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Zero Subsums and Their Weighted Generalization

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Introduction

Denote by $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ the smallest number of elements such that in any sequence in $\mathbb{Z}_n^d$ of length at least $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ one can find a subsequence of length n whose sum is zero modulo n. In Section 1 (The Number $\mathrm{s}\left(\mathbb{Z}_n^d\right)$) the state of the art is collected and edited in a uniform notation. Particularly, Section 1.2.1 (Edel Lifting) is devoted to the clever construction of sequences due to Edel [Ede08]. The subsequent main part of this thesis, Section 2 (Weighted Generalization), deals with the following weighted generalization of $\mathrm{s}\left(\mathbb{Z}_n^d\right)$. A sequence contains an A-weighted zero subsum of length n, if it contains a subsequence $v_1, \ldots, v_n$ of length n and corresponding weights $a_1, \ldots, a_n \in A$ such that

$$\sum_{i=1}^{n} a_i v_i = 0.$$

Accordingly, denote by $\mathrm{s}_A\left(\mathbb{Z}_n^d\right)$ the smallest number of elements such that any sequence in $\mathbb{Z}_n^d$ of length at least $\mathrm{s}_A\left(\mathbb{Z}_n^d\right)$ contains an A-weighted zero subsum of length n.

Trivially, $\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) \leq \mathrm{s}\left(\mathbb{Z}_n^d\right)$. In Example 2.9 (Dimension of ones) a new construction is presented that relates these two numbers the other way round

$$\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) \geq \mathrm{s}\left(\mathbb{Z}_n^{d-1}\right).$$

This enables to reuse the lower bounds Theorem 1.54 (Edel) and Theorem 1.73 (Edel) from the unweighted case.

In contrast to the unweighted case, one can mutually interchange some vectors in a sequence in the sense of Observation 2.45 (A-weighted transformation) without affecting A-weighted zero subsums. This effect has been defined in Definition 2.4 (A-distinct) and has been discussed in Lemma 2.16 (A-distinct).

Another difference in the weighted case is the zero vector. Although the zero vector is present in all maximum sequences regarding the unweighted number $\mathrm{s}\left(\mathbb{Z}_n^d\right)$, the contrary seems to be true in the weighted case, see Lemma 2.21 (Zero vector). Therefore, a new Notation 2.22 (Allowed vectors) has been introduced to explicitly exclude the zero vector. With the help of this Notation the currently best known upper bound $\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) \leq \frac{n^d-1}{2}(n-1)+1$ by Godinho, Lemos, and Marques [GLM13, Theorem 1] has been improved in Lemma 2.51 (±-weighted upper bound)

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) \leq \frac{n^d - 1}{n - 1} \left(n + \lfloor \log_2 n \rfloor - 1\right) + n.$$

Furthermore, the zero vector has been investigated in the ternary case $\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right)$ in Lemma 2.60 (Zero vector) which enabled to preserve most of the proofs of the unweighted case. Thus, those values are known as well, especially their relation to ternary projective caps, see Corollary 2.64 (±-weighted ternary values in low dimensions).

In dimension $d = 1$, the Cauchy–Davenport Theorem applies to A-weighted zero subsums. In this way, an upper bound to sensible weights A has been found in Corollary 2.48 (Maximal weights)

$$\mathrm{s}_A(\mathbb{Z}_p) = p + 1$$

where $|A| \geq \frac{p-1}{2}$ and Property D has been disproved regarding $\mathrm{s}_A(\mathbb{Z}_p)$ where $|A| \geq 2$ in Lemma 2.49 (Weighted Property D).

Finally, a computer has been consulted, see Remark 1.28 (Computer implementation). Not only known sequences have been rediscovered but also new ones, see Examples 2.75 (±-weighted dimension 4) and 2.80 (Maximal ±-weighted quinary sequence). Let $n \geq 3$, odd.

$$\begin{aligned} \mathrm{s}_{\pm}\left(\mathbb{Z}_n^4\right) &\geq 4\,(n - 1) + 1 \\ \mathrm{s}_{\pm}\left(\mathbb{Z}_5^5\right) &\geq 21\,(5 - 1) + 1. \end{aligned}$$

The latter quinary sequence is interesting insofar as all other known maximum sequences so far are ternary sequences. Moreover, some observations of computer results have been made, see Observation 2.34 and Conjectures 2.53 (Weighted dimension 2), 2.54 (±-weighted dimension 3), and 2.78 (±-weighted dimension 4).

1 The Number $\mathrm{s}\left(\mathbb{Z}_n^d\right)$

Let $(G,+)$ be a finite abelian group. Find the smallest number ℓ, such that any sequence (multiset) S in G of length $|S| \geq \ell$ contains a zero subsum with certain desired properties. Typically, such desired properties concern the length (number of elements) of a zero subsum.

Definition 1.1 (Zero subsum)**.** A sequence S in a finite abelian group contains a zero subsum of length k, if it contains a subsequence T of length k such that $\sum_{v \in T} v = 0$

Notation 1.2 (Exponent)**.** The exponent $\exp G$ of a finite abelian group G is the least common multiple of the orders of the elements of G, respectively written multiplicatively, the least positive integer such that $\forall x \in G : x^{\exp G} = 1$.

Notation 1.3 (Zero subsum)**.** Let $(G,+)$ be a finite abelian group. The number $\mathrm{s}(G)$ denotes the smallest integer, such that every sequence S in G of length $|S| \geq \mathrm{s}(G)$ contains a zero subsum of length $\exp G$.

The fundamental theorem of finitely generated abelian groups states that every finite abelian group decomposes into cyclic groups. Therefore, the most fundamental case is the study of cyclic groups $(\mathbb{Z}_n,+)$ and more generally $\left(\mathbb{Z}_n^d,+\right)$ where $\mathbb{Z}_n^d$ denotes the d-dimensional vector space of vectors modulo n: $\left(\mathbb{Z}/n\mathbb{Z}\right)^d$.

In the literature notation varies. Initially number theorists used a standard notation $f(n,d)$ [Har73]. Later, algebraists reformulated the problem for general finite abelian groups thereby introducing $\mathrm{s}(G)$. There, cyclic groups are written multiplicatively, this is $\mathrm{s}(C_n^r)$.

The one-dimensional problem $\mathrm{s}(\mathbb{Z}_n)$ originated from Erdős, Ginzburg, and Ziv [EGZ61]. Initially, Harborth [Har73] posed it as lattice point problem. The number $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ reformulated reads as follows: What is the minimum integer such that for every set of at least $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ many lattice points in the d-dimensional Euclidean space one can find n among them whose centroid is also a lattice point?

The ternary case $\mathrm{s}\left(\mathbb{Z}_3^d\right)$ relates to ternary affine caps, see Section 1.1 (Ternary Case).

Notation 1.4 (Zero subsums)**.** Along the same lines as $\mathrm{s}(G)$ there is a variety of similar interesting questions, see [EEG+07].

$\mathrm{s}(G)$ denotes the smallest number of elements such that any sequence S in G of length $|S| \geq \mathrm{s}(G)$ contains a zero subsum of length $\exp G$.

$\eta(G)$ denotes the smallest number of elements such that any sequence S in G of length $|S| \geq \eta(G)$ contains a short zero subsum, that is a zero subsum of length between $[1, \exp G]$.

$\mathrm{g}(G)$ denotes the smallest number of elements such that any sequence S of distinct elements (also called square-free) in G of length $|S| \geq \mathrm{g}(G)$, contains a zero subsum of length $\exp G$.

$\mathrm{D}(G)$ the Davenport constant denotes the smallest number of elements such that any sequence S in G of length $|S| \geq \mathrm{D}(G)$ contains a nonempty zero subsum (of arbitrary length).

$\mathrm{E}(G)$ denotes the smallest number of elements such that any sequence S in G of length $|S| \geq \mathrm{E}(G)$ contains a zero subsums of length $|G|$.

Remark 1.5 (Exponent). In dimension $d = 1$ the numbers $\mathrm{s}(\mathbb{Z}_n)$ and $\mathrm{E}(\mathbb{Z}_n)$ coincide as $\exp \mathbb{Z}_n = n = |\mathbb{Z}_n|$.

$$\mathrm{s}(\mathbb{Z}_n) = \mathrm{E}(\mathbb{Z}_n)$$

Remark 1.6 (Gao). [Gao96, Theorem 1] The Davenport constant $\mathrm{D}(G)$ and its sibling $\mathrm{E}(G)$ are closely related:

$$\mathrm{E}(G) = \mathrm{D}(G) + |G| - 1.$$

In this thesis, mainly $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ is discussed, namely the smallest number ℓ, such that among any ℓ vectors in $\mathbb{Z}_n^d$ one can find n of them whose sum is zero modulo n.

In the study of both lower and upper bounds of $\mathrm{s}(G)$, one has to deal with sequences S handily.

Notation 1.7 (Free monoid)**.** Let G be a set. The free monoid of G is the monoid of all finite sequences in G together with string concatenation as monoid operation written multiplicatively.

A sequence can be written as product $S = \prod_{v \in S} v$ and furthermore powers are interpreted as repetition $v^k = \underbrace{v, \ldots, v}_{k \text{ times}}$. Note that the order of a sequence is irrelevant in the study of $\mathrm{s}(G)$. Therefore, one identifies sequences with different orderings $S^k = \underbrace{v_1, \ldots, v_1}_{k \text{ times}}, \ldots, \underbrace{v_{|S|}, \ldots, v_{|S|}}_{k \text{ times}}$.

Example 1.8 (Dimension 1)**.** Already one of the simplest sequence one can imagine, namely $S = 0, 1$ respectively S^{n-1} is quite interesting. Assume there is a zero subsum of length n modulo n in S^{n-1}. But then solely zeros or solely ones have been taken n times despite they are only available $n-1$ times. This proves $\mathrm{s}(\mathbb{Z}_n) \geq |S^{n-1}| + 1 = 2\,(n-1) + 1$.

Example 1.9 (Affine basis)**.** The previous example can be trivially generalized to higher dimensions to show $\mathrm{s}\left(\mathbb{Z}_n^d\right) \geq (d+1)\,(n-1) + 1$. Let S in $\mathbb{Z}_n^d$ consist of the zero vector together with the standard basis. For example in dimension 3

$$S = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Example 1.10 (Harborth [Har73])**.** Being a bit more careful one can even take all 0-1-vectors $n-1$ times, which establishes the lower bound $\mathrm{s}\left(\mathbb{Z}_n^d\right) \geq 2^d\,(n-1) + 1$. Again, for example in dimension 3

$$S = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The proof generalizes from the one in Example 1.8 (Dimension 1) naturally.

Proof. Assume there is a zero subsum of length n modulo n in S^{n-1}. Then in each coordinate solely zeros or solely ones have been taken n times. Hence a single vector has been taken n times despite they are all available just $n-1$ times. $\square$

Another way to prove this is by lifting Example 1.8 (Dimension 1) to higher dimensions as in Lemma 1.56 (Product construction).

Example 1.11 (Edel)**.** [Ede08, Lemma 11] Not alone sequences without zero subsums can be useful. Consider the following sequence

$$S = 0, 1, 2.$$

How do possible zero subsums in S^{n-1} look like? Let a_0, a_1, a_2 denote the number of occurrences of zeros, ones, and twos in a zero subsum of length n in S^{n-1}. Expressing the zero subsum:

$$a_0 \cdot 0 + a_1 \cdot 1 + a_2 \cdot 2 \equiv 0 \pmod{n}$$

This sum cannot be $0n$ or $2n$ as they can only be achieved by n zeros or n twos, though so many are not available. The congruency simplifies to an equation

$$a_1 + 2a_2 = n$$

Expressing a_0 and a_1 in terms of a_2 results in $a_1 = n - 2a_2$, $a_0 = n - a_1 - a_2 = a_2$. Summarized, zero subsums T in S^{n-1} are of shape

$$T = \left(0\right)^a, \left(1\right)^{n-2a}, \left(2\right)^a, \quad 0 < a \leq \frac{n-1}{2}.$$

This sequence comes in handy later in Edel's product construction, see Example 1.67 (Edel).

Remark 1.12 (Maximal vs. maximum)*.* There is a subtle distinction between maximal sequence S (cannot be extended) and maximum sequence S (of maximum cardinality). For example, a maximum sequence regarding $\mathrm{s}(G)$ means a sequence S in G of length $|S| = \mathrm{s}(G) - 1$ without zero subsums of length $\exp G$. An illustration example: $S = 0, 0, 0, 1, 2$ is a maximal sequence regarding $\mathrm{s}(\mathbb{Z}_4)$ because extending it by any number permits a zero subsum of length 4. However, S is not a maximum sequence, since $0, 0, 0, 1, 1, 1$ is a longer sequence without zero subsums of length 4.

In order to reshape and compare different sequences, it is useful to apply affine transformations. This does not affect zero subsums.

Lemma 1.13 (Affine transformation)**.** *Let S be a sequence in $\mathbb{Z}_n^d$ and f be an affine transformation*

$$\begin{aligned} f : \mathbb{Z}_n^d &\to \mathbb{Z}_n^d \\ v &\mapsto Mv + b \end{aligned}$$

for some $M \in \mathrm{GL}\left(\mathbb{Z}_n^d\right), b \in \mathbb{Z}_n^d$. Then S contains a zero subsum of length n if and only if $f(S) = \prod_{v \in S} f(v)$ does.

Proof. A subsequence T of length n in S is a zero subsum if and only if $f(T)$ is a zero subsum in $f(S)$.

$$\sum_{\tilde{v}\in f(T)} \tilde{v} = \sum_{v\in T} f(v) = \sum_{v\in T} (Mv+b) = M\sum_{v\in T} v + nb \equiv M\sum_{v\in T} v \pmod{n}$$

Moreover, M is invertible, hence,

$$\sum_{\tilde{v}\in f(T)} \tilde{v} \equiv 0 \Leftrightarrow \sum_{v\in T} v \equiv 0 \pmod{n}$$

□

Example 1.14 (Affine transformation)**.** Besides simple affine transformations like row transpositions one can also scale an entire dimension by a factor α coprime to n. Furthermore, an affine transformation may also just be a shift $f(v) = v + b$. For instance Example 1.9 (Affine basis) in dimension 2

$$S = \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$$

shifting by $b = \begin{pmatrix}1 & 1\end{pmatrix}^\intercal$ changes into

$$f(S) = \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$$

Remark 1.15 (Affine basis). As a further consequence of Lemma 1.13 (Affine transformation), when constructing large sequences one may assume that a maximal sequence contains an affine basis such as Example 1.9 (Affine basis). Why is this? The zero vector is obtained by shifting an arbitrary vector. Applying Gaussian elimination also the basis vectors are attained. Note that all three types of elementary matrices in Gaussian elimination are invertible and thus are legitimate affine transformations.

Studying centroids of lattice points, Harborth established basic lower and upper bounds.

Lemma 1.16 (Harborth)**.** *[Har73, Hilfssatz 1]*

$$2^d(n-1)+1 \le \mathrm{s}\left(\mathbb{Z}_n^d\right) \le n^d(n-1)+1$$

Proof. The lower bound has already been discussed in Example 1.10 (Harborth [Har73]). For the upper bound apply the pigeonhole principle: $\mathbb{Z}_n^d$ consists of n^d vectors. Accordingly, among more than $n^d(n-1)$ vectors, one of them must occur at least n times, which gives a zero subsum. □

For large n or d one can do much better: Either replace the term n^d by a (large) constant depending only on d or think of n as constant and replace n^d by something smaller.

Theorem 1.17 (Alon and Dubiner)**.** *[AD95, Theorem 1.1,3 Open problems]*

$$\begin{aligned} \mathrm{s}\left(\mathbb{Z}_n^d\right) &= \mathrm{O}(n)\,, \quad \text{as } n \to \infty \\ \mathrm{s}\left(\mathbb{Z}_n^d\right) &= \mathrm{o}\left(n^d\right)\,, \quad \text{as } d \to \infty \end{aligned}$$

The latter has been improved further. The result had previously been proved by Silke Kubertin (unpublished manuscript, around 2005).

Theorem 1.18 (Liu and Spencer)**.** *[LS09, Theorem 1]*

$$\mathrm{s}\left(\mathbb{Z}_n^d\right) = \mathrm{O}\left(\frac{n^d}{d^{n-2}}\right), \quad \text{as } d \to \infty$$

The number $\mathrm{s}(G)$ is closely related with $\eta(G)$ and $\mathrm{g}(G)$.

Observation 1.19 (Gao)*.* [Gao03, Lemma 2.2]

$$\mathrm{s}\left(\mathbb{Z}_n^d\right) \geq \eta\left(\mathbb{Z}_n^d\right) + n - 1.$$

Proof. Let T be a maximum sequence regarding $\eta\left(\mathbb{Z}_n^d\right)$. Then $S = \underbrace{0, \ldots, 0}_{n-1 \text{ times}}, T$ does not contain zero subsums of length n since otherwise by removing all zeros from a zero subsum in S a short zero subsum in T is obtained. In particular,

$$\mathrm{s}\left(\mathbb{Z}_n^d\right) \geq |T| + 1 = |S| + n - 1 + 1 = \eta\left(\mathbb{Z}_n^d\right) + n - 1$$

□

So far, all known maximum sequences regarding $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ fulfill the other direction as well. This is assumed to be true for all maximum sequences.

Conjecture 1.20 (Gao)**.** *[Gao03, Conjecture 2.3]*

$$\mathrm{s}(G) = \eta(G) + \exp G - 1.$$

Geroldinger and Halter-Koch [GH06, Theorem 5.8.3] proved this Conjecture for groups of rank at most 2.

The sequences in Examples 1.8 to 1.10 are all of the shape S^{n-1}, which means every vector occurs exactly $n-1$ times. It seems that this is true for all maximum sequences. Van Emde Boas [vEmd69] formulated this for $\eta(\mathbb{Z}_n^2)$ and later Gao and Thangadurai [GT03, Definition 1.1] stated it for general dimensions $\mathrm{s}\left(\mathbb{Z}_n^d\right)$.

Conjecture 1.21 (Property D)**.** *Every maximum sequence S regarding $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ is of shape $S = T^{n-1}$ for some sequence T.*

Gao and Thangadurai [GT03, Theorem 1] proved that Property D is multiplicative with respect to n. Property D has been verified for $\mathrm{s}\left(\mathbb{Z}_2^d\right)$ in Theorem 1.36 (Harborth), $\mathrm{s}\left(\mathbb{Z}_3^d\right)$ in Theorem 1.44 (Harborth), $\mathrm{s}(\mathbb{Z}_5^2)$ by Gao [Gao00], and $\mathrm{s}(\mathbb{Z}_5^3)$ by Gao *et al.* [GHST07, Proposition 5.6].

Remark 1.22 (Gao and Thangadurai). [GT03, Corollary 1.2(i)] Property D implies Conjecture 1.20 (Gao).

Proof. As mentioned in the motivating Observation 1.19 (Gao) it suffices to prove $\mathrm{s}\left(\mathbb{Z}_n^d\right) \leq \eta\left(\mathbb{Z}_n^d\right) + n - 1$. Let S be a maximum sequence regarding $\mathrm{s}\left(\mathbb{Z}_n^d\right)$. We may not only assume that the zero vector is part of this sequence, see Remark 1.15 (Affine basis), but by assumption even that it is present $n-1$ times. Then the subsequence of S without the zero vectors does not contain short zero subsums since otherwise by adding up to $n-1$ many zero vectors we would obtain a zero subsum of length n in S. In particular,

$$\eta\left(\mathbb{Z}_n^d\right) \geq |S| - (n-1) + 1 = \mathrm{s}\left(\mathbb{Z}_n^d\right) - (n-1)\,.$$

□

The following property is closely related to Property D.

Definition 1.23 (Property G)**.**

$$\mathrm{s}\left(\mathbb{Z}_n^d\right) = \left(\mathrm{g}\left(\mathbb{Z}_n^d\right) - 1\right)(n-1) + 1$$

Remark 1.24 (Property G). Note that the direction "$\leq$" is always true:

$$\mathrm{s}\left(\mathbb{Z}_n^d\right) \leq \left(\mathrm{g}\left(\mathbb{Z}_n^d\right) - 1\right)(n-1)+1$$

Proof. Let S be a sequence in $\mathbb{Z}_n^d$ of length at least $\left(\mathrm{g}\left(\mathbb{Z}_n^d\right) - 1\right)(n-1)+1$. To show: S contains a zero subsum of length n. By the pigeonhole principle, either a vector occurs n times or there are $\mathrm{g}\left(\mathbb{Z}_n^d\right)$ distinct vectors. In both cases we have found a zero subsum of length n as requested. □

Remark 1.25. Property G implies Property D.

Proof. Let S be a sequence in $\mathbb{Z}_n^d$ of length $|S| = \mathrm{s}\left(\mathbb{Z}_n^d\right) - 1 = \left(\mathrm{g}\left(\mathbb{Z}_n^d\right) - 1\right)(n-1)$ without zero subsums of length n and $T \subseteq S$ be its subsequence of distinct vectors. As n times the same vector gives a zero subsum, $T^{n-1} \supseteq S$. On the contrary, T does not contain zero subsums as well, which limits its length by $|T| \leq \mathrm{g}\left(\mathbb{Z}_n^d\right) - 1 = \frac{|S|}{n-1}$, whence, $|T^{n-1}| \leq |S|$. It follows $T^{n-1} = S$. □

Frequently, it remains to consider $\mathrm{s}\left(\mathbb{Z}_p^d\right)$ only for primes p, when dealing with upper bounds. This is due to the following Lemma by Erdős, Ginzburg, and Ziv [EGZ61] and Harborth [Har73, Hilfssatz 2].

Lemma 1.26 (Multiplicativity)**.**

$$\begin{aligned}\mathrm{s}\left(\mathbb{Z}_{nm}^d\right) \leq \min\Big\{ &\mathrm{s}\left(\mathbb{Z}_n^d\right) + n\left(\mathrm{s}\left(\mathbb{Z}_m^d\right) - 1\right), \\ &\mathrm{s}\left(\mathbb{Z}_m^d\right) + m\left(\mathrm{s}\left(\mathbb{Z}_n^d\right) - 1\right)\Big\}\end{aligned}$$

Proof. It suffices to consider the first upper bound since n and m can be swapped. The proof constructs a zero subsum as required.
As long as there are at least $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ many vectors, remove a zero subsum modulo n of length n. Assume k subsums have already been removed.

$$\mathrm{s}\left(\mathbb{Z}_n^d\right) + n\left(\mathrm{s}\left(\mathbb{Z}_m^d\right) - 1\right) - kn \geq \mathrm{s}\left(\mathbb{Z}_n^d\right) \Leftrightarrow k \leq \mathrm{s}\left(\mathbb{Z}_m^d\right) - 1$$

Consequently, one ends up with $\mathrm{s}\left(\mathbb{Z}_m^d\right)$ many zero subsums T_i. Among their well-defined means

$$\bar{x}_i = \frac{1}{n}\sum_{x \in T_i} x \in \mathbb{Z}_n^d$$

we can again find m of them which sum to zero modulo m.

$$\sum_{i \in I} \bar{x}_i \equiv 0 \pmod{m}$$

Accordingly, a zero subsum $T = \biguplus_{i \in I} T_i$ modulo nm of length nm has been obtained.

$$|T| = \sum_{i \in I} |T_i| = mn$$
$$\sum_{x \in T} x = \sum_{i \in I} \sum_{x \in T_i} x = n \sum_{i \in I} \bar{x}_i \equiv 0 \pmod{nm}$$

□

Corollary 1.27 (Multiplicativity)**.** *It suffices to prove upper bounds of type* $\mathrm{s}\left(\mathbb{Z}_n^d\right) \leq c_d\,(n-1) + 1$ *just for primes* $n = p$*.*

Proof. Assume the upper bound has already been verified for n and m. Then

$$\begin{aligned} \mathrm{s}\left(\mathbb{Z}_{nm}^d\right) &\leq \mathrm{s}\left(\mathbb{Z}_n^d\right) + n\left(\mathrm{s}\left(\mathbb{Z}_m^d\right) - 1\right) \\ &\leq c_d\,(n-1) + 1 + n\,(c_d\,(m-1) + 1 - 1) \\ &= c^d\,(nm-1) + 1 \end{aligned}$$

□

Remark 1.28 (Computer implementation). It helps to verify small examples with a computer. A method is needed that takes a bunch of vectors in $\mathbb{Z}_n^d$ and looks at all zero subsums of length n. There are usually a lot of possible sequences S to test, therefore this method should be as fast as possible. Often, one assumes Conjecture 1.21 (Property D) which reduces $|S|$ by a factor of $n-1$. Lemma 1.16 (Harborth) tells us to cope with $|S| \geq 2^d$ many different vectors.

Iterativity. The naive brute force approach of enumerating all $\binom{|S|}{n} \geq \binom{2^d}{n}$ combinations of vectors does not seem to be the right choice. Throughout this thesis, an iterative approach has been chosen: Keep track of all possible values of subsums of length k for $k = 2, \ldots, n-1$. The idea behind this is that a lot of subsums of length k will have the same value already for small k and thus reduce the number of operations. Why is this? Compare the number of subsums $\binom{|S|}{k} \geq \binom{2^d}{k}$ to the number of possible values $\left|\mathbb{Z}_3^d\right| = 3^d$. For example, already in the ternary case $\mathbb{Z}_3^d$ there are collisions of subsums of length $k = 2$ starting at dimension $d = 3$ since $3^3 = 27 < 28 = \binom{2^3}{2} \leq \binom{|S|}{2}$ and much more as d grows. Let $s_k \subseteq \mathbb{Z}_n^d$ be the set of values of subsums of length k. The two essential methods are:

A test whether a vector $v \in \mathbb{Z}_n^d$ can be safely added to S without creating zero subsums of length n.

function Sequence.Test(v)
  return $-v \in s_{n-1}$
end function

The procedure that adds such a vector v to the sequence S and updates the corresponding sets s_k where $s_k + v$ denotes element-wise addition.

procedure Sequence.Add(v)
  for $k = n-1, n-2, \ldots, 3$ **do**
    $s_k \leftarrow s_k \cup (s_{k-1} + v)$
  end for
  $s_2 \leftarrow s_2 \cup (S + v)$
  $S \leftarrow S \cup \{v\}$
end procedure

With these two methods greedy and complete enumeration is possible. In the greedy approach, vectors $v \in \mathbb{Z}_n^d$ are greedily added to obtain maximal sequences which serve as lower bounds for $\mathrm{s}\left(\mathbb{Z}_n^d\right)$, see Remarks 1.35 (Greedy confirmation of dimension 3), 1.49 (Ternary greedy),1.63 (Odd greedy), 2.66 ($\pm$-weighted ternary greedy), and Example 2.80 (Maximal $\pm$-weighted quinary sequence). For not too small n and dimension d it is also possible to enumerate all maximal sequences of length m for $m = 1, \ldots, \mathrm{s}\left(\mathbb{Z}_n^d\right) - 1$. This way, not only $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ can be determined but also the distribution of maximal sequences and shapes of maximum sequences, see Remarks 1.48 (Ternary enumeration), 2.65 ($\pm$-weighted ternary enumeration), 2.77 ($\pm$-weighted dimension 4 greedy), and Appendix A.1 (Maximum Sequences).

Addition tables. Modulo operations in multiple dimensions can be speed up dramatically by precalculated tables. A straightforward initialization of these tables can take quite a while. For example, in $\mathbb{Z}_3^6$ the addition table has $(3^6)^2 = 531\,441$ entries. Partially initialized addition tables can be reused already during the initialization: Two lookups and no further addition suffices in most of the cases

$$u + v = (u + (v - 1)) + 1.$$

Vector representation. Now that we have talked about precalculated addition tables, we should think of an efficient choice of vector representation. Efficient here means regarding the memory size and possibility to store. We don't have to care about addition speed as this is covered by the addition tables. An optimal choice of storing vectors in $\mathbb{Z}_n^d$ are n-ary numbers with d digits. Consequently, vectors are represented by plain integer numbers. For example, in $\mathbb{Z}_7^4$

$$\begin{pmatrix} 3 \\ 1 \\ 6 \\ 2 \end{pmatrix} \mapsto 2613_7 = 990_{10} = 1111011110_2.$$

Bit array The execution speed of the presented methods Test and Add depends on a suitable set data structure. A bit array is the simplest and by far the fastest one, provided the values are represented as integers.

procedure BitArray.Add($i \in \mathbb{N}$)
    $a_i \leftarrow$ **true**
end procedure
function BitArray.Contains($i \in \mathbb{N}$)
    return a_i
end function

Sequence representation. When generating maximal sequences, the same sequence may appear a lot of times "disguised", because there are a lot of symmetries. We aim at a unique sequence representation that is invariant under these symmetries. Because of Remark 1.15 (Affine basis) we can assume that any sequence contains an affine basis. Additionally, sort the vectors in lexicographic order. The dimensions can be reordered as well even further reducing the degree of freedom.

Dimension 1 has been solved by Erdős, Ginzburg, and Ziv in their influential paper [EGZ61].

Theorem 1.29 (Erdős, Ginzburg, and Ziv)**.** *[EGZ61] Any $2n-1$ integers contain n of them whose sum is divisible by n.*

Proof idea. Induction on the number of prime factors of n. □

In other words $\mathrm{s}(\mathbb{Z}_n) \leq 2n-1$. Together with Example 1.8 (Dimension 1) this yields equality:

$$\mathrm{s}(\mathbb{Z}_n) = 2\,(n-1)+1.$$

Here, a simpler proof is presented as in the original paper that generalizes to dimension $d=2$ Theorem 1.31 (Reiher), as well as to the weighted case Theorem 2.52 (Adhikari *et al.*). Yet another proof relies on Theorem 2.42 (Cauchy–Davenport).

Theorem 1.30 (Chevalley's theorem, [Che35])**.** *Let p be a prime and let*

$$\begin{aligned}&f_1(X_1,\dots,X_n)\\&\vdots\\&f_m(X_1,\dots,X_n)\end{aligned}$$

be some polynomials in the ring $\mathbb{Z}_p[X_1,\dots,X_n]$ such that the sum of their total degrees $\sum_{i=1}^m \deg(f_i)$ is strictly less than the number of variables n. If the trivial solution $x_1=\cdots=x_n=0$ is a common zero of the polynomials $f_1,\dots,f_m$ then they share another one.

Proof of Theorem 1.29 (Erdős, Ginzburg, and Ziv). The lower bound has already been established in Example 1.8 (Dimension 1). It suffices to consider the upper bound of $\mathrm{s}(\mathbb{Z}_p)$ for primes p because of Corollary 1.27 (Multiplicativity). Let $n=2\,(p-1)+1$ and $S=a_1,\dots,a_n$ be a sequence of length n in $\mathbb{Z}_p$. Consider the following clever polynomials $f_1,f_2\in\mathbb{Z}_p[X_1,\dots,X_n]$.

$$\begin{aligned}f_1\,(X_1,\dots,X_n) &= \sum_{i=1}^{n} a_i X_i^{p-1}\\ f_2\,(X_1,\dots,X_n) &= \sum_{i=1}^{n} X_i^{p-1}.\end{aligned}$$

What is the purpose of these two polynomials? $X_i^{p-1}\in\{0,1\}$ indicates whether an element has been selected, $f_1=0$ ensures that they form a zero subsum, and $f_2 = 0$ validates the number of selected elements to be a multiple of p between

$$0 \leq \sum_{i=1}^{n} X_i^{p-1} \leq n = 2p - 1 < 2p.$$

Altogether, nontrivial solutions of $f_1 = f_2 = 0$ correspond to zero subsums in S of length p. The polynomials share the trivial solution and the sum of their total degrees is $2\,(p-1) < n$ as required, thus Theorem 1.30 (Chevalley's theorem, [Che35]) applies and ensures a nontrivial solution to $f_1 = f_2 = 0$. ☐

Dimension 2 turned out to be much more difficult. From Example 1.10 (Harborth [Har73]), namely the sequence S^{n-1} where

$$S = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

it follows the lower bound $\mathrm{s}(\mathbb{Z}_n^2) \geq 4\,(n-1) + 1$. In the 1980s Kemnitz [Kem82; Kem83] proved in his doctoral dissertation that this also matches the upper bound in the special cases $n = p = 2, 3, 5, 7$ and therefore conjectured it to be true for general n. In 2003, Reiher finally proved Kemnitz's conjecture.

Theorem 1.31 (Reiher)**.** *[Rei07]*

$$\mathrm{s}\left(\mathbb{Z}_n^2\right) = 4\,(n-1) + 1$$

The following statement tells us that it suffices to look for zero subsums of length p or $3p$.

Proposition 1.32 (Alon and Dubiner)**.** *[AD93, Lemma 3.2] Let S be a zero subsum in $\mathbb{Z}_p^2$ of length $3p$. Then S contains a zero subsum of length p.*

Proof. Let $n = 3p - 2$ and $S = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_{3p} \\ b_{3p} \end{pmatrix}$ be a zero subsum of length $3p$ in $\mathbb{Z}_p^2$. Similar to the presented proof of Theorem 1.29 (Erdős, Ginzburg, and Ziv) the polynomials are

$$f_1\,(X_1, \dots, X_n) = \sum_{i=1}^{n} a_i X_i^{p-1}$$

$$f_2\,(X_1, \dots, X_n) = \sum_{i=1}^{n} b_i X_i^{p-1}$$

$$f_3\,(X_1, \dots, X_n) = \sum_{i=1}^{n} X_i^{p-1}.$$

Again, Theorem 1.30 (Chevalley's theorem, [Che35]) ensures a nontrivial solution of $f_1 = f_2 = f_3 = 0$ which corresponds to a zero subsum in S, however, this time of length p or $2p$. In the latter case choose the complement to obtain a zero subsum of length $3p - 2p = p$. □

In order to prove Theorem 1.31 (Reiher), a refinement of Theorem 1.30 (Chevalley's theorem, [Che35]) is needed.

Theorem 1.33 (Chevalley–Warning theorem [Che35; War35])**.** *Same setting as in Theorem 1.30 (Chevalley's theorem, [Che35]). Then p divides the number of common zeros of $f_1, \ldots, f_m$.*

Proof idea of Theorem 1.31 (Reiher). The lower bound has already been established in Example 1.10 (Harborth [Har73]) and the case $n = 2$ in Theorem 1.36 (Harborth). Reiher applied Theorem 1.33 (Chevalley–Warning theorem [Che35; War35]) to slightly modified polynomials as in the proof of Proposition 1.32 (Alon and Dubiner) in order to relate the number of solutions of zero subsums of different lengths. Eventually, he obtained that there is a zero subsum of length p or $3p$, henceforth, Proposition 1.32 (Alon and Dubiner) ensures a zero subsum of length p. □

Presently, dimension 3 remains open, though the following is assumed, which has been verified for some special cases.

Conjecture 1.34 (Gao and Thangadurai [GT06] and Gao *et al.* [GHST07])**.**

$$\mathrm{s}\left(\mathbb{Z}_n^3\right) = \begin{cases} 8\,(n-1)+1, & n \text{ even} \\ 9\,(n-1)+1, & n \text{ odd} \end{cases}$$

Remark 1.35 (Greedy confirmation of dimension 3). Implementing the greedy approach described in Remark 1.28 (Computer implementation) and assuming Conjecture 1.21 (Property D) the computer corroborated (not proved) Conjecture 1.34 (Gao and Thangadurai [GT06] and Gao *et al.* [GHST07]) up to $n \leq 26$.

Instead of restricting to a specific dimension d, one can also think about fixing the modulus n. Starting with $n = 2$, as well as powers of two, this has been treated by Harborth in the early 70s.

Theorem 1.36 (Harborth)**.** *[Har73, Korollar 1]*

$$\mathrm{s}\left(\mathbb{Z}_{n=2^a}^d\right) = 2^d\,(n-1)+1$$

Proof. The lower bound is established in Example 1.10 (Harborth [Har73]). For the upper bound, because of Corollary 1.27 (Multiplicativity) it suffices to consider the case $n = p = 2$, which has already been covered in Lemma 1.16 (Harborth). $\square$

1.1 Ternary Case

The study of the ternary case $\mathrm{s}\left(\mathbb{Z}_3^d\right)$ boils down to finding large ternary affine caps.

Definition 1.37 (Projective space)**.** [BR98, chapter 1.2] A projective space is a set called points together with a set of subsets of points called lines that satisfy the following four axioms.

(i) (Line Axiom) There is a unique line passing through any two distinct points.

(ii) (Veblen–Young) Let A, B, C, D be four distinct points. If the lines through AB and CD share a common point, then so do the lines through AC and BD. Broadly speaking, there are no parallel lines.

(iii) Every line contains at least three points.

(iv) (Nondegenerate) There are at least two lines.

The affine geometry introduces the concept of parallel lines. Instead of stating another bunch of axioms, the affine geometry is usually obtained from the projective geometry.

Definition 1.38 (Affine geometry)**.** [MS77, appendix B §2] An affine or Euclidean geometry is obtained by deleting the points of an arbitrary fixed hyperplane $\mathcal{H}_\infty$ called the hyperplane at infinity from the subspaces of a projective geometry.

The next step is to give concrete examples of projective and affine geometries.

Definition 1.39 (Finite geometry)**.** [MS77, appendix B §2] Let $\mathbb{F}_q$ be the finite field of order q and let $d \geq 2$.

The affine geometry/space $\mathrm{AG}(d, q)$ consists of the points $\mathbb{F}_q^d$. The line through two distinct points $x, y \in \mathbb{F}_q^d$ consists of the points $\alpha x + (1 - \alpha)\, y$ where $\alpha \in \mathbb{F}_q$.

The projective geometry/space $\mathrm{PG}(d, q)$ consists of the nonzero points $\mathbb{F}_q^{d+1} \setminus \{0\}$ with the rule that x and αx are the same point where $\alpha \in \mathbb{F}_q \setminus \{0\}$, called homogeneous coordinates. The line through two distinct points $x, y \in \mathbb{F}_q^{d+1} \setminus \{0\}$ consists of the points $\alpha x + \beta y$ where $\alpha, \beta \in \mathbb{F}_q \setminus \{0\}$.

Remark 1.40 (Desarguesian)*.* One might already feel uncomfortable because in the Definition 1.39 (Finite geometry) it says "the" affine/projective geometry but there might also be other constructions. In a so-called Desarguesian geometry one can introduce coordinates which then can be related to a finite field. For $d \geq 3$ any finite geometry is Desarguesian, consequently, any finite projective/affine geometry is $\mathrm{PG}(d, q)$ or $\mathrm{AG}(d, q)$ respectively (see [MS77, Appendix B §3, Theorem 1] for further references). This justifies the uniqueness term "the" for $d \geq 3$. However, for projective/affine planes $d = 2$ there are nondesarguesian planes known as well, see [MS77, Appendix B §4, Theorem 11].

Definition 1.41 (Cap)**.** [Hir98, chapter 3.3] A cap is a set of points no three of which are collinear.

In the literature, a consistent notation for maximum caps has been established only for projective spaces (see for example [Hir98, chapter 3.3]). In this thesis, a generalization of this notation $\mathrm{m}_2(d, q)$ is used that equally suits both affine and projective spaces.

Notation 1.42 (Caps)**.** The maximum size of a cap is denoted by $\mathrm{m}_2(\mathrm{PG}(d, q))$ or $\mathrm{m}_2(\mathrm{AG}(d, q))$ respectively.

Observation 1.43 (Ternary equivalences)*.* [EEG+07, chapter 5] Let S be a sequence of distinct elements in $\mathbb{Z}_3^d$. The following statements are equivalent:

(i) S contains a zero subsum of length 3.

(ii) S contains three collinear points.

(iii) S contains an arithmetic progression of length 3.

In particular,

$$\mathrm{m}_2(\mathrm{AG}(d, q)) = \mathrm{g}\left(\mathbb{Z}_3^d\right) - 1.$$

Proof. **Arithmetic progression.** Three different points form a (proper) arithmetic progression if

$$x + z = 2y.$$

Collinearity. Three different points are collinear if $\alpha x + (1-\alpha)\, y = z$. As these points should be different, $\alpha \notin \{0, 1\}$, and therefore only $\alpha = 2$ remains, which means

$$2x - y = z.$$

Zero subsum. A zero subsum of length 3 of distinct vectors is given when

$$x + y + z = 0.$$

Note that all these equations are equivalent modulo 3. $\square$

It remains to connect $\mathrm{s}\left(\mathbb{Z}_3^d\right)$ with $\mathrm{g}\left(\mathbb{Z}_3^d\right)$ in order to relate $\mathrm{s}\left(\mathbb{Z}_3^d\right)$ with ternary affine caps.

Theorem 1.44 (Harborth)**.** *[Har73, Hilfssatz 3] In the ternary case* $\mathrm{s}\left(\mathbb{Z}_3^d\right)$ *Conjecture 1.21 (Property D) is fulfilled. In fact,*

$$\mathrm{s}\left(\mathbb{Z}_3^d\right) = 2\,\mathrm{g}\left(\mathbb{Z}_3^d\right) - 1.$$

Proof. Recalling Remark 1.25: Property D is implied by Property G, namely

$$\mathrm{s}\left(\mathbb{Z}_3^d\right) = \left(\mathrm{g}\left(\mathbb{Z}_3^d\right) - 1\right)(3-1) + 1 = 2\,\mathrm{g}\left(\mathbb{Z}_3^d\right) - 1.$$

Moreover, because of Remark 1.24 (Property G) it suffices to prove $\mathrm{s}\left(\mathbb{Z}_3^d\right) \overset{!}{\geq} \left(\mathrm{g}\left(\mathbb{Z}_3^d\right) - 1\right) 2 + 1$. Let S be a maximum sequence regarding $\mathrm{g}\left(\mathbb{Z}_3^d\right)$, that is a sequence of distinct vectors of length $|S| = \mathrm{g}\left(\mathbb{Z}_3^d\right) - 1$ without zero subsums of length 3. Claim: $S^2 = S, S$ does not contain zero subsums of length 3 either, which then concludes the proof

$$\mathrm{s}\left(\mathbb{Z}_3^d\right) \geq \left|S^2\right| + 1 = 2|S| + 1 = \left(\mathrm{g}\left(\mathbb{Z}_3^d\right) - 1\right) 2 + 1.$$

Assume for a contradiction, there is a zero subsum in S^2. As S does not contain zero subsums, a zero subsum cannot consist of three distinct vectors. So suppose $v + 2u \equiv 0 \pmod 3$. To fulfill this equation also $v = u$, yet so many are not available in S^2. $\square$

Remark 1.45 (Ternary implementation)*.* Theorem 1.44 (Harborth) speeds up computer results by working with distinct sequences regarding $\mathrm{g}\left(\mathbb{Z}_3^d\right)$ instead of sequences twice as big regarding $\mathrm{s}\left(\mathbb{Z}_3^d\right)$.

Remark 1.46 (Largest ternary affine caps). Up to dimension $d = 6$ maximum ternary affine caps in $\mathrm{AG}(d, 3)$ are known. The latter two are due to Edel *et al.* [EFLS02] and Potechin [Pot08]. The corresponding sequence $(\mathrm{m}_2(\mathrm{AG}(d, 3)))_{d\in\mathbb{N}} = 2, 4, 9, 20, 45, 112, \ldots$ can be found in The On-Line Encyclopedia of Integer Sequences [Hav04a].

Corollary 1.47 (Ternary values in low dimensions)**.**

$$\begin{aligned}
\mathrm{s}(\mathbb{Z}_3) &= 2\,(3-1)+1 = 5\\
\mathrm{s}\left(\mathbb{Z}_3^2\right) &= 4\,(3-1)+1 = 9\\
\mathrm{s}\left(\mathbb{Z}_3^3\right) &= 9\,(3-1)+1 = 19\\
\mathrm{s}\left(\mathbb{Z}_3^4\right) &= 20\,(3-1)+1 = 41\\
\mathrm{s}\left(\mathbb{Z}_3^5\right) &= 45\,(3-1)+1 = 91\\
\mathrm{s}\left(\mathbb{Z}_3^6\right) &= 112\,(3-1)+1 = 225.
\end{aligned}$$

Proof. Combine Remark 1.46 (Largest ternary affine caps), Observation 1.43 (Ternary equivalences), and Theorem 1.44 (Harborth)

$$\mathrm{s}\left(\mathbb{Z}_3^d\right) = 2\,\mathrm{g}\left(\mathbb{Z}_3^d\right) - 1 = 2\,\mathrm{m}_2(\mathrm{AG}(d, 3)) + 1.$$

□

Remark 1.48 (Ternary enumeration). Implementing the approach described in Remarks 1.28 (Computer implementation) and 1.45 (Ternary implementation) all maximal sequences regarding $\mathrm{g}\left(\mathbb{Z}_3^d\right)$ have been enumerated up to dimension $d = 3$. Regarding $\mathrm{g}(\mathbb{Z}_3^2)$ there are just 2 maximal and at the same time maximum sequences of length 4. One dimension higher there are 52 different maximal sequences of length 8 and 7 maximum sequences of length 9.

Remark 1.49 (Ternary greedy). Additionally to Remark 1.48 (Ternary enumeration), the computer was able to greedily find all ternary maximum sequences up to dimension $d = 5$, although it takes some time to find the 45 vectors of a maximum sequence regarding $\mathrm{g}(\mathbb{Z}_3^5)$. In higher dimensions the computer only found inferior sequences. For example, in dimension $d = 6$ the best found sequence was $|S| = 80$ which is far from the maximum 112. Even lifting the maximum sequence in dimension $d = 3$ yields a better sequence in dimension $d = 6$ of length $|S| = 9 \cdot 9 = 81$, see Lemma 1.56 (Product construction).

In higher dimensions some bounds have been established.

Theorem 1.50 (Meshulam)**.** *[Mes95, Theorem 1.2]*

$$\mathrm{g}\left(\mathbb{Z}_3^d\right) - 1 \leq 2\frac{3^d}{d}.$$

In particular,

$$\mathrm{s}\left(\mathbb{Z}_3^d\right) = \mathrm{O}\left(\frac{3^d}{d}\right), \quad \text{as } d \to \infty.$$

Remark 1.51 (Liu and Spencer). [LS09, Remark 6] Retrospectively, Theorem 1.50 (Meshulam) is a special case of Theorem 1.18 (Liu and Spencer) by calculating the hidden constant explicitly.

Only recently, a dramatic advance happened due to Croot, Lev, and Pach [CLP16] which has been quickly adapted by Ellenberg and Gijswijt [EG16] for arithmetic progressions.

Theorem 1.52 (Ellenberg and Gijswijt)**.** *[EG16, Corollary 5] There exists a constant $c < 3$ such that*

$$\mathrm{s}\left(\mathbb{Z}_3^d\right) = \mathrm{O}\left(c^d\right) = \mathrm{o}\left(2.76^d\right), \quad \text{as } d \to \infty.$$

Proof idea. Let $A \subseteq \mathbb{Z}_3^d$ be a maximum subset without arithmetic progressions. Consider the $\mathbb{F}_3$-vector space V of polynomials in d variables over the field $\mathbb{F}_3$ and denote by $V_{\leq k}$ the subspace generated by monomials with total degree at most k. A natural basis of this subspace $V_{\leq k}$ are the monomials in d variables with degree at most 2 in each variable and total degree at most k. Ellenberg and Gijswijt examined the subspace of V of polynomials vanishing on the complement of A, showing that there exists a constant $c < 2.756$ such that (assuming d is a multiple of 3)

$$\begin{aligned} \mathrm{g}\left(\mathbb{Z}_3^d\right) - 1 = |A| &\leq 2 \dim V_{\leq \frac{k}{2}} + \left(3^d - \dim V_{\leq k}\right), \quad \forall k \\ &\leq 3 \dim V_{\leq \frac{2}{3}d} = \mathrm{O}\left(c^d\right), \quad \text{as } d \to \infty. \end{aligned}$$

□

Remark 1.53 (Zeilberger). [Zei16, Asymptotics] A more detailed asymptotic expression.

$$c = \frac{\sqrt[3]{5589 + 891\sqrt{33}}}{8} = 2.755\ldots$$
$$\mathrm{g}\left(\mathbb{Z}_3^d\right) - 1 \leq 3.33\frac{c^d}{\sqrt{d}}\left(1 + \mathrm{O}\left(\frac{1}{d}\right)\right), \quad \text{as } d \to \infty.$$

A quantitative comparison in Table 1 (Comparison of asymptotic bounds) shows that Theorem 1.50 (Meshulam) is superseded by Theorem 1.52 (Ellenberg and Gijswijt) from dimension $d = 27$ on, though both upper bounds are far away from the truth, at least for small dimensions.

The currently best-known asymptotic lower bound uses a ternary affine cap in $\mathrm{AG}(62, 3)$ which gets lifted to higher dimensions. This so-called product construction is discussed in the following Section 1.2 (Odd Case).

Theorem 1.54 (Edel)**.** *[Ede04, 5. Asymptotic Results] There is a ternary affine cap in* $\mathrm{AG}(62, 3)$ *of size* $2\,573\,417\,086\,913\,773\,305\,856$*. In particular,*

$$\mathrm{s}\left(\mathbb{Z}_3^d\right) = \Omega\left(\sqrt[62]{2\,573\,417\,086\,913\,773\,305\,856}^{\,d}\right) = \Omega\left(2.21^d\right), \quad \text{as } d \to \infty.$$

Proof idea. Similarly to Theorem 1.61 (Elsholtz). □

d	[Ede04]	$\mathrm{g}(\mathbb{Z}_d^3) - 1$	[Mes95]	[EG16]
3		9	18	30
4		20	40	
5		56	97	
6		112	243	504
9			4 374	9 183
⋮				
24			$2.36 \cdot 10^{10}$	$2.54 \cdot 10^{10}$
27			$5.65 \cdot 10^{11}$	$5.05 \cdot 10^{11}$
⋮				
63	$5.14 \cdot 10^{21}$		$3.64 \cdot 10^{28}$	$2.45 \cdot 10^{27}$

Table 1: Comparison of asymptotic bounds

1.2 Odd Case

In order to obtain general lower bounds one can lift sequences without zero subsums to higher dimensions.

Lemma 1.55 (Trivial lifting)**.**

$$\eta\left(\mathbb{Z}_n^d\right) \geq d\left(\eta(\mathbb{Z}_n) - 1\right) + 1$$
$$\mathrm{s}\left(\mathbb{Z}_n^d\right) \geq n + d\left(\eta(\mathbb{Z}_n) - 1\right)$$

Proof. Let S be a maximum sequence regarding $\eta(\mathbb{Z}_n)$. The idea is to take S once in every dimension. Let S_i be S in dimension i and zero otherwise. Then $S_1, \dots, S_d$ does not contain short zero subsums either since otherwise there would be a short zero subsum in some dimension i. In particular, see Observation 1.19 (Gao)

$$\eta\left(\mathbb{Z}_n^d\right) \geq |S_1, \dots, S_d| + 1 = d|S| + 1 = d\left(\eta(\mathbb{Z}_n) - 1\right) + 1$$
$$\mathrm{s}\left(\mathbb{Z}_n^d\right) \geq \eta\left(\mathbb{Z}_n^d\right) + n - 1 \geq n + d\left(\eta(\mathbb{Z}_n) - 1\right).$$

□

Assuming Conjecture 1.21 (Property D) much better results are obtained.

Lemma 1.56 (Product construction)**.** *Let R^{n-1}, S^{n-1} be sequences in $\mathbb{Z}_n^r$ respectively $\mathbb{Z}_n^s$ which both do not contain zero subsums of length n. Define a new sequence T^{n-1} in $\mathbb{Z}_n^{r+s}$ by*

$$T = \begin{pmatrix} R \\ S \end{pmatrix} := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Z}_n^{r+s} : u \in R, v \in S \right\}.$$

Then T^{n-1} does not contain zero subsums too. In particular,

$$\mathrm{s}\left(\mathbb{Z}_n^{r+s}\right) \geq \left|T^{n-1}\right| + 1 = |R||S|\,(n-1) + 1.$$

Proof. Note that due to the product construction a vector $u \in R$ or $v \in S$ may be available more than $n-1$ times in the projection of T^{n-1} to the corresponding upper r or lower s dimensions. By assumption, R^{n-1} does not contain zero subsums. Thus, any zero subsum in the upper dimensions of T^{n-1} of length n consists of a single vector $u \in R$ taken n times. But then in the lower dimensions all vectors $v \in S$ are available at most $n-1$ times and therefore cannot contain a zero subsum since by assumption S^{n-1} does not contain zero subsums either. □

Example 1.57 (Product construction)**.** An illustration example: Suppose your favorite computer program found two interesting sequences R^6, S^6 without zero subsums of length 7 in $\mathbb{Z}_7^2$.

$$R = \begin{pmatrix}3\\6\end{pmatrix}, \begin{pmatrix}5\\4\end{pmatrix}, S = \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}$$

Then these can be combined to obtain a sequence T^6 in $\mathbb{Z}_7^4$ without zero subsums either.

$$T = \begin{pmatrix}3\\6\\1\\1\end{pmatrix}, \begin{pmatrix}3\\6\\2\\3\end{pmatrix}, \begin{pmatrix}5\\4\\1\\1\end{pmatrix}, \begin{pmatrix}5\\4\\2\\3\end{pmatrix}$$

To gain something significantly, one needs good extremal sequences to start with.

Observation 1.58 (Elsholtz)*.* [Els04, 2. Proof] Let $n \geq 3$, odd. There are maximum sequences in $\mathbb{Z}_3^d$, which incidentally also work in general in $\mathbb{Z}_n^d$. In particular,

$$\begin{aligned} \mathrm{s}\left(\mathbb{Z}_3^3\right) &= 9\,(3-1)+1 \\ \mathrm{s}\left(\mathbb{Z}_n^3\right) &\geq 9\,(n-1)+1. \end{aligned}$$

Elsholtz lifted the following sequence to higher dimensions.

Example 1.59 (Elsholtz)**.** [Els04, 2. Proof] A maximum sequence S^2 in $\mathbb{Z}_3^3$ has already been discovered by Harborth [Har73], which is essentially three times an affine basis in dimension 2, the latter one shifted by 1.

$$S = \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}$$

Claim: Let $n \geq 3$, odd. S^{n-1} in $\mathbb{Z}_n^3$ still does not contain a zero subsum.

Proof. Assume there is a zero subsum in S^{n-1}. Denote by $a_1, \ldots, a_9$ the number of occurrences of the vectors of S in this zero subsum. Written as linear system:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 &= n \\ a_2 + a_5 + a_7 + 2a_8 + a_9 &\equiv 0 \pmod{n} \\ a_3 + a_6 + a_7 + a_8 + 2a_9 &\equiv 0 \pmod{n} \\ a_4 + a_5 + a_6 + 2a_7 + 2a_8 + 2a_9 &\equiv 0 \pmod{n} \end{aligned}$$

First, replace the equivalences by equalities. The coefficients range between 0 and 2, therefore the candidates are $= 0n$, $= 1n$, and $= 2n$. We want to argue that the extremal cases cannot happen.

Assume one of the coordinates equals $0n$. Then all appearing a_i must be zero since they are nonnegative. In particular $a_7 = a_8 = a_9 = 0$. However, the remaining vectors are a subset of the 0-1-vectors in Example 1.10 (Harborth [Har73]) thence do not contain a zero subsum.

Next, assume one of the coordinates equals $2n$. This means solely vectors with the coefficient 2 in this coordinate have been taken. Again, these vectors are a subset of Example 1.14 (Affine transformation).

After this prelude, we are left with a linear equation system.

$$\begin{aligned}
a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \ \ a_7 + \ \ a_8 + \ \ a_9 &= n \\
a_2 \qquad\qquad + a_5 \qquad + \ \ a_7 + 2a_8 + \ \ a_9 &= n \\
a_3 \qquad\qquad + a_6 + \ \ a_7 + \ \ a_8 + 2a_9 &= n \\
a_4 + a_5 + a_6 + 2a_7 + 2a_8 + 2a_9 &= n
\end{aligned}$$

Adding the first and last row whilst subtracting the second and third one yields

$$a_1 + 2a_4 + a_5 + a_6 = 0.$$

which implies $a_1 = a_4 = a_5 = a_6 = 0$. Looking again at the last row:

$$2a_7 + 2a_8 + 2a_9 = n.$$

A contradiction to n odd. $\square$

Remark 1.60 (Algorithmic approach). The proof of Example 1.59 (Elsholtz) is quite interesting in the following sense: It can be accomplished by a computer. There are plenty of integer equation system solvers available, which makes it possible to test any sequence for zero subsums for general n within seconds.

Theorem 1.61 (Elsholtz)**.** *[Els04, 1. Introduction] Let $n \geq 3$, odd.*

$$\begin{aligned}
&\mathrm{s}\left(\mathbb{Z}_n^d\right) \geq 1.125^{\lfloor \frac{d}{3} \rfloor} 2^d (n-1) + 1 \\
&\mathrm{s}\left(\mathbb{Z}_n^d\right) = \Omega\left(\sqrt[3]{9}^d\right) = \Omega\left(2.08^d\right), \quad \text{as } d \to \infty
\end{aligned}$$

Proof. Let $d = 3k + r, 0 \leq r \leq 2$. Lifting Example 1.59 (Elsholtz) to higher dimensions, sequences in $\mathbb{Z}_n^{3k}$ of length 9^k are obtained. In the remaining r

dimensions further lift the sequence using all 2^d 0-1-vectors of Example 1.10 (Harborth [Har73]). In total $|S| = 9^k 2^r$. Therefore, by Lemma 1.56 (Product construction)

$$\begin{aligned} s\left(\mathbb{Z}_n^d\right) &\geq \left|S^{n-1}\right| + 1 = 9^k 2^r (n-1) + 1 \\ &= \left(\frac{9}{8}\right)^k 2^{3k+r} (n-1) + 1 = 1.125^{\lfloor \frac{d}{3} \rfloor} 2^d (n-1) + 1. \end{aligned}$$

On the other hand, fixing n, as $d \to \infty$

$$s\left(\mathbb{Z}_n^d\right) \geq \left|S^{n-1}\right| + 1 = 9^k 2^r (n-1) + 1 \geq 9^k \geq 9^{\frac{d-2}{3}} = \Omega\left(\sqrt[3]{9}^d\right).$$

□

This has been improved with a sequence in dimension 4 of length 20.

Theorem 1.62 (Edel *et al.*)**.** *[EEG+07] Let $n \geq 3$, odd.*

$$s\left(\mathbb{Z}_n^d\right) = \Omega\left(\sqrt[4]{20}^d\right) = \Omega\left(2.11^d\right), \quad \text{as } d \to \infty$$

Remark 1.63 (Odd greedy)*.* Implementing Remark 1.60 (Algorithmic approach) assuming Conjecture 1.21 (Property D) using Gurobi [Gur16] to solve linear programs, all best-known sequences regarding $s\left(\mathbb{Z}_n^d\right)$ for all odd n have been greedily found up to dimension $d = 3$. In dimension $d = 4$ Gurobi only manages to find a sequence of length 18 instead of the best-known 20 vectors of Theorem 1.62 (Edel *et al.*). It is not even able to verify these 20 vectors. It seems that the tools equipped with Gurobi are not enough to solve the linear programs of "good" sequences. In dimension $d = 5$ it just finds 33 vectors which is almost as bad as the trivial lower bound Lemma 1.16 (Harborth).

Having in mind Example 1.59 (Elsholtz), it turns out to be more profitable to search for maximal ternary sequences simultaneously regarding $s\left(\mathbb{Z}_3^d\right)$, $s\left(\mathbb{Z}_5^d\right)$, and higher odd n. This way one finds 19 vectors verified by Gurobi. Just verifying for small odd n without Gurobi even all 20 vectors have been found. One dimension higher in dimension $d = 5$ this approach finds 36 vectors.

One might think, maybe searching within ternary sequences is not enough to satisfy all odd n. Taking quinary sequences instead, the computer also found the 20 vectors regarding $s(\mathbb{Z}_n^4)$, however, much slower. In dimension $d = 5$ it found 35 vectors.

Anyways, we will see in the next Section 1.2.1 (Edel Lifting) that one can do much better.

1.2.1 Edel Lifting

The best known lower bound so far is $\mathrm{s}\left(\mathbb{Z}_n^d\right) = \Omega\left(2.13^d\right)$ due to ţteauthorEdel.2008. He uses a nontrivial product construction to obtain better sequences in dimensions 5,6, and 7. The idea is to group sequences by weight modulo 2 in order to combine them more efficiently.

Definition 1.64 (Weight)**.** The weight of a vector $v \in \mathbb{Z}_n^d$ is the sum of its entries $\sum_{i=1}^d v_i$. The total weight of a sequence S in $\mathbb{Z}_n^d$ is the sum of all its vectors' weights $\sum_{v \in S} \sum_{i=1}^d v_i$. A sequence is said to have weight k if all its vectors have weight k.

Essentially, Edel is exploiting the following observation to be able to pack more vectors into sequences nonetheless avoiding zero subsums.

Observation 1.65 (Edel). [Ede08, Lemma 13] Let $n \geq 3$, odd and S be a sequence in $\mathbb{Z}_n^d$ of weight $k \pmod 2$. Then any subsequence in S of length n has total weight $k \pmod 2$.

Proof. The total weight of a subsequence of length n is the sum of its vectors' weights:

$$n \cdot k \equiv k \pmod 2$$

□

This observation motivates the following definition of sequences, which then later are used in a more sophisticated product construction to yield sequences without zero subsums.

Definition 1.66 (Edel sequence)**.** Let S be a sequence in $\mathbb{Z}_n^d$ grouped by weight modulo 2 into $S = S_d, S_{d+1}$. Then S is said to be an Edel sequence if the groups S_d^{n-1} and S_{d+1}^{n-1} do not contain zero subsums of length n, though the entire sequence S^{n-1} is allowed to contain zero subsums of length n, albeit only of total weight d.

Example 1.67 (Edel)**.** [Ede08, Lemma 19] Revising Example 1.11 (Edel)

$$S = \underbrace{1}_{S_1}, \underbrace{0, 2}_{S_2}$$

We already worked out that zero subsums in S^{n-1} of length n are of shape

$$\left(1\right)^{n-2a}, \left(0\right)^a, \left(2\right)^a, \quad 0 < a \leq \frac{n-1}{2}.$$

Let $n \geq 3$, odd. Then zero subsums in S^{n-1} in $\mathbb{Z}_n$ have total weight

$$(n-2a) \cdot 1 + a \cdot 0 + a \cdot 2 = n \equiv 1 = d \pmod 2.$$

Moreover, S_1^{n-1} and S_2^{n-1} do not contain zero subsums by Example 1.10 (Harborth [Har73]) and Lemma 1.13 (Affine transformation). Hence, S is an Edel sequence in $\mathbb{Z}_n$ for all odd n.

Similar to the trivial Lemma 1.56 (Product construction), two Edel sequences can be put together in a more sophisticated way.

Proposition 1.68 (Edel). *[Ede08, Theorem 16] Let $n \geq 3$, odd and $R = R_r, R_{r+1}, S = S_s, S_{s+1}$ be two Edel sequences in $\mathbb{Z}_n^r$ and $\mathbb{Z}_n^s$ respectively. Let*

$$\begin{aligned} T &= \begin{pmatrix} R_r \\ S_{s+1} \end{pmatrix}, \begin{pmatrix} R_{r+1} \\ S_s \end{pmatrix} \\ &= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{Z}_n^{r+s} : (u \in R_r, v \in S_{s+1}) \vee (u \in R_{r+1}, v \in S_s) \right\}. \end{aligned}$$

Then T^{n-1} does not contain a zero subsum of length n. In particular,

$$\mathrm{s}\left(\mathbb{Z}_n^{r+s}\right) \geq \left|T^{n-1}\right| + 1 = \left(|R_r||S_{s+1}| + |R_{r+1}||S_s|\right)(n-1) + 1$$

Proof. Note that due to the product construction a vector $u \in R$ or $v \in S$ may be available more than $n-1$ times in the projection of T^{n-1} to the corresponding upper r or lower s dimensions. Nevertheless, a zero subsum of length n cannot consist of solely one vector taken n times in the upper or lower dimensions, because then in the remaining dimensions one has found a zero subsum in one of the four parts $R_r^{n-1}, R_{r+1}^{n-1}, S_s^{n-1}, S_{s+1}^{n-1}$. However, by assumption neither of them contains zero subsums. Accordingly, any vector $u \in R$ or $v \in S$ occurs at most $n-1$ times in the corresponding dimensions. By assumption, R^{n-1} and S^{n-1} only contain zero subsums of total weight r and s modulo 2. Therefore, any zero subsum in T^{n-1} has total weight $r+s \pmod 2$. On the contrary, every vector in T has weight $r+s+1 \pmod 2$ and thus, by Observation 1.65 (Edel) any zero subsum in T^{n-1} has total weight $r+s+1 \pmod 2$. □

In order to construct Edel sequences in higher dimensions, the following lifting is expedient.

Proposition 1.69 (Edel)**.** *[Ede08, Lemma 15] Let $n \geq 3$, odd and $S = S_d, S_{d+1}$ be an Edel sequence. The following sequence $T = T_{d+1}, T_{d+2}$ in $\mathbb{Z}_n^{d+1}$ is an Edel sequence of length $|T_{d+1}| = |T_{d+2}| = |S_d| + |S_{d+1}|$.*

$$T = \underbrace{\begin{pmatrix} S_{d+1} \\ 0 \end{pmatrix}, \begin{pmatrix} S_d \\ 1 \end{pmatrix}}_{T_{d+1}}, \underbrace{\begin{pmatrix} S_{d+1} \\ 1 \end{pmatrix}, \begin{pmatrix} S_d \\ 2 \end{pmatrix}}_{T_{d+2}}$$

Proof. Assume there is a zero subsum in T_{d+1}^{n-1}. In the bottom dimension we meet Example 1.8 (Dimension 1). Consequently, a zero subsum must lie either within S_d or S_{d+1} though both, by assumption, do not contain zero subsums. Analogously nor T_{d+2}^{n-1} contains zero subsums. It remains to consider the total weight of zero subsums in the entire sequence T^{n-1}. Observe that in a zero subsum of length n in the upper d dimensions of T^{n-1} any vector $v \in S$ occurs at most $n-1$ times. By assumption, S^{n-1} only contains zero subsums of total weight $d \pmod 2$. Example 1.67 (Edel) in the bottom dimension is an Edel sequence, which means it contains only zero subsums of total weight $1 \pmod 2$. Thus, zero subsums in T^{n-1} have total weight $d+1 \pmod 2$ as required. □

Specifically in dimension 3, the following construction yields a larger Edel sequence.

Proposition 1.70 (Edel)**.** *[Ede08, Definition 17] Let $n \geq 3$, odd. The following sequence $S = S_d, S_{d+1}$ is an Edel sequence in $\mathbb{Z}_n^d$.*
S_d consists of all vectors in $\{0,1\}^d$ with exactly m zeros and all vectors in $\{1,2\}^d$ with exactly m twos for a fixed even m.
S_{d+1} is composed of all vectors in $\{0,1\}^d$ with exactly k zeros and all vectors in $\{1,2\}^d$ with exactly k twos for all odd k.
Moreover,

$$|S_d| = 2\binom{d}{m}, |S_{d+1}| = 2^d$$

is maximal for $m = 2\left\lfloor \frac{d+2}{4} \right\rfloor$.

Proof. There are multiple things to check. Before, let us characterize zero subsums of length n in S^{n-1}. By Example 1.10 (Harborth [Har73]) the part within $\{0,1\}^d$ does not contain zero subsums and furthermore by Lemma 1.13 (Affine transformation) neither does the other part within $\{1,2\}^d$. It follows that in each coordinate a zero subsum equals n since otherwise to gain $0n$ or $2n$ solely zeros or twos are necessary, which restricts to either $\{0,1\}^d$ or $\{1,2\}^d$. Summing over all coordinates, the total weight of a zero subsum is $d \cdot n \equiv d \pmod 2$ as required. Before we prove that S_{d+1}^{n-1} does not contain zero subsums, note that the weights are as claimed

$$\begin{aligned} k \cdot 0 + (d-k)\,1 &\equiv d+1 \pmod 2 \\ k \cdot 2 + (d-k)\,1 &\equiv d+1 \pmod 2. \end{aligned}$$

The total weight of a zero subsum of length n in S_{d+1}^{n-1} is $n\,(d+1) \equiv d+1 \pmod 2$ whereas we already observed that all zero subsums in S^{n-1} have total weight $d \pmod 2$. Similarly to S_{d+1} the weights of S_d are as claimed $d \pmod 2$. Assume there is a zero subsum of length n within S_d^{n-1}. We already excluded the degenerate cases $0n$ and $2n$. Therefore, as characterized in Example 1.11 (Edel) in each coordinate there is the same number of zeros and twos. Hence, also the total numbers of zeros and twos are equal. By construction every vector consists of exactly m zeros or twos, thus a zero subsum consists of an even number of vectors in contradiction to its odd length n. Finally, the cardinality: In S_d there are m out of d positions to choose the ones or twos, hence $|S_d| = 2\binom{d}{m}$. In S_{d+1} the same is done for all odd k

$$|S_{d+1}| = 2 \sum_{k \text{ odd}} \binom{d}{k} = 2 \cdot 2^{d-1} = 2^d.$$

The even parameter m is chosen nearest possible to $\frac{d}{2} = 2\frac{d}{4}$ in order to maximize $\binom{d}{m}$. For example, $m = 2\left\lfloor \frac{d+2}{4} \right\rfloor$ works. $\square$

Before stating the final results, the largest Edel sequences obtained for each dimension are arranged.

Proposition 1.71 (Edel). *[Ede08, Lemma 19] Let $n \geq 3$, odd. There are Edel sequences $S = S_d, S_{d+1}$ of size*

- $|S_1| = 1, |S_2| = 2$ *in* $\mathbb{Z}_n$
- $|S_2| = |S_3| = 3$ *in* $\mathbb{Z}_n^2$
- $|S_3| = 6, |S_4| = 8$ *in* $\mathbb{Z}_n^3$
- $|S_4| = |S_5| = 14$ *in* $\mathbb{Z}_n^4$.

Proof. An Edel sequence in dimension 1 of size $|S_1| = 1, |S_2| = 2$ has already been presented in Example 1.67 (Edel). Applying Proposition 1.69 (Edel) this sequence gets lifted to an Edel sequence in dimension 2 of size $|S_2| = |S_3| = 1 + 2 = 3$. In dimension 3, the best results are obtained by Proposition 1.70 (Edel), that is $|S_3| = 2\binom{3}{2} = 6, |S_4| = 2^3 = 8$. Another lifting by one dimension using Proposition 1.69 (Edel) yields an Edel sequence in dimension 4 of size $|S_4| = |S_5| = 6 + 8 = 14$. □

Example 1.72 (Edel sequences)**.** Now, what are those Edel sequences actually?

Dimension 1. Example 1.67 (Edel)

$$S_1 = 1, \quad S_2 = 0, 2$$

Dimension 2. This then is lifted via Proposition 1.69 (Edel).

$$S_2 = \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \quad S_3 = \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\2\end{pmatrix}$$

Dimension 3. Proposition 1.70 (Edel) yields a bigger example specifically in dimension 3.

$$S_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$S_4 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}$$

Theorem 1.73 (Edel)**.** *[Ede08, Lemma 19] Let $n \geq 3$, odd.*

$$\begin{aligned}
\mathrm{s}\left(\mathbb{Z}_n^3\right) &\geq 9\,(n-1)+1\\
\mathrm{s}\left(\mathbb{Z}_n^4\right) &\geq 20\,(n-1)+1\\
\mathrm{s}\left(\mathbb{Z}_n^5\right) &\geq 42\,(n-1)+1\\
\mathrm{s}\left(\mathbb{Z}_n^6\right) &\geq 96\,(n-1)+1\\
\mathrm{s}\left(\mathbb{Z}_n^7\right) &\geq 196\,(n-1)+1
\end{aligned}$$

Proof. Combining Edel sequences from Proposition 1.71 (Edel) using Proposition 1.68 (Edel) yields

$$\begin{aligned}
\mathrm{s}\left(\mathbb{Z}_n^3\right) &= \mathrm{s}\left(\mathbb{Z}_n^{1+2}\right) \geq (1 \cdot 3 + 2 \cdot 3)\,(n-1) + 1 = 9\,(n-1) + 1 \\
\mathrm{s}\left(\mathbb{Z}_n^4\right) &= \mathrm{s}\left(\mathbb{Z}_n^{1+3}\right) \geq (1 \cdot 8 + 2 \cdot 6)\,(n-1) + 1 = 20\,(n-1) + 1 \\
\mathrm{s}\left(\mathbb{Z}_n^5\right) &= \mathrm{s}\left(\mathbb{Z}_n^{2+3}\right) \geq (3 \cdot 8 + 3 \cdot 6)\,(n-1) + 1 = 42\,(n-1) + 1 \\
\mathrm{s}\left(\mathbb{Z}_n^6\right) &= \mathrm{s}\left(\mathbb{Z}_n^{3+3}\right) \geq (6 \cdot 8 + 8 \cdot 6)\,(n-1) + 1 = 96\,(n-1) + 1 \\
\mathrm{s}\left(\mathbb{Z}_n^7\right) &= \mathrm{s}\left(\mathbb{Z}_n^{3+4}\right) \geq (6 \cdot 14 + 8 \cdot 14)\,(n-1) + 1 = 196\,(n-1) + 1
\end{aligned}$$

□

Corollary 1.74 (Asymptotic lower bound)**.** *Let $n \geq 3$, odd.*

$$\mathrm{s}\left(\mathbb{Z}_n^d\right) = \Omega\left(\sqrt[6]{96}^{\,d}\right) = \Omega\left(2.13^d\right), \quad \text{as } d \to \infty$$

Proof idea. Similar to the proof of Theorem 1.61 (Elsholtz) using the sequence in dimension 6 of length 96. □

Example 1.75 (Edel's sequences)**.** Finally, what are those sequences? For example in dimensions 2, 3, and 4:

$$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}$$

$$S = \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}$$

$$\begin{aligned}
S = {} & \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\2\\2\\2\end{pmatrix}, \\
& \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\2\\2\end{pmatrix}. \begin{pmatrix}2\\0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\2\\2\end{pmatrix}
\end{aligned}$$

2 Weighted Generalization

There are many generalizations of Theorem 1.29 (Erdős, Ginzburg, and Ziv). In 2006, Adhikari and Rath [AR06] proposed another generalization that includes weights.

Definition 2.1 (Weighted zero subsum)**.** Let $A \subseteq \mathbb{Z} \setminus \{0\}$ be some nonzero weights. A sequence S in a finite abelian group contains an A-weighted zero subsum of length k, if it contains a subsequence T of length k and corresponding weights $a_v \in A$ such that

$$\sum_{v \in T} a_v v = 0$$

Notation 2.2 ($\pm$-weighted)**.** Instead of $\{1, -1\}$-weighted one simply writes $\pm$-weighted.

Notation 2.3 (Weighted zero subsums)**.** The unweighted issues in Notation 1.4 (Zero subsums) are intuitively generalized to weighted zero subsums.

$\mathrm{s}_A(G)$ denotes the smallest number of elements such that any sequence S in G of length $|S| \geq \mathrm{s}_A(G)$ contains an A-weighted zero subsum of length $\exp G$.

$\eta_A(G)$ denotes the smallest number of elements such that any sequence S in G of length $|S| \geq \eta_A(G)$ contains a short A-weighted zero subsum, that is an A-weighted zero subsum of length between $[1, \exp G]$.

$\mathrm{D}_A(G)$ the Davenport constant denotes the smallest number of elements such that any sequence S in G of length $|S| \geq \mathrm{D}_A(G)$ contains a nonempty A-weighted zero subsum (of arbitrary length).

$\mathrm{E}_A(G)$ denotes the smallest number of elements such that any sequence S in G of length $|S| \geq \mathrm{E}_A(G)$ contains an A-weighted zero subsums of length $|G|$.

Maybe you are wondering what happened to the number $\mathrm{g}(G)$ about sequences of distinct elements. We will see in Observation 2.45 (A-weighted transformation) that under certain weights A some vectors are mutually interchangeable without affecting A-weighted zero subsums. For example, it does not matter which one of v or $-v$ to choose regarding $\pm$-weights. Let us formalize this.

Definition 2.4 (A-distinct)**.** Let $A \subseteq \mathbb{Z}_n \setminus \{0\}$ be some weights. Two vectors $v, w \subseteq \mathbb{Z}_n^d$ are said to be A-equivalent if they are interchangeable under the weights A

$$u \sim_A v \Leftrightarrow Au = Av$$

where $Av = \{av : a \in A\}$ denotes the set of element-wise multiplication. Note that in order to be well-defined this is indeed an equivalence relation.

Now we are able to define a consistent weighted generalization of the number $\mathrm{g}(G)$ minding Observation 2.45 (A-weighted transformation). In contrast to the straight forward generalization $\mathrm{g}_A(G)$ which can lead to misleading results, see Observation 2.24 (Godinho, Lemos, and Marques), the term A-distinct used in the definition of $\mathrm{h}_A(G)$ additionally takes the weights into account.

Notation 2.5 (Weighted zero subsums)**.** The unweighted number $\mathrm{g}(G)$ is generalized to the weighted case in two ways.

$\mathrm{g}_A(G)$ denotes the smallest number of elements such that any sequence S of distinct elements in G of length $|S| \geq \mathrm{g}_A(G)$ contains an A-weighted zero subsum of length $\exp G$.

$\mathrm{h}_A(G)$ denotes the smallest number of elements such that any sequence S of A-distinct elements in G of length $|S| \geq \mathrm{h}_A(G)$ contains an A-weighted zero subsum of length $\exp G$.

Observation 2.6 (Transitivity of weights)*.* Let $B \subseteq A$ be some weights. Then it is easier to avoid B-weighted zero subsums than A-weighted ones. Consequently,

$$\mathrm{s}_B(G) \geq \mathrm{s}_A(G)\,.$$

In particular, connecting the weighted with the unweighted case

$$\mathrm{s}(G) = \mathrm{s}_{\{1\}}(G) \geq \mathrm{s}_{\{-1,1\}}(G) = \mathrm{s}_{\pm}(G)\,.$$

The weighted generalizations of many statements of the unweighted case just need some extra effort or additional constraints.

Remark 2.7 (Grynkiewicz, Marchan, and Ordaz)*.* [GMO12] The relation between the Davenport constant $\mathrm{D}(G)$ and $\mathrm{E}(G)$ in Remark 1.6 (Gao) still holds in the weighted case

$$\mathrm{E}_A(G) = \mathrm{D}_A(G) + |G| - 1.$$

We start off with some examples.

Example 2.8 (Basis)**.** A trivial sequence is a standard basis. Trivially, any sequence S with exactly one nonzero value in each dimension cannot admit A-weighted zero subsums. This proves, see Observation 1.19 (Gao)

$$\eta_A\left(\mathbb{Z}_n^d\right) \geq |S| + 1 = d + 1$$
$$\mathrm{s}_A\left(\mathbb{Z}_n^d\right) \geq n - 1 + \eta_A\left(\mathbb{Z}_n^d\right) \geq n + d.$$

For example in $\mathbb{Z}_n^3$

$$S = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

In the $\pm$-weighted case we can even take the basis $n-1$ times. Let n be odd.

$$\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) \geq \left|S^{n-1}\right| + 1 = d\,(n-1) + 1$$

Proof sketch. Assume to the contrary that there is a $\pm$-weighted zero subsum of length n in S^{n-1}. In order to reach 0, every vector occurs as many times with weight 1 as with weight -1 which is a contradiction to the assumption that n is odd. □

A similar parity argument enables reuse of sequences from the unweighted case.

Example 2.9 (Dimension of ones)**.** Let $n \geq 3$, odd. For a moment, forget about the weights and consider your favorite sequence $S = v_1, \ldots, v_k$ in $\mathbb{Z}_n^d$ without any (unweighted) zero subsum of length n. What happens if we append a dimension out of sheer 1s?

$$\begin{pmatrix}S\\1\end{pmatrix} = \begin{pmatrix}v_1\\1\end{pmatrix}, \begin{pmatrix}v_2\\1\end{pmatrix}, \ldots, \begin{pmatrix}v_k\\1\end{pmatrix}$$

Claim: The obtained sequence does not contain any $\pm$-weighted zero subsum. In particular,

$$\mathrm{s}_\pm\left(\mathbb{Z}_n^{d+1}\right) \geq \mathrm{s}\left(\mathbb{Z}_n^d\right)$$
$$\mathrm{h}_\pm\left(\mathbb{Z}_n^{d+1}\right) \geq \mathrm{g}\left(\mathbb{Z}_n^d\right)$$

Proof. What is the use of the row of 1s? All weights in a $\pm$-weighted zero subsum of length n are the same and therefore we are back in the familiar unweighted case. To see this, consider a zero subsum of length n as well as corresponding weights $a_i \in \{1, -1\}$ and look at the last row:

$$\sum_{i=1}^{n} a_i \cdot 1 \equiv 0 \pmod{n}.$$

As often before, we are going to examine the different cases of the equivalence sign. To reach n or $-n$ all weights a_i have to be the same as desired. In the remaining case 0 we would have $\frac{n}{2}$ many weights 1 and -1, though n was assumed to be odd. Note that the row of 1s also ensures that the obtained sequence is $\pm$-distinct whenever S is distinct. $\square$

Example 2.10 (Adhikari *et al.*)**.** [ACF+06, 1. Introduction] A superincreasing sequence yields the lower bound in dimension $d = 1$.

$$S = 2^0, 2^1, \dots, 2^{\lfloor \log_2 n \rfloor - 1}$$

Claim: This sequence does not contain $\pm$-weighted zero subsums. In particular, see Observation 1.19 (Gao)

$$\begin{aligned} \eta_{\pm}(\mathbb{Z}_n) &\geq |S| + 1 = \lfloor \log_2 n \rfloor + 1 \\ \mathrm{s}_{\pm}(\mathbb{Z}_n) &\geq \eta_{\pm}(\mathbb{Z}_n) + n - 1 = n + \lfloor \log_2 n \rfloor. \end{aligned}$$

Proof. Any $\pm$-weighted subsum in S cannot equal 0 due to the uniqueness of the binary expansion and multiples of n are also not possible since the absolute value of any sum is bounded by $2^{\lfloor \log_2 n \rfloor - 1 + 1} - 1 \leq n - 1 < n$. $\square$

Example 2.11 (Adhikari, Grynkiewicz, and Sun)**.** [AGS12, Theorem 1.3] By taking a superincreasing sequence in every dimension Example 2.10 (Adhikari *et al.*) can be lifted to general dimensions d using the construction of Lemma 1.55 (Trivial lifting)

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) \geq n + d\left(\eta_{\pm}(\mathbb{Z}_n) - 1\right) \geq n + d\lfloor \log_2 n \rfloor.$$

Example 2.12 (Adhikari, Ambily, and Sury)**.** [AAS10, Proposition 3 (ii)] Let $Q \subseteq \mathbb{Z}_p^\times$ be the subgroup of quadratic residues and let $t \in \mathbb{Z}_p^\times \setminus Q$ be an arbitrary nonresidue. Then

$$S = 1, -t$$

does not contain Q-weighted zero subsums. In particular, see Lemma 1.55 (Trivial lifting)

$$\begin{aligned}
\eta_Q(\mathbb{Z}_p) &\geq |S| + 1 = 3 \\
\eta_Q\left(\mathbb{Z}_p^d\right) &\geq d\left(\eta_Q(\mathbb{Z}_p) - 1\right) + 1 \geq 2d + 1 \\
\mathrm{s}_Q\left(\mathbb{Z}_p^d\right) &\geq p + d\left(\eta_Q(\mathbb{Z}_p) - 1\right) \geq p + 2d.
\end{aligned}$$

Proof. Assume to the contrary that there is a zero subsum, that is for some $q_1, q_t \in Q$

$$q_1 1 - q_t t \equiv 0 \Leftrightarrow q_1 \equiv q_t t. \pmod p$$

This is a contradiction as the product of a residue and a nonresidue is a nonresidue. □

Similarly to Lemma 1.13 (Affine transformation) one can reshape sequences without affecting zero subsums.

Lemma 2.13 (Linear transformation)**.** *Let A be some weights, S be a sequence in $\mathbb{Z}_n^d$ and f be a linear transformation, actually an automorphism,*

$$\begin{aligned}
f : \mathbb{Z}_n^d &\to \mathbb{Z}_n^d \\
v &\mapsto Mv
\end{aligned}$$

for some $M \in \mathrm{GL}\left(\mathbb{Z}_n^d\right)$. Then S contains an A-weighted zero subsum of length n if and only if $f(S) = \prod_{v \in S} f(v)$ does.

Proof. A subsequence T of length n in S forms a weighted zero subsum if and only if $f(T)$ is a weighted zero subsum in $f(S)$. Let $a_v \in A$.

$$\sum_{\tilde{v} \in f(T)} a_v \tilde{v} = \sum_{v \in T} a_v f(v) = \sum_{v \in T} a_v M v = M \sum_{v \in T} a_v v$$

Moreover, M is invertible, hence,

$$\sum_{\tilde{v} \in f(T)} a_v \tilde{v} \equiv 0 \Leftrightarrow \sum_{v \in T} a_v v \equiv 0 \pmod n$$

□

Remark 2.14 (Basis). Lemma 2.13 (Linear transformation) enables in the same way as in Remark 1.15 (Affine basis) that we may assume that a maximal sequence regarding $\mathrm{s}_A(G)$ contains a basis like Example 2.8 (Basis).

Lemma 2.15 (Godinho, Lemos, and Marques)**.** *[GLM13, Theorem 1] Let $n \geq 3$, odd.*

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) \geq 2^{d-1}(n-1)+1$$

In their original proof, Godinho, Lemos, and Marques constructed an explicit sequence of size $|S^{n-1}| = 2^{d-1}(n-1)$. Whereas, the presented proof here is radically simpler. Note that this new proof even enables a stronger statement, compare Remark 2.32 (Weighted asymptotic bounds).

Proof. This is an immediate consequence of Lemma 1.16 (Harborth) and Example 2.9 (Dimension of ones).

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) \geq \mathrm{s}\left(\mathbb{Z}_n^{d-1}\right) \geq 2^{d-1}(n-1)+1$$

$\square$

In the prime case there is a nice characterization of Definition 2.4 (A-distinct).

Lemma 2.16 (A-distinct)**.** *Let $A \subseteq \mathbb{Z}_p \setminus \{0\}$ be some weights. Then for any vectors $v, w \in \mathbb{Z}_p^d \setminus \{0\}$, $v \sim_A w$ if and only if $w = \alpha v$ and $A = \alpha A$ for some scalar $\alpha \in \mathbb{Z}_p \setminus \{0\}$ where $\alpha A = \{\alpha a : a \in A\}$ denotes element-wise multiplication.*

Proof. **Direction "if".** Let $w = \alpha v$ and $A = \alpha A$. Then

$$Av = A\alpha v = Aw$$

whence by definition $v \sim_A w$.

Direction "only if". Let $v \sim_A w$. For the first part let $a_v \in A$. Since $v \sim_A w$, there exists an $a_w \in A$ such that $a_v v = a_w w$ whence

$$w = \underbrace{a_w^{-1} a_v}_{\alpha} v.$$

The second part is a consequence of the finite field $\mathbb{Z}_p$ where every nonzero element is invertible.

$$Av = Aw = A\alpha v \Leftrightarrow A = A\alpha.$$

$\square$

Remark 2.17 (Subgroup condition). Let $A \leq \mathbb{Z}_p^\times$ be a subgroup of $\mathbb{Z}_p^\times$. Then the condition $A = \alpha A$ in the previous Lemma 2.16 (A-distinct) is trivially fulfilled by any $\alpha \in A$.

Observation 2.18 (Adhikari and Rath). [AR06, Remarks] Let $A \subseteq \mathbb{Z}_n \setminus \{0\}$ be some weights. Scaling the weights by an invertible scalar $\alpha \in \mathbb{Z}_n^\times$ does not influence zero subsums. In particular,

$$\mathrm{s}_A\left(\mathbb{Z}_n^d\right) = \mathrm{s}_{\alpha A}\left(\mathbb{Z}_n^d\right)$$

where $\alpha A = \{\alpha a : a \in A\}$ denotes element-wise multiplication.

Proof. Let S be a sequence and a_v be corresponding weights of an A-weighted zero subsum T in the sense of Definition 2.1 (Weighted zero subsum). Then T is an A-weighted zero subsum if and only if T is an αA-weighted zero subsum

$$\sum_{v \in T} a_v v = 0 \Leftrightarrow \alpha \sum_{v \in T} a_v v = \sum_{v \in T} \alpha a_v v = 0.$$

□

Remark 2.19 (Weights representation). Let $A \subseteq \mathbb{Z}_n^\times$ be some weights. The previous Observation 2.18 (Adhikari and Rath) tells us that we may assume $1 \in A$. Note that this representation is not unique. For example, in $\mathbb{Z}_7^\times$

$$\{1, 2, 3\} = 4\{1, 4, 5\} = 5\{1, 3, 5\}.$$

In the unweighted case the zero vector is part of all maximal sequences, see Remark 1.15 (Affine basis). However, in the $\pm$-weighted case the zero vector stands in contrast to Conjecture 2.33 (Property D). Thus, it is unlikely that the zero vector is part of a large sequence without $\pm$-weighted zero subsums.

Example 2.20 (Zero vector)**.** Let $n \geq 3$, odd. Claim: The sequence

$$v^{n-1}, 0 = \underbrace{v, \ldots, v}_{n-1 \text{ times}}, 0, \quad v \in \mathbb{Z}_n^d$$

forms a $\pm$-weighted zero subsum of length n.

Proof. Sum up $\frac{n-1}{2}$ many pairs of vs and $-v$s

$$v - v + \cdots + v - v + 0 = 0.$$

□

Not even too often two times the same vector is possible with the zero vector.

Lemma 2.21 (Zero vector)**.** *Let $n \geq 3$, odd and let $A = -A = \{-a : a \in A\}$ be some additively invertible weights. If the zero vector is part of some maximum sequence regarding $\mathrm{s}_A\left(\mathbb{Z}_n^d\right)$ then*

$$\mathrm{s}_A\left(\mathbb{Z}_n^d\right) \leq \mathrm{h}_A\left(\mathbb{Z}_n^d\right) + n - 2.$$

Proof. Let S be a maximum sequence regarding $\mathrm{s}_A\left(\mathbb{Z}_n^d\right)$ and assume the zero vector is present in S. Since there is no A-weighted zero subsum of length n in S, there are no $\frac{n-1}{2}$ many pairs of A-equivalent vectors in addition to the zero vector since otherwise we could form an A-weighted zero subsum of length n like in Example 2.20 (Zero vector). Consequently, by subtracting at most $\frac{n-1}{2} - 1$ many pairs and a possibly further zero vector, an A-distinct subsequence is obtained, still without A-weighted zero subsums of length n. Therefore, we obtain the lower bound

$$\mathrm{h}_A\left(\mathbb{Z}_n^d\right) \geq \mathrm{s}_A\left(\mathbb{Z}_n^d\right) - \left(2\left(\frac{n-1}{2} - 1\right) + 1\right) = \mathrm{s}_A\left(\mathbb{Z}_n^d\right) - (n-2)\,.$$

$\square$

The previous Lemma is an indication that the zero vector is not useful for large sequences, see Lemma 2.60 (Zero vector). Let us prepare some notation in order to formalize such statements.

Notation 2.22 (Allowed vectors)**.** Let $H \subseteq G$ be a subset of an abelian group G. $\mathrm{s}_A(H)$ denotes the smallest number of elements such that any sequence S in H of length $|S| \geq \mathrm{s}_A(H)$ contains an A-weighted zero subsum of length $\exp G$ in G. Analogously for all the other numbers defined in Notations 2.3 and 2.5 (Weighted zero subsums).

In Section 2.1 (Ternary Case) the following number will be of importance: $\mathrm{h}_\pm\left(\mathbb{Z}_n^d \setminus \{0\}\right)$ denotes the smallest number of elements such that any sequence S of $\pm$-distinct nonzero vectors in $\mathbb{Z}_n^d$ of length $|S| \geq \mathrm{h}_\pm\left(\mathbb{Z}_n^d \setminus \{0\}\right)$ contains a $\pm$-weighted zero subsum of length n.

Observation 2.23 (Dimension of ones).

$$\mathrm{h}_\pm\left(\mathbb{Z}_n^{d+1} \setminus \{0\}\right) \geq \mathrm{g}\left(\mathbb{Z}_n^d\right)$$

Proof. The constructed sequence in Example 2.9 (Dimension of ones) does not contain the zero vector. □

Sequences can be disguised to appear to be distinct provided there are enough weights and no multiple zero vectors. This has been proved by Godinho, Lemos, and Marques just for the case $\mathrm{s}_\pm\left(\mathbb{Z}_3^d\right)$.

Observation 2.24 (Godinho, Lemos, and Marques). [GLM13, Proposition 2] Let $H \subseteq \mathbb{Z}_p^d \setminus \{0\}$.

$$\mathrm{s}_{\mathbb{Z}_p^\times}(H) = \mathrm{g}_{\mathbb{Z}_p^\times}(H)$$

Proof. Trivially, $\mathrm{s}_A(H) \geq \mathrm{g}_A(H)$ as $\mathrm{g}_A(H)$ is more restrictive than $\mathrm{s}_A(H)$. As for the other direction, let S be a maximum sequence regarding $\mathrm{s}_\pm(H)$. Some vectors might occur up to $p-1$ times in S (though already p times the same vector gives a zero subsum). In order to transform S into a distinct sequence replace these duplicate nonzero vectors $v, \ldots, v$ by $v, 2v, 3v, \ldots, (p-1)\, v$ which by Observation 2.45 (A-weighted transformation) does not affect zero subsums. Consequently, $\mathrm{g}_{\mathbb{Z}_p^\times}(H) \geq |S| + 1 = \mathrm{s}_{\mathbb{Z}_p^\times}(H)$. □

Example 2.25 (Scaled copy)**.** Let $v \in \mathbb{Z}_n^d \setminus \{0\}$ be a nonzero vector. The sequence of scaled copies of v

$$S = \alpha_1 v, \ldots, \alpha_{|S|} v$$

where $\alpha_1, \ldots, \alpha_{|S|} \in \mathbb{Z}_n \setminus \{0\}$ are some nonzero scalars, contains an A-weighted zero subsum of length n if $|S| \geq \mathrm{s}_A(\mathbb{Z}_n \setminus \{0\})$.

Proof. Consider the scalars themselves as sequence in $\mathbb{Z}_n \setminus \{0\}$. Since $|S| \geq \mathrm{s}_A(\mathbb{Z}_n \setminus \{0\})$ there is an A-weighted zero subsum of length n within $\alpha_1, \ldots, \alpha_{|S|}$. Without loss of generality let $T = \alpha_1, \ldots, \alpha_n$ be this subsequence. Consequently, S contains an A-weighted zero subsum of length n

$$\sum_{i=1}^{n} a_i \alpha_i v = v \sum_{i=1}^{k} a_i \alpha_i \equiv 0. \pmod{n}$$

□

This previous Example enables to lift upper bounds in dimension $d = 1$ to general dimensions d.

Lemma 2.26 (*A*-weighted upper bound)**.**

$$\mathrm{s}_A\left(\mathbb{Z}_n^d\right) \leq \frac{n^d-1}{n-1}\left(\mathrm{s}_A(\mathbb{Z}_n \setminus \{0\})-1\right)+n$$

Proof. Each of the n^d-1 many nonzero vectors and their $n-1$ multiples occur at most $\mathrm{s}_A(\mathbb{Z}_n \setminus \{0\})-1$ times and the zero vector at most $n-1$ times

$$\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) \leq \frac{n^d-1}{n-1}\left(\mathrm{s}_A(\mathbb{Z}_n \setminus \{0\})-1\right)+(n-1)+1.$$

□

Lemma 2.27 (Adhikari, Ambily, and Sury)**.** *[AAS10, Proposition 3 (ii)] Let $A \leq \mathbb{Z}_p^\times$ be a subgroup. Then*

$$\mathrm{s}_A\left(\mathbb{Z}_p^d\right) \leq \left(\frac{d}{|A|}+1\right)(p-1)+1, \quad \forall d < \frac{|A|}{p-1}p.$$

Proof. Let $n = \left(\frac{d}{|A|}+1\right)(p-1)+1 = d\frac{p-1}{|A|}+p$ and $S = \begin{pmatrix} s_{11} \\ \vdots \\ s_{n1} \end{pmatrix}, \ldots, \begin{pmatrix} s_{1n} \\ \vdots \\ s_{nn} \end{pmatrix}$ be a sequence of length n in $\mathbb{Z}_p^d$. Similarly to the presented proof of Theorem 1.29 (Erdős, Ginzburg, and Ziv), consider suitable polynomials.

$$\begin{aligned} f_i\left(X_1, \ldots, X_n\right) &= \sum_{j=1}^n s_{ij} X_j^{\frac{p-1}{|A|}}, \quad 1 \leq i \leq d \\ f_{d+1}\left(X_1, \ldots, X_n\right) &= \sum_{j=1}^n X_j^{p-1} \end{aligned}$$

This time, $X_i^{\frac{p-1}{|A|}} \in \{0\} \uplus A$ not only indicates whether an element has been selected but also includes its weight. Again, $f_1 = \cdots = f_d = 0$ ensures that they form an A-weighted zero subsum and $f_{d+1} = 0$ validates the number of selected elements provided

$$n = d\frac{p-1}{|A|}+p < 2p \Leftrightarrow d < \frac{p|A|}{p-1}.$$

Altogether, nontrivial solutions of $f_1 = \cdots = f_{d+1} = 0$ correspond to A-weighted zero subsums in S of length p. The polynomials share the trivial solution and the sum of their total degrees is $d\frac{p-1}{|A|}+p-1 < n$ as required, thus Theorem 1.30 (Chevalley's theorem, [Che35]) ensures a nontrivial solution. □

The following Corollary generalizes Theorem 2.52 (Adhikari *et al.*).

Corollary 2.28 (Adhikari, Ambily, and Sury)**.** *[AAS10, Proposition 3 (i)]*

$$\mathrm{s}_{\mathbb{Z}_p^\times}\left(\mathbb{Z}_p^{p-1}\right) = 2\,(p-1) + 1$$

Proof sketch. Applying the previous Lemma 2.27 (Adhikari, Ambily, and Sury) proves the upper bound. The lower bound comes from Example 2.8 (Basis). □

The second Corollary has been confirmed by complete enumeration in Appendix A.1 (Maximum Sequences) for the cases $\mathrm{s}_{\{1,4\}}\left(\mathbb{Z}_5^d\right)$ where $d \leq 2$, $\mathrm{s}_{\{1,2,4\}}\left(\mathbb{Z}_7^d\right)$ where $d \leq 2$, and $\mathrm{s}_{\{1,3,4,5,9\}}(\mathbb{Z}_{11})$. From these computer results also note that the condition $d \leq \frac{p-1}{2}$ is necessary: For example, $\mathrm{s}_{\{1,4\}}(\mathbb{Z}_5^3) = 4\,(5-1) + 1 > 5 + 2 \cdot 3$.

Corollary 2.29 (Adhikari, Ambily, and Sury)**.** *[AAS10, Proposition 3 (ii)] Let $Q \leq \mathbb{Z}_p^\times$ be the subgroup of quadratic residues. Then*

$$\mathrm{s}_Q\left(\mathbb{Z}_p^d\right) = p + 2d, \quad \forall d \leq \frac{p-1}{2}.$$

Proof. The upper bound is another application of Lemma 2.27 (Adhikari, Ambily, and Sury). The lower bound has been presented in Example 2.12 (Adhikari, Ambily, and Sury). □

Generalizations of Corollary 2.29 (Adhikari, Ambily, and Sury) for arbitrary n have been proved in dimension $d = 1$ by Adhikari, David, and Jiménez Urroz [ADJ08], Chintamani and Moriya [CM12], Grundman and Owens [GO13], and Grynkiewicz and Hennecart [GH15].

Observation 2.30 (Trivial lifting)*.* The lower bound in Corollary 2.29 (Adhikari, Ambily, and Sury) comes from Lemma 1.55 (Trivial lifting). For small dimensions d this seems already to be optimal for several different weights, see computer results in Appendix A.1 (Maximum Sequences), especially compare the maximum sequences in Table 5 (Maximal septenary sequences in dimension $d = 1$) with those in Table 6 (Maximal septenary sequences in dimension $d = 2$).

Observation 2.31 (Subgroup weights)*.* Even though better upper bounds are known when $A \leq \mathbb{Z}_p^\times$ is a subgroup, computer results in Appendix A.1 (Maximum Sequences) did not show any particular differences to general $A \subseteq \mathbb{Z}_p \setminus \{0\}$.

Remark 2.32 (Weighted asymptotic bounds). The asymptotic upper bounds from Theorems 1.17 (Alon and Dubiner) and 1.18 (Liu and Spencer) are trivially inherited.

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) \leq \mathrm{s}\left(\mathbb{Z}_n^d\right) = \mathrm{O}(n)\,, \quad \text{as } n \to \infty$$
$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) \leq \mathrm{s}\left(\mathbb{Z}_n^d\right) = \mathrm{O}\left(\frac{n^d}{d^{n-2}}\right), \quad \text{as } d \to \infty$$

Lifting the asymptotic lower bound of Corollary 1.74 (Asymptotic lower bound) is yet another application of Example 2.9 (Dimension of ones). Fix $n \geq 3$, odd.

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) \geq \mathrm{s}\left(\mathbb{Z}_n^{d-1}\right) = \Omega\left(2.13^{d-1}\right) = \Omega\left(2.13^d\right), \quad \text{as } d \to \infty$$

The fact that a sequence can have many A-equivalent representations in the sense of Observation 2.45 (A-weighted transformation) requires Conjecture 1.21 (Property D) to be adapted for the A-weighted case.

Conjecture 2.33 (Property D)**.** *Every maximum sequence S' regarding $\mathrm{s}_A\left(\mathbb{Z}_n^d\right)$ has an A-equivalent representation S in the sense of Observation 2.45 (A-weighted transformation) that is of shape $S = T^{n-1}$ for some sequence T.*

Observation 2.34. It seems that at least one of Conjecture 1.20 (Gao) or Conjecture 2.33 (Property D) is fulfilled for at least one maximum sequence regarding $\mathrm{s}_A\left(\mathbb{Z}_n^d\right)$ where n is odd, see computer results in Appendix A.1 (Maximum Sequences).

Next, we adapt Definition 1.23 (Property G) for the weighted case. Later, in Section 2.1 (Ternary Case) we will have to treat the zero vector separately, that is why we formulate it via the flexible Notation 2.22 (Allowed vectors).

Definition 2.35 (Property H)**.** Let $H \subseteq \mathbb{Z}_n^d$.

$$\mathrm{s}_A(H) = (\mathrm{h}_A(H) - 1)\,(n-1) + 1$$

Remark 2.36 (Property H). Note that like in the unweighted case the direction "$\leq$" is always true: Let $H \subseteq \mathbb{Z}_n^d$.

$$\mathrm{s}_A(H) \leq (\mathrm{h}_A(H) - 1)\,(n-1) + 1$$

Proof. The proof is similar to the one of Remark 1.24 (Property G). Let S be a maximum sequence regarding $\mathrm{s}_A(H)$. By the Pigeonhole principle, either there are at least $\mathrm{h}_A(H)$ many A-distinct vectors or a vector occurs n times A-weighted. In both cases we have found an A-weighted zero subsum of length n. □

Remark 2.37. Likewise, Property H implies Conjecture 2.33 (Property D).

Proof. The proof is similar to the one of Remark 1.25 (). Let S' be a maximum sequence regarding $\mathrm{s}_A(H)$ of length $|S'| = \mathrm{s}_A(H) - 1 = (\mathrm{h}_A(H) - 1)(n-1)$ and let $T \subseteq S'$ be a subsequence of A-distinct vectors of maximum length. As n times the same vector gives a zero subsum, $T^{n-1} \supseteq S$ for some A-equivalent representation S of S' in the sense of Observation 2.45 (A-weighted transformation). On the contrary, T does not contain zero subsums of length n, which limits its length by $|T| \leq \mathrm{h}_A(H) - 1 = \frac{|S'|}{n-1}$, whence, $|T^{n-1}| \leq |S'| = |S|$. It follows $T^{n-1} = S$. □

Remark 2.38 (A-weighted implementation). The approach described in Remark 1.28 (Computer implementation) still works in the weighted case with one obvious modification: Keep track of all possible values of A-weighted zero subsums of length k. The number of candidate vectors can be reduced by a factor of $|A|$ whenever Observation 2.45 (A-weighted transformation) applies, which is useful for enumerating all maximal sequences done in Remark 2.65 ($\pm$-weighted ternary enumeration).

It turns out that in dimension $d = 1$ the lower bound by Example 2.10 (Adhikari *et al.*) is best possible.

Theorem 2.39 (Adhikari *et al.*)**.** *[ACF+06, Theorem 1.1]*

$$\mathrm{s}_{\pm}(\mathbb{Z}_n) = n + \lfloor \log_2 n \rfloor$$

Other weights than $\pm$-weights have been treated as well. The following Theorem has also been confirmed for $p \leq 11$ by computer enumeration in Appendix A.1 (Maximum Sequences).

Theorem 2.40 (Adhikari and Rath)**.** *[AR06, Theorem 2]*

$$\mathrm{s}_{\{1,2,\dots,k\}}(\mathbb{Z}_p) = \left\lceil \frac{p}{k} \right\rceil + p - 1.$$

In dimension $d = 1$ Theorem 2.42 (Cauchy–Davenport) is a useful tool. For example, Theorem 1.29 (Erdős, Ginzburg, and Ziv) can be deduced.

Notation 2.41 (Sumset)**.** [Nat96, p. 1] The element-wise addition of two sets is denoted by

$$A + B = \{a + b : a \in A, b \in B\}.$$

The following Theorem has been independently proven by Cauchy [Cau13] and Davenport [Dav35; Dav47], see Adhikari *et al.* [ACGM10].

Theorem 2.42 (Cauchy–Davenport)**.** *[Nat96, Theorem 2.2] Let p be a prime number and $\emptyset \neq A, B \subseteq \mathbb{Z}_p$ be two nonempty subsets. Then*

$$|A + B| \geq \min\{|A| + |B| - 1, p\}.$$

Corollary 2.43 (Cauchy–Davenport)**.** *[Nat96, Theorem 2.3] Let p be a prime number and $\emptyset \neq A_1, \ldots, A_k \subseteq \mathbb{Z}_p$ be some nonempty subsets. Then*

$$|A_1 + \cdots + A_k| \geq \min\left\{\sum_{i=1}^{k} |A_i| - k + 1, p\right\}.$$

Proof by induction on k.

Induction basis $k = 2$**.** Theorem 2.42 (Cauchy–Davenport).

Induction step. Assume the statement holds for $k - 1$.

$$\begin{aligned} |A_1 + \cdots + A_k| &\geq \min\{|A_1 + \cdots + A_{k-1}| + |A_k| - 1, p\} \\ &\geq \min\left\{\min\left\{\sum_{i=1}^{k-1} |A_i| - (k-1) + 1, p\right\} + |A_k| - 1, p\right\} \\ &= \min\left\{\sum_{i=1}^{k} |A_i| - k + 1, p\right\}. \end{aligned}$$

□

Observation 2.44 (Weighted sums sumset). Definition 2.1 (Weighted zero subsum) can be expressed as sumset: A sequence contains an A-weighted zero subsum of length k, if it contains a subsequence T of length k such that

$$0 \in \sum_{v \in T} Av$$

where $Av = \{av : a \in A\}$ denotes element-wise multiplication.

The following Observation is the motivation for defining the number $\mathrm{h}_A(G)$. It could have been proved earlier, nevertheless Notation 2.41 (Sumset) offers a very neat proof.

Observation 2.45 (A-weighted transformation)*.* Let $v \sim_A w$ for some vectors $v, w \in \mathbb{Z}_n^d$, see Lemma 2.16 (A-distinct). Then exchanging v and w in a sequence does not affect A-weighted zero subsums.

Proof. This follows immediately from Observation 2.44 (Weighted sums sumset) since by Definition 2.4 (A-distinct) $Av = Aw$. □

Proposition 2.46 (A-weighted sumset)**.** *Let $A \subseteq \mathbb{Z}_p \setminus \{0\}$ be some weights of cardinality at least $|A| \geq \frac{p-1}{k} + 1$. Then any k nonzero numbers in $\mathbb{Z}_p$ contain an A-weighted subsum of length k of arbitrary value, in particular a zero subsum.*

Proof. Let $x_1, \dots, x_k \in \mathbb{Z}_p \setminus \{0\}$ be k nonzero numbers. Utilizing the previous Observation 2.44 (Weighted sums sumset), the set of values of A-weighted subsums of length k can be written as sumset $Ax_1 + \cdots + Ax_k$. We will show that $Ax_1 + \cdots + Ax_k = \mathbb{Z}_p$ via Corollary 2.43 (Cauchy–Davenport).

$$|Ax_1 + \cdots + Ax_k| \geq \min\left\{\sum_{i=1}^{k} |Ax_i| - k + 1, p\right\} = \min\{p, p\} = p$$

$$\sum_{i=1}^{k} |Ax_i| = \sum_{i=1}^{k} |A| \geq k\left(\frac{p-1}{k} + 1\right) = p - 1 + k$$

□

Lemma 2.47 (A-weighted upper bound)**.** *Let $|A| \geq 2$ be some weights. Then*

$$\eta_A(\mathbb{Z}_p) \leq \left\lceil \frac{p-1}{|A|-1} \right\rceil$$

$$\mathrm{s}_A(\mathbb{Z}_p) \leq p + \left\lceil \frac{p-1}{|A|-1} \right\rceil - 1.$$

Proof. Let S be a sequence in $\mathbb{Z}_p$ of $k = \left\lceil \frac{p-1}{|A|-1} \right\rceil$ nonzero vectors, then

$$|A| \geq \frac{p-1}{k} + 1 \Leftrightarrow k \geq \frac{p-1}{|A|-1}$$

hence by the previous Proposition 2.46 (A-weighted sumset) they contain a short A-weighted zero subsum of length $k \leq p$, showing that $\eta_A(\mathbb{Z}_p) \leq k$. In order to show $\mathrm{s}_A(\mathbb{Z}_p) \leq p-1+k$, let T be a sequence in $\mathbb{Z}_p$ of $p-1+k$ vectors. This sequence contains a subsequence of at least k nonzero vectors. As before, they contain A-weighted subsums of length k of arbitrary value. Adding $p-k$ many of the priorly dismissed vectors, an A-weighted zero subsum of length p is obtained. $\square$

Corollary 2.48 (Maximal weights). *Let $A \subseteq \mathbb{Z}_p \setminus \{0\}$ of cardinality at least $|A| \geq \frac{p+1}{2}$. Then*

$$\eta_A(\mathbb{Z}_p) = 2$$
$$\mathrm{s}_A(\mathbb{Z}_p) = p + 1.$$

Proof. The lower bound is the trivial Example 2.8 (Basis). Lemma 2.47 (A-weighted upper bound) proves the upper bound

$$\left\lceil \frac{p-1}{|A|-1} \right\rceil \leq \left\lceil \frac{p-1}{\frac{p+1}{2}-1} \right\rceil = 2.$$

$\square$

Lemma 2.49 (Weighted Property D). *Let $A \subseteq \mathbb{Z}_p \setminus \{0\}$ be some weights of cardinality at least $|A| \geq 2$. Then in dimension $d = 1$, Conjecture 2.33 (Property D) is not fulfilled regarding $\mathrm{s}_A(\mathbb{Z}_p)$ and $\eta_A(\mathbb{Z}_p)$.*

Proof. By Proposition 2.46 (A-weighted sumset), any $p-1$ nonzero vectors contain a zero subsum of length $p-1$. A fortiori, by Example 2.8 (Basis) the sequence of $p-1$ zeros is not maximal. $\square$

Despite the negative result in dimension $d = 1$ it is nonetheless reasonable to assume Property D for d large enough, compare with Section 2.1 (Ternary Case).

Proposition 2.50. *Let p be a prime. Then*

$$\mathrm{s}_\pm(\mathbb{Z}_p \setminus \{0\}) = p.$$

Proof. The lower bound is the trivial Example 2.8 (Basis). If $n = p$ is a prime, then the upper bound follows from Proposition 2.46 (A-weighted sumset) as any $p-1$ or more nonzero vectors contain a $\pm$-weighted zero subsum of arbitrary length. $\square$

Lemma 2.51 (±-weighted upper bound)**.**

$$\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) \le \frac{n^d - 1}{n - 1}\left(n + \lfloor \log_2 n \rfloor - 1\right) + n$$

Furthermore, if $n = p$ is a prime or Conjecture 2.33 (Property D) is fulfilled, then

$$\mathrm{s}_\pm\left(\mathbb{Z}_p^d\right) \le n^d + n - 1.$$

Proof. This follows from Lemma 2.26 (A-weighted upper bound) and Theorem 2.39 (Adhikari *et al.*)

$$\begin{aligned}
\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) &\le \frac{n^d - 1}{n - 1}\left(\mathrm{s}_\pm\left(\mathbb{Z}_n^d \setminus \{0\}\right) - 1\right) + n \\
&\le \frac{n^d - 1}{n - 1}\left(\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) - 1\right) + n \\
&\le \frac{n^d - 1}{n - 1}\left(n + \lfloor \log_2 n \rfloor - 1\right) + n.
\end{aligned}$$

The “furthermore” part follows from Proposition 2.50

$$\begin{aligned}
\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) &\le \frac{n^d - 1}{n - 1}\left(\mathrm{s}_\pm\left(\mathbb{Z}_n^d \setminus \{0\}\right) - 1\right) + n \\
&= \frac{n^d - 1}{n - 1}(n - 1) + n = n^d + n - 1.
\end{aligned}$$

or assuming Property D the statement of Lemma 2.26 (A-weighted upper bound) can be strengthened as the multiplicity of any vector must be a multiple of $n - 1$

$$\begin{aligned}
\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right) &\le \frac{n^d - 1}{n - 1}\left\lfloor \frac{\mathrm{s}_\pm\left(\mathbb{Z}_n^d \setminus \{0\}\right) - 1}{n - 1} \right\rfloor (n - 1) + n \\
&\le \frac{n^d - 1}{n - 1}\left\lfloor \frac{n + \lfloor \log_2 n \rfloor - 1}{n - 1} \right\rfloor (n - 1) + n \\
&= \frac{n^d - 1}{n - 1}(n - 1) + n = n^d + n - 1.
\end{aligned}$$

□

This improves for $n \geq 5$, odd the currently best known upper bound $\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) \leq \frac{n^d-1}{2}(n-1)+1$ by Godinho, Lemos, and Marques [GLM13, Theorem 1].

The Polynomial method proof of dimension $d = 1$ in the unweighted case Theorem 1.29 (Erdős, Ginzburg, and Ziv) generalizes surprisingly naturally to dimension $d = 2$ in the weighted case.

Theorem 2.52 (Adhikari *et al.*)**.** *[ABPR08, Theorem 3] Let $n \geq 3$, odd.*

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^2\right) = 2(n-1)+1$$

Proof. The lower bound has already been established in Example 2.8 (Basis). It suffices to consider the upper bound for odd primes p due to Corollary 1.27 (Multiplicativity) as Lemma 1.26 (Multiplicativity) still holds in the A-weighted case provided A is multiplicatively closed. This has already been presented more generally in Corollary 2.28 (Adhikari, Ambily, and Sury). □

It seems that Theorem 2.52 (Adhikari *et al.*) also holds for different weights of cardinality $|A| = 2$, see Appendix A.1 (Maximum Sequences).

Conjecture 2.53 (Weighted dimension 2)**.** *Let $n \geq 3$, odd and $|A| = 2$.*

$$\mathrm{s}_A\left(\mathbb{Z}_n^2\right) = 2(n-1)+1$$

In dimension $d = 3$ only a lower bound is known. A quick construction is Example 2.9 (Dimension of ones)

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^3\right) \geq \mathrm{s}\left(\mathbb{Z}_n^2\right) = 4(n-1)+1.$$

Unfortunately, the proof of Theorem 1.31 (Reiher) does not directly generalize to the weighted case in dimension $d = 3$. The argument in Proposition 1.32 (Alon and Dubiner) is broken: A $\pm$-weighted zero subsum of length $3p$ which contains a $\pm$-weighted zero subsum of length $2p$ does not necessarily contain an $\pm$-weighted zero subsum of length $3p - 2p = p$ as the weights could differ.

Conjecture 2.54 ($\pm$-weighted dimension 3)**.** *Let $n \geq 3$, odd.*

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^3\right) = 4(n-1)+1$$

This has been confirmed for $n = 5$ in Table 4 (Maximal quinary sequences in dimension $d = 3$).

In the even case Example 2.11 (Adhikari, Grynkiewicz, and Sun) determines the asymptotic behavior up to a small error term

Theorem 2.55 (Adhikari, Grynkiewicz, and Sun)**.** *[AGS12] Let n be even and dimension d fixed.*

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^d\right) = n + d\log_2 n + \mathrm{O}(\log_2\log_2 n) \quad \text{as } n \to \infty.$$

2.1 Ternary Case

In the ternary case $n = 3$ the only nontrivial weights are $A = \{1, -1\} = \mathbb{Z}_3^\times$. Like in the unweighted case, the ternary case boils down to caps, this time, ternary projective caps. Recall Notation 2.22 (Allowed vectors).

Observation 2.56 (Marchan *et al.*)*.* [MOSS15, Lemma 5.2] A cap in $\mathrm{PG}(d, 3)$ corresponds to a sequence regarding $\mathrm{h}_{\pm}\left(\mathbb{Z}_3^{d+1} \setminus \{0\}\right)$. In particular, their maximum sizes coincide

$$\mathrm{m}_2(\mathrm{PG}(d, 3)) = \mathrm{h}_{\pm}\left(\mathbb{Z}_3^{d+1} \setminus \{0\}\right) - 1.$$

Proof. In addition to Observation 1.43 (Ternary equivalences) note that in the ternary case Definition 1.39 (Finite geometry) matches Definition 2.4 (A-distinct) in $\mathbb{Z}_3^{d+1} \setminus \{0\}$, see Lemma 2.16 (A-distinct). □

In order to relate $\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right)$ with projective caps, we need to relate $\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right)$ with $\mathrm{h}_{\pm}\left(\mathbb{Z}_3^{d+1} \setminus \{0\}\right)$.

Remark 2.57 (Ternary notation)*.* In the ternary case the numbers $\eta_{\pm}\left(\mathbb{Z}_3^d\right)$ and $\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right)$ coincide.

$$\eta_{\pm}\left(\mathbb{Z}_3^d\right) = \mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right)$$

Proof. No zero subsums of length 1 corresponds to nonzero vectors and forbidden zero subsums of length 2 corresponds to $\pm$-distinct vectors, see Lemma 2.16 (A-distinct)

$$u \pm v \equiv 0 \Leftrightarrow u \equiv \pm v \pmod 3.$$

□

Even though $\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right)$ is the same as $\eta_\pm\left(\mathbb{Z}_3^d\right)$, only $\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right)$ incorporates the right interpretation and sometimes generalizes statements for general n. However, Godinho, Lemos, and Marques [GLM13]misuse $\eta_\pm\left(\mathbb{Z}_3^d\right)$ to represent $\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right)$.

In order to relate $\mathrm{s}_\pm\left(\mathbb{Z}_3^d\right)$ with ternary projective caps we need to connect $\mathrm{s}\left(\mathbb{Z}_3^d\right)$ with $\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right)$. In the weighted case the zero vector behaves differently to the other nonzero vectors. For example, it is the only vector, that stays the same under any weights A. That is the reason why the zero vector is treated separately afterwards.

Lemma 2.58 (Ternary Property H)**.** *In the ternary case* $\mathrm{s}_\pm\left(\mathbb{Z}_3^d\right) \setminus \{0\}$ *Definition 2.35 (Property H) is fulfilled. In fact,*

$$\mathrm{s}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) = 2\,\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1.$$

Proof. The proof is similar to the one of Theorem 1.44 (Harborth). Because of Remark 2.36 (Property H) it suffices to prove $\mathrm{s}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) \geq 2\,\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1$. Let S be a maximum sequence regarding $\mathrm{h}_\pm\left(\mathbb{Z}_3^d\right)$, that is a sequence of $\pm$-distinct vectors of length $|S| = \mathrm{h}_\pm\left(\mathbb{Z}_3^d\right) - 1$ without $\pm$-weighted zero subsums of length 3. Claim: $S^2 = S, S$ does not contain $\pm$-weighted zero subsums of length 3 either, which then concludes the proof

$$\begin{aligned} \mathrm{s}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) &\geq \left|S^2\right| + 1 = 2|S| + 1 = 2\left(\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1\right) + 1 \\ &= 2\,\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1. \end{aligned}$$

Assume for a contradiction, there is a $\pm$-weighted zero subsum in S^2. As S does not contain $\pm$-weighted zero subsums, a $\pm$-weighted zero subsum cannot consist of three $\pm$-distinct vectors. So suppose $v \pm u \pm u \equiv 0 \pmod 3$. To fulfill this equation either $v = 0 \notin S$ or also $v = \pm u$, yet so many are not available in S^2. $\square$

It remains to connect $\mathrm{s}_\pm\left(\mathbb{Z}_3^d\right)$ with $\mathrm{s}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right)$. This happens in two steps.

Lemma 2.59 (Zero vector)**.** *The zero vector is part of all maximal sequences regarding* $\mathrm{h}_\pm\left(\mathbb{Z}_3^d\right)$. *In particular,*

$$\mathrm{h}_\pm\left(\mathbb{Z}_3^d\right) = \mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) + 1$$

Proof. The zero vector can be safely added to any sequence of nonzero $\pm$-distinct vectors without $\pm$-weighted zero subsums of length 3 because $0 \pm u \pm v \equiv 0 \pmod 3$ implies that u and v are not $\pm$-distinct. $\square$

Lemma 2.60 (Zero vector)**.** *The zero vector is not part of any maximum sequence regarding* $\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right)$ *for all dimensions* $d \geq 3$. *In contrast, in dimension* $d = 1$ *all maximum sequences contain the zero vector. In between in dimension* $d = 2$ *both cases occur. In particular, for* $d \geq 2$

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) = \mathrm{s}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right).$$

Proof. **Case** $d = 1$**.** By Theorem 2.39 (Adhikari *et al.*) $\mathrm{s}_{\pm}(\mathbb{Z}_3) = 4$. Assume to the contrary that there is a maximum sequence of length $|S| = \mathrm{s}_{\pm}(\mathbb{Z}_3) - 1 = 3$ avoiding the zero vector. Note that 1s and 2s are the same under $\pm$-weights, see Observation 2.45 (A-weighted transformation) and Lemma 2.16 (A-distinct). However, the sequence $S = 1, 1, 1$ contains a zero subsums of length 3.

Case $d = 2$**.** Theorem 2.52 (Adhikari *et al.*) tells us that the length of a maximum sequences is $\mathrm{s}_{\pm}(\mathbb{Z}_3^2) - 1 = 4$. Example 2.9 (Dimension of ones) and Example 2.11 (Adhikari, Grynkiewicz, and Sun) provide both maximum sequences with and without the zero vector.

$$\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$$
$$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$$

Case $d \geq 3$**.** Assume for a contradiction that the zero vector is present in a maximum sequence regarding $|S|$. Then Lemma 2.21 (Zero vector) implies

$$\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d\right) \geq \mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) - 1.$$

Concatenating this with Lemma 2.59 (Zero vector) and Lemma 2.58 (Ternary Property H) yields

$$\begin{aligned}\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) = \mathrm{h}_{\pm}\left(\mathbb{Z}_3^d\right) - 1 \geq \mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) - 2 \geq \mathrm{s}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 2 \\ = 2\,\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 3\end{aligned}$$

whence $3 \geq \mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right)$ contradicting Observation 2.23 (Dimension of ones) recalling Section 1.1 (Ternary Case)

$$\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) \geq \mathrm{h}_{\pm}\left(\mathbb{Z}_3^3 \setminus \{0\}\right) \geq \mathrm{g}\left(\mathbb{Z}_3^2\right) = 5.$$

□

Theorem 2.61 (Godinho, Lemos, and Marques)**.** *[GLM13, Proposition 1] Let $d \geq 3$. In the $\pm$-weighted ternary case $\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right)$ Conjecture 2.33 (Property D) is fulfilled. In fact, even for $d \geq 2$*

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) = 2\,\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1.$$

Proof. By Lemma 2.60 (Zero vector) the numbers $\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right)$ and $\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right)$ coincide for $d \geq 3$ and their values are equal even starting at $d \geq 2$. Regarding the latter number $\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right)$ it has been proved in Lemma 2.58 (Ternary Property H) that Property H is fulfilled which by Remark 2.37 implies Property D

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) = \mathrm{s}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) = 2\,\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1.$$

□

Remark 2.62 ($\pm$-weighted ternary implementation)*.* Similarly to Remark 1.45 (Ternary implementation), exploiting Theorem 2.61 (Godinho, Lemos, and Marques) enables working with sequences half as large. Moreover, in the $\pm$-weighted case one can assume that a maximal sequence contains a basis, see Remark 2.14 (Basis).

Remark 2.63 (Davis and Maclagan)*.* [DM03, Table 3] Large projective caps in low dimensions have already been tackled in the 1970s by Pellegrino [Pel70] and Hill [Hil73]. The corresponding sequence $(\mathrm{m}_2(\mathrm{PG}(d,3)))_{d\in\mathbb{N}} = 2, 4, 10, 20, 56, \ldots$ can be found in The On-Line Encyclopedia of Integer Sequences [Hav04b].

Corollary 2.64 ($\pm$-weighted ternary values in low dimensions)**.**

$$\begin{aligned}
\mathrm{s}_\pm(\mathbb{Z}_3) &= 3 + \lfloor \log_2 3 \rfloor = 4 \\
\mathrm{s}_\pm\left(\mathbb{Z}_3^2\right) &= 2\,(3-1) + 1 = 5 \\
\mathrm{s}_\pm\left(\mathbb{Z}_3^3\right) &= 4\,(3-1) + 1 = 9 \\
\mathrm{s}_\pm\left(\mathbb{Z}_3^4\right) &= 10\,(3-1) + 1 = 21 \\
\mathrm{s}_\pm\left(\mathbb{Z}_3^5\right) &= 20\,(3-1) + 1 = 41 \\
\mathrm{s}_\pm\left(\mathbb{Z}_3^6\right) &= 56\,(3-1) + 1 = 113
\end{aligned}$$

Proof. Except for dimension $d = 1$ which is due to Theorem 2.39 (Adhikari *et al.*), relating Theorem 2.61 (Godinho, Lemos, and Marques) with Observation 2.56 (Marchan *et al.*) also the ternary $\pm$-weighted number $\mathrm{s}_\pm\left(\mathbb{Z}_3^d\right)$ is determined by projective caps, see Remark 2.63 (Davis and Maclagan)

$$\mathrm{s}_\pm\left(\mathbb{Z}_3^d\right) = 2\,\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1 = 2\,\mathrm{m}_2(\mathrm{PG}(d-1,q)) + 1.$$

$\square$

Remark 2.65 ($\pm$-weighted ternary enumeration)*.* Analogously to Remark 1.48 (Ternary enumeration), implementing the greedy approach described in Remarks 2.38 (A-weighted implementation) and 2.62 ($\pm$-weighted ternary implementation) all maximal sequences have been enumerated regarding $\mathrm{h}_\pm\left(\mathbb{Z}_3^d\right) \setminus \{0\}$ up to dimension $d = 4$. Regarding $\mathrm{h}_\pm(\mathbb{Z}_3^3) \setminus \{0\}$ there are 3 maximal and at the same time maximum sequences of length 4. One dimension higher there are 181 different maximal sequences of length 8 and 18 maximum sequences of length 10.

Remark 2.66 ($\pm$-weighted ternary greedy)*.* Analogously to Remark 1.48 (Ternary enumeration), maximum sequences regarding $\mathrm{h}_\pm\left(\mathbb{Z}_3^d \setminus \{0\}\right)$ have been found up to dimension $d = 6$.

Example 2.67 (Ternary weighted dimension 4)**.** Apart from the fact the the value of $\mathrm{s}_\pm(\mathbb{Z}_3^4) = 21$ is known, exhaustive computer search shows that the weighted adaption of Example 1.59 (Elsholtz) via Example 2.9 (Dimension of ones) can be (uniquely) extended by $\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^\intercal$. The full sequence is

$$S = \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\0\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\1\\2\\1 \end{pmatrix}, \begin{pmatrix} 2\\1\\2\\1 \end{pmatrix}, \begin{pmatrix} 1\\2\\2\\1 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}.$$

In particular,

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^4\right) \geq 10\,(3-1)+1 = 21.$$

Example 2.68 (Ternary weighted dimension 6)**.** Apart from the fact the the value of $\mathrm{s}_{\pm}(\mathbb{Z}_3^6) = 113$ is known as well, it is nevertheless surprising a large maximum sequence can still be quickly found by a computer as described in Remarks 2.38 (A-weighted implementation), 2.62 ($\pm$-weighted ternary implementation), and 2.66 ($\pm$-weighted ternary greedy). The full sequence S consists of the vectors of the following matrix.

$$S = \begin{pmatrix} 10100022010201210212212010200211011021210220210201021012 \\ 01120201111200200011121212120012112101010020120100220220 \\ 11111122001200120000012201222222111200110002001111120002 \\ 00001111000011112222000011112222000011112222000011112222 \\ 00000000111111111111000000000000111111111111222222222222 \\ 00000000000000000000111111111111111111111111111111111111 \end{pmatrix}$$

In particular,

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^6\right) \geq 56\,(3-1)+1 = 113.$$

In higher dimensions better caps are known than found with this greedy computer approach.

Remark 2.69 (Edel; Bierbrauer and Edel). [Ede04; BE14] The best known projective caps are available on Edel's homepage [Ede10].

$$\begin{aligned} \mathrm{m}_2(\mathrm{PG}(6,3)) &\geq 112 \\ \mathrm{m}_2(\mathrm{PG}(7,3)) &\geq 248 \\ \mathrm{m}_2(\mathrm{PG}(8,3)) &\geq 541 \\ \mathrm{m}_2(\mathrm{PG}(9,3)) &\geq 1216 \\ \mathrm{m}_2(\mathrm{PG}(10,3)) &\geq 2744 \\ \mathrm{m}_2(\mathrm{PG}(11,3)) &\geq 6464. \end{aligned}$$

The following approach is a different way of obtaining most of the known upper bounds.

Lemma 2.70 (Godinho, Lemos, and Marques)**.** *[GLM13, Proposition 5] Let $d \geq 2$.*

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) \leq 2\left\lceil \frac{\mathrm{s}\left(\mathbb{Z}_3^d\right)}{4} \right\rceil - 1$$

Proof. Concatenating Lemma 2.58 (Ternary Property H), Observation 2.24 (Godinho, Lemos, and Marques), and Observation 2.6 (Transitivity of weights)

$$2\,\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1 = \mathrm{s}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) = \mathrm{g}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) \leq \mathrm{g}_{\pm}\left(\mathbb{Z}_3^d\right) \leq \mathrm{g}\left(\mathbb{Z}_3^d\right)$$

whence (note that since $\mathrm{h}_A(H) \in \mathbb{N}$ we are allowed to round down)

$$\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) \leq \left\lfloor \frac{\mathrm{g}\left(\mathbb{Z}_3^d\right) + 1}{2} \right\rfloor.$$

Eventually, in terms of $\mathrm{s}_A\left(\mathbb{Z}_n^d\right)$ and $\mathrm{s}\left(\mathbb{Z}_n^d\right)$ utilizing Lemma 2.60 (Zero vector) and Theorem 1.44 (Harborth)

$$\begin{aligned}\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) &= \mathrm{s}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) = 2\,\mathrm{h}_{\pm}\left(\mathbb{Z}_3^d \setminus \{0\}\right) - 1 \leq 2\left\lfloor \frac{\mathrm{g}\left(\mathbb{Z}_3^d\right) + 1}{2} \right\rfloor - 1 \\ &= 2\left\lfloor \frac{\frac{\mathrm{s}\left(\mathbb{Z}_3^d\right)+1}{2} + 1}{2} \right\rfloor - 1 = 2\left\lfloor \frac{\mathrm{s}\left(\mathbb{Z}_3^d\right) + 3}{4} \right\rfloor - 1 = 2\left\lceil \frac{\mathrm{s}\left(\mathbb{Z}_3^d\right)}{4} \right\rceil - 1.\end{aligned}$$

□

Example 2.71. Plugging in the values of Corollary 1.47 (Ternary values in low dimensions) into the previous Lemma 2.70 (Godinho, Lemos, and Marques), the following upper bounds are obtained, which are tight except for dimension $d = 5$.

$$\begin{aligned}\mathrm{s}_{\pm}\left(\mathbb{Z}_3^2\right) &\leq 5 \\ \mathrm{s}_{\pm}\left(\mathbb{Z}_3^3\right) &\leq 9 \\ \mathrm{s}_{\pm}\left(\mathbb{Z}_3^4\right) &\leq 21 \\ \mathrm{s}_{\pm}\left(\mathbb{Z}_3^5\right) &\leq 45 \\ \mathrm{s}_{\pm}\left(\mathbb{Z}_3^6\right) &\leq 113\end{aligned}$$

Combining Observation 2.24 (Godinho, Lemos, and Marques) with Observation 2.6 (Transitivity of weights) and Theorem 1.50 (Meshulam) yields

Lemma 2.72 (Godinho, Lemos, and Marques)**.** *[GLM13, Remark 2] Let $d \geq 2$.*

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) = \mathrm{g}_{\pm}\left(\mathbb{Z}_3^d\right) \leq \mathrm{g}\left(\mathbb{Z}_3^d\right) \leq 2\frac{3^d}{d}$$

Similarly, Theorem 1.52 (Ellenberg and Gijswijt) is trivially inherited from the unweighted case.

Lemma 2.73 (Weighted ternary asymptotic upper bound)**.**

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) \leq \mathrm{s}\left(\mathbb{Z}_3^d\right) = \mathrm{o}\left(2.76^d\right), \quad \text{as } d \to \infty$$

Finally, Theorem 1.54 (Edel) is inherited from the unweighted case via Example 2.9 (Dimension of ones).

Theorem 2.74 (Asymptotic weighted lower bound)**.**

$$\begin{aligned}\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right) \geq \mathrm{s}\left(\mathbb{Z}_3^{d-1}\right) &= \Omega\left(\sqrt[62]{2573417086913773305856}^{d-1}\right) \\ &= \Omega\left(\sqrt[62]{2573417086913773305856}^{d}\right) \\ &= \Omega\left(2.21^d\right), \quad \text{as } d \to \infty.\end{aligned}$$

2.2 Odd Case

Example 2.75 ($\pm$-weighted dimension 4)**.** Let $n \geq 3$, odd. Let S be the sequence in $\mathbb{Z}_n^4$ formed by the columns of one of the two matrices.

$$\begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 & 2 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Claim: S^{n-1} does not contain $\pm$-weighted zero subsums of length n. In particular,

$$\mathrm{s}_{\pm}\left(\mathbb{Z}_n^4\right) \geq 10\,(n-1) + 1.$$

Proof. The proof is only performed for the sequence with $\pm$-entries. Besides, it can be accomplished by a computer anyway, see Remark 2.76 (Algorithmic approach). Similarly to Example 1.59 (Elsholtz) formulate zero subsums as congruency system modulo n. Denote by a_i^+, a_i^- the number of occurrences of the vector v_i with weight 1 or -1 respectively. Express the total weight

$a_i = a_i^+ - a_i^-$ and the total number of occurrences $|a_i| = a_i^+ + a_i^-$.

$$\begin{aligned}
|a_1| + |a_2| + |a_3| + |a_4| + |a_5| + |a_6| + |a_7| + |a_8| + |a_9| + |a_{10}| &= n \\
a_1 - a_2 - a_3 - a_4 + a_5 \qquad\qquad + a_8 \qquad\qquad &\equiv 0 \\
-a_1 + a_2 - a_3 - a_4 \qquad + a_6 \qquad\qquad + a_9 \qquad &\equiv 0 \\
-a_1 - a_2 + a_3 - a_4 \qquad\qquad + a_7 \qquad\qquad + a_{10} &\equiv 0 \\
a_5 + a_6 + a_7 - a_8 - a_9 - a_{10} &\equiv 0
\end{aligned}$$

Assume one of the first three coordinates equals $\pm n$. Then solely vectors with the coefficient $\pm 1s$ haven been taken. Consider without loss of generality the first coordinate (the first three coordinates are symmetric). Furthermore, scale each vector in the sense of Observation 2.45 (A-weighted transformation) to normalize the first coordinate. Then, recalling Example 2.9 (Dimension of ones) we are back in the unweighted case (all a_i^- are zero).

$$\begin{aligned}
a_1 + a_2 + a_3 + a_4 + a_5 + a_8 &= n \\
-a_1 - a_2 + a_3 + a_4 \qquad\qquad &\equiv 0 \pmod n \\
-a_1 + a_2 - a_3 + a_4 \qquad\qquad &\equiv 0 \pmod n \\
a_5 - a_8 &\equiv 0 \pmod n
\end{aligned}$$

The congruency in the last coordinate can be easily resolved to equals 0 since $0 \leq a_i \leq n-1$. Assume one of the other coordinates does not equal zero. This limits the linear system to just two variables in order to reach $\pm n$. Without loss of generality consider the first coordinate equals n.

$$\begin{aligned}
a_3 + a_4 &= n \\
-a_3 + a_4 &\equiv 0 \pmod n
\end{aligned}$$

Then, since $0 \leq a_i \leq n-1$ the other coordinate equals 0. Adding them up yields to a contradiction to n odd. Therefore all coordinates equal 0.

$$\begin{aligned}
a_1 + a_2 + a_3 + a_4 + a_5 + a_8 &= n \\
-a_1 - a_2 + a_3 + a_4 \qquad\qquad &= 0 \\
-a_1 + a_2 - a_3 + a_4 \qquad\qquad &= 0 \\
a_5 - a_8 &= 0
\end{aligned}$$

Again, adding all rows (the second one twice) yields a contradiction to n odd.

After this prelude, we may assume that the first three coordinates all equal 0 in the weighted case. The last row is not needed anymore and henceforth omitted . We are left with

$$\begin{array}{rrrrrrrrrrl}
|a_1| + & |a_2| + & |a_3| + & |a_4| + & |a_5| + & |a_6| + & |a_7| + & |a_8| + & |a_9| + & |a_{10}| & = n \\
a_1 - & a_2 - & a_3 - & a_4 + & a_5 & & + & a_8 & & & = 0 \\
-a_1 + & a_2 - & a_3 - & a_4 & & + \; a_6 & & & + \; a_9 & & = 0 \\
-a_1 - & a_2 + & a_3 - & a_4 & & & + \; a_7 & & & + \; a_{10} & = 0
\end{array}$$

Once again, adding all rows yields a contradiction to n odd since $|x| \pm x$ is even. □

Remark 2.76 (Algorithmic approach). As seen before in Example 2.75 ($\pm$-weighted dimension 4) and analogously to the unweighted case Remark 1.60 (Algorithmic approach) we can express A-weighted zero subsums as linear congruency system as well.

Remark 2.77 ($\pm$-weighted dimension 4 greedy). Computer search for lower bounds regarding $\mathrm{s}_\pm\left(\mathbb{Z}_p^4\right)$ where $p = 5, 7$ did also just find sequences of length 10 but no longer. This motivates the following Conjecture.

Conjecture 2.78 ($\pm$-weighted dimension 4)**.** *Let $n \geq 3$, odd.*

$$\mathrm{s}_\pm\left(\mathbb{Z}_n^4\right) = 10\,(n-1) + 1$$

Lemma 2.79 ($\pm$-weighted lower bound in low dimensions)**.** *Let $n \geq 3$, odd.*

$$\begin{aligned}
\mathrm{s}_\pm\left(\mathbb{Z}_n^3\right) &\geq 4\,(n-1) + 1 \\
\mathrm{s}_\pm\left(\mathbb{Z}_n^4\right) &\geq 10\,(n-1) + 1 \\
\mathrm{s}_\pm\left(\mathbb{Z}_n^5\right) &\geq 20\,(n-1) + 1 \\
\mathrm{s}_\pm\left(\mathbb{Z}_n^6\right) &\geq 42\,(n-1) + 1 \\
\mathrm{s}_\pm\left(\mathbb{Z}_n^7\right) &\geq 96\,(n-1) + 1 \\
\mathrm{s}_\pm\left(\mathbb{Z}_n^8\right) &\geq 196\,(n-1) + 1
\end{aligned}$$

Proof. Except for Example 2.75 ($\pm$-weighted dimension 4), these lower bounds are inherited from the unweighted case Theorem 1.73 (Edel) using Example 2.9 (Dimension of ones). □

So far, good sequences that work for general odd n are all ternary sequences. Moreover, there are better ternary sequences known than for general n, compare Corollary 2.64 ($\pm$-weighted ternary values in low dimensions) with Lemma 2.79 ($\pm$-weighted lower bound in low dimensions). It turns out that this is not the case in dimension $d = 5$ where there are better quinary sequences than ternary sequences.

Example 2.80 (Maximal $\pm$-weighted quinary sequence)**.** Let S be the sequence in $\mathbb{Z}_5^5$ formed by the columns of one of the following matrices.

$$\begin{pmatrix} 4 & 2 & 2 & 4 & 4 & 4 & 2 & 2 & 3 & 2 & 4 & 1 & 4 & 2 & 0 & 1 & 1 & 0 & 4 & 0 & 3 \\ 0 & 1 & 0 & 4 & 0 & 4 & 4 & 3 & 1 & 3 & 4 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 1 & 3 & 4 \\ 1 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 0 & 2 & 1 & 2 & 2 & 0 & 0 & 1 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 & 3 & 4 & 4 & 3 & 4 & 0 & 0 & 0 & 2 & 2 & 3 & 3 & 4 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 & 0 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 & 4 & 1 & 4 & 1 & 4 & 1 & 4 & 1 & 4 & 1 & 4 \\ 1 & 0 & 4 & 0 & 0 & 1 & 4 & 0 & 0 & 4 & 4 & 1 & 1 & 4 & 4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 4 & 4 & 0 & 0 & 4 & 4 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 4 & 4 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 4 & 4 & 0 & 0 & 4 & 4 \\ 4 & 0 & 1 & 0 & 4 & 0 & 1 & 0 & 4 & 0 & 1 & 4 & 0 & 1 & 1 & 0 & 4 & 0 & 1 & 0 & 4 \end{pmatrix}$$

Claim: S^4 does not contain $\pm$-weighted zero subsums of length 5. In particular,

$$\mathrm{s}_\pm\left(\mathbb{Z}_5^5\right) \geq 21\,(5-1)+1.$$

Proof sketch. Let the computer do the work, see Remarks Remark 1.28 (Computer implementation) and Remark 2.38 (A-weighted implementation). $\square$

Conclusion

The majority of Section 1 (The Number $\mathrm{s}\left(\mathbb{Z}_n^d\right)$) could be recovered for the weighted case $\mathrm{s}_\pm\left(\mathbb{Z}_n^d\right)$, alas not Theorem 1.31 (Reiher). Considering other weights A, there are less results known, particularly weights A that are not subgroups of $\mathbb{Z}_n^\times$ are rarely covered by any of the known statements.

Apart from that, there are much more related questions that are beyond the scope of this thesis and thus have not been mentioned like for example $\mathrm{s}(G)$ where G is a different abelian group than $\mathbb{Z}_n^d$.

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A Appendix

A.1 Maximum Sequences

Implementing the enumeration approach described in Remarks 1.28 (Computer implementation) and 2.38 (A-weighted implementation) all maximum sequences regarding $\mathrm{s}_A\left(\mathbb{Z}_p^d\right)$ have been enumerated for small dimensions d and $p \leq 11$.

Remark A.1 (How to read these tables)*.* As already stated in Observation 2.34 it turned out that all maximum sequences either fulfill Conjecture 2.33 (Property D) or are determined by Conjecture 1.20 (Gao). That means that there are subsequences S such that

$$\mathrm{s}_A\left(\mathbb{Z}_n^d\right) = \left|S^{n-1}\right| + 1 = |S|\,(n-1) + 1$$

or

$$\mathrm{s}_A\left(\mathbb{Z}_n^d\right) = \underbrace{(p-1)}_{\text{zeros}} + \eta_A\left(\mathbb{Z}_n^d\right) + 1.$$

or both. This is indicated in the tables and only the (shorter) regarding sequences S are given.

Recalling Remark 2.19 (Weights representation) the tables have been slimmed further: Only one of the equivalent weights is given. For example, $2\{1,3\} \equiv \{1,2\} \pmod 5$ hence it suffices to look at $\mathrm{s}_{\{1,2\}}\left(\mathbb{Z}_5^d\right)$.

Corollary 2.48 (Maximal weights) enables further less weights A in dimension $d = 1$, because $\mathrm{s}_A(\mathbb{Z}_p) = p + 1$ where $|A| \geq \frac{p+1}{2}$. In these cases there is just one maximum sequence Example 2.8 (Basis). This is why they are omitted.

The last convention: Repeating rows are indicated by “—” dashes.

d	$\mathrm{s}_{\pm}\left(\mathbb{Z}_3^d\right)$	shape of maximum sequences S	
1	4	$\eta_{\pm}(\mathbb{Z}_3) = 1 + 1$	$S = 1$
2	5	$\eta_{\pm}(\mathbb{Z}_3^2) = 2 + 1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
		$\mathrm{s}_{\pm}(\mathbb{Z}_3^2) = 2\,(3-1) + 1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
3	9	$\mathrm{s}_{\pm}(\mathbb{Z}_3^3) = 4\,(3-1) + 1$	$S = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, v \quad v \in \{1,2\}^d$
4	21	$\mathrm{s}_{\pm}(\mathbb{Z}_3^4) = 10\,(3-1) + 1$	18 examples

Table 2: Maximal ternary sequences

d	A	$\mathrm{s}_A\left(\mathbb{Z}_5^d\right)$	shape of maximum sequences S	
1	$\pm$	7	$\eta_A(\mathbb{Z}_5) = 2 + 1$	$S = 1, 2$
1	$\{1, 2\}$	—	—	$S = 1, 1$
2	$\pm$	9	$\eta_A(\mathbb{Z}_5^2) = 4 + 1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-2\\1\end{pmatrix}$
				$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}0\\x\end{pmatrix}, \quad x \in \mathbb{Z}_5$
			$\mathrm{s}_A(\mathbb{Z}_5^2) = 2\,(5-1) + 1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
2	$\{1, 2\}$	—	$\eta_A(\mathbb{Z}_5^2) = 4 + 1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}x\\1\end{pmatrix}, \quad x \in \mathbb{Z}_5$
				$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, v, v, \quad v \in \left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right\}$
			$\mathrm{s}_A(\mathbb{Z}_5^2) = 2\,(5-1) + 1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
2	$\{1, 2, 3\}$	7	$\eta_A(\mathbb{Z}_5^2) = 2 + 1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
2	$\mathbb{Z}_5^\times$	—	—	—

Table 3: Maximal quinary sequences in dimension $d \leq 2$

A	$\mathrm{s}_A(\mathbb{Z}_5^3)$	shape of maximum sequences S	
$\pm$	17	$\mathrm{s}_A(\mathbb{Z}_5^3) = 4\,(5-1)+1$	$S = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, v, \quad v \in \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\4\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\3\\3\end{pmatrix} \right\}$
$\{1,2\}$	17	$\mathrm{s}_A(\mathbb{Z}_5^3) = 4\,(5-1)+1$	$S = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, v, \quad v \in \left\{ \begin{pmatrix}1\\1\\4\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\}$
$\{1,2,3\}$	9	$\eta_A(\mathbb{Z}_5^3) = 4+1$	$S = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}$
$\mathbb{Z}_5^\times$	8	$\eta_A(\mathbb{Z}_5^3) = 3+1$	$S = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

Table 4: Maximal quinary sequences in dimension $d = 3$

A	$\mathrm{s}_A(\mathbb{Z}_7)$	shape of maximum sequences S		
$\{1,2\}$	10	$\eta_A(\mathbb{Z}_7) = 3+1$	$S = 1,1,1$	
$\pm$	9	$\eta_A(\mathbb{Z}_7) = 2+1$	$S = 1, v,$	$v \in \{2,3,4,5\}$
$\{1,3\}$	—	—	$S = 1, v,$	$v \in \{1,3,5\}$
$\{1,2,3\}$	—	—	$S = 1,1$	
$\{1,2,4\}$	—	—	$S = 1, v,$	$v \in A$
$\{1,2,5\}$	8	$\eta_A(\mathbb{Z}_7) = 1+1$	$S = 1$	
$\{1,2,6\}$	—	—	—	

Table 5: Maximal septenary sequences in dimension $d = 1$

A	$\mathrm{s}_A(\mathbb{Z}_7^2)$	shape of maximum sequences S	
$\pm$	13	$\mathrm{s}_A(\mathbb{Z}_5^2) = 2\,(7-1)+1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
$\{1,2\}$	—	$\eta_A(\mathbb{Z}_7) = 6+1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}x\\1\end{pmatrix}, \begin{pmatrix}y\\1\end{pmatrix}, \quad x,y \in \mathbb{Z}_7$
		$\mathrm{s}_A(\mathbb{Z}_7^2) = 2\,(7-1)+1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
$\{1,3\}$	—	$\mathrm{s}_A(\mathbb{Z}_7^2) = 2\,(7-1)+1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
$\{1,2,3\}$	11	$\eta_A(\mathbb{Z}_7^2) = 4+1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}x\\1\end{pmatrix}, \quad x \in \mathbb{Z}_7$
$\{1,2,4\}$	—	—	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}x\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}y\\z\end{pmatrix}, \quad x,z \in A, y \in \mathbb{Z}_7$
$\{1,2,5\}$	10	$\eta_A(\mathbb{Z}_7^2) = 3+1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}$
$\{1,2,6\}$	—	—	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}$
$\{1,2,3,5\}$	—	—	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$
$\{1,2,5,6\}$	—	—	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, v, \quad v \in \left\{\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}\right\}$
$\{1,2,3,4\}$	9	$\eta_A(\mathbb{Z}_7^2) = 2+1$	$S = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$
$\{1,2,3,4,5\}$	—	—	—
$\mathbb{Z}_7^\times$	—	—	—

Table 6: Maximal septenary sequences in dimension $d = 2$

A	$\mathrm{s}_A(\mathbb{Z}_{11})$	shape of maximum sequences S	
$\{1,2\}$	16	$\eta_A(\mathbb{Z}_{11}) = 5+1$	$S = 1,1,1,1,1$
$\{1,3\}$	15	$\eta_A(\mathbb{Z}_{11}) = 4+1$	$S = 1,1,1,1$
$\pm$	14	$\eta_A(\mathbb{Z}_{11}) = 3+1$	$S = 1,2,4$
			$S = 1,2,5$
			$S = 1,3,5$
$\{1,5\}$	—	—	$S = 1,1,v, \quad v \in \{3,7,8\}$
			$S = 1,3,4$
$\{1,7\}$	—	—	$S = 1,1,v, \quad v \in \{1,5\}$
$\{1,2,3\}$	—	—	$S = 1,1,1$
$\{1,2,4\}$	—	—	—
$\{1,2,5\}$	13	$\eta_A(\mathbb{Z}_{11}) = 2+1$	$S = 1,v, \quad v \in \{1,7,8\}$
$\{1,2,7\}$	—	—	—
$\{1,2,8\}$	—	—	$S = 1,v, \quad v \in \{1,2,6\}$
$\{1,2,9\}$	—	—	$S = 1,v, \quad v \in \{3,4,7,8\}$
$\{1,2,10\}$	—	—	—
$\{1,3,4\}$	—	—	$S = 1,v, \quad v \in \{1,3,4,5,9\}$
$\{1,3,5\}$	—	—	—
$\{1,3,8\}$	—	—	$S = 1,v, \quad v \in \{2,5,6,9\}$
$\{1,3,10\}$	—	—	—
$\{1,5,7\}$	—	—	$S = 1,v, \quad v \in \{1,5,7,8,9\}$

Table 7: Maximal undenary sequences where $|A| \leq 3$

A	$\mathrm{s}_A(\mathbb{Z}_{11})$	shape of maximum sequences S		
$\{1,3,4,5\}$	13	$\eta_A(\mathbb{Z}_{11}) = 2+1$	$S = 1, v,$	$v \in \{1,3,4,5,9\}$
$\{1,2,3,6\}$	—	—	$S = 1, v,$	$v \in \{1,2,6\}$
$\{1,2,5,7\}$	—	—	$S = 1, v,$	$v \in \{1,7,8\}$
$\{1,2,3,8\}$	—	—	$S = 1, v,$	$v \in \{2,6\}$
$\{1,2,4,9\}$	—	—	$S = 1, v,$	$v \in \{3,4\}$
$\{1,2,9,10\}$	—	—	—	
$\{1,2,5,10\}$	—	—	$S = 1, v,$	$v \in \{7,8\}$
$\{1,3,4,10\}$	—	—	$S = 1, v,$	$v \in \{5,9\}$
$\{1,3,8,10\}$	—	—	$S = 1, v,$	$v \in \{2,5\}$
$\{1,2,3,4\}$	—	—	$S = 1, 1$	
$\{1,2,3,5\}$	—	—	—	
$\{1,2,3,7\}$	—	—	—	
$\{1,2,4,5\}$	—	—	—	
$\{1,2,7,8\}$	—	—	—	
$\{1,2,3,9\}$	12	$\eta_A(\mathbb{Z}_{11}) = 1+1$	$S = 1$	
$\{1,2,3,10\}$	—	—	—	
$\{1,2,4,7\}$	—	—	—	
$\{1,2,4,10\}$	—	—	—	
$\{1,2,5,9\}$	—	—	—	
$\{1,2,7,10\}$	—	—	—	
$\{1,2,8,9\}$	—	—	—	
$\{1,2,8,10\}$	—	—	—	

Table 8: Maximal undenary sequences where $|A| = 4$

A	$\mathrm{s}_A(\mathbb{Z}_{11})$	shape of maximum sequences S		
$\{1,2,3,4,5\}$	13	$\eta_A(\mathbb{Z}_{11}) = 2+1$	$S = 1,1$	
$\{1,2,3,4,6\}$	—	—	—	
$\{1,2,3,5,8\}$	—	—	—	
$\{1,3,4,5,9\}$	12	$\eta_A(\mathbb{Z}_{11}) = 1+1$	$S = 1, v,$	$v \in A$
$\{1,2,3,5,8\}$	—	—	$S = 1$	
$\{1,2,3,5,8\}$	—	—	—	
$\{1,2,3,5,8\}$	—	—	—	
$\{1,2,3,5,8\}$	—	—	—	
$\{1,2,3,5,8\}$	—	—	—	
$\{1,2,3,5,8\}$	—	—	—	
$\{1,2,3,5,10\}$	—	—	—	
$\{1,2,3,5,10\}$	—	—	—	
$\{1,2,3,6,8\}$	—	—	—	
$\{1,2,3,6,10\}$	—	—	—	
$\{1,2,3,7,8\}$	—	—	—	
$\{1,2,3,7,9\}$	—	—	—	
$\{1,2,3,8,9\}$	—	—	—	
$\{1,2,3,8,10\}$	—	—	—	
$\{1,2,3,9,10\}$	—	—	—	
$\{1,2,4,5,7\}$	—	—	—	
$\{1,2,4,5,9\}$	—	—	—	
$\{1,2,4,7,9\}$	—	—	—	
$\{1,2,4,7,10\}$	—	—	—	
$\{1,2,4,9,10\}$	—	—	—	
$\{1,2,5,7,10\}$	—	—	—	
$\{1,2,7,8,10\}$	—	—	—	

Table 9: Maximal undenary sequences where $|A| = 5$