

By substituting Eq. 2 and Eq. 3 into the continuity equation (Eq. 1), a general form of routing equation of reservoirs with orifice outlets will be obtained, as shown in Eq. 4.

$$A\frac{dh}{dt} = I - \lambda Ca\sqrt{2gh} \tag{4}$$

The inflow hydrograph may have various types [14]. The solution of outflow hydrograph for 4 types of inflow hydrograph is proposed which are triangular, abrupt wave, flood pulse and broad peak. In the following, the analytical equations for hydraulic head and outflow of these inflow hydrograph are derived:

2.1. TRIANGULAR INFLOW HYDROGRAPH

The triangular inflow hydrograph (Fig. 1) function is expressed as follows:

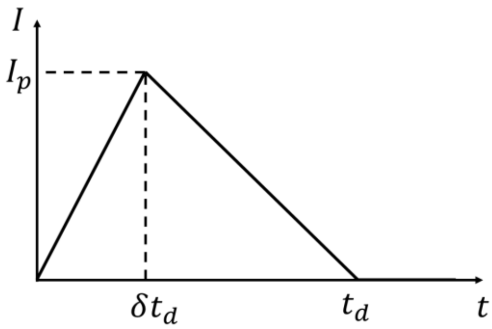

Figure 1. Triangular inflow hydrograph

$$I = \begin{cases} \dfrac{I_p}{\delta t_d}t & 0 \le t < \delta t_d \\ \dfrac{I_p}{1-\delta}\left(1-\dfrac{t}{t_d}\right) & \delta t_d \le t < t_d \\ 0 & t \le t_d \end{cases} \tag{5}$$

Thus, the outflow hydrograph can be divided into 3 parts: $0 \le t < \delta t_d$, $\delta t_d \le t < t_d$ and $t_d \le t$. By substituting inflow for the first part into Eq. 4, the governing equation obtained as:

$$A\frac{dh}{dt} = \frac{I_p}{\delta t_d}t - \lambda Ca\sqrt{2gh} \tag{6}$$

Eq. 6 is in form of Abel differential equation [15]. By applying the initial condition of no initial storage ($h = 0$ at $t = 0$), the analytical solution of water head can be achieved as:

$$h = \frac{g}{8A^2}\left[\sqrt{(\lambda Ca)^2 + \frac{4AI_p}{\delta t_d\, g}} - \lambda Ca\right]^2 t^2 \tag{7}$$

Implementing Eq. 7 into Eq. 3 yields the analytical solution of first part of outflow hydrograph as:

$$Q = \frac{(\lambda Ca)^2 g}{2A}\left[\sqrt{1 + \frac{4AI_p}{\delta t_d\, g\,(\lambda Ca)^2}} - 1\right] \times t \tag{8}$$

Eq. 8 reveals that the discharge from the orifice outlet increases linearly with respect to time t in the ascending limb of inflow hydrograph.

For the second part of outflow hydrograph, the governing equation is

$$A\frac{dh}{dt} = \frac{I_p}{1-\delta}\left(1 - \frac{t}{t_d}\right) - \lambda\,\mathrm{Ca}\sqrt{2gh} \tag{9}$$

The initial condition of this part is the head water at time $t = \delta t_d$ which is:

$$h_{(t=\delta t_d)} = \frac{g\,\delta^2 t_d^2}{8A^2}\left(\sqrt{(\lambda Ca)^2 + \frac{4AI_p}{\delta t_d\, g}} - \lambda Ca\right)^2 \tag{10}$$

The analytical solution of this equation is not available but routing results from numerical methods show that the outflow hydrograph is nonlinear in this part.

Finally, the analytical solution of outflow hydrograph of third part is obtained by putting $I = 0$ in Equ. 4 as

$$h^{1/2} = -\sqrt{\frac{g}{2}}\,\frac{\lambda Ca}{A}t + J \tag{11}$$

where J is a constant that can be found by the initial condition of the third part, i.e., the head water at time $t = t_d$. Substituting Eq. 11 into Eq. 3 will result in a linear hydrograph similar to the first part.

2.2. ABRUPT WAVE INFLOW HYDROGRAPH

The mathematical expression of this type of hydrograph, which is shown in Fig. 2, is as follows

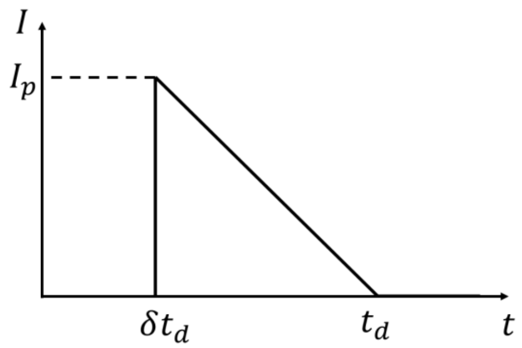

Figure 2. Abrupt wave inflow hydrograph

$$I = \begin{cases} 0 & 0 \le t < \delta t_d \\ \dfrac{I_p}{1-\delta}\left(1 - \dfrac{t}{t_d}\right) & \delta t_d \le t < t_d \\ 0 & t \le t_d \end{cases} \tag{12}$$

First and third parts of this hydrograph are similar to the third part of triangular inflow hydrograph. Thus,

$$h^{1/2} = -\sqrt{\frac{g}{2}}\,\frac{\lambda Ca}{A}t + J \tag{13}$$

where J can be determined by initial condition of each part which are $h_{(t=0)} = 0$ and $h_{(t=t_d)} = h_d$, respectively.

The head water at time t_d can be found by solving the equation for second part of hydrograph. Similar to the second part of triangular hydrograph, there is no analytical solution for this part of hydrograph and it can be determined by numerical methods.

2.3. FLOOD PULSE INFLOW HYDROGRAPH

Third inflow hydrograph discussed in this paper is flood pulse hydrograph (Fig. 3). Similar to the previous types, it can be divided into 3 parts.

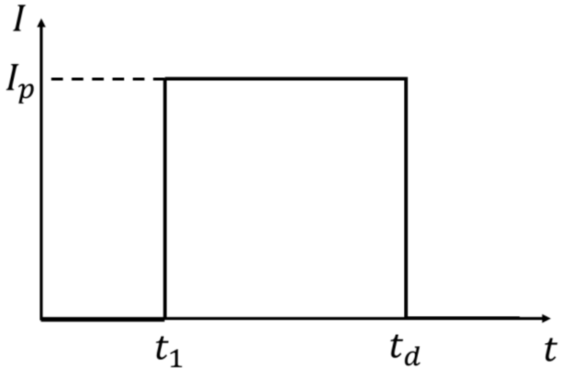

Figure 3. Flood pulse inflow hydrograph

$$I = \begin{cases} 0 & 0 \le t < t_1 \\ I_p & t_1 \le t < t_d \\ 0 & t_d \le t \end{cases} \tag{14}$$

Substituting Eq. 14 into Eq. 4 will yield to analytical solution of each part of outflow. The first and third parts of this hydrograph are same as last part of triangular hydrograph with different initial conditions. Thus the analytical solution of head water is similar to Equ. 13. The initial conditions of these parts are $h_{(t=t_1)} = h_1$ and $h_{(t=t_d)} = h_d$. Eq. 15 and Eq. 16 are simplified equations of head water in the first and third part of hydrograph, respectively.

$$h^{1/2} = \sqrt{\frac{g}{2}}\frac{\lambda Ca}{A}(t_1 - t) + \mathrm{h}_1^{1/2} \tag{15}$$

$$h^{1/2} = \sqrt{\frac{g}{2}}\frac{\lambda Ca}{A}(t_d - t) + \mathrm{h}_d^{1/2} \tag{16}$$

Substituting the above equations into Eq. 3 will result in the linear relations for outflow which increase with time. The middle part of hydrograph with constant inflow results in Abel differential equation of the second kind [12]. Solving this equation results in the following equation

$$\frac{-\lambda Ca\sqrt{2g}}{2A} \times t = h^{1/2} + \frac{I_p}{\lambda Ca\sqrt{2g}} \ln\left(h^{1/2} - \frac{I_p}{\lambda Ca\sqrt{2g}}\right) + J \tag{17}$$

where J is a constant that can be determined by the one of the initial conditions mentioned in this section. However, because the natural logarithm argument is always negative, there is no analytical solution for this part of outflow hydrograph and should be determined by numerical methods.

2.4. BROAD PEAK INFLOW HYDROGRAPH

The last type of hydrographs discussed in this study is broad peak hydrograph which is shown in Fig. 4. Unlike three other types, this hydrograph should be divided into 4 parts (Equ. 18).

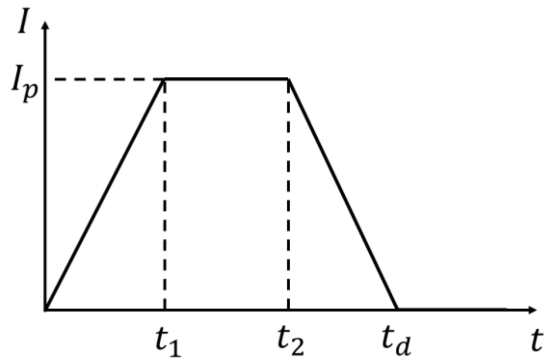

Figure 4. Broad peak inflow hydrograph

$$I = \begin{cases} \frac{I_p}{t_1} t & 0 \le t < t_1 \\ I_p & t_1 \le t < t_2 \\ \frac{I_p}{\left(1-\frac{t_2}{t_d}\right)}\left(1-\frac{t}{t_d}\right) & t_2 \le t < t_d \\ 0 & t_d \le t \end{cases} \tag{18}$$

By substituting inflow relation of each part into Eq. 4, the relation of outflow hydrograph can be obtained. The first part is same as the first part of triangular hydrograph with initial condition of $h_{(t=0)} = 0$ (Equ. 19).

$$h = \frac{g}{8A^2}\left[\sqrt{(\lambda Ca)^2 + \frac{4AI_p}{t_1 g}} - \lambda Ca\right]^2 t^2 \tag{19}$$

The second part is similar to the middle part of flood pulse hydrograph. Thus, there is no analytical solution for that.

As mentioned earlier, the descending limb of triangular hydrograph has not analytical solution. The descending part of Fig. 4 is same as the one in Fig. 1. Thus, the analytical solution of this part is not available and its outflow hydrograph is highly nonlinear.

Finally, the last part of inflow hydrograph which lies down on zero line, is like the last part of triangular hydrograph. The analytical solution of this part can be obtained by implementing the initial condition $h_{(t=t_d)} = 0$ to Eq. 13 as

$$h^{1/2} = \sqrt{\frac{g}{2}}\frac{\lambda Ca}{A}\left(t_d - t\right) \tag{20}$$

3. BENCHMARKS

In this section, the applicability of proposed formulations is demonstrated using four examples, one for each of inflow hydrographs. The analytical solutions are compared with results of a robust numerical method named as fourth order Runge-Kutta method which has shown good results in the literature [17]. Parameters of the orifice outlet and reservoir are listed in Table 1.

Table 1- Parameters of the model

Parameter	Value
λ	1
C	0.6
a (m^2)	2
A (m^2)	12000
Δt (s)	200

First, the triangular hydrograph with peak of $50^{m^3/s}$ and total time of 7200^s and $\delta = 0.5$ is assumed as input of the model (Gray dots in Fig. 5). As described in section 2, the ascending and zero limbs of inflow hydrograph result in linear outflow. Since there is no analytical solution for the second part of hydrograph, the results of the numerical solution in the result chart are used.

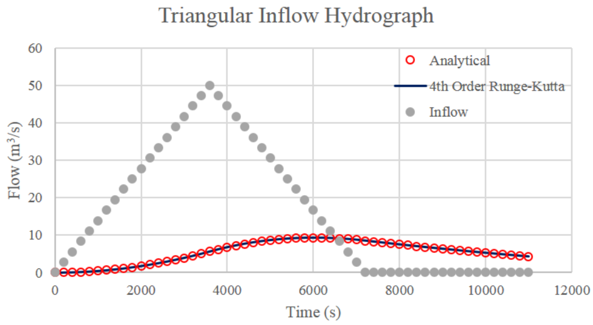

Figure 5. The outflow of triangular inflow hydrograph

Thereafter, hydrograph has zero value until 3600^s. In the second part, it reduced linearly from peak of $50^{m^3/s}$ to 0 and again it has zero value. According to the section 2, the first and third parts have analytical solution (Equ. 11) and there is no analytical solution for the second part.

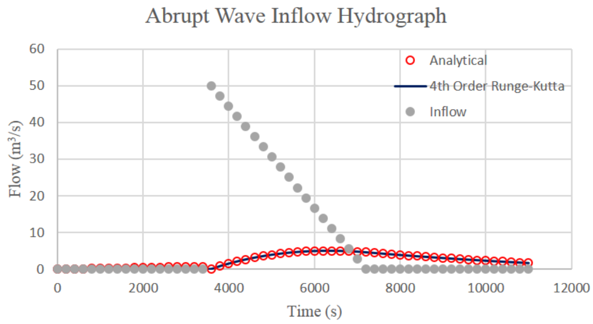

Figure 6. The outflow of abrupt wave inflow hydrograph

Flood pulse hydrograph composed of three constant parts. However, because the second part has non zero value, no analytical solution can be established and the numerical result is used for this part.

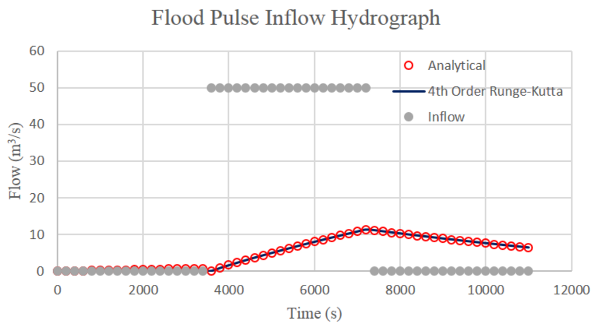

Figure 7. The outflow of flood pulse inflow hydrograph

The last inflow hydrograph shape is the broad peak hydrograph. It is a general trapezoid with an extended peak of $100^{m^3/s}$ between a linear rise and descending parts. As an example, the hydrograph of two reservoirs in the historical 1997 flood of California can be approximated into this type of hydrograph [18].

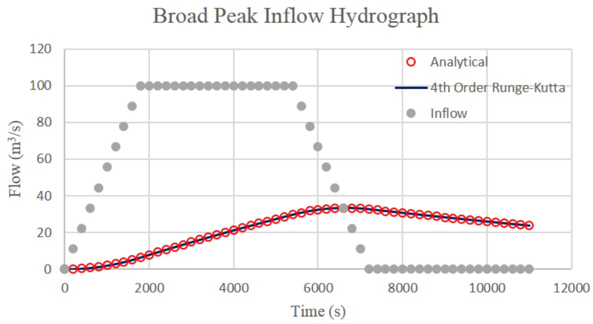

Figure 8. The outflow of broad peak inflow hydrograph

As illustrated in the above figures, the analytical solutions have very good agreement with numerical results. This good agreement indicates that the presented analytical solutions can be confidentially utilized for reservoir routing in cases in which the inflow hydrograph has similar shapes.

4. CONCLUSIONS

In this paper, analytical solutions for reservoir routing for several types of inflow hydrograph have been developed. According to the results, those parts of outflow hydrograph which has analytical solution are linear while those with no analytical solution are highly nonlinear. Compared with the results of Runge-Kutta method for four numerical examples, the analytical solutions have good agreement with the numerical method used for validation. It is obvious that general form of outflow hydrograph cannot be derived in practice. However, it may be possible to obtain analytical solution for special cases of inflow hydrograph and outlet conditions. Although, analytical solutions for four common types of hydrographs have been derived in this paper, seeking for analytical solution of other especial inflow hydrograph cases is suggested for future researches.

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