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Space-time boundary element methods for the heat equation

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Abstract

In this thesis we describe the boundary element method for the discretization of the time-dependent heat equation. In contrast to standard time-stepping schemes we consider an arbitrary decomposition of the boundary of the space-time cylinder into boundary elements. Besides adaptive refinement strategies this approach allows us to parallelize the computation of the global solution of the whole space-time system. In addition to the analysis of the boundary integral operators and the derivation of boundary element methods for the Dirichlet initial boundary value problem we state convergence properties and error estimates of the approximations. Those estimates are based on the approximation properties of boundary element spaces in anisotropic Sobolov spaces, in particular in $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. The systems of linear equations, which arise from the discretization of the integral equations, are solved with the GMRES method. For an efficient computation of the solution we need preconditioners. Based on the mapping properties of the single layer- and the hypersingular boundary integral operator we construct and analyse a preconditioner for the discretization of the first boundary integral equation. Moreover we describe the FEM-BEM coupling method for parabolic transmission problems. Finally we present numerical examples for the one-dimensional heat equation to confirm the theoretical results.

Kurzfassung

In dieser Arbeit wird die Randelementmethode zur Diskretisierung von zeitabhängigen Anfangsrandwertproblemen am Modell der Wärmeleitungsgleichung beschrieben. Anders als bei klassischen Zeitschrittverfahren wird eine beliebige Zerlegung des Randes des Raum-Zeit-Zylinders betrachtet, was adaptive Verfeinerungsstrategien und eine Parallelisierung des iterativen Lösungsverfahrens bezüglich des gesamten Raum-Zeit-Zylinders in einem Schritt ermöglicht. Neben der Analysis der Randintegraloperatoren und der Herleitung von Randelementmethoden für das Dirichlet-Anfangsrandwertproblem werden auch Konvergenzeigenschaften und Fehlerabschätzungen der Näherungslösungen angegeben. Diese basieren auf den Approximationseigenschaften von Ansatzräumen in anisotropen Sobolev-Räumen, insbesondere in $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ bzw. $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. Die linearen Gleichungssysteme, welche sich aus der Diskretisierung der Randintegralgleichungen ergeben, werden mittels GMRES-Verfahren gelöst. Für ein effizientes Lösen sind Vorkonditionierer notwendig. Ausgehend von den Abbildungseigenschaften des Einfachschicht- und des hypersingulären Randintegraloperators und der Projektionseigenschaften des Calderón-Operators wird ein Vorkonditionierer für die Diskretisierung der ersten Randintegralgleichung konstruiert und analysiert. Zusätzlich wird die Methode der FEM-BEM-Kopplung für parabolische Transmissionsprobleme beschrieben. Die erarbeiteten theoretischen Aussagen werden anhand von numerischen Beispielen für die eindimensionale Wärmeleitungsgleichung überprüft.

Vorwort

Die vorliegende Arbeit baut auf meiner Bachelorarbeit und meiner Projektarbeit auf. Seit Beginn meiner Bachelorarbeit beschäftige ich mich mit der Raum-Zeit-Diskretisierung der Wärmeleitungsgleichung mit der Randelementmethode und mein Interesse hierfür ist in den letzten Jahren kontinuierlich gewachsen. Deshalb möchte ich mich in erster Linie bei Univ.-Prof. Dr. O. Steinbach bedanken, der mir dieses Thema vorgeschlagen und mich beim Verfassen der Masterarbeit stets unterstützt hat. Außerdem möchte ich mich beim gesamten Institut für Numerische Mathematik der TU Graz für die Unterstützung im Studium und die angenehme Atmosphäre bedanken.

Contents

Introduction

In this thesis we describe the boundary element method for the discretization of time-dependent initial boundary value problems using the heat equation as a model problem. There are different numerical methods in order to compute an approximate solution of time-dependent initial boundary value problems. Besides standard time-stepping schemes [32] and decomposition of the whole space-time domain into finite elements [16] we can use the boundary element method to get an approximation of the solution of the model problem. As for stationary problems [27] one can use the fundamental solution of the partial differential equation and the given boundary and initial conditions to derive a representation formula for the solution of the time-dependent model problem. The problem is reduced to the boundary and we can apply the trace operators to the representation formula to get boundary integral equations and use some discretization method to compute an approximate solution of those equations.
The presented analysis of the boundary integral operators and boundary integral equations is well established and mainly based on the work of Costabel [2] and Noon [17]. There is already a variety of papers regarding different applications of the discretization of the heat equation with the boundary element method [2, 8, 15, 17, 18, 20] using space-time tensor product spaces to discretize the variational formulations of the boundary integral equations. This method refers to a separate triangulation of the boundary Γ of the domain Ω and the time interval $(0,T)$ and uses tensor product spaces as trial spaces. In contrast to this approach we consider an arbitrary decomposition of the boundary Σ of the space-time domain $Q = \Omega \times (0,T)$ into boundary elements. Besides adaptive refinement strategies and in contrast to standard time-stepping schemes this approach allows us to parallelize the computation of the global solution of the whole space-time system.
In this work we consider a Dirichlet initial boundary value problem for the heat equation. After the derivation of the fundamental solution and the representation formula for the solution of the model problem we discuss the mapping properties of the heat potentials and the boundary integral operators, based on [2]. Since Dirichlet boundary conditions are given, it remains to determine the conormal derivative of the solution of the model problem. This can be done by solving the boundary integral equations. The analysis of the operators is done in anisotropic Sobolev spaces [12]. In this setting the single layer boundary integral operator V and the hypersingular operator D are not only bounded but also elliptic. Due to the ellipticity of V we can conclude unique solvability of the first boundary integral equation using the standard theory of elliptic operators [27]. Analoguously we get unique solvability of the integral equation

related to an indirect approach using the single layer potential. After discussing the different approaches of determining the unknown conormal derivative of the solution we consider a Galerkin-Bubnov variational formulation in order to discretize the first boundary integral equation and compute an approximation of the conormal derivative. Our goal is to use an arbitrary triangulation of the boundary of the space-time domain Q. However, we first consider a separate triangulation of the boundary Γ and the time interval $(0, T)$ and derive approximation properties of the corresponding space-time tensor product spaces. Afterwards we consider an arbitrary decomposition of the space-time boundary for $n = 1, 2$ and state approximation properties of the space of piecewise constant basis functions by using the approximation properties of the space-time tensor product spaces. The system of linear equations corresponding to the Galerkin-Bubnov variational formulation is solved with the GMRES method. Since the condition number of the system matrix depends on the mesh size of the triangulation of the space-time boundary, the iteration number of the iterative solver increases with each refinement step. To get rid of this dependency we have to apply suitable preconditioning strategies. We present a preconditioning technique using operators of opposite order such as V and D. The presented concept is based on [31].
An advantage of the boundary element method is the handling of exterior problems in a natural way. This allows us to apply the FEM-BEM coupling method to transmission problems for the heat equation. In Chapter 9 we introduce the concept of a non-symmetric FEM-BEM coupling method for parabolic transmission problems, based on [24], and use a Galerkin method to compute an approximate solution of the problem. In the last chapter we present numerical examples for the one-dimensional heat equation to confirm the theoretical results.

1 Basics

In order to study the unique solvability of boundary integral equations and their discretizations we need some results in operator theory. In this chapter we summarize the main statements, based on [27].

1.1 Operator theory

Let X be a Hilbert space with norm $\|\cdot\|_X = \sqrt{\langle\cdot,\cdot\rangle}$ and X' be the dual space of X. The norm of an element $f \in X'$ is given by

$$\|f\|_{X'} = \sup_{0 \neq v \in X} \frac{\langle f, v\rangle}{\|v\|_X}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing on $X' \times X$. Let $A : X \to X'$ be a linear and bounded operator, i.e. there exists a constant $c_2^A > 0$ such that

$$\|Av\|_{X'} \leq c_2^A \|v\|_X \quad \text{for all } v \in X.$$

We want to find a solution $u \in X$ of the operator equation

$$Au = f \tag{1.1}$$

where $f \in X'$ is given. This problem is equivalent to its variational formulation, which is to find $u \in X$ such that

$$\langle Au, v\rangle = \langle f, v\rangle \quad \text{for all } v \in X. \tag{1.2}$$

If the operator A is X-elliptic, i.e. there exists a constant $c_1^A > 0$ such that

$$\langle Av, v\rangle \geq c_1^A \|v\|_X^2 \quad \text{for all } v \in X,$$

then the following theorem ensures the unique solvability of the operator equation (1.1).

Theorem 1.1 (Lemma of Lax-Milgram)**.** *[27, Theorem 3.2] Let the linear operator $A : X \to X'$ be bounded and X-elliptic. For any $f \in X'$ there exists a unique solution $u \in X$ of the operator equation*

$$Au = f.$$

The solution u satisfies

$$\|u\|_X \leq \frac{1}{c_1^A} \|f\|_{X'}.$$

Since (1.2) is equivalent to (1.1) the ellipticity of the operator A ensures the unique solvability of the variational formulation of the operator equation as well. According to Theorem 1.1 the inverse operator $A^{-1} : X' \to X$ is well defined and satisfies

$$\left\|A^{-1} f\right\| \leq \frac{1}{c_1^A} \|f\|_{X'} \quad \text{for all } f \in X'.$$

Now let us consider a linear and bounded operator $B : X \to X$. For a given $g \in X$ we want to find a solution of the operator equation

$$(I - B)u = g. \tag{1.3}$$

The following theorem ensures unique solvability of equation (1.3) when assuming, that the operator B is a contraction in X, i.e.

$$\|B\| = \sup_{0 \neq v \in X} \frac{\|Bv\|_X}{\|v\|_X} < 1$$

Theorem 1.2 (Neumann series)**.** *[35, Theorem II.1.11] Let X be a Banach-space and let $B : X \to X$ be a linear and bounded operator satisfying $\|B\| < 1$. Then $I - B$ is invertible and*

$$(I - B)^{-1} = \sum_{k=0}^{\infty} B^k.$$

The inverse operator satisfies

$$\left\|(I - B)^{-1}\right\| \leq \frac{1}{1 - \|B\|}.$$

1.2 Galerkin methods

We want to compute an approximate solution of the variational problem (1.2). Hence we need unique solvability of the discretized problem as well. Let $N \in \mathbb{N}$. We consider a finite dimensional subspace $X_h := \operatorname{span}\{\varphi_k\}_{k=1}^N \subset X$ and want to find an approximation u_h of the solution u of (1.2) where

$$u_h = \sum_{k=1}^{N} u_k \varphi_k \in X_h. \tag{1.4}$$

The Galerkin-Bubnov variational formulation is to find $u_h \in X_h$ such that

$$\langle A u_h, v_h \rangle = \langle f, v_h \rangle \quad \text{for all } v_h \in X_h. \tag{1.5}$$

This problem is equivalent to solving the system of linear equations

$$A_h \underline{u} = \underline{f} \tag{1.6}$$

where

$$A_h[l,k] := \langle A\varphi_k, \varphi_l \rangle, \quad \underline{f}[l] := \langle f, \varphi_l \rangle$$

for $l,k = 1,...,N$ and $\underline{u}$ is the vector of coefficients regarding (1.4). For $\underline{u}, \underline{v} \in \mathbb{R}^N$ we have

$$(A_h \underline{u}, \underline{v}) = \langle A u_h, v_h \rangle.$$

Hence we get

$$(A_h \underline{v}, \underline{v}) = \langle A v_h, v_h \rangle \geq c_1^A \left\| v_h \right\|_X^2 \quad \text{for all } \underline{v} \in \mathbb{R}^N$$

which implies the positive definiteness and therefore the invertibility of the matrix A_h. Thus, the system of linear equations (1.6) and the variational problem (1.5) are uniquely solvable.

Since $X_h \subset X$, we have

$$\langle A u, v_h \rangle = \langle f, v_h \rangle \quad \text{for all } v_h \in X_h$$

where u is the unique solution of the variational problem (1.2). By subtracting (1.5) from this equation we get the Galerkin orthogonality

$$\langle A(u - u_h), v_h \rangle = 0 \quad \text{for all } v_h \in X_h. \tag{1.7}$$

The following theorem states a stability estimate for the approximate solution u_h as well as an error estimate with respect to the solution u of the variational problem (1.2).

Theorem 1.3 (Cea's Lemma)**.** *[27, Theorem 8.1] Let $A : X \to X'$ be a bounded and X-elliptic linear operator. For the unique solution $u_h \in X_h$ of the variational problem* (1.5) *there holds*

$$\left\| u_h \right\|_X \leq \frac{1}{c_1^A} \left\| f \right\|_{X'}$$

and

$$\left\| u - u_h \right\|_X \leq \frac{c_2^A}{c_1^A} \inf_{v_h \in X_h} \left\| u - v_h \right\|_X .$$

Hence we have quasi-optimality of the Galerkin approximation u_h and we can use approximation properties of the finite dimensional subspace X_h to derive error estimates and study the convergence of the Galerkin method.

2 Model problem

Let $\Omega \subset \mathbb{R}^n$ $(n = 1, 2, 3)$ be a bounded domain with Lipschitz-boundary $\Gamma := \partial\Omega$, $T \in \mathbb{R}$ with $T > 0$ and $\alpha \in \mathbb{R}$ with $\alpha > 0$. We consider the initial boundary value problem

$$\begin{aligned} \alpha\partial_t u(x,t) - \Delta_x u(x,t) &= f(x,t) \quad && \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) &= g(x,t) \quad && \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) &= u_0(x) \quad && \text{for } x \in \Omega \end{aligned} \tag{2.1}$$

with given source term f and boundary- and initial conditions g and u_0 satisfying the compatibility condition $g(x,0) = u_0(x)$ for $x \in \Gamma$. Our aim is to represent the solution of the problem in terms of the given data f, g and u_0. Before we derive the representation formula for the solution of the model problem we first consider two special cases of the heat equation and compute analytical solutions of those two problems.

2.1 Series representation of the solution

In this section we derive a series representation of the solution of the homogeneous heat equation with boundary condition $g = 0$ and initial condition u_0. Due to the compatibility condition we have $u_{0|\Gamma} = 0$.

2.1.1 One-dimensional heat equation

Without loss of generality we choose $\Omega = (0,1) \subset \mathbb{R}$. Let $u_0 \in C(\overline{\Omega})$ be a given initial condition with $u_0(0) = u_0(1) = 0$. We consider the initial boundary value problem

$$\begin{aligned} \alpha\partial_t u(x,t) - \partial_{xx} u(x,t) = 0 \quad & \text{for } (x,t) \in (0,1) \times (0,T), \\ u(0,t) = u(1,t) = 0 \quad & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) \quad & \text{for } x \in (0,1). \end{aligned} \tag{2.2}$$

By using separation of variables, i.e. $u(x,t) = X(x)T(t)$, we get

$$\alpha X(x)T'(t) = X''(x)T(t)$$

which is equivalent to

$$\alpha\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda = \text{const.}$$

Together with the boundary conditions in (2.2) we get the Dirichlet eigenvalue problem

$$\begin{aligned} X''(x) - \lambda X(x) &= 0 \quad \text{for } x \in (0,1), \\ X(0) = X(1) &= 0 \end{aligned} \tag{2.3}$$

and the initial value problem

$$\begin{aligned} T'(t) - \frac{1}{\alpha}\lambda T(t) &= 0 \quad \text{for } t > 0, \\ T(0) &= a \end{aligned} \tag{2.4}$$

for some constant $a \in \mathbb{R}$. The eigenvalues and eigenfunctions of (2.3) are given by

$$\lambda_k = -(k\pi)^2$$

and

$$X_k(x) = \sin(k\pi x)$$

where $k \in \mathbb{N}$. The solution of (2.4) is then given by

$$T(t) = a_k \exp\left(\frac{\lambda_k t}{\alpha}\right).$$

We conclude that the functions

$$u_k(x,t) = a_k \exp\left(\frac{\lambda_k t}{\alpha}\right) \sin(k\pi x)$$

are solutions of the homogeneous heat equation in (2.2) satisfying the boundary conditions $u_k(0,t) = u_k(1,t) = 0$. The series representation is given by

$$u(x,t) = \sum_{k=1}^{\infty} a_k \exp\left(\frac{-(k\pi)^2 t}{\alpha}\right) \sin(k\pi x).$$

It remains to determine the coefficients $a_k \in \mathbb{R}$. Since u has to satisfy the initial condition we have

$$u(x,0) = \sum_{k=1}^{\infty} a_k \sin(k\pi x) \overset{!}{=} u_0(x).$$

Multiplying the equation with $\sin(l\pi x)$ and integrating over $(0,1)$ gives

$$\sum_{k=1}^{\infty} a_k \int_0^1 \sin(k\pi x)\sin(l\pi x)dx = \int_0^1 u_0(x)\sin(l\pi x)dx.$$

Since

$$\int_0^1 \sin(k\pi x)\sin(l\pi x)dx = \begin{cases} \frac{1}{2} & \text{if } l = k, \\ 0 & \text{if } l \neq k, \end{cases}$$

we get

$$a_k = 2\int_0^1 u_0(x)\sin(k\pi x)dx.$$

2.1.2 Heat equation in the unit square

Let $\Omega = (0,1)\times(0,1)$ be the unit square and $u_0 \in C(\overline{\Omega})$ be some given initial condition satisfying the compatibility condition

$$u_0(x,0) = u_0(x,1) = u_0(0,y) = u_0(1,y) = 0.$$

As in the spatially one-dimensional case we use separation of variables , i.e.

$$u(x,t) = X_1(x_1)X_2(x_2)T(t),$$

to get

$$\alpha\frac{T''(t)}{T(t)} = \frac{X_1''(x_1)}{X_1(x_1)} + \frac{X_2''(x_2)}{X_2(x_2)} = \lambda = \text{const.}$$

In consideration of the homogeneous boundary conditions this relation leads to the Dirichlet eigenvalue problems

$$\begin{aligned} X_i''(x_i) - \lambda_i X_i(x_i) &= 0 \quad \text{for } x_i \in (0,1), \\ X_i(0) = X_i(1) &= 0 \end{aligned} \tag{2.5}$$

for $i = 1,2$ and the initial value problem

$$\begin{aligned} T'(t) - \frac{1}{\alpha}\lambda T(t) &= 0 \quad \text{for } t > 0, \\ T(0) &= a \end{aligned} \tag{2.6}$$

for some constant $a \in \mathbb{R}$ and $\lambda = \lambda_1 + \lambda_2$. The eigenvalues and eigenfunctions of (2.5) are given by

$$\lambda_{i,k} = -(k\pi)^2$$

and

$$X_{i,k}(x_i) = \sin(k\pi x_i)$$

where $k \in \mathbb{N}$. The solution of (2.6) is then given by

$$T(t) = a_{kl}\exp\left(\frac{\lambda_{kl}t}{\alpha}\right)$$

with $\lambda_{kl} = -\left(k^2 + l^2\right)\pi^2$. Thus, the functions

$$u_{kl}(x,t) = a_{kl}\exp\left(\frac{\lambda_{kl}t}{\alpha}\right)\sin(k\pi x_1)\sin(l\pi x_2)$$

are solutions of the homogeneous heat equation with boundary condition $u_{|\Sigma} = 0$. The general solution is given by the series

$$u(x,t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}\exp\left(\frac{-(k^2+l^2)\pi^2 t}{\alpha}\right)\sin(k\pi x_1)\sin(l\pi x_2).$$

It remains to determine the coefficients a_{kl}. Similarly to the one-dimensional case we get

$$a_{kl} = 4 \int_0^1 \int_0^1 u_0(x_1, x_2) \sin(k\pi x_1) \sin(l\pi x_2) dx_2 dx_1.$$

2.2 Representation formula and fundamental solution

In order to state a solution of the initial boundary value problem (2.1) and to derive boundary integral equations the existence of a fundamental solution of the heat equation is essential. In this section we will derive the representation formula for the solution of the model problem (2.1) including the fundamental solution of the heat equation.

Theorem 2.1 (Green's first formula)**.** *[1, Corollary 7.8] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-boundary $\Gamma = \partial\Omega$ and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then there holds*

$$\int_\Omega [\Delta u(x) v(x) + \nabla u(x) \cdot \nabla v(x)]\, dx = \int_\Gamma \frac{\partial}{\partial n} u(x) v(x) ds_x \quad \text{for all } v \in C^1(\Omega) \cap C(\overline{\Omega}).$$

Assume that $u \in C^2(\overline{Q})$ is a solution of the partial differential equation in (2.1). Multiplying the partial differential equation in (2.1) with a function $v \in C^2(\overline{Q})$ and integration over Q leads to

$$\int_0^T \int_\Omega [\alpha \partial_s u(y,s) v(y,s) - \Delta_y u(y,s) v(y,s)]\, dyds = \int_0^T \int_\Omega f(y,s) v(y,s) dyds.$$

Applying Theorem 2.1 to the second term of the left hand side gives

$$\begin{aligned} &\int_0^T \int_\Omega [\alpha \partial_s u(y,s) v(y,s) + \nabla_y u(y,s) \cdot \nabla_y v(y,s)]\, dyds \\ &\qquad = \int_0^T \int_\Omega f(y,s) v(y,s) dyds + \int_0^T \int_\Gamma \frac{\partial}{\partial n_y} u(y,s) v(y,s) ds_y ds. \end{aligned} \tag{2.7}$$

This equation is called Green's first formula for the heat equation. By using integration by parts regarding the first term of the left hand side and by rearranging the terms we get

$$\begin{aligned} \alpha \int_\Omega u(y,T) v(y,T) dy = {}& \alpha \int_\Omega u(y,0) v(y,0) dy + \int_0^T \int_\Omega f(y,s) v(y,s) dyds \\ &+ \int_0^T \int_\Omega [\alpha u(y,s) \partial_s v(y,s) - \nabla_y u(y,s) \cdot \nabla_y v(y,s)]\, dyds \\ &+ \int_0^T \int_\Gamma \frac{\partial}{\partial n_y} u(y,s) v(y,s) ds_y ds. \end{aligned}$$

Again, by using Theorem 2.1 we get the equation

$$
\begin{aligned}
\alpha \int_\Omega u(y,T)v(y,T)dy = \alpha \int_\Omega u(y,0)v(y,0)dy + \int_0^T \int_\Omega f(y,s)v(y,s)dyds \\
- \int_0^T \int_\Omega \left[-\alpha\partial_s v(y,s) - \Delta_y v(y,s)\right] u(y,s)dyds \\
+ \int_0^T \int_\Gamma \frac{\partial}{\partial n_y} u(y,s)v(y,s)ds_y ds - \int_0^T \int_\Gamma \frac{\partial}{\partial n_y} v(y,s)u(y,s)ds_y ds.
\end{aligned} \tag{2.8}
$$

This equation is called Green's second formula for the heat equation. We want the third integral of the right hand side to be zero, i.e. we search for a function v which is a solution of the adjoint homogeneous heat equation

$$-\alpha\partial_s v(y,s) - \Delta_y v(y,s) = 0 \quad \text{for } (y,s) \in Q.$$

Since we want to find a representation of the solution $u = u(x,t)$ of the model problem (2.1) we define v as

$$v(y,s) := U(y-x, t-s)$$

where $(x,t) \in Q$ is fixed. In this case we have

$$\partial_s v(y,s) = \partial_s U(y-x, t-s) = -\partial_\tau U(y-x, \tau)$$

where $\tau = t - s$, thus

$$\alpha\partial_\tau U(y-x,\tau) - \Delta_y U(y-x,\tau) = 0 \quad \text{for } (y,s) \in Q.$$

We assume the function U to be spherically symmetric, i.e. $U(y-x,\tau) = \widetilde{U}(r,\tau)$ where $r = |y-x|$. For $r \neq 0$ we get

$$\alpha\partial_\tau \widetilde{U}(r,\tau) - \partial_{rr}\widetilde{U}(r,\tau) - (n-1)\frac{1}{r}\partial_r \widetilde{U}(r,\tau) = 0. \tag{2.9}$$

With $\widetilde{U}(r,\tau) = \tau^\gamma g(z)$, where $z = \dfrac{r}{\sqrt{\tau}}$, $\gamma \in \mathbb{R}$ and $\tau > 0$ ($\Leftrightarrow s < t$) we get

$$
\begin{aligned}
\partial_\tau \widetilde{U}(r,\tau) &= \gamma\tau^{\gamma-1}g(z) - \frac{1}{2}\tau^{\gamma-1}zg'(z), \\
\partial_r \widetilde{U}(r,\tau) &= g'(z)\tau^{\gamma-\frac{1}{2}}, \\
\partial_{rr}\widetilde{U}(r,\tau) &= g''(z)\tau^{\gamma-1}.
\end{aligned}
$$

Therefore equation (2.9) changes to

$$\alpha\left[\gamma\tau^{\gamma-1}g(z) - \frac{1}{2}\tau^{\gamma-1}zg'(z)\right] - g''(z)\tau^{\gamma-1} - (n-1)\frac{1}{r}g'(z)\tau^{\gamma-\frac{1}{2}} = 0$$

which is equivalent to

$$\alpha\left[\gamma g(z) - \frac{1}{2} z g'(z)\right] - g''(z) - (n-1)\frac{1}{z} g'(z) = 0. \tag{2.10}$$

It remains to solve this ordinary differential equation. First we consider the one-dimensional case $n = 1$, i.e. we have

$$\alpha\gamma g(z) - \alpha\frac{1}{2} z g'(z) - g''(z) = 0$$

which can be written as

$$\alpha\left(\gamma + \frac{1}{2}\right) g(z) - \frac{d}{dz}\left[\alpha\frac{1}{2} z g(z) + g'(z)\right] = 0.$$

By choosing $\gamma = -\frac{1}{2}$ we get

$$\frac{d}{dz}\left[\alpha\frac{1}{2} z g(z) + g'(z)\right] = 0.$$

Hence

$$\alpha\frac{1}{2} z g(z) + g'(z) = c_0$$

with $c_0 \in \mathbb{R}$. By setting $c_0 = 0$ and using separation of variables we get

$$\ln g = -\alpha\frac{1}{4} z^2 + c_1$$

and with $c_1 = 0$ we conclude

$$g(z) = \exp\left(-\frac{\alpha}{4} z^2\right) \tag{2.11}$$

which is a solution of the differential equation (2.10) for $n = 1$. When inserting (2.11) into (2.10) for general n we get

$$\begin{aligned} 0 &= \alpha\left[\gamma \exp\left(-\frac{\alpha}{4} z^2\right) + \frac{\alpha}{4} z^2 \exp\left(-\frac{\alpha}{4} z^2\right)\right] + \frac{\alpha}{2}\exp\left(-\frac{\alpha}{4} z^2\right) \\ &\quad - \frac{\alpha^2}{4} z^2 \exp\left(-\frac{\alpha}{4} z^2\right) + (n-1)\frac{\alpha}{2}\exp\left(-\frac{\alpha}{4} z^2\right) \\ &= \exp\left(-\frac{\alpha}{4} z^2\right)\alpha\left[\gamma + \frac{n}{2}\right]. \end{aligned}$$

Thus, (2.11) is also a solution in the two- and three dimensional case if $\gamma = -\frac{n}{2}$. Reconsidering the definition of the functions U and $\widetilde{U}$ we have

$$U(y - x, t - s) = (t - s)^{-n/2} \exp\left(-\frac{\alpha |y - x|^2}{4(t - s)}\right) \quad \text{for } s < t.$$

Due to the singularity of the function U at $t = s$ we consider the space-time-cylinder $Q_{t-\varepsilon} := \Omega \times (0, t-\varepsilon)$ where $0 < \varepsilon < t$. Analogously to (2.8) we get

$$\begin{aligned}\alpha \int_\Omega u(y,t-\varepsilon)v(y,t-\varepsilon)dy &= \alpha \int_\Omega u(y,0)v(y,0)dy + \int_0^{t-\varepsilon}\int_\Omega f(y,s)v(y,s)dyds \\ &- \int_0^{t-\varepsilon}\int_\Omega \left[-\alpha\partial_s v(y,s) - \Delta_y v(y,s)\right] u(y,s)dyds \\ &+ \int_0^{t-\varepsilon}\int_\Gamma \frac{\partial}{\partial n_y} u(y,s)v(y,s)ds_y ds - \int_0^{t-\varepsilon}\int_\Gamma \frac{\partial}{\partial n_y} v(y,s)u(y,s)ds_y ds.\end{aligned}$$

With $v(y,s) = U(y-x, t-s)$ we have

$$\begin{aligned}\alpha \int_\Omega u(y,t-\varepsilon)U(y-x,\varepsilon)dy &= \alpha \int_\Omega u(y,0)U(y-x,t)dy \\ &+ \int_0^{t-\varepsilon}\int_\Omega f(y,s)U(y-x,t-s)dyds + \int_0^{t-\varepsilon}\int_\Gamma \frac{\partial}{\partial n_y} u(y,s)U(y-x,t-s)ds_y ds \\ &- \int_0^{t-\varepsilon}\int_\Gamma \frac{\partial}{\partial n_y} U(y-x,t-s)u(y,s)ds_y ds.\end{aligned} \tag{2.12}$$

Let us consider the integral of the left hand side that is

$$\alpha \int_\Omega u(y,t-\varepsilon)U(y-x,\varepsilon)dy = \alpha \int_\Omega \varepsilon^{-n/2} u(y,t-\varepsilon)\exp\left(-\frac{\alpha|y-x|^2}{4\varepsilon}\right)dy.$$

By using the Taylor expansion $u(y,t-\varepsilon) = u(x,t) + (y-x)^\top \nabla_x u(\xi_x,\xi_t) - \varepsilon\partial_t u(\xi_x,\xi_t)$ with

$$\begin{pmatrix}\xi_x \\ \xi_t\end{pmatrix} = \begin{pmatrix} x + \theta(y-x) \\ t - \theta\varepsilon\end{pmatrix}$$

where $\theta \in (0,1)$ we get

$$\begin{aligned}\frac{\alpha}{\varepsilon^{n/2}}\int_\Omega u(y,t-\varepsilon)\exp\left(-\frac{\alpha|y-x|^2}{4\varepsilon}\right)dy &= u(x,t)\frac{\alpha}{\varepsilon^{n/2}}\int_\Omega \exp\left(-\frac{\alpha|y-x|^2}{4\varepsilon}\right)dy \\ &+ \frac{\alpha}{\varepsilon^{n/2}}\int_\Omega (y-x)^T\nabla_x u(\xi_x,\xi_t)\exp\left(-\frac{\alpha|y-x|^2}{4\varepsilon}\right)dy \\ &- \frac{\alpha}{\varepsilon^{n/2-1}}\int_\Omega \partial_t u(\xi_x,\xi_t)\exp\left(-\frac{\alpha|y-x|^2}{4\varepsilon}\right)dy.\end{aligned} \tag{2.13}$$

Next we are going to show the convergence of the first integral of the right hand side. First we consider the spatially one-dimensional case $n = 1$, i.e. $\Omega = (a,b)$ with $a, b \in \mathbb{R}$

and $x \in (a,b)$. We have

$$\begin{aligned} A :&= \frac{\alpha}{\varepsilon^{1/2}} \int_a^b \exp\left(-\frac{\alpha(y-x)^2}{4\varepsilon}\right) dy \\ &= \frac{\alpha}{\varepsilon^{1/2}} \int_a^x \exp\left(-\frac{\alpha(y-x)^2}{4\varepsilon}\right) dy + \frac{\alpha}{\varepsilon^{1/2}} \int_x^b \exp\left(-\frac{\alpha(y-x)^2}{4\varepsilon}\right) dy. \end{aligned}$$

By using the substitution $z = \frac{x-y}{x-a}$ for the first integral and $z = \frac{y-x}{b-x}$ for the second one we get

$$\begin{aligned} A = &\frac{\alpha}{\varepsilon^{1/2}}(x-a) \int_0^1 \exp\left(-\frac{\alpha(x-a)^2 z^2}{4\varepsilon}\right) dz \\ &+ \frac{\alpha}{\varepsilon^{1/2}}(b-x) \int_0^1 \exp\left(-\frac{\alpha(b-x)^2 z^2}{4\varepsilon}\right) dz. \end{aligned}$$

The substitution $\frac{\alpha(x-a)^2 z^2}{4\varepsilon} = \eta^2$ for the first and $\frac{\alpha(b-x)^2 z^2}{4\varepsilon} = \eta^2$ for the second integral leads to

$$A = 2\sqrt{\alpha} \int_0^{\frac{(x-a)}{2}\sqrt{\frac{\alpha}{\varepsilon}}} \exp\left(-\eta^2\right) d\eta + 2\sqrt{\alpha} \int_0^{\frac{(b-x)}{2}\sqrt{\frac{\alpha}{\varepsilon}}} \exp\left(-\eta^2\right) d\eta$$

and we finally get

$$A \longrightarrow 4\sqrt{\alpha} \int_0^\infty \exp\left(-\eta^2\right) d\eta = 2\sqrt{\alpha\pi}$$

as $\varepsilon \to 0$. In the two-dimensional case we choose $R > 0$ such that $B_R(x) \subset \Omega$ and consider

$$A := \frac{\alpha}{\varepsilon} \int_{B_R(x)} \exp\left(-\frac{\alpha|y-x|^2}{4\varepsilon}\right) dy.$$

The integral over $\Omega \setminus B_R(x)$ converges to 0, since $\frac{\alpha}{\varepsilon} \exp\left(-\frac{\alpha|y-x|^2}{4\varepsilon}\right) \to 0$ for $y \neq x$ as $\varepsilon \to 0$. By using polar coordinates we get

$$\begin{aligned} A &= \frac{\alpha}{\varepsilon} \int_0^R \int_0^{2\pi} \exp\left(-\frac{\alpha r^2}{4\varepsilon}\right) r\, d\varphi\, dr = \frac{2\pi\alpha}{\varepsilon} \int_0^R \exp\left(-\frac{\alpha r^2}{4\varepsilon}\right) r\, dr \\ &= 4\pi \left[1 - \exp\left(-\frac{\alpha R^2}{4\varepsilon}\right)\right] \longrightarrow 4\pi \end{aligned}$$

as $\varepsilon \to 0$. In the three-dimensional case we also choose $R > 0$ such that $B_R(x) \subset \Omega$ and consider

$$A := \frac{\alpha}{\varepsilon^{3/2}} \int_{B_R(x)} \exp\left(-\frac{\alpha|y-x|^2}{4\varepsilon}\right) dy.$$

As in the two-dimensional case the integral over $\Omega \setminus B_R(x)$ vanishes. By using spherical coordinates we get

$$A = \frac{\alpha}{\varepsilon^{3/2}} \int_0^R \int_0^{2\pi} \int_0^{\pi} \exp\left(-\frac{\alpha r^2}{4\varepsilon}\right) r^2 \sin\theta\, d\theta\, d\varphi\, dr = \frac{4\pi\alpha}{\varepsilon^{3/2}} \int_0^R \exp\left(-\frac{\alpha r^2}{4\varepsilon}\right) r^2\, dr.$$

The substitution $\eta^2 = \frac{\alpha r^2}{4\varepsilon}$ leads to

$$A = \frac{32\pi}{\sqrt{\alpha}} \int\limits_0^{\sqrt{\frac{\alpha}{4\varepsilon}}R} \exp\left(-\eta^2\right)\eta^2 d\eta \longrightarrow \frac{32\pi}{\sqrt{\alpha}} \int_0^\infty \exp\left(-\eta^2\right)\eta^2 d\eta = \frac{8\pi^{3/2}}{\sqrt{\alpha}}$$

as $\varepsilon \to 0$. The other two integrals in (2.13) vanish as $\varepsilon \to 0$ due to the boundedness of $\nabla_x u$ and $\partial_t u$. We finally get the representation formula by taking the limit $\varepsilon \to 0$ in (2.12), i.e. we have

$$\begin{aligned} u(x,t) &= \int_\Omega u(y,0)U^\star(x-y,t)dy + \frac{1}{\alpha}\int_0^t \int_\Omega f(y,s)U^\star(x-y,t-s)dyds \\ &\quad + \frac{1}{\alpha}\int_0^t \int_\Gamma \frac{\partial}{\partial n_y} u(y,s)U^\star(x-y,t-s)ds_y ds \\ &\quad - \frac{1}{\alpha}\int_0^t u(y,s)\frac{\partial}{\partial n_y}U^\star(x-y,t-s)ds_y ds \end{aligned} \tag{2.14}$$

where

$$U^\star(x-y,t-s) = \left(\frac{\alpha}{4\pi(t-s)}\right)^{n/2} \exp\left(\frac{-\alpha|x-y|^2}{4(t-s)}\right) \quad \text{for } s < t.$$

The function

$$U^\star(x-y,t-s) = \begin{cases} \left(\frac{\alpha}{4\pi(t-s)}\right)^{n/2} \exp\left(\frac{-\alpha|x-y|^2}{4(t-s)}\right), & s < t, \\ 0\,, & \text{else} \end{cases} \tag{2.15}$$

is called the fundamental solution of the heat equation. Due to the definition of $U^\star$ equation (2.14) can be written as

$$\begin{aligned} u(x,t) &= \int_\Omega u(y,0)U^\star(x-y,t)dy + \frac{1}{\alpha}\int_0^T \int_\Omega f(y,s)U^\star(x-y,t-s)dyds \\ &\quad + \frac{1}{\alpha}\int_0^T \int_\Gamma \frac{\partial}{\partial n_y} u(y,s)U^\star(x-y,t-s)ds_y ds \\ &\quad - \frac{1}{\alpha}\int_0^T u(y,s)\frac{\partial}{\partial n_y}U^\star(x-y,t-s)ds_y ds. \end{aligned} \tag{2.16}$$

Remark 2.1. *Let $\varepsilon > 0$. Due to construction the fundamental solution $U^\star$ is a solution of the homogeneous heat equation*

$$\left(\alpha\partial_t - \Delta_x\right)U^\star(x-y,t-s) = 0 \quad \textit{for } (x,t) \in Q \textit{ and } (y,s) \in \Omega \times (0, t-\varepsilon).$$

Moreover the fundamental solution $U^\star$ defined by (2.15) has the following properties.

Lemma 2.2. *For $t > 0$ there holds*

$$\int_{\mathbb{R}^n} U^\star(x,t)dx = 1.$$

Proof. Let $t > 0$. We have

$$\begin{aligned}\int_{\mathbb{R}^n} U^\star(x,t) &= \left(\frac{\alpha}{4\pi t}\right)^{n/2} \int_{\mathbb{R}^n} \exp\left(\frac{-\alpha|x|^2}{4t}\right) dx \\ &= \pi^{-n/2} \int_{\mathbb{R}^n} \exp\left(-|z|^2\right) dz \\ &= \pi^{-n/2} \prod_{i=1}^{n} \int_{\mathbb{R}^n} \exp\left(-z_i^2\right) dz_i = 1.\end{aligned}$$

□

Lemma 2.3. *Let $u_0 \in C(\Omega) \cap L^\infty(\Omega)$. For $x \in \Omega$ there holds*

$$\lim_{t\to 0} \int_\Omega U^\star(x-y,t)u_0(y)dy = u_0(x).$$

Proof. Let $\varepsilon > 0$ and $u_0 \in C(\Omega) \cap L^\infty(\Omega)$. The function $\widetilde{u}_0$ is defined as

$$\widetilde{u}_0(x) = \begin{cases} u_0(x) & \text{for } x \in \Omega, \\ 0 & \text{else.} \end{cases}$$

Due to Lemma 2.2 and since $U^\star(x,t) > 0$ for $(x,t) \in \mathbb{R}^n \times (0,T)$ we have

$$\begin{aligned}\left| \int_\Omega U^\star(x-y,t)u_0(y)dy - u_0(x) \right| &= \left| \int_{\mathbb{R}^n} U^\star(x-y,t)[\widetilde{u}_0(y) - \widetilde{u}_0(x)]dy \right| \\ &\le \int_{\mathbb{R}^n} U^\star(x-y,t)\left|\widetilde{u}_0(y) - \widetilde{u}_0(x)\right| dy.\end{aligned}$$

Since u_0 ist continuous, there exists a constant $\delta > 0$ such that $|\widetilde{u}_0(y) - \widetilde{u}_0(x)| < \varepsilon/2$ if $|y - x| < \delta$. Hence we can write the last integral as

$$\begin{aligned}\int\limits_{\mathbb{R}^n} U^\star(x-y,t)\left|\widetilde{u}_0(y) - \widetilde{u}_0(x)\right| dy &= \int\limits_{\mathbb{R}^n\setminus B_\delta(x)} U^\star(x-y,t)\left|\widetilde{u}_0(y) - \widetilde{u}_0(x)\right| dy \\ &\quad + \int\limits_{B_\delta(x)} U^\star(x-y,t)\left|\widetilde{u}_0(y) - \widetilde{u}_0(x)\right| dy.\end{aligned}$$

The second integral can be estimated from above by

$$\int\limits_{B_\delta(x)} U^\star(x-y,t)\,\underbrace{|\widetilde{u}_0(y)-\widetilde{u}_0(x)|}_{<\varepsilon/2}\,dy < \frac{\varepsilon}{2}\int_{\mathbb{R}^n} U^\star(x-y,t)dy = \frac{\varepsilon}{2}.$$

Considering the first integral we have

$$\int\limits_{\mathbb{R}^n\setminus B_\delta(x)} U^\star(x-y,t)\,|\widetilde{u}_0(y)-\widetilde{u}(x)|\,dy \le 2\,\|u\|_{L^\infty(\Omega)} \int\limits_{\mathbb{R}^n\setminus B_\delta(x)} U^\star(x-y,t)dy,$$

since $u \in L^\infty(\Omega)$. By using the substitution $z = x-y$ we get

$$\int\limits_{\mathbb{R}^n\setminus B_\delta(x)} U^\star(x-y,t)dy = \int\limits_{\mathbb{R}^n\setminus B_\delta(0)} U^\star(z,t)dz = \left(\frac{\alpha}{4\pi t}\right)^{n/2} \int\limits_{\mathbb{R}^n\setminus B_\delta(0)} \exp\left(\frac{-|z|^2\alpha}{4t}\right)dz.$$

By using polar coordinates we get the estimate

$$\int\limits_{\mathbb{R}^n\setminus B_\delta(x)} U^\star(x-y,t)dy \le Ct^{-n/2}\int\limits_{\delta}^{\infty} r^{n-1}\exp\left(\frac{-r^2\alpha}{4t}\right)dr = C' \int\limits_{at^{-1/2}}^{\infty} \rho^{n-1}\exp\left(-\rho^2\right)d\rho$$

with suitable constants $C, C' > 0$ and $a = \delta\left(\frac{\alpha}{4}\right)^{1/2}$. The last integral converges to zero as $t \to 0$, i.e. for t small enough there holds

$$\int\limits_{\mathbb{R}^n\setminus B_\delta(x)} U^\star(x-y,t)\,|\widetilde{u}_0(y)-\widetilde{u}_0(x)|\,dy < \varepsilon/2.$$

Altogether we have

$$\left|\int_\Omega U^\star(x-y,t)u_0(y)dy - u_0(x)\right| < \varepsilon$$

for t small enough. Since $\varepsilon > 0$ was arbitrarily chosen, the assertion is proven. □

Due to the representation formula (2.16) it suffices to know the Cauchy data $\partial_n u_{|\Sigma}$ and $u_{|\Sigma}$ to compute the solution of the model problem (2.1). Thus, the problem is reduced to the boundary. Since the boundary datum $u_{|\Sigma} = g$ is given, it remains to determine the unknown conormal derivative $\partial_n u_{|\Sigma}$. By applying the Dirichlet and Neumann trace operators (see Chapter 3) to the representation formula (2.14) we get boundary integral equations, which have to be solved. To study the existence and uniqueness of solutions of the boundary integral equations we have to define suitable function spaces, which will be discussed in the following chapter.

3 Function spaces

Solutions of the heat equation show different behaviour in temporal and spatial direction. This leads to the concept of anisotropic Sobolev spaces, which we introduce and discuss in this chapter. Under certain conditions we can define trace operators acting on those spaces and therefore provide conditions for the given Dirichlet datum g and the unkown Neumann datum $\partial_n u_{|\Sigma}$ of the solution, which result in existence and uniqueness theorems of solutions of the model problem (2.1). The presented results are based on [12] and [13].
Before introducing the concept of anisotropic Sobolev spaces we recall definitions and properties of standard Sobolev spaces, based on [14] and [27] .

3.1 Standard Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\Gamma := \partial\Omega$. For $k \in \mathbb{N}_0$ the Sobolev space $W_2^k(\Omega)$ is defined as

$$W_2^k(\Omega) := \{v \in L_2(\Omega) : D^\alpha v \in L_2(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n : |\alpha| \leq k\}$$

where $D^\alpha v$ denotes the weak derivative of v of order α [27, Chapter 2.2]. The Sobolev space $W_2^k(\Omega)$ equipped with the scalar product

$$\langle v, w\rangle_{W_2^k(\Omega)} := \sum_{|\alpha|\leq k} \int_\Omega D^\alpha v D^\alpha w \, dx$$

defines a Hilbert space. Let $\kappa \in (0, 1)$ and $s := k + \kappa \in \mathbb{R} \setminus \mathbb{N}$. Then

$$\|u\|_{W_2^s(\Omega)} := \left\{ \|u\|^2_{W_2^k(\Omega)} + |u|^2_{W_2^s(\Omega)} \right\}^{1/2}$$

with

$$|u|^2_{W_2^s(\Omega)} := \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\kappa}} dxdy$$

defines a norm called the Sobolev-Slobodeckii norm. The Sobolev space $W_2^s(\Omega)$ defined as

$$W_2^s(\Omega) := \left\{ v \in W_2^k(\Omega) : \|v\|_{W_2^s(\Omega)} < \infty \right\}$$

is a Hilbert space with respect to the scalar product

$$\langle v, w\rangle_{W_2^s(\Omega)} := \langle v, w\rangle_{W_2^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{(D^\alpha v(x) - D^\alpha v(y))\,(D^\alpha w(x) - D^\alpha w(y))}{|x-y|^{n+2\kappa}} dxdy.$$

The Sobolev spaces $H^s(\mathbb{R}^n)$ are introduced using distributions [27, Chapter 2.4]. The norm of a function $u \in H^s(\mathbb{R}^n)$ is given by

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{u}(\xi)|^2 \, d\xi$$

where $\hat{u}$ denotes the Fourier transform of u. For $s \in \mathbb{R}_0^+$ we have [27, Theorem 2.4]

$$H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n).$$

For a domain $\Omega \subset \mathbb{R}^n$ the Sobolev space $H^s(\Omega)$ is defined as

$$H^s(\Omega) := \left\{ v = \widetilde{v}_{|\Omega} : \widetilde{v} \in H^s(\mathbb{R}^n) \right\}$$

with norm

$$\|v\|_{H^s(\Omega)} := \inf_{\widetilde{v} \in H^s(\mathbb{R}^n), \widetilde{v}_{|\Omega} = v} \|\widetilde{v}\|_{H^s(\mathbb{R}^n)} .$$

If Ω is a Lipschitz domain we have [27, Theorem 2.6]

$$H^s(\Omega) = W_2^s(\Omega) \text{ for all } s > 0.$$

Moreover we consider the spaces

$$\widetilde{H}^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^n)}}, \quad H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}.$$

Again, if Ω is a Lipschitz domain we have [27, Theorem 2.5]

$$\widetilde{H}^s(\Omega) = \left[H^{-s}(\Omega)\right]', \quad H^s(\Omega) = \left[\widetilde{H}^{-s}(\Omega)\right]' \quad \text{for all } s \in \mathbb{R}.$$

From now on let Ω be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$. Similarly to what we have seen above we can define Sobolev spaces $H^s(\Gamma)$ on the boundary Γ for $|s| \leq 1$. For $s \in (0,1)$ the Sobolev-Slobodeckii norm is defined as

$$\|v\|_{H^s(\Gamma)} := \left\{ \|v\|^2_{L^2(\Gamma)} + |v|^2_{H^s(\Gamma)} \right\}^{1/2} \tag{3.1}$$

with

$$|v|^2_{H^s(\Gamma)} = \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x-y|^{n-1+2s}} ds_x ds_y.$$

The Sobolev space $H^s(\Gamma)$ for $s \in (0,1)$ is defined as the completion of $C(\Gamma)$ with respect of the Sobolev-Slobodeckii norm (3.1) and is a Hilbert space with scalar product

$$\langle u,v\rangle_{H^s(\Gamma)} := \langle u,v\rangle_{L_2(\Gamma)} + \int_\Gamma\int_\Gamma \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{n-1+2s}}dxdy.$$

For $s=0$ we have $H^0(\Gamma)=L^2(\Gamma)$.
Sobolev spaces with negative order $s \in (-1,0)$ are defined as the dual spaces of $H^{-s}(\Gamma)$, i.e. we have

$$H^s(\Gamma) := \left[H^{-s}(\Gamma)\right]'$$

with norm

$$\|w\|_{H^s(\Gamma)} := \sup_{0\neq v\in H^{-s}(\Gamma)} \frac{\langle w,v\rangle_\Gamma}{\|v\|_{H^{-s}(\Gamma)}}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality product

$$\langle w,v\rangle_\Gamma := \int_\Gamma w(x)v(x)ds_x.$$

3.2 Anisotropic Sobolev spaces

In this section we introduce anisotropic Sobolev spaces. The definitions and the main results are based on [12] and [13].
For $r,s \geq 0$ the anisotropic Sobolev space $H^{r,s}(\mathbb{R}^n \times \mathbb{R})$ is defined as

$$H^{r,s}(\mathbb{R}^n \times \mathbb{R}) := L^2(\mathbb{R}, H^r(\mathbb{R}^n)) \cap H^s(\mathbb{R}, L^2(\mathbb{R}^n))$$

where

$$v \in H^s(\mathbb{R}, L^2(\mathbb{R}^n)) \Leftrightarrow \left(1+|\tau|^2\right)^{s/2}\hat{v} \in L^2(\mathbb{R}^n \times \mathbb{R})$$

and $\hat{v}$ denotes the Fourier transform of v with respect to the time variable t. The space $L^2(\mathbb{R}, H^r(\mathbb{R}^n))$ can be characterized in a similar way by using the Fourier transform with respect to the spatial variable x. The norm of a function $u \in H^{r,s}(\mathbb{R}^n \times \mathbb{R})$ is given by

$$\|u\|_{H^{r,s}(\mathbb{R}^n\times\mathbb{R})} := \int_{\mathbb{R}}\int_{\mathbb{R}^n}\left[\left(1+|\xi|^2\right)^r + \left(1+|\tau|^2\right)^s\right]|\hat{u}(\xi,\tau)|^2 d\xi d\tau.$$

For $Q = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $T>0$, the space $H^{r,s}(Q)$ is defined as the space of restrictions of functions in $H^{r,s}(\mathbb{R}^n \times \mathbb{R})$ to Q equipped with the quotient norm. We write

$$H^{r,s}(Q) = L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega)).$$

Moreover we define the space of functions in $H^{r,s}(Q)$ with zero initial conditions

$$\widetilde{H}^{r,s}(Q) := \left\{u = \widetilde{u}_{|Q} : u \in H^{r,s}(\Omega \times (-\infty, T)) : u(x,t) = 0 \text{ for } t < 0\right\}.$$

Another important space in terms of existence and uniqueness analysis is the space

$$\mathcal{V}(Q) := L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$$

with norm

$$\|u\|^2_{\mathcal{V}(Q)} := \|u\|^2_{L^2(0,T;H^1(\Omega))} + \|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} \tag{3.2}$$

and its subspaces

$$\begin{aligned}\widetilde{\mathcal{V}}(Q) &:= \left\{u = \widetilde{u}_{|Q} : u \in \mathcal{V}(\Omega \times (-\infty, T)) : u(x,t) = 0 \text{ for } t < 0\right\},\\ \mathcal{V}_0(Q) &:= L^2(0,T; H^1_0(\Omega)) \cap H^1(0,T; H^{-1}(\Omega)).\end{aligned}$$

The space $\widetilde{\mathcal{V}}(Q)$ is the space of functions in $\mathcal{V}(Q)$ having zero initial conditions, whereas $\mathcal{V}_0(Q)$ denotes the space of functions in $\mathcal{V}(Q)$ having zero boundary conditions. The norms in (3.2) are given by

$$\|u\|^2_{L^2(0,T;H^1(\Omega))} := \int_0^T \|u(\cdot,t)\|^2_{H^1(\Omega)}\, dt$$

and

$$\|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} := \sup_{0 \neq v \in L^2(0,T;H^1_0(\Omega))} \frac{\langle \alpha\partial_t u, v\rangle_Q}{\|v\|_{L^2(0,T;H^1_0(\Omega))}}$$

with

$$\|v\|^2_{L^2(0,T;H^1_0(\Omega))} := \int_0^T \int_\Omega |\nabla_x v(x,t)|^2 dx dt.$$

The space $\mathcal{V}(Q)$ is a dense subspace of $H^{1,\frac{1}{2}}(Q)$ [12, Theorem 12.4]. An important property of functions $u \in \mathcal{V}(Q)$ is, that they are in some sense continuous in time. More precisely we have

$$u \in C([0,T]; L^2(\Omega)). \tag{3.3}$$

If $u \in L^2(0,T; H^2(\Omega) \cap H^1(0,T; L^2(\Omega))$, then $u \in C([0,T]; H^1(\Omega))$.

3.3 Anisotropic Sobolev spaces on Σ

The spaces $H^{r,s}(\Sigma)$ for $r, s \geq 0$ are defined analogously. We have

$$H^{r,s}(\Sigma) := L^2(0,T; H^r(\Gamma) \cap H^s(0,T; L^2(\Gamma)).$$

For $r, s \in (0,1)$ an equivalent norm is given by

$$\|u\|^2_{H^{r,s}(\Sigma)} := \|u\|^2_{L^2(\Sigma)} + |u|^2_{L^2(0,T;H^r(\Gamma))} + |u|^2_{H^s(0,T;L^2(\Gamma))}$$

with

$$|u|^2_{L^2(0,T;H^r(\Gamma))} := \int_0^T \int_\Gamma \int_\Gamma \frac{|u(x,t)-u(y,t)|^2}{|x-y|^{n-1+2r}} ds_y ds_x dt$$

and

$$|u|^2_{H^s(0,T;L^2(\Gamma))} := \int_0^T \int_0^T \frac{\|u(\cdot,t)-u(\cdot,\tau)\|^2_{L^2(\Gamma)}}{|t-\tau|^{1+2s}} d\tau dt.$$

Moreover we define the subspace

$$H^{r,s}_{,0}(\Sigma) := L^2(0,T;H^r(\Gamma)) \cap H^s_0(0,T;L^2(\Gamma))$$

which is the closure in $H^{r,s}(\Sigma)$ of the subspace of functions vanishing in a neighborhood of $t=0$ and $t=T$. For $0 \le s < \frac{1}{2}$ we have $H^{r,s}_{,0}(\Sigma) = H^{r,s}(\Sigma)$.
Anisotropic Sobolev spaces on Σ with negative order $r,s<0$ are defined as the dual spaces of $H^{-r,-s}_{,0}(\Sigma)$, i.e. we have

$$H^{r,s}(\Sigma) := \left[H^{-r,-s}_{,0}(\Sigma)\right]'$$

with norm

$$\|w\|_{H^{r,s}(\Sigma)} := \sup_{0\neq v\in H^{-r,-s}_{,0}(\Sigma)} \frac{\langle w,v\rangle_\Sigma}{\|v\|_{H^{-r,-s}(\Sigma)}}$$

where $\langle\cdot,\cdot\rangle_\Sigma$ denotes the duality product

$$\langle w,v\rangle_\Sigma := \int_\Sigma w(y,s)v(y,s) ds_y ds.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$. For a function $u \in C(\overline{Q})$ we define the interior Dirichlet trace

$$\gamma_0^{\text{int}} u(x,t) := \lim_{\Omega\ni\widetilde{x}\to x\in\Gamma} u(\widetilde{x},t) \quad \text{for } (x,t)\in\Sigma.$$

Hence $\gamma_0^{\text{int}}u$ coincides with the restriction of u to the space-time boundary Σ, i.e. we have $\gamma_0^{\text{int}}u = u_{|\Sigma}$. The following theorem provides a relation of the Dirichlet trace $\gamma_0^{\text{int}}u$ and u in case of functions in anisotropic Sobolev spaces.

Theorem 3.1. *[13, Theorem 2.1] Let $u \in H^{r,s}(Q)$ with $r > \frac{1}{2}$, $s \ge 0$. Then there exists a linear, bounded operator $\gamma_0^{int} : H^{r,s}(Q) \to H^{\mu,\nu}(\Sigma)$ with*

$$\left\|\gamma_0^{int}u\right\|_{H^{\mu,\nu}(\Sigma)} \le c_T \|u\|_{H^{r,s}(Q)} \quad \textit{for all } u \in H^{r,s}(Q),$$

where $\mu = r - \frac{1}{2}$, $\nu = s - \frac{s}{2r}$ and γ_0 is an extension of $\gamma_0^{int}u = u_{|\Sigma}$ for $u \in C(\overline{Q})$.

If $r=1$ and $s=\frac{1}{2}$ we have $\gamma_0^{\text{int}} : H^{1,\frac{1}{2}}(Q) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. The Dirichlet trace operator also satisfies a surjectivity property in the following setting.

Lemma 3.2. *[2, Corollary 2.12] The mapping $\gamma_0^{int} : \widetilde{\mathcal{V}}(Q) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ is surjective.*

Since we want to compute the conormal derivative of solutions of initial boundary value problems, we have to define some Neumann trace operator as well. Let $u \in C^1(\overline{Q})$. The interior Neumann trace of u is defined as

$$\gamma_1^{\mathrm{int}} u(x,t) := \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}} u(\widetilde{x},t) \quad \text{for } (x,t) \in \Sigma.$$

Hence $\gamma_1^{\mathrm{int}} u$ coincides with the conormal derivative of u, i.e we have $\gamma_1^{\mathrm{int}} u = \partial_n u_{|\Sigma}$. Again, we want to define the Neumann trace of functions in anisotropic Sobolev spaces as well. Four our purposes it suffices to restrict to the space

$$H^{1,\frac{1}{2}}(Q, \alpha\partial_t - \Delta_x) := \left\{ u \in H^{1,\frac{1}{2}}(Q) : (\alpha\partial_t - \Delta) u \in L^2(Q) \right\}.$$

Theorem 3.3. *[2, Proposition 2.8] The mapping $\gamma_1^{int} : H^{1,\frac{1}{2}}(Q, \alpha\partial_t - \Delta_x) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ is linear and bounded. If $u \in C^1(\overline{Q})$ then $\gamma_1^{int} u = \partial_n u_{|\Sigma}$ in the distributional sense.*

3.4 Piecewise smooth functions on Σ

Let $\Gamma_0 \subset \Gamma = \partial\Omega$ be an open part of the boundary of Ω. For $r \geq 0$ we define the space

$$H^r(\Gamma_0) := \left\{ v = \widetilde{v}_{|\Gamma_0} : \widetilde{v} \in H^r(\Gamma) \right\}$$

equipped with the quotient norm. For a closed, piecewise smooth boundary $\Gamma = \bigcup_{i=1}^J \overline{\Gamma}_i$ with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$ and $r \geq 0$ we define the space of piecewise smooth functions on Γ as

$$H^r_{\mathrm{pw}}(\Gamma) := \left\{ v \in L^2(\Gamma) : v_{|\Gamma_i} \in H^r(\Gamma_i) \text{ for } i = 1, ..., J \right\}$$

with norm

$$\|v\|_{H^r_{\mathrm{pw}}(\Gamma)} := \left(\sum_{i=1}^J \left\| v_{|\Gamma_i} \right\|^2_{H^r(\Gamma_i)} \right)^{1/2}.$$

With $\Sigma_j := \Gamma_j \times (0,T)$ for $j = 1, ..., J$ we have $\overline{\Sigma} = \bigcup_{j=1}^J \overline{\Sigma}_j$. For $r \geq 0$ and $s \geq 0$ we define anisotropic Sobolev spaces on the open part Σ_j of the space-time boundary Σ as

$$\begin{aligned} H^{r,s}(\Sigma_j) &:= \left\{ v = \widetilde{v}_{|\Sigma_j} : \widetilde{v} \in H^{r,s}(\Sigma) \right\}, \\ H^{r,s}_0(\Sigma_j) &:= \left\{ v = \widetilde{v}_{|\Sigma_j} : \widetilde{v} \in H^{r,s}_{,0}(\Sigma) : \text{ supp } \widetilde{v} \subset \Sigma_j \right\} \end{aligned}$$

and the space of piecewise smooth functions on Σ as

$$H^{r,s}_{\mathrm{pw}}(\Sigma) := \left\{ v \in L^2(\Sigma) : v_{|\Sigma_j} \in H^{r,s}(\Sigma_j) \text{ for } j = 1, ..., J \right\}$$

with norm

$$\|v\|_{H^{r,s}_{\mathrm{pw}}(\Sigma)} := \left(\sum_{j}^{J} \left\| v_{|\Sigma_j} \right\|^2_{H^{r,s}(\Sigma_j)} \right)^{1/2} .$$

For $r, s < 0$ the anisotropic Sobolev spaces on Σ_j are defined as the dual spaces

$$H^{r,s}(\Sigma_j) := \left[H_0^{-r,-s}(\Sigma_j) \right]', \quad \widetilde{H}^{r,s}(\Sigma_j) := \left[H^{-r,-s}(\Sigma_j) \right]'$$

and

$$H^{r,s}_{\mathrm{pw}}(\Sigma) := \prod_{j=1}^{J} \widetilde{H}^{r,s}(\Sigma_j)$$

with norm

$$\|w\|_{H^{r,s}_{\mathrm{pw}}(\Sigma)} := \sum_{j=1}^{J} \left\| w_{|\Sigma_j} \right\|_{\widetilde{H}^{r,s}(\Sigma_j)} .$$

Lemma 3.4. *For $r, s < 0$ and $w \in H^{r,s}_{pw}(\Sigma)$ there holds*

$$\|w\|_{H^{r,s}(\Sigma)} \leq \|w\|_{H^{r,s}_{pw}(\Sigma)} .$$

Proof. For $w \in H^{r,s}_{\mathrm{pw}}(\Sigma)$ we have

$$\begin{aligned} \|w\|_{H^{r,s}(\Sigma)} &= \sup_{0 \neq v \in H_{,0}^{-r,-s}(\Sigma)} \frac{|\langle w, v \rangle_{\Sigma}|}{\|v\|_{H^{-r,-s}(\Sigma)}} \leq \sup_{0 \neq v \in H_{,0}^{-r,-s}(\Sigma)} \sum_{j=1}^{J} \frac{|\langle w, v \rangle_{\Sigma_j}|}{\|v\|_{H^{-r,-s}(\Sigma)}} \\ &\leq \sup_{0 \neq v \in H_{,0}^{-r,-s}(\Sigma)} \sum_{j=1}^{J} \frac{|\langle w_{|\Sigma_j}, v_{|\Sigma_j} \rangle_{\Sigma_j}|}{\left\| v_{|\Sigma_j} \right\|_{H^{-r,-s}(\Sigma_j)}} . \end{aligned}$$

Since $H_{,0}^{-r,-s}(\Sigma) \subset H^{-r,-s}(\Sigma)$ we get

$$\begin{aligned} \|w\|_{H^{r,s}(\Sigma)} &\leq \sup_{0 \neq v \in H^{-r,-s}(\Sigma)} \sum_{j=1}^{J} \frac{|\langle w_{|\Sigma_j}, v_{|\Sigma_j} \rangle_{\Sigma_j}|}{\left\| v_{|\Sigma_j} \right\|_{H^{-r,-s}(\Sigma_j)}} \\ &\leq \sum_{j=1}^{J} \sup_{0 \neq v_j \in H^{-r,-s}(\Sigma_j)} \frac{|\langle w_{|\Sigma_j}, v_j \rangle_{\Sigma_j}|}{\|v_j\|_{H^{-r,-s}(\Sigma_j)}} = \|w\|_{H^{r,s}_{\mathrm{pw}}(\Sigma)} . \end{aligned}$$

□

4 Existence theorems

Using the concept of anisotropic Sobolev spaces we can prove existence and uniqueness of solutions of the initial boundary value problem (2.1). We consider different settings of the model problem.

Lemma 4.1. *[2, Lemma 2.3] Let $f \in L^2(0,T;H^{-1}(\Omega))$. Then the initial boundary value problem*

$$\begin{aligned} \alpha\partial_t u - \Delta_x u &= f \quad \text{in } Q, \\ u &= 0 \quad \text{on } \Sigma, \\ u &= 0 \quad \text{on } \Omega \times \{0\} \end{aligned}$$

has a unique solution $u \in \mathcal{V}_0(Q)$.

Theorem 4.2. *[5, Chapter 7, Theorem 3]. Let $u_0 \in L^2(\Omega)$ and $f \in L^2(Q)$. Then the initial boundary value problem*

$$\begin{aligned} \alpha\partial_t u - \Delta_x u &= f \quad \text{in } Q, \\ u &= 0 \quad \text{on } \Sigma, \\ u &= u_0 \quad \text{on } \Omega \times \{0\} \end{aligned} \tag{4.1}$$

has a unique solution $u \in \mathcal{V}_0(Q)$.

Regarding the unique solution $u \in \mathcal{V}_0(Q) \subset H^{1,\frac{1}{2}}(Q)$ of the initial boundary value problem (4.1) we have $\gamma_0^{\mathrm{int}} u \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and since $(\alpha\partial_t - \Delta_x)u = 0$ we conclude $u \in H^{1,\frac{1}{2}}(Q, \alpha\partial_t - \Delta_x)$ and therefore $\gamma_1^{\mathrm{int}} u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. Higher regularity of the initial condition u_0 leads to higher regularity of the solution u.

Theorem 4.3. *[5, Chapter 7, Theorem 5] Let $u_0 \in H_0^1(\Omega)$, $f \in L^2(Q)$ and $u \in \mathcal{V}_0(Q)$ be the unique solution of problem (4.1). Then $u \in L^2(0,T;H^2(U)) \cap H^1(0,T;L^2(U))$.*

Theorem 4.4. *[2, Theorem 2.9] Let $g \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. Then the initial boundary value problem*

$$\begin{aligned} \alpha\partial_t u - \Delta_x u &= 0 \quad \text{in } Q, \\ u &= g \quad \text{on } \Sigma, \\ u &= 0 \quad \text{on } \Omega \times \{0\} \end{aligned} \tag{4.2}$$

has a unique solution $u \in \widetilde{H}^{1,1/2}(Q)$.

The unique solution $u \in \widetilde{H}^{1,\frac{1}{2}}(Q)$ of (4.2) satisfies $\alpha\partial_t u - \Delta_x u = 0$. We conclude $\partial_t u \in L^2(0,T;H^{-1}(Q))$, since

$$\partial_t u = \frac{1}{\alpha}\Delta_x u \in L^2(0,T;H^{-1}(\Omega)).$$

Hence we have $u \in \widetilde{\mathcal{V}}(Q)$.

Theorem 4.5. *Let $u_0 \in L^2(\Omega)$ and $g \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. Then the initial boundary value problem*

$$\begin{aligned} \alpha\partial_t u - \Delta_x u &= 0 \quad \text{in } Q, \\ u &= g \quad \text{on } \Sigma, \\ u &= u_0 \quad \text{on } \Omega \times \{0\} \end{aligned} \tag{4.3}$$

has a unique solution $u \in \mathcal{V}(Q)$.

Proof. Let $\widetilde{u} \in \mathcal{V}_0(Q)$ be the unique solution of the initial boundary value problem (4.1) with $f = 0$ and initial condition $u_0 \in L^2(\Omega)$ which exists according to Theorem 4.2. For $u := \widetilde{u} + \hat{u}$ we have

$$\begin{aligned} \alpha\partial_t \hat{u} - \Delta_x \hat{u} &= 0 \quad \text{in } Q, \\ \hat{u} &= g \quad \text{on } \Sigma, \\ \hat{u} &= 0 \quad \text{on } \Omega \times \{0\} \end{aligned}$$

since $\widetilde{u} \in C([0,T], L^2(\Omega))$. Theorem 4.4 implies, that there exists a unique solution $\hat{u} \in \widetilde{\mathcal{V}}(Q)$. Hence $u = \widetilde{u} + \hat{u} \in \mathcal{V}(Q)$ is a unique solution of (4.3). $\square$

5 Boundary integral operators

We consider the Dirichlet initial boundary value problem (2.1) with source term $f \in L^2(Q)$, boundary condition $g \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and initial condition $u_0 \in L^2(\Omega)$. The solution for $(x,t) \in Q$ is given by the representation formula

$$\begin{aligned} u(x,t) = &\int_\Omega U^\star(x-y,t)u_0(y)dy + \frac{1}{\alpha}\int_0^T \int_\Omega U^\star(x-y,t-s)f(y,s)dyds \\ &+ \frac{1}{\alpha}\int_0^T \int_\Gamma U^\star(x-y,t-s)\frac{\partial}{\partial n_y}u(y,s)ds_y ds \\ &- \frac{1}{\alpha}\int_0^T \int_\Gamma \frac{\partial}{\partial n_y}U^\star(x-y,t-s)g(y,s)ds_y ds. \end{aligned}$$

By applying the trace operators we get boundary integral equations. To study the unique solvability of those equations we have to analyse the mapping properties of the heat potentials. The presented theory on boundary integral operators for the heat equation is mainly based on [2].

5.1 Initial potential

Let $u_0 \in L^2(\Omega)$. The function

$$(\widetilde{M}_0 u_0)(x,t) := \int_\Omega U^\star(x-y,t)u_0(y)dy \quad \text{for } (x,t) \in Q \tag{5.1}$$

is called initial potential of the heat equation with initial condition u_0. For $x \in \Gamma$ and $t \in (0,T)$ we define the boundary integral operators

$$(M_0 u_0)(x,t) := \gamma_0^{\text{int}}(\widetilde{M}_0 u_0)(x,t) = \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} (\widetilde{M}_0 u_0)(\widetilde{x},t)$$

and

$$(M_1 u_0)(x,t) := \gamma_1^{\text{int}}(\widetilde{M}_0 u_0)(x,t) = \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}}(\widetilde{M}_0 u_0)(\widetilde{x},t).$$

The initial potential has the following properties.

Lemma 5.1. *The initial potential $\widetilde{M}_0 u_0$ with $u_0 \in L^2(\Omega)$ is a solution of the homogeneous heat equation in Q.*

Proof. Let $u(x,t) := (\widetilde{M}_0 u_0)(x,t)$ for $(x,t) \in Q$ be the intial potential with initial condition $u_0 \in L^2(\Omega)$. We have

$$\alpha\partial_t u(x,t) - \Delta_x u(x,t) = (\alpha\partial_t - \Delta_x u)\int_\Omega U^\star(x-y,t)u_0(y)dy.$$

The fundamental solution $U^\star(x-y,t)$ is smooth for $t>0$ and $x \in \Omega$. Thus, we can exchange integration and differentiation. Using the fact, that $U^\star(x-y,t)$ is a solution of the homogeneous heat equation for all $y \in \Omega$ we get

$$\alpha\partial_t u(x,t) - \Delta_x u(x,t) = \int_\Omega \underbrace{[\alpha\partial_t U^\star(x-y,t) - \Delta_x U^\star(x-y,t)]}_{=0} u_0(y)dy = 0.$$

□

Lemma 5.2. *For $u_0 \in C(\Omega) \cap L^\infty(\Omega)$ and $x \in \Omega$ it holds*

$$\lim_{t\to 0}(\widetilde{M}_0 u_0)(x,t) = u_0(x).$$

Proof. Follows with Lemma 2.3. □

Hence the initial potential satisfies the initial condition.

Lemma 5.3. *[18, Lemma 5.4] The mapping $\widetilde{M}_0 : L^2(\Omega) \to H^{1,\frac{1}{2}}(Q, \alpha\partial_t - \Delta_x)$ is linear and bounded.*

Hence according to Lemma 3.1 and Lemma 3.3 the integral operators $M_0 = \gamma_0^{\text{int}}\widetilde{M}_0$ and $M_1 = \gamma_1^{\text{int}}\widetilde{M}_0$ are well defined and bounded. We have

$$M_0 : L^2(\Omega) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma),$$
$$M_1 : L^2(\Omega) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

5.2 Volume potential

Let $f \in L^2(Q)$. The function

$$(\widetilde{N}_0 f)(x,t) := \frac{1}{\alpha}\int_0^t \int_\Omega U^\star(x-y,t-s)f(y,s)dyds \quad \text{for } (x,t) \in Q \tag{5.2}$$

is called volume potential of the heat equation with source term f. For $x \in \Gamma$ and $t \in (0,T)$ we define the boundary integral operators

$$(N_0 f)(x,t) := \gamma_0^{\text{int}}(\widetilde{N}_0 f)(x,t) = \lim_{\Omega\ni\widetilde{x}\to x\in\Gamma}(\widetilde{N}_0 f)(\widetilde{x},t)$$

and

$$(N_1 f)(x,t) := \gamma_1^{\text{int}}(\widetilde{N}_0 f)(x,t) = \lim_{\Omega\ni\widetilde{x}\to x\in\Gamma} n_x \cdot \nabla_{\widetilde{x}}(\widetilde{N}_0 f)(\widetilde{x},t).$$

Lemma 5.4. *Let $f \in C(\overline{Q})$. The volume potential $\widetilde{N}_0 f$ is a solution of the partial differential equation*

$$\alpha\partial_t u(x,t) - \Delta_x u(x,t) = f(x,t) \quad \text{for } (x,t) \in Q.$$

Proof. Let $u(x,t) := (\widetilde{N}_0 f)(x,t)$ for $(x,t) \in Q$ be the volume potential with source term $f \in C(\overline{\Omega})$. We have

$$\begin{aligned}\alpha\partial_t u(x,t) - \Delta_x u(x,t) &= (\alpha\partial_t - \Delta_x)(\widetilde{N}_0 f)(x,t) \\ &= (\alpha\partial_t - \Delta_x) \lim_{\varepsilon\to 0} \frac{1}{\alpha} \int_0^{t-\varepsilon} \int_\Omega U^\star(x-y,t-s) f(y,s) dyds.\end{aligned}$$

Let $\varepsilon > 0$. The fundamental solution $U^\star(x-y,t-s)$ is smooth for $t > 0$ and $s \in (0, t-\varepsilon)$. Hence we can apply the Leibniz integral rule and exchange integration and differentiation. We have

$$\begin{aligned}&(\alpha\partial_t - \Delta_x)\frac{1}{\alpha}\int_0^{t-\varepsilon}\int_\Omega U^\star(x-y,t-s)f(y,s)dyds \\ &\qquad = \int_0^{t-\varepsilon}\int_\Omega \partial_t U^\star(x-y,t-s)f(y,s)dyds + \int_\Omega U^\star(x-y,\varepsilon)f(y,t-\varepsilon)dy \\ &\qquad\qquad - \frac{1}{\alpha}\int_0^{t-\varepsilon}\int_\Omega \Delta_x U^\star(x-y,t-s)f(y,s)dyds \\ &\qquad = \int_0^{t-\varepsilon}\int_\Omega \left[\partial_t - \frac{1}{\alpha}\Delta_x\right] U^\star(x-y,t-s)f(y,s)dyds \\ &\qquad\qquad + \int_\Omega U^\star(x-y,\varepsilon)f(y,t-\varepsilon)dy.\end{aligned}$$

Since $U^\star(x-y,t-s)$ is a solution of the homogeneous heat equation for $y \in \Omega$ and $s \in (0,t-\varepsilon)$, the first integral of the right hand side vanishes. Additionally Lemma 2.3 implies

$$\int_\Omega U^\star(x-y,\varepsilon)f(y,t-\varepsilon)dy \longrightarrow f(x,t)$$

as $\varepsilon \to 0$. Since $\varepsilon > 0$ was arbitrarily chosen we get

$$(\alpha\partial_t - \Delta_x)u(x,t) = f(x,t).$$

□

By definition the volume potential satisfies $(\widetilde{N}_0 f)(x,0) = 0$ for $x \in \Omega$. Due to Lemma 4.1 and Theorem 4.4 we conclude $\widetilde{N}_0 : L^2(\Omega) \to H^{1,\frac{1}{2}}(Q, \alpha\partial_t - \Delta)$ is linear and bounded. Hence the integral operators $N_0 = \gamma_0^{\text{int}}\widetilde{N}_0$ and $N_1 = \gamma_1^{\text{int}}\widetilde{N}_0$ are bounded as well. We have

$$\begin{aligned}N_0 &: L^2(Q) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma), \\ N_1 &: L^2(Q) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).\end{aligned}$$

5.3 Single layer potential

The single layer potential of the heat equation with density $w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ is defined as

$$(\widetilde{V}w)(x,t) := \frac{1}{\alpha}\int_0^t \int_\Gamma U^\star(x-y,t-s)w(y,s)ds_y ds \quad \text{for } (x,t) \in Q. \tag{5.3}$$

For $x \in \Gamma$ and $t \in (0,T)$ we define the single layer boundary integral operator V with density w as

$$(Vw)(x,t) := \gamma_0^{\text{int}}(\widetilde{V}w)(x,t) = \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma}(\widetilde{V}w)(\widetilde{x},t).$$

Lemma 5.5. *The single layer potential $\widetilde{V}w$ with density w is a solution of the homogeneous heat equation in Q.*

Proof. Let $(x,t) \in \Omega \times (0,T)$ and $w \in L^2(\Sigma)$. For $u(x,t) := (\widetilde{V}w)(x,t)$ we have

$$\begin{aligned}\alpha\partial_t u(x,t) - \Delta_x u(x,t) &= (\alpha\partial_t - \Delta_x)(\widetilde{V}w)(x,t)\\ &= (\alpha\partial_t - \Delta_x)\lim_{\varepsilon\to 0}\frac{1}{\alpha}\int_0^{t-\varepsilon}\int_\Gamma U^\star(x-y,t-s)w(y,s)ds_y ds.\end{aligned}$$

Let $\varepsilon > 0$. The fundamental solution $U^\star(x-y,t-s)$ is smooth for $y \in \Omega$ and $s \in (0,t-\varepsilon)$. Thus, we can apply the Leibniz integral rule and exchange integration and differentiation. We have

$$\begin{aligned}&(\alpha\partial_t - \Delta_x)\frac{1}{\alpha}\int_0^{t-\varepsilon}\int_\Gamma U^\star(x-y,t-s)w(y,s)ds_y ds\\ &\quad= \int_0^{t-\varepsilon}\int_\Gamma \partial_t U^\star(x-y,t-s)w(y,s)ds_y ds + \int_\Gamma U^\star(x-y,\varepsilon)w(y,t-\varepsilon)ds_y\\ &\qquad - \frac{1}{\alpha}\int_0^{t-\varepsilon}\int_\Gamma \Delta_x U^\star(x-y,t-s)w(y,s)ds_y ds\\ &\quad= \int_0^{t-\varepsilon}\int_\Gamma \left[\partial_t - \frac{1}{\alpha}\Delta_x\right]U^\star(x-y,t-s)w(y,s)ds_y ds\\ &\qquad + \int_\Gamma U^\star(x-y,\varepsilon)w(y,t-\varepsilon)ds_y\end{aligned}$$

The fundamental solution $U^\star(x-y,t-s)$ is a solution of the homogeneous heat equation for $(x,t) \in \Omega \times (0,T)$ and $(y,s) \in \Gamma \times (0,t-\varepsilon)$. Thus, the first integral of the right hand side vanishes. For $x \in \Omega$ and $y \in \Gamma$ we have $x \neq y$ and therefore the dominated convergence theorem [4, Theorem 5.2] implies

$$\int_\Gamma U^\star(x-y,\varepsilon)w(y,t-\varepsilon)ds_y \longrightarrow 0$$

as $\varepsilon \to 0$. Hence we get

$$(\alpha\partial_t - \Delta_x)u(x,t) = 0.$$

By using a density argument we conclude that the statement holds for $w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ as well. $\square$

Lemma 5.6. *[2, Remark 3.2] The mapping $\widetilde{V} : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{1,\frac{1}{2}}(Q, \alpha\partial_t - \Delta_x)$ is linear and bounded.*

Hence $Vw = \gamma_0^{\text{int}}(\widetilde{V}w)$ is well defined and due to Lemma 3.1 the single layer boundary integral operator

$$V := \gamma_0^{\text{int}}\widetilde{V} : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

is linear and bounded, i.e. there exsists a constant $c_2^V > 0$ such that

$$\|Vw\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq c_2^V \, \|w\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \quad \text{for all } w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

For the exterior trace we have

$$(Vw)(x,t) = \gamma_0^{\text{ext}}(\widetilde{V}w)(x,t) = \lim_{\mathbb{R}^n\setminus\overline{\Omega}\ni\widetilde{x}\to x\in\Gamma} (\widetilde{V}w)(\widetilde{x},t) \quad \text{for } (x,t) \in \Sigma$$

and therefore we obtain the jump relation

$$\left[\gamma_0\widetilde{V}w\right]_{|\Sigma} = \gamma_0^{\text{ext}}(\widetilde{V}w)(x,t) - \gamma_0^{\text{int}}(\widetilde{V}w)(x,t) = 0 \quad \text{for } (x,t) \in \Sigma. \tag{5.4}$$

5.4 Adjoint double layer potential

Due to Lemma 3.3 the operator $\gamma_1^{\text{int}}\widetilde{V} : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ is linear and bounded.

Lemma 5.7. *[7, Chapter 5, Theorem 1]. Let $w \in C(\overline{\Sigma})$ and $(x,t) \in \Sigma = \Gamma \times (0,T)$. The single layer potential $\widetilde{V}w$ satisfies the relation*

$$\gamma_1^{int}(\widetilde{V}w)(x,t) = \lim_{\Omega\ni\widetilde{x}\to x\in\Gamma} n_x \cdot \nabla_{\widetilde{x}}(\widetilde{V}w)(\widetilde{x},t) = \frac{1}{2}w(x,t) + (K'w)(x,t)$$

where

$$(K'w)(x,t) = \frac{1}{\alpha}\int_0^t \int_\Gamma \frac{\partial}{\partial n_x} U^\star(x-y,t-s)w(y,s)ds_y ds \quad \textit{for } (x,t) \in \Sigma$$

is called the adjoint double layer potential with density w.

The exterior conormal derivative of the single layer potential satisfies

$$\gamma_1^{\text{ext}}(\widetilde{V}w)(x,t) = \lim_{\mathbb{R}^n\setminus\overline{\Omega}\ni\widetilde{x}\to x\in\Gamma} n_x \cdot \nabla_{\widetilde{x}}(\widetilde{V}w)(\widetilde{x},t) = -\frac{1}{2}w(x,t) + (K'w)(x,t).$$

Thus, we get the jump relation

$$\left[\gamma_1\widetilde{V}w\right]_{|\Sigma} = \gamma_1^{\text{ext}}(\widetilde{V}w)(x,t) - \gamma_1^{\text{int}}(\widetilde{V}w)(x,t) = -w(x,t) \quad \text{for } (x,t)\in\Sigma.$$

Using this relation, we can define the adjoint double boundary integral operator for functions $w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ as

$$K'w := \frac{1}{2}\left(\gamma_1^{\text{int}}(\widetilde{V}w) + \gamma_1^{\text{ext}}(\widetilde{V}w)\right).$$

Due to the linearity and boundedness of $\widetilde{V}$ and the Neumann trace operators the mapping

$$K' : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$$

is linear and bounded. The jump relation for the conormal derivative of the single layer potential holds for functions $w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ as well, i.e. we have

$$\left[\gamma_1\widetilde{V}w\right]_{|\Sigma} = -w(x,t) \quad \text{for } (x,t)\in\Sigma. \tag{5.5}$$

5.5 Double layer potential

The double layer potential of the heat equation with density $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ is defined as

$$(Wv)(x,t) := \frac{1}{\alpha}\int_0^T\int_\Gamma \frac{\partial}{\partial n_y}U^\star(x-y,t-s)v(y,s)ds_y ds \quad \text{for } (x,t)\in Q. \tag{5.6}$$

Lemma 5.8. *The double layer potential Wv with density v is a solution of the homogeneous heat equation in Q.*

Proof. Let $(x,t)\in\Omega\times(0,T)$ and $v\in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. For $u(x,t) := (Wv)(x,t)$ we have

$$\begin{aligned}\alpha\partial_t u(x,t) - \Delta_x u(x,t) &= (\alpha\partial_t - \Delta_x)(Wv)(x,t)\\ &= (\alpha\partial_t - \Delta_x)\lim_{\varepsilon\to 0}\frac{1}{\alpha}\int_0^{t-\varepsilon}\int_\Gamma \frac{\partial}{\partial n_y}U^\star(x-y,t-s)v(y,s)ds_y ds.\end{aligned}$$

Let $\varepsilon > 0$. As in the case of the single layer potential we can apply the Leibniz integral rule and exchange integration and differentiation. We get

$$
\begin{aligned}
(\alpha\partial_t - \Delta_x)\frac{1}{\alpha}\int_0^{t-\varepsilon}\int_\Gamma \frac{\partial}{\partial n_y}U^\star(x-y,t-s)v(y,s)ds_y ds \\
= \int_0^{t-\varepsilon}\int_\Gamma \frac{\partial}{\partial n_y}\left[\partial_t - \frac{1}{\alpha}\Delta_x\right]U^\star(x-y,t-s)v(y,s)ds_y ds \\
+ \int_\Gamma \frac{\partial}{\partial n_y}U^\star(x-y,\varepsilon)v(y,t-\varepsilon)ds_y.
\end{aligned}
$$

The first integral of the right hand side vanishes, since $U^\star(x-y,t-s)$ is a solution of the homogeneous heat equation. For $x \in \Omega$ and $y \in \Gamma$ we have $x \neq y$ and therefore the dominated convergence theorem [4, Theorem 5.2] implies

$$
\int_\Gamma \frac{\partial}{\partial n_y}U^\star(x-y,\varepsilon)v(y,t-\varepsilon)ds_y \longrightarrow 0
$$

as $\varepsilon \to 0$. Hence we have

$$
(\alpha\partial_t - \Delta_x)u(x,t) = 0.
$$

□

Lemma 5.9. *[2, Proposition 3.3] The mapping $W : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{1,\frac{1}{2}}(Q,\alpha\partial_t - \Delta_x)$ is linear and bounded.*

Thus, the linear operator $\gamma_0^{\mathrm{int}}W : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ is well defined and bounded.

Lemma 5.10. *[7, Chapter 5] Let $v \in C(\overline{\Sigma})$ and $(x,t) \in \Sigma$. The double layer potential Wv satisfies the relation*

$$
\gamma_0^{int}(Wv)(x,t) = \lim_{\Omega \ni \widetilde{x} \to x \in \Gamma}(Wv)(\widetilde{x},t) = -\frac{1}{2}v(x,t) + (Kv)(x,t)
$$

where K denotes the double layer boundary integral operator defined as

$$
(Kv)(x,t) = \frac{1}{\alpha}\int_0^T\int_\Gamma \frac{\partial}{\partial n_y}U^\star(x-y,t-s)v(y,s)ds_y ds \quad \textit{for } (x,t) \in \Sigma.
$$

The exterior Dirichlet trace of the double layer potential satisfies

$$
\gamma_0^{\mathrm{ext}}(Wv)(x,t) = \lim_{\mathbb{R}^n \setminus \overline{\Omega} \ni \widetilde{x} \to x \in \Gamma}(Wv)(\widetilde{x},t) = \frac{1}{2}v(x,t) + (Kv)(x,t) \quad \text{for } (x,t) \in \Sigma.
$$

Hence we have the following jump relation for the Dirichlet trace of the double layer potential

$$
[\gamma_0 Wv]_{|\Sigma} = \gamma_0^{\mathrm{ext}}(Wv)(x,t) - \gamma_0^{\mathrm{int}}(Wv)(x,t) = v(x,t) \quad \text{for } (x,t) \in \Sigma. \tag{5.7}
$$

Using this relation we can define the double layer boundary integral operator K for functions $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ as

$$Kv := \frac{1}{2}\left(\gamma_0^{\text{int}}(Wv) + \gamma_0^{\text{ext}}(Wv)\right).$$

The boundedness of the mapping

$$K : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$$

follows from the boundedness of the operator W and the Dirichlet trace operators. The jump relation (5.7) holds for functions $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ as well, i.e. we have

$$[\gamma_0 Wv]_{|\Sigma} = v(x,t) \quad \text{for } (x,t) \in \Sigma. \tag{5.8}$$

5.6 Hypersingular boundary integral operator

The hypersingular operator D with density $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ is defined as

$$(Dv)(x,t) := -\gamma_1^{\text{int}}(Wv)(x,t) = -\lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}}(Wv)(\widetilde{x},t) \quad \text{for } (x,t) \in \Sigma.$$

Due to the boundedness of the operator W and the Neumann trace operator the hypersingular operator

$$D : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$$

is bounded as well, i.e. there exists a constant $c_2^D > 0$ such that

$$\|Dv\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq c_2^D \|v\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \text{for all } v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

The conormal derivative of the double layer potential satisfies the jump relation [2, Theorem 3.4]

$$[\gamma_1 Wv]_{|\Sigma} = \gamma_1^{\text{ext}}(Wv)(x,t) - \gamma_0^{\text{int}}(Wv)(x,t) = 0 \quad \text{for } (x,t) \in \Sigma.$$

6 Boundary integral equations

Let us consider the representation formula (2.16) for the solution of the model problem (2.1). For $(\widetilde{x}, t) \in Q$ we have

$$u(\widetilde{x}, t) = (\widetilde{V}\gamma_1^{\text{int}} u)(\widetilde{x}, t) - (W\gamma_0^{\text{int}} u)(\widetilde{x}, t) + (\widetilde{M_0} u_0)(\widetilde{x}, t) + (\widetilde{N_0} f)(\widetilde{x}, t).$$

By applying the Dirichlet trace operator and recalling the jump relations of the heat potentials we get the first boundary integral equation

$$\begin{aligned} \gamma_0^{\text{int}} u(x,t) =& (V\gamma_1^{\text{int}} u)(x,t) + \frac{1}{2}\gamma_0^{\text{int}} u(x,t) - (K\gamma_0^{\text{int}} u)(x,t) \\ &+ (M_0 u_0)(x,t) + (N_0 f)(x,t) \end{aligned} \tag{6.1}$$

for $(x, t) \in \Sigma$. Similarly by applying the Neumann trace operator we get the second boundary integral equation

$$\begin{aligned} \gamma_1^{\text{int}} u(x,t) =& \frac{1}{2}\gamma_1^{\text{int}} u(x,t) + (K'\gamma_1^{\text{int}} u)(x,t) + (D\gamma_0^{\text{int}} u)(x,t) \\ &+ (M_1 u_0)(x,t) + (N_1 f)(x,t) \end{aligned} \tag{6.2}$$

for $(x, t) \in \Sigma$. Together these equations lead to the Calderón system of boundary integral equations. We have

$$\begin{pmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix}}_{=:\, \mathcal{C}} \begin{pmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{pmatrix} + \begin{pmatrix} M_0 u_0 \\ M_1 u_0 \end{pmatrix} + \begin{pmatrix} N_0 f \\ N_1 f \end{pmatrix}. \tag{6.3}$$

The operator $\mathcal{C}$ is called the Calderón projection operator.

Lemma 6.1. *$\mathcal{C}$ is a projection, i.e. $\mathcal{C} = \mathcal{C}^2$.*

Proof. Let $(\psi, \varphi) \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \times H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. Then the function

$$u(\widetilde{x}, t) := (\widetilde{V}\psi)(\widetilde{x}, t) - (W\varphi)(\widetilde{x}, t) \quad \text{for } (\widetilde{x}, t) \in Q$$

is a solution of the homogeneous heat equation. By applying the trace operators we get the boundary integral equations

$$\begin{aligned} \gamma_0^{\text{int}} u &= V\psi + \left(\frac{1}{2}I - K\right)\varphi \\ \gamma_1^{\text{int}} u &= \left(\frac{1}{2}I + K'\right)\psi + D\varphi. \end{aligned} \tag{6.4}$$

Additionally u is a solution of the homogeneous heat equation with Cauchy data $\gamma_0^{\text{int}}u, \gamma_1^{\text{int}}u$ and inital condition $u_0 = 0$, i.e. we have

$$\begin{pmatrix} \gamma_0^{\text{int}}u \\ \gamma_1^{\text{int}}u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \gamma_0^{\text{int}}u \\ \gamma_1^{\text{int}}u \end{pmatrix}.$$

Inserting (6.4) leads to

$$\begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix}^2 \begin{pmatrix} \psi \\ \varphi \end{pmatrix}.$$

Since the functions ψ, φ were arbitrarily chosen, we conclude $\mathcal{C} = \mathcal{C}^2$. □

In consequence of the projection property of the Calderón operator we have the following relations.

Corollary 6.2. *The boundary integral operators satisfy*

$$\begin{aligned} VD &= \left(\frac{1}{2}I - K\right)\left(\frac{1}{2}I + K\right), \\ DV &= \left(\frac{1}{2}I + K'\right)\left(\frac{1}{2}I - K'\right), \\ VK' &= KV, \\ K'D &= DK. \end{aligned}$$

Proof. Follows from $\mathcal{C} = \mathcal{C}^2$. □

Let us recall the mapping properties of the boundary integral operators. We have

$$\begin{aligned} V &: H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma), \\ K &: H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma), \\ K' &: H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma), \\ D &: H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma). \end{aligned} \tag{6.5}$$

Theorem 6.3. *[2, Corollary 3.10, Theorem 3.11] The operator*

$$\mathcal{A} : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \times H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \times H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$$

defined as

$$\mathcal{A} := \begin{pmatrix} -K & V \\ D & K' \end{pmatrix}$$

is an isomorphism and there exists a constant $c_1 > 0$ such that

$$\langle \begin{pmatrix} \psi \\ \varphi \end{pmatrix}, \begin{pmatrix} V & -K \\ K' & D \end{pmatrix} \begin{pmatrix} \psi \\ \varphi \end{pmatrix} \rangle \geq c_1 \left(\|\psi\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} + \|\varphi\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \right)$$

for all $(\psi, \varphi) \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \times H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$.

By using this theorem we can prove the ellipticity of the operators V and D.

6.1 Ellipticity of the single layer boundary integral operator V

Lemma 6.4. *The single layer boundary integral operator V defines an isomorphism and there exists a constant $c_1^V > 0$ such that*

$$\langle Vw, w\rangle \geq c_1^V \|w\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \quad \text{for all } w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

Proof. Follows from Theorem 6.3 with $\varphi = 0$. □

Hence the single layer boundary integral operator $V : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ is bounded and $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$-elliptic and therefore invertible. Thus, the inverse operator $V^{-1} : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ is well defined and bounded according to the Lemma of Lax-Milgram (Theorem 1.1). We have

$$\left\|V^{-1}v\right\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq \frac{1}{c_1^V} \|v\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \text{for all } v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

6.2 Ellipticity of the hypersingular boundary integral operator D

Lemma 6.5. *The hypersingular boundary integral operator D defines an isomorphism and there exists a constant $c_1^D > 0$ such that*

$$\langle Dv, v\rangle \geq c_1^D \|v\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \text{for all } v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

Proof. Follows from Theorem 6.3 with $\psi = 0$. □

Hence D is invertible and according to the Lemma of Lax-Milgram (Theorem 1.1) we have

$$\left\|D^{-1}w\right\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \leq \frac{1}{c_1^D} \|w\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \quad \text{for all } w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

According to Corollary 6.2 we have the relations

$$\begin{aligned} VD &= \left(\frac{1}{2}I - K\right)\left(\frac{1}{2}I + K\right), \\ DV &= \left(\frac{1}{2}I + K'\right)\left(\frac{1}{2}I - K'\right). \end{aligned}$$

Since V and D define isomorphisms, the operators

$$\frac{1}{2}I - K, \frac{1}{2}I + K : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma),$$
$$\frac{1}{2}I - K', \frac{1}{2}I + K' : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$$

define isomorphisms as well. Hence these operators are invertible. It is well known that under certain circumstances these operators define contractions, see [17] and [19]. For example if the boundary Γ is C^2, then the double layer boundary integral operator K defines a contraction for sufficiently small T.

6.3 Steklov-Poincaré operator

We consider the system of boundary integral equations corresponding to the homogeneous heat equation with initial condition $u_0 = 0$. We have

$$\begin{pmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{pmatrix}.$$

Using the first integral equation we can define the Dirichlet to Neumann map

$$\gamma_1^{\text{int}} u = V^{-1} \left(\frac{1}{2}I + K \right) \gamma_0^{\text{int}} u. \tag{6.6}$$

The operator

$$S := V^{-1} \left(\frac{1}{2}I + K \right) : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \tag{6.7}$$

is called Steklov-Poincaré operator for the heat equation. When inserting (6.6) into the second boundary integral equation we obtain

$$\gamma_1^{\text{int}} u = \left[D + \left(\frac{1}{2}I + K' \right) V^{-1} \left(\frac{1}{2}I + K \right) \right] \gamma_0^{\text{int}} u.$$

Hence we get a symmetric representation of the Steklov-Poincaré operator. We have

$$S = D + \left(\frac{1}{2}I + K' \right) V^{-1} \left(\frac{1}{2}I + K \right). \tag{6.8}$$

Due to the boundedness of the operators K, K', D and V^{-1} the operator S is bounded as well.

Lemma 6.6. *The Steklov-Poincaré operator S is $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$-elliptic, i.e. there exists a constant $c_1^S > 0$ such that*

$$\langle Sv, v \rangle \geq c_1^S \, \|v\|^2_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} \quad \textit{for all } v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

Proof. For $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ we define $\psi := V^{-1}\left(\frac{1}{2}I + K\right)v \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and get

$$\begin{aligned}
\langle \begin{pmatrix} \psi \\ v \end{pmatrix}, \begin{pmatrix} V & -K \\ K' & D \end{pmatrix}\begin{pmatrix} \psi \\ v \end{pmatrix}\rangle &= \frac{1}{2}\langle V^{-1}\left(\frac{1}{2}I + K\right)v, v\rangle + \langle v, K'V^{-1}\left(\frac{1}{2}I + K\right)v + Dv\rangle \\
&= \langle v, \left(\frac{1}{2}I + K'\right)V^{-1}\left(\frac{1}{2}I + K\right)v + Dv\rangle \\
&= \langle v, Sv\rangle.
\end{aligned}$$

The statement follows with Theorem 6.3. □

6.4 Dirichlet initial boundary value problem

Let us consider the initial boundary value problem

$$\begin{aligned}
\alpha\partial_t u(x,t) - \Delta_x u(x,t) &= 0 && \text{for } (x,t) \in Q = \Omega \times (0,T), \\
u(x,t) &= g(x,t) && \text{for } (x,t) \in \Sigma = \Gamma \times (0,T), \\
u(x,0) &= u_0(x) && \text{for } x \in \Omega
\end{aligned}$$

with boundary condition $g \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and intial condition $u_0 \in L^2(\Omega)$. Then the solution is given by the representation formula

$$u(\widetilde{x},t) = (\widetilde{V}\gamma_1^{\text{int}}u)(\widetilde{x},t) - (Wg)(\widetilde{x},t) + (\widetilde{M_0}u_0)(\widetilde{x},t) \quad \text{for } (\widetilde{x},t) \in Q.$$

It remains to determine the unknown conormal derivative $\gamma_1^{\text{int}}u \in H^{-\frac{1}{2},\frac{1}{4}}(\Sigma)$. There are different ways to accomplish this. For example we can use the boundary integral equations (6.3) to solve the problem. This is called direct approach. First let us consider the boundary integral equation (6.1), i.e. we have

$$\gamma_0^{\text{int}}u(x,t) = (V\gamma_1^{\text{int}}u)(x,t) + \frac{1}{2}\gamma_0^{\text{int}}u(x,t) - (K\gamma_0^{\text{int}}u)(x,t) + (M_0u_0)(x,t)$$

for $(x,t) \in \Sigma$. We have to find $\gamma_1^{\text{int}}u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ such that

$$V\gamma_1^{\text{int}}u = \left(\frac{1}{2}I + K\right)g - M_0u_0 \quad \text{on } \Sigma.$$

Since the boundary integral operators $K : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, $M_0 : L^2(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ and $V : H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma) \to H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ are bounded and V is $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$-elliptic, there exists a unique solution $\gamma_1^{\text{int}}u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ according to the Lemma of Lax-Milgram (Theorem 1.1). The solution $\gamma_1^{\text{int}}u$ satisfies

$$\begin{aligned}
\left\|\gamma_1^{\text{int}}u\right\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} &\leq \frac{1}{c_1^V}\left(\left\|\left(\frac{1}{2}I + K\right)g\right\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} + \|M_0u_0\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)}\right) \\
&\leq \frac{1}{c_1^V}\left(c_2^W\|g\|_{H^{\frac{1}{2},\frac{1}{4}}(\Sigma)} + c_2^{M_0}\|u_0\|_{L^2}\right).
\end{aligned}$$

The variational formulation of the problem is to find $\gamma_1^{\text{int}} u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ such that

$$\langle V\gamma_1^{\text{int}} u, \tau\rangle_\Sigma = \langle \left(\frac{1}{2}I + K\right) g, \tau\rangle_\Sigma - \langle M_0 u_0, \tau\rangle_\Sigma \quad \text{for all } \tau \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

We could also use the second boundary integral equation to determine the unknown Neumann datum $\gamma_1^{\text{int}} u$, which is given by

$$\gamma_1^{\text{int}} u(x,t) = \frac{1}{2}\gamma_1^{\text{int}} u(x,t) + (K'\gamma_1^{\text{int}} u)(x,t) + (D\gamma_0^{\text{int}} u)(x,t) + (M_1 u_0)(x,t)$$

for $(x,t) \in \Sigma$. We have to find $\gamma_1^{\text{int}} u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ such that

$$\left(\frac{1}{2}I - K'\right) \gamma_1^{\text{int}} u = Dg + M_1 u_0 \quad \text{on } \Sigma.$$

The operator $\frac{1}{2}I - K'$ is invertible and therefore there exists a unique solution of the problem.
Another approach is using an indirect formulation with the single layer potential $\widetilde{V}$. A solution of the homogeneous heat equation with initial condition u_0 is given by

$$u(\widetilde{x},t) = (\widetilde{V}w)(\widetilde{x},t) + (\widetilde{M_0} u_0)(\widetilde{x},t) \quad \text{for } (\widetilde{x},t) \in Q$$

with density $w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$. By applying the Dirichlet trace operator to this equation we get

$$g(x,t) = (Vw)(x,t) + (M_0 u_0)(x,t) \quad \text{for } (x,t) \in \Sigma. \tag{6.9}$$

Thus, we have to find $w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ such that

$$Vw = g - M_0 u_0 \quad \text{on } \Sigma.$$

As in the case of the direct formulation with the first boundary integral equation the unique solvability follows with the Lemma of Lax-Milgram (Theorem 1.1). The variational formulation is to find $w \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ such that

$$\langle Vw, \tau\rangle_\Sigma = \langle g - M_0 u_0, \tau\rangle_\Sigma \quad \text{for all } \tau \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

In the same way we can use the double layer potential W. The function

$$u(\widetilde{x},t) = -(Wv)(\widetilde{x},t) + (\widetilde{M_0} u_0)(\widetilde{x},t) \quad \text{for } (\widetilde{x},t) \in Q$$

with density $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ solves the homogeneous heat equation with initial condition u_0. Applying the Dirichlet trace operator leads to the boundary integral equation

$$g(x,t) = \frac{1}{2}v(x,t) - (Kv)(x,t) + (M_0 u_0)(x,t) \quad \text{for } (x,t) \in \Sigma.$$

Hence we have to find $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ such that

$$\left(\frac{1}{2}I - K\right) v = g - M_0 u_0 \quad \text{on } \Sigma.$$

Again, this problem is uniquely solvable since the operator $\frac{1}{2}I - K$ is invertible. The variational formulation of this problem is to find $v \in H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ such that

$$\langle \left(\frac{1}{2}I - K\right) v, \tau\rangle_\Sigma = \langle g - M_0 u_0, \tau\rangle_\Sigma \quad \text{for all } \tau \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

Hence the ansatz space and the test space differ in this setting.

7 Boundary element methods

The solution of the Dirichlet initial boundary value problem (2.1) with initial condition u_0, boundary data g and source term $f = 0$ is given by

$$u(\widetilde{x}, t) = (\widetilde{V}\gamma_1^{\text{int}} u)(\widetilde{x}, t) - (Wg)(\widetilde{x}, t) + (\widetilde{M_0} u_0)(\widetilde{x}, t) \quad \text{for } (\widetilde{x}, t) \in Q.$$

In the previous chapter we have shown, that we can determine the unknown conormal derivative $\gamma_0^{\text{int}} u$ by solving the first boundary integral equation (6.1), which is equivalent to solving the corresponding variational formulation. We have to find $w := \gamma_1^{\text{int}} u \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma)$ such that

$$\langle Vw, \tau \rangle_\Sigma = \langle \left(\frac{1}{2} I + K \right) g, \tau \rangle_\Sigma - \langle M_0 u_0, \tau \rangle_\Sigma \quad \text{for all } \tau \in H^{-\frac{1}{2}, -\frac{1}{4}}(\Sigma). \tag{7.1}$$

In this section we discretize this problem by considering a Galerkin-Bubnov variational formulation. First we have to define suitable trial spaces with respect to an admissible triangulation of the space-time boundary Σ.

7.1 Discretization

We consider two different triangulation approaches. The first one is a separate triangulation of the boundary Γ and the time interval $(0, T)$. In this case we can use space-time tensor product spaces to discretize the variational formuation (7.1) and we are able to state error estimates simply by combining approximation properties of the spatial and temporal discretization. The second approach is the triangluation of the space-time boundary $\Sigma = \Gamma \times (0, T)$. When using piecewise constant basis functions for the discretization of the variational formulation (7.1) we can derive error estimates by using the approximation properties of the space-time tensor product spaces of the first method.

Spatial and temporal triangulation

We assume, that the Lipschitz boundary $\Gamma = \partial\Omega$ is piecewise smooth with $\overline{\Gamma} = \bigcup_{j=1}^{J} \overline{\Gamma}_j$. Let $\{\Gamma_{N_X}\}_{N_X \in \mathbb{N}}$ be a family of admissible triangulations of the boundary Γ into boundary elements γ_l, i.e. we have

$$\Gamma_{N_X} = \bigcup_{l=1}^{N_X} \overline{\gamma}_l. \tag{7.2}$$

Moreover we assume, that there are no curved elements and that there is no approximation of the boundary. For each boundary element γ_l there exists $j \in \{1, ..., J\}$ such that $\gamma_l \subset \Gamma_j$. The boundary elements γ_l can be described as $\gamma_l = \chi_l(\gamma)$, where γ is some reference element in $\mathbb{R}^{n-1}$. The boundary elements γ_l are line segments in the spatially two-dimensional case $n = 2$ and plane triangles in the three-dimensional case $n = 3$. In the one-dimensional case the boundary Γ is a set of two points and therefore we do not have a triangulation of Γ for $n = 1$. Let $\{x_k\}_{k=1}^{M}$ be the set of boundary nodes x_k of Γ_{N_X}. For each boundary element γ_l we define its volume

$$\Delta_l := \int_{\gamma_l} ds_x,$$

its local mesh size

$$h_l := \Delta_l^{1/(n-1)}$$

and its diameter

$$d_l := \sup_{x,y \in \gamma_l} |x - y|.$$

The global mesh size is given by

$$h_x := \max_{l=1,...,N_X} h_l.$$

The family $\{\Gamma_{N_X}\}_{N_X \in \mathbb{N}}$ of triangulations is said to be globally quasi-uniform, if there exists a constant $c_{G,x} \geq 1$ independent of Γ_{N_X} such that

$$\frac{h_{x,\max}}{h_{x,\min}} \leq c_{G,x}.$$

We assume that the boundary elements are shape regular, i.e. there exists a constant c_B independent of Γ_{N_X} such that

$$d_l \leq c_B h_l \quad \text{for all } l = 1, ..., N_X.$$

As mentioned before, in the spatially one-dimensional case there is no triangulation of the boundary Γ. Whereas for $n = 2, 3$ we need a parametrization of the boundary elements γ_l.

In the two-dimensional case the boundary elements γ_l are line segments with nodes $x_{l_1}, x_{l_2} \in \mathbb{R}^2$. Thus, γ_l can be described as

$$x(\xi) = x_{l_1} + \xi(x_{l_2} - x_{l_1}) \quad \text{for } \xi \in \gamma = (0, 1).$$

Hence we have

$$h_l = d_l = \Delta_l = \int_{\gamma_l} ds_x = |x_{l_2} - x_{l_1}|.$$

In the three-dimensional case the boundary elements γ_l are plane triangles with nodes $x_{l_1}, x_{l_2}, x_{l_3} \in \mathbb{R}^3$. Therefore the boundary element γ_l can be described as

$$x(\xi) = x_{l_1} + \xi_1(x_{l_2} - x_{l_1}) + \xi_2(x_{l_3} - x_{l_1}) = x_{l_1} + J_l \xi$$

where $\xi = (\xi_1, \xi_2)^T \in \gamma = \{\xi \in \mathbb{R}^2 : 0 < \xi_1 < 1, 0 < \xi_2 < 1 - \xi_1\}$ and

$$J_l = \begin{pmatrix} x_{l_2} - x_{l_1} & x_{l_3} - x_{l_1} \end{pmatrix}.$$

The volume of a boundary element γ_l is given by

$$\Delta_l = \int_{\gamma_l} ds_x = \frac{1}{2}|\det J_l|.$$

Since time is one-dimensional we consider a family $\{I_{N_T}\}_{N_T \in \mathbb{N}}$ of decompositions of the time interval $I := (0, T)$ into line segments τ_k, i.e. we have

$$I_{N_T} = \bigcup_{k=1}^{N_T} \overline{\tau}_k. \tag{7.3}$$

The elements $\tau_k = (t_{k_1}, t_{k_2})$ can be described as $\tau_k = \psi_k(\tau)$, where τ is the reference element $\tau := (0, 1)$. In our case we have

$$\psi_k(\hat{t}) = t_{k_1} + \hat{t}(t_{k_2} - t_{k_1}) \quad \text{for } \hat{t} \in (0, 1).$$

As for the boundary elements γ_l we can define the local mesh size of an element τ_k, which coincides with its volume, as

$$h_{t_k} := t_{k_2} - t_{k_1}$$

and the global mesh size as

$$h_t := \max_{k=1,\ldots,N_T} h_{t_k}.$$

Again, the family $\{I_{N_T}\}_{N_T \in \mathbb{N}}$ of triangulations is said to be globally quasi-uniform, if there exists a constant $c_{G,t} \geq 1$ independent of I_{N_T} such that

$$\frac{h_{t,\max}}{h_{t,\min}} \leq c_{G,t}.$$

Trial spaces

In order to find an approximation of the solution of (7.1) we have to define suitable finite dimensional function spaces. Since the conormal derivative of functions could be discontinuous depending on the domain Ω, it is reasonable to approximate the

conormal derivative $w = \gamma_1^{\text{int}} u$ by discontinuous functions. In this work we consider the space of piecewise constant basis functions.
Let $S_{h_t}^0(I)$ be the space of piecewise constant basis functions on $I = (0,T)$ with respect to the temporal triangulation I_{N_T}, i.e. we have

$$S_{h_t}^0(I) := \operatorname{span}\left\{\psi_k^0\right\}_{k=1}^{N_T}$$

with

$$\psi_k^0(t) := \begin{cases} 1 & \text{for } t \in \tau_k, \\ 0 & \text{else} \end{cases}$$

and $S_{h_x}^0(\Gamma)$ be the space of piecewise constant basis functions on $\Gamma = \partial\Omega$ with respect to the spatial triangulation Γ_{N_X}, i.e.

$$S_{h_x}^0(\Gamma) := \operatorname{span}\left\{\varphi_l^0\right\}_{l=1}^{N_X}$$

with

$$\varphi_l^0(x) := \begin{cases} 1 & \text{for } x \in \gamma_l, \\ 0 & \text{else.} \end{cases}$$

The space-time tensor product space of piecewise constant basis functions on $\Sigma = \Gamma \times (0,T)$ is then given by

$$S_{h_x,h_t}^{0,0}(\Sigma) := S_{h_x}^0(\Gamma) \otimes S_{h_t}^0(I). \tag{7.4}$$

We can use these trial spaces to search for an approximation w_h of the conormal derivative $w = \gamma_1^{\text{int}} u$. Since we want to derive error estimates for the approximation w_h, we have to examine approximation properties of functions $v_h \in S_{h_x,h_t}^{0,0}(\Sigma)$.

Approximation properties

In order to derive approximation properties of the trial space $S_{h_x,h_t}^{0,0}(\Sigma)$ we first recall some approximation properties of the spaces $S_{h_x}^0(\Gamma)$ and $S_{h_t}^0(I)$.
The $L^2(\Gamma)$-projection $Q_{h_x} u \in S_{h_x}^0(\Gamma)$ of a function $u \in L^2(\Gamma)$ is defined as the unique solution of the variational problem

$$\langle Q_{h_x} u, v_h \rangle_{L^2(\Gamma)} = \langle u, v_h \rangle_{L^2(\Gamma)} \quad \text{for all } v_h \in S_{h_x}^0(\Gamma). \tag{7.5}$$

The $L^2(\Gamma)$-projection operator Q_{h_x} satisfies the stability esimate

$$\|Q_{h_x} u\|_{L^2(\Gamma)} \le \|u\|_{L^2(\Gamma)} \quad \text{for all } u \in L^2(\Gamma). \tag{7.6}$$

Theorem 7.1. *[27, Theorem 10.2] Let $u \in H^s(\Gamma)$ with $s \in [0,1]$ and $Q_{h_x} u \in S_{h_x}^0(\Gamma)$ be the $L^2(\Gamma)$-projection of u. Then there hold the error estimates*

$$\begin{aligned} \|u - Q_{h_x} u\|_{L^2(\Gamma)} &\le \|u\|_{L^2(\Gamma)}, \\ \|u - Q_{h_x} u\|_{L^2(\Gamma)} &\le c h_x^s |u|_{H^s(\Gamma)}. \end{aligned} \tag{7.7}$$

Corollary 7.2. *[27, Corollary 10.3] Let $u \in H^s(\Gamma)$ with $s \in [0,1]$. For $\sigma \in [-1,0)$ there holds the error estimate*

$$\|u - Q_{h_x}u\|_{H^\sigma(\Gamma)} \le c h_x^{s-\sigma} |u|_{H^s(\Gamma)}. \tag{7.8}$$

The following lemma states an global inverse inequalitiy for functions $w_h \in S^0_{h_x}(\Gamma)$. The proof with respect to the $H^{-\frac{1}{2}}(\Gamma)$-norm is given in [27, Lemma 10.10] and uses interpolation arguments. In the same way one can show the estimate in the $H^\sigma(\Gamma)$-norm for $\sigma \in (-1,0]$.

Lemma 7.3. *Assume that the boundary decomposition Γ_{N_X} is globally quasi-uniform and let $\sigma \in (-1,0]$. Then there holds*

$$\|w_h\|_{L^2(\Gamma)} \le c_I h_x^\sigma \, \|w_h\|_{H^\sigma(\Gamma)} \qquad \textit{for all } w_h \in S^0_{h_x}(\Gamma). \tag{7.9}$$

Analogously the $L^2(I)$-projection $Q_{h_t}u \in S^0_{h_t}(I)$ of a function $u \in L^2(I)$ is defined as the unique solution of the variational problem

$$\langle Q_{h_t}u, v_h\rangle_{L^2(I)} = \langle u, v_h\rangle_{L^2(I)} \qquad \text{for all } v_h \in S^0_{h_t}(I). \tag{7.10}$$

The estimates (7.7), (7.8) and (7.9) hold for functions $u \in H^s(I)$ and $w_h \in S^0_{h_t}(I)$ as well.
By using those approximation properties we can derive estimates for the $L^2(\Sigma)$-projection $Q_{h_x,h_t}u \in S^{0,0}_{h_x,h_t}(\Sigma)$ of a function $u \in L^2(\Sigma)$ where $Q_{h_x,h_t}u$ is the unique solution of the variational problem

$$\langle Q_{h_x,h_t}u, v_h\rangle_{L^2(\Sigma)} = \langle u, v_h\rangle_{L^2(\Sigma)} \qquad \text{for all } v_h \in S^{0,0}_{h_x,h_t}(\Sigma). \tag{7.11}$$

The projections $Q^\Sigma_{h_x}u$ and $Q^\Sigma_{h_t}u$ for $u \in L^2(\Sigma)$ are defined as

$$\begin{aligned} (Q^\Sigma_{h_x}u)(x,t) &:= (Q_{h_x}u(\cdot,t))(x), \\ (Q^\Sigma_{h_t}u)(x,t) &:= (Q_{h_t}u(x,\cdot))(t). \end{aligned} \tag{7.12}$$

Let $u \in L^2(\Sigma)$. By using the stability estimate (7.6) we get

$$\begin{aligned} \left\|Q^\Sigma_{h_x}u\right\|^2_{L^2(\Sigma)} &= \int_0^T \int_\Gamma \left[(Q^\Sigma_{h_x}u)(x,t)\right]^2 ds_x dt = \int_0^T \int_\Gamma \left[(Q_{h_x}u(\cdot,t))(x)\right]^2 ds_x dt \\ &= \int_0^T \|(Q_{h_x}u(\cdot,t))\|^2_{L^2(\Gamma)}\, dt \le \int_0^T \|u(\cdot,t)\|^2_{L^2(\Gamma)}\, dt = \|u\|^2_{L^2(\Sigma)}. \end{aligned}$$

Hence we have

$$\left\|Q^\Sigma_{h_x}u\right\|_{L^2(\Sigma)} \le \|u\|_{L^2(\Sigma)} \quad \text{for all } u \in L^2(\Sigma). \tag{7.13}$$

Let $v_h \in S^{0,0}_{h_x,h_t}(\Sigma)$ and $u \in L^2(\Sigma)$. Since $v_h \in S^{0,0}_{h_x,h_t}(\Sigma)$ we have the representation

$$v_h(x,t) = \sum_{i=1}^{N_X}\sum_{j=1}^{N_T} v_{ij}\varphi_i^0(x)\psi_j^0(t).$$

According to (7.11) we have

$$\begin{aligned}\langle Q_{h_x,h_t}u, v_h\rangle_{L^2(\Sigma)} = \langle u, v_h\rangle_{L^2(\Sigma)} &= \int_0^T \int_\Gamma u(x,t)v_h(x,t)ds_x dt \\ &= \sum_{i=1}^{N_X}\sum_{j=1}^{N_T} v_{ij}\int_\Gamma \varphi_i^0(x)\int_0^T u(x,t)\psi_j^0(t)dtds_x.\end{aligned}$$

By using the projection property (7.10) we get

$$\begin{aligned}\langle Q_{h_x,h_t}u, v_h\rangle_{L^2(\Sigma)} &= \sum_{i=1}^{N_X}\sum_{j=1}^{N_T} v_{ij}\int_\Gamma \varphi_i^0(x)\int_0^T (Q^\Sigma_{h_t}u)(x,t)\psi_j^0(t)dtds_x \\ &= \sum_{i=1}^{N_X}\sum_{j=1}^{N_T} v_{ij}\int_0^T \psi_j^0(t)\int_\Gamma (Q^\Sigma_{h_t}u)(x,t)\varphi_i^0(x)ds_x dt.\end{aligned}$$

Again, the projection property (7.5) gives

$$\begin{aligned}\langle Q_{h_x,h_t}u, v_h\rangle_{L^2(\Sigma)} &= \sum_{i=1}^{N_X}\sum_{j=1}^{N_T} v_{ij}\int_0^T \psi_j^0(t)\int_\Gamma (Q^\Sigma_{h_x}Q^\Sigma_{h_t}u)(x,t)\varphi_i^0(x)ds_x dt \\ &= \int_0^T\int_\Gamma (Q^\Sigma_{h_x}Q^\Sigma_{h_t}u)(x,t)v_h(x,t)ds_x dt \\ &= \langle Q^\Sigma_{h_x}Q^\Sigma_{h_t}u, v_h\rangle_{L^2(\Sigma)}.\end{aligned}$$

We conclude

$$\langle Q_{h_x,h_t}u - Q^\Sigma_{h_x}Q^\Sigma_{h_t}u, v_h\rangle_{L^2(\Sigma)} = 0 \quad \text{for all } v_h \in S^{0,0}_{h_x,h_t}(\Sigma)$$

and since $Q^\Sigma_{h_x}Q^\Sigma_{h_t}u \in S^{0,0}_{h_x,h_t}(\Sigma)$ we can choose $v_h = Q_{h_x,h_t}u - Q^\Sigma_{h_x}Q^\Sigma_{h_t}u$ and get

$$\left\|Q_{h_x,h_t}u - Q^\Sigma_{h_x}Q^\Sigma_{h_t}u\right\|_{L^2(\Sigma)} = 0.$$

This is true for all $u \in L^2(\Sigma)$. Hence the operators coincide and we have the representation $Q_{h_x,h_t} = Q^\Sigma_{h_x}Q^\Sigma_{h_t}$. Due to the definition of the $L^2(\Sigma)$-projections $Q^\Sigma_{h_x}u$ and $Q^\Sigma_{h_t}u$ (7.12) we can use the approximation properties of the operators Q_{h_x} and Q_{h_t} to derive estimates for the $L^2(\Sigma)$-projection $Q_{h_x,h_t}u$.

Lemma 7.4. *Let $u \in H^{r,s}(\Sigma)$ with $r, s \in [0,1]$ and $Q_{h_x,h_t}u \in S^{0,0}_{h_x,h_t}(\Sigma)$ be the $L^2(\Sigma)$-projection of u. Then there hold the estimates*

$$\|u - Q_{h_x,h_t}u\|_{L^2(\Sigma)} \leq \|u\|_{L^2(\Sigma)}\,,$$
$$\|u - Q_{h_x,h_t}u\|_{L^2(\Sigma)} \leq c\,(h_x^r + h_t^s)\,|u|_{H^{r,s}(\Sigma)}$$

where

$$|u|^2_{H^{r,s}(\Sigma)} = |u|^2_{L^2(0,T;H^r(\Gamma))} + |u|^2_{H^s(0,T;L^2(\Gamma))}.$$

Proof. For $u \in H^{r,s}(\Sigma) \subset L^2(\Sigma)$ we have

$$\langle u - Q_{h_x,h_t}u, v_h\rangle_{L^2(\Sigma)} = 0 \quad \text{for all } v_h \in S^{0,0}_{h_x,h_t}(\Sigma)$$

and therefore

$$\begin{aligned}\|u - Q_{h_x,h_t}u\|^2_{L^2(\Sigma)} &= \langle u - Q_{h_x,h_t}u, u - Q_{h_x,h_t}u\rangle_{L^2(\Sigma)} = \langle u - Q_{h_x,h_t}u, u\rangle_{L^2(\Sigma)}\\ &\leq \|u - Q_{h_x,h_t}u\|_{L^2(\Sigma)}\,\|u\|_{L^2(\Sigma)}\end{aligned}$$

The first estimate follows by multiplying this inequality with $\|u - Q_{h_x,h_t}u\|^{-1}_{L^2(\Sigma)}$ if $u - Q_{h_x,h_t}u \neq 0$. The estimate is also true for $u - Q_{h_x,h_t}u = 0$. By using the triangle inequality, the stability estimate (7.13) and Theorem 7.1 we get

$$\begin{aligned}\|u - Q_{h_x,h_t}u\|_{L^2(\Sigma)} &= \left\|u - Q^{\Sigma}_{h_x}Q^{\Sigma}_{h_t}u\right\|_{L^2(\Sigma)} = \left\|u - Q^{\Sigma}_{h_x}u + Q^{\Sigma}_{h_x}(u - Q^{\Sigma}_{h_t}u)\right\|_{L^2(\Sigma)}\\ &\leq \left\|u - Q^{\Sigma}_{h_x}u\right\|_{L^2(\Sigma)} + \left\|u - Q^{\Sigma}_{h_t}u\right\|_{L^2(\Sigma)}\\ &\leq \widetilde{c}\left(h_x^r|u|_{L^2(0,T;H^r(\Gamma))} + h_t^s|u|_{H^s(0,T;L^2(\Gamma))}\right).\end{aligned}$$

Thus, we have

$$\|u - Q_{h_x,h_t}\|_{L^2(\Sigma)} \leq c\,(h_x^r + h_t^s)\,|u|_{H^{r,s}(\Sigma)}.$$

□

The following lemma states an error estimate in norms of anisotropic Sobolev spaces with negative order.

Lemma 7.5. *Let $u \in H^{r,s}(\Sigma)$ with $r, s \in [0,1]$. For $\sigma, \mu \in [-1, 0)$ there holds the error estimate*

$$\|u - Q_{h_x,h_t}u\|_{H^{\sigma,\mu}(\Sigma)} \leq c\left(h_x^{-\sigma} + h_t^{-\mu}\right)(h_x^r + h_t^s)\,|u|_{H^{r,s}(\Sigma)}.$$

Proof. By duality and using (7.11) we get

$$\begin{aligned}\|u - Q_{h_x,h_t}u\|_{H^{\sigma,\mu}(\Sigma)} &= \sup_{0\neq v\in H^{-\sigma,-\mu}_{,0}(\Sigma)} \frac{\langle u - Q_{h_x,h_t}u, v\rangle_{\Sigma}}{\|v\|_{H^{-\sigma,-\mu}(\Sigma)}}\\ &= \sup_{0\neq v\in H^{-\sigma,-\mu}_{,0}(\Sigma)} \frac{\langle u - Q_{h_x,h_t}u, v - Q_{h_x,h_t}v\rangle_{\Sigma}}{\|v\|_{H^{-\sigma,-\mu}(\Sigma)}}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and using Lemma 7.4 leads to

$$\begin{aligned}\|u - Q_{h_x,h_t}u\|_{H^{\sigma,\mu}(\Sigma)} &\leq \|u - Q_{h_x,h_t}u\|_{L^2(\Sigma)} \sup_{0\neq v\in H_{,0}^{-\sigma,-\mu}(\Sigma)} \frac{\|v - Q_{h_x,h_t}v\|_{L^2(\Sigma)}}{\|v\|_{H^{-\sigma,-\mu}(\Sigma)}} \\ &\leq c\,(h_x^r + h_t^s)\left(h_x^{-\sigma} + h_t^{-\mu}\right)|u|_{H^{r,s}(\Sigma)}.\end{aligned}$$

□

Triangulation of the space-time boundary Σ

Recall that the boundary Γ is piecewise smooth with $\overline{\Gamma} = \bigcup_{j=1}^{J} \overline{\Gamma}_j$. With $\Sigma_j := \Gamma_j \times (0,T)$ for $j = 1, ..., J$ we get $\overline{\Sigma} = \bigcup_{j=1}^{J} \overline{\Sigma}_j$. We consider a family $\{\Sigma_N\}_{N\in\mathbb{N}}$ of admissible triangulations of Σ into boundary elements σ_l, i.e. we have

$$\Sigma_N = \bigcup_{l=1}^{N} \overline{\sigma}_l. \tag{7.14}$$

Again, we assume that there are no curved elements and that there is no approximation of the space-time boundary Σ. For each boundary element σ_l there exists exactly one $j \in \{1, ..., J\}$ such that $\sigma_l \subset \Sigma_j$. The boundary elements σ_l can be described as $\sigma_l = \chi_l(\sigma)$, where σ is some reference element in $\mathbb{R}^n$. The elements σ_l are line segments in the one-dimensional case $n = 1$, plane triangles in the two-dimensional case $n = 2$, and tetrahedra in the three-dimensional case $n = 3$. Let $\{(x_k, t_k)\}_{k=1}^{M}$ be the set of boundary nodes (x_k, t_k) of Σ_N. For each boundary element σ_l we define its volume

$$\Delta_l := \int_{\sigma_l} ds_x,$$

its local mesh size

$$h_l := \Delta_l^{1/n}$$

and its diameter

$$d_l := \sup_{(x,t),(y,s)\in\sigma_l} |(x,t) - (y,s)|\,.$$

The global mesh size is defined as

$$h := \max_{l=1,...,N} h_l.$$

The family $\{\Sigma_N\}_{N\in\mathbb{N}}$ of triangulations is said to be globally quasi-uniform, if there exists a constant $c_G \geq 1$ independent of N such that

$$\frac{h_{\max}}{h_{\min}} \leq c_G.$$

We assume that the boundary elements are shape regular, i.e. there exists a constant c_B independent of N such that

$$d_l \le c_B h_l \quad \text{for } l = 1, ..., N.$$

In the spatially one-dimensional case $n = 1$ the boundary $\Gamma = \partial\Omega$ is a set of two points $x_1, x_2 \in \mathbb{R}$. Thus, the boundary elements σ_l are line segments in temporal direction with fixed spatial coordinate $x_l \in (x_1, x_2)$. Let (x_l, t_{l_1}) and (x_l, t_{l_2}) be the nodes of the boundary element σ_l. Then σ_l can be described as

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x_l \\ t_{l_1} + \xi(t_{l_2} - t_{l_1}) \end{pmatrix}$$

where $\xi \in \sigma = (0, 1)$. Hence we have

$$h_l = d_l = \Delta_l = \int_{\sigma_l} ds_x dt = |t_{l_2} - t_{l_1}|.$$

In the two-dimensional case the boundary elements σ_l are plane triangles with nodes (x_{l_1}, t_{l_1}), (x_{l_2}, t_{l_2}) and (x_{l_3}, t_{l_3}). We assume that the boundary elements are rectangular triangles, where one of the edges adjacent to the right angle is parallel to the domain Ω as shown in Figure 7.1 . The boundary element σ_l can be described as

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x_{l_1} + \xi_1(x_{l_2} - x_{l_1}) + \xi_2(x_{l_3} - x_{l_1}) \\ t_{l_1} + \xi_1(t_{l_2} - t_{l_1}) + \xi_2(t_{l_3} - t_{l_1}) \end{pmatrix} = \begin{pmatrix} x_{l_1} \\ t_{l_1} \end{pmatrix} + J_l \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

where $(\xi_1, \xi_2)^T \in \sigma = \{\xi \in \mathbb{R}^2 : 0 < \xi_1 < 1, 0 < \xi_2 < 1 - \xi_1\}$ and

$$J_l = \begin{pmatrix} x_{l_2} - x_{l_1} & x_{l_3} - x_{l_1} \\ t_{l_2} - t_{l_1} & t_{l_3} - t_{l_1} \end{pmatrix}.$$

The volume of a boundary element σ_l is given by

$$\Delta_l = \int_{\sigma_l} ds_x dt = \frac{1}{2}|\det J_l|.$$

In the three-dimensional-case the boundary elements σ_l are tetrahedra with nodes (x_{l_i}, t_{l_i}), $i = 1, ..., 4$. Similarly to the two-dimensional case we assume, that the boundary elements are trirectangular tetrahedra where one of the sides adjacent to the right angle is parallel to the boundary Γ. The boundary element σ_l can be described as

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} x_{l_1} \\ t_{l_1} \end{pmatrix} + J_l \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

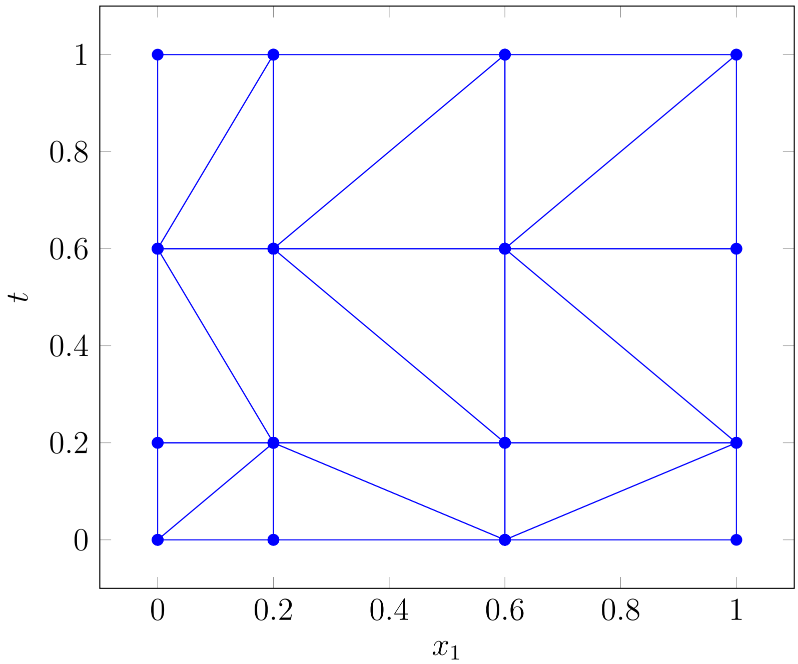

Figure 7.1: Sample triangulation of a part of Σ for $n = 2$.

where $(\xi_1, \xi_2, \xi_3)^T \in \sigma = \{\xi \in \mathbb{R}^3 : 0 < \xi_1 < 1, 0 < \xi_2 < 1 - \xi_1, 0 < \xi_3 < 1 - \xi_1 - \xi_2\}$ and

$$J_l = \begin{pmatrix} x_{l_2} - x_{l_1} & x_{l_3} - x_{l_1} & x_{l_4} - x_{l_1} \\ t_{l_2} - t_{l_1} & t_{l_3} - t_{l_1} & t_{l_4} - t_{l_1} \end{pmatrix}.$$

The volume of a boundary element σ_l is given by

$$\Delta_l = \int_{\sigma_l} ds_x dt = \frac{1}{6} |\det J_l|.$$

Trial spaces

For the approximation of the conormal derivative $w = \gamma_1^{\text{int}} u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ we consider the space of piecewise constant basis function $S_h^0(\Sigma)$ with respect to the triangulation Σ_N defined as

$$S_h^0(\Sigma) := \text{span}\left\{\varphi_l^0\right\}_{l=1}^N$$

with

$$\varphi_l^0(x,t) := \begin{cases} 1 & \text{for } (x,t) \in \sigma_l, \\ 0 & \text{else.} \end{cases}$$

Approximation properties

In the spatially one-dimensional case the form of the boundary elements and therefore the trial spaces coincide with the space-time tensor product spaces introduced at the beginning of this section. Hence we have the same approximation properties.
Due to the assumption, that the boundary elements in the two-dimensional case are rectangular triangles where one of the edges adjacent to the right angle is parallel to Ω, see Figure 7.1, we can derive error estimates of the $L^2(\Sigma)$-projection $Q_h u \in S_h^0(\Sigma)$ for $u \in L^2(\Sigma)$ by using the approximation properties of functions $v_h \in S_{h_x,h_t}^{0,0}(\Sigma)$.
Due to the structure of the decomposition of Σ, two boundary elements $\sigma_{l_1}, \sigma_{l_2}$, $l_1 \neq l_2$, form a rectangle, which can be represented as $\gamma_{l_1} \times \tau_{l_1}$ where γ_{l_1} is an element of the spatial triangulation (7.2) and τ_{l_1} is an element of the temporal triangulation (7.3). The decomposition Σ_N of Σ induces a spatial decomposition Γ_{N_X} of the boundary Γ and a temporal decomposition I_{N_T} of $I = (0, T)$. Let $S_{h_x,h_t}^{0,0}(\Sigma)$ be the corresponding space-time tensor product space (7.4), where h_x denotes the maximum size of the boundary elements in spatial direction and h_t the maximum size of the boundary elements in temporal direction. For each boundary element σ_l there exists an adjacent element σ_{k_l} and elements γ_{i_l}, τ_{j_l} such that

$$\overline{\sigma_l \cup \sigma_{k_l}} = \overline{\gamma_{i_l} \times \tau_{j_l}}.$$

In other words, for each rectangle $\varsigma_{ij} := \gamma_i \times \tau_j$ there exist exactly two boundary elements $\sigma_{l(ij)_1}$, $\sigma_{l(ij)_2}$ such that

$$\overline{\varsigma_{ij}} = \overline{\sigma_{l(ij)_1} \cup \sigma_{l(ij)_2}}.$$

Hence we have $S_{h_x,h_t}^{0,0}(\Sigma) \subset S_h^0(\Sigma)$.

The $L^2(\Sigma)$-projection $Q_h u \in S_h^0(\Sigma)$ of a function $u \in L^2(\Sigma)$ is defined as the unique solution of the variational problem

$$\langle Q_h u, v_h \rangle_{L^2(\Sigma)} = \langle u, v_h \rangle_{L^2(\Sigma)} \quad \text{for all } v_h \in S_h^0(\Sigma). \tag{7.15}$$

For $u \in L^2(\Sigma)$ we have

$$\begin{aligned} \|u - Q_h u\|_{L^2(\Sigma)}^2 &= \langle u - Q_h u, u - Q_h u \rangle_{L^2(\Sigma)} = \langle u - Q_h u, u - Q_{h_x,h_t} u \rangle_{L^2(\Sigma)} \\ &\leq \|u - Q_h u\|_{L^2(\Sigma)} \|u - Q_{h_x,h_t} u\|_{L^2(\Sigma)} \end{aligned}$$

since $Q_{h_x,h_t} u \in S_{h_x,h_t}^0(\Sigma) \subset S_h^0(\Sigma)$. We get

$$\|u - Q_h u\|_{L^2(\Sigma)} \leq \|u - Q_{h_x,h_t} u\|_{L^2(\Sigma)} \quad \text{for all } u \in L^2(\Sigma) \tag{7.16}$$

and obtain the following approximation properties.

Lemma 7.6. *Let $u \in H^{r,s}(\Sigma)$ with $r, s \in [0,1]$ and $Q_h u \in S_h^0(\Sigma)$ be the $L^2(\Sigma)$-projection of u. Then there hold the error estimates*

$$\|u - Q_h u\|_{L^2(\Sigma)} \leq \|u\|_{L^2(\Sigma)}\,,$$
$$\|u - Q_h u\|_{L^2(\Sigma)} \leq c\,(h_x^r + h_t^s)\,|u|_{H^{r,s}(\Sigma)}.$$

Proof. Follows with (7.16) and Lemma 7.4. □

Lemma 7.7. *Let $u \in H^{r,s}(\Sigma)$ with $r, s \in [0,1]$. For $\sigma, \mu \in [-1, 0)$ there holds*

$$\|u - Q_h u\|_{H^{\sigma,\mu}(\Sigma)} \leq c\left(h_x^{-\sigma} + h_t^{-\mu}\right)(h_x^r + h_t^s)\,|u|_{H^{r,s}(\Sigma)}.$$

Proof. Analogously to the proof of Lemma 7.5. □

The three-dimensional case needs further examination. By using similiar arguments as for $n = 2$ we can assume, that we have the same approximation properties for $n = 3$ as well.

7.2 BEM for the Dirichlet initial boundary value problem

For the discretization of the variational formulation (7.1) we consider the space of piecewise constant basis function $S_h^0(\Sigma) \subset H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ corresponding to the triangulation (7.14) of the space-time boundary Σ. The Galerkin-Bubnov variational formulation of (7.1) is to find $w_h \in S_h^0(\Sigma)$ such that

$$\langle V w_h, \tau_h\rangle_\Sigma = \langle\left(\frac{1}{2}I + K\right) g, \tau_h\rangle_\Sigma - \langle M_0 u_0, \tau_h\rangle_\Sigma \quad \text{for all } \tau_h \in S_h^0(\Sigma). \tag{7.17}$$

Considering $w_h(x,t) = \sum_{k=1}^N w_k \varphi_k^0(x,t)$ this problem is equivalent to

$$\sum_{k=1}^N w_k \langle V\varphi_k^0, \varphi_l^0\rangle_\Sigma = \langle\left(\frac{1}{2}I + K\right) g, \varphi_l^0\rangle_\Sigma - \langle M_0 u_0, \varphi_l^0\rangle_\Sigma \quad \text{for } l = 1, ..., N.$$

This system of linear equations can be written as

$$V_h \underline{w} = \underline{f} \tag{7.18}$$

where

$$V_h[l,k] = \langle V\varphi_k^0, \varphi_l^0\rangle_\Sigma$$

and

$$\underline{f}[l] = \langle\left(\frac{1}{2}I + K\right) g, \varphi_l^0\rangle_\Sigma - \langle M_0 u_0, \varphi_l^0\rangle_\Sigma$$

for $l, k = 1, ..., N$. Due to the ellipticity of V this system is uniquely solvable, see Chapter 1.

7.3 Error estimates

Since the single layer boundary integral operator V is $H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$-elliptic and bounded we can use Cea's Lemma (Theorem 1.3) to conclude quasi-optimality of the Galerkin approximation $w_h \in S_h^0(\Sigma)$, i.e. we have

$$\|w - w_h\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq \frac{c_2^V}{c_1^V} \inf_{\tau_h \in S_h^0(\Sigma)} \|w - \tau_h\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} . \tag{7.19}$$

Now we can use the approximation properties of the trial space $S_h^0(\Sigma)$ to derive error esimates for the solution w_h of (7.17). By applying Lemma 3.4 we get

$$\|w - w_h\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq \frac{c_2^V}{c_1^V} \sum_{j=1}^{J} \inf_{\tau_h^j \in S_h^0(\Sigma_j)} \left\|w_{|\Sigma_j} - \tau_h^j\right\|_{\widetilde{H}^{-\frac{1}{2},-\frac{1}{4}}(\Sigma_j)} \tag{7.20}$$

and obtain the following error estimates.

Theorem 7.8. *Let $w_h \in S_h^0(\Sigma)$ be the unique solution of the Galerkin variational problem* (7.17)*. For $w \in H_{pw}^{r,s}(\Sigma)$ with $r, s \in [0,1]$ there holds*

$$\|w - w_h\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq c \left(h_x^{1/2} + h_t^{1/4}\right) (h_x^r + h_t^s) \, |w|_{H_{pw}^{r,s}(\Sigma)}.$$

Proof. The assertion follows by applying Lemma 7.5 to the estimate (7.20). □

Since the boundary elements are assumed to be shape regular, we have $h_x \leq c_B h$ and $h_t \leq c_B h$ and get

$$\|w - w_h\|_{H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)} \leq \tilde{c} h^{\frac{1}{4}+\alpha} |w|_{H_{\mathrm{pw}}^{r,s}(\Sigma)}$$

where $\alpha = \min(r, s)$.
In the one-dimensional case we can identify the space $S_h^0(\Sigma)$ with $S_h^0(I)$ where $I = (0, T)$. The terms with h_x in Theorem 7.8 vanish and we get

$$\|w - w_h\|_{H^{-\frac{1}{4}}(I)} \leq c h^{\frac{1}{4}+s} |w|_{H_{\mathrm{pw}}^{s}(I)}.$$

For $n = 1$ it is quite easy to derive an error estimate in the $L^2(I)$ norm, assuming that the family of triangulations of Σ is globally quasi-uniform.

Theorem 7.9. *Assume that the boundary decomposition Σ_N is globally quasi-uniform. For $n = 1$ let $w_h \in S_h^0(I)$ be the unique solution of the Galerkin variational problem* (7.17)*. For $w \in H_{pw}^s(I)$ with $s \in [0,1]$ there holds*

$$\|w - w_h\|_{L^2(I)} \leq h^s |w|_{H_{pw}^s(I)}.$$

Proof. By using the triangle inequality, Lemma 7.4 and Lemma 7.3 we get

$$\begin{aligned} \|w - w_h\|_{L^2(I)} &\le \|w - Q_h w\|_{L^2(I)} + \|Q_h w - w_h\|_{L^2(I)} \\ &\le \tilde{c} h^s |w|_{H^s_{\mathrm{pw}}(I)} + c_I h^{-1/4} \|Q_h w - w_h\|_{H^{-\frac{1}{4}}(I)} . \end{aligned}$$

The assertion follows with

$$\|Q_h w - w_h\|_{H^{-\frac{1}{4}}(I)} \le \|Q_h w - w\|_{H^{-\frac{1}{4}}(I)} + \|w - w_h\|_{H^{-\frac{1}{4}}(I)} ,$$

Lemma 7.5 and Theorem 7.8. $\square$

8 Preconditioning

Since the matrix V_h is positive definite we can apply the GMRES method to solve the system of linear equations

$$V_h \underline{w} = \underline{f}$$

derived in the previous chapter. The number of iterations depends on the condition number of the matrix V_h. In our case the condition number increases as the mesh size h decreases. Thus, it is necessary to apply preconditioning strategies. Let us recall the basic concept of preconditioners.

Let X be a reflexive Banach space and X' its dual space. We consider the problem

$$Au = f$$

with a linear, bounded and X-elliptic operator $A : X \to X'$ and $f \in X'$. According to the Lemma of Lax-Milgram (Theorem 1.1) this operator equation is uniquely solvable. Let $X_h = \operatorname{span}\{\varphi_i\}_{i=1}^N \subset X$ be a finite dimensional subspace. Then the discrete problem

$$A_h \underline{u} = \underline{f} \tag{8.1}$$

is uniquely solvable as well, see Chapter 1. By multiplying this equation with the inverse of a regular matrix $C_A \in \mathbb{R}^{N \times N}$ we get

$$C_A^{-1} A_h \underline{u} = C_A^{-1} \underline{f}. \tag{8.2}$$

Due to the regularity of the matrix C_A this problem is equivalent to (8.1). The idea is to choose the matrix C_A in a such way, that the condition number of the preconditioned matrix $C_A^{-1} A_h$ is independent of the mesh size h. Moreover we want to be able to compute the inverse of the matrix C_A efficiently.

When using boundary element methods we can use preconditioning techniques based on boundary integral operators of opposite order, such as the single layer boundary integral operator V and the hypersingular boundary integral operator D.

8.1 Calderón preconditioning

The presented preconditioning strategy is based on [31] and [9]. Let $B : X' \to X$ be a linear and bounded operator and $Y_h = \operatorname{span}\{\psi_{j=1}\}_{j=1}^M \subset X'$ be a finite dimensional subspace. Moreover B satisfies the inf-sup condition

$$\sup_{0 \neq w_h \in Y_h} \frac{|\langle B q_h, w_h \rangle|}{\|w_h\|_{X'}} \geq c_1^B \|q_h\|_{X'} \quad \text{for all } q_h \in Y_h.$$

The ellipticity of the operator A implies

$$\sup_{0\neq v_h\in X_h}\frac{|\langle Au_h,v_h\rangle|}{\|v_h\|_X}\geq c_1^A\|u_h\|_X\quad\text{for all } u_h\in X_h.$$

Theorem 8.1. *[9, Theorem 2.1] If $N=dimX_h=dimY_h=M$ and there exists a constant $c_1^M>0$ such that*

$$\sup_{0\neq w_h\in Y_h}\frac{(v_h,w_h)}{\|w_h\|_{X'}}\geq c_1^M\|v_h\|_X\quad\textit{for all } v_h\in X_h,$$

then

$$\kappa\left(M_h^{-1}B_hM_h^{-T}A_h\right)\leq\frac{c_2^Ac_2^B}{c_1^Ac_1^B(c_1^M)^2}$$

where

$$A_h[i,j]=\langle A\varphi_j,\varphi_i\rangle,\quad B_h[i,j]=\langle B\psi_j,\psi_i\rangle,\quad M_h[i,j]=(\varphi_j,\psi_i)$$

for $i,j=1,...,N$.

Let us consider the single layer boundary integral operator V and the hypersingular boundary integral operator D and finite dimensional subspaces $X_h\subset H^{-\frac12,-\frac14}(\Sigma)$, $Y_h\subset H^{\frac12,\frac14}(\Sigma)$. Both operators are elliptic. Hence the operators satisfy the inf-sup conditions

$$\sup_{0\neq\tau_h\in X_h}\frac{|\langle Vt_h,\tau_h\rangle_\Sigma|}{\|\tau_h\|_{H^{-\frac12,-\frac14}(\Sigma)}}\geq c_1^V\|t_h\|_{H^{-\frac12,-\frac14}(\Sigma)}\quad\text{for all } t_h\in X_h$$

and

$$\sup_{0\neq v_h\in Y_h}\frac{|\langle Du_h,v_h\rangle_\Sigma|}{\|v_h\|_{H^{\frac12,\frac14}(\Sigma)}}\geq c_1^D\|u_h\|_{H^{\frac12,\frac14}(\Sigma)}\quad\text{for all } u_h\in Y_h.$$

If we choose suitable subspaces X_h,Y_h with $\dim X_h=\dim Y_h$ which satisfy the inf-sup condition

$$\sup_{0\neq v_h\in Y_h}\frac{\langle\tau_h,v_h\rangle_{L^2(\Sigma)}}{\|v_h\|_{H^{\frac12,\frac14}(\Sigma)}}\geq c_1^M\|\tau_h\|_{H^{-\frac12,-\frac14}(\Sigma)}\quad\text{for all } \tau_h\in X_h,\tag{8.3}$$

then Theorem 8.1 implies

$$\kappa\left(M_h^{-1}D_hM_h^{-T}V_h\right)\leq\frac{c_2^Vc_2^D}{c_1^Vc_1^D(c_1^M)^2}$$

where

$$V_h[l,k]=\langle V\varphi_k,\varphi_l\rangle_\Sigma,\quad D_h[l,k]=\langle D\psi_k,\psi_l\rangle_\Sigma,\quad M_h[l,k]=\langle\varphi_k,\psi_l\rangle_{L^2(\Sigma)}$$

for $l,k=1,...,N$. Hence if we use the preconditioner

$$C_V^{-1}=M_h^{-1}D_hM_h^{-T}$$

the condition number of the preconditioned matrix is bounded. Since M_h is a sparse matrix, the inverse M_h^{-1} can be computed efficiently. It remains to choose the finite dimensional subspaces X_h, Y_h.
In this work we present different approaches for the spatially one-dimensional case. Some of the results can easily be extended to $n = 2, 3$.

8.1.1 Piecewise constant basis functions

For $n = 1$ the boundary elements are line segments in temporal direction. Thus, we can identify the boundary of the space-time cylinder with the time interval $I := (0, T)$. Since $S_h^0(I) \subset H^{\frac{1}{4}}(I)$ we can choose $S_h^0(I)$ as the trial space for the discretization of the single layer operator V and the hypersingular operator D, i.e. we have $X_h := S_h^0(I)$ and $Y_h := S_h^0(I)$ and therefore $\dim X_h = \dim Y_h$. In order to prove the inf-sup condition (8.3) for $X_h = Y_h = S_h^0(I)$ we need stability of the $L^2(I)$-projection operator Q_h defined by (7.15) in the Sobolev space $H^{\frac{1}{4}}(I)$. We consider the boundary decomposition Σ_N given by (7.14), i.e. we have

$$\Sigma_N = \bigcup_{l=1}^{N} \overline{\sigma}_l.$$

For $l = 1, ..., N$ we define $I(l)$ to be the index set of the boundary element σ_l and all its adjacent elements. The local mesh size associated with the boundary element σ_l is then given by

$$\hat{h}_l := \frac{1}{|I(l)|} \sum_{k \in I(l)} h_k \quad \text{for } l = 1, ...N. \tag{8.4}$$

We assume the boundary decomposition Σ_N to be locally quasi-uniform, i.e. there exists a constant $c_L \geq 1$ independent of N such that

$$\frac{1}{c_L} \leq \frac{\hat{h}_l}{h_k} \leq c_L \quad \text{for all } k \in I(l), \quad l = 1, ..., N.$$

Moreover we define

$$\omega_l := \bigcup_{k \in I(l)} \sigma_k \quad \text{for } l = 1, ..., N.$$

The proof of the stability of the $L^2(I)$-projection onto the space of piecewise linear and continuous basis fuctions $S_h^1(I) = \text{span}\,\{\varphi_i^1\}_{i=1}^M$ in fractional Sobolev spaces can be found in [25]. It is based on the assumption, that there exists a constant $c_0 > 0$ such that

$$(H_l G_l H_l^{-1} \underline{x}_l, \underline{x}_l) \geq c_0 (D_l \underline{x}_l, \underline{x}_l) \quad \text{for all } \underline{x}_l \in \mathbb{R}^{|J(l)|}$$

where $J(l)$ denotes the index set of all nodes adjacent to the element σ_l and the local matrices are defined as

$$G_l[j,i] := \langle \varphi_i^1, \varphi_j^1 \rangle_{L^2(\sigma_l)}, \quad D_l := \text{diag}\left(\left\| \varphi_i^1 \right\|^2_{L^2(\sigma_l)} \right), \quad H_l := \text{diag}\left(\hat{h}_i^{\frac{1}{4}} \right)$$

for $i, j \in J(l)$ and $l = 1, ...N$. In this context $\hat{h}_i$ is the local mesh size associated with the node (x_i, t_i) of the mesh. When using piecewise constant basis functions we define the local matrices as

$$G_l[j,i] := \langle \varphi_i^0, \varphi_j^0 \rangle_{L^2(\omega_l)}, \quad D_l := \operatorname{diag}\left(\left\| \varphi_i^0 \right\|^2_{L^2(\omega_l)} \right), \quad H_l := \operatorname{diag}\left(\hat{h}_i^{\frac{1}{4}} \right)$$

for $i, j \in I(l)$ and $l = 1, ..., N$ where $\hat{h}_i$ is the local mesh size defined by (8.4). Hence we have $G_l = D_l$ and get

$$(H_l G_l H_l^{-1} \underline{x}_l, \underline{x}_l) = (D_l \underline{x}_l, \underline{x}_l) \quad \text{for all } \underline{x}_l \in \mathbb{R}^{|I(l)|}$$

and $l = 1, ..., N$. The remaining steps to prove the stability of the $L^2(I)$-projection onto the space of piecewise constant basis functions in $H^{\frac{1}{4}}(I)$ are the same as in the case of piecewise linear and continuous basis functions described in [25]. We conclude that there exists a constant $c_Q > 0$ such that

$$\left\| Q_h v \right\|_{H^{\frac{1}{4}}(I)} \le c_Q \left\| v \right\|_{H^{\frac{1}{4}}(I)} \quad \text{for all } v \in H^{\frac{1}{4}}(I). \tag{8.5}$$

Lemma 8.2. *For a locally quasi-uniform mesh there holds the inf-sup condition*

$$\frac{1}{c_Q} \left\| \tau_h \right\|_{H^{-\frac{1}{4}}(I)} \le \sup_{0 \neq v_h \in S_h^0(I)} \frac{\langle \tau_h, v_h \rangle_{L^2(I)}}{\left\| v_h \right\|_{H^{\frac{1}{4}}(I)}} \quad \textit{for all } \tau_h \in S_h^0(I).$$

Proof. For a locally quasi-uniform mesh the operator $Q_h : H^{\frac{1}{4}}(I) \to S_h^0(I) \subset H^{\frac{1}{4}}(I)$ is bounded according to (8.5). For $\tau \in S_h^0(I)$ we have

$$\begin{aligned}
\left\| \tau_h \right\|_{H^{-\frac{1}{4}}(I)} &= \sup_{0 \neq v \in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, v \rangle_{L^2(I)}}{\left\| v \right\|_{H^{\frac{1}{4}}(I)}} = \sup_{0 \neq v \in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, Q_h v \rangle_{L^2(I)}}{\left\| v \right\|_{H^{\frac{1}{4}}(I)}} \\
&\le c_Q \sup_{0 \neq v \in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, Q_h v \rangle_{L^2(I)}}{\left\| Q_h v \right\|_{H^{\frac{1}{4}}(I)}} \\
&\le c_Q \sup_{0 \neq v_h \in S_h^0(I)} \frac{\langle \tau_h, v_h \rangle_{L^2(I)}}{\left\| v_h \right\|_{H^{\frac{1}{4}}(I)}}
\end{aligned}$$

which concludes the proof. □

Hence the inf-sup-condition (8.3) is satisfied and we can use the discretization of the hypersingular operator D with respect to the space of piecewise constant basis functions $S_h^0(I)$ as a preconditioner.

8.1.2 Piecewise linear and continuous basis functions

As a second approach we choose the space of piecewise linear and continuous basis functions $S_h^1(I) = \text{span}\,\{\varphi_i^1\}_{i=1}^M$ for the discretization of the operators V and D, since $X_h := S_h^1(I) \subset H^{-\frac{1}{4}}(I)$ and $Y_h := S_h^1(I) \subset H^{\frac{1}{4}}(I)$. Clearly we have $\dim X_h = \dim Y_h$. Before proving the inf-sup condition (8.3) in this setting let us recall some properties of the space $S_h^1(I)$, based on [27, Chapter 10.2].

Lemma 8.3. *Assume that the boundary decomposition Σ_N is globally quasi-uniform and let $s \in [0,1]$. Then there holds the inverse inequality*

$$\|v_h\|_{H^s(I)} \le c_I h^{-s} \|v_h\|_{L^2(I)} \quad \textit{for all } v_h \in S_h^1(I).$$

The $L^2(I)$-projection $Q_h u \in S_h^1(I)$ of a function $u \in L^2(I)$ is defined as the unique solution of the variational problem

$$\langle Q_h u, v_h \rangle_{L^2(I)} = \langle u, v_h \rangle_{L^2(I)} \quad \text{for all } v_h \in S_h^1(I).$$

The operator Q_h satisfies the stability estimate

$$\|Q_h u\|_{L^2(I)} \le \|u\|_{L^2(I)} \quad \text{for all } u \in L^2(I). \tag{8.6}$$

Let $s \in [0,1]$. According to [25, Theorem 3.2] there exists a constant $c_S > 0$ such that

$$\|Q_h u\|_{H^s(I)} \le c_S \|u\|_{H^s(I)} \quad \text{for all } u \in H^s(I) \tag{8.7}$$

assuming appropriate mesh conditions locally, see [25, Section 4]. This estimate is also satisfied for a globally quasi-uniform decomposition of I.

Theorem 8.4. *Let $u \in H^s(I)$ with $s \in [0,1]$ and $Q_h u \in S_h^1(I)$ be the $L^2(I)$-projection of u. Then there holds the estimate*

$$\|u - Q_h u\|_{L^2(I)} \le c h^s |u|_{H^s(I)}.$$

By using the stability estimate (8.7) we can prove an inf-sup-condition for the finite-dimensional function spaces X_h and Y_h.

Lemma 8.5. *Assume that the stability estimate* (8.7) *is satisfied. Then there holds*

$$\frac{1}{c_S} \|\tau_h\|_{H^{-\frac{1}{4}}(I)} \le \sup_{0 \ne v_h \in S_h^1(I)} \frac{\langle \tau_h, v_h \rangle_{L^2(I)}}{\|v_h\|_{H^{\frac{1}{4}}(I)}} \quad \textit{for all } \tau_h \in S_h^1(I).$$

Proof. According to (8.7) the $L^2(I)$-projection operator $Q_h : H^{\frac{1}{4}}(I) \to S_h^1(I) \subset H^{\frac{1}{4}}(I)$ is bounded, i.e. there exists $c_S > 0$ such that

$$\|Q_h v\|_{H^{\frac{1}{4}}(I)} \le c_S \|v\|_{H^{\frac{1}{4}}(I)} \quad \text{for all } v \in H^{\frac{1}{4}}(I).$$

For $\tau_h \in S_h^1(I)$ we have

$$\begin{aligned}
\|\tau_h\|_{H^{-\frac{1}{4}}(I)} &= \sup_{0\neq v\in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, v\rangle_{L^2(I)}}{\|v\|_{H^{\frac{1}{4}}(I)}} = \sup_{0\neq v\in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, Q_h v\rangle_{L^2(I)}}{\|v\|_{H^{\frac{1}{4}}(I)}} \\
&\leq c_S \sup_{0\neq v\in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, Q_h v\rangle_{L^2(I)}}{\|Q_h v\|_{H^{\frac{1}{4}}(I)}} \\
&\leq c_S \sup_{0\neq v_h\in S_h^1(I)} \frac{\langle \tau_h, v_h\rangle_{L^2(I)}}{\|v_h\|_{H^{\frac{1}{4}}(I)}}
\end{aligned}$$

which concludes the proof. □

Thus, according to Theorem 8.1 the condition number of the preconditioned matrix is bounded when using the space $S_h^1(I)$ for the discretization of V and D.
This strategy can be extended to the two- and three-dimensional case, since we have $S_h^1(\Sigma) \subset H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ and $S_h^1(\Sigma) \subset H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. If one can prove the boundedness of the $L^2(\Sigma)$-projection $Q_h : H^{\frac{1}{2},\frac{1}{4}}(\Sigma) \to S_h^1(\Sigma) \subset H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$, for example by using interpolation arguments [12, 13], then the proof of the inf-sup-condition (8.3) is the same as in Lemma 8.5.

8.1.3 Dual mesh

The third approach is to use $X_h := S_h^0(\widetilde{I})$ corresponding to a dual mesh for the discretization of V and $Y_h := S_h^1(I)$ for D. For details concerning the construction of the dual mesh, see [28] and [10]. We have $\dim X_h = \dim Y_h$. Figure 8.1 shows a sample dual mesh for the one-dimensional case. It remains to prove the inf-sup condition (8.3). We assume, that the boundary decomposition Σ_N is globally quasi-uniform.

Lemma 8.6. *There holds the inf-sup-condition*

$$\frac{1}{c_{DM}} \|\tau_h\|_{H^{-\frac{1}{4}}(I)} \leq \sup_{0\neq v_h\in S_h^1(I)} \frac{\langle \tau_h, v_h\rangle_{L^2(I)}}{\|v_h\|_{H^{\frac{1}{4}}(I)}} \quad \textit{for all } \tau_h \in S_h^0(\widetilde{I}).$$

Proof. According to [26, Section 3] we have

$$\tilde{c}\,\|v_h\|_{L^2(I)} \leq \sup_{0\neq \tau_h\in S_h^0(\widetilde{I})} \frac{\langle \tau_h, v_h\rangle_{L^2(I)}}{\|\tau_h\|_{L^2(I)}} \quad \text{for all } v_h \in S_h^1(I). \tag{8.8}$$

Let the $L^2(I)$-projection $\widetilde{Q}_h : L^2(I) \to S_h^1(I) \subset L^2(I)$ be defined as

$$\langle \widetilde{Q}_h u, \tau_h\rangle_{L^2(I)} = \langle u, \tau_h\rangle_{L^2(I)} \quad \text{for all } \tau_h \in S_h^0(\widetilde{I}). \tag{8.9}$$

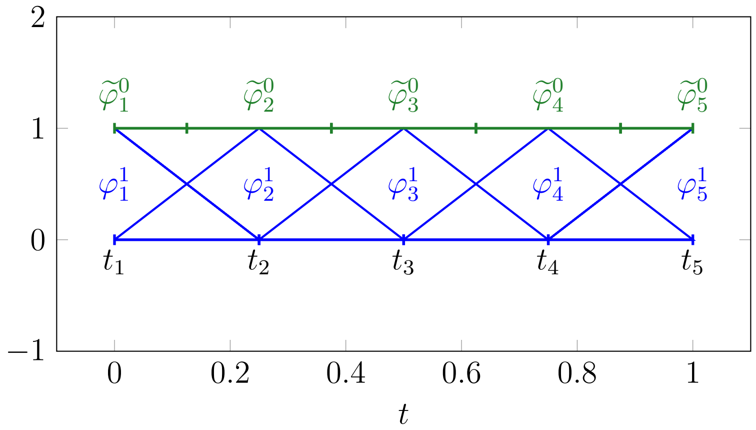

Figure 8.1: Sample dual mesh for $n = 1$.

Due to (8.8) this variational problem is well defined and uniquely solvable. For $u \in L^2(I)$ the triangle inequality implies

$$\left\| u - \widetilde{Q}_h u \right\|_{L^2(I)} \leq \| u - Q_h u \|_{L^2(I)} + \left\| Q_h u - \widetilde{Q}_h u \right\|_{L^2(I)}$$

where $Q_h u \in S_h^1(I)$ is the standard $L^2(I)$-projection of u. Since $Q_h u - \widetilde{Q}_h u \in S_h^1(I)$ we can use (8.8), (8.9) and the Cauchy-Schwarz inequality to get

$$\begin{aligned} \left\| u - \widetilde{Q}_h u \right\|_{L^2(I)} &\leq \| u - Q_h u \|_{L^2(I)} + \frac{1}{\tilde{c}} \sup_{0 \neq \tau_h \in S_h^0(\widetilde{I})} \frac{\langle \tau_h, Q_h u - u \rangle_{L^2(I)}}{\| \tau_h \|_{L^2(I)}} \\ &\leq c \| u - Q_h u \|_{L^2(I)} \end{aligned} \tag{8.10}$$

with some constant $c > 0$. The triangle inequality, the global inverse inequality (Lemma 8.3) and the stability estimate (8.7) imply

$$\begin{aligned} \left\| \widetilde{Q}_h u \right\|_{H^{\frac{1}{4}}(I)} &\leq \left\| \widetilde{Q}_h u - Q_h u \right\|_{H^{\frac{1}{4}}(I)} + \| Q_h u \|_{H^{\frac{1}{4}}(I)} \\ &\leq c_I h^{-\frac{1}{4}} \left\| \widetilde{Q}_h u - Q_h u \right\|_{L^2(I)} + c_S \| u \|_{H^{\frac{1}{4}}(I)} . \end{aligned}$$

Since $\widetilde{Q}_h u \in S_h^1(I)$ we have $\widetilde{Q}_h u - Q_h u = Q_h \left(\widetilde{Q}_h u - u \right)$. Thus, applying the estimates (8.6), (8.10) and Theorem 8.4 leads to

$$\left\| \widetilde{Q}_h u \right\|_{H^{\frac{1}{4}}(I)} \leq \hat{c} h^{-\frac{1}{4}} \| u - Q_h u \|_{L^2(I)} + c_S \| u \|_{H^{\frac{1}{4}}(I)} \leq c_{DM} \| u \|_{H^{\frac{1}{4}}(I)}$$

with some constant $c_{DM} > 0$. Hence for $\tau_h \in S_h^0(\widetilde{I})$ we have

$$\begin{aligned}\|\tau_h\|_{H^{-\frac{1}{4}}(I)} &= \sup_{0\neq v\in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, v\rangle_{L^2(I)}}{\|v\|_{H^{\frac{1}{4}}(I)}} = \sup_{0\neq v\in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, \widetilde{Q}_h v\rangle_{L^2(I)}}{\|v\|_{H^{\frac{1}{4}}(I)}} \\ &\leq c_{DM} \sup_{0\neq v\in H^{\frac{1}{4}}(I)} \frac{\langle \tau_h, \widetilde{Q}_h v\rangle_{L^2(I)}}{\left\|\widetilde{Q}_h v\right\|_{H^{\frac{1}{4}}(I)}} \leq c_{DM} \sup_{0\neq v_h\in S_h^1(I)} \frac{\langle \tau_h, v_h\rangle_{L^2(I)}}{\|v_h\|_{H^{\frac{1}{4}}(I)}}\end{aligned}$$

which concludes the proof. □

Consequently, when discretizing V in the space $S_h^0(\widetilde{I})$ and D in the space $S_h^1(I)$ the preconditioner $C_V^{-1} = M_h^{-1} D_h M_h^{-T}$ leads to a bounded condition number of the preconditioned system matrix. As in the case of piecewise linear basis functions this approach is a suitable preconditioning technique for $n = 2, 3$ as well. Since the boundary elements of the dual mesh are of arbitrary form (polygonial), we need approximation properties of trial spaces corresponding to the dual mesh in anisotropic Sobolev spaces.

9 FEM-BEM coupling

In this chapter we present and discuss a FEM-BEM coupling method for parabolic transmission problems, based on [3]. As in the case of stationary transmission problems [24] we can derive boundary integral equations for the exterior problem and use a coupling method to solve the integral equations in combination with a finite element discretization of the interior problem [29]. We consider a non-symmetric FEM-BEM coupling method. In addition to the derivation of a variational formulation for the coupled problem we consider a Galerkin method in order to discretize the problem and compute an approximation of the solution.

9.1 Model problem

Let $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in H_0^1(\Omega)$. We consider the transmission problem

$$\begin{aligned} \alpha\partial_t u_i(x,t) - \operatorname{div}_x\left[A(x,t)\nabla_x u_i(x,t)\right] &= f(x,t) && \text{for } (x,t) \in \Omega \times (0,T), \\ \alpha\partial_t u_e(x,t) - \Delta u_e(x,t) &= 0 && \text{for } (x,t) \in \Omega^{\text{ext}} \times (0,T), \\ u_i(x,0) &= u_0(x) && \text{for } x \in \Omega, \\ u_e(x,0) &= 0 && \text{for } x \in \Omega^{\text{ext}} \end{aligned} \tag{9.1}$$

with $\Omega^{\text{ext}} := \mathbb{R}^n \setminus \overline{\Omega}$ and transmission conditions

$$u_i(x,t) = u_e(x,t), \quad n_x \cdot \left[A(x,t)\nabla_x u_i(x,t)\right] = \frac{\partial}{\partial n_x} u_e(x,t) =: w_e(x,t) \tag{9.2}$$

for $(x,t) \in \Sigma$. We assume, that the coefficient matrix $A(x,t) \in \mathbb{R}^{n\times n}$ is symmetric and uniform positive definite, i.e. there exists $\theta > 0$ such that

$$\theta|\xi|^2 \leq [A(x,t)\,\xi]\cdot\xi$$

for all $(x,t) \in Q$ and all $\xi \in \mathbb{R}^n$. The solution u_e of the exterior problem satisfies a radiation condition for $|x| \to \infty$ and $t \in (0,T)$. Similarly to the initial boundary value problem (2.1) we can derive a representation formula for the solution u_e of the exterior problem [3]. For $(\widetilde{x}, t) \in \Omega^{\text{ext}} \times (0,T)$ we have

$$\begin{aligned} u_e(\widetilde{x}, t) = &- \frac{1}{\alpha} \int\limits_0^T \int\limits_\Gamma U^*(\widetilde{x} - y, t - s) \frac{\partial}{\partial n_y} u_e(y,s) ds_y ds \\ &+ \frac{1}{\alpha} \int\limits_0^T \int\limits_\Gamma \frac{\partial}{\partial n_y} U^*(\widetilde{x} - y, t - s) u_e(y,s) ds_y ds. \end{aligned} \tag{9.3}$$

By applying the Dirichlet trace operator and using the jump relations (5.4) and (5.7) we get the first boundary integral equation for the exterior problem

$$\gamma_0^{\text{ext}} u_e = -V \gamma_1^{\text{ext}} u_e + \left(\frac{1}{2} I + K \right) \gamma_0^{\text{ext}} u_e \quad \text{on } \Sigma. \tag{9.4}$$

9.2 Coupling

First we consider the initial boundary value problem

$$\begin{aligned} \alpha \partial_t u_i(x,t) - \text{div}_x \left[A(x,t) \nabla_x u_i(x,t) \right] &= f(x,t) \quad \text{for } (x,t) \in \Omega \times (0,T) \\ u_i(x,0) &= u_0(x) \qquad \text{for } x \in \Omega \end{aligned} \tag{9.5}$$

with $f \in L_2(0,T; H^{-1}(\Omega)), u_0 \in H_0^1(\Omega)$ and the Neumann boundary condition

$$n_x \cdot \left[A(x,t) \nabla_x u_i(x,t) \right] = w_i(x,t) \quad \text{for } (x,t) \in \Gamma \times (0,T).$$

The variational formulation is to find $u_i \in L_2(0,T; H^1(\Omega)) \cap H^1(0,T; \widetilde{H}^{-1}(\Omega))$ with $u_i(x,0) = u_0(x)$ for $x \in \Omega$ such that

$$a(u_i, v) = \int_0^T \int_\Omega f(x,t), v(x,t) dx dt + \int_0^T \int_\Gamma w_i(x,t) v(x,t) ds_x dt$$

for all $v \in L_2(0,T; H^1(\Omega))$. The bilinear form $a(\cdot, \cdot)$ is given by

$$a(u,v) := \int_0^T \int_\Omega \alpha \partial_t u_i(x,t) v(x,t) dx dt + \int_0^T \int_\Omega \left[A(x,t) \nabla_x u_i(x,t) \right] \cdot \nabla_x v(x,t) dx dt$$

for $u \in L_2(0,T; H^1(\Omega)) \cap H^1(0,T; \widetilde{H}^{-1}(\Omega))$ und $v \in L_2(0,T; H^1(\Omega))$. Due to the initial condition in (9.5) we consider the decomposition $u_i(x,t) = \overline{u}_i(x,t) + \overline{u}_0(x,t)$ for $(x,t) \in Q$ where $\overline{u}_0 \in L_2(0,T; H_0^1(\Omega)) \cap H^1(0,T; \widetilde{H}^{-1}(\Omega))$ is an extension of the initial condition $u_0 \in H_0^1(\Omega)$ into the space-time cylinder Q. Hence we want to find $\overline{u}_i \in L_2(0,T; H^1(\Omega)) \cap H^1(0,T; \widetilde{H}^{-1}(\Omega))$ with $\overline{u}_i(x,0) = 0$ for $x \in \Omega$ such that

$$a(\overline{u}_i, v) - \langle w_i, \gamma_0^{\text{int}} v \rangle_\Sigma = \langle f, v \rangle_Q - a(\overline{u}_0, v) \quad \text{for all } v \in L_2(0,T; H^1(\Omega)).$$

For details regarding existence and uniqueness of solutions, see [29] and [30]. Let X be the Dirichlet trace space of $L_2(0,T;H^1(\Omega)) \cap H^1(0,T;\widetilde{H}^{-1}(\Omega))$. The variational formulation of the boundary integral equation (9.4) is to find $w_e \in X'$ such that

$$\langle V w_e, \tau\rangle_\Sigma + \langle \left(\frac{1}{2}I - K\right) \gamma_0^{\text{ext}} u_e, \tau\rangle_\Sigma = 0 \quad \text{for all } \tau \in X'.$$

Together with the transmission conditions (9.2) we get the variational formulation of the coupled problem. We have to find $\overline{u}_i \in L_2(0,T;H^1(\Omega)) \cap H^1(0,T;\widetilde{H}^{-1}(\Omega))$ with $\overline{u}_i(x,0) = 0$ for $x \in \Omega$ and $w_e \in X'$ such that

$$\begin{aligned} a(\overline{u}_i, v) - \langle w_e, \gamma_0^{\text{int}} v\rangle_\Sigma &= \langle f, v\rangle_Q - a(\overline{u}_0, v), \\ \langle V w_e, \tau\rangle_\Sigma + \langle \left(\frac{1}{2}I - K\right) \gamma_0^{\text{int}} \overline{u}_i, \tau\rangle_\Sigma &= 0 \end{aligned} \tag{9.6}$$

for all $v \in L_2(0,T;H^1(\Omega))$ and $\tau \in X'$.

9.3 Discretization

We consider an admissible triangulation $\mathcal{T}_h = \{q_l\}_{l=1}^{N_Q}$ of the space-time cylinder Q into finite elements q_l. Let $\{(x_k,t_k)\}_{k=1}^{M_Q}$ be the set of nodes of the triangulation. We define I_0 to be the index set of the nodes, which do not belong to $\overline{\Omega} \times \{0\}$ and $M_0 := |I_0|$. Moreover I_I is the index set of the nodes, which do not belong to $\overline{\Sigma} \cup \left(\overline{\Omega} \times \{0\}\right)$ and $M_I := |I_I|$. The nodes are sorted in such a way, that $I_0 \subset \{1,...,M_0\}$ and $I_I \subset \{1,...,M_I\}$. The boundary elements $\mathcal{E}_h = \{\sigma_k\}_{k=1}^{N_\Sigma}$ of the induced decomposition of Σ are given by

$$\mathcal{E}_h := \left\{\sigma \subset \overline{\Sigma} : \exists\, q \in \mathcal{T}_h : \sigma = \partial q \cap \overline{\Sigma}\right\}.$$

Figure 9.1 shows a sample triangulation for the spatially one-dimensional problem. Let $S_h^0(\Sigma) = \operatorname{span}\{\varphi_k^0\}_{k=1}^{N_\Sigma}$ be the space of piecewise constant basis function with respect to the triangulation $\mathcal{E}_h$ of the boundary Σ, i.e. we have

$$\varphi_k^0(x,t) := \begin{cases} 1 & \text{for } (x,t) \in \sigma_k, \\ 0 & \text{else} \end{cases}$$

for $k = 1,...,N_\Sigma$ and $S_h^1(Q) = \operatorname{span}\{\varphi_i^1\}_{i=1}^{M_Q}$ be the space of piecewise linear and continuous basis functions φ_i^1 with respect to the triangulation $\mathcal{T}_h$ of the space-time cylinder Q, i.e.

$$\varphi_i^1(x_j,t_j) = \delta_{ij} \quad \text{for } i,j = 1,...,M_Q,$$

where $(x_j,t_j) \in \mathbb{R}^{n+1}$ are the coordinates of the j-th node of $\mathcal{T}_h$, see [34]. Moreover we define $S_{h,0}^1(Q)$ to be the space of functions in $S_h^1(Q)$, which vanish on $\overline{\Omega} \times \{0\}$. Due

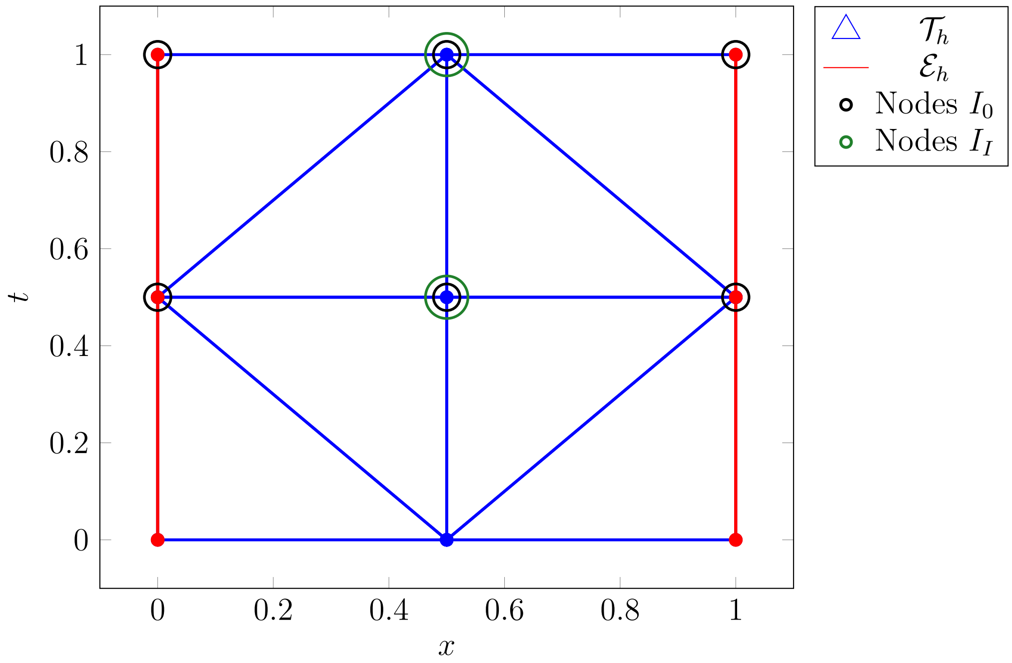

Figure 9.1: Sample triangulation of $Q = (0,1) \times (0,1)$.

to the sorting of the nodes we have $S^1_{h,0}(Q) = \operatorname{span}\{\varphi^1_i\}_{i=1}^{M_0}$. We approximate w_e and $\overline{u}_i$ by

$$w_{e,h} = \sum_{k=1}^{N_\Sigma} w^k \varphi^0_k \in S^0_h(\Sigma), \quad \overline{u}_{i,h} = \sum_{j=1}^{M_0} u^j \varphi^1_j \in S^1_{h,0}(Q).$$

Hence it remains to compute the unknown coefficients w^k and u^j.

9.4 Galerkin method

Let $\overline{u}_{0,h}$ be the interpolation of $\overline{u}_0$ in $S^1_h(Q)$, i.e. we have

$$\overline{u}_{0,h} = \sum_{j=1}^{M_Q} u^j_0 \varphi^1_j$$

with $u^j_0 = \overline{u}_0(x_j, t_j)$ for $j = 1, ..., M_Q$. Since $\overline{u}_0 \in L_2(0,T; H^1_0(\Omega)) \cap H^1(0,T; \widetilde{H}^{-1}(\Omega))$ we have $u^j_0 = 0$ for $x_j \in \Gamma$. The Galerkin variational formulation of (9.6) is to find $\overline{u}_{i,h} \in S^1_{h,0}(Q)$ and $w_{e,h} \in S^0_h(\Sigma)$ such that

$$\begin{aligned} a(\overline{u}_{i,h}, v_h) - \langle w_{e,h}, v_h\rangle_\Sigma &= \langle f, v_h\rangle_Q - a(\overline{u}_{0,h}, v_h), \\ \langle V w_{e,h}, \tau_h\rangle_\Sigma + \langle \left(\frac{1}{2}I - K\right) \overline{u}_{i,h}, \tau_h\rangle_\Sigma &= 0 \end{aligned}$$

for all $v \in S^1_{h,0}(Q)$ and for all $\tau_h \in S^0_h(\Sigma)$. This formulation is equivalent to the system of linear equations

$$\begin{pmatrix} A_{QQ} & A_{Q\Sigma} & \\ A_{\Sigma Q} & A_{\Sigma\Sigma} & -M_h^T \\ & \frac{1}{2}M_h - K_h & V_h \end{pmatrix} \begin{pmatrix} \underline{\hat{u}}^Q \\ \underline{\hat{u}}^\Sigma \\ \underline{w} \end{pmatrix} = \begin{pmatrix} \underline{f}^Q \\ \underline{f}^\Sigma \\ \underline{0} \end{pmatrix} \tag{9.7}$$

with

$$A[j,i] = a(\varphi_i^1, \varphi_j^1), \quad \underline{\hat{u}}[j] = u^j + u_0^j, \quad \underline{f}[j] = \langle f, \varphi_j^1 \rangle_Q - \sum_{r=M_0+1}^{M_Q} u_0^r \, a(\varphi_r^1, \varphi_j^1)$$

for $i, j = 1, ..., M_0$, where $u_0^j = 0$ for $x_j \in \Gamma$, and

$$M_h[l,i] = \langle \varphi^1_{M_I+i}, \varphi^0_l \rangle_\Sigma, \quad K_h[l,i] = \langle K\varphi^1_{M_I+i}, \varphi^0_l \rangle_\Sigma, \quad V_h[l,k] = \langle V\varphi^0_k, \varphi^0_l \rangle_\Sigma$$

for $i = 1, ..., M_0 - M_I$ and $k, l = 1, ..., N_\Sigma$.

Consequently we can compute the unkown coefficients w^k and u^j and the corresponding approximations $w_{e,h}$ and $\overline{u}_{i,h}$. The solution $u_{i,h}$ of the interior transmission problem (9.1) is then given by $u_{i,h} = \overline{u}_{i,h} + \overline{u}_{0,h}$, whereas the solution $u_{e,h}$ of the exterior problem is given by the representation formula (9.3), i.e. we have

$$u_{e,h}(\widetilde{x}, t) = -(\widetilde{V} w_{e,h})(\widetilde{x}, t) + (W \gamma_0^{\text{int}} \overline{u}_{i,h})(\widetilde{x}, t) \quad \text{for } (\widetilde{x}, t) \in \Omega^{\text{ext}} \times (0, T).$$

10 Numerical examples

In this chapter we present numerical examples regarding the convergence properties of the Galerkin approximations and the preconditioning techniques for the one-dimensional heat equation as well as examples for the FEM-BEM coupling method.

10.1 Preconditioning

Let $\Omega = (0,1)$ and $T = 1$. In general we consider the initial boundary value problem

$$\begin{aligned} \alpha\partial_t u(x,t) - \partial_{xx} u(x,t) &= 0 \qquad \text{for } (x,t) \in (0,1) \times (0,1), \\ u(0,t) = u(1,t) &= 0 \qquad \text{for } t \in (0,1), \\ u(x,0) &= u_0(x) \quad \text{for } x \in (0,1) \end{aligned} \tag{10.1}$$

where u_0 is some given initial condition satisfying $u_0(0) = u_0(1) = 0$. The solution u is given by the representation formula

$$u(\widetilde{x},t) = (\widetilde{V}\gamma_1^{\text{int}} u)(\widetilde{x},t) + (\widetilde{M_0} u_0)(\widetilde{x},t) \qquad \text{for } (\widetilde{x},t) \in (0,1) \times (0,1).$$

We use the variational formulation of the first boundary integral equation (7.1) to determine the unknown conormal derivative $\gamma_1^{\text{int}} u$, i.e. we have to find $\gamma_1^{\text{int}} u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ such that

$$\langle V\gamma_1^{\text{int}} u, \tau\rangle_\Sigma = -\langle M_0 u_0, \tau\rangle_\Sigma \qquad \text{for all } \tau \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma).$$

For the discretization of this formulation we consider the space of piecewise constant basis functions $S_h^0(\Sigma)$. This leads to the system of linear equations

$$V_h \underline{w} = \underline{f} \tag{10.2}$$

where

$$\begin{aligned} V_h[l,k] = \langle V\varphi_k^0, \varphi_l^0\rangle_\Sigma &= \int_\Sigma (V\varphi_k^0)(x,t)\varphi_l^0(x,t)ds_x dt \\ &= \frac{1}{\alpha} n_l n_k \int_{t_{l_1}}^{t_{l_2}} \int_{t_{k_1}}^{t_{k_2}} U^\star(x_l - x_k, t-s) ds dt \end{aligned}$$

and

$$\begin{aligned} \underline{f}[l] = -\langle M_0 u_0, \varphi_l^0\rangle_\Sigma &= -\int_\Sigma (M_0 u_0)(x,t)\varphi_l^0(x,t) ds_x dt \\ &= -n_l \int_{t_{l_1}}^{t_{l_2}} \int_0^1 U^\star(x_l - y, t) u_0(y) dy dt \end{aligned}$$

for $l, k = 1, ..., N$. The Galerkin approximation w_h of the conormal derivative $w = \gamma_1^{\text{int}} u$ is then given by

$$w_h(x,t) = \sum_{l=1}^{N} w_l \varphi_l^0(x,t) \quad \text{for } (x,t) \in \Sigma.$$

The system (10.2) is solved with the GMRES method. We want the condition number to be independent of the mesh size h. Therefore we use the preconditioning technique introduced in Chapter 8. In the one-dimensional case we can use the piecewise constant basis functions to discretize the hypersingular operator D. The preconditioner is given by

$$C_V^{-1} = M_h^{-1} D_h M_h^{-T}$$

where

$$D_h[l,k] = \langle D\varphi_k^0, \varphi_l^1 \rangle_\Sigma = \int_\Sigma (D\varphi_k^0)(x,t)\varphi_l^0(x,t) ds_x dt = n_l \int_{t_{l_1}}^{t_{l_2}} (D\varphi_k^0)(x_l,t) dt$$

and

$$\begin{aligned} M_h[l,k] = \langle \varphi_k^0, \varphi_l^0 \rangle_{L^2(\Sigma)} &= \int_\Sigma \varphi_k^0(x,t)\varphi_l^0(x,t) ds_x dt \\ &= \begin{cases} |t_{l_2} - t_{l_1}| & \text{if } l = k, \\ 0 & \text{if } l \neq k \end{cases}. \end{aligned}$$

Since M_h is a diagonal matrix, the computation of the inverse of M_h and the computation of the preconditioning matrix C_V^{-1} respectively is quite fast and very simple.

Uniform refinement

First we consider the initial boundary value problem (10.1) with initial condition $u_0(x) = \sin(2\pi x) \in H_0^1(\Omega)$ und use a uniform refinement strategy to compute the approximation w_h. The unique solution of this problem is shown in Figure 10.1. Table 10.1 shows the $L^2(\Sigma)$-error and the corrensponding convergence, the condition numbers of the system matrix and the preconditioned matrix, as well as the iteration numbers of the GMRES method in both cases. We obtain linear convergence of the approximation, which is what we expected according to Theorem (7.9). While we can see that the condition number of the matrix V_h is increasing with a factor of approximately $\sqrt{2}$, the condition number of the preconditioned matrix $C_V^{-1} V_h$ is bounded. Hence the iteration number of the preconditioned system is bounded as well, while the iteration number corresponding to (10.2) is increasing with a factor of approximately $\sqrt[4]{2}$.

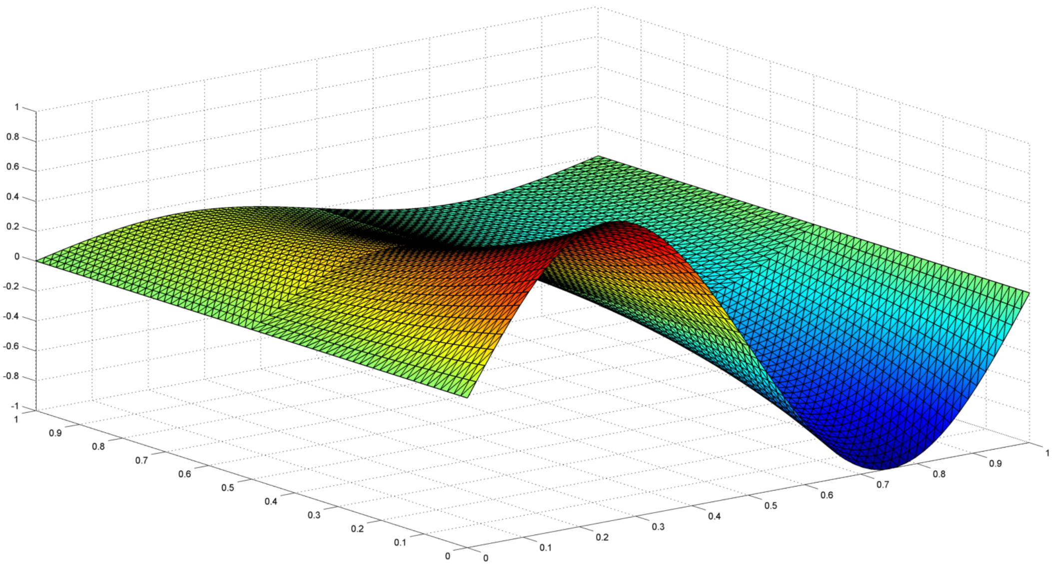

Figure 10.1: Uniform refinement. Approximation u_h at level 11.

L	N	$\|w - w_h\|_{L_2(\Sigma)}$	eoc	$\kappa(V_h)$	Factor	It.	Factor	$\kappa(C_V^{-1}V_h)$	It.
0	2	2,249	0	1,001	0	1	0	1,002	1
1	4	1,311	0,778	2,808	2,807	2	2	1,279	2
2	8	0,658	0,996	4,905	1,746	4	2	1,422	4
3	16	0,324	1,021	7,548	1,539	8	2	1,486	8
4	32	0,16	1,017	11,14	1,476	16	2	1,541	14
5	64	0,079	1,01	16,724	1,501	31	1,938	1,563	13
6	128	0,04	1,006	13,47	0,805	41	1,323	1,59	13
7	256	0,02	1,003	22,053	1,637	50	1,22	1,615	12
8	512	0,01	1,001	32,043	1,453	59	1,18	1,636	12
9	1024	0,005	1,001	60,957	1,902	70	1,186	1,777	11
10	2048	0,002	1,000	88,488	1,452	82	1,171	1,762	11
11	4096	0,001	1,000	125,957	1,423	96	1,171	1,765	10

Table 10.1: Uniform refinement. Condition and iteration numbers.

The unique solution of the initial boundary value problem (10.1) with initial condition $u_0 = 0$ is $u = 0$. Thus, we have $\gamma_1^{\text{int}} u = 0$. Table 10.2 shows the iteration numbers of the GMRES method using a random initial guess. Again, we obtain boundedness of the iteration numbers corresponding to the preconditioned system. We can conclude that the results concerning the iteration numbers in the previous example do not depend on the given data.

L	N	It.	Factor	It. Prec.
0	2	2	0	2
1	4	3	1,5	3
2	8	4	1,333	4
3	16	8	2	8
4	32	16	2	13
5	64	31	1,938	14
6	128	42	1,355	13
7	256	52	1,238	13
8	512	64	1,231	13
9	1024	77	1,203	13
10	2048	92	1,195	13
11	4096	109	1,185	12

Table 10.2: Iteration numbers for a random initial guess of the GMRES.

Adaptive refinement

We consider problem (10.1) with initial condition $u_0 = 5\exp(-10x)\sin(\pi x)$ and use an adaptive refinement strategy to compute the approximation w_h.
Let $N \in \mathbb{N}$ and Σ_N be a decomposition of Σ into N boundary elements σ_l as given by (7.14), i.e. we have

$$\Sigma_N = \bigcup_{l=1}^{N} \overline{\sigma}_l.$$

Let $w = \gamma_1^{\text{int}} u \in H^{-\frac{1}{2},-\frac{1}{4}}(\Sigma)$ be the exact solution and $w_h = \sum_{l=1}^{N} w_l \varphi_l^0 \in S_h^0(\Sigma)$ be the Galerkin approximation of the problem. The local $L^2(\Sigma)$-error on a boundary element σ_l is then given by

$$e_l := \|w - w_h\|_{L^2(\sigma_l)}.$$

The strategy is to refine each boundary element σ_l whose error satisfies

$$e_l \geq \theta \max_{k=1,\dots,N} e_k$$

with parameter $\theta \in (0,1)$. In our case we divide those elements into two elements of same size. We continue to refine the mesh until the global error

$$e := \left(\sum_{l=1}^{N} e_l^2 \right)^{\frac{1}{2}}$$

is less than some given value. Of course, in general we do not know the exact solution of the problem. Thus, we have to use a posteriori error estimators, see for example [6] and [23].

According to Section 2.1.1 the exact solution of the problem (10.1) is given by the series

$$u(x,t) = \sum_{k=1}^{\infty} a_k \exp\left(\frac{-(k\pi)^2 t}{\alpha} \right) \sin(k\pi x) \quad \text{for } (x,t) \in Q$$

with

$$a_k = 2 \int_0^1 u_0(x) \sin(k\pi x) dx.$$

Hence the conormal derivative $w = \partial_n u_{|\Sigma}$ is given by

$$\partial_n u(x,t) = n_x \sum_{k=1}^{\infty} a_k \exp\left(\frac{-(k\pi)^2 t}{\alpha} \right) k\pi \cos(k\pi x) \quad \text{for } (x,t) \in \Sigma$$

where $n_x = -1$ for $x = 0$ and $n_x = 1$ for $x = 1$. The solution u_h of the problem (10.1) is shown in Figure 10.2. Figure 10.3 shows the approximation w_h of the conormal derivative. In Table 10.3 you can see the $L^2(\Sigma)$-error, the condition numbers of the system matrix and the preconditioned matrices, as well as the iteration numbers of the GMRES method. In addition to the Calderón preconditioner the condition and iteration numbers of the diagonally scaled system matrix is listed in the table. As in the case of uniform refinement we obtain boundedness of the condition number of the Calderón-preconditioned matrix $C_V^{-1} V_h$, as well as the boundedness of the corresponding iteration numbers of the GMRES method. Diagonal scaling causes an improvement concerning the condition numbers of the system matrix and the iteration numbers of the GMRES method.

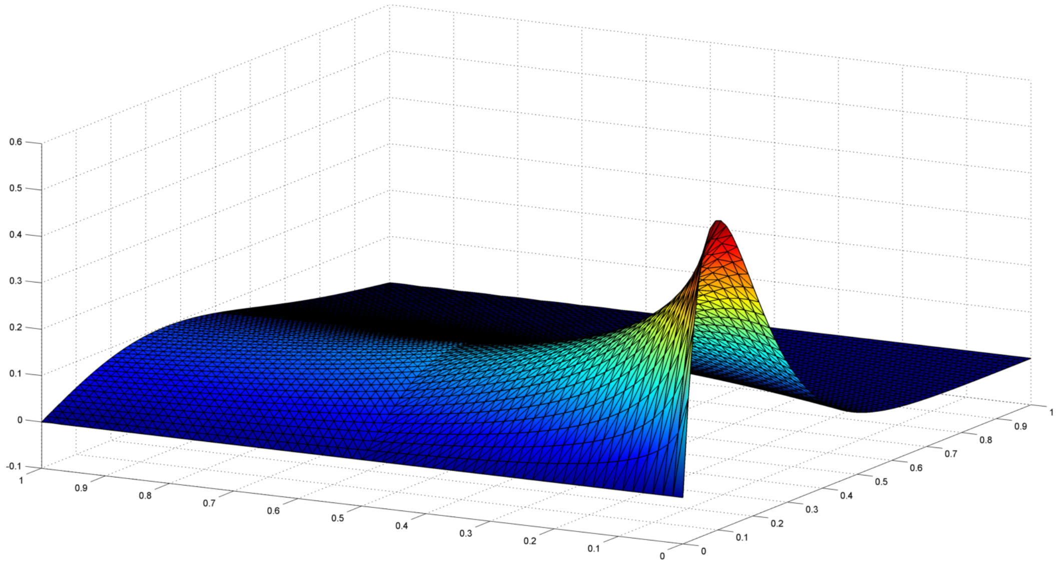

Figure 10.2: Adaptive refinement. Approximation u_h at level 10.

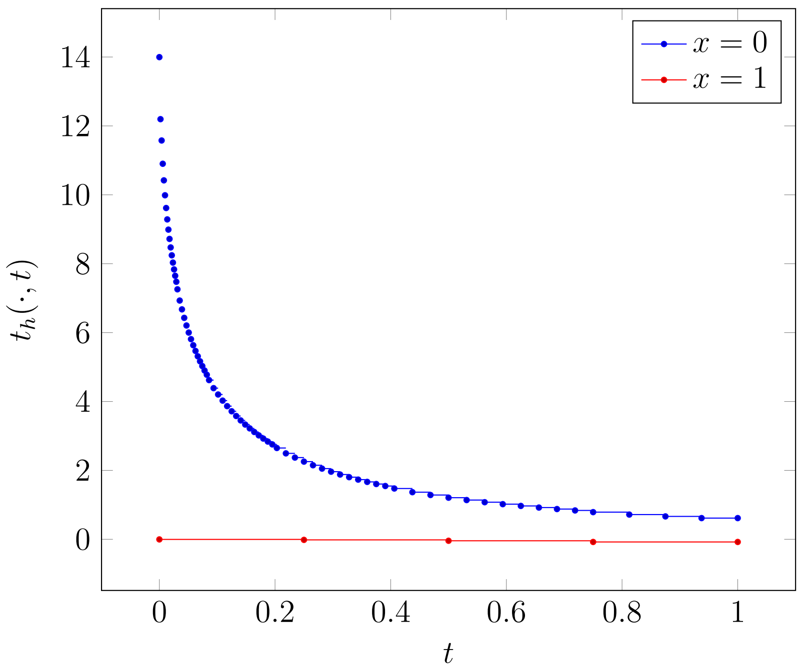

Figure 10.3: Adaptive refinement. Approximation w_h at level 10.

					$C_V = \text{diag}V_h$		$C_V = M_h D_h^{-1} M_h$	
L	N	$\|w - w_h\|_{L_2(\Sigma)}$	$\kappa(V_h)$	It.	$\kappa(C_V^{-1}V_h)$	It.	$\kappa(C_V^{-1}V_h)$	It.
0	2	1,886	1,001	2	1,001	2	1,002	2
1	3	1,637	3,972	3	2,553	3	1,16	3
2	5	1,272	12,225	5	4,055	4	1,166	4
3	7	0,914	34,212	7	3,611	6	1,156	6
4	9	0,615	92,081	9	3,164	8	1,149	8
5	11	0,401	118,586	11	2,945	10	1,224	10
6	13	0,267	338,26	13	2,803	12	1,21	12
7	20	0,166	621,773	20	3,524	18	1,197	13
8	31	0,101	1608,08	31	4,457	27	1,252	12
9	47	0,063	2344,9	47	5,779	32	1,574	11
10	74	0,039	6141,47	74	8,348	37	1,692	11
11	114	0,024	8409,92	114	10,95	42	1,561	10
12	177	0,015	23007,6	173	14,324	47	1,716	10
13	278	0,01	27528,3	200	21,094	53	1,677	10

Table 10.3: Adaptive refinement. Condition and iteration numbers.

10.2 FEM-BEM coupling

For $\Omega = (0,1)$ and $T = 1$ we consider the one-dimensional transmission problem

$$\begin{aligned} \alpha\partial_t u_i(x,t) - \partial_x\left[A(x)\partial_x u_i(x,t)\right] &= f(x,t) && \text{for } (x,t) \in \Omega \times (0,1), \\ \alpha\partial_t u_e(x,t) - \partial_{xx} u_e(x,t) &= 0 && \text{for } (x,t) \in \Omega^{\text{ext}} \times (0,1), \\ u_i(x,0) &= u_0(x) && \text{for } x \in \Omega, \\ u_e(x,0) &= 0 && \text{for } x \in \Omega^{\text{ext}} \end{aligned} \tag{10.3}$$

where $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in H^1(\Omega)$ are given. The following examples refer to the initial triangulation shown in Figure 9.1 and a uniform refinement strategy. The tables show the $L^2(Q)$-error $\|u_i - u_{i,h}\|_{L^2(Q)}$ and the corresponding convergence of the approximation of the interior problem. The $L^2(Q)$-error was computed with the 7-point rule [33, Section C.1]. The system of linear equations (9.7) was solved with `PARDISO` Version 5.0.0 [11, 21, 22].

Initial condition u_0 continuously differentiable

We consider the transmission problem (10.3) with $A \equiv 1$, $f \equiv 0$ and initial condition

$$u_0(x) = \begin{cases} \exp\left(\dfrac{1}{(2x-1)^2-1}\right)\sin(\pi x) & \text{for } x \in (0,1), \\ 0 & \text{else.} \end{cases}$$

Figure 10.4 shows the approximation u_h of the solution of the transmission problem (10.3). The $L^2(Q)$-errors and the estimated orders of convergence are listed in Table 10.4. As expected we get an order of convergence of 2 for the Galerkin approximation of the interior problem.

Level	M_Q	N_Q	N_Σ	$\lVert u_i - u_{i,h}\rVert_{L_2(Q)}$	eoc
0	9	8	4	0.0316182	0
1	25	32	8	0.015779	1.00275
2	81	128	16	0.00422301	1.90166
3	289	512	32	0.00111947	1.91545
4	1089	2048	64	0.000289581	1.95078
5	4225	8192	128	7.3735e-05	1.97355

Table 10.4: Error and convergence of $u_{i,h}$.

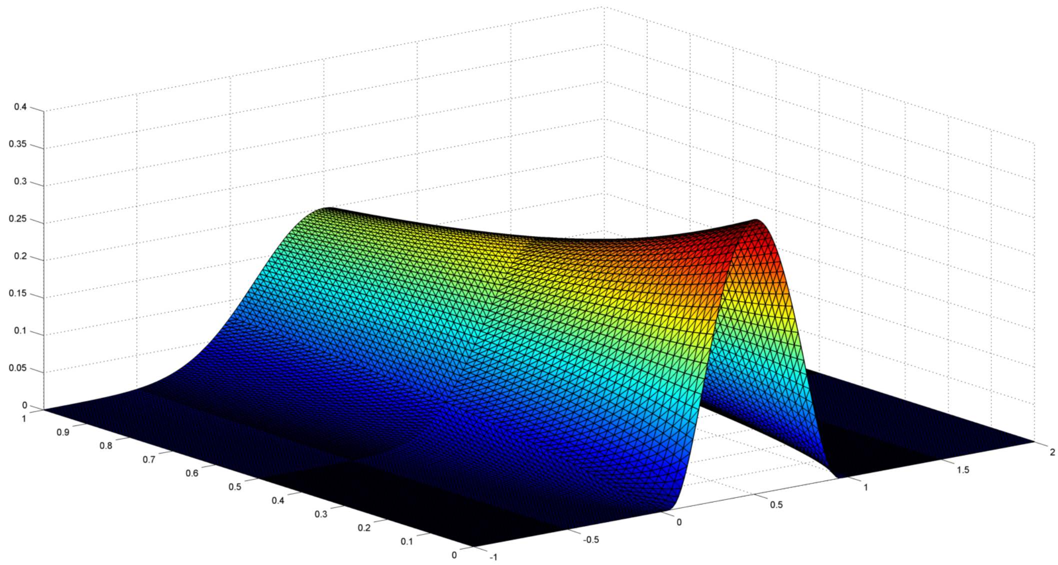

Figure 10.4: Approximation u_h at level 5.

Initial condition u_0 not continuously differentiable

We consider the transmission problem (10.3) with $A \equiv 1$, $f \equiv 0$ and initial condition

$$u_0(x) = \begin{cases} \frac{1}{2} \exp\left(\frac{1}{(2x-1)^2 - 1}\right)(1 - |1 - 2x|) & \text{for } x \in (0,1), \\ 0 & \text{else.} \end{cases}$$

The solution u_h of the corresponding transmission problem (10.3) is shown in Figure 10.5. In this case we get a reduced order of convergence, as you can see in Table 10.5, which is what we expected due to the given initial condition.

Level	M_Q	N_Q	N_Σ	$\|u_i - u_{i,h}\|_{L_2(Q)}$	eoc
0	9	8	4	0.0299642	0
1	25	32	8	0.00684995	2.12907
2	81	128	16	0.00187444	1.86963
3	289	512	32	0.000584336	1.68159
4	1089	2048	64	0.000202621	1.52801
5	4225	8192	128	7.33331e-05	1.46625

Table 10.5: Error and convergence of $u_{i,h}$.

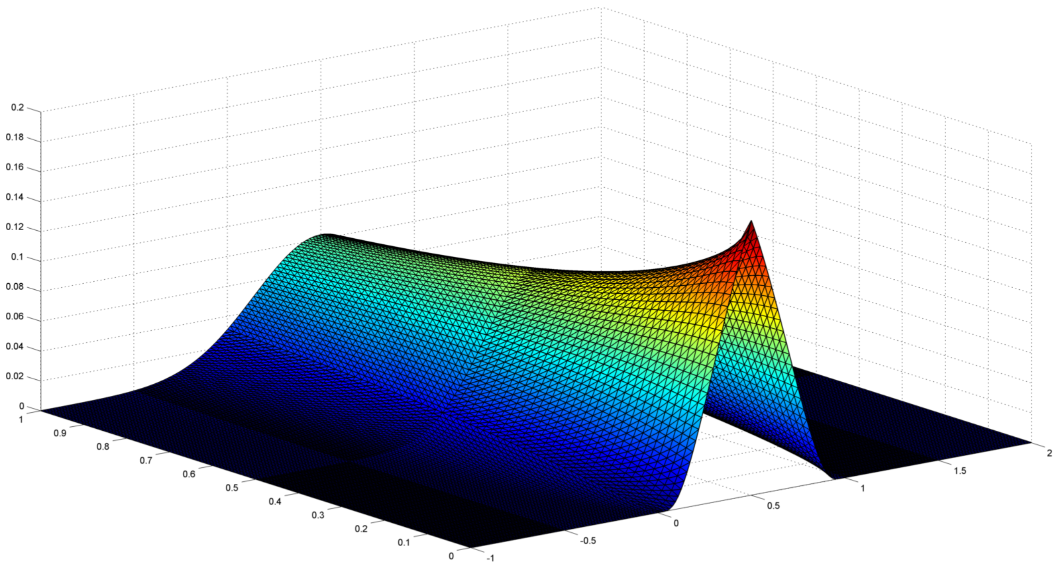

Figure 10.5: Approximate solution u_h at level 5.

11 Conclusion and Outlook

In this work we presented the boundary element method for the discretization of the time-dependent heat equation. After the derivation of the representation formula for the solution of the model problem (2.1) we analysed the heat potentials and the boundary integral operators in the setting of anisotropic Sobolev spaces. We have shown, that the single layer boundary integral operator V and the hypersingular operator D are elliptic and bounded and therefore invertible. We presented four different approaches for the computation of the unkown conormal derivative $\partial_n u_{|\Sigma}$. We discussed direct approaches using the boundary integral equations (6.3) as well as indirect formulations with the single layer potential and the double layer potential. All four formulations are uniquely solvable.

We considered a Galerkin-Bubnov variational formulation of the first boundary integral equation (7.17) and concluded unique solvability of the discrete problem due to the ellipticity and boundedness of the operator V. As a trial space we used the space of piecewise constant basis functions $S_h^0(\Sigma)$ corresponding to an arbitrary triangulation of the space-time boundary Σ for the one- and two-dimensional problem, see Section 7.1. In the two-dimensional case we made an assumption on the form and orientation of the boundary elements, see Figure 7.1, and derived the approximation properties of $S_h^0(\Sigma)$ by using the approximation properties of space-time tensor product spaces. The extension of those approximation properties to trial spaces with respect to triangulations with boundary elements of arbitrary form and orientation for $n = 2$ as well as for $n = 3$ is still open.

We estimated the error of the Galerkin approximations in the energy norm and stated an estimate for the $L^2(\Sigma)$-error of the approximations for the one-dimensional problem by using the inverse inequality (7.9). If we can prove an inverse inequality for functions in $S_h^0(\Sigma)$ for $n = 2, 3$ in the setting of anisotropic Sobolev spaces, we are able to give an $L^2(\Sigma)$-error estimate for $n = 2, 3$ as well.

We used the GMRES method to solve the system of linear equations (7.18). Since the condition number of the matrix V_h depends on the mesh size h of the decomposition of Σ we applied the Calderón preconditioning strategy and used the discretization of the hypersingular operator D as a preconditioner for V_h, see Chapter 8. We discussed the one-dimensional problem and presented three different approaches for the discretization of V and D. If we can prove stability of the $L^2(\Sigma)$-projection onto the space of piecewise linear and continuous basis functions in the anisotropic Sobolev space $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ some of the results can be extended to $n = 2, 3$. However, this is still an open issue.

In Chapter 9 we presented the concept of a non-symmetric FEM-BEM coupling method for the parabolic transmission problem (9.1) and described a Galerkin method for the discretization of the coupled problem. It is still an open question how to characterize the trace spaces of the Bochner spaces used for the variational formulation of the interior problem and if the operator V is elliptic in this setting as well.
In Chapter 10 we introduced an adaptive refinement strategy but used the exact solution to get an error estimator. Of course in general we do not know the exact solution. Thus, we have to establish a posteriori error estimators for the anisotropic BEM in order to define adaptive refinement strategies.

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