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# The inverse problem for the dielectric conductivity

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# Contents

# Abstract

This thesis deals with the inverse problem for the dielectric conductivity $\sigma$, which was first posed by Calderón in 1980, see [2]. We start with proving the uniqueness of this problem and go through similar steps as in [10]. Next we derive different algorithms to compute $\sigma$ from the given boundary data. Then we look at a posed reconstruction problem and discretize the algorithms. In the end we compare the numerical results.

In dieser Arbeit behandeln wir das inverse Problem zur Bestimmung der dielektrischen Leitfähigkeit $\sigma$, welches erstmals von Calderón in [2] gestellt wurde. Zunächst zeigen wir die Eindeutigkeit des gestellten Problems und gehen dabei ähnlich vor wie in [10]. Als nächstes leiten wir verschiedene Algorithmen zur Bestimmung von $\sigma$ her. Danach beschäftigen wir uns mit einem Rekonstruktionsproblem und diskretisieren die einzelnen Algorithmen. Am Ende vergleichen die numerischen Ergebnisse.

# Posing the problem

In this thesis we work on the inverse problem for the dielectric conductivity and compare algorithms to solve it. This inverse problem finds application for example in the Electrical impedance tomography (EIT). The idea behind the problem is to compute the dielectric conductivity, which is substance specific, by sending electricity into the body and measuring it's outcome on the boundary.
Therefore we consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with the boundary $\Gamma = \partial\Omega$. Furthermore we assume for the dielectric conductivity $\sigma$:

- $\sigma \in C^2(\overline{\Omega})$.
- There exists a $\sigma_0 \in \mathbb{R}_+$ such that $\sigma_0^{-1} \geq \sigma(x) \geq \sigma_0 > 0$ for all $x \in \overline{\Omega}$.
- Let $\sigma$ be known on the boundary.

In case of $d = 2$ we have additionally the restriction $diam(\Omega) < 1$. The differential equation which we want to use to compute $\sigma$ is one of the Maxwell equations namely

$$\begin{aligned} -\operatorname{div}(\sigma \nabla u) &= 0 \qquad && \text{in } \Omega, \\ u &= g && \text{on } \Gamma, \end{aligned}$$

where $\nabla u$ is the electrical field. For the boundary data $g$ we will assume $g \in H^{1/2}(\Gamma)$. We will derive a method to compute $\sigma$ from the Dirichlet-to-Neumann operator $S$ which assigns the boundary data $g$ to the Neumann data of the solution of the differential equation above. Therefore we begin with proving the uniqueness of this procedure in the first two chapters. Then we will proceed to finding this Dirichlet-to-Neumann operator $S$ in chapter three. After that the goal is to compute $\sigma$ from the Dirichlet-to-Neumann operator $S$. From then on we will consider only $d = 2$ and $d = 3$. In chapter four we derive several algorithms to get $\sigma$. Following that we work on the discretization of the algorithms for a posed reconstruction problem in chapter five and end up with some numerical results in chapter six.

An overview on this topic is given in [13] for $d = 2$ and in [14] for $d = 3$. However they are using one more assumption for $\sigma$ which is $\sigma \equiv 1$ in a neighborhood of $\Gamma$. For some proofs of convergence rates in [13] $\sigma \in C^\infty(\Omega)$ is required and $\Omega$ needs a $C^\infty$ boundary. In [14] they use as well that $\Omega = B_a(0)$, which is the ball around 0 with radius $a$.
The uniqueness of this problem was first shown by Sylvester and Uhlmann in [4] for $\sigma \in C^\infty(\Omega)$, $d \geq 3$ and $\Omega \subset \mathbb{R}^d$ with a smooth boundary. One of the weakest

assumptions to show uniqueness of this problem was taken by Brown und Torres in [12] who only needed $\sigma \in W^{3/2,p}(\Omega)$ for $p \geq 2d$ and $\Omega \subset \mathbb{R}^d$.

# 1. Introduction

## 1.1. Solvability of the original problem

We start with the problem: Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot \sigma \nabla u &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

holds in a weak sense. Therefore we take an extension $\tilde{g} \in H^1(\Omega)$ of $g$ in $\Omega$. Then we consider $u_0 := u - \tilde{g}$. After putting the differential equation into the $L^2(\Omega)$ duality pairing with a test function $v \in H_0^1(\Omega)$ and applying integration by parts, we get the variational formulation for this problem: Find $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega \sigma \nabla u_0 \cdot \nabla v \; dx = - \int_\Omega \sigma \nabla \tilde{g} \cdot \nabla v \; dx \quad \forall v \in H_0^1(\Omega) \tag{1.2}$$

with $u = u_0 + \tilde{g}$. Therefore we are looking at the bilinear form

$$a(u, v) := \int_\Omega \sigma \nabla u \cdot \nabla v \; dx. \tag{1.3}$$

From the assumption $\sigma \geq \sigma_0 > 0$ we get that

$$\sum_{i=1}^{d} \sigma(x) \eta_i \eta_i \geq \sigma_0 \sum_{i=1}^{d} \eta_i^2 \quad \forall \eta \in \mathbb{R}^d$$

and therefore

$$a(u, u) \geq \sigma_0 \|\nabla u\|_{L^2(\Omega)}^2.$$

With the Poincaré inequality (B.1) we get that $a(.,.)$ is $H_0^1(\Omega)$-elliptic, i.e. there exists a constant $c > 0$ such that

$$a(u, u) \geq c \|u\|_{H^1(\Omega)}^2 \qquad \forall u \in H_0^1(\Omega).$$

Since $a(.,.)$ is as well bounded due to the assumption

$$\sigma(x) \leq \sigma_0^{-1} \qquad \forall x \in \Omega,$$

we get from the lemma of Lax-Milgram, see Thm. (B.3), unique solvability and the estimate

$$\|u\|_{H^1(\Omega)} \leq c_1 \|g\|_{H^{1/2}(\Gamma)}.$$

## 1.2. Definition of the Dirichlet-to-Neumann Operator

The Dirichlet-to-Neumann operator (DtN operator) is defined as the Poincaré-Steklov Operator $S$. We write the DtN operator for (1.1) as

$$S_\sigma g(x) := \lim_{\Omega \ni \tilde{x} \to x} \sigma(\tilde{x}) \nabla u_g(\tilde{x}) \cdot n_x$$

for $x \in \Gamma$, where $u_g$ solves (1.2) and $n$ is the normal which is pointing outside of $\Omega$ almost everywhere. $S_\sigma : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is well defined because for every $g \in H^{1/2}(\Gamma)$ there exists a unique solution $u_g \in H^1(\Omega)$ that solves the equation (1.2). Now we find a weak formulation for $S_\sigma$. For that let $v_h \in H^1(\Omega)$ be a bounded extension for a given $h \in H^{1/2}(\Gamma)$ which suffices $\gamma_0^{int} v_h = h$ and

$$\|v_h\|_{H^1(\Omega)} \leq c_2 \|h\|_{H^{1/2}(\Gamma)}$$

for a $c_2 > 0$ and solves the corresponding variational formulation (1.2). If we apply the first Green's identity (B.2) to $u$ and $v_h$, we get

$$\begin{aligned} \langle S_\sigma g, h \rangle_\Gamma &= \int_\Gamma \sigma \frac{\partial u_g}{\partial n} h \, ds_x \\ &\overset{(B.2)}{=} \int_\Omega \left[ (\nabla \cdot \sigma \nabla u_g) v_h + \sigma \nabla u_g \cdot \nabla v_h \right] dx \\ &\overset{(1.2)}{=} \int_\Omega \sigma \nabla u_g \cdot \nabla v_h \, dx. \end{aligned}$$

Now we have a weak definition of the operator $S_\sigma$ for $g, h \in H^{1/2}(\Gamma)$ :

$$\langle S_\sigma g, h \rangle_\Gamma = \int_\Omega \sigma \nabla u_g \cdot \nabla v_h \, dx, \tag{1.4}$$

where $u_g$ and $v_h$ are the extensions of $g$ and $h$ which solve the corresponding equation (1.2). With this the self-adjointness of the operator $S_\sigma$ follows as well.
We get with the Cauchy–Schwarz inequality and the theorem of Fréchet-Riesz, see

Thm. B.4, that

$$\begin{aligned}
\|S_\sigma g\|_{H^{-1/2}(\Gamma)} &= \sup_{\substack{h\in H^{1/2}(\Gamma)\\ h\neq 0}} \frac{|\langle S_\sigma g, h\rangle_\Gamma|}{\|h\|_{H^{1/2}(\Gamma)}} = \sup_{\substack{h\in H^{1/2}(\Gamma)\\ h\neq 0}} \frac{|\left(\sigma\nabla u_g, \nabla v_h\right)_{L^2(\Omega)}|}{\|h\|_{H^{1/2}(\Gamma)}} \\
&\leq \sup_{\substack{h\in H^{1/2}(\Gamma)\\ h\neq 0}} \frac{\|\sigma\|_{L^\infty(\Omega)}\|u_g\|_{H^1(\Omega)}\|v_h\|_{H^1(\Omega)}}{\|h\|_{H^{1/2}(\Gamma)}} \\
&\leq c_2\|\sigma\|_{L^\infty(\Omega)}\|u_g\|_{H^1(\Omega)} \\
&\overset{\text{Thm.}A.13}{\leq} c_2c_1\|\sigma\|_{L^\infty(\Omega)}\|g\|_{H^{1/2}(\Gamma)},
\end{aligned}$$

where we get the last inequality from the trace operator, see Thm. A.13. So the DtN operator $S_\sigma$ is linear and bounded as well as not dependent on the extension $u_g$, since it holds true that for $u_0 \in H_0^1(\Omega)$

$$\langle S_\sigma g, \gamma_0^{int} u_0\rangle_\Gamma = \int_\Gamma \sigma \frac{\partial u_g}{\partial n}\; 0 \; ds_x = 0.$$

Therefore we can substitute $u_g$ with $(u_g + u_0)$ and still get the same result for the integral. If we take $u_g, u_g' \in H^1(\Omega)$ with $\gamma_0^{int} u_g = g = \gamma_0^{int} u_g'$, then $u_g' - u_g \in H_0^1(\Omega)$ and it follows that

$$\langle S_\sigma g, \gamma_0^{int} u_g'\rangle_\Gamma = \langle S_\sigma g, \gamma_0^{int} u_g\rangle_\Gamma.$$

Our original goal is to derive the dielectric conductivity $\sigma$ from the boundary values $g$ or furthermore from the DtN operator $S$. However we first have to consider the unique solvability of this problem. In the first three chapters we want to show the following theorem:

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain, $\sigma_1$ and $\sigma_2$ two positive functions in $C^2(\overline{\Omega})$. If $S_{\sigma_1} = S_{\sigma_2}$, then it holds true that $\sigma_1 = \sigma_2$ almost everywhere in $\Omega$.*

This theorem implies uniqueness to this problem, that was first posed by Calderon in [2]. In the case $d = 2$, under the condition that there exists a $R \in \mathbb{R}$ such that $R > \sigma_i > R^{-1}$ for $i = 1, 2$ Astala and Päivärinta proved this theorem in [9]. We will prove the uniqueness for $d \geq 3$ as it was done in [10] to get a better understanding of the nature of this problem. However to prove this theorem, we first must transform the problem into another one.

## 1.3. Transformation to the Schrödinger equation

Since the solution of the differential equation (1.1) is unique we can transform it to another uniquely solvable equation which will give us more information about $\sigma$.
Let $\sigma, u \in C^2(\overline{\Omega})$. For $v_\sigma = \sigma^{1/2}u$ it holds true that

$$\begin{aligned} 0 &\overset{(1.1)}{=} -\nabla \cdot (\sigma \nabla u) = -\nabla \cdot \left(\sigma \nabla \left(\sigma^{-1/2} v_\sigma\right)\right) \\ &= -\nabla \cdot \left(-\sigma \frac{1}{\sigma}(\nabla \sigma^{1/2}) v_\sigma + \frac{\sigma}{\sigma^{1/2}} \nabla v_\sigma\right) \\ &= -\nabla \cdot \left(-(\nabla \sigma^{1/2}) v_\sigma + \sigma^{1/2} \nabla v_\sigma\right) \\ &= (\Delta \sigma^{1/2}) v_\sigma - \sigma^{1/2} \Delta v_\sigma. \end{aligned}$$

Therefore we get

$$\begin{aligned} \left(-\Delta + q\right) v_\sigma &= 0 \qquad \text{in } \Omega, \\ v_\sigma &= \sigma^{1/2} g \quad \text{on } \Gamma \end{aligned} \tag{1.5}$$

with

$$q := \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}}. \tag{1.6}$$

This transformation can be done as well for $u \in H^1(\Omega)$:

**Lemma 1.2.** *Let $\sigma \in C^2(\overline{\Omega})$, $\sigma \geq \sigma_0 > 0$ for $\sigma_0 \in \mathbb{R}$ and $u \in H^1(\Omega)$. Then it holds true for $v_\sigma = \sigma^{1/2}u$*

$$\nabla \cdot \sigma \nabla (\sigma^{-1/2} v_\sigma) = \sigma^{1/2} \left(-\Delta + q\right) v_\sigma$$

*in a weak sense.*

**Proof.**
The idea of the proof was given in [10, p.16].
We know that for the dual of $H_0^1(\Omega)$ holds true that

$$H^{-1}(\Omega) := [H_0^1(\Omega)]^*$$

with the norm

$$\|F\|_{H^{-1}(\Omega)} := \sup_{\|u\|_{H_0^1(\Omega)}=1} |F(u)|, \tag{1.7}$$

see Thm. A.11 and Thm. A.9 for more details. Remember that $H^{-1}(\Omega)$ is a Banach space, for the proof see [3, Cor. II.2.2, p.58].

First we show that for $v_\sigma \in H^1(\Omega)$ there exists a sequence $(v_{\sigma,k})_{k\in\mathbb{N}} \subset C^2(\overline{\Omega})$ that converges in $H^1(\Omega)$ and satisfies

$$\|\nabla \cdot \nabla \sigma^{-1/2}(v_{\sigma,k} - v_\sigma)\|_{H^{-1}(\Omega)} \to 0 \text{ as } k \to \infty.$$

Since $\Omega$ is a Lipschitz domain $C_0^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$ with respect to the $H^1(\Omega)$-norm, for the proof see [15, p.77]. Hence $C^2(\overline{\Omega})$ is dense in $H^1(\Omega)$ since $C_0^\infty(\overline{\Omega}) \subset C^2(\overline{\Omega}) \subset H^1(\Omega)$. We choose $(v_{\sigma,k})_{k\in\mathbb{N}} \subset C^2(\overline{\Omega})$ with $\|v_{\sigma,k} - v_\sigma\|_{H^1(\Omega)} \to 0$ for $k \to \infty$. It holds true that

$$\sigma^{-1/2} \leq \sigma_0^{-1/2}.$$

With $2a^2 + 2b^2 \geq (a+b)^2$, $a, b \in \mathbb{R}$, we get

$$\begin{aligned}
\|\sigma^{-1/2}v_{\sigma,k} - \sigma^{-1/2}v_\sigma\|_{H^1(\Omega)}^2 &= \|\sigma^{-1/2}(v_{\sigma,k} - v_\sigma)\|_{L^2(\Omega)}^2 + \|\nabla(\sigma^{-1/2}(v_{\sigma,k} - v_\sigma))\|_{L^2(\Omega)}^2 \\
&\leq \sigma_0^{-1}\|v_{\sigma,k} - v_\sigma\|_{L^2(\Omega)}^2 \\
&\quad + \|\nabla(\sigma^{-1/2})(v_{\sigma,k} - v_\sigma) + \sigma^{-1/2}\nabla(v_{\sigma,k} - v_\sigma)\|_{L^2(\Omega)}^2 \\
&\leq \sigma_0^{-1}\|v_{\sigma,k} - v_\sigma\|_{L^2(\Omega)}^2 \\
&\quad + \left(\|\nabla(\sigma^{-1/2})(v_{\sigma,k} - v_\sigma)\|_{L^2(\Omega)} + \|\sigma^{-1/2}\nabla(v_{\sigma,k} - v_\sigma)\|_{L^2(\Omega)}\right)^2 \\
&\leq \sigma_0^{-1}\left(\|v_{\sigma,k} - v_\sigma\|_{L^2(\Omega)}^2 + 2\|\nabla(v_{\sigma,k} - v_\sigma)\|_{L^2(\Omega)}^2\right) \\
&\quad + 2\|\nabla\sigma^{-1/2}\|_{L^\infty(\Omega)}^2\|v_{\sigma,k} - v_\sigma\|_{L^2(\Omega)}^2.
\end{aligned}$$

Due to $\sigma \in C^2(\overline{\Omega})$ we know that $\nabla\sigma$ is bounded in $\Omega$. Therefore it follows that

$$\begin{aligned}
\|\nabla\sigma^{-1/2}\|_{L^\infty(\Omega)} &= \|-\frac{1}{2}\sigma^{-3/2}\nabla\sigma\|_{L^\infty(\Omega)} \\
&\leq \frac{1}{2}\sigma_0^{-3/2}\|\nabla\sigma\|_{L^\infty(\Omega)} =: C < \infty.
\end{aligned}$$

As a consequence we get

$$\begin{aligned}
\|\sigma^{-1/2}v_{\sigma,k} - \sigma^{-1/2}v_\sigma\|_{H^1(\Omega)} &\leq (2\sigma_0^{-1} + 2C^2)^{1/2}\|v_{\sigma,k} - v_\sigma\|_{H^1(\Omega)} \\
&\to 0 \quad \text{ for } v_{\sigma,k} \to v_\sigma.
\end{aligned}$$

In the end we have

$$\begin{aligned}
\|\sigma\nabla\sigma^{-1/2}(v_{\sigma,k} - v_\sigma)\|_{L^2(\Omega)} &\leq \|\sigma\|_{L^\infty}\|\sigma^{-1/2}(v_{\sigma,k} - v_\sigma)\|_{H^1(\Omega)} \\
&\to 0 \quad \text{ for } v_{\sigma,k} \to v_\sigma.
\end{aligned} \tag{1.8}$$

For the next step we need the definition of derivatives in the $H^{-1}(\Omega)$ duality pairing: Let $u \in L^2(\Omega)$, $w \in H_0^1(\Omega)$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = 1$, then we define $D^\alpha u \in H^{-1}(\Omega)$ with

$$\langle D^\alpha u, w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} := -\langle u, D^\alpha w\rangle_{L^2(\Omega)}$$

see [1, p. 22]. The Cauchy Schwartz inequality implies

$$\begin{aligned}
\left|\langle\frac{\partial}{\partial x_i}\,\sigma\frac{\partial}{\partial x_i}\,\sigma^{-1/2}(v_{\sigma,k}-v_\sigma),w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\right| &= \left|-\langle\sigma\frac{\partial}{\partial x_i}\,\sigma^{-1/2}(v_{\sigma,k}-v_\sigma),\frac{\partial}{\partial x_i}w\rangle_{L^2(\Omega)}\right|\\
&\leq \|\sigma\frac{\partial}{\partial x_i}\,\sigma^{-1/2}(v_{\sigma,k}-v_\sigma)\|_{L^2(\Omega)}\ \|\frac{\partial}{\partial x_i}w\|_{L^2(\Omega)}\\
&\leq \|\sigma\frac{\partial}{\partial x_i}\,\sigma^{-1/2}(v_{\sigma,k}-v_\sigma)\|_{L^2(\Omega)}\ \|w\|_{H_0^1(\Omega)}
\end{aligned}$$

for $i=1,...,d$. Therefore we get

$$\|\frac{\partial}{\partial x_i}\,\sigma\frac{\partial}{\partial x_i}\,\sigma^{-1/2}(v_{\sigma,k}-v_\sigma)\|_{H^{-1}(\Omega)}\leq\|\sigma\frac{\partial}{\partial x_i}\,\sigma^{-1/2}(v_{\sigma,k}-v_\sigma)\|_{L^2(\Omega)}$$

with the theorem of Fréchet-Riesz, see Thm B.4, which tells us that we can rewrite the norm that was defined in (1.7) as

$$\|\frac{\partial}{\partial x_i}\,\sigma\frac{\partial}{\partial x_i}\,\sigma^{-1/2}(v_{\sigma,k}-v_\sigma)\|_{H^{-1}(\Omega)}=\sup_{0\neq w\in H_0^1(\Omega)}\frac{\langle\frac{\partial}{\partial x_i}\,\sigma\frac{\partial}{\partial x_i}\,\sigma^{-1/2}(v_{\sigma,k}-v_\sigma),w\rangle}{\|w\|_{H_0^1(\Omega)}}.$$

With (1.8) we get for $v_{\sigma,k}\to v_\sigma$ that

$$\|\nabla\cdot\sigma\nabla\sigma^{-1/2}(v_{\sigma,k}-v_\sigma)\|_{H^{-1}(\Omega)}\to 0.$$

By using integration by parts and Cauchy Schwartz inequality we get for $w\in H_0^1(\Omega)$

$$\begin{aligned}
&\left|\langle\sigma^{1/2}(-\Delta+q)(v_{\sigma,k}-v_\sigma),w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\right|\\
&\qquad\leq\|\sigma^{1/2}\|_{L^\infty(\Omega)}\sum_{i=1}^d\left|\langle\frac{\partial}{\partial x_i}(v_{\sigma,k}-v_\sigma),\frac{\partial}{\partial x_i}w\rangle_{L^2(\Omega)}\right|\\
&\qquad\quad+\|\sigma^{1/2}\|_{L^\infty(\Omega)}\|q\|_{L^\infty(\Omega)}\|v_{\sigma,k}-v_\sigma\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)}\\
&\qquad\leq\|\sigma^{1/2}\|_{L^\infty(\Omega)}\left(1+\|q\|_{L^\infty(\Omega)}\right)\|v_{\sigma,k}-v_\sigma\|_{H^1(\Omega)}\|w\|_{H_0^1(\Omega)}.
\end{aligned}$$

Due to $\sigma\in C^2(\overline{\Omega})$ the function $\sigma$ is bounded as well as $q$ that was defined in (1.6). Since $v_{\sigma,k}$ is converging to $v_\sigma$ in $H^1(\Omega)$, we get

$$\left|\langle\sigma^{1/2}(-\Delta+q)(v_{\sigma,k}-v_\sigma),w\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\right|\to 0\qquad\text{for } v_{\sigma,k}\to v_\sigma.$$

Because $v_{\sigma,k}\in C^2(\Omega)$, it holds true that

$$\nabla\cdot\sigma\nabla(\sigma^{-1/2}v_{\sigma,k})=\sigma^{1/2}\left(-\Delta+q\right)v_{\sigma,k},$$

see the arguments above, and we get

$$
\begin{aligned}
&\|\nabla\cdot\sigma\nabla(\sigma^{-1/2}v_\sigma) - \sigma^{1/2}\,(-\Delta+q)\,v_\sigma\|_{H^{-1}(\Omega)} \\
&\qquad\qquad\le \|\nabla\cdot\sigma\nabla(\sigma^{-1/2}(v_\sigma - v_{\sigma,k}))\|_{H^{-1}(\Omega)} \\
&\qquad\qquad\quad + \|\sigma^{1/2}\,(-\Delta+q)\,(v_\sigma - v_{\sigma,k})\|_{H^{-1}(\Omega)} \\
&\qquad\qquad\quad + \|\nabla\cdot\sigma\nabla(\sigma^{-1/2}v_{\sigma,k}) - \sigma^{1/2}\,(-\Delta+q)\,v_{\sigma,k}\|_{H^{-1}(\Omega)} \\
&\qquad\qquad\to 0 \qquad \text{for } v_{\sigma,k}\to v_\sigma.
\end{aligned}
$$

□

**Remark 1.1.** *To get the variational formulation of the Schrödinger equation (1.5), we consider $\tilde g$ as an extension of the boundary data $\sigma^{1/2}g$. With this we get*

$$
\int_\Omega \nabla v_\sigma\cdot\nabla w\;dx + \int_\Omega q v_\sigma\;w\;dx = -\int_\Omega \nabla\tilde g\cdot\nabla w\;dx - \int_\Omega q\tilde g\;w\;dx \qquad \forall w\in H_0^1(\Omega). \tag{1.9}
$$

*The lemma 1.2 also implies the equivalence of the variational formulation (1.2) of the original problem and this variational formulation of the Schrödinger equation (1.5). Due to the discussion in the beginning of this chapter we know of the unique solvability of the first variational formulation (1.2). Therefore we know that (1.9) is uniquely solvable.*

## 1.4. Definition of the DtN operator for the Schrödinger equation

Now we can look at the Dirichlet-to-Neumann operator for the Schrödinger equation as well. For every boundary data $g\in H^{1/2}(\Gamma)$ there exists a unique weak solution $v_\sigma$ of (1.9) and $c_4>0$ with

$$\|v_\sigma\|_{H^1(\Omega)} \le c_4\|g\|_{H^{1/2}(\Gamma)}.$$

Formally we define $\tilde S_q$ as

$$\tilde S_q g(x) := \lim_{\Omega\ni\tilde x\to x} \nabla v_\sigma(\tilde x)\cdot n_{\tilde x}$$

for $x\in\Gamma$, where $v_\sigma$ solves the variational formulation (1.9) of the Schrödinger equation (1.5) with boundary data $g$. For the weak definition of $\tilde S_q$ we look at $g,h\in H^{1/2}(\Gamma)$ and the extensions $v_g, w_h\in H^1(\Omega)$ which solve the variational formulation (1.9) with boundary data $g$ and $h$ respectively, where $\gamma_0^{int}w_h = h$ and

$$\|w_h\|_{H^1(\Omega)} \le c_5\|h\|_{H^{1/2}(\Gamma)}$$

for a $c_5 > 0$. Such $v_g$ and $w_h$ exist due to the trace theorem A.13 and previous theorems. With the first Green's identity (B.2) for $v_g$ and $w_h$ we get

$$\begin{aligned}
\langle \tilde{S}_q g, h\rangle_\Gamma &= \int\limits_\Gamma \frac{\partial v_g}{\partial n}\ h\ ds_x \\
&\overset{(B.2)}{=} \int\limits_\Omega \{\Delta v_g\ w_h + \nabla v_g \cdot \nabla w_h\}\ dx \\
&\overset{\text{Rem.1.1}}{=} \int\limits_\Omega \{q v_g\ w_h + \nabla v_g \cdot \nabla w_h\}\ dx. \qquad (1.10)
\end{aligned}$$

With this we get the self-adjointness of the operator $\tilde{S}_q$. $\tilde{S}_q$ is linear and bounded because $q$ is bounded in $\Omega$ and with the trace Thm. A.13 it follows that

$$\begin{aligned}
\|\tilde{S}_q g\|_{H^{-1/2}(\Gamma)} &= \sup_{\substack{h\in H^{1/2}(\Gamma)\\ h\neq 0}} \frac{|\langle \tilde{S}_q g, h\rangle_\Gamma|}{\|h\|_{H^{1/2}(\Gamma)}} = \sup_{\substack{h\in H^{1/2}(\Gamma)\\ h\neq 0}} \frac{|(q v_g, w_h)_{L^2(\Omega)} + (\nabla v_g, \nabla w_h)_{L^2(\Omega)}|}{\|h\|_{H^{1/2}(\Gamma)}} \\
&\overset{\text{Thm}A.13}{\leq} \sup_{\substack{h\in H^{1/2}(\Gamma)\\ h\neq 0}} \left\{ \frac{(\|q\|_{L^\infty(\Omega)} + 1)\|v_g\|_{H^1(\Omega)}\|w_h\|_{H^1(\Omega)}}{\|h\|_{H^{1/2}(\Gamma)}} \right\} \\
&\leq \left(\|q\|_{L^\infty(\Omega)} + 1\right) c_5 \|v_g\|_{H^1(\Omega)} \\
&\leq \left(\|q\|_{L^\infty(\Omega)} + 1\right) c_4 c_5\ \|g\|_{H^{1/2}(\Gamma)}.
\end{aligned}$$

With the same argumentation as in Chapter 1.3 we get that $\tilde{S}_q$ is as well not dependent on the extension $v_g$. Therefore we get that $\tilde{S}_q$ is well defined, linear, bounded and independent of the extension $v_g$.

Next we will look at the dependency between the two DtN operators. Let $g \in H^{1/2}(\Gamma)$ and $v_\sigma$ the corresponding solution of the variational formulation (1.9) with the boundary value $g$. Remember that $v_\sigma = \sigma^{1/2} u$ and $S_\sigma u = \sigma \frac{\partial u}{\partial n}$. Then it holds true

$$\begin{aligned}
\int\limits_\Gamma \tilde{S}_q g h\ ds_x &= \int\limits_\Gamma \frac{\partial}{\partial n}(\sigma^{1/2} u) h\ ds_x \\
&= \int\limits_\Gamma \frac{1}{2}\sigma^{-1/2}\frac{\partial \sigma}{\partial n} u h\ ds_x + \int\limits_\Gamma \sigma^{1/2}\frac{\partial u}{\partial n} h\ ds_x \\
&= \int\limits_\Gamma \frac{1}{2}\sigma^{-1/2}\frac{\partial \sigma}{\partial n}\sigma^{-1/2} v_\sigma h\ ds_x + \int\limits_\Gamma \sigma^{-1/2} S_\sigma(u) h\ ds_x \\
&= \int\limits_\Gamma \frac{1}{2}\sigma^{-1}\frac{\partial \sigma}{\partial n} g h\ ds_x + \int\limits_\Gamma \sigma^{-1/2} S_\sigma(\sigma^{-1/2} g) h\ ds_x
\end{aligned}$$

for every $h \in H^{1/2}(\Gamma)$. Therefore we get

$$\tilde{S}_q g = \frac{1}{2}\sigma^{-1}\frac{\partial \sigma}{\partial n} g + \sigma^{-1/2} S_\sigma(\sigma^{-1/2} g) \tag{1.11}$$

in the weak sense for every $g \in H^{1/2}(\Gamma)$.

We now look at the slightly different uniqueness theorem.

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $g_1, g_2 \in H^{1/2}(\Gamma)$ and $q_1$ and $q_2$ two functions in $L^\infty(\Omega)$, for which*

$$\begin{aligned} (-\Delta + q_1)u_1 &= 0 \qquad && \text{in } \Omega, \\ u_1 &= g_1 && \text{on } \Gamma, \end{aligned}$$

*and*

$$\begin{aligned} (-\Delta + q_2)u_2 &= 0 \qquad && \text{in } \Omega, \\ u_2 &= g_2 && \text{on } \Gamma, \end{aligned}$$

*are uniquely solvable in a weak sense. If $\tilde{S}_{q_1} = \tilde{S}_{q_2}$, then it holds true that $q_1 = q_2$ almost everywhere in $\Omega$.*

This theorem implies Thm. 1.1 due to the following reasoning:
Let $\sigma_1, \sigma_2 \in C^2(\overline{\Omega})$ be two positive functions and $S_{\sigma_1} = S_{\sigma_2}$. So it holds true for any $g \in H^{1/2}(\Gamma)$ that

$$S_{\sigma_1} g = \sigma_1 \frac{\partial g}{\partial n} = \sigma_2 \frac{\partial g}{\partial n} = S_{\sigma_2} g$$

on $\Gamma$. With this we follow with the right choice of $g$

$$\begin{aligned} \gamma_0^{int}\sigma_1(x) &= \gamma_0^{int}\sigma_2(x) =: \gamma_0^{int}\sigma(x), \\ \frac{\partial \sigma_1}{\partial n}(x) &= \frac{\partial \sigma_2}{\partial n}(x) \end{aligned}$$

for $x \in \Gamma$. For $q_i = \frac{\Delta \sigma_i^{1/2}}{\sigma_i^{1/2}}$ holds $q_i \in L^\infty(\Omega)$. We get for $g \in H^{1/2}(\Gamma)$

$$\begin{aligned} \tilde{S}_{q_1} g &= \frac{1}{2}\sigma_1^{-1}\frac{\partial \sigma_1}{\partial n} g + \sigma_1^{-1/2} S_{\sigma_1}(\sigma_1^{-1/2} g) \\ &= \frac{1}{2}\sigma_2^{-1}\frac{\partial \sigma_2}{\partial n} g + \sigma_2^{-1/2} S_{\sigma_2}(\sigma_2^{-1/2} g) \\ &= \tilde{S}_{q_2} g \end{aligned}$$

and that the equations

$$\begin{aligned} (-\Delta + q_1)u_1 &= 0 && \text{in } \Omega, \\ u_1 &= g_1 && \text{on } \Gamma, \end{aligned}$$

and

$$\begin{aligned} (-\Delta + q_2)u_2 &= 0 && \text{in } \Omega, \\ u_2 &= g_2 && \text{on } \Gamma, \end{aligned}$$

are well posed and uniquely solvable in a weak sense for $g_1, g_2 \in H^{1/2}(\Gamma)$ due to the reasoning of chapter 1.3 and 1.4.
Thm. 1.3 now implies that

$$q_1 = \frac{\Delta\sigma_1^{1/2}}{\sigma_1^{1/2}} = \frac{\Delta\sigma_2^{1/2}}{\sigma_2^{1/2}} = q_2 =: q$$

almost everywhere in $\Omega$. However both $\sigma_i$, $i = 1, 2$, solve the equation

$$\begin{aligned} (-\Delta + q)\sigma_i^{1/2} &= 0 && \text{in } \Omega, \\ \sigma_i &= \gamma_0^{int}\sigma && \text{on } \Gamma, \end{aligned}$$

and we know that this equation is uniquely solvable in a weak sense. Hence we get

$$\sigma_1 = \sigma_2$$

almost everywhere in $\Omega$. Therefore the equivalence is proven.

# 2. Uniqueness of the inverse problem

We look at the Schrödinger equation

$$\begin{aligned} (-\Delta + q)\, v_\sigma &= 0 \qquad && \text{in } \Omega, \\ v_\sigma &= \sigma^{1/2} g && \text{on } \Gamma \end{aligned}$$

with

$$q := \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}}. \tag{2.1}$$

Our goal is to derive $\sigma$ from $\tilde{S}_q v_\sigma$. However we do not know anything about the uniqueness of this problem. Therefore we are going to prove Thm. 1.3, which tells us that if we have two DtN operators $\tilde{S}_{q_1}$ and $\tilde{S}_{q_2}$ of the Schrödinger equation and $\tilde{S}_{q_1} = \tilde{S}_{q_2}$, then $q_1$ and $q_2$ are identical almost everywhere in $\Omega$. As the first step in this direction we prove the following lemma.

**Lemma 2.1.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $g_1, g_2 \in H^{1/2}(\Gamma)$ and $q_1, q_2 \in L^\infty(\Omega)$ for which*

$$\begin{aligned} (-\Delta + q_1) u_1 &= 0 \qquad && \textit{in } \Omega, \\ u_1 &= g_1 && \textit{on } \Gamma, \end{aligned}$$

*and*

$$\begin{aligned} (-\Delta + q_2) u_2 &= 0 \qquad && \textit{in } \Omega, \\ u_2 &= g_2 && \textit{on } \Gamma, \end{aligned}$$

*are uniquely solvable in a weak sense. Then*

$$\left\langle \left( \tilde{S}_{q_1} - \tilde{S}_{q_2} \right) g_1, g_2 \right\rangle_\Gamma = \int\limits_\Omega (q_1 - q_2) u_1 u_2 \; dx,$$

*holds true with $u_1, u_2 \in H^1(\Omega)$ as the weak solutions of the differential equations above.*

**Proof.**
This statement follows directly from the weak definition (1.10) of $\tilde{S}_q$. □

Since we consider the case $\tilde{S}_{q_1} = \tilde{S}_{q_2}$ we get

$$0 = \int_\Omega (q_1 - q_2) u_1 u_2 \, dx.$$

To show that $q_1 = q_2$ we need that the products $u_1 u_2$ of all solutions $u_i$ of the Schrödinger equation above are dense in $L^2(\Omega)$. Since we know that $\{e^{ik\cdot x}\}_{k\in\mathbb{Z}^d}$ form a orthonormal basis in $L^2(\Omega)$, see Thm. B.7, we would like the product $u_1 u_2$ to behave like $e^{ik\cdot x}$, where $k \in \mathbb{Z}^d$. In the following chapter we will discuss such solutions.

## 2.1. Solution of Schrödinger equation

In this chapter we prove a special behavior of solutions of the Schrödinger equation:

**Theorem 2.2.** *Let $q \in L^\infty(\Omega)$. Then there is a constant $C_0(\Omega, d)$, such that for every $\eta \in \mathbb{C}^d \backslash \{0\}$ with $\eta \cdot \eta = 0$ and $|\eta| \geq \max(C_0 \|q\|_{L^\infty(\Omega)}, 1)$ and for any $a \in H^2(\Omega)$ with*

$$\eta \cdot \nabla a = 0$$

*almost everywhere, the equation $(-\Delta + q)u = 0$ in $\Omega$ has a solution*

$$u(x) = e^{i\eta\cdot x}(a(x) + r(x)) \tag{2.2}$$

*almost everywhere, where $r \in H^1(\Omega)$ and*

$$\|r\|_{L^2(\Omega)} \leq \frac{C_0}{|\eta|} \|(-\Delta + q)a\|_{L^2(\Omega)}, \tag{2.3}$$

$$\|\nabla r\|_{L^2(\Omega)} \leq C_0 \|(-\Delta + q)a\|_{L^2(\Omega)}. \tag{2.4}$$

**Proof.**
This proof was given in [10, p. 22-25].
If

$$u(x) = e^{i\eta\cdot x}(a(x) + r(x))$$

and $u$ solves $(-\Delta + q)u = 0$ in $\Omega$ then it holds true that

$$e^{-i\eta\cdot x}(-\Delta + q(x))e^{i\eta\cdot x}(a(x) + r(x)) = 0.$$

It holds true that

$$
\begin{aligned}
e^{-i\eta\cdot x}\nabla\cdot\nabla\left(e^{i\eta\cdot x}(a(x)+r(x))\right) &= e^{-i\eta\cdot x}\nabla\cdot\left(i\eta e^{i\eta\cdot x}(a(x)+r(x))+e^{i\eta\cdot x}\nabla(a(x)+r(x))\right)\\
&= e^{-i\eta\cdot x}\{-\eta\cdot\eta e^{i\eta\cdot x}(a(x)+r(x))\\
&\quad+2ie^{i\eta\cdot x}\eta\cdot\nabla(a(x)+r(x))\\
&\quad+e^{i\eta\cdot x}\nabla\cdot\nabla(a(x)+r(x))\}\\
&= (-\eta\cdot\eta(a(x)+r(x))+2i\eta\cdot\nabla(a(x)+r(x))\\
&\quad+\nabla\cdot\nabla(a(x)+r(x)))\\
&\overset{\eta\cdot\eta=0}{=} (2i\eta\cdot\nabla(a(x)+r(x))+\nabla\cdot\nabla(a(x)+r(x))).
\end{aligned}
$$

Therefore we study the equation

$$(-\Delta-2i\eta\cdot\nabla+q)(a+r)=0.$$

Since $\eta\cdot\nabla a=0$ almost everywhere, we now look for a $r\in H^1(\Omega)$ which solves the differential equation

$$(-\nabla\cdot\nabla-2i\eta\cdot\nabla+q)r=(-\nabla\cdot\nabla+q)a,$$

and for which the two estimates (2.3) and (2.4) hold true. We split this problem into two. First we study the equation

$$(\nabla\cdot\nabla+2i\eta\cdot\nabla)r=f \text{ in } \Omega$$

and it's solutions for $f\in L^2(\Omega)$ in Lem. 2.3. Afterwards we consider

$$(-\Delta-2i\eta\cdot\nabla+q)r=f \text{ in } \Omega$$

for $f\in L^2(\Omega)$ and solve the equation in Lem. 2.5. Finally we sum up and end with the proof of Thm. 2.2.

**Lemma 2.3.** *There exists a constant $C_0(\Omega,d)$ such that for any $\eta\in\mathbb{C}^d\backslash\{0\}$ with $\eta\cdot\eta=0$ and $|\eta|>1$ and for any $f\in L^2(\Omega)$ the differential equation*

$$(\nabla\cdot\nabla+2i\eta\cdot\nabla)r=f \text{ in } \Omega$$

*has a solution $r\in H^1(\Omega)$ satisfying*

$$\|r\|_{L^2(\Omega)}\le\frac{C_0}{|\eta|}\|f\|_{L^2(\Omega)} \tag{2.5}$$

$$\|\nabla r\|_{L^2(\Omega)}\le C_0\|f\|_{L^2(\Omega)} \tag{2.6}$$

**Proof.**
This proof was given in [10, Thm 3.7]. Since we have constant coefficients and a linear differential equation we might think of the Fourier transform first. With

$$\mathcal{F}(\nabla_j u)(\xi) = \xi_j \mathcal{F}u(\xi)$$

for $\xi \in \mathbb{R}^d$, we get the transformed differential equation

$$(\xi \cdot \xi + 2i\eta \cdot \xi)\mathcal{F}r(\xi) = \mathcal{F}f(\xi)$$

for $\xi \in \mathbb{R}^d$. However the term $(\xi \cdot \xi + 2i\eta \cdot \xi)$ is zero for certain $\xi \in \mathbb{R}^d$. Therefore we try to use Fourier series instead.
Write $\eta = s(\omega_1 + i\omega)$ with $s = \frac{|\eta|}{\sqrt{2}}$, where $\omega_1$ and $\omega_2$ are orthogonal unit vectors in $\mathbb{R}^d$. Without loss of generality we can assume that $\omega_1 = e_1$ is the first coordinate vector and $\omega_2 = e_2$ is the second coordinate vector, because otherwise we can rotate and shift the coordinate system. It follows that

$$(\Delta + 2is(\partial_1 + i\partial_2))r = f,$$

and therefore

$$(\Delta + 2is\partial_1 - 2s\partial_2)r = f.$$

Let $x_0 \in \overset{\circ}{\Omega}$. For simplicity we assume that $\Omega \subset [-\pi, \pi]^d =: Q$. Otherwise we scale and shift $\Omega$ by an bijective function $h_{x_0} : \Omega \to \Omega_{scale}$

$$\begin{aligned} d &:= \frac{1}{\pi} \max\{|x - x_0| : x \in \overline{\Omega}\}, \\ h_{x_0}(x) &:= \frac{x - x_0}{d}, \end{aligned}$$

and look instead at the functions

$$\begin{aligned} \tilde{r}(\tilde{x}) &:= r(h_{x_0}^{-1}(\tilde{x})), \\ \tilde{f}(\tilde{x}) &:= f(h_{x_0}^{-1}(\tilde{x})), \end{aligned}$$

where $\tilde{x} \in \Omega_{scale}$.
Let $w_k(x) := e^{i(k+\frac{1}{2}e_2)\cdot x}$ for $k \in \mathbb{Z}^d$. Consider the Fourier series in the lattice $\mathbb{Z}^d + \frac{1}{2}e_2$. We know that the $\{w_k\}_{k\in\mathbb{Z}^d}$ form an orthonormal basis in $L^2(Q)$ with respect to scalar product

$$(u, v) = (2\pi)^{-d} \int\limits_{\Omega} u\overline{v}\, dx,$$

$u, v \in L^2(\Omega)$, see Thm. B.8. The Parseval equality from Thm. B.8 tells us that for

$$f = \sum_{k\in\mathbb{Z}^d} f_k w_k, \qquad f_k := (f, w_k),$$

the equation

$$\|f\|_{L^2(\Omega)} = \sum_{k\in\mathbb{Z}^d} |f_k|^2$$

holds true. Because we search for an $r \in L^2(\Omega)$, we can rewrite $r$ as

$$r = \sum_{k\in\mathbb{Z}^d} r_k w_k, \qquad r_k := (r, w_k).$$

With

$$\nabla w_k = i(k + \frac{1}{2}e_2)w_k$$

we get

$$p_k r_k = f_k \qquad \forall k \in \mathbb{Z}^d, \tag{2.7}$$

where

$$p_k := -(k + \frac{1}{2}e_2)^2 - 2sk_1 - i\ 2s(k_2 + \frac{1}{2}).$$

Therefore the imaginary part of $p_k$ is never zero for $s > 0$, which is the reason why we were looking at the shifted lattice $\mathbb{Z}^d + \frac{1}{2}e_2$ and $\eta \neq 0$. Now we can compute

$$r_k = \frac{f_k}{p_k}.$$

It follows that

$$|r_k| \leq \frac{1}{|p_k|}|f_k| \leq \frac{1}{|2s(k_2 + \frac{1}{2})|}|f_k| \leq \frac{1}{s}|f_k|. \tag{2.8}$$

Therefore we get

$$\|r\|_{L^2(Q)} = \left(\sum_{k\in\mathbb{Z}^d} |r_k|^2\right)^{1/2} \leq \frac{1}{s}\left(\sum_{k\in\mathbb{Z}^d} |f_k|^2\right)^{1/2} = \frac{1}{s}\|f\|_{L^2(Q)} = \frac{\sqrt{2}}{|\eta|}\|f\|_{L^2(Q)}.$$

It remains to show that $\nabla r \in L^2(Q)$ and the estimate (2.6). We know that

$$\nabla r = \sum_{k\in\mathbb{Z}^d} i(k + \frac{1}{2}e_2)\ r_k\ w_k.$$

So we show that

$$|(k + \frac{1}{2}e_2)r_k| \leq 4|f_k| \quad \forall k \in \mathbb{Z}^d,$$

which implies

$$\|\nabla r\|_{L^2(Q)} = \left( \sum_{k\in\mathbb{Z}^d} |(k + \frac{1}{2}e_2)r_k|^2 \right)^{1/2} \leq 4 \left( \sum_{k\in\mathbb{Z}^d} |f_k|^2 \right)^{1/2} = 4\|f\|_{L^2(Q)}.$$

For this we consider two cases:

- $|k + \frac{1}{2}e_2| \leq 4s$,
- $|k + \frac{1}{2}e_2| > 4s$.

In the first case we have

$$|(k + \frac{1}{2}e_2)r_k| \leq 4s|r_k| \overset{(2.8)}{\leq} \frac{4s}{s}|f_k| \leq 4|f_k|.$$

In the second case we have

$$\frac{1}{2}|k + \frac{1}{2}e_2| - 2s > 0$$

and therefore

$$\begin{aligned}
\left| |k + \frac{1}{2}e_2|^2 + 2sk_1 \right| &\geq |k + \frac{1}{2}e_2|^2 - 2s|k + \frac{1}{2}e_2| \\
&= \left( \frac{1}{2}|k + \frac{1}{2}e_2| + \frac{1}{2}|k + \frac{1}{2}e_2| - 2s \right) |k + \frac{1}{2}e_2| \\
&\geq \frac{1}{2}|k + \frac{1}{2}e_2|^2.
\end{aligned}$$

Now we have an estimate for the real part of $p_k$ and we get

$$\begin{aligned}
|(k + \frac{1}{2}e_2)r_k| &= \frac{|k + \frac{1}{2}e_2|}{|p_k|}|f_k| \\
&\leq \frac{|k + \frac{1}{2}e_2|}{\frac{1}{2}|k + \frac{1}{2}e_2|^2}|f_k| \\
&\leq \frac{1}{2s}|f_k|.
\end{aligned}$$

With the condition $s > \frac{1}{\sqrt{2}}$, which is full filled if $|\eta| > 1$, we get that $C_0 = 4$. Therefore we have proven this lemma. □

**Continuation of the proof of Thm. 2.2**
And so we have a $r$ that behaves quit similar to the desired solution.
Consider

**Definition 2.4.** *Define*

$$G_\eta : L^2(\Omega) \subset H^{-1}(\Omega) \to H^1(\Omega), f \mapsto r$$

*where $r$ is the constructed solution of the last lemma and solves*

$$(\nabla \cdot \nabla + 2i\eta \cdot \nabla)r = f \ \textit{in } \Omega.$$

Now we want to consider the case that $q \neq 0$:

**Lemma 2.5.** *Let $q \in L^\infty(\Omega)$, $\eta \in \mathbb{C}^d \backslash \{0\}$ with $\eta \cdot \eta = 0$. Then there exists an $C_0(\Omega, d)$ such that for $|\eta| \geq \max(2\ C_0 \|q\|_{L^\infty(\Omega)}, 1)$ and for any $f \in L^2(\Omega)$ the equation*

$$(-\Delta - 2i\eta \cdot \nabla + q)r = f \ \textit{in } \Omega, \tag{2.9}$$

*has a solution $r \in H^1(\Omega)$ almost everywhere which satisfies*

$$\begin{aligned} \|r\|_{L^2(\Omega)} &\leq \frac{C_0}{|\eta|}\|f\|_{L^2(\Omega)}, \\ \|\nabla r\|_{L^2(\Omega)} &\leq C_0\|f\|_{L^2(\Omega)}. \end{aligned}$$

**Proof.**
This proof was given in [10, Thm. 3.8]. For $q \equiv 0$ we already have the solution $r = G_\eta f$.
Consider that $q$ is not 0 everywhere. Let $r = G_\eta g$, where $g$ has to be determined. If we put this into the differential equation, we get

$$\begin{aligned} f &= (\nabla \cdot \nabla + 2i\eta \cdot \nabla + q)G_\eta g \\ &= (\nabla \cdot \nabla + 2i\eta \cdot \nabla)G_\eta g + qG_\eta g \\ &\overset{\text{Lem2.3}}{=} (I + qG_\eta)g. \end{aligned}$$

Now we have to verify that $(I + qG_\eta)$ is invertible. We know from Lem. 2.3

$$\|qG_\eta g\|_{L^2(\Omega)} \leq \|q\|_{L^\infty(\Omega)}\|G_\eta g\|_{L^2(\Omega)} \leq \|q\|_{L^\infty(\Omega)}\frac{C_0}{|\eta|}\|g\|_{L^2(\Omega)}$$

for $g \in L^2(\Omega)$. Therefore we get

$$\|qG_\eta\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{C_0}{|\eta|}\|q\|_{L^\infty(\Omega)}.$$

For $|\eta| \geq \max(2\ C_0 \|q\|_{L^\infty(\Omega)}, 1)$ it holds true that

$$\|qG_\eta\|_{L^2(\Omega)\to L^2(\Omega)} \leq \frac{1}{2}.$$

Due to the Neumann series $(I + qG_\eta)$ is invertible and it holds true that

$$\|(I + qG_\eta)^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \leq (1 - \|qG_\eta\|_{L^2(\Omega)\to L^2(\Omega)})^{-1} \leq 2,$$

for the proof of this equivalence see [3, Thm II.1.11, p.56]. Hence we can compute

$$g = (I + qG_\eta)^{-1} f.$$

This implies

$$\begin{aligned}
(-\Delta - 2i\eta \cdot \nabla + q)r &= (-\Delta - 2i\eta \cdot \nabla)G_\eta g + qG_\eta g \\
&\overset{\text{Lem.2.3}}{=} g + qG_\eta g \\
&= (I + qG_\eta)g \\
&= f
\end{aligned}$$

and therefore $r$ solves the equation (2.9). Since

$$\|(I + qG_\eta)^{-1}\|_{L^2(\Omega)\to L^2(\Omega)} \leq 2$$

we get that

$$\|g\|_{L^2(\Omega)} \leq 2\|f\|_{L^2(\Omega)}.$$

With lemma 2.3 it follows that

$$\begin{aligned}
\|G_\eta g\|_{L^2(\Omega)} &\leq \frac{C_0}{|\eta|}\|g\|_{L^2(\Omega)} \\
&\leq \frac{C_0}{|\eta|}\|(I + qG_\eta)^{-1} f\|_{L^2(\Omega)} \\
&\leq 2\frac{C_0}{|\eta|}\|f\|_{L^2(\Omega)}, \\
\|\nabla G_\eta g\|_{L^2(\Omega)} &\leq C_0\|g\|_{L^2(\Omega)} \\
&\leq 2C_0\|f\|_{L^2(\Omega)}.
\end{aligned}$$

$\square$

**Continuation of the proof of Thm. 2.2**
Due to lemma 2.5 we know now, that

$$(-\nabla \cdot \nabla - 2i\eta \cdot \nabla + q)r = (-\nabla \cdot \nabla + q)a,$$

has a solution $r$ with

$$\begin{aligned} \|r\|_{L^2(\Omega)} &\leq \frac{C_0}{|\eta|} \|(-\Delta + q)a\|_{L^2(\Omega)}, \\ \|r\|_{L^2(\Omega)} &\leq 2C_0 \|(-\Delta + q)a\|_{L^2(\Omega)}. \end{aligned}$$

Therefore

$$u(x) = e^{i\eta \cdot x}(a(x) + r(x)$$

is the required solution to $(-\Delta + q)u = 0$ in $\Omega$. $\square$

## 2.2. Uniqueness of the inverse problem

In chapter one we have already proven the equivalence of the uniqueness of the original problem as it was stated in Thm. 1.1 to the following one:

**Theorem 1.3** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $g_1, g_2 \in H^{1/2}(\Gamma)$ and $q_1$ and $q_2$ two functions in $L^\infty(\Omega)$, for which*

$$\begin{aligned} (-\Delta + q_1)u_1 &= 0 \qquad \textit{in } \Omega, \\ u_1 &= g_1 \qquad \textit{auf } \Gamma \end{aligned}$$

*and*

$$\begin{aligned} (-\Delta + q_2)u_2 &= 0 \qquad \textit{in } \Omega, \\ u_2 &= g_2 \qquad \textit{auf } \Gamma \end{aligned}$$

*are uniquely solvable in a weak sense. If $\tilde{S}_{q_1} = \tilde{S}_{q_2}$ then it holds true that $q_1 = q_2$ in $\Omega$ almost everywhere.*

In addition we haven proven:
**Lemma 2.1** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $g_1, g_2 \in H^{1/2}(\Gamma)$ and $q_1, q_2 \in L^\infty(\Omega)$ for which*

$$\begin{aligned} (-\Delta + q_1)u_1 &= 0 \qquad \textit{in } \Omega, \\ u_1 &= g_1 \qquad \textit{auf } \Gamma \end{aligned}$$

*and*

$$\begin{aligned}(-\Delta + q_2)u_2 &= 0 \qquad && \textit{in } \Omega,\\ u_2 &= g_2 && \textit{auf } \Gamma\end{aligned}$$

*are uniquely solvable in a weak sense. Then*

$$\left\langle \left(\tilde{S}_{q_1} - \tilde{S}_{q_2}\right) g_1, g_2 \right\rangle_\Gamma = \int\limits_\Omega (q_1 - q_2)u_1 u_2 \; dx,$$

*holds true, where* $u_1, u_2 \in H^1(\Omega)$ *solve the differential equations above.*

We will now proof the Thm. 1.3:
This proof was given in [10, Thm 3.2]. Since we consider the case $\tilde{S}_{q_1} = \tilde{S}_{q_2}$ we get with Lem. 2.1

$$0 = \int\limits_\Omega (q_1 - q_2)u_1 u_2 \; dx. \tag{2.10}$$

To show that $q_1 = q_2$ we need that the products $u_1u_2$ of all solutions $u_i$ of the differential equations above are dense in $L^2(\Omega)$. The following proof comes from [10]. We consider only the case of $d \geq 3$. For the case $d = 2$ please read [9] from Astala and Päivärinta. Choose for an arbitrary fixed value $\xi \in \mathbb{R}^d$, where $d \geq 3$, two orthogonal vectors $\omega_1, \omega_2 \in \mathbb{R}^d$ which are as well orthogonal to $\xi$ and full fill $\omega_i \cdot \omega_i = 0$ for $i = 1, 2$. Then we get an orthogonal set $\{\xi, \omega_1, \omega_2\}$ in $\mathbb{R}^d$. Let $s \in \mathbb{R}$ be arbitrary and pick

$$\eta = s(\omega_1 + i\omega_2).$$

It follows that $\eta \cdot \eta = 0$. Due to $\sigma_i \in C^2(\overline{\Omega})$ and $q_i = \frac{\Delta\sigma_i^{1/2}}{\sigma_i^{1/2}}$, we get $q_i \in L^\infty(\Omega)$ for $i = 1, 2$. Now we want to use the Thm. 2.2. It holds true that

$$\eta \cdot \nabla e^{i\xi\cdot x} = i\eta \cdot \xi e^{i\xi\cdot x} = 0.$$

From Thm. 2.2 we know for $s$ sufficiently large there exist functions $r_1$ and $r_2$ such that

$$\begin{aligned}w_1(x) &:= e^{-i\eta\cdot x}(e^{i\xi\cdot x} + r_1(x)),\\ w_2(x) &:= e^{i\eta\cdot x}(1 + r_2(x))\end{aligned}$$

solve the equation $(-\Delta + q_i)w_i = 0$ in $\Omega$ with boundary data $g_1 := \gamma_0^{int} w_1$ and $g_2 := \gamma_0^{int} w_2$. Additionally we get

$$\|r_i\|_{L^2(\Omega)} \leq \frac{\tilde{c}}{s} \tag{2.11}$$

for $i = 1,2$ and for $\tilde{c} > 0$. If we insert $w_1, w_2$ in the equation (2.10), we see that

$$0 = \int_\Omega (q_1 - q_2)(x)(e^{i\xi\cdot x} + r_1(x))(1 + r_2(x))\ dx$$

for $s$ large enough. We get with the Cauchy-Schwarz inequality

$$\begin{aligned}
&\left| \int_\Omega (q_1 - q_2)(x) \left( r_1(x)(1 + r_2(x)) + e^{i\xi\cdot x} r_2(x) \right)\ dx \right| \\
&\qquad \le \|q_1 - q_2\|_{L^\infty(\Omega)} \left( \int_\Omega |r_1(x)(1 + r_2(x))|\ dx + \int_\Omega \left| e^{i\xi\cdot x} r_2(x) \right|\ dx \right) \\
&\qquad \le \|q_1 - q_2\|_{L^\infty(\Omega)} \left( \int_\Omega |r_1(x)|\ dx + \int_\Omega |r_1(x) r_2(x)|\ dx + \int_\Omega \left| e^{i\xi\cdot x} r_2(x) \right|\ dx \right) \\
&\qquad \le \|q_1 - q_2\|_{L^\infty(\Omega)} (\|\mathbf{1}_\Omega\|_{L^2(\Omega)} \|r_1\|_{L^2(\Omega)} + \|r_1\|_{L^2(\Omega)} \|r_2\|_{L^2(\Omega)} \\
&\qquad\qquad\qquad\qquad + \|e_\xi\|_{L^2(\Omega)} \|r_2\|_{L^2(\Omega)}) \\
&\qquad = \|q_1 - q_2\|_{L^\infty(\Omega)} \left( \|r_1\|_{L^2(\Omega)} \|r_2\|_{L^2(\Omega)} + |\Omega| (\|r_1\|_{L^2(\Omega)} + \|r_2\|_{L^2(\Omega)}) \right) \\
&\qquad \overset{s\to\infty}{\to} 0,
\end{aligned}$$

because of (2.11). Hence we get

$$0 = \int_\Omega (q_1 - q_2)(x) e^{i\xi\cdot x}\ dx.$$

We did this for an arbitrary $\xi \in \mathbb{R}^d$. Since $q_1, q_2 \in L^\infty(\Omega)$ we know that the Fourier transform of $(q_1 - q_2)$ exists, see Thm A.6, and is zero for every frequency $\xi$. Therefore $q_1 = q_2$ must hold, which we can derive from the Plancherel formula, as it was stated in Thm. B.5. With this we get the uniqueness of the inverse problem. $\square$

# 3. An integral equation to compute $\tilde{S}_q\psi_\eta$

## 3.1. A formula for $\psi_\eta$ on $\Gamma$

We know now of some behavior and the existence of the solution of the equation

$$\begin{aligned}(-\Delta + q)v_\sigma &= 0 \qquad \text{in } \Omega,\\ v_\sigma &= \sigma^{1/2}g \quad \text{on } \Gamma.\end{aligned}$$

Choose a $\eta \in \mathbb{C}^d\backslash\{0\}$ with $\eta \cdot \eta = 0$ and consider the continuation $\psi_\eta$ of $v_\sigma$ that solves the Laplace operator in the exterior of $\Omega$:

$$\begin{aligned}(-\Delta + \tilde{q})\psi_\eta &= 0 && \text{in } \mathbb{R}^d\backslash\Gamma,\\ \psi_\eta &= \sigma^{1/2}g && \text{on } \Gamma,\\ \psi_\eta(x) &= e^{i\eta\cdot x} + O(\frac{1}{|x|}) && \text{for } |x| \to \infty,\end{aligned} \tag{3.1}$$

where $\eta \cdot \eta = 0$, $\eta \in \mathbb{C}^d\backslash\{0\}$ and

$$\tilde{q} = \begin{cases} q & \text{in } \Omega,\\ 0 & \text{in } \mathbb{R}^d\backslash\overline{\Omega}.\end{cases}$$

Additionally we know about the unique solvability of the inverse problem. In this chapter we will derive a method to compute the DtN operator $\tilde{S}_q$ from the Dirichlet data of $\psi_\eta$ so that we can compute $\sigma$ from $\tilde{S}_q$ in the next chapter. For that we first prove the following theorem which tells us about the behavior of the solution $\psi_\eta$ of the equation (3.1) for $x \in \Gamma$.

**Theorem 3.1.** *Let $d \geq 2$, $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\eta \in \mathbb{C}^2\backslash\{0\}$ with $\eta \cdot \eta = 0$. Then the solution of the equation (3.1) has the form*

$$\psi_\eta(x) = e^{i\eta\cdot x} - \int\limits_\Omega G_0(x,y)q(y)\psi_\eta(y)\ dy.$$

*for $x \in \Gamma$, where $G_0$ is the fundamental solution of the Laplace operator.*

We will later see in Thm. 3.3 under what conditions the integral exists. To prove this theorem we consider $w = e^{-i\eta\cdot x}\psi_\eta$. Then it holds true that

$$\begin{aligned} -\tilde{q}\psi_\eta(x) &\overset{(3.1)}{=} -\Delta\psi_\eta(x) = -\nabla\cdot\left(\nabla\left(e^{i\eta\cdot x}w(x)\right)\right) \\ &= -\nabla\cdot\left(i\eta e^{i\eta\cdot x} + e^{i\eta\cdot x}\nabla w(x)\right) \\ &= -(i)^2\eta\cdot\eta e^{i\eta\cdot x}w(x) - 2e^{i\eta\cdot x}i\eta\cdot\nabla w(x) - e^{i\eta\cdot x}\Delta w(x). \end{aligned}$$

Let $e_{-\eta}(x) := e^{-i\eta\cdot x}$. Under the condition $\eta\cdot\eta = 0$ we get for $w$

$$\begin{aligned} -\Delta w - 2i\eta\cdot\nabla w &= -e_{-\eta}\,\tilde{q}\,\psi_\eta && \text{in } \mathbb{R}^d\backslash\Gamma, \\ w &= e_{-\eta}\,\sigma^{1/2}g && \text{on } \Gamma, \\ w &= 1 + O(\frac{1}{|x|}) && \text{for } |x|\to\infty. \end{aligned} \tag{3.2}$$

To solve this equation we first look at the homogeneous equation

$$-\Delta w - 2i\eta\cdot\nabla w = 0. \tag{3.3}$$

Since $w(x) = e^{-i\eta\cdot x}$ does not solve the equation (3.3) we consider $w(x) = e^{-i\eta\cdot x}r(x)$, where $r$ is a correction term to get the exact solution. It holds true that

$$\begin{aligned} \frac{\partial}{\partial x_j}w(x) &= -i\eta_j e^{-i\eta\cdot x}r(x) + e^{-i\eta\cdot x}\frac{\partial}{\partial x_j}r(x), \\ \frac{\partial^2}{\partial x_j^2}w(x) &= -\eta_j^2 e^{-i\eta\cdot x}r(x) - 2i\eta_j e^{-i\eta\cdot x}\frac{\partial}{\partial x_j}r(x) + e^{-i\eta\cdot x}\frac{\partial^2}{\partial x_j^2}r(x), \end{aligned}$$

for $j = 1, ..., d$. If we put this result into (3.3) we get:

$$\begin{aligned} 0 &\overset{(3.3)}{=} \eta\cdot\eta\; e^{-i\eta\cdot x}r(x) + 2ie^{-i\eta\cdot x}\eta\cdot\nabla r(x) - e^{-i\eta\cdot x}\Delta r(x) - 2\eta\cdot\eta\; e^{-i\eta\cdot x}r(x) \\ &\quad - 2ie^{-i\eta\cdot x}\eta\cdot\nabla r(x) \\ &\overset{\eta\cdot\eta=0}{=} 2ie^{-i\eta\cdot x}\eta\cdot\nabla r(x) - e^{-i\eta\cdot x}\Delta r(x) - 2ie^{-i\eta\cdot x}\eta\cdot\nabla r(x) \\ &= -e^{-i\eta\cdot x}\Delta r(x). \end{aligned}$$

So we need $\Delta r(x) = 0$. For that reason we take the fundamental solution of the Laplace operator

$$r(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & \text{if } d = 2, \\ \frac{1}{d(d-2)\alpha(d)}\frac{1}{|x|^{d-2}} & \text{if } d \geq 3 \end{cases}$$

where $\alpha(d)$ is the volume of the unit ball in $\mathbb{R}^d$, see [8, p.22] for more details. Hence $w(x) = e^{-i\eta\cdot x}r(x)$, $x \neq 0$, solves the equation (3.3). The fundamental solution of (3.2) for which we were searching, is therefore

$$G_\eta(x,y) := \begin{cases} -\frac{e^{-i\eta\cdot(x-y)}}{2\pi}\log|x-y| & \text{if } d = 2, \\ \frac{1}{d(d-2)\alpha(d)}\frac{e^{-i\eta\cdot(x-y)}}{|x-y|^{d-2}} & \text{if } d \geq 3 \end{cases} \tag{3.4}$$

for $x \neq y$, $x, y \in \mathbb{R}^d$.

**Lemma 3.2.** *Let the same conditions as in Thm. 3.1 hold true. Then the solution of the equation (3.2) has the form*

$$w(x) = 1 - \int_\Omega G_\eta(x,y)(e^{-i\eta\cdot y} q(y)\psi_\eta(y))\ dy.$$

*for $x \in \Gamma$.*

**Proof.**
We first look at the interior problem

$$\begin{aligned} -\Delta w - 2i\eta\cdot\nabla w &= -e_{-\eta}\ q\ \psi_\eta && \text{in } \Omega, \\ w &= e_{-\eta}\ \sigma^{1/2} g && \text{on } \Gamma. \end{aligned}$$

For the homogeneous equation

$$-\Delta w - 2i\eta\cdot\nabla w = 0, \tag{3.5}$$

we already have a fundamental solution $G_\eta(x,y)$. Next look at the dual problem of the homogeneous differential equation, which is

$$-\Delta w + 2i\eta\cdot\nabla w = 0. \tag{3.6}$$

With the same steps as above, we can show that

$$G^*_\eta(x,y) := \begin{cases} -\frac{e^{i\eta\cdot(x-y)}}{2\pi}\log|x-y| & \text{if } d = 2, \\ \frac{1}{d(d-2)\alpha(d)}\frac{e^{i\eta\cdot(x-y)}}{|x-y|^{d-2}} & \text{if } d \geq 3 \end{cases} \tag{3.7}$$

is the fundamental solution of the dual problem. With the representation formula for the interior Dirichlet problem, which was proven in [15, Thm 7.5, p.226], we get

$$\begin{aligned} w(x) = &-\int_\Omega G_\eta(x,y)\left(e_{-\eta}(y)q(y)\psi_\eta(y)\right)\ dy - \int_\Gamma \left(\partial_{n_y} G^*_\eta(x,y)\right) e_{-\eta}\sigma^{1/2} g\ ds_y \\ &+ \int_\Gamma G_\eta(x,y)\partial_{n_y}\left(e_{-\eta}(y)\psi_\eta(y)\right)\ ds_y \end{aligned}$$

for $x \in \Omega$, where $n_y$ is the normal vector which is pointing outside of $\Omega$ almost everywhere. As the next step we look at the problem in the exterior

$$\begin{aligned} -\Delta w - 2i\eta\cdot\nabla w = 0 & && \text{in } \mathbb{R}^d\backslash\overline{\Omega}, \\ w &= e_{-\eta}\ \sigma^{1/2} g && \text{on } \Gamma, \\ w &= 1 + O(\frac{1}{|x|}) && \text{for } |x| \to \infty. \end{aligned}$$

With the representation formula for the exterior Dirichlet problem, which was proven in [15, Thm 7.12, p.235], we get

$$w(x) = 1 + \int_\Gamma \left(\partial_{n_y} G_\eta^*(x,y)\right) e_{-\eta}\sigma^{1/2} g \; ds_y - \int_\Gamma G_\eta(x,y)\partial_{n_y}\left(e_{-\eta}(y)\psi_\eta(y)\right) \; ds_y$$

for $x \in \mathbb{R}^d \backslash \overline{\Omega}$. By using the super position principle and traces we get for $x \in \Gamma$

$$w(x) = 1 - \int_\Omega G_\eta(x,y)\left(e_{-\eta}(y)q(y)\psi_\eta(y)\right) \; dy.$$

□

If we transform this result back to $\psi_\eta$ we get

$$\begin{aligned}\psi_\eta(x) &= e^{i\eta\cdot x} - \int_\Omega G_\eta(x,y)e^{i\eta\cdot(x-y)}q(y)\;\psi_\eta(y)\;dy \\ &= e^{i\eta\cdot x} - \int_\Omega G_0(x,y)q(y)\;\psi_\eta(y)\;dy,\end{aligned}$$

for $x \in \Gamma$, where $G_0$ is the fundamental solution of the Laplace operator as well as $G_\eta$ with $\eta = 0$. With this Thm. 3.1 is proven.

The next step to compute the DtN operator $\tilde{S}_q$ is to transform this equation in a way such that it is independent of $q$. But before we solve this question we look at a special case of the Schrödinger equation (1.5) where $\sigma \equiv 1$. In this case we get the Laplace operator

$$\begin{aligned} -\Delta v_1 &= 0 && \text{in } \Omega, \\ v_1 &= \sigma^{1/2} g && \text{on } \Gamma. \end{aligned} \tag{3.8}$$

Now we go back to the representation of $\psi_\eta$ on $\Gamma$. For $x \in \Gamma$ we get with the Schrödinger equation (3.1)

$$\begin{aligned}\psi_\eta(x) &= e^{i\eta\cdot x} - \int_\Omega G_0(x,y)\;q(y)\;\psi_\eta(y)\;dy \\ &= e^{i\eta\cdot x} - \int_\Omega G_0(x,y)\;\Delta_y\psi_\eta(y)\;dy \\ &\overset{(3.8)}{=} e^{i\eta\cdot x} - \int_\Omega G_0(x,y)\;\left(\Delta_y\psi_\eta(y) - \Delta_y v_1(y)\right)\;dy.\end{aligned}$$

In the following theorem we look at the well posedness of this problem:

**Theorem 3.3.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain. If*

$$(\Delta_y \psi_\eta - \Delta_y v_1) \in L^\infty(\overline{\Omega}),$$

*then the integral*

$$\int\limits_\Omega G_0(x,y) \;\; (\Delta_y \psi_\eta(y) - \Delta_y v_1(y)) \;\; dy$$

*exists for every $x \in \overline{\Omega}$.*

**Proof.**
The idea comes from [11, Lem. 6.7, p.119].
Consider first $d \geq 3$:
Let $x \in \overline{\Omega}$ and $0 < r_0 < 1$, then it holds true for all $y \in \Omega$ with $|x - y| < r_0$

$$\begin{aligned}\left| \frac{1}{d(d-2)\alpha(d)} \frac{1}{|x-y|^{d-2}} \;\; (\Delta_y \psi_\eta(y) - \Delta_y v_1(y)) \right| \\ &\leq \frac{1}{d(d-2)\alpha(d)} \frac{1}{|x-y|^{d-2}} \sup_{y \in B_{r_0}(x)} |\Delta_y \psi_\eta(y) - \Delta_y v_1(y)| \\ &\leq \frac{C}{|x-y|^{d-2}}\end{aligned}$$

where $B_{r_0}(x) = \{y \in \mathbb{R}^d : |x - y| < r_0\}$ and

$$C > \frac{1}{d(d-2)\alpha(d)} \|\Delta_y \psi_\eta(y) - \Delta_y v_1(y)\|_{L^\infty(B_{r_0}(x))}.$$

For this proof of existence it is enough if we show for $0 < r_1 < r_2$

$$\frac{1}{d(d-2)\alpha(d)} \int\limits_{r_1 < |x-y| < r_2} \left| \frac{1}{|x-y|^{d-2}} \, (\Delta_y \psi_\eta(y) - \Delta_y v_1(y)) \right| \; dy \to 0, \;\text{ for } r_2 \to 0,$$

since the rest of the integral is bounded. We get for $0 < r_1 < r_2$

$$\begin{aligned}\frac{1}{d(d-2)\alpha(d)} \int\limits_{r_1 < |x-y| < r_2} \left| \frac{1}{|x-y|^{d-2}} \, (\Delta_y \psi_\eta(y) - \Delta_y v_1(y)) \right| \; dy \\ &< \int\limits_{r_1 < |x-y| < r_2} \frac{C}{|x-y|^{d-2}} \; dy.\end{aligned}$$

By applying substitution with spherical coordinates on the integral, we get with $\phi_1, .., \phi_{d-2} \in [0, \pi)$ and $\phi_{d-1} \in [0, 2\pi)$

$$\int\limits_{r_1<|x-y|<r_2} \frac{C}{|x-y|^{d-2}}\, dy$$
$$= \int\limits_{r_1}^{r_2} \int\limits_{0}^{\pi} ... \int\limits_{0}^{2\pi} \frac{C}{r^{d-2}} r^{d-1} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) ... \sin(\phi_{d-1})\, d\phi_{d-1} \,...d\phi_1\, dr.$$

With the theorem of Fubini and $\alpha(d)$ as the volume of the unit ball in $\mathbb{R}^d$, we get

$$\begin{aligned}
&\int\limits_{r_1}^{r_2} \int\limits_{0}^{\pi} ... \int\limits_{0}^{2\pi} \frac{C}{r^{d-2}} r^{d-1} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) ... \sin(\phi_{d-1})\, d\phi_{d-1} \,...d\phi_1\, dr \\
&= C \int\limits_{r_1}^{r_2} r\, dr \underbrace{\int\limits_{0}^{\pi} ... \int\limits_{0}^{2\pi} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) ... \sin(\phi_{d-1})\, d\phi_{d-1} \,...d\phi_1}_{=\alpha(d)} \\
&= C\alpha(d) \left( \frac{r_2^2}{2} - \frac{r_1^2}{2} \right) \\
&\to 0, \text{ for } r_2 \to 0.
\end{aligned}$$

Therefore the integral exists.
Now consider $d = 2$:

$$\begin{aligned}
\left| -\frac{1}{2\pi} \log|x-y| \ (\Delta_y \psi_\eta(y) - \Delta_y v_1(y)) \right| &\leq \frac{1}{2\pi} \log|x-y| \sup_{y \in B_{r_0}(x)} |\Delta_y \psi_\eta(y) - \Delta_y v_1(y)| \\
&\leq C \log|x-y|.
\end{aligned}$$

with

$$C > \frac{1}{2\pi} \|\Delta_y \psi_\eta(y) - \Delta_y v_1(y)\|_{L^\infty(B_{r_0}(x))}.$$

We get with the same reasoning as above

$$\begin{aligned}\int\limits_{r_1<|x-y|<r_2} C\log|x-y|\;dy &= \int\limits_{r_1}^{r_2}\int\limits_{0}^{2\pi} C\log r\; r\; d\phi\; dr\\ &= 2\pi C\int\limits_{r_1}^{r_2}\log r\; r\; dr\\ &= 2\pi C\left[\frac{1}{4}r^2(2\log(r)-1)\right]_{r=r_1}^{r_2}\\ &= \pi C\left[r_2^2(\log(r_2)-1)-r_1^2(\log(r_1)-1)\right].\end{aligned}$$

With L'Hopital's rule we can easily see, that the last term tends to zero as $r_2 \to 0$. Therefore the integral exists. $\square$

## 3.2. An integral equation for $\tilde{S}_q\psi_\eta$

In this chapter we derive an equation for the DtN operator $\tilde{S}_q$ and discuss the solvability of this equation. For this let $\eta \in \mathbb{C}^d\backslash\{0\}$ with $\eta\cdot\eta = 0$. In the beginning we have the boundary data of $\psi_\eta$ and therefore as well for $v_1$. Now we want to transform the integral so that we just need the boundary data as input to solve an integral equation to get $\tilde{S}_q\psi_\eta$.
Consider the second Green's formula

$$\int\limits_\Omega \Delta u\; v\; dx - \int\limits_\Omega u\;\Delta v\; dx = \int\limits_\Gamma \frac{\partial u}{\partial n}\; v\; ds - \int\limits_\Gamma u\;\frac{\partial v}{\partial n}\; ds. \tag{3.9}$$

Let $u = G_\eta$ and $v = \psi_\eta - v_1$. Then we get for $x \in \Gamma$

$$\begin{aligned}\psi_\eta(x) &= e^{i\eta\cdot x} - \int\limits_\Omega G_0(x,y)\;\left(\Delta_y\psi_\eta(y) - \Delta_y v_1(y)\right)\; dy\\ &\overset{(3.9)}{=} e^{i\eta\cdot x} + \int\limits_\Gamma \frac{\partial}{\partial n_y}\left(G_0(x,y)\right)\left(\psi_\eta(y)-v_1(y)\right)\; ds_y\\ &\quad - \int\limits_\Gamma G_0(x,y)\frac{\partial(\psi_\eta - v_1)}{\partial n_y}(y)\; ds_y\\ &\quad - \int\limits_\Omega \Delta_y\left(G_0(x,y)\right)\left(\psi_\eta(y)-v_1(y)\right)\; dy\end{aligned}$$

Since $v_1$ and $\psi_\eta$ have the same boundary data on $\Gamma$ and $G_0(x,y)$ solves the Laplace equation for $x \neq y$, we get that for $x \in \Gamma$ and $\eta \in \mathbb{C}^d \backslash \{0\}$ with $\eta \cdot \eta = 0$

$$\psi_\eta(x) = e^{i\eta\cdot x} - \int_\Gamma G_0(x,y) \frac{\partial(\psi_\eta - v_1)}{\partial n_y}(y)\ ds_y.$$

With the definition of $\tilde{S}_q$ we have

$$\psi_\eta(x) = e^{i\eta\cdot x} - \int_\Gamma G_0(x,y) \left(\tilde{S}_q - \tilde{S}_0\right) \psi_\eta(y)\ ds_y \tag{3.10}$$

for $x \in \Gamma$, where $\tilde{S}_0$ is the DtN operator for the Laplace equation. Hence we want to solve this integral equation to get $\tilde{S}_q$. As the next step we look at the solvability of this equation.

### 3.2.1. Solvability

In this subsection we only consider $d = 3$ and $d = 2$. Remember the assumptions in the beginning:

- $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain
- $\sigma \in C^2(\overline{\Omega})$
- There exists a $\sigma_0 \in \mathbb{R}_+$ such that $\sigma_0^{-1} \geq \sigma(x) \geq \sigma_0 > 0$ for all $x \in \overline{\Omega}$.
- Let $\sigma$ be known on the boundary $\Gamma := \partial\Omega$.

In $d = 2$ we have additionally the restriction $diam(\Omega) < 1$. Now we look at the solvability of the integral equation (3.10). For the DtN operator $\tilde{S}_0$ we have on $\Gamma$

$$\tilde{S}_0 = V^{-1}\left(\frac{1}{2}I + K\right), \tag{3.11}$$

for details see [11, p.148], where $I$ is the identity, $V$ is the single layer potential operator and $K$ the double layer potential operator defined as follows:

$$V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$
$$K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

with

$$Vv(x) := \int_\Gamma G_0(x,y)v(y)\ ds_y,$$
$$Kw(x) := \int_\Gamma \frac{\partial}{\partial n_y} G_0(x,y)w(y)\ ds_y$$

where $x \in \Gamma$, $w \in H^{1/2}(\Gamma)$, $v \in H^{-1/2}(\Gamma)$ and $G_0(x,y)$ is the fundamental solution of the Laplace equation with

$$G_0(x,y) = \begin{cases} -\frac{1}{2\pi}\log|x-y| & \text{if } d = 2, \\ \frac{1}{4\pi}\frac{1}{|x-y|} & \text{if } d = 3. \end{cases}$$

We know that $V$ and $K$ are linear and that

$$\begin{aligned} \|Kw\|_{H^{1/2}(\Gamma)} &\le c_2^K \|w\|_{H^{1/2}(\Gamma)} \\ \|Vv\|_{H^{-1/2}(\Gamma)} &\le c_2^V \|v\|_{H^{1/2}(\Gamma)} \end{aligned}$$

hold true for all $w \in H^{1/2}(\Gamma)$ and $v \in H^{-1/2}(\Gamma)$, see [11, Thm. 6.34, p.154] for more details. If we use the theorem of Fubini, see [5, Chapter 5.2., p.175 et seq.], we can easily see that $V$ is self-adjoint. It holds true as well that $V$ is elliptic

$$\langle Vv, v\rangle_\Gamma \ge c_1^V \|v\|^2_{H^{-1/2}(\Gamma)},$$

see [11, Thm. 6.23, p. 143] for $d = 2$ with the restriction, that $diam(\Omega) < 1$ and [11, Thm. 6.22, p.141] for $d = 3$. Therefore $V^{-1}$ is bounded and elliptic, see [11, p.144] and [11, Lem. 3.5, p.47] for the proof. When we look at the integral equation

$$\psi_\eta(x) = e^{i\eta\cdot x} - \int_\Gamma G_0(x,y)\left(\tilde{S}_q - \tilde{S}_0\right)\psi_\eta(y)\, ds_y$$

for $x \in \Gamma$, we can rewrite it as

$$\psi_\eta(x) + V\tilde{S}_q\psi_\eta(x) - V\tilde{S}_0\psi_\eta(x) = e^{i\eta\cdot x}.$$

Let $e_\eta(x) := e^{i\eta\cdot x}$. With (3.11) we get

$$\left(\frac{1}{2}I - K\right)\psi_\eta + V\tilde{S}_q\psi_\eta = e_\eta$$

on $\Gamma$. If we take into consideration that $\psi_\eta = \sigma^{1/2}g$ on $\Gamma$, see the equation (3.1), we get

$$V\tilde{S}_q\psi_\eta = e_\eta - \left(\frac{1}{2}I - K\right)\sigma^{1/2}g \tag{3.12}$$

on $\Gamma$. We had as assumption for $\sigma$ is bounded by $\sigma_0^{-1}$ in $\overline{\Omega}$ and therefore $\sigma^{1/2}g \in H^{1/2}(\Gamma)$ for $g \in H^{1/2}(\Gamma)$ due to the norm equivalence in Lem. A.17. Since $\Omega$ is bounded $e_\eta$ is as well in $H^{1/2}(\Gamma)$. Hence the right hand side of this equation is in

$H^{1/2}(\Gamma)$. If we look at the equation in the $L^2(\Gamma)$ duality pairing with $w \in H^{-1/2}(\Gamma)$, we get the variational formulation

$$\int_\Gamma V(\tilde{S}_q\psi_\eta)\ w\ ds_y = \int_\Gamma e_\eta\ w\ ds_y - \int_\Gamma \left(\frac{1}{2}I - K\right)(\sigma^{1/2}g)\ w\ ds_y$$

for all $w \in H^{-1/2}(\Gamma)$. Since $V$ is elliptic and bounded, see the argumentation above, we know that this variational formulation is uniquely solvable due to the lemma of Lax Milgram B.3.

# 4. Solution of the inverse problem

## 4.1. Computation of $q$

Let $d = 2$ or $d = 3$ and $\eta \in \mathbb{C}^d \backslash \{0\}$ with $\eta \cdot \eta = 0$. In the last three chapters we considered $\psi_\eta$ which solves

$$\begin{aligned} (-\Delta + \tilde{q})\psi_\eta &= 0 && \text{in } \mathbb{R}^d, \\ \psi_\eta &= \sigma^{1/2} g && \text{on } \Gamma, \\ \psi_\eta(x) &= e^{i\eta\cdot x} + O(\frac{1}{|x|}) && \text{for } |x| \to \infty, \end{aligned} \tag{4.1}$$

where

$$\tilde{q} = \begin{cases} \frac{\Delta\sigma^{1/2}}{\sigma^{1/2}} & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^d \backslash \overline{\Omega}. \end{cases}$$

We concluded that $\psi_\eta$ solves the equation 3.12

$$V\tilde{S}_q\psi_\eta = e_\eta - \left(\frac{1}{2}I - K\right)\sigma^{1/2} g$$

on $\Gamma$, where $K$ and $V$ are defined as in chapter 3.2.1 and $e_\eta(x) := e^{i\eta\cdot x}$. This boundary integral equation is uniquely solvable for $\tilde{S}_q\psi_\eta$ and so we are able to compute the Neumann data $\tilde{S}_q\psi_\eta$ from the boundary data of $\psi_\eta$.

With this information we now try to compute $q$ from $\tilde{S}_q\psi_\eta$. Therefore we use the second Green's identity (B.2). If we take $v \in H^2(\Omega)$ arbitrary and $u = \psi_\eta$, we get with the Schrödinger equation

$$\int_\Omega (v\ q\psi_\eta - \psi_\eta\Delta v)\ dx = \int_\Gamma \left(v\ \tilde{S}_q\psi_\eta - \sigma^{1/2} g\ \gamma_1^{int} v\right)\ ds_x.$$

On the other hand if we take $u = v_1$ with

$$\begin{aligned} -\Delta v_1 &= 0 && \text{in } \Omega, \\ v_1 &= \sigma^{1/2} g && \text{on } \Gamma, \end{aligned}$$

we get with the same Green's identity (B.2)

$$-\int_\Omega v_1 \Delta v \ dx = \int_\Gamma \left( v \ \tilde{S}_0 v_1 - \sigma^{1/2} g \ \gamma_1^{int} v \right) \ ds_x.$$

To compute this equation we have to calculate $\tilde{S}_0 v_1$ as well. For this we use the equation (3.11) and have to solve

$$V \tilde{S}_0 v_1 = \left( \frac{1}{2} I + K \right) \sigma^{1/2} g.$$

Next we subtract both integral equation from each other and get

$$\int_\Omega v \ q\psi_\eta \ dx = \int_\Gamma v \ \left( \tilde{S}_q \psi_\eta - \tilde{S}_0 v_1 \right) \ ds_x + \int_\Omega (\psi_\eta - v_1) \Delta v \ dx. \tag{4.2}$$

Since we are able to compute $\tilde{S}_q \psi_\eta$ with the equation (3.12), we can compute the right hand side as long as the second integral vanishes. Therefore we have to choose our $v$ wisely to compute q from this equation. Now we will discuss several possible $v$s to choose from and in chapter 6 we will compare the results.

### 4.1.1. Computation of $q$ with the Fourier transform

We choose $v(x) = e^{-i(\xi+\eta)\cdot x}$, where $\xi \in \mathbb{R}^d$ and $\eta \in V_\xi$ with $V_\xi := \{\eta \in \mathbb{C}^d \backslash \{0\} | \eta \cdot \eta = (\eta + \xi) \cdot (\eta + \xi) = 0\}$. Next we compute $\tilde{S}_q \psi_\eta$ accordingly. Since $v$ is harmonic, we get

$$\int_\Omega e_{-(\xi+\eta)} \ q\psi_\eta \ dx = \int_\Gamma e_{-(\xi+\eta)} \ \left( \tilde{S}_q \psi_\eta - \tilde{S}_0 v_1 \right) \ ds_x.$$

The left hand side approximates the Fourier transform $\mathcal{F}\tilde{q}(\xi)$ of $\tilde{q}$ at $\xi \in \mathbb{R}^d$ for $|\eta| \to \infty$ because

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \tilde{q}(x) \psi_\eta(x) e^{-i(\xi+\eta)\cdot x} \ dx - \int_{\mathbb{R}^d} \tilde{q}(x) e^{-i\xi\cdot x} \ dx \right| \\
&\qquad = \left| \int_\Omega q(x) \psi_\eta(x) e^{-i(\xi+\eta)\cdot x} \ dx - \int_\Omega q(x) e^{-i\xi\cdot x} \ dx \right| \\
&\qquad = \left| \int_\Omega q(x) e^{-i\xi\cdot x} \ (e^{-i\eta\cdot x} \psi_\eta(x) - 1) \ dx \right| \\
&\qquad \leq \| e^{-i\eta\cdot x} \psi_\eta(x) - 1 \|_{L^2(\Omega)} \ \| q(x) e^{-i\xi\cdot x} \|_{L^2(\Omega)} \\
&\qquad \xrightarrow[|\eta|\to\infty]{(4.1)} 0.
\end{aligned}$$

Since $\sigma \in C^2(\overline{\Omega})$ we get $q \in C^\infty(\Omega)$. Hence $\tilde{q} \in L^1(\mathbb{R}^d, \mathbb{C}) \cap L^2(\mathbb{R}^d, \mathbb{C})$ and $\mathcal{F}\tilde{q} \in L^2(\mathbb{R}^d)$ due to Thm. A.6. On the other hand we have a bound from Thm. 2.9

$$\begin{aligned} \|e^{-i\eta\cdot x}\psi_\eta - 1\|_{L^2(\Omega)} &\leq \frac{C_0}{|\eta|} \|(-\Delta + q)e^{i\eta\cdot}\|_{L^2(\Omega)}, \\ &= \frac{C_0}{|\eta|} \|qe^{i\eta\cdot}\|_{L^2(\Omega)}. \end{aligned}$$

So we get a linear convergence if $e_\eta$ is bounded. For example in the case of $\Im(\eta) \geq 0$ or $\Im(\eta)$ fixed, $e_\eta$ would be bounded.
In the end, if we want to get $\tilde{q}$, we have to take the inverse Fourier transform of

$$\mathcal{F}\tilde{q}(\xi) = \lim_{\substack{|\eta| \to \infty \\ \eta \in V_\xi}} \int_\Gamma e_{-(\xi+\eta)} \left( \tilde{S}_q \psi_\eta - \tilde{S}_0 v_1 \right) \, ds_x.$$

However we have to be careful with $d = 2$. In this case the space $V_\xi$ is finite. So we cannot take the limit.

### 4.1.2. Computation of $q$ with $G_0$

Let $y \in \mathbb{R}^d \backslash \Omega$. If we choose $v(x) = G_0(x, y)$, we get the integral equation

$$\begin{aligned} \int_\Omega G_0(x,y) \; q\psi_\eta \; dx &= \int_\Gamma G_0(x,y) \; \left( \tilde{S}_q \psi_\eta - \tilde{S}_0 v_1 \right) \; ds_x + \int_\Omega (\psi_\eta - v_1) \, \Delta G_0(x,y) \; dx \\ &= \int_\Gamma G_0(x,y) \; \left( \tilde{S}_q \psi_\eta - \tilde{S}_0 v_1 \right) \; ds_x. \end{aligned}$$

For $\eta \in \mathbb{C}^d \backslash \{0\}$ with very large absolute value $|\eta|$ and $\eta \cdot \eta = 0$ we get

$$\int_\Omega G_0(x,y) \; qe_\eta \; dx \approx \int_\Gamma G_0(x,y) \; \left( \tilde{S}_q \psi_\eta - \tilde{S}_0 v_1 \right) \; ds_x.$$

On the left side we have the Newton potential used on $qe_\eta$. We know it is bounded for $qe_\eta \in \tilde{H}^s(\Omega)$ with $s \in [-2, 0]$, see [11, p.153]. However, we can only choose $y \in \mathbb{R}^d \backslash \Omega$, since we do not know how $\psi_\eta$ is behaving inside of $\Omega$. If we still choose $y \in \Omega$ we get the term $-(\psi_\eta(y) - v_1(y))$ added on the right hand side. The only way to approximate $\psi_\eta$ in $\Omega$ we know of is to use $e_\eta$ instead. We will see in chapter 6, that this approximation does not work very well.

### 4.1.3. Computation of $q$ with basis functions

Let $y \in \mathbb{R}^d$. If we choose $v(x) = \phi_i^1(x) \in S_h^1(\Omega)$ where $S_h^1(\Omega)$ is the space of piecewise linear and continuous functions. Then we get the integral equation

$$\begin{aligned}\int\limits_{\Omega} \phi_i^1 \; q\psi_\eta \; dx &= \int\limits_{\Gamma} \phi_i^1 \; \left(\tilde{S}_q\psi_\eta - \tilde{S}_0 v_1\right) \; ds_x + \int\limits_{\Omega} \left(\psi_\eta - v_1\right) \Delta\phi_i^1 \; dx \\ &= \int\limits_{\Gamma} \phi_i^1 \; \left(\tilde{S}_q\psi_\eta - \tilde{S}_0 v_1\right) \; ds_x.\end{aligned}$$

For $\eta \in \mathbb{C}^d \backslash \{0\}$ with a very large absolute value $|\eta|$ and $\eta \cdot \eta = 0$, we get

$$\int\limits_{\Omega} \phi_i^1 \; qe_\eta \; dx \approx \int\limits_{\Gamma} \phi_i^1 \; \left(\tilde{S}_q\psi_\eta - \tilde{S}_0 v_1\right) \; ds_x.$$

We will see in chapter 5 that this equation is uniquely solvable if we choose the discretization of $q$ and the function $\phi_i^1$ wisely. However we will see in chapter 6 that this method does not yield very good results either.

## 4.2. Computation of $\sigma$

If we know $q$ in $\Omega$, there is nothing to hold us back from computing $\sigma$, because

$$(-\Delta + q)\sigma^{1/2} = 0 \text{ in } \Omega$$

since $q = \frac{\Delta\sigma^{1/2}}{\sigma^{1/2}}$. The boundary data of $\sigma$ is known as a condition to the whole inverse problem. Let this boundary data be $\sigma_\Gamma$ and $\tilde{\sigma}$ the extension of $\sigma_\Gamma$ in $\Omega$. Then we try to find a $\sigma_0^{1/2} = \sigma^{1/2} - \tilde{\sigma}^{1/2} \in H_0^1(\Omega)$ with

$$\int\limits_{\Omega} \nabla\sigma_0^{1/2} \cdot \nabla v \; dx + \int\limits_{\Omega} q \; \sigma_0^{1/2} v \; dx = -\int\limits_{\Omega} \nabla\tilde{\sigma}^{1/2} \cdot \nabla v \; dx - \int\limits_{\Omega} q \; \tilde{\sigma}^{1/2} v \; dx \quad \forall v \in H_0^1(\Omega).$$

This equation is uniquely solvable due to the work we did in chapter 1.

# 4.3. The algorithm

We will be working with the equation

$$\begin{aligned}(-\Delta + \tilde{q})\psi_\eta &= 0 && \text{in } \mathbb{R}^d, \\ \psi_\eta &= \sigma^{1/2} g && \text{on } \Gamma, \\ \psi_\eta(x) &= e^{i\eta\cdot x} + O(\frac{1}{|x|}) && \text{for } |x| \to \infty.\end{aligned} \tag{4.3}$$

In the beginning we have the boundary data $g$ and $\sigma_\Gamma$ as the boundary data of $\sigma$ on $\Gamma$. Our goal is to compute $\sigma$ by knowing the boundary data $g$ and $\sigma_\Gamma$. The three steps to that goal are

1. Computing $\tilde{S}_q\psi_\eta$ and $\tilde{S}_0 v_1$.
2. Computing $q$.
3. Computing $\sigma$.

## 4.3.1. Compute the Neumann data

Now we want to compute

- the Neumann data $t := \tilde{S}_q\psi_\eta$.
- the Neumann data $t_0 := \tilde{S}_0 v_1 = \frac{\partial v_1}{\partial n}$ .

First we have to solve

$$Vt = e_\eta - (\frac{1}{2}I - K)\sigma_\Gamma^{1/2} g \qquad \text{on } \Gamma.$$

If we put this equation into the duality pairing with $w \in H^{1/2}(\Gamma)$ we get the variational formulation to find $t \in H^{-1/2}(\Gamma)$ with

$$\int_\Gamma V(t)\ w\ ds_y = \int_\Gamma (-\frac{1}{2}I + K)(\sigma_\Gamma^{1/2} g)\ w\ ds_y + \int_\Gamma e_\eta w\ ds_y \qquad \forall w \in H^{-1/2}(\Gamma). \tag{4.4}$$

This equation is uniquely solvable, because $V$ is bounded and elliptic, for more details see the discussion we had in chapter 3.2.1. Next we compute

$$Vt_0 = (\frac{1}{2}I + K)\sigma_\Gamma^{1/2} g \qquad \text{on } \Gamma.$$

We use again the duality pairing and get to the variational formulation: Find $t_0 \in H^{-1/2}(\Gamma)$ which solves

$$\int_\Gamma V(t_0)\ w\ ds_y = \int_\Gamma (\frac{1}{2}I + K)(\sigma_\Gamma^{1/2} g)\ w\ ds_y \qquad \forall w \in H^{-1/2}(\Gamma). \tag{4.5}$$

This equation is as well uniquely solvable.

### 4.3.2. Compute $q$

With this Neumann data we can solve

$$\int_\Omega v\ q\psi_\eta\ dx = \int_\Gamma v\ \ (t - t_0)\ \ ds_x + \int_\Omega (\psi_\eta - v_1)\,\Delta v\ dx.$$

by using

1. $v_1(x) = e^{-i(\xi+\eta)\cdot x}$ with $\xi \in \mathbb{R}^d$ and $\eta \in V_\xi$.
2. $v_2(x) = G_0(x, y)$ with $y \in \mathbb{R}^d \backslash \Omega$.
3. $v_3(x) = \phi_i^1(x)$ with $\phi_i^1 \in S_h^1(\Omega)$.

In chapter 6 we will compare the results for the different $v_i$, $i = 1, 2, 3$.

### 4.3.3. Compute $\sigma$

Afterwards we solve

$$\begin{aligned} (-\Delta + q)\sigma^{1/2} &= 0 && \text{in } \Omega, \\ \sigma^{1/2} &= \sigma_\Gamma^{1/2} && \text{on } \Gamma. \end{aligned}$$

with $\sigma_\Gamma$ as the given boundary data of $\sigma$ on $\Gamma$. We use a continuation $\tilde{\sigma}$ of the boundary data. Then we try to find a $\sigma_0^{1/2} = \sigma^{1/2} - \tilde{\sigma}^{1/2} \in H_0^1(\Omega)$ with

$$\int_\Omega \nabla\sigma_0^{1/2} \cdot \nabla v\ dx + \int_\Omega q\sigma_0^{1/2} v\ dx = -\int_\Omega \nabla\tilde{\sigma}^{1/2} \cdot \nabla v\ dx - \int_\Omega q\tilde{\sigma}^{1/2} v\ dx \quad \forall v \in H_0^1(\Omega). \tag{4.6}$$

### 4.3.4. Summary

We start with the boundary data $g$ and $\sigma_\Gamma$. First we solve the two variational formulations

$$\begin{aligned} \int_\Gamma V(t)\ w\ ds_y &= -\int_\Gamma (\frac{1}{2}I - K)(\sigma_\Gamma^{1/2} g)\ w\ ds_y + \int_\Gamma e_\eta w\ ds_y && \forall w \in H^{-1/2}(\Gamma), \\ \int_\Gamma V(t_0)\ w\ ds_y &= \int_\Gamma (\frac{1}{2}I + K)(\sigma_\Gamma^{1/2} g)\ w\ ds_y && \forall w \in H^{-1/2}(\Gamma) \end{aligned}$$

to get the Neumann data $t, t_0 \in H^{-1/2}(\Gamma)$. Next we solve

$$\int_\Omega v\ q\psi_\eta\ dx = \int_\Gamma v\ \ (t - t_0)\ \ ds_x + \int_\Omega (\psi_\eta - v_1)\,\Delta v\ dx$$

by using one of the methods which were explained above. In the end we solve

$$\int\limits_{\Omega} \nabla\sigma_0^{1/2} \cdot \nabla v \; dx + \int\limits_{\Omega} q\sigma_0^{1/2} v \; dx = -\int\limits_{\Omega} \nabla\tilde{\sigma}^{1/2} \cdot \nabla v \; dx - \int\limits_{\Omega} q\tilde{\sigma}^{1/2} v \; dx \quad \forall v \in H_0^1(\Omega)$$

to get the desired dielectric conductivity $\sigma$.

# 5. The reconstruction problem

## 5.1. Introduction

Let $d = 2$. Consider $\Omega = B((0.75, 0.75), 0.25\sqrt{2})$. If we try to solve the reconstruction problem we first take $\sigma$ as a given function. In this case we use

$$\sigma_1(x) = \begin{cases} (1 + 10\exp(\frac{1}{|x-x_m|^2 - \frac{8}{9}r^2}))^2 & \text{if } |x - x_m|^2 < \frac{8}{9}r^2, \\ 1 & \text{otherwise,} \end{cases}$$

where $r = 0.25\sqrt{2}$ and $x_m = (0.75, 0.75)$. The function $\sigma_1$ full fills our assumptions for $\sigma$ since $\sigma \in C^2(\overline{\Omega})$.
To be able to use the algorithm of chapter four we first need to get the boundary data of $u$. For this we solve

$$\int_\Omega \sigma_1 \nabla u \cdot \nabla v \; dx = \int_\Gamma \frac{\partial u}{\partial n} v \; ds_y \qquad \forall v \in H^1(\Omega).$$

However this equation is not uniquely solvable without the boundary data $g$ of $u$, so we need another equation. We get from the equivalence of the DtN operators (1.11)

$$\tilde{S}_q(\gamma_0^{int}\psi_\eta) = S_\sigma(\gamma_0^{int}u),$$

and that

$$g = \gamma_0^{int}u = \gamma_0^{int}\psi_\eta$$

holds true, since $\sigma_1 \equiv 1$ in a neighborhood of $\Gamma$. If we put this into the equation (3.12), we have

$$Vt = e_\eta - (\frac{1}{2}I - K)g \qquad \text{on } \Gamma,$$

with $t := \tilde{S}_q\psi_\eta = S_\sigma u$. So we need to solve the variational formulation

$$\int_\Gamma V(t) \; w \; ds_y = \int_\Gamma e_\eta w \; ds_y - \int_\Gamma (\frac{1}{2}I - K)(g) \; w \; ds_y \qquad \forall w \in H^{-1/2}(\Gamma),$$

as well as the variational formulation for $u$. After that we go trough the algorithm as explained in the previous chapter. In short we start by solving the two variational equations

$$\begin{aligned} \int_\Omega \sigma_1 \nabla u \cdot \nabla v \; dx &= \int_\Gamma t \; v \; ds_y && \forall v \in H^1(\Omega), \\ \int_\Gamma V(t) \; w \; ds_y &= \int_\Gamma e_\eta w \; ds_y - \int_\Gamma (\frac{1}{2} I - K)(g) \; w \; ds_y && \forall w \in H^{-1/2}(\Gamma). \end{aligned}$$

With the data we get from solving them, we compute $t_0$ by solving

$$\int_\Gamma V(t_0) \; w \; ds_y = \int_\Gamma (\frac{1}{2} I + K)(g) \; w \; ds_y \qquad \forall w \in H^{-1/2}(\Gamma).$$

Afterwards we solve the equation

$$\int_\Omega v \; q\psi_\eta \; dx = \int_\Gamma v \;\; (t - t_0) \;\; ds_x + \int_\Omega (\psi_\eta - v_1) \, \Delta v \; dx$$

with different $v$. The last step is to compute $\sigma$. Since $\sigma_1 \equiv 1$ on $\Gamma$, we use $\mathbf{1}_\Gamma$ as the continuation of the boundary data. For this we try to find a $\sigma_0^{1/2} = \sigma^{1/2} - \mathbf{1}_\Gamma \in H_0^1(\Omega)$ with

$$\int_\Omega \nabla \sigma_0^{1/2} \cdot \nabla v \; dx + \int_\Omega q \sigma_0^{1/2} v \; dx = - \int_\Omega \nabla \mathbf{1}_\Gamma \cdot \nabla v \; dx - \int_\Omega q \mathbf{1}_\Gamma v \; dx \quad \forall v \in H_0^1(\Omega).$$

## 5.2. Discretization

Now we need to solve the two equations:

$$\begin{aligned} \int_\Omega \sigma_1 \nabla u \cdot \nabla v \; dx &= \int_\Gamma t \; v \; ds_y && \forall v \in H^1(\Omega), \\ \int_\Gamma V(t) \; w \; ds_y &= \int_\Gamma e_\eta w \; ds_y - \int_\Gamma (\frac{1}{2} I - K)(g) \; w \; ds_y && \forall w \in H^{-1/2}(\Gamma). \end{aligned}$$

To discretize this problem we use as the solution space for $u$ the space of piecewise linear and continuous functions $S_{h_u}^1(\Omega)$ with the basis functions $\{\phi_i^1\}_{i=1}^N$ and $h_u$ as the step in space. For the solution space for $t$ we use the space of piecewise constant functions $S_{h_u}^0(\Gamma)$ with the basis functions $\{\phi_k^0\}_{k=1}^M$ and as well the space step $h_u$. For the test function space we choose $S^1_{h_u}(\Omega)$ for the integral equation with $u$ and $S^0_{h_u}(\Gamma)$ for the boundary integral equation with $t$. From this discretization we get the equation

$$\begin{pmatrix} A_{II} & A_{\Gamma I} & \\ A_{I\Gamma} & A_{\Gamma\Gamma} & -M_h^\top \\ & (\frac{1}{2}M_h - K_h) & V_h \end{pmatrix} \begin{pmatrix} u_I \\ u_\Gamma \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_\eta \end{pmatrix}, \tag{5.1}$$

where

$$\begin{aligned} u[j] &:= \langle u, \phi_j^1 \rangle_{L^2(\Omega)} && \forall j = 1, ..., N, \\ t[l] &:= \langle \frac{\partial u}{\partial n}, \phi_l^0 \rangle_\Gamma && \forall l = 1, ..., M, \\ f_\eta[k] &:= \langle e_\eta, \phi_k^0 \rangle_\Gamma && \forall k = 1, ..., M, \\ A[i,j] &:= \langle \sigma \nabla \phi_j^1, \nabla \phi_i^1 \rangle_{L^2(\Omega)} && \forall i,j = 1, ..., N, \\ V_h[k,l] &:= \langle V \phi_l^0, \phi_k^0 \rangle_\Gamma && \forall k,l = 1, ...M. \end{aligned}$$

Let $\{\phi_i^1\}_{i=1}^N$ be ordered by the basis functions in $\mathring{\Omega}$ and functions on $\Gamma$. Choose $1 \leq P \leq N$ so, that the first P basis functions are the functions of points in $\mathring{\Omega}$ and $(N-P) = M$. Hence we can write

$$\begin{aligned} M_h[k, j-P] &:= \langle \phi_j^1, \phi_k^0 \rangle_\Gamma && \forall j = P+1, ...N, \forall k = 1, ..., M, \\ K_h[k, j-P] &:= \langle K\phi_j^1, \phi_k^0 \rangle_\Gamma && \forall j = P+1, ...N, \forall k = 1, ..., M, \\ u_I[j] &:= u[j] && \forall j = 1, ..., P, \\ u_\Gamma[j-P] &:= u[j] && \forall j = P+1, ..., N, \\ A_{II}[i,j] &:= A[i,j] && \forall i,j = 1, ..., P, \\ A_{\Gamma I}[i, j-P] &:= A[i,j] && \forall i = 1, ..., P, \forall j = P+1, ..., N, \\ A_{I\Gamma}[i-P, j] &:= A[i,j] && \forall j = 1, ..., P, \forall i = P+1, ..., N, \\ A_{\Gamma\Gamma}[i-P, j-P] &:= A[i,j] && \forall i,j = P+1, ..., N. \end{aligned}$$

Since we know now the boundary data $g = u_\Gamma$, we are able to compute $t_0$ by solving

$$\int_\Gamma V(t_0)\ w\ ds_y = \int_\Gamma (\frac{1}{2}I + K)(g)\ w\ ds_y \qquad \forall w \in H^{-1/2}(\Gamma).$$

To solve this variational formulation we take $S^0_{h_u}(\Gamma)$ as test function and solution space. With this we get the equation

$$V_h t_0 = (\frac{1}{2}M_h + K_h) u_\Gamma. \tag{5.2}$$

where

$$t_0[l] := \langle t_0, \phi_l^0 \rangle_\Gamma \qquad \forall l = 1, ..., M.$$

With all the data we have computed, we now solve

$$\int_\Omega v\ q\psi_\eta\ dx = \int_\Gamma v\ \ (t-t_0)\ \ ds_x + \int_\Gamma \Delta v\ \ (\psi_\eta - v_1)(x)dx.$$

We will discuss several methods to get q by solving this equation in chapter 5.2.1, 5.2.2, 5.2.3 and 5.2.4.
After we have computed $q$, we solve the following variational formulation to find a $\sigma_0^{1/2} = \sigma^{1/2} - \mathbf{1}_\Gamma \in H_0^1(\Omega)$ with

$$\int_\Omega \nabla\sigma_0^{1/2}\cdot\nabla v\ dx + \int_\Omega q\sigma_0^{1/2} v\ dx = -\int_\Omega \nabla\mathbf{1}_\Gamma\cdot\nabla v\ dx - \int_\Omega q\mathbf{1}_\Gamma v\ dx \qquad \forall v\in H_0^1(\Omega).$$

For this we choose $S^1_{h_\sigma}(\overset{\circ}{\Omega})$, the space of piecewise linear and continuous functions which have support in $\Omega$ and the space step $h_\sigma$, as the solution and the test function space with the basis functions $\{\phi_i^1\}_{i=1}^{N_\sigma}$, $N_\sigma \le N$. Hence we get

$$\tilde{A}_{II}\underline{\sigma}_0^{1/2} = \tilde{f}, \tag{5.3}$$

where

$$\begin{aligned}
\tilde{A}_{II}[i,j] &:= \int_\Omega \nabla\phi_j^1\cdot\nabla\phi_i^1\ dx + \int_\Omega q\phi_j^1\phi_i^1\ dx && \forall i,j = 1,...,P_\sigma\\
\tilde{A}_{\Gamma I}[i,j-P_\sigma] &:= \int_\Omega \nabla\phi_j^1\cdot\nabla\phi_i^1\ dx + \int_\Omega q\phi_j^1\phi_i^1\ dx && \forall i = 1,...,P_\sigma, \forall j = P_\sigma+1,...,N_\sigma,\\
\tilde{f} &:= -\tilde{A}_{\Gamma I}\mathbf{1},
\end{aligned}$$

and the first $P_\sigma$ basis functions are the functions of inner points of $\Omega$.

### 5.2.1. Compute $q$ with the Fourier transform

For arbitrary $\xi\in\mathbb{R}^2$ and $\eta \in V_\xi := \{\eta\in\mathbb{C}^d\backslash\{0\} : \eta\cdot\eta = (\xi+\eta)\cdot(\xi+\eta) = 0\}$, it holds true that

$$\int_\Omega e_{-(\xi+\eta)}\ q\psi_\eta\ dx = \int_\Gamma e_{-(\xi+\eta)}\ \left(\tilde{S}_q\psi_\eta - \tilde{S}_0 v_1\right)\ ds_x.$$

For $\eta$ with very large absolute value $|\eta|$ it holds true that

$$\mathcal{F}\tilde{q}(\xi) \approx \int_\Gamma e_{-(\xi+\eta)}\ \ (t-t_0)\ \ ds_x.$$

We approximate the integration domain of the inverse Fourier transform by a $\tilde{\Omega} \subset \mathbb{R}^2$ big enough and it should hold true that $\Omega \subset \tilde{\Omega}$. For $\tilde{\Omega}$ we choose $B((0.75, 0.75), 10.25\sqrt{2})$ in this example. In chapter D we explain this choice and compare different $\tilde{\Omega}$ based on some results of [6].

Now consider $q_h \in S^1_{h_q}(\overset{\circ}{\Omega})$ with the basis functions $\{\phi^1_j\}_{j=1}^{N_q}$ with $N_\sigma \leq N_q \leq N$ of $\Omega$ and the supporting points $\{x_j\}_{k=1}^{N_q}$. Then we get

$$q_h(x) = \sum_{j=1}^{N_q} q_h^j \phi^1_j(x),$$

where $q_h^j \approx q(x_j)$. We will look at the Fourier transform $\hat{q}$ of $\tilde{q}$ in the space $S^1_{\tilde{h}}(\tilde{\Omega})$ with the basis functions $\{\phi^1_k\}_{k=1}^{\tilde{N}}$ and the supporting points $\{\xi_k\}_{k=1}^{\tilde{N}}$ so that

$$\hat{q}_h(\xi) = \sum_{k=1}^{\tilde{N}} \hat{q}_h^k \phi^1_k(\xi).$$

with $\hat{q}_h^k \approx \hat{q}(\xi_k)$. Now we have to do the following step for every $k = 1, ..., \tilde{N}$:

Take $\eta \in V_{\xi_k} = \{\eta \in \mathbb{C}^d \backslash \{0\} : \eta \cdot \eta = (\xi_k + \eta) \cdot (\xi_k + \eta) = 0\}$ with maximum absolute value of all $\eta$ in $V_{\xi_k}$. In chapter C we discuss how $V_{\xi_k}$ looks like in more detail. For this example we choose $\eta = x + iy$, $x, y \in \mathbb{R}^2$, as follows:

If the first component $\xi_{k,1}$ of $\xi_k$ is not zero, we choose

$$\begin{aligned}
x_2 &= \frac{-a}{b}, \\
y_2 &= \sqrt{\frac{1}{b}((\frac{\xi_{k,1}^2 + \xi_{k,2}^2}{2\xi_{k,1}} - \frac{a}{b}\frac{\xi_{k,2}}{\xi_{k,1}})^2 + \frac{a^2}{b^2})}, \\
x_1 &= \frac{\xi_{k,1}^2 + \xi_{k,2}^2 + 2x_2\xi_{k,2}}{-2\xi_{k,1}}, \\
y_1 &= \frac{-\xi_{k,2} y_2}{\xi_{k,1}},
\end{aligned}$$

where

$$\begin{aligned}
a &:= \frac{\xi_{k,1}^2 + \xi_{k,2}^2}{2\xi_{k,1}} \frac{\xi_{k,2}}{\xi_{k,1}}, \\
b &:= (1 + \frac{\xi_{k,2}^2}{\xi_{k,1}^2}).
\end{aligned}$$

For $\xi_{k,1} = 0$ and $\xi_{k,2} \neq 0$ we choose

$$\begin{aligned} x_2 &= -\frac{\xi_{k,2}}{2}, \\ y_2 &= 0, \\ x_1 &= 0, \\ y_1 &= -\frac{\xi_{k,2}}{2}. \end{aligned}$$

If $\xi_i = 0$, we take

$$\begin{aligned} x_2 &= 10, \\ y_2 &= -10, \\ x_1 &= 10, \\ y_1 &= 10. \end{aligned}$$

These $\eta$ are in $V_\xi$ and have maximal norm except for the last case with $\xi_k = 0$. For more details about the space $V_\xi$ see [6].

In the end we compute

$$\hat{q}_h^k = \sum_{l=1}^{M} (t[l] - t_0[l]) \int\limits_{\Gamma_l} e_{-(\xi_k+\eta)} \; ds_x.$$

When we are done computing $\hat{q}_h^k$ for $k = 1, ..., \tilde{N}$, we can approximate

$$q_h^j = \frac{1}{4\pi^2} \int\limits_{\tilde{\Omega}} \hat{q}_h(\xi) e^{i\xi \cdot x_j} \; d\xi$$

for $j = 1, .., N_q$. In chapter six we will look at the error of this approximation and compare the results.

### 5.2.2. Compute $q$ with $G_0$

Again we start with the integral equation

$$\int\limits_{\Omega} v \; q\psi_\eta \; dx = \int\limits_{\Gamma} v \; \left(\tilde{S}_q\psi_\eta - \tilde{S}_0 v_1\right) \; ds_x + \int\limits_{\Omega} (\psi_\eta - v_1) \, \Delta v \; dx.$$

For $\eta \in \mathbb{C}^d \backslash \{0\}$ with very large absolute value $|\eta|$ and $\eta \cdot \eta = 0$ we get

$$\int\limits_{\Omega} v \; qe_\eta \; dx \approx \int\limits_{\Gamma} v \; \left(\tilde{S}_q\psi_\eta - \tilde{S}_0 v_1\right) \; ds_x + \int\limits_{\Omega} (\psi_\eta - v_1) \, \Delta v \; dx.$$

Let $\{x_k\}_{k=1}^{N_q}$ be the inner supporting points of $\Omega$, where $N_\sigma \leq N_q \leq N$. Furthermore we assume that $q^h \in S^1_{h_q}(\overset{\circ}{\Omega})$, which is the space of piecewise linear functions with support in $\Omega$ and the space step $h_q$. Let $\{\phi_k^1\}_{k=1}^{N_q}$ be the basis of $S^1_{h_q}(\overset{\circ}{\Omega})$. We will break this problem into two problems. First we consider $w := qe_\eta$ and solve

$$\int_\Omega v\ w\ dx = \int_\Gamma v\ \left(\tilde{S}_q\psi_\eta - \tilde{S}_0 v_1\right)\ ds_x + \int_\Omega (\psi_\eta - v_1)\,\Delta v\ dx.$$

After that we will solve

$$\int_\Omega q e_\eta \phi_j^1\ dx = \int_\Omega w\phi_j^1\ dx$$

for $j = 1,..,N_q$. Let us assume that $w \in S^1_{h_q}(\overset{\circ}{\Omega})$. Then we can rewrite $w$ and $q$ as

$$w(x) = \sum_{k=1}^{N_q} w^k \phi_k^1(x),$$
$$q(x) = \sum_{k=1}^{N_q} (\Re(q^k) + i\Im(q^k))\phi_k^1(x)$$

with $w^i = w(x_i)$ and $q^i = q(x_i)$. Hence we get

$$\sum_{k=1}^{N_q} w^k \int_\Omega v\ \phi_k^1\ dx = \int_\Gamma v\ \left(\tilde{S}_q\psi_\eta - \tilde{S}_0 v_1\right)\ ds_x + \int_\Omega (\psi_\eta - v_1)\,\Delta v\ dx \tag{5.4}$$

and

$$\sum_{k=1}^{N_q} \int_\Omega \left(\Re(q^k)\Re(e_\eta) - \Im(q^k)\Im(e_\eta) + i(\Re(q^k)\Im(e_\eta) + \Im(q^k)\Re(e_\eta))\right) \phi_k^1\phi_j^1\ dx \tag{5.5}$$
$$= \sum_{k=1}^{N_q} (\Re(w^k) + i\Im(w^k)) \int_\Omega \phi_k^1\phi_j^1\ dx$$

for $j = 1,..,N_q$. Now we need $N_q$ different test functions $v \in H^2(\Omega)$ to get an equation for $\{w^k\}_{k=1}^{N_q}$.

In this example we choose the fundamental solution of the Laplace operator $v = G_0(x, y_l)$, defined in (3.4), $y_l \in \mathbb{R}^d \backslash \Omega$ for $l = 1, ..., N_q$. Consider

$$y_l = (0.25\sqrt{2} + h)\begin{pmatrix}\sin(\alpha_l)\\ \cos(\alpha_l)\end{pmatrix} + \begin{pmatrix}0.75\\ 0.75\end{pmatrix}$$

with

$$\alpha_l = \frac{2\pi l}{N_q}$$

for $l = 1, ..., N_q$ and $h > 0$ as step in space. Then we get

$$\sum_{k=1}^{N_q} w^k \int_\Omega G_0(x, y_l)\ \phi_k^1(x)\ dx = \int_\Gamma G_0(x, y_l)\ \left(\tilde{S}_q \psi_\eta(x) - \tilde{S}_0 v_1(x)\right)\ ds_x,$$

for $l = 1, ..., N_q$. Therefore we end up with the equation

$$A\ \underline{w} = f,$$

where

$$\begin{aligned}
A[l, k] &:= \int_\Omega G_0(x, y_l)\ \phi_k^1(x)\ dx && \forall l, k = 1, ..., N_q,\\
\underline{w}[k] &:= w^k && \forall k = 1, ..., N_q,\\
f[l] &:= \int_\Gamma G_0(x, y_l)\ \left(\tilde{S}_q \psi_\eta^{\tilde{h}}(x) - \tilde{S}_0 v_1(x)\right)\ ds_x && \forall l = 1, ..., N_q.
\end{aligned}$$

With this we want to solve

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \Re(\underline{w}) \\ \Im(\underline{w}) \end{pmatrix} = \begin{pmatrix} \Re(f) \\ \Im(f) \end{pmatrix}. \tag{5.6}$$

The integral in $f$ is the single layer potential $V$ applied on $(\tilde{S}_q \psi_\eta - \tilde{S}_0 v_1)$. On the other hand the integral in $A$ is the Newton potential applied on $\phi_k^1$. Hence we know of the existence of these integrals and how they behave on the discrete level, see [11] for more details. After we solved the first equation, we have to calculate

$$\begin{aligned}
K^{Re}[j, k] &:= \int_\Omega \Re(e_\eta)(x) \phi_k^1 \phi_j^1\ dx && \forall j, k = 1, ..., N_q,\\
K^{Im}[j, k] &:= \int_\Omega \Im(e_\eta)(x) \phi_k^1 \phi_j^1\ dx && \forall j, k = 1, ..., N_q,\\
g^{Re}[j] &:= \sum_{k=1}^{N_q} \Re(w^k) \int_\Omega \phi_k^1 \phi_j^1\ dx && \forall j = 1, ..., N_q,\\
g^{Im}[j] &:= \sum_{k=1}^{N_q} \Im(w^k) \int_\Omega \phi_k^1 \phi_j^1\ dx && \forall j = 1, ..., N_q
\end{aligned}$$

to be able to compute the second equation

$$\begin{pmatrix} K^{Re} & -K^{Im} \\ K^{Im} & K^{Re} \end{pmatrix} \begin{pmatrix} \Re(\underline{q}) \\ \Im(\underline{q}) \end{pmatrix} = \begin{pmatrix} g^{Re} \\ g^{Im} \end{pmatrix}$$

with

$$\underline{q}[k] := q^k \qquad \forall k = 1, ..., N_q.$$

Furthermore let

$$\begin{aligned} K &:= \begin{pmatrix} K^{Re} & -K^{Im} \\ K^{Im} & K^{Re} \end{pmatrix}, \\ g &:= \begin{pmatrix} g^{Re} \\ g^{Im} \end{pmatrix}. \end{aligned} \tag{5.7}$$

The matrix $K$ will make us some problems since the function $e_\eta$ can be hard to approximate for large $\eta$. This we see from the numerical results in chapter six. Therefore we look at a second way to solve $e_\eta q = w$.

### 5.2.3. Second way to compute $q$ with $G_0$

Again let $\{x_k\}_{k=1}^{N_q}$ be the inner supporting points of $\Omega$ and we are staying with the same solution and test function spaces. Hence we can write

$$\begin{aligned} w(x) &= \sum_{k=1}^{N_q} w^i \phi_k^1(x), \\ q^h(x) &= \sum_{k=1}^{N_q} (\Re(q^k) + i\Im(q^k))\phi_k^1(x) \end{aligned}$$

with $w^i = w(x_i)$ and $q^i = q(x_i)$. We remain with first equation (5.6) and solve

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} \Re(\underline{w}) \\ \Im(\underline{w}) \end{pmatrix} = \begin{pmatrix} \Re(f) \\ \Im(f) \end{pmatrix}$$

with

$$\begin{aligned} A[l,k] &:= \int_\Omega G_0(x, y_l)\ \phi_k^1(x)\ dx && \forall l, k = 1, ..., N_q, \\ \underline{w}[k] &:= w^k && \forall k = 1, ..., N_q, \\ f[l] &:= \int_\Gamma G_0(x, y_l)\ \left(\tilde{S}_q \psi_\eta^{\tilde{h}}(x) - \tilde{S}_0 v_1(x)\right)\ ds_x && \forall l = 1, ..., N_q. \end{aligned}$$

However we take a different approach for the second step to solve

$$q\psi_\eta = w.$$

Now we use the equation

$$\int_\Omega q\phi_j^1 \; dx = \int_\Omega \frac{w}{e_\eta}\phi_j^1 \; dx \qquad \forall j = 1, ..., N_q \tag{5.8}$$

and we get for $j = 1, ..., N_q$

$$\begin{aligned}
&\sum_{k=1}^{N_q} \int_\Omega (\Re(q^k) + i\Im(q^k))\phi_k^1(x)\phi_j^1(x) \; dx \\
&\qquad = \sum_{k=1}^{N_q} \int_\Omega \frac{\Re(w^k) + i\Im(w^k)}{\Re(e_\eta)(x) + i\Im(e_\eta)(x)}\phi_k^1(x)\phi_j^1(x) \; dx \\
&\qquad = \sum_{k=1}^{N_q} \int_\Omega \frac{\Re(w^k)\Re(e_\eta)(x) + \Im(w^k)\Im(e_\eta)(x)}{\Re(e_\eta)(x)^2 + \Im(e_\eta)(x)^2}\phi_k^1(x)\phi_j^1(x) \; dx \\
&\qquad\quad + i \int_\Omega \frac{\Im(w^k)\Re(e_\eta)(x) - \Re(w^k)\Im(e_\eta)(x)}{\Re(e_\eta)(x)^2 + \Im(e_\eta)(x)^2}\phi_k^1(x)\phi_j^1(x) \; dx.
\end{aligned}$$

Therefore we want to solve

$$\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} \Re(\underline{q}) \\ \Im(\underline{q}) \end{pmatrix} = \begin{pmatrix} \Re(f) \\ \Im(f) \end{pmatrix}$$

with

$$\begin{aligned}
K[j,k] &:= \int_\Omega \phi_k^1\phi_j^1 \; dx && \forall j,k = 1, ..., N_q, \\
g^{Re}[j] &:= \sum_{k=1}^{N_q} \int_\Omega \frac{\Re(w^k)\Re(e_\eta)(x) + \Im(w^k)\Im(e_\eta)(x)}{\Re(e_\eta)(x)^2 + \Im(e_\eta)(x)^2}\phi_k^1(x)\phi_j^1(x) \; dx && \forall j = 1, ..., N_q, \\
g^{Im}[j] &:= \sum_{k=1}^{N_q} \int_\Omega \frac{\Im(w^k)\Re(e_\eta)(x) - \Re(w^k)\Im(e_\eta)(x)}{\Re(e_\eta)(x)^2 + \Im(e_\eta)(x)^2}\phi_k^1(x)\phi_j^1(x) \; dx && \forall j = 1, ..., N_q, \\
\underline{q}[k] &:= q^k && \forall k = 1, ..., N_q.
\end{aligned}$$

In this case $K$ is the Mass matrix and we know that this matrix is invertible, see [11] for more details. Hence we can solve this equation even though we won't get very good results since the function $e_\eta$ is still behaving badly.

### 5.2.4. Compute $q$ with basis functions

For $\eta \in \mathbb{C}^d \backslash \{0\}$ with very large absolute value $|\eta|$ and $\eta \cdot \eta = 0$ we get

$$\int_\Omega v\ q e_\eta\ dx \approx \int_\Gamma v\ \left(\tilde{S}_q \psi_\eta - \tilde{S}_0 v_1\right)\ ds_x + \int_\Omega (\psi_\eta - v_1)\, \Delta v\ dx.$$

Furthermore we assume that $q \in S^1_{h_q}(\mathring{\Omega})$ and let $\{\phi^1_k\}_{k=1}^{N_q}$ be the basis of $S^1_{h_q}(\mathring{\Omega})$. Now we consider $v(x) = \phi^1_k$ and look at the equation

$$\int_\Omega \phi^i_k\ q e_\eta\ dx \approx \int_\Gamma \phi^1_k\ \left(\tilde{S}_q \psi_\eta - \tilde{S}_0 v_1\right)\ ds_x.$$

Hence we want to solve

$$\sum_{j=1}^{N_q} q_j \int_\Omega \phi^1_k\ \phi^1_j e_\eta\ dx = \int_\Gamma \phi^1_k\ \left(\tilde{S}_q \psi_\eta - \tilde{S}_0 v_1\right)\ ds_x \qquad \forall k = 1, ..N_q.$$

So we end up with the equation

$$A\underline{q} = f,$$

where

$$\begin{aligned} A[k,j] &:= \int_\Omega \phi^1_k\ \phi^1_j e_\eta\ dx && \forall k, j = 1, ..., N_q, && (5.9)\\ f[k] &:= \int_\Gamma \phi^1_k\ \left(\tilde{S}_q \psi_\eta - \tilde{S}_0 v_1\right)\ ds_x && \forall k = 1, ..., N_q, \\ \underline{q}[j] &:= q^j && \forall j = 1, ..., N_q. && (5.10)\end{aligned}$$

Here we will have the same problems with the matrix $A$ as with the matrix $K$ in chapter 5.2.2. The function $e_\eta$ can be hard to approximate for large $\eta$ and this we will see in the numerical results of the next chapter.

# 6. Numerical results

In this chapter we work with the same $\Omega$ and $\sigma_1$ as in chapter 5, therefore we have

$$\Omega = B((0.75, 0.75), 0.25\sqrt{2})$$

and

$$\sigma_1(x) = \begin{cases} (1 + 10\exp(\frac{1}{r^2 - \frac{8}{9}r_M^2}))^2 & \text{if } r^2 < \frac{8}{9}r_M^2, \\ 1 & \text{otherwise.} \end{cases}$$

with

$$\Delta\sigma_1^{1/2}(x) = \begin{cases} 2 \cdot 10\exp(\frac{1}{r^2-\frac{8}{9}r_M^2}) \left[\frac{4r^2}{(r^2-\frac{8}{9}r_M^2)^3}(\frac{1}{r^2-\frac{8}{9}r_M^2} + 2) - \frac{4}{(r^2-\frac{8}{9}r_M^2)^2}\right] & \\ \quad +2 \cdot 10^2\exp(\frac{2}{r^2-\frac{8}{9}r_M^2}) \left[\frac{8r_M^2}{(r^2-\frac{8}{9}r_M^2)^3}(\frac{1}{r^2-\frac{8}{9}r_M^2} + 1) - \frac{4}{(r^2-\frac{8}{9}r_M^2)^2}\right] & \text{if } r^2 < \frac{8}{9}r_M^2, \\ 0.0 & \text{otherwise.} \end{cases}$$

where $r := |x - x_m|$, $r_M := 0.25\sqrt{2}$ and $x_m := (0.75, 0.75)$ is the radius of $\Omega$. We first look at the approximation of $q$ and compare the different methods. In the end we look at numerical results for the approximation of $\sigma$.

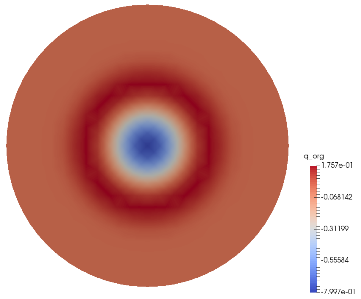
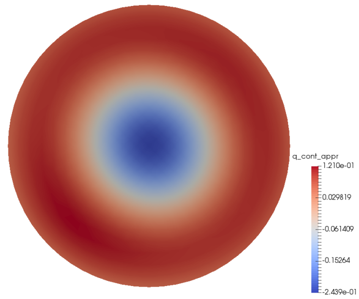
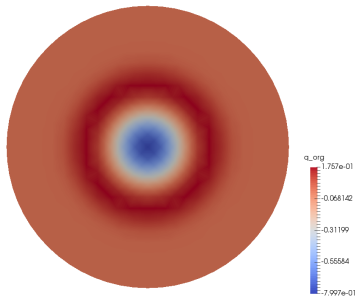
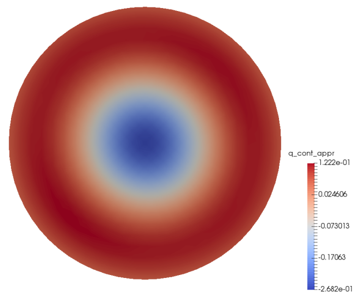
## 6.1. Approximation of q

### 6.1.1. Results for approximating $q$ with the Fourier transform

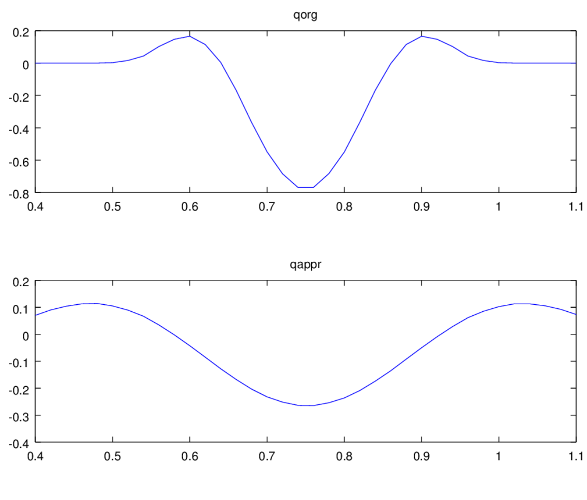
We now test the approximation which was given in the previous chapter. For the forth level of refinement for $\Omega$ and different levels for $\tilde{\Omega} = B((0.75, 0.75), 10.25\sqrt{2})$

| Level | $\|q_h - q\|_{L^2(\Omega)}$ |
|---|---|
| 0 | 0.101697 |
| 1 | 0.087418 |
| 2 | 0.0882527 |
| 3 | 0.0825344 |
| 4 | 0.085346 |
| 5 | 0.0519215 |

If we take a look at the graphical approximation, we see

| Level | q original | $\Re(\underline{q})$ |
|---|---|---|
| 4 |  |  |
| 5 |  |  |

Before we make a decision about the quality of this approximation, we look at the profile of this plot to get a better understanding of it.




We see an approximation which is quit good. The range of the values is almost in the same range as in the original. It rather looks like $q$ was smoothed by the approximation. However we will see, that this is the best method compared to to the others.

### 6.1.2. Results for approximating $q$ with $G_0$

Let $\eta = \begin{pmatrix} -100 \\ i100 \end{pmatrix}$. We refine the neighborhood of $y_l$ twice to compute the entries of $A$, which was defined in (5.6). Then we look at the smallest eigenvalues of $A$. If we take $h = 0$ in the definition of $y_l$, we get

| Dof of q | smallest Eigen value |
|---|---|
| 5 | -1.72464e-34 + i 2.56731e-34 |
| 25 | 5.08604e-18 + i 0 |
| 113 | 6.24598e-19 + i 0 |

For $h > 0$ as the discretization step we get

| Dof of q | smallest Eigen value |
|---|---|
| 5 | -6.20407e-08 + i 0 |
| 25 | -9.42247e-18 + i 0 |
| 113 | -1.39933e-19 + i 0 |

Therefore matrices are singular and we can't compute the following steps. However if we look at the inner points of $\Omega$ $y_l = x_l$, $l = 1, ..., N_q$, the equation $Aw = f$ as it was stated in (5.6) is solvable. Which means as well that we ignore the term $-(\psi_\eta(y_l) + v_1(y)$ on the right hand side. But now $K$ is singular, what we see from the following results:
For the smallest eigenvalue of $K$ in each refinement step, we get

| Dof of q | smallest Eigen value |
|---|---|
| 5 | -1.72464e-34 + i 2.56731e-34 |
| 25 | 3.37291e-44 + i 1.82075e-44 |
| 113 | -2.05844e-49 + i 0 |

We conclude that we can't compute $\sigma$ with this method. Therefore we use a slightly different method:

### 6.1.3. Results for the second way to approximate $q$ with $G_0$

In the last chapter we tried to solve

$$\int_\Omega q e_\eta \phi_j^1 \, dx = \int_\Omega w \phi_j^1 \, dx$$

for $j = 1, ..., N_q$. Now we solve instead the equation (5.8)

$$\int_\Omega q \phi_j^1 \, dx = \int_\Omega \frac{w}{e_\eta} \phi_j^1 \, dx \qquad \forall j = 1, ..., N_q.$$

The matrix we get by discretizing this equation is uniquely solvable.

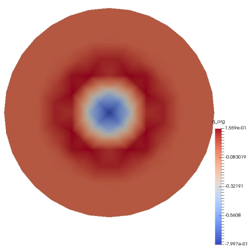
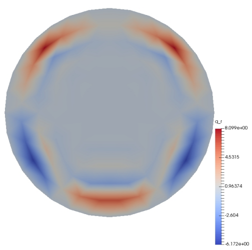
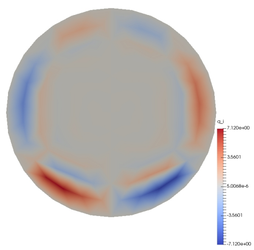
Let $\eta = \begin{pmatrix} -10 \\ i10 \end{pmatrix}$. We get for the system error between the matrix applied on the original values of $q$ and the right hand side for different degrees of freedom of $q$

| Dof of q | $\|Kq - g\|_{L^2(\Omega)}$ |
|---|---|
| 5 | 0.54235 |
| 25 | 0.268102 |
| 113 | 0.0622128 |

Here we see a convergence. Therefore we look at the approximation error and we get

| Dof of q | $\|q - \underline{q}\|_{L^2(\Omega)}$ |
|---|---|
| 5 | 4.19236 |
| 25 | 3.30176 |
| 113 | 1.37379 |

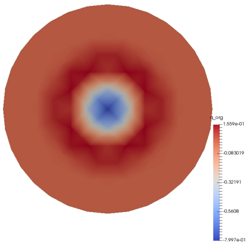
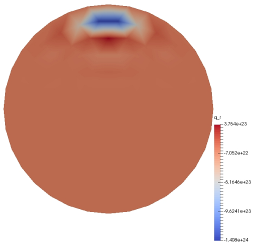
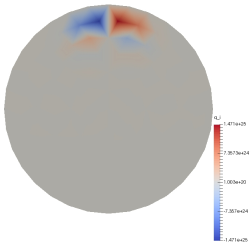
Here we can see a convergence as well. However we take a look at the graphical approximation get a better understanding how good the approximation actually is.

| Dof of q | q original | $\Re(\underline{q})$ | $\Im(\underline{q})$ |
|---|---|---|---|
| 113 |  |  |  |

Now we see, that $q$ is not approximated at all, since the pictures show a completely different function.
Therefore we take another $\eta$ and hope for better results. If we choose $\eta = \begin{pmatrix} -100 \\ i100 \end{pmatrix}$, then we get for 113 degrees of freedom for $q$ the errors

| Dof of q | $\|Kq - g\|_{L^2(\Omega)}$ | $\|q - \underline{q}\|_{L^2(\Omega)}$ |
|---|---|---|
| 113 | 4.28126e+22 | 1.00813e+24 |

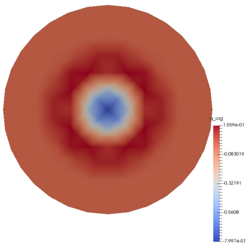
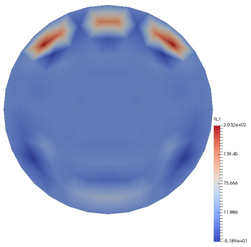
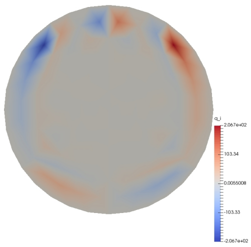
We already see that this is a very bad approximation and the pictures tell us the same:

| Dof of q | q original | $\Re(\underline{q})$ | $\Im(\underline{q})$ |
|---|---|---|---|
| 113 |  |  |  |

As a last choice we take $\eta = \begin{pmatrix} -20 \\ i20 \end{pmatrix}$ and compare the results. We get for 113 degrees of freedom for $q$ the errors

| Dof of q | $\|Kq - g\|_{L^2(\Omega)}$ | $\|q - \underline{q}\|_{L^2(\Omega)}$ |
|---|---|---|
| 113 | 1.10034 | 25.3053 |

We already expect bad results in the graphical approximation from these errors and the pictures tell us that this guess it right:

| Dof of q | q original | $\Re(\underline{q})$ | $\Im(\underline{q})$ |
|---|---|---|---|
| 113 |  |  |  |

Since we were not able to produce good results, we yet again take a different approach. Now we try to use the Least square method by taking $(N_q + N_{q,bound})$ points on the boundary $\Gamma$, where $N_{q,bound}$ is the number of boundary points of the refinement step with the space step $h_q$ and which gives us $N_q$ inner points. With this we get the following results:
For the smallest Eigen value of the matrix $A^T A$ we get

| Dof of q | smallest Eigen value |
|---|---|
| 5 | 1.00319e-10 + i 0 |
| 25 | 1.07312e-19 + i 0 |
| 113 | -7.82646e-22 + i 4.32274e-21 |
| 481 | 1.10415e-21 + i 0 |

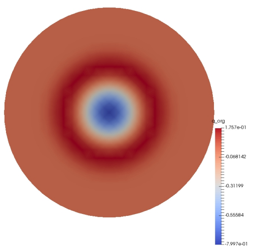
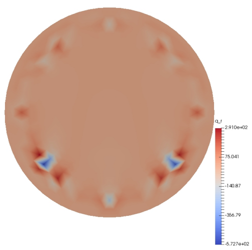
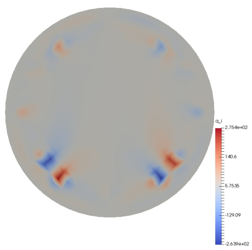
The smallest Eigen value of $K$ is for different degrees of freedom for $q$

| Dof of q | smallest Eigen value |
|---|---|
| 25 | 2.67125e-07 + i 1.80736e-07 |
| 113 | 1.49145e-09 + i 4.62069e-08 |
| 481 | 1.59458e-09 + i 8.13272e-09 |

For the error of the solution we get

| Dof of q | $\\|q - \underline{q}\\|_{L^2(\Omega)}$ |
|---|---|
| 5 | 60357.6 |
| 25 | 107712 |
| 113 | 7263.13 |
| 481 | 29.7206 |

Here we do not have a good convergence and in the pictures we do not have a graphical approximation either

| Dof of q | q original | $\Re(\underline{q})$ | $\Im(\underline{q})$ |
|---|---|---|---|
| 481 |  |  |  |

Therefore we can conclude that this method does not work well. So we look at the next method and hope for better results.

### 6.1.4. Results for approximating $q$ with basis functions

Now we look the third method to approximate $q$ in which we use basis functions for $v$ in the equation (4.2).

Let $\eta = \begin{pmatrix} -100 \\ i100 \end{pmatrix}$. Again we are unable to solve the equation $Aw = f$, which was posed in (5.9), because $A$ is singular as we see from the smallest eigenvalues of $A$, which are

| Dof of q | smallest Eigen value |
|---|---|
| 5 | -2.472e-35 + i 0 |
| 25 | -2.73198e-45 + i 0 |
| 113 | -7.91374e-50 + i 0 |

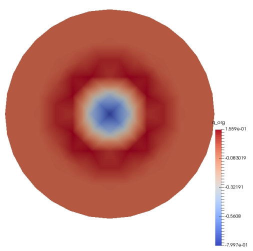
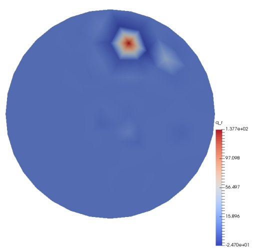
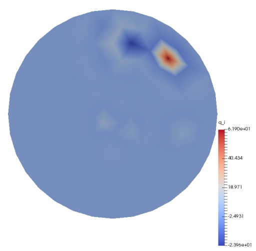
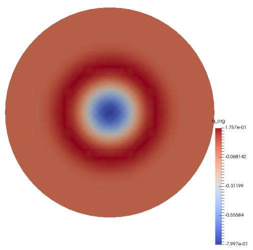
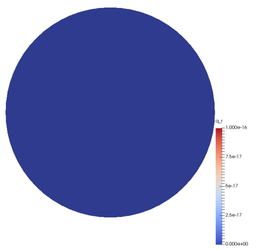
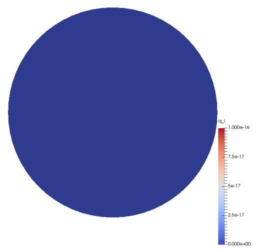
Therefore we choose another $\eta = \begin{pmatrix} -10 \\ i10 \end{pmatrix}$. We yet again start with the smallest eigenvalues

| Dof of q | smallest Eigen value |
|---|---|
| 25 | 2.67125e-07 + i 1.80736e-07 |
| 113 | 1.49145e-09 + i 4.62069e-08 |
| 481 | 1.59458e-09 + i 8.13272e-09 |

The Eigen values are small but still big enough so we can solve the equation $Aw = f$. So we look at the error of the approximation $\underline{q}$

| Dof of q | $\|q - \underline{q}\|_{L^2(\Omega)}$ |
|---|---|
| 5 | 76.5604 |
| 25 | 6.26376 |
| 113 | 6.0203 |
| 481 | 0.0890929 |

In case of 481 degrees of freedom for $q$ we get the zero solution, which we see in the pictures below. We also see that there is no approximation of $q$.

| Dof of q | q original | $\Re(\underline{q})$ | $\Im(\underline{q})$ |
|---|---|---|---|
| 113 |  |  |  |
| 481 |  |  |  |

Since we do not have any approximation of $q$ with this method, we a different approach. Now we try the Least square method to get better results. We test with all the $N$ basis functions, which we used to compute $\psi_\eta$ and $u_\Gamma$, to get an over determined equation.

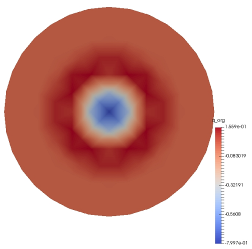
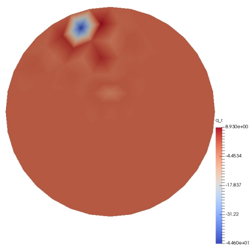
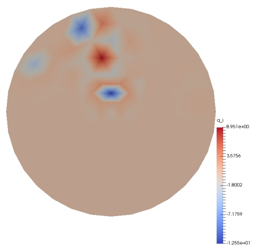
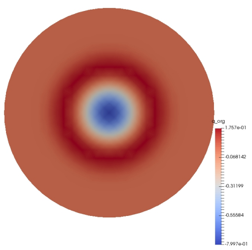
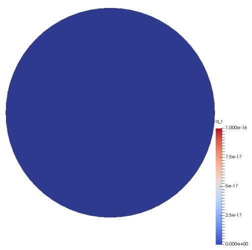
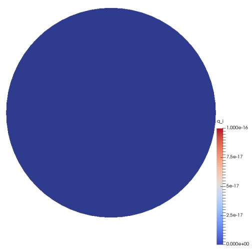
For the smallest Eigen value of $A^T A$ we end up with

| Dof of q | smallest Eigen value of $A^T A$ |
|---|---|
| 5 | 1.04245e-11 + i 0 |
| 25 | 4.70264e-14 + i 0 |
| 113 | 7.71787e-16 + i 0 |
| 481 | 2.51413e-17 + i 0 |

Especially in the case of 481 degrees of freedom the Eigen value with the smallest norm is very small and we can't expect to get a good approximation. When we calculate the error of the solution we get

| Dof of q | $\|q - \underline{q}\|_{L^2(\Omega)}$ |
|---|---|
| 5 | 1.95721 |
| 25 | 1.09349 |
| 113 | 1.96717 |
| 481 | 0.0890929 |

In case of 481 degrees of freedom for $q$ we get the zero solution again. This we could have expected from the very small Eigen values. If we now look at the results in pictures, we get:

| Dof of q | q original | $\Re(\underline{q})$ | $\Im(\underline{q})$ |
|---|---|---|---|
| 113 |  |  |  |
| 481 |  |  |  |

We see as well in the pictures, that we get the zero solution in case of 481 degrees of freedom for $q$. With this we can conclude that this method does not show any good results. Even if we try to use the Least square method we don't end up with sufficient solutions.

## 6.2. Approximation of $\sigma$

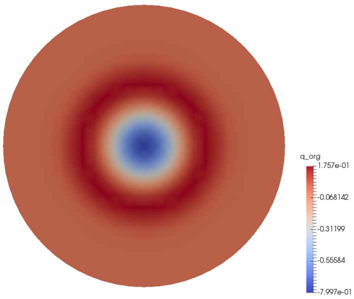
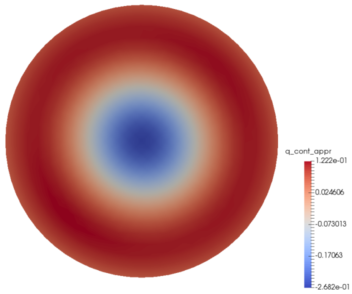
We look at the equation

$$\begin{aligned}(-\Delta + q)\sigma^{1/2} &= 0 && \text{in } \Omega,\\ \sigma^{1/2} &= 1 && \text{on } \Gamma,\end{aligned}$$

and solve the variational formulation (4.6) which was discretized in (5.3) in chapter 5. Let $\Omega$ be refined four times. Now we use the results of the approximation of $q$ with the Fourier transform from chapter 6.1.1 on the fifth level of refinement for $\tilde{\Omega}$ which yielded the error

| Level | $\|q_h - q\|_{L^2(\Omega)}$ |
|---|---|
| 5 | 0.0519215 |

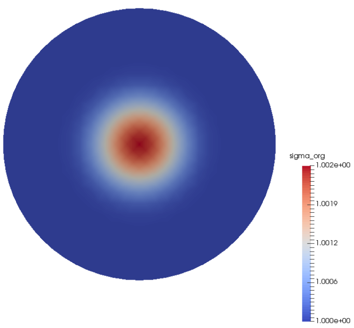
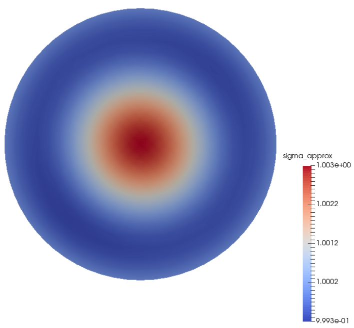
and the graphical approximation

| Level | q original | $\Re(\underline{q})$ |
|---|---|---|
| 5 |   |   |

If we solve the equation (5.3), we get the error

| $\|\sigma_h - \sigma\|_{L^2(\Omega)}$ |
|---|
| 0.000149404 |

and the graphical approximation

| $\sigma$ original | $\sigma_h$ |
|---|---|
|   |   |

We see that the values of the approximation $\sigma_h$ are almost in the same range as the original function. In the same way as the function $q_h$ the approximation $\sigma_h$ seems to be smoothed. This happened probably when we chose to cut the integral domain of the inverse Fourier transform from $\mathbb{R}^2$ to $\tilde{\Omega}$.

# 7. Conclusion

In this thesis we derived several methods to compute $\sigma$. We tested them and concluded that we get the best approximation for $q$ if we use the method with the Fourier transform as it was described in chapter 4.1.1. On the other hand we did not take computational time into consideration. The method with the Fourier transform does take the most time, because we need to compute $\tilde{S}_q\psi_\eta$ for every discretization point $\xi_k$ of $\tilde{\Omega}$ since $\eta \in V_{\xi_k}$, as it was explained in chapter 5.2.1. But in the end we see that this computation time is well spend since we can see in chapter 6.2 that the results are acceptable and much better compared to the other methods. Therefore we end up with the algorithm:

We start with the boundary data $g$ and $\sigma_\Gamma$ and solve

$$\int_\Gamma V(t_0)\; w\; ds_y = \int_\Gamma (\frac{1}{2}I + K)(\sigma_\Gamma^{1/2} g)\; w\; ds_y \qquad \forall w \in H^{-1/2}(\Gamma)$$

to get $t_0 \in H^{-1/2}(\Gamma)$. Then we choose the integration domain $\tilde{\Omega} \subset \mathbb{R}^d$ for the inverse Fourier transform. For every discretization point $\xi_k$, $k = 1, ..., \tilde{N}$, of $\tilde{\Omega}$ we choose $\eta_k \in V_{\xi_k}$ with a large enough absolute value. Next we do the following two calculations for every $k = 1, ..., \tilde{N}$:

First we solve the equation

$$\int_\Gamma V(t_k)\; w\; ds_y = \int_\Gamma e_{\eta_k} w\; ds_y - \int_\Gamma (\frac{1}{2}I - K)(\sigma_\Gamma^{1/2} g)\; w\; ds_y \qquad \forall w \in H^{-1/2}(\Gamma),$$

to get the Neumann data $t_k \in H^{-1/2}(\Gamma)$. Secondly we compute

$$\mathcal{F}^h \tilde{q}(\xi_k) = \int_\Gamma e_{-(\xi_k+\eta_k)}\; (t_k - t_0)\; ds_x.$$

By interpolating the approximation of the Fourier transform $\mathcal{F}^h\tilde{q}$ we can compute the approximated inverse Fourier transform and get

$$q^h(x) = \frac{1}{(2\pi)^d} \int_{\tilde{\Omega}} \mathcal{F}^h \tilde{q}(\xi) e^{i\xi\cdot x}\; d\xi.$$

Now we are able to solve the variational formulation

$$\int_\Omega \nabla \sigma_0^{1/2} \cdot \nabla v \; dx + \int_\Omega q^h \sigma_0^{1/2} v \; dx = - \int_\Omega \nabla \tilde{\sigma}^{1/2} \cdot \nabla v \; dx - \int_\Omega q^h \tilde{\sigma}^{1/2} v \; dx \qquad \forall v \in H_0^1(\Omega)$$

for $\tilde{\sigma}$ as the extension of the boundary data $\sigma_\Gamma$ to get the desired dielectric conductivity $\sigma = (\sigma_0^{1/2} + \tilde{\sigma}^{1/2})^2$.

# Appendices

# A. Definitions

First we will state some definitions, that are used in this thesis. The following two definitions for Lipschitz domains come from the book [15, p.89]:

**Definition A.1.** *The open set $\Omega \subset \mathbb{R}^d$ is called a Lipschitz hypograph if there exists a function $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}^d$ such that*

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^d : x_n < \zeta(x'), \forall x' = (x_1, ..., x_{n-1}) \in \mathbb{R}^{d-1}\}$$

*and $\zeta$ is Lipschitz, i.e. if there is a constant $M > 0$ such that*

$$|\zeta(x') - \zeta(y')| \leq M|x' - y'| \qquad \forall x', y' \in \mathbb{R}^{d-1}.$$

**Definition A.2.** *The open set $\Omega \subset \mathbb{R}^d$ is called a Lipschitz domain if its boundary $\Gamma = \partial\Omega$ is compact and if there exist finite families $\{W_j\}$ and $\{\Omega_j\}$ having the following properties:*

- *The family $\{W_j\}$ is a finite open cover of $\Gamma$, i.e., each $W_j$ is an open subset of $\mathbb{R}^d$, and $\Gamma \subset \bigcup\limits_j W_j$.*
- *Each $\Omega_j$ can be transformed to a Lipschitz hypograph by a rigid motion, i.e. by a rotation plus a translation.*
- *The set $\Omega$ satisfies $W_j \cap \Omega = W_j \cap \Omega_j$ for each $j$.*

From now on let $\Omega \subset \mathbb{R}^d$ open and bounded. The following definition of the $L^p(\Omega)$ space comes from [15, p.58]:

**Definition A.3.** *The space $L^p(\Omega)$ is defined by the norm*

$$\|u\|_{L^p(\Omega)} := \left( \int\limits_\Omega |u(x)|^p \, dx \right)^{1/p}$$

*for $1 \leq p < \infty$. We define*

$$\langle u, v \rangle_\Omega := \int\limits_\Omega u(x)v(x) \, dx.$$

The definition of $C^k(\Omega)$ is from [15, p.61]:

**Definition A.4.** *The space of $k$-times continuous differentiable functions, $k \in \mathbb{N}$, is called*

$$C^k(\Omega) := \{f | \partial^\alpha f \text{ exists and is continuous in } \Omega \text{ for any } \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leq k\}$$

*and*

$$C^\infty(\Omega) = \bigcap_{k \geq 0} C^k(\Omega).$$

The following definition for compact supported function spaces comes from [15, p. 61,65,72]:

**Definition A.5.**
- *Let $K$ be a compact subset of $\Omega$ and $r \in \mathbb{N}_0$, then define the space of compactly supported $C^r$- functions as*

$$C^r_K(\Omega) := \{u \in C^r(\Omega) : supp(u) \subseteq K\},$$
$$\mathcal{D}(\Omega) := \{u : u \in C^\infty_K(\Omega) \text{ for some } K \text{ which is a compact subset of } \Omega\}$$

- *We define the Schwartz as the space of rapidly decreasing $C^\infty$-functions with*

$$S(\mathbb{R}^d) := \{\psi \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta \psi(x)| < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^d\}.$$

For the definition of the Fourier transform we need the theorem of Plancherel as stated as in [7, p.188, Thm 6.44]:

**Theorem A.6** (Theorem of Plancherel)**.** *There is a unique operator $\mathcal{F} : L^2(\mathbb{R}^d, \mathbb{C}) \to L^2(\mathbb{R}^d, \mathbb{C})$ with*

$$(\mathcal{F}f, \mathcal{F}g)_{L^2(\Omega)} = (f, g)_{L^2(\Omega)} \qquad \forall f, g \in L^2(\mathbb{R}^d, \mathbb{C}),$$

*such that*

$$\mathcal{F}f(.) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot .} u(x)\, dx,$$

*for $f \in L^1(\mathbb{R}^d, \mathbb{C}) \cap L^2(\mathbb{R}^d, \mathbb{C})$. It holds true that*

$$(\mathcal{F}^{-1}f)(x) = (\mathcal{F}f)(-x)$$

*almost everywhere for all $f \in L^2(\mathbb{R}^d, \mathbb{C})$.*

The following definition is from [15, p.70,75]:

**Definition A.7.** • *The operator*

$$\mathcal{F}u(\xi) := (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{-i\xi\cdot x} u(x)\ dx$$

*is called the Fourier transform for $\xi \in \mathbb{R}^d$.*

- *We define the Bessel potential of order $s \in \mathbb{R}$ by*

$$\mathcal{J}^s u(x) := (2\pi)^{d/2} \int\limits_{\mathbb{R}^d} (1+|\xi|^2)^{s/2} \mathcal{F}u(\xi) e^{i\xi\cdot x}\ d\xi$$

*for $x \in \mathbb{R}^d$.*

With this we can now define $H$ spaces as follows:

**Definition A.8.** *We define for $s \in \mathbb{R}$*

$$H^s(\mathbb{R}^d) := \{u \in S^*(\mathbb{R}^d) : \mathcal{J}^s u \in L^2(\mathbb{R}^d)\},$$

*and*

$$H^s(\Omega) := \{u \in \mathcal{D}^*(\Omega) : u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^d)\}.$$

*Furthermore we define*

$$\begin{aligned} \tilde{H}^s(\Omega) &:= \text{ closure of } \mathcal{D}(\Omega) \text{ in } H^s(\mathbb{R}^d), \\ H_0^s(\Omega) &:= \text{ closure of } \mathcal{D}(\Omega) \text{ in } H^s(\Omega). \end{aligned}$$

This definition is from [15, p.76-78]. For Lipschitz domains we get a relation between $\tilde{H}^s$ and $H_0^1$:

**Theorem A.9.** *Let $s \geq 0$. If $\Omega$ is a Lipschitz domain, then*

$$\tilde{H}^s(\Omega) = \{u \in L^2(\Omega) : \tilde{u} \in H^s(\mathbb{R}^n)\} \subset H_0^s(\Omega)$$

*where $\tilde{u}$ denotes the extension of $u$ by zero:*

$$\tilde{u} = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^d \backslash \Omega. \end{cases}$$

*In fact*

$$\tilde{H}^s(\Omega) = H_0^s(\Omega) \text{ provided } s \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...\}.$$

for the proof see [15, Thm. 3.33, p.95]. In comparison we define $W$ spaces as in [15, p.74]:

**Definition A.10.** *We define the Sobolev space $W_p^k(\Omega)$, $k, p \in \mathbb{N}$, as follows*

$$W_p^k(\Omega) := \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq k\}$$

*with*

$$\|u\|_{W_p^k(\Omega)} := \left( \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u(x)|^p \, dx \right)^{1/p}.$$

*We denote the Slobodechiĭ seminorm by*

$$|u|_{\mu,p,\Omega} := \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{d+p\mu}} \, dx \, dy \right)^{1/p}$$

*for $0 < \mu < 1$. For $s = k + \mu$, $k \in \mathbb{N}$, $0 < \mu < 1$, we define*

$$W_p^s(\Omega) := \{u \in W_p^k(\Omega) : |\partial^\alpha u|_{\mu,p,\Omega} < \infty \text{ for } |\alpha| = k\}$$

*and equip it with the norm*

$$\|u\|_{W_p^s(\Omega)} := \left( \|u\|^p_{W_p^k(\Omega)} + \sum_{|\alpha|=k} |\partial^\alpha u|^p_{\mu,p,\Omega} \right)^{1/p}.$$

The equivalence we get from the following theorem, as was stated in [15, p.92].

**Theorem A.11.** *If $\Omega$ is a Lipschitz domain, then*

- $H^s(\Omega)^* = \tilde{H}^{-s}(\Omega)$ *and* $\tilde{H}^s(\Omega)^* = H^{-s}(\Omega)$ *for all* $s \in \mathbb{R}$;

- $W^s(\Omega) = H^s(\Omega)$ *for all* $s > 0$.

**Theorem A.12.** *For any non-empty open set $\Omega \subset \mathbb{R}^d$, and for any integer $r \geq 0$, it holds true that*

$$H^{-r}(\Omega) = W^{-r}(\Omega)$$

*with equivalent norms.*

**Proof.**
Look at [15, p.81]. □

One of the most important theorems in this work is the trace theorem:

**Theorem A.13.** *Define the trace operator $\gamma : \mathcal{D}(\overline{\Omega}) \to \mathcal{D}(\Gamma)$ by*

$$\gamma u = u|_\Gamma.$$

*If $\Omega$ is a Lipschitz domain, and if $\frac{1}{2} < s < \frac{3}{2}$, then $\gamma$ has a unique extension to a bounded linear operator*

$$\gamma : H^s(\Omega) \to H^{s-1/2}(\Gamma),$$

*and this extension has a continuous right inverse.*

**Proof.**
See [15, Thm. 3.37, p.102] and [15, Thm. 3.38, p.102]. □

With this theorem, we can write a definition of the traces on $\Omega$, which comes from [11]:

**Definition A.14.** *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then we define for $u \in H^1(\Omega)$ and $x \in \Gamma = \partial\Omega$*

$$\gamma_0^{int} u(x) := \lim_{\tilde{x} \to x, \tilde{x} \in \Omega} u(\tilde{x}),$$
$$\gamma_0^{ext} u(x) := \lim_{\tilde{x} \to x, \tilde{x} \in \mathbb{R}^d \backslash \overline{\Omega}} u(\tilde{x}).$$

*If further $u \in H^2(\Omega)$ holds, we define*

$$\gamma_1^{int} u(x) := \lim_{\tilde{x} \to x, \tilde{x} \in \Omega} \frac{\partial u}{\partial n_{int}}(\tilde{x}),$$
$$\gamma_1^{ext} u(x) := \lim_{\tilde{x} \to x, \tilde{x} \in \mathbb{R}^d \backslash \overline{\Omega}} \frac{\partial u}{\partial n_{ext}}(\tilde{x}),$$

*as the normal derivative for the Laplace operator, where $n_{int}$ is the normal of $\Omega$ and $n_{ext}$ the normal of $\mathbb{R}^d \backslash \overline{\Omega}$.*

These limits are to be understand as the use of the trace operator.
With this definition we can define the $L^2(\Gamma)$ product:

**Definition A.15.** *For $u, v \in H^1(\Omega)$ then we define*

$$\langle u, v \rangle_\Gamma := \int_\Gamma \gamma_0^{int} u \; \gamma_0^{int} v \; ds_x.$$

We will also need the Sobolev space $H^{1/2}(\Gamma)$ for the weak formulation of the differential equations:

**Definition A.16.** *Let $\Omega$ be a Lipschitz domain and $0 \leq s \leq 1$*

$$H^s(\Gamma) := \{u \in L^2(\Gamma) : u_\zeta(x') = u(x', \zeta(x')) \text{ for some } u_\zeta \in H^s(\mathbb{R}^{d-1})\}$$

*with the norm*

$$\|w\|^2_{H^s(\Gamma)} := \|u_\zeta\|^2_{H^s(\mathbb{R}^{d-1})}.$$

See [15, p.98].

**Lemma A.17.** *It holds true that*

$$\|w\|^2_{H^{1/2}(\Gamma)} \sim \|u_\zeta\|^2_{L^2(\Gamma)} + |u_\zeta|^2_{1/2,2,\mathbb{R}^{d-1}}.$$

**Proof.**
By using the theorem of norm equivalence for $W^{1/2}(\mathbb{R}^{d-1})$ and $H^{1/2}(\mathbb{R}^{d-1})$, [15, Thm. 3.16,p.80], we get

$$\begin{aligned}
\|w\|^2_{H^{1/2}(\Gamma)} &:= \|u_\zeta\|^2_{H^{1/2}(\mathbb{R}^{d-1})} \\
&\sim \|u_\zeta\|^2_{W^{1/2}(\mathbb{R}^{d-1})} \\
&= \|u_\zeta\|^2_{W^0(\mathbb{R}^{d-1})} + \sum_{|\alpha|=0} |\partial^\alpha u_\zeta|^2_{1/2,2,\mathbb{R}^{d-1}} \\
&\overset{(A.16)}{=} \|u_\zeta\|^2_{L^2(\Gamma)} + |u_\zeta|^2_{1/2,2,\mathbb{R}^{d-1}}.
\end{aligned}$$

And so this lemma is proven. $\square$

# B. Tools

Let furthermore $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists an interesting inequality for $H_0^1(\Omega)$ functions:

**Theorem B.1** (Poincaré inequality)**.** *There exists a constant $c > 0$ depending on $\Omega$ such that*

$$\|u\|_{L^2(\Omega)} \leq c\|\nabla u\|_{L^2(\Omega)}$$

*for all $u \in H_0^1(\Omega)$.*

**Proof.**
Consider [8, Thm. 3, p.279] for the proof. $\square$

Next we look at the Green's identities, which were proven in [15, p.4] :

**Theorem B.2** (Green's identities)**.** *Let $u, v \in H^2(\Omega)$. Then it holds true that*
*1.*

$$\int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Gamma v \; \gamma_1^{int} u \; ds_x - \int_\Omega v \Delta u \; dx.$$

*2.*

$$\int_\Omega (v\Delta u - u\Delta v) \; dx = \int_\Gamma \left(v \; \gamma_1^{int} u - u \; \gamma_1^{int} v\right) \; ds_x.$$

We will be needing the lemma of Lax-Milgram as it was stated in [15, Lem. 2.32, p.43]:

**Theorem B.3** (Lemma of Lax-Milgram)**.** *Let $H$ be a Hilbert space, $H^*$ its dual and $\langle .,. \rangle$ the corresponding duality product. Let the operator $A : H \to H^*$ be linear, bounded and elliptic, i.e. there exists a constant $c > 0$ such that*

$$|\langle Au, u \rangle| \geq c\|u\|_H^2$$

*for all $u \in H$. Then $A$ has a bounded inverse $A^{-1} : H^* \to H$.*

Later we use the Fréchet-Riesz representation as it was stated in [3, Thm. V.3.6, p.226]:

**Theorem B.4** (Fréchet-Riesz representation theorem)**.** *Let $H$ be a Hilbert space. Then the operator $A : H \to H^*$, $y \mapsto \langle ., y\rangle$ is bijective, isometric and conjungated linear, i.e.*

$$A(\lambda y) = \overline{\lambda} A(y).$$

*That means that for every $x^* \in H^*$ exists exactly one $y \in H$ with $x^*(x) = \langle x, y\rangle$ for $x \in H$ and it holds true that $\|x^*\|_{H^*} = \|y\|_H$.*

We will be also in need of some properties of the Fourier series which were proven in [10, p.6-11].

**Theorem B.5.** *Consider $L^2(Q)$ with $Q := [-\pi, \pi]^d$ and the inner product*

$$(u, v) := (2\pi)^{-d} \int\limits_Q u\overline{v}\, dx, \qquad u, v \in L^2(Q).$$

*If $f \in L^2(Q)$, then one has the Fourier series*

$$f(x) = \sum_{k\in\mathbb{Z}^d} f_k e^{ik\cdot x}$$

*with convergence in $L^2(Q)$, where the Fourier coefficients are given by*

$$f_k = (f, e^{ik\cdot x})$$

*One has the Plancherel formula*

$$\|f\|^2_{L^2(Q)} = \sum_{k\in\mathbb{Z}^d} |f_k|^2.$$

The following definition is from [3, Def. V.4.1, p.230]:

**Definition B.6.** *Let $H$ be a Hilbertspace.*

- *A subset $S \subset H$ is called an orthonormal system if for all $e, f \in S$, $e \neq f$ holds*

$$\|e\|_H = 1,$$
$$\langle e, f\rangle_H = 0.$$

- *An orthonormal system $S \subset H$ is called an orthonormal basis if for every orthonormal set $T$ with $S \subset T \subset H$ follows $T = S$.*

With this definition we can prove the next Corollary:

**Corollary B.7.** *Under the assumptions of the previous theorem* $\{e^{ik\cdot x}\}_{k\in\mathbb{Z}^d}$ *is an orthonormal basis in* $L^2(Q)$.

**Proof.**
We simply need to show, that $\{e^{ik\cdot x}\}_{k\in\mathbb{Z}^d}$ is an orthonormal set in $L^2(Q)$. The rest will follow due to the equivalence of the Placherel formula and the Parseval identity, which on the other hand is equivalent to the condition that $\{e^{ik\cdot x}\}_{k\in\mathbb{Z}^d}$ is a basis, see [3, Thm. V.4.9, p.234].
Let $k, j \in \mathbb{Z}^d$. If $k \neq j$ then there exists a $n \in \mathbb{N}$ so that $k_n \neq j_n$. For simplicity let $n = d$. With the theorem of Fubini, see [5, Chapter 5.2., p.175 et seq.] for it's proof, we get

$$\begin{aligned}(e^{ij\cdot x}, e^{ik\cdot x}) &\overset{Fubini}{=} (2\pi)^{-d} \int\limits_{[-\pi,\pi]^{d-1}} \int\limits_{-\pi}^{\pi} \cos\left((j_d - k_d)x_d + \sum_{n=1}^{d-1}(j_n - k_n)x_n\right) \\ &\qquad + i\sin\left((j_d - k_d)x_d + \sum_{n=1}^{d-1}(j_n - k_n)x_n\right)\ dx_d\ d[x_{d-1},..x_1] \\ &= (2\pi)^{-d} \int\limits_{[-\pi,\pi]^{d-1}} 0\ d[x_{d-1},..x_1] = 0.\end{aligned}$$

The last equation holds because we integrate the shifted cos and sin over $(j_d - k_d)\ 2\pi$. However if $k = j$ then

$$(e^{ij\cdot x}, e^{ik\cdot x}) = (2\pi)^{-d} \int\limits_Q 1\ dx = 1.$$

Therefore we have proven the statement. $\square$

In this thesis we need as well some properties of $e^{i(k+\frac{1}{2}e_2)\cdot x}$, where $e_2$ is the second basis unit vector, with $k \in \mathbb{Z}^d$. Therefore we look at the shifted lattice $\mathbb{Z}^d + \frac{1}{2}e_2$:

**Theorem B.8.** *Consider* $L^2(Q)$ *with* $Q := [-\pi, \pi]^d$ *and the inner product*

$$(u, v) := (2\pi)^{-d} \int\limits_Q u\bar{v}\ dx, \qquad u, v \in L^2(Q).$$

*The functions*

$$w_k(x) = e^{i(k+\frac{1}{2}e_2)\cdot x}, k \in \mathbb{Z}^d$$

*form an orthonormal basis. which implies the Parseval identity*

$$\|u\|^2_{L^2(\Omega)} = \sum_{k\in\mathbb{Z}^d} |(u, w_k)|^2.$$

**Proof.**
First we show that $\{w_k\}_{k\in Z^d}$ is an orthonormal set which means

$$(w_j, w_k) = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Then we show the following property which is equivalent to the statement that $\{w_k\}_{k\in\mathbb{Z}^d}$ is an orthonormal basis:

For any $v \in L^2(Q)$ with

$$(v, w_k) = 0, \qquad \forall k \in \mathbb{Z}^d$$

it holds that

$$v = 0.$$

The proof of the last statement is shown in [3, Thm. V.4.9, p.234].

1. Now show that $\{w_k\}_{k\in\mathbb{N}^d}$ is an orthonormal system:
Consider

$$\begin{aligned} (w_j, w_k) &= (2\pi)^{-d} \int\limits_Q e^{i(j+\frac{1}{2}e_2)\cdot x} e^{-i(k+\frac{1}{2}e_2)\cdot x}\, dx \\ &= (e^{ij\cdot x}, e^{ik\cdot x}) \\ &\overset{Cor.B.7}{=} \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases} \end{aligned}$$

2. Let $v \in L^2(\Omega)$ and

$$(v, w_k) = 0, \qquad \forall k \in \mathbb{Z}^d.$$

We know that $\{e^{ik\cdot x}\}_{k\in\mathbb{Z}^d}$ is an orthonormal basis in $L^2(Q)$. It holds that

$$\begin{aligned} 0 = (v, w_k) &= (v, e^{i(k+\frac{1}{2}e_2)\cdot x}) \\ &= (e^{-i\frac{1}{2}e_2\cdot x} v, e^{ik\cdot x}), \qquad \forall k \in \mathbb{Z}^d. \end{aligned}$$

We get with the Plancherel formula from Thm. B.5 that

$$\begin{aligned}\|v\|_{L^2(Q)}^2 &= \|e^{-i\frac{1}{2}e_2\cdot x}v\|_{L^2(Q)}^2 \\ &= \sum_{k\in\mathbb{Z}^d} |(e^{-i\frac{1}{2}e_2\cdot x}v, e^{ik\cdot x})|^2. \\ &= 0,\end{aligned}$$

Therefore $v \equiv 0$ and we get that $\{w_k\}_{k\in\mathbb{Z}^d}$ are indeed an orthonormal basis in $L^2(Q)$. The Parseval identity follows because $\{w_k\}_{k\in\mathbb{Z}^d}$ is an orthonormal basis and the identity is equivalent to that statement, see [3, Thm. V.4.9, p. 234] for more details. $\square$

# C. The space $V_\xi$

Let $\xi \in \mathbb{R}^2$. The space $V_\xi$ is defined as

$$\{\eta \in \mathbb{C}^2 \backslash \{0\} : \eta \cdot \eta = (\xi + \eta) \cdot (\xi + \eta) = 0\}.$$

For $d = 2$ and $\xi \neq 0$ the two equations $\eta \cdot \eta = 0$ and $(\xi + \eta) \cdot (\xi + \eta) = 0$ yield only two solutions. In case the first component $\xi_1$ of $\xi$ is not zero, we get for $\eta = x + iy$, $x, y \in \mathbb{R}^2$,

$$\begin{aligned}
x_2 &= \frac{-a}{b},\\
y_2 &= \pm\sqrt{\frac{1}{b}\Big(\Big(\frac{\xi_1^2 + \xi_2^2}{2\xi_1} - \frac{a}{b}\frac{\xi_2}{\xi_1}\Big)^2 + \frac{a^2}{b^2}\Big)},\\
x_1 &= \frac{\xi_1^2 + \xi_2^2 + 2x_2\xi_2}{-2\xi_1},\\
y_1 &= \frac{-\xi_2 y_2}{\xi_1},
\end{aligned}$$

where

$$\begin{aligned}
a &:= \frac{\xi_1^2 + \xi_2^2}{2\xi_1}\frac{\xi_2}{\xi_1},\\
b &:= \Big(1 + \frac{\xi_2^2}{\xi_1^2}\Big).
\end{aligned}$$

In case of $\xi_2 \neq 0$, but $\xi_1 = 0$, we get

$$\begin{aligned}
x_2 &= \frac{-\xi_2}{2},\\
y_2 &= 0,\\
x_1 &= 0,\\
y_1 &= \pm\frac{-\xi_2}{2}.
\end{aligned}$$

So for $\xi \neq 0$ the space $V_\xi$ is finite. To see the full computation see [6]. For $\xi = 0$ we are left with only one complex equation which yields the two conditions

$$\begin{aligned}
x_1^2 - y_1^2 + x_2^2 - y_2^2 &= 0,\\
2x_1 y_2 + 2x_2 y_2 &= 0.
\end{aligned}$$

Therefore $V_0$ is not finite.

# D. Comparison between different $\tilde{\Omega}$

In [6] we compared different $\tilde{\Omega}$ to see how good the approximation of the inverse Fourier transform is. All of the results in this chapter are from [6]. For a more detailed explanation see [6].
Let $\Omega = B((0.75, 0.75), 0.25\sqrt{2})$ and consider the function

$$w(x) = \begin{cases} \exp(\frac{1}{|x-x_m|^2 - \frac{8}{9}r^2}) & \text{if } |x - x_m|^2 < \frac{8}{9}r^2, \\ 0.0 & \text{otherwise,} \end{cases}$$

where $r = 0.25\sqrt{2}$ and $x_m = (0.75, 0.75)$. We compute the Fourier transform

$$\mathcal{F}w(\xi) = \int\limits_{\Omega} w(x)e^{-ix\cdot\xi}\, dx$$

after that we compute an approximation of the inverse Fourier transform

$$w_h^i(x) = \frac{1}{4\pi^2} \int\limits_{\tilde{\Omega}_i} \mathcal{F}w(\xi)e^{i\xi\cdot x}\, d\xi.$$

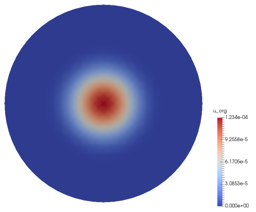
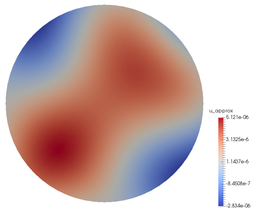
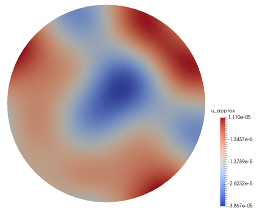
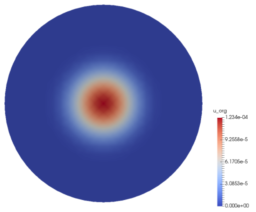
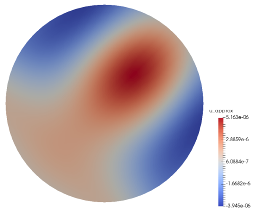
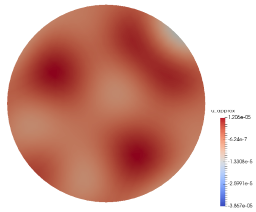
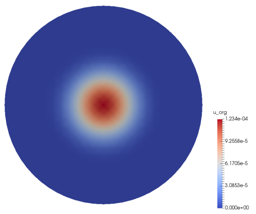
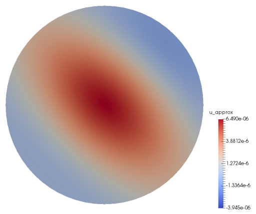
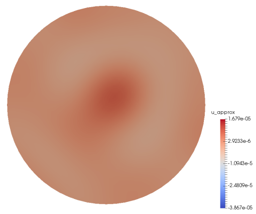
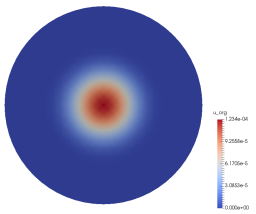
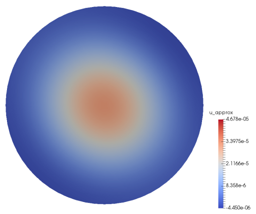
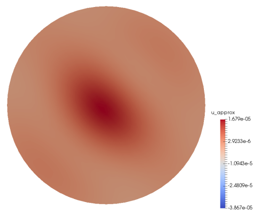
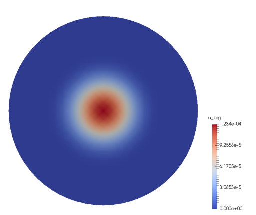
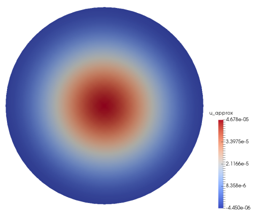
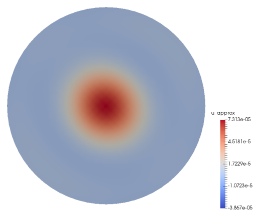
for different $\tilde{\Omega}_i$, $i = 1, ..., 4$. In the end we compare the result $w_h^i$ with the original $w$. All the following results are for the forth refinement level for $\Omega$ and different refinement levels for $\tilde{\Omega}_i$:
For $\tilde{\Omega}_1 = B((0.75, 0.75), 10.25\sqrt{2})$ we get the errors

| Lvl | $\|w_h^1 - w\|_{L^2(\Omega)}$ | eoc |
|---|---|---|
| 0 | 1.55798e-05 | |
| 1 | 1.56934e-05 | -0.0118285 |
| 2 | 1.50754e-05 | 0.0596308 |
| 3 | 1.08646e-05 | 0.475869 |
| 4 | 9.4266e-06 | 0.205185 |

Next we look at $\tilde{\Omega}_2 = B((0.75, 0.75), 20.25\sqrt{2})$ and get

| Lvl | $\|w_h^2 - w\|_{L^2(\Omega)}$ | eoc |
|---|---|---|
| 0 | 2.40924e-05 | |
| 1 | 1.64794e-05 | 0.618569 |
| 2 | 1.53221e-05 | 0.108073 |
| 3 | 1.37403e-05 | 0.158306 |
| 4 | 5.94448e-06 | 1.2109 |

To get a better understanding of the approximation we look at the graphics:

| Lvl | $w$ original | $\Re(w_h^1)$ | $\Re(w_h^2)$ |
|---|---|---|---|
| 0 |  |  |  |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

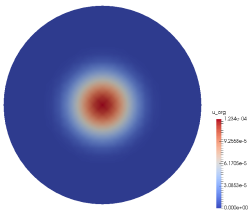
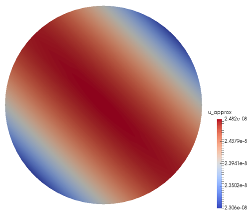
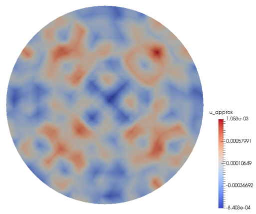
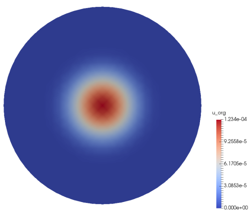
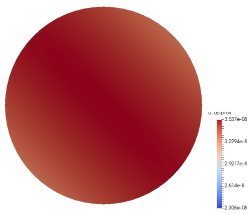
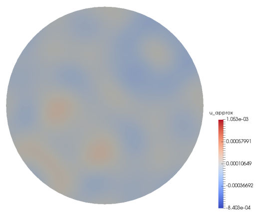
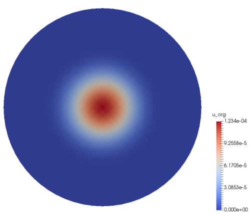
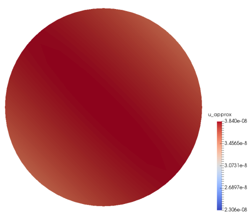
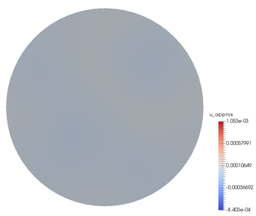
The graphical approximation for $\tilde{\Omega}_2$ is more fitting, but the error is better for $\tilde{\Omega}_1$. Next we compare this results with a very small and a very big $\tilde{\Omega}$ to get a better understanding for the choice of $\tilde{\Omega}$. Therefore we choose $\tilde{\Omega}_3 = B((0.75, 0.75), 0.25\sqrt{2})$ and get

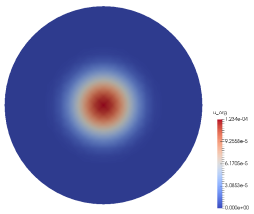
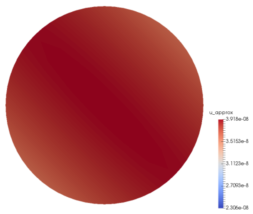
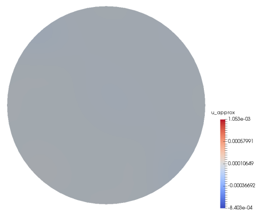
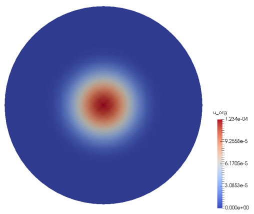
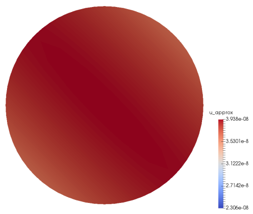
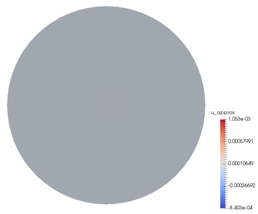
| Lvl | $\|w_h^3 - w\|_{L^2(\Omega)}$ | eoc |
|---|---|---|
| 0 | 1.63354e-05 | |
| 1 | 1.63328e-05 | 0.000254511 |
| 2 | 1.63321e-05 | 6.65223e-05 |
| 3 | 1.63319e-05 | 1.6857e-05 |
| 4 | 1.63319e-05 | 4.22949e-06 |

In comparison we choose $\tilde{\Omega}_4 = B((0.75, 0.75), 100.25\sqrt{2})$ and get the error

| Lvl | $\|w_h^4 - w\|_{L^2(\Omega)}$ | eoc |
|---|---|---|
| 0 | 0.00017215 | |
| 1 | 4.70614e-05 | 2.11233 |
| 2 | 2.34675e-05 | 1.03279 |
| 3 | 1.85175e-05 | 0.344169 |
| 4 | 1.46407e-05 | 0.339497 |

Again we compare the graphical approximation:

| Lvl | $w$ original | $\Re(w_h^3)$ | $\Re(w_h^4)$ |
|---|---|---|---|
| 0 |  |  |  |
| 1 |  |  |  |
| 2 |  |  |  |

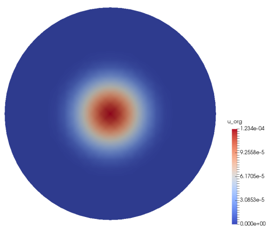
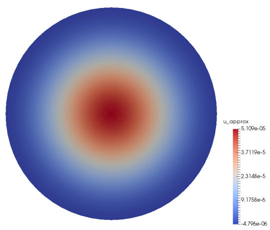
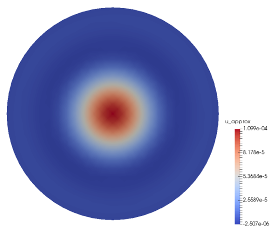
| Lvl | $w$ original | $\Re(w_h^3)$ | $\Re(w_h^4)$ |
|---|---|---|---|
| 3 |  |  |  |
| 4 |  |  |  |

We see that both $\tilde{\Omega}_3$ and $\tilde{\Omega}_4$ don't yield any good results..

## D.1. With refinement steps beforehand

For the forth refinement level of $\Omega$ we refine $\tilde{\Omega}_1$ five times and $\tilde{\Omega}_2$ six times beforehand, so that we get $h < 1$ in the beginning of our computations.

| $\|w - w_h^1\|_{L^2(\Omega)}$ | $\|w - w_h^2\|_{L^2(\Omega)}$ |
|---|---|
| 9.18703e-06 | 8.98132e-06 |

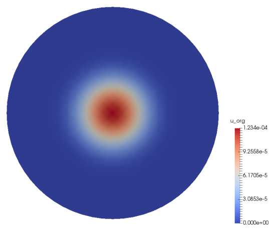
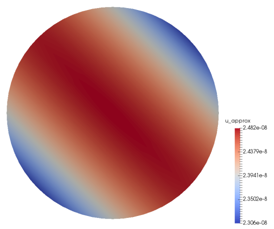
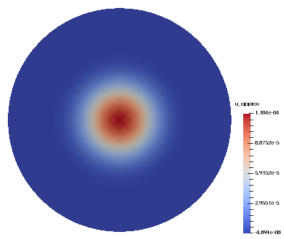
| $w$ original | $\Re(w_h^1)$ | $\Re(w_h^2)$ |
|---|---|---|
|  |  |  |

Here we see a better approximation in case of $\tilde{\Omega}_2$.

We don't refine the space $\tilde{\Omega}_3$ beforehand, because $h < 1$ is already full filled in the beginning. $\tilde{\Omega}_4$ we refine eight time beforehand to get $h < 1$ before we start the computations. As a result we get

| $\|w - w_h^3\|_{L^2(\Omega)}$ | $\|w - w_h^4\|_{L^2(\Omega)}$ |
|---|---|
| 9.58595e-06 | 8.98035e-06 |

For the graphical approximation we get

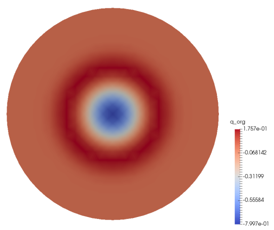
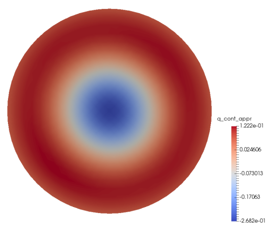
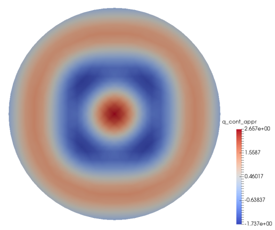
| $w$ original | $\Re(w_h^3)$ | $\Re(w_h^4)$ |
|---|---|---|
|  |  |  |

We see that we still get a bad approximation for $\tilde{\Omega}_3$ since it is just too small. In comparison we get a very good graphical approximation for $\tilde{\Omega}_4$. However we needed to first refine the space eight times to get this approximation, what is quit costly in computational time.
In case of the approximation of $q$ which was explained in chapter 4.1.1 we will get a different result:

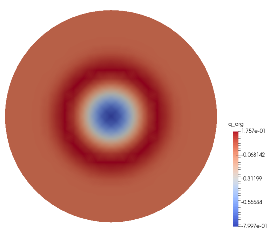
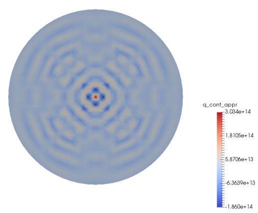
With the same refinement steps beforehand that were stated above we get

| $L^2$- error for $q$ in $\tilde{\Omega}_1$ | $L^2$- error for $q$ in $\tilde{\Omega}_2$ | $L^2$- error for $q$ in $\tilde{\Omega}_4$ |
|---|---|---|
| 0.0519215 | 0.110233 | 1.09258e+12 |

and for the graphical approximation

| $q$ original | $\Re(q_h^1)$ | $\Re(q_h^2)$ |
|---|---|---|
|  |  |  |

We see a much a better approximation for $\tilde{\Omega}_1$. On the other hand we have a very bad approximation for $\tilde{\Omega}_4$:

| $q$ original | $\Re(q_h^4)$ |
|---|---|
|  |  |

Here we do not see any kind of approximation. Therefore we see that $\tilde{\Omega}_1$ is the best choice we can make for $\tilde{\Omega}$ when we want to approximate $q$. That is the reason why we chose $\tilde{\Omega}_1$ for $\tilde{\Omega}$ in chapter 5.2.1.

# Bibliography

[1] A.Jüngel. Das kleine finite-elemente-skript, lecture notes. *http://www.asc.tuwien.ac.at/~juengel/scripts/femscript.pdf)*, Version 29.05.2014, Summer 2001.

[2] A.P.Calderón. On an inverse value problem. *In Seminar on Numerical Analysis and its Applications to Continuum Physics*, pages 65–73, 1980.

[3] D.Werner. *Funktionalanalysis.* Springer, Berlin Heidelberg New York, 2007.

[4] J.Sylvester G.Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2) 125 no. 1*, pages 153–169, 1987.

[5] J.Elstrodt. *Maß- und Integrationstheorie.* Springer, Berlin [u.a.], 2009.

[6] J.I.M.Hauser. The inverse problem for the dielectric conductivity. Projekt TM, Institut für Numerische Mathematik, Technische Universität Graz, 2017.

[7] W.Arendt K.Urban. *Partielle Differenzialgleichungen - Eine Einführung in analytische und numerische Methoden* . Spektrum Akademischer Verlag, Heidelberg, 2010.

[8] L.C.Evans. *Partial Differential Equations.* American Math. Soc, Providence, RI, 2010.

[9] K.Astala L.Päivärinta. Calderón's inverse conductivity problem in the plane. *Annals of Mathematics*, 163:265–299, 2006.

[10] M.Salo. Calderón problem, lecture notes. *http://www.rni.helsinki.fi/~msa/lecturenotes/calderon_lectures.pdf*, Version 29.05.2014, Spring 2008.

[11] O.Steinbach. *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements.* Springer, New York, NY, 2008.

[12] R.M.Brown R.H.Torres. Uniquness in the inverse conductivity problem for conductivities with 3/2 derivatives in $l^p$ ; $p > 2n$. *J. Fourier Anal. Appl. 9 no. 6*, pages 563–574, 2003.

[13] S.Siltanen. *Electrical Impedance Tomography and Fadeev's Green functions, Thesis (Ph.D.)-Teknillinen Korkeakoulu.* Helsini, 1999.

[14] H.Cornean S.Siltanen, K.Knudsen. Towards a d-bar reconstruction method for thee-dimensional eit. *J. Inverse Ill-Posed Probl. 14*, pages 111–134, 2006.

[15] W.McLean. *Strongly Elliptic Systems and Boundary Integral Equations.* Cambridge University Press, Cambridge, UK, 2000.

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