Caroline Moosmüller, MSc

Smoothness analysis of linear and nonlinear Hermite subdivision schemes

DISSERTATION

zur Erlangung des akademischen Grades

Doktorin der Naturwissenschaften

eingereicht an der

Technischen Universität Graz

Betreuer:

Univ.-Prof. Dr. Johannes Wallner

Institut für Geometrie

Graz, Dezember 2016

EIDESSTATTLICHE ERKLÄRUNG
AFFIDAVIT

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit selbständig verfasst, andere als die angegebenen Quellen/Hilfsmittel nicht benutzt, und die den benutzten Quellen wörtlich und inhaltlich entnommenen Stellen als solche kenntlich gemacht habe. Das in TUGRAZonline hochgeladene Textdokument ist mit der vorliegenden Dissertation identisch.

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly indicated all material which has been quoted either literally or by content from the sources used. The text document uploaded to TUGRAZonline is identical to the present doctoral dissertation.

Datum/*Date*

Unterschrift/*Signature*

Contents

Abstract

Hermite subdivision schemes are iterative methods for refining discrete point-vector data in order to obtain, in the limit, a function together with its derivatives. In this thesis we study the convergence behavior of such subdivision schemes as well as the regularity of the functions which arise as their limits. Furthermore, we establish properties of Hermite schemes in nonlinear situations, especially of schemes whose definition is solely via intrinsic properties of the geometry the data are contained in.

The first part of this thesis addresses Hermite subdivision schemes in the setting of manifolds. We present two adaptations of linear schemes to operate on manifold-valued data using intrinsic constructions such as geodesics and parallel transport. In the case of submanifolds of $\mathbb{R}^n$, we also consider manifold-valued schemes which are defined from linear ones by applying a projection. This approach is not intrinsic, since projections depend on the submanifold's embedding in $\mathbb{R}^n$. If the submanifold in question is invariant with respect to certain transformations, however, the projection approach can be structure-preserving. For example, this is the case for the special orthogonal group $\mathrm{SO}_3 \subset \mathbb{R}^{3\times 3}$.

Furthermore, we present a framework for analyzing nonlinear Hermite schemes with respect to convergence and C^1 smoothness. This is based on a so-called proximity condition, which allows us to conclude convergence and smoothness properties of a nonlinear scheme from its linear counterpart.

In the second part we present a method for constructing both vector and Hermite subdivision schemes with limits of high regularity. This is inspired by a similar method in scalar subdivision and works by manipulating symbols. Via the iterated application of the smoothing procedure we developed, an Hermite scheme with limits of regularity at least C^1 can be transformed to a new scheme of arbitrarily high regularity. In particular, this method gives rise to new linear Hermite schemes.

Acknowledgments

I would like to express my gratitude to everybody who supported me during the time this thesis was accomplished. In particular, I would like to thank my supervisor Johannes Wallner who was very helpful and encouraging. He dedicated a lot of time to our discussions and always provided me with useful comments for my papers and presentations.

The suggestions by Costanza Conti and Tomas Sauer were very valuable to me. They helped to improve this thesis and gave me ideas for extending my results.

I am grateful to Nira Dyn who invited me to Tel Aviv and made me feel at home during my stay. I benefited greatly from our discussions and from her enthusiasm for Mathematics.

Furthermore I would like to thank my colleagues from the Institute of Geometry and from the DK "Discrete Mathematics" for the friendly and encouraging atmosphere.

Special thanks go to my family, in particular to my parents and to Florian, for their continued support and motivation.

1 Introduction

Originally, subdivision schemes refer to the successive refinement of control polygons for the generation of smooth curves [17] and were mainly used for the design of geometric models [4]. Due to the simplicity and locality of the resulting algorithms, subdivision schemes also became a standard method for interpolation and approximation [16, 18, 33]. Today they find application in the areas of computer aided geometric design, geometric modeling, approximation theory and for 2D data with irregular combinatorics, in computer graphics and animation [1, 26, 32, 68, 88]. Furthermore, subdivision schemes can be used for the construction of wavelets and for the numerical solution of PDEs [14, 55].

Linear subdivision

Linear subdivision schemes have been studied extensively over the years. As a result, today's literature provides a compact framework for their systematic analysis, see e.g. the classical works [3, 26, 32]. Starting from a control polygon, a linear subdivision scheme produces refined polygons by iteratively applying *linear* rules. Via this process, a smooth curve is obtained in the limit. If, instead of a control polygon, a mesh of potentially irregular combinatorics is refined, we obtain a smooth surface in the limit. Prominent examples are the Lane-Riesenfeld algorithm [57], which produces B-Splines in the limit, and the 4-point scheme [33] in the curve case, and the classical algorithms by Catmull-Clark and Doo-Sabin in the surface case [2, 21]. For a comprehensive overview, see [1, 68].

Most examples of linear subdivision schemes, including the ones mentioned above, are so-called scalar schemes. This means that the linear rule applied to the control polygon resp. mesh uses real-valued coefficients (called *mask*). Also, many schemes use the same set of coefficients in every refinement step (stationary schemes), even though there has been quite some progress in analyzing level-dependent (i.e. non-stationary) scalar schemes [6, 10, 12, 29].

From the viewpoint of approximation theory, a linear subdivision scheme refines discrete point data, which are attached to the grid $2^{-n}\mathbb{Z}^s$ in the n-th refinement step and produces parametrized univariate ($s = 1$) or multivariate ($s > 1$) functions in the limit [3, 50, 51, 73]. Hermite subdivision schemes, on the other hand, refine discrete point-vector data in order to obtain, in the limit, a function together with its derivatives. Setting aside the specific interpretation of the input data, Hermite schemes also differ in other aspects from the schemes we discussed above: They use matrix-valued masks, which means that the refinement of the point data, for example, is also influenced by the vector data. Furthermore, since data at the n-th level constitute a function and its consecutive derivatives at $2^{-n}\mathbb{Z}^s$, by the chain rule, the subdivision scheme becomes level-dependent. Note also that a convergent Hermite subdivision scheme necessarily requires the limit function to possess a certain regularity.

A well-studied class of Hermite subdivision schemes was proposed by J.-L. Merrien in [58]. This class also includes the scheme producing the piecewise cubic interpolant of given point-vector data.

The convergence analysis of linear Hermite schemes is often transferred to the analysis of stationary scalar or vector subdivision schemes, which are easier to handle. This was first proposed by [30, 31] and then pursued by other authors in [22, 23, 25, 60]. We would like to point out the approach of [60] (resp. the more general version [59]), which allows to analyze Hermite schemes in the same manner as stationary scalar schemes.

In this thesis we study linear and nonlinear univariate Hermite schemes, which are *inherently stationary* [11]. This means that the level-dependence arises only from the specific interpretation of the input data (as opposed to *inherently non-stationary* schemes studied in e.g. [9]). Furthermore, we restrict ourselves to schemes which produce a function and its first derivative, even though we believe that our results can be generalized to schemes refining more than one derivative (see Section "Future research").

Nonlinear subdivision

In recent years, the adaption of linear subdivision schemes to operate on data lying in nonlinear spaces (such as manifolds) has become a topic of high interest, also from the viewpoint of applications [20, 70, 77, 80]. Many adaptions proposed in the literature are of intrinsic nature and are defined solely in terms of the structure imposed on the manifold, like a Riemannian metric, or a Lie group structure. They do not depend on auxiliary data like an embedding or a parametrization of the manifold in question. A commonly used method replaces elementary linear operations, such as binary averaging or point-vector addition, by analogous ones in manifolds, see e.g. [35, 46, 77, 78, 79]. The approach via projection [41, 77, 84], on the other hand, required the manifold to be embedded in an ambient space where subdivision is already defined. Other adaptions reinterpret a subdivision rule as the minimizer of a function [36, 79] or via the expected values of random variables [37]. A comprehensive overview of different manifold-valued analogues of linear subdivision schemes and their properties can be found in [43].

The schemes mentioned in the paragraph above operate on manifold-valued data and create limits in the same manifold. Such schemes are necessarily nonlinear. Nonlinearities may also arise in contexts not related to manifolds, for example, from the aim of preserving certain structures, such as circles [5]. In this paper the dependence of the limit on the input data is nonlinear, but the topic is different from the one studied here. The interested reader is referred to [15, 38, 39, 66, 67] and references therein.

Subdivision schemes in manifolds behave in a manner different from their linear counterparts. In general manifolds, certain geometric construction might not be globally defined or not uniquely defined (think of antipodal points on the sphere and their geodesic midpoint). Also, convergence results in general manifolds are only available for very dense input data. These problems can be avoided, for example, by restricting to suitable geometries, such as Cartan-Hadamard manifolds or by restricting to simpler schemes, like interpolating ones.

Many results for subdivision schemes in general manifolds are available for "dense enough" input data. Usually, their proofs are based on a so-called proximity condition, which allows to conclude properties of a nonlinear scheme from the linear scheme it is derived from. Topics studied include the convergence and smoothness of univariate [43, 46, 75, 77, 78, 85, 86] and multivariate schemes [40, 42, 83] (also with irregular combinatorics [81]), approximation order and stability [28, 44, 87], wavelets and multiscale transforms [45, 47, 48, 53, 82] and others.

Results concerning the convergence of subdivision schemes for all input data can be found for example in complete Riemannian manifolds [35, 34], in Cartan-Hadamard spaces for schemes with nonnegative mask coefficients [36, 37, 79] and in the space of positive-definite matrices [74]. Also, certain interpolatory schemes in complete manifolds converge for all input data [76].

Many papers are concerned with proving the "smoothness equivalence conjecture", see [41, 43, 84, 85, 86]. It has been formulated in [70] and states that a manifold subdivision scheme possesses exactly the same smoothness as its linear counterpart. For a certain big class of manifold subdivision algorithms it is now known exactly under which circumstances their limits are as smooth as the limits of the linear scheme which has the same mask [43].

The constructions and results mentioned in this section apply to scalar subdivision schemes of stationary-type, i.e. to schemes refining point data by using the same set of real coefficients in every subdivision step. Therefore, there is potential for further research, some of which we discuss in the next sections.

Contributions of this thesis

This thesis has two main contributions to the theory of subdivision schemes. Firstly we construct intrinsic Hermite subdivision schemes in the manifold setting and analyze their convergence and smoothness Chapters 2 and 3. Secondly we introduce a framework for creating new linear Hermite schemes whose limits enjoy high regularity (Chapter 4).

We would like to mention that the results obtained in this thesis essentially coincide with the author's publications listed below. In fact, Chapters 2 to 4 reproduce the papers (I) to (III). For the sake of simplicity, the bibliographies of these papers have been added to the general bibliography of the thesis.

List of publications

(I) C. Moosmüller. *C^1 analysis of Hermite subdivision schemes on manifolds.* SIAM J. Numer. Anal., 54(5): 3003–3031, 2016.

(II) C. Moosmüller. *Hermite subdivision on manifolds via parallel transport.* Submitted, April 2016.

(III) C. Moosmüller and N. Dyn. *Smoothing of vector and Hermite subdivision schemes.* In preparation, 2016.

Hermite subdivision for manifold-valued data (Chapters 2 and 3)

Since Hermite subdivision schemes refine point-vector data, which are interpreted as function values and first derivatives, the space to sample data from is not the manifold itself, but its tangent bundle. We present three adaptions of linear Hermite subdivision schemes to operate on point-vector data lying in the tangent bundle: Log-exp, parallel transport and projection analogues.

The Log-exp and parallel transport analogues are similar in the sense that both are defined via an intrinsic construction which was essentially first suggested by [70] and then pursued by [46, 78] for Lie groups and by [79] for general Riemannian manifolds. The main idea is to move the whole subdivision process from the manifold to a linear space attached to it, namely to a tangent space. Within this tangent space, a linear subdivision rule is applied. Then, the refined data are moved back to the manifold.

But how to move from the manifold to a tangent space and back? For point data, this is realized by the exponential map of the manifold, resp. by its inverse [19, 79]. For point-vector data, that is, Hermite data, we suggest two approaches:

1. The Log-exp analogue reinterprets point-vector data (p, v) as point-point data (p, q) via the transformation $q = p + v$ (resp. by an analogous point-vector addition on the manifold). Now we are in the situation of pure point data, which can be moved by the exponential map.

2. The parallel transport analogue moves point data by the exponential map and vector data by a parallel transport operator of the manifold [19].

These constructions are described in more detail in Section 2.3 resp. Section 3.4.

The projection approach, on the other hand, applies to submanifolds of $\mathbb{R}^n$. It is of extrinsic nature, since it depends on auxiliary data, such as on the submanifold's embedding in Euclidean space. Adaptions of scalar subdivision schemes that use projections have been studied in e.g. [41, 77, 84]. In this approach, data in the submanifold are refined by a linear subdivision rule (which, in general, does not produce new data in the submanifold) and then projected onto the submanifold. The generalization to Hermite data is straightforward: Point data is mapped via a projection, vector data via its derivative (see Section 2.3).

Having these adaptions of linear Hermite schemes at hand, a natural question addresses their convergence and smoothness properties. In particular, it is desirable for nonlinear adaptions to have the same properties as the linear schemes they are derived from ("smoothness equivalence"). For reasons discussed in the section above, in general manifolds, this is only possible by restricting to dense input data.

We follow the ideas of [77] and define a proximity condition for Hermite schemes. This condition requires a linear scheme and its nonlinear analogue to be "not too far" from each other. We prove that whenever "proximity" is fulfilled, convergence and C^1 smoothness of nonlinear Hermite schemes can be established from their linear counterparts, provided that the input data are dense enough (Section 2.6).

In Section 2.7 and Section 3.5 we prove that the Log-exp, parallel transport and projection analogues are C^1 convergent if they are constructed from a C^1 linear scheme by verifying that the proximity condition is fulfilled. Therefore, we prove that C^1 convergence carries over from linear schemes to these three analogues.

New Hermite schemes with limits of high regularity (Chapter 4)

The Hermite schemes we study in this thesis refine discrete point-vector data in order to obtain a function together with its first derivative. Therefore, a convergent Hermite scheme requires the limit function to be at least C^1. This is the *minimal* regularity the scheme has to have. Certainly, we may also consider limit functions with higher regularity. In [11], for example, it is proved that in some cases the de Rham transform of an Hermite scheme [24] produces C^2 limit functions.

The aim of this chapter is to construct new linear Hermite schemes of arbitrarily high regularity from given ones. For example, starting from the Merrien schemes [58], which are known to produce C^1 limits, we construct new Hermite schemes with limits of regularity C^2, C^3, etc.

This is motivated by the well-known "smoothing" procedure for scalar subdivision schemes: A scalar scheme which has C^ℓ ($\ell \geq 0$) limits can be transformed to a new scheme with $C^{\ell+1}$ limits via the additional insertion of midpoints in every subdivision step [32]. This procedure can be described by multiplying the symbol with the *smoothing factor* $\frac{z+1}{2}$.

A prominent example are the B-Spline functions, which are obtained by the Lane-Riesenfeld algorithm. For $\ell \geq 1$, the symbol of the ℓ-th Lane-Riesenfeld algorithm is given by

$$a_\ell(z) = \frac{(z+1)^{\ell+1}}{(2z)^\ell}.$$

Therefore, $a_\ell(z)$ is obtained from $a_1(z)$ by multiplying $(\ell-1)$ times with the smoothing factor (and an additional index shift). Since $a_1(z) = \frac{(z+1)^2}{2}$ is the symbol of the scheme generating the piecewise linear interpolant of given input data, the ℓ-th Lane-Riesenfeld algorithm produces limits of regularity $C^{\ell-1}$.

Similar to the scalar case, we approach the smoothing of Hermite schemes by manipulating symbols. In Theorem 4.41 we present a formula for computing the new, smoothed mask from a given one in terms of symbols. This procedure is more involved than the simple multiplication with a smoothing factor, but still presents a generalization of the scalar case. It increases the support of a scheme by a maximum of 5 (Corollary 4.42), the maximum being attained by the schemes presented in Example 4.46 and Example 4.47.

Future research

In this section we would like to comment on possible generalizations of the results presented here and on other topics which are interesting for further research.

1. For Hermite subdivision operating on manifold-valued data, it would be natural to consider schemes producing a function and more than one derivative (this has been studied in the linear case, see e.g. [60]). We believe that a generalization to higher derivatives becomes quite technical: Available results from manifold subdivision suggest that the case of more than 2 derivatives is more involved compared to the case of 1 derivative [43, 85]. Also, the data now have to be sampled from the jet bundle of the manifold.

2. We expect that the results obtained in (I) can be generalized to the (regular) multivariate setting.

3. Convergence and smoothness of level-dependent manifold-valued schemes still needs to be analyzed, even in the scalar case. We believe, that this can be achieved by combining [75, 77] and [10]. Recent results [9] show a similar factorization as in [60] for linear inherently non-stationary Hermite schemes. This probably allows us to generalize our results from (I) to the non-stationary setting.

4. Similar to [28, 44] one could study approximation order and stability of manifold-valued Hermite subdivision schemes.

5. It would be desirable to obtain convergence results for manifold-valued Hermite schemes for all input data. As previous results suggest [79], Cartan-Hadarmard manifolds are suitable spaces to work in. By transforming point-vector data to point-point data as in the Log-exp analogue, we believe that [79] can be applied to Hermite data.

6. Recently it was suggested to study linear Hermite schemes which produce limit functions of higher regularity than the minimal one (that is, the number of derivatives), see [11]. For example, [11] proves that in some cases the de Rham transform [24] produces C^2 limit curves while refining data consisting only of function values and first derivatives. Our paper (III), on the other hand, is capable of producing linear schemes with limits of arbitrarily high regularity while refining this kind of data. It would be interesting to show that the limit functions of our manifold-valued analogues constructed in (I)-(II) inherit regularity higher than C^1 from their linear counterparts.

2 C^1 analysis of Hermite subdivision schemes on manifolds

This chapter comprises the paper (I).

Abstract. We propose two adaptations of linear Hermite subdivision schemes to operate on manifold-valued data. Our approach is based on a Log-exp analogue and on projection, respectively and can be applied to both interpolatory and non-interpolatory Hermite schemes. Furthermore, we introduce a new proximity condition, which bounds the difference between a linear Hermite subdivision scheme and its manifold-valued analogue. Verification of this condition gives the main result: The manifold-valued Hermite subdivision scheme constructed from a C^1- convergent linear scheme is also C^1, if certain technical conditions are met.

Keywords. Hermite subdivision · manifold subdivision · proximity · C^1 smoothness

AMS Subject Classification. 65D05 · 65D17 · 41A05 · 41A25

2.1 Introduction

In this paper we continue recent work on adapting linear subdivision schemes to operate on manifold-valued data. We are treating Hermite schemes, which are iterative methods for refining discrete point-vector data in order to obtain, in the limit, a function together with its derivatives.

Linear Hermite schemes are widely studied, see e.g. [11, 22, 23, 25, 30, 31, 52, 58, 60] and others. The C^1 analysis of linear Hermite schemes is often related to the convergence analysis of scalar-valued or vector-valued stationary subdivision schemes, which are easier to handle. This approach was first suggested by [30, 31]. In this paper we are particularly interested in the approach of [60]: The authors introduce the *Taylor operator* and the *Taylor scheme* for linear Hermite schemes, which play the same role as the forward difference operator and derived scheme, respectively, in the analysis of ordinary subdivision schemes [3, 26, 27]. We use the Taylor operator to define a *smoothness condition* for linear Hermite schemes (inspired by a similar condition in [77]), which is sufficient for C^1 convergence.

We present two adaptations of linear Hermite schemes to the manifold setting. The first one is based on the Log-exp approach of [46, 79]; the second one is a so-called projection analogue as suggested by [41, 84]. The C^1 convergence of these nonlinear Hermite schemes is established from their linear counterparts by means of a new proximity condition, following the ideas of [77]. Like almost all previous work on subdivision in general Riemannian manifolds we show convergence only for dense enough input data, and we show C^1 smoothness of all limits which exist. There has been some progress in showing convergence for any input data, see e.g. [34, 36, 37, 76].

Our results imply C^1 convergence of nonlinear analogues of linear Hermite schemes, in particular analogues of the examples listed in [58].

The paragraph above mentioned previous work on (non-Hermite) subdivision in manifolds, but there is also previous work on nonlinear Hermite subdivision: The paper [13] gives a detailed discussion of shape-preserving subdivision on the basis of linear Hermite schemes. Here the dependence of the limit function on the data is nonlinear, but this topic is different from the one studied in the present paper.

The paper is organized as follows. In Section 2.2 we give a short survey on linear Hermite subdivision and introduce notation used throughout the text. Section 2.3 presents our adaptation of linear Hermite schemes to the manifold setting. The Log-exp approach as well as the projection analogue are discussed in detail. The Taylor operator and the Taylor scheme are defined in Section 2.4, which, together with convergence results of Section 2.6.1, are important ingredients for the C^1 analysis of linear and nonlinear Hermite schemes. In Section 2.6 we prove C^1 convergence results, first in the linear Hermite case by means of a smoothness condition and then in the manifold case using a proximity condition. Section 2.7 concludes the paper by proving that a proximity condition applies to both the Log-exp analogue and the projection analogue.

We would like to mention that some results concerning linear Hermite subdivision in Sections 2.4 and 2.6 are already presented in the literature. We reprove these results in order to extend them more easily to the manifold-valued case.

2.2 Linear Hermite subdivision

We begin by introducing the notation and recalling some known facts about linear Hermite subdivision. The data to be refined by a linear Hermite subdivision scheme consists of a point-vector sequence, where we consider both the point and vector component to have values in the same real vector space V. In the course of our analysis we also encounter refinement of vector-data, which cannot be interpreted as points. To cover all cases we therefore use the notation f for elements in V^2, where f can be interpreted as point and vector, vector and vector, or as point and point. If the particular interpretation of the components is of interest, we use $\binom{p}{v}$ for point-vector, $\binom{v}{w}$ for vector-vector, and $\binom{p}{q}$ for point-point.

Sequences of elements in V^2 are denoted by boldface letters, that is $\mathbf{f} = \{f(\alpha) : \alpha \in \mathbb{Z}\}$. If we are interested in the components, we use the notation $\binom{\mathbf{p}}{\mathbf{v}} = \{\binom{p(\alpha)}{v(\alpha)} : \alpha \in \mathbb{Z}\}$, etc. The space of all sequences with values in V^2 is denoted by $\ell(V^2)$.

We also consider the space $\ell(L(V)^{2\times 2})$, where $L(V)$ is the vector space of linear functions on V. Uppercase letters A are used for elements in $L(V)^{2\times 2}$ and boldface uppercase letters $\mathbf{A}$ for elements in $\ell(L(V)^{2\times 2})$.

A finitely supported sequence $\mathbf{A} \in \ell(L(V)^{2\times 2})$ is called *mask* and with it we associate a *linear subdivision operator* $S_{\mathbf{A}} : \ell(V^2) \to \ell(V^2)$ by

$$(S_{\mathbf{A}}\mathbf{f})(\alpha) = \sum_{\beta\in\mathbb{Z}} A(\alpha - 2\beta) f(\beta), \qquad \alpha \in \mathbb{Z}, \quad \mathbf{f} \in \ell(V^2). \tag{2.1}$$

We associate two types of linear schemes to a linear subdivision operator $S_{\mathbf{A}}$:

- A *linear Hermite subdivision scheme* is the procedure of constructing $\mathbf{f}^1, \mathbf{f}^2, \ldots$ from $\mathbf{f}^0 \in \ell(V^2)$ by the rule

  $$D^n \mathbf{f}^n = S_{\mathbf{A}}^n \mathbf{f}^0,$$

  where $D \in L(V)^{2\times 2}$ is the block-diagonal dilation operator

  $$D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

  Here a constant c is to be understood as $c\,\mathrm{id}_V$. This notation will be used throughout the text.

- The procedure of constructing $\mathbf{g}^1, \mathbf{g}^2, \ldots$ from $\mathbf{g}^0 \in \ell(V^2)$ by the rule

  $$\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$$

  will be called a *linear point subdivision scheme*. This is because here the two components of $\mathbf{g}^n \in \ell(V^2)$ are not interpreted as a point-vector sequence, but as a point-point sequence.

Note that if $\mathbf{f}^0 = \mathbf{g}^0$, then the two schemes are related via $D^n \mathbf{f}^n = \mathbf{g}^n$. Therefore, the refined sequences $\mathbf{f}^n$ and $\mathbf{g}^n$ only differ in the second component by the factor 2^n.

A linear Hermite subdivision scheme is called *interpolatory* if the mask satisfies $A(0) = D$ and $A(2\alpha) = 0$ for all $\alpha \in \mathbb{Z}\backslash 0$.

We always assume a linear subdivision operator $S_{\mathbf{A}}$ of a linear Hermite scheme to reproduce a degree 1 polynomial and its derivative

$$\mathbf{f} = \left\{ \begin{pmatrix} v + \alpha w \\ w \end{pmatrix} : \alpha \in \mathbb{Z} \right\} \quad \text{for } v, w \in V,$$

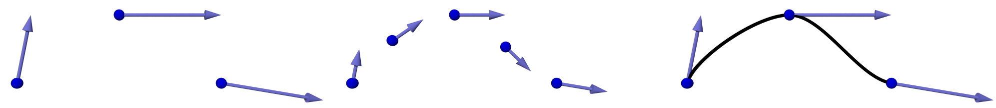

Figure 2.1: The point subdivision scheme of Example 2.1

apart from a parameter shift. This means that we require that there is $\varphi \in \mathbb{R}$ such that the shifted sequence $\mathbf{f}_\varphi = \left\{ \left(\begin{smallmatrix} v+(\alpha+\varphi)w \\ w \end{smallmatrix} \right) : \alpha \in \mathbb{Z} \right\}$ satisfies

$$(S_{\mathbf{A}}\mathbf{f}_\varphi)(\alpha) = \left(\begin{smallmatrix} v+\frac{\alpha+\varphi}{2}w \\ \frac{1}{2}w \end{smallmatrix} \right), \quad \text{for } v, w \in V, \alpha \in \mathbb{Z}.$$

This condition is called the *spectral condition* and is equivalent to the requirement that there is $\varphi \in \mathbb{R}$ such that both the constant sequence $\mathbf{k_0} = \{ \left(\begin{smallmatrix} w \\ 0 \end{smallmatrix} \right) : \alpha \in \mathbb{Z}\}$ and the linear sequence $\boldsymbol{\ell} = \{ \left(\begin{smallmatrix} (\alpha+\varphi)w \\ w \end{smallmatrix} \right) : \alpha \in \mathbb{Z}\}$ for $w \in V$ obey the rule

$$S_{\mathbf{A}}\mathbf{k_0} = \mathbf{k_0}, \quad S_{\mathbf{A}}\boldsymbol{\ell} = \tfrac{1}{2}\boldsymbol{\ell}. \tag{2.2}$$

The spectral condition has been introduced in [25] and is crucial for the C^1 analysis of linear Hermite subdivision schemes.

By means of the components of the mask $\mathbf{A} = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix} \right)$ the spectral condition reads

$$\sum_{\beta\in\mathbb{Z}} a(\alpha - 2\beta) = 1, \quad \sum_{\beta\in\mathbb{Z}} c(\alpha - 2\beta) = 0, \tag{2.3}$$

$$\sum_{\beta\in\mathbb{Z}} a(\alpha - 2\beta)\beta + b(\alpha - 2\beta) = \frac{1}{2}(\alpha - \varphi), \quad \sum_{\beta\in\mathbb{Z}} c(\alpha - 2\beta)\beta + d(\alpha - 2\beta) = \frac{1}{2}, \tag{2.4}$$

for some φ which indicates a parameter transform, and for all $\alpha \in \mathbb{Z}$.

Example 2.1. As a model example we consider one of the interpolatory linear Hermite subdivision schemes introduced in [58], see Figure 2.1. Its mask is given by

$$A(-1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{pmatrix}, \quad A(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad A(1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} \end{pmatrix}.$$

It is easy to see that it satisfies the spectral condition eq. (2.2). It is well known that this scheme produces the piecewise cubic interpolant of given point-vector input data.

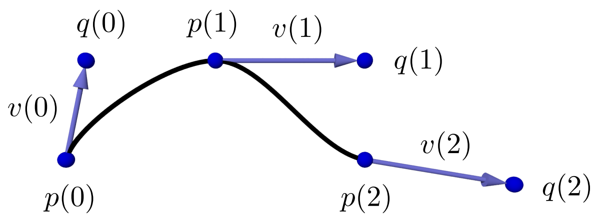

Figure 2.2: Transformation of input data from point-vector data $\binom{\mathbf{p}}{\mathbf{v}}$ to point-point data $\binom{\mathbf{p}}{\mathbf{q}}$, where $\mathbf{q} = \mathbf{p} + \mathbf{v}$.

2.2.1 Transformation of input data

To a subdivision operator $S_{\mathbf{A}}$ we associate a subdivision operator $S_{\tilde{\mathbf{A}}}$ by transformation of input data. We change the point-vector input data $\binom{\mathbf{p}}{\mathbf{v}}$ to point-point input data $\binom{\mathbf{p}}{\mathbf{q}}$ via the transformation $\mathbf{q} = \mathbf{p} + \mathbf{v}$, see Figure 2.2. Hence $\binom{\mathbf{p}}{\mathbf{q}} = \mathcal{T}\binom{\mathbf{p}}{\mathbf{v}}$, where $\mathcal{T} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Then we let

$$S_{\tilde{\mathbf{A}}}\binom{\mathbf{p}}{\mathbf{q}} = \mathcal{T} S_{\mathbf{A}} \mathcal{T}^{-1}\binom{\mathbf{p}}{\mathbf{q}},$$

i.e., the mask $\tilde{\mathbf{A}}$ is computed from the mask $\mathbf{A}$ by the relation

$$\tilde{A}(\alpha) = \mathcal{T} A(\alpha) \mathcal{T}^{-1},$$

for $\alpha \in \mathbb{Z}$.

Note that $\mathbf{A}$ satisfies eq. (2.3) if and only if $\tilde{\mathbf{A}} = \left(\begin{smallmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} \\ \tilde{\mathbf{c}} & \tilde{\mathbf{d}} \end{smallmatrix}\right)$ satisfies

$$\sum_{\beta \in \mathbb{Z}} \tilde{a}(\alpha - 2\beta) + \tilde{b}(\alpha - 2\beta) = 1, \quad \sum_{\beta \in \mathbb{Z}} \tilde{c}(\alpha - 2\beta) + \tilde{d}(\alpha - 2\beta) = 1. \tag{2.5}$$

This is the reproduction property $S_{\tilde{\mathbf{A}}}\mathbf{k_2} = \mathbf{k_2}$, where $\mathbf{k_2}$ is the constant sequence $\mathbf{k_2} = \{\binom{w}{w} : \alpha \in \mathbb{Z}\}$ for $w \in V$. In Section 2.3 the subdivision operator $S_{\tilde{\mathbf{A}}}$ as well as property eq. (2.5) will be useful.

In Example 2.1, the mask $\tilde{\mathbf{A}}$ associated to $\mathbf{A}$ is given by

$$\tilde{A}(-1) = \begin{pmatrix} \frac{5}{8} & -\frac{1}{8} \\ \frac{3}{2} & -\frac{1}{4} \end{pmatrix}, \quad \tilde{A}(0) = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \tilde{A}(1) = \begin{pmatrix} \frac{3}{8} & \frac{1}{8} \\ -\frac{1}{4} & 0 \end{pmatrix}. \tag{2.6}$$

2.3 Hermite subdivision on manifolds

This section presents two methods for deriving nonlinear Hermite subdivision schemes from linear ones. The first one is an intrinsic construction. It works on any manifold that has an exponential mapping. The main instances of manifolds we consider here are finite-dimensional Riemannian manifolds, Lie groups and symmetric spaces. The second method invokes a projection and can be defined on submanifolds. We start by generalizing the notions of *subdivision operator*, *point subdivision scheme*, and *Hermite subdivision scheme.*

Definition 2.2 (Subdivision operator)**.** A *subdivision operator* is a map U which takes as argument a sequence $\mathbf{f}$ and produces a new sequence $U\mathbf{f}$. It must satisfy

(i) $L^2U = UL$, where L is the left shift operator, and

(ii) U has compact support, that is, there exists $N \in \mathbb{N}$ such that both $U\mathbf{f}(2\alpha)$ and $U\mathbf{f}(2\alpha+1)$ only depend on $f(\alpha - N), \dots, f(\alpha + N)$ for all $\alpha \in \mathbb{Z}$ and sequences $\mathbf{f}$.

Note that a linear subdivision operator with finitely supported mask satisfies these conditions.

While for linear subdivision schemes both point-point and point-vector data can be taken from the same vector space V^2, this is no longer the case in the manifold setting. On a manifold M, point-point data is sampled from the space M^2. For point subdivision we therefore consider the associated sequence space $\ell(M^2)$.

Definition 2.3 (Point subdivision scheme)**.** Let U be a subdivision operator which takes arguments in $\ell(M^2)$ and again produces a sequence in M^2. We associate a point subdivision scheme to U:

A point subdivision scheme is the procedure of constructing $\mathbf{g}^1, \mathbf{g}^2, \dots$ from input data $\mathbf{g}^0 \in \ell(M^2)$ by the rule

$$\mathbf{g}^n = U^n \mathbf{g}^0.$$

The derivative in a point of a manifold-valued curve $c : \mathbb{R} \to M$ lies in a tangent space of M, namely $c'(t) \in T_{c(t)}M$. Therefore tangent vectors serve as point-vector input data for Hermite subdivision. Let $TM = \bigcup_{p \in M} T_pM$ be the tangent bundle of M and $\ell(TM)$ its associated sequence space. We consider an element of $\ell(TM)$ as a point-vector pair $\binom{\mathbf{p}}{\mathbf{v}}$, where $\mathbf{p}$ is a sequence in M and $\mathbf{v}$ a sequence in the appropriate tangent space, that is $v(\alpha) \in T_{p(\alpha)}M$ for all $\alpha \in \mathbb{Z}$ (strictly speaking $v(\alpha)$ carries the information which tangent space it is contained in, but we want to retain the analogy to the linear case). In this notation let $D : \ell(TM) \to \ell(TM)$ be the dilation operator

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{p} \\ \frac{1}{2}\mathbf{v} \end{pmatrix},$$

which is an analogue of the block-diagonal operator D defined in Section 2.2.

Definition 2.4 (Hermite subdivision scheme)**.** Let U be a subdivision operator which takes arguments in $\ell(TM)$ and again produces a sequence in TM. We associate an Hermite subdivision scheme to U:

An Hermite subdivision scheme is the procedure of constructing $\mathbf{f}^1, \mathbf{f}^2, \dots$ from input data $\mathbf{f}^0 \in \ell(TM)$ by the rule

$$D^n \mathbf{f}^n = U^n \mathbf{f}^0.$$

An Hermite subdivision scheme is called *interpolatory* if $(U\mathbf{f})(2\alpha) = (D\mathbf{f})(\alpha)$ for all $\mathbf{f}$ and $\alpha \in \mathbb{Z}$. Note that if U is a linear subdivision operator, this is equivalent to the definition given in Section 2.2.

2.3.1 The Log-exp analogue of a linear subdivision scheme

The idea of using the exponential mapping for transferring linear operations to manifold-valued data has been proposed by [20, 70]. Analysis of subdivision schemes has been done by [43, 46, 79, 85, 86] and others.

Constructing a subdivision rule by the Log-exp method requires operations $q = p \oplus v$ and $v = q \ominus p$, which are similar to point-vector addition and the difference vector of points. We recall their definition which is found e.g. in [79].

Let N be a Riemannian manifold. For $p \in N$ and a tangent vector $v \in T_pN$, the exponential mapping $\exp_p(v)$ gives the endpoint of the geodesic line of length $\|v\|$ emanating from p in direction v. It is a local diffeomorphism around $0 \in T_pN$ and hence posses a local inverse $\exp_p^{-1}$.

We define

$$p \oplus v = \exp_p(v) \quad \text{and} \quad q \ominus p = \exp_p^{-1}(q). \tag{2.7}$$

While $\oplus$ is always smooth, and is often defined for all tangent vectors v (for example, this is the case on complete Riemannian manifolds, see [54, Theorem 10.3]), $\ominus$ in general is definable as a smooth mapping only for p, q close to each other. Since the convergence and smoothness analysis of Section 2.6.3 only considers “dense enough” input data, we may assume that $\ominus$ is always smooth.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra, which is the tangent space at the identity element. By $\exp : \mathfrak{g} \to G$ we denote the exponential mapping in the group [54, §2, Chapter II]. Define $\exp_p : T_pG \to G$ by

$$\exp_p(v) = p \exp(p^{-1} \cdot v), \quad p \in G, v \in T_pG.$$

Here $p^{-1} \cdot v$ is the transfer of v to the vector space $\mathfrak{g}$ by left multiplication with p^{-1}. The map $\exp_p$ is a local diffeomorphism, hence possesses a local inverse $\exp_p^{-1}$. Define

$$p \oplus v = \exp_p(v) \quad \text{and} \quad q \ominus p = \exp_p^{-1}(q), \quad \text{for } p, q \in G \text{ and } v \in T_pG. \tag{2.8}$$

Note that $\exp_p^{-1}(q) = p \cdot \exp^{-1}(p^{-1}q)$. On Lie groups $\oplus$ is globally smooth, while this is generally not the case for $\ominus$.

Equation (2.8) is invariant with respect to left translations in G, that is, for $g, p, q \in G$ and $v \in T_pG$, we have $gp \oplus (g \cdot v) = g(p \oplus v)$ and $(gq) \ominus (gp) = g \cdot (q \ominus p)$.

We follow the construction of [79] to define an exponential mapping on symmetric spaces. Let $X = G/K$ be a symmetric space, which is a Lie group G factorized by a closed subgroup K meeting certain conditions which ensure that an exponential mapping can be defined. By $\exp_p : T_pX \to X$ we denote its exponential mapping. We define

$$p \oplus v = \exp_p(v) \quad \text{and} \quad q \ominus p = \exp_p^{-1}(q), \tag{2.9}$$

where $p, q \in X$ and $v \in T_pX$. As in the Lie group case, $\oplus$ is globally smooth, while $\ominus$ is a diffeomorphism in a neighborhood of p. Furthermore, eq. (2.9) is invariant with respect to the action of the group G.

Let M be a Riemannian manifold, Lie group or symmetric space. For $p, q \in M$ we define their *mean* by

$$\operatorname{mean}(p, q) := p \oplus \frac{1}{2}(q \ominus p).$$

Again in the case of Lie groups and symmetric spaces we have invariance of the mean with respect to the action of the group, i.e. $\operatorname{mean}(gp, gq) = g \operatorname{mean}(p, q)$.

In the following, we derive a subdivision operator $U_{\mathbf{A}}$ on TM from a linear subdivision operator $S_{\mathbf{A}}$ which satisfies the spectral condition eq. (2.2).

We write $S_{\mathbf{A}}$ in the form

$$S_{\mathbf{A}} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} (\alpha) = \sum_{\beta \in \mathbb{Z}} \begin{pmatrix} a(\alpha - 2\beta) & b(\alpha - 2\beta) \\ c(\alpha - 2\beta) & d(\alpha - 2\beta) \end{pmatrix} \begin{pmatrix} p(\beta) \\ v(\beta) \end{pmatrix},$$

where $\left(\begin{smallmatrix} \mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right)$ is point-vector input data.

In Section 2.2.1 we associated a linear subdivision operator $S_{\tilde{\mathbf{A}}}$ to $S_{\mathbf{A}}$ by changing the input data from point-vector form to point-point form:

$$\begin{aligned} S_{\tilde{\mathbf{A}}} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} (\alpha) &= \sum_{\beta \in \mathbb{Z}} \begin{pmatrix} \tilde{a}(\alpha - 2\beta) & \tilde{b}(\alpha - 2\beta) \\ \tilde{c}(\alpha - 2\beta) & \tilde{d}(\alpha - 2\beta) \end{pmatrix} \begin{pmatrix} p(\beta) \\ q(\beta) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{\beta \in \mathbb{Z}} \tilde{a}(\alpha - 2\beta) p(\beta) + \tilde{b}(\alpha - 2\beta) q(\beta) \\ \sum_{\beta \in \mathbb{Z}} \tilde{c}(\alpha - 2\beta) p(\beta) + \tilde{d}(\alpha - 2\beta) q(\beta) \end{pmatrix}, \end{aligned} \tag{2.10}$$

where $\mathbf{q} = \mathbf{p} + \mathbf{v}$ and $\tilde{\mathbf{A}} = \left(\begin{smallmatrix} \tilde{\mathbf{a}} & \tilde{\mathbf{b}} \\ \tilde{\mathbf{c}} & \tilde{\mathbf{d}} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. By using the reproduction property eq. (2.5), the operator eq. (2.10) can be equivalently written as

$$S_{\tilde{\mathbf{A}}} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} (\alpha) = \begin{pmatrix} m_0(\alpha) + \sum_{\beta \in \mathbb{Z}} \tilde{a}(\alpha - 2\beta)(p(\beta) - m_0(\alpha)) + \tilde{b}(\alpha - 2\beta)(q(\beta) - m_0(\alpha)) \\ m_1(\alpha) + \sum_{\beta \in \mathbb{Z}} \tilde{c}(\alpha - 2\beta)(p(\beta) - m_1(\alpha)) + \tilde{d}(\alpha - 2\beta)(q(\beta) - m_1(\alpha)) \end{pmatrix}, \tag{2.11}$$

for any base point sequences $\mathbf{m_0}, \mathbf{m_1}$. This definition is useful for transferring the linear refinement rule to the manifold setting, since it consists of point-vector addition and point-point subtraction. As we have seen in the beginning of this section, these operations are defined on manifolds. We therefore use eq. (2.11) to define a subdivision operator $U_{\mathbf{A}}$ for sequences of tangent vectors:

Consider input data $\binom{\mathbf{p}}{\mathbf{v}} \in \ell(TM)$ and define $\mathbf{r} \in \ell(M)$ by $\mathbf{r} = \{p(\alpha) \oplus v(\alpha) : \alpha \in \mathbb{Z}\}$. As to base point sequences, the following possibilities are an obvious choice:

(i) $\mathbf{m_0} = \mathbf{p}$ or $\mathbf{m_0} = \{\mathrm{mean}(p(\alpha), p(\alpha+1)) : \alpha \in \mathbb{Z}\}$,

(ii) $\mathbf{m_1} = \mathbf{r}$, $\mathbf{m_1} = \{\mathrm{mean}(r(\alpha), r(\alpha+1)) : \alpha \in \mathbb{Z}\}$ or $\mathbf{m_1} = \mathbf{m_0}$.

Define $\mathbf{s}_1, \mathbf{r}_1 \in \ell(M)$ using eq. (2.11):

$$\begin{pmatrix} s_1(\alpha) \\ r_1(\alpha) \end{pmatrix} = \begin{pmatrix} m_0(\alpha) \oplus \sum_{\beta\in\mathbb{Z}} \tilde{a}(\alpha-2\beta)(p(\beta)\ominus m_0(\alpha)) + \tilde{b}(\alpha-2\beta)(r(\beta)\ominus m_0(\alpha)), \\ m_1(\alpha) \oplus \sum_{\beta\in\mathbb{Z}} \tilde{c}(\alpha-2\beta)(p(\beta)\ominus m_1(\alpha)) + \tilde{d}(\alpha-2\beta)(r(\beta)\ominus m_1(\alpha)) \end{pmatrix},$$

for $\alpha \in \mathbb{Z}$. Therefore, the operator $U_{\mathbf{A}}$ defined by

$$U_{\mathbf{A}} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} (\alpha) = \begin{pmatrix} s_1(\alpha) \\ r_1(\alpha) \ominus s_1(\alpha) \end{pmatrix}, \tag{2.12}$$

is a subdivision operator on TM. In Section 2.6 we show that the Hermite scheme $\binom{\mathbf{p}}{\mathbf{v}}, D^{-1}U_{\mathbf{A}}\binom{\mathbf{p}}{\mathbf{v}}, D^{-2}U_{\mathbf{A}}^2\binom{\mathbf{p}}{\mathbf{v}}, \ldots$ converges to a curve and its derivative.

In the case of symmetric spaces and Lie groups, $U_{\mathbf{A}}$ is invariant with respect to the group action.

Remark 2.5. While for the smoothness analysis of manifold-valued Hermite schemes it does not matter if we use base point sequences (i) or (ii) from above, there is a preferred choice if the linear Hermite scheme has additional properties, such as symmetry. A linear Hermite scheme $S_{\mathbf{A}}$ is called *symmetric*, if its mask satisfies $A(\gamma-\alpha) = A(\gamma+\alpha)$ (case 1) or $A(\gamma-\alpha) = A(\gamma+1-\alpha)$ (case 2), for a fixed index γ and for all $\alpha \in \mathbb{Z}$. Then the following relation is fulfilled

$$S_{\mathbf{A}} M = L^2 M S_{\mathbf{A}}, \tag{2.13}$$

where L is the left shift operator and M operates on sequences $\mathbf{q}$ by $(M\mathbf{q})(\alpha) = q(-\alpha)$ for all $\alpha \in \mathbb{Z}$. A natural question addresses the preservation of symmetry: Is the manifold-valued analogue of a symmetric linear scheme also symmetric? That is, does it satisfy eq. (2.13)? Depending on whether we consider case 1 or case 2, the symmetry is preserved if the base points are chosen as the data points ($\mathbf{m_0} = \mathbf{p}, \mathbf{m_1} = \mathbf{r}$) or as the geodesic midpoints ($\mathbf{m_0} = \{\mathrm{mean}(p(\alpha), p(\alpha+1)) : \alpha \in \mathbb{Z}\}$, $\mathbf{m_1} = \{\mathrm{mean}(r(\alpha), r(\alpha+1)) : \alpha \in \mathbb{Z}\}$), respectively. For more details, including examples, see [46].

Example 2.6. Consider the 2-dimensional sphere S^2 in $\mathbb{R}^3$. It is a Riemannian manifold with metric induced from the ambient space $\mathbb{R}^3$. The operations $\oplus, \ominus$ on $p, q \in S^2$ and $v \in T_pS^2 = \{w \in \mathbb{R}^3 : \langle w, p\rangle = 0\}$ are given by

$$p \oplus v = \cos(\|v\|)p + \sin(\|v\|)\frac{v}{\|v\|}, \quad q \ominus p = \arccos(\langle q, p\rangle)\frac{q - \langle q, p\rangle p}{\|q - \langle q, p\rangle p\|}. \tag{2.14}$$

We consider the Log-exp analogue of the linear interpolatory Hermite scheme introduced in Example 2.1 on the sphere. In Figure 2.3 the input data, the first step and the limit curve of the Log-exp analogue are shown.

For input data $\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right) \in \ell(TS^2)$ and $\mathbf{r} = \{p(\alpha) \oplus v(\alpha) : \alpha \in \mathbb{Z}\}$, define the base point sequences $\mathbf{m_0}, \mathbf{m_1}$ by

$$m_0(2\alpha) = m_0(2\alpha+1) = \mathrm{mean}(p(\alpha+1), p(\alpha)) = p(\alpha+1) \oplus \Big(\frac{1}{2}p(\alpha) \ominus p(\alpha+1)\Big),$$
$$m_1(2\alpha) = m_1(2\alpha+1) = \mathrm{mean}(r(\alpha+1), r(\alpha)) = r(\alpha+1) \oplus \Big(\frac{1}{2}r(\alpha) \ominus r(\alpha+1)\Big).$$

Furthermore, we introduce the sequences $\mathbf{v}^0, \mathbf{v}^1, \mathbf{w}^0, \mathbf{w}^1$:

$$\begin{aligned} v^0_{\alpha,\beta} &= p(\alpha) \ominus m_0(\beta), & v^1_{\alpha,\beta} &= p(\alpha) \ominus m_1(\beta),\\ w^0_{\alpha,\beta} &= r(\alpha) \ominus m_0(\beta), & w^1_{\alpha,\beta} &= r(\alpha) \ominus m_1(\beta), \end{aligned}$$

for $\alpha, \beta \in \mathbb{Z}$. Therefore, the Log-exp version of the Hermite subdivision scheme $\left(\begin{smallmatrix}\mathbf{s}_1\\ \mathbf{r}_1 \ominus \mathbf{s}_1\end{smallmatrix}\right) = U_{\mathbf{A}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)$ is given by the sequences

$$\begin{aligned} s_1(2\alpha) &= p(\alpha),\\ r_1(2\alpha) &= m_1(2\alpha) \oplus \Big(\frac{1}{2}v^1_{\alpha,2\alpha} + \frac{1}{2}w^1_{\alpha,2\alpha}\Big),\\ s_1(2\alpha+1) &= m_0(2\alpha+1) \oplus \Big(\frac{5}{8}v^0_{\alpha+1,2\alpha+1} + \frac{3}{8}v^0_{\alpha,2\alpha+1} - \frac{1}{8}w^0_{\alpha+1,2\alpha+1} + \frac{1}{8}w^0_{\alpha,2\alpha+1}\Big),\\ r_1(2\alpha+1) &= m_1(2\alpha+1) \oplus \Big(\frac{3}{2}v^1_{\alpha+1,2\alpha+1} - \frac{1}{4}v^1_{\alpha,2\alpha+1} - \frac{1}{4}w^1_{\alpha+1,2\alpha+1}\Big), \end{aligned}$$

where the coefficients are taken from eq. (2.6).

The sphere is also a symmetric space, namely $S^2 = \mathrm{SO}_3/\mathrm{SO}_2$. The exponential map of S^2 as a Riemannian manifold coincides with the exponential map of S^2 as a symmetric space [56, Chapter XI, Theorem 3.3] and therefore the above calculations are valid in both cases. Furthermore, it can be checked immediately that eq. (2.14) is invariant with respect to SO_3, which implies the invariance of $\mathbf{s}_1$ respectively $\mathbf{r}_1$.

2.3.2 The projection analogue of a linear subdivision scheme

Projections onto submanifolds have been used to create subdivision schemes for manifold-valued data by various authors, see [77, 75, 41, 84]. We generalize this method to the Hermite case.

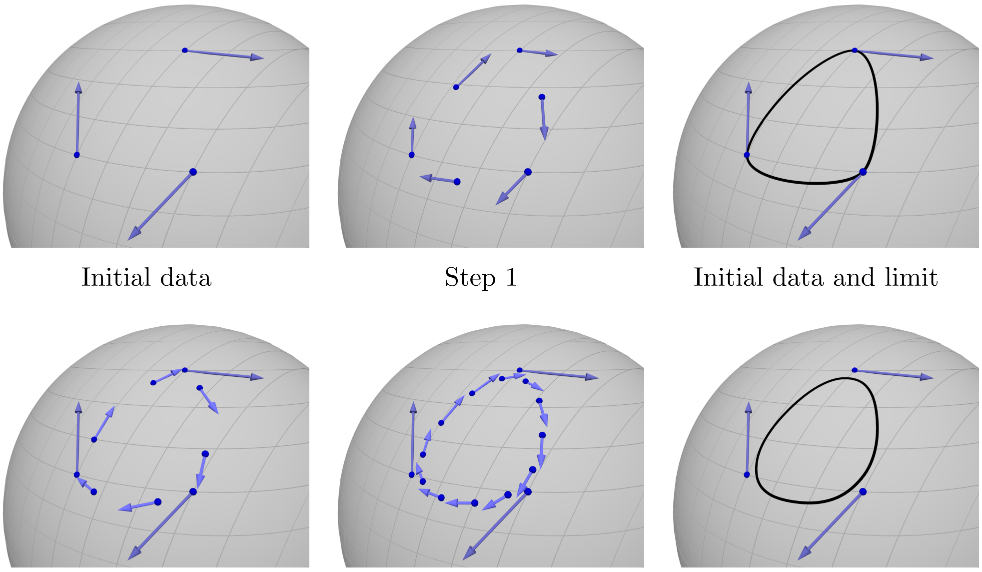

Figure 2.3: Log-exp analogue of an interpolatory Hermite scheme (first row) vs. a non-interpolatory one (second row). Both schemes are applied to the same initial data on the sphere. *First row*: Initial data, first step and limit curve of the interpolatory scheme presented in Example 2.6. *Second row*: Initial data together with the first step, second step and limit curve of a non-interpolatory scheme. This scheme is the log-exp analogue of a linear Hermite scheme constructed as the de Rham transform [24] of a scheme proposed by [58]. The mask is the special case $\lambda = -\frac{1}{8}, \mu = \frac{1}{2}$ in [24, Section 4].

Let M be a submanifold of Euclidean space $\mathbb{R}^n$. A *projection* π is a smooth mapping onto M defined in a neighborhood of M such that $\pi|_M = \mathrm{id}$. Its derivative is denoted by $d\pi$.

Let $S_{\mathbf{A}}$ be a linear subdivision operator. Define a subdivision operator $U_{\mathbf{A}}$ that operates on $\ell(TM)$ by

$$U_{\mathbf{A}}\mathbf{f}(\alpha) = d\pi(S_{\mathbf{A}}\mathbf{f}(\alpha)), \quad \text{for } \alpha \in \mathbb{Z}.$$

In Section 2.6 we show that the sequence of refined data $\mathbf{f}, D^{-1}U_{\mathbf{A}}\mathbf{f}, D^{-2}U_{\mathbf{A}}^2\mathbf{f}, \ldots$ converges to a curve and its derivative.

2.4 Derived schemes and factorization results

For the convergence and C^1 analysis of linear subdivision schemes, the derived schemes with respect to the forward difference operator Δ are of importance, see for example [3, 27]. In the Hermite case, the Taylor operator T is the natural analogue of Δ, see [60]. It is introduced as follows:

Definition 2.7. We define operators T, Δ_0, Δ_1 acting on $\ell(V^2)$ in block operator notation by

$$T := \begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix}, \quad \Delta_0 := \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta_1 := \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix},$$

where Δ is the forward difference operator $(\Delta \mathbf{f})(\alpha) = f(\alpha+1) - f(\alpha)$. T is called *Taylor operator.*

The next proposition considers derived schemes of $S_\mathbf{A}$ with respect to the operators Δ_0, T, $\Delta_1 T$. In order to keep an overview of these derived schemes, we introduce the notation $\partial_T S_\mathbf{A}$ to mean the derived scheme of $S_\mathbf{A}$ with respect to the operator T.

Proposition 2.8. *Let* $\mathbf{A}$ *be a mask that satisfies the spectral condition eq.* (2.2). *Then there exist derived schemes of* $S_\mathbf{A}$ *with respect to* $\Delta_0, T, \Delta_1 T$, *i.e. subdivision operators* $\partial_T S_\mathbf{A}, \partial_{\Delta_0} S_\mathbf{A}$ *and* $\partial_{\Delta_1 T} S_\mathbf{A}$ *which satisfy*

(i) $2TS_\mathbf{A} = (\partial_T S_\mathbf{A})T$,

(ii) $\Delta_0 S_\mathbf{A} = (\partial_{\Delta_0} S_\mathbf{A})\Delta_0$ *and* $2\Delta_1 T S_\mathbf{A} = (\partial_{\Delta_1 T} S_\mathbf{A})\Delta_1 T$.

Furthermore, if we define a mask $\mathbf{E}$ *by*

$$E(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E(-1) = E(1) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad E(\alpha) = 0 \quad \text{for } \alpha \neq -1, 0, 1,$$

then there exist subdivision operators $S_\mathbf{B}, S_\mathbf{C}$ *such that*

(iii) $S_\mathbf{A} - S_\mathbf{E} = S_\mathbf{B}\Delta_0$,

(iv) $\partial_T S_\mathbf{A} - S_\mathbf{E} = S_\mathbf{C}\Delta_1$.

Part (i) is a result of [60]. The second factorization of (ii) is also proved in [60] (in this paper $\Delta_1 T$ is called *complete Taylor operator*), but will follow from results we derive here. The proof of Proposition 2.8 can be found at the end of this section.

The essential ingredient in the proof is the reproduction of constants property eq. (2.2): $S_\mathbf{A}\mathbf{k_0} = \mathbf{k_0}$, where $\mathbf{k_0}$ is the constant sequence $\mathbf{k_0} = \{\left(\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right) : \alpha \in \mathbb{Z}\}$ for $w \in V$. The operator $\partial_T S_\mathbf{A}$ satisfies a different reproduction property, namely

Lemma 2.9. *Let* $\mathbf{A}$ *be a mask that satisfies the spectral condition eq.* (2.2) *and let* $\partial_T S_\mathbf{A}$ *be the derived scheme of* $S_\mathbf{A}$ *w.r.t.* T *(which exists by [60]). Denote by* $\mathbf{k_1}$ *the constant sequence* $\mathbf{k_1} = \{\left(\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right) : \alpha \in \mathbb{Z}\}$ *for* $w \in V$. *Then we have*

$$(\partial_T S_\mathbf{A})\mathbf{k_1} = \mathbf{k_1}. \tag{2.15}$$

Proof. By property eq. (2.2), there exists $\varphi \in \mathbb{R}$ such that $\boldsymbol{\ell} = \{\left(\begin{smallmatrix}(\alpha+\varphi)w\\ w\end{smallmatrix}\right) : \alpha \in \mathbb{Z}\}$ is reproduced: $S_{\mathbf{A}}\boldsymbol{\ell} = \frac{1}{2}\boldsymbol{\ell}$. Furthermore for all $\alpha \in \mathbb{Z}$

$$(T\boldsymbol{\ell})(\alpha) = T\left(\begin{smallmatrix}(\alpha+t)w\\ w\end{smallmatrix}\right) = \left(\begin{smallmatrix}\Delta & -1\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}(\alpha+t)w\\ w\end{smallmatrix}\right) = \left(\begin{smallmatrix}(\alpha+1+t)w-(\alpha+t)w-w\\ w\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\ w\end{smallmatrix}\right),$$

hence $T\boldsymbol{\ell} = \mathbf{k_1}$. This implies $(\partial_T S_{\mathbf{A}})\mathbf{k_1} = (\partial_T S_{\mathbf{A}})T\boldsymbol{\ell} = 2TS_{\mathbf{A}}\boldsymbol{\ell} = T\boldsymbol{\ell} = \mathbf{k_1}$ □

Note that the mask $\mathbf{E}$ defined in Proposition 2.8 reproduces both $\mathbf{k_0}$ and $\mathbf{k}_1$, since

$$S_{\mathbf{E}}\mathbf{k_0}(2\alpha) = E(0)\left(\begin{smallmatrix}w\\ 0\end{smallmatrix}\right) = k_0(2\alpha), \quad S_{\mathbf{E}}\mathbf{k_0}(2\alpha+1) = (E(-1)+E(1))\left(\begin{smallmatrix}w\\ 0\end{smallmatrix}\right) = k_0(2\alpha+1)$$

and similarly for $\mathbf{k}_1$. Therefore, in order to prove Proposition 2.8 we are going to study masks which reproduce either $\mathbf{k}_0$ or $\mathbf{k}_1$ or both.

The mask of a linear subdivision scheme is often analyzed by considering its symbol, which is a vector-valued or matrix-valued Laurent polynomial. We recall some basic facts concerning symbols, as they frequently appear in the sequel.

To a sequence $\mathbf{f}$ we associate its *symbol* by

$$\mathbf{f}^*(z) := \sum_{\alpha\in\mathbb{Z}} f(\alpha)z^\alpha.$$

By eq. (2.1) we have the following identities:

$$(\Delta\mathbf{f})^*(z) = (z^{-1}-1)\mathbf{f}^*(z), \quad (S_{\mathbf{A}}\mathbf{f})^*(z) = \mathbf{A}^*(z)\mathbf{f}^*(z^2), \tag{2.16}$$

where $\mathbf{A}$ has finite support. The operators of Definition 2.7, acting from the left on symbols $\mathbf{f}^*(z)$, are given by

$$\Delta_0^*(z) = \begin{pmatrix} z^{-1}-1 & 0\\ 0 & 1\end{pmatrix}, \ \Delta_1^*(z) = \begin{pmatrix} 1 & 0\\ 0 & z^{-1}-1\end{pmatrix}, \ T^*(z) = \begin{pmatrix} z^{-1}-1 & -1\\ 0 & 1\end{pmatrix}. \tag{2.17}$$

Furthermore, we have the following well known result for a finitely supported symbol:

$$\sum_{\beta\in\mathbb{Z}} f(\alpha-2\beta) = 0 \quad \forall\alpha\in\mathbb{Z} \quad \text{implies} \quad \mathbf{f}^*(z) = (z^{-2}-1)\mathbf{h}^*(z), \tag{2.18}$$

for a symbol $\mathbf{h}^*(z)$. This follows from the fact that $\sum_{\beta\in\mathbb{Z}} f(\alpha-2\beta) = 0$ for all $\alpha \in \mathbb{Z}$ implies $\sum_{\beta\in\mathbb{Z}} f(1-2\beta) = 0$ and $\sum_{\beta\in\mathbb{Z}} f(2\beta) = 0$. Therefore -1 as well as 1 are zeros of $\mathbf{f}^*(z)$. This yields that $1-z^2$ is a factor of $\mathbf{f}^*(z)$.

We say that a mask $\mathbf{M}$ *annihilates* $\mathbf{k_0}$ (resp. $\mathbf{k_1}$) if $S_{\mathbf{M}}\mathbf{k_0} = 0$ (resp. $S_{\mathbf{M}}\mathbf{k_1} = 0$).

Proposition 2.10. *Let* $\mathbf{M}$ *be a mask that annihilates* $\mathbf{k_0}$ *(resp.* $\mathbf{k_1}$*). Then there exists a linear subdivision operator* $S_{\mathbf{N}}$ *such that*

$$S_{\mathbf{M}} = S_{\mathbf{N}}\Delta_0 \quad (\textit{resp. } S_{\mathbf{M}} = S_{\mathbf{N}}\Delta_1).$$

Proof. We prove the case where $\mathbf{k_0}$ is annihilated, the other case is analogous. In terms of symbols, we want to prove that $(S_{\mathbf{M}}\mathbf{f})^*(z) = (S_{\mathbf{N}}\Delta_0\mathbf{f})^*(z)$. By eqs. (2.16) and (2.17), this is equivalent to

$$\mathbf{M}^*(z)\mathbf{f}^*(z^2) = \mathbf{N}^*(z)(\Delta_0\mathbf{f})^*(z^2) = \mathbf{N}^*(z)\begin{pmatrix} z^{-2}-1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{f}^*(z^2).$$

Let $\mathbf{M} = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right)$. Annihilation of $\mathbf{k_0}$ implies

$$\sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta) = 0 \quad \text{and} \quad \sum_{\beta\in\mathbb{Z}} c(\alpha-2\beta) = 0.$$

Thus eq. (2.18) implies that there exist symbols $\tilde{\mathbf{a}}^*(z), \tilde{\mathbf{c}}^*(z)$ such that $\mathbf{a}^*(z) = (z^{-2} - 1)\tilde{\mathbf{a}}^*(z)$ and $\mathbf{c}^*(z) = (z^{-2}-1)\tilde{\mathbf{c}}^*(z)$. Therefore

$$\mathbf{M}^*(z) = \begin{pmatrix} \mathbf{a}^*(z) & \mathbf{b}^*(z) \\ \mathbf{c}^*(z) & \mathbf{d}^*(z) \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{a}}^*(z) & \mathbf{b}^*(z) \\ \tilde{\mathbf{c}}^*(z) & \mathbf{d}^*(z) \end{pmatrix}\begin{pmatrix} z^{-2}-1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The result follows with $\mathbf{N}^*(z) := \left(\begin{smallmatrix} \tilde{\mathbf{a}}^*(z) & \mathbf{b}^*(z) \\ \tilde{\mathbf{c}}^*(z) & \mathbf{d}^*(z) \end{smallmatrix}\right)$. □

The proof of Proposition 2.8 follows from Proposition 2.10:

Proof of Proposition 2.8. Let $\mathbf{A}$ be a mask that satisfies the spectral condition eq. (2.2) and $\partial_T S_{\mathbf{A}}$ the derived scheme w.r.t. T. We prove (ii)–(iv) of Proposition 2.8.

(ii): The subdivision operator $S_{\mathbf{A}}$ reproduces $\mathbf{k_0}$ and $\Delta_0\mathbf{k_0} = 0$. Therefore $\Delta_0 S_{\mathbf{A}}\mathbf{k_0} = 0$ and by Proposition 2.10 there exists an operator $\partial_{\Delta_0}S_{\mathbf{A}}$ such that $\Delta_0 S_{\mathbf{A}} = (\partial_{\Delta_0}S_{\mathbf{A}})\Delta_0$. Similarly, $\partial_T S_{\mathbf{A}}$ reproduces $\mathbf{k_1}$ (by Lemma 2.9) and $\Delta_1\mathbf{k_1} = 0$. Thus there exists $\partial_{\Delta_1}\partial_T S_{\mathbf{A}}$ such that $\Delta_1\partial_T S_{\mathbf{A}} = (\partial_{\Delta_1}\partial_T S_{\mathbf{A}})\Delta_1$ and hence

$$2\Delta_1 T S_{\mathbf{A}} = \Delta_1(\partial_T S_{\mathbf{A}})T = (\partial_{\Delta_1}\partial_T S_{\mathbf{A}})\Delta_1 T.$$

Therefore $\partial_{\Delta_1 T}S_{\mathbf{A}}$ exists and is given by $\partial_{\Delta_1 T}S_{\mathbf{A}} = \partial_{\Delta_1}\partial_T S_{\mathbf{A}}$.

(iii): The equation $(S_{\mathbf{A}} - S_{\mathbf{E}})\mathbf{k_0} = 0$ and Proposition 2.10 implies that there exists a subdivision operator $S_{\mathbf{B}}$ such that $(S_{\mathbf{A}} - S_{\mathbf{E}}) = S_{\mathbf{B}}\Delta_0$.

(iv): is proved analogously to (iii). □

2.5 Norms

In the following sections we will encounter different types of norms, which are collected here. On a finite-dimensional vector space V we choose a norm $\|\cdot\|$. From this norm, we induce a norm on V^2 by

$$\left\|\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)\right\| = \max\{\|v\|, \|w\|\}, \quad \text{for } v, w \in V. \tag{2.19}$$

The norm of an element $A \in L(V)^{2\times 2}$ is the operator norm

$$\|A\| = \sup\{\|A\left(\begin{smallmatrix} v_0 \\ v_1 \end{smallmatrix}\right)\|, \text{ where } \|\left(\begin{smallmatrix} v_0 \\ v_1 \end{smallmatrix}\right)\| = 1\}.$$

For simplicity we use the same symbol in all three cases; it will be clear from the context which one is used.

We write $\ell^\infty(V)$ resp. $\ell^\infty(V^2)$ resp. $\ell^\infty(L(V)^{2\times 2})$ for the Banach spaces of all bounded sequences equipped with the norms

$$\|\mathbf{p}\|_\infty = \sup_{\alpha\in\mathbb{Z}} \|p(\alpha)\| \text{ resp. } \|\mathbf{f}\|_\infty = \sup_{\alpha\in\mathbb{Z}} \|f(\alpha)\| \text{ resp. } \|\mathbf{A}\|_\infty = \sup_{\alpha\in\mathbb{Z}} \|A(\alpha)\|.$$

It is easy to see that if $\mathbf{f} \in \ell^\infty(V^2)$ has the components $\mathbf{f} = \left(\begin{smallmatrix} \mathbf{v} \\ \mathbf{w} \end{smallmatrix}\right)$, then $\|\mathbf{f}\|_\infty = \max\{\|\mathbf{v}\|_\infty, \|\mathbf{w}\|_\infty\}$.

A linear subdivision operator $S_\mathbf{A}$ as defined in eq. (2.1) maps $\ell(V^2)$ to $\ell(V^2)$. Since the linear combination in eq. (2.1) is finite, it is clear that $\|S_\mathbf{A}\mathbf{f}\|_\infty \le d\|\mathbf{f}\|_\infty$ for some $d > 0$. It is not difficult to see that $d \le \sup_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} \|A(\alpha - 2\beta)\| \le N\|\mathbf{A}\|_\infty$, where N is a positive integer such that the support of $\mathbf{A}$ is contained in the interval $[-N, N]$. Therefore, $S_\mathbf{A}$ restricts to an operator $\ell^\infty(V^2) \to \ell^\infty(V^2)$, hence has an induced operator norm $\|S_\mathbf{A}\|_\infty$.

Also the operators of Definition 2.7 restrict to operators $\ell^\infty(V^2) \to \ell^\infty(V^2)$, since it is easy to see that

$$\|\Delta_0\|_\infty = \|\Delta_1\|_\infty = 2 \quad \text{and} \quad \|T\|_\infty = 3. \tag{2.20}$$

We choose the norm eq. (2.19) on V^2 only for technical reasons. We would like to mention that the results presented in the next sections are valid for *any* norm on V^2. Suppose that $\|\cdot\|'$ is another norm on V^2. Since in every finite dimensional vector space any two norms are equivalent, there exist constants $c_0, c_1 > 0$ such that

$$c_0\|\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)\| \le \|\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)\|' \le c_1\|\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)\|, \quad \forall v, w \in V.$$

For a sequence $\mathbf{f} \in \ell(V^2)$, we define $\|\mathbf{f}\|'_\infty = \sup_{\alpha\in\mathbb{Z}} \|f(\alpha)\|'$. It follows immediately that $\|\mathbf{f}\|_\infty$ and $\|\mathbf{f}\|'_\infty$ are equivalent with the same constants $c_0, c_1 > 0$ from above:

$$c_0\|\mathbf{f}\|_\infty \le \|\mathbf{f}\|'_\infty \le c_1\|\mathbf{f}\|_\infty. \tag{2.21}$$

Therefore, a sequence $\mathbf{f}^n$ converges with respect to $\|\cdot\|_\infty$ if and only if it converges with respect to $\|\cdot\|'_\infty$. Furthermore, every inequality of the form $\|\mathbf{f}\|_\infty \le c\|\mathbf{g}\|_\infty$ (with $c > 0$) induces an inequality $\|\mathbf{f}\|'_\infty \le c'\|\mathbf{g}\|'_\infty$ (with $c' > 0$) and vice versa. The results in the following sections will either be concerned with convergence or with this type of inequalities. Therefore, they hold for any norm on V^2.

2.6 Convergence analysis

In this section we derive the following results concerning C^1 convergence of Hermite subdivision schemes:

- We define a "smoothness condition" for linear Hermite subdivision schemes which is sufficient for C^1 convergence.
- We introduce a "proximity condition", which bounds the difference between a nonlinear Hermite subdivision scheme and a linear one.
- We show that the nonlinear Hermite subdivision scheme is C^1 convergent if its linear counterpart is, and certain technical conditions are met.

We start by repeating the definition of convergence for point sequences and point-vector sequences. This needs the following notion: By $\mathcal{F}_n(\mathbf{g}^n)$ we denote the piecewise linear interpolant of the sequence $\mathbf{g}^n$ on the grid $2^{-n}\mathbb{Z}$. If $\mathbf{g}^n$ has more than one component, $\mathcal{F}_n(\mathbf{g}^n)$ is constructed componentwise.

Definition 2.11 (Convergence of point and point-vector sequences)**.** Let V be a vector space. Then we define:

(i) A point sequence $\mathbf{g}^n \in \ell(V^2)$ is said to be convergent if $\mathcal{F}_n(\mathbf{g}^n)$ converges uniformly on compact intervals to a continuous curve $\Psi \in C(\mathbb{R}, V^2)$.

(ii) A point-vector sequence $\binom{\mathbf{p}^n}{\mathbf{v}^n} \in \ell(V^2)$ is said to be C^1 convergent if $\mathcal{F}_n(\mathbf{p}^n)$ converges uniformly on compact intervals to a continuously differentiable curve $\varphi \in C^1(\mathbb{R}, V)$ and $\mathcal{F}_n(\mathbf{v}^n)$ converges uniformly on compact intervals to its derivative φ'.

In the case of manifold-valued sequences, we require that (i) resp. (ii) is satisfied in a chart of the manifold.

In this paper, the sequences $\mathbf{g}^n$ and $\binom{\mathbf{p}^n}{\mathbf{v}^n}$ are produced by subdivision. Due to the compact support of a subdivision operator U, the limit curve on compact intervals depends only on finitely many points of the initial data. In order to prove convergence and smoothness results for subdivision schemes it is therefore sufficient to consider finite input data. In particular we can assume that the input data is bounded.

For bounded input data Definition 2.11 is equivalent to the following:

(i) A point sequence $\mathbf{g}^n \in \ell^\infty(V^2)$ is convergent if there exists a continuous curve $\Psi \in C(\mathbb{R}, V^2)$ such that

$$\sup_{\alpha\in\mathbb{Z}} \|g^n(\alpha) - \Psi\Big(\frac{\alpha}{2^n}\Big)\| \to 0 \quad \text{as} \quad n \to \infty.$$

(ii) A point-vector sequence $\binom{\mathbf{p}^n}{\mathbf{v}^n} \in \ell^\infty(V^2)$ is C^1 convergent if there exists a continuously differentiable curve $\varphi \in C^1(\mathbb{R}, V)$ such that

$$\sup_{\alpha\in\mathbb{Z}} \|p^n(\alpha) - \varphi\Big(\frac{\alpha}{2^n}\Big)\| \to 0 \quad \text{and} \quad \sup_{\alpha\in\mathbb{Z}} \|v^n(\alpha) - \varphi'\Big(\frac{\alpha}{2^n}\Big)\| \to 0 \quad \text{as} \quad n\to\infty.$$

Remark 2.12. In Definition 2.11 we define C^1 *convergence* of a point-vector sequence $\binom{\mathbf{p}^n}{\mathbf{v}^n}$. By this we mean that the limit curve φ enjoys *at least* C^1 smoothness. There exist linear Hermite schemes producing a point-vector sequence $\binom{\mathbf{p}^n}{\mathbf{v}^n}$ and a C^2 limit curve φ (see [11]). We nevertheless use the terminology "C^1 convergence", since we do not prove anything about higher derivatives in this paper.

Remark 2.13. Recall from Section 2.3 that the Hermite scheme $D^n\mathbf{f}^n = U^n\mathbf{f}^0$ and the point scheme $\mathbf{g}^n = U^n\mathbf{g}^0$ only differ in the second component by a factor 2^n if $\mathbf{f}^0 = \mathbf{g}^0$. That is, for $\mathbf{f}^n = \binom{\mathbf{p}^n}{\mathbf{v}^n}$ and $\mathbf{g}^n = \binom{\mathbf{p}^n}{\mathbf{u}^n}$ we have $2^n\mathbf{u}^n = \mathbf{v}^n$.

If the Hermite scheme $\mathbf{f}^n$ is C^1 convergent with limit φ, then the point scheme $\mathbf{g}^n$ is convergent with limit $\Psi = \binom{\varphi}{0}$. In general, the converse is not true. Verification of the following, however, implies C^1 convergence of $\mathbf{f}^n$:

- the point sequence $\mathbf{p}^n$ converges to a continuously differentiable curve ψ_1,
- the point sequence $2^n\mathbf{u}^n$ converges to a continuous curve ψ_2 and
- $\psi_1' = \psi_2$.

In Sections 2.6.2 and 2.6.3 we will use this line of arguments to prove C^1 convergence of Hermite schemes.

Example 2.14. As already mentioned, the interpolatory linear scheme defined in Example 2.1 produces the piecewise cubic interpolant of given point-vector input data. Therefore the Hermite scheme is C^1 convergent in the sense of Definition 2.11 (and hence also the point scheme converges).

The scheme associated to the mask $\mathbf{E}$ we defined in Proposition 2.8 is interpolatory and adds midpoints between two consecutive points in every step. Therefore it produces a continuous limit which in general is not C^1. Hence the point scheme associated with it is convergent, but the associated Hermite scheme is not.

2.6.1 Convergence results for derived schemes

This section deals solely with the convergence of linear point schemes. We treat the *Taylor scheme* $\mathbf{g}^n = (\partial_T S_{\mathbf{A}})^n\mathbf{g}^0$ as well as $\mathbf{g}^n = (\partial_{\Delta_0} S_{\mathbf{A}})^n\mathbf{g}^0$ and $\mathbf{g}^n = (\partial_{\Delta_1 T} S_{\mathbf{A}})^n\mathbf{g}^0$, where we use the notation for derived schemes introduced by Proposition 2.8. Results obtained here will be useful in Sections 2.6.2 and 2.6.3 for the C^1 analysis of Hermite schemes.

Define distance functions D_0, D_1 in $\ell(V^2)$ by

$$D_0(\mathbf{g}) = \|\Delta_0 \mathbf{g}\|_\infty \quad \text{and} \quad D_1(\mathbf{g}) = \|\Delta_1 \mathbf{g}\|_\infty. \tag{2.22}$$

Equation (2.20) implies that both $D_0(\mathbf{g})$ and $D_1(\mathbf{g})$ are finite if $\mathbf{g} \in \ell^\infty(V^2)$.

We already mentioned in Section 2.5 that a linear subdivision operator $S_\mathbf{A}$ has an operator norm. It is given by

$$\|S_\mathbf{A}\|_\infty = \sup\{\|S_\mathbf{A}\mathbf{g}\|_\infty, \text{ where } \|\mathbf{g}\|_\infty = 1\}.$$

Proposition 2.15. *Let $S_\mathbf{A}$ satisfy the spectral condition eq. (2.2) and let $\partial_T S_\mathbf{A}$ be its derived scheme w.r.t. the Taylor operator T. Then there exist constants c_0, c_1 such that for all $\mathbf{g} \in \ell^\infty(V^2)$*

$$\begin{aligned} \|\mathcal{F}_{n+1}(S_\mathbf{A}\mathbf{g}) - \mathcal{F}_n(\mathbf{g})\| &\le c_0 D_0(\mathbf{g}), \\ \|\mathcal{F}_{n+1}(\partial_T S_\mathbf{A}\mathbf{g}) - \mathcal{F}_n(\mathbf{g})\| &\le c_1 D_1(\mathbf{g}), \end{aligned}$$

where we use the notation $\|\varphi\| = \sup_{t\in\mathbb{R}} \|\varphi(t)\|$ for a continuous curve $\varphi \in C(\mathbb{R}, V^2)$. Furthermore the constant c_0 (resp. c_1) only depends on $S_\mathbf{A}$ (resp. $\partial_T S_\mathbf{A}$) and neither on n nor on $\mathbf{g}$.

Proof. Observe that

$$\|\mathcal{F}_n(\mathbf{g}) - \mathcal{F}_n(\mathbf{h})\| = \sup_{\alpha\in\mathbb{Z}} \|g(\alpha) - h(\alpha)\| = \|\mathbf{g} - \mathbf{h}\|_\infty,$$

for $n = 0, 1, \ldots$ and $\mathbf{g}, \mathbf{h} \in \ell^\infty(V^2)$. Therefore using Proposition 2.8 and $\mathcal{F}_n(\mathbf{g}) = \mathcal{F}_{n+1}(S_\mathbf{E}\mathbf{g})$, with $\mathbf{E}$ from Proposition 2.8,

$$\begin{aligned} \|\mathcal{F}_{n+1}(S_\mathbf{A}\mathbf{g}) - \mathcal{F}_n(\mathbf{g})\| &= \|\mathcal{F}_{n+1}(S_\mathbf{A}\mathbf{g}) - \mathcal{F}_{n+1}(S_\mathbf{E}\mathbf{g})\| \\ &= \|(S_\mathbf{A} - S_\mathbf{E})\mathbf{g}\|_\infty = \|S_\mathbf{G}\Delta_0\mathbf{g}\|_\infty \\ &\le \|S_\mathbf{G}\|_\infty \|\Delta_0\mathbf{g}\|_\infty = c_0 D_0(\mathbf{g}). \end{aligned}$$

A similar argument proves the result for $\partial_T S_\mathbf{A}$. □

Proposition 2.16. *Let $S_\mathbf{A}$ satisfy the spectral condition eq. (2.2) and let $\partial_{\Delta_0} S_\mathbf{A}$ be the derived scheme w.r.t. to the operator Δ_0. Then the following are equivalent:*

(i) The point scheme $\mathbf{g}^n = S_\mathbf{A}^n \mathbf{g}^0$ is convergent with continuous limit curve $\binom{\psi}{0}$ for all input data $\mathbf{g}^0$.

(ii) The point scheme $\mathbf{h}^n = (\partial_{\Delta_0} S_\mathbf{A})^n \mathbf{h}^0$ converges to 0 for all input data $\mathbf{h}^0 = \Delta_0 \mathbf{g}^0$.

(iii) There exists a positive constant c_0 and $\alpha \in (0, 1]$ such that

$$D_0(S_\mathbf{A}^n \mathbf{g}) \le c_0 2^{-\alpha n} D_0(\mathbf{g}) \quad \textit{for } n = 0, 1, \ldots \textit{ and } \mathbf{g} \in \ell^\infty(V^2).$$

Proof. In fact this proposition follows from similar results in [3, 7, 26]. For the convenience of the reader we give a proof.

Recall that it is sufficient to prove convergence for bounded input data. We therefore assume that $\mathbf{g}^0 \in \ell^\infty(V^2)$.

(i) $\Longrightarrow$ (ii): Let $\mathbf{g}^n = \binom{\mathbf{p}^n}{\mathbf{u}^n}$. Convergence to $\binom{\psi}{0}$ implies that

$$\sup_{\alpha\in\mathbb{Z}} \|p^n(\alpha) - \psi\Big(\frac{\alpha}{2^n}\Big)\| \to 0 \quad \text{and} \quad \sup_{\alpha\in\mathbb{Z}} \|u^n(\alpha)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.23}$$

Furthermore $\mathbf{h}^n = (\partial_{\Delta_0} S_{\mathbf{A}})^n \mathbf{h}^0 = (\partial_{\Delta_0} S_{\mathbf{A}})^n \Delta_0 \mathbf{g}^0 = \Delta_0 S_{\mathbf{A}}^n \mathbf{g}^0 = \Delta_0 \mathbf{g}^n = \binom{\Delta \mathbf{p}^n}{\mathbf{u}^n}$, which immediately implies that the second component of $\mathbf{h}^n$ converges to 0. As to the first component, consider

$$\begin{aligned}
&\|(\Delta\mathbf{p}^n)(\alpha)\| = \|p^n(\alpha+1) - p^n(\alpha)\| \\
&\quad \le \Big\|p^n(\alpha+1) - \psi\Big(\frac{\alpha+1}{2^n}\Big)\Big\| + \Big\|\psi\Big(\frac{\alpha}{2^n}\Big) - p^n(\alpha)\Big\| + \Big\|\psi\Big(\frac{\alpha+1}{2^n}\Big) - \psi\Big(\frac{\alpha}{2^n}\Big)\Big\|.
\end{aligned}$$

The continuity of ψ together with eq. (2.23) implies that $\sup_{\alpha\in\mathbb{Z}} \|(\Delta\mathbf{p}^n)(\alpha)\|$ converges to 0 and hence $\mathbf{h}^n$ converges to 0.

(ii) $\Longrightarrow$ (iii): Suppose that $\mathbf{h}^n = S_{\mathbf{D}}^n \mathbf{h}^0$ converges to 0 with $\mathbf{h}_0 = \Delta_0 \mathbf{g}$. Then there exists a positive integer N such that

$$\|S_{\mathbf{D}}^N \Delta_0 \mathbf{g}\|_\infty \le \frac{1}{2} \|\Delta_0 \mathbf{g}\|_\infty.$$

Write a positive integer n as $n = mN + r$, where $m = \lfloor \frac{n}{N} \rfloor$ and $0 \le r < N$. Then we have

$$\begin{aligned}
\|\Delta_0 S_{\mathbf{A}}^n \mathbf{g}\|_\infty &= \|S_{\mathbf{D}}^n \Delta_0 \mathbf{g}\|_\infty = \|S_{\mathbf{D}}^r S_{\mathbf{D}}^{mN} \Delta_0 \mathbf{g}\|_\infty \\
&\le 2^{-m} \|S_{\mathbf{D}}^r\|_\infty \|\Delta_0 \mathbf{g}\|_\infty = 2^{-\frac{n}{N}} 2^{\frac{r}{N}} \|S_{\mathbf{D}}^r\|_\infty \|\Delta_0 \mathbf{g}\|_\infty \\
&\le 2^{-n\alpha} \max_{0\le r<N} 2^{\frac{r}{N}} \|S_{\mathbf{D}}^r\|_\infty \|\Delta_0 \mathbf{g}\|_\infty \\
&\le c_0 2^{-n\alpha} \|\Delta_0 \mathbf{g}\|_\infty,
\end{aligned}$$

where $\alpha = \frac{1}{N}$ and $c_0 = \max_{0\le r<N} 2^{\frac{r}{N}} \|S_{\mathbf{D}}^r\|_\infty$. This proves inequality (iii).

(iii) $\Longrightarrow$ (i): Let $\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$. Since we want to prove convergence, we may assume that $\mathbf{g}^0$ is bounded. By Proposition 2.15 and assumption (i) we have

$$\|\mathcal{F}_{n+1}(S_{\mathbf{A}}^{n+1}\mathbf{g}^0) - \mathcal{F}_n(S_{\mathbf{A}}^n \mathbf{g}^0)\| \le c_0 D_0(S_{\mathbf{A}}^n \mathbf{g}^0) \le c_0 2^{-\alpha n} D_0(\mathbf{g}^0),$$

for all $n = 0, 1, \ldots$. This shows that $\mathcal{F}_n(S_{\mathbf{A}}^n \mathbf{g}^0) = \mathcal{F}_n(\mathbf{g}^n)$ is a Cauchy sequence. The sequence $\mathbf{g}^n$ takes values in the finite dimensional vector space V^2 which implies that the space of continuous curves on V^2, equipped with the ∞-norm, is a Banach space. Therefore $\mathcal{F}_n(\mathbf{g}^n)$ converges to a continuous curve.

Let $\mathbf{g}^n = \binom{\mathbf{p}^n}{\mathbf{u}^n}$. Inequality (iii) implies that $\|\Delta\mathbf{p}^n\|_\infty, \|\mathbf{u}^n\|_\infty$ converge to 0. Therefore the second component of the limit curve equals 0. $\square$

The following result can be proved analogously to Proposition 2.16:

Proposition 2.17. *Let $S_{\mathbf{D}}$ satisfy the reproduction property eq.* (2.15) *and let $S_{\mathbf{F}}$ be such that $\Delta_1 S_{\mathbf{D}} = S_{\mathbf{F}}\Delta_1$ ($S_{\mathbf{F}}$ exists by Proposition 2.8). Then the following are equivalent:*

(i) The point scheme $\mathbf{g}^n = S_{\mathbf{D}}^n \mathbf{g}^0$ is convergent with continuous limit curve $\binom{0}{\psi}$ for all input data $\mathbf{g}^0$.

(ii) The point scheme $\mathbf{h}^n = S_{\mathbf{F}}^n \mathbf{h}^0$ converges to 0 for all input data $\mathbf{h}^0 = \Delta_1 \mathbf{g}^0$.

(iii) There exists a positive constant c_1 and $\alpha \in (0,1]$ such that

$$D_1(S_{\mathbf{D}}^n \mathbf{g}) \leq c_1 2^{-\alpha n} D_1(\mathbf{g}) \quad \textit{for } n = 0,1,\dots \textit{ and } \mathbf{g} \in \ell^\infty(V^2).$$

Note that if $S_{\mathbf{A}}$ satisfies the spectral condition eq. (2.2), then Proposition 2.17 can be applied to $S_{\mathbf{D}} = \partial_T S_{\mathbf{A}}$. The operator $S_{\mathbf{F}}$ is then given by $\partial_{\Delta_1 T} S_{\mathbf{A}}$.

2.6.2 C^1 results for linear Hermite schemes

The C^1 analysis of linear Hermite subdivision schemes is often transferred to the convergence analysis of the respective Taylor schemes, see [22, 23, 25, 60]. The main theorem of this section (Theorem 2.21) is similar in this regard, but differs somewhat in the proof. Instead of constructing the limit function explicitly, we invoke convergence and smoothness conditions which we later adapt to the manifold-valued case.

Definition 2.18 (Convergence and smoothness conditions)**.** Consider a linear subdivision operator $S_{\mathbf{A}} : \ell(V^2) \to \ell(V^2)$. We use the following terminology:

$S_{\mathbf{A}}$ satisfies a convergence condition, if there exists $\gamma_0 < 1$ and a positive constant c_0 such that

$$D_0(S_{\mathbf{A}}^n \mathbf{g}) \leq c_0 \gamma_0^n D_0(\mathbf{g}) \qquad \text{for all } \mathbf{g} \in \ell^\infty(V^2).$$

$S_{\mathbf{A}}$ satisfies a smoothness condition, if there exists $\gamma_1 < 1$ and a positive constant c_1 such that

$$D_1(2^n T S_{\mathbf{A}}^n \mathbf{g}) \leq c_1 \gamma_1^n D_1(T\mathbf{g}) \qquad \text{for all } \mathbf{g} \in \ell^\infty(V^2).$$

Define $\alpha = -\log_2(\gamma_0)$ and $\beta = -\log_2(\gamma_1)$. We are content to find contraction factors γ_i with $\frac{1}{2} \leq \gamma_i$ for $i = 0,1$, which is the optimum for linear functions. Therefore we w.l.o.g. assume that $\alpha, \beta \in (0,1]$. We will always work with $\gamma_0 = 2^{-\alpha}$ and $\gamma_1 = 2^{-\beta}$.

Definition 2.18 is based on similar conditions in [77, 75, 85]. Note that the convergence condition is exactly our condition (iii) of Proposition 2.16. Furthermore, the smoothness condition is our condition (iii) of Proposition 2.17 for input data of the form $T\mathbf{g}$ applied to $\partial_T S_{\mathbf{A}}$. The following Lemma is a preparation for Theorem 2.21.

Lemma 2.19. *Let $S_{\mathbf{A}}$ satisfy the spectral condition eq. (2.2). If $S_{\mathbf{A}}$ satisfies the smoothness condition of Definition 2.18, then it also satisfies the convergence condition.*

Proof. Let $\mathbf{g}^0 = \binom{\mathbf{p}^0}{\mathbf{u}^0}$ be bounded input data. Define $\binom{\mathbf{p}^n}{\mathbf{u}^n} = S_{\mathbf{A}}^n \binom{\mathbf{p}^0}{\mathbf{u}^0}$ by iterated subdivision. By Proposition 2.8 the derived scheme $\partial_{\Delta_0} S_{\mathbf{A}}$ exists. We will prove that the point scheme $\mathbf{h}^n = (\partial_{\Delta_0} S_{\mathbf{A}})^n \mathbf{h}^0$ with $\mathbf{h}^0 = \Delta_0 \mathbf{g}^0$ converges to 0. Then Proposition 2.16 implies the convergence condition.

In order to prove that $\mathbf{h}^n$ converges to 0, note that

$$\mathbf{h}^n = (\partial_{\Delta_0} S_{\mathbf{A}})^n \mathbf{h}^0 = (\partial_{\Delta_0} S_{\mathbf{A}})^n \Delta_0 \mathbf{g}^0 = \Delta_0 S_{\mathbf{A}}^n \mathbf{g}^0 = \binom{\Delta \mathbf{p}^n}{\mathbf{u}^n}.$$

Therefore, we have to prove that

(i) $\sup_{\alpha\in\mathbb{Z}} \|(\Delta \mathbf{p}^n)(\alpha)\| \to 0$ and

(ii) $\sup_{\alpha\in\mathbb{Z}} \|u^n(\alpha)\| \to 0$.

This will follow from the smoothness condition:

By Proposition 2.8, the Taylor scheme $\partial_T S_{\mathbf{A}}$ exists. By Proposition 2.17, the smoothness condition implies that the point scheme $\mathbf{k}^n = (\partial_T S_{\mathbf{A}})^n \mathbf{k}^0$ converges to a curve $\binom{0}{\psi}$ for all input data $\mathbf{k}^0 = T\mathbf{g}^0$. Since $\mathbf{k}^n = 2^n T S_{\mathbf{A}}^n \mathbf{g}^0 = 2^n \binom{\Delta \mathbf{p}^n - \mathbf{u}^n}{\mathbf{u}^n}$ we have

$$\begin{aligned} &\sup_{\alpha\in\mathbb{Z}} \|2^n (\Delta \mathbf{p}^n)(\alpha) - 2^n u^n(\alpha)\| \to 0, \\ &\sup_{\alpha\in\mathbb{Z}} \|2^n u^n(\alpha) - \psi\big(\frac{\alpha}{2^n}\big)\| \to 0, \end{aligned}$$

as $n \to \infty$. Therefore

$$\sup_{\alpha\in\mathbb{Z}} \|u^n(\alpha)\| \le \frac{1}{2^n} \sup_{\alpha\in\mathbb{Z}} \|2^n u^n(\alpha) - \psi\big(\frac{\alpha}{2^n}\big)\| + \frac{1}{2^n} \sup_{\alpha\in\mathbb{Z}} \|\psi\big(\frac{\alpha}{2^n}\big)\| \to 0,$$

as $n \to \infty$. This proves (ii). We use (ii) to prove (i):

$$\sup_{\alpha\in\mathbb{Z}} \|(\Delta \mathbf{p}^n)(\alpha)\| \le \frac{1}{2^n} \sup_{\alpha\in\mathbb{Z}} \|2^n (\Delta \mathbf{p}^n)(\alpha) - 2^n u^n(\alpha)\| + \sup_{\alpha\in\mathbb{Z}} \|u^n(\alpha)\| \to 0,$$

as $n \to \infty$. This proves that $\mathbf{h}^n$ converges to 0. Thus, by Proposition 2.16, $S_{\mathbf{A}}$ satisfies the convergence condition. □

As a preparation for the next theorem we state a well-known result, which will also be useful for the rest of the paper:

Lemma 2.20. *[77, Lemma 8] Consider a sequence of polygons $\mathbf{p}^n$ such that $\mathcal{F}_n(\mathbf{p}^n)$ converges to φ as $n \to \infty$ and φ is continuous. If $\mathcal{F}_n(2^n \Delta \mathbf{p}^n)$ is a Cauchy sequence and $\|2^n \Delta^2 \mathbf{p}^n\|_\infty \to 0$ as $n \to \infty$, then φ is continuously differentiable and $\mathcal{F}_n(2^n \Delta \mathbf{p}^n)$ converges to φ' as $n \to \infty$.*

Theorem 2.21. *If $S_{\mathbf{A}}$ satisfies the spectral condition eq.* (2.2) *and the smoothness condition of Definition 2.18, then the linear Hermite scheme $\mathbf{f}^n = D^{-n} S_{\mathbf{A}}^n \mathbf{f}^0$ is C^1 convergent for all input data $\mathbf{f}^0 \in \ell(V^2)$.*

Proof. As suggested in Remark 2.13, we will use the linear point scheme $\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$ to prove C^1 convergence of the Hermite scheme $\mathbf{f}^n = D^{-n} S_{\mathbf{A}}^n \mathbf{f}^0$.

Let $\mathbf{f}^n = \binom{\mathbf{p}^n}{\mathbf{v}^n}$ and $\mathbf{g}^n = \binom{\mathbf{p}^n}{\mathbf{u}^n}$ with $2^n \mathbf{u}^n = \mathbf{v}^n$. By Definition 2.11, we have to prove that $\mathbf{p}^n$ converges to a C^1 curve φ and that $\mathbf{v}^n = 2^n \mathbf{u}^n$ converges to its derivative φ'.

By Proposition 2.8, the Taylor scheme $\partial_T S_{\mathbf{A}}$ exists. By Proposition 2.17, the point scheme $\mathbf{k}^n = (\partial_T S_{\mathbf{A}})^n \mathbf{k}^0$ converges to a continuous curve $\binom{0}{\psi}$ for all input data $\mathbf{k}^0 = T\mathbf{g}^0$. Since $\mathbf{k}^n = 2^n T S_{\mathbf{A}}^n \mathbf{g}^0 = 2^n T\binom{\mathbf{p}^n}{\mathbf{u}^n} = 2^n \binom{\Delta\mathbf{p}^n - \mathbf{u}^n}{\mathbf{u}^n}$, also $2^n \binom{\Delta\mathbf{p}^n - \mathbf{u}^n}{\mathbf{u}^n}$ converges to $\binom{0}{\psi}$. This implies that both $\mathbf{v}^n = 2^n \mathbf{u}^n$ and $2^n \Delta \mathbf{p}^n$ converge to ψ and that

$$\sup_{\alpha\in\mathbb{Z}} \|2^n (\Delta \mathbf{p}^n)(\alpha) - v^n(\alpha)\| \to 0 \quad \text{as } n \to \infty. \tag{2.24}$$

By Lemma 2.19, the convergence condition applies and by Proposition 2.16, the linear point scheme $\mathbf{g}^n = S_{\mathbf{A}}^n \mathbf{g}^0$ converges to a continuous curve $\binom{\varphi}{0}$. Hence, the sequence $\mathbf{p}^n$ converges to φ.

By Proposition 2.8, the derived scheme $\partial_{\Delta_1 T} S_{\mathbf{A}}$ exists. By Proposition 2.17, the linear point scheme $\mathbf{h}^n = (\partial_{\Delta_1 T} S_{\mathbf{A}})^n \mathbf{h}^0$ converges to 0 for all input data $\mathbf{h}^0 = \Delta_1 \mathbf{k}^0 = \Delta_1 T \mathbf{g}^0$. Since

$$\mathbf{h}^n = (\partial_{\Delta_1 T} S_{\mathbf{A}})^n \mathbf{h}^0 = (\partial_{\Delta_1 T} S_{\mathbf{A}})^n \Delta_1 T \mathbf{g}^0 = 2^n \Delta_1 T S_{\mathbf{A}}^n \mathbf{g}^0 = 2^n \Delta_1 T \binom{\mathbf{p}^n}{\mathbf{u}^n} = 2^n \binom{\Delta\mathbf{p}^n - \mathbf{u}^n}{\Delta \mathbf{u}^n},$$

both $2^n(\Delta \mathbf{p}^n - \mathbf{u}^n)$ and $2^n \Delta \mathbf{u}^n$ converge to 0. Therefore

$$\sup_{\alpha\in\mathbb{Z}} \|(\Delta \mathbf{v}^n)(\alpha)\| = \sup_{\alpha\in\mathbb{Z}} \|2^n (\Delta \mathbf{u}^n)(\alpha)\| \to 0 \quad \text{as } n \to \infty.$$

This together with eq. (2.24) and $\|\Delta\|_\infty = 2$ indicates that

$$\begin{aligned}
\sup_{\alpha\in\mathbb{Z}} \|2^n (\Delta^2 \mathbf{p}^n)(\alpha)\| &\le \sup_{\alpha\in\mathbb{Z}} \|2^n (\Delta^2 \mathbf{p}^n)(\alpha) - (\Delta \mathbf{v}^n)(\alpha)\| + \sup_{\alpha\in\mathbb{Z}} \|(\Delta \mathbf{v}^n)(\alpha)\| \\
&\le \sup_{\alpha\in\mathbb{Z}} 2 \|2^n (\Delta \mathbf{p}^n)(\alpha) - v^n(\alpha)\| + \sup_{\alpha\in\mathbb{Z}} \|(\Delta \mathbf{v}^n)(\alpha)\| \to 0,
\end{aligned}$$

as $n \to \infty$. To summarize, we showed that

(i) $\mathbf{p}^n$ converges to a continuous function φ,

(ii) $2^n \Delta \mathbf{p}^n$ as well as $\mathbf{v}^n$ converge to a continuous function ψ and

(iii) $2^n \Delta^2 \mathbf{p}^n$ converges to 0.

Lemma 2.20 implies that φ is continuously differentiable and that $\varphi' = \psi$, which gives the result. □

Corollary 2.22. *If $S_{\mathbf{A}}$ satisfies the spectral condition eq.* (2.2) *and there exists a positive integer N such that the derived scheme $\partial_{\Delta_1 T} S_{\mathbf{A}}$ satisfies $\|\partial_{\Delta_1 T} S_{\mathbf{A}}^N\|_\infty < 1$, then the linear Hermite scheme $\mathbf{f}^n = D^{-n} S_{\mathbf{A}}^n \mathbf{f}^0$ is C^1 convergent for all input data $\mathbf{f}^0 \in \ell(V^2)$.*

Proof. A similar argument as in the proof of Proposition 2.16 shows that $\|\partial_{\Delta_1 T} S_{\mathbf{A}}^N\|_\infty < 1$ implies the smoothness condition. The rest then follows from Theorem 2.21. □

Remark 2.23. We would like to comment on the independence of the chosen norm, continuing the discussion of Section 2.5. Certainly, if $\|\partial_{\Delta_1 T} S_{\mathbf{A}}^N\|_\infty < 1$ holds for $\|\cdot\|_\infty$, then in general it does not hold for an equivalent norm $\|\cdot\|'_\infty$. So in this case, the choice of norm is relevant. Nevertheless, the result is independent of the chosen norm in the following sense: If we can prove that $\|\partial_{\Delta_1 T} S_{\mathbf{A}}^N\|_\infty < 1$ for *any* norm $\|\cdot\|_\infty$, then the linear Hermite scheme is C^1 convergent with respect to this norm, and therefore, it is C^1 convergent with respect to every other norm.

2.6.3 C^1 convergence from proximity

In this section we are going to derive C^1 convergence for nonlinear Hermite subdivision schemes, which are *close enough* to linear ones. The comparison of a subdivision operator $U_{\mathbf{A}}$ on TM to a linear subdivision operator $S_{\mathbf{A}}$ only makes sense in a chart or in an embedding of M. This paper uses charts. Therefore, from now on, all results are to be understood in a chart of M. Hence we w.l.o.g. assume that $TM \subset V^2$. Furthermore, since the particular mask the subdivision operator is not important, we write S instead of $S_{\mathbf{A}}$ and U instead of $U_{\mathbf{A}}$.

We say that a subdivision operator U on TM and a linear subdivision operator S satisfy the *proximity condition* if there exists a constant c_0 such that

$$\|(U - S)\mathbf{g}\|_\infty \leq c_0 D_0(\mathbf{g})^2. \tag{2.25}$$

This is analogous to the proximity condition defined in [77].

We start with two theorems similar to [77, Theorem 2 and 5] and [85, Theorem 2.4]. By $\ell^\epsilon(V^2)$ we denote the set of sequences $\mathbf{g} : \mathbb{Z} \to V^2$ which satisfy $D_0(\mathbf{g}) < \epsilon$.

Theorem 2.24. *Let U be a subdivision operator on TM and let S be a linear subdivision operator. Suppose that S and U satisfy the proximity condition eq.* (2.25) *for all $\mathbf{g} \in \ell^\epsilon(TM)$ and that S satisfies the convergence condition*

$$D_0(S^n \mathbf{g}) \leq c_0 2^{-\alpha n} D_0(\mathbf{g}). \tag{2.26}$$

Then there exists a positive integer m such that for every $\beta \in \left(0, \alpha - \frac{\log_2(c)}{m}\right)$ there exists $\epsilon' \in (0, \epsilon)$ such that $\bar{U} = U^m$ satisfies

$$D_0(\bar{U}^n \mathbf{g}) \leq 2^{-\beta m n} D_0(\mathbf{g}), \tag{2.27}$$

for all $\mathbf{g} \in \ell^{\epsilon'}(TM)$.

Proof. Choose m such that $\mu = \alpha - \frac{\log_2(c_0)}{m} > 0$. Then $\bar{S} = S^m$ satisfies

$$D_0(\bar{S}\mathbf{g}) \leq 2^{-\mu m} D_0(\mathbf{g}), \tag{2.28}$$

for all $\mathbf{g} \in \ell^\epsilon(TM)$. Furthermore, it is proved in [75, Lemma 3] that the iterates $\bar{U}, \bar{S}$ satisfy a proximity condition as well. Hence

$$\|(\bar{U} - \bar{S})\mathbf{g}\|_\infty \leq \bar{c} D_0(\mathbf{g})^2.$$

Choose $\beta \in (0, \mu)$ and $\epsilon' \in (0, \epsilon)$ such that ϵ' satisfies $2\bar{c}\epsilon' + 2^{-\mu m} < 2^{-\beta m}$. Note that we can choose such an ϵ' since $2^{-\beta m} - 2^{-\mu m} > 0$. Then for $\mathbf{g} \in \ell^{\epsilon'}(TM)$ we have

$$\begin{aligned} D_0(\bar{U}^n \mathbf{g}) &\leq \|(\Delta_0 \bar{U}^n - \Delta_0 \bar{S}\bar{U}^{n-1})\mathbf{g}\|_\infty + \|\Delta_0 \bar{S}\bar{U}^{n-1}\mathbf{g}\|_\infty \\ &\leq 2\|(\bar{U} - \bar{S})\bar{U}^{n-1}\mathbf{g}\|_\infty + D_0(\bar{S}\bar{U}^{n-1}\mathbf{g}) \\ &\leq 2\bar{c} D_0(\bar{U}^{n-1}\mathbf{g})^2 + 2^{-\mu m} D_0(\bar{U}^{n-1}\mathbf{g}). \end{aligned}$$

Using this recursion, we will prove by induction that eq. (2.27) holds. For $n = 1$ we have

$$D_0(\bar{U}\mathbf{g}) \leq 2\bar{c} D_0(\mathbf{g})^2 + 2^{-\mu m} D_0(\mathbf{g}) < (2\bar{c}\epsilon' + 2^{-\mu m}) D_0(\mathbf{g}) < 2^{-\beta m} D_0(\mathbf{g}).$$

Assume that eq. (2.27) holds for $1, \ldots, n-1$. Then

$$\begin{aligned} D_0(\bar{U}^n \mathbf{g}) &\leq 2\bar{c} D_0(\bar{U}^{n-1}\mathbf{g})^2 + 2^{-\mu m} D_0(\bar{U}^{n-1}\mathbf{g}) \\ &\leq 2\bar{c} 2^{-2\beta m(n-1)} D_0(\mathbf{g})^2 + 2^{-\mu m} 2^{-\beta m(n-1)} D_0(\mathbf{g}) \\ &\leq \left(2\bar{c}\epsilon' + 2^{-\mu m}\right) 2^{-\beta m(n-1)} D_0(\mathbf{g}) \\ &\leq 2^{-\beta m n} D_0(\mathbf{g}), \end{aligned}$$

which concludes the induction. □

Theorem 2.25. *Let U be a subdivision operator on TM and let S be a linear subdivision operator. Suppose that S and U satisfy the proximity condition eq. (2.25) for all $\mathbf{g} \in \ell^\epsilon(TM)$ and that S satisfies the smoothness condition*

$$D_1(2^n T S^n \mathbf{g}) \leq c_1 2^{-\alpha_1 n} D_1(T\mathbf{g}). \tag{2.29}$$

Then there exists a positive integer m such that for every $\beta_1 \in \left(0, \alpha_1 - \frac{\log_2(c_1)}{m}\right)$ there exist $\epsilon' \in (0, \epsilon)$ and a linear polynomial P such that $\bar{U} = U^m$ satisfies

$$D_1(2^{mn} T \bar{U}^n \mathbf{g}) \leq 2^{-\beta_1 m n} \Big(D_1(T\mathbf{g}) + P(n) D_0(\mathbf{g}) \Big), \tag{2.30}$$

for $\mathbf{g} \in \ell^{\epsilon'}(TM)$.

Proof. The linear subdivision operator S satisfies the smoothness condition eq. (2.29) and hence by Lemma 2.19 it also satisfies the convergence condition eq. (2.26). Hence there exist $c_0 > 0$ and $\alpha_0 \in (0,1]$ such that

$$D_0(S_A^n \mathbf{g}) \leq c_0 2^{-\alpha_0 n} D_0(\mathbf{g})$$

and for all $\mathbf{g} \in \ell^\epsilon(TM)$.

For $i = 0, 1$ choose m_i such that $\mu_i = \alpha_i - \frac{\log_2(c_i)}{m_i} > 0$ and set $m = \max\{m_0, m_1\}$. Decrease α_1 until $0 < \alpha_1 < 2\alpha_0$ and increase c_1 until $2^m c_0^2 < c_1$. Note that these modifications do not affect the correctness of eq. (2.29).

Then $\bar{S} = S^m$ satisfies

$$\begin{aligned} D_0(\bar{S}\mathbf{g}) &\leq 2^{-\mu_0 m} D_0(\mathbf{g}) \\ D_1(T\bar{S}\mathbf{g}) &\leq 2^{(-\mu_1 - 1)m} D_1(T\mathbf{g}), \end{aligned}$$

for $\mathbf{g} \in \ell^\epsilon(TM)$. As in the proof of Theorem 2.24, the proximity condition for the iterates $\bar{S} = S^m, \bar{U} = U^m$ is satisfied:

$$\|(\bar{U} - \bar{S})\mathbf{g}\|_\infty \leq \bar{c} D_0(\mathbf{g})^2.$$

Furthermore, by Theorem 2.24, for $\beta_0 = \left(\alpha_0 - \frac{\log_2(c_0)}{m}\right) - \eta$ with $\eta \in \left(0, \frac{2\alpha_0 - \alpha_1}{2}\right)$ there exists $\epsilon' \in (0, \epsilon)$ such that

$$D_0(\bar{U}^n \mathbf{g}) \leq 2^{-\beta_0 m n} D_0(\mathbf{g}), \tag{2.31}$$

for $\mathbf{g} \in \ell^{\epsilon'}(TM)$. Note that the modifications of c_1 and α_1 now imply that $1 + \mu_1 - 2\beta_0 < 0$.

We have for all $\mathbf{g} \in \ell^{\epsilon'}(TM)$

$$\begin{aligned} D_1(T\bar{U}^n\mathbf{g}) = \|\Delta_1 T\bar{U}^n \mathbf{g}\|_\infty &\leq \|\Delta_1 T\bar{S}\bar{U}^{n-1}\mathbf{g}\|_\infty + \|\Delta_1 T(\bar{S} - \bar{U})U^{n-1}\mathbf{g}\|_\infty \\ &\leq 2^{(-\mu_1 - 1)m} D_1(T\bar{U}^{n-1}\mathbf{g}) + 4\bar{c} D_0(\bar{U}^{n-1}\mathbf{g})^2. \end{aligned}$$

Choose β_1 such that $\beta_1 \in (0, \mu_1)$. Then iteration of this recursion gives

$$\begin{aligned} D_1(T\bar{U}^n\mathbf{g}) &\leq 2^{(-\mu_1-1)mn} D_1(T\mathbf{g}) + 4\bar{c} \sum_{i=0}^{n-1} 2^{m(-\mu_1-1)(n-1-i)} D_0(\bar{U}^i\mathbf{g})^2 \\ &= 2^{(-\mu_1-1)mn} \left(D_1(T\mathbf{g}) + 4\bar{c} 2^{m(1+\mu_1)} \sum_{i=0}^{n-1} 2^{m(1+\mu_1)i} D_0(\bar{U}^i\mathbf{g})^2 \right) \\ &\overset{eq.\ (2.31)}{\leq} 2^{(-\beta_1-1)mn} \left(D_1(T\mathbf{g}) + 4\bar{c} 2^{m(1+\mu_1)} \sum_{i=0}^{n-1} 2^{m(1+\mu_1)i + 2(-\beta_0 m i)} D_0(\mathbf{g})^2 \right) \\ &= 2^{(-\beta_1-1)mn} \left(D_1(T\mathbf{g}) + 4\bar{c} 2^{m(1+\mu_1)} \sum_{i=0}^{n-1} 2^{(1+\mu_1-2\beta_0)mi} D_0(\mathbf{g})^2 \right) \\ &\leq 2^{(-\beta_1-1)mn} \left(D_1(T\mathbf{g}) + 4\bar{c} 2^{m(1+\mu_1)} \epsilon' n D_0(\mathbf{g}) \right). \end{aligned}$$

Defining $P(n) = 4\bar{c} 2^{m(1+\mu_1)} \epsilon' n$ gives the result. □

Theorem 2.26. *Let U be a subdivision operator on TM and let S be a linear subdivision operator which satisfies the spectral condition eq.* (2.2)*. Suppose that S satisfies the smoothness condition of Definition 2.18 and that S and U satisfy the proximity condition eq.* (2.25) *for all input from $\ell^\epsilon(TM)$ for some $\epsilon > 0$. Then there exists $\epsilon' > 0$ such that the Hermite scheme $\mathbf{f}^n = D^{-n}U^n\mathbf{f}^0$ is C^1 convergent for all input $\mathbf{f}^0 \in \ell^{\epsilon'}(TM)$.*

Proof. As suggested in Remark 2.13, we use the point scheme $\mathbf{g}^n = U^n\mathbf{g}^0$ to prove C^1 convergence of the Hermite scheme $\mathbf{f}^n = D^{-n}U^n\mathbf{f}^0$. Let $\binom{\mathbf{p}^n}{\mathbf{v}^n} = \mathbf{f}^n$ and $\binom{\mathbf{p}^n}{\mathbf{u}^n} = \mathbf{g}^n$ with $2^n\mathbf{u}^n = \mathbf{v}^n$. We want to show that $\mathbf{p}^n$ converges to a continuously differentiable curve φ and $\mathbf{v}^n = 2^n\mathbf{u}^n$ converges to its derivative φ'.

Since S satisfies the smoothness condition for all $\mathbf{f}^0 \in \ell^\epsilon(TM)$, theorem 2.24 and theorem 2.25 imply that there exist a positive integer m such that $\bar{U} = U^m$ satisfies both the convergence eq. (2.27) and the smoothness condition eq. (2.30).

We start by considering the case $m = 1$. Then by eqs. (2.27) and (2.30) there exists an $\epsilon' \in (0, \epsilon)$ such that U satisfies

$$\begin{aligned} D_0(U^n\mathbf{f}^0) &\leq 2^{-\beta_0 n} D_0(\mathbf{f}^0), \\ D_1(2^n T U^n \mathbf{f}^0) &\leq 2^{-\beta_1 n}\Big(D_1(T\mathbf{f}^0) + P(n) D_0(\mathbf{f}^0)\Big), \end{aligned} \tag{2.32}$$

for all $\mathbf{f}^0 \in \ell^{\epsilon'}(TM)$ and β_0, β_1 as in Theorem 2.24 and Theorem 2.25.

We prove C^1 convergence of $\mathbf{f}^n$ for all input data $\mathbf{f}^0 \in \ell^{\epsilon'}(TM)$. By arguments given at the beginning of this section, we may assume that $\mathbf{f}^0$ is bounded.

For simplicity, we denote the input data by $\mathbf{f}$. We first prove that the sequence $\mathbf{p}^n$ converges to a continuous curve:

$$\begin{aligned} \|\mathcal{F}_{n+1}(U^{n+1}\mathbf{f}) - \mathcal{F}_n(U^n\mathbf{f})\| &\leq \|\mathcal{F}_{n+1}(U^{n+1}\mathbf{f}) - \mathcal{F}_{n+1}(SU^n\mathbf{f})\| \\ &\quad + \|\mathcal{F}_{n+1}(SU^n\mathbf{f}) - \mathcal{F}_n(U^n\mathbf{f})\| \\ &\leq \|(U - S)(U^n\mathbf{f})\|_\infty + \|\mathcal{F}_{n+1}(SU^n\mathbf{f}) - \mathcal{F}_n(U^n\mathbf{f})\| \\ &\leq c_0 D_0(U^n\mathbf{f})^2 + c_1 D_0(U^n\mathbf{f}) \qquad (2.33) \\ &\leq c_0 2^{-2\beta_0 n} D_0(\mathbf{f})^2 + c_1 2^{-\beta_0 n} D_0(\mathbf{f}). \end{aligned}$$

Inequality eq. (2.33) uses the proximity condition eq. (2.25) for S and U and Proposition 2.15. We have proved that the point scheme $\mathbf{g}^n = U^n\mathbf{f}$ is convergent. In particular, $\mathbf{p}^n$ converges to a continuous curve φ.

We continue by proving that $\mathbf{v}^n$ converges to a continuous function. By Proposition 2.8

the Taylor scheme $\partial_T S$ exists. We compute

$$\begin{aligned}
&\|\mathcal{F}_{n+1}(2^{n+1}TU^{n+1}\mathbf{f}) - \mathcal{F}_n(2^nTU^n\mathbf{f})\| \leq \\
&\|\mathcal{F}_{n+1}(2^{n+1}TU^{n+1}\mathbf{f}) - \mathcal{F}_{n+1}(2^{n+1}TSU^n\mathbf{f})\| + \|\mathcal{F}_{n+1}(2^{n+1}TSU^n\mathbf{f}) - \mathcal{F}_n(2^nTU^n\mathbf{f})\| \\
&\leq \|2^{n+1}T(U-S)U^n\mathbf{f}\|_\infty + \|\mathcal{F}_{n+1}(\partial_T S(2^nTU^n\mathbf{f})) - \mathcal{F}_n(2^nTU^n\mathbf{f})\| \\
&\leq 2^{n+1}3c_0D_0(U^n\mathbf{f})^2 + c_1\|2^n\Delta_1 TU^n\mathbf{f}\|_\infty \\
&= 2^{n+1}3c_0D_0(U^n\mathbf{f})^2 + c_1D_1(2^nTU^n\mathbf{f}) \\
&\leq 2^{n+1-2\beta_0 n}3c_0D_0(\mathbf{f})^2 + c_1 2^{-\beta_1 n}\Big(D_1(T\mathbf{f}) + P(n)D_0(\mathbf{f})\Big) \\
&\leq 2^{(1-2\beta_0)n}6c_0D_0(\mathbf{f})^2 + c_1 2^{-\beta_1 n}\Big(D_1(T\mathbf{f}) + P(n)D_0(\mathbf{f})\Big) =: b_n.
\end{aligned}$$

In this computation we used $\|T\|_\infty = 3$, the proximity condition eq. (2.25) for S and U and Proposition 2.15. By performing modifications as in the proof of Theorem 2.25, we can achieve that $1 + \mu_1 - 2\beta_0 < 0$ for some $\mu_1 > 0$. This implies that $1 - 2\beta_0 < 0$. Therefore $\sum b_n < \infty$, which shows that $2^nTU^n\mathbf{f}$ is convergent. Since $2^nTU^n\mathbf{f} = 2^nT\mathbf{g}^n = 2^nT\binom{\mathbf{p}^n}{\mathbf{u}^n} = 2^n\binom{\Delta\mathbf{p}^n - \mathbf{u}^n}{\mathbf{u}^n}$, we have proved that $2^n\mathbf{u}^n = \mathbf{v}^n$ converges to a continuous curve ψ.

The rest of the proof is analogous to the proof of Theorem 2.21, that is, inequality eq. (2.32) implies that

- $2^n\Delta\mathbf{p}^n - 2^n\mathbf{u}^n$ converges to 0 and hence $2^n\Delta\mathbf{p}^n$ converges to ψ,
- $2^n\Delta\mathbf{u}^n$ converges to 0 and hence $2^n\Delta^2\mathbf{p}^n$ converges to 0.

Therefore, we have proved that

(i) $\mathbf{p}^n$ converges to a continuous curve φ,

(ii) $\mathbf{v}^n$ and $2^n\Delta\mathbf{p}^n$ converge to a continuous curve ψ,

(iii) $2^n\Delta^2\mathbf{p}^n$ converges to 0.

It follows from Lemma 2.20 that φ is C^1 with $\varphi' = \psi$.

For general m, the proof is analogous. We only have to replace U by U^m, 2 by 2^m, etc. □

2.7 Verification of proximity conditions

In the following, we verify that the proximity condition eq. (2.25) holds between a linear subdivision operator and its nonlinear analogues we constructed in Section 2.3. In particular, this will imply the following result:

Theorem 2.27. *Let M be a Riemannian manifold, Lie group or symmetric space (resp. a submanifold of Euclidean space $\mathbb{R}^n$). Let $S_{\mathbf{A}}$ be a linear subdivision operator that satisfies the spectral condition eq. (2.2), let $\partial_{\Delta_1 T} S_{\mathbf{A}}$ be its derived scheme w.r.t. to $\Delta_1 T$ and let $U_{\mathbf{A}}$ be the Log-exp analogue (resp. the projection analogue) of $S_{\mathbf{A}}$ on TM. If there exists a positive integer N such that $\|(\partial_{\Delta_1 T} S_{\mathbf{A}})^N\|_\infty < 1$, then the Hermite scheme $\mathbf{f}^n = D^{-n} U_{\mathbf{A}}^n \mathbf{f}^0$ is C^1 convergent for dense enough input data $\mathbf{f}^0$.*

Proof. A similar argument as in the proof of Proposition 2.16 shows that $\|(\partial_{\Delta_1 T} S_{\mathbf{A}})^N\|_\infty < 1$ implies the smoothness condition of Definition 2.18. In Sections 2.7.1 and 2.7.2 we will show that the proximity condition holds between $S_{\mathbf{A}}$ and $U_{\mathbf{A}}$. Theorem 2.26 then implies that the Hermite scheme $\mathbf{f}^n$ is C^1 convergent for dense enough input data. □

Note that the result of the theorem is independent of the norm $\|\cdot\|_\infty$. This follows from arguments in Remark 2.23.

Example 2.28. We consider the linear subdivision operator $S_{\mathbf{A}}$ defined in example 2.1. In [60] it is shown that mask $\mathbf{F}$ of $\partial_{\Delta_1 T} S_{\mathbf{A}}$ is given by

$$F(0) = \begin{pmatrix} 1 & -\frac{1}{4} \\ \frac{3}{2} & -\frac{1}{4} \end{pmatrix} \quad \text{and} \quad F(1) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{5}{4} \end{pmatrix}$$

and that $\|\partial_{\Delta_1 T} S_{\mathbf{A}}\|_\infty < 1$. This implies that the Log-exp analogue (resp. the projection analogue) on any Riemannian manifold, Lie group or symmetric space (resp. submanifold of Euclidean space) of $S_{\mathbf{A}}$ is C^1 convergent for dense enough input data. In particular, this includes Example 2.6.

2.7.1 Proximity for the Log-exp analogue

Let M denote a Riemannian manifold, Lie group or symmetric space and $\exp_p$ its exponential map. Using Taylor expansion, in a chart we have

$$\begin{aligned} p \oplus v &= \exp_p(v) = p + v + O(\|v\|^2), \\ q \ominus p &= \exp_p^{-1}(q) = q - p + O(\|q-p\|^2), \end{aligned} \tag{2.34}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$, for $n = \dim M$.

Let $S_{\mathbf{A}}$ be a linear subdivision operator with mask $\mathbf{A} = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right)$ that satisfies the spectral condition eq. (2.2) and let $U_{\mathbf{A}}$ be its Log-exp analogue constructed in Section 2.3.1. As mentioned in Section 2.6, we can restrict the analysis to finite input data $\left(\begin{smallmatrix} \mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right)$. We aim to prove that the proximity condition eq. (2.25) is satisfied in a chart of M:

$$\|(S_{\mathbf{A}} - U_{\mathbf{A}}) \left(\begin{smallmatrix} \mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right)\|_\infty \le c_0 \|\left(\begin{smallmatrix} \Delta\mathbf{p} \\ \mathbf{v} \end{smallmatrix}\right)\|_\infty^2, \tag{2.35}$$

for some constant c_0. In order to prove this, we introduce some notation.

Notation

Let $\mathbf{q} = \{p(\alpha) + v(\alpha) : \alpha \in \mathbb{Z}\}$ and $\mathbf{r} = \{p(\alpha) \oplus v(\alpha) : \alpha \in \mathbb{Z}\}$. Recall that in Section 2.3.1 we introduced $S_{\tilde{\mathbf{A}}}$, which is constructed from $S_{\mathbf{A}}$ by transforming point-vector $\left(\begin{smallmatrix}\mathbf{p}\\\mathbf{v}\end{smallmatrix}\right)$ to point-point data $\left(\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}\right)$:

$$S_{\tilde{\mathbf{A}}}\begin{pmatrix}\mathbf{p}\\\mathbf{q}\end{pmatrix}(\alpha) = \begin{pmatrix}\sum_{\beta\in\mathbb{Z}} \tilde{a}(\alpha-2\beta)p(\beta) + \tilde{b}(\alpha-2\beta)q(\beta)\\ \sum_{\beta\in\mathbb{Z}} \tilde{c}(\alpha-2\beta)p(\beta) + \tilde{d}(\alpha-2\beta)q(\beta)\end{pmatrix},$$

with mask given by $\tilde{A}(\alpha) = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)A(\alpha)\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$ for $\alpha \in \mathbb{Z}$. Furthermore, we introduced the sequences $\mathbf{s}_1, \mathbf{r}_1$ by

$$\begin{pmatrix}s_1(\alpha)\\r_1(\alpha)\end{pmatrix} = \begin{pmatrix}m_0(\alpha) \oplus \sum_{\beta\in\mathbb{Z}} \tilde{a}(\alpha-2\beta)\,(p(\beta)\ominus m_0(\alpha)) + \tilde{b}(\alpha-2\beta)\,(r(\beta)\ominus m_0(\alpha))\\ m_1(\alpha) \oplus \sum_{\beta\in\mathbb{Z}} \tilde{c}(\alpha-2\beta)\,(p(\beta)\ominus m_1(\alpha)) + \tilde{d}(\alpha-2\beta)\,(r(\beta)\ominus m_1(\alpha))\end{pmatrix}$$

in order to define $U_{\mathbf{A}}$ by

$$U_{\mathbf{A}}\begin{pmatrix}\mathbf{p}\\\mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{s}_1\\\mathbf{r}_1\ominus\mathbf{s}_1\end{pmatrix}.$$

Let $\left(\begin{smallmatrix}\mathbf{p^1}\\\mathbf{v^1}\end{smallmatrix}\right) = S_{\mathbf{A}}\left(\begin{smallmatrix}\mathbf{p}\\\mathbf{v}\end{smallmatrix}\right)$. It is easy to see that $S_{\tilde{A}}\left(\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)S_{\mathbf{A}}\left(\begin{smallmatrix}\mathbf{p}\\\mathbf{v}\end{smallmatrix}\right)$, which shows that the first component is also $\mathbf{p^1}$. This justifies the definition $\left(\begin{smallmatrix}\mathbf{p}^1\\\tilde{\mathbf{q}}^1\end{smallmatrix}\right) = S_{\tilde{A}}\left(\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}\right)$ with $\mathbf{v^1} = \tilde{\mathbf{q}}^{\mathbf{1}} - \mathbf{p^1}$. Furthermore, we set and $\left(\begin{smallmatrix}\tilde{\mathbf{p}}^1\\\tilde{\mathbf{r}}^1\end{smallmatrix}\right) = S_{\tilde{A}}\left(\begin{smallmatrix}\mathbf{p}\\\mathbf{r}\end{smallmatrix}\right)$.

What we want to prove

Using the notation we just introduced, the proximity condition eq. (2.35) reads:

$$\begin{aligned}\|\mathbf{p^1}-\mathbf{s_1}\|_\infty &\le c_0\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\\mathbf{v}\end{smallmatrix}\right)\|_\infty^2,\\ \|\mathbf{v^1}-\mathbf{r_1}\ominus\mathbf{s_1}\|_\infty &\le c_0\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\\mathbf{v}\end{smallmatrix}\right)\|_\infty^2.\end{aligned}$$

In order to prove this, we show that every term in the following inequalities is less or equal to $c\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\\mathbf{v}\end{smallmatrix}\right)\|_\infty^2$:

$$\begin{aligned}\|\mathbf{p^1}-\mathbf{s_1}\|_\infty \le& \|\mathbf{p^1}-\tilde{\mathbf{p}}^{\mathbf{1}}\|_\infty + \|\tilde{\mathbf{p}}^{\mathbf{1}}-\mathbf{s_1}\|_\infty,\\ \|\mathbf{v^1}-\mathbf{r_1}\ominus\mathbf{s_1}\|_\infty \le& \|\tilde{\mathbf{q}}^{\mathbf{1}}-\tilde{\mathbf{r}}^{\mathbf{1}}\|_\infty + \|\tilde{\mathbf{p}}^{\mathbf{1}}-\mathbf{p^1}\|_\infty + \|\tilde{\mathbf{r}}^{\mathbf{1}}-\mathbf{r_1}\|_\infty\\ &+ \|\mathbf{s_1}-\tilde{\mathbf{p}}^{\mathbf{1}}\|_\infty + \|\mathbf{r_1}-\mathbf{s_1}-\mathbf{r_1}\ominus\mathbf{s_1}\|_\infty.\end{aligned}$$

A closer look at the above terms shows that in fact we only have to analyze three of them:

(i) $\|\mathbf{p^1}-\tilde{\mathbf{p}}^{\mathbf{1}}\|_\infty$. The term $\|\tilde{\mathbf{q}}^{\mathbf{1}}-\tilde{\mathbf{r}}^{\mathbf{1}}\|_\infty$ can be estimated analogously.

(ii) $\|\tilde{\mathbf{p}}^{\mathbf{1}}-\mathbf{s_1}\|_\infty$. The term $\|\tilde{\mathbf{r}}^{\mathbf{1}}-\mathbf{r_1}\|$ can be estimated analogously.

(iii) $\|\mathbf{r_1}-\mathbf{s_1}-\mathbf{r_1}\ominus\mathbf{s_1}\|_\infty$.

Estimation of (i)

The estimation of this term reduces to estimating $\|\mathbf{q}-\mathbf{r}\|_\infty$. At every index $\alpha \in \mathbb{Z}$ of the sequences $\mathbf{q}$ and $\mathbf{r}$ we have

$$q - r = p + v - p \oplus v \overset{eq.\ (2.34)}{=} p + v - (p + v + O(\|v\|^2)) = O(\|v\|^2),$$

and therefore $\|\mathbf{q}-\mathbf{r}\|_\infty \leq c\|\mathbf{v}\|_\infty^2$.

Estimation of (ii)

In order to estimate $\|\tilde{\mathbf{p}}^{\mathbf{1}} - \mathbf{s_1}\|_\infty$, we introduce some notation. For $\tilde{a}(\alpha - 2\beta)$ write $\tilde{a}_\beta$ and similar with other indices of the form $\alpha - 2\beta$. Furthermore, we generally suppress the index α. Define $u_\beta := p(\beta) \ominus m_0$ and $t_\beta := r(\beta) \ominus m_0$. Note that $p(\beta) = \exp_{m_0}(u_\beta)$ and $r(\beta) = \exp_{m_0}(t_\beta)$. We rewrite $\|\tilde{p}^1 - s_1\|$ (the index α is suppressed) as follows:

$$\begin{aligned}
&\|m_0 + \textstyle\sum_\beta \tilde{a}_\beta(p(\beta) - m_0) + \tilde{b}_\beta(r(\beta) - m_0) \\
&\qquad\qquad -m_0 \oplus \Big(\textstyle\sum_\beta \tilde{a}_\beta(p(\beta) \ominus m_0) + \tilde{b}_\beta(r(\beta) \ominus m_0)\Big)\| \\
&= \|m_0 + \textstyle\sum_\beta \tilde{a}_\beta(\exp_{m_0}(u_\beta) - m_0) + \tilde{b}_\beta(\exp_{m_0}(t_\beta) - m_0) \\
&\qquad\qquad - \exp_{m_0}\Big(\textstyle\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta\Big)\| \\
&= \|\textstyle\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta + O(\|u_\beta\|^2) + O(\|t_\beta\|^2) \\
&\qquad\qquad -\Big(\textstyle\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta + O(\|\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta\|^2)\Big)\| \\
&\leq O(\sup_\beta \|u_\beta\|^2) + O(\sup_\beta \|u_\beta\| \sup_\beta \|t_\beta\|) + O(\sup_\beta \|t_\beta\|^2).
\end{aligned}$$

Define $w := \max\{\sup_\beta \|u_\beta\|, \sup_\beta \|t_\beta\|\}$. Then we have the estimate

$$\|\tilde{p}^1 - s_1\| \leq c_0 w^2. \tag{2.36}$$

By Lemma 2.29 (see below), $w \leq c_0\|\Delta\mathbf{p}\|_\infty + c_1\|\mathbf{v}\|_\infty \leq c_0\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty$, which gives the right estimate.

Estimation of (iii)

We use notation as introduced for the estimation of (ii). The Taylor expansion eq. (2.34) at every index α implies

$$\|r_1 - s_1 - r_1 \ominus s_1\| \leq c_0\|r_1 - p_1\|^2.$$

Define $U_\beta := p(\beta) \ominus m_1, T_\beta := r(\beta) \ominus m_1$ and $W := \max\{\sup_\beta \|U_\beta\|, \sup_\beta \|T_\beta\|\}$. Then we have

$$\begin{aligned}
\|r_1 - s_1\| =& \| \exp_{m_1}\Big(\sum_\beta \tilde{c}_\beta U_\beta + \tilde{d}_\beta T_\beta\Big) - \exp_{m_0}\Big(\sum_\beta \tilde{a}_\beta u_\beta + \tilde{b}_\beta t_\beta\Big)\| \\
\leq& \|m_1 - m_0\| + O(\sup_\beta \|U_\beta\|) + O(\sup_\beta \|T_\beta\|) + O(\sup_\beta \|u_\beta\|) + O(\sup_\beta \|t_\beta\|) \\
\leq& \|m_1 - m_0\| + c_0 W + c_1 w \\
\leq& c_0 \|\Delta \mathbf{p}\|_\infty + c_1 \|\mathbf{v}\|_\infty,
\end{aligned}$$

where w is taken from above and the last inequality follows from Lemma 2.29 below. This shows that $\|\mathbf{r_1} - \mathbf{s_1} - \mathbf{r_1} \ominus \mathbf{s_1}\|_\infty \leq c_0 \|\binom{\Delta \mathbf{p}}{\mathbf{v}}\|_\infty^2$.

Lemma 2.29. *In a chart of M consider finite initial data $\binom{\mathbf{p}}{\mathbf{v}}$ and $\mathbf{r} = \{p(\alpha) \oplus v(\alpha) : \alpha \in \mathbb{Z}\}$. Let $\mathbf{m_0} = \mathbf{p}$ or $\mathbf{m_0} = \{\mathrm{mean}(p(\alpha+1), p(\alpha)) : \alpha \in \mathbb{Z}\}$ and $\mathbf{m_1} = \mathbf{r}$ or $\mathbf{m_1} = \{\mathrm{mean}(r(\alpha+1), r(\alpha)) : \alpha \in \mathbb{Z}\}$. Then there exist constants $c_0, \ldots, c_8 > 0$ such that*

$$\begin{aligned}
\|p(\beta) \ominus m_0(\alpha)\| &\leq c_0 \|\Delta \mathbf{p}\|_\infty, \\
\|r(\beta) \ominus m_0(\alpha)\| &\leq c_1 \|\Delta \mathbf{p}\|_\infty + c_2 \|\mathbf{v}\|_\infty, \\
\|p(\beta) \ominus m_1(\alpha)\| &\leq c_3 \|\Delta \mathbf{p}\|_\infty + c_4 \|\mathbf{v}\|_\infty, \\
\|r(\beta) \ominus m_1(\alpha)\| &\leq c_5 \|\Delta \mathbf{p}\|_\infty + c_6 \|\mathbf{v}\|_\infty, \\
\|m_1(\alpha) - m_0(\alpha)\| &\leq c_7 \|\Delta \mathbf{p}\|_\infty + c_8 \|\mathbf{v}\|_\infty,
\end{aligned}$$

for all $\alpha, \beta \in \mathbb{Z}$.

Proof. In a sufficiently bounded neighborhood any smooth function is Lipschitz, hence $\|F(x) - F(x_0)\| \leq c\|x - x_0\|$. Encode the finite input data $\binom{\mathbf{p}}{\mathbf{v}}$ in a vector $x = \begin{pmatrix} p(i_0) & \ldots & p(i_n) \\ v(i_0) & \ldots & v(i_n) \end{pmatrix}$ and define $x_0 = \begin{pmatrix} p(i_0) & \ldots & p(i_0) \\ 0 & \ldots & 0 \end{pmatrix}$. In order to prove the second inequality, set $F(x) = r(\beta) \ominus m_0(\alpha)$, where the indices α and β appear among $i_0, \ldots, i_n$. Evaluation at x_0 gives $F(x_0) = 0$ and hence by the Lipschitz condition

$$\|r(\beta) \ominus m_0(\alpha)\| = \|F(x)\| \leq c\|x - x_0\| \leq c_1 \|\Delta \mathbf{p}\| + c_2 \|\mathbf{v}\|.$$

The other inequalities are proved analogously. □

2.7.2 Proximity for the projection analogue

Let M be a submanifold of Euclidean space $\mathbb{R}^n$, π a projection map onto M and $S_\mathbf{A}$ be a linear subdivision operator with mask $\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ that satisfies the spectral condition eq. (2.2). We want to prove that the proximity condition eq. (2.35) holds between $S_\mathbf{A}$ and its projection analogue $U_\mathbf{A}$, which is defined by $U_\mathbf{A}\mathbf{f}(\alpha) = d\pi(S_\mathbf{A}\mathbf{f}(\alpha))$.

For $\alpha \in \mathbb{Z}$ and finite input data $\mathbf{f} = \binom{\mathbf{p}}{\mathbf{v}}$, define sequences $\mathbf{p^1}$ and $\mathbf{v^1}$ by

$$\begin{pmatrix} p^1(\alpha) \\ v^1(\alpha) \end{pmatrix} = S_{\mathbf{A}} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} (\alpha) = \begin{pmatrix} \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)p(\beta) + b(\alpha-2\beta)v(\beta) \\ \sum_{\beta\in\mathbb{Z}} c(\alpha-2\beta)p(\beta) + d(\alpha-2\beta)v(\beta) \end{pmatrix}.$$

Hence in order to prove the proximity condition, we have to show the following inequalities

$$\sup_{\alpha\in\mathbb{Z}} \|p^1(\alpha) - \pi(p^1(\alpha))\| \le c_0 \|\binom{\Delta\mathbf{p}}{\mathbf{v}}\|_\infty^2, \tag{2.37}$$

$$\sup_{\alpha\in\mathbb{Z}} \|v^1(\alpha) - d_{p^1(\alpha)}\pi(v^1(\alpha))\| \le c_0 \|\binom{\Delta\mathbf{p}}{\mathbf{v}}\|_\infty^2, \tag{2.38}$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^n$.

Preparation

Before we prove eqs. (2.37) and (2.38), we rewrite the terms $p^1, \pi(p^1), v^1$ and $d_{p^1}\pi(v^1)$ at an index α. The main ingredient is the following Taylor expansion of the projection π:

$$\pi(p+v) = \pi(p) + d_p\pi(v) + O(\|v\|^2), \tag{2.39}$$

where p is a point in M and v is a tangent vector. This implies

$$\begin{aligned} p(\beta) &= \pi(p(\beta)) = \pi(p(\alpha)) + d_{p(\alpha)}\pi(p(\beta)-p(\alpha)) + O(\|p(\beta)-p(\alpha)\|^2), \\ v(\beta) &= d_{p(\beta)}\pi(v(\beta)), \end{aligned}$$

for $\alpha, \beta \in \mathbb{Z}$. Therefore

$$\begin{aligned} p^1(\alpha) &= \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)p(\beta) + b(\alpha-2\beta)v(\beta) \\ &= \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)\Big(\pi(p(\alpha)) + d_{p(\alpha)}\pi(p(\beta)-p(\alpha)) + O(\|p(\beta)-p(\alpha)\|^2)\Big) \\ &\qquad + b(\alpha-2\beta)d_{p(\beta)}\pi(v(\beta)). \end{aligned} \tag{2.40}$$

Using $\sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta) = 1$, which is the reproduction of constants property eqs. (2.3) and (2.39) we have

$$\begin{aligned} \pi(p^1(\alpha)) &= \pi\Big(p(\alpha) + \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)(p(\beta)-p(\alpha)) + b(\alpha-2\beta)v(\beta)\Big) \\ &= \pi(p(\alpha)) + \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)d_{p(\alpha)}\pi(p(\beta)-p(\alpha)) + b(\alpha-2\beta)d_{p(\alpha)}\pi(v(\beta)) \\ &\qquad + O(\sup_\beta \|p(\beta)-p(\alpha)\|^2) + O(\sup_\beta \|p(\beta)-p(\alpha)\| \sup_\beta \|v(\beta)\|) + O(\sup_\beta \|v(\beta)\|^2) \\ &= \sum_{\beta\in\mathbb{Z}} a(\alpha-2\beta)\Big(\pi(p(\alpha)) + d_{p(\alpha)}\pi(p(\beta)-p(\alpha))\Big) + b(\alpha-2\beta)d_{p(\alpha)}\pi(v(\beta)) \\ &\qquad + c_1\|\Delta\mathbf{p}\|_\infty^2 + c_2\|\Delta\mathbf{p}\|_\infty\|\mathbf{v}\|_\infty + c_3\|\mathbf{v}\|_\infty^2. \end{aligned} \tag{2.41}$$

Note that $\sup \|p(\beta) - p(\alpha)\| = O(\|\Delta \mathbf{p}\|_\infty)$ since p is finite. Using $\sum_\beta c(\alpha - 2\beta) = 0$, which is the reproduction of constants property eq. (2.3), we have

$$\begin{aligned} v^1(\alpha) &= \sum_{\beta \in \mathbb{Z}} c(\alpha - 2\beta) p(\beta) + d(\alpha - 2\beta) v(\beta) \\ &= \sum_{\beta \in \mathbb{Z}} c(\alpha - 2\beta) \Big(p(\beta) - p^1(\alpha) \Big) + d(\alpha - 2\beta) d_{p(\beta)} \pi(v(\beta)). \end{aligned} \tag{2.42}$$

Furthermore

$$d_{p^1(\alpha)} \pi(v^1(\alpha)) = \sum_{\beta \in \mathbb{Z}} c(\alpha - 2\beta) d_{p^1(\alpha)} \Big(p(\beta) - p^1(\alpha) \Big) + d(\alpha - 2\beta) d_{p^1(\alpha)} \pi(v(\beta)). \tag{2.43}$$

Proving the proximity condition

The first part of proximity eq. (2.37) can be estimated using eq. (2.40) and eq. (2.41):

$$\begin{aligned} \|p^1(\alpha) - \pi(p^1(\alpha))\| \leq & c_0 \sup_\beta \|d_{p(\beta)} \pi(v(\beta)) - d_{p(\alpha)} \pi(v(\beta))\| + c_1 \|\Delta \mathbf{p}\|_\infty^2 \\ & + c_2 \|\Delta \mathbf{p}\|_\infty \|\mathbf{v}\|_\infty + c_3 \|\mathbf{v}\|_\infty^2. \end{aligned}$$

Similarly, the second part of the proximity condition eq. (2.38) can be estimated using eq. (2.42) and eq. (2.43):

$$\begin{aligned} \|v^1(\alpha) - d_{p^1(\alpha)} \pi(v^1(\alpha))\| \leq & c_0 \sup_\beta \|(p(\beta) - p^1(\alpha)) - d_{p^1(\alpha)} \pi(p(\beta) - p^1(\alpha))\| \\ & + c_1 \sup_\beta \|d_{p(\beta)} \pi(v(\beta)) - d_{p^1(\alpha)} \pi(v(\beta))\|. \end{aligned}$$

Therefore, in order to prove the proximity condition, we have to show that the following terms are bounded by quadratic terms in $\|\Delta \mathbf{p}\|_\infty$ and $\|\mathbf{v}\|_\infty$:

(i) $\sup_\beta \|d_{p(\beta)} \pi(v(\beta)) - d_{p(\alpha)} \pi(v(\beta))\|$,

(ii) $\sup_\beta \|d_{p(\beta)} \pi(v(\beta)) - d_{p^1(\alpha)} \pi(v(\beta))\|$,

(iii) $\sup_\beta \|(p(\beta) - p^1(\alpha)) - d_{p^1(\alpha)} \pi(p(\beta) - p^1(\alpha))\|$.

This is done below.

Proof of (i)-(iii)

For fixed y consider the map $\varphi : x \mapsto d_x\pi(y)$. Using Taylor expansion one can conclude that $\varphi(z) - \varphi(x) = \varphi(x + (z - x)) - \varphi(x) = d_x\varphi(z - x) + O(\|z - x\|^2)$. Hence

$$d_z\pi(y) - d_x\pi(y) = d_x^2\pi(y, z - x) + O(\|z - x\|^2).$$

With $z = p(\beta), y = v(\beta)$ and $x = p(\alpha)$, we have

$$\begin{aligned}\|d_{p(\beta)}\pi(v(\beta)) - d_{p(\alpha)}\pi(v(\beta))\| &\leq \|d^2_{p(\alpha)}\pi(v(\beta), p(\beta) - p(\alpha))\| + O(\|p(\beta) - p(\alpha)\|^2) \\ &\leq c_0\|v(\beta)\|\|p(\beta) - p(\alpha)\| + O(\|p(\beta) - p(\alpha)\|^2).\end{aligned} \tag{2.44}$$

The last inequality follows from the fact that in a sufficiently small neighborhood there exists a uniform constant c_0 such that

$$\|d^2_{p(\alpha)}\pi(v(\beta), p(\beta) - p(\alpha))\| \leq c_0\|v(\beta)\|\|p(\beta) - p(\alpha)\|.$$

Taking the supremum over inequality eq. (2.44) proves (i). In particular this proves eq. (2.37). The second inequality (ii) can be proved analogously. Note that $\|p(\beta) - p^1(\alpha)\| \leq c_0\|\Delta\mathbf{p}\|_\infty + c_1\|\mathbf{v}\|_\infty$. For part (iii) we have

$$\begin{aligned}&\|p(\beta) - p^1(\alpha) - d_{p^1(\alpha)}\pi(p(\beta) - p^1(\alpha))\| \\ &\leq \|\pi(p(\beta)) - \pi(p^1(\alpha)) - d_{p^1(\alpha)}\pi(p(\beta) - p^1(\alpha))\| + \|\pi(p^1(\alpha)) - p^1(\alpha)\| \\ &\leq \|\pi(p^1(\alpha) + (p(\beta) - p^1(\alpha))) - \pi(p^1(\alpha)) - d_{p^1(\alpha)}\pi(p(\beta) - p^1(\alpha))\| + \|\pi(p^1(\alpha)) - p^1(\alpha)\| \\ &\leq O(\|p(\beta) - p^1(\alpha)\|^2) + \|\pi(p^1(\alpha)) - p^1(\alpha)\|,\end{aligned}$$

where the last inequality follows from eq. (2.39). The bounds on (iii) follow from eq. (2.37). This proves eq. (2.38) and concludes the proof of Theorem 2.27.

Conclusion

We have studied two natural nonlinear analogues of Hermite subdivision schemes: One is an intrinsic construction on manifolds equipped with an exponential mapping, the other one applies to embedded manifolds (i.e., surfaces) and uses projections. Furthermore, a full C^1 analysis for Hermite schemes was presented which allows to deduce C^1 convergence from the verification of a proximity condition. This is used to prove the result stated in Theorem 2.27: Both nonlinear analogues of a linear Hermite scheme are C^1 convergent, if the derived scheme is appropriately bounded and the input data is dense enough.

Generalizations

We would like to comment on generalizations of the results presented in this paper:

It would be natural to consider sequences with more than 2 components, with the $(k+1)$-st component representing k-th derivatives. Judging from the behavior of point-subdivision in manifolds we expect that the case $k = 2$ is similar to the case $k = 1$, setting aside the technicalities of the jet bundle, but there is a big difference between $k \leq 2$ and $k > 2$, see [43, 85].

In this paper we are concerned with the analysis of stationary Hermite schemes, i.e. schemes, whose mask $\mathbf{A}$ (except for the dilation matrix D) is independent of the subdivision level. Recent results [9] provide a factorization of non-stationary Hermite subdivision operators generalizing the Taylor factorization. We believe that this allows to extend our results to the non-stationary setting. Non-stationary masks are a topic of future research.

Acknowledgments. This research is supported by the Austrian Science Fund under grant No. W1230. The author would like to thank Johannes Wallner for helpful discussions and gratefully acknowledges the suggestions and remarks of the anonymous reviewers.

3 Hermite subdivision on manifolds via parallel transport

This chapter comprises the paper (II).

Abstract. We propose a new adaption of linear Hermite subdivision schemes to the manifold setting. Our construction is intrinsic, as it is based solely on geodesics and on the parallel transport operator of the manifold. The resulting nonlinear Hermite subdivision schemes are analyzed with respect to convergence and C^1 smoothness. Similar to previous work on manifold-valued subdivision, this analysis is carried out by proving that a so-called proximity condition is fulfilled. This condition allows to conclude convergence and smoothness properties of the manifold-valued scheme from its linear counterpart, provided that the input data are dense enough. Therefore the main part of this paper is concerned with showing that our nonlinear Hermite scheme is "close enough", i.e., in proximity, to the linear scheme it is derived from.

Keywords. Hermite subdivision · manifold subdivision · C^1 analysis · proximity

AMS Subject Classification. 41A25 · 65D17 · 53A99

3.1 Introduction

Hermite subdivision is an iterative method for constructing a curve together with its derivatives from discrete point-vector data. It has mainly been studied in the linear setting, where many results concerning convergence and smoothness are available, such as [30, 31, 23, 22, 25, 60] and others.

In a recent paper [64] we propose an analogue of linear Hermite schemes in manifolds which are equipped with an exponential map. This construction works via conversion of vector data to point data, and makes use of the well-established methods of non-Hermite subdivision in manifold, see [43] for an overview. The present paper investigates manifold analogues of Hermite subdivision rules which work directly with vectors and employ the parallel transport operators available in Riemannian manifolds and also in Lie groups. In this way subdivision works directly on Hermite data in an intrinsic way.

The C^1 convergence analysis of the nonlinear schemes we obtain by the parallel transport approach is provided from their linear counterparts by means of a proximity condition for Hermite schemes introduced by [64]. This condition allows to conclude

C^1 convergence of the manifold-valued scheme if it is "close enough" to a C^1 convergent linear one. Similar to most previous results on manifold subdivision, C^1 convergence can only be deduced if the input data are dense enough.

The paper is organized as follows: In Section 3.2 we recapitulate Hermite subdivision on both linear spaces and manifolds. Section 3.3 discusses parallel transport and geodesics, which we use in Section 3.4 to define the parallel transport analogue of a linear Hermite scheme. The main part of this paper is concerned with proving that the proximity condition holds between the parallel transport Hermite scheme and the linear scheme it is derived from (Section 3.5). The results are then stated in Section 3.6.

Throughout this paper we use as an instructive example a certain non-interpolatory Hermite scheme which is the de Rham transform [24] of a scheme proposed by [58].

3.2 Hermite subdivision: Basic concepts

In this section we recall some known facts about linear Hermite subdivision and introduce a generalized concept of Hermite subdivision for manifold-valued data.

3.2.1 Linear Hermite subdivision

The data to be refined by a linear Hermite subdivision scheme consists of a point-vector sequence, where we assume that both point and vector sequence take values in the same finite dimensional vector space V. The space of all such point-vector sequences is denoted by $\ell(V^2)$, and an element of this space is written as $\binom{\mathbf{p}}{\mathbf{v}} = \{\binom{p_i}{v_i} : i \in \mathbb{Z}\}$.

A *linear subdivision operator* $S_{\mathbf{A}}$ is a map $\ell(V^2) \to \ell(V^2)$, which is defined by

$$S_{\mathbf{A}}\binom{\mathbf{p}}{\mathbf{v}}_i = \sum_{j\in\mathbb{Z}} A_{i-2j}\binom{p_j}{v_j}, \qquad i \in \mathbb{Z}, \quad \binom{\mathbf{p}}{\mathbf{v}} \in \ell(V^2), \tag{3.1}$$

where the finitely supported sequence $\mathbf{A} \in \ell(L(V)^{2\times 2})$ is called *mask*.

A *linear Hermite subdivision scheme* is the procedure of constructing $\binom{\mathbf{p}^1}{\mathbf{v}^1}, \binom{\mathbf{p}^2}{\mathbf{v}^2}, \ldots$ from input data $\binom{\mathbf{p}^0}{\mathbf{v}^0} \in \ell(V^2)$ by the rule

$$D^n\binom{\mathbf{p}^n}{\mathbf{v}^n} = S_{\mathbf{A}}^n\binom{\mathbf{p}^0}{\mathbf{v}^0},$$

where $D \in L(V)^{2\times 2}$ is the block-diagonal dilation operator

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Here a constant c is to be understood as $c\,\mathrm{id}_V$.

A linear Hermite subdivision operator or scheme is called *interpolatory* if its mask satisfies $A_0 = D$ and $A_{2i} = 0$ for all $i \in \mathbb{Z}\backslash 0$.

We always assume a linear Hermite scheme to satisfy the *spectral condition*, which is a useful assumption for the analysis of linear Hermite schemes [23, 22, 25, 60]. We require that up to a parameter shift the subdivision operator reproduces a degree 1 polynomial and its derivative

$$\begin{pmatrix} v + iw \\ w \end{pmatrix}_{i\in\mathbb{Z}} \quad \text{for } v, w \in V.$$

To be precise, we require that there exists $\varphi \in \mathbb{R}$ such that

$$S_{\mathbf{A}} \begin{pmatrix} v + (i+\varphi)w \\ w \end{pmatrix}_{i\in\mathbb{Z}} = \begin{pmatrix} v + \frac{i+\varphi}{2} w \\ \frac{1}{2} w \end{pmatrix}_{i\in\mathbb{Z}},$$

for all $v, w \in V$. This condition is equivalent to the requirement that there exists $\varphi \in \mathbb{R}$ such that the constant sequence $\mathbf{c} = \{\binom{v}{0} : i \in \mathbb{Z}\}$ and the linear sequence $\boldsymbol{\ell} = \{\binom{(i+\varphi)v}{v} : i \in \mathbb{Z}\}$ for $v \in V$ respectively obey the rules

$$S_{\mathbf{A}}\mathbf{c} = \mathbf{c} \quad \text{and} \quad S_{\mathbf{A}}\boldsymbol{\ell} = \frac{1}{2}\boldsymbol{\ell}. \tag{3.2}$$

The spectral condition can also be expressed by means of the mask $\mathbf{A} = \left(\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right)$. It is equivalent to

$$\sum_{j\in\mathbb{Z}} a_{i-2j} = 1, \quad \sum_{j\in\mathbb{Z}} c_{i-2j} = 0, \tag{3.3}$$

$$\sum_{j\in\mathbb{Z}} a_{i-2j}j + b_{i-2j} = \frac{1}{2}(i - \varphi), \quad \sum_{j\in\mathbb{Z}} c_{i-2j}j + d_{i-2j} = \frac{1}{2}, \tag{3.4}$$

for all $i \in \mathbb{Z}$ and some $\varphi \in \mathbb{R}$, which indicates the parameter transform. Equation (3.3) is equivalent to the reproduction of constants, whereas (3.4) expresses the reproduction of linear functions.

3.2.2 Hermite subdivision on manifolds

We would like to consider Hermite subdivision in the more general setting of manifolds. In this context, tangent vectors serve as point-vector input data for Hermite subdivision. Therefore, the input data is sampled from the tangent bundle $TM = \bigcup_{x\in M} T_xM$ of a manifold M. Its associated sequence space is denoted by $\ell(TM)$. In order to retain the analogy to the linear case, an element of $\ell(TM)$ is written as a pair $\binom{\mathbf{p}}{\mathbf{v}}$ consisting of an M-valued point sequence $\mathbf{p}$ and a vector sequence $\mathbf{v}$ which takes values in the appropriate tangent space, i.e., $v_i \in T_{p_i}M$ for $i \in \mathbb{Z}$.

A *subdivision operator* U on TM is a map that takes arguments in $\ell(TM)$ and produces again a point-vector sequence. It must satisfy

(i) $L^2U = UL$, where L is the left shift operator, and

(ii) U has compact support, that is, there exists $N \in \mathbb{N}$ such that both $U\binom{\mathbf{p}}{\mathbf{v}}_{2i}$ and $U\binom{\mathbf{p}}{\mathbf{v}}_{2i+1}$ only depend on $\binom{p_{i-N}}{v_{i-N}}, \dots, \binom{p_{i+N}}{v_{i+N}}$ for all $i \in \mathbb{Z}$ and sequences $\binom{\mathbf{p}}{\mathbf{v}}$.

Let $D : \ell(TM) \to \ell(TM)$ be the dilation operator

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{p} \\ \frac{1}{2}\mathbf{v} \end{pmatrix},$$

which is an analogue of the block-diagonal operator D defined in Section 3.2.1.

An *Hermite subdivision scheme* is the procedure of constructing $\binom{\mathbf{p}^1}{\mathbf{v}^1}, \binom{\mathbf{p}^2}{\mathbf{v}^2}, \dots$ from input data $\binom{\mathbf{p}^0}{\mathbf{v}^0} \in \ell(TM)$ by the rule

$$D^n \binom{\mathbf{p}^n}{\mathbf{v}^n} = U^n \binom{\mathbf{p}^0}{\mathbf{v}^0}.$$

An Hermite subdivision operator or scheme is called *interpolatory* if $U\binom{\mathbf{p}}{\mathbf{v}}_{2i} = D\binom{\mathbf{p}}{\mathbf{v}}_i$ for all $\binom{\mathbf{p}}{\mathbf{v}}$ and $i \in \mathbb{Z}$.

Note that these definitions are direct generalizations of the concepts introduced in Section 3.2.1: Every linear subdivision operator satisfies conditions (i) and (ii) from above. If U is linear then the definition of (interpolatory) Hermite subdivision scheme is equivalent to the one given in Section 3.2.1.

3.2.3 Different types of norms

Since we need a variety of norms in the following sections, we summarize all of them here.

The notation $\|v\|$, where v is an element of $V = \mathbb{R}^n$, means that we use the Euclidean norm. On matrix groups we use the Frobenius norm $\|g\|^2 = \operatorname{trace}(gg^T)$, which corresponds to the Euclidean norm, if the matrix entries are put into a column vector. From this norm on V we induce the Euclidean norm $\|\binom{v_0}{v_1}\| = (\|v_0\|^2 + \|v_1\|^2)^{\frac{1}{2}}$ on V^2. On the space $L(V)^{2\times 2}$ we use the operator norm

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \sup \left\{ \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right\|, \text{ where } \left\| \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right\| = 1 \right\},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L(V)^{2\times 2}$ and $\binom{v_0}{v_1} \in V^2$. We equip the space of sequences $\ell(V^2)$ with the norm

$$\left\| \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} \right\|_\infty = \sup_{i \in \mathbb{Z}} \left\| \begin{pmatrix} p_i \\ v_i \end{pmatrix} \right\|$$

and denote by $\ell^\infty(V^2)$ the space of all sequences which are bounded with respect to this norm. Similarly we define a norm for $\mathbf{A} \in \ell(L(V)^{2\times 2})$:

$$\|\mathbf{A}\|_\infty = \sup_{i \in \mathbb{Z}} \|A_i\|$$

and denote by $\ell^\infty(L(V)^{2\times 2})$ the space of bounded sequences.

A linear subdivision operator $S_{\mathbf{A}}$ as defined in (3.1) restricts to an operator $\ell^\infty(V^2) \to \ell^\infty(V^2)$. This follows from $\|S_{\mathbf{A}}\binom{\mathbf{p}}{\mathbf{v}}\|_\infty \leq d\|\mathbf{A}\|_\infty \|\binom{\mathbf{p}}{\mathbf{v}}\|_\infty$, where d is a positive integer such that the support of $\mathbf{A}$ is contained in $[-d, d]$. Therefore $S_{\mathbf{A}}$ has an induced operator norm, which we denote by $\|S_{\mathbf{A}}\|_\infty$.

We mention that for the proofs of Section 3.5, the particular choices of the norms on V and V^2 are not important. We will only need the Euclidean norm in Example 3.9. What we will use, however, are the following facts concerning the equivalence of norms: Since in every finite dimensional vector space, any two norms are equivalent, the Euclidean norm $\|\binom{v_0}{v_1}\|$ on V^2 is equivalent to $\|\binom{v_0}{v_1}\|' = \max\{\|v_0\|, \|v_1\|\}$. That is, there exist constants $c_1, c_1 > 0$ such that

$$c_1\|\binom{v_0}{v_1}\|' \leq \|\binom{v_0}{v_1}\| \leq c_2\|\binom{v_0}{v_1}\|'.$$

It follows immediately that also the norms $\|\binom{\mathbf{p}}{\mathbf{v}}\|'_\infty = \sup_i \|\binom{p_i}{v_i}\|'$ and $\|\binom{\mathbf{p}}{\mathbf{v}}\|_\infty$ on $\ell(V^2)$ are equivalent with the same constants:

$$c_1\|\binom{\mathbf{p}}{\mathbf{v}}\|'_\infty \leq \|\binom{\mathbf{p}}{\mathbf{v}}\|_\infty \leq c_2\|\binom{\mathbf{p}}{\mathbf{v}}\|'_\infty. \tag{3.5}$$

3.2.4 C^1 convergence

To a sequence $\mathbf{p}^n$ of points in a vector space we associate a curve $\mathcal{F}_n(\mathbf{p}^n)$, which is the piecewise linear interpolant of $\mathbf{p}^n$ on the grid $2^{-n}\mathbb{Z}$.

We say that a point-vector sequence $\binom{\mathbf{p}^n}{\mathbf{v}^n}$ is C^1 convergent, if $\mathcal{F}_n(\mathbf{p}^n)$ resp. $\mathcal{F}_n(\mathbf{v}^n)$ converge uniformly on compact intervals to a continuously differentiable curve resp. its derivative. If the point-vector sequence is manifold-valued, then we require that the above is true in a chart.

An Hermite scheme defined by the subdivision operator U is said to be C^1 convergent, if the point-vector sequence $\binom{\mathbf{p}^n}{\mathbf{v}^n}$ constructed via $D^n\binom{\mathbf{p}^n}{\mathbf{v}^n} = U^n\binom{\mathbf{p}^0}{\mathbf{v}^0}$ is C^1 convergent. Due to the compact support of a subdivision operator U, the limit curve on compact intervals depends only on finitely many points of the initial data. It is therefore sufficient to consider finite input data and we can assume that the input data are bounded. Thus we have the following formal definition of C^1 convergence:

Definition 3.1. An Hermite subdivision scheme is C^1 convergent if for all input data $\binom{\mathbf{p}^0}{\mathbf{v}^0} \in \ell^\infty(V^2)$ there exists a continuously differentiable curve $\varphi \in C^1(\mathbb{R}, V)$ such that the point-vector sequence $\binom{\mathbf{p}^n}{\mathbf{v}^n}$ constructed via $D^n\binom{\mathbf{p}^n}{\mathbf{v}^n} = U^n\binom{\mathbf{p}^0}{\mathbf{v}^0}$ satisfies

$$\sup_{i\in\mathbb{Z}} \|p_i^n - \varphi\Big(\frac{i}{2^n}\Big)\| \to 0 \quad \text{and} \quad \sup_{i\in\mathbb{Z}} \|v_i^n - \varphi'\Big(\frac{i}{2^n}\Big)\| \to 0 \quad \text{as} \quad n \to \infty.$$

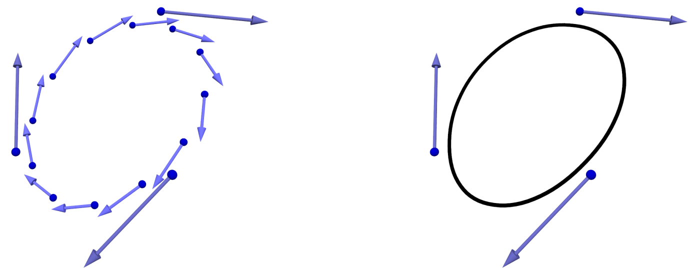

Figure 3.1: The linear non-interpolatory Hermite subdivision scheme of Example 3.2. *Left*: Input data and second iteration step. *Right*: Input data and limit curve.

Example 3.2. We consider the de Rham transform [24] of one of the interpolatory linear Hermite schemes introduced by [58]. It is a non-interpolatory scheme with mask

$$A_{-2} = \frac{1}{8}\begin{pmatrix} \frac{48}{25} & -\frac{29}{25} \\ \frac{29}{50} & \frac{13}{20} \end{pmatrix}, \quad A_{-1} = \frac{1}{8}\begin{pmatrix} \frac{152}{25} & -\frac{31}{25} \\ \frac{29}{50} & \frac{277}{100} \end{pmatrix},$$

$$A_{0} = \frac{1}{8}\begin{pmatrix} \frac{152}{25} & \frac{31}{25} \\ -\frac{29}{50} & \frac{277}{100} \end{pmatrix}, \quad A_{1} = \frac{1}{8}\begin{pmatrix} \frac{48}{25} & \frac{29}{25} \\ -\frac{29}{50} & \frac{13}{20} \end{pmatrix}.$$

In [24] it is shown that the spectral condition is satisfied and that this scheme is C^1 convergent.

3.3 Parallel transport and geodesics

Using parallel transport and geodesics, we are going to adapt linear Hermite subdivision to work on manifold data. We here discuss these concepts for submanifolds of Euclidean space (i.e., surfaces) and for matrix groups, even though they belong to the more general classes of Riemannian manifolds resp. Lie groups. The reason is that we first prove convergence and smoothness results in the special cases of surfaces and matrix groups (Section 3.5). In Theorem 3.7 we then generalize all our results to Riemannian manifolds and Lie groups.

3.3.1 Surfaces

On a surface M in $\mathbb{R}^n$ we consider vector fields $V(t)$ along a curve $g(t)$, i.e., we require that $V(t) \in T_{g(t)}M$ for all t. We say that such a vector field V is *parallel along g* if its derivative is orthogonal to M. Equivalently, the projection of $\dot{V}$ to the tangent space $T_{g(t)}M$ vanishes for all t, i.e.

$$\frac{DV}{dt} := (\dot{V})^{\text{tang}} = 0. \tag{3.6}$$

Therefore, parallel vector fields are the solutions of the linear differential equation (3.6).

Let the curve g connect the points p and m on M, i.e., $g(0) = p$ and $g(1) = m$. The *parallel transport along* g, denoted by $\mathrm{P}_p^m : T_pM \to T_mM$, is defined as follows: $\mathrm{P}_p^m(v)$ means $V(1)$, where V is the parallel vector field along g with initial value $V(0) = v$.

Parallel transport along g satisfies

$$\mathrm{P}_q^m \circ \mathrm{P}_p^q = \mathrm{P}_p^m, \tag{3.7}$$

where q is any point on the curve. Furthermore, it is an isometry, that is $\|\mathrm{P}_p^m(v)\| = \|v\|$. This is not difficult to show: For two vector fields V, W along g a product rule holds:

$$\frac{d}{dt}\Big\langle V, W\Big\rangle = \Big\langle \frac{DV}{dt}, W\Big\rangle + \Big\langle V, \frac{DW}{dt}\Big\rangle. \tag{3.8}$$

If V is parallel along g, then (3.8) implies that $\frac{d}{dt}\langle V(t), V(t)\rangle = 0$, i.e., $\|V(t)\|$ is constant for all t. So P_p^m is an isometry.

In addition to parallel transport, we need the concept of geodesics. A *geodesic* is a curve g on M such that $\dot{g}$ is parallel along g, i.e., a curve which satisfies the differential equation

$$\frac{D\dot{g}}{dt} = 0.$$

It is useful to express geodesics by means of the *exponential mapping*, which is defined as follows: $\exp_p(v)$ means $g(1)$, where g is the geodesic starting at the point p with tangent vector v. A geodesic g can then be written as $g(t) = \exp_p(tv)$.

We mention that $\frac{D}{dt}$, parallel transport, geodesics and exponential mapping are actually concepts of Riemannian geometry. Here they are described only for the special case of surfaces in Euclidean space. For details we refer to textbooks on differential geometry, e.g. [19].

3.3.2 Matrix groups

This section discusses parallel transport and geodesics in matrix groups, i.e., subgroups of $\mathrm{GL}(n, \mathbb{R})$.

We use the matrix exponential function $\exp(v) = \sum_{k=0}^{\infty} \frac{1}{k!} v^k$ to define an *exponential mapping* by $\exp_p(v) = p \exp(p^{-1}v)$. Then a *geodesic*[1] g starting at the point p and tangent vector v is defined by

$$g(t) = \exp_p(tv). \tag{3.9}$$

The curve $g(t)$ is a left translate of the 1-parameter subgroup $\exp(tp^{-1}v)$, and it is also a right translate, since $p\exp(p^{-1}v) = \exp(vp^{-1})p$. We define three different parallel transports ${}^{+}\mathrm{P}_p^m$, ${}^{-}\mathrm{P}_p^m$ and ${}^{0}\mathrm{P}_p^m$ on G, which are mappings of T_pG to T_mG. The first two are given by left resp. right multiplication, that is

$${}^{+}\mathrm{P}_p^m(v) = mp^{-1}v \quad \text{and} \quad {}^{-}\mathrm{P}_p^m(v) = vp^{-1}m.$$

Let $g(t) = \exp_p(tv)$ be the geodesic connecting p and m. Denote by $\mu_{p,m}$ the geodesic midpoint of p and m, i.e., $\mu_{p,m} = g(\frac{1}{2})$. Then the third kind of parallel transport is defined by

$${}^{0}\mathrm{P}_p^m(v) = \mu_{p,m}p^{-1}vp^{-1}\mu_{p,m}. \tag{3.10}$$

Therefore, as in the Riemannian case, an exponential mapping, geodesics and parallel transport can be defined in matrix groups.[2]

3.3.3 Unified notation

The following sections treat surfaces and matrix groups simultaneously. Therefore we introduce a unified notation.

M means either a surface or a matrix group. The exponential mapping of M is denoted by $\exp_p(v)$. In the surface case, P_p^m denotes the parallel transport along the *geodesic* connecting p and m. If M is a matrix group, P_p^m refers to one of the parallel transports introduced in Section 3.3.2.

Following [79], we introduce the symbols $\oplus$ and $\ominus$ which are analogues of point-vector addition and difference. For $p, q \in M$ and $v \in T_pM$, let

$$p \oplus v = \exp_p(v) \quad \text{and} \quad q \ominus p = \exp_p^{-1}(q). \tag{3.11}$$

Note that in the matrix group case the $\oplus$ and $\ominus$ operations are invariant w.r.t. both left and right multiplication.

[1] We call these curves geodesics to emphasize the analogy to the Riemannian case. Note that in the group case we define geodesics via the exponential map, but in the Riemannian case, we define the exponential map via geodesics.

[2] In fact a more general statement is true, which also gives a connection to the case of Riemannian manifolds: On Lie groups, three operators ${}^{+}\frac{D}{dt}$, ${}^{-}\frac{D}{dt}$ and ${}^{0}\frac{D}{dt}$ can be defined, which map a vector field along a curve to another vector field along the same curve. They all define the same geodesics, namely (3.9) and induce the three parallel transports from above. While ${}^{+}\mathrm{P}_p^m$ and ${}^{-}\mathrm{P}_p^m$ are independent of the curve connecting p and m, Definition (3.10) is only valid if the curve under consideration is the geodesic connecting p and m. For details see e.g. [69]. Furthermore, if the group G carries a bi-invariant metric, then the Riemannian covariant derivative $\frac{D}{dt}$ on G coincides with ${}^{0}\frac{D}{dt}$ [56, Chapter X].

While $\oplus$ is always smooth and often globally defined (this is the case in both matrix groups and complete surfaces [65, 54]), $\ominus$ is in general only smooth in some neighborhood of p. Our results in Section 3.5 are based on [64], which only considers "dense enough" input data. We therefore assume that $\ominus$ is always smooth. As in the matrix group case, we define the midpoint of two points p, q on M: If g is the geodesic connecting p and q, then

$$\mu_{p,q} = g\Big(\frac{1}{2}\Big) = p \oplus \frac{1}{2}\Big(q \ominus p\Big).$$

3.4 Hermite subdivision on manifolds via parallel transport

Starting with a linear Hermite subdivision operator $S_{\mathbf{A}}$ satisfying the spectral condition (3.2), we define a subdivision operator U in a surface or a matrix group M.

Recall that we can write $S_{\mathbf{A}}$ in the form

$$S_{\mathbf{A}} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix}_i = \sum_{j\in\mathbb{Z}} \begin{pmatrix} a_{i-2j} & b_{i-2j} \\ c_{i-2j} & d_{i-2j} \end{pmatrix} \begin{pmatrix} p_j \\ v_j \end{pmatrix} = \begin{pmatrix} \sum_{j\in\mathbb{Z}} a_{i-2j}p_j + b_{i-2j}v_j \\ \sum_{j\in\mathbb{Z}} c_{i-2j}p_j + d_{i-2j}v_j \end{pmatrix}. \tag{3.12}$$

The reproduction of constants (3.3) is characterized by the conditions $\sum_{j\in\mathbb{Z}} a_{i-2j} = 1$ and $\sum_{j\in\mathbb{Z}} c_{i-2j} = 0$. This allows us to rewrite (3.12) as

$$S_{\mathbf{A}} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix}_i = \begin{pmatrix} m_i + \sum_{j\in\mathbb{Z}} a_{i-2j}(p_j - m_i) + b_{i-2j}v_j \\ \sum_{j\in\mathbb{Z}} c_{i-2j}(p_j - m_i) + d_{i-2j}v_j \end{pmatrix}, \tag{3.13}$$

for any base point sequence $\mathbf{m}$. We use (3.13) to define a subdivision operator U that takes arguments in $\ell(TM)$.

Consider input data $\binom{\mathbf{p}}{\mathbf{v}} \in \ell(TM)$. For the base point sequence $\mathbf{m} \in \ell(M)$ we either choose

$$m_i = p_i \quad \text{or} \quad m_i = \mu_{p_i,p_{i+1}} \quad \text{for } i \in \mathbb{Z}.$$

In [79] these base point sequences have been used for the C^1 and C^2 analysis of manifold-valued subdivision rules. It was shown in [43, 85], however, that base point sequences have to be chosen in a more sophisticated manner if one wants to obtain higher smoothness results.

Based on (3.13) we now define the subdivision operator U for manifold-valued data:

$$U \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix}_i = \begin{pmatrix} r_i \\ \mathrm{P}_{m_i}^{r_i}(w_i) \end{pmatrix}, \tag{3.14}$$

$$\text{where } \begin{cases} r_i = m_i \oplus \sum_{j\in\mathbb{Z}} a_{i-2j}(p_j \ominus m_i) + b_{i-2j}\mathrm{P}_{p_j}^{m_i}(v_j), \\ w_i = \sum_{j\in\mathbb{Z}} c_{i-2j}(p_j \ominus m_i) + d_{i-2j}\mathrm{P}_{p_j}^{m_i}(v_j). \end{cases}$$

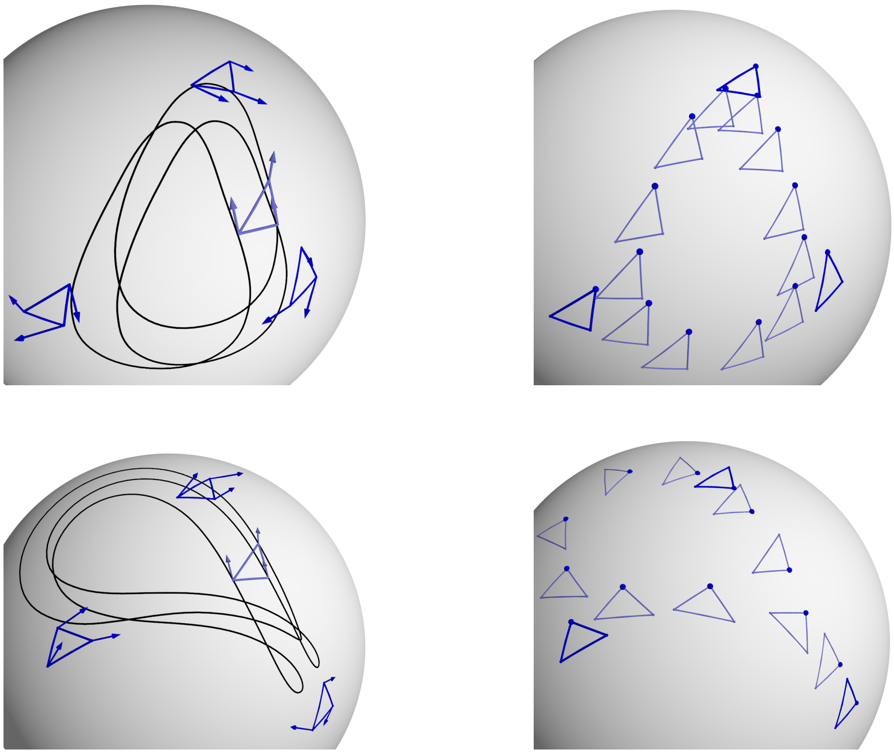

Figure 3.2: The SO(3)-valued Hermite subdivision scheme of Example 3.3 with respect to the (0) parallel transport. Input data are represented by spherical triangles. *Upper and lower left figures:* Limit curves of point-vector input data and one triangle of the second iteration step. *Upper and lower right figures:* second iteration step (tangent vectors are omitted).

In Section 3.6 we show that the successively generated data $\binom{\mathbf{p}}{\mathbf{v}}, D^{-1}U\binom{\mathbf{p}}{\mathbf{v}}$, $D^{-2}U^2\binom{\mathbf{p}}{\mathbf{v}}, \ldots$ converge to a curve and its derivative.

Note that if M is a matrix group, then U is invariant w.r.t. both left and right multiplication. Furthermore, if the linear operator $S_{\mathbf{A}}$ is interpolatory, then obviously so is U.

We mention that U can be defined analogously in the more general cases of Riemannian manifolds and Lie groups.

Example 3.3. Consider the matrix group $\mathrm{SO}(3) = \{p \in \mathbb{R}^{3\times 3} : p \text{ is orthogonal and } \det(p) > 0\}$. The tangent space at $p \in \mathrm{SO}(3)$ is given by $T_p\mathrm{SO}(3) = \{v \in \mathbb{R}^{3\times 3} : p^{-1}v \text{ is skew-symmetric}\}$.

We consider the parallel transport version of the linear Hermite scheme introduced in Example 3.2. Recall from Section 3.3.3 that for $p, q \in \mathrm{SO}(3)$ and $v \in T_p\mathrm{SO}(3)$ the operators $\oplus, \ominus$ are given by

$$p \oplus v = p \exp(p^{-1}v) \quad \text{and} \quad q \ominus p = p \log(p^{-1}q),$$

where exp is the matrix exponential and log is the matrix logarithm.

For input data $\binom{\mathbf{p}}{\mathbf{v}} \in \ell(T\mathrm{SO}(3))$ we choose the base point sequence $\mathbf{m}$ as the midpoints of consecutive points of $\mathbf{p}$:

$$m_{2i} = m_{2i+1} = \mu_{p_{i+1},p_i} = p_{i+1} \oplus \frac{1}{2}\Big(p_i \ominus p_{i+1}\Big).$$

Furthermore, for $i, j \in \mathbb{Z}$ we introduce the following sequences:

$$\begin{aligned}
u_{i,j} &= p_i \ominus m_j, \\
z_{j,i} &= \mathrm{P}_{p_j}^{m_i}(v_j) = \begin{cases} m_i p_j^{-1} v_j & \text{for the } (+) \text{ parallel transport,} \\ v_j p_j^{-1} m_i & \text{for the } (-) \text{ parallel transport,} \\ \mu_{p_j,m_i} p_j^{-1} v_j p_j^{-1} \mu_{p_j,m_i} & \text{for the } (0) \text{ parallel transport.} \end{cases}
\end{aligned}$$

The operator U of (3.14) is given by

$$U \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix}_i = \begin{pmatrix} r_i \\ \mathrm{P}_{m_i}^{r_i}(w_i) \end{pmatrix},$$

where

$$\begin{aligned}
r_{2i} &= m_{2i} \oplus \frac{1}{8}\Big(\frac{48}{25} u_{i+1,2i} + \frac{152}{25} u_{i,2i} - \frac{29}{25} z_{i+1,2i} + \frac{31}{25} z_{i,2i}\Big), \\
w_{2i} &= \frac{1}{8}\Big(\frac{29}{50} u_{i+1,2i} - \frac{29}{50} u_{i,2i} + \frac{13}{20} z_{i+1,2i} + \frac{277}{100} z_{i,2i}\Big), \\
r_{2i+1} &= m_{2i+1} \oplus \frac{1}{8}\Big(\frac{152}{25} u_{i+1,2i+1} + \frac{48}{25} u_{i,2i+1} - \frac{31}{25} z_{i+1,2i+1} + \frac{29}{25} z_{i,2i+1}\Big), \\
w_{2i+1} &= \frac{1}{8}\Big(\frac{29}{50} u_{i+1,2i+1} - \frac{29}{50} u_{i,2i+1} + \frac{277}{100} z_{i+1,2i+1} + \frac{13}{20} z_{i,2i+1}\Big).
\end{aligned}$$

The coefficients are taken from Example 3.2.

We consider the bi-invariant inner product $\langle u, v \rangle = \mathrm{trace}(uv^T)$ on SO(3). This bi-invariant inner product coincides with the standard inner product induced by $\mathbb{R}^9$, since $\mathrm{trace}(uv^T) = \sum_{i,j} u_{ij} v_{ij}$. Therefore, SO(3) is a surface which carries a bi-invariant inner product. It is known that the (0) parallel transport defined above coincides with the surface parallel transport (the same is true for the exponential mapping). Therefore, the above calculations are also valid if SO(3) is viewed as a surface.

3.5 Proximity inequalities

In order to conclude convergence and smoothness of ordinary manifold-valued subdivision rules, the proximity method was introduced, see [77, 75] and others. This method requires to establish inequalities on the difference between linear subdivision rules and manifold-valued subdivision rules.

3.5.1 The proximity condition for Hermite schemes

Consider a linear Hermite subdivision operator $S_{\mathbf{A}}$ and a manifold-valued Hermite subdivision operator U. Then the proximity condition, introduced by [64] for Hermite schemes, is given by

$$\|(U - S_{\mathbf{A}})\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty \le c\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty^2. \tag{3.15}$$

Here c is a constant and Δ denotes the forward difference operator $(\Delta\mathbf{p})_i = p_{i+1} - p_i$ for $i \in \mathbb{Z}$.

To conclude C^1 convergence of U from convergence of $S_{\mathbf{A}}$, it is required that condition (3.15) is fulfilled whenever $\|\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty$ is bounded and $\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty$ is small enough.

In the following we prove that the proximity condition (3.15) holds between a linear operator $S_{\mathbf{A}}$ and the TM-valued operator U constructed from $S_{\mathbf{A}}$ (3.14), where M is a surface or matrix group.

Recall from Equation (3.14) that we defined sequences r, w by

$$r_i = m_i \oplus \sum_j a_{i-2j}(p_j \ominus m_i) + b_{i-2j}\mathrm{P}_{p_j}^{m_i}(v_j), \tag{3.16}$$
$$w_i = \sum_j c_{i-2j}(p_j \ominus m_i) + d_{i-2j}\mathrm{P}_{p_j}^{m_i}(v_j),$$

for $i \in \mathbb{Z}$. We also define $\mathbf{r}^{\mathrm{lin}}$ and $\mathbf{w}^{\mathrm{lin}}$, which are the linear versions of $\mathbf{r}$ and $\mathbf{w}$. This means that $\oplus$ and $\ominus$ are replaced by $+$ and $-$ respectively and $\mathrm{P}_{p_j}^{m_i}(v_j)$ is replaced by v_j. Therefore, in order to prove (3.15), we have to show the inequalities:

$$\|\mathbf{r} - \mathbf{r}^{\mathrm{lin}}\|_\infty \le c\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty^2, \tag{3.17}$$
$$\|\mathrm{P}_{\mathbf{m}}^{\mathbf{r}}(\mathbf{w}) - \mathbf{w}^{\mathrm{lin}}\|_\infty \le c\|\left(\begin{smallmatrix}\Delta\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\|_\infty^2. \tag{3.18}$$

The main ingredient in the proof is the following lemma:

Lemma 3.4. *Let M be a surface or matrix group. Then for $p, m \in M$ and tangent vectors v the following linearizations hold:*

$$p \oplus v = p + v + O(\|v\|^2) \quad as \quad v \to 0, \tag{3.19}$$
$$m \ominus p = m - p + O(\|m - p\|^2) \quad as \quad m \to p, \tag{3.20}$$
$$\mathrm{P}_m^p(v) = v + O(\|m - p\|\,\|v\|) \quad as \quad m \to p. \tag{3.21}$$

In the case that M is a surface, P_m^p denotes the parallel transport along the geodesic connecting p and m. If M is a matrix group, then P_m^p denotes one of the $(+), (-)$, or (0) parallel transports.

Proof. In a chart of M, (3.19) and (3.20) are exactly the well-known linearization of the exponential map. In order to prove (3.21), we first observe that $(m, v) \mapsto \mathrm{P}_m^p(v)$ is smooth. On a surface, this can be deduced from the fact that the solution of an ordinary differential equation depends smoothly on the initial data. In the matrix group case, the smoothness of this map follows from the definition of the parallel transport. Restricting to a unit vector v and using Taylor expansion in a chart at $m = p$, we obtain

$$\mathrm{P}_m^p(v) = \mathrm{P}_p^p(v) + O(\|m - p\|) = v + O(\|m - p\|) \quad \text{as} \quad m \to p, \ v = \text{const.}$$

Since P_m^p is a linear map, for a general v, we obtain $\mathrm{P}_m^p(v) = v + O(\|m - p\| \, \|v\|)$ as $m \to p$. This completes the proof. $\square$

Corollary 3.5 (Proximity inequalities)**.** *Let M be a surface or matrix group. Consider bounded input data $\binom{\mathbf{p}}{\mathbf{v}}$ on TM and a base point sequence $\mathbf{m}$, which is either given by $m_i = p_i$ or $m_i = \mu_{p_i, p_{i+1}}$ for $i \in \mathbb{Z}$. Then the sequences $\mathbf{r}$ and $\mathbf{w}$ as defined in (3.16) satisfy*

$$\begin{aligned} r_i &= r_i^{\mathrm{lin}} + O(\sup_j \|m_i - p_j\|^2) + O(\sup_j \|m_i - p_j\| \sup_j \|v_j\|) + O(\sup_j \|v_j\|^2), \\ w_i &= w_i^{\mathrm{lin}} + O(\sup_j \|m_i - p_j\| \sup_j \|v_j\|), \\ \mathrm{P}_{m_i}^{r_i}(w_i) &= w_i^{\mathrm{lin}} + O(\sup_j \|m_i - p_j\|^2) + O(\sup_j \|m_i - p_j\| \sup_j \|v_j\|) + O(\sup_j \|v_j\|^2), \end{aligned}$$

for $\mathbf{m} \to \mathbf{p}$ and $\mathbf{v} \to 0$ and $i \in \mathbb{Z}$. In particular, the proximity inequalities (3.17) and (3.18) follow.

Proof. Using Lemma 3.4, the results for $\mathbf{r}$ and $\mathbf{w}$ immediately follow. Similarly, we can show that $\|r_i - m_i\| = O(\sup_j \|p_j - m_i\|) + O(\sup_j \|v_j\|)$. This implies

$$\begin{aligned} \mathrm{P}_{m_i}^{r_i}(w_i) &= w_i + O(\|r_i - m_i\| \, \|w_i\|) \\ &= w_i^{\mathrm{lin}} + O(\sup_j \|m_i - p_j\| \sup_j \|v_j\|) + O(\|r_i - m_i\| \, \|w_i\|) \\ &= w_i^{\mathrm{lin}} + O(\sup_j \|m_i - p_j\|^2) + O(\sup_j \|m_i - p_j\| \sup_j \|v_j\|) + O(\sup_j \|v_j\|^2), \end{aligned}$$

Furthermore, Lemma 3.4 implies $\sup_j \|m_i - p_j\| \le c\|\Delta \mathbf{p}\|_\infty$. Thus the above equations show that $\|\mathbf{r} - \mathbf{r}^{\mathrm{lin}}\|_\infty \le c \max\{\|\Delta \mathbf{p}\|_\infty^2, \|\mathbf{v}\|_\infty^2\}$ and $\|\mathrm{P}_{\mathbf{m}}^{\mathbf{r}}(\mathbf{w}) - \mathbf{w}^{\mathrm{lin}}\|_\infty \le c \max\{\|\Delta \mathbf{p}\|_\infty^2, \|\mathbf{v}\|_\infty^2\}$. By the equivalence of norms (3.5), the proximity inequality (3.17) and (3.18) are proved. This completes the proof. $\square$

3.6 Results

In the previous section we have gathered all proximity inequalities we need to prove C^1 convergence of the manifold-valued Hermite scheme defined in Section 3.4. Our main theorem (Theorem 3.7) is analogous to Theorem 27 of [64].

Before we state the theorem, we have to introduce the *Taylor operator*. In linear Hermite subdivision, the Taylor operator is the natural analogue to the forward difference operator $(\Delta \mathbf{p})_i = p_{i+1} - p_i$ for $i \in \mathbb{Z}$, see [60]. It acts on $\ell(V^2)$ and is defined by

$$T = \begin{pmatrix} \Delta & -1 \\ 0 & \Delta \end{pmatrix}.$$

In [60] this operator is called *complete Taylor operator*. We have the following result:

Theorem 3.6 (Merrien and Sauer, 2012)**.** *Let $S_{\mathbf{A}}$ be a linear subdivision operator which satisfies the spectral condition (3.2). Then we have the following*

1. *There exists a linear subdivision operator $S_{\mathbf{B}}$ such that*

$$2TS_{\mathbf{A}} = S_{\mathbf{B}}T.$$

   We call $S_{\mathbf{B}}$ the Taylor scheme of $S_{\mathbf{A}}$.

2. *If there exists $N \in \mathbb{N}$ such that $\|S_{\mathbf{B}}^N\|_\infty < 1$, then the linear Hermite scheme associated to $S_{\mathbf{A}}$ is C^1 convergent.*

Now we can state the main result of our paper:

Theorem 3.7. *Let $S_{\mathbf{A}}$ be a linear subdivision operator whose mask $\mathbf{A}$ satisfies the spectral condition (3.2), and let $S_{\mathbf{B}}$ be the Taylor scheme of $S_{\mathbf{A}}$ (Theorem 3.6). Let M be a surface or a matrix group and let U be the manifold-valued analogue of $S_{\mathbf{A}}$ given by (3.14). Then we have the following result:*

If there exists $N \in \mathbb{N}$ such that $\|S_{\mathbf{B}}^N\|_\infty < 1$, then the Hermite scheme $\binom{\mathbf{p}}{\mathbf{v}}, D^{-1}U\binom{\mathbf{p}}{\mathbf{v}}$, $D^{-2}U^2\binom{\mathbf{p}}{\mathbf{v}}, \dots$ is C^1 convergent whenever $\binom{\mathbf{p}}{\mathbf{v}}$ are dense enough.

The statement of the theorem remains true if "surface" is replaced by "Riemannian manifold" and "matrix group" by "Lie group".

Proof. It is proved in [64] that $\|S_{\mathbf{A}}^N\|_\infty < 1$ for some integer N together with the proximity condition implies C^1 convergence of the manifold-valued Hermite scheme. Therefore, the result follows from Section 3.5 and [64].

Note that the input data does not have to be bounded. This follows from the fact that on any compact interval the limit curve only depends on finitely many points of the input data. We can therefore w.l.o.g. assume that $\|\binom{\mathbf{p}}{\mathbf{v}}\|_\infty$ is bounded.

The global embedding theorem states that any Riemannian manifold can be isometrically embedded as a surface into a Euclidean space of sufficiently high dimension. The smoothness is preserved by this embedding. Our result applies to this surface. Furthermore, by Ado's theorem, any Lie group is locally isomorphic to a matrix group. Therefore, the generalized statement is also true. This completes the proof. $\square$

Remark 3.8. We would like to remark on a possible generalization of this result, which is a topic of future research. It would be natural to consider schemes which produce more than one derivative, i.e. schemes refining sequences with more than two components, with the kth component representing the $(k-1)$st derivative. This has been studied in the linear case, see e.g. [60].

We believe that such a generalization becomes quite technical: Available results from manifold subdivision suggest that the case of more than two derivatives is more involved compared to the case of one derivative [43, 85]. Also, the data now have to be sampled from the jet bundle of the manifold.

Example 3.9. We consider the linear subdivision operator $S_{\mathbf{A}}$ whose mask is defined in Example 3.2. In [60] it is shown that the operator $S_{\mathbf{B}}$ satisfying $2TS_{\mathbf{A}} = S_{\mathbf{B}}T$ has the mask

$$B_{-1} = \frac{1}{4}\begin{pmatrix} \frac{48}{25} & -\frac{29}{25} \\ \frac{29}{50} & \frac{13}{20} \end{pmatrix}, \quad B_0 = \frac{1}{4}\begin{pmatrix} \frac{179}{50} & -\frac{73}{100} \\ 0 & \frac{53}{25} \end{pmatrix}, \quad B_1 = \frac{1}{4}\begin{pmatrix} \frac{67}{50} & \frac{47}{100} \\ -\frac{29}{50} & \frac{123}{100} \end{pmatrix}.$$

We prove $\|S_{\mathbf{B}}\|_\infty < 1$. The norm of a subdivision operator is given by

$$\|S_{\mathbf{B}}\|_\infty = \sup\left\{\left\|S_{\mathbf{B}}\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\right\|_\infty : \left\|\left(\begin{smallmatrix}\mathbf{p}\\ \mathbf{v}\end{smallmatrix}\right)\right\|_\infty = 1\right\}.$$

It is well known that

$$\|S_{\mathbf{B}}\|_\infty = \max\Big\{\sum_{j\in\mathbb{Z}} \|B_{-2j}\|, \sum_{j\in\mathbb{Z}} \|B_{-2j+1}\|\Big\}.$$

Therefore, we have to prove that $\max\{\|B_0\|, \|B_{-1}\| + \|B_1\|\} < 1$. The operator norm of a matrix w.r.t. to the Euclidean norm equals the spectral norm, therefore

$$\|B_i\| = \sqrt{\lambda_{\max}(B_i^T B_i)},$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $B_i^T B_i$ for $i = -1, 0, 1$. This yields

$$\begin{aligned}
\lambda_{\max}(0) &= \frac{178437 + 73\sqrt{1651145}}{320000} < 1, \\
\lambda_{\max}(-1) &= \frac{57909 + 5\sqrt{75106529}}{320000} < \frac{36}{100}, \\
\lambda_{\max}(1) &= \frac{19329 + 11\sqrt{38537}}{160000} < \frac{16}{100}.
\end{aligned}$$

This implies that $\|S_{\mathbf{B}}\|_\infty < 1$ and therefore the C^1 convergence of the linear Hermite scheme defined by $S_{\mathbf{A}}$. Furthermore, Theorem 3.7 shows that its parallel transport version on any Riemannian manifold or Lie group is C^1 convergent for dense enough input data. In particular this includes SO(3), i.e., our Example 3.3.

3.6.1 Conclusion

We have studied a manifold-valued analogue of linear Hermite subdivision schemes which is defined by using the parallel transport operator of the manifold. This construction is intrinsic and gives rise to a C^1 convergent nonlinear subdivision scheme, if the input data are dense enough and the Taylor scheme is appropriately bounded (Theorem 3.7). Similar to most convergence and smoothness results of subdivision rules in general manifolds, the main ingredient of the proof is the method of proximity.

Acknowledgments. The author would like to thank Johannes Wallner for helpful discussions on earlier versions of this paper and gratefully acknowledges the suggestions of the anonymous reviewers. This research is supported by the doctoral program "Discrete Mathematics", funded by the Austrian Science Fund FWF under grant agreement W1230.

4 Smoothing of vector and Hermite subdivision schemes

This chapter comprises the paper (III) and is joint work with Nira Dyn.

Abstract. In this paper we study the regularity of curves which arise as limits of subdivision schemes. In particular, we are interested in increasing this regularity. In scalar subdivision, it is well known that a scheme which produces C^ℓ limit curves can be transformed to a new scheme producing $C^{\ell+1}$ limit curves by taking the midpoints in each round of iteration. This procedure can be described by multiplying the scheme's symbol with the *smoothing factor* $\frac{z+1}{2}$. We present a similar smoothing procedure for vector and Hermite subdivision schemes, approaching this problem algebraically by manipulating the symbol of a given scheme. The algorithms presented in this paper allow to construct vector and Hermite subdivision schemes of arbitrarily high regularity from a convergent vector scheme respectively from an Hermite scheme whose Taylor scheme is at least C^0.

Keywords. Hermite subdivision · vector subdivision · high regularity · smoothing

AMS Subject Classification. 65D10 · 65D15 · 65D17 · 41A05

4.1 Introduction

Subdivision schemes are algorithms which iteratively refine discrete input data and produce smooth curves or surfaces in the limit. The regularity of the limit curve resp. surface is a topic of high interest.

In this paper we are concerned with the *stationary* and *univariate* case, i.e. with subdivision schemes using the same set of coefficients (called *mask*) in every refinement step and which have curves as limits. We study three types of such schemes: *scalar*, *vector* and *Hermite* subdivision schemes.

In scalar subdivision the mask is a real-valued sequence and thus it is in fact a special case of vector subdivision, which uses matrix-valued masks. These schemes have been studied intensively over the years and many results are available, including (but not limited to) the analysis of convergence and smoothness. For a non-complete list of references see [3, 27, 26, 32, 63, 71, 7].

In Hermite subdivision, on the other hand, the input data is interpreted as function values and derivatives. This results in a level-dependent case of vector subdivision, where the convergence of a scheme already includes the regularity of the limit curve. Corresponding literature can be found in [30, 31, 23, 22, 52, 49, 60] and references therein. Note that we consider *inherently stationary* Hermite schemes [11], which means that the level-dependence arises only from the specific interpretation of the input data. Inherently non-stationary Hermite schemes are discussed e.g. in [9].

The convergence and smoothness analysis of subdivision schemes is strongly connected to the existence of the *derived scheme* resp. the *Taylor scheme*, which arise from factorizing the original scheme w.r.t. a difference operator Δ [32, 7] or w.r.t. the *Taylor operator* in the Hermite case [60]. In all cases we have the following result: If the derived scheme (resp. Taylor scheme) produces C^ℓ ($\ell \geq 0$) limit curves, then the original scheme produces $C^{\ell+1}$ limit curves, see [32, 7, 60, 11]. This result is the essential tool in our smoothing approach.

We use a scheme which is known to have a certain regularity as the derived scheme resp. Taylor scheme of a new, to be computed scheme. By the above result, the regularity of the new scheme is increased by 1. This idea comes from scalar subdivision, where it is well known that a scheme with symbol $a(z)$ is the derived scheme of $b(z) = \frac{1+z}{2}z^{-1}a(z)$ [32]. The scheme with symbol $b(z)$ is then the new scheme mentioned above.

It is possible to iterate this process to obtain vector and Hermite subdivision schemes of arbitrarily high smoothness from a convergent vector scheme respectively from an Hermite scheme whose Taylor scheme produces at least C^0 limits.

We would like to mention other approaches which increase the regularity of subdivision schemes: It is known that the de Rham transform [24] of some Hermite schemes increases the regularity by 1, see [11]. In contrast to our approach, it is not clear if this procedure can be iterated to obtain schemes of higher regularity. Nevertheless, in the examples listed in [11], the de Rham approach increases the support only by 1, whereas our smoothing procedure has the drawback of producing Hermite schemes with large supports, see Corollary 4.42, Example 4.46 and Example 4.47. Also, the authors of [24] use geometric ideas, such as corner cutting. Our approach, on the other hand, is of an algebraic nature as it manipulates symbols.

A recent result which increases the regularity of an Hermite scheme is contained in [61]. This is different from our approach, as it also increases the *dimension* of the scheme in question.

We would also like to point to the paper [72], which gives a detailed discussion of generalizing the smoothness procedure to the scalar multivariate setting (with general dilation). Naturally, vector subdivision schemes appear in this approach, but the aim is to smoothen scalar schemes.

Our paper is organized as follows. In Section 4.2 we introduce the notation used throughout this text and recall some definitions concerning subdivision schemes. Section 4.3 presents the well known smoothing procedure for scalar subdivision schemes. This is mainly taken from [32]. We introduce new notation, however, to underline the analogy to the smoothing procedures for vector and Hermite schemes presented in Sections 4.4 and 4.5. We conclude by applying our smoothing algorithm to an interpolatory Hermite scheme of [58] and to an Hermite scheme of de Rham-type [24]. This results in limit curves of regularity C^2 resp. C^3.

4.2 Notation and background

In this section we introduce the notation which is used throughout this paper and recall some known facts about scalar, vector and Hermite subdivision schemes.

Vectors in $\mathbb{R}^p$ will be labeled by lowercase letters c. The standard basis is denoted by $e_1, \ldots, e_p$. Sequences of elements in $\mathbb{R}^p$ are denoted by boldface letters $\mathbf{c} = \{c_i \in \mathbb{R}^p : i \in \mathbb{Z}\}$. The space of all such sequences is $\ell^p(\mathbb{Z})$.

We define a *subdivision operator* $S_{\mathbf{a}} : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ by

$$(S_{\mathbf{a}}\mathbf{c})_i = \sum_{j \in \mathbb{Z}} a_{i-2j} c_j, \quad i \in \mathbb{Z},\ \mathbf{c} \in \ell^p(\mathbb{Z}), \tag{4.1}$$

where the multiplication is to be understood componentwise. The sequence of coefficients $\mathbf{a} \in \ell(\mathbb{Z})$ is called *mask*. We study the case of finitely supported masks, that is, masks whose support

$$\operatorname{supp}(\mathbf{a}) = \{i \in \mathbb{Z} : a_i \neq 0\}$$

is finite. In this case also the sum in eq. (4.1) is finite.

We also consider matrix-valued masks. To distinguish them from the scalar case, we denote matrices in $\mathbb{R}^{p \times p}$ by uppercase letters. Sequences of matrices are denoted by boldface letters $\mathbf{A} = \{A_i \in \mathbb{R}^{p \times p} : i \in \mathbb{Z}\}$.

We define a *subdivision operator* $S_{\mathbf{A}} : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ by

$$(S_{\mathbf{A}}\mathbf{c})_i = \sum_{j \in \mathbb{Z}} A_{i-2j} c_j, \quad i \in \mathbb{Z},\ \mathbf{c} \in \ell^p(\mathbb{Z}), \tag{4.2}$$

where the finitely supported sequence of coefficients $\mathbf{A} \in \ell^{p \times p}(\mathbb{Z})$ is called mask. We define three kinds of subdivision schemes:

Definition 4.1.

1. A *scalar subdivision scheme* is the procedure of constructing $\mathbf{c}^n$ $(n \geq 1)$ from input data $\mathbf{c}^0 \in \ell^p(\mathbb{Z})$ by the rule $\mathbf{c}^n = S_{\mathbf{a}}\mathbf{c}^{n-1}$, where $\mathbf{a} \in \ell(\mathbb{Z})$ is a mask.
2. A *vector subdivision scheme* is the procedure of constructing $\mathbf{c}^n$ $(n \geq 1)$ from input data $\mathbf{c}^0 \in \ell^p(\mathbb{Z})$ by the rule $\mathbf{c}^n = S_{\mathbf{A}}\mathbf{c}^{n-1}$, where $\mathbf{A}$ is a matrix-valued mask.

3. An *Hermite subdivision scheme* is the procedure of constructing $\mathbf{c}^n$ ($n \geq 1$) from $\mathbf{c}^0 \in \ell^p(\mathbb{Z})$ by the rule $D^n \mathbf{c}^n = S_{\mathbf{A}} D^{n-1} \mathbf{c}^{n-1}$, where $\mathbf{A}$ is a matrix-valued mask and D is the dilation matrix

$$D = \begin{pmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{2^{p-1}} \end{pmatrix}.$$

The difference between scalar and vector subdivision lies in the dimension of the mask. In scalar subdivision the components of $\mathbf{c}$ are refined *independently* of each other. This is not the case in vector subdivision. Note also that scalar schemes are a special case of vector schemes with mask $A_i = a_i I_p$, where I_p is the $(p \times p)$ unit matrix. In Hermite subdivision, on the other hand, the components of $\mathbf{c}$ are interpreted as function values and derivatives up to order $p-1$. This is represented by the matrix D. In particular, Hermite subdivision is a level-dependent case of vector subdivision: $\mathbf{c}^n = S_{\tilde{\mathbf{A}}_n} \mathbf{c}^{n-1}$ with $\tilde{\mathbf{A}}_n = \{D^{-n} A_i D^{n-1} : i \in \mathbb{Z}\}$.

On the space $\ell^p(\mathbb{Z})$ we define a norm by

$$\|\mathbf{c}\|_\infty = \sup_{i \in \mathbb{Z}} \|c_i\|,$$

where $\|\cdot\|$ is a norm on $\mathbb{R}^p$. The Banach space of all bounded sequences is denoted by $\ell^p_\infty(\mathbb{Z})$. Using this norm, we define convergence of scalar, vector and Hermite subdivision schemes. We start with scalar and vector schemes:

Definition 4.2. A scalar (resp. vector) subdivision scheme associated with the mask $\mathbf{a}$ (resp. $\mathbf{A}$) is *convergent in* $\ell^p_\infty(\mathbb{Z})$ *(also called* C^0*)*, if for all input data $\mathbf{c}^0 \in \ell^p_\infty(\mathbb{Z})$ there exists a function $\Psi \in C(\mathbb{R}, \mathbb{R}^p)$, such that the sequences $\mathbf{c}^n = S^n_{\mathbf{a}} \mathbf{c}^0$ (resp. $\mathbf{c}^n = S^n_{\mathbf{A}} \mathbf{c}^0$) satisfy

$$\sup_{i \in \mathbb{Z}} \|c^n_i - \Psi(\tfrac{i}{2^n})\| \to 0, \quad \text{as } n \to \infty.$$

and $\Psi \neq 0$ for some $\mathbf{c}^0 \in \ell^p_\infty(\mathbb{Z})$. We say that the scheme is C^ℓ, if in addition Ψ is ℓ-times continuously differentiable.

In Section 4.5 we only consider Hermite subdivision schemes which refine function values and first derivatives. This point-vector data is subdivided componentwise and therefore it is sufficient to treat convergence for data in $\ell^2(\mathbb{Z})$.

In order to distinguish between the convergence of vector subdivision schemes and the convergence of Hermite subdivision schemes, we use notation as introduced in [11]:

Definition 4.3. An Hermite subdivision scheme associated with the mask $\mathbf{A}$ is said to be HC^ℓ *convergent* with $\ell \geq 1$, if for all input data $\mathbf{c}^0 \in \ell^2_\infty(\mathbb{Z})$, there exists a function

$\Psi = \binom{\Psi^0}{\Psi^1}$ with $\Psi^0 \in C^\ell(\mathbb{R},\mathbb{R})$ and $\frac{d\Psi^0}{dt} = \Psi^1$ such that the sequences $D^n\mathbf{c}^n = S^n_{\mathbf{A}}\mathbf{c}^0$ satisfy

$$\sup_{i\in\mathbb{Z}} \|c_i^n - \Psi(\tfrac{i}{2^n})\| \to 0, \quad \text{as } n \to \infty.$$

Note that in contrast to the vector case, an Hermite subdivision scheme is convergent only if the limit curve already possesses a certain degree of regularity.

Before we start with introducing the smoothing procedure, we would like to recall some facts about the *symbol* of a sequence $\mathbf{c}$. The symbol of $\mathbf{c}$ is the formal Laurent series defined by

$$\mathbf{c}^*(z) = \sum_{i\in\mathbb{Z}} c_i z^i.$$

It is easy to see that the symbol has the following properties:

Lemma 4.4. *Let* $\mathbf{c}$ *be a sequence and let* $\mathbf{a}$ *be a mask. By* Δ *we denote the forward-difference operator* $(\Delta\mathbf{c})_i = c_{i+1} - c_i$. *Then we have:*

$$(\Delta\mathbf{c})^*(z) = (z^{-1} - 1)\mathbf{c}^*(z) \text{ and } (S_{\mathbf{a}}\mathbf{c})^*(z) = \mathbf{a}^*(z)\mathbf{c}^*(z^2).$$

Furthermore, for finite sequences we have the equalities

$$\mathbf{c}^*(1) = \sum_{i\in\mathbb{Z}} c_{2i} + \sum_{i\in\mathbb{Z}} c_{2i+1} \quad \text{and} \quad \mathbf{c}^*(-1) = \sum_{i\in\mathbb{Z}} c_{2i} - \sum_{i\in\mathbb{Z}} c_{2i+1},$$

$$\mathbf{c}^{*\prime}(1) = \sum_{i\in\mathbb{Z}} c_{2i}(2i) + \sum_{i\in\mathbb{Z}} c_{2i+1}(2i+1) \quad \text{and} \quad \mathbf{c}^{*\prime}(-1) = \sum_{i\in\mathbb{Z}} c_{2i+1}(2i+1) - \sum_{i\in\mathbb{Z}} c_{2i}(2i).$$

4.3 Smoothing of scalar subdivision schemes

In this section we recall the smoothing procedure in scalar subdivision which is realized by the smoothing factor $\frac{z+1}{2}$. The results of this section are taken from [32]. We introduce notation in order to illustrate the analogy to the smoothing procedures we will present for vector schemes in Section 4.4.

Recall that in scalar subdivision, the mask is a finitely supported sequence $\mathbf{a} = \{a_i \in \mathbb{R} : i \in \mathbb{Z}\}$ and the refinement procedure is obtained by iteratively applying a subdivision operator $S_{\mathbf{a}}$ as in eq. (4.1). It is well known that the condition $\sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1$ for the mask $\mathbf{a}$ is necessary for convergence of the subdivision scheme, see e.g. [32]. In this case $\mathbf{a}^*(z)$ has a factor $(z+1)$ and there exists a mask $\partial\mathbf{a}$ such that

$$\Delta S_{\mathbf{a}} = \tfrac{1}{2} S_{\partial\mathbf{a}} \Delta. \tag{4.3}$$

The scalar scheme of $\partial\mathbf{a}$ is called the *derived scheme*. It is easy to see that $(\partial\mathbf{a})^*(z) = 2z\frac{\mathbf{a}^*(z)}{z+1}$.

The following result allows us to define a smoothing procedure:

Theorem 4.5 (Theorem 4.11 and Theorem 4.13 of [32]). *Let* $\mathbf{a}$ *be a mask which satisfies* $\mathbf{a}^*(1) = 2$ *and* $\mathbf{a}^*(-1) = 0$.

1. *The scalar scheme associated with* $\mathbf{a}$ *is convergent if and only if the scalar scheme of* $\frac{1}{2}\partial\mathbf{a}$ *is contractive.*
2. *If the scalar scheme of* $\partial\mathbf{a}$ *is* C^ℓ $(\ell \geq 0)$ *then the scalar subdivision scheme associated with* $\mathbf{a}$ *is* $C^{\ell+1}$.

In view of this theorem, a smoothing procedure can be defined as follows: For a mask $\mathbf{a}$, define a new mask $\mathcal{I}\mathbf{a}$ by $(\mathcal{I}\mathbf{a})^*(z) = \frac{(1+z)}{2} z^{-1}\mathbf{a}^*(z)$. Then $(\mathcal{I}\mathbf{a})^*(-1) = 0$ and $\partial(\mathcal{I}\mathbf{a}) = \mathbf{a}$ (Note that if $\partial\mathbf{a}$ is well-defined, then also $\mathcal{I}(\partial\mathbf{a}) = \mathbf{a}$).

Corollary 4.6. *Let* $\mathbf{a}$ *be a mask with* C^ℓ $(\ell \geq 0)$ *convergent scalar subdivision scheme. Then the mask* $\mathcal{I}\mathbf{a}$ *gives rise to a* $C^{\ell+1}$ *convergent scheme.*

Therefore, by iterative application of $\mathcal{I}$, a scalar subdivision scheme which is at least C^0 convergent, can be transformed to a new scheme of arbitrarily high regularity. We call $\mathcal{I}$ a *smoothing operator* resp. and $\frac{z+1}{2}$ a *smoothing factor*. Note that the factor z^{-1} in $\mathcal{I}$ is an index shift.

Example 4.7 (B-Spline curves). An example of the above mentioned smoothing procedure are B-Spline curves, which are obtained from the Lane-Riesenfeld (L-R) algorithm, see e.g. [32].

Let $\ell \geq 1$. One step of the ℓ-th L-R algorithm is given by an initial doubling of the input data, followed by ℓ rounds of inserting midpoints. Its mask is given by

$$\mathbf{a}_\ell^*(z) = \left(\tfrac{(z+1)}{2} z^{-1}\right)^\ell (z+1).$$

Therefore, $\mathbf{a}_\ell = \mathcal{I}^{\ell-1}\mathbf{a}_1$, where $\mathbf{a}_1^*(z) = \frac{(z+1)^2}{2}$ is the symbol of the subdivision scheme generating the piecewise linear interpolant of the input data. Thus the limit curves of the ℓ-th L-R algorithm are $C^{\ell-1}$.

4.4 Smoothing of vector subdivision schemes

In this section we describe a smoothing procedure for vector schemes similar to the scalar case. It is more involved since we consider masks consisting of matrix sequences.

We would like to mention that the vector smoothing procedure we present in this section essentially follows from results by C. Michelli and T. Sauer [62, 63, 71]. We reprove these results here in order to extend the smoothing procedure more easily to the Hermite case.

4.4.1 Convergence and smoothness analysis

In this section we collect results concerning the convergence and smoothness of vector subdivision schemes. Their proofs can be found in [7, 62, 63, 71].

Denote by A^0 and A^1 the sum of even and odd entries of a mask $\mathbf{A}$, i.e.

$$A^0 = \sum_{i\in\mathbb{Z}} A_{2i}, \quad A^1 = \sum_{i\in\mathbb{Z}} A_{2i+1}. \tag{4.4}$$

Following [63], by $\mathcal{E}_{\mathbf{A}}$ we denote the common eigenspace of A^0 and A^1 with respect to the eigenvalue 1:

$$\mathcal{E}_{\mathbf{A}} = \{v \in \mathbb{R}^p : A^0 v = v \text{ and } A^1 v = v\}. \tag{4.5}$$

Let $k = \dim \mathcal{E}_{\mathbf{A}}$. A priori, $0 \le k \le p$. It is well known, however, that for convergent vector subdivision schemes, $\mathcal{E}_{\mathbf{A}} \neq \{0\}$, i.e. $1 \le k \le p$. Therefore, the existence of a common eigenvector of A^0 and A^1 w.r.t. the eigenvalue 1 is a necessary condition for convergence.

In this paper we are mainly concerned with the special case of vector schemes satisfying $\mathcal{E}_{\mathbf{A}} = \operatorname{span}\{e_1, \dots, e_k\}$. In [63] it is shown that any vector subdivision scheme can be transformed to this special type and also that the convergence analysis can be reduced to this case. We collect this result in the following easy lemma:

Lemma 4.8. *Let $S_{\mathbf{A}}$ be a C^ℓ $(\ell \ge 0)$ convergent vector subdivision scheme.*

(a) Let $R \in \mathbb{R}^{p\times p}$ be invertible and define a new mask $\overline{\mathbf{A}}$ by $\overline{A}_i = R^{-1} A_i R$ for $i \in \mathbb{Z}$. Then the vector subdivision scheme associated with $\overline{\mathbf{A}}$ is also C^ℓ.

(b) There exists an invertible matrix $R \in \mathbb{R}^{p\times p}$ such that (using the same notation) $\overline{\mathbf{A}}$ satisfies $\mathcal{E}_{\overline{\mathbf{A}}} = \operatorname{span}\{e_1, \dots, e_k\}$, where $k = \dim \mathcal{E}_{\mathbf{A}}, 1 \le k \le p$.

We introduce a generalization of the forward-difference operator Δ, by letting

$$\Delta_k = \begin{pmatrix} \Delta I_k & 0 \\ 0 & I_{p-k} \end{pmatrix}.$$

Here I_k is the $(k \times k)$ unit matrix. It is shown in [71] that if $\mathcal{E}_{\mathbf{A}} = \operatorname{span}\{e_1, \dots, e_k\}$, then there exists a mask $\partial_k \mathbf{A}$ such that

$$\Delta_k S_{\mathbf{A}} = \tfrac{1}{2} S_{\partial_k \mathbf{A}} \Delta_k. \tag{4.6}$$

We denote by $\partial_k \mathbf{A}$ any mask satisfying eq. (4.6). The vector scheme associated with $\partial_k \mathbf{A}$ is called the *derived scheme* of $\mathbf{A}$ with respect to Δ_k. Furthermore, we have the following result concerning the convergence of $\mathbf{A}$ in terms of $\partial_k \mathbf{A}$:

Theorem 4.9 (Theorem 4 and Corollary 5 of [7])**.** *Let $\mathbf{A}$ be a mask such that $\mathcal{E}_{\mathbf{A}} = \operatorname{span}\{e_1, \dots, e_k\}$. If $\|\tfrac{1}{2} S_{\partial_k \mathbf{A}}\| < 1$ (that is, $\tfrac{1}{2} S_{\partial_k \mathbf{A}}$ is contractive), then the vector scheme associated with $\mathbf{A}$ is C^0.*

In fact the authors of [7] show a stronger result, but we only need this special case. Furthermore we have the following result concerning smoothness:

Theorem 4.10 (Theorem 6 and Corollary 7 of [7])**.** *Let* $\mathbf{A}$ *be a mask such that* $\mathcal{E}_{\mathbf{A}} = \operatorname{span}\{e_1, \ldots, e_k\}$. *If the vector scheme associated with* $\partial_k \mathbf{A}$ *is* C^ℓ *for* $\ell \geq 0$*, then the vector scheme associated with* $\mathbf{A}$ *is* $C^{\ell+1}$.

Remark 4.11. In the formulation of the result of Theorem 4.10 in [7], $S_{\mathbf{A}}$ is required to be convergent. However, if the scheme associated with $\partial_k \mathbf{A}$ converges to a C^ℓ function, then the scheme associated with $\frac{1}{2}\partial_k \mathbf{A}$ is contractive. This implies the convergence of $\mathbf{A}$.

Before we present the smoothing procedure, we prove a lemma which is useful later.

Lemma 4.12. *Let* $\mathbf{A}$ *be a mask. Then we have*

$$\mathcal{E}_{\mathbf{A}} = \{v \in \mathbb{R}^p : \mathbf{A}^*(1)v = 2v \text{ and } \mathbf{A}^*(-1)v = 0\}.$$

Proof. It follows immediately from eq. (4.4) and the definition of a symbol that $A^0 = \frac{1}{2}\big(\mathbf{A}^*(1) + \mathbf{A}^*(-1)\big)$ and $A^1 = \frac{1}{2}\big(\mathbf{A}^*(1) - \mathbf{A}^*(-1)\big)$. This, together with eq. (4.5), implies the claim of the lemma. □

4.4.2 Preparation for smoothing

We would like to modify a given mask $\mathbf{B}$ of a C^ℓ vector subdivision scheme to obtain a new scheme $S_{\mathbf{A}}$ which is $C^{\ell+1}$. The idea is to define $\mathbf{A}$ such that $\partial_k \mathbf{A} = \mathbf{B}$, i.e. such that $\Delta_k S_{\mathbf{A}} = \frac{1}{2} S_{\mathbf{B}} \Delta_k$ is satisfied for some k. If we can prove that $\mathcal{E}_{\mathbf{A}} = \operatorname{span}\{e_1, \ldots, e_k\}$, then by Theorem 4.9, the scheme $S_{\mathbf{A}}$ is $C^{\ell+1}$. There are some immediate questions:

(I) Under what conditions on a mask $\mathbf{B}$ can we define a mask $\mathbf{A}$ such that $\partial_k \mathbf{A} = \mathbf{B}$?

(II) How to choose k?

In order to answer these questions, we have to study in more details the derived scheme $\partial_k \mathbf{A}$ and its "inverse".

Definition 4.13. Given a mask $\mathbf{A}$, we denote by p its dimension, i.e. $A_i \in \mathbb{R}^{p \times p}$ for $i \in \mathbb{Z}$. For $k \in \{1, \ldots, p\}$, we use the block notation

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

with $\mathbf{A}_{11}$ of size $(k \times k)$.

The following result is the main tool for our smoothing procedure. It also gives an answer to question (I).

Lemma 4.14. *Let* $\mathbf{A}, \mathbf{B}$ *be masks of dimension* p. *With the notation of Definition 4.13 we have*

(a) If there exists $k \in \{1, \dots, p\}$ *such that* $\mathbf{A}_{11}^*(-1) = 0, \mathbf{A}_{21}^*(-1) = 0$ *and* $\mathbf{A}_{21}^*(1) = 0$, *then we can define a mask* $\partial_k \mathbf{A}$ *by* $\Delta_k S_{\mathbf{A}} = \frac{1}{2} S_{\partial_k \mathbf{A}} \Delta_k$.

(b) If there exists $k \in \{1, \dots, p\}$ *such that* $\mathbf{B}_{12}^*(1) = 0$, *then we can define a mask* $\mathcal{I}_k \mathbf{B}$ *by*

$$\Delta_k S_{\mathcal{I}_k \mathbf{B}} = \tfrac{1}{2} S_{\mathbf{B}} \Delta_k. \tag{4.7}$$

(c) Under the conditions of (a), the mask $\mathcal{I}_k(\partial_k \mathbf{A})$ *exists. In this case,* $\mathcal{I}_k(\partial_k \mathbf{A}) = \mathbf{A}$.

(d) Under the conditions of (b), the mask $\partial_k(\mathcal{I}_k \mathbf{B})$ *exists. In this case,* $\partial_k(\mathcal{I}_k \mathbf{B}) = \mathbf{B}$.

Proof. Under the assumptions of (a), the matrix

$$2 \begin{pmatrix} \mathbf{A}_{11}^*(z)/(z^{-1}+1) & (z^{-1}-1)\mathbf{A}_{12}^*(z) \\ \mathbf{A}_{21}^*(z)/(z^{-2}-1) & \mathbf{A}_{22}^*(z) \end{pmatrix},$$

is a well-defined symbol. If we denote it by $(\partial_k \mathbf{A})^*(z)$, then the equation $\Delta_k S_{\mathbf{A}} = \frac{1}{2} S_{\partial_k \mathbf{A}} \Delta_k$ is satisfied. Indeed, if we write this equation in terms of symbols,

$$\begin{aligned} &\begin{pmatrix} (z^{-1}-1)I_k & 0 \\ 0 & I_{p-k} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{11}^*(z) & \mathbf{A}_{12}^*(z) \\ \mathbf{A}_{21}^*(z) & \mathbf{A}_{22}^*(z) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} (\partial_k \mathbf{A})_{11}^*(z) & (\partial_k \mathbf{A})_{12}^*(z) \\ (\partial_k \mathbf{A})_{21}^*(z) & (\partial_k \mathbf{A})_{22}^*(z) \end{pmatrix} \begin{pmatrix} (z^{-2}-1)I_k & 0 \\ 0 & I_{p-k} \end{pmatrix}. \end{aligned}$$

then it is satisfied in view of the definition of $\partial_k \mathbf{A}$.

Similarly, under the assumptions of (b), the matrix

$$(\mathcal{I}_k \mathbf{B})^*(z) = \frac{1}{2} \begin{pmatrix} (z^{-1}+1)\mathbf{B}_{11}^*(z) & \mathbf{B}_{12}^*(z)/(z^{-1}-1) \\ (z^{-2}-1)\mathbf{B}_{21}^*(z) & \mathbf{B}_{22}^*(z) \end{pmatrix}, \tag{4.8}$$

is a well-defined symbol which satisfies $\Delta_k S_{\mathcal{I}_k \mathbf{B}} = \frac{1}{2} S_{\mathbf{B}} \Delta_k$.

We continue by proving (c). Under the conditions of (a), the symbol $(\partial_k \mathbf{A})^*(z)$ is well-defined. Since $(\partial_k \mathbf{A})_{12}^*(1) = 0$, from (b) it follows that also $\mathcal{I}_k(\partial_k \mathbf{A})$ is well-defined. It is easy to see that $\mathcal{I}_k(\partial_k \mathbf{A}) = \mathbf{A}$.

Statement (d) is proved in a similar way. $\square$

Remark 4.15. Note that if $k = p$ in Lemma 4.14 then $\mathcal{I}_p \mathbf{B} = \frac{z^{-1}+1}{2} I_p$, where $\frac{z^{-1}+1}{2}$ is the smoothing factor in the scalar case.

We have constructed two operators ∂_k and $\mathcal{I}_k$, which (under some conditions) are inverse to each other. Denote by

$\ell_a^k \ldots$ the set of all masks satisfying the conditions of Lemma 4.14 (a).

$\ell_b^k \ldots$ the set of all masks satisfying the conditions of Lemma 4.14 (b).

Then we have

$$\begin{aligned} \partial_k : \quad & \ell_a^k \to \ell_b^k & \qquad \mathcal{I}_k : \quad & \ell_b^k \to \ell_a^k \\ & \mathbf{A} \mapsto \partial_k \mathbf{A}, & & \mathbf{B} \mapsto \mathcal{I}_k \mathbf{B} \end{aligned} \tag{4.9}$$

such that

$$\partial_k(\mathcal{I}_k \mathbf{B}) = \mathbf{B} \quad \text{and} \quad \mathcal{I}_k(\partial_k \mathbf{A}) = \mathbf{A}. \tag{4.10}$$

Remark 4.16. The algebraic conditions in Lemma 4.14 are not sufficient to define a smoothing procedure for a mask $\mathbf{B}$, based on Theorem 4.10. The application of Theorem 4.10 to $\mathcal{I}_k\mathbf{B}$ is based on $\partial_l(\mathcal{I}_k\mathbf{B})$, where l is the dimension of $\mathcal{E}_{\mathcal{I}_k\mathbf{B}}$, while Lemma 4.14 guarantees the existence of $\partial_k(\mathcal{I}_k\mathbf{B})$. Thus the smoothing procedure is possible if $k = l$. In the next section we define a class of masks for which this condition is satisfied if k is chosen as the dimension of $\mathcal{E}_{\mathbf{B}}$. This answers question (II).

Corollary 4.17. *If $\mathbf{A}$ is a mask such that $\mathcal{E}_{\mathbf{A}} = \operatorname{span}\{e_1, \ldots, e_k\}$, then there exists a mask $\partial_k\mathbf{A}$.*

Proof. From Lemma 4.12 we know that $\mathcal{E}_{\mathbf{A}} = \operatorname{span}\{e_1, \ldots, e_k\}$ implies the properties of $\mathbf{A}$ required in Lemma 4.14 (a). □

Corollary 4.18. *For any $\mathbf{B} \in \ell_b^k$ let $\mathbf{C}_{12}^*(z)$ be a symbol such that $\mathbf{B}_{12}^*(z) = (z^{-1} - 1)$ $\mathbf{C}_{12}^*(z)$. Then*

$$(\mathcal{I}_k\mathbf{B})^*(1) = \begin{pmatrix} \mathbf{B}_{11}^*(1) & \frac{1}{2}\mathbf{C}_{12}^*(1) \\ 0 & \frac{1}{2}\mathbf{B}_{22}^*(1) \end{pmatrix}, \quad (\mathcal{I}_k\mathbf{B})^*(-1) = \begin{pmatrix} 0 & \frac{1}{2}\mathbf{C}_{12}^*(-1) \\ 0 & \frac{1}{2}\mathbf{B}_{22}^*(-1) \end{pmatrix}.$$

Proof. This follows directly from eq. (4.8). □

4.4.3 Transformation to the standard basis

Let $\mathbf{B}$ be a mask of a convergent vector subdivision scheme $S_{\mathbf{B}}$. Denote by $k = \dim \mathcal{E}_{\mathbf{B}}$. We define a new mask $\overline{\mathbf{B}}$ such that $\dim \mathcal{E}_{\overline{\mathbf{B}}} = k, \overline{\mathbf{B}} \in \ell_b^k$ and such that $\mathcal{E}_{\mathcal{I}_k\overline{\mathbf{B}}} = \operatorname{span}\{e_1, \ldots, e_k\}$. This is achieved by considering a kind of Jordan normal form of $M_{\mathbf{B}} = \frac{1}{2}(B^0 + B^1)$. First we cite a theorem which is of importance to our analysis:

Theorem 4.19 (Theorem 2.2 of [8]). *Let $\mathbf{B}$ be a mask of a convergent vector subdivision scheme. A basis of $\mathcal{E}_{\mathbf{B}}$ is also a basis of the eigenspace of $M_{\mathbf{B}} = \frac{1}{2}(B^0 + B^1)$ corresponding to the eigenvalue 1. Moreover, the eigenvalues of $M_{\mathbf{B}}$ which are not 1 have modulus less than 1.*

In particular, this implies that if $S_{\mathbf{B}}$ is a convergent vector scheme, then $\mathcal{E}_{\mathbf{B}}$ is the eigenspace of $B^*(1)$ w.r.t. to the eigenvalue 2.

We define a class of feasible subdivision schemes for our smoothing procedure:

Definition 4.20. Let $\mathbf{B}$ be a mask of a convergent vector scheme. We term such a mask *admissible* if the algebraic multiplicity of the eigenvalue 1 of $M_{\mathbf{B}} = \frac{1}{2}(B^0 + B^1)$ equals its geometric multiplicity.

Let $\mathbf{B}$ be an admissible mask and let $\mathcal{V} = \{v_1, \ldots, v_k\}$ be a basis of $\mathcal{E}_{\mathbf{B}}$ (and therefore also a basis of the eigenspace w.r.t. 1 of $M_{\mathbf{B}}$). We define a matrix

$$R = [v_1, \ldots, v_k | Q], \tag{4.11}$$

where the columns of Q span the invariant $(p-k)$-dimensional subspace complementary to $\mathcal{E}_{\mathbf{B}}$. Upon complexification, that space is spanned by the eigenvectors and possibly generalized eigenvectors corresponding to the eigenvalues different from 1. Q completes $\mathcal{V}$ to a basis of $\mathbb{R}^p$ and R is a transformation as in (b) of Lemma 4.8. Define a modified mask $\overline{\mathbf{B}}$ by

$$\overline{B}_i = R^{-1} B_i R, \quad \text{for } i \in \mathbb{Z}. \tag{4.12}$$

Then, by Lemma 4.8 we have that $\mathcal{E}_{\overline{\mathbf{B}}} = \operatorname{span}\{e_1, \ldots, e_k\}$. In particular $\dim \mathcal{E}_{\overline{\mathbf{B}}} = k$. Furthermore

$$M_{\overline{\mathbf{B}}} = \tfrac{1}{2}(\overline{B}^0 + \overline{B}^1) = R^{-1} M_{\mathbf{B}} R = \begin{pmatrix} I_k & 0 \\ 0 & J \end{pmatrix}. \tag{4.13}$$

We have the following result for $\overline{\mathbf{B}}$:

Theorem 4.21. *Let $\mathbf{B}$ be an admissible mask and let $k = \dim \mathcal{E}_{\mathbf{B}}$. Define $\overline{\mathbf{B}}$ by eq. (4.12). Then $\overline{\mathbf{B}}$ has the following properties:*

(a) The mask $\overline{\mathbf{B}}$ is admissible.

(b) If $\mathbf{B}$ is C^ℓ for $\ell \geq 1$ then also $\overline{\mathbf{B}}$ is C^ℓ.

(c) $\mathcal{E}_{\overline{\mathbf{B}}} = \operatorname{span}\{e_1, \ldots, e_k\}$ and $\overline{\mathbf{B}} \in \ell_b^k$.

(d) $\mathcal{E}_{\mathcal{I}_k \overline{\mathbf{B}}} = \operatorname{span}\{e_1, \ldots, e_k\}$.

Proof. By admissibility and convergence, the eigenvalues of J in eq. (4.13), have modulus less than 1. Therefore the eigenspace of $M_{\overline{\mathbf{B}}}$ w.r.t. the eigenvalue 1 is $\operatorname{span}\{e_1, \ldots, e_k\}$ and by Lemma 4.8, $\overline{\mathbf{B}}$ is admissible. This proves (a). Part (b) follows from Lemma 4.8.

We just proved that $\mathcal{E}_{\overline{\mathbf{B}}} = \operatorname{span}\{e_1, \dots, e_k\}$. Since $\overline{\mathbf{B}}^*(1) = \overline{B}^0 + \overline{B}^1 = 2M_{\overline{\mathbf{B}}}$, it follows from eq. (4.13) that $\overline{\mathbf{B}}_{11}^*(1) = 2I_k$ and $\overline{\mathbf{B}}_{12}^*(1) = 0$. In particular, $\overline{\mathbf{B}} \in \ell_b^k$ and $\mathcal{I}_k\overline{\mathbf{B}}$ is well-defined. This proves (c).

In order to prove (d), we use Lemma 4.12 and prove that $\mathcal{E}_{\mathcal{I}_k\overline{\mathbf{B}}} = \{v \in V^n : (\mathcal{I}_k\overline{\mathbf{B}})^*(1)v = 2v \text{ and } (\mathcal{I}_k\overline{\mathbf{B}})^*(-1)v = 0\}$ is spanned by $e_1, \dots, e_k$. From eq. (4.13) it follows that $\overline{\mathbf{B}}_{11}^*(1) = 2I_k$ and $\overline{\mathbf{B}}_{22}^*(1) = 2J$. Now use Corollary 4.18 to gain the block form:

$$(\mathcal{I}_k\overline{\mathbf{B}})^*(1) = \begin{pmatrix} 2I_k & \frac{1}{2}\overline{\mathbf{C}}_{12}^*(1) \\ 0 & J \end{pmatrix}, \quad (\mathcal{I}_k\overline{\mathbf{B}})^*(-1) = \begin{pmatrix} 0 & \frac{1}{2}\overline{\mathbf{C}}_{12}^*(-1) \\ 0 & \frac{1}{2}\overline{\mathbf{B}}_{22}^*(-1) \end{pmatrix}, \tag{4.14}$$

where $\overline{\mathbf{C}}_{12}^*(z)$ is such that $\overline{\mathbf{B}}_{12}^*(z) = (z^{-1} - 1)\overline{\mathbf{C}}_{12}^*(z)$. From the form of these matrices we see that $\operatorname{span}\{e_1, \dots, e_k\} \subseteq \mathcal{E}_{\mathcal{I}_k\overline{\mathbf{B}}}$. Since the eigenspace of $(\mathcal{I}_k\overline{\mathbf{B}})^*(1)$ w.r.t. the eigenvalue 2 is exactly $\operatorname{span}\{e_1, \dots, e_k\}$ (the matrix J only contributes eigenvalues with modulus less than 1), we see that in fact $\operatorname{span}\{e_1, \dots, e_k\} = \mathcal{E}_{\mathcal{I}_k\overline{\mathbf{B}}}$. □

4.4.4 The smoothing procedure for vector schemes

Theorem 4.21 allows us to define the following smoothing procedure which increases the smoothness of a vector scheme with an admissible mask:

Algorithm 4.22. *Let $\mathbf{B}$ be an admissible mask such that $S_{\mathbf{B}}$ is C^ℓ ($\ell \geq 0$). Let $k = \dim \mathcal{E}_{\mathbf{B}}$. We define a mask $\mathbf{A}$ as follows:*

(a) Choose a basis $\mathcal{V}$ of $\mathcal{E}_{\mathbf{B}}$ and define R as in eq. (4.11).

(b) Define $\overline{\mathbf{B}} = R^{-1}\mathbf{B}R$.

(c) Define $\overline{\mathbf{A}} = \mathcal{I}_k\overline{\mathbf{B}}$ as in eq. (4.8).

(d) Define $\mathbf{A} = R\overline{\mathbf{A}}R^{-1}$.

Then $S_{\mathbf{A}}$ is $C^{\ell+1}$. Furthermore, a basis of $\mathcal{E}_{\mathbf{B}}$ is also a basis of $\mathcal{E}_{\mathbf{A}}$. In particular, $\dim \mathcal{E}_{\mathbf{A}} = k$ and $\mathbf{A}$ is admissible.

Proof. In step (b) we obtain a mask $\overline{\mathbf{B}}$ with properties listed in Theorem 4.21. In particular $\overline{\mathbf{A}}$ of step (c) is well-defined and $\mathcal{E}_{\overline{\mathbf{A}}} = \operatorname{span}\{e_1, \dots, e_k\}$. By Lemma 4.14, the derived mask $\partial_k\overline{\mathbf{A}}$ exists and $\partial_k\overline{\mathbf{A}} = \overline{\mathbf{B}}$. Now we apply Theorem 4.10: The vector scheme associated with $\partial_k\overline{\mathbf{A}} = \overline{\mathbf{B}}$ is C^ℓ by assumption and therefore the scheme associated with $\overline{\mathbf{A}}$ is $C^{\ell+1}$. Applying the transformation R does not change the smoothness of the limit function, see Lemma 4.8. Therefore, also the vector scheme $S_{\mathbf{A}}$ is $C^{\ell+1}$.

To prove the last claim, let $\mathcal{V} = \{v_1, \dots, v_k\}$ be the basis of $\mathcal{E}_{\mathbf{B}}$. By Theorem 4.21, $\{e_1, \dots, e_k\}$ is a basis of both $\mathcal{E}_{\overline{\mathbf{B}}}$ and $\mathcal{E}_{\overline{\mathbf{A}}}$. From $Re_i = v_i$ for $i = 1, \dots, k$ it follows immediately that $\mathcal{V} \subseteq \mathcal{E}_{\mathbf{A}}$. Furthermore, from $M_{\overline{\mathbf{A}}} = \frac{1}{2}\overline{\mathbf{A}}^*(1) = \frac{1}{2}(\mathcal{I}_k\overline{\mathbf{B}})^*(1)$ and eq. (4.14) we get

$$M_{\overline{\mathbf{A}}} = \begin{pmatrix} I_k & \frac{1}{4}\overline{\mathbf{C}}_{12}^*(1) \\ 0 & \frac{1}{2}J \end{pmatrix}. \tag{4.15}$$

Thus the eigenspace of $M_{\overline{\mathbf{A}}}$ w.r.t. to the eigenvalue 1 is spanned by $e_1, \ldots, e_k$. All $v \in \mathcal{E}_{\mathbf{A}}$ satisfy $M_{\mathbf{A}}v = v$. Since $M_{\mathbf{A}} = RM_{\overline{\mathbf{A}}}R^{-1}$, the vector $R^{-1}v$ is an eigenvector of $M_{\overline{\mathbf{A}}}$ w.r.t. 1. Thus $R^{-1}v \in \operatorname{span}\{e_1, \ldots, e_k\}$ which gives $v \in \operatorname{span}\{v_1, \ldots, v_k\}$. Therefore $\mathcal{E}_{\mathbf{A}} = \operatorname{span}\mathcal{V}$ and $\dim\mathcal{E}_{\mathbf{A}} = k$.

Since $M_{\overline{\mathbf{A}}}$ and $M_{\mathbf{A}}$ have the same eigenvalues, we see from eq. (4.15) that the algebraic multiplicity of 1 of $M_{\mathbf{A}}$ is also k. Therefore, $\mathbf{A}$ is admissible. □

Corollary 4.23. *Assume that* $\mathbf{B}$ *and* $\mathbf{A}$ *are masks as in Algorithm 4.22. If the support of* $\mathbf{B}$ *is contained in* $[-N_1, N_2]$ *with* $N_1, N_2 \in \mathbb{N}$*, then the support of* $\mathbf{A}$ *is contained in* $[-N_1 - 2, N_2]$*. Therefore the smoothing procedure for vector schemes (Algorithm 4.22) increases the support length at most by* 2.

Corollary 4.24. *Assume that* $\mathbf{B}$ *and* $\mathbf{A}$ *are masks as in Algorithm 4.22. Then their symbols* $\mathbf{B}^*(z), \mathbf{A}^*(z)$ *are related as follows: If* $\mathbf{B}^*(1)$ *has eigenvalues* $2, \lambda_1, \ldots, \lambda_{p-k}$*, then* $\mathbf{A}^*(1)$ *has eigenvalues* $2, \frac{1}{2}\lambda_1, \ldots, \frac{1}{2}\lambda_{p-k}$*. Furthermore, they have the same eigenspace w.r.t. the eigenvalue* 2.

Proof. It is clear that $\overline{\mathbf{B}}^*(1)$ has the same eigenvalues as $\mathbf{B}^*(1)$. Therefore $M_{\overline{\mathbf{B}}} = \frac{1}{2}\overline{\mathbf{B}}^*(1)$ has eigenvalues $1, \frac{1}{2}\lambda_1, \ldots, \frac{1}{2}\lambda_{p-k}$ and J has eigenvalues $\frac{1}{2}\lambda_1, \ldots, \frac{1}{2}\lambda_{p-k}$. From eq. (4.15) we see that $M_{\overline{\mathbf{A}}}$ has eigenvalues $1, \frac{1}{4}\lambda_1, \ldots, \frac{1}{4}\lambda_{p-k}$ and hence $\overline{\mathbf{A}}^*(1)$ has eigenvalues $2, \frac{1}{2}\lambda_1, \ldots, \frac{1}{2}\lambda_{p-k}$. It is clear that $\mathbf{A}^*(1)$ has the same eigenvalues as $\overline{\mathbf{A}}^*(1)$. The statement about the eigenspace follows directly from Algorithm 4.22. □

Note that Corollary 4.24 is in general not true for $\mathbf{B}^*(-1)$ and $\mathbf{A}^*(-1)$ in place of $\mathbf{B}^*(1)$ and $\mathbf{A}^*(1)$. However, Example 4.28 shows that this can well be the case.

An overview of Algorithm 4.22 can be found in Figure 4.1. Algorithm 4.22 allows us to extend the operator $\mathcal{I}_k$ to the set of admissible masks $\mathbf{B}$ with $k = \dim\mathcal{E}_{\mathbf{B}}$. We let

$$\mathcal{I}_k\mathbf{B} := R(\mathcal{I}_k\overline{\mathbf{B}})R^{-1}. \tag{4.16}$$

We call $\mathcal{I}_k$ a *smoothing operator.*

Remark 4.25. Note that if $\mathbf{B}$ is a mask of dimension p and also $\dim\mathcal{E}_{\mathbf{B}} = p$, then $(\mathcal{I}_p\mathbf{B})^*(z) = \frac{z^{-1}+1}{2}\mathbf{B}^*(z)$, independent of the matrix R. Compare also Remark 4.15.

Theorem 4.26. *Let* $\mathbf{B}$ *be a mask of a convergent vector subdivision scheme* $S_{\mathbf{B}}$*. Then* $\mathbf{B}$ *is admissible. Thus the smoothing procedure can be applied to any convergent vector subdivision scheme* $S_{\mathbf{B}}$.

$$
\begin{tikzcd}
S_{\mathbf{B}},\ C^{\ell} \arrow[rr, "\overline{\mathbf{B}} = R^{-1}\mathbf{B}R"] \arrow[d, dashed, "\text{smoothing}"] & & S_{\overline{\mathbf{B}}},\ \text{also } C^{\ell} \arrow[d, "\overline{\mathbf{A}} = \mathcal{I}_k \overline{\mathbf{B}}"] \\
S_{\mathbf{A}},\ \text{also } C^{\ell+1} & & S_{\overline{\mathbf{A}}},\ C^{\ell+1} \arrow[ll, "\mathbf{A} = R\overline{\mathbf{A}}R^{-1}"']
\end{tikzcd}
$$

Figure 4.1: Smoothing procedure for vector subdivision schemes.

This theorem can be seen as a corollary to [8, Theorem 2.2]. In the statement of [8, Theorem 2.2] it is implied that our Theorem 4.26 is true, but it is not proved. We provide a proof here:

Proof of Theorem 4.26. Denote by k the geometric multiplicity of the eigenvalue 1 of $M_{\mathbf{B}} = \frac{1}{2}(B^0 + B^1)$ and by l its algebraic multiplicity. Then $l \geq k$. We want to prove that actually, $l = k$. There is a change of basis T such that $\mathcal{J} = TM_{\mathbf{B}}T^{-1}$ has the partial Jordan normal form

$$\mathcal{J} = \begin{pmatrix} J_1 & & & \\ & \ddots & & \\ & & J_k & \\ & & & R \end{pmatrix},$$

where R has eigenvalues with modulus strictly smaller than 1 and the Jordan blocks $J_1, \ldots, J_k$ correspond to the eigenvalue 1. Since the total size of the blocks $J_1, \ldots, J_k$ is l, the size of one block J_i, for $i = 1, \ldots, k$, lies between 1 and $l - (k - 1)$. It has the form

$$J_i = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}.$$

In [8, Theorem 2.2] it is proved that $M_{\mathbf{B}}^n$, for $n \in \mathbb{N}$, converges as $n \to \infty$. Therefore also $\mathcal{J}^n$ converges as $n \to \infty$. Since

$$\mathcal{J}^n = \begin{pmatrix} J_1^n & & & \\ & \ddots & & \\ & & J_k^n & \\ & & & R^n \end{pmatrix} \quad \text{and} \quad J_i^n = \begin{pmatrix} 1 & n & & * \\ & 1 & \ddots & \\ & & \ddots & n \\ & & & 1 \end{pmatrix},$$

we see that $\mathcal{J}^n$ converges only if all J_i are of size 1, for $i = 1, \ldots, k$. Therefore $l = k$ and every convergent scheme is admissible. In particular, our smoothing procedure can be applied to every convergent subdivision scheme. $\square$

Conclusion 4.27. By iterative application of the smoothing operator $\mathcal{I}_k$, a convergent vector subdivision scheme can be transformed to an arbitrarily smooth vector subdivision scheme.

Example 4.28 (Double-knot cubic spline subdivision)**.** We consider the vector subdivision scheme with symbol

$$\mathbf{B}^*(z) = \frac{1}{8}\begin{pmatrix} 2+6z+z^2 & 2z+5z^2 \\ 5+2z & 1+6z+2z^2 \end{pmatrix}.$$

It is known that this scheme produces C^1 limit curves, see [32]. We apply Algorithm 4.22 to $\mathbf{B}$ to obtain a vector subdivision scheme $\mathbf{A}$ of regularity C^2:

(a) First we find a basis of $\mathcal{E}_\mathbf{B}$ in order to compute the transformation R. The matrices $\mathbf{B}^*(1)$ and $\mathbf{B}^*(-1)$ are given by

$$\mathbf{B}^*(1) = \frac{1}{8}\begin{pmatrix} 9 & 7 \\ 7 & 9 \end{pmatrix}, \quad \mathbf{B}^*(-1) = \frac{1}{8}\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$$

and have the following eigenvalues and eigenvectors

$$\mathbf{B}^*(1)\ldots \quad \text{eigenvalues}: 2, \tfrac{1}{4}, \quad \text{eigenvectors:} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ resp. } \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

$$\mathbf{B}^*(-1)\ldots \quad \text{eigenvalues}: 0, -\tfrac{3}{4}, \quad \text{eigenvectors:} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ resp. } \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore $\mathcal{E}_\mathbf{B}$ is spanned by $\binom{1}{1}$. Since $M_\mathbf{B} = \frac{1}{2}\mathbf{B}^*(1)$, the transformation R is given by the eigenvectors of $\mathbf{B}^*(1)$:

$$R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad R^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(b) We continue by computing $\overline{\mathbf{B}} = R^{-1}\mathbf{B}R$. It has the symbol

$$\overline{\mathbf{B}}^*(z) = \frac{1}{8}\begin{pmatrix} 4(1+z)^2 & 3(z^2-1) \\ -2(z^2-1) & -1+4z-z^2 \end{pmatrix}.$$

(c) From (a) we know that $k = \dim \mathcal{E}_\mathbf{B} = 1$. Therefore, we compute $\overline{\mathbf{A}} = \mathcal{I}_1\overline{\mathbf{B}}$:

$$\overline{\mathbf{A}}^*(z) = \frac{1}{16}\begin{pmatrix} 4z^{-1}(1+z)^3 & -3z^{-1}(z+1) \\ 2z^{-2}(z^2-1)^2 & -1+4z-z^2 \end{pmatrix}.$$

(d) In the last step we go back to the original basis $\mathbf{A} = R\overline{\mathbf{A}}R^{-1}$:

$$\mathbf{A}^*(z) = \frac{1}{32}z^{-2}\begin{pmatrix} z^4+16z^3+18z^2+7z-2 & 3z^4+8z^3+14z^2+z-2 \\ 7z^4+8z^3+12z^2+7z+2 & 5z^4+16z^3+4z^2+z+2 \end{pmatrix}.$$

From the smoothing procedure it follows that $\mathbf{A}$ is C^2 convergent.

We verify that $\mathcal{E}_{\mathbf{A}}$ has indeed the same basis as $\mathcal{E}_{\mathbf{B}}$. We compute

$$\mathbf{A}^*(1) = \frac{1}{8}\begin{pmatrix} 10 & 6 \\ 9 & 7 \end{pmatrix}, \quad \mathbf{A}^*(-1) = \frac{1}{16}\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$$

and their eigenvalues and eigenvectors:

$$\mathbf{A}^*(1)\ldots \quad \text{eigenvalues}: 2, \tfrac{1}{8}, \quad \text{eigenvectors: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ resp. } \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

$$\mathbf{A}^*(-1)\ldots \quad \text{eigenvalues}: 0, -\tfrac{3}{8}, \quad \text{eigenvectors: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ resp. } \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore $\mathcal{E}_{\mathbf{A}}$ is also spanned by $\binom{1}{1}$ and $\dim \mathcal{E}_{\mathbf{A}} = 1$.

Note that the eigenvector with resp. to $\frac{1}{8}$ of $\mathbf{A}^*(1)$ is different from the eigenvector with resp. to $\frac{1}{4}$ of $\mathbf{B}^*(1)$. Therefore we have to use a new transformation matrix R (defined by the eigenvectors of $\mathbf{A}^*(1)$) if we want to apply a second round of smoothing.

Also, comparing the eigenvalues of $\mathbf{A}^*(1)$ and $\mathbf{B}^*(1)$ we see that Corollary 4.24 is satisfied. Note that in this example, also the eigenvalues of $\mathbf{A}^*(-1)$ and $\mathbf{B}^*(-1)$ satisfy Corollary 4.24. This follows from the fact that $\mathbf{B}^*(1)$ and $\mathbf{B}^*(-1)$ have the same eigenvectors and thus R also transforms $\mathbf{B}^*(-1)$ to its Jordan form.

Note that the smoothing procedure increases the support of the mask by 2.

4.5 Smoothing of Hermite subdivision schemes

In this section we describe a similar smoothing procedure for Hermite schemes. We consider Hermite subdivision schemes which operate on data $\mathbf{c} \in \ell^2(\mathbb{Z})$, i.e. on function values and first derivatives. As in the vector case, Hermite subdivision uses matrix-valued masks $\mathbf{A} = \{A_i \in \mathbb{R}^{2\times 2} : i \in \mathbb{Z}\}$ and subdivision operators $S_{\mathbf{A}}$ as defined in eq. (4.2). Input data $\mathbf{c}^0 \in \ell^2(\mathbb{Z})$ is refined via $D^n \mathbf{c}^n = S^n_{\mathbf{A}} \mathbf{c}^0$, where D is the dilation matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

An Hermite subdivision scheme is called *interpolatory* if its mask $\mathbf{A}$ satisfies $A_0 = D$ and $A_{2i} = 0$ for all $i \in \mathbb{Z}\backslash 0$.

We always assume that an Hermite scheme satisfies the *spectral condition*. This condition requires that there is $\varphi \in \mathbb{R}$ such that both the constant sequence $\mathbf{k} = \{\binom{1}{0} : i \in \mathbb{Z}\}$ and the linear sequence $\boldsymbol{\ell} = \{\binom{i+\varphi}{1} : i \in \mathbb{Z}\}$ obey the rule

$$S_{\mathbf{A}}\mathbf{k} = \mathbf{k}, \quad S_{\mathbf{A}}\boldsymbol{\ell} = \tfrac{1}{2}\boldsymbol{\ell}. \tag{4.17}$$

The spectral condition was introduced in [25] and is crucial for the convergence and smoothness analysis of linear Hermite subdivision schemes. If the Hermite scheme is interpolatory we can choose $\varphi = 0$.

We would like to characterize the spectral condition in terms of the symbol of the mask $\mathbf{A}$. As in Section 4.4 we introduce the notation

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix}, \tag{4.18}$$

where $\mathbf{a}_{ij} \in \mathbb{R}$ for $i, j \in \{1, 2\}$.

Lemma 4.29. *A mask $\mathbf{A}$ satisfies the spectral condition eq.* (4.17) *with $\varphi \in \mathbb{R}$ if and only if its symbol $\mathbf{A}^*(z)$ satisfies*

(a) $\mathbf{a}_{11}^*(1) = 2$, $\mathbf{a}_{11}^*(-1) = 0$.

(b) $\mathbf{a}_{21}^*(1) = 0$, $\mathbf{a}_{21}^*(-1) = 0$.

(c) ${\mathbf{a}_{11}^*}'(1) - 2\mathbf{a}_{12}^*(1) = 2\varphi$, ${\mathbf{a}_{11}^*}'(-1) + 2\mathbf{a}_{12}^*(-1) = 0$.

(d) ${\mathbf{a}_{21}^*}'(1) - 2\mathbf{a}_{22}^*(1) = -2$, ${\mathbf{a}_{21}^*}'(-1) + 2\mathbf{a}_{22}^*(-1) = 0$.

Part (a) and (b) relate to the reproduction of constants, whereas (c) and (d) is the reproduction of linear functions.

Proof. The spectral condition eq. (4.17) is equivalent to

$$\sum_{j\in\mathbb{Z}} a_{11}(i-2j) = 1, \quad \sum_{j\in\mathbb{Z}} a_{21}(i-2j) = 0, \tag{4.19}$$

$$\sum_{j\in\mathbb{Z}} a_{11}(i-2j)j + a_{12}(i-2j) = \frac{1}{2}(i-\varphi), \quad \sum_{j\in\mathbb{Z}} a_{21}(i-2j)j + a_{22}(i-2j) = \frac{1}{2}, \tag{4.20}$$

for all $i \in \mathbb{Z}$.

If i is even resp. odd, then eq. (4.19) for $\mathbf{a}_{11}$ becomes $\sum_{j\in\mathbb{Z}} a_{11}(2j) = \sum_{j\in\mathbb{Z}} a_{11}(2j+1) = 1$. This is equivalent to $\mathbf{a}_{11}^*(1) = 2$ and $\mathbf{a}_{11}^*(-1) = 0$. The proof for $\mathbf{a}_{21}$ works analogously. This shows the equivalence of (a) and (b) to the reproduction of constants.

We continue with the first part in eq. (4.20). For i we insert $2i$ and, using eq. (4.19), we get

$$\begin{aligned} \sum_{j\in\mathbb{Z}} a_{11}(2j)(i-j) + a_{12}(2j) &= \frac{1}{2}(2i-\varphi), \\ \sum_{j\in\mathbb{Z}} a_{11}(2j)(-j) + a_{12}(2j) &= -\frac{\varphi}{2}, \\ -\frac{1}{2}\sum_{j\in\mathbb{Z}} a_{11}(2j)(2j) + \sum_{j\in\mathbb{Z}} a_{12}(2j) &= -\frac{\varphi}{2}. \end{aligned} \tag{4.21}$$

Similarly, for $i \to 2i+1$ we get

$$-\frac{1}{2}\sum_{j\in\mathbb{Z}} a_{11}(2j+1)(2j+1) + \sum_{j\in\mathbb{Z}} a_{12}(2j+1) = -\frac{\varphi}{2}. \tag{4.22}$$

From Lemma 4.4 we know that

$$\begin{aligned}
\mathbf{a}_{11}^{*}{}'(1) &= \sum_{j\in\mathbb{Z}} a_{11}(2j+1)(2j+1) + a_{11}(2j)(2j),\\
\mathbf{a}_{11}^{*}{}'(-1) &= \sum_{j\in\mathbb{Z}} a_{11}(2j+1)(2j+1) - a_{11}(2j)(2j).
\end{aligned}$$

Therefore, by adding resp. subtracting eq. (4.21) and eq. (4.22) we gain

$$\begin{aligned}
-\frac{1}{2}\mathbf{a}_{11}^{*}{}'(1) + \mathbf{a}_{12}^{*}(1) &= -\varphi,\\
\frac{1}{2}\mathbf{a}_{11}^{*}{}'(-1) + \mathbf{a}_{12}^{*}(-1) &= 0,
\end{aligned}$$

which is equivalent to (c). Part (d) is proved analogously using the second part in eq. (4.20). □

4.5.1 Convergence and smoothness analysis

In this section we collect results on the HC^ℓ smoothness of Hermite schemes. This follows the lines of [11]. We define the *Taylor operator* T, which was first suggested in [60]:

$$T = \begin{pmatrix} \Delta & -1 \\ 0 & 1 \end{pmatrix}.$$

The Taylor operator is a natural analogue of the operator Δ_1 in vector subdivision resp. the forward difference operator Δ in scalar subdivision. Similar to eq. (4.6), we have the following result:

Lemma 4.30 ([60]). *If the Hermite subdivision scheme associated with a mask* $\mathbf{A}$ *satisfies the spectral condition eq.* (4.17)*, then there exists a mask* $\partial_t\mathbf{A}$ *such that*

$$TS_{\mathbf{A}} = \tfrac{1}{2}S_{\partial_t\mathbf{A}}T. \tag{4.23}$$

The vector scheme *associated with* $\partial_t\mathbf{A}$ *is called* Taylor scheme*.*

Theorem 4.31 ([11]). *Consider an Hermite subdivision scheme which satisfies the spectral condition eq.* (4.17)*. If its Taylor scheme is* C^ℓ *convergent, for* $\ell \geq 0$*, then the Hermite scheme is* $HC^{\ell+1}$ *convergent.*

We would like to mention that as in the vector case (see [7]), this condition is only *sufficient*, not necessary.

4.5.2 Properties of the Taylor scheme

In order to increase the regularity of an Hermite subdivision scheme, the obvious idea is to pass to its Taylor scheme eq. (4.23), smoothen this scheme by the vector smoothing algorithm (Algorithm 4.22) and then use the resulting vector scheme as the Taylor scheme of a new Hermite scheme. The first question which arises in this process is if the last step is always possible, i.e., if the smoothing operator $\mathcal{I}_k$ of eq. (4.16) maps Taylor schemes to Taylor schemes. In order to answer this question we have to study in more detail the Taylor scheme $\partial_t \mathbf{A}$.

Definition 4.32. A mask $\mathbf{B}$ satisfies the *Taylor condition*, if it satisfies

(a) $\mathbf{b}^*_{12}(1) = 0, \mathbf{b}^*_{12}(-1) = 0$.

(b) $\mathbf{b}^*_{22}(1) = 2, \mathbf{b}^*_{22}(-1) = 0$.

(c) $\mathbf{b}^*_{11}(1) + \mathbf{b}^*_{21}(1) = 2$.

Here we use the notation of eq. (4.18).

We prove in Theorem 4.34 that the mask $\partial_t \mathbf{A}$ obtained via eq. (4.23) satisfies this condition. This justifies the name *Taylor condition.*

Remark 4.33. Conditions (a) and (b) of Definition 4.32 relate to $S_{\mathbf{B}}\mathbf{c} = \mathbf{c}$, where $\mathbf{c}$ is the constant sequence $c_i = e_2 = \binom{0}{1}, i \in \mathbb{Z}$. Also, $e_2 \in \mathcal{E}_{\mathbf{B}}$ and therefore $\dim \mathcal{E}_{\mathbf{B}} \geq 1$.

Theorem 4.34. *We have the following connection between masks satisfying the spectral condition eq.* (4.17) *and masks satisfying the Taylor condition (Definition 4.32):*

(a) Let $\mathbf{A}$ *be a mask satisfying the spectral condition. Then we can define a mask* $\partial_t \mathbf{A}$ *such that* $TS_{\mathbf{A}} = \frac{1}{2} S_{\partial_t \mathbf{A}} T$ *is satisfied. Also,* $\partial_t \mathbf{A}$ *satisfies the Taylor condition.*

(b) Let $\mathbf{B}$ *be a mask satisfying the Taylor condition. Then we can define a mask* $\mathcal{I}_t \mathbf{B}$ *such that* $TS_{\mathcal{I}_t \mathbf{B}} = \frac{1}{2} S_{\mathbf{B}} T$ *is satisfied. Also,* $\mathcal{I}_t \mathbf{B}$ *satisfies the spectral condition.*

(c) The mask $\mathcal{I}_t(\partial_t \mathbf{A})$ *resp.* $\partial_t(\mathcal{I}_t \mathbf{B})$ *is well-defined if* $\partial_t \mathbf{A}$ *resp.* $\mathcal{I}_t \mathbf{B}$ *is well-defined. In this case* $\mathcal{I}_t(\partial_t \mathbf{A}) = \mathbf{A}$ *resp.* $\partial_t(\mathcal{I}_t \mathbf{B}) = \mathbf{B}$.

Proof. We consider the general equation $TS_{\mathbf{C}} = \frac{1}{2} S_{\mathbf{D}} T$ in terms of symbols:

$$\begin{pmatrix} z^{-1}-1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{c}^*_{11}(z) & \mathbf{c}^*_{12}(z) \\ \mathbf{c}^*_{21}(z) & \mathbf{c}^*_{22}(z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{d}^*_{11}(z) & \mathbf{d}^*_{12}(z) \\ \mathbf{d}^*_{21}(z) & \mathbf{d}^*_{22}(z) \end{pmatrix} \begin{pmatrix} z^{-2}-1 & -1 \\ 0 & 1 \end{pmatrix},$$

which results in the following equations:

$$(z^{-1}-1)\mathbf{c}^*_{11}(z) - \mathbf{c}^*_{21}(z) = \tfrac{1}{2}\mathbf{d}^*_{11}(z)(z^{-2}-1), \tag{4.24}$$

$$(z^{-1}-1)\mathbf{c}^*_{12}(z) - \mathbf{c}^*_{22}(z) = \tfrac{1}{2}(\mathbf{d}^*_{12}(z) - \mathbf{d}^*_{11}(z)), \tag{4.25}$$

$$\mathbf{c}^*_{21}(z) = \tfrac{1}{2}\mathbf{d}^*_{21}(z)(z^{-2}-1), \tag{4.26}$$

$$\mathbf{c}^*_{22}(z) = \tfrac{1}{2}(\mathbf{d}^*_{22}(z) - \mathbf{d}^*_{21}(z)). \tag{4.27}$$

We also differentiate eqs. (4.24) to (4.26) as it will be useful in the proof. We obtain

$$
\begin{aligned}
& - z^{-2}\mathbf{c}_{11}^*(z) + (z^{-1}-1){\mathbf{c}_{11}^*}'(z) - {\mathbf{c}_{21}^*}'(z) = \tfrac{1}{2}({\mathbf{d}_{11}^*}'(z)(z^{-2}-1) - 2\mathbf{d}_{11}^*(z)z^{-3}), && (4.28)\\
& - z^{-2}\mathbf{c}_{12}^*(z) + (z^{-1}-1){\mathbf{c}_{12}^*}'(z) - {\mathbf{c}_{22}^*}'(z) = \tfrac{1}{2}({\mathbf{d}_{12}^*}'(z) - {\mathbf{d}_{11}^*}'(z)), && (4.29)\\
& {\mathbf{c}_{21}^*}'(z) = \tfrac{1}{2}({\mathbf{d}_{21}^*}'(z)(z^{-2}-1) - 2z^{-3}\mathbf{d}_{21}^*(z)). && (4.30)
\end{aligned}
$$

Proof of (a): In eqs. (4.24) to (4.26), we set $\mathbf{C} = \mathbf{A}$ and $\mathbf{D} = \partial_t\mathbf{A}$. Then we get

$$
\begin{aligned}
(\partial_t\mathbf{A})_{11}^*(z) &= 2\Big(\frac{\mathbf{a}_{11}^*(z)}{z^{-1}+1} - \frac{\mathbf{a}_{21}^*(z)}{z^{-2}-1}\Big),\\
(\partial_t\mathbf{A})_{12}^*(z) &= 2\Big((z^{-1}-1)\mathbf{a}_{12}^*(z) - \mathbf{a}_{22}^*(z) + \frac{\mathbf{a}_{11}^*(z)}{z^{-1}+1} - \frac{\mathbf{a}_{21}^*(z)}{z^{-2}-1}\Big),\\
(\partial_t\mathbf{A})_{21}^*(z) &= 2\frac{\mathbf{a}_{21}^*(z)}{z^{-2}-1},\\
(\partial_t\mathbf{A})_{22}^*(z) &= 2\Big(\mathbf{a}_{22}^*(z) + \frac{\mathbf{a}_{21}^*(z)}{z^{-2}-1}\Big),
\end{aligned}
$$

which is well-defined by the spectral condition (Lemma 4.29). Note that we only need the first two conditions of Lemma 4.29 (reproduction of constants) to define $\partial_t\mathbf{A}$.

We now show that $\partial_t\mathbf{A}$ satisfies the Taylor condition. Setting $z = 1$ resp. $z = -1$ in eq. (4.28) and using the spectral condition for $\mathbf{A}$ we get

$$
(\partial_t\mathbf{A})_{11}^*(1) = 2\mathbf{a}_{22}^*(1), \quad (\partial_t\mathbf{A})_{11}^*(-1) = 4\mathbf{a}_{12}^*(-1) + 2\mathbf{a}_{22}^*(-1).
$$

Setting $z = 1$ resp. $z = -1$ in (4.25) we obtain

$$
\begin{aligned}
(\partial_t\mathbf{A})_{12}^*(1) &= -2\mathbf{a}_{22}^*(1) + (\partial_t\mathbf{A})_{11}^*(1) = 0,\\
(\partial_t\mathbf{A})_{12}^*(-1) &= -4\mathbf{a}_{12}^*(-1) - 2\mathbf{a}_{22}^*(-1) + (\partial_t\mathbf{A})_{11}^*(-1) = 0.
\end{aligned}
$$

This proves that part (a) of Definition 4.32 is satisfied.

Equation (4.30) implies $(\partial_t\mathbf{A})_{21}^*(1) = 2 - 2\mathbf{a}_{22}^*(1)$ and $(\partial_t\mathbf{A})_{21}^*(-1) = -2\mathbf{a}_{22}^*(-1)$. From eq. (4.27) we obtain

$$
\begin{aligned}
(\partial_t\mathbf{A})_{22}^*(1) &= 2\mathbf{a}_{22}^*(1) + (\partial_t\mathbf{A})_{21}^*(1) = 2,\\
(\partial_t\mathbf{A})_{22}^*(-1) &= 2\mathbf{a}_{22}^*(-1) + (\partial_t\mathbf{A})_{21}^*(-1) = 0.
\end{aligned}
$$

This concludes part (b) of Definition 4.32. We come to part (c) of Definition 4.32:

$$
(\partial_t\mathbf{A})_{11}^*(1) + (\partial_t\mathbf{A})_{21}^*(1) = 2\mathbf{a}_{22}^*(1) + (2 - 2\mathbf{a}_{22}^*(1)) = 2.
$$

Therefore, we have proved (a) of Theorem 4.34.

Proof of (b): Suppose that $\mathbf{B}$ satisfies the Taylor condition. We let $\mathbf{D} = \mathbf{B}$ and $\mathbf{C} = \mathcal{I}_t\mathbf{B}$ in the above equations. Then

$$\begin{aligned}
(\mathcal{I}_t\mathbf{B})^*_{11}(z) =& \tfrac{1}{2}(z^{-1}+1)(\mathbf{b}^*_{11}(z)+\mathbf{b}^*_{21}(z)),\\
(\mathcal{I}_t\mathbf{B})^*_{12}(z) =& \tfrac{1}{2}\Big(\mathbf{b}^*_{12}(z)-\mathbf{b}^*_{11}(z)-\mathbf{b}^*_{21}(z)+\mathbf{b}^*_{22}(z)\Big)\Big/(z^{-1}-1),\\
(\mathcal{I}_t\mathbf{B})^*_{21}(z) =& \tfrac{1}{2}\mathbf{b}^*_{21}(z)(z^{-2}-1),\\
(\mathcal{I}_t\mathbf{B})^*_{22}(z) =& \tfrac{1}{2}(\mathbf{b}^*_{22}(z)-\mathbf{b}^*_{21}(z)),
\end{aligned}$$

which is well-defined by the Taylor condition.

We continue by showing that $\mathcal{I}_t\mathbf{B}$ satisfies the spectral condition. It is immediately clear from the definition of $\mathcal{I}_t\mathbf{B}$ that (a) and (b) of Lemma 4.29 are satisfied. Furthermore, it is easy to see that

$$\begin{aligned}
(\mathcal{I}_t\mathbf{B})^{*}_{21}{}'(1) - 2(\mathcal{I}_t\mathbf{B})^*_{22}(1) &= -\mathbf{b}^*_{21}(1) - \mathbf{b}^*_{22}(1) + \mathbf{b}^*_{21}(1) = -2,\\
(\mathcal{I}_t\mathbf{B})^{*}_{21}{}'(-1) + 2(\mathcal{I}_t\mathbf{B})^*_{22}(-1) &= \mathbf{b}^*_{21}(1) + \mathbf{b}^*_{22}(-1) - \mathbf{b}^*_{21}(-1) = 0,
\end{aligned}$$

which proves (d) of Lemma 4.29.

From the definition of $\mathcal{I}_t\mathbf{B}$ we see that

$$\begin{aligned}
(\mathcal{I}_t\mathbf{B})^{*}_{11}{}'(-1) + 2(\mathcal{I}_t\mathbf{B})^*_{12}(-1) =& -\tfrac{1}{2}(\mathbf{b}^*_{11}(-1)+\mathbf{b}^*_{21}(-1))\\
& -\tfrac{1}{2}(\mathbf{b}^*_{12}(-1)-\mathbf{b}^*_{11}(-1)-\mathbf{b}^*_{21}(-1)+\mathbf{b}^*_{22}(-1))\\
&= 0.
\end{aligned}$$

Furthermore from eq. (4.29) we obtain

$$(\mathcal{I}_t\mathbf{B})^*_{12}(1) = -\tfrac{1}{2}\Big(\mathbf{b}^{*}_{12}{}'(1) - \mathbf{b}^{*}_{11}{}'(1) + \mathbf{b}^{*}_{22}{}'(1) - \mathbf{b}^{*}_{21}{}'(1)\Big),$$

which implies that

$$(\mathcal{I}_t\mathbf{B})^{*}_{11}{}'(1) - 2(\mathcal{I}_t\mathbf{B})^*_{12}(1) = 2\varphi$$

is fulfilled with $\varphi = \frac{1}{2}(\mathbf{b}^{*}_{12}{}'(1) + \mathbf{b}^{*}_{22}{}'(1) - 1)$. This proves property (c) of Lemma 4.29, concluding the proof of part (b).

Proof of (c): Note that the masks $\partial_t\mathbf{A}$ resp. $\mathcal{I}_t\mathbf{B}$ are well defined iff $\mathbf{a}^*_{11}(-1) = \mathbf{a}^*_{21}(-1) = \mathbf{a}^*_{21}(1) = 0$ resp. iff $\mathbf{b}^*_{12}(1) - \mathbf{b}^*_{11}(1) - \mathbf{b}^*_{21}(1) + \mathbf{b}^*_{22}(1) = 0$. Therefore in order to merely define $\partial_t\mathbf{A}$ and $\mathcal{I}_t\mathbf{B}$ the spectral resp. the Taylor condition are not necessary.

If $\partial_t\mathbf{A}$ is well-defined, i.e. if $\mathbf{a}^*_{11}(-1) = \mathbf{a}^*_{21}(-1) = \mathbf{a}^*_{21}(1) = 0$ then it is easy to see that

$$(\partial_t\mathbf{A})^*_{12}(z) - (\partial_t\mathbf{A})^*_{11}(z) - (\partial_t\mathbf{A})^*_{21}(z) + (\partial_t\mathbf{A})^*_{12}(z) = 2(z^{-1}-1)\mathbf{a}^*_{12}(z),$$

which evaluates to 0 at $z = 1$. Thus $\mathcal{I}_t(\partial_t\mathbf{A})$ is well-defined.

On the other hand, it is immediately clear from the definition of $\mathcal{I}_t\mathbf{B}$ that $(\mathcal{I}_t\mathbf{B})^*_{11}(-1) = (\mathcal{I}_t\mathbf{B})^*_{21}(-1) = (\mathcal{I}_t\mathbf{B})^*_{21}(1) = 0$ and thus $\partial_t(\mathcal{I}_t\mathbf{B})$ is well-defined. The rest of the statement follows from the definition of $\partial_t\mathbf{A}$ and $\mathcal{I}_t\mathbf{B}$. $\square$

Similarly to Section 4.4 we define

$\ell_t^k \dots$ the set of masks $\mathbf{B}$ satisfying the Taylor condition (Definition 4.32) with $k = \dim \mathcal{E}_{\mathbf{B}}$.

We would like to find out under what condition the smoothing operator $\mathcal{I}_k$ for vector schemes (4.16) satisfies $\mathcal{I}_k(\ell_t^k) \subseteq \ell_t^k$. Note that $k \in \{1, 2\}$, since in this section we consider the space $\mathbb{R}^2$. We start by finding the transformation R of (4.11) of a mask in ℓ_t^k:

Lemma 4.35. *Let $\mathbf{B} \in \ell_t^k$. As in Section 4.4 denote by $M_{\mathbf{B}} = \frac{1}{2}(B^0 + B^1)$. Then $M_{\mathbf{B}}$ has the following eigenvalues and eigenvectors:*

$$1 \quad \textit{with eigenvector} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\tfrac{1}{2}\mathbf{b}_{11}^*(1) \quad \textit{with eigenvector} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Furthermore, the algebraic multiplicity of the eigenvalue 1 of $M_{\mathbf{B}}$ equals its geometric multiplicity. Therefore, the matrix

$$R = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \quad \textit{with inverse} \quad R^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

is a transformation of the form (4.11).

Proof. From Definition 4.32 we immediately get

$$M_{\mathbf{B}} = \tfrac{1}{2}(B^0 + B^1) = \tfrac{1}{2}\mathbf{B}^*(1) = \begin{pmatrix} \frac{1}{2}\mathbf{b}_{11}^*(1) & 0 \\ \frac{1}{2}\mathbf{b}_{21}^*(1) & 1 \end{pmatrix}.$$

The eigenvalues of $M_{\mathbf{B}}$ can now be read from the diagonal. Also, it is clear that $\binom{0}{1}$ is an eigenvector with eigenvalue 1. For the other eigenvector use property (c) of Definition 4.32:

$$\begin{aligned} M_{\mathbf{B}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2}\mathbf{b}_{11}^*(1) & 0 \\ \frac{1}{2}\mathbf{b}_{21}^*(1) & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mathbf{b}_{11}^*(1) \\ \frac{1}{2}\mathbf{b}_{21}^*(1) - 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mathbf{b}_{11}^*(1) \\ -\frac{1}{2}\mathbf{b}_{11}^*(1) \end{pmatrix} \\ &= \tfrac{1}{2}\mathbf{b}_{11}^*(1)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

If the eigenvalue 1 of $M_{\mathbf{B}}$ has algebraic multiplicity 2, then $\mathbf{b}_{11}^*(1) = 2$. The Taylor condition then implies $\mathbf{b}_{21}^*(1) = 0$ and therefore $M_{\mathbf{B}}$ is the identity matrix. Therefore, the geometric multiplicity of 1 is also 2.

The case of algebraic multiplicity 1 is clear. □

A direct consequence of these observations is that the eigenvalue condition for admissibility (Definition 4.20) is automatically fulfilled for schemes in ℓ_t^k.

Corollary 4.36. *Let* $\mathbf{B}$ *be a mask satisfying the Taylor condition (Definition 4.32) and let* $S_{\mathbf{B}}$ *be convergent. Then* $\mathbf{B} \in \ell_t^2 \Leftrightarrow \mathbf{b}_{21}^*(1) = 0$*. Similarly,* $\mathbf{B} \in \ell_t^1 \Leftrightarrow \mathbf{b}_{21}^*(1) \neq 0$*.*

Proof. The "only if" part follows from Lemma 4.35 in both cases. On the other hand, if $\mathbf{b}_{21}^*(1) = 0$, then $M_{\mathbf{B}}$ is the identity matrix and the eigenspace of 1 is spanned by $\{e_1, e_2\}$. Since $\mathbf{B}$ is convergent, also $\mathcal{E}_{\mathbf{B}}$ is spanned by $\{e_1, e_2\}$ and thus $\mathbf{B} \in \ell_t^2$. The case for $\mathbf{b}_{21}^*(1) \neq 0$ works analogously. □

Theorem 4.37. *Let* $\mathbf{B} \in \ell_t^k$ *and let its vector scheme be convergent. Let* $\mathcal{I}_k$ *be the smoothing operator for vector schemes, see Section 4.4. Then we have*

(i) If $k = 2$*, then* $\mathcal{I}_2\mathbf{B} \in \ell_t^2$*.*

(ii) If $k = 1$*, then* $\mathcal{I}_1\mathbf{B} \in \ell_t^1$ *iff* $\mathbf{b}_{11}^*(z) + \mathbf{b}_{21}^*(z) - \mathbf{b}_{12}^*(z) - \mathbf{b}_{22}^*(z)$ *has a root at* 1 *of multiplicity at least* 2*.*

Note that the Taylor condition (Definition 4.32) implies that there exists a root at 1. Here we need that the multiplicity is at least 2.

Proof. Note that from Theorem 4.26 we know that $\mathbf{B}$ is admissible and therefore we can apply $\mathcal{I}_k$. We start with proving (i). This is the trivial case. If $k = 2$ then $(\mathcal{I}_2\mathbf{B})^*(z) = \frac{z^{-1}+1}{2}\mathbf{B}^*(z)$ (see Remark 4.25). In particular, $(\mathcal{I}_2\mathbf{B})^*(1) = \mathbf{B}^*(1)$ and $(\mathcal{I}_2\mathbf{B})^*(-1) = 0$. Therefore, $\mathcal{I}_2\mathbf{B}$ satisfies the Taylor condition (Definition 4.32).

We continue with (ii): Recall from Section 4.4 that $\mathcal{I}_1\mathbf{B} = R(\mathcal{I}_1\overline{\mathbf{B}})R^{-1}$ with $\overline{\mathbf{B}} = R^{-1}\mathbf{B}R$. In Lemma 4.35 the matrix R is computed. Therefore $\overline{\mathbf{B}}$ is given by

$$\overline{\mathbf{B}} = \begin{pmatrix} \overline{\mathbf{b}}_{11} & \overline{\mathbf{b}}_{12} \\ \overline{\mathbf{b}}_{21} & \overline{\mathbf{b}}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{12} + \mathbf{b}_{22} & \mathbf{b}_{11} + \mathbf{b}_{21} - \mathbf{b}_{12} - \mathbf{b}_{22} \\ \mathbf{b}_{12} & \mathbf{b}_{11} - \mathbf{b}_{12} \end{pmatrix}.$$

Combining this with Corollary 4.18 we obtain

$$(\mathcal{I}_1\overline{\mathbf{B}})^*(1) = \begin{pmatrix} 2 & \frac{1}{2}\overline{\mathbf{c}}_{12}^*(1) \\ 0 & \frac{1}{2}\mathbf{b}_{11}^*(1) \end{pmatrix} \quad \text{and} \quad (\mathcal{I}_1\overline{\mathbf{B}})^*(-1) = \begin{pmatrix} 0 & \frac{1}{2}\overline{\mathbf{c}}_{12}^*(-1) \\ 0 & \frac{1}{2}\mathbf{b}_{11}^*(-1) \end{pmatrix},$$

where $\overline{\mathbf{c}}_{12}^*(z)$ is such that $\overline{\mathbf{b}}_{12}^*(z) = (z^{-1} - 1)\overline{\mathbf{c}}_{12}^*(z)$. Therefore

$$(\mathcal{I}_1\mathbf{B})^*(1) = R(\mathcal{I}_1\overline{\mathbf{B}})^*(1)R^{-1} = \begin{pmatrix} \frac{1}{2}\mathbf{b}_{11}^*(1) & 0 \\ 2 + \frac{1}{2}(\overline{\mathbf{c}}_{12}^*(1) - \mathbf{b}_{11}^*(1)) & 2 \end{pmatrix} \quad \text{and} \tag{4.31}$$

$$(\mathcal{I}_1\mathbf{B})^*(-1) = R(\mathcal{I}_1\overline{\mathbf{B}})^*(-1)R^{-1} = \begin{pmatrix} \frac{1}{2}\mathbf{b}_{11}^*(-1) & 0 \\ \frac{1}{2}(\overline{\mathbf{c}}_{12}^*(-1) - \mathbf{b}_{11}^*(-1)) & 0 \end{pmatrix}.$$

Therefore $\mathcal{I}_1\mathbf{B}$ satisfies the Taylor condition if and only if $\overline{\mathbf{c}}_{12}^*(1) = 0$. This is equivalent to $\overline{\mathbf{b}}_{12}^*(z) = \mathbf{b}_{11}^*(z) + \mathbf{b}_{21}^*(z) - \mathbf{b}_{12}^*(z) - \mathbf{b}_{22}^*(z)$ having a root of multiplicity 2 at 1. □

Therefore, in general, $\mathcal{I}_1(\ell_t^1) \nsubseteq \ell_t^1$. Nevertheless, we have the following lemma:

Lemma 4.38. *Let* $\mathbf{B}$ *be a mask with* $\mathcal{E}_{\mathbf{B}} = \operatorname{span}\{e_2\}$ *and* $\mathbf{b}_{11}^*(1) \neq 2$. *Then there exists a transformation* S *such that* $\widetilde{\mathbf{B}} = S^{-1}\mathbf{B}S$ *satisfies the Taylor condition (Definition 4.32) and* $\mathcal{E}_{\widetilde{\mathbf{B}}} = \operatorname{span}\{e_2\}$.

Proof. If $\mathbf{B}$ satisfies $\mathcal{E}_{\mathbf{B}} = \operatorname{span}\{e_2\}$ it follows that

$$\mathbf{B}^*(1) = \begin{pmatrix} a & 0 \\ b & 2 \end{pmatrix}, \quad \mathbf{B}^*(-1) = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix},$$

with $a, b, c, d \in \mathbb{R}$ and $a \neq 2$. Define S by

$$S = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix},$$

with $e = 1 + \frac{b}{a-2}$. Then we obtain

$$\widetilde{\mathbf{B}}^*(1) = \begin{pmatrix} a & 0 \\ 2-a & 2 \end{pmatrix}, \quad \widetilde{\mathbf{B}}^*(-1) = \begin{pmatrix} c & 0 \\ d-ec & 0 \end{pmatrix}.$$

Therefore $\widetilde{\mathbf{B}}$ satisfies the Taylor condition. Also, it is clear that $e_2 \in \mathcal{E}_{\widetilde{\mathbf{B}}}$. Let $v \in \mathcal{E}_{\widetilde{\mathbf{B}}}$. Then $Sv \in \mathcal{E}_{\mathbf{B}} = \operatorname{span}\{e_2\}$ and $v \in \operatorname{span}\{S^{-1}e_2\} = \operatorname{span}\{e_2\}$. Therefore also $\mathcal{E}_{\widetilde{\mathbf{B}}} = \operatorname{span}\{e_2\}$. Note that $\widetilde{\mathbf{b}}_{11}^*(1) \neq 2$. $\square$

Corollary 4.39. *If* $\mathbf{B} \in \ell_t^1$ *and the vector scheme* $S_{\mathbf{B}}$ *is convergent then* $\widetilde{\mathcal{I}_1(\mathbf{B})} \in \ell_t^1$. *Furthermore, the vector scheme of* $\widetilde{\mathcal{I}_1(\mathbf{B})}$ *is convergent.*

Proof. Since $\mathbf{B} \in \ell_t^1$, we know that $\mathcal{E}_{\mathbf{B}} = \operatorname{span}\{e_2\}$. By Algorithm 4.22 it follows that $\mathcal{E}_{\mathcal{I}_1\mathbf{B}}$ has the same basis as $\mathcal{E}_{\mathbf{B}}$ and is thus also spanned by e_2. Equation (4.31) implies $(\mathcal{I}_1\mathbf{B})_{11}^*(1) = \frac{1}{2}\mathbf{b}_{11}^*(1)$. By Lemma 4.35 we know that $\frac{1}{2}\mathbf{b}_{11}^*(1)$ is an eigenvalue of $M_{\mathbf{B}}$. By Theorem 4.19 this eigenvalue is either 1 or has modulus less than 1 (In fact it cannot be 1 since $\dim \mathcal{E}_{\mathbf{B}} = 1$). In particular $(\mathcal{I}_1\mathbf{B})_{11}^*(1) \neq 2$. Therefore, $(\mathcal{I}_1\mathbf{B})^*(z)$ satisfies the conditions of Lemma 4.38.

By Theorem 4.26, the mask $\mathbf{B}$ is admissible. Smoothing with $\mathcal{I}_1$ results in admissible masks (Algorithm 4.22). Therefore $\mathcal{I}_1(\mathbf{B})$ is admissible. Since $\widetilde{\mathcal{I}_1(\mathbf{B})} = S^{-1}\mathcal{I}_1(\mathbf{B})S$, also this mask is admissible and thus convergent. $\square$

4.5.3 The smoothing procedure for Hermite schemes

Theorem 4.37 and Corollary 4.39 allow us to define a smoothing procedure for Hermite schemes:

$$\begin{tikzcd}
\mathbf{A},\, HC^{\ell} \arrow[rr, "\text{compute Taylor scheme}"] \arrow[dd, dashed, "\text{Hermite smoothing}"'] & & \partial_t \mathbf{A},\, C^{\ell-1} \arrow[dd, dashed, "\text{vector smoothing}"] \\
& \widetilde{\mathcal{I}_1(\partial_t \mathbf{A})},\, C^{\ell} \arrow[dl, "\text{apply } \mathcal{I}_t"'] & \\
\mathbf{C},\, HC^{\ell+1} & & \mathcal{I}_k(\partial_t \mathbf{A}), C^{\ell} \arrow[ul, "k=1"'] \arrow[ll, "{k=2,\ \text{apply } \mathcal{I}_t}"]
\end{tikzcd}$$

Figure 4.2: Smoothing algorithm for Hermite subdivision schemes.

Algorithm 4.40. *Let* $\mathbf{A}$ *be a mask satisfying the spectral condition eq.* (4.17) *and assume that the vector scheme associated with* $\partial_t \mathbf{A}$ *is convergent. Let* $m \geq 1$ *be the multiplicity of the root at* 1 *of* $\mathbf{a}^*_{21}(z)$. *We define a new mask* $\mathbf{C}$ *by the following algorithm:*

(a) Compute the Taylor scheme $\partial_t \mathbf{A}$*, see Theorem 4.34.*

(b) If $m \geq 2$ *apply Algorithm 4.22 to obtain* $\mathbf{B} = \mathcal{I}_2(\partial_t \mathbf{A})$. *If* $m = 1$ *apply Algorithm 4.22 and Lemma 4.38 to obtain* $\mathbf{B} = \widetilde{\mathcal{I}_1(\partial_t \mathbf{A})}$.

(c) Define $\mathbf{C} = \mathcal{I}_t(\mathbf{B})$*, see Theorem 4.34.*

Then the mask $\mathbf{C}$ *satisfies the spectral condition eq.* (4.17). *Furthermore, if the vector scheme of* $\partial_t \mathbf{A}$ *is* $C^{\ell-1}$ $(\ell \geq 1)$ *convergent (hence the Hermite scheme of* $\mathbf{A}$ *is* HC^{ℓ}*), then the Hermite scheme of* $\mathbf{C}$ *is* $HC^{\ell+1}$.

Proof. Since $\mathbf{A}$ satisfies the spectral condition, the mask $\partial_t \mathbf{A}$ is well-defined and satisfies the Taylor condition (see Theorem 4.34). We know that $\dim \mathcal{E}_{\partial_t \mathbf{A}} \in \{1, 2\}$. Corollary 4.36 and the definition of $\partial_t \mathbf{A}$ in Theorem 4.34 imply

$$\dim \mathcal{E}_{\partial_t \mathbf{A}} = 2 \Leftrightarrow \partial_t \mathbf{A} \in \ell^2_t \;\Leftrightarrow\; (\partial_t \mathbf{A})^*_{21}(1) = 0 \;\Leftrightarrow\; \mathbf{a}^{*\prime}_{21}(1) = 0 \;\Leftrightarrow\; m \geq 2.$$

Similarly, $\dim \mathcal{E}_{\partial_t \mathbf{A}} = 1 \;\Leftrightarrow\; m = 1$.

If $m = 2$, then $\partial_t \mathbf{A} \in \ell^2_t$ and by Theorem 4.37 also $\mathbf{B} = \mathcal{I}_2(\partial_t \mathbf{A}) \in \ell^2_t$. Similarly, if $m = 1$, then $\partial_t \mathbf{A} \in \ell^1_t$ and by Lemma 4.38 the mask $\mathbf{B} = \widetilde{\mathcal{I}_1(\partial_t \mathbf{A})}$ is also in ℓ^1_t.

Now we can define $\mathbf{C} = \mathcal{I}_t(\mathbf{B})$ by using Theorem 4.34. Also by this theorem, the mask $\mathbf{C}$ satisfies the spectral condition. If $\partial_t \mathbf{A}$ is $C^{\ell-1}$ then $\mathbf{B}$ is C^{ℓ} by Algorithm 4.22 and Lemma 4.8. Therefore, the Hermite scheme associated with $\mathbf{C}$ is $HC^{\ell+1}$ since its Taylor scheme is $\partial_t \mathbf{C} = \mathbf{B}$ and therefore C^{ℓ}. $\square$

Theorem 4.41. *Let* $\mathbf{A}$ *be a mask satisfying the spectral condition eq.* (4.17) *and let the vector scheme associated with* $\partial_t \mathbf{A}$ *be convergent. Let* $m \geq 1$ *be the multiplicity of the root at* 1 *of* $\mathbf{a}_{21}^*(z)$. *Through Algorithm 4.40 we obtain a new mask* $\mathbf{C}$ *with symbol*

$$\mathbf{C}^*(z) = \frac{z^{-1}+1}{2}\mathbf{A}^*(z)$$

if $m \geq 2$. *If* $m = 1$ *then* $\mathbf{a}_{22}^*(1) \neq 2$ *and* $\mathbf{C}$ *is given by*

$$\begin{aligned}
\mathbf{c}_{11}^*(z) = {} & \tfrac{1}{2}(z^{-1}+1)\Big(\mathbf{a}_{12}^*(z)\Big((s-s^2)z^{-3} + s^2z^{-2} + (s^2-1)z^{-1} - (s^2+s)\Big) \\
& + \mathbf{a}_{11}^*(z)\Big(s(z^{-1}-1)(1-s)+s\Big) + \mathbf{a}_{22}^*(z)(s(z^{-2}-1)-1)(s-1) \\
& + \mathbf{a}_{21}^*(z)(s^2-s)\Big), \\
\mathbf{c}_{12}^*(z) = {} & \tfrac{1}{2}\Big(\mathbf{a}_{12}^*(z)\Big((1-s)^2z^{-3} + s(1-s)z^{-2} + s(1-s)z^{-1} + s^2\Big) \\
& + \mathbf{a}_{22}^*(z)\Big(-(z^{-2}-1)(1-s)^2 + s - 1\Big) \\
& + \mathbf{a}_{11}^*(z)\Big((z^{-1}-1)(1-s)^2 + 1 - s\Big) - \mathbf{a}_{21}^*(z)(1-s)^2\Big)\Big/(z^{-1}-1), \\
\mathbf{c}_{21}^*(z) = {} & \tfrac{1}{2}(z^{-2}-1)\Big(\mathbf{a}_{12}^*(z)\Big(-s^2z^{-3} + (s+s^2)(z^{-2}+z^{-1}) - (s+1)^2\Big) \\
& + \mathbf{a}_{11}^*(z)s(1-s(z^{-1}-1)) + \mathbf{a}_{22}^*(z)s(s(z^{-2}-1)-1) + s^2\mathbf{a}_{21}^*(z)\Big), \\
\mathbf{c}_{22}^*(z) = {} & \tfrac{1}{2}\Big(\mathbf{a}_{12}^*(z)\Big((s^2-s)z^{-3} + (1-s^2)z^{-2} - s^2z^{-1} + (s^2+s)\Big) \\
& + \mathbf{a}_{11}^*(z)(1-s)(1-s(z^{-1}-1)) + \mathbf{a}_{22}^*(z)s((1-s)(z^{-2}-1)+1) \\
& + \mathbf{a}_{21}^*(z)(s-s^2)\Big).
\end{aligned}$$

where $s = 1 + \frac{\mathbf{a}_{12}^*(1)}{2-\mathbf{a}_{22}^*(1)}$.

In the special case of $\mathbf{a}_{12}^*(1) = 0$, *i.e.* $s = 1$, *this reduces to*

$$\begin{aligned}
\mathbf{c}_{11}^*(z) &= \tfrac{1}{2}(z^{-1}+1)\Big((z^{-2}-2)\mathbf{a}_{12}^*(z) + \mathbf{a}_{11}^*(z)\Big), \\
\mathbf{c}_{12}^*(z) &= \tfrac{1}{2}\frac{\mathbf{a}_{12}^*(z)}{(z^{-1}-1)}, \\
\mathbf{c}_{21}^*(z) &= \tfrac{1}{2}(z^{-2}-1)\Big(\mathbf{a}_{21}^*(z) - \mathbf{a}_{11}^*(z)(z^{-1}-2) \\
&\qquad + \mathbf{a}_{22}^*(z)(z^{-2}-2) - \mathbf{a}_{12}^*(z)(z^{-1}-2)(z^{-2}-2)\Big), \\
\mathbf{c}_{22}^*(z) &= \tfrac{1}{2}(\mathbf{a}_{22}^*(z) - (z^{-1}-2)\mathbf{a}_{12}^*(z)).
\end{aligned}$$

Furthermore, if $\mathbf{A}$ *satisfies the spectral condition with* $\varphi \in \mathbb{R}$, *then* $\mathbf{C}$ *satisfies the spectral condition with* $\varphi - \frac{1}{2}$. *In particular, smoothing of interpolatory schemes does not result in interpolatory schemes.*

Proof. We start with the case $m \geq 2$. Then from Remark 4.25 we know that $(\mathcal{I}_2(\partial_t \mathbf{A}))^*(z) = \frac{z^{-1}+1}{2}(\partial_t \mathbf{A})^*(z)$. From the definition of $\mathbf{C}$ we know that $(\partial_t \mathbf{C})^*(z) = \frac{z^{-1}+1}{2}(\partial_t \mathbf{A})^*(z)$. Using the definition of ∂_t in Theorem 4.34 it follows immediately that $\mathbf{C}^*(z)$ has the above form. In order to prove the part concerning φ note that ${\mathbf{c}_{11}^*}'(1) = -1 + {\mathbf{a}_{11}^*}'(1)$. Therefore

$$
{\mathbf{c}_{11}^*}'(1) - 2\mathbf{c}_{12}^*(1) = -1 + {\mathbf{a}_{11}^*}'(1) - 2\mathbf{a}_{12}^*(1) = -1 + 2\varphi = 2(\varphi - \tfrac{1}{2}).
$$

We continue with $m = 1$. From the proof of Theorem 4.34 we know that $(\partial_t \mathbf{A}_{11})^*(1) = 2\mathbf{a}_{22}^*(1)$. Lemma 4.35 implies that $\mathbf{a}_{22}^*(1)$ is an eigenvalue of $M_{\partial_t \mathbf{A}}$. Therefore, by Theorem 4.19, $\mathbf{a}_{22}^*(1) \neq 2$.

Before we prove the above form of $\mathbf{C}$, we have to show that $\mathbf{c}_{12}^*(z)$ is well-defined. Evaluating the numerator at $z = 1$, we obtain:

$$
\begin{aligned}
&\mathbf{a}_{12}^*(1)((1-s)^2 + 2s(1-s) + s^2) + \mathbf{a}_{22}^*(1)(s-1) + 2(1-s) \\
&= \mathbf{a}_{12}^*(1) - \mathbf{a}_{22}^*(1) + 2 + (\mathbf{a}_{22}^*(1) - 2)s = \mathbf{a}_{12}^*(1) - \mathbf{a}_{22}^*(1) + 2 + (\mathbf{a}_{22}^*(1) - 2) - \mathbf{a}_{12}^*(1) \\
&= 0.
\end{aligned}
$$

In order to go from $\mathbf{A}$ to $\mathbf{B} = \widetilde{\mathcal{I}_1(\partial_t \mathbf{A})}$ in Algorithm 4.40 we have to follow these steps:

$$
\mathbf{A} \to \partial_t \mathbf{A} \to \overline{\partial_t \mathbf{A}} = R^{-1}(\partial_t \mathbf{A})R \to \mathcal{I}_1(\overline{\partial_t \mathbf{A}}) \to \mathbf{B} = S^{-1}R\mathcal{I}_1(\overline{\partial_t \mathbf{A}})R^{-1}S
$$

with R from Lemma 4.35 and S from Lemma 4.38.

From the proofs of Theorem 4.34, Theorem 4.37 and Lemma 4.14 we obtain:

$$
\begin{aligned}
\mathcal{I}_1(\overline{\partial_t \mathbf{A}})_{11}^*(z) &= (z^{-2} - 1)\mathbf{a}_{12}^*(z) + \mathbf{a}_{11}^*(z), \\
\mathcal{I}_1(\overline{\partial_t \mathbf{A}})_{12}^*(z) &= -\mathbf{a}_{12}^*(z), \\
\mathcal{I}_1(\overline{\partial_t \mathbf{A}})_{21}^*(z) &= (z^{-1} - 1)(z^{-2} - 1)\mathbf{a}_{12}^*(z) - (z^{-2} - 1)\mathbf{a}_{22}^*(z) \\
&\quad + (z^{-1} - 1)\mathbf{a}_{11}^*(z) - \mathbf{a}_{21}^*(z), \\
\mathcal{I}_1(\overline{\partial_t \mathbf{A}})_{22}^*(z) &= -(z^{-1} - 1)\mathbf{a}_{12}^*(z) + \mathbf{a}_{22}^*(z).
\end{aligned}
$$

In order to get $\mathbf{B}$ from $\mathcal{I}_1(\overline{\partial_t \mathbf{A}})$ we have to transform with the matrix

$$
S^{-1}R = \begin{pmatrix} 0 & 1 \\ 1 & -(e+1) \end{pmatrix}, \quad \text{where } S = \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}.
$$

We continue by computing the matrix S of Lemma 4.38, resp. the value $e = 1 + \frac{b}{a-2}$. Thus we have to find the values $a = \mathcal{I}_1(\partial_t \mathbf{A})_{11}^*(1)$ and $b = \mathcal{I}_1(\partial_t \mathbf{A})_{21}^*(1)$. From eq. (4.31) we obtain

$$
\begin{aligned}
a &= \mathcal{I}_1(\partial_t \mathbf{A})_{11}^*(1) = \frac{1}{2}(\partial_t \mathbf{A})_{11}^*(1) = \mathbf{a}_{22}^*(1), \\
b &= \mathcal{I}_1(\partial_t \mathbf{A})_{21}^*(1) = 2 + \frac{1}{2}(\overline{\mathbf{c}}_{12}^*(1) - (\partial_t \mathbf{A})_{11}^*(1)),
\end{aligned}
$$

where $(z^{-1}-1)\overline{\mathbf{c}}_{12}^*(z) = \overline{\partial_t \mathbf{A}}_{12}^*(z)$. It is easy to see that $\overline{\mathbf{c}}_{12}^*(z) = -2\mathbf{a}_{12}^*(z)$, which results in

$$b = \mathcal{I}_1(\partial_t \mathbf{A})_{21}^*(1) = 2 - \mathbf{a}_{12}^*(1) - \mathbf{a}_{22}^*(1).$$

Therefore

$$e = 1 + \frac{b}{a-2} = 1 + \frac{2 - \mathbf{a}_{12}^*(1) - \mathbf{a}_{22}^*(1)}{\mathbf{a}_{22}^*(1) - 2} = \frac{\mathbf{a}_{12}^*(1)}{2 - \mathbf{a}_{22}^*(1)}.$$

Note that the value s in the statement of Theorem 4.41 is exactly $e+1$. Applying the transformation $S^{-1}R$ we obtain:

$$\begin{aligned}
\mathbf{b}_{11}^*(z) &= \mathbf{a}_{12}^*(z)(z^{-1}-1)(s(z^{-2}-1)-1) + \mathbf{a}_{22}^*(z)(-s(z^{-2}-1)+1) \\
&\quad + \mathbf{a}_{11}^*(z)s(z^{-1}-1) - s\,\mathbf{a}_{21}^*(z), \\
\mathbf{b}_{12}^*(z) &= \mathbf{a}_{12}^*(z)(z^{-1}-1)(z^{-2}-1) - \mathbf{a}_{22}^*(z)(z^{-2}-1) + \mathbf{a}_{11}^*(z)(z^{-1}-1) - \mathbf{a}_{21}^*(z), \\
\mathbf{b}_{21}^*(z) &= \mathbf{a}_{12}^*(z)\Big((z^{-2}-1)s - s^2(z^{-2}-1)(z^{-1}-1) + s(z^{-1}-1) - 1\Big) \\
&\quad + \mathbf{a}_{22}^*(z)s(s(z^{-2}-1)-1) + \mathbf{a}_{11}^*(z)s(1-s(z^{-1}-1)) + s^2\mathbf{a}_{21}^*(z), \\
\mathbf{b}_{22}^*(z) &= \mathbf{a}_{12}^*(z)(z^{-2}-1)(1-s(z^{-1}-1)) + \mathbf{a}_{22}^*(z)s(z^{-2}-1) \\
&\quad + \mathbf{a}_{11}^*(z)(1-s(z^{-1}-1)) + s\,\mathbf{a}_{21}^*(z).
\end{aligned}$$

Now applying the operator $\mathcal{I}_t$ to $\mathbf{B}$ we obtain $\mathbf{C}$ as in the statement of the theorem.

We come to the part involving φ. Deriving the equations of $\mathbf{c}_{11}^*(z)$ and $\mathbf{c}_{12}^*(z)$ and evaluating at $z=1$ we obtain:

$$\begin{aligned}
{\mathbf{c}_{11}^*}'(1) - 2\mathbf{c}_{12}^*(1) &= {\mathbf{a}_{11}^*}'(1) - 2\mathbf{a}_{12}^*(1) + (s-1)({\mathbf{a}_{21}^*}'(1) - 2\mathbf{a}_{22}^*(1)) \\
&\quad + 2(s-1) + \frac{1}{2}\mathbf{a}_{12}^*(1) - s - \frac{1}{2}\mathbf{a}_{22}^*(1)(1-s) \\
&= 2\varphi + \frac{1}{2}(\mathbf{a}_{12}^*(1) - \mathbf{a}_{22}^*(1)) - \frac{1}{2}s(2 - \mathbf{a}_{22}^*(1)) \\
&= 2(\varphi - \frac{1}{2}).
\end{aligned}$$

□

Corollary 4.42. *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be masks as in Theorem 4.41. If* $\mathbf{A}$ *has support contained in* $[-N_1, N_2]$ *with* $N_1, N_2 \in \mathbb{N}$*, then the support of* $\mathbf{C}$ *is contained in* $[-N_1 - 1, N_2]$ *if* $m \geq 2$ *and in* $[-N_1 - 5, N_2]$ *if* $m = 1$*. Therefore the smoothing procedure for Hermite schemes (Algorithm 4.40) increases the support length at most by 5.*

Corollary 4.43. *Let* $\mathbf{A}$ *be a mask satisfying the spectral condition eq.* (4.17) *and let the vector scheme associated with* $\partial_t \mathbf{A}$ *be convergent. Denote by* m *the multiplicity of the root at* 1 *of* $\mathbf{a}_{21}^*(z)$*. Denote by* $\mathbf{C}$ *the mask obtained via Algorithm 4.40. Then the root at* 1 *of* $\mathbf{c}_{21}^*(z)$ *also has multiplicity* m*.*

Proof. If $m \geq 2$ this is clear from the definition of $\mathbf{C}$. We continue with $m = 1$. By the spectral condition ${\mathbf{c}_{21}^*}'(1) = -2 + 2\mathbf{c}_{22}^*(1) = -2 + \mathbf{a}_{22}^*(1)$. Therefore ${\mathbf{c}_{21}^*}'(1) = 0$ iff $\mathbf{a}_{22}^*(1) = 2$. By Theorem 4.41 this can not happen if $m = 1$. □

Corollary 4.44. *Let* $\mathbf{A}$ *be a mask satisfying the spectral condition eq.* (4.17) *and let the vector scheme associated with* $\partial_t \mathbf{A}$ *be convergent. Assume that the multiplicity of the root at* 1 *of* $\mathbf{a}_{21}^*(z)$ *is* 1 *and that* $\mathbf{a}_{12}^*(1) = 0$ *(i.e.* $s = 1$*). Denote by* $\mathbf{C}$ *the mask obtained via Algorithm 4.40. Then* $\mathbf{c}_{12}^*(1) = 0$ *iff* $\mathbf{a}_{12}^{*\prime}(1) = 0$.

Proof. From the definition of $\mathbf{C}$ in Theorem 4.41 it is easy to see that $\mathbf{c}_{12}^*(1) = -\frac{1}{2}\mathbf{a}_{12}^{*\prime}(1)$. Therefore $\mathbf{c}_{12}^*(1) = 0$ iff $\mathbf{a}_{12}^{*\prime}(1) = 0$. □

Conclusion 4.45. Consider an Hermite subdivision scheme with mask $\mathbf{A}$ satisfying the spectral condition eq. (4.17) and suppose that its Taylor scheme is of regularity at least C^0 (and hence the Hermite scheme is of regularity at least HC^1).

Via the iterated application of Algorithm 4.40, this Hermite scheme can be transformed to a new Hermite scheme of arbitrarily high regularity.

If the multiplicity of the root at 1 of $\mathbf{a}_{21}^*(z)$ is m, then the root at 1 of every new mask obtained by iterated application of Algorithm 4.40 has the multiplicity m in this component.

If $m = 1$ and $\mathbf{A}$ satisfies $\mathbf{a}_{12}^*(1) = 0$ (i.e. special case $s = 1$), then $r - 1$ iterations of the smoothing process stay within this special case, where r denotes the multiplicity of the root at 1 of $\mathbf{a}_{12}^*(z)$.

Example 4.46. We consider one of the interpolatory Hermite subdivision scheme introduced in [58]. Its mask is given by

$$A_{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{8} \\ \frac{3}{4} & -\frac{1}{8} \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} \\ -\frac{3}{4} & -\frac{1}{8} \end{pmatrix}.$$

It is easy to see that it satisfies the spectral condition eq. (4.17) with $\varphi = 0$. It is well known that this scheme produces the piecewise cubic interpolant of given point-vector input data. In [60] it is proved that its Taylor scheme is C^0 (and thus the original Hermite scheme is HC^1).

We would like to apply Algorithm 4.40 to this scheme to obtain a new Hermite scheme $\mathbf{C}$ of regularity HC^2.

First we compute the symbol:

$$\mathbf{A}^*(z) = \begin{pmatrix} \frac{1}{2}(1+z)^2 z^{-1} & -\frac{1}{8}(1-z^2)z^{-1} \\ \frac{3}{4}(1-z^2)z^{-1} & -\frac{1}{8}z^{-1} + \frac{1}{2} - \frac{1}{8}z \end{pmatrix}.$$

Note that $\mathbf{a}_{21}^*(1) = 0$ and $\mathbf{a}_{12}^*(1) = 0$ both with multiplicity 1. Therefore we are in the special case of $m = 1$ and $s = 1$ of Theorem 4.41.

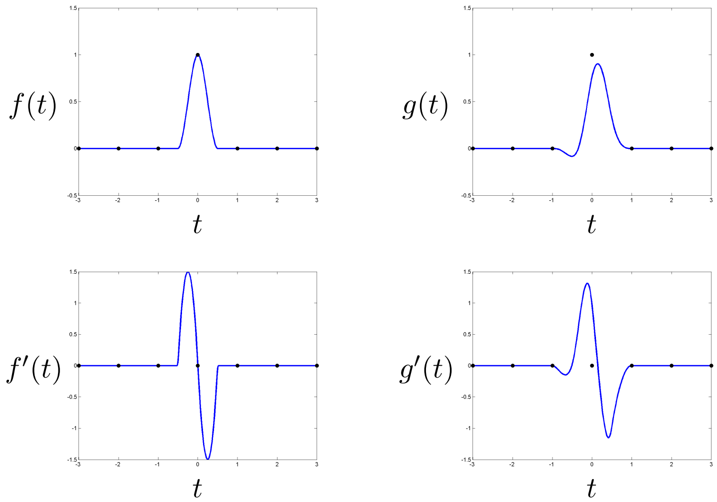

Figure 4.3: Basic limit functions and their first derivatives of the Hermite schemes of Example 4.46. *First column:* interpolatory HC^1 scheme $\mathbf{A}$ with limit function f. *Second column:* non-interpolatory HC^2 scheme $\mathbf{C}$ with limit function g.

We apply Theorem 4.41 to gain the symbol of $\mathbf{C}$:

$$\mathbf{C}^*(z) = \frac{1}{16}\begin{pmatrix} (z^{-1}+1)^2(-z^{-2}+z^{-1}+6+2z) & -z-1 \\ (z^{-2}-1)\Big(z^{-4}-3z^{-3}-3z^{-2}+13z^{-1}+6\Big) & z^{-2}-3z^{-1}+3+z \end{pmatrix}.$$

From Theorem 4.41 we also know that $\mathbf{C}$ satisfies the spectral condition with $\varphi = -\frac{1}{2}$. Therefore the Hermite scheme associated with $\mathbf{C}$ is an HC^2 scheme which is not interpolatory (for the basic limit function of this scheme see Figure 4.3). Note that the support of $\mathbf{C}$ is $[-6, 1]$ and has thus increased from length of 3 to the length of 8.

If we want to apply another round of smoothing, we have to use Theorem 4.41 with $m = 1$ and $s = \frac{14}{15}$.

Example 4.47. We consider one of the de Rham-type Hermite schemes of [24]. Its mask is given by

$$A_{-2} = \frac{1}{8}\begin{pmatrix} \frac{5}{4} & -\frac{3}{8} \\ \frac{9}{2} & -\frac{5}{4} \end{pmatrix}, \quad A_{-1} = \frac{1}{8}\begin{pmatrix} \frac{27}{4} & -\frac{9}{8} \\ \frac{9}{2} & \frac{3}{4} \end{pmatrix},$$

$$A_0 = \frac{1}{8}\begin{pmatrix} \frac{27}{4} & \frac{9}{8} \\ -\frac{9}{2} & \frac{3}{4} \end{pmatrix}, \quad A_1 = \frac{1}{8}\begin{pmatrix} \frac{5}{4} & \frac{3}{8} \\ -\frac{9}{2} & -\frac{5}{4} \end{pmatrix}.$$

This is the de Rham transform of the scheme discussed in Example 4.46. It is easy to see that it satisfies the spectral condition eq. (4.17) with $\varphi = -\frac{1}{2}$. In [11] it is proved that its Taylor scheme is C^1 (and thus the original Hermite scheme is HC^2).

We would like to apply Algorithm 4.40 to this scheme to obtain a new Hermite scheme $\mathbf{C}$ of regularity HC^3.

First we compute the symbol:

$$\mathbf{A}^*(z) = \frac{1}{16} \begin{pmatrix} \frac{1}{2}(z^{-1}+1)(5z+2z+5z^{-1}) & -\frac{3}{4}(z^{-1}-1)(z+4+z^{-1}) \\ 9(z^{-2}-1)(z+1) & \frac{1}{2}(z^{-1}+1)(-5z+8-5z^{-1}) \end{pmatrix}.$$

Note that $\mathbf{a}_{21}^*(1) = 0$ and $\mathbf{a}_{12}^*(1) = 0$ both with multiplicity 1. Therefore, as in Example 4.46, we are in the special case $m = 1$ and $s = 1$ of Theorem 4.41. We apply Theorem 4.41 to gain the symbol of $\mathbf{C}$:

$$\begin{aligned}
\mathbf{c}_{11}^*(z) &= \frac{1}{128}(z^{-1}+1)(-3z^{-4}-9z^{-3}+25z^{-2}+75z^{-1}+36+4z), \\
\mathbf{c}_{12}^*(z) &= -\frac{3}{128}(z+4+z^{-1}), \\
\mathbf{c}_{21}^*(z) &= \frac{1}{128}(z^{-2}-1)\Big(3z^{-5}-7z^{-4}-37z^{-3}+37z^{-2}+128z^{-1}+20-8z\Big), \\
\mathbf{c}_{22}^*(z) &= \frac{1}{128}(3z^{-3}-7z^{-2}-21z^{-1}+21-4z).
\end{aligned}$$

From Theorem 4.41 we also know that $\mathbf{C}$ satisfies the spectral condition with $\varphi = -1$. Therefore the Hermite scheme associated with $\mathbf{C}$ is an HC^3 scheme which is not interpolatory (for the basic limit function of this scheme see Figure 4.4). Note that the support of $\mathbf{C}$ is $[-7, 1]$ and has thus increased from length of 4 to the length of 9.

If we want to apply another round of smoothing, we have to use Theorem 4.41 with $m = 1$ and $s = \frac{41}{44}$.

Remark 4.48. The Examples 4.46 and 4.47 show that the basic limit functions of smoothened schemes are no longer symmetric. This deficiency can be remedied by replacing the smoothened scheme by the average of itself and its mirror reflection.

Conclusion

In this paper we studied a method to construct both vector and Hermite schemes with limit curves of high regularity. This method is a direct generalization of the well known smoothing procedure in scalar subdivision and works by manipulating symbols.

In the Hermite case it is possible to construct schemes of arbitrarily high regularity from an Hermite scheme whose Taylor scheme is at least C^0 (Algorithm 4.40). Our smoothing algorithm increases the support of the mask by a maximum of 5, see Corollary 4.42. This maximum is attained in Example 4.46 and Example 4.47, where we obtain Hermite schemes of regularity HC^2 resp. HC^3.

In the vector case our smoothing procedure (Algorithm 4.22) is restricted to schemes satisfying a certain eigenvalue-condition (these schemes are termed “admissible”, see

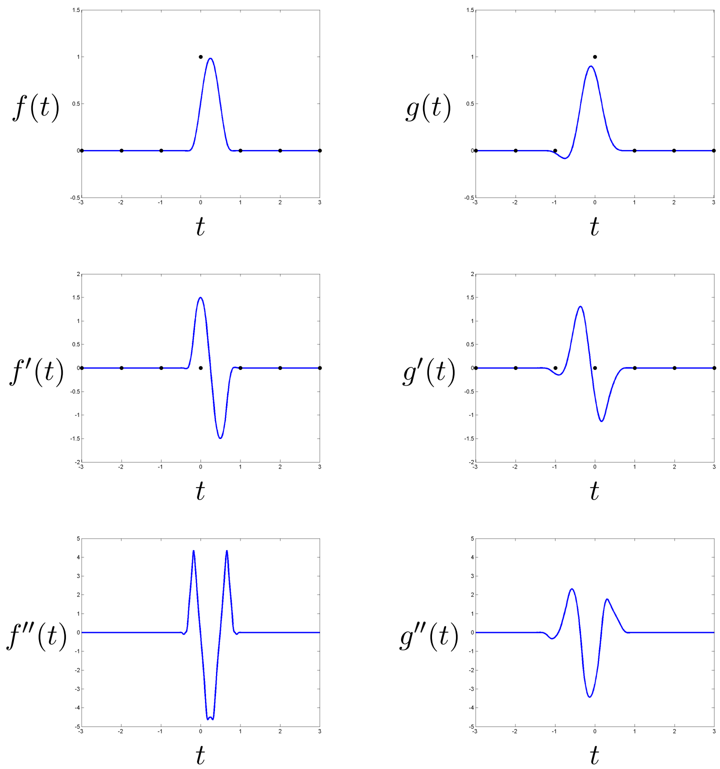

Figure 4.4: Basic limit functions, their first and second derivatives of the Hermite schemes of Example 4.47. *First column:* non-interpolatory HC^2 scheme $\mathbf{A}$ with limit function f. *Second column:* non-interpolatory HC^3 scheme $\mathbf{C}$ with limit function g.

Definition 4.20). We prove in Theorem 4.26, however, that every convergent scheme is admissible. Therefore, a convergent vector scheme can be raised to arbitrarily high smoothness. In contrast to the Hermite case, the support length is only increased by 2 (see Corollary 4.23).

Acknowledgments. The first author gratefully acknowledges the support by the Austrian Science Fund FWF under grant agreements W1230 (doctoral program "Discrete Mathematics") and I705.

Bibliography

[1] L.-E. Andersson and N. F. Stewart. *Mathematics of subdivision surfaces.* SIAM, 2010.

[2] E. Catmull and J. Clark. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer-Aided Design*, 10(6):350–355, 1978.

[3] A. Cavaretta, W. Dahmen, and C. Michelli. Stationary subdivision. *Mem. Amer. Math. Soc.*, 93(453), 1991.

[4] G. M. Chaikin. An algorithm for high-speed curve generation. *Computer Graphics and Image Processing*, 3:346–349, 1974.

[5] P. Chalmovianský and B. Jüttler. A non-linear circle-preserving subdivision scheme. *Adv. Comp. Math.*, 27(4):375–400, 2007.

[6] M. Charina, C. Conti, and L. Romani. Reproduction of exponential polynomials by multivariate non-stationary subdivision schemes with general dilation matrix. *Numerische Mathematik*, 127(2):223–254, 2014.

[7] M. Charina, C. Conti, and T. Sauer. Regularity of multivariate vector subdivision schemes. *Numerical algorithms*, 39(1-3):97–113, 2005.

[8] A. Cohen, N. Dyn, and D. Levin. Stability and rank properties of matrix subdivision schemes. In K. Jetter F. Fontanella and P.-J. Laurent, editors, *Advanced Topics in Multivariate Approximation*, pages 1–13. World Scientific Publishing, 1996.

[9] C. Conti, M. Cotronei, and T. Sauer. Factorisation of Hermite subdivision operators preserving exponentials and polynomials. *Advances in Computational Mathematics*, 42(5):1055–1079, 2016.

[10] C. Conti, N. Dyn, C. Manni, and M.-L. Mazure. Convergence of univariate non-stationary subdivision schemes via asymptotic similarity. *Computer Aided Geometric Design*, 37:1–8, 2015.

[11] C. Conti, J.-L. Merrien, and L. Romani. Dual Hermite subdivision schemes of de Rham-type. *BIT Numer. Math.*, 54:955–977, 2014.

[12] C. Conti and L. Romani. Algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction. *Journal of Computational and Applied Mathematics*, 236(4):543–556, 2011.

[13] P. Costantini and C. Manni. On constrained nonlinear Hermite subdivision. *Constructive Approximation*, 28(3):291–331, 2008.

[14] I. Daubechies. *Ten lectures on wavelets.* SIAM, 1992.

[15] I. Daubechies, O. Runborg, and W. Sweldens. Normal multiresolution approximation of curves. *Constr. Approx.*, 20(3):399–463, 2004.

[16] C. de Boor. Cutting corners always works. *Computer Aided Geometric Design*, 4(1–2):125–131, 1987.

[17] G. de Rham. Sur une courbe plane. *J. de Mathématiques Pures et Appliquées*, 39:25–42, 1956.

[18] G. Deslauriers and S. Dubuc. Symmetric iterative interpolation processes. *Constr. Approx.*, 5:49–68, 1989.

[19] M. P. do Carmo. *Riemannian geometry.* Birkhäuser Verlag, 1992.

[20] D. L. Donoho. Wavelet-type representation of Lie-valued data. Talk at the IMI Approximation and Computation Meeting, Charleston, 2001.

[21] D. Doo and M. Sabin. Behaviour of recursive division surfaces near extraordinary points. *Computer-Aided Design*, 10(6):356–360, 1978.

[22] S. Dubuc. Scalar and Hermite subdivision schemes. *Applied and Computational Harmonic Analysis*, 21(3):376–394, 2006.

[23] S. Dubuc and J.-L. Merrien. Convergent vector and Hermite subdivision schemes. *Constructive Approximation*, 23(1):1–22, 2005.

[24] S. Dubuc and J.-L. Merrien. de Rham transform of a Hermite subdivision scheme. In M. Neamtu and L. L. Schumaker, editors, *Approximation Theory XII*, pages 121–132, Nashville, TN, 2008. Nashboro Press.

[25] S. Dubuc and J.-L. Merrien. Hermite subdivision schemes and Taylor polynomials. *Constructive Approximation*, 29(2):219–245, 2009.

[26] N. Dyn. Subdivision schemes in computer-aided geometric design. In *Advances in Numerical Analysis*, pages 36–104. Oxford University Press, 1992.

[27] N. Dyn, J. A. Gregory, and D. Levin. Analysis of uniform binary subdivision schemes for curve design. *Constructive Approximation*, 7(1):127–147, 1991.

[28] N. Dyn, P. Grohs, and J. Wallner. Approximation order of interpolatory subdivision schemes. *Journal of Computational and Applied Mathematics*, 233(7):1697–1703, 2010.

[29] N. Dyn and D. Levin. Analysis of asymptotic equivalent binary subdivision schemes. *Journal of Mathematical Analysis and Applications*, 193(2):594–621, 1995.

[30] N. Dyn and D. Levin. Analysis of Hermite-type subdivision schemes. In C. K. Chui and L. L. Schumaker, editors, *Approximation Theory VIII. Vol 2: Wavelets and Multilevel Approximation*, pages 117–124. World Sci., 1995.

[31] N. Dyn and D. Levin. Analysis of Hermite-interpolatory subdivision schemes. In S. Dubuc and G. Deslauriers, editors, *Spline Functions and the Theory of Wavelets*, pages 105–113, Providence, RI, 1999. Amer. Math. Soc.

[32] N. Dyn and D. Levin. Subdivision schemes in geometric modelling. *Acta Numerica*, 11:73–144, 2002.

[33] N. Dyn, D. Levin, and J. A. Gregory. A 4-point interpolatory subdivision scheme for curve design. *Computer Aided Geometric Design*, 4:257–268, 1987.

[34] N. Dyn and N. Sharon. A global approach to the refinement of manifold data. *Math. Comp.*, 2016. `doi:10.1090/mcom/3087`.

[35] N. Dyn and N. Sharon. Manifold-valued subdivision schemes based on geodesic inductive averaging. *Journal of Computational and Applied Mathematics*, 311:54–67, 2017.

[36] O. Ebner. Convergence of refinement schemes on metric spaces. *Proceedings of the American Mathematical Society*, 141(2):677–686, 2013.

[37] O. Ebner. Stochastic aspects of nonlinear refinement schemes. *SIAM Journal on Numerical Analysis*, 52(2):717–734, 2014.

[38] T. Ewald, U. Reif, and M. Sabin. Hölder regularity of geometric subdivision schemes. *Constructive Approximation*, 42(3):425–458, 2015.

[39] R. Goldman, S. Schaefer, and E. Vouga. Nonlinear subdivision through nonlinear averaging. *Comput. Aided Geom. Des.*, 25(3):162–180, 2008.

[40] P. Grohs. *Smoothness analysis of nonlinear subdivision schemes on regular grids*. PhD thesis, TU Wien, 2007.

[41] P. Grohs. Smoothness equivalence properties of univariate subdivision schemes and their projection analogues. *Numerische Mathematik*, 113(2):163–180, 2009.

[42] P. Grohs. Smoothness of interpolatory multivariate subdivision in Lie groups. *IMA Journal of Numerical Analysis*, 29(3):760–772, 2009.

[43] P. Grohs. A general proximity analysis of nonlinear subdivision schemes. *SIAM Journal on Mathematical Analysis*, 42(2):729–750, 2010.

[44] P. Grohs. Approximation order from stability for nonlinear subdivision schemes. *Journal of Approximation Theory*, 162(5):1085–1094, 2010.

[45] P. Grohs. Stability of manifold-valued subdivision schemes and multiscale transformations. *Constr. Approx.*, 32(3):569–596, 2010.

[46] P. Grohs and J. Wallner. Log-exponential analogues of univariate subdivision schemes in Lie groups and their smoothness properties. In M. Neamtu and L. L. Schumaker, editors, *Approximation Theory XII*, pages 181–190. Nashboro Press, 2008.

[47] P. Grohs and J. Wallner. Interpolatory wavelets for manifold-valued data. *Applied and Computational Harmonic Analysis*, 27(3):325–333, 2009.

[48] P. Grohs and J. Wallner. Definability and stability of multiscale decompositions for manifold-valued data. *Journal of the Franklin Institute*, 349(5):1648–1664, 2012.

[49] N. Guglielmi, C. Manni, and D. Vitale. Convergence analysis of C^2 Hermite interpolatory subdivision schemes by explicit joint spectral radius formulas. *Linear Algebra and its Applications*, 434:884–902, 2011.

[50] B. Han. Computing the smoothness exponent of a symmetric multivariate refinable function. *SIAM Journal on Matrix Analysis and Applications*, 3:693–714, 2003.

[51] B. Han and R.-Q. Jia. Multivariate refinement equations and convergence of subdivision schemes. *SIAM Journal on Mathematical Analysis*, 29(5):1177–1199, 1998.

[52] B. Han, T. Yu, and Y. Xue. Noninterpolatory Hermite subdivision schemes. *Mathematics of Computation*, 74(251):1345–1367, 2005.

[53] S. Harizanov and P. Oswald. Stability of nonlinear subdivision and multiscale transforms. *Constr. Approx.*, 31(3):359–393, 2010.

[54] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces.* Academic Press, 1979.

[55] M. Holmström. Solving hyperbolic PDEs using interpolating wavelets. *SIAM J. Sci. Comp.*, 21(2):405–420, 1999.

[56] S. Kobayashi and K. Nomizu. *Foundations of differential geometry, Volume 2.* New York: John Wiley & Sons, 1969.

[57] J. M. Lane and R. F. Riesenfeld. A theoretical development for the computer generation and display of piecewise polynomial surfaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2(1):35–46, 1980.

[58] J.-L. Merrien. A family of Hermite interpolants by bisection algorithms. *Numerical Algorithms*, 2(2):187–200, 1992.

[59] J.-L. Merrien and T. Sauer. A generalized Taylor factorisation for Hermite subdivision schemes. *Journal of Computational and Applied Mathematics*, 236(4):565–574, 2011.

[60] J.-L. Merrien and T. Sauer. From Hermite to stationary subdivision schemes in one and several variables. *Advances in Computational Mathematics*, 36(4):547–579, 2012.

[61] J.-L. Merrien and T. Sauer. Extended Hermite subdivision schemes. Preprint available online from `https://hal.archives-ouvertes.fr/hal-01250509v3/`, 2016.

[62] C. Michelli and T. Sauer. Regularity of multiwavlets. *Advances in Computational Mathematics*, 7:455–545, 1997.

[63] C. Michelli and T. Sauer. On vector subdivision. *Mathematische Zeitschrift*, 229:621–674, 1998.

[64] C. Moosmüller. C^1 analysis of Hermite subdivision schemes on manifolds. *SIAM J. Numer. Anal.*, 54(5):3003–3031, 2016.

[65] A. L. Onishchik and E. B. Vinberg. *Lie groups and Lie algebras I: Foundations of Lie theory.* Springer, 1993.

[66] P. Oswald. Smoothness of a nonlinear subdivision scheme. In A. Cohen, J.-L. Merrien, and L. L. Schumaker, editors, *Curve and Surface Fitting: Saint-Malo 2002*, pages 323–332. Nashboro Press, Brentwood, 2003.

[67] P. Oswald. Smoothness of nonlinear median-interpolation subdivision. *Adv. Comp. Math.*, 20(4):401–423, 2004.

[68] J. Peters and U. Reif. *Subdivision surfaces.* Springer, 2008.

[69] M. M. Postnikov. *Geometry VI: Riemannian geometry.* Springer, 2001.

[70] I. Ur Rahman, I. Drori, V. C. Stodden, D. L. Donoho, and P. Schröder. Multiscale representations for manifold-valued data. *SIAM Journal on Multiscale Modeling & Simulation*, 4(4):1201–1232, 2005.

[71] T. Sauer. Stationary vector subdivision: quotient ideals, differences and approximation power. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas. RACSAM*, 96(2):257–277, 2002.

[72] T. Sauer. How to generate smoother refinable functions from given ones. In W. Haussmann et al., editors, *Modern Developments in Multivariate Approximation*, volume 145 of *Int. Series of Numerical Mathematics*, pages 279–293. Birkhäuser, 2003.

[73] T. Sauer. Differentiability of multivariate refinable functions and factorisation. *Advances in Computational Mathematics*, 25:211–235, 2006.

[74] N. Sharon and U. Itai. Approximation schemes for functions of positive-definite matrix values. *IMA Journal of Numerical Analysis*, 33(4):1436–1468, 2013.

[75] J. Wallner. Smoothness analysis of subdivision schemes by proximity. *Constructive Approximation*, 24:289–318, 2006.

[76] J. Wallner. On convergent interpolatory subdivision schemes in Riemannian geometry. *Constructive Approximation*, 40:473–486, 2014.

[77] J. Wallner and N. Dyn. Convergence and C^1 analysis of subdivision schemes on manifolds by proximity. *Computer Aided Geometric Design*, 22(7):593–622, 2005.

[78] J. Wallner, E. Nava Yazdani, and P. Grohs. Smoothness properties of Lie group subdivision schemes. *Multiscale Modeling & Simulation*, 6(2):493–505, 2007.

[79] J. Wallner, E. Nava Yazdani, and A. Weinmann. Convergence and smoothness analysis of subdivision rules in Riemannian and symmetric spaces. *Advances in Computational Mathematics*, 34(2):201–218, 2011.

[80] J. Wallner and H. Pottmann. Intrinsic subdivision with smooth limits for graphics and animation. *ACM Transactions on Graphics (TOG)*, 25(2):356–374, 2006.

[81] A. Weinmann. Nonlinear Subdivision schemes on irregular meshes. *Constr. Approx.*, 31(3):395–415, 2010.

[82] A. Weinmann. Interpolatory multiscale representation for functions between manifolds. *SIAM J. Math. Anal.*, 44(1):162–191, 2012.

[83] A. Weinmann. Subdivision schemes with general dilation in the geometric and nonlinear setting. *Journal of Approximation Theory*, 164(1):105–137, 2012.

[84] G. Xie and T. P-Y Yu. Smoothness equivalence properties of manifold-valued data subdivision schemes based on the projection approach. *SIAM Journal on Numerical Analysis*, 45(3):1200–1225, 2007.

[85] G. Xie and T. P-Y Yu. Smoothness equivalence properties of general manifold-valued data subdivision schemes. *SIAM Journal on Multiscale Modeling & Simulation*, 7(3):1073–1100, 2009.

[86] G. Xie and T. P-Y Yu. Smoothness equivalence properties of interpolatory Lie group subdivision schemes. *IMA Journal of Numerical Analysis*, 30(3):731–750, 2010.

[87] G. Xie and T. P-Y Yu. Approximation order equivalence properties of manifold-valued data subdivision schemes. *IMA Journal of Numerical Analysis*, 32(2):687–700, 2012.

[88] D. Zorin et al. Subdivision for modeling and animation. *SIGGRAPH 2000 Course Notes*, 2000. Available online from `https://mrl.nyu.edu/publications/subdiv-course2000/coursenotes00.pdf`.