Determination of Optimum Cross Section of Earth Dams by Using Artificial Intelligence

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Abstract

Earth Dams are one of the most important and expensive civil engineering structures. Cost of construction of these structures directly depends on the needed volume of embankments which in turn depends on their cross section area. Reduction in the cross section area of earth dam causes decreasing in the embankment volume and leads to a significant reduction in the construction cost of these structures. Obtaining optimum cross section for earth dams including traditional design method utilizations in addition to stability and construction aspects, is too time-consuming and almost impossible. In this paper, an Artificial Neural Network and ant colony optimization algorithm have been used to solve this complicated problem which is known as one of the most important problems in geotechnical engineering.

Keywords: Earth dams, Slope stability, Artificial intelligence, Artificial neural network, Ant colony optimization algorithm.

1. INTRODUCTION

The purpose of this paper is finding the cross section of earth dam which attains satisfying slope stability conditions and accounting for construction aspects and furthermore reaches a minimum volume of earth works. This purpose is gained by creating some berms with various widths and heights in the body of an earth dam, as shown in Figure. 1.

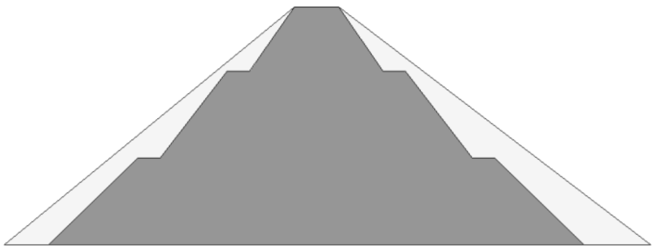

Figure.1. Employing berms in order to reduce fill volume in earth dams

In traditional methods of designing earth dams, obtaining the optimum cross section with suitable configuration of berms is rigorous and time consuming. Optimization methods are required for solving this problem and finding the optimum section for earth dams. Since there are many variables of the problem, employing the classic optimization methods is difficult. Therefore, it is advantageous to use heuristic methods for solving such problems. Although finding the global optimum solution in heuristic methods is not granted, but with optimum solution. These methods are being used for solving the combinational optimization problems, especially in engineering optimization problems. Despite their success in various fields of engineering, however, these methods have not been applied significantly in geotechnical problems. In this paper, the ant colony optimization algorithm tool was used for finding the optimum section for an earth dam. The ant colony algorithm (ACO) was originally inspired by the behavior of real ants. Dorigo [1] first developed ant ACO and successfully applied it to the traveling salesman problem (TSP). Several variations of the original ACA including ant system (AS) [1], elitist ant system (ASelite) [1], max-min ant system (MMAS) [2-4] and ranked ant system (ASrank) [5, 6] have been introduced recently. ACO has enjoyed important success in various fields of engineering. Despite its success, however, the method has found little success in geotechnical engineering applications. As a frontier in application of modern metaheuristic optimization algorithms to slope stability analysis, Cheng et al. [7] have evaluated the performance of six heuristic global optimization methods in the location of critical slip surface in slope stability analysis, including ACO.

The problem under consideration is one conditional optimization problem and its object is minimization; namely, finding a section with minimum cross section area. The program includes the following steps:

First of all, a connected graph is created by problem variables, as shown in Figure. 2. This graph provides a searching space for artificial ants for finding the shorter path, and it is combined by some decision points which are variables of the problem. Each decision point is constituted by decision components created from discrete variables. At second step, artificial ants are directed to this graph. after preparing the connected graph. They find solutions by randomized walk across the connected graph. They choose some paths on this graph which each of them presents a cross section of an earth dam. Once all the ants survey their path and touch to the destination node, they will go back to the initial point after obtaining a solution. At third step, pheromone trail is deposited on the nodes of every path. Pouring pheromone trail on every path depends on quality of the path (cross section area of earth dam section). Paths with higher quality (or with lower cross section area) are smeared with more pheromone trial. Therefore, these paths have more chance for selecting in next iterations. Also, paths that cannot satisfy conditions of the problem are called impossible solutions. These paths are then forgotten by applying heavy penalties to them through attributing high safety factors.

At second iteration, once again artificial ants are directed to the graph and they build solutions by performing randomized walk on the graph. Thus, new paths are chosen in this graph, and paths with higher quality at first iteration have more chance of being chosen in the second iteration. This process was continued until more ants tended to more optimum path (shorter path). Finally, the shorter path is found. In this problem the shorter path is the optimum cross section for certain earth dam.

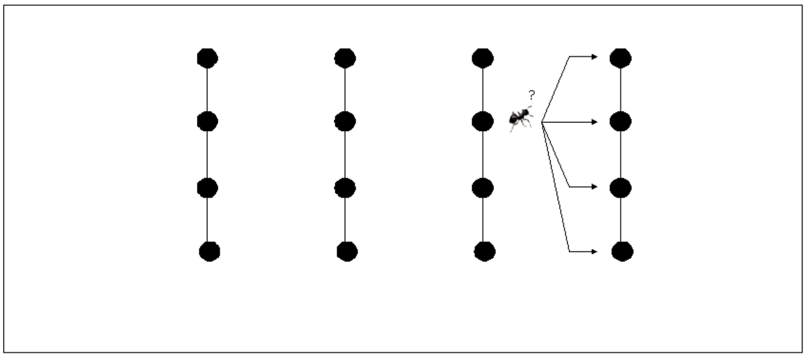

Figure.2. Graphical representation of the optimization problem in ACO

2. Characteristic of Determining Optimum Cross Section Model

The optimization problem of finding optimum cross section is represented by Eq.1:

$$C = \min A(x) \tag{1}$$

Where C is the objective function of the problem, $A(x)$ is the cross section area for a certain section of the earth dam, x parameter is a set of variables of this problem which are defined subsequently.

In determining optimum cross section problem, the objective function is the cross section area for certain section which should be minimized. It means that the objective is finding one cross section with minimum area and thereupon minimum volume of earth works for certain earth dam. The variables of determining optimum cross section problem consist of n, n', $b1_i$, $b2_i$, $h1_i$, $h2_i$, $I1_i$ and $I2_i$, where are defined in Figure.3. As shown in Figure.3, n and n' are the number of berms at upstream and downstream of earth dam. $b1_i$ and $b2_i$ are the width of berm *i-th* at upstream and downstream of the earth dam. $h1_i$ and $h2_i$ are the height of berm *i-th* at upstream and downstream of the earth dam from foundation level. $I1_i$ and $I2_i$ are the slope of zone *i-th* at upstream and downstream of the earth dam. Also at Figure.3, B, H and H_f are constant parameters of cross section of the earth dam which are respectively, the width of crest of earth dam, the height of the dam and the foundation depth.

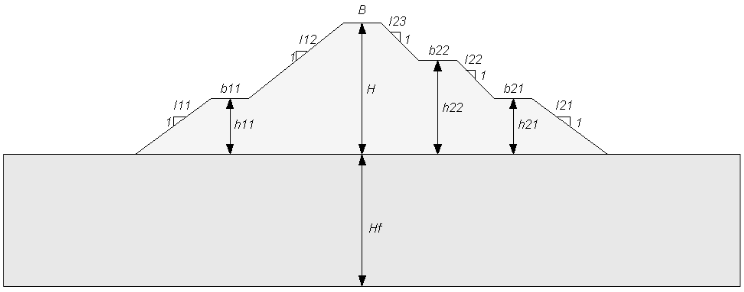

Figure.3. optimization of cross section of an earth dam

The mentioned variables are represented in a graph, as shown in Figure.4, which involves finding the optimum values of cross section with *n* berm at upstream and *n'* berm at downstream variables by passing the artificial ants on this graph and deposit pheromone trail on the nodes of every path.

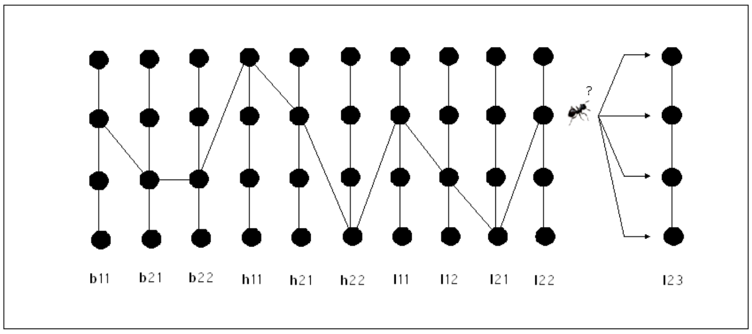

Figure.4. Graph of the optimization problem involving minimization of the cross section of an earth dam.

There are two group of independent constraints in this problem. First group includes constraints which define the boundary of ants search space. These constraints are provided the search space which includes the possible solutions (possible paths). It should be avoided of zones which include the impossible solutions to determine the search space. In the problem, $b1_i$ and $b2_i$ were chosen among 4 meter up to 10 meter randomly. Minimum value of $b1_i$ and $b2_i$ was determined due to practical conditions and the maximum value was determined due to maximum value of normal berms width. Also the values of $h1_i$ and $h2_i$ are chosen among 10 meter and *H* minus10 meter randomly. $I1_i$ and $I2_i$ are randomly selected through a reasonable range which is between response angle of soil material and 5:1 slope.

Second group of constraints includes conditional constraints. They separate possible solution from impossible solutions that are not satisfied. A method for applying impossibility of these solutions and inattention at these decisions is fining these paths (solutions) in order to offend of conditional constraints. Thus, instead of pheromone trial depositing as many as the value of their objective function on these paths, pheromone trail depositing is avoided by applying a penalty. This problem subjected the conditional constraints which are presented at Eq.2 to Eq.4.

$$h1_{i+1} - h1_i \geq a \tag{2}$$

$$h2_{i+1} - h2_i \geq a \tag{3}$$

$$Fs \geq Fs_{allow} \tag{4}$$

Eq.2 and Eq.3 suggest that the height of each berm must be more than height of previous berm. Also, height difference of each berm with the previous berm must be more than, or equal to *a*. Moreover, minimum height of first berm (lower berm) must be greater than or equal the foundation level *a*. Unless, this section is forgotten with applying penalty, because it can not satisfy geometrical conditions. Eq.4 also indicates that each section of earth dam must satisfy slope stability conditions for all load cases of earth dam. These load cases, include the end of construction step, steady state seepage of mid level step, steady state seepage of maximum level step and rapid drawdown of normal level step. For all the load cases, minimum factor of safety of certain cross section must be more than allowable factor of safety for this load case. If the certain section factor of safety, even at one of load cases, was below the allowable factor of safety, it would be forgotten by applying penalty. The penalty for sections which are offensive of the conditional constraints is applying value of cross section area equal a large number as 10^{10} m^2. This large value led to less than zero pouring pheromone deposit trail on unsuitable paths. So, impossible paths (solutions) were forgotten, and they could not be effective in attracting artificial ants' selection at future iterations. Figure.5 show flowchart for the ant colony algorithm.

3. Review of Ant Colony Optimization Algorithms

The basic idea of ant colony optimization (ACO), inspired by the behavior of real ants, is to use artificial ants to search for *good solutions* of a combinatorial optimization problem. ACO is therefore a metaheuristic in the sense that the absolute optimum solution is not found, but *good solutions* practically close enough to the optimum are found. Real ants coordinate their activities through stigmergy, which is a form of indirect communication [8]. Specifically, in the search for food, ants deposit chemicals along the path they travel which is recognized by other ants, and will increase the probability of the path being travelled by other ants of the colony. The chemical is called *Pheromone* [8].

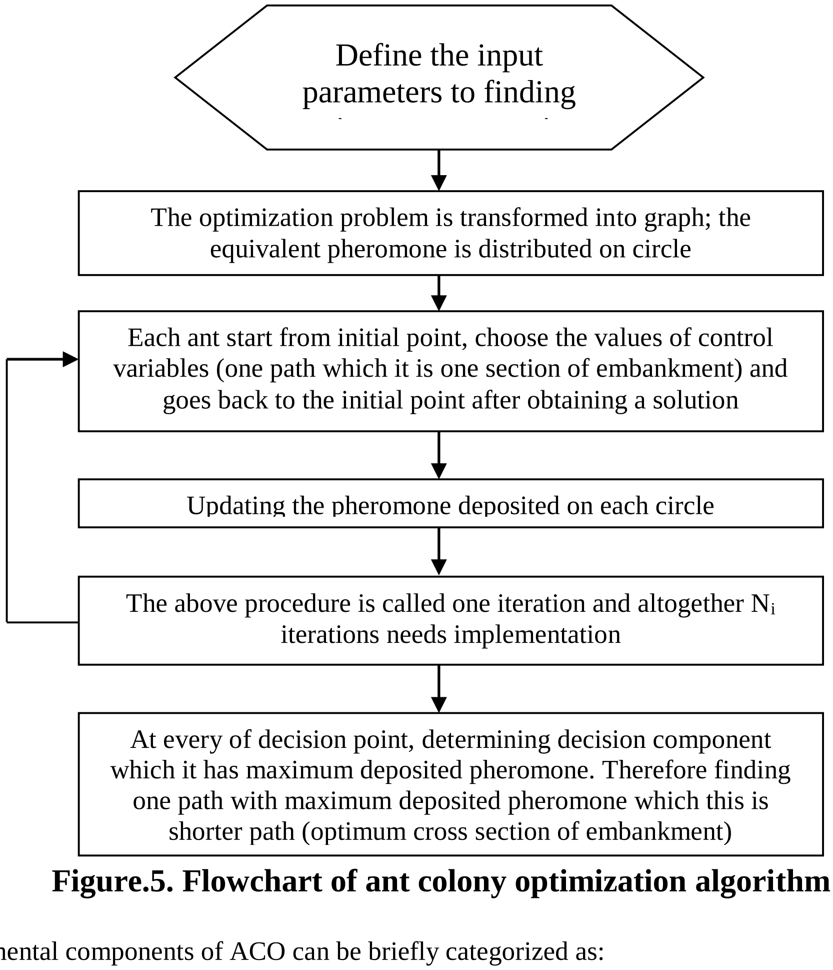

Figure.5. Flowchart of ant colony optimization algorithm

The fundamental components of ACO can be briefly categorized as:

I- Construct a graph of the problem: In the case of finding the optimum cross section, this step involves dividing the two dimensions in the *x* and *y* direction into m and n discrete subdivisions, respectively. The resulting graph will have m *columns*, and each column has n *nodes*.

II- Define the objective function and the restraints: When finding the optimum cross section, the objective function is the cross section area, i.e. the function to be optimized. Also, some restraints are placed on the variables in this stage. The number of ants and the number of attempts for solving the problem are also specified in this step.

III- Move artificial ants on the graph in order to construct admissible solutions to the problem: In this step, artificial ants placed on the initial point start moving on the grid from left to right, randomly selecting a node on each consecutive column in order to build incremental solutions to the problem under consideration. In finding the optimum cross section, ants move from the start point of the graph towards the end point, following the rule described in section 2 above. Every time an ant reaches the end point of the graph, a cross section is formed. The more ants placed on the graph, the more cross sections produced, and the higher the chances are that the best solution is approached. In selecting the nodes of a column to move to, the probability of an ant selecting the *j*-th node of the *i*-th column is described by the following relation:

$$P_{i,j} = \frac{\tau_{i,j}}{\sum \tau_{i,j}} \tag{5}$$

In Eq.5, $\tau_{i,j}$ is the sum of the pheromone placed on node (i,j) from previous attempts. In the first attempt, all nodes have an equal pheromone of τ_0, and therefore in the first attempt, all nodes have an equal chance of being selected by the ants.

IV- Evaluate the solutions obtained by each ant in the first attempt: Once all the ants complete the first attempt, the objective function *f* is calculated for each ant. The objective function here, as mentioned previously, is the cross section area. Next, pheromone is deposited along the trail which each ant has chosen in forming an incremental solution. The amount of pheromone deposited on each node is reversely related to the objective function of the path being considered, i.e., $\tau = 1/f$. As the rule states, the lower the objective function (cross section area) of a path (cross section), the more pheromone will be deposited on the components of the path.

V- Update the pheromone value of each node in the graph: After calculating the pheromone value of every node for the present attempt, the updated pheromone value of each node is obtained through the following relation:

$$\tau_{i,j}(t+1) = (1-\rho)\tau_{i,j}(t) + \Delta\tau_{i,j}(t) \tag{6}$$

In Eq.6, $\Delta\tau_{i,j}$ is the difference between the deposited pheromone in the present attempt and the previous attempt, $\tau_{i,j}{}^{*}$ is the updated pheromone value, and ρ is the evaporation index which takes a value between zero and one. Pheromone evaporation is a useful form of forgetting, preventing the algorithm from rapidly converging towards local optima. The term $(1-\rho)$ thus determines how much of the pheromone accumulated from previous attempts is evaporated.

VI- Repeat steps III through IV in the next attempts in order to reach the optimum solution: in the next attempts, the decision making process of the artificial ants is no longer completely random; as stated by Eq. 5, nodes with more pheromone have a higher chance of being selected by the ants. After each attempt, pheromone values are updated and some pheromone is evaporated. The combined action of pheromone deposit and evaporation enables a constant exploration of the search space towards a global optimum in ACO.

The above mentioned steps form the fundamental framework of the ACO algorithm. Various improvements have been introduced to the original algorithm in recent years, aiming to make the search algorithm both more effective and more efficient. Accordingly, in addition to the ants system (AS) algorithm discussed above, three other algorithms have been more successful, and have been used in the present study: ranked ant system (AS_{rank}), elite ant system (AS_{elite}), maximum-minimum ant system (MMAS). The principle features of these algorithms are briefly discussed herein.

Ants System (AS): This is the simplest form of ACO first introduced by Dorigo et al. [1]. In AS, artificial ants choose their path according to the following probabilistic relation:

$$\rho_{i,j}(k,t) = \frac{\left[\tau_{i,j}(k,t)\right]^{\alpha}\left[\eta_{i,j}(k,t)\right]^{\beta}}{\sum_{j=1}^{j}\left[\tau_{i,j}(k,t)\right]^{\alpha}\left[\eta_{i,j}(k,t)\right]^{\beta}} \tag{7}$$

in which $\rho_{i,j}(k,t)$ is the probability of selecting i-th node of the j-th column, by the k-th ant in the t-th attempt. $\eta_{i,j}(k,t)$ in Eq.7 represents the heuristic information and the determination of its value is problem-specific. In some problems, the value of $\eta_{i,j}(k,t)$ is hard to determine, and is therefore omitted from equation. α and β in Eq.7 are constants which determine the role of pheromone and heuristic information in the artificial ants' decision making process. If $\alpha >> \beta$, the role of pheromone is emphasized and heuristic information has less effect on the decision of the ants. Adversely, $\beta >> \alpha$ means that the ants decide which node to move to based on the heuristic information, paying less attention to the pheromone deposited from previous attempts.

Another important characteristic of ant colony algorithms is the way that pheromone update is defined in these algorithms. AS defines pheromone using Eq.6. $\Delta\tau_{i,j}$ is determined as:

$$\Delta\tau_{i,j}(t) = \sum_{k=1}^{m}\frac{Q}{f(S_k(t))} I_{S_k(t)}\left\{(i,j)\right\} \tag{8}$$

in which m is the number of artificial ants, or the number of solutions produced; Q is a constant named the pheromone return index and its value depends on the amount of pheromone deposited; $S_k(t)$ represents all the nodes which the *k*-th ant has chosen on the *t*-th attempt; $I_{S_k(t)}\left\{(i,j)\right\}$ is a coefficient which is either zero or one, depending respectively on whether the *k*-th ant has chosen the node (*i, j*) or not. In other words, $I_{S_k(t)}$ ensures that only the nodes on which the *k*-th ant has moved to will be considered in depositing pheromone. It can be deduced from Eq.8 that in AS, solutions with a lower objective function will have more pheromone deposited, and vice versa.

Elitist Ants System (AS_{elite}): In this algorithm, more attention is focused on the elite ant of the colony. The elite ant is the one which has produced the best answer in all previous attempts. Specifically, in AS_{elite} extra pheromone is deposited on the path which the elite ant has produced. The ants decide which node to move to using Eq.7. The pheromone update rule in AS_{elite} is

$$\tau_{i,j}(t+1) = (1-\rho)\tau_{i,j}(t) + \Delta\tau_{i,j}(t) + \sigma\Delta\tau_{i,j}^{gb}(t) \tag{9}$$

Where $\sigma\Delta\tau_{i,j}^{gb}(t)$ is the extra pheromone deposited by the elite ant, and σ is the weight of the extra pheromone. AS_{elite} is an attempt to balance between exploration and exploitation in the algorithm.

Ranked Ants System (AS_{rank}): The ranked ants system was first introduced by Bullnheimer et al [5, 6] as an extension of the elitist ants system. In this algorithm, unlike the AS_{elite} in which all ants participate in the pheromone update process, only $\sigma - 1$ elite ants which have created better solutions are chosen to update the pheromone of the paths they have chosen. In AS_{rank}, following each attempt, the ants are lined up according to the solutions they have obtained, and pheromone update values are assigned to each ant, the most pheromone being assigned to the best solution and decreasing thereafter to the last ant in the line. Thus, the pheromone update rule in AS_{rank} can be stated as

$$\Delta\tau^{rank}{}_{i,j}(t) = \sum_{k=1}^{\sigma-1}(\sigma-k)\frac{Q}{f(S_k(t))} I_{S_k(t)}\left\{(i,j)\right\} \tag{10}$$

Minimum-Maximum Ants System (MMAS): Stutzle and Hoos [2-4] first reported the MMAS algorithm in a successful attempt to improve the efficiency of AS. The general structure of MMAS is similar to AS. However, only the path with the best solution in each attempt is chosen to deposit pheromone on its trail. In this way, the solution rapidly converges to the optimum. The danger always exists that the ants quickly move towards the first optimum solution achieved, before having the chance to explore other possibly better solutions in the search space. In order to prevent this from occurring, a restriction is placed on the minimum and maximum allowable net pheromone deposit on the trails, i.e., the deposited pheromone value is limited to $[\tau_{\min}, \tau_{\max}]$. Following each pheromone deposition step, all pheromone values are controlled to fit within the mentioned limit, and any node for which the pheromone value exceeds the limits is adjusted to the allowable limit. This is a way to promote the ants to explore new solutions in the search space. The maximum and minimum allowable pheromone values of the *t-th* attempt is calculated as

$$\begin{cases} \tau_{\max}(t) = \dfrac{1}{1-\rho}\dfrac{Q}{f\,(s^{gb}(t))} & (11) \\[2ex] \tau_{\min}(t) = \dfrac{\tau_{\max}(t)(1-\sqrt[n]{P_{best}})}{(NO_{avg}-1)\sqrt[n]{P_{best}}} & (12) \end{cases}$$

where $f\,(s^{gb}(t))$ is the value of the objective function up to the *t-th* attempt, P_{best} is the probability of the ants choosing the best solution once again, NO_{avg} is the average of the number of decision choices in the decision points. It is noteworthy to mention that the initial pheromone value associated with the nodes, τ_0 is $\tau_{\max}(t)$.

The above discussed four algorithms were employed in the present study in searching for the critical failure surface in slope stability analysis. The implementation procedure is described in the next section.

4. Implementation of ACO in Optimization of Earth Dam and Numerical Examples

The basic components of ACO applied to the problem of optimization the earth dam is shown as a flowchart in Figure.5. The aim of the present study was to assess the ability of four aforementioned ACOs in finding the optimum cross section of earth dam.

The first example represents the homogeneous earth dams composed of coarse soils with different heights. The soil parameters are: unit weight 19.0 kN/m^3, cohesion 15.0 kpa and effective friction angle 20º. The results obtained by this example show the effect of berms on earth dams which decrease the volume of the earth dams. In this example it was assumed the earth dam is founded on bedrock. Figure.6 shows the layout of the embankments. The upstream shape of embankments is similar to downstream. In the example, different height was considered for earth dams and the effect of different number of berms was considered for each earth dams. Thus not only effect of berms at decreasing in embankment volume was considered, but also the optimum number of berms for each specific dam height was determined. Furthermore, four different ant colony algorithms namely the ant system, elite ant system, ranked ant system and maximum-minimum ants system were employed for studying the efficiency of ant colony optimization algorithms in the finding optimum cross section problem. For different earth dams, the cross section areas obtained by each algorithm are tabulated in Table 1 to 7. In comparison to the cases which there are no berms, when there are berms, the percentage of volume reduction for different height earth dams is presented in Table 8.

According to table 8, for homogeneous earth dam, if height of earth dam is less than 40m, using of berms with suitable number, level and width in body of earth dam will reduce embankment volume more than 10 percent in contrast to an earth dam section without berms. Moreover, if height of earth dam exceeds 40m, in case of increasing in height of earth dam, the effect of berms will reduce. Also, according to Tables 1 to 7, there is an optimum number of berms that can be considered in maximum reduction of fill volume in the earth dam. This number is three berms for 80m height earth dam and is one berm for 40m, 30m, 20m, 10m and 5m earth dams.

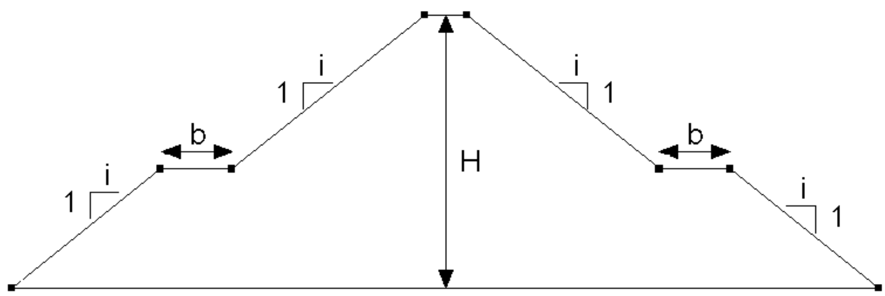

Figure.6. configuration of variables controlling the section fill volume.

Table 1. Results of ACO calculations for minimum fill volume for 160m earth dam

Fill volume six	five	four	three	two	one	zero	Number of berms / Algorithm
68340	52745	49278	43592	45339	43499	63840	AS
43456	43444	43118	44236	42602	43971	43360	AS_{elite}
42135	43523	43227	42316	42317	44336	43360	AS_{rank}
42638	42877	42616	43071	42834	42306	43360	MMAS

Table 2. Results of ACO calculations for minimum fill volume for 80m earth dam

Four	three	two	one	zero	Number of berms / Algorithm
12314	12163	11135	11596	12080	AS
11007	11112	10960	11580	11440	AS_{elite}
11363	10842	11316	11121	11440	AS_{rank}
11731	11426	11177	11213	11440	MMAS

Table 3. Results of ACO calculations for minimum fill volume for 40m earth dam

two	one	zero	Number of berms / Algorithm
2737	2739	2960	AS
2775	2593	2800	AS_{elite}
2681	2723	2800	AS_{rank}
2636	2507	2800	MMAS

Table 4. Results of ACO calculations for minimum fill volume for 30m earth dam

two	one	zero	Number of berms / Algorithm
1437	1501	1740	AS
1488	1382	1560	AS_{elite}
1492	1350	1560	AS_{rank}
1550	1390	1560	MMAS

Table 5. Results of ACO calculations for minimum fill volume for 20m earth dam

two	one	zero	Number of berms / Algorithm
697	656	760	AS
686	672	760	AS_{elite}
787	728	760	AS_{rank}
740	710	760	MMAS

Table 6. Results of ACO calculations for minimum fill volume for 10m earth dam

two	*one*	*zero*	*Number of berms* / *Algorithm*
243	*170*	*200*	AS
220	*157*	*200*	AS_{elite}
182	*166*	*190*	AS_{rank}
212	*164*	*190*	MMAS

Table 7. Results of ACO calculations for minimum fill volume for 5m earth dam

two	*one*	*zero*	*Number of berms* / *Algorithm*
59	*50.5*	*55*	AS
52	*51.5*	*55*	AS_{elite}
54.25	*47.5*	*55*	AS_{rank}
54	*50*	*55*	MMAS

Table 8. Percentage of reduction of fill volume for various earth dam heights

Reduction of embankment volume than without berm cross sections (%)	*Height of earth dam (m)*
14	*5*
17	*10*
13.7	*20*
13.5	*30*
10.6	*40*
5	*80*
3	*160*

5. CONCLUSIONS

This paper dealt with the evaluation of the effectiveness and accuracy of ant colony optimization (ACO) algorithms in finding optimum cross section of earth dams. Four ant colony algorithms were studied, such as ants system (AS), elite ants system (ASelite), ranked ants system (ASrank) and maximum-minimum ants system (MMAS). In order to evaluate the performance of the four mentioned algorithms, two illustrative examples were considered. The following conclusions were drawn from the results of this study:

(1) For homogeneous earth dams with height of 40m, on resistant foundation, suitable number could decrease embankment volume more of 10 percent in contrast of cross section which has not berm. But for earth dam than 40m, when the height increase then the rate of reduction of embankment volume with berms will decrease. In this case, using of berms at body of earth dams usually due to construction aspects.

(2) In earth dams, for each height there are the optimum number of berms which if it uses further or lesser number of berms then it will increase embankment volume.

(3) For determining optimum cross section of earth dams, MMAS, ASrank and ASelite are more efficiency than AS which is weakest and initial algorithm of ant colony optimization algorithms. Also in this

problem weak efficiency of AS algorithm has shown. It is due to low support of this algorithm from surveying optimum paths.

(4) Comparing various height homogeneous earth dam shows whatever the berms located in lower levels of the body, more optimum cross sections will be created. The wider the bottom of the berms, the more effective decreasing volume will be at top levels.

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