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# Differential Geometric Aspects of Semidiscrete Surfaces

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# Contents

# Abstract

Semidiscrete surfaces, which constitute our object of study, are represented by parametrizations that possess one discrete and one continuous variable. Building a bridge between smooth surfaces on the one hand and purely discrete surfaces (meshes) on the other hand, they enjoy enough geometric properties to deserve separate study.

The first part of our work is concerned with Laplace operators on semidiscrete surfaces. Laplacians on both smooth and discrete surfaces have been an object of interest for a long time, also from the viewpoint of applications. As a first approach, we define a semidiscrete Laplace operator to be the limit of a discrete Laplacian on a quadrilateral mesh which converges to the semidiscrete surface. In a second paper, we use notions and methods from calculus of variations to derive an entire family of semidiscrete Laplace operators by variation of appropriate Dirichlet energy functionals. In both cases we establish several core properties of the Laplacian, like symmetry, positive semidefiniteness, and linear precision. Moreover, we discuss its relation to the mean curvature normal and pointwise convergence toward the Laplace-Beltrami operator on smooth surfaces.

In the second part we investigate semidiscrete surfaces with constant mean curvature along with their associated families. The notion of mean curvature introduced here is motivated by a recently developed curvature theory for quadrilateral meshes, and extends previous work on semidiscrete surfaces. In the situation of vanishing mean curvature, the associated families are defined via a Weierstraß representation. For the general cmc case, we introduce a Lax pair representation that directly defines associated families of cmc surfaces, and is connected to a semidiscrete sinh-Gordon equation.

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*In memory of my mother, Waltraud Carl.*

# Overview

The present dissertation studies differential geometric aspects of semidiscrete surfaces, which are represented by parametrizations that possess one discrete and one continuous variable. They can be seen as semidiscretizations of smooth surfaces or as partial limit cases of purely discrete surfaces (meshes).

Historically, the analysis of smooth curves and surfaces in three-dimensional Euclidean space formed the basis for the development of differential geometry during the 18th and 19th centuries. Over the last decades, the study of discrete surfaces has gained significant attention, not only within pure mathematics, but also from the viewpoint of applications, e.g., in geometry processing, computer graphics, and architectural design. In particular, the field of discrete differential geometry has emerged on the border between differential and discrete geometry. Instead of smooth curves and surfaces, it deals with polygons and meshes, and aims at the development of a self-contained discrete theory that respects fundamental aspects of the smooth one. A first systematic approach toward this goal was initiated by Sauer [35], whose work is summarized in his monograph [36]. From a modern viewpoint, an important aspect of discrete differential geometry is the study of surface transformations in the sense of Eisenhart [18]. A sequence of surfaces (obtained, e.g., by iterating Bäcklund-Darboux type transformations) is seen as the limit of a higher-dimensional discrete net where only some parameters converge to continuous ones, while others remain discrete. For a comprehensive overview of this topic, and especially of the important concepts of consistency and integrability, we refer to the textbook by Bobenko and Suris [10].

As a matter of fact, the low-dimensional case of parametrizations with only one discrete and one continuous variable, i.e., sequences of curves, has not received much attention within the aforementioned transformation theory, but nevertheless is rich enough in geometry to deserve separate study. A more thorough investigation of such parametrizations was initiated around the year 2008 by Pottmann et al. [31], who discussed the problem of approximating smooth surfaces by piecewise-developable ones, motivated by applications in architecture and manufacturing. For that purpose they investigated semidiscrete incarnations of conjugate nets, in particular conical and circular ones, which enjoy elegant geometric properties. Their results motivated further research in that direction.

For instance, Müller and Wallner [27] considered semidiscrete isothermic surfaces, conformal mappings, and dualizability in the sense of Christoffel. As a matter of course, their observations lead to the investigation of semidiscrete constant mean curvature surfaces in three-dimensional Euclidean space (see Müller [26]). Similar to the purely discrete case, the definition of these special surfaces is actually not based on a notion of mean curvature. Instead, a semidiscrete isothermic surface is called minimal, if its Christoffel dual is contained in the unit sphere, and it is termed a cmc surface, if its Christoffel dual is at constant distance. In accordance with these definitions, Rossman and Yasumoto [34] have established a semidiscrete version of the Weierstraß representation of isothermic minimal surfaces. In the PhD thesis of Yasumoto [46] also semidiscrete maximal surfaces in Minkowski three-space and their singularities have been investigated. Taking another point of view, in the recent work of Burstall et al. [11] semidiscrete isothermic surfaces are described as sequences of Darboux transforms of curves, and their transformation theory is studied.

Semidiscrete asymptotic parametrizations (A-surfaces) and especially semidiscrete constant negative Gauß curvature surfaces (K-surfaces) have been analyzed by Wallner [40]. It has been shown that, in contrast to the discrete situation, it is possible to define Gauß curvature via the Lelieuvre normal vector field of a semidiscrete asymptotic surface. The definition is meaningful in the sense that an A-surface turns out to be a K-surface if and only if it has the Chebyshev property.

However, as different kinds of parametrizations have their own way of discretization, the development of a unifying (semi)discrete curvature theory is still an active topic of research. As a first approach, Karpenkov and Wallner [22] introduced curvatures for semidiscrete conjugate surfaces based on the concept of offsets (i.e., parallel surfaces at constant distance), in analogy to the situation of polyhedral meshes considered by Bobenko et al. [8].

As can be seen, there are various different approaches toward the development of semidiscrete equivalents of notions and methods of smooth surface theory. The present work constitutes a contribution to the ongoing research on a deeper understanding of the relation between purely discrete, semidiscrete, and smooth objects in (discrete) differential geometry. The first two chapters are concerned with Laplace operators on semidiscrete surfaces. In Chapter 1 we utilize the fact that any semidiscrete surface can be seen as a partial limit case of a quadrilateral mesh to derive a semidiscrete Laplacian from the discrete Laplace operator described by Alexa and Wardetzky [1]. In the second chapter we use notions and methods from calculus of variations to derive an entire family of semidiscrete Laplace operators by variation of appropriate Dirichlet energy functionals. Surprisingly, the operator obtained as limit in the first chapter is contained in the latter family of variational Laplacians. Last but not least, the third chapter deals with a completely different topic, namely with semidiscrete constant mean curvature surfaces and their associated families. The introduced notion of mean curvature is motivated by a curvature theory for quadrilateral meshes recently developed by Hoffmann et al. [20], and extends previous work on semidiscrete surfaces. We provide a more detailed overview of the individual topics below.

At this point we want to highlight the fact that the three chapters which constitute the present dissertation essentially coincide with the corresponding journal articles itemized in the following list of publications. Thus, throughout this work the terms “chapter” and “paper” are used synonymously. To avoid multiply defined references, the respective bibliographies have been merged into the general list starting on page 77.

## List of publications

[i] W. Carl. A Laplace Operator on Semi-Discrete Surfaces. *Found. Comput. Math.*, 2015, DOI 10.1007/s10208-015-9271-y.

[ii] W. Carl and J. Wallner. Variational Laplacians for semidiscrete surfaces. Submitted: Dec 2014 / Revised: Oct 2015.

[iii] W. Carl. On semidiscrete constant mean curvature surfaces and their associated families. Submitted: Nov 2015.

## Chapter 1: A Laplace operator on semidiscrete surfaces

The Laplace-Beltrami operator $\Delta = -\mathrm{div} \circ \mathrm{grad}$ on smooth surfaces and Riemannian manifolds is an extremely well investigated differential operator which plays an essential role in many fields including applications. A main strength lies in Riemannian geometry, but it is also relevant to the elementary differential geometry of surfaces in three-dimensional space, e.g., via the equation $\Delta\,\mathrm{id} = -2H\mathbf{n}$ that relates the Laplacian to the mean curvature and unit normal vector field. Its intrinsic nature makes it very useful for computational applications, e.g., in geometry processing, and it has therefore been extensively discretized.

In this paper, we use the discrete Laplacian described by Alexa and Wardetzky [1] to derive a Laplace operator on semidiscrete surfaces. Our approach is based on the fact that any semidiscrete surface can be seen as a limit case of a purely discrete one. More precisely, we may discretize a semidiscrete surface $x : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^3$ and a real-valued function $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ near a point of interest $(k, t) \in \mathbb{Z} \times \mathbb{R}$ by letting

$$x^{\varepsilon}_{i,j} := x(k + i, t + \varepsilon j) \quad \text{and} \quad u^{\varepsilon}_{i,j} := u(k + i, t + \varepsilon j).$$

This defines the vertices $x^{\varepsilon}_{i,j}$ of a quadrilateral mesh with regular combinatorics and function values $u^{\varepsilon}_{i,j}$ on these vertices. Denoting the discrete Laplace operator on that mesh by $\mathbb{L}^{\varepsilon}$, we define the semidiscrete Laplacian of $u$ at the point $(k, t)$ to be the limit

$$(\Delta_{\lim} u)(k, t) := \lim_{\varepsilon\to 0} (\mathbb{L}^{\varepsilon} u^{\varepsilon})\Big|_{0,0}.$$

The main result of this paper is stated in Theorem 1.1, where we show that this limit exists under very mild regularity assumptions. Along with the proof of this theorem we derive a closed-form expression for the semidiscrete Laplacian, which we summarize in Corollary 1.1. Moreover, we reveal a direct relation between the discrete Laplacian from Alexa and Wardetzky [1] and the discrete scheme described by Liu at al. [24] (see Remark 1.1).

Subsequently, we show that the semidiscrete Laplacian inherits several important properties from the discrete operator. We define an area element and verify that the semidiscrete Laplacian is symmetric and positive semidefinite with respect to the corresponding $L^2$ inner product (see Lemma 1.3). Moreover, we prove that it converges pointwise to the Laplace-Beltrami operator, if the semidiscrete surface converges to a smooth one (see Theorem 1.2). This result particularly implies that the corresponding discrete Laplacian on quadrilateral meshes is a consistent discretization (see Remark 1.3). Additionally, we show that the semidiscrete Laplacian inherits the "linear precision" property from its discrete counterpart, meaning that on planar semidiscrete surfaces the Laplacian of linear functions vanishes.

## Chapter 2: Variational Laplacians for semidiscrete surfaces

In the second paper we demonstrate a variational approach toward a semidiscrete Laplace operator. We utilize the well-known fact that the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ on a Riemannian manifold $\mathcal{M}$ can be defined via the Dirichlet energy functional

$$E(u) = \frac{1}{2}\int_{\mathcal{M}} \|\nabla u\|^2 \, dV, \quad u \in C^2(\mathcal{M}, \mathbb{R}).$$

Indeed, it is given as the gradient of the Dirichlet energy functional,

$$\Delta_{\mathcal{M}} = \nabla E,$$

which means that for smooth test functions $u$ and all smooth one-parameter variations $u_\xi$ of $u$, with the property that $\frac{\partial}{\partial\xi} u_\xi|_{\xi=0}$ is compactly supported, we have

$$\frac{d}{d\xi} E(u_\xi)\Big|_{\xi=0} = \Big\langle \Delta_{\mathcal{M}} u, \ \frac{\partial u_\xi}{\partial \xi}\Big|_{\xi=0}\Big\rangle_{L^2},$$

with the usual definition $\langle f, g\rangle_{L^2} = \int_{\mathcal{M}} f(x) g(x) \, dV(x)$ (see, e.g., [21, pp. 89–94]). This relation is basic to the generalization of the Laplace-Beltrami operator to discrete surfaces and will also be used in this chapter. More precisely, in Section 2.2 we define a Laplace operator on semidiscrete surfaces as gradient of an appropriate Dirichlet energy functional. We show that this gradient exists and provide a closed-form expression for the semidiscrete Laplacian in Theorem 2.1. It turns out that there is quite some freedom in the choice of the particular $L^2$ space which is basic to the concepts of both gradient and Dirichlet energy. Surprisingly, using as an area measure a simple numerical integration rule yields precisely the semidiscrete Laplacian $\Delta_{\lim}$ from the first chapter (see Section 2.2.2).

Moreover we recall that, on a surface $\mathcal{M}$ embedded in $\mathbb{R}^3$, the mean curvature normal $\mathbf{H} = H\mathbf{n}$ likewise allows for a variational definition, namely

$$-2\mathbf{H} = \nabla \text{area}(\mathcal{M}), \ \text{ i.e.,} \qquad \frac{d}{d\xi}\text{area}(p_\xi(\mathcal{M}))\Big|_{\xi=0} = \Big\langle -2\mathbf{H}, \ \frac{\partial p_\xi}{\partial \xi}\Big|_{\xi=0}\Big\rangle_{L^2(\mathcal{M},\mathbb{R}^3)}$$

for every smooth one-parameter variation $p_\xi : \mathcal{M} \to \mathbb{R}^3$ with $p_0 = \text{id}_{\mathcal{M}}$ (see, e.g., [13, p. 7]). Here, $\text{area}(\mathcal{M}) = \int_{\mathcal{M}} 1 \, dV$ and $\langle f, g\rangle_{L^2(\mathcal{M},\mathbb{R}^3)} = \int_{\mathcal{M}} \langle f(x), g(x)\rangle \, dV(x)$. Consequently, in Section 2.3 we investigate the gradient of the area functional to gain a semidiscrete mean curvature normal, and establish the relation $\Delta\, \text{id} = -2\mathbf{H}$ for the semidiscrete case (see Theorem 2.2). In turn, this relation implies that linear functions on flat surfaces are in the kernel of the Laplacian (i.e., the linear precision property). In Section 2.4 we discuss further properties of the semidiscrete Laplace operator like locality, symmetry, positive semidefiniteness, and lack of a maximum principle. The last section deals with pointwise convergence of the semidiscrete Laplacian toward the Laplace-Beltrami operator on smooth surfaces (cf. Theorem 2.3).

## Chapter 3: Semidiscrete cmc surfaces and their associated families

As opposed to the first two chapters, this paper investigates semidiscrete constant mean curvature surfaces in three-dimensional Euclidean space within the framework of integrable systems. Surfaces with constant mean curvature $H$ or constant Gauß curvature $K$ have been of particular interest in differential geometry for a long time. Typically, the investigation of constant curvature surfaces is tied to specific parametrizations, like isothermic parametrizations for constant mean curvature surfaces. An interesting feature of these surfaces is that they possess one-parameter families of deformations preserving the respective curvature, while changing the type of parametrization. As different kinds of parametrizations have their own way of discretization, it has been a challenge to receive similar results in the discrete and semidiscrete situations. Only recently Hoffmann et al. [20] presented a unifying curvature theory for quadrilateral meshes equipped with unit normal vectors at the vertices. Their theory encompasses a remarkably large class of existing discrete special parametrizations and, in particular, provides a deeper insight into the associated families of discrete constant curvature surfaces.

Accordingly, at the beginning of this chapter we translate the discrete curvatures introduced by Hoffmann et al. [20] to the semidiscrete setting (see Section 3.2). We also highlight the intersection with the curvature theory for semidiscrete conjugate parametrizations of Karpenkov and Wallner [22]. In Section 3.3, we recapitulate the notion of isothermic parametrizations and show that a semidiscrete surface is isothermic if and only if its quaternionic cross ratio allows for a specific factorization (cf. Lemma 3.5). Subsequently, in Section 3.4, we investigate semidiscrete isothermic minimal surfaces. Their Weierstraß representation, established by Rossman and Yasumoto [34], immediately gives rise to their associated families, whose members are however no longer isothermic. The main result of this section is that all the members of these associated families are minimal as well (cf. Theorem 3.1). Moreover, we show that the conjugate surface of an isothermic minimal surface is asymptotically parametrized. In Section 3.5, we introduce a Lax pair representation for semidiscrete isothermic cmc surfaces, which directly contains the definition of their associated families. We prove that the members of these associated families, which again are no longer isothermic, all have the same constant mean curvature (cf. Theorem 3.2). We conclude the paper by investigating the Lax pair representation of semidiscrete rotational symmetric cmc surfaces (see Section 3.6). We demonstrate that the discrete version of the classical Delaunay rolling ellipse construction, obtained by Bobenko et al. [8], also applies to the semidiscrete setting.

# Chapter 1

# A Laplace operator on semidiscrete surfaces

**Abstract**

This paper studies a Laplace operator on semidiscrete surfaces. A semidiscrete surface is represented by a mapping into three-dimensional Euclidean space possessing one discrete variable and one continuous variable. It can be seen as a limit case of a quadrilateral mesh, or as a semidiscretization of a smooth surface. Laplace operators on both smooth and discrete surfaces have been an object of interest for a long time, also from the viewpoint of applications. There are a wealth of geometric objects available immediately once a Laplacian is defined, e.g., the mean curvature normal. We define our semidiscrete Laplace operator to be the limit of a discrete Laplacian on a quadrilateral mesh, which converges to the semidiscrete surface. The main result of this paper is that this limit exists under very mild regularity assumptions. Moreover, we show that the semidiscrete Laplace operator inherits several important properties from its discrete counterpart, like symmetry, positive semidefiniteness, and linear precision. We also prove consistency of the semidiscrete Laplacian, meaning that it converges pointwise to the Laplace-Beltrami operator, when the semidiscrete surface converges to a smooth one. This result particularly implies consistency of the corresponding discrete scheme.

**Keywords:** Semidiscrete surface, Quadrilateral mesh, Laplace operator, Consistency.

**Mathematics Subject Classification (2010):** Primary 53B20; Secondary 53A05, 41A25.

## 1.1 Introduction

The Laplace-Beltrami operator $\Delta = -\operatorname{div} \circ \operatorname{grad}$ on smooth surfaces and Riemannian manifolds is a well-studied differential operator with many applications inside and outside mathematics. It plays an important role in a variety of areas and applications, such as physical simulation, parametrization, geometric modeling, shape analysis, and surface optimization. It is also

This chapter comprises the research article [i].

relevant to elementary differential geometry, especially since, on embedded surfaces in $\mathbb{R}^3$, the Laplace operator is connected to the mean curvature normal $H\mathbf{n}$ via the equation $\Delta \operatorname{id} = -2H\mathbf{n}$.

## Discrete Laplace operators

For practical computations, smooth surfaces are often approximated by discrete meshes. Therefore, it is necessary to establish discrete Laplace operators that ideally maintain as many of the core properties of their smooth counterpart as possible. Regarding the applications, it is also important to analyze the convergence behavior of the discrete schemes under appropriate mesh refinement.

There is by now a well-developed theory of discrete Laplacians on triangle meshes. A famous example is the so-called cotangent formula, which has already been explored from various different viewpoints (see, e.g., MacNeal [25], Duffin [16], Dziuk [17], Pinkall and Polthier [29], and Desbrun et al. [14]). The convergence behavior of this scheme was analyzed by Xu [45] and Wardetzky [41] among others. Several variants of the cotangent Laplacian have also been considered (see Bobenko and Springborn [9] for an example). Another discrete Laplace operator on triangle meshes was proposed by Belkin et al. [2], who could prove consistency of their scheme, meaning that it converges pointwise to the smooth Laplacian, as the mesh converges to a smooth surface.

For the situation of quadrilateral or even general polygonal meshes, there are far fewer results. The obvious approach of triangulating the given mesh and using one of the above-mentioned operators is not adequate in this more general case, as different triangulations in general lead to different results. Nevertheless, this idea motivated Xiong et al. [44] to average the cotangent formula over all possible triangulations in order to obtain a discrete Laplace operator on quadrilateral meshes. Earlier, Liu et al. [24] described a Laplacian on quadrilateral meshes based on a bilinear interpolation of each face. They achieve consistency under some special, but not too restrictive conditions.

Only recently the cotangent formula has been extended to the case of general polygonal meshes by Alexa and Wardetzky [1]. Their discretization of the Laplace operator enjoys several important properties, some of which will be of particular interest within the course of the present paper. However, the convergence behavior of their scheme has not been analyzed so far. In this regard, we are contributing a first result (see Remark 1.3).

## A semidiscrete Laplace operator

In this paper, we utilize the discrete Laplacian by Alexa and Wardetzky [1] to derive a Laplace operator on semidiscrete surfaces. A semidiscrete surface is represented by a function

$$x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3 : (k,t) \mapsto x(k,t)$$

depending on one discrete variable and one continuous variable.

The study of semidiscrete objects has been of high relevance in different branches of mathematics for a long time. A great deal of their attraction lies in the fact that they can lead us to a deeper understanding of both the discrete and the continuous cases.

A good example of this phenomenon is the modern viewpoint on surface transformation theory, where sequences of surfaces (obtained, e.g., by iterating Bäcklund-Darboux type transformations) are seen as semidiscrete objects, preferably at the same time interpreted as partial limits of a discrete master object, governed by an integrable system (see Bobenko and Suris [10]).

Semidiscretization also plays an important role in the field of computational mathematics, e.g., for transforming a partial differential equation into a system of ordinary differential equations.

The present work on a semidiscrete Laplace operator is a fundamental contribution to the ongoing research on a deeper understanding of the relation between purely discrete, semidiscrete, and smooth objects in (discrete) differential geometry. In particular, the Laplacian and its connection to mean curvature is of interest with respect to topics like semidiscrete minimal surfaces, or semidiscrete conformal mappings (see Müller and Wallner [27]).

Our approach toward a semidiscrete Laplacian is based on the fact that any semidiscrete surface can be seen as a limit case of a purely discrete surface. Indeed, for any semidiscrete surface $x$ and $\varepsilon > 0$ the quadrilateral mesh

$$M_\varepsilon : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^3 : (k, m) \mapsto x(k, t + \varepsilon m)$$

fulfills $M_\varepsilon(k, m) \to x(k, t)$ as $\varepsilon \to 0$. Motivated by this observation, we study the discrete Laplacian and its action on the mesh $M_\varepsilon$ and carry out the limit $\varepsilon \to 0$ to gain a semidiscrete Laplace operator.

## Results

Given a semidiscrete surface $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ and a function $u : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ we define the semidiscrete Laplacian of $u$ at a point $(k, t)$ to be the limit

$$(\Delta_{\lim} u)(k, t) := \lim_{\varepsilon \to 0} (\mathbb{L}_\varepsilon u)(x(k, t)),$$

where $\mathbb{L}_\varepsilon$ denotes the discrete Laplace operator described by Alexa and Wardetzky [1] on the quadrilateral mesh $M_\varepsilon$. The main result of this paper is stated in Theorem 1.1, where we show that this limit exists at every point of the semidiscrete surface under very mild regularity assumptions. Along with the proof of this theorem we derive a closed-form expression for the semidiscrete Laplacian, which we summarize in Corollary 1.1. Moreover, we reveal a direct relation between the discrete Laplacian described by Alexa and Wardetzky [1] and the discrete scheme described by Liu at al. [24] (see Remark 1.1).

Subsequently, we show that the semidiscrete Laplacian inherits several important properties from the discrete operator. We define an area element and verify that the semidiscrete Laplacian is symmetric and positive semidefinite with respect to the corresponding $L^2$ inner product (see Lemma 1.3). Moreover, we prove that it converges pointwise to the Laplace-Beltrami operator, if the semidiscrete surface converges to a smooth one (see Theorem 1.2). This result particularly implies that the corresponding discrete Laplacian on quadrilateral meshes is a consistent discretization (see Remark 1.3). Additionally, we show that the semidiscrete Laplacian inherits the "linear precision" property from its discrete counterpart, meaning that on planar semidiscrete surfaces, the Laplacian of linear functions vanishes.

The paper is organized as follows. In Section 1.2 we give an overview of Alexa and Wardetzky's Laplacian on general polygonal meshes. In Section 1.3 we specify a semidiscrete Laplacian and state our main result (Theorem 1.1). Its proof is split into two separate sections. First, we thoroughly investigate the restriction of the discrete Laplace operator to the special case of quadrilateral meshes (Section 1.4). Then, we study the discrete Laplacian and its action on the mesh $M_\varepsilon$ and analyze its convergence behavior as $\varepsilon \to 0$ (Section 1.5). In Section 1.6 we study the properties of our semidiscrete Laplace operator. As an example of a possible application, the last section demonstrates the corresponding semidiscrete mean curvature vector field.

## 1.2 A discrete Laplace operator

Alexa and Wardetzky [1] investigate discrete Laplace operators on general polygonal meshes. Their approach is based on the following definition of the Laplace operator on an oriented two-dimensional Riemannian manifold $\mathcal{M}$. Denote by $\Omega^\ell(\mathcal{M})$ the vector space of differential $\ell$-forms on $\mathcal{M}$ and let $d : \Omega^0(\mathcal{M}) \to \Omega^1(\mathcal{M})$ be the exterior derivative. Furthermore, let $d^* : \Omega^1(\mathcal{M}) \to \Omega^0(\mathcal{M})$ be the codifferential, which is the formal adjoint of $d$ with respect to the inner products induced on $\Omega^0(\mathcal{M})$ and $\Omega^1(\mathcal{M})$ by the Riemannian metric (see, e.g., Rosenberg [33] for details). Then, the Laplacian on 0-forms, i.e., real-valued functions, can be defined as

$$\Delta_{\mathcal{M}} := d^* d. \tag{1.1}$$

In the discrete case, $M$ is an oriented 2-manifold mesh with vertex set $V$, edge set $E$, and face set $F$. Here, oriented means that all faces carry an orientation such that any two adjacent faces induce opposite orientations on their common edge. To distinguish between these two orientations, one has to work with oriented half-edges. As $M$ may possess some boundary, one further has to distinguish between the set $E_I$ of inner edges and the set $E_B$ of boundary edges. Thus, one has a total number of $|E| = 2|E_I| + |E_B|$ oriented half-edges associated with $M$.

Using the same notation as in the smooth case, we denote by $\Omega^\ell(M)$ the vector space of discrete $\ell$-forms. Here 0-forms are real values associated with vertices and can therefore be written as elements of $\mathbb{R}^{|V|}$. Analogously, elements of $\mathbb{R}^{|E|}$ can be interpreted as discrete 1-forms on $M$. The discrete counterpart of the exterior derivative is the so-called coboundary operator, which we also denote by $d : \Omega^0(M) \to \Omega^1(M)$. It can be represented by a matrix of dimension $|E| \times |V|$ and acts via

$$(du)(e_{pq}) = u(q) - u(p),$$

for a function $u \in \Omega^0(M)$ and an oriented half-edge $e_{pq}$ from $p$ to $q$. Inner products on the spaces $\Omega^0(M)$ and $\Omega^1(M)$ can be represented by symmetric positive definite matrices $M_0$ and $M_1$ of dimensions $|V| \times |V|$ and $|E| \times |E|$, respectively. Now, for any choice of inner products $M_0$ and $M_1$, the adjoint $d^* : \Omega^1(M) \to \Omega^0(M)$ of $d$ is given by

$$d^* = M_0^{-1} d^T M_1.$$

Motivated by the smooth case, a discrete Laplacian is then defined as

$$\mathbb{L} := d^* d = M_0^{-1} L, \quad \text{with} \quad L := d^T M_1 d.$$

By choosing particular inner products, Alexa and Wardetzky [1] specify discrete Laplacians that satisfy certain properties analogous to the properties of the smooth operator. The present paper is especially concerned with the following features:

- *Locality:* As a differential operator, the smooth Laplacian is a local operator. In the discrete case, locality corresponds to the desirable property of sparsity.
- *Symmetry:* The Laplace-Beltrami operator on a manifold without boundary is self-adjoint with respect to the $L^2$ inner product induced by the Riemannian metric. This property translates to $\mathbb{L}$ being self-adjoint with respect to the inner product induced by $M_0$, i.e., to $L^T = L$.
- *Semidefiniteness:* On a Riemannian manifold without boundary, the Laplacian is positive semidefinite with one-dimensional kernel consisting of the constants. In the discrete case, a corresponding property is achieved by requiring that the matrices $M_0$ and $M_1$ are positive definite, since the kernel of the coboundary operator $d$ is exactly given by the constant functions.
- *Linear precision:* In the smooth case, $\Delta_{\mathcal{M}} u \equiv 0$, whenever $\mathcal{M}$ is contained in a plane and $u : \mathcal{M} \to \mathbb{R}$ is linear. Thus, we require that if all vertices of the mesh $M$ lie in a single plane and the function $u : M \to \mathbb{R}$ is linear, then $\mathbb{L}u(p) = 0$ at each interior vertex $p$. In applications, this property is important for, e.g., mesh parametrization, where an already planar mesh should remain unaltered.

In order to receive a Laplace operator with these properties, Alexa and Wardetzky [1] first introduce the vector area and the maximal projection of a (possibly non-planar) polygon $f$ with $k_f$ vertices. Using their notation, we write

$X_f := \left(x_1^f, \ldots, x_{k_f}^f\right)^T \in \mathbb{R}^{k_f \times 3}$ for the matrix of cyclically ordered vertices along the boundary $\partial f$,

$E_f := \left(\mathbf{e}_1^f, \ldots, \mathbf{e}_{k_f}^f\right)^T \in \mathbb{R}^{k_f \times 3}$ for the matrix of oriented and cyclically ordered half-edges along $\partial f$,

$B_f := \left(\mathbf{b}_1^f, \ldots, \mathbf{b}_{k_f}^f\right)^T \in \mathbb{R}^{k_f \times 3}$ for the matrix of midpoints of each edge.

Now, the vector area $[A_f]$ of $f$ coincides with the Darboux vector of the skew-symmetric $3 \times 3$ matrix $A_f := E_f^T B_f$, meaning that $[A_f] \times x = A_f x$ for all $x \in \mathbb{R}^3$. The magnitude of the vector area, denoted by $|f|$, is the largest area over all orthogonal projections of $f$ to planes of $\mathbb{R}^3$. Thus, a planar polygon $\bar{f}$ is called a maximal projection of $f$, if it is an orthogonal projection of $f$ that has the same vector area as $f$ (cf. Figure 1.4.2). Furthermore, one can show that the vector area is orthogonal to the plane in which $\bar{f}$ lies. Therefore, if we define $\mathbf{n}_f := [A_f]/|f|$ and require that the mentioned plane contains the origin, the vertices $\bar{x}_i^f$ of the maximal projection $\bar{f}$ of $f$ can be calculated as

$$\bar{x}_i^f := x_i^f - \langle x_i^f, \mathbf{n}_f \rangle \mathbf{n}_f, \quad \forall\, i \in \{1, \ldots, k_f\}.$$

After these preparations, we proceed with the construction of the inner product matrix $M_1$. For the sake of locality, Alexa and Wardetzky [1] require that $M_1$ is defined per face. This means that

$$\alpha^T M_1 \beta = \sum_{f \in F} (\alpha|_f)^T M_f \beta|_f, \quad \forall\, \alpha, \beta \in \Omega^1(M),$$

where $M_f \in \mathbb{R}^{k_f \times k_f}$ are symmetric and positive definite matrices and $\alpha|_f \in \mathbb{R}^{k_f}$ denotes the restriction of a 1-form $\alpha \in \Omega^1(M)$ to the $k_f$ oriented half-edges incident with the face $f \in F$.

In particular, they start with the matrices

$$\widetilde{M}_f := \frac{1}{|f|} B_f B_f^T,$$

which are symmetric, but in general only positive semidefinite. In order to obtain positive definite matrices $M_f$, they define the matrices $C_{\bar{f}}$ as $k_f \times (k_f - 2)$ matrices consisting of an orthonormal basis of the null space of $E_{\bar{f}}^T \in \mathbb{R}^{3 \times k_f}$, where $\bar{f}$ is again the maximal projection of $f$. Then, an admissible choice of $M_f$ is given by

$$M_f := \widetilde{M}_f + \lambda\, C_{\bar{f}} C_{\bar{f}}^T,$$

for any $\lambda > 0$. Furthermore, they define the inner product matrix $M_0$ to be a diagonal matrix with

$$(M_0)_{pp} := \sum_{f \ni p} \frac{|f|}{k_f}, \quad \forall\, p \in V. \tag{1.2}$$

Hence, their Laplacian is the composition of two parts encoded by the matrices $\widetilde{M}_f$ and $\lambda\, C_{\bar{f}} C_{\bar{f}}^T$, respectively. In order to analyze these two components separately, we define the decomposition $M_1 := \widetilde{M}_1 + \widehat{M}_1$ by

$$\alpha^T \widetilde{M}_1 \beta := \sum_{f \in F} (\alpha|_f)^T \widetilde{M}_f \beta|_f \quad \text{and} \quad \alpha^T \widehat{M}_1 \beta := \sum_{f \in F} (\alpha|_f)^T \lambda\, C_{\bar{f}} C_{\bar{f}}^T \beta|_f,$$

for all discrete 1-forms $\alpha, \beta \in \Omega^1(M)$. The discrete Laplace operator that we will use henceforth can now be written as

$$\mathbb{L} := \widetilde{\mathbb{L}} + \widehat{\mathbb{L}}, \tag{1.3}$$

where

$$\widetilde{\mathbb{L}} := M_0^{-1} \widetilde{L}, \text{ with } \widetilde{L} := d^T \widetilde{M}_1 d \quad \text{and} \quad \widehat{\mathbb{L}} := M_0^{-1} \widehat{L}, \text{ with } \widehat{L} := d^T \widehat{M}_1 d.$$

## 1.3 A semidiscrete Laplace operator

In this section, we derive a Laplace operator on semidiscrete surfaces from the discrete Laplacian described in Section 1.2. A semidiscrete surface is given by a function

$$x : \mathbb{Z} \times \mathbb{R} \supseteq D \to \mathbb{R}^3 : (k,t) \to x(k,t)$$

possessing one discrete variable and one continuous variable. To illustrate a semidiscrete surface, we connect corresponding points on successive curves by line segments $[x(k,t), x(k+1,t)]$ (see Figure 1.5, left). Throughout this paper, we assume that $x$ is sufficiently often differentiable in the second variable. Moreover, since most of the following considerations are local, we assume that the domain $D$ of $x$ is the entire space $\mathbb{Z} \times \mathbb{R}$. In order to make the upcoming formulas shorter and thus better readable, we use the abbreviations

$$x_1(k,t) := x(k+1,t) \quad \text{and} \quad x_{\bar{1}}(k,t) := x(k-1,t).$$

For the partial derivatives of $x$ with respect to the continuous variable we write $x'$, $x''$, and $x^{(n)}$, $n \in \mathbb{N}$. Finite differences with respect to the discrete variable are denoted by

$$\delta x := x_1 - x.$$

Note that these derivatives commute, so it is natural to use a notation like

$$\delta x'(k,t) = \frac{\partial}{\partial t} x(k+1,t) - \frac{\partial}{\partial t} x(k,t).$$

In addition to these definitions we need the following concepts. A semidiscrete surface is called *regular*, if the sets

$$\{\delta x, x'\}, \quad \{\delta x, x_1'\} \quad \text{and} \quad \{\delta x, x_1' + x'\}$$

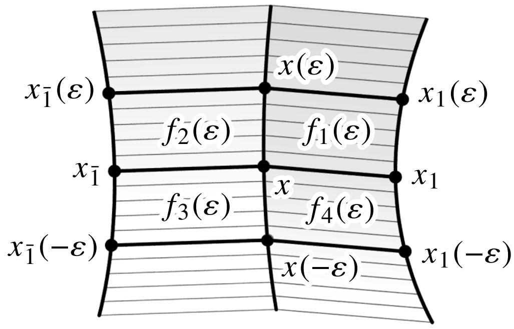


Figure 1.1: The quadrilateral mesh $M_\varepsilon$ around the point $x = x(k,t)$.

are linearly independent for all $(k,t) \in \mathbb{Z}\times\mathbb{R}$. Moreover, a real-valued function $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ is called *smooth* if it is sufficiently often differentiable in the second argument. For the partial derivatives and the finite differences of the mapping $u$ we use the same notation as for the function $x$.

Our aim is to deduce a formula for the Laplacian of a function $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ at every point $x = x(k,t)$ of the semidiscrete surface from the discrete Laplace operator (1.3). For this purpose, we first define a quadrilateral mesh on the semidiscrete surface around the point $x$, then we evaluate the discrete scheme at this mesh, and finally we carry out a limit process in the smooth direction of the semidiscrete surface.

For the above-mentioned quadrilateral mesh around the point $x = x(k,t)$, we define the functions (cf. Figure 1.1)

$$x_{\bar{1}}(\varepsilon) := x(k-1, t+\varepsilon), \quad x(\varepsilon) := x(k, t+\varepsilon), \quad \text{and} \quad x_1(\varepsilon) := x(k+1, t+\varepsilon).$$

Furthermore, for $\varepsilon > 0$, we consider the four quadrilaterals

$$\begin{aligned} f_2(\varepsilon) &:= \big(x, x(\varepsilon), x_{\bar{1}}(\varepsilon), x_{\bar{1}}\big), & f_1(\varepsilon) &:= \big(x, x_1, x_1(\varepsilon), x(\varepsilon)\big), \\ f_3(\varepsilon) &:= \big(x, x_{\bar{1}}, x_{\bar{1}}(-\varepsilon), x(-\varepsilon)\big), & f_4(\varepsilon) &:= \big(x, x(-\varepsilon), x_1(-\varepsilon), x_1\big). \end{aligned}$$

A discrete mesh $M_\varepsilon$ consisting of these quadrilaterals is now well defined. Consequently we consider the discrete Laplace operator of Section 1.2 on the mesh $M_\varepsilon$, which we denote by $\mathbb{L}_\varepsilon$. Obviously any mapping $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ can be interpreted as a real-valued function on the mesh $M_\varepsilon$. Finally we define the semidiscrete Laplacian of a function $u$ at a point $(k,t)$ to be the limit of $(\mathbb{L}_\varepsilon u)(x(k,t))$ as $\varepsilon$ tends to zero.

**Definition 1.1.** Let $x : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^3$ be a regular semidiscrete surface, and let $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ be a smooth function. Then the *semidiscrete Laplacian* of $u$ at a point $(k,t) \in \mathbb{Z}\times\mathbb{R}$ is given by the limit

$$(\Delta_{\lim} u)(k,t) := \lim_{\varepsilon \searrow 0} (\mathbb{L}_\varepsilon u)(x(k,t)), \tag{1.4}$$

where $\mathbb{L}_\varepsilon$ denotes the discrete Laplacian (1.3) on the quadrilateral mesh $M_\varepsilon$.

Under the assumptions of Definition 1.1 the limit $\lim_{\varepsilon\searrow 0}(\mathbb{L}_\varepsilon u)(x)$ exists at every point $x = x(k,t)$ of the semidiscrete surface. We get the following results regarding the convergence rates of the two parts $\widetilde{\mathbb{L}}_\varepsilon$ and $\widehat{\mathbb{L}}_\varepsilon$ of the discrete Laplacian (1.3).

**Theorem 1.1.** *Let $x : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^3$ be a regular semidiscrete surface, and let the function $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ be smooth. Then the limit*

$$(\Delta_{\lim} u)(k,t) = \lim_{\varepsilon\searrow 0}(\mathbb{L}_\varepsilon u)(x(k,t)) \in \mathbb{R}$$

*exists at every point $(k,t) \in \mathbb{Z}\times\mathbb{R}$. In particular, we have*

$$(\widehat{\mathbb{L}}_\varepsilon u)(x(k,t)) = O(\varepsilon), \quad \text{as } \varepsilon \to 0,$$

*and*

$$(\widetilde{\mathbb{L}}_\varepsilon u)(x(k,t)) = (\Delta_{\lim} u)(k,t) + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

*where $\mathbb{L}_\varepsilon = \widetilde{\mathbb{L}}_\varepsilon + \widehat{\mathbb{L}}_\varepsilon$ denotes the discrete Laplace operator* (1.3) *on the quadrilateral mesh $M_\varepsilon$.*

In order to prove Theorem 1.1 we first investigate the discrete Laplacian (1.3) and its action on quadrilateral meshes (see Section 1.4). After that we study the discrete Laplacian on the mesh $M_\varepsilon$ and analyze its convergence behavior as $\varepsilon \to 0$ (see Section 1.5).

## 1.4 The discrete Laplacian on quadrilateral meshes

As a first step toward the proof of Theorem 1.1, we restrict the discrete Laplacian of Section 1.2 to the special case of quadrilateral meshes and analyze its action in detail. Henceforth, let $M = (V, E, F)$ be a quadrilateral mesh in $\mathbb{R}^3$. For a point $x \in V$ with valence 4 and its incident faces $f_j = (x, x_j, y_j, x_{j+1})$, where the indices are to be understood modulo 4 (cf. Figure 1.2), we have

$$E_{f_j} = \left(\mathbf{e}_1^{f_j}, \ldots, \mathbf{e}_4^{f_j}\right)^T = (x_j - x,\ y_j - x_j,\ x_{j+1} - y_j,\ x - x_{j+1})^T \in \mathbb{R}^{4\times 3}, \quad \text{and}$$
$$B_{f_j} = \left(\mathbf{b}_1^{f_j}, \ldots, \mathbf{b}_4^{f_j}\right)^T = \frac{1}{2}(x_j + x,\ y_j + x_j,\ x_{j+1} + y_j,\ x + x_{j+1})^T \in \mathbb{R}^{4\times 3}.$$

The vector area of $f_j$ is, in this case, given by

$$[A_{f_j}] = \frac{1}{2}(x_{j+1} - x_j) \times (x - y_j).$$

Moreover, the coboundary operator for one quadrilateral $f_j$ can be represented by

$$d|_{f_j} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

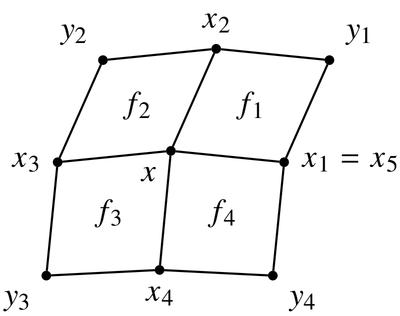


Figure 1.2: The faces $f_j = (x, x_j, y_j, x_{j+1})$ incident with $x$.

We are now going to derive a more explicit formula for the discrete Laplacian of a function $u \in \Omega^0(M)$ at one point $x \in V$. That is, we want to expand

$$(\mathbb{L}u)(x) = (\widetilde{\mathbb{L}}u)(x) + (\widehat{\mathbb{L}}u)(x).$$

### 1.4.1 The first part of the discrete Laplace operator

We begin with the first term $(\widetilde{\mathbb{L}}u)(x)$, and compute

$$\widetilde{L} = d^T \widetilde{M}_1 d = \sum_{f\in F} (d|_f)^T \widetilde{M}_f (d|_f) = \sum_{f\in F} \frac{1}{|f|} (d|_f)^T B_f B_f^T (d|_f).$$

For a quadrilateral $f_j = (x, x_j, y_j, x_{j+1})$ incident with $x$, we have

$$(d|_{f_j})^T B_{f_j} = \frac{1}{2}(x_{j+1} - x_j,\ x - y_j,\ x_{j+1} - x_j,\ y_j - x)^T.$$

Since we are interested in the Laplacian at the point $x$, we do not need to calculate the entire matrix $\frac{1}{|f_j|}(d|_{f_j})^T B_{f_j} B_{f_j}^T d|_{f_j}$, but only the row associated with $x$. This row is given by

$$((d|_{f_j})^T \widetilde{M}_{f_j} d|_{f_j})_x = \frac{1}{4|f_j|}\Big(\|x_{j+1} - x_j\|^2,\ \langle x_{j+1} - x_j,\ x - y_j\rangle, \\ -\|x_{j+1} - x_j\|^2,\ -\langle x_{j+1} - x_j, x - y_j\rangle\Big).$$

Denoting by $\widetilde{L}|_{f_j} \in \mathbb{R}^{4\times 4}$ the restriction of $\widetilde{L}$ to the quadrilateral $f_j$, we arrive at

$$(\widetilde{L}|_{f_j} u|_{f_j})(x) = \frac{1}{4|f_j|}\Big(\|x_{j+1} - x_j\|^2 (u(x) - u(y_j)) + \langle x_{j+1} - x_j, x - y_j\rangle (u(x_j) - u(x_{j+1}))\Big).$$

This formula holds true for all four quadrilaterals $f_j$ incident with the point $x$, so

$$(\widetilde{L}u)(x) = \sum_{j=1}^{4} \frac{1}{4|f_j|}\Big(\|x_{j+1} - x_j\|^2 (u(x) - u(y_j)) + \langle x_{j+1} - x_j, x - y_j\rangle (u(x_j) - u(x_{j+1}))\Big), \tag{1.5}$$

for all functions $u \in \Omega^0(M)$. Recalling definition (1.2) of the inner product matrix $M_0$, we get

$$(\widetilde{\mathbb{L}}u)(x) = \frac{1}{\sum_{j=1}^4 |f_j|} \sum_{j=1}^{4} \frac{1}{|f_j|}\Big(\|x_{j+1} - x_j\|^2 (u(x) - u(y_j)) + \\ +\langle x_{j+1} - x_j, x - y_j\rangle (u(x_j) - u(x_{j+1}))\Big). \tag{1.6}$$

*Remark* 1.1. We would like to point out that this formula agrees exactly with the simplified scheme for the discrete Laplace operator described by Liu et al. [24]. Their approach is based on the following formula for the mean curvature normal of a surface. Fix a point $p$ of a sufficiently smooth surface $\mathcal{M} \subset \mathbb{R}^3$ and denote by $\mathcal{A}_R = \text{area}(R)$ the area of a region $R$ of the surface around $p$ with diameter $\text{diam}(R)$. Then the mean curvature normal at $p$ can be calculated as

$$\mathbf{H}(p) = -\lim_{\text{diam}(R)\to 0} \frac{\int_{q\in R} \nabla\text{area}(q)}{2\mathcal{A}_R}, \tag{1.7}$$

where $\nabla$area is the gradient vector field of the area functional. Furthermore, they use the identity

$$\Delta_{\mathcal{M}}\,\text{id} = -2\mathbf{H} \tag{1.8}$$

in order to produce a discrete scheme for the Laplacian. To gain an approximation of the mean curvature normal in the discrete case, Liu et al. [24] take the four quadrilaterals incident with the point $x \in M$ as an appropriate region around $x$. As these quadrilaterals are in general non-planar, they interpolate each quadrilateral with a bilinear function given by

$$S_{f_j}(u, v) = (1-u)(1-v)x + v(1-u)x_j + u(1-v)x_{j+1} + uvy_j, \quad (u, v) \in [0, 1]^2.$$

Using a one-point numerical integration formula, the area $\mathcal{A}_{f_j}$ of one quadrilateral is approximated by

$$\mathcal{A}_{f_j} = \sqrt{\|S^u_{f_j}\|^2 \|S^v_{f_j}\|^2 - \langle S^u_{f_j}, S^v_{f_j}\rangle^2}, \tag{1.9}$$

where

$$S^u_{f_j} = \frac{x_{j+1} - x_j}{2} + \frac{y_j - x}{2} \quad \text{and} \quad S^v_{f_j} = \frac{x_j - x_{j+1}}{2} + \frac{y_j - x}{2}.$$

An easy calculation shows that (1.9) is exactly the magnitude of the vector area of $f_j$ discussed above. Via equation (1.7) one gets

$$\mathbf{H}(x) = \frac{-1}{2\sum_{j=1}^4 \mathcal{A}_{f_j}} \sum_{j=1}^4 \frac{1}{\mathcal{A}_{f_j}} \Big( \|x_{j+1} - x_j\|^2 (x - y_j) + \langle x_{j+1} - x_j, x - y_j\rangle (x_j - x_{j+1}) \Big).$$

Recalling the relation (1.8), Liu et al. [24] end up with the following formula for the Laplacian of a function $u$ evaluated at one point $x$ of a quadrilateral mesh $M$:

$$\begin{aligned}(\Delta_M u)(x) = \frac{1}{\sum_{j=1}^4 \mathcal{A}_{f_j}} \sum_{j=1}^4 \frac{1}{\mathcal{A}_{f_j}} \Big( &\|x_{j+1} - x_j\|^2 (u(x) - u(y_j)) + \\ &+ \langle x_{j+1} - x_j, x - y_j\rangle (u(x_j) - u(x_{j+1})) \Big).\end{aligned}$$

This scheme agrees with the representation of $(\widetilde{\mathbb{L}}u)(x)$ given in Equation (1.6).

### 1.4.2 The second part of the discrete Laplace operator

Here we expand the second term $(\widehat{\mathbb{L}u})(x)$ of the discrete Laplacian on $M$. That is, we investigate the matrix

$$\widehat{L} = d^T \widehat{M}_1 d = \sum_{f \in F} (d|_f)^T \lambda\, C_{\bar{f}} C_{\bar{f}}^T (d|_f).$$

For every quadrilateral $f_j = (x, x_j, y_j, x_{j+1})$ incident with $x$, the matrix $C_{\bar{f}_j} \in \mathbb{R}^{4\times 2}$ consists of an orthonormal basis of the nullspace of

$$E_{\bar{f}_j}^T = \left(\mathbf{e}_1^{\bar{f}_j}, \ldots, \mathbf{e}_4^{\bar{f}_j}\right) = (\bar{x}_j - \bar{x},\ \bar{y}_j - \bar{x}_j,\ \bar{x}_{j+1} - \bar{y}_j,\ \bar{x} - \bar{x}_{j+1}) \in \mathbb{R}^{3\times 4},$$

where $\bar{f}_j = (\bar{x}, \bar{x}_j, \bar{y}_j, \bar{x}_{j+1})$ is the maximal projection of $f_j$. For the definition of $\bar{f}_j$ we recall that the vector area $[A_{f_j}]$ of $f_j$ is given by

$$[A_{f_j}] = \frac{1}{2}(x_{j+1} - x_j) \times (x - y_j),$$

and that $\mathbf{n}_j = [A_{f_j}]/|f_j|$ is the unit normal vector of the plane in which $\bar{f}_j$ lies. Thus, the vertices of $\bar{f}_j$ can be obtained as

$$\bar{x} = x - \langle x, \mathbf{n}_j\rangle \mathbf{n}_j, \quad \bar{x}_j = x_j - \langle x_j, \mathbf{n}_j\rangle \mathbf{n}_j, \quad \text{and} \quad \bar{y}_j = y_j - \langle y_j, \mathbf{n}_j\rangle \mathbf{n}_j.$$

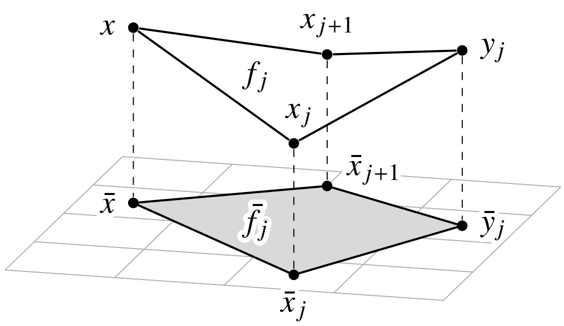


Figure 1.3: A non-planar quadrilateral $f_j$ and its maximal projection $\bar{f}_j$.

Since the edges of $\bar{f}_j$ form a closed loop, the vector $v_1^j := (1, 1, 1, 1)^T$ is obviously in the kernel of $E_{\bar{f}_j}^T$. To receive a basis of this kernel, let us assume henceforth that the edges $\mathbf{e}_1^{\bar{f}_j}$ and $\mathbf{e}_4^{\bar{f}_j}$ are linearly independent, which apparently is not always the case. However, we will see later that this assumption holds true in the setting of Theorem 1.1 for sufficiently small $\varepsilon > 0$. Thus, we can uniquely determine real values $\sigma_j$ and $\omega_j$, such that

$$\sigma_j \mathbf{e}_1^{\bar{f}_j} + \omega_j \mathbf{e}_4^{\bar{f}_j} = -\mathbf{e}_3^{\bar{f}_j},$$

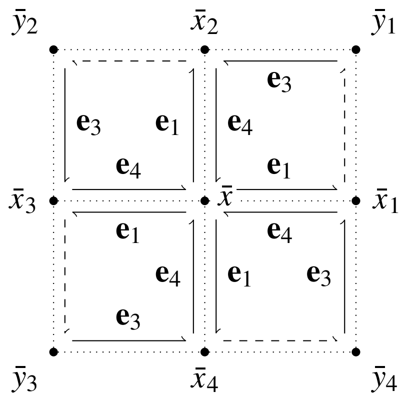


Figure 1.4: Scheme of the half-edges involved in the definitions of $\sigma_j$ and $\omega_j$.

since these edges lie in the same plane (see Figure 1.4). Using the notation

$$\mathrm{e}^j_{k\ell} := \langle \mathbf{e}^{\bar{f}_j}_k, \mathbf{e}^{\bar{f}_j}_\ell \rangle, \quad \forall\, k, \ell \in \{1, \dots, 4\},$$

we get

$$\sigma_j := \frac{\mathrm{e}^j_{14}\mathrm{e}^j_{34} - \mathrm{e}^j_{13}\mathrm{e}^j_{44}}{\mathrm{e}^j_{11}\mathrm{e}^j_{44} - (\mathrm{e}^j_{14})^2} \quad \text{and} \quad \omega_j := \frac{\mathrm{e}^j_{13}\mathrm{e}^j_{14} - \mathrm{e}^j_{11}\mathrm{e}^j_{34}}{\mathrm{e}^j_{11}\mathrm{e}^j_{44} - (\mathrm{e}^j_{14})^2}.$$

A second vector in ker $(E^T_{\bar{f}_j})$ is now given by $v^j_2 := (\sigma_j, 0, 1, \omega_j)^T$.

In the next step, we apply Gram-Schmidt orthonormalization to the basis $\{v^j_1, v^j_2\}$ of the nullspace of $E^T_{\bar{f}_j}$. Normalizing $v^j_1$ yields $\kappa^j_1 := \frac{1}{2}(1,1,1,1)^T$. The second vector transforms to

$$\kappa^j_2 := \frac{1}{2\sqrt{N_j}} \begin{pmatrix} 3\sigma_j - \omega_j - 1 \\ -\sigma_j - \omega_j - 1 \\ -\sigma_j - \omega_j + 3 \\ -\sigma_j + 3\omega_j - 1 \end{pmatrix},$$

where $N_j := 3(\sigma_j^2 + \omega_j^2 + 1) - 2(\sigma_j + \omega_j + \sigma_j\omega_j)$. Hence, one possible choice for the matrix $C_{\bar{f}_j}$ is given by $(\kappa^j_1, \kappa^j_2) \in \mathbb{R}^{4\times 2}$. Notice that the expression $C_{\bar{f}_j} C^T_{\bar{f}_j}$ remains invariant under choosing different orthonormal bases of the nullspace of $E^T_{\bar{f}_j}$, since it represents the orthogonal projection onto the latter subspace. Therefore, also the matrix $\widehat{M}_1$ remains invariant.

In order to calculate the matrix $\widehat{L}$, we first compute

$$C^T_{\bar{f}_j} d|_{f_j} = \frac{2}{\sqrt{N_j}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega_j - \sigma_j & \sigma_j & -1 & 1 - \omega_j \end{pmatrix}.$$

As before, we need only one row of the matrix $(d|_{f_j})^T \lambda\, C_{\bar{f}_j} C_{\bar{f}_j}^T d|_{f_j}$, namely the row associated with $x$. This row is given by

$$((d|_{f_j})^T \lambda\, C_{\bar{f}_j} C_{\bar{f}_j}^T d|_{f_j})_x = \frac{4\lambda}{N_j}\Big((\omega_j - \sigma_j)^2,\ \sigma_j(\omega_j - \sigma_j),\ \sigma_j - \omega_j,\ (1-\omega_j)(\omega_j - \sigma_j)\Big)\,.$$

Writing $\widehat{L}|_{f_j} \in \mathbb{R}^{4\times 4}$ for the restriction of $\widehat{L}$ to the face $f_j$, we get

$$(\widehat{L}|_{f_j} u|_{f_j})(x) = \frac{4\lambda}{N_j}(\omega_j - \sigma_j)\Big(\omega_j\,[u(x) - u(x_{j+1})] + \sigma_j\,[u(x_j) - u(x)] + u(x_{j+1}) - u(y_j)\Big).$$

As this holds for all quadrilaterals $f_j$ incident with $x$, it follows that

$$(\widehat{L}u)(x) = \sum_{j=1}^{4} \frac{4\lambda}{N_j}(\omega_j - \sigma_j)\Big(\omega_j\,[u(x) - u(x_{j+1})] + \sigma_j\,[u(x_j) - u(x)] + u(x_{j+1}) - u(y_j)\Big).$$

Bearing in mind definition (1.2) of the inner product matrix $M_0$, we arrive at

$$\begin{aligned}(\widehat{\mathbb{L}}u)(x) = \frac{16\lambda}{\sum_{j=1}^4 |f_j|} \sum_{j=1}^{4} \frac{\omega_j - \sigma_j}{N_j}\Big(\omega_j\,[u(x) - u(x_{j+1})] + \\ + \sigma_j\,[u(x_j) - u(x)] + u(x_{j+1}) - u(y_j)\Big).\end{aligned} \tag{1.10}$$

As we are going to use this formula in the proof of Theorem 1.1, we now want to study the values $\sigma_j$ and $\omega_j$ (cf. Figure 1.4) in detail. First observe that

$$\mathbf{e}_k^{\bar{f}_j} = \mathbf{e}_k^{f_j} - \langle \mathbf{e}_k^{f_j}, \mathbf{n}_j\rangle \mathbf{n}_j, \quad \forall\, k \in \{1,\ldots,4\}$$

and thus

$$\mathrm{e}_{k\ell}^{j} = \langle \mathbf{e}_k^{\bar{f}_j}, \mathbf{e}_\ell^{\bar{f}_j}\rangle = \langle \mathbf{e}_k^{f_j}, \mathbf{e}_\ell^{f_j}\rangle - \langle \mathbf{e}_k^{f_j}, \mathbf{n}_j\rangle\langle \mathbf{e}_\ell^{f_j}, \mathbf{n}_j\rangle, \quad \forall\, k,\ell \in \{1,\ldots,4\}.$$

By inserting the normal vector

$$\mathbf{n}_j = \frac{1}{2|f_j|}(x_{j+1} - x_j) \times (x - y_j)$$

and using the identity $\langle \mathbf{a}, \mathbf{b}\times\mathbf{c}\rangle = \det(\mathbf{a},\mathbf{b},\mathbf{c})$, we gain

$$\begin{aligned}\langle \mathbf{e}_1^{f_j}, \mathbf{n}_j\rangle &= \frac{1}{2|f_j|}\det\,(x_j - x, y_j - x, x_{j+1} - x), & \langle \mathbf{e}_3^{f_j}, \mathbf{n}_j\rangle &= \frac{1}{2|f_j|}\det\,(x_j - x, y_j - x, x_{j+1} - x),\\ \langle \mathbf{e}_2^{f_j}, \mathbf{n}_j\rangle &= \frac{-1}{2|f_j|}\det\,(x_j - x, y_j - x, x_{j+1} - x), & \langle \mathbf{e}_4^{f_j}, \mathbf{n}_j\rangle &= \frac{-1}{2|f_j|}\det\,(x_j - x, y_j - x, x_{j+1} - x).\end{aligned}$$

Notice that we have to calculate only one determinant per face, which will be very useful later. By setting

$$\det{}_j := \frac{1}{4|f_j|^2}\det\,(x_j - x, y_j - x, x_{j+1} - x)^2$$

the values $\mathrm{e}_{k\ell}^{j}$ can now be written as

$$\mathrm{e}_{k\ell}^{j} = \langle \mathbf{e}_k^{f_j}, \mathbf{e}_\ell^{f_j}\rangle + (-1)^{k+\ell+1}\det{}_j, \quad \forall\, k,\ell \in \{1,\ldots,4\}. \tag{1.11}$$

## 1.5 Proofs of the convergence results

This section provides the proof of Theorem 1.1. As it is rather technical and long, we split it into two Lemmas that deal with the first and the second part of the discrete Laplace operator (1.3) separately. Since the method of proof is the same in both situations, we begin with some preparations.

Recall that for a semidiscrete surface $x$ and $\varepsilon > 0$, we use the abbreviations

$$x_{\bar{1}}(\varepsilon) = x(k-1, t+\varepsilon), \quad x(\varepsilon) = x(k, t+\varepsilon), \quad \text{and} \quad x_1(\varepsilon) = x(k+1, t+\varepsilon).$$

Correspondingly, we define $u_{\bar{1}}(\varepsilon), u(\varepsilon)$, and $u_1(\varepsilon)$ for any function $u : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$.

We consider the Taylor polynomials of degree 4 of $x_{\bar{1}}(\pm\varepsilon)$, $x(\pm\varepsilon)$, and $x_1(\pm\varepsilon)$ at the points $x_{\bar{1}}$, $x$, and $x_1$, respectively. We have

$$\begin{aligned} x(\pm\varepsilon) &= x \pm \varepsilon x' + \frac{1}{2}\varepsilon^2 x'' \pm \frac{1}{6}\varepsilon^3 x^{(3)} + \frac{1}{24}\varepsilon^4 x^{(4)} + O(\varepsilon^5), \\ x_1(\pm\varepsilon) &= x_1 \pm \varepsilon x_1' + \frac{1}{2}\varepsilon^2 x_1'' \pm \frac{1}{6}\varepsilon^3 x_1^{(3)} + \frac{1}{24}\varepsilon^4 x_1^{(4)} + O(\varepsilon^5), \end{aligned} \tag{1.12}$$

and similar terms for $x_{\bar{1}}(\pm\varepsilon)$. Likewise, we expand $u_{\bar{1}}(\pm\varepsilon), u(\pm\varepsilon)$, and $u_1(\pm\varepsilon)$ into Taylor polynomials around $u_{\bar{1}}$, $u$, and $u_1$. In order to prove the desired convergence results, we plug these Taylor polynomials into the discrete schemes (1.6) and (1.10), and expand the resulting expressions. Since the areas $|f_j(\varepsilon)|$ of the four faces

$$\begin{aligned} f_2(\varepsilon) &= (x, x(\varepsilon), x_{\bar{1}}(\varepsilon), x_{\bar{1}}), & f_1(\varepsilon) &= (x, x_1, x_1(\varepsilon), x(\varepsilon)), \\ f_3(\varepsilon) &= (x, x_{\bar{1}}, x_{\bar{1}}(-\varepsilon), x(-\varepsilon)), & f_4(\varepsilon) &= (x, x(-\varepsilon), x_1(-\varepsilon), x_1). \end{aligned}$$

of the quadrilateral mesh $M_\varepsilon$ (cf. Figure 1.1) occur in both parts of the discrete Laplacian, we investigate them separately.

**Proposition 1.1.** *Let $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ be a semidiscrete surface and let $f_j(\varepsilon)$ be the four faces of the quadrilateral mesh $M_\varepsilon$. Then,*

$$\begin{aligned} |f_2(\varepsilon)| &= \frac{\varepsilon}{2}\sqrt{a_{\bar{1}} + \varepsilon b_{\bar{1}} + \varepsilon^2 c_{\bar{1}} + O(\varepsilon^3)}, & |f_1(\varepsilon)| &= \frac{\varepsilon}{2}\sqrt{a_1 + \varepsilon b_1 + \varepsilon^2 c_1 + O(\varepsilon^3)}, \\ |f_3(\varepsilon)| &= \frac{\varepsilon}{2}\sqrt{a_{\bar{1}} - \varepsilon b_{\bar{1}} + \varepsilon^2 c_{\bar{1}} + O(\varepsilon^3)}, & |f_4(\varepsilon)| &= \frac{\varepsilon}{2}\sqrt{a_1 - \varepsilon b_1 + \varepsilon^2 c_1 + O(\varepsilon^3)}, \end{aligned}$$

*where*

$$\begin{aligned}
a_1 &:= \|(\delta x) \times (x_1' + x')\|^2,\\
b_1 &:= \|\delta x\|^2 \langle x_1' + x', x_1'' + x''\rangle - \langle \delta x, x_1' + x'\rangle\langle \delta x, x_1'' + x''\rangle -\\
&\quad - 2\langle \delta x, x'\rangle\langle x_1', x_1' + x'\rangle + 2\langle \delta x, x_1'\rangle\langle x', x_1' + x'\rangle,\\
c_1 &:= \|\delta x\|^2\Big(\frac{1}{4}\|x_1'' + x''\|^2 + \frac{1}{3}\langle x_1' + x', x_1^{(3)} + x^{(3)}\rangle\Big) + \|x_1' \times x'\|^2 +\\
&\quad + \langle \delta x, x_1'\rangle\Big(\langle x', x_1'' + x''\rangle + \langle x'', x_1' + x'\rangle - \frac{1}{3}\langle \delta x, x_1^{(3)} + x^{(3)}\rangle\Big) -\\
&\quad - \langle \delta x, x'\rangle\Big(\langle x_1', x_1'' + x''\rangle + \langle x_1'', x_1' + x'\rangle + \frac{1}{3}\langle \delta x, x_1^{(3)} + x^{(3)}\rangle\Big) +\\
&\quad + \langle \delta x, x_1''\rangle\langle x', x_1' + x'\rangle - \langle \delta x, x''\rangle\langle x_1', x_1' + x'\rangle - \frac{1}{4}\langle \delta x, x_1'' + x''\rangle^2.
\end{aligned}$$

*The terms $a_{\bar 1}, b_{\bar 1}$, and $c_{\bar 1}$ are obtained form $a_1, b_1$, and $c_1$, respectively, by replacing $x_1$ with $x_{\bar 1}$ and $x_1^{(n)}$ with $x_{\bar 1}^{(n)}$ for all $n \in \{1, 2, 3\}$. In particular, we have*

$$\sum_{j=1}^{4} |f_j(\varepsilon)| = \varepsilon\big((a_1)^{\frac{1}{2}} + (a_{\bar 1})^{\frac{1}{2}}\big) + O(\varepsilon^3). \tag{1.13}$$

*Proof.* The area of the first quadrilateral $f_1(\varepsilon) = (x, x_1, x_1(\varepsilon), x(\varepsilon))$ is given by

$$\begin{aligned}
|f_1(\varepsilon)| &= \frac{1}{2}\|(x(\varepsilon) - x_1) \times (x - x_1(\varepsilon))\| =\\
&= \frac{1}{2}\sqrt{\|x(\varepsilon) - x_1\|^2\|x - x_1(\varepsilon)\|^2 - \langle x(\varepsilon) - x_1, x - x_1(\varepsilon)\rangle^2}.
\end{aligned}$$

We insert the Taylor polynomials (1.12) of $x(\varepsilon)$ and $x_1(\varepsilon)$ into this equation, and expand the occurring inner products to receive

$$\begin{aligned}
|f_1(\varepsilon)| &= \frac{1}{2}\varepsilon\sqrt{a_1 + \varepsilon b_1 + \varepsilon^2 c_1 + O(\varepsilon^3)} =\\
&= \frac{1}{2}\varepsilon(a_1)^{1/2} + \frac{1}{4}\varepsilon^2 b_1 (a_1)^{-1/2} + \frac{1}{16}\varepsilon^3(4a_1c_1 - b_1b_1)(a_1)^{-3/2} + O(\varepsilon^4),
\end{aligned}$$

with the stated coefficients $a_1, b_1$, and $c_1$.

Analogously, the area of the second face $f_2(\varepsilon) = (x, x(\varepsilon), x_{\bar 1}(\varepsilon), x_{\bar 1})$ is given by

$$\begin{aligned}
|f_2(\varepsilon)| &= \frac{1}{2}\varepsilon\sqrt{a_{\bar 1} + \varepsilon b_{\bar 1} + \varepsilon^2 c_{\bar 1} + O(\varepsilon^3)} =\\
&= \frac{1}{2}\varepsilon(a_{\bar 1})^{1/2} + \frac{1}{4}\varepsilon^2 b_{\bar 1} (a_{\bar 1})^{-1/2} + \frac{1}{16}\varepsilon^3(4a_{\bar 1}c_{\bar 1} - b_{\bar 1}b_{\bar 1})(a_{\bar 1})^{-3/2} + O(\varepsilon^4),
\end{aligned}$$

where $a_{\bar 1}, b_{\bar 1}$, and $c_{\bar 1}$ are obtained form $a_1, b_1$, and $c_1$ by replacing $x_1$ with $x_{\bar 1}$ and $x_1^{(n)}$ with $x_{\bar 1}^{(n)}$ for all $n \in \{1, 2, 3\}$.

By inserting the Taylor polynomials of $x_{\bar{1}}(-\varepsilon), x(-\varepsilon)$, and $x_1(-\varepsilon)$, the areas of the faces $f_3(\varepsilon)$ and $f_4(\varepsilon)$ expand to

$$\begin{aligned}
|f_3(\varepsilon)| &= \frac{1}{2}\varepsilon\sqrt{a_{\bar{1}} - \varepsilon b_{\bar{1}} + \varepsilon^2 c_{\bar{1}} + O(\varepsilon^3)} = \\
&= \frac{1}{2}\varepsilon(a_{\bar{1}})^{1/2} - \frac{1}{4}\varepsilon^2 b_{\bar{1}}(a_{\bar{1}})^{-1/2} + \frac{1}{16}\varepsilon^3(4a_{\bar{1}}c_{\bar{1}} - b_{\bar{1}}b_{\bar{1}})(a_{\bar{1}})^{-3/2} + O(\varepsilon^4), \\
|f_4(\varepsilon)| &= \frac{1}{2}\varepsilon\sqrt{a_1 - \varepsilon b_1 + \varepsilon^2 c_1 + O(\varepsilon^3)} = \\
&= \frac{1}{2}\varepsilon(a_1)^{1/2} - \frac{1}{4}\varepsilon^2 b_1(a_1)^{-1/2} + \frac{1}{16}\varepsilon^3(4a_1c_1 - b_1b_1)(a_1)^{-3/2} + O(\varepsilon^4).
\end{aligned}$$

These computations immediately yield equation (1.13). □

*Remark* 1.2. Note that the term $\frac{1}{2}(a_1)^{1/2} = \frac{1}{2}\|(\delta x) \times (x_1' + x')\|$ also appears in the work of Karpenkov and Wallner [22], where the authors refer to it as the area of an infinitesimal quadrilateral. On regular semidiscrete surfaces, this expression does not vanish.

After all these preparations, we are ready prove the convergence results of Theorem 1.1.

### 1.5.1 Convergence of the second part

In this subsection, we show that the second part $\widehat{\mathbb{L}}_\varepsilon$ of the discrete Laplacian (1.3) on the quadrilateral mesh $M_\varepsilon$ vanishes as $\varepsilon \to 0$.

**Lemma 1.1.** *Let $x : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^3$ be regular, and let $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ be smooth. Then, at every point $x = x(k,t)$, we have*

$$(\widehat{\mathbb{L}}_\varepsilon u)(x) = O(\varepsilon), \quad \text{as } \varepsilon \to 0,$$

*where $\widehat{\mathbb{L}}_\varepsilon$ denotes the second part of the discrete Laplacian* (1.3) *on the mesh $M_\varepsilon$.*

*Proof.* We insert the Taylor polynomials (1.12) and the corresponding polynomials for the function $u$ into the discrete scheme (1.10) and expand the resulting terms.

First of all we use Equation (1.11) to analyze

$$\sigma_j = \frac{e^j_{14}e^j_{34} - e^j_{13}e^j_{44}}{e^j_{11}e^j_{44} - (e^j_{14})^2} \quad \text{and} \quad \omega_j = \frac{e^j_{13}e^j_{14} - e^j_{11}e^j_{34}}{e^j_{11}e^j_{44} - (e^j_{14})^2}.$$

Then, we investigate the denominators

$$N_j = 3(\sigma_j^2 + \omega_j^2 + 1) - 2(\sigma_j + \omega_j + \sigma_j\omega_j),$$

as well as the terms

$$T_j := (\omega_j - \sigma_j)\Big(\omega_j\left[u(x) - u(x_{j+1})\right] + \sigma_j\left[u(x_j) - u(x)\right] + u(x_{j+1}) - u(y_j)\Big),$$

where we interpret $u$ as a mapping defined on the vertices of the mesh $M_\varepsilon$.

For $j = 1$, i.e., for the first quadrilateral $f_1(\varepsilon) = (x, x_1, x_1(\varepsilon), x(\varepsilon))$, we start by expanding

$$\det_1(\varepsilon) = \frac{1}{4|f_1(\varepsilon)|^2} \det\left(x_1 - x, x_1(\varepsilon) - x, x(\varepsilon) - x\right)^2.$$

Using the same notation as in Proposition 1.1, we have

$$4|f_1(\varepsilon)|^2 = \varepsilon^2 a_1 + \varepsilon^3 b_1 + \varepsilon^4 c_1 + O(\varepsilon^5).$$

Expanding the determinant yields

$$\det\left(x_1 - x, x_1(\varepsilon) - x, x(\varepsilon) - x\right) = \varepsilon^2 d_1 + \varepsilon^3 d_2 + \varepsilon^4 d_3 + O(\varepsilon^5),$$

where

$$\begin{aligned} d_1 &:= \det(x_1 - x, x_1', x'), \quad d_2 := \frac{1}{2}\Big(\det(x_1 - x, x_1'', x') + \det(x_1 - x, x_1', x'')\Big), \\ d_3 &:= \frac{1}{4}\det(x_1 - x, x_1'', x'') + \frac{1}{6}\Big(\det(x_1 - x, x_1', x^{(3)}) + \det(x_1 - x, x_1^{(3)}, x')\Big). \end{aligned}$$

With these expressions at hand, an easy calculation shows that

$$\det_1(\varepsilon) = \varepsilon^2 D_1 + \varepsilon^3 D_2 + \varepsilon^4 D_3 + O(\varepsilon^5),$$

with

$$D_1 := \frac{d_1^2}{a_1}, \quad D_2 := \frac{2d_1 d_2 - b_1 D_1}{a_1}, \quad D_3 := \frac{2d_1 d_3 + d_2^2 - b_1 D_2 - c_1 D_1}{a_1}.$$

Next, we investigate the values $\mathrm{e}^1_{k\ell} = \langle \mathbf{e}_k^{\bar{f}_1}, \mathbf{e}_\ell^{\bar{f}_1} \rangle$ for $k, \ell \in \{1, \ldots, 4\}$. By inserting the Taylor polynomials (1.12), we obtain

$$\begin{aligned} \mathrm{e}^1_{11}(\varepsilon) &= \|\delta x\|^2 - \det_1(\varepsilon) + O(\varepsilon^5), \\ \mathrm{e}^1_{13}(\varepsilon) &= -\|\delta x\|^2 - \varepsilon\langle \delta x, \delta x'\rangle - \frac{\varepsilon^2}{2}\langle \delta x, \delta x''\rangle - \frac{\varepsilon^3}{6}\langle \delta x, \delta x^{(3)}\rangle - \det_1(\varepsilon) + O(\varepsilon^4), \\ \mathrm{e}^1_{14}(\varepsilon) &= -\varepsilon\langle \delta x, x'\rangle - \frac{\varepsilon^2}{2}\langle \delta x, x''\rangle - \frac{\varepsilon^3}{6}\langle \delta x, x^{(3)}\rangle - \frac{\varepsilon^4}{24}\langle \delta x, x^{(4)}\rangle + \det_1(\varepsilon) + O(\varepsilon^5), \\ \mathrm{e}^1_{34}(\varepsilon) &= \varepsilon\langle \delta x, x'\rangle + \varepsilon^2\Big(\langle \delta x', x'\rangle + \frac{1}{2}\langle \delta x, x''\rangle\Big) + \frac{\varepsilon^3}{2}\left(\langle \delta x', x''\rangle + \langle x', \delta x''\rangle\right) + \det_1(\varepsilon) + O(\varepsilon^4), \\ \mathrm{e}^1_{44}(\varepsilon) &= \varepsilon^2\|x'\|^2 + \varepsilon^3\langle x', x''\rangle + \varepsilon^4\Big(\frac{1}{3}\langle x', x^{(3)}\rangle + \frac{1}{4}\|x''\|^2\Big) - \det_1(\varepsilon) + O(\varepsilon^5). \end{aligned}$$

We are now ready to expand $\sigma_1$ and $\omega_1$. Their common denominator expands to

$$\mathrm{cd}_1(\varepsilon) := \mathrm{e}^1_{11}(\varepsilon)\mathrm{e}^1_{44}(\varepsilon) - \left(\mathrm{e}^1_{44}(\varepsilon)\right)^2 = \varepsilon^2\varrho_1 + \varepsilon^3\varrho_2 + O(\varepsilon^4),$$

where

$$\varrho_1 := \|\delta x\|^2(\|x'\|^2 - D_1) - \langle \delta x, x'\rangle^2,\ \ \varrho_2 := \|\delta x\|^2(\langle x', x''\rangle - D_2) + \langle \delta x, x'\rangle(2D_1 - \langle \delta x, x''\rangle).$$

For the nominator of $\sigma_1$ we compute

$$\widetilde{\sigma}_1(\varepsilon) := \mathrm{e}^1_{14}(\varepsilon)\mathrm{e}^1_{34}(\varepsilon) - \mathrm{e}^1_{13}(\varepsilon)\mathrm{e}^1_{44}(\varepsilon) = \varepsilon^2\lambda_1 + \varepsilon^3\lambda_2 + O(\varepsilon^4),$$

with

$$\begin{aligned}
\lambda_1 &:= \|\delta x\|^2(\|x'\|^2 - D_1) - \langle \delta x, x'\rangle^2,\\
\lambda_2 &:= \|\delta x\|^2(\langle x', x''\rangle - D_2) - \langle \delta x, \delta x'\rangle D_1 - \langle \delta x, x'\rangle(\langle \delta x, x''\rangle + \langle x', x_1'\rangle) + \|x'\|^2\langle \delta x, x_1'\rangle.
\end{aligned}$$

Notice that $\lambda_1 = \varrho_1$, which we will use later. Moreover, the nominator of $\omega_1$ expands to

$$\widetilde{\omega}_1(\varepsilon) := \mathrm{e}^1_{13}(\varepsilon)\mathrm{e}^1_{14}(\varepsilon) - \mathrm{e}^1_{11}(\varepsilon)\mathrm{e}^1_{34}(\varepsilon) = \varepsilon^2\mu_1 + \varepsilon^3\mu_2 + O(\varepsilon^4),$$

where

$$\begin{aligned}
\mu_1 &:= -\|\delta x\|^2(\langle \delta x', x'\rangle + 2D_1) + \langle \delta x, x'\rangle\langle \delta x, \delta x'\rangle,\\
\mu_2 &:= -\|\delta x\|^2\left(\frac{\langle \delta x', x''\rangle + \langle x', \delta x''\rangle}{2} + 2D_2\right) - \langle \delta x, \delta x'\rangle D_1 +\\
&\quad + \frac{1}{2}\Big(\langle \delta x, x'\rangle\langle \delta x, \delta x''\rangle + \langle \delta x, x''\rangle\langle \delta x, \delta x'\rangle\Big).
\end{aligned}$$

We can now investigate the first summand of the discrete scheme (1.10). For this purpose, we extract the common denominator $\mathrm{cd}_1(\varepsilon)$ of $\sigma_1(\varepsilon)$ and $\omega_1(\varepsilon)$, and write

$$N_1(\varepsilon) = 3(\sigma_1(\varepsilon)^2 + \omega_1(\varepsilon)^2 + 1) - 2(\sigma_1(\varepsilon) + \omega_1(\varepsilon) + \sigma_1(\varepsilon)\omega_1(\varepsilon)) = \frac{1}{\mathrm{cd}_1(\varepsilon)^2}\widetilde{N}_1(\varepsilon),$$

with

$$\widetilde{N}_1(\varepsilon) := 3\Big(\widetilde{\sigma}_1(\varepsilon)^2 + \widetilde{\omega}_1(\varepsilon)^2 + \mathrm{cd}_1(\varepsilon)^2\Big) - 2\Big(\mathrm{cd}_1(\varepsilon)\widetilde{\sigma}_1(\varepsilon) + \mathrm{cd}_1(\varepsilon)\widetilde{\omega}_1(\varepsilon) + \widetilde{\sigma}_1(\varepsilon)\widetilde{\omega}_1(\varepsilon)\Big).$$

Likewise, we write

$$T_1(\varepsilon) = (\omega_1(\varepsilon) - \sigma_1(\varepsilon))\Big(\omega_1(\varepsilon)[u - u(\varepsilon)] + \sigma_1(\varepsilon)[u_1 - u] + u(\varepsilon) - u_1(\varepsilon)\Big) = \frac{1}{\mathrm{cd}_1(\varepsilon)^2}\widetilde{T}_1(\varepsilon),$$

with

$$\widetilde{T}_1(\varepsilon) := (\widetilde{\omega}_1(\varepsilon) - \widetilde{\sigma}_1(\varepsilon))\Big(\widetilde{\omega}_1(\varepsilon)[u - u(\varepsilon)] + \widetilde{\sigma}_1(\varepsilon)[u_1 - u] + \mathrm{cd}_1(\varepsilon)[u(\varepsilon) - u_1(\varepsilon)]\Big).$$

Therefore, we have

$$\frac{1}{N_1(\varepsilon)}T_1(\varepsilon) = \frac{1}{\widetilde{N}_1(\varepsilon)}\widetilde{T}_1(\varepsilon).$$

Expanding $\widetilde{N}_1(\varepsilon)$ yields $\widetilde{N}_1(\varepsilon) = \varepsilon^4\eta_1 + \varepsilon^5\eta_2 + O(\varepsilon^6)$, where

$$\eta_1 := 3(\lambda_1^2 + \mu_1^2 + \varrho_1^2) - 2(\varrho_1\lambda_1 + \varrho_1\mu_1 + \lambda_1\mu_1),$$
$$\eta_2 := 6(\lambda_1\lambda_2 + \mu_1\mu_2 + \varrho_1\varrho_2) - 2(\varrho_1\lambda_2 + \varrho_2\lambda_1 + \varrho_1\mu_2 + \varrho_2\mu_1 + \lambda_1\mu_2 + \lambda_2\mu_1).$$

For $\widetilde{T}_1(\varepsilon)$ we additionally have to insert the corresponding Taylor polynomials for $u$ to gain $\widetilde{T}_1(\varepsilon) = \varepsilon^4\theta_1 + \varepsilon^5\theta_2 + O(\varepsilon^6)$, with

$$\theta_1 := ((\mu_1 - \lambda_1)(\lambda_1 - \varrho_1))\delta u = 0,$$
$$\theta_2 := ((\mu_1 - \lambda_1)(\lambda_2 - \varrho_2))\delta u + (\lambda_1^2 - \lambda_1\mu_1)u_1' - (\lambda_1 - \mu_1)^2 u',$$

where we have used the fact that $\lambda_1 = \varrho_1$. This finally leads us to

$$\frac{1}{N_1(\varepsilon)}T_1(\varepsilon) = \varepsilon\frac{\theta_2}{\eta_1} + O(\varepsilon^2).$$

Analogous computations have to be done for the three remaining quadrilaterals. As the method is exactly the same, we are going to omit the details and write down the results only.

For the second quadrilateral, $f_2(\varepsilon) = (x, x(\varepsilon), x_{\bar{1}}(\varepsilon), x_{\bar{1}})$, we expand

$$N_2(\varepsilon) = 3(\sigma_2(\varepsilon)^2 + \omega_2(\varepsilon)^2 + 1) - 2(\sigma_2(\varepsilon) + \omega_2(\varepsilon) + \sigma_2(\varepsilon)\omega_2(\varepsilon))$$

and

$$T_2(\varepsilon) = (\omega_2(\varepsilon) - \sigma_2(\varepsilon))\Big(\omega_2(\varepsilon)[u(x) - u(x_{\bar{1}})] + \sigma_2(\varepsilon)[u(x(\varepsilon)) - u(x)] + u(x_{\bar{1}}) - u(x_{\bar{1}}(\varepsilon))\Big)$$

to obtain

$$\frac{1}{N_2(\varepsilon)}T_2(\varepsilon) = \varepsilon\frac{\theta_{\bar{2}}}{\eta_{\bar{1}}} + O(\varepsilon^2),$$

where $\theta_{\bar{2}}$ and $\eta_{\bar{1}}$ are obtained from $\theta_2$ and $\eta_1$, respectively, by replacing $x_1^{(n)}$ with $x_{\bar{1}}^{(n)}$, as well as $u_1^{(n)}$ with $u_{\bar{1}}^{(n)}$ for all $n \in \{0, 1, 2\}$. Here, we have to remark that the individual coefficients of the terms $\sigma_2(\varepsilon)$ and $\omega_2(\varepsilon)$ cannot be obtained from the corresponding coefficients of $\sigma_1(\varepsilon)$ and $\omega_1(\varepsilon)$ by the aforementioned substitutions. The reason for this is that the half-edges used in the definitions of $\sigma_j(\varepsilon)$ and $\omega_j(\varepsilon)$ are not placed symmetrically (cf. Figure 1.4). Nevertheless, the coefficients of the terms $N_2(\varepsilon)$ and $T_2(\varepsilon)$ can indeed be obtained in this manner.

Likewise, for the third face $f_3(\varepsilon)$ and the fourth face $f_4(\varepsilon)$, we gain

$$\frac{1}{N_3(\varepsilon)}T_3(\varepsilon) = -\varepsilon\frac{\theta_{\bar{2}}}{\eta_{\bar{1}}} + O(\varepsilon^2) \quad \text{and} \quad \frac{1}{N_4(\varepsilon)}T_4(\varepsilon) = -\varepsilon\frac{\theta_2}{\eta_1} + O(\varepsilon^2).$$

Combining all these results, we finally conclude that

$$\sum_{j=1}^{4} \frac{1}{N_j(\varepsilon)}T_j(\varepsilon) = O(\varepsilon^2).$$

Together with equation (1.13) this yields

$$(\widehat{\mathbb{L}}_\varepsilon u)(x) = \frac{16\lambda}{\sum_{j=1}^4 |f_j(\varepsilon)|}\sum_{j=1}^{4}\frac{1}{N_j(\varepsilon)}T_j(\varepsilon) = O(\varepsilon). \qquad \square$$

### 1.5.2 Convergence of the first part

Here, we analyze the convergence behavior of the first part $\widetilde{\mathbb{L}}_\varepsilon$ of the discrete Laplace operator (1.3) on the quadrilateral mesh $M_\varepsilon$. By inserting the Taylor polynomials (1.12) into the discrete scheme (1.6) we gain the proposed convergence result and simultaneously derive a closed-form expression for our semidiscrete Laplace operator (see Corollary 1.1).

**Lemma 1.2.** *Let $x : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^3$ be regular, and let $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ be smooth. Then, the limit*

$$(\Delta_{\lim} u)(k,t) = \lim_{\varepsilon\searrow 0} (\mathbb{L}_\varepsilon u)(x(k,t)) = \lim_{\varepsilon\searrow 0} (\widetilde{\mathbb{L}}_\varepsilon u)(x(k,t)) \in \mathbb{R}$$

*exists at every point $(k,t) \in \mathbb{Z}\times\mathbb{R}$. In particular, we have*

$$(\widetilde{\mathbb{L}}_\varepsilon u)(x(k,t)) = (\Delta_{\lim} u)(k,t) + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

*where $\widetilde{\mathbb{L}}_\varepsilon$ is the first part of the discrete Laplace operator* (1.3) *on the mesh $M_\varepsilon$.*

*Proof.* Recall that the discrete Laplace operator (1.3) on the mesh $M_\varepsilon$ is given by the decomposition $\mathbb{L}_\varepsilon = \widetilde{\mathbb{L}}_\varepsilon + \widehat{\mathbb{L}}_\varepsilon$. According to Lemma 1.1 we have

$$\lim_{\varepsilon\searrow 0} (\widehat{\mathbb{L}}_\varepsilon u)(x(k,t)) = 0$$

at every point $x = x(k,t)$. Hence

$$(\Delta_{\lim} u)(k,t) = \lim_{\varepsilon\searrow 0} (\mathbb{L}_\varepsilon u)(x(k,t)) = \lim_{\varepsilon\searrow 0} (\widetilde{\mathbb{L}}_\varepsilon u)(x(k,t)),$$

if the latter limit exists. In order to show the stated convergence behavior, we first investigate the four summands of the discrete scheme (1.6) separately.

For $j = 1$, i.e., for the face $f_1(\varepsilon) = (x, x_1, x_1(\varepsilon), x(\varepsilon))$, we expand

$$S_1(\varepsilon) := \|x(\varepsilon) - x_1\|^2(u - u_1(\varepsilon)) + \langle x(\varepsilon) - x_1, x - x_1(\varepsilon)\rangle(u_1 - u(\varepsilon)).$$

Using the Taylor polynomials (1.12) and the corresponding polynomials for $u$, we get

$$S_1(\varepsilon) = \varepsilon\alpha_1 + \varepsilon^2\beta_1 + \varepsilon^3\gamma_1 + O(\varepsilon^4),$$

where

$$\begin{aligned}
\alpha_1 &:= \langle \delta x, x_1' + x'\rangle\delta u - \|\delta x\|^2(u_1' + u'),\\
\beta_1 &:= \Big(\frac{1}{2}\langle \delta x, x_1'' + x''\rangle - \langle x', x_1' + x'\rangle\Big)\delta u +\\
&\quad + 2\langle \delta x, x'\rangle u_1' - \langle \delta x, x_1' - x'\rangle u' - \frac{1}{2}\|\delta x\|^2(u_1'' + u''),\\
\gamma_1 &:= \Big(\frac{1}{6}\langle \delta x, x_1^{(3)} + x^{(3)}\rangle - \frac{1}{2}\langle x_1' + x', x''\rangle - \frac{1}{2}\langle x', x_1'' + x''\rangle\Big)\delta u +\\
&\quad + (\langle \delta x, x''\rangle - \|x'\|^2)u_1' + \Big(\langle x', x_1'\rangle - \frac{1}{2}\langle \delta x, x_1'' - x''\rangle\Big)u' +\\
&\quad + \langle \delta x, x'\rangle u_1'' - \frac{1}{2}\langle \delta x, x_1' - x'\rangle u'' - \frac{1}{6}\|\delta x\|^2(u_1^{(3)} + u^{(3)}).
\end{aligned}$$

For the second quadrilateral, $f_2(\varepsilon) = \big(x, x(\varepsilon), x_{\bar{1}}(\varepsilon), x_{\bar{1}}\big)$, we expand

$$S_2(\varepsilon) := \|x_{\bar{1}} - x(\varepsilon)\|^2 (u - u_{\bar{1}}(\varepsilon)) + \langle x_{\bar{1}} - x(\varepsilon), x - x_{\bar{1}}(\varepsilon)\rangle (u(\varepsilon) - u_{\bar{1}})$$

and receive

$$S_2(\varepsilon) = \varepsilon\alpha_{\bar{1}} + \varepsilon^2\beta_{\bar{1}} + \varepsilon^3\gamma_{\bar{1}} + O(\varepsilon^4),$$

where $\alpha_{\bar{1}}, \beta_{\bar{1}}, \gamma_{\bar{1}}$ are obtained form $\alpha_1, \beta_1, \gamma_1$ by replacing $x_1^{(n)}$ with $x_{\bar{1}}^{(n)}$, as well as $u_1^{(n)}$ with $u_{\bar{1}}^{(n)}$ for all $n \in \{0, 1, 2, 3\}$. For the third and fourth face we define

$$\begin{aligned} S_3(\varepsilon) &:= \|x(-\varepsilon) - x_{\bar{1}}\|^2 (u - u_{\bar{1}}(-\varepsilon)) + \langle x(-\varepsilon) - x_{\bar{1}}, x - x_{\bar{1}}(-\varepsilon)\rangle (u_{\bar{1}} - u(-\varepsilon)) \\ S_4(\varepsilon) &:= \|x_1 - x(-\varepsilon)\|^2 (u - u_1(-\varepsilon)) + \langle x_1 - x(-\varepsilon), x - x_1(-\varepsilon)\rangle (u(-\varepsilon) - u_1) \end{aligned}$$

and get

$$\begin{aligned} S_3(\varepsilon) &= -\varepsilon\alpha_{\bar{1}} + \varepsilon^2\beta_{\bar{1}} - \varepsilon^3\gamma_{\bar{1}} + O(\varepsilon^4), \\ S_4(\varepsilon) &= -\varepsilon\alpha_1 + \varepsilon^2\beta_1 - \varepsilon^3\gamma_1 + O(\varepsilon^4). \end{aligned}$$

The first part of the discrete Laplacian (1.3) on the mesh $M_\varepsilon$ now reads

$$(\widetilde{\mathbb{L}_\varepsilon} u)(x) = \frac{1}{\sum_{j=1}^4 |f_j(\varepsilon)|} \sum_{j=1}^4 \frac{1}{|f_j(\varepsilon)|} S_j(\varepsilon).$$

To show the desired convergence result, we write

$$\begin{aligned} \sum_{j=1}^4 \frac{1}{|f_j(\varepsilon)|} S_j(\varepsilon) = \frac{1}{|f_1(\varepsilon)||f_4(\varepsilon)|}\Big(|f_4(\varepsilon)|S_1(\varepsilon) + |f_1(\varepsilon)|S_4(\varepsilon)\Big) + \\ + \frac{1}{|f_2(\varepsilon)||f_3(\varepsilon)|}\Big(|f_3(\varepsilon)|S_2(\varepsilon) + |f_2(\varepsilon)|S_3(\varepsilon)\Big) \end{aligned}$$

and compute

$$|f_1(\varepsilon)||f_4(\varepsilon)| = \frac{1}{4}\varepsilon^2 a_1 + O(\varepsilon^4), \qquad |f_2(\varepsilon)||f_3(\varepsilon)| = \frac{1}{4}\varepsilon^2 a_{\bar{1}} + O(\varepsilon^4),$$

and

$$\begin{aligned} |f_4(\varepsilon)|S_1(\varepsilon) + |f_1(\varepsilon)|S_4(\varepsilon) &= \varepsilon^3 \left( (a_1)^{1/2}\beta_1 - \frac{b_1\alpha_1}{2(a_1)^{1/2}} \right) + O(\varepsilon^5), \\ |f_3(\varepsilon)|S_2(\varepsilon) + |f_2(\varepsilon)|S_3(\varepsilon) &= \varepsilon^3 \left( (a_{\bar{1}})^{1/2}\beta_{\bar{1}} - \frac{b_{\bar{1}}\alpha_{\bar{1}}}{2(a_{\bar{1}})^{1/2}} \right) + O(\varepsilon^5). \end{aligned}$$

Thus, we have

$$\sum_{j=1}^4 \frac{1}{|f_j(\varepsilon)|} S_j(\varepsilon) = 4\varepsilon \left( \frac{\beta_1}{(a_1)^{1/2}} + \frac{\beta_{\bar{1}}}{(a_{\bar{1}})^{1/2}} - \frac{1}{2}\left( \frac{b_1\alpha_1}{(a_1)^{3/2}} + \frac{b_{\bar{1}}\alpha_{\bar{1}}}{(a_{\bar{1}})^{3/2}} \right) \right) + O(\varepsilon^3). \tag{1.14}$$

Together with Equation (1.13) we finally gain

$$(\widetilde{\mathbb{L}}_\varepsilon u)(x) = \frac{4}{(a_1)^{1/2} + (a_{\bar{1}})^{1/2}} \left( \frac{\beta_1}{(a_1)^{1/2}} + \frac{\beta_{\bar{1}}}{(a_{\bar{1}})^{1/2}} - \frac{1}{2} \left( \frac{b_1 \alpha_1}{(a_1)^{3/2}} + \frac{b_{\bar{1}} \alpha_{\bar{1}}}{(a_{\bar{1}})^{3/2}} \right) \right) + O(\varepsilon^2). \qquad \square$$

The proof of Lemma 1.2 immediately yields the following closed-form expression for the semidiscrete Laplacian.

**Corollary 1.1.** *Let $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ be a regular semidiscrete surface, and let the function $u : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be smooth. Then, the semidiscrete Laplacian of $u$ at the point $(k,t) \in \mathbb{Z} \times \mathbb{R}$ is given by*

$$(\Delta_{\lim} u)(k,t) = \frac{4}{A_1 + A_{\bar{1}}} \left( \frac{\beta_1}{A_1} + \frac{\beta_{\bar{1}}}{A_{\bar{1}}} - \frac{1}{2} \left( \frac{\alpha_1 \cdot B_1}{(A_1)^3} + \frac{\alpha_{\bar{1}} \cdot B_{\bar{1}}}{(A_{\bar{1}})^3} \right) \right), \tag{1.15}$$

*where*

$$\begin{aligned} A_1 &:= \|(\delta x) \times (x_1' + x')\|, \\ B_1 &:= \|\delta x\|^2 \langle x_1' + x', x_1'' + x'' \rangle - \langle \delta x, x_1' + x' \rangle \langle \delta x, x_1'' + x'' \rangle - \\ &\quad - 2 \langle \delta x, x' \rangle \langle x_1', x_1' + x' \rangle + 2 \langle \delta x, x_1' \rangle \langle x', x_1' + x' \rangle \end{aligned}$$

*and*

$$\begin{aligned} \alpha_1 &:= \langle \delta x, x_1' + x' \rangle \delta u - \|\delta x\|^2 (u_1' + u'), \\ \beta_1 &:= \left( \frac{1}{2} \langle \delta x, x_1'' + x'' \rangle - \langle x', x_1' + x' \rangle \right) \delta u - \frac{1}{2} \|\delta x\|^2 (u_1'' + u'') - \langle \delta x, x_1' - x' \rangle u' + 2 \langle \delta x, x' \rangle u_1'. \end{aligned}$$

*The terms $A_{\bar{1}}$ and $B_{\bar{1}}$ are obtained from $A_1$ and $B_1$ by replacing $x_1, x_1'$, and $x_1''$ with $x_{\bar{1}}, x_{\bar{1}}'$, and $x_{\bar{1}}''$, respectively. To obtain $\alpha_{\bar{1}}$ and $\beta_{\bar{1}}$ from $\alpha_1$ and $\beta_1$ one further has to replace $u_1, u_1'$, and $u_1''$ with $u_{\bar{1}}, u_{\bar{1}}'$, and $u_{\bar{1}}''$.*

**Example 1.1.** As an example, we want to demonstrate the convergence behavior of the discrete Laplacian $\mathbb{L}_\varepsilon$ on the semidiscrete surface

$$x(k,t) := \left( k/5,\ t,\ \sin(\pi k/5) \sin(\pi t) e^{-(k/5)^2 - t^2} \right)^T$$

(see Figure 1.5, left). For $\varepsilon \in \{2^{-9}, 2^{-10}, \dots\}$ we compute the maximal error between the discrete Laplacian $\mathbb{L}_\varepsilon$ and the semidiscrete Laplacian $\Delta_{\lim}$ of the function

$$u := p_1(x)^3 + p_2(x)^3 + p_3(x)^3,$$

where $p_n(x)$ denotes the $n$-th coordinate function of $x$. We evaluate the error at the points $(k_i, t_j)$ with $k_i \in \{-4, \dots, 4\}$ and $t_j \in \{-\frac{19}{20}, -\frac{18}{20}, \dots, \frac{18}{20}, \frac{19}{20}\}$. For the computation of the discrete Laplace operator $\mathbb{L}_\varepsilon$ we use the quadrilateral meshes $M_\varepsilon$ as defined at the beginning of Section 1.3 (cf. Figure 1.1) around the points $x(k_i, t_j)$ . Besides $\mathbb{L}_\varepsilon$ we also compute $\widetilde{\mathbb{L}}_\varepsilon$ in order to observe the different convergence rates. As one can see in Figure 1.5, we achieve exactly the expected behavior.

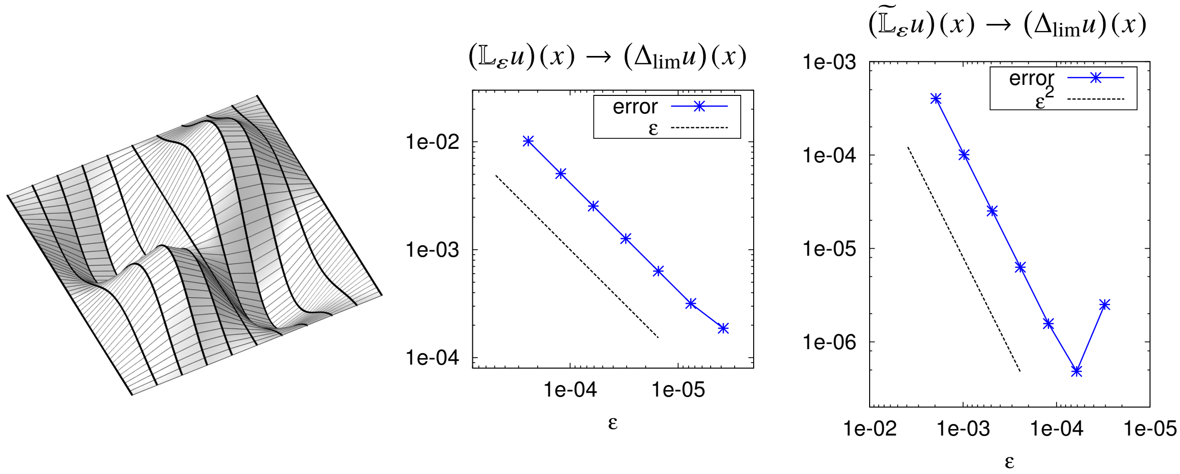


Figure 1.5: *Left:* The semidiscrete surface $x(k,t)$ for $k \in \{-5,\dots,5\}$ and $t \in [-1,1]$. *Center:* The maximal error $|(\mathbb{L}_\varepsilon u)(x(k_i,t_j)) - (\Delta_{\lim} u)(k_i,t_j)|$ for $\varepsilon \in \{2^{-12},\dots,2^{-18}\}$, where $u = p_1(x)^3 + p_2(x)^3 + p_3(x)^3$. *Right:* The maximal error $|(\widetilde{\mathbb{L}}_\varepsilon u)(x(k_i,t_j)) - (\Delta_{\lim} u)(k_i,t_j)|$ for $\varepsilon \in \{2^{-9},\dots,2^{-15}\}$.

## 1.6 Properties of the semidiscrete Laplacian

There are quite a few important properties which the classical Laplacian enjoys. It is a very interesting question which of these carry over to discrete or semidiscrete Laplace operators. For instance, Wardetzky et al. [42] show that certain of these properties are incompatible for Laplacians on triangle meshes. In the following, we discuss corresponding properties of the semidiscrete Laplace operator.

### 1.6.1 Locality

The smooth Laplace operator on a Riemannian manifold $\mathcal{M}$ is *local* in the sense that the value $(\Delta_{\mathcal{M}} u)(p)$ is independent of both the values of $u$ and the properties of $\mathcal{M}$ outside an arbitrarily small open neighborhood of $p$. The semidiscrete Laplacian fulfills an analogous property: For any $(k,t)$ the term $(\Delta_{\lim} u)(k,t)$ only depends on the values of $x$ and $u$ at arbitrarily small pieces of the consecutive lines $\{(k-1,s) : s \in \mathbb{R}\}$, $\{(k,s) : s \in \mathbb{R}\}$, and $\{(k+1,s) : s \in \mathbb{R}\}$.

### 1.6.2 Linear precision

If the manifold $\mathcal{M}$ is contained in a plane and the function $u : \mathcal{M} \to \mathbb{R}$ is linear, then $\Delta_{\mathcal{M}} u \equiv 0$. The discrete Laplace operator has the same property, i.e., if all vertices of the mesh $M$ lie in a single plane and the function $u : M \to \mathbb{R}$ is linear, then $(\mathbb{L}u)(x) = 0$ at each interior point $x \in M$.

This property immediately carries over to the semidiscrete case. Indeed, let $x : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^3$ be a planar regular semidiscrete surface and let $u : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ be linear. Then, at every point

$x(k,t)$ and for every $\varepsilon > 0$ the quadrilateral mesh $M_\varepsilon$ (cf. Figure 1.1) is contained in the same plane. Since $x$ is an interior point of the mesh $M_\varepsilon$ and the restriction of $u$ to $M_\varepsilon$ is linear as well, we have $(\mathbb{L}_\varepsilon u)(x(k,t)) = 0$. Hence,

$$(\Delta_{\lim} u)(k,t) = \lim_{\varepsilon \searrow 0} (\mathbb{L}_\varepsilon u)(x(k,t)) = 0.$$

Moreover, the kernel of the Laplace operator on an arbitrary Riemannian manifold always contains the constant functions. It is straightforward to see that this property is maintained by the semidiscrete Laplacian, i.e., $\Delta_{\lim} u \equiv 0$, whenever $u$ is constant.

### 1.6.3 Symmetry and positive semidefiniteness

On a Riemannian manifold $\mathcal{M}$ without boundary, the Laplacian is symmetric and positive semidefinite with respect to the $L^2$ inner product induced on $\Omega^0(\mathcal{M})$ by the Riemannian metric (see, e.g., Rosenberg [33]). In this subsection, we show that the semidiscrete Laplace operator has an equivalent property.

Recalling the definitions

$$A_1 = \|(x_1 - x) \times (x_1' + x')\| \quad \text{and} \quad A_{\bar{1}} = \|(x_{\bar{1}} - x) \times (x_{\bar{1}}' + x')\|$$

form Corollary 1.1, we define the area element (cf. Remark 1.2)

$$d\mathfrak{A} := \frac{A_1 + A_{\bar{1}}}{4}\, d(\mu \otimes \lambda),$$

where $\mu$ denotes the counting measure on $\mathbb{Z}$ and $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$. Next, we consider the corresponding $L^2$ inner product

$$\langle u, v\rangle_{L^2(x)} := \int_{\mathbb{Z}\times\mathbb{R}} u(k,t)v(k,t)\, d\mathfrak{A}(k,t).$$

This scalar product is intrinsic, because it is easy to verify that for all functions $u, v$ and all parameter transformations $\varphi(k,t) = (k_0 \pm k, \phi(t))$ we have

$$\langle u \circ \varphi, v \circ \varphi\rangle_{L^2(x\circ\varphi)} = \langle u, v\rangle_{L^2(x)}.$$

The following Lemma tells us that the semidiscrete Laplace operator is symmetric and positive semidefinite with respect to this inner product.

**Lemma 1.3.** *Let $x : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}^3$ be regular and let $u, v : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ be smooth and compactly supported. Then,*

$$\langle \Delta_{\lim} u, v\rangle_{L^2(x)} = \langle u, \Delta_{\lim} v\rangle_{L^2(x)} \quad \textit{and} \quad \langle \Delta_{\lim} u, u\rangle_{L^2(x)} \geq 0,$$

*i.e., the semidiscrete Laplace operator is symmetric and positive semidefinite.*

*Proof.* Since $u$ and $v$ are compactly supported, there exist $K \in \mathbb{N}$ and $T \in \mathbb{R}_+$ such that $\operatorname{supp}(u), \operatorname{supp}(v) \subset \{-K, \dots, K\} \times [-T, T]$. Hence, by Fubini's theorem, we have

$$\langle \Delta_{\lim} u, v \rangle_{L^2(x)} = \sum_{k=-K}^{K} \int_{-T}^{T} (\Delta_{\lim} u)(k,t) v(k,t) \frac{A_1(k,t) + A_{\bar{1}}(k,t)}{4} \, dt.$$

Using the same notation as in Corollary 1.1 we define

$$(\Lambda_{\lim} u)(k,t) := \left( \frac{\beta_1}{A_1} + \frac{\beta_{\bar{1}}}{A_{\bar{1}}} - \frac{1}{2} \left( \frac{\alpha_1 \cdot B_1}{(A_1)^3} + \frac{\alpha_{\bar{1}} \cdot B_{\bar{1}}}{(A_{\bar{1}})^3} \right) \right),$$

so that

$$\int_{-T}^{T} (\Delta_{\lim} u)(k,t) v(k,t) \frac{A_1(k,t) + A_{\bar{1}}(k,t)}{4} \, dt = \int_{-T}^{T} (\Lambda_{\lim} u)(k,t) v(k,t) \, dt.$$

From the boundedness of the considered functions, it follows that for every $h > 0$ there exists an $\varepsilon > 0$ such that for every $k \in \{-K, \dots, K\}$

$$\int_{-T}^{T} (\Lambda_{\lim} u)(k,t) v(k,t) \, dt = \sum_{n=-N_\varepsilon}^{N_\varepsilon} \varepsilon (\Lambda_{sd} u)(k, \varepsilon n) v(k, \varepsilon n) + O(h),$$

with $N_\varepsilon := \left\lceil \frac{T}{\varepsilon} \right\rceil$. Let us now consider the quadrilateral mesh

$$\mathfrak{M}_\varepsilon : \{-K, \dots, K\} \times \{-N_\varepsilon, \dots, N_\varepsilon\} \to \mathbb{R}^3 : (k, n) \mapsto x(k, \varepsilon n),$$

which is a global version of the mesh $M_\varepsilon$ of Section 1.3.

From the proof of Lemma 1.2 we know that

$$(\widetilde{L}_\varepsilon u)(x(k, \varepsilon n)) = \sum_{j=1}^{4} \frac{1}{4 |f_j(\varepsilon)|} S_j(\varepsilon) = \varepsilon (\Lambda_{\lim} u)(k, \varepsilon n) + O(\varepsilon^3),$$

where $\widetilde{\mathbb{L}}_\varepsilon = M_{0,\varepsilon}^{-1} \widetilde{L}_\varepsilon$ is the first part of the discrete Laplace operator (1.3) on the mesh $\mathfrak{M}_\varepsilon$ (cf. Equations (1.5) and (1.14)).

Putting it all together, we get

$$\langle \Delta_{\lim} u, v \rangle_{L^2(x)} = \sum_{k=-K}^{K} \sum_{n=-N_\varepsilon}^{N_\varepsilon} (\widetilde{L}_\varepsilon u)(x(k, \varepsilon n)) v(k, \varepsilon n) + O(h) = v^T \widetilde{L}_\varepsilon u + O(h),$$

where we identify the functions $u$ and $v$ with elements of $\Omega^0(\mathfrak{M}_\varepsilon)$, i.e., with vectors of length $(2K + 1) \cdot (2N_\varepsilon + 1)$. Analogously, we receive

$$\langle u, \Delta_{\lim} v \rangle_{L^2(x)} = u^T \widetilde{L}_\varepsilon v + O(h) \quad \text{and} \quad \langle u, \Delta_{\lim} u \rangle_{L^2(x)} = u^T \widetilde{L}_\varepsilon u + O(h).$$

Since the matrix $\widetilde{L}_\varepsilon$ is symmetric and positive semidefinite this concludes the proof. □

### 1.6.4 Consistency

Here, we prove that the semidiscrete Laplacian (1.4) is consistent, meaning that it converges pointwise to the smooth Laplace operator (1.1) if the semidiscrete surface converges to a smooth one. This result particularly implies consistency of Alexa and Wardetzky's Laplacian on quadrilateral meshes (see Remark 1.3).

First of all, we set up some notation. For now, let us assume that the curves $\{x(k,t) : t \in \mathbb{R}\}$, $k \in \mathbb{Z}$, of the semidiscrete surface $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ lie on a surface $\mathcal{M} \subset \mathbb{R}^3$ with a sufficiently smooth (local) parametrization $\psi : \mathbb{R}^2 \to \mathcal{M} : (\xi_1, \xi_2) \mapsto \psi(\xi_1, \xi_2)$. We denote the partial derivatives of the parametrization $\psi$ by $\partial_i\psi, \partial_{ij}\psi$ and so forth. For the coefficient matrix of the first fundamental form, we write

$$G := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad \text{with } g_{ij} := \langle \partial_i\psi, \partial_j\psi \rangle.$$

We denote the entries of $G^{-1}$ by $g^{ij}$, i.e., $G^{-1} = (g^{ij})_{i,j=1}^2$. Furthermore, for $i, j, k, \ell \in \{1, 2\}$, we set

$$g_{ijk} := \langle \partial_i\psi, \partial_{jk}\psi \rangle, \quad g_{ijk\ell} := \langle \partial_{ij}\psi, \partial_{k\ell}\psi \rangle, \quad \text{and} \quad e_{ijk\ell} := \langle \partial_i\psi, \partial_{jk\ell}\psi \rangle.$$

Using local coordinates, the formula for the Laplace operator $\Delta_{\mathcal{M}}$, acting on a scalar function $u : \mathcal{M} \to \mathbb{R}$, reads (see, e.g., Rosenberg [33, p. 18])

$$\Delta_{\mathcal{M}} u = -\frac{1}{\sqrt{\det(G)}} \sum_{i,j=1}^{2} \frac{\partial}{\partial \xi_i} \left( g^{ij} \sqrt{\det(G)} \frac{\partial}{\partial \xi_j} (u \circ \psi) \right).$$

By applying the appropriate rules of differentiation, this formula expands to

$$\begin{aligned} \Delta_{\mathcal{M}} u = \frac{1}{\det(G)^2} \big[ &\tau_1 \partial_1 (u \circ \psi) + \tau_2 \partial_2 (u \circ \psi) + \\ &+ \tau_{11} \partial_{11} (u \circ \psi) + \tau_{12} \partial_{12} (u \circ \psi) + \tau_{22} \partial_{22} (u \circ \psi) \big], \end{aligned} \tag{1.16}$$

with

$$\begin{aligned} \tau_1 &= g_{11}(g_{22}g_{122} - g_{12}g_{222}) - 2g_{12}(g_{22}g_{112} - g_{12}g_{212}) + g_{22}(g_{22}g_{111} - g_{12}g_{211}), \\ \tau_2 &= g_{11}(g_{11}g_{222} - g_{12}g_{122}) - 2g_{12}(g_{11}g_{212} - g_{12}g_{112}) + g_{22}(g_{11}g_{211} - g_{12}g_{111}), \\ \tau_{11} &= -g_{22} \det(G), \quad \tau_{12} = 2g_{12} \det(G), \quad \text{and} \quad \tau_{22} = -g_{11} \det(G). \end{aligned}$$

After these preparations, we can prove the following convergence result.

**Theorem 1.2.** *Let $p$ be a point of a surface $\mathcal{M} \subset \mathbb{R}^3$ with a sufficiently smooth parametrization $\psi : \mathbb{R}^2 \to \mathcal{M}$, such that $p = \psi(0, 0)$. Then, for every $u \in C^2(\mathcal{M})$, we have*

$$(\Delta^{\varepsilon}_{\lim}(u \circ \psi))(0, 0) = (\Delta_{\mathcal{M}} u)(p) + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

*where $\Delta^{\varepsilon}_{\lim}$ is the semidiscrete Laplacian* (1.4) *on the semidiscrete surface*

$$x^{\varepsilon} : \mathbb{Z} \times \mathbb{R} \to \mathcal{M} : (k, t) \mapsto \psi(\varepsilon k, t).$$

*Proof.* We begin by expanding $x^\varepsilon_{\bar{1}} = \psi(-\varepsilon, 0)$ and $x^\varepsilon_1 = \psi(\varepsilon, 0)$ into Taylor polynomials around $p = \psi(0, 0)$, that is,

$$\psi(\pm\varepsilon, 0) = \psi \pm \varepsilon\partial_1\psi + \frac{1}{2}\varepsilon^2\partial_{11}\psi \pm \frac{1}{6}\varepsilon^3\partial_{111}\psi + O(\varepsilon^4).$$

Moreover, we expand the partial derivatives $(x^\varepsilon_{\bar{1}})' = \partial_2\psi(-\varepsilon, 0)$ and $(x^\varepsilon_1)' = \partial_2\psi(\varepsilon, 0)$ into Taylor polynomials around $\partial_2\psi(0, 0)$, i.e.,

$$\partial_2\psi(\pm\varepsilon, 0) = \partial_2\psi \pm \varepsilon\partial_{12}\psi + \frac{1}{2}\varepsilon^2\partial_{112}\psi + \frac{1}{6}\varepsilon^3\partial_{1112}\psi + O(\varepsilon^4).$$

Likewise, for the second-order partial derivatives $(x^\varepsilon_{\bar{1}})'' = \partial_{22}\psi(-\varepsilon, 0)$ and $(x^\varepsilon_1)'' = \partial_{22}\psi(\varepsilon, 0)$, we have

$$\partial_{22}\psi(\pm\varepsilon, 0) = \partial_{22}\psi \pm \varepsilon\partial_{122}\psi + \frac{1}{2}\varepsilon^2\partial_{1122}\psi + \frac{1}{6}\varepsilon^3\partial_{11122}\psi + O(\varepsilon^4).$$

Analogously, we expand $u(x^\varepsilon_{\bar{1}}) = (u \circ \psi)(-\varepsilon, 0)$ and $u(x^\varepsilon_1) = (u \circ \psi)(\varepsilon, 0)$ and the corresponding partial derivatives into the following Taylor polynomials

$$\begin{aligned}
(u \circ \psi)(\pm\varepsilon, 0) &= (u \circ \psi) \pm \varepsilon\partial_1(u \circ \psi) + \frac{1}{2}\varepsilon^2\partial_{11}(u \circ \psi) \pm \frac{1}{6}\varepsilon^3\partial_{111}(u \circ \psi) + O(\varepsilon^4),\\
\partial_2(u \circ \psi)(\pm\varepsilon, 0) &= \partial_2(u \circ \psi) \pm \varepsilon\partial_{12}(u \circ \psi) + \frac{1}{2}\varepsilon^2\partial_{112}(u \circ \psi) \pm \frac{1}{6}\varepsilon^3\partial_{1112}(u \circ \psi) + O(\varepsilon^4),\\
\partial_{22}(u \circ \psi)(\pm\varepsilon, 0) &= \partial_{22}(u \circ \psi) \pm \varepsilon\partial_{122}(u \circ \psi) + \frac{1}{2}\varepsilon^2\partial_{1122}(u \circ \psi) \pm \frac{1}{6}\varepsilon^3\partial_{11122}(u \circ \psi) + O(\varepsilon^4).
\end{aligned} \tag{1.17}$$

To show the desired convergence results, we insert these Taylor polynomials into the semidiscrete scheme (1.15) and expand the resulting terms.

First of all, we expand the expressions $A_1$ and $A_{\bar{1}}$ (see Corollary 1.1). In our current notation, we have

$$A_1 = \|[\delta x^\varepsilon] \times [(x^\varepsilon_1)' + (x^\varepsilon)']\| = \|[\psi(\varepsilon, 0) - \psi(0, 0)] \times [\partial_2\psi(\varepsilon, 0) + \partial_2\psi(0, 0)]\|.$$

Inserting the corresponding Taylor polynomials and expanding yield

$$\begin{aligned}
A_1 &= \varepsilon\sqrt{\lambda_1 + \varepsilon\lambda_2 + \varepsilon^2\lambda_3 + O(\varepsilon^3)} =\\
&= \varepsilon(\lambda_1)^{1/2} + \frac{1}{2}\varepsilon^2\lambda_2(\lambda_1)^{-1/2} + \frac{1}{8}\varepsilon^3(4\lambda_1\lambda_3 - \lambda_2\lambda_2)(\lambda_1)^{-3/2} + O(\varepsilon^4),
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1 &:= 4\det(G), \quad \lambda_2 := 4(g_{11}g_{212} + g_{22}g_{111} - g_{12}[g_{112} + g_{211}]),\\
\lambda_3 &:= g_{11}g_{1212} + g_{22}g_{1111} - g_{112}^2 - g_{211}^2 + 4g_{111}g_{212} +\\
&\quad + 2(g_{11}e_{2112} - g_{12}e_{1112} - g_{12}g_{1112} - g_{112}g_{211}) + \frac{4}{3}(g_{22}e_{1111} - g_{12}e_{2111}).
\end{aligned}$$

Analogously, we get

$$A_{\bar{1}} = \varepsilon\sqrt{\lambda_1 - \varepsilon\lambda_2 + \varepsilon^2\lambda_3 + O(\varepsilon^3)} =$$
$$= \varepsilon(\lambda_1)^{1/2} - \frac{1}{2}\varepsilon^2\lambda_2(\lambda_1)^{-1/2} + \frac{1}{8}\varepsilon^3(4\lambda_1\lambda_3 - \lambda_2\lambda_2)(\lambda_1)^{-3/2} + O(\varepsilon^4).$$

By inserting the Taylor polynomials of $x_1^\varepsilon$, $(x_1^\varepsilon)'$, and $(x_1^\varepsilon)''$ into $B_1$ we receive

$$B_1 = \varepsilon^2\mu_1 + \varepsilon^3\mu_2 + O(\varepsilon^4),$$

where

$$\begin{aligned}
\mu_1 &:= 4(g_{11}g_{222} + g_{22}g_{112} - g_{12}[g_{122} + g_{212}]),\\
\mu_2 &:= 2(g_{11}[e_{2122} + g_{1222}] - g_{12}[e_{1122} + g_{1122} + e_{2112} + g_{1212}]+\\
&\qquad + g_{22}[e_{1112} + g_{1112}] - [g_{122} + g_{212}][g_{112} + g_{211}] + 2g_{111}g_{222}).
\end{aligned}$$

In the same way, we get

$$B_{\bar{1}} = \varepsilon^2\mu_1 - \varepsilon^3\mu_2 + O(\varepsilon^4).$$

Next, we expand the expressions $\alpha_1$, $\beta_1$ and $\alpha_{\bar{1}}$, $\beta_{\bar{1}}$ of Corollary 1.1. Here, we additionally need the Taylor polynomials (1.17) to receive

$$\alpha_1 = \varepsilon^2\eta_1 + \varepsilon^3\eta_2 + O(\varepsilon^4),$$

where

$$\begin{aligned}
\eta_1 &:= 2(g_{12}\partial_1(u\circ\psi) + g_{11}\partial_2(u\circ\psi)),\\
\eta_2 &:= (g_{112} + g_{211})\partial_1(u\circ\psi) - 2g_{111}\partial_2(u\circ\psi) + g_{12}\partial_{11}(u\circ\psi) - g_{11}\partial_{12}(u\circ\psi).
\end{aligned}$$

Similarly, we have

$$\alpha_{\bar{1}} = \varepsilon^2\eta_1 - \varepsilon^3\eta_2 + O(\varepsilon^4).$$

For the terms $\beta_1$ and $\beta_{\bar{1}}$ we get

$$\beta_1 = \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3\theta_3 + O(\varepsilon^4)$$

with

$$\begin{aligned}
\theta_1 &:= 2(g_{12}\partial_2(u\circ\psi) - g_{22}\partial_1(u\circ\psi)),\\
\theta_2 &:= (g_{122} - g_{212})\partial_1(u\circ\psi) + (g_{211} - g_{112})\partial_2(u\circ\psi)-\\
&\qquad - g_{22}\partial_{11}(u\circ\psi) + 2g_{12}\partial_{12}(u\circ\psi) - g_{11}\partial_{22}(u\circ\psi),\\
\theta_3 &:= \frac{1}{2}(g_{1122} + e_{1122} - e_{2112})\partial_1(u\circ\psi) + g_{211}\partial_{12}(u\circ\psi) - g_{111}\partial_{22}(u\circ\psi)+\\
&\qquad + \Big(\frac{1}{3}e_{2111} - \frac{1}{2}(e_{1112} + g_{1112})\Big)\partial_2(u\circ\psi) + \frac{1}{2}(g_{122} - g_{212})\partial_{11}(u\circ\psi),
\end{aligned}$$

and

$$\beta_{\bar{1}} = -\varepsilon\theta_1 + \varepsilon^2\theta_2 - \varepsilon^3\theta_3 + O(\varepsilon^4).$$

To complete the proof, we substitute all these expressions into the formula for the semidiscrete Laplacian

$$(\Delta^{\varepsilon}_{\mathrm{lim}}(u \circ \psi))(0,0) = \frac{4}{A_1 + A_{\bar{1}}}\left(\frac{\beta_1}{A_1} + \frac{\beta_{\bar{1}}}{A_{\bar{1}}} - \frac{1}{2}\left(\frac{\alpha_1 B_1}{(A_1)^3} + \frac{\alpha_{\bar{1}} B_{\bar{1}}}{(A_{\bar{1}})^3}\right)\right).$$

We start by noticing that

$$A_1 + A_{\bar{1}} = 2\varepsilon(\lambda_1)^{1/2} + O(\varepsilon^3).$$

Next, we compute

$$\frac{\beta_1}{A_1} + \frac{\beta_{\bar{1}}}{A_{\bar{1}}} = \frac{1}{A_1 A_{\bar{1}}}(\beta_1 A_{\bar{1}} + \beta_{\bar{1}} A_1) = \frac{1}{\varepsilon^2\lambda_1 + O(\varepsilon^4)}\left(\varepsilon^3(\theta_1\lambda_2(\lambda_1)^{-1/2} + \theta_2(\lambda_1)^{1/2}) + O(\varepsilon^5)\right)$$

and obtain

$$\frac{4}{A_1 + A_{\bar{1}}}\left(\frac{\beta_1}{A_1} + \frac{\beta_{\bar{1}}}{A_{\bar{1}}}\right) = \frac{2}{(\lambda_1)^2}(\theta_1\lambda_2 + \theta_2\lambda_1) + O(\varepsilon^2).$$

For the remaining part, we compute

$$\begin{aligned}\frac{\alpha_1 B_1}{(A_1)^3} + \frac{\alpha_{\bar{1}} B_{\bar{1}}}{(A_{\bar{1}})^3} &= \frac{1}{(A_1 A_{\bar{1}})^3}(\alpha_1 B_1 (A_{\bar{1}})^3 + \alpha_{\bar{1}} B_{\bar{1}} (A_1)^3) = \\ &= \frac{1}{\varepsilon^6(\lambda_1)^3 + O(\varepsilon^8)}\left(2\varepsilon^7\eta_1\mu_1(\lambda_1)^{\frac{3}{2}} + O(\varepsilon^9)\right)\end{aligned}$$

and receive

$$-\frac{2}{A_1 + A_{\bar{1}}}\left(\frac{\alpha_1 B_1}{(A_1)^3} + \frac{\alpha_{\bar{1}} B_{\bar{1}}}{(A_{\bar{1}})^3}\right) = -\frac{2}{(\lambda_1)^2}(\eta_1\mu_1) + O(\varepsilon^2).$$

Putting it all together, we have

$$(\Delta^{\varepsilon}_{\mathrm{lim}}(u \circ \psi))(0,0) = \frac{2}{(\lambda_1)^2}(\theta_1\lambda_2 + \theta_2\lambda_1 - \eta_1\mu_1) + O(\varepsilon^2).$$

Finally, we insert the above-defined terms $\lambda_1, \lambda_2, \theta_1, \theta_2, \mu_1$, and $\eta_1$ into this equation to gain

$$(\Delta^{\varepsilon}_{\mathrm{lim}}(u \circ \psi))(0,0) = (\Delta_{\mathcal{M}} u)(p) + O(\varepsilon^2),$$

where $(\Delta_{\mathcal{M}} u)(p)$ is given in the form (1.16). □

*Remark* 1.3. Theorems 1.1 and 1.2 when combined yield consistency of the discrete Laplacian described by Alexa and Wardetzky [1] on quadrilateral meshes in the following sense: Firstly, their discrete construction is consistent with our semidiscrete one, and secondly, the semidiscrete construction is consistent with the smooth Laplacian. In other words, the discrete scheme (1.6) converges pointwise to the Laplace-Beltrami operator, when the quadrilateral mesh converges to a smooth surface in the two stages just mentioned.

## 1.7 Mean curvature

Now that we have a semidiscrete Laplace operator, we can define a semidiscrete mean curvature vector field on semidiscrete surfaces. We again mimic the smooth case of a two-dimensional submanifold $\mathcal{M} \subset \mathbb{R}^3$.

At a point $p \in \mathcal{M}$ the mean curvature vector is defined as

$$\mathbf{H}(p) := H(p)\mathbf{n}(p),$$

where $H(p)$ is the mean curvature and $\mathbf{n}(p)$ is the corresponding locally defined unit normal vector of the surface $\mathcal{M}$ at $p$. It is a well-known fact that the mean curvature vector is related to the Laplace-Beltrami operator via

$$\mathbf{H}(p) = -\frac{1}{2}(\Delta_{\mathcal{M}} \operatorname{id})(p),$$

where $\operatorname{id} : \mathcal{M} \to \mathbb{R}^3$ is the identity to which we apply the Laplacian component-wise (see Colding and Minicozzi [13, p. 22]). This motivates the following definition.

**Definition 1.2.** For a regular semidiscrete surface $x : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}^3$ the expression

$$\mathbf{H}_{\mathrm{lim}}(k,t) := -\frac{1}{2}(\Delta_{\mathrm{lim}} x)(k,t) \in \mathbb{R}^3$$

is called *semidiscrete mean curvature vector*.

With this definition at hand, we also have an unsigned semidiscrete mean curvature $H_{\mathrm{lim}}(k,t)$ given by

$$H_{\mathrm{lim}}(k,t) := \|\mathbf{H}_{\mathrm{lim}}(k,t)\|.$$

Moreover, at points with non-vanishing mean curvature, we can define a unit normal vector via

$$\mathbf{n}_{\mathrm{lim}}(k,t) := \frac{1}{H_{\mathrm{lim}}(k,t)}\mathbf{H}_{\mathrm{lim}}(k,t).$$

Notice that the convergence results of Subsection 1.6.4 can directly be applied to our current setting. Therefore, in the situation of Theorem 1.2, the semidiscrete mean curvature vector field converges pointwise to the smooth one at a quadratic rate.

Analogous to the smooth case, we also consider the mean curvature flow. A family of smooth surfaces $\{\mathcal{M}_s\}_{s\in\mathbb{R}}$ with corresponding parametrizations $\{\varphi_s\}_{s\in\mathbb{R}}$ evolves under the mean curvature flow, if each point of a surface $\mathcal{M}_\sigma$ moves with speed and direction given by the mean curvature vector at that point, i.e., if

$$\frac{\partial}{\partial s}\varphi_s(\xi_1,\xi_2)\big|_{s=\sigma} = \mathbf{H}(\varphi_\sigma(\xi_1,\xi_2)), \quad \forall\, \xi_1,\xi_2.$$

Note that, except in special cases, the mean curvature flow develops singularities. An analogous notion can be defined in the semidiscrete case, namely by evolution of surfaces $x_s(k,t)$ via

$$\frac{\partial}{\partial s} x_s(k,t)\big|_{s=\sigma} = \mathbf{H}_{\mathrm{lim}}^{x_\sigma}(k,t), \quad \forall\ (k,t) \in \mathbb{Z} \times \mathbb{R}.$$

We conclude this paper with a numerical example.

**Example 1.2.** We examine the action of the mean curvature flow on the semidiscrete cylinder

$$x_0(k,t) := (r\cos(\pi k/5),\ r\sin(\pi k/5), t), \quad k \in \mathbb{Z}_{10},\ t \in [-1,1].$$

For its numerical solution, we discretize the mean curvature flow equation as

$$x_{n+1}(k,t) = x_n(k,t) + h\mathbf{H}_{\mathrm{lim}}^{x_n}(k,t),$$

for small $h > 0$. Since this recursion would merely produce cylinders with decreasing radii, we additionally fix the boundary of $x_0$ to make the outcome more interesting.

For our experiments, we approximate each curve $\{x_n(k,t) : t \in [-1,1]\}$ by a cubic spline through some equidistant sample points and calculate the mean curvature vector at those points according to Definition 1.2. To obtain the results of Figure 1.6 we fix $h = \frac{1}{100}$ and set $r = \cosh(1)$.

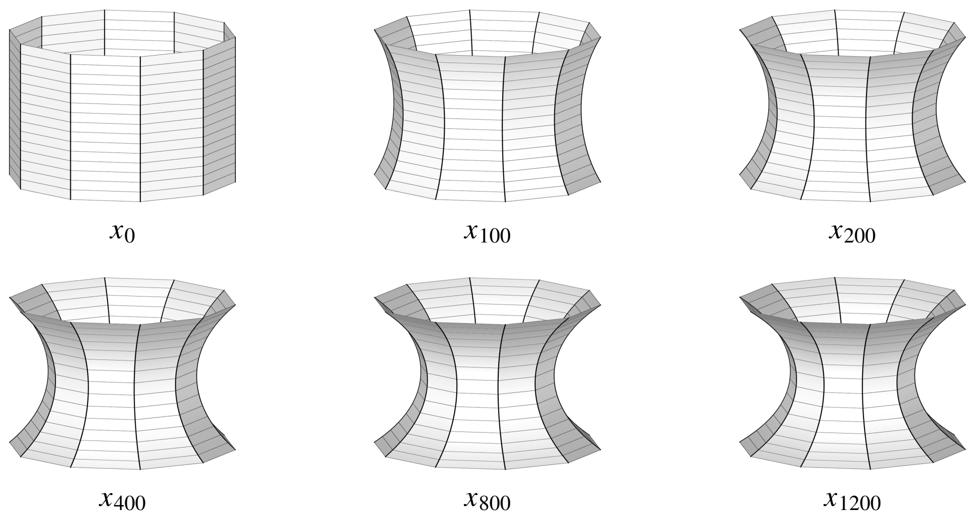


Figure 1.6: The action of the mean curvature flow on a semidiscretized cylinder.

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# Chapter 2

# Variational Laplacians for semidiscrete surfaces

Joint work with J. Wallner

**Abstract**

We study a Laplace operator on semidiscrete surfaces that is defined by variation of the Dirichlet energy functional. We show existence and its relation to the mean curvature normal, which is itself defined via variation of area. We establish several core properties like linear precision (closely related to the mean curvature of flat surfaces), and pointwise convergence. It is interesting to observe how a certain freedom in choosing area measures yields different kinds of Laplacians: it turns out that using as a measure a simple numerical integration rule yields a Laplacian previously studied as the pointwise limit of geometrically meaningful Laplacians on polygonal meshes.

**Keywords:** Semidiscrete surface, Variational Laplace operator, Mean curvature normal, Consistency.

**Mathematics Subject Classification (2010):** 53A05, 58E30, 49Q20, 41A25.

## 2.1 Introduction and Preliminaries

### 2.1.1 Introduction

The Laplace-Beltrami operator $\Delta = -\operatorname{div} \circ \operatorname{grad}$ on smooth surfaces and Riemannian manifolds is an extremely well investigated differential operator, which plays an essential role in many fields including applications. A main strength lies in Riemannian geometry, but it is also relevant to the elementary differential geometry of surfaces in three-dimensional space, e.g., via the equation $\Delta \operatorname{id} = -2H\mathbf{n}$ that relates the Laplacian to the mean curvature and unit normal vector field. Its intrinsic nature makes it very useful for computational applications in

This chapter comprises the research article [ii].

geometry processing, see, e.g., [37], and it has therefore been extensively discretized. Discrete Laplace operators defined on triangulations share characteristics with graph Laplacians, but ideally maintain as many of the core properties of the original Laplace-Beltrami operator as possible. For contributions to this topic see, e.g., [29, 9, 2, 24, 1]. Another important aspect of discretizations is a suitable convergence behavior, see, e.g., [45, 41, 43].

A powerful tool to derive Laplace operators on more general surfaces arises from the calculus of variations. The Laplacian of Riemannian geometry can be seen as gradient of the Dirichlet energy, which leads to the famous “cotangent formula” Laplacian on triangle meshes, see, e.g., [17, 29]. The variational approach is also particularly suited to study the mean curvature normal $\mathbf{H} = H\mathbf{n}$, which has an interpretation as the gradient vector field of the area functional.

In this paper we follow the variational approach. Our aim is to define meaningful Laplacians on semidiscrete parametric surfaces, which are represented by a point depending on one continuous and one discrete variable. The reader is reminded that semidiscrete objects occur in the classical theory of transformations of surfaces. For a systematic and unified treatment of continuous and semidiscrete surfaces as limits of a discrete master theory we refer to the textbook [10]. The lowest-dimensional case, i.e., two-dimensional surfaces, has been investigated from various viewpoints. The semidiscrete incarnation of conjugate surfaces is studied by [31] where piecewise-developable surfaces (including circular and conical semidiscrete surfaces) are considered from the computational viewpoint. Curvatures, in analogy to polyhedral surfaces, are the topic of [22]. Asymptotic surfaces and especially K-surfaces are investigated by [40]. The present paper however, is not concerned with any special class of semidiscrete surfaces.

### Outline and results

In Section 2.2 we define a Laplace operator on semidiscrete surfaces by a variational principle, namely as gradient of an appropriate Dirichlet energy functional. We show that this gradient exists and provide a closed-form expression for the semidiscrete Laplacian in Theorem 2.1. It turns out that there is quite some freedom in the choice of the particular $L^2$ space which is basic to the concepts of both gradient and Dirichlet energy. Section 2.3 investigates the gradient of the area functional to gain a semidiscrete mean curvature normal, and establishes the relation $\Delta\,\mathrm{id} = -2\mathbf{H}$ for the semidiscrete case (Theorem 2.2), which in turn implies that linear functions on flat surfaces are in the kernel of the Laplacian (i.e., the linear precision property). Section 2.4 discusses further properties like locality, symmetry, positive semidefiniteness, and lack of a maximum principle. The last section deals with pointwise convergence of the semidiscrete Laplacian toward the Laplace-Beltrami operator on smooth surfaces (Theorem 2.3).

### 2.1.2 Variational properties of the Laplacian

The Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ on a Riemannian manifold $\mathcal{M}$ can be defined via the Dirichlet energy functional

$$E(u) = \frac{1}{2}\int_{\mathcal{M}} \|\nabla u\|^2 \, dV, \quad u \in C^2(\mathcal{M}, \mathbb{R}).$$

It is then given as the gradient of the Dirichlet energy,

$$\Delta_{\mathcal{M}} = \nabla E,$$

which means that for smooth test functions $u$, and all smooth one-parameter variations $u_\xi$ of $u$, with the property that $\frac{\partial}{\partial\xi} u_\xi|_{\xi=0}$ is compactly supported, we have

$$\frac{d}{d\xi} E(u_\xi)\Big|_{\xi=0} = \Big\langle \Delta_{\mathcal{M}} u,\ \frac{\partial u_\xi}{\partial \xi}\Big|_{\xi=0}\Big\rangle_{L^2}$$

(with the usual definition $\langle f, g\rangle_{L^2} = \int_{\mathcal{M}} f(x)g(x)\, dV(x)$; see [21, pp. 89–94]). This relation is basic to the generalization of the Laplace-Beltrami operator to discrete surfaces and will also be used in the present paper. Recall that for a surface $\mathcal{M}$ embedded in $\mathbb{R}^3$, the Laplace operator has a remarkable connection to the mean curvature normal. Applying the Laplacian component-wise to the identity mapping $\mathrm{id}_{\mathcal{M}}$, we get

$$\Delta_{\mathcal{M}}\, \mathrm{id}_{\mathcal{M}} = -2\mathbf{H}$$

(see [13, p. 22]), where the mean curvature normal $\mathbf{H} = H\mathbf{n}$ is a unit normal vector $\mathbf{n}$ on $\mathcal{M}$ scaled by the corresponding mean curvature $H$. Observe that $\mathbf{H}$ is independent of the particular choice of $\mathbf{n}$, as the sign of $H$ depends on the direction of $\mathbf{n}$. This vector field likewise has a variational definition, namely

$$-2\mathbf{H} = \nabla \mathrm{area}(\mathcal{M}), \text{ i.e., } \quad \frac{d}{d\xi}\, \mathrm{area}(p_\xi(\mathcal{M}))\Big|_{\xi=0} = \Big\langle -2\mathbf{H},\ \frac{\partial p_\xi}{\partial \xi}\Big|_{\xi=0}\Big\rangle_{L^2(\mathcal{M},\mathbb{R}^3)}$$

for every smooth one-parameter variation $p_\xi : \mathcal{M} \to \mathbb{R}^3$ with $p_0 = \mathrm{id}_{\mathcal{M}}$ (see [13, p. 7]). Here, $\mathrm{area}(\mathcal{M}) = \int_{\mathcal{M}} 1\, dV$ and $\langle f, g\rangle_{L^2(\mathcal{M},\mathbb{R}^3)} = \int_{\mathcal{M}} \langle f(x), g(x)\rangle\, dV(x)$.

### 2.1.3 Semidiscrete surfaces

The semidiscrete surfaces which constitute our object of study are mappings of the form

$$x : D \to V : (k, t) \to x(k, t), \quad \text{with } D \subset \mathbb{Z} \times \mathbb{R} \text{ open},$$

and where $V$ is a vector space equipped with a positive definite scalar product $\langle \cdot, \cdot \rangle_V$. Throughout this paper we assume that $x$ is at least twice continuously differentiable in the second argument, and denote the corresponding set of mappings by $C^2_{\mathrm{sd}}(D, V)$. Accordingly, the set of semidiscrete functions that are merely continuous in the second argument is denoted by $C_{\mathrm{sd}}(D, V)$. With the help of the canonical hat function

$$\varphi(s) := \max\{1 - |s|, 0\},$$

we extend $x$ to a mapping, again called $x$,

$$x : \widehat{D} \to V : (s, t) \mapsto \sum_{k\colon (k,t)\in D} \varphi(s - k)x(k, t), \tag{2.1}$$

where the domain $\widehat{D}$ is constructed as a disjoint union of strips $D_k \subset \mathbb{R}^2$, each strip being defined as

$$D_k := \bigcup\nolimits_{t\colon (k,t)\in D \,\wedge\, (k+1,t)\in D} [k, k + 1] \times \{t\}. \tag{2.2}$$

In the non-degenerate case, this procedure converts a sequence of curves into a piece-wise ruled surface, connecting corresponding points $x(k, t)$ and $x(k + 1, t)$ by straight line segments. For each pair of successive curves $x(k, \cdot)$ and $x(k + 1, \cdot)$ there is a ruled surface strip, which is treated separately from the others as far as the domain of definition is concerned. This procedure does not alter the values $x(s, t)$ where $s$ happens to equal an integer $k \in \mathbb{Z}$; $x(s, t)$ has the same value regardless of the question if $s$ is considered as an element of $[k - 1, k]$ or as an element of $[k, k + 1]$. We call the procedure of converting a semidiscrete surface $x(k, t)$ to a piecewise-ruled surface $x(s, t)$ an "extension", even if $D$ is not a subset of $\widehat{D}$.

In order to make the upcoming formulas shorter and thus better readable, we set up the following notation. For the derivatives of $x(k, t)$ with respect to the variable $t$, we write $x'$, $x''$, and so forth. Finite differences in the discrete direction are denoted by

$$\delta x(s, t) := x(k + 1, t) - x(k, t), \quad \text{for } s \in [k, k + 1].$$

Note that in contrast to $x$ itself, the discrete derivative $\delta x$ does have different values for $s = k \in \mathbb{Z}$, depending on whether $s$ is thought to be contained in $[k - 1, k]$ or in $[k, k + 1]$. We resolve this ambiguity by always making it clear which of the two corresponding surface strips we are considering.

We call a semidiscrete surface *regular*, if all its surface strips are regular in the usual sense, i.e., if the set

$$\{\delta x(s, t), x'(s, t)\}, \quad s \in [k, k + 1],$$

is linearly independent throughout. Moreover, we call $(k, t) \in D$ an *inner point*, if

$$\{k - 1, k, k + 1\} \times (t - \varepsilon, t + \varepsilon) \subset D,$$

for some $\varepsilon > 0$. Otherwise it is called a *boundary point*. The set of inner points of $D$ will be denoted by $D^{\mathrm{inn}}$.

Note that we do not make any assumptions on the embeddedness of the surfaces we study. Later, when considering a real-valued function $u$ on a semidiscrete surface $x$, we regard it as defined in $D$ rather than in $x(D)$. Such a function $u$ therefore formally is a semidiscrete surface in its own right and we use the same notation as for the surface $x$. We call $u$ *smooth*, if it is at least twice continuously differentiable in the second argument, i.e., if $u \in C^2_{\mathrm{sd}}(D, \mathbb{R})$.

*Remark* 2.1. It is easy to see that a semidiscrete surface $x(s,t)$ is regular for all $s \in [k, k+1]$ if $x'(k,t)$, $x'(k+1,t)$ and $\delta x(k,t)$ are linearly independent (in which case the ruled surface strip corresponding to $s \in [k, k+1]$ is a regular skew ruled surface). In case those vectors are linearly dependent, regularity in that interval is equivalent to $|\delta x(k,t), x'(k,t)| \cdot |\delta x(k,t), x'(k+1,t)| > 0$, for any determinant form $|\cdot,\cdot|$ on a plane containing these three vectors (the strip then has a torsal generator whose singular point $x(s^*,t)$ obeys $s^* \notin [k, k+1]$; cf. [32, §5.1.1]).

## 2.2 Variational definition of a semidiscrete Laplace operator

This section aims at a meaningful definition of a Laplace operator on semidiscrete surfaces. Mimicking the smooth case, we define a semidiscrete Laplacian as gradient of an appropriate Dirichlet energy functional. For this purpose we first discuss area measures.

### 2.2.1 Integration and Laplacian on semidiscrete surfaces.

Consider a semidiscrete surface $x$ with open domain $D \subset \mathbb{Z} \times \mathbb{R}$, which has been extended to a piecewise-ruled surface defined in the domain $\widehat{D}$, as described above (cf. Equation (2.1)).

A reasonable definition of its area obviously is given by the sum of the areas of individual ruled surface strips, which in terms of the matrix I of the first fundamental form is expressed as

$$\operatorname{area}(x) = \iint_{\widehat{D}} \sqrt{\det \mathrm{I}(s,t)}\, ds\, dt, \qquad \text{with } \mathrm{I} = \begin{pmatrix} \|\delta x\|^2 & \langle \delta x, x' \rangle \\ \langle \delta x, x' \rangle & \|x'\|^2 \end{pmatrix} \tag{2.3}$$

(see [15, p. 98]). Note that, in order to resolve the ambiguity in the definition of $\delta x$, the double integral over $\widehat{D}$ has to be interpreted as the sum of double integrals over the individual strips $D_k$ stated in Equation (2.2).

It makes sense to generalize this definition by replacing Lebesgue measure $ds\,dt$ by other measures. We start with a Borel measure $\mu_0$ supported on the unit interval $[0,1]$, whose zeroth and first moments have the following values:

$$m_0 = \int_{[0,1]} 1\, d\mu_0(s) = 1 \quad \text{and} \quad m_1 = \int_{[0,1]} s\, d\mu_0(s) = \frac{1}{2}. \tag{2.4}$$

That is, we require integration of polynomials up to degree 1 to coincide with integration w.r.t. Lebesgue measure. A stronger property is *symmetry* of the measure, meaning that

$$\int_{[0,1]} f(s)\, d\mu_0(s) = \int_{[0,1]} f(1-s)\, d\mu_0(s), \quad \text{for all } f \in L^1([0,1], \mu_0). \tag{2.5}$$

Together with $m_0 = 1$, symmetry implies $m_1 = \frac{1}{2}$. This symmetry property is not required except in Theorem 2.3, where it is explicitly mentioned.

We will see that these assumptions are crucial for some important properties of the Laplacian, and also for convergence. We actually construct an entire family of semidiscrete Laplace operators, depending on the type of integration we employ. Note that in particular the measure $\mu_0$ might be a numerical integration rule, like the midpoint rule or the trapezoidal rule. As it turns out, a particular choice of measure leads to the semidiscrete Laplacian introduced in the first paper (Chapter 1) as a pointwise limit of the discrete construction of Alexa and Wardetzky [1]. We discuss this connection in §2.2.2.

Now, by translation, $\mu_0$ acts as a measure on each interval $[k, k+1]$, and we denote the sum of measures on the disjoint union of intervals $[k, k+1]$ by $\mu$. With the Lebesgue measure $\lambda$ on the reals, we consider the product measure $\mu \otimes \lambda$ on the disjoint union of strips $[k, k+1] \times \mathbb{R}$. It is precisely this measure which we use for integration in the domain $\widehat{D}$:

**Definition 2.1.** Consider a semidiscrete surface $x : D \to V$, extended to a piecewise-ruled surface $x : \widehat{D} \to V$. We use its first fundamental form I and the measure $\mu \otimes \lambda$ on $\widehat{D}$ to define the surface integral of a function $u : \widehat{D} \to \mathbb{R}$:

$$\int_x u\, dA := \int_{\widehat{D}} u(s,t)\sqrt{\det \mathrm{I}(s,t)}\, d(\mu \otimes \lambda)(s,t).$$

The surface area is given by $\mathrm{area}_\mu(x) := \int_x 1\, dA$.

Again, by the integral over $\widehat{D}$ we mean the sum of integrals over the individual strips $D_k$ given by Equation (2.2).

**Example 2.1.** This definition in particular applies to a semidiscrete function $u : D \to \mathbb{R}$, which has been extended to a piecewise-linear function $u : \widehat{D} \to \mathbb{R}$ by linear interpolation:

$$u(s,t) = \sum_{k\colon (k,t)\in D} \varphi(s-k)u(k,t).$$

If $u$ vanishes at the boundary of $D$, we can write its surface integral as

$$\int_x u\, dA = \int_D u(k,t)\, \mathbf{a}(k,t)\, dt,$$

where $\int_D dt$ means integration with respect to Lebesgue measure on each straight line segment contained in $D$, and the semidiscrete function $\mathbf{a}$ is defined by

$$\mathbf{a}(k,t) := \int_{[k-1,k]\,\sqcup\,[k,k+1]} \varphi(s-k)\sqrt{\det \mathrm{I}(s,t)}\, d\mu(s).$$

Here, the integral over $[k-1, k] \sqcup [k, k+1]$ represents the sum of the integrals over the intervals $[k-1, k]$ and $[k, k+1]$. This formula follows from computing the left hand side by the iterated integral $\int_D \left( \int_{[k,k+1]} u \cdot \sqrt{\det \mathrm{I}}\ d\mu(s)\right) dt$, and using $u(s,t) = \sum_j \varphi(s-j)u(j,t)$ to express the interior integral as $u(k,t) \int_{[k,k+1]} \varphi(s-k)\sqrt{\det \mathrm{I}}\, d\mu(s) + u(k+1,t) \int_{[k,k+1]} \varphi(s-k-1)\sqrt{\det \mathrm{I}}\, d\mu(s)$. An index shift yields the formula given above.

**Definition 2.2.** Given a semidiscrete surface $x$ with domain $D$, we define $L^2$ inner products for semidiscrete real-valued (resp. $V$-valued) functions $u$, $v$ with the same domain by letting

$$\langle u, v\rangle_{L^2(x)} := \int_x uv\, dA, \quad \text{resp.} \quad \langle u, v\rangle_{L^2(x,V)} := \int_x \langle u, v\rangle_V\, dA.$$

The integrals in the previous formulas mean that the semidiscrete functions $u$, $v$ are multiplied to create a semidiscrete product function $(u \cdot v)(k, t)$ (resp. $\langle u, v\rangle_V(k, t)$), which for the purpose of integration undergoes linear interpolation. The inner products are, for instance, well defined for semidiscrete functions that are continuous in the second argument and have finite $L^2$ norm.

For the Dirichlet energy of a semidiscrete function we use the following definition:

**Definition 2.3.** Let $x$ be a regular semidiscrete surface defined on $D$. Then, the Dirichlet energy $E_\mu(u)$ of a smooth semidiscrete function $u : D \to \mathbb{R}$, considered as a function on $x$, is the Dirichlet energy, in the smooth sense, of the extended function $u(s,t)$ over the extended surface $x(s,t)$. Since $E_\mu$ is a quadratic form, we also use the corresponding symmetric bilinear form $\mathcal{E}_\mu$ which is uniquely characterized by $\mathcal{E}_\mu(u, u) = E_\mu(u)$:

$$E_\mu(u) = \frac{1}{2}\int_x \|\nabla u\|^2\, dA, \quad \mathcal{E}_\mu(u, v) = \frac{1}{2}\int_x \langle \nabla u, \nabla v\rangle\, dA.$$

It is tempting to employ $L^2$ notation for the definition of the Dirichlet energy. We will not do that, since the integrand is not generated by extending a semidiscrete function, and therefore does not fit Definition 2.2.

As to the gradient of a real-valued function $u(s,t)$ on a parametric surface $x(s,t)$, recall that $\|\nabla u\|^2 = \binom{\partial_s u}{\partial_t u}^T \cdot \mathrm{I}^{-1} \cdot \binom{\partial_s u}{\partial_t u}$, where $\mathrm{I}(s,t)$ is the matrix of the first fundamental form. This leads to the following explicit expression for the Dirichlet energy in case I is regular:

$$E_\mu(u) = \frac{1}{2}\int_{\widehat{D}} \det \mathrm{I}^{-1/2} \left(\|x'\|^2(\delta u)^2 - 2\langle \delta x, x'\rangle(\delta u)(u') + \|\delta x\|^2(u')^2\right) d(\mu \otimes \lambda).$$

Next, we generalize the notion of the gradient of an energy functional to the semidiscrete case. For that we consider "admissible" variations of semidiscrete functions:

**Definition 2.4.** An admissible variation $x_\xi(k,t)$ of a smooth semidiscrete mapping $x : D \to V$ is a $V$-valued function of arguments $(\xi, k, t) \in (-\varepsilon, \varepsilon) \times D$, which depends smoothly on $\xi$ and $t$, coincides with $x(k,t)$ for $\xi = 0$, and such that $x_\xi(k,t)$ does not depend on $\xi$ outside a compact subset of $D^{\mathrm{inn}}$. We use the notation

$$\dot{x}(k,t) := \frac{\partial x_\xi}{\partial \xi}(k,t)\Big|_{\xi=0}.$$

This definition in particular applies to the admissible variations $u_\xi(k,t)$ of smooth semidiscrete functions $u : D \to \mathbb{R}$. Now the definition of the gradient of an energy functional reads as follows.

**Definition 2.5.** Let $x : D \to V$ be a regular semidiscrete surface and let $E$ be a functional on $C^2_{\mathrm{sd}}(D,\mathbb{R})$, with the property that there exists an operator $\nabla E : C^2_{\mathrm{sd}}(D,\mathbb{R}) \to C_{\mathrm{sd}}(D^{\mathrm{inn}},\mathbb{R})$, such that for every $u \in C^2_{\mathrm{sd}}(D,\mathbb{R})$ and all admissible one-parameter variations $u_\xi$ of $u$, we have

$$\frac{d}{d\xi}E(u_\xi)\Big|_{\xi=0} = \langle \nabla E(u), \dot{u}\rangle_{L^2(x)}.$$

Then $\nabla E$ is called the *gradient* of $E$. In particular, we define the *semidiscrete Laplace operator* $\Delta_{\mathrm{sd}}$ on $x$ as the gradient of the Dirichlet energy functional $E_\mu$, i.e., $\Delta_{\mathrm{sd}} := \nabla E_\mu$.

**Theorem 2.1.** *If $x : D \to V$ is a regular semidiscrete surface, then the semidiscrete Laplacian $\Delta_{\mathrm{sd}}u$ exists for all smooth semidiscrete functions $u$ defined in the same domain:*

$$(\Delta_{\mathrm{sd}}u)(k,t) = \frac{1}{\mathbf{a}(k,t)}\Big(\delta\mathbf{b}(k,t) + \mathbf{c}'(k,t)\Big), \tag{2.6}$$

*with* $\mathbf{a}$ *as in Example 2.1, and with semidiscrete functions* $\mathbf{b}$, $\mathbf{c}$ *defined by*

$$\mathbf{b}(k,\cdot) := \int_{[k-1,k]} \det\Gamma^{-1/2}\,(\langle\delta x, x'\rangle u' - \|x'\|^2\delta u)\,d\mu,$$
$$\mathbf{c}(k,\cdot) := \int_{[k-1,k]\sqcup[k,k+1]} \det\Gamma^{-1/2}\,\varphi(s-k)(\langle\delta x, x'\rangle\delta u - \|\delta x\|^2 u')\,d\mu.$$

*Proof.* Let $u_\xi$ be an admissible variation of $u$ with derivative $\dot{u}$. We compute the derivative of the Dirichlet energy by using the Leibniz rule (which applies because all occurring functions are smooth in the variables $\xi$ and $t$, and $\dot{u}$ has compact support):

$$\frac{d}{d\xi}E_\mu(u_\xi)\Big|_{\xi=0} = \int_{\widehat{D}} \det\Gamma^{-1/2}\Big(\|x'\|^2\delta u\delta\dot{u} - \langle\delta x, x'\rangle(\delta u\dot{u}' + u'\delta\dot{u}) + \|\delta x\|^2 u'\dot{u}'\Big)\,d(\mu\otimes\lambda) =$$
$$= \int_{\widehat{D}} (-b(s,t)\delta\dot{u}(s,t) - c(s,t)\dot{u}'(s,t))\,d(\mu\otimes\lambda)(s,t), \quad \text{where}$$
$$b := \det\Gamma^{-1/2}(\langle\delta x, x'\rangle u' - \|x'\|^2\delta u), \quad c := \det\Gamma^{-1/2}(\langle\delta x, x'\rangle\delta u - \|\delta x\|^2 u'). \tag{2.7}$$

Next we apply integration by parts w.r.t. $t$ to the second summand:

$$\frac{d}{d\xi}E_\mu(u_\xi)\Big|_{\xi=0} = \int_{\widehat{D}}\Big(-b(s,t)\delta\dot{u}(s,t) + c'(s,t)\dot{u}(s,t)\Big)\,d(\mu\otimes\lambda)(s,t)$$
$$= \int_{(k,t)\in D}\int_{s\in[k,k+1]} b(s,t)(\dot{u}(k,t) - \dot{u}(k+1,t)) +$$
$$+\ c'(s,t)(\varphi(s-k)\dot{u}(k,t) + \varphi(s-k-1)\dot{u}(k+1,t))\,d\mu(s)\,d\lambda(t).$$

Observe that the boundary terms vanish, since the support of $\dot{u}$ is contained in $D^{\mathrm{inn}}$. Finally, an index shift yields

$$\frac{d}{d\xi}E_\mu(u_\xi)\Big|_{\xi=0} = \int_{(k,t)\in D}(\delta\mathbf{b}(k,t) + \mathbf{c}'(k,t))\dot{u}(k,t)\,d\lambda(t) = \langle\Delta_{\mathrm{sd}}u, \dot{u}\rangle_{L^2(x)},$$

with $\mathbf{b}$, $\mathbf{c}$, and $\Delta_{\mathrm{sd}}u$ as stated above (cf. also Example 2.1). □

### 2.2.2 Example: Semidiscrete Laplacians arising as limits of discrete ones.

As demonstrated in Chapter 1, the discrete Laplace operator $\mathbb{L}$ constructed by Alexa and Wardetzky [1] for functions defined on the vertices of a polygonal mesh gives rise to a Laplace operator on semidiscrete surfaces via pointwise limits. We may discretize a regular semidiscrete surface $x : D \to \mathbb{R}^3$ and a smooth function $u : D \to \mathbb{R}$ near a point of interest $(k,t) \in D^{\mathrm{inn}}$ by letting

$$x^\varepsilon_{ij} := x(k+i, t+\varepsilon j), \quad u^\varepsilon(x^\varepsilon_{ij}) := u(k+i, t+\varepsilon j).$$

This defines the vertices $x^\varepsilon_{ij}$ of a quad mesh with regular combinatorics, and function values on these vertices. The discrete Lapace operator on that mesh is denoted by $\mathbb{L}^\varepsilon$, and we let

$$(\Delta_{\mathrm{lim}} u)(k,t) := \lim_{\varepsilon \searrow 0} (\mathbb{L}^\varepsilon u^\varepsilon)\Big|_{0,0}.$$

Existence and properties of this limit were investigated in Chapter 1, in particular independence of the limit from the still remaining degrees of freedom in the construction of $\mathbb{L}$. There is a remarkable connection between the semidiscrete Laplacian $\Delta_{\mathrm{sd}}$ and the semidiscrete Laplacian $\Delta_{\mathrm{lim}}$ which arises by pointwise limits. In fact, if the measure $\mu_0$ used to construct $\Delta_{\mathrm{sd}}$ is taken as the midpoint rule for numerical integration (i.e., $\int_{[0,1]} f(s)\, d\mu_0(s) = f(\frac{1}{2})$), then they are equal:

$$\mu_0\left(\left\{\tfrac{1}{2}\right\}\right) = 1 \implies \Delta_{\mathrm{sd}} u = \Delta_{\mathrm{lim}} u, \quad \forall\, u \in C^2_{\mathrm{sd}}(U, \mathbb{R}).$$

This claim is easily verified by comparing the formulae for $\Delta_{\mathrm{lim}} u$ given in Corollary 1.1 with the explicit expressions stated in Theorem 2.1 of the present chapter: If $\mu_0(\{\frac{1}{2}\}) = 1$,

$$\begin{aligned}\mathbf{a}(k,t) &= \int_{[k-1,k]\,\sqcup\,[k,k+1]} \varphi(s-k)\|\delta x(s,t) \times x'(s,t)\|\, d\mu(s) = \\ &= \frac{1}{4}\Big(\|(x_1 - x) \times (x_1' + x')\| + \|(x_{\bar{1}} - x) \times (x_{\bar{1}}' + x')\|\Big) = \frac{A_1 + A_{\bar{1}}}{4},\end{aligned}$$

where we adopt the notation from Corollary 1.1. In particular, $x_1(k,t) = x(k+1,t)$ and $x_{\bar{1}}(k,t) = x(k-1,t)$. Likewise, we get

$$\begin{aligned}\mathbf{b}(k,t) &= \frac{1}{2A_1}\Big(\langle x_1 - x, x_1' + x'\rangle(u_1' + u') - \|x_1' + x'\|^2(u_1 - u)\Big), \quad \text{and} \\ \mathbf{c}(k,t) &= \frac{1}{2A_1}\Big(\|x_1 - x\|^2(u_1' + u') - \langle x_1 - x, x_1' + x'\rangle(u_1 - u)\Big) = -\frac{\alpha_1}{2A_1}.\end{aligned}$$

By inserting these functions into Equation (2.6) and comparing the resulting expression with the formula stated in Corollary 1.1, we see that, for this particular choice of $\mu_0$, we have $\Delta_{\mathrm{sd}} u = \Delta_{\mathrm{lim}} u$.

## 2.3 Semidiscrete mean curvature normals

Before we analyze further properties of the semidiscrete Laplace operator, we discuss its connection to the mean curvature normal. Recall from the introductory section the relations between the Laplacian and the mean curvature normal, which hold for smooth surfaces embedded in $\mathbb{R}^3$: On the one hand, $\Delta_{\mathcal{M}}\,\mathrm{id}_{\mathcal{M}} = -2\mathbf{H}$, on the other hand the mean curvature normal itself has the variational definition $-2\mathbf{H} = \nabla \mathrm{area}(\mathcal{M})$. Here we consider the semidiscrete version of these objects and the relations between them. Our notation is not entirely the same as in §2.1.2, because we deal with parametric surfaces.

### 2.3.1 Variational properties of mean curvature

**Definition 2.6.** Let $F$ be a functional on $C^2_{\mathrm{sd}}(D, V)$ and let $x : D \to V$ be a semidiscrete surface with the property that there exists a function $\nabla F(x) \in C_{\mathrm{sd}}(D^{\mathrm{inn}}, V)$, such that for all admissible one-parameter variations $x_\xi$ of $x$, we have

$$\frac{d}{d\xi}F(x_\xi)\Big|_{\xi=0} = \langle \nabla F(x), \dot{x}\rangle_{L^2(x,V)}.$$

Then $\nabla F(x)$ is called the *gradient* of $F$ at $x$. In particular, the *semidiscrete mean curvature normal* $\mathbf{H}_{\mathrm{sd}}$ of a regular semidiscrete surface $x$ is defined as

$$\mathbf{H}_{\mathrm{sd}} := -\frac{1}{2}\nabla\,\mathrm{area}(x).$$

**Theorem 2.2.** *For regular semidiscrete surfaces $x$, the mean curvature normal vector field exists and can be computed by applying the Laplacian componentwise to the identity mapping on $x$:*

$$\Delta_{\mathrm{sd}} x = -2\mathbf{H}_{\mathrm{sd}}.$$

*Proof.* Let $x_\xi(k,t)$ be an admissible variation of $x$. Each semidiscrete surface $x_\xi(k,t)$ is extended to a piecewise-ruled surface $x_\xi(s,t)$, having first fundamental form $\mathrm{I}_\xi(s,t)$ (cf. Equation (2.3)). By definition, $x_\xi(k,t)$ is independent of $\xi$ outside a compact subset of $D^{\mathrm{inn}}$. Thus, by a standard argument, the piecewise-ruled surfaces $x_\xi(s,t)$ are regular for all $\xi$ in some interval $(-h,h)$, because $\sqrt{\det \mathrm{I}_\xi}$, i.e., the area spanned by the partial derivatives of $x_\xi$, is positive in a compact set $\{0\} \times K \subset \mathbb{R} \times \widehat{D}$, thus positive in a neighborhood of this set, and consequently positive in a product set $(-h,h) \times K$.

Thus, we may compute

$$\frac{\partial}{\partial\xi}\sqrt{\det \mathrm{I}_\xi} = \frac{1}{2\sqrt{\det \mathrm{I}_\xi}}\frac{\partial}{\partial\xi}\left[\|\delta x_\xi\|^2\|x'_\xi\|^2 - \langle \delta x_\xi, x'_\xi\rangle^2\right].$$

For $\xi = 0$, this expression is simplified by computing the individual derivatives $\frac{\partial}{\partial\xi}\|\delta x_\xi\|^2 = 2\langle \delta x, \delta\dot{x}\rangle$, $\frac{\partial}{\partial\xi}\|x'_\xi\|^2 = 2\langle x', \dot{x}'\rangle$, and $\frac{\partial}{\partial\xi}\langle \delta x_\xi, x'_\xi\rangle^2 = 2(\langle x', \delta\dot{x}\rangle + \langle \delta x, \dot{x}'\rangle)\langle \delta x, x'\rangle$. We get

$$\frac{d}{d\xi}\,\mathrm{area}_\mu(x_\xi)\Big|_{\xi=0} = \int_{\widehat{D}} \big(-\langle b(s,t), \delta\dot{x}(s,t)\rangle - \langle c(s,t), \dot{x}'(s,t)\rangle\big)\, d(\mu\otimes\lambda)(s,t),$$

where the functions $b(s,t)$ and $c(s,t)$ are the same as in Equation (2.7), and the previous formula is the same as the expression for the derivative of the Dirichlet energy in the proof of Theorem 2.1, only with $x$ instead of $u$, and scalar products of $V$-valued functions instead of products of real-valued ones. It follows that the gradient of $\mathrm{area}_\mu$ evaluated at $x$ indeed equals $\Delta_{\mathrm{sd}} x$. □

### 2.3.2 Mean curvature of extrinsically flat surfaces.

We show that the mean curvature normal of a semidiscrete surface vanishes, if that surface is contained in a two-dimensional plane. Besides constituting a sanity check on our definitions, this fact is of importance later when we show the "linear precision" property of the semidiscrete Laplacian.

**Lemma 2.1.** *If the regular semidiscrete surface $x : D \to V$ is contained in a two-dimensional plane $\Pi$, then its mean curvature normal $\mathbf{H}_{\mathrm{sd}}$ vanishes.*

*Proof.* The general idea of the proof is to show that $\|\mathbf{H}_{\mathrm{sd}}\|_{L^2(x,V)} = 0$ by constructing a variation whose derivative equals $\mathbf{H}_{\mathrm{sd}}$. This can be done in the following way. Choose a smooth function $v : D \to \mathbb{R}$ with compact support contained in $D^{\mathrm{inn}}$. Then

$$x_\xi(k,t) := x(k,t) + \xi v(k,t)^2 \mathbf{H}_{\mathrm{sd}}(k,t)$$

is a well-defined one-parameter variation of $x$ with velocity $\dot{x} = v^2 \mathbf{H}_{\mathrm{sd}}$.

Moreover, without loss of generality $0 \in \Pi$, so $\Pi$ is a linear subspace and therefore $\delta x, x' \in \Pi$. It follows from Theorem 2.1 and Theorem 2.2 that $\mathbf{H}_{\mathrm{sd}}(k,t) \in \Pi$, and consequently, $x_\xi(k,t) \in \Pi$. Since $\dim \Pi = 2$, we can express the above-mentioned area in terms of an appropriate determinant form $|\cdot,\cdot|$:

$$\begin{aligned}\sqrt{\det \mathrm{I}_\xi(s,t)} = |\delta x_\xi(s,t),\ x'_\xi(s,t)| = (1-s+k)|\delta x_\xi(k,t),\ x'_\xi(k,t)| + \\ + (s-k)|\delta x_\xi(k,t),\ x'_\xi(k+1,t)|, \quad \text{for } s \in [k,k+1],\ t \text{ fixed.}\end{aligned}$$

By Equation (2.4), integrating $\sqrt{\det \mathrm{I}_\xi}$ over $[k,k+1]$ w.r.t. $d\mu(s)$ is the same as integrating w.r.t. Lebesgue measure. Thus, $\mathrm{area}_\mu(x_\xi)$ equals the unsigned Euclidean area. Since the variation $x_\xi$ leaves the boundary of the surface unchanged, $\mathrm{area}_\mu(x_\xi)$ does not depend on $\xi$, and we get

$$\begin{aligned}\|v\mathbf{H}_{\mathrm{sd}}\|^2_{L^2(x,V)} &= \left\langle \mathbf{H}_{\mathrm{sd}}, v^2\mathbf{H}_{\mathrm{sd}} \right\rangle_{L^2(x,V)} = -\frac{1}{2}\left\langle \nabla \,\mathrm{area}(x), \dot{x} \right\rangle_{L^2(x,V)} \\ &= -\frac{1}{2}\frac{d}{d\xi}\mathrm{area}(x_\xi)\Big|_{\xi=0} = 0.\end{aligned}$$

We conclude that $v\mathbf{H}_{\mathrm{sd}}$ vanishes for all $v$, i.e., $\mathbf{H}_{\mathrm{sd}} = \text{const.} = 0$. □

## 2.4 Properties of the semidiscrete Laplacian

The classical Laplace operator enjoys several properties like linear precision, symmetry, positive semidefiniteness, and an associated maximum principle for harmonic functions. It is natural to ask if they carry over to the purely discrete or semidiscrete cases (for triangle meshes, these core properties turn out to be incompatible for Laplacians whose definition involves the one-ring neighborhood of individual vertices; see [42]). We start by investigating the kernel of the Laplacian. Surely it contains the constant functions. As to linear functions, we have the following result:

**Lemma 2.2.** *For a regular semidiscrete surface $x$ and its corresponding Laplacian $\Delta_{\mathrm{sd}}$ and mean curvature normal field $\mathbf{H}_{\mathrm{sd}}$, the following statements are equivalent:*

*(a) All functions $u(k,t) = L(x(k,t))$ with $L : V \to \mathbb{R}$ linear are contained in the kernel of $\Delta_{\mathrm{sd}}$.*

*(b) $x$ is harmonic, i.e., $\Delta_{\mathrm{sd}} x = \text{const.} = 0$.*

*(c) $x$ is a minimal surface, i.e., $\mathbf{H}_{\mathrm{sd}} = \text{const.} = 0$.*

*Proof.* The equivalence of (b) and (c) is Theorem 2.2. Since the coordinate components of $x$ are linear functions of $x$, (a) implies (b). Conversely, any linear function is a linear combination of coordinate functions and a constant, so (b) implies (a). □

**Corollary 2.1.** *The semidiscrete Laplacian enjoys the linear precision property, i.e., for a regular semidiscrete surface lying in a two-dimensional plane, all linear functions are contained in the kernel of the Laplacian.*

*Proof.* Combine Lemmas 2.1 and 2.2. □

We show that our semidiscrete Laplacian is symmetric and positive semidefinite in the $L^2$ sense, in a way analogous to the well known Laplace-Beltrami operator (see, e.g., [33]). This follows directly from the variational definition of the Laplacian.

**Lemma 2.3.** *The semidiscrete Laplace operator $\Delta_{\mathrm{sd}}$ is symmetric and positive semidefinite. More precisely, for semidiscrete functions $u$ and $v$, with compact support contained in $D^{\mathrm{inn}}$, we have*

$$\langle \Delta_{\mathrm{sd}} u, v\rangle_{L^2(x)} = \langle u, \Delta_{\mathrm{sd}} v\rangle_{L^2(x)} \quad \textit{and} \quad \langle \Delta_{\mathrm{sd}} u, u\rangle_{L^2(x)} = 2E_\mu(u) \geq 0.$$

*Proof.* We use the quadratic form $\mathcal{E}_\mu$ corresponding to the Dirichlet energy (see Definition 2.3) and compute $\langle \Delta_{\mathrm{sd}} u, v\rangle_{L^2(x)} = \langle \nabla E_\mu(u), v\rangle_{L^2(x)} = \frac{d}{d\xi} E_\mu(u+\xi v)|_{\xi=0} = \frac{d}{d\xi}(\mathcal{E}_\mu(u,u) + 2\xi\mathcal{E}_\mu(u,v) + \xi^2\mathcal{E}_\mu(v,v))|_{\xi=0} = 2\mathcal{E}_\mu(u,v)$, where we have used the relations given in Definition 2.5. This implies symmetry and, for $u = v$, semidefiniteness. □

Unfortunately, the maximum principle is not valid for the semidiscrete Laplacian, even for functions on very simple surfaces. This is in contrast to the smooth case, where the maximum principle holds in general; and it is also in contrast to the cotan-Laplacian on triangle meshes (likewise found as gradient of the Dirichlet energy), where a maximum principle holds, e.g., if all angles are acute. A counterexample is as follows.

**Example 2.2.** Here we construct a semidiscrete harmonic function $u$ with a maximum at the inner point $(0,0)$ of the semidiscrete surface $x(k,t) := (k,t)$. For this purpose we first derive a more explicit expression for the Laplacian $\Delta_{\mathrm{sd}}u$ of a semidiscrete function $u$ on $x$. We extend $x$ to $x(s,t) = (s,t)$ and $u$ to the piecewise-linear function $u(s,t) = \sum_{(k,t)\in\mathbb{Z}\times\mathbb{R}} \varphi(s-k)u(k,t)$. Then $\mathrm{I} = \mathrm{diag}(1,1)$, so by the assumptions (2.4), we get

$$\begin{aligned}
\mathbf{a}(k,t) &= \int_{[k-1,k]\sqcup[k,k+1]} \varphi(s-k)\,d\mu(s) = 1,\\
\mathbf{b}(k,t) &= -\int_{[k-1,k]} \delta u(s,t)\,d\mu(s) = -\delta u(k-1,t) = u(k,t) - u(k-1,t),\\
\mathbf{c}(k,t) &= -\int_{[k-1,k]\sqcup[k,k+1]} \varphi(s-k)u'(s,t)\,d\mu(s) =\\
&= -2m_2u'(k,t) - \left(\tfrac{1}{2} - m_2\right)\left(u'(k-1,t) + u'(k+1,t)\right),
\end{aligned}$$

where $m_2 = \int_{[0,1]} s^2\,d\mu_0$ is the second moment of the measure $\mu_0$. Hence, in this situation, the Laplacian of $u$ is given by

$$\begin{aligned}
\Delta_{\mathrm{sd}}u(k,t) = 2u(k,t) &- u(k-1,t) - u(k+1,t)-\\
&- 2m_2u''(k,t) - \left(\tfrac{1}{2} - m_2\right)\left(u''(k-1,t) + u''(k+1,t)\right).
\end{aligned}$$

The harmonicity condition $\Delta_{\mathrm{sd}}u = 0$ thus becomes a system of *linear* ODEs for the functions $t \mapsto u(k,t)$, where $k$ runs through the integers. Observe that the assumptions (2.4) imply $\frac{1}{4} \leq m_2 \leq \frac{1}{2}$. The maximum principle obviously holds if $m_2 = \frac{1}{2}$, which applies, e.g., to the trapezoidal rule. Otherwise, for $m_2 < \frac{1}{2}$, we can construct a harmonic function $u$ on $x$ with a maximum at $(0,0)$ as follows. Choosing $u(0,t) := -t^2$ and assuming symmetry $u(\pm1,t) := \phi(t)$, we find $\phi(t)$ easily as $\phi(t) = 1-t^2+\gamma_1\cos\left(\left(\frac{1}{2}-m_2\right)^{-1/2}t\right)+\gamma_2\sin\left(\left(\frac{1}{2}-m_2\right)^{-1/2}t\right)$. An appropriate choice of constants, e.g., $\gamma_1 = -2$, $\gamma_2 = 0$, yields a function $u(k,t)$, which undoubtedly has a local maximum in $u(0,0) = 1$. We have thus created a locally defined counterexample to the maximum principle. It can be turned into a globally defined example by constructing $u(\pm2,t), u(\pm3,t), \ldots$ such that overall $\Delta_{\mathrm{sd}}u = 0$: one has to iteratively solve linear ODEs.

## 2.5 Pointwise convergence / consistency

In this section we show that the semidiscrete Laplace operator converges pointwise to its smooth counterpart, as the semidiscrete surface converges to a smooth one. In the Finite

Elements literature this kind of convergence is called *consistency*, while *convergence* would be reserved for the situation where the solutions of equations involving the semidiscrete Laplacian converge to solutions of equations which involve the continuous Laplacian.

More precisely, the situation in the following theorem is as follows. We fix a point $p$ on a regular surface $\mathcal{M}$, which is assumed to have a local parametrization $f$. Without loss of generality, $p = f(0,0)$. Next, we consider the semidiscrete surface

$$x_\varepsilon : (k,t) \mapsto f(\varepsilon k, t), \quad \varepsilon > 0,$$

which obviously contains the point $p = x_\varepsilon(0,0)$ and is inscribed in the surface $\mathcal{M}$. Then we analyze the semidiscrete Laplace operator associated with $x_\varepsilon$ and its action on functions $u_\varepsilon$, and let $\varepsilon \to 0$.

**Theorem 2.3.** *Consider a smooth regular surface $\mathcal{M}$ with parametrization $f$ and a real-valued function $u(s,t)$ which represents a function defined on the surface $\mathcal{M}$. Let $p = f(0,0)$.*

*Semidiscretize these objects by defining a semidiscrete surface $x_\varepsilon(k,t) := f(\varepsilon k, t)$ and a semidiscrete function $u_\varepsilon(k,t) := u(\varepsilon k, t)$. Then the corresponding semidiscrete Laplace operator $\Delta^\varepsilon_{\mathrm{sd}}$ converges to the Laplace-Beltrami operator $\Delta_\mathcal{M}$ defined on $\mathcal{M}$:*

$$f, u \in C^2 \implies (\Delta^\varepsilon_{\mathrm{sd}} u_\varepsilon)(0,0) = (\Delta_\mathcal{M} u)(p) + o(1), \quad \text{as } \varepsilon \to 0.$$

*In case the measure $d\mu_0$ is symmetric in the sense of Equ.* (2.5)*, convergence is improved:*

$$\begin{aligned} f, u \in C^3 &\implies (\Delta^\varepsilon_{\mathrm{sd}} u_\varepsilon)(0,0) = (\Delta_\mathcal{M} u)(p) + \mathrm{o}(\varepsilon), \quad \text{as } \varepsilon \to 0, \\ f, u \in C^4 &\implies (\Delta^\varepsilon_{\mathrm{sd}} u_\varepsilon)(0,0) = (\Delta_\mathcal{M} u)(p) + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Theorem 2.2 immediately implies a convergence statement concerning mean curvature:

**Corollary 2.2.** *In the situation of Theorem 2.3, the semidiscrete mean curvature normal $\mathbf{H}^\varepsilon_{\mathrm{sd}}$ on $x_\varepsilon$ converges pointwise to its smooth counterpart (with the rate of convergence depending on the smoothness of $x$).*

*Proof of Theorem 2.3.* We first set up some notation. For differentiation with respect to $s$ and $t$ we use the notation $\partial_1$ and $\partial_2$, respectively. The coefficients of the first fundamental form are denoted by $g_{ij} := \langle \partial_i f, \partial_j f \rangle$. Their determinant is denoted by $\det \mathrm{I} = g_{11} g_{22} - g_{12}^2$. We also use the symbols $\rho_{ijk} := \langle \partial_i f, \partial_{jk} f \rangle$.

- *Step 1: Overview of the proof.* In local coordinates, the Laplacian is expressed as

$$\begin{aligned} &\Delta_\mathcal{M} u = \frac{1}{\mathbf{a}^{\mathrm{cont}}} (\partial_1 \mathbf{b}^{\mathrm{cont}} + \partial_2 \mathbf{c}^{\mathrm{cont}}), \quad \text{where} \\ &\mathbf{a}^{\mathrm{cont}} = \sqrt{\det \mathrm{I}}, \quad \mathbf{b}^{\mathrm{cont}} = \frac{g_{12} \partial_2 u - g_{22} \partial_1 u}{\sqrt{\det \mathrm{I}}}, \quad \mathbf{c}^{\mathrm{cont}} = \frac{g_{12} \partial_1 u - g_{11} \partial_2 u}{\sqrt{\det \mathrm{I}}} \end{aligned}$$

(see, e.g., [33, p. 18]). On the other hand, the semidiscrete Laplacian $\Delta^{\varepsilon}_{\mathrm{sd}}$ associated with $x_\varepsilon$ is computed, using notation $\mathbf{a}_\varepsilon$, $\mathbf{b}_\varepsilon$, $\mathbf{c}_\varepsilon$ analogous to Theorem 2.1, as

$$\Delta^{\varepsilon}_{\mathrm{sd}} u_\varepsilon(0,0) = \frac{1}{\mathbf{a}_\varepsilon}\Big(\delta \mathbf{b}_\varepsilon + \mathbf{c}'_\varepsilon\Big)\Big|_{(0,0)}.$$

We compute Taylor polynomials around $(0,0)$ for $x_\varepsilon(\pm 1, 0) = f(\pm\varepsilon, 0)$ and $u_\varepsilon(\pm 1, 0) = u(\pm\varepsilon, 0)$, and insert them into this formula. Long computations yield

$$\frac{\mathbf{a}_\varepsilon(0,0)}{\varepsilon} \approx \mathbf{a}^{\mathrm{cont}}(0,0), \quad \frac{\delta\mathbf{b}_\varepsilon(0,0)}{\varepsilon} \approx \partial_1 \mathbf{b}^{\mathrm{cont}}(0,0), \quad \frac{\mathbf{c}'_\varepsilon(0,0)}{\varepsilon} \approx \partial_2 \mathbf{c}^{\mathrm{cont}}(0,0),$$

where the $\approx$ symbol means equality up to $O(\varepsilon^2)$ in the $C^4$ case, resp. $o(\varepsilon)$ in the $C^3$ case, resp. $o(1)$ in the $C^2$ case. Having obtained these convergence rates, the proof is complete. It remains to perform the above-mentioned long computations.

• *Step 2: Taylor expansion of $x_\varepsilon$, $u_\varepsilon$ and their derivatives.* Note that

$$x_\varepsilon(s,0) = \sum_k \varphi(s-k) x_\varepsilon(k,0) = \begin{cases} (1+s) f(0,0) - s f(-\varepsilon, 0), & \text{if } s \in [-1,0], \\ (1-s) f(0,0) + s f(\varepsilon, 0), & \text{if } s \in [0,1]. \end{cases} \tag{2.8}$$

The expression for $u_\varepsilon$ in terms of $u(\pm\varepsilon, 0)$ is analogous. The Taylor polynomials of $f(\pm\varepsilon,0)$ and its derivatives around $\varepsilon = 0$ are in the $C^4$ case given by

$$\begin{aligned} f(\pm\varepsilon,0) &= f(0,0) \pm \varepsilon\partial_1 f(0,0) + \tfrac{\varepsilon^2}{2}\partial_{11} f(0,0) \pm \tfrac{\varepsilon^3}{6}\partial_{111} f(0,0) + O(\varepsilon^4), \\ \partial_2 f(\pm\varepsilon,0) &= \partial_2 f(0,0) \pm \varepsilon\partial_{12} f(0,0) + \tfrac{\varepsilon^2}{2}\partial_{112} f(0,0) + O(\varepsilon^3). \\ \partial_{22} f(\pm\varepsilon,0) &= \partial_{22} f(0,0) \pm \varepsilon\partial_{122} f(0,0) + O(\varepsilon^2). \end{aligned}$$

The remainder terms $O(\varepsilon^j)$ in the individual formulas have to be replaced by $o(\varepsilon^{j-1})$ in the $C^3$ case. In the $C^2$ case, the terms containing third order partial derivatives of $f$ have to be replaced by $o(\varepsilon^{j-2})$. There are analogous expressions for $u(\pm\varepsilon,0)$ and its derivatives.

• *Step 3: Taylor expansion of the area element.* For sufficiently small $\varepsilon > 0$, we consider the first fundamental form $\mathrm{I}_\varepsilon$ associated with the piecewise-ruled surface $x_\varepsilon$ and look at the quantity

$$\alpha(\varepsilon, s) = \sqrt{\det \mathrm{I}_\varepsilon(s,0)}, \qquad s \in [-1,1].$$

Note that, for $s \in [0,1]$, $\det \mathrm{I}_\varepsilon(s,0)$ is the Gram determinant of vectors

$$\delta x_\varepsilon(s,0) = f(\varepsilon,0) - f(0,0), \quad x'_\varepsilon(s,0) = (1-s)\partial_2 f(0,0) + s\partial_2 f(\varepsilon,0).$$

In the $C^4$ case, a simple computation and taking square roots by means of the binomial series yields

$$\begin{aligned} &\det \mathrm{I}_\varepsilon(s,0) = \varepsilon^2\left(\alpha_1 + \varepsilon(\alpha_2 + s\alpha_3) + \varepsilon^2\alpha_4 + O(\varepsilon^3)\right), \text{ as } \varepsilon \to 0, \quad \text{with} \\ &\alpha_1 = \det \mathrm{I}\,|_{(0,0)},\ \alpha_2 = (g_{22}\rho_{111} - g_{12}\rho_{211})|_{(0,0)},\ \alpha_3 = 2(g_{11}\rho_{212} - g_{12}\rho_{112})|_{(0,0)}, \\ &\alpha(\varepsilon,s) = |\varepsilon|\Big(\alpha_1^{1/2} + \frac{\varepsilon}{2}\frac{\alpha_2 + s\alpha_3}{\alpha_1^{1/2}} + \frac{\varepsilon^2}{8}\frac{4\alpha_1\alpha_4 - (\alpha_2 + s\alpha_3)^2}{\alpha_1^{3/2}} + O(\varepsilon^3)\Big). \end{aligned}$$

In the $C^3$ case, the remainder term is $o(\varepsilon^2)$, while in the $C^2$ case, the terms involving $\varepsilon^2$ have to be replaced by $o(\varepsilon)$. For $s \in [-1,0]$, the situation is analogous.

• *Step 4: The relation between* $\mathbf{a}_\varepsilon$ *and* $\mathbf{a}^{\mathrm{cont}}$. Our aim is to give a proof of $\frac{1}{\varepsilon}\mathbf{a}_\varepsilon(0,0) \approx \mathbf{a}^{\mathrm{cont}}(0,0)$, where the meaning of "≈" is equality up to an error term depending on the differentiability class of the objects involved. The relation $\alpha(\varepsilon,-s) = \alpha(-\varepsilon,s)$ yields

$$\begin{aligned}\mathbf{a}_\varepsilon(0,0) &= \int_{[-1,0]} (1+s)\alpha(\varepsilon,s)\,d\mu(s) + \int_{[0,1]} (1-s)\alpha(\varepsilon,s)\,d\mu(s) = \\ &= \int_{[0,1]} s\alpha(-\varepsilon,1-s) + (1-s)\alpha(\varepsilon,s)\,d\mu(s) = \\ &= |\varepsilon| \int_{[0,1]} \alpha_1^{1/2} + \frac{\varepsilon}{2}(1-2s)\alpha_2\alpha_1^{-1/2} + O(\varepsilon^2)\,d\mu(s)\end{aligned}$$

in the $C^3$ and $C^4$ cases, and the same formula with remainder term $o(\varepsilon)$ in the $C^2$ case. Now Equation (2.4) yields $\int_{[0,1]}(1-2s)\,d\mu(s) = 0$, so the result follows.

• *Step 5: Relation between* $\mathbf{b}_\varepsilon$ *and* $\mathbf{b}^{\mathrm{cont}}$. With computations similar to those of the previous Step 4, it is not difficult to see that

$$\delta\mathbf{b}_\varepsilon(0,0) = \mathbf{b}_\varepsilon(1,0) - \mathbf{b}_\varepsilon(0,0) = \int_{[0,1]} \frac{\beta(-\varepsilon,1-s)}{\alpha(-\varepsilon,1-s)} + \frac{\beta(\varepsilon,s)}{\alpha(\varepsilon,s)}\,d\mu_0(s), \tag{2.9}$$

where $\beta(\varepsilon,s) := \langle \delta x_\varepsilon, x'_\varepsilon\rangle u'_\varepsilon - \|x'_\varepsilon\|^2\delta u_\varepsilon|_{(0,0)}$ is expressed as

$$\begin{aligned}\beta(\varepsilon,s) = \big\langle f(\varepsilon,0) - f(0,0),\ (1-s)\partial_2 f(0,0) + s\partial_2 f(\varepsilon,0)\big\rangle\big((1-s)\partial_2 u(0,0) + s\partial_2 u(\varepsilon,0)\big) - \\ - \|(1-s)\partial_2 f(0,0) + s\partial_2 f(\varepsilon,0)\|^2\big(u(\varepsilon,0) - u(0,0)\big),\ \text{for } s \in [0,1].\end{aligned}$$

Note that, for $s \in [0,1]$, $\beta(\varepsilon,-s) = -\beta(-\varepsilon,s)$ and $\alpha(\varepsilon,-s) = \alpha(-\varepsilon,s)$. Assuming symmetry of the measure $\mu_0$, this simplifies to

$$\delta\mathbf{b}_\varepsilon(0,0) = \int_{[0,1]} \frac{\beta(-\varepsilon,s)}{\alpha(-\varepsilon,s)} + \frac{\beta(\varepsilon,s)}{\alpha(\varepsilon,s)}\,d\mu_0(s).$$

Inserting Taylor polynomials yields the expansion (for the $C^3$ case)

$$\begin{aligned}&\beta(\varepsilon,s) = \varepsilon\beta_1 + \varepsilon^2(\beta_2 + s\beta_3) + \varepsilon^3\beta_4 + o(\varepsilon^3), \quad \text{where} \\ &\quad \beta_1 = g_{12}\partial_2 u - g_{22}\partial_1 u|_{0,0}, \quad \beta_2 = \frac{1}{2}(\rho_{211}\partial_2 u - g_{22}\partial_{11} u)|_{0,0}, \\ &\quad \beta_3 = (\rho_{112}\partial_2 u - 2\rho_{212}\partial_1 u + g_{12}\partial_{12} u)|_{0,0}.\end{aligned}$$

This leads to

$$\frac{\beta(-\varepsilon,s)}{\alpha(-\varepsilon,s)} + \frac{\beta(\varepsilon,s)}{\alpha(\varepsilon,s)} = \frac{\varepsilon^3\big(2(\beta_2 + s\beta_3)\alpha_1^{1/2} - \beta_1(\alpha_2 + s\alpha_3)\alpha_1^{-1/2}\big) + o(\varepsilon^4)}{\varepsilon^2\alpha_1 + o(\varepsilon^3)}.$$

Integration with respect to $d\mu(s)$ and substituting the definitions of $\alpha_j$, $\beta_j$ eventually yields

$$\frac{\delta \mathbf{b}_\varepsilon(0,0)}{\varepsilon} = \frac{2\beta_2 + \beta_3}{\alpha_1^{1/2}} - \frac{\beta_1(\alpha_2 + \frac{1}{2}\alpha_3)}{\alpha_1^{3/2}} + \mathrm{o}(\varepsilon) = \partial_1 \mathbf{b}^{\mathrm{cont}}(0,0) + \mathrm{o}(\varepsilon).$$

In the $C^4$ case the remainder term is $O(\varepsilon^2)$, whereas in the $C^2$ case, where symmetry of the measure is not required, the integral on the right hand side of Equation (2.9) does not simplify as shown above, and we only get a remainder term of $o(1)$.

• *Step 6: Computing the derivative of* $\mathbf{c}(k,t)$. The following explicit formula, which is found by differentiating the definition of $\mathbf{c}(k,t)$, is needed later:

$$\begin{aligned} \frac{d\mathbf{c}(k,t)}{dt} &= \int_{[k-1,k]\sqcup[k,k+1]} \varphi(s-k)\frac{\partial}{\partial t}\left(\frac{\langle \delta x, x'\rangle \delta u - \|\delta x\|^2 u'}{\det \mathrm{I}^{1/2}}\right) d\mu(s) \\ &= \int_{[k-1,k]\sqcup[k,k+1]} \varphi(s-k)\left(\frac{c^*(s,t)}{\det \mathrm{I}^{1/2}} - \frac{c^{**}(s,t)}{\det \mathrm{I}^{3/2}}\right) d\mu(s), \quad \text{where} \end{aligned}$$

$$\begin{aligned} c^* &:= (\langle \delta x', x'\rangle + \langle \delta x, x''\rangle)\delta u + \langle \delta x, x'\rangle \delta u' - 2\langle \delta x, \delta x'\rangle u' - \|\delta x\|^2 u'', \\ c^{**} &:= (\langle \delta x, x'\rangle \delta u - \|\delta x\|^2 u')\left[\langle x', x''\rangle \|\delta x\|^2 - (\langle \delta x', x'\rangle + \langle \delta x, x''\rangle)\langle \delta x, x'\rangle + \langle \delta x, \delta x'\rangle \|x'\|^2\right]. \end{aligned}$$

• *Step 7: Relation between* $\mathbf{c}_\varepsilon$ *and* $\mathbf{c}^{\mathrm{cont}}$. We use the notation of Step 6 to introduce the symbols $\gamma^*(\varepsilon,s)$, $\gamma^{**}(\varepsilon,s)$, which arise from the functions $c^*$, $c^{**}$, resp., by substituting $x_\varepsilon$ for $x$ and $u_\varepsilon$ for $u$, and letting $t=0$. Note that $\gamma^*(\varepsilon,-s) = \gamma^*(-\varepsilon,s)$ and the same for $\gamma^{**}$, for $s \in [0,1]$. With a computation similar to Step 4, it is easy to see that $\mathbf{c}'_\varepsilon(0,0)$ is expressed as

$$\int_{[0,1]} s\frac{\gamma^*(-\varepsilon,1-s)}{\alpha(-\varepsilon,1-s)} + (1-s)\frac{\gamma^*(\varepsilon,s)}{\alpha(\varepsilon,s)} - s\frac{\gamma^{**}(-\varepsilon,1-s)}{\alpha(-\varepsilon,1-s)^3} - (1-s)\frac{\gamma^{**}(\varepsilon,s)}{\alpha(\varepsilon,s)^3}\, d\mu_0(s).$$

Assuming symmetry of the measure $\mu_0$, this expression simplifies to

$$\int_{[0,1]} (1-s)\left(\frac{\gamma^*(-\varepsilon,s)}{\alpha(-\varepsilon,s)} + \frac{\gamma^*(\varepsilon,s)}{\alpha(\varepsilon,s)}\right) - (1-s)\left(\frac{\gamma^{**}(-\varepsilon,s)}{\alpha(-\varepsilon,s)^3} + \frac{\gamma^{**}(\varepsilon,s)}{\alpha(\varepsilon,s)^3}\right) d\mu_0(s).$$

In the same manner as before we get the expansions

$$\gamma^*(\varepsilon,s) = \varepsilon^2\gamma_1^* + \varepsilon^3(\gamma_2^* + s\gamma_3^*) + \mathrm{o}(\varepsilon^3), \qquad \gamma^{**}(\varepsilon,s) = \varepsilon^4\gamma_1^{**} + \varepsilon^5(\gamma_2^{**} + s\gamma_3^{**}) + \mathrm{o}(\varepsilon^5),$$

where

$$\begin{aligned} \gamma_1^* &= ((\rho_{122} + \rho_{212})\partial_1 u + g_{12}\partial_{12}u - 2\rho_{112}\partial_2 u - g_{11}\partial_{22}u)\big|_{(0,0)}, \\ \gamma_1^{**} &= (g_{12}\partial_1 u - g_{11}\partial_2 u)(\rho_{112}g_{22} + \rho_{222}g_{11} - (\rho_{212} + \rho_{122})g_{12})\big|_{(0,0)}. \end{aligned}$$

This leads to

$$\frac{\gamma^*(-\varepsilon,s)}{\alpha(-\varepsilon,s)} + \frac{\gamma^*(\varepsilon,s)}{\alpha(\varepsilon,s)} = \frac{1}{\varepsilon^2\alpha_1 + \mathrm{o}(\varepsilon^3)}\Big(\varepsilon^3 2\gamma_1^*\alpha_1^{1/2} + \mathrm{o}(\varepsilon^4)\Big) = \varepsilon\frac{2\gamma_1^*}{\alpha_1^{1/2}} + \mathrm{o}(\varepsilon^2),$$
$$\frac{\gamma^{**}(-\varepsilon,s)}{\alpha(-\varepsilon,s)^3} + \frac{\gamma^{**}(\varepsilon,s)}{\alpha(\varepsilon,s)^3} = \frac{1}{\varepsilon^6\alpha_1^3 + \mathrm{o}(\varepsilon^7)}\Big(\varepsilon^7 2\gamma_1^{**}\alpha_1^{3/2} + \mathrm{o}(\varepsilon^8)\Big) = \varepsilon\frac{2\gamma_1^{**}}{\alpha_1^{3/2}} + \mathrm{o}(\varepsilon^2).$$

Using property (2.4) of the measure $\mu_0$ for integration, and substituting the definitions of $\alpha_1, \gamma_1^*, \gamma_1^{**}$, one eventually gets

$$\frac{\mathbf{c}'_\varepsilon(0,0)}{\varepsilon} = \frac{\gamma_1^*}{\alpha_1^{1/2}} - \frac{\gamma_1^{**}}{\alpha_1^{3/2}} + \mathrm{o}(\varepsilon) = \partial_2\mathbf{c}^{\mathrm{cont}}(0,0) + o(\varepsilon).$$

This result applies to the $C^3$ case. In the $C^4$ case, one more term in the Taylor polynomials becomes available, and the remainder term in the formula above becomes $O(\varepsilon^2)$ instead of $o(\varepsilon)$. In the $C^2$ case, where symmetry of the measure $\mu_0$ is not required, we only get $\mathrm{o}(1)$ (the details are omitted). The estimates of Steps 4, 5, and 7 together conclude the proof. □

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# Chapter 3

# Semidiscrete constant mean curvature surfaces and their associated families

**Abstract**

The present paper studies semidiscrete surfaces in three-dimensional Euclidean space within the framework of integrable systems. In particular, we investigate semidiscrete surfaces with constant mean curvature along with their associated families. The notion of mean curvature introduced in this paper is motivated by a recently developed curvature theory for quadrilateral meshes equipped with unit normal vectors, and extends previous work on semidiscrete surfaces. In the situation of vanishing mean curvature, the associated families are defined via a Weierstraß representation. For the general cmc case, we introduce a Lax pair representation that directly defines associated families of cmc surfaces, and is connected to a semidiscrete sinh-Gordon equation. Utilizing this theory we investigate semidiscrete Delaunay surfaces and their connection to elliptic billiards.

**Keywords:** Semidiscrete surface, Constant mean curvature, Associated family, Weierstraß representation, Lax pair representation.

**Mathematics Subject Classification (2010):** 53A05, 53A10, 39A12.

## 3.1 Introduction

Surfaces with constant mean curvature $H$ or constant Gauß curvature $K$ have been of particular interest in differential geometry for a long time. In a modern viewpoint, these special geometries are associated with the theory of integrable systems, not least due to rather recent developments in discrete differential geometry (cf. Bobenko and Suris [10]). Typically the investigation of constant curvature surfaces is tied to specific parametrizations, like isothermic parametrizations for constant mean curvature surfaces.

Over the last decades, various discrete versions of these special parametrizations have been established. For a comprehensive overview see Bobenko and Pinkall [7] or Bobenko and Suris

This chapter comprises the research article [iii].

[10]. Generally, different kinds of parametrizations (conjugate, asymptotic, ...) have their own way of discretization. For this reason, discretizing entire families of smooth surfaces is a challenge, if the type of parametrization changes. Accordingly, a unifying discrete curvature theory is still an active topic of research. As a first step toward this direction, Bobenko et al. [8] introduced a general curvature theory for polyhedral meshes with planar faces based on mesh parallelity. Their theory is capable of unifying notable previously defined classes of surfaces, such as discrete isothermic minimal or constant mean curvature surfaces. More recently, Hoffmann et al. [20] presented a discrete parametrized surface theory for quadrilateral meshes equipped with unit normal vectors at the vertices, permitting non-planar faces. Their theory encompasses a remarkably large class of existing discrete special parametrizations. In addition it provides a deeper insight into the associated families of discrete constant curvature surfaces.

For semidiscrete surfaces, represented by parametrizations possessing one discrete variable and one continuous variable, the situation is quite similar to the discrete case. The analysis of semidiscrete surfaces with $H$ = const. respectively $K$ = const. is bound to isothermic resp. asymptotic parametrizations (cf. Müller [26] resp. Wallner [40]). However, to the author's knowledge, results concerning their associated families have been missing so far.

### 3.1.1 Outline and results

In the present paper we investigate two distinct situations: (i) semidiscrete surfaces with vanishing mean curvature (*minimal surfaces*), and (ii) semidiscrete surfaces with constant but non-vanishing mean curvature (*cmc surfaces*). Since we are especially interested in the associated families of these surfaces, we do not restrict ourselves to isothermic parametrizations. Thus, at the beginning (see Section 3.2), we translate the discrete curvature theory introduced by Hoffmann et al. [20] to the semidiscrete setting. We also highlight the intersection with the curvature theory for semidiscrete conjugate parametrizations previously considered by Karpenkov and Wallner [22].

In Section 3.3, we recapitulate the notion of isothermic parametrizations. In particular, we show that a semidiscrete surface is isothermic if and only if its quaternionic cross ratio allows for a specific factorization (cf. Lemma 3.5).

Subsequently, in Section 3.4, we investigate semidiscrete isothermic minimal surfaces. Their Weierstraß representation, established by Rossman and Yasumoto [34], immediately gives rise to their associated families, whose members are however no longer isothermic. The main result of this section is that all the members of these associated families are minimal as well (cf. Theorem 3.1). Moreover, we show that the conjugate surface of an isothermic minimal surface is asymptotically parametrized.

In Section 3.5, we introduce a Lax pair representation for semidiscrete isothermic cmc surfaces, which directly contains the definition of their associated families. We prove that the members of these associated families, which again are no longer isothermic, all have the same constant mean curvature (cf. Theorem 3.2).

We conclude the paper by investigating the Lax pair representation of semidiscrete rotational symmetric cmc surfaces (see Section 3.6). It turns out that the discrete version of the classical Delaunay rolling ellipse construction, obtained by Bobenko et al. [8], also applies to the semidiscrete setting.

## 3.2 A curvature theory for semidiscrete surfaces

Our main object of study are two-dimensional semidiscrete surfaces in three-dimensional Euclidean space represented by parametrizations

$$x : \mathbb{Z} \times \mathbb{R} \supseteq D \to \mathbb{R}^3 : (k,t) \mapsto x(k,t)$$

possessing one discrete variable and one continuous variable. Throughout this paper we assume that $x$ is at least once continuously differentiable w.r.t. the second argument. We abbreviate the corresponding derivative by $\partial x$. The forward difference w.r.t. the discrete parameter is denoted by

$$\delta x := x_1 - x,$$

where the notation $x_1$ indicates an index shift: $x_1(k,t) := x(k+1,t)$. We only consider *regular* semidiscrete surfaces having the property that the sets $\{\delta x, \partial x\}$, $\{\delta x, \partial x_1\}$, and $\{\delta x, \partial x + \partial x_1\}$ are linearly independent throughout.

Just like a smooth parametrized surface can be viewed as built of its *contact elements* (consisting of a point together with the surface normal at this point), we henceforth consider a semidiscrete surface to be represented by a *pair* of weakly coupled parametrizations. Translating the relation between a surface and its Gauß map to the semidiscrete setting, we define:

**Definition 3.1.** A pair of semidiscrete surfaces $(x,n) : \mathbb{Z}\times\mathbb{R} \supseteq D \to \mathbb{R}^3 \times \mathbb{S}^2$ is called *coupled*, if

$$\delta x \perp (n + n_1) \quad \text{and} \quad \partial x \perp n \tag{3.1}$$

throughout the parameter domain.

The following definition contains a limit version of the "midpoint connectors" of a quadrilateral considered by Hoffmann et al. [20] as replacements of the first order partial derivatives of a smooth parametrization.

**Definition 3.2.** For a semidiscrete surface $(x,n)$ we define the *partial derivatives*

$$\partial_1 x := \delta x = x_1 - x, \quad \partial_2 x := \frac{\partial x + \partial x_1}{2},$$

as well as the *strip normal*

$$N := \frac{\partial_1 n \times \partial_2 n}{\|\partial_1 n \times \partial_2 n\|} = \frac{\delta n \times (\partial n + \partial n_1)}{\|\delta n \times (\partial n + \partial n_1)\|}.$$

In classical surface theory the principal curvatures of a surface at a point are defined as the eigenvalues of the shape operator that lives on the tangent plane at this point. In the semidiscrete case the fundamental forms and the shape operator live on the plane perpendicular to the strip normal $N$.

**Definition 3.3.** Let $(x, n)$ be a semidiscrete surface with strip normal $N$ and let $\pi$ denote the orthogonal projection onto the plane perpendicular to $N$, i.e., $\pi(x) := x - \langle x, N\rangle N$. Mimicking the smooth case, we define the *fundamental forms* I, II, III, and the *shape operator* $S$ by

$$\mathrm{I} := \begin{pmatrix} \|\pi(\partial_1 x)\|^2 & \langle \pi(\partial_1 x), \pi(\partial_2 x)\rangle \\ \text{symm.} & \|\pi(\partial_2 x)\|^2 \end{pmatrix}, \qquad \mathrm{III} := \begin{pmatrix} \|\partial_1 n\|^2 & \langle \partial_1 n, \partial_2 n\rangle \\ \text{symm.} & \|\partial_2 n\|^2 \end{pmatrix},$$

$$\mathrm{II} := -\begin{pmatrix} \langle \partial_1 x, \partial_1 n\rangle & \langle \partial_1 x, \partial_2 n\rangle \\ \langle \partial_2 x, \partial_1 n\rangle & \langle \partial_2 x, \partial_2 n\rangle \end{pmatrix}, \qquad S := \mathrm{I}^{-1}\,\mathrm{II}\,.$$

The following observation is crucial for the definition of the mean and Gauß curvatures via the shape operator.

**Lemma 3.1.** *If the pair* $(x, n)$ *is coupled, the second fundamental form* II *is symmetric.*

*Proof.* Differentiating the equation $\langle \delta x, n + n_1\rangle = 0$ yields $\langle \delta x, \partial n + \partial n_1\rangle = -\langle \partial\delta x, n + n_1\rangle$. Using the assumptions $\partial x \perp n$ and $\partial x_1 \perp n_1$ completes the proof. □

Symmetry of the second fundamental form is equivalent to the selfadjointness of the shape operator $S$ w.r.t. the inner product induced by the first fundamental form. In case of symmetry, the eigenvalues of $S$ are real.

**Definition 3.4.** Let $(x, n)$ be a coupled semidiscrete surface and let $\kappa_1, \kappa_2 \in \mathbb{R}$ be the eigenvalues of the shape operator $S$. Then the *mean curvature* $H$ and the *Gauß curvature* $K$ are defined as

$$H := \frac{1}{2}\,\mathrm{tr}(S) = \frac{\kappa_1 + \kappa_2}{2} \quad \text{and} \quad K := \det(S) = \frac{\det \mathrm{II}}{\det \mathrm{I}} = \kappa_1\kappa_2.$$

Another approach toward a meaningful curvature theory for discrete or semidiscrete surfaces uses the concept of offsets and their connection to the mean and Gauß curvatures via the Steiner formula. This viewpoint has already been examined, e.g., by Bobenko et al. [8] in the purely discrete setting, and by Karpenkov and Wallner [22] in the semidiscrete case. We are going to demonstrate that the curvatures given in Definition 3.4 can just as well be gained via the Steiner formula. First we note that coupled semidiscrete surfaces naturally feature offsets.

**Lemma 3.2.** *A pair of semidiscrete surfaces* $(x, n)$ *is coupled if and only if for some (and hence for all)* $r \in \mathbb{R}$ *the offset* $(x^r, n) := (x + r\,n, n)$ *is coupled.*

*Proof.* Since $n \in \mathbb{S}^2$, we have $\langle \partial x^r, n\rangle = \langle \partial x, n\rangle$ and $\langle \delta x^r, n + n_1\rangle = \langle \delta x, n + n_1\rangle$, for all $r \in \mathbb{R}$. □

The relation between offsets and curvatures is established by the so-called mixed area form. The following definition is motivated by the work of Hoffmann et al. [20]. Also note the similarities to the mixed area form for parallel conjugate semidiscrete surfaces previously investigated by Karpenkov and Wallner [22].

**Definition 3.5.** For two semidiscrete surfaces $(x, n)$, $(y, n)$ with the same Gauß map $n$ and strip normal $N$, the *mixed area form* is given by

$$\begin{aligned} A(x, y) &:= \frac{1}{2}\Big( \det(\partial_1 x, \partial_2 y, N) + \det(\partial_1 y, \partial_2 x, N)\Big) = \\ &= \frac{1}{4}\Big( \det(\delta x, \partial y + \partial y_1, N) + \det(\delta y, \partial x + \partial x_1, N)\Big). \end{aligned}$$

It turns out that, for a coupled semidiscrete surface $(x, n)$, the mean and Gauß curvatures from Definition 3.4 can be expressed in terms of the mixed area forms of the parametrization $x$ and its Gauß map $n$ in a way completely analogous to the smooth setting. In particular, this observation shows that the curvatures given in Definition 3.4 coincide with those discussed by Karpenkov and Wallner [22] in the case of circular surfaces (see Definition 3.6).

**Lemma 3.3.** *Let $(x, n)$ be a coupled semidiscrete surface, then*

$$(i)\ \det \mathrm{I} = A(x, x)^2, \quad (ii)\ K = \frac{A(n, n)}{A(x, x)}, \quad (iii)\ H = -\frac{A(x, n)}{A(x, x)}, \quad (iv)\ \mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I} = 0.$$

*Proof.* $(i)$ We have

$$\det \mathrm{I} = \|\pi(\partial_1 x) \times \pi(\partial_2 x)\|^2 = \det(\pi(\partial_1 x), \pi(\partial_2 x), N)^2 = \det(\partial_1 x, \partial_2 x, N)^2 = A(x, x)^2.$$

$(ii)$ Using the Binet-Cauchy identity, we compute

$$\det \mathrm{II} = \langle \pi(\partial_1 x) \times \pi(\partial_2 x), \partial_1 n \times \partial_2 n\rangle = \det(\pi(\partial_1 x), \pi(\partial_2 x), N)\|\partial_1 n \times \partial_2 n\| = A(x, x)A(n, n).$$

$(iii)$ Likewise, we obtain

$$\begin{aligned} A(x, n) &= \frac{1}{2}\Big( \det(\pi(\partial_1 x), \partial_2 n, N) + \det(\partial_1 n, \pi(\partial_2 x), N)\Big) = \\ &= \frac{1}{2A(x, x)}\Big(\|\pi(\partial_1 x)\|^2\langle\pi(\partial_2 x), \partial_2 n\rangle - \langle\pi(\partial_1 x), \pi(\partial_2 x)\rangle\langle\pi(\partial_1 x), \partial_2 n\rangle + \\ &\quad + \|\pi(\partial_2 x)\|^2\langle\pi(\partial_1 x), \partial_1 n\rangle - \langle\pi(\partial_1 x), \pi(\partial_2 x)\rangle\langle\pi(\partial_2 x), \partial_1 n\rangle\Big) = -\frac{A(x, x)}{2}\operatorname{tr}(S). \end{aligned}$$

$(iv)$ By the Cayley-Hamilton theorem $S^2 - \operatorname{tr}(S)S + \det(S)E = 0$, which yields the last equation. □

**Corollary 3.1** (Semidiscrete Steiner formula)**.** *Let $(x, n)$ be a coupled semidiscrete surface with offset $(x^r, n) = (x + r\,n, n)$, $r \in \mathbb{R}$. Then,*

$$A(x^r, x^r) = (1 - 2Hr + Kr^2)A(x, x).$$

Next, we recapitulate the notion of semidiscrete isothermic parametrizations.

## 3.3 Semidiscrete isothermic surfaces

A smooth parametrization is called isothermic, if it is a conformal curvature line parametrization, possibly upon a reparametrization of independent variables.

A discrete analog of curvature line parametrizations is given, for example, by circular nets, i.e., quadrilateral meshes with the property that each face possesses a circumcircle. They have been the topic of various contributions from the perspective of discrete differential geometry and integrable systems (see, e.g., [6, 12, 23, 5, 10]). Among all quadrilateral meshes, circular nets are the only ones which posses nontrivial vertex offsets, i.e., parallel meshes at constant vertexwise distance (cf. Pottmann et al. [30]). In particular, choosing an arbitrary offset direction resp. normal vector at one vertex determines the normal vectors at all other vertices.

The following semidiscrete version of circular nets was first investigated by Pottmann et al. [31]. They can be understood as semidiscrete curvature line parametrizations in exactly the same manner as their purely discrete counterparts.

**Definition 3.6.** A semidiscrete surface $(x, n)$ is called *circular*, if

(a) for each corresponding pair of points $x$, $x_1$ there is a circle $C$ passing through these points and being tangent to $\partial x$, $\partial x_1$ there, and

(b) the Gauß map $n$ is parallel to $x$, i.e., $\delta n \parallel \delta x$ and $\partial n \parallel \partial x$ throughout.

*Remark* 3.1. Similar to the discrete case, a parallel Gauß map $n$ of a semidiscrete surface $x$ with the property (a) is completely determined by choosing *one* normal vector $n(k_0, t_0)$ arbitrarily in $\mathbb{S}^2 \cap \partial x(k_0, t_0)^\perp$ (see Karpenkov and Wallner [22, Theorem 1.12]). Due to the parallelity (b), the Gauß map $n$ also enjoys the property (a), and the pair $(x, n)$ is coupled.

*Remark* 3.2. For planar semidiscrete surfaces $x : D \to \mathbb{R}^2 \cong \mathbb{C}$, circularity is equivalent to the existence of a function $s : D \to \mathbb{R}^*$, with

$$\delta x = is \left( \frac{\partial x}{\|\partial x\|} + \frac{\partial x_1}{\|\partial x_1\|} \right).$$

We adopt the following definition of semidiscrete isothermic surfaces from Müller and Wallner [27].

**Definition 3.7.** A circular semidiscrete surface $(x, n)$ is called *isothermic*, if there exist positive semidiscrete functions $\nu$, $\sigma$, and $\tau$, such that

$$\|\delta x\|^2 = \sigma \nu \nu_1, \quad \|\partial x\|^2 = \tau \nu^2, \quad \text{and} \quad \partial \sigma = \delta \tau = 0.$$

An isothermic function $g : D \to \mathbb{C}$ is called *holomorphic*.

In analogy to the smooth and purely discrete settings, for circular semidiscrete surfaces $x$, isothermicity is equivalent to the existence of a Christoffel dual (see Müller and Wallner [27, Theorem 11]). Recall that a semidiscrete surface $x$ is called *conjugate*, if $\{\delta x, \partial x, \partial x_1\}$ is linearly dependent throughout.

**Definition 3.8.** Two conjugate semidiscrete surfaces $x$, $x^*$ are *dual* to each other, if there exists a positive semidiscrete function $\nu$, such that

$$\delta x^* = \frac{1}{\nu\nu_1}\delta x \quad \text{and} \quad \partial x^* = -\frac{1}{\nu^2}\partial x.$$

In this case, $x^*$ is called the *Christoffel dual* of $x$.

*Remark* 3.3. Using the notation of Definition 3.7, dual semidiscrete surfaces $x$, $x^*$ fulfill

$$\delta x^* = \frac{\sigma}{\|\delta x\|^2}\delta x, \quad \partial x^* = -\frac{\tau}{\|\partial x\|^2}\partial x, \quad \text{and} \quad A(x, x^*) = 0.$$

### 3.3.1 Quaternionic description of semidiscrete isothermic surfaces

Here we provide a characterization of semidiscrete isothermic surfaces in terms of quaternions, which we will use for the study of cmc surfaces. In particular, we demonstrate that, similar to the discrete situation, a semidiscrete surface is isothermic if and only if its cross ratio allows for a specific factorization.

Consider the algebra of quaternions $\mathbb{H}$ equipped with the basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, where $\mathbf{ij} = \mathbf{k}$, $\mathbf{jk} = \mathbf{i}$, $\mathbf{ki} = \mathbf{j}$. Using the standard matrix representation of $\mathbb{H}$, this basis is related to the Pauli matrices $\sigma_1$, $\sigma_2$, $\sigma_3$ via

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{j} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

where $i = \sqrt{-1} \in \mathbb{C}$. We embed $\mathbb{R}^3$ into $\mathbb{H}$ by

$$p = (p_1, p_2, p_3)^T \in \mathbb{R}^3 \quad \longleftrightarrow \quad p = p_1\,\mathbf{i} + p_2\,\mathbf{j} + p_3\,\mathbf{k} = \begin{pmatrix} -ip_3 & -ip_1 - p_2 \\ -ip_1 + p_2 & ip_3 \end{pmatrix} \in \operatorname{Im}\mathbb{H}. \quad (3.2)$$

Then, the scalar product is expressed as $\langle p, q\rangle = -\frac{1}{2}\operatorname{tr}(pq)$.

The identification (3.2) can be used to define a cross ratio of four possibly non-coplanar points in three-dimensional space up to inner automorphisms. It is known that discrete isothermic surfaces can be defined by the property that the cross ratio of each face allows for a special factorization (cf. Bobenko and Pinkall [6]). We are going to analyze how this property translates to the semidiscrete case.

**Definition 3.9.** For a semidiscrete surface $x$ in $\mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$, we define the function $Q[x] : D \to \mathbb{H}$ via

$$Q[x] := (\partial x)(\delta x)^{-1}(\partial x_1)(\delta x)^{-1},$$

and call the unordered pair

$$\{q[x], \bar{q}[x]\} := \operatorname{Re} Q[x] \pm i\,\|\operatorname{Im} Q[x]\|$$

the *cross ratio* of $x$.

The cross ratio of four points in $\mathbb{R}^3$ is known to be Möbius invariant (see, e.g., Bobenko and Pinkall [6, Lemma 1]). By a limit argument this property immediately carries over to $q[x]$, $\bar{q}[x]$. Another important feature is that the cross ratio of four points is real if and only if they lie on a circle. An analogous property holds in the semidiscrete case.

**Lemma 3.4.** *A semidiscrete surface $x$ is circular if and only if its cross ratio is real. In this case, the vectors $\partial x$, $\partial x_1$ lie to the same side of the line spanned by $\delta x$ if and only if $Q[x] < 0$.*

*Proof.* At each point $(k,t) \in D$ there is a Möbius transformation $\mu$, such that $x \overset{\mu}{\mapsto} (0,0,0)^T$, $x_1 \overset{\mu}{\mapsto} (1,0,0)^T = \mathbf{i}$, and $\partial x \overset{d\mu}{\longmapsto} (0,1,0)^T = \mathbf{j}$. Thus, by the Möbius invariance of the cross ratio, we have

$$\begin{aligned} \operatorname{Re} Q[x] &= \operatorname{Re}\left(\mathbf{j}\mathbf{i}^{-1}\partial(\mu\circ x)_1\mathbf{i}^{-1}\right) = \operatorname{Re}\left(\mathbf{i}\mathbf{j}\partial(\mu\circ x)_1\mathbf{i}^{-1}\right) = \\ &= \operatorname{Re}\left(\mathbf{j}\partial(\mu\circ x)_1\right) = -\langle\partial(\mu\circ x), \partial(\mu\circ x)_1\rangle, \\ \|\operatorname{Im} Q[x]\| &= \|\operatorname{Im}\left(\mathbf{j}\partial(\mu\circ x)_1\right)\| = \|\partial(\mu\circ x)\times\partial(\mu\circ x)_1\|\,. \end{aligned}$$

Hence, the cross ratio is real iff $\partial(\mu\circ x) \parallel \partial(\mu\circ x)_1$, which means that the the vector $\partial(\mu\circ x)_1$ anchored at $\mu(x_1)$ is tangent to the circle defined by $\mu(x)$, $\mu(x_1)$, and $\partial(\mu\circ x)$. Moreover, the cross ratio is negative iff the vectors $\partial(\mu\circ x)$ and $\partial(\mu\circ x)_1$ point to the same direction. By applying the inverse Möbius transformation $\mu^{-1}$, the statement follows immediately. □

The following lemma provides us with a characterization of semidiscrete isothermic surfaces in terms of their cross ratios.

**Lemma 3.5.** *A semidiscrete surface $x$ is isothemic if and only if there exist positive semidiscrete functions $\sigma$ and $\tau$, such that*

$$Q[x] = -\frac{\tau}{\sigma} \quad \text{and} \quad \partial\sigma = \delta\tau = 0.$$

*In this case, $Q[x] = -\frac{\|\partial x\|\|\partial x_1\|}{\|\delta x\|^2}$.*

*Proof.* Let $x$ be an isothermic semidiscrete surface with $\nu$, $\sigma$, and $\tau$ as in Definition 3.7. Moreover, for each fixed $(k,t) \in D$, let the Möbius transformation $\mu$ be defined by $x \overset{\mu}{\mapsto} (0,0,0)^T$, $x_1 \overset{\mu}{\mapsto} (1,0,0)^T$, and $\partial x \overset{d\mu}{\longmapsto} (0,1,0)^T$. Then, for each $\mu$, there exists $\rho > 0$, such that $\|\mu(x) - \mu(y)\|^2 = \rho(x)\rho(y)\|x-y\|^2$, for all $x, y \in \mathbb{R}^3$. This also implies that $\|d_x\mu(v)\|^2 = \rho(x)^2\|v\|^2$, for a tangent vector $v$ attached to $x$. Thus, $1 = \|\mu(x)-\mu(x_1)\|^2 = \rho(x)\rho(x_1)\|\delta x\|^2$, and by the previous lemma we get

$$Q[x] = -\langle\partial(\mu\circ x), \partial(\mu\circ x)_1\rangle = -\|\partial(\mu\circ x)\|\|\partial(\mu\circ x)_1\| = -\frac{\|\partial x\|\|\partial x_1\|}{\|\delta x\|^2} = -\frac{\tau}{\sigma}.$$

Conversely, assume that $Q[x] = -\frac{\tau}{\sigma}$, with $\partial\sigma = \delta\tau = 0$. By the previous lemma $x$ is circular and the vectors $\partial x$, $\partial x_1$ lie to the same side of the line spanned by $\delta x$. Hence, by the observations above, we have $\frac{\|\partial x\|\|\partial x_1\|}{\|\delta x\|^2} = -Q[x] = \frac{\tau}{\sigma}$. Setting $\nu := \frac{1}{\sqrt{\tau}}\|\partial x\|$ completes the proof. □

## 3.4 Semidiscrete minimal surfaces

Smooth minimal surfaces in $\mathbb{R}^3$ can be defined in several equivalent ways, e.g., by locally minimizing the surface area or by having vanishing mean curvature. An isothermic mimimal surface is determined by the property of being Christoffel dual to its Gauß map, giving rise to their well known Weierstraß-Enneper representation. This section is concerned with *semidiscrete* minimal surfaces, which do not fully enjoy these properties.

**Definition 3.10.** A coupled semidiscrete surface $(x, n)$ is called *minimal*, if its mean curvature $H$ vanishes identically.

It has already been noted by Müllner and Wallner [27] that semidiscrete *isothermic* minimal surfaces are Christoffel dual to their Gauß map. Similar to the smooth case, this observation leads to a Weierstraß type representation, as demonstrated by Rossman and Yasumoto [34]. In turn, this representation gives rise to a one-parameter family of associated surfaces. These are however no longer isothermic, which has made it difficult to understand their minimality in the discrete and semidiscrete settings so far.

Let us recall the Weierstraß representation. Let $g : \mathbb{Z} \times \mathbb{R} \supseteq D \to \mathbb{C}$ be a semidiscrete holomorphic function with $\nu_g$, $\sigma_g$, and $\tau_g$ as in Definition 3.7. It is straightforward to show that the composition of $g$ with the inverse of the stereographic projection, given by

$$n := \frac{1}{|g|^2 + 1}(2\operatorname{Re}(g),\ 2\operatorname{Im}(g),\ |g|^2 - 1)^T,$$

is isothermic with $\nu = \frac{2\nu_g}{|g|^2+1}$, $\tau = \tau_g$, and $\sigma = \sigma_g$. Now, the Christoffel dual $x$ of $n$ is uniquely determined, up to translation, as solution of the system

$$\delta x = \frac{\sigma}{\|\delta n\|^2}\delta n \quad \text{and} \quad \partial x = -\frac{\tau}{\|\partial n\|^2}\partial n.$$

We see immediately that $A(x, n) = 0$, so the semidiscrete surface $(x, n)$ is minimal. Moreover, it has been verified by Rossman and Yasumoto [34] that any semidiscrete isothermic minimal surface can be described in this way by some semidiscrete holomorphic function $g$.

As already mentioned before, the Weierstraß representation immediately gives rise to the associated family of an isothermic minimal surface.

**Definition 3.11.** Let $(x, n)$ be a semidiscrete isothermic minimal surface arising from a semidiscrete holomorphic function $g$ with $\sigma$ and $\tau$ as in Definition 3.7. Then, the *associated family* $(x^\alpha, n)$, $\alpha \in \mathbb{R}$, of $(x, n)$ is defined, up to translation, as solution of the system

$$\delta x^\alpha = \frac{\sigma}{2}\operatorname{Re}(\lambda\phi), \quad \text{with } \phi := \frac{1}{\delta g}(1 - g g_1,\ i(1 + g g_1),\ g + g_1)^T, \text{ and}$$
$$\partial x^\alpha = -\frac{\tau}{2}\operatorname{Re}(\lambda\psi), \text{ with } \psi := \frac{1}{\partial g}(1 - g^2,\ i(1 + g^2),\ 2g)^T, \text{ where } \lambda := e^{i\alpha}.$$

**Lemma 3.6.** *For every semidiscrete isothermic minimal surface $(x, n)$ the members of its associated family $(x^\alpha, n)$ are well defined and coupled.*

*Proof.* To show the existence of $x^\alpha$, we check the compatibility condition $\partial(\delta x^\alpha) = \delta(\partial x^\alpha)$. Using the abbreviation

$$\omega := \left(g^2\partial g_1 - g_1^2\partial g - \partial\delta g,\ \ i(g_1^2\partial g - g^2\partial g_1 - \partial\delta g),\ \ 2(g_1\partial g - g\partial g_1)\right)^T,$$

and the fact that $\partial\sigma = \delta\tau = 0$, one can compute

$$\begin{aligned}\partial\left(\delta x^\alpha\right) &= \frac{\sigma}{2}\operatorname{Re}\left(\frac{\lambda}{(\delta g)^2}\,\omega\right) = \frac{\tau|\delta g|^2}{2|\partial g||\partial g_1|}\operatorname{Re}\left(\frac{\lambda}{(\delta g)^2}\,\omega\right) = \\ &= \frac{\tau}{2}\operatorname{Re}\left(\frac{\lambda\delta\bar{g}}{|\partial g||\partial g_1|\delta g}\,\omega\right) \overset{(*)}{=} -\frac{\tau}{2}\operatorname{Re}\left(\frac{\lambda}{\partial g\partial g_1}\,\omega\right) = \delta\left(\partial x^\alpha\right).\end{aligned}$$

Note that the equality $(*)$ follows from the circularity of the mapping $g$ (cf. Remark 3.2):

$$\frac{\delta\bar{g}}{|\partial g||\partial g_1|\delta g} = \frac{-is\left(\frac{\partial\bar{g}}{|\partial g|} + \frac{\partial\bar{g}_1}{|\partial g_1|}\right)}{|\partial g||\partial g_1|is\left(\frac{\partial g}{|\partial g|} + \frac{\partial g_1}{|\partial g_1|}\right)} = -\frac{\frac{\partial\bar{g}}{|\partial g|} + \frac{\partial\bar{g}_1}{|\partial g_1|}}{\partial g|\partial g_1| + \partial g_1|\partial g|}\cdot\frac{\partial g\partial g_1}{\partial g\partial g_1} = -\frac{1}{\partial g\partial g_1}.$$

Finally, direct computations show $\langle\partial x^\alpha, n\rangle = 0$ and $\langle\delta x^\alpha, n\rangle = -\langle\delta x^\alpha, n_1\rangle = -\frac{\sigma}{2}\operatorname{Re}(\lambda)$. This concludes the proof. □

In order to show that the members of the associated family are indeed minimal, we follow Hoffmann et al. [20]. The key observation is as follows: Consider an (infinitesimal) quadrilateral of any member of such a family and orthogonally project it in direction of the face normal $N$. Then the resulting (infinitesimal) quadrilateral is a rotated and scaled version of the corresponding (infinitesimal) quadrilateral of the original isothermic surface (cf. Figure 3.1). As a first step toward this result we provide a semidiscrete version of [20, Lemma 24].

**Lemma 3.7.** *Let $(x^\alpha, n)$ denote the associated family of a semidiscrete isothermic minimal surface $(x, n)$. Then, for each $\alpha \in \mathbb{R}$, we have*

$$\begin{aligned}\delta x^\alpha &= \frac{\|\delta x\|^2}{\sigma}\big(\cos\alpha\,\delta n - \sin\alpha\,\delta n\times n\big), \ \textit{and} \\ \partial x^\alpha &= -\frac{\|\partial x\|^2}{\tau}\big(\cos\alpha\,\partial n - \sin\alpha\,\partial n\times n\big).\end{aligned}$$

*Proof.* Lemma 3.6 implies $\delta x^\alpha \perp (n + n_1)$ and $\partial x^\alpha \perp n$. Hence, $\delta x^\alpha$ is a linear combination of $\delta n$ and $\delta n \times n$, whereas $\partial x^\alpha$ is a linear combination of $\partial n$ and $\partial n \times n$. Moreover,

$$\begin{aligned}\delta x^\alpha &= \frac{\sigma}{2}\operatorname{Re}(\lambda\phi) = \frac{\sigma}{2}\big(\cos\alpha\ \operatorname{Re}(\phi) - \sin\alpha\ \operatorname{Im}(\phi)\big), \text{ and} \\ \partial x^\alpha &= -\frac{\tau}{2}\operatorname{Re}(\lambda\psi) = -\frac{\tau}{2}\big(\cos\alpha\ \operatorname{Re}(\psi) - \sin\alpha\ \operatorname{Im}(\psi)\big).\end{aligned}$$

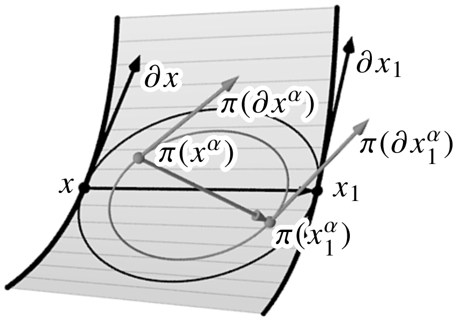


Figure 3.1: An infinitesimal quadrilateral $\{x, x_1, \partial x, \partial x_1\}$ (black) of a semidiscrete isothermic minimal surface and the corresponding projected infinitesimal quadrilateral $\{\pi(x^\alpha), \pi(x_1^\alpha), \pi(\partial x^\alpha), \pi(\partial x_1^\alpha)\}$ (gray) of a member of the associated family.

Since by construction the surfaces $(x, n)$ and $(n, n)$ are dual to each other, we know that $\mathrm{Re}(\phi) = \frac{2}{\sigma \nu \nu_1}\delta n = \frac{2}{\|\delta n\|^2}\delta n = \frac{2\|\delta x\|^2}{\sigma^2}\delta n$ and $\mathrm{Re}(\psi) = \frac{2}{\tau \nu^2}\partial n = \frac{2}{\|\partial n\|^2}\partial n = \frac{2\|\partial x\|^2}{\tau^2}\partial n$.

It remains to show that $\mathrm{Im}(\phi) = \frac{2\|\delta x\|^2}{\sigma^2}(\delta n \times n)$ and $\mathrm{Im}(\psi) = \frac{2\|\partial x\|^2}{\tau^2}(\partial n \times n)$. Firstly, it is easy to verify that $\langle \phi, \bar{\phi}\rangle_{\mathbb{C}^3} = 1$ and $\langle \psi, \bar{\psi}\rangle_{\mathbb{C}^3} = 0$, which implies that $\mathrm{Im}(\phi)$ and $\mathrm{Im}(\psi)$ are perpendicular to $\mathrm{Re}(\phi)$ and $\mathrm{Re}(\psi)$, respectively. Furthermore, we check that $\langle \mathrm{Im}(\phi), n\rangle = 0$ and $\langle \mathrm{Im}(\psi), n\rangle = 0$, so $\mathrm{Im}(\phi) \parallel \delta n \times n$ and $\mathrm{Im}(\psi) \parallel \partial n \times n$. Finally, we compute

$$\begin{aligned}
\langle \mathrm{Im}(\phi), \delta n \times n\rangle &= \det(\mathrm{Im}(\phi), \delta n, n) = \mathrm{Im}\left(\det(\phi, n_1, n)\right) = \\
&= \frac{2(|g|^2|g_1|^2 + g_1\bar{g} + g\bar{g}_1 + 1)}{(1+|g|^2)(1+|g_1|^2)} = 2 - \frac{2|\delta g|^2}{(1+|g|^2)(1+|g_1|^2)} = \\
&= 2 - \frac{\|\delta n\|^2}{2} = \frac{\|n_1 + n\|^2}{2} = \frac{2\|\delta n \times n\|^2}{\|\delta n\|^2} = \frac{2\|\delta x\|^2}{\sigma^2}\|\delta n \times n\|^2, \\
\langle \mathrm{Im}(\psi), \partial n \times n\rangle &= \det(\mathrm{Im}(\psi), \partial n, n) = \frac{2|\partial g|^2}{(1+|g|^2)^2}\det(\mathrm{Im}(\psi), \mathrm{Re}(\psi), n) = \\
&= \frac{|\partial g|^2}{(1+|g|^2)^2}\mathrm{Im}\left(\det(\psi, \bar{\psi}, n)\right) = 2,
\end{aligned}$$

where we have used the fact that $n$ maps to $\mathbb{S}^2$. Thus, we have $\mathrm{Im}(\phi) = \frac{2\|\delta x\|^2}{\sigma^2}(\delta n \times n)$ and $\mathrm{Im}(\psi) = \frac{2}{\|\partial n \times n\|^2}(\partial n \times n) = \frac{2}{\|\partial n\|^2}(\partial n \times n) = \frac{2\|\partial x\|^2}{\tau^2}(\partial n \times n)$. This concludes the proof. □

The rotation property mentioned above is stated as follows:

**Lemma 3.8.** *Let $(x^\alpha, n)$ be the associated family of a semidiscrete isothermic minimal surface $(x, n)$ and let $\pi$ denote the orthogonal projection in direction of the strip normal $N$. Then, for all $\alpha$, the infinitesimal quadrilateral $\{\pi(x^\alpha), \pi(x_1^\alpha), \pi(\partial x^\alpha), \pi(\partial x_1^\alpha)\}$ is a rotated and scaled version of the infinitesimal quadrilateral $\{x, x_1, \partial x, \partial x_1\}$ (cf. Figure 3.1).*

*Proof.* By the previous lemma,

$$\pi(\delta x^\alpha) = \frac{\|\delta x\|^2}{\sigma}\big(\cos\alpha\,\delta n - \sin\alpha\,\pi(\delta n \times n)\big), \text{ and}$$
$$\pi(\partial x^\alpha) = -\frac{\|\partial x\|^2}{\tau}\big(\cos\alpha\,\partial n - \sin\alpha\,\pi(\partial n \times n)\big).$$

The orthogonality $\pi(\delta n \times n) \perp \delta n$ implies

$$\begin{aligned}
\|\pi(\delta x^\alpha)\|^2 &= \frac{\|\delta x\|^4}{\sigma^2}\big(\cos(\alpha)^2\|\delta n\|^2 - \sin(\alpha)^2\|\pi(\delta n \times n)\|^2\big) = \\
&= \frac{\|\delta x\|^4}{\sigma^2}\big(\cos(\alpha)^2\|\delta n\|^2 - \sin(\alpha)^2\cos(\mu)^2\|(\delta n \times n)\|^2\big) = \\
&= \|\delta x\|^2\Big(\cos(\alpha)^2 - \sin(\alpha)^2\cos(\mu)^2\Big\|\frac{n+n_1}{2}\Big\|^2\Big) = \|\delta x\|^2\big(\cos(\alpha)^2 - \sin(\alpha)^2 d^2\big),
\end{aligned}$$

where $\mu := \angle(\delta n \times n, \pi(\delta n, \times n)) = \angle(\frac{n+n_1}{2}, N)$, and $d$ denotes the distance between the origin and the center of the circle $C$ determined by $\{n, n_1, \partial n, \partial n_1\}$ in the same manner as in Definition 3.6 (a). Likewise, $\pi(\partial n \times n) \perp \partial n$ implies

$$\begin{aligned}
\|\pi(\partial x^\alpha)\|^2 &= \frac{\|\partial x\|^4}{\tau^2}\big(\cos(\alpha)^2\|\partial n\|^2 - \sin(\alpha)^2\|\pi(\partial n \times n)\|\big) = \\
&= \frac{\|\partial x\|^4}{\tau^2}\big(\cos(\alpha)^2\|\partial n\|^2 - \sin(\alpha)^2\cos(\xi)^2\|\partial n \times n\|^2\big) = \\
&= \|\partial x\|^2\big(\cos(\alpha)^2 - \sin(\alpha)^2\cos(\xi)^2\|n\|^2\big) = \|\partial x\|^2\big(\cos(\alpha)^2 - \sin(\alpha)^2 d^2\big),
\end{aligned}$$

where $\xi := \angle(\partial n \times n, \pi(\partial n \times n)) = \angle(n, N)$. Analogously, we obtain

$$\|\pi(\partial x_1^\alpha)\|^2 = \|\partial x_1\|^2\big(\cos(\alpha)^2 - \sin(\alpha)^2 d^2\big).$$

Finally, we observe that

$$\frac{\langle \pi(\delta x^\alpha), \delta x\rangle}{\|\pi(\delta x^\alpha)\|\|\delta x\|} = \frac{\langle \pi(\partial x^\alpha), \partial x\rangle}{\|\pi(\partial x^\alpha)\|\|\partial x\|} = \frac{\langle \pi(\partial x_1^\alpha), \partial x_1\rangle}{\|\pi(\partial x_1^\alpha)\|\|\partial x_1\|} = \frac{\cos(\alpha)}{\sqrt{\cos(\alpha)^2 - \sin(\alpha)^2 d^2}}.$$

Thus, the infinitesimal quadrilateral $\{\pi(x^\alpha),\ \pi(x_1^\alpha),\ \pi(\partial x^\alpha),\ \pi(\partial x_1^\alpha)\}$ arises from the infinitesimal quadrilateral $\{x, x_1, \partial x, \partial x_1\}$ by scaling with factor $\rho_\alpha$ and rotating by the angle $\theta_\alpha$, with

$$\rho_\alpha = \sqrt{\cos(\alpha)^2 - \sin(\alpha)^2 d^2} \quad \text{and} \quad \cos\theta_\alpha = \frac{\cos\alpha}{\rho_\alpha}. \qquad \square$$

We are now able to prove the main result of the present section.

**Theorem 3.1.** *Every member $(x^\alpha, n)$ of the associated family of a semidiscrete isothermic minimal surface $(x, n)$ is minimal, i.e., has vanishing mean curvature.*

*Proof.* Recall that the rotation by an angle $\theta$ about the axis in direction of $N$ can be written as

$$R_{N,\theta}(x) = \langle N, x\rangle N + \cos\theta\,(N \times x) \times N + \sin\theta\, N \times x.$$

According to Lemma 3.8, we thus have

$$\pi(\delta x^\alpha) = \rho_\alpha R_{N,\theta_\alpha}(\delta x) = \rho_\alpha\big(\cos\theta_\alpha\,\delta x + \sin\theta_\alpha\, N \times \delta x\big), \text{ and}$$
$$\pi(\partial x^\alpha + \partial x_1^\alpha) = \rho_\alpha R_{N,\theta_\alpha}(\partial x + \partial x_1) = \rho_\alpha\big(\cos\theta_\alpha\,(\partial x + \partial x_1) + \sin\theta_\alpha\, N \times (\partial x + \partial x_1)\big).$$

Since

$$N \times \delta x = \frac{1}{\|\delta n \times (\partial n + \partial n_1)\|}\Big(\langle \delta x, \delta n\rangle(\partial n + \partial n_1) - \langle \delta x, \partial n + \partial n_1\rangle \delta n\Big),$$
$$N \times (\partial x + \partial x_1) = \frac{1}{\|\delta n \times (\partial n + \partial n_1)\|}\Big(\langle \delta n, \partial x + \partial x_1\rangle(\partial n + \partial n_1) - \langle \partial n + \partial n_1, \partial x + \partial x_1\rangle \delta n\Big),$$

the term $4A(x^\alpha, n) = \det\big(\pi(\delta x^\alpha), \partial n + \partial n_1, N\big) + \det\big(\delta n, \pi(\partial x^\alpha + \partial x_1^\alpha), N\big)$ vanishes for all $\alpha \in \mathbb{R}$ if and only if $A(x, n) = 0$ and $\langle \delta x, \partial n + \partial n_1\rangle = \langle \delta n, \partial x + \partial x_1\rangle$. Both equations hold since $(x, n)$ is an isothermic minimal surface (cf. Remark 3.3 and Lemma 3.1). □

In the smooth setting, the Gauß curvature of the members of the associated family of a minimal surface is independent of the parameter $\alpha$ as well. This is no longer the case in the discrete and semidiscrete situations.

**Lemma 3.9.** *Under the assumptions of Theorem 3.1, the Gauß curvature $K^\alpha$ of $(x^\alpha, n)$ obeys*

$$K^\alpha = \frac{K^0}{\cos(\alpha)^2 + \sin(\alpha)^2 d^2},$$

*where $d$ is the distance between the origin and the center of the circle $C$ determined by $\{n, n_1, \partial n, \partial n_1\}$.*

*Proof.* From the proof of Lemma 3.8 it follows that $A(x^\alpha, x^\alpha) = \rho_\alpha^2 A(x, x)$, with $\rho_\alpha^2 = \cos(\alpha)^2 + \sin(\alpha)^2 d^2$. □

We conclude this section by proving that the *conjugate surface* $(x^{\pi/2}, n)$ of a semidiscrete isothermic minimal surface $(x, n)$ is an asymptotic parametrization, in analogy to the smooth and discrete cases. Semidiscrete asymptotic parametrizations have been studied, e.g., by Wallner [40]. Here, the notation $x_{\bar{1}}$ indicates an index shift in the opposite direction: $x_{\bar{1}}(k, t) := x(k-1, t)$.

**Lemma 3.10.** *Let $(x, n)$ be a semidiscrete isothermic minimal surface with associated family $(x^\alpha, n)$. Then, the conjugate surface $(x^{\pi/2}, n)$ is an asymptotic parametrization, i.e., the vectors*

$$\partial x^{\pi/2},\ \partial^2 x^{\pi/2},\ \delta x^{\pi/2} = x_1^{\pi/2} - x^{\pi/2},\ \textit{and}\ \delta x_{\bar{1}}^{\pi/2} = x^{\pi/2} - x_{\bar{1}}^{\pi/2}$$

*lie in a plane with unit normal vector $n$.*

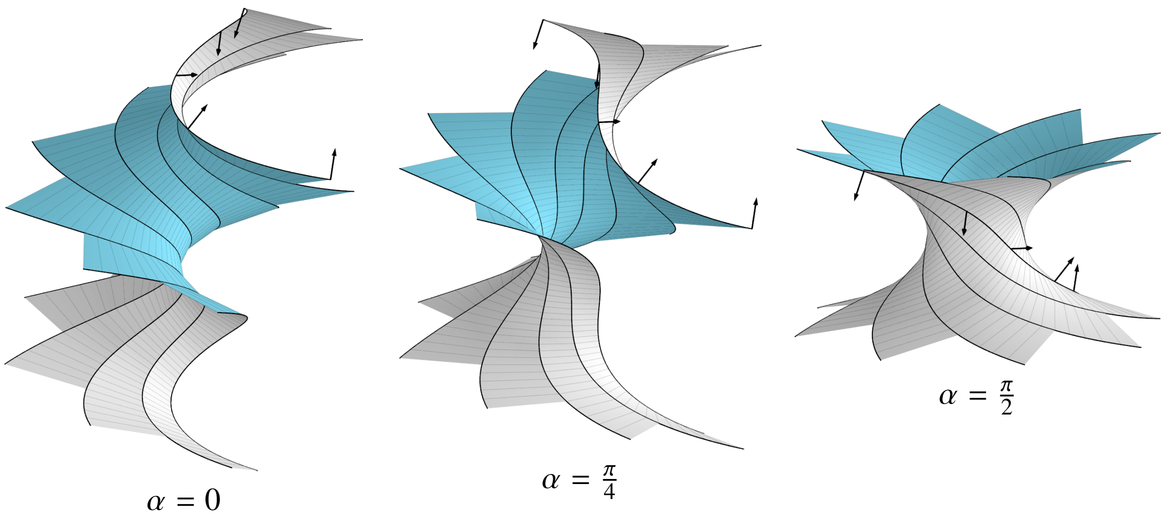


Figure 3.2: A semidiscrete helicoid (left) and two members of its associated family. The corresponding semidiscrete holomorphic function $g$ described in Example 3.1 takes $r = \sqrt{2}$, $\beta = \pi/4$, and $\varphi = \pi/8$ as parameters. For $\alpha = \pi/2$ we obtain a semidiscrete asymptotic parametrization of a catenoid (right).

*Proof.* From Lemma 3.7, we have

$$\delta x^{\pi/2} = -\frac{\|\delta x\|^2}{\sigma}\delta n \times n \quad \text{and} \quad \partial x^{\pi/2} = \frac{\|\partial x\|^2}{\tau}\partial n \times n.$$

The computation $\partial^2 x^{\pi/2} = \partial\left(\frac{\|\partial x\|^2}{\tau}\right)\partial n \times n + \frac{\|\partial x\|^2}{\tau}(\partial^2 n) \times n$ concludes the proof. □

**Example 3.1.** As an example we investigate the associated family of a semidiscrete helicoid (cf. Figure 3.2). In classical differential geometry, an isothermic parametrization of the helicoid is gained from the Weierstraß data $f(z) = 1/(1+i)$ and $g(z) = \exp((1+i)z)$. Its conjugate surface is an asymtotically parametrized catenoid. A semidiscrete analog of the holomorphic map $z \mapsto \exp(az)$, $a \in \mathbb{C}$, has been proposed by Müller [26, Theorem 7] and is given by

$$g(k,t) = \exp\left(r\exp(i\beta)t + (i\varphi + \log\mu)k\right),$$

with $r \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, and $\varphi \in \mathbb{R}^*$, such that $\mu := \frac{\cos(\beta+\varphi/2)}{\cos(\beta-\varphi/2)} > 0$. It is straightforward to check that $g$ is holomorphic with

$$\nu_g = \mu^k \exp(r\cos(\beta)t), \quad \sigma_g = \frac{2\sin(\varphi)^2}{\cos(2\beta)+\cos(\varphi)}, \quad \text{and} \quad \tau_g = r^2.$$

## 3.5 Semidiscrete cmc surfaces

This section focuses on semidiscrete cmc surfaces, which enjoy non-zero constant mean curvature. In contrast to minimal surfaces, isothermic cmc surfaces are characterized by having a Christoffel dual at constant distance. This observation immediately follows from the fact that, in agreement with the smooth case, cmc surfaces are linear Weingarten surfaces.

**Lemma 3.11.** *Let $(x, n)$ be coupled. Then, the mean and Gauß curvatures of the offsets $(x^r, n) = (x + r\,n, n)$, $r \in \mathbb{R}$, are given by*

$$H^r = \frac{H - Kr}{1 - 2Hr + Kr^2} \quad \text{and} \quad K^r = \frac{K}{1 - 2Hr + Kr^2}.$$

*If $H = \text{const.} \neq 0$, $(x^r, n)$ is a linear Weingarten surface, i.e., there exist $a, b \in \mathbb{R}$ only depending on $r$ and $H$, such that $aH^r + bK^r = 1$. An analogous result applies to constant Gauß curvature surfaces.*

*Proof.* In case $H = \text{const.} \neq 0$, we set $a := \frac{1}{H} - 2r$ and $b := \frac{r}{H} - r^2$. If $K = \text{const.} \neq 0$, we set $a := -2r$ and $b := \frac{1}{K} - r^2$. □

**Corollary 3.2.** *If the surface $(x, n)$ has constant mean curvature $H = \frac{1}{h}$, then the offset $(x^h, n) = (x + h\,n, n)$ has constant mean curvature $H^h = -H$, and the central surface $(x^{h/2}, n) = (x + \frac{h}{2}\,n, n)$ has constant positive Gauß curvature $K^{h/2} = 4H^2$.*

**Corollary 3.3.** *For a coupled semidiscrete surface $(x, n)$ and its offset $(\hat{x}, n) := (x + n, n)$, we have the equivalence*

$$A(x, \hat{x}) = 0 \iff H = -\frac{A(x, n)}{A(x, x)} = 1 \iff \hat{H} = -\frac{A(\hat{x}, n)}{A(\hat{x}, \hat{x})} = -1.$$

We dedicate the rest of this paper to the description of semidiscrete isothermic cmc surfaces in terms of a pair of linear first-order matrix partial differential equations called a Lax pair. Similar to the case of minimal surfaces, this representation directly includes the definition of a one-parameter family of associated surfaces. We consider only the case $H = \pm 1$, since it can always be achieved by scaling.

### 3.5.1 The Lax pair representation of smooth cmc surfaces

We briefly recapitulate the smooth situation. For details see Bobenko [4] or Fujimori et al. [19]. Consider a smooth conformal immersion

$$x : \mathbb{C} \supseteq D \to \mathbb{R}^3 : z \to x(z),$$

with complex coordinate $z = s + it$. Conformality means that

$$\langle \partial_z x, \partial_z x \rangle = \langle \partial_{\bar{z}} x, \partial_{\bar{z}} x \rangle = 0$$

throughout the parameter domain, where $\langle\cdot,\cdot\rangle$ denotes the bilinear complex extension of the standard Euclidean inner product and $\partial_z$, $\partial_{\bar z}$ are the Wirtinger derivatives $\partial_z = \frac{1}{2}(\partial_s - i\partial_t)$ and $\partial_{\bar z} = \frac{1}{2}(\partial_s + i\partial_t)$.

As initiated in Section 3.3.1, we identify $\mathbb{R}^3$ with the set of purely imaginary quaternions $\operatorname{Im}\mathbb{H}$. Thereby, rotating a point $x \in \mathbb{R}^3$ translates to the conjugation of $x \in \operatorname{Im}\mathbb{H}$ by a unit quaternion $q$. In the matrix representation of $\mathbb{H}$, the set of unit quaternions $\{q \in \mathbb{H} : \|q\| = 1\}$ coincides with the Lie group $\mathrm{SU}_2 = \{A \in \mathbb{C}^{2\times 2} : A^H = A^{-1},\ \det(A) = 1\}$. The corresponding Lie algebra is $\mathfrak{su}_2 = \{A \in \mathbb{C}^{2\times 2} : A^H = -A,\ \operatorname{tr}(A) = 0\}$. In this manner, $\mathrm{SU}_2$ is a double covering of $\mathrm{SO}_3$, which we identify with the set of positively oriented orthonormal frames.

Now, let $\Psi = \Psi(z) \in \mathrm{SU}_2$ represent the frame $(\frac{\partial_s x}{\|\partial_s x\|}, \frac{\partial_t x}{\|\partial_t x\|}, n)^T \in \mathrm{SO}_3$, where $n = \frac{\partial_s x \times \partial_t x}{\|\partial_s x \times \partial_t x\|}$. Then,

$$\partial_s x = e^{u/2}\Psi^{-1}\mathbf{i}\Psi, \quad \partial_t x = e^{u/2}\Psi^{-1}\mathbf{j}\Psi, \quad \text{and} \quad n = \Psi^{-1}\mathbf{k}\Psi, \tag{3.3}$$

with $e^u = 2\langle\partial_z x, \partial_{\bar z} x\rangle$. It turns out that the frame $\Psi$ moves according to

$$\partial_z \Psi = \begin{pmatrix} \frac{\partial_z u}{4} & -Qe^{-u/2} \\ \frac{1}{2}He^{u/2} & -\frac{\partial_z u}{4} \end{pmatrix}\Psi, \qquad \partial_{\bar z}\Psi = \begin{pmatrix} -\frac{\partial_{\bar z} u}{4} & -\frac{1}{2}He^{u/2} \\ \bar{Q}e^{-u/2} & \frac{\partial_{\bar z} u}{4} \end{pmatrix}\Psi, \tag{3.4}$$

where the so-called Hopf differential $Q$ and the mean curvature $H$ satisfy $Q = \langle\partial_z\partial_z x, n\rangle$ and $\frac{1}{2}He^u = \langle\partial_z\partial_{\bar z} x, n\rangle$. The integrability condition of this system, i.e., $\partial_z(\partial_{\bar z}\Psi) = \partial_{\bar z}(\partial_z\Psi)$, is equivalent to

$$\partial_z\partial_{\bar z} u = 2Q\bar{Q}e^{-u} - \frac{1}{2}H^2 e^u \quad \text{and} \quad \partial_{\bar z} Q = \frac{1}{2}e^u \partial_z H. \tag{3.5}$$

Thus, if we assume constant mean curvature, the Hopf differential is holomorphic. If in addition the surface has no umbilic points, then $Q \neq 0$ and we can achieve that $Q = \text{const.} \neq 0$ by a holomorphic change of coordinates. Moreover, Equations (3.5) then are invariant with respect to the transformation $Q \mapsto \Lambda Q$, with $\Lambda = e^{2i\alpha}$, $\alpha \in \mathbb{R}$. In particular, we may assume that the Hopf differential is real, in which case $x$ is isothermic. By integrating Equations (3.4) and (3.3) with $Q$ replaced by $\Lambda Q$, we obtain a one-parameter family of surfaces $x^\alpha$ with the same constant mean curvature.

Remarkably, the solution $x^\alpha$ of the system (3.3) can be obtained without integration, by a formula first suggested by Sym [38] for K-surfaces and later translated by Bobenko [3, 4] to numerous other cases, including cmc surfaces in various space forms. Indeed, for any solution $\Psi = \Psi(z, \alpha)$ of the system (3.4) with $Q$ replaced by $\Lambda Q$, the parametrization

$$x^\alpha := -\frac{1}{H}\Psi^{-1}\frac{\partial}{\partial\alpha}\Psi + \Psi^{-1}\mathbf{k}\Psi,$$

describes a cmc surface with metric $e^u$, mean curvature $H$, and Hopf differential $\Lambda Q$ (see [4, Theorem 5]).

For the sake of simplicity, we henceforth assume without loss of generality that $H = 1$ and $Q = 1/2$. Furthermore, we introduce the gauge equivalent frame

$$\widetilde{\Psi} := \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \Psi = \begin{pmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \Psi, \quad \text{with } \lambda := \sqrt{\Lambda} = e^{i\alpha}.$$

Using the relations $\partial_s = \partial_z + \partial_{\bar{z}}$ and $\partial_t = i(\partial_z - \partial_{\bar{z}})$, the frame equations (3.4) with $H = 1$ and $Q = \Lambda/2$ translate to

$$\begin{aligned} \partial_s \widetilde{\Psi} &= \mathcal{U}\widetilde{\Psi}, \quad \text{with } \mathcal{U} = \frac{1}{2} \begin{pmatrix} -i\frac{\partial_t u}{2} & -\frac{e^{u/2}}{\lambda} - \frac{\lambda}{e^{u/2}} \\ \lambda e^{u/2} + \frac{1}{\lambda e^{u/2}} & i\frac{\partial_t u}{2} \end{pmatrix}, \text{ and} \\ \partial_t \widetilde{\Psi} &= \mathcal{V}\widetilde{\Psi}, \quad \text{with } \mathcal{V} = \frac{1}{2} \begin{pmatrix} i\frac{\partial_s u}{2} & -\frac{i\lambda}{e^{u/2}} + \frac{ie^{u/2}}{\lambda} \\ i\lambda e^{u/2} - \frac{i}{\lambda e^{u/2}} & -i\frac{\partial_s u}{2} \end{pmatrix}. \end{aligned}$$

Here, the integrability condition $\partial_s(\partial_t \widetilde{\Psi}) = \partial_t(\partial_s \widetilde{\Psi}) \iff \partial_s \mathcal{V} + \mathcal{V}\mathcal{U} = \partial_t \mathcal{U} + \mathcal{U}\mathcal{V}$ is equivalent to the elliptic sinh-Gordon equation:

$$-\partial_{ss} u - \partial_{tt} u = 4\sinh(u).$$

Finally, we note that the matrices $\mathcal{U}$ and $\mathcal{V}$ belong to the loop algebra

$$\Lambda\mathfrak{su}_2 := \{A : \mathbb{S}^1 \to \mathfrak{su}_2 : A(-\lambda) = \sigma_3 A(\lambda)\sigma_3\},$$

and accordingly $\widetilde{\Psi}$ lies in the corresponding loop group

$$\Lambda\mathrm{SU}_2 := \{A : \mathbb{S}^1 \to \mathrm{SU}_2 : A(-\lambda) = \sigma_3 A(\lambda)\sigma_3\}.$$

The condition $A(-\lambda) = \sigma_3 A(\lambda)\sigma_3$ states that the elements of $\Lambda\mathrm{SU}_2$ and $\Lambda\mathfrak{su}_2$ have even functions of $\lambda$ on their diagonals and odd functions of $\lambda$ on their off-diagonals.

### 3.5.2 A Lax pair representation of semidiscrete cmc surfaces

As demonstrated by Bobenko and Pinkall [7], the observations above can be utilized to derive a Lax pair representation of *discrete* isothermic cmc surfaces along with their associated families. However, only recently it has been verified by Hoffmann et al. [20] that the members of these associated families, which are no longer isothermic, indeed have the same constant mean curvature. In this subsection we explore similar results for *semidiscrete* surfaces.

Mimicking the smooth and discrete cases, we seek a solution $\Phi(k, t, \alpha) \in \Lambda\mathrm{SU}_2$ of the system

$$\Phi_1 = U\Phi, \quad \partial\Phi = V\Phi, \quad \Phi(0, 0, \alpha) = \mathbf{1}, \tag{3.6}$$

with the Lax matrices

$$U := \frac{1}{\eta} \begin{pmatrix} a & \frac{i}{u\lambda} - iu\lambda \\ \frac{i\lambda}{u} - \frac{iu}{\lambda} & \bar{a} \end{pmatrix} \in \Lambda\mathrm{SU}_2, \quad V := \frac{1}{\vartheta} \begin{pmatrix} ib & \frac{1}{v\lambda} + v\lambda \\ -\frac{\lambda}{v} - \frac{v}{\lambda} & -ib \end{pmatrix} \in \Lambda\mathfrak{su}_2, \tag{3.7}$$

where $\lambda := e^{i\alpha}$, $\alpha \in \mathbb{R}$, $a : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$, $b, \vartheta : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, $u, v : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}_+$, and $\eta^2 := |a|^2 + u^2 + u^{-2} - \lambda^2 - \lambda^{-2}$, such that $\det(U) = 1$.

The compatibility condition $\partial(\delta\Phi) = \delta(\partial\Phi)$ of the system (3.6) is equivalent to

$$\partial U + UV = V_1 U, \tag{3.8}$$

which expands to

$$\begin{aligned} &\partial\eta = \delta\vartheta = 0, \quad u^2 = vv_1, \\ &i\vartheta\partial u + (b_1 + b)u = av - \bar{a}v_1, \quad \text{and} \\ &i\vartheta\partial a + (b_1 - b)a = uv + uv_1 - \frac{1}{uv} - \frac{1}{uv_1}. \end{aligned} \tag{3.9}$$

To resolve the relation $u^2 = vv_1$, we introduce a function $w : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ and set $v = e^{2w}$ and $u = e^{w+w_1}$. Then, taking the real resp. imaginary parts of the Equations (3.9) leads to $\mathrm{Im}(a) = \frac{\vartheta(\partial w + \partial w_1)}{2\cosh(w - w_1)}$, $b_1 = 2\,\mathrm{Re}(a)\sinh(w - w_1) - b$, $\partial\,\mathrm{Re}(a) = -\frac{\mathrm{Im}(a)}{\vartheta}(b_1 - b)$, and

$$-\vartheta\partial\,\mathrm{Im}(a) + (b_1 - b)\,\mathrm{Re}(a) = 2\sinh(3w + w_1) + 2\sinh(w + 3w_1),$$

which is a semidiscrete version of the elliptic sinh-Gordon equation. The analogy to the smooth case is not obvious at first glance. For a purely discrete version of this equation we refer to Pedit and Wu [28, Theorem 4.1].

As in the smooth and discrete cases, we use the Sym-Bobenko formula to gain a parametrization of the semidiscrete surface related to the frame $\Phi$. In particular, we are going to investigate the following three parallel surfaces.

**Definition 3.12.** Let $\Phi(k, t, \alpha) \in \Lambda\mathrm{SU}_2$, $\alpha \in \mathbb{R}$, be a solution of the system $\Phi_1 = U\Phi$, $\partial\Phi = V\Phi$, $\Phi(0, 0, \alpha) = \mathbf{1}$, where $U \in \Lambda\mathrm{SU}_2$ and $V \in \Lambda\mathfrak{su}_2$ are Lax matrices of the form (3.7) satisfying the compatibility condition (3.8). Then, we define the following families of parallel surfaces

$$\check{x}^\alpha := -\Phi^{-1}\frac{\partial}{\partial\alpha}\Phi - \frac{1}{2}n^\alpha, \quad x^\alpha := -\Phi^{-1}\frac{\partial}{\partial\alpha}\Phi, \quad \hat{x}^\alpha := -\Phi^{-1}\frac{\partial}{\partial\alpha}\Phi + \frac{1}{2}n^\alpha,$$

together with their common Gauß map $n^\alpha := \Phi^{-1}\mathbf{k}\Phi$.

We will see later (cf. Corollary 3.6) that, for $\alpha = 0$, the surfaces $(\check{x}^0, n^0)$ and $(\hat{x}^0, n^0)$ constructed as in Definition 3.12 are Christoffel dual isothermic cmc surfaces. Consequently, the families $(\check{x}^\alpha, n^\alpha)$ and $(\hat{x}^\alpha, n^\alpha)$ represent their *associated families*.

At first we show that the pairs $(\check{x}^\alpha, n^\alpha)$, $(x^\alpha, n^\alpha)$, and $(\hat{x}^\alpha, n^\alpha)$ from Definition 3.12 are coupled. In fact, we prove the following slightly more general result.

**Lemma 3.12.** *Let $\Phi(k, t, \alpha) \in \mathrm{SU}_2$ be a moving frame defined by $\Phi_1 = U\Phi$, $\partial\Phi = V\Phi$, $\Phi(0, 0, \alpha) = \mathbf{1}$, where $U \in \mathrm{SU}_2$ and $V \in \mathfrak{su}_2$ satisfy the compatibility condition* (3.8). *Moreover,* *let $p, q \in \mathbb{R}$ be arbitrary coefficients. Then the semidiscrete surface $(x^\alpha, n^\alpha)$ defined by the Sym-Bobenko formula*

$$x^\alpha := p\Phi^{-1}\frac{\partial}{\partial\alpha}\Phi + qn^\alpha, \qquad n^\alpha := \Phi^{-1}\mathbf{k}\Phi,$$

*fulfills the constraint* (3.1) *if and only if $U$ satisfies $U_{22}\frac{\partial}{\partial\alpha}U_{11} = U_{11}\frac{\partial}{\partial\alpha}U_{22}$, and $V$ satisfies* $\operatorname{tr}(\frac{\partial}{\partial\alpha}V\mathbf{k}) = 0$, *i.e., $\frac{\partial}{\partial\alpha}V_{11} = \frac{\partial}{\partial\alpha}V_{22}$.*

*Proof.* The condition $\delta x^\alpha \perp (n_1^\alpha + n^\alpha)$ holds iff $\operatorname{tr}\left((x_1^\alpha - x^\alpha)(n^\alpha + n_1^\alpha)\right) = 0$. Thus, we compute

$$\begin{aligned}(x_1^\alpha - x^\alpha)n^\alpha &= \Phi^{-1}\left(pU^{-1}\frac{\partial}{\partial\alpha}U + qU^{-1}\mathbf{k}U - q\mathbf{k}\right)\Phi\Phi^{-1}\mathbf{k}\Phi = \\ &= \Phi^{-1}U^{-1}\left(p\frac{\partial}{\partial\alpha}U + q\mathbf{k}U - qU\mathbf{k}\right)\mathbf{k}\Phi, \quad \text{and} \\ n_1^\alpha(x_1^\alpha - x^\alpha) &= \Phi_1^{-1}\mathbf{k}\Phi_1\Phi^{-1}\left(pU^{-1}\frac{\partial}{\partial\alpha}U + qU^{-1}\mathbf{k}U - q\mathbf{k}\right)\Phi = \\ &= \Phi^{-1}U^{-1}\mathbf{k}\left(p\frac{\partial}{\partial\alpha}U + q\mathbf{k}U - qU\mathbf{k}\right)\Phi.\end{aligned}$$

Therefore, $\operatorname{tr}\left((x_1^\alpha - x^\alpha)(n^\alpha + n_1^\alpha)\right) = 0 \iff \operatorname{tr}\left(U^{-1}(\frac{\partial}{\partial\alpha}U\mathbf{k} + \mathbf{k}\frac{\partial}{\partial\alpha}U)\right) = 0$, which is equivalent to $U_{22}\frac{\partial}{\partial\alpha}U_{11} = U_{11}\frac{\partial}{\partial\alpha}U_{22}$.

To complete the proof, we show that $\operatorname{tr}(\partial x^\alpha\, n^\alpha) = p\operatorname{tr}\left(\frac{\partial}{\partial\alpha}V\mathbf{k}\right)$. Since $\|n^\alpha\|^2 = -\frac{1}{2}\operatorname{tr}(n^\alpha\, n^\alpha) = 1$, we have $\operatorname{tr}(\partial n^\alpha n^\alpha) = 0$. Moreover,

$$\begin{aligned}\partial\left(\Phi^{-1}\frac{\partial}{\partial\alpha}\Phi\right) &= \partial\left(\Phi^{-1}\right)\frac{\partial}{\partial\alpha}\Phi + \Phi^{-1}\frac{\partial^2}{\partial t\partial\alpha}\Phi = (V\Phi)^H\frac{\partial}{\partial\alpha}\Phi + \Phi^{-1}\frac{\partial}{\partial\alpha}(V\Phi) = \\ &= \Phi^{-1}V^H\frac{\partial}{\partial\alpha}\Phi + \Phi^{-1}\frac{\partial}{\partial\alpha}(V)\Phi + \Phi^{-1}V\frac{\partial}{\partial\alpha}\Phi = \Phi^{-1}\frac{\partial}{\partial\alpha}(V)\Phi,\end{aligned}$$

where we have used that $\Phi^{-1} = \Phi^H$ and $V^H + V = 0$. Hence, $\partial x^\alpha \perp n^\alpha \iff \operatorname{tr}\left(\frac{\partial}{\partial\alpha}V\mathbf{k}\right) = 0$. □

**Corollary 3.4.** *The semidiscrete surfaces from Definition 3.12 are coupled.*

*Proof.* We have $U_{22}\frac{\partial}{\partial\alpha}U_{11} = U_{11}\frac{\partial}{\partial\alpha}U_{22} = |a|^2\frac{\partial}{\partial\alpha}\left(\frac{1}{\eta}\right)$, and $\frac{\partial}{\partial\alpha}V_{11} = \frac{\partial}{\partial\alpha}V_{22} = 0$. □

The main result of the present section is that the surfaces $(\check{x}^\alpha, n^\alpha)$ and $(\hat{x}^\alpha, n^\alpha)$ have constant mean curvature in the sense of Definition 3.4.

**Theorem 3.2.** *Let $(\check{x}^\alpha, n^\alpha)$ and $(\hat{x}^\alpha, n^\alpha)$ be given as in Definition 3.12. Then, for every $\alpha \in \mathbb{R}$, we have $A(\check{x}^\alpha, \hat{x}^\alpha) = 0$ throughout the parameter domain.*

*Proof.* The statement can be verified by direct computations. However, since the involved expressions are rather lengthy, we defer the proof to the appendix (see Section 3.A). □

**Corollary 3.5.** *The semidiscrete surfaces* $(\check{x}^\alpha, n^\alpha)$ *and* $(\hat{x}^\alpha, n^\alpha)$ *from Definition 3.12 have constant mean curvatures* $\check{H} = 1$ *and* $\hat{H} = -1$*, respectively. The central surface* $(x^\alpha, n^\alpha)$ *has constant Gauß curvature* $K = 4$.

Just like in the smooth case, the surfaces $(\check{x}^0, n^0)$ and $(\hat{x}^0, n^0)$ turn out to be isothermic and dual to each other.

**Lemma 3.13.** *Consider the families* $(\check{x}^\alpha, n^\alpha)$, $(x^\alpha, n^\alpha)$, *and* $(\hat{x}^\alpha, n^\alpha)$ *from Definition 3.12. Then, for* $j \in \mathbb{Z}$, *we have*

$$\begin{aligned}
Q[\check{x}^{j\pi/2}] &= Q[\hat{x}^{j\pi/2}] = -\frac{\eta^2}{\vartheta^2}, \\
Q[x^{j\pi/2}] &= -\frac{\eta^2}{\vartheta^2} \frac{\left(v - (-1)^j v^{-1}\right)\left(v_1 - (-1)^j v_1^{-1}\right)}{\left(u + (-1)^j u^{-1}\right)^2}, \\
Q[n^{j\pi/2}] &= -\frac{\eta^2}{\vartheta^2} \frac{\left(v + (-1)^j v^{-1}\right)\left(v_1 + (-1)^j v_1^{-1}\right)}{\left(u - (-1)^j u^{-1}\right)^2}.
\end{aligned}$$

*Proof.* Inserting the respective expressions derived in the proof of Theorem 3.2 into the formula for the cross ratio (cf. Definition 3.9) immediately yields the stated results. □

**Corollary 3.6.** *For every fixed* $j \in \mathbb{Z}$, *the semidiscrete surfaces* $(\check{x}^{j\pi/2}, n^{j\pi/2})$ *and* $(\hat{x}^{j\pi/2}, n^{j\pi/2})$ *are isothermic and dual to each other.*

*Proof.* Isothermicity immediately follows from the previous lemma (cf. also Lemma 3.5). Duality is a consequence of Theorem 3.2. □

## 3.6 Semidiscrete Delaunay surfaces and elliptic billiards

In this section we construct semidiscrete cmc surfaces of rotational symmetry with discrete profile curves. For this purpose, we assume that the Lax matrices $U$ and $V$ of the form (3.7) are independent of the continuous parameter $t$. In this case, the compatibility condition (3.8) resp. the Equations (3.9) are given by

$$\vartheta = \text{const.}, \quad u^2 = vv_1, \quad \operatorname{Im}(a) = 0,$$
$$(b_1 - b)\operatorname{Re}(a) = uv + uv_1 - \frac{1}{uv} - \frac{1}{uv_1}, \quad \text{and} \quad (b_1 + b)u = \operatorname{Re}(a)(v - v_1).$$

To resolve the relation $u^2 = vv_1$, we introduce a function $w : \mathbb{Z} \to \mathbb{R}_+$ and set $v = w^2$ and $u = ww_1$. Next we try to solve the resulting equations

$$(b_1 - b)\operatorname{Re}(a) = w^3 w_1 + w w_1^3 - \frac{1}{w^3 w_1} - \frac{1}{w w_1^3}, \quad (b_1 + b) w w_1 = \operatorname{Re}(a)(w^2 - w_1^2)$$

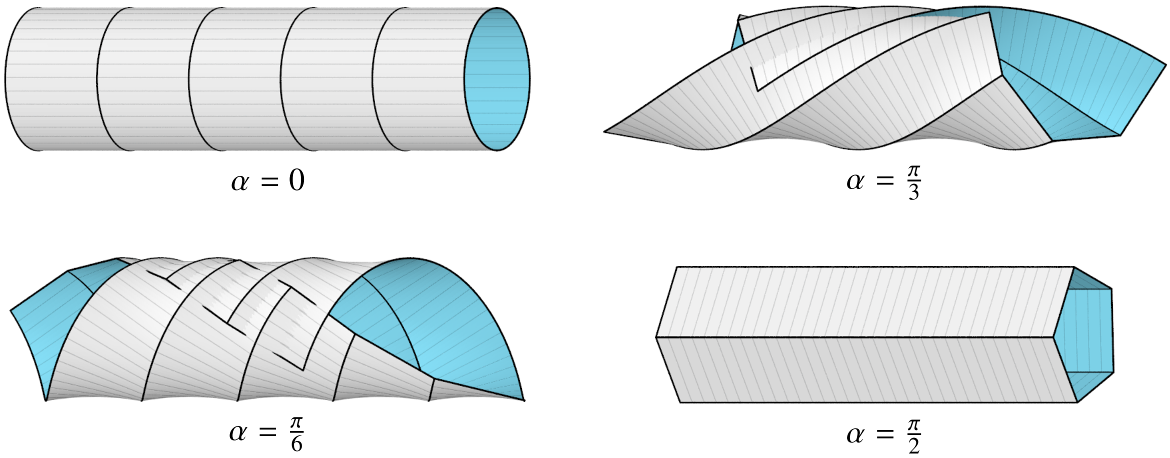


Figure 3.3: A semidiscrete cylinder with radius $r = 1/2$ and three members of its associated family. The parameters $a = \text{const.} = 2\sqrt{1 + 2/\sqrt{5}}$, $w(0) = 1$, $b(0) = 0$, and $\vartheta = 4$ have been chosen such that we get a periodic surface for $\alpha = \pi/2$.

for the successors $b_1$ and $w_1$ of $b$ and $w$, respectively. From the equation on the right hand side we get $b_1 = \operatorname{Re}(a)\frac{w^2-w_1^2}{ww_1} - b$. Inserting this expression into the left hand equation yields the following condition for $w_1$:

$$w^4 w_1^6 + (w^6 + \operatorname{Re}(a)^2 w^2) w_1^4 + 2\operatorname{Re}(a) b w^3 w_1^3 - \left(\operatorname{Re}(a)^2 w^4 + 1\right) w_1^2 - w^2 = 0.$$

Due to Descartes' rule of signs there exists a unique positive solution $w_1$ of the latter equation.

Hence, for any given sequence $a : \mathbb{Z} \to \mathbb{R}$ and initial values $w(0) \in \mathbb{R}_+$, $b(0) \in \mathbb{R}$, the values $w(k)$ and $b(k)$ can be determined recursively for all $k \in \mathbb{Z}_+$. In this way we obtain Lax matrices $U(k, \alpha)$ and $V(k, \alpha)$ of the form (3.7) fulfilling the compatibility condition (3.8). Consequently, there exists a solution $\Phi = \Phi(k, t, \alpha)$ of the corresponding system (3.6). Given that $\Phi_1 = U\Phi$ and $\Phi(0, 0, \alpha) = \mathbf{1}$, we have

$$\Phi(k, 0, \alpha) = U(k-1, \alpha) U(k-2, \alpha) \cdots U(1, \alpha) U(0, \alpha).$$

Solving $\partial\Phi = V\Phi$ finally yields

$$\Phi(k, t, \alpha) = \exp\left(V(k, \alpha) t\right) \Phi(k, 0, \alpha).$$

By inserting this frame into the Sym-Bobenko formula (see Definition 3.12), we obtain semidiscrete Delaunay surfaces together with their associated families. For example, the initial values $w(0) = 1$ and $b(0) = 0$ yield $w(k) = 1$ and $b(k) = 0$ for all $k \in \mathbb{Z}_+$. The corresponding surfaces are semidiscrete cylinders with radius $r = 1/2$ (see Figure 3.3). By choosing $b(0) \neq 0$, we obtain more general semidiscrete Delaunay surfaces (see Figure 3.4).

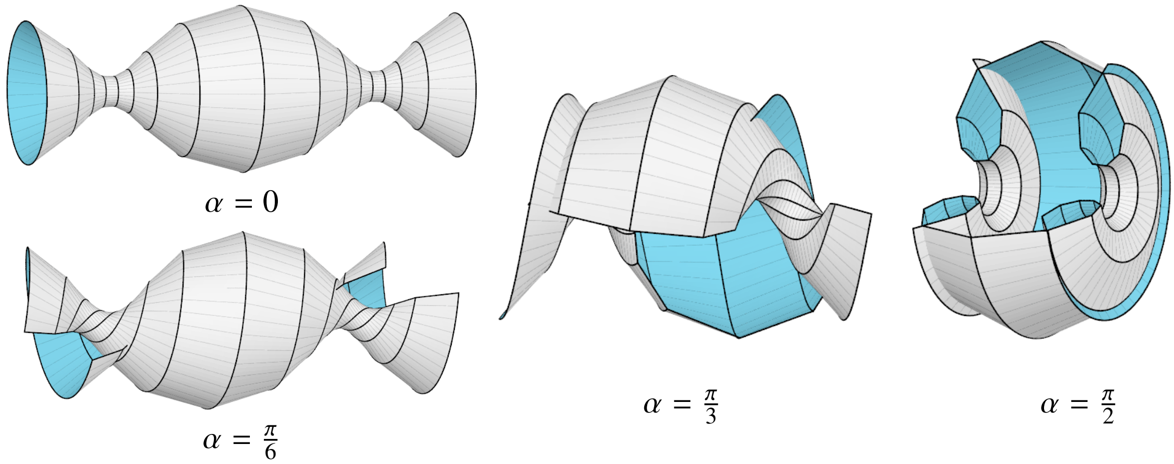


Figure 3.4: A semidiscrete unduloid (top left) and three members of its associated family. The corresponding parameters are $a$ = const. $\approx 6.29$, $w(0) = 1$, $b(0) = 2$, and $\vartheta = 4\sqrt{5}$. For $\alpha = \pi/2$ we obtain a semidiscrete nodoid (right).

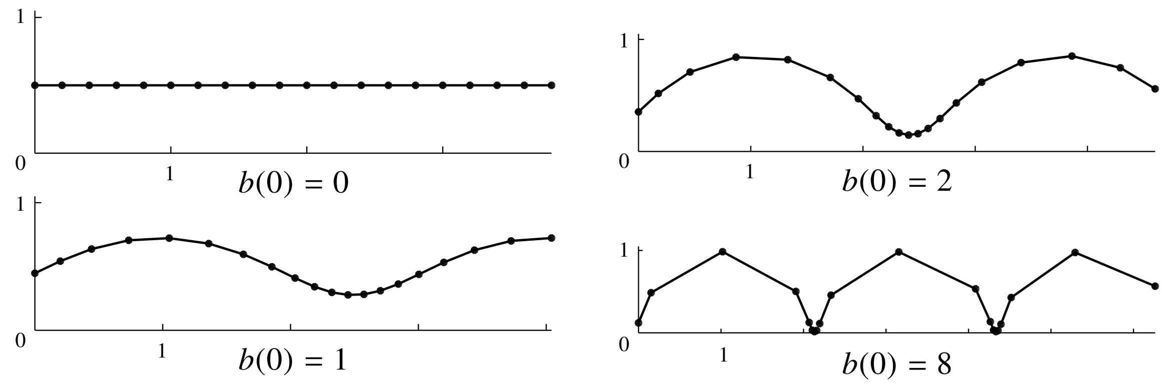


Figure 3.5: Profile curves of semidiscrete rotational symmetric cmc surfaces for different choices of the initial value $b(0)$, which controls the oscillation of the meridean polygon. Here, $\alpha = 0$, $a$ = const. = 10, and $w(0) = 1$.

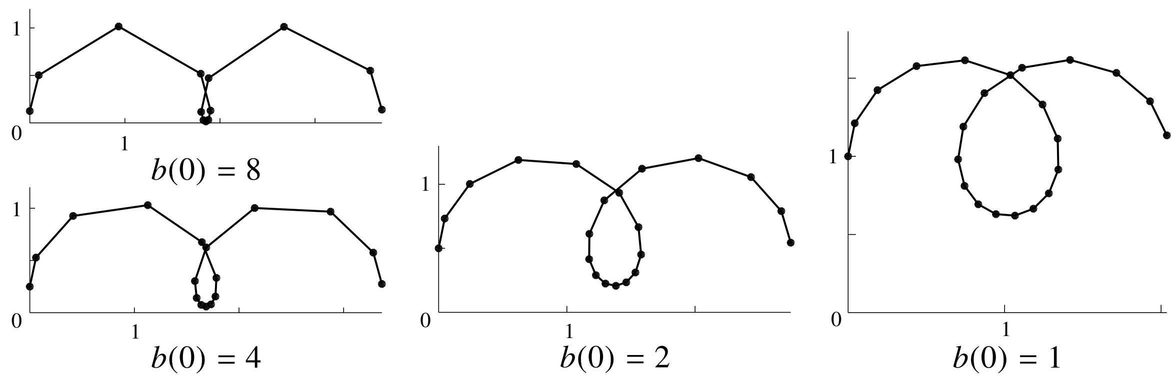


Figure 3.6: Profile curves of semidiscrete rotational symmetric cmc surfaces for different choices of the initial value $b(0)$. Here, $\alpha = \pi/2$, $a$ = const. = 10, and $w(0) = 1$.

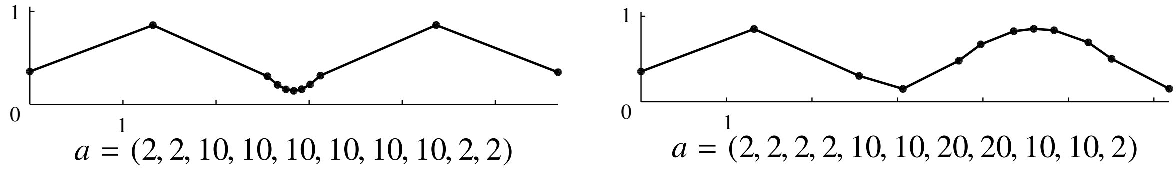


Figure 3.7: Profile curves of semidiscrete rotational symmetric cmc surfaces for different values of the sequence $a$, which controls the step size of the polygon. Here, $\alpha = 0$, $w(0) = 1$, and $b(0) = 2$.

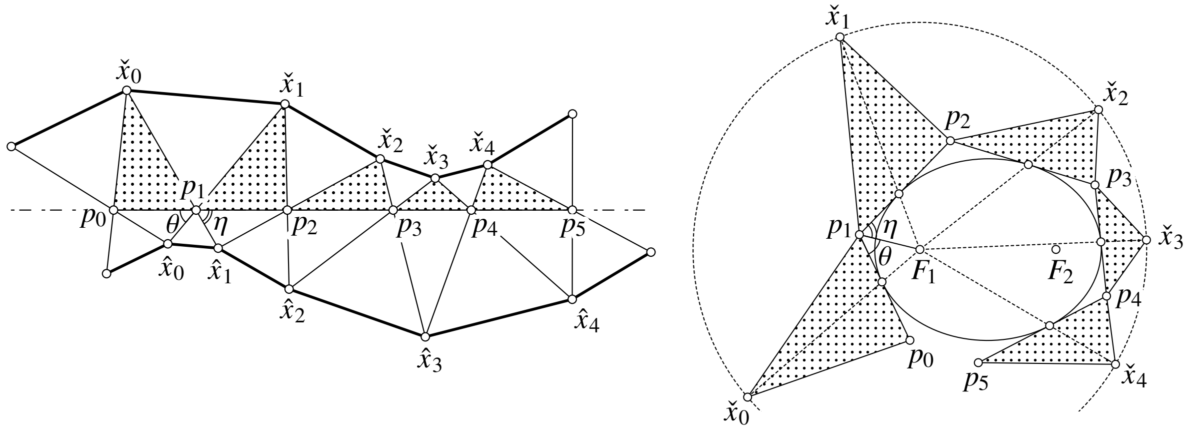


Figure 3.8: *Left:* Profile curves of semidiscrete rotational symmetric cmc surfaces $\check{x}$ and $\hat{x}$. *Right:* An external elliptic billiard $\{p_k\}_{k\in\mathbb{Z}}$. Each dotted triangle $\{\check{x}_k, p_k, p_{k+1}\}$ on the left is mapped isometrically to the corresponding triangle on the right. However, for illustrational reasons, the figure on the right hand side has been scaled up uniformly.

Observe that the initial value $b(0)$ regulates the shape of the profile curve of the isothermic rotational symmetric cmc surface gained for $\alpha = 0$. Setting $b(0) = 0$ yields a straight line and in the limit $b(0) \to \infty$ we end up with consecutive half circles (cf. Figure 3.5). The resulting surfaces are semidiscrete unduloids. Simultaneously, for $\alpha = \pi/2$, we obtain the profile curves of semidiscrete nodoids (cf. Figure 3.6).

Similarly, the sequence $a : \mathbb{Z} \to \mathbb{R}$ can be used to regulate the spacing between the vertices of the profile polygons. More precisely, the value $a(k)$ is inversely proportional to the length of the edge $[\check{x}^0(k,t), \check{x}^0(k+1,t)]$. For an illustration see Figure 3.7.

It turns out that there exists a nice geometric construction of the profile curves of semidiscrete rotational symmetric cmc surfaces. In fact, the discrete version of the classical Delaunay rolling ellipse construction for cmc surfaces of revolution, described by Bobenko et al. [8, §7.3], also applies to the semidiscrete setting (cf. Figure 3.8). This has to be so, since for discrete surfaces of rotational symmetry the mean curvature is independent of the angle of rotation (see Bobenko et al. [8, §7.2]). For this reason the notions of discrete and semidiscrete mean curvatures coincide in this particular case. For a comprehensive overview of mathematical billiards we refer to Tabachnikov [39].

# 3.A Appendix: Proof of Theorem 3.2

Here we provide the proof of Theorem 3.2. We show that the coupled semidiscrete surfaces $(\check{x}^\alpha, n^\alpha)$ and $(\hat{x}^\alpha, n^\alpha)$ from Definition 3.12 satisfy $A(\check{x}^\alpha, \hat{x}^\alpha) = 0$. Applying the Binet-Cauchy identity to the determinants occurring in the mixed area form yields

$$\begin{aligned} A(\check{x}^\alpha, \hat{x}^\alpha) = 0 \iff &\langle \delta\check{x}^\alpha, \delta n^\alpha\rangle\langle \partial\hat{x}^\alpha + \partial\hat{x}_1^\alpha, \partial n^\alpha + \partial n_1^\alpha\rangle - \\ &- \langle \delta\check{x}^\alpha, \partial n^\alpha + \partial n_1^\alpha\rangle\langle \partial\hat{x}^\alpha + \partial\hat{x}_1^\alpha, \delta n^\alpha\rangle + \\ &+ \langle \delta\hat{x}^\alpha, \delta n^\alpha\rangle\langle \partial\check{x}^\alpha + \partial\check{x}_1^\alpha, \partial n^\alpha + \partial n_1^\alpha\rangle - \\ &- \langle \delta\hat{x}^\alpha, \partial n^\alpha + \partial n_1^\alpha\rangle\langle \partial\check{x}^\alpha + \partial\check{x}_1^\alpha, \delta n^\alpha\rangle = 0. \end{aligned} \tag{3.10}$$

Moreover, we observe that, for every coupled semidiscrete surface $(x, n)$, we have

$$\langle \delta x, \partial n + \partial n_1\rangle = \langle \partial x, n_1\rangle - \langle \partial x_1, n\rangle = \langle \partial x + \partial x_1, \delta n\rangle.$$

Now, direct computations yield

$$\begin{aligned} \delta\check{x}^\alpha &= \frac{1}{\eta^2}\Phi^{-1}\begin{pmatrix} -i(2u^{-2} - \lambda^2 - \lambda^{-2}) & -2\bar{a}\frac{1}{u\lambda} \\ 2a\frac{\lambda}{u} & i(2u^{-2} - \lambda^2 - \lambda^{-2}) \end{pmatrix}\Phi, \\ \partial\check{x}^\alpha &= \frac{2}{v\vartheta}\Phi^{-1}\begin{pmatrix} 0 & i\lambda^{-1} \\ i\lambda & 0 \end{pmatrix}\Phi, \quad \partial\check{x}_1^\alpha = \frac{2}{v_1\vartheta_1}\Phi^{-1}U^{-1}\begin{pmatrix} 0 & i\lambda^{-1} \\ i\lambda & 0 \end{pmatrix}U\Phi, \end{aligned}$$

and

$$\begin{aligned} \delta\hat{x}^\alpha &= \frac{1}{\eta^2}\Phi^{-1}\begin{pmatrix} i(2u^2 - \lambda^2 - \lambda^{-2}) & -2\bar{a}u\lambda \\ 2a\frac{u}{\lambda} & -i(2u^2 - \lambda^2 - \lambda^{-2}) \end{pmatrix}\Phi, \\ \partial\hat{x}^\alpha &= -\frac{2v}{\vartheta}\Phi^{-1}\begin{pmatrix} 0 & i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}\Phi, \quad \partial\hat{x}_1^\alpha = -\frac{2v_1}{\vartheta_1}\Phi^{-1}U^{-1}\begin{pmatrix} 0 & i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}U\Phi. \end{aligned}$$

Further, for the Gauß map $n^\alpha$, we obtain

$$\begin{aligned} n^\alpha &= \Phi^{-1}\mathbf{k}\Phi = \Phi^{-1}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\Phi, \\ n_1^\alpha &= \Phi^{-1}U^{-1}\mathbf{k}U\Phi = \frac{1}{\eta^2}\Phi^{-1}\begin{pmatrix} -i|a|^2 - i\left(u\lambda - \frac{1}{u\lambda}\right)\left(\frac{\lambda}{u} - \frac{u}{\lambda}\right) & -2\bar{a}\left(u\lambda - \frac{1}{u\lambda}\right) \\ 2a\left(\frac{u}{\lambda} - \frac{\lambda}{u}\right) & i|a|^2 + i\left(\frac{\lambda}{u} - \frac{u}{\lambda}\right)\left(u\lambda - \frac{1}{u\lambda}\right) \end{pmatrix}\Phi, \\ \delta n^\alpha &= \frac{2}{\eta^2}\Phi^{-1}\begin{pmatrix} i\left(u^2 + u^{-2} - \lambda^2 - \lambda^{-2}\right) & \bar{a}\left(\frac{1}{u\lambda} - u\lambda\right) \\ a\left(\frac{u}{\lambda} - \frac{\lambda}{u}\right) & -i\left(u^2 + u^{-2} - \lambda^2 - \lambda^{-2}\right) \end{pmatrix}\Phi, \\ \partial n_1^\alpha &= -\frac{2i}{\vartheta_1}\Phi^{-1}U^{-1}\begin{pmatrix} 0 & v_1\lambda + \frac{1}{v_1\lambda} \\ \frac{\lambda}{v_1} + \frac{v_1}{\lambda} & 0 \end{pmatrix}U\Phi, \quad \partial n^\alpha = -\frac{2i}{\vartheta}\Phi^{-1}\begin{pmatrix} 0 & v\lambda + \frac{1}{v\lambda} \\ \frac{\lambda}{v} + \frac{v}{\lambda} & 0 \end{pmatrix}\Phi. \end{aligned}$$

Next, we compute

$$U^{-1}\begin{pmatrix}0 & i\lambda^{-1}\\ i\lambda & 0\end{pmatrix}U = \frac{1}{\eta^2}\begin{pmatrix} a\left(\frac{1}{u}-u\lambda^2\right)-\bar{a}\left(\frac{1}{u}-\frac{u}{\lambda^2}\right) & i\frac{\bar{a}^2}{\lambda}+i\lambda\left(\frac{1}{u\lambda}-u\lambda\right)^2\\ ia^2\lambda+\frac{i}{\lambda}\left(\frac{\lambda}{u}-\frac{u}{\lambda}\right)^2 & \bar{a}\left(\frac{1}{u}-\frac{u}{\lambda^2}\right)-a\left(\frac{1}{u}-u\lambda^2\right)\end{pmatrix},$$

$$U^{-1}\begin{pmatrix}0 & i\lambda\\ i\lambda^{-1} & 0\end{pmatrix}U = \frac{1}{\eta^2}\begin{pmatrix} a\left(\frac{1}{u\lambda^2}-u\right)-\bar{a}\left(\frac{\lambda^2}{u}-u\right) & i\bar{a}^2\lambda+\frac{i}{\lambda}\left(\frac{1}{u\lambda}-u\lambda\right)^2\\ i\frac{a^2}{\lambda}+i\lambda\left(\frac{\lambda}{u}-\frac{u}{\lambda}\right)^2 & \bar{a}\left(\frac{\lambda^2}{u}-u\right)-a\left(\frac{1}{u\lambda^2}-u\right)\end{pmatrix},$$

and observe that

$$iU^{-1}\begin{pmatrix}0 & v_1\lambda+\frac{1}{v_1\lambda}\\ \frac{\lambda}{v_1}+\frac{v_1}{\lambda} & 0\end{pmatrix}U = U^{-1}\left[v_1\begin{pmatrix}0 & i\lambda\\ i\lambda^{-1} & 0\end{pmatrix}+\frac{1}{v_1}\begin{pmatrix}0 & i\lambda^{-1}\\ i\lambda & 0\end{pmatrix}\right]U.$$

Finally, we get

$$\begin{aligned}
\langle\delta\check{x}^\alpha,\delta n^\alpha\rangle &= -\frac{2}{\eta^2}\left(2u^{-2}-\lambda^2-\lambda^{-2}\right), & \langle\delta\hat{x}^\alpha,\delta n^\alpha\rangle &= \frac{2}{\eta^2}\left(2u^2-\lambda^2-\lambda^{-2}\right),\\
\langle\partial\check{x}_1^\alpha,n^\alpha\rangle &= -\frac{4}{v_1\vartheta_1\eta^2}\operatorname{Im}\left(a\left(u^{-1}-u\lambda^2\right)\right), & \langle\partial\check{x}^\alpha,\partial n^\alpha\rangle &= -\frac{2}{\vartheta^2}\left(2v^{-2}+\lambda^2+\lambda^{-2}\right),\\
\langle\partial\check{x}^\alpha,n_1^\alpha\rangle &= -\frac{4}{v\vartheta\eta^2}\operatorname{Im}\left(a\left(u^{-1}-u\lambda^{-2}\right)\right), & \langle\partial\check{x}_1^\alpha,\partial n_1^\alpha\rangle &= -\frac{2}{\vartheta_1^2}\left(2v_1^{-2}+\lambda^2+\lambda^{-2}\right),\\
\langle\partial\hat{x}_1^\alpha,n^\alpha\rangle &= -\frac{4v_1}{\vartheta_1\eta^2}\operatorname{Im}\left(a\left(u-u^{-1}\lambda^{-2}\right)\right), & \langle\partial\hat{x}^\alpha,\partial n^\alpha\rangle &= \frac{2}{\vartheta^2}\left(2v^2+\lambda^2+\lambda^{-2}\right),\\
\langle\partial\hat{x}^\alpha,n_1^\alpha\rangle &= -\frac{4v}{\vartheta\eta^2}\operatorname{Im}\left(a\left(u-u^{-1}\lambda^2\right)\right), & \langle\partial\hat{x}_1^\alpha,\partial n_1^\alpha\rangle &= \frac{2}{\vartheta_1^2}\left(2v_1^2+\lambda^2+\lambda^{-2}\right),
\end{aligned}$$

as well as

$$\begin{aligned}
\langle\partial\check{x}^\alpha,\partial n_1^\alpha\rangle &= -\frac{4}{v\vartheta\vartheta_1\eta^2}\operatorname{Re}\left(a^2\left(v_1^{-1}+v_1\lambda^{-2}\right)+\left(v_1+v_1^{-1}\lambda^{-2}\right)\left(u^{-1}\lambda-u\lambda^{-1}\right)^2\right),\\
\langle\partial\hat{x}^\alpha,\partial n_1^\alpha\rangle &= \frac{4v}{\vartheta\vartheta_1\eta^2}\operatorname{Re}\left(a^2\left(v_1+v_1^{-1}\lambda^2\right)+\left(v_1^{-1}+v_1\lambda^2\right)\left(u^{-1}\lambda-u\lambda^{-1}\right)^2\right),\\
\langle\partial\check{x}_1^\alpha,\partial n^\alpha\rangle &= -\frac{4}{v_1\vartheta\vartheta_1\eta^2}\operatorname{Re}\left(a^2\left(v^{-1}+v\lambda^2\right)+\left(v+v^{-1}\lambda^{-2}\right)\left(u^{-1}\lambda-u\lambda^{-1}\right)^2\right),\\
\langle\partial\hat{x}_1^\alpha,\partial n^\alpha\rangle &= \frac{4v_1}{\vartheta\vartheta_1\eta^2}\operatorname{Re}\left(a^2\left(v+v^{-1}\lambda^{-2}\right)+\left(v^{-1}+v\lambda^2\right)\left(u^{-1}\lambda-u\lambda^{-1}\right)^2\right).
\end{aligned}$$

We complete the proof by substituting these expressions into Equation (3.10) and using the fact that $\vartheta_1 = \vartheta$ and $u^2 = vv_1$. □

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