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Potential theory on infinite trees and the unit disk

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Abstract

Consider an arbitrary homogeneous tree together with its boundary. In this work we investigate several questions concerning the behavior of functions at the boundary and its relation to the behavior of these functions on the set of vertices of the tree. First, we introduce the notion of Blashke-type condition for subharmonic functions on homogeneous trees and show that this condition is essentially linked to the choice of a zero measure subset of the boundary and the growth rate of a subharmonic function on this subset. Our second goal is to develop the generalization of the mean value theory for the case of nonharmonic functions on homogeneous trees. In order to do this we introduce the notion of Laplace operator power series on the boundary. We prove that the value of the function at the root vertex of the tree equals the integral of such series for the function over the boundary. In both cases, we also present the analogous results on the unit disk, respectively in Euclidean space. The spirit of this thesis is that of elaborating the close analogies between the discrete and continuous settings.

Contents

1 Introduction

The modern potential theory on homogeneous trees has many deep relations with the classical potential theory on the unit disc. Various objects and properties of the classical theory have a counterpart on the homogeneous trees and vice versa. In this thesis we investigate several questions that have traces in potential theory on the unit disk from one side and potential theory on the homogeneous tree on the other side.

Let us start with a small historical overview of potential theory.

First we would like to mention a famous I. Newton's work "Philosophiæ Naturalis Principia Mathematica" ("Mathematical Principles of Natural Philosophy"), which is traditionally considered as the first contribution to potential theory (see Brackenridge, 1996). This work was published in 1687, and it was dedicated to the gravitational laws. I. Newton studied the classical model of gravity, in which one observes a collection of particles, and additionally for every pair of particles there is a gravity force acting between them. In fact this force is a force of attraction which is proportional to the product of masses of these particles from the one hand and which is inverse to the square of the distance between these particles. From the point of view of celestial mechanics and the theory of geodesy the first important subject related to potential theory was the subject of attraction forces for material points and for finite solid bodies. After several preliminary results by I. Newton and several other researchers, the investigation was carried out by J.L. Lagrange, A. Legendre and P.S. Laplace who realized the importance of the original work of I. Newton and essentially attracted the attention of the broad research community for many years.

One of the most important contributions to the origins of potential theory was made by J.L. Lagrange who established that a field generated by gravitation forces is a *potential field*. He has introduced a function which was later referred by G. Green (1828) as a *potential function* and by C.F. Gauss (1840) as a *potential*. The above mentioned results form the basis of the classical potential theory. For further details on the foundation of potential theory we refer to Kellogg, 1929 and Landkof, 1972.

C.F. Gauss and his followers established that the method of potentials can be applied to a wide range of problems in mathematical physics; for instance, they are broadly used in electrostatics and magnetism. From that time onwards the potentials started to be studied in the contents of the physically realistic problems concerning the mutual attraction of particles whose masses have arbitrary signs (not necessarily positive). A little later major principal boundary value problems were formulated, including the Dirichlet problem, the Neumann problem, the electrostatic problem of the static distribution of charges on conductors (i.e., the Robin problem), the problem of sweeping-out mass (Balayage method), etc. Certain types of potentials turned out to be efficient to solve the mentioned above problems in the case of sufficiently smooth domains: the most famous of them are the volume potentials of distributed mass, single- and double-layer potentials, logarithmic potentials, Green potentials, etc.

At the end of the 19th century A.M. Lyapunov made an exhaustive study of several important aspects of potential theory. One of his most valuable contributions to the theory of potentials was his work "On some questions connected with Dirichlet's problem" (see Lyapunov, 1954) written in 1897. Here he studied a number of the basic properties of the potentials of simple and double layers. A.M. Lyapunov obtained important results concerning the behavior of the derivatives of the Dirichlet problem solution, when approaching the boundary. His research in this field was further followed his student V.A. Steklov. From that time onwards potential theory started to be considered as an independent branch of mathematics.

In the first half of the past century the ideas of potential theory were greatly extended and generalized to many different cases. In particular some developments were based on the general notions of a Radon measure, capacity and generalized functions. Nowadays potential theory also includes the problems concerning harmonic and subharmonic functions (which we will be studied within current work), the Dirichlet problem, the harmonic measure, Green's functions, potentials and capacity. Modern potential theory is closely related to probability theory, since many potential theoretic concepts have natural interpretations in the probabilistic terms.

In his paper of 1933 A.N. Kolmogorov introduced an axiomatic approach to probability theory which was based on measure theory (see Kolmogorov, 1933). Since the mid 1940' the research in probability and potential theory started to have more and more interaction points (see Kakutani, 1944). One of the key points here is the correspondence between the mean value property of harmonic functions and the fact that in a Brownian motion the probability of the motion of a point is the same in all admissible directions. J. Doob founded and developed the modern field of probabilistic potential theory. This is subsumed in his remarkable book Doob, 2001. Let us also mention the work by M. Brelot in which the author developed potential theory using an axiomatic approach. This results are consistent with the viewpoint of the classical theory (see Brelot, 1967). The relation between Markov processes and probabilistic interpretation of some key tools of potential theory was studied by Bauer, 1966. The fundamental global theory of kernels was developed by Hunt, 1957, 1958. For further development of discrete potential theory and random walks on infinite graphs and groups we refer the reader to the survey by W. Woess (see Woess, 1994).

In this work we bring together various concepts of classical potential theory and the corresponding concepts on the homogeneous tree (whenever possible we analyze the observed similarities and distinctions). In fact, one may think of the tree as a discrete a model for the unit disk in the complex plane. This viewpoint has its origins in the seminal work of Cartier, 1972, see also Koranyi, M. Picardello, and Taibleson, 1984 and a considerable amount of further works.

Organization of the thesis. Chapter 1 of current work contains an introduction to both the classical potential theory and to potential theory on homogeneous trees. In this chapter we give all necessary definition, fix the notation and introduce some important facts that are essential for the further reading.

In Chapter 2 of this thesis we are interested in functions that grow exponentially fast near a subset of the boundary. In both cases of the unit disk and of homogeneous trees we mostly work with subharmonic functions. It turns out that it is possible to give quantitative estimates to the non-harmonicity of such functions. There is a natural way to do this via the Riesz measure. The main results of this chapter are a necessary condition and a sufficient condition for the growth rate of the Riesz measure for certain subharmonic functions. This part of the thesis is based on the preprint Boiko and Woess, 2014.

Finally in Chapter 3, we investigate the mean value property for non-harmonic functions. In this work we show that the mean value property has a natural extension to the case non-harmonic case. In order to do so for to general non-harmonic functions, we introduce some Laplace operator series (that are very similar to Taylor expansions). We write explicitly such series in the Euclidean case and in the case of infinite homogeneous trees. Most of the results presented in this chapter are based on the preprint Boiko and Karpenkov, 2014.

1.0.1 On the correspondents between the potential theory on the unit disk and on the homogeneous tree

From many points of view an infinite homogeneous tree is a discrete analogue of the hyperbolic plane. These are two basic examples of Gromov-hyperbolic metric spaces. Consider the Poincaré model of the hyperbolic plane: i.e., we consider the open unit disk $\mathbb{D}$ as a topological space with the hyperbolic metric on it. Its natural geometric compactification is obtained by passing from the hyperbolic to the Euclidean metric and taking the closure of the open unit disc, as a result we obtain the closed unit disc. In a similar way, any homogeneous tree admits a natural compactification. It is obtained by passing from the original graph metric to a new (bounded) metric, which can be obtained by embedding of the tree to some compact subset of the Euclidean space (say to the unit disc). In this new metric we analogously consider all limit points of the vertices of the tree. As a result we get the compactification of the tree with respect to the induced metric.

Various objects on $\mathbb{D}$ have counterparts on homogeneous trees and vice versa. It is not always immediately apparent that simply by examining the unit disk $\mathbb{D}$ both with Euclidean and with hyperbolic point of view one may provide some additional insight. But this is essential for the interplay between the unit disk $\mathbb{D}$ and homogeneous trees. The purpose of this section is to exhibit some potential theoretic aspects of that correspondence.

1.1 Potential theory in the complex plane and $\mathbb{R}^n$

We start with several general definitions. In what follows we denote by $S^{d-1}(r)$ the $(d-1)$-dimensional *sphere* in the Euclidean space $\mathbb{R}^d$ with radius r and center at the origin.

In the next subsection we deal with the two-dimensional case (which one also can consider as a one-dimensional complex case). A good references here are Ransford, 1995 and Helms, 1969. The term *unit disc* will be used for the open unit disk centered at the origin with unit radius. Let 0 be the origin of the complex plane $\mathbb{C}$, we write

$$\mathbb{D} = \{z \in \mathbb{C} : |z - 0| < 1\}.$$

For the *unit circle* we write

$$\partial\mathbb{D} = \{z \in \mathbb{C} : |z - 0| = 1\}.$$

Let us now continue with the notion of harmonic functions.

1.1.1 Harmonic functions

We will define harmonic functions as solutions of Laplace's equation. In fact, there is a strong connection between analytic functions and holomorphic functions. We start with the formal definition and then give some examples.

The *Laplace operator* (or *Laplacian*) is the sum of all the unmixed second partial derivatives in the Cartesian coordinates x_i

$$\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

It is named after the French mathematician Pierre-Simon de Laplace.

Definition 1. Let U be an open subset of the Euclidean space $\mathbb{R}^n$. A function $h : U \to \mathbb{R}$ is called *harmonic* if $h \in C^2(U)$ and $\triangle h = 0$ on U.

In fact, we will mostly consider harmonic functions on some subsets of $\mathbb{R}^2$ which is traditionally identified with the complex plane $\mathbb{C}$.

Further in chapter 3 we also use the Laplace operator in polar coordinates. In polar coordinates we have the following expression for the Laplace operator

$$\triangle(f) = \triangle_r(f) + \frac{1}{r^2}\triangle_{S^{d-1}} f, \quad \text{where} \quad \triangle_r(f) = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial f}{\partial r}\right),$$

the radial part, and $\triangle_{S^{d-1}}$ is the Laplace–Beltrami operator on the $(d-1)$-sphere.

1.1.2 Examples of harmonic functions

A nice source of examples of harmonic functions is the following classical result.

Theorem 1.1.1. *Let U be a domain in $\mathbb{C}$.*

- *If a function f is holomorphic on the domain U, then the real-valued functions* $\mathrm{Re} f$ *and* $\mathrm{Im} f$ *are harmonic.*

- *If a function h is harmonic on a simply connected domain U, then $h = \mathrm{Re} f$ for some holomorphic function f of U. Up to an additive constant, the function f is uniquely defined by h.*

Let us use this theorem to give some examples of harmonic functions.

Example 1.1.2. Consider the holomorphic function $f(x,y) = e^{x+iy}$ on $\mathbb{C}$. Its real part $\mathrm{Re} f = e^x \cos(y)$ and imaginary part $\mathrm{Im} f = e^x \sin(y)$ are harmonic.

Example 1.1.3. Consider the holomorphic function

$$f(z) = \ln z = \ln |z| + i \mathrm{Arg} z$$

defined on $\mathrm{Re} > 0$. Its real and imaginary parts are respectively

$$\mathrm{Re} f = \frac{1}{2} \ln(x^2 + y^2) \quad \text{and} \quad \mathrm{Im} f = \mathrm{Arg}(z).$$

By the above theorem the functions $\mathrm{Re} f$ and $\mathrm{Im} f$ are harmonic.

Let us now give an example of multivariate harmonic function.

Example 1.1.4. The function

$$h(x_1, x_2, \ldots x_n) = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1-n}{2}} \quad \text{with} \quad n > 2$$

is harmonic on $\mathbb{R}^n \backslash \{0\}$.

Example 1.1.5. The function $f(x,y) = x^2 + y^2$ is not harmonic. Indeed,

$$\triangle f(x,y) = 4 \neq 0.$$

1.1.3 Mean value property

The *mean value property* (also known as *Gauss' mean value theorem*) will be considered in the last part of this thesis. We will be aiming to extend this property to the case of non-harmonic functions. Let us now recall the classical result.

Theorem 1.1.6 (Mean value property). *Let h be a harmonic function on the domain $U \subset \mathbb{R}^d$, and r be an arbitrary positive integer number. Let the ball bounded by the sphere $S^{d-1}(r)$ be completely contained in U. Then*

$$h(0) = \frac{1}{\mathrm{Vol}(S^{d-1}(r))} \int\limits_{S^{d-1}(r)} h \, d\lambda.$$

We give an example of a harmonic function and check the mean value property for this function.

Example 1.1.7. Consider the harmonic function

$$h(x, y) = x^2 - y^2.$$

This function is harmonic, since

$$\triangle h = 2 - 2 = 0.$$

The mean value is

$$\int\limits_{S^1} (x^2 - y^2) d\mu = \int\limits_0^{2\pi} \cos^2(\varphi) - \sin^2(\varphi) d\varphi = \int\limits_0^{2\pi} \cos(2\varphi) d\varphi = 0.$$

1.1.4 Subharmonic functions

The idea of a subharmonic function was expounded in essence by H. Poincarè in the balayage method. Subharmonic functions are also considered in the work of Hartogs, 1906 on the theory of analytic functions of several complex variables; the systematic study of subharmonic functions began with the work of Riesz, 1926 (see also Hartig, 1983).

A function $v \in C^2(\mathbb{R})$ is *subharmonic* if its Laplacian satisfies inequality

$$\triangle u \geq 0.$$

However if we would like to define the notion of subharmonicity for functions that are not in the class $C^2(\mathbb{R})$, we cannot use the Laplacian. It turns out, that there is a way to avoid this assumption, by analogy with convex functions on $\mathbb{R}^n$. Recall that convexity is actually defined via a sub-mean value property. This allows us to consider convex functions, which are non-smooth (for instance, the function $|x|$ is an example of such a function).

Before giving a formal definition of a subharmonic function, we take a look at upper semicontinuous functions.

Definition 2. Let X be a topological space. A function $f : X \to (-\infty, \infty]$ is said to be *upper semicontinuous* if the set $\{x \in X : f(x) < \alpha\}$ is open in X for every $\alpha \in \mathbb{R}$.

Equivalently it can be expressed as

$$\limsup_{x \to x_0} f(x) \leq f(x_0), \quad x \in X.$$

Now we are ready to give the following general definition.

Definition 3. A function v on an open subset U of the complex plane to the union of the real line and $-\infty$ is called a *subharmonic function* if v is upper semicontinuous and satisfies the local sub-mean inequality. The sub-mean inequality here is written as follows. Given $z \in U$, there exists $\rho > 0$ such that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{it})dt, \quad (0 \leq r < \rho)$$

We say that a function $u : U \to (-\infty, \infty]$ is *superharmonic* if $-u$ is subharmonic.

Remark 1.1.8. Subharmonicity is a local property, since it is defined via the local submean inequality (i.e., the value of ρ actually depends of the point z).

Remark 1.1.9. According to our definition, the function $u \equiv -\infty$ is a subharmonic function.

We note also that the set at which a subharmonic function attain the value $-\infty$ should either coincide with the domain of the definition of this function or this set is not "too large". The following result holds for this set. Further we use the following definition.

Definition 4. The set on which a subharmonic function attains the value $-\infty$ is called the $-\infty$-*set*:

$$\{z \in \mathbb{D} : u(z) = -\infty\}$$

Theorem 1.1.10. *Let u be a subharmonic function on the domain $U \subset \mathbb{D}$ and let u be not identically equal $-\infty$ on U. Then the $-\infty$-set has Lebesgue measure zero.*

Remark 1.1.11. A stronger result on the structure of the $-\infty$-set can be found for instance in Ransford, 1995 [Th 3.5.1].

1.1.5 Examples of subharmonic functions

The following example is especially important in our consideration. It shows a deep relation between subharmonic functions and analytic functions. Moreover it shows the way of possible use of subharmonic functions for the study of analytic functions of one or several variables.

Example 1.1.12. Let f be a holomorphic function on an open set U in the complex plane. Then the function $\log|f|$ is subharmonic on the set U.

From Theorem 1.1.10 it follows that subharmonic functions from the last example have $-\infty$-set precisely in the zero set of f.

Example 1.1.13. Every harmonic function is subharmonic.

Moreover, a function is harmonic if and only if it is both subharmonic and superharmonic.

1.1.6 Maximum principle

The maximum of a subharmonic function cannot be achieved in the interior of its domain unless the function is constant, this is so-called the *maximum principle* for subharmonic functions. However, the minimum of a subharmonic function can be achieved in the interior of its domain.

Theorem 1.1.14 (Maximum principle)**.** *Let u be a subharmonic function on a domain $U \subset \mathbb{C}$.*

- *If u attains a global maximum on U, then it is constant.*
- *If we have*
$$\limsup_{z \to \zeta} u(x) \leq 0$$
for all $\zeta \in \partial U$, then $u \leq 0$ on U.

Remark 1.1.15. The validity of the second part of the theorem is due to our convention that $\infty \in \partial U$ whenever U is unbounded.

1.1.7 The Dirichlet problem on the disk

The Dirichlet problem is designed to find a harmonic function in a given domain with given values on the boundary of the domain. One of the great advantages of harmonic functions as compared with holomorphic functions is that for "nice" domains, a solution for the Dirichlet problem always exists. We proceed with the precise formulation of the problem.

Definition 5. Let U be a domain of the complex plane and $\phi : \partial U \to \mathbb{R}$ be a continuous function. Then the *Dirichlet problem* is as follows: find a harmonic function h on the domain U such that
$$\lim_{z \to \zeta} h(z) = \phi(\zeta), \quad \text{for all} \quad \zeta \in \partial \mathbb{D}.$$

The uniqueness of the solution of the Dirichlet problem follows directly from the maximum principle for harmonic functions.

The question of existence is more complicated for a general domain U. In this work we consider essentially the case of the unit disc. We start with the following definition.

Definition 6. The *Poisson kernel* $P : \mathbb{D} \times \partial\mathbb{D} \to \mathbb{R}^1$ is defined as follows

$$P(z, \zeta) = \frac{1 - |z|^2}{|\zeta - z|^2},$$

where $|z| < 1$ and $|\zeta| = 1$.

Let us list some basic properties of the Poisson kernel.

Theorem 1.1.16. *The Poisson kernel $P : \mathbb{D} \times \partial\mathbb{D} \to \mathbb{R}^1$ satisfies the following conditions.*

1. *For arbitrary $|z| < 1$ and $|\zeta| = 1$ we have $P(z, \zeta) > 0$.*

2. *Let $|z| < 1$ then*

$$\frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) d\theta = 1.$$

3. *Let $|\zeta_0| = 1$ and $\delta > 0$, then*

$$\sup_{|\zeta - \zeta_0| \geq \delta} P(z, \zeta) \to 0 \quad as \quad z \to \zeta.$$

By $S(z, r)$ we denote the circle of radius r centered in z.

Definition 7. Let $\phi : S(z, r) \to \mathbb{R}$ be a Lebesgue-integrable function. Then its *Poisson integral* is defined by the function

$$P_{S(z,r)}\phi(w) = \frac{1}{2\pi} \int_0^{2\pi} P\Big(\frac{w - z}{r}, e^{i\theta}\Big) \phi(z + re^{i\theta}) d\theta \tag{1.1}$$

for every $w \in S(z, r)$.

Now we formulate the fundamental result on the Poisson integral.

Theorem 1.1.17. *The following statements hold.*

1. *The function $P_{S(z,r)}\phi$ is harmonic on the unit disk $\mathbb{D}$.*

2. *If the function ϕ is continuous at $\zeta \in \partial\mathbb{D}$, then*

$$\lim_{z\to\zeta} P_{S(z,r)}\phi(z) = \phi(\zeta).$$

This theorem implies the following important statement.

Corollary 1.1.18. *If the function ϕ is continuous on the whole boundary of the unit disk $\mathbb{D}$, then the Poisson integral $P_{\mathbb{D}}\phi$ solves the Dirichlet problem on the unit disc.*

1.1.8 Potentials

Potentials provide a rich source of examples of subharmonic functions, for instance, one can construct subharmonic functions that possess some prescribed properties. Moreover, as we shall see, the potentials are in some sense as general as arbitrary subharmonic functions, namely for some purposes the cases of potentials and of subharmonic functions coincide.

We start with the following definition

Definition 8. Let d be a positive integer. The *Newtonian kernel* of dimension d is the following function:

$$K(z-\zeta) = \begin{cases} \log|z-\zeta|, & \text{if} \quad n=2, \\ -|z-\zeta|^{2-n}, & \text{if} \quad n \geq 3. \end{cases} \tag{1.2}$$

Let us now define potentials for finite measures with a compact support on the space $\mathbb{R}^d$.

Definition 9. Let μ be a finite Borel measure on $\mathbb{R}^d$ with a compact support. The function $p_\mu : \mathbb{C} \to [-\infty, \infty)$ defined as

$$p_\mu(z) = \int K(z-\zeta)d\mu(\zeta), \quad z \in \mathbb{C}. \tag{1.3}$$

is called the *potential* (or sometimes *Newtonian* or *logarithmic potential*) of the measure μ.

In the framework of subharmonic functions we have the following general result on potentials.

Theorem 1.1.19. *A potential p_μ is subharmonic on the complex plane and harmonic on the set $\mathbb{C} \setminus \operatorname{supp}(\mu)$.*

1.1.9 Riesz representation theorem

One of the central theorems in the theory of subharmonic functions is the Riesz local representation theorem (see Riesz, 1926). This theorem states, that a subharmonic (or superharmonic) function is locally equal to a Newtonian potential of a measure (respectively ≤ 0 or ≥ 0) up to a harmonic function. In fact, many problems related to general subharmonic functions can be reformulated in terms of the appropriate potentials and vice versa.

Theorem 1.1.20 (Riesz representation theorem)**.** *Let D be a domain of the space $\mathbb{R}^n$. Consider a subharmonic function u on the domain D that is not identically equal to $-\infty$. Then there exists a unique non-negative Borel measure μ_u on the domain D (i.e., a measure associated with u, it is called the Riesz measure) such that for any compact set $U \subset D$ the following representation holds*

$$u(z) = h(z) + \int\limits_U K(z-\zeta)d\mu_u(\zeta),$$

where h is a harmonic function in the interior of U and K is the Newtonian kernel.

In a particular case of the unit disk (i.e., $D = \mathbb{D}$) the *Riesz measure* associated with the function u is as follows

$$\mu_u = \frac{1}{2\pi} \triangle u\big|_U,$$

where $\triangle u$ is to be understood in the distributional sense.

In other words, for every function f of class C^∞ with compact support in the unit disk $\mathbb{D}$

$$\int_{\mathbb{D}} f d\mu_u = \frac{1}{2\pi} \int_{\mathbb{D}} u(z) \triangle f(z) d\lambda(z).$$

A Green function is a family of fundamental solutions of the Laplace operator with value zero on the boundary. Here is the formal definition.

Definition 10. The *Green function* of the Laplace operator is

$$G_{\mathbb{D}}(z,w) = \log \frac{|1 - z\overline{w}|}{z - w}, \quad z, w \in \mathbb{D}. \tag{1.4}$$

The Green function is related to another important concept, the harmonic measure. Using Green function one can find estimates of the harmonic measure.

1.1.10 Harmonic measure

While describing the Dirichlet problem on the unit disc, we have introduced an explicit formula for it. Now we extend it to the cases of more general domains. Let us formulate the notion of harmonic measure.

Definition 11. Let U be a bounded open domain in the n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$), and let ∂U be the boundary of U. Consider a continuous function $f : \partial U \to \mathbb{R}$. It determines a unique solution of the Dirichlet problem in the domain U, which we denote by $H_U f$.

For a fixed point $z \in U$ the function $H_U f$ determines a Borel probability measure $\omega_U(z)$, which is called the *harmonic measure* at the point z. The formula for the representation of the generalized solution of the Dirichlet problem given by

$$H_U f(z) = \int_{\partial U} f(\zeta) d\omega_U(z, \zeta).$$

The formula for the representation of the generalized solution of the Dirichlet problem was obtained by Ch.J. de la Vallée-Poussin by the balayage method (see de la Vallée-Poussin, 1949). It is valid for all functions which are continuous on ∂U. Usually explicit computations of harmonic measures are possible only for the simplest domains (mainly for discs, spheres, half-planes and half-spaces).

Example 1.1.21. The harmonic measure in the case of the unit disk is as follows

$$d\omega_{\mathbb{D}}(z,\zeta) = \frac{1}{2\pi}P(z,\zeta)|d\zeta| = \frac{1}{2\pi}\frac{1-|z|^2}{|\zeta - z|^2}|d\zeta|$$

As it was noticed by Kakutani, 1944, the harmonic measure is closely related to Brownian motion. That is, if a solution $H_U f$ exists, then $H_U f(z)$ is the expected value of f(z) at the first exit point from U for a canonical Brownian motion starting at z.

1.2 Potential theory on homogeneous trees

In this chapter we present basic definitions and facts related to the theory of Markov chains, random walks on graphs and potential theory on graphs, in particular, on homogeneous trees. We shall follow the notation of Woess, 2009. For more information see Kemeny, Snell, and Knapp, 1976, Dynkin, 1969 and Spitzer, 2001.

As we said in the introduction of this work, we are linking various concepts of classical potential theory with the corresponding concepts on the homogeneous tree.

Let us recall some general terminology of graph theory. Let G be an locally finite, connected graph. Denote by $E(G)$ and $V(G)$ the edge and vertex sets of G respectively. We say that a pair of vertices (v_1, v_2) of a graph G are *neighbors* if they are connected by some edge, and write $v_1 \sim v_2$.

In this thesis we will work with a special kind of graphs, with so-called homogeneous trees. First we give the definition of a tree.

Definition 12. A *tree* is a simply connected, undirected graph without circles.

Now we specify what is a homogeneous tree. The *degree* of a vertex in a graph is the number of edges adjacent to this vertex. The *homogeneous tree* $\mathbb{T}_q$ is the tree whose vertices all have degree $(q+1)$.

Remark 1.2.1. All homogeneous trees have infinitely many vertices.

A tree is called *rooted* if one vertex is marked; we call this vertex the *root* (or *origin*) and denote it by o. The edges of a rooted tree can be given a natural orientation, say away from the root.

1.2.1 Graph distance

A *path* on a graph G is a finite sequence of vertices $v_0, v_1, \dots, v_n$ of G such that $v_{i-1} \sim v_i$ for $i = 1, \dots, n$. We denote this path by $\pi = [v_0, v_1, \dots, v_n]$ and say that it is connecting v_0 and v_n.

Definition 13. Let v and w be two vertices of a connected graph G. The *graph distance* between v and w is the least possible value of the lengths of all paths connecting v and w. We denote it by

$$d(v, w). \tag{1.5}$$

For an arbitrary rooted tree we write $|v| = d(o, v)$.

Note that according to this definition, the distance between two vertices on a homogeneous tree can be arbitrarily large. In the case of the unit disc, distance between two points is at most 2. So the graph $\mathbb{T}_q$ is not a good analogue of the Euclidean distance on $\mathbb{D}$.

Later we will compare some notions on the homogeneous tree with notions on the Poincarè disk. Note that $d(v, w)$ corresponds to the hyperbolic distance.

In case of an arbitrary tree for any two vertices v and w there exists a unique minimal path between v and w. It is called the *geodesic* path between v and w and denoted by $[v, w]$.

1.2.2 Closure of homogeneous trees and corresponding metrics

From now on we restrict ourselves to the case of homogeneous trees. Let us fix an integer $q \geq 2$ and consider the homogeneous tree $\mathbb{T}_q$. A one-sided infinite sequence of vertices $v_0, v_1, \ldots$ on $\mathbb{T}_q$ is called a *ray* if for any $i > 0$ we have $v_i \sim v_{i-1}$. We denote this ray by $[v_0, v_1, \ldots]$.

Let us introduce an equivalence relation on the set of all rays. Two rays $[v_0, v_1, ...]$ and $[w_0, w_1, ...]$ on $\mathbb{T}_q$ are *equivalent* if they differ only by a finite number of elements.

Definition 14. An *end* of $\mathbb{T}_q$ is an equivalence class of infinite rays on $\mathbb{T}_q$.

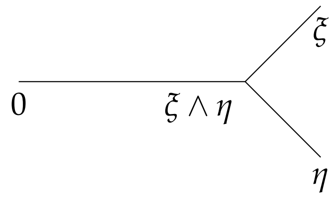
Definition 15. The *boundary* of $\mathbb{T}_q$ is the set of all its ends, we denote it by $\partial\mathbb{T}_q$. Denote also $\widehat{T}_q = \mathbb{T}_q \bigcup \partial\mathbb{T}_q$.

Consider an arbitrary end ξ of the tree $\mathbb{T}_q$. As an equivalence class of infinite rays, it contains a unique ray $[v_0, v_1, ...]$ without repetitions of vertices and $v_0 = o$. This ray is said to be the *geodesic ray* between o and ξ; we denote it by $[o, \xi]$.

In order to define a metric on $\mathbb{T}_q$ we need the definition of the confluent.

Definition 16. Let v, w be distinct elements of $\widehat{\mathbb{T}}$, then their *confluent* $v \wedge w$ is the last common vertex of the geodesic paths $[o, v]$ and $[o, w]$.

We illustrate this definition by a figure.



In this work we consider the following *metric* on the closure $\widehat{\mathbb{T}}_q$:

$$\rho_{\mathbb{T}_q}(v,w) = \begin{cases} q^{-|v\wedge w|}, & if\ v \neq w, \\ 0, & if\ v = w. \end{cases} \tag{1.6}$$

Remark 1.2.2. Notice that $\rho_{\mathbb{T}}$ is an *ultra metric*, i.e., for an arbitrary $v, w \in \widehat{\mathbb{T}}$ it holds

$$\rho_{\mathbb{T}}(v,w) \leq \max_{u\in\mathbb{T}_q}\{\rho_{\mathbb{T}}(v,u), \rho_{\mathbb{T}}(w,u)\}.$$

Further we also use the following definition.

Definition 17. The *distance* from an arbitrary vertex v of $\mathbb{T}_q$ to a subset E of $\partial\mathbb{T}_q$ is

$$\rho_{\mathbb{T}}(v,E) = \inf_{\xi\in E} \rho_{\mathbb{T}}(v,\xi).$$

1.2.3 Measure on the boundary

For an arbitrary vertex v of the tree $\mathbb{T}_q$ we define the *sector* $S(v) \subset \partial\mathbb{T}_q$ as follows

$$S(v) = \{\xi \in \partial\mathbb{T}_q \mid v \in [o,\xi]\}.$$

(recall that $[o,\xi]$ is the geodesic ray between o and ξ).

Consider the minimal topology of the boundary $\partial\mathbb{T}_q$ containing the sectors $S(v)$ for all vertices of the tree $\mathbb{T}_q$. The boundary $\partial\mathbb{T}_q$ with this topology is compact and totally disconnected. It is induced by the restriction of Let $\mathfrak{S}$ denote the corresponding σ-algebra defined by open sets of this topology (and therefore it is defined by all sectors

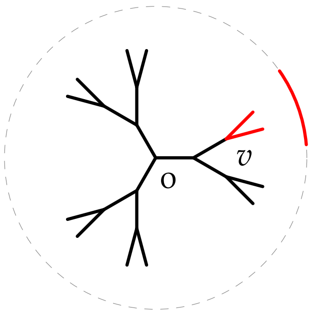
$S(v)$). We work with the following measure ν_o on $\mathfrak{S}$. For an arbitrary sector $S(v)$ we put by definition

$$\nu_o(S(v)) = \frac{1}{(q+1)q^{|v|-1}}.$$

We illustrate this definition with the following example.

Example 1.2.3. Take the homogeneous tree $\mathbb{T}_2$. Consider a sector with vertex v, such that $d(v,0) = 2$.

$$\nu_0(S(v)) = \frac{1}{3 \cdot 2^{2-1}} = \frac{1}{6}.$$



Remark 1.2.4. The measure ν_0 is rotation invariant. It is the *limit distribution* of simple random walk starting at o (for more information see Woess, 2009).

1.2.4 Simple random walks, Green kernels, potentials

Since the averaging in the definition of harmonic function can be interpreted as expectation after one move, harmonic functions is an important tool in studying of random walks. Thereby harmonic functions appear also in the theory of electrical networks, and in statics, providing a connection between these fields. In particular, various methods and results from the theory of electricity and statics (that are often motivated by physics) can be applied to provide results about random walks and vice versa (see Nash-Williams, 1959, Doyle and Snell, 1984, Soardi, 1994).

The *simple random walk* on a homogeneous tree $\mathbb{T}_q$ is a Markov chain with state space $V(\mathbb{T}_q)$ and transition probabilities

$$p(v,w) = \begin{cases} \dfrac{1}{q+1}, & \text{if } v \sim w, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $p^{(n)}(v,w)$ the probability to travel from vertex v to vertex w in n steps. It is clear that on any tree there exists a positive integer n such that $p^{(n)}(v,w)$ is strictly positive.

For more information about simple random walks we refer, first of all, to the book Doyle and Snell, 1984, which contributed to popularization of the topic, and Saloff-Coste, 1997.

Definition 18. The *Green kernel* $G_{\mathbb{T}}$ associated with the simple random walk on $\mathbb{T}_q$ is defined by

$$G_{\mathbb{T}}(v,w) = \frac{\mathsf{q}}{\mathsf{q}-1}\,\mathsf{q}^{-d(v,w)}, \quad v,w \in \mathbb{T}_q. \tag{1.7}$$

A function $f : V(\mathbb{T}_q) \to \mathbb{R}$ is said to be $G_{\mathbb{T}}$*-integrable* if for any vertex v

$$\sum_{w\in T} G_{\mathbb{T}}(v,w)|f(w)| < \infty.$$

Definition 19. Let f be a $G_{\mathbb{T}}$-integrable function. The function

$$G_{\mathbb{T}}f(v) = \sum_{w\in T} G_{\mathbb{T}}(v,w)f(w)$$

is called the *potential* of f.

1.2.5 Harmonic and subharmonic functions

Consider the following operator P acting on the space of all real-valued functions defined on the set of vertices $V(\mathbb{T}_q)$. For an arbitrary function f we write Pf for $P(f)$, then the value of Pf at an arbitrary vertex v is defined as

$$Pf(v) = \sum_{v:v\sim w} p(v,w)f(w).$$

Definition 20. A function $f : V(\mathbb{T}_q) \to \mathbb{R}$ is called

- *harmonic* if $Pf(v) = f(v)$ for every $v \in \mathbb{T}_q$,
- *subharmonic* if $Pf(v) \geq f(v)$ for every $v \in \mathbb{T}_q$,
- *superharmonic* if $Pf(v) \leq f(v)$ for every $v \in \mathbb{T}_q$.

Remark 1.2.5. It is important to emphasize the following local geometric behavior of such function. Harmonic functions are locally linear at each vertex, subharmonic functions are locally convex, and superharmonic functions are locally concave.

1.2.6 Riesz decompositions

Let us start with the following important definition.

Definition 21. We say that a superharmonic function f has a *Riesz decomposition* if there exist a harmonic function h and G-integrable function r such that

$$f = h + G_{\mathbb{T}}r.$$

In the proof of our main results we use the following fundamental theorem.

Theorem 1.2.6 (Riesz decomposition theorem)**.** *Consider a superharmonic function f. Let the function $f - Pf$ be $G_{\mathbb{T}}$-integrable. Then f possesses the unique Riesz decomposition $h + G\mu_f$. The functions h and μ_f of this decomposition are explicitly defined as follows:*

$$\mu_f = f - Pf \qquad \textit{and} \qquad h = f - G_{\mathbb{T}}\mu_f.$$

□

The measure on the set of vertices $V(\mathbb{T}_q)$ whose distribution coincides with the function $\mu_f = f - Pf$ is called the *Riesz measure associated to f*.

Proposition 1.2.7. *If f is a positive superharmonic function f then the function $\mu_f = f - Pf$ is $G_{\mathbb{T}}$-integrable.* □

For more information about the Riesz decomposition we refer e.g. to Cohen, Colonna, and Singman, 2008 and Cohen, Colonna, and Singman, 2007.

1.2.7 Martin kernel

By definition the *Martin Kernel* is the following function

$$K(v,w) = \frac{G_{\mathbb{T}}(v,w)}{G_{\mathbb{T}}(o,w)} \quad v,w \in \mathbb{T}_q.$$

In case of $\xi \in \partial\mathbb{T}$, the Martin kernel is defined as

$$K(v,\xi) = \lim_{w\to\xi} \frac{G_{\mathbb{T}}(v,w)}{G_{\mathbb{T}}(o,w)} = \mathsf{q}^{-\mathfrak{h}_{\mathbb{T}}(v,\xi)} \tag{1.8}$$

with the *Busemann function*

$$\mathfrak{h}_{\mathbb{T}}(v,\xi) = \lim_{w\to\xi}\Big(d(w,v) - d(w,o)\Big) = d(v\wedge\xi, v) - d(v\wedge\xi, o). \tag{1.9}$$

The Poisson-Martin representation theorem states that for every positive harmonic function h there exists unique positive Borel measure ν on $\partial\mathbb{T}_q$ such that

$$h(v) = \int\limits_{\partial\mathbb{T}_q} K(v,\xi)d\nu(\xi).$$

1.2.8 On the correspondence between potential theory on $\mathbb{D}$ and $\mathbb{T}_q$

Consider the Poincaré disk model of the hyperbolic plane, denote it by $\mathbb{H}$ (see Stoll, 2001 and the introductory chapter of Helgason, 2000). The Poincaré length element and metric are given by

$$d_{\mathbb{H}}s = \frac{2\sqrt{dx^2+dy^2}}{1-|z|^2} \quad \text{and} \quad \rho_{\mathbb{H}}(z,w) = \log\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}. \tag{1.10}$$

Lebesgue measure $\mathsf{m}_{\mathbb{D}}$ on $\mathbb{H}$ can be expressed on $\mathbb{H}$ as

$$d\mathsf{m}_{\mathbb{D}}(z) = \frac{1}{4\cosh^4\big(\rho_{\mathbb{H}}(z,0)/2\big)}\, d\mathsf{m}_{\mathbb{H}}(z) \approx e^{-2\rho_{\mathbb{H}}(z,0)}\, d\mathsf{m}_{\mathbb{H}}(z)\,, \quad \text{as } \rho_{\mathbb{H}}(z,0)\to\infty\,. \tag{1.11}$$

The hyperbolic Laplace operator in the variable $z = x + \mathrm{i}\, y$ is

$$\Delta_{\mathbb{H}} = \frac{(1-|z|)^2}{4}\Delta_{\mathbb{D}}\,. \tag{1.12}$$

In particular, its harmonic functions are the same as the $\Delta_{\mathbb{D}}$-harmonic functions.

Above, we defined the Euclidean average over a circle in $\mathbb{D}$. Now, we let $r > 0$ and $z \in \mathbb{H}$ and consider the hyperbolic circle

$$C^{\mathbb{H}}(z,r) = \{w \in \mathbb{H} : \rho_{\mathbb{H}}(z,w) = r\}.$$

This is also a Euclidean circle: $C^{\mathbb{H}}(z,r) = C^{\mathbb{D}}(z',r')$, where

$$z' = \frac{1-\tanh^2(r/2)}{1-|z|^2\tanh^2(r/2)}\, z \quad \text{and} \quad r' = \frac{1-|z|^2}{1-|z|^2\tanh^2(r/2)}\tanh(r/2)\,.$$

Its hyperbolic length is $2\pi\sinh r$, see Beardon, 1983 [page 132].

Now, a function $u : \mathbb{H} \to [-\infty,+\infty)$ is *subharmonic on* $\mathbb{H}$ if it is lower semicontinuous and for every $z \in \mathbb{H}$ and $r > 0$, one has

$$u(z) \le \frac{1}{2\pi\sinh r}\int_{C^{\mathbb{H}}(z,r)} u\; d_{\mathbb{H}}s\,.$$

Lemma 1.2.8. *A function u is hyperbolically superharmonic if and only if it is superharmonic on $\mathbb{D}$ in the Euclidean sense.*

The Green function of $\Delta_{\mathbb{H}}$ is the same as the one for $\Delta_{\mathbb{D}}$ given in (1.4), and will henceforth also be denoted by $G_{\mathbb{H}}(\cdot,\cdot)$. Using the hyperbolic metric,

$$G_{\mathbb{H}}(z,w) = -\log\tanh\big(\rho_{\mathbb{H}}(z,w)/2\big). \tag{1.13}$$

Consequently, the hyberbolic Riesz decomposition and the Riesz measure of a superharmonic function u are the same as the Euclidean one.

The natural hyperbolic *compactification* $\widehat{\mathbb{H}}$ of $\mathbb{H}$ arises from the identification of $\mathbb{H}$ with $\mathbb{D}$ and taking the Euclidean closure. The *boundary at infinity* $\partial\mathbb{H}$ of $\mathbb{H}$ is then the unit circle $\mathbb{S}$. It is instructive to interpret this as follows: we first transform the metric $\rho_{\mathbb{H}}$ of the hyperbolic plane into a new metric, namely the Euclidean metric. For use in the subsection on trees, note that on the large scale, the change of the metric is quantified by

$$\begin{gathered} \mathsf{d}_{\mathbb{D}}(z,\mathbb{S}) = 1-|z| = \frac{2}{1+e^{\rho_{\mathbb{H}}(z,0)}} \approx 2e^{-\rho_{\mathbb{H}}(z,0)} \\ \text{as } |z|\to 1\,,\ \text{or equivalently, as } \rho_{\mathbb{H}}(z,0)\to\infty\,. \end{gathered} \tag{1.14}$$

In order to get used to the two geometric views on the same object, we shall freely switch back and forth: $\mathbb{D} \leftrightarrow \mathbb{H}$ and $\mathbb{S} \leftrightarrow \partial\mathbb{H}$.

The *Poisson kernel* on $\mathbb{H}\times\partial\mathbb{H} = \mathbb{D}\times\mathbb{S}$ is defined for $z\in\mathbb{H}$, $\xi\in\mathbb{S}$ as

$$P(z,\xi) = \frac{1-|z|^2}{|\xi-z|^2} = \lim_{w\to\xi}\frac{G_{\mathbb{H}}(z,w)}{G_{\mathbb{H}}(0,w)} = e^{-\mathfrak{h}_{\mathbb{H}}(z,\xi)}. \tag{1.15}$$

with the *Busemann function*

$$\mathfrak{h}_{\mathbb{H}}(z,\xi) = \lim_{w\to\xi}\Big(\rho_{\mathbb{H}}(w,z)-\rho_{\mathbb{H}}(w,0)\Big). \tag{1.16}$$

It also has a probabilistic interpretation: we start Euclidean Brownian motion (BM) at $z\in\mathbb{D}$ and consider its hitting distribution ν_z on the boundary $\mathbb{S}$. That is, if $B\subset\mathbb{S}$ is a Borel set, then $\nu_z(B)$ is the probability that the first visit of BM to $\mathbb{S}$ occurs in a point of B. Denoting by $\lambda_{\mathbb{S}}$ the normalized Lebesgue arc measure on the unit circle, we have

$$\frac{d\nu_z}{d\lambda_{\mathbb{S}}}(\xi) = P(z,\xi)\,,\quad \xi\in\mathbb{S}\,. \tag{1.17}$$

Note that $\nu_0 = \lambda_{\mathbb{S}}$.

Theorem 1.2.9. (a) *For every $\xi\in\mathbb{S}$, the function $z\mapsto P(z,\xi)$ is harmonic on $\mathbb{D}\equiv\mathbb{H}$.*

(b) [Poisson representation] *For every positive harmonic function h on $\mathbb{D}\equiv\mathbb{H}$, there is a unique Borel measure ν^h on $\mathbb{S}\equiv\partial\mathbb{H}$ such that*

$$h(z) = \int_{\mathbb{S}} P(z,\cdot)\,d\nu^h\,.$$

(c) *For every continuous function φ on $\mathbb{S} \equiv \partial\mathbb{H}$,*

$$h(z) = \int_{\mathbb{S}} P(z,\cdot)\,\varphi\, d\lambda_{\mathbb{S}}$$

is the unique harmonic function h on $\mathbb{D} \equiv \mathbb{H}$ such that

$$\lim_{z\to\xi} h(z) = \varphi(\xi) \quad \textit{for every } \xi \in \mathbb{S}\,.$$

We now pursue the line followed above by an exponential change the metric of $\mathbb{H}$, see (1.14). A natural choice is as follows. We fix a root vertex $o \in \mathbb{T}$, define an ultra-metric

$$\rho_{\mathbb{T}}(w,z) = \begin{cases} q^{-|w\wedge z|}\,, \text{if } z \neq w\,, \\ 0\,, \text{if } z = w\,. \end{cases} \tag{1.18}$$

In the induced topology, $\widehat{\mathbb{T}}$ is compact, and $\mathbb{T}$ is discrete and dense. Convergence in this topology is as follows: if $\xi \in \partial\mathbb{T}$ then a sequence (z_n) in $\widehat{\mathbb{T}}$ converges to ξ if and only if $|\xi \wedge z_n| \to \infty$.

At this point, we underline that in the "translation" from disk to tree, the graph metric $\mathsf{d}_{\mathbb{T}}$ corresponds to the hyperbolic metric $\rho_{\mathbb{H}}$, while the metric $\rho_{\mathbb{T}}$ is the one that may be interpreted to correspond to the Euclidean metric $\mathsf{d}_{\mathbb{D}}$. The next identity should be compared with (1.14).

$$\rho_{\mathbb{T}}(x,\partial\mathbb{T}) = \mathsf{q}^{-|x|} \quad \text{for } x \in \mathbb{T}\,. \tag{1.19}$$

We remind, that the *Martin kernel* on $\mathbb{T} \times \partial\mathbb{T}$ is defined for $x \in \mathbb{T}$, $\xi \in \partial\mathbb{T}$ as

$$K(x,\xi) = \lim_{y\to\xi} \frac{G_{\mathbb{T}}(x,y)}{G_{\mathbb{T}}(o,y)} = \mathsf{q}^{-\mathfrak{h}_{\mathbb{T}}(x,\xi)}. \tag{1.20}$$

Again, we have a probabilistic interpretation. It is a well-known exercise to show that SRW on $\mathbb{T}$ converges alsmost surely in the topology of $\widehat{\mathbb{T}}$ to a limit random variable Z_∞ that takes its values in $\partial\mathbb{T}$. Let ν_x be the distribution of Z_∞, when SRW starts at vertex x. Then $\nu_o = \lambda_{\partial\mathbb{T}}$ is the tree-analogue of the normalized Lebesgue measure $\lambda_{\mathbb{S}}$ on the unit circle: $\lambda_{\partial\mathbb{T}}$ is the unique probability measure on $\partial\mathbb{T}$ which is invariant under "rotations" of $\mathbb{T}$, that is, self-isometries of the graph $\mathbb{T}$ which fix the root vertex o. Connectedness of $\mathbb{T}$ implies that ν_x is absolutely continuous with respect to $\lambda_{\partial\mathbb{T}}$, and the Radon-Nikodym-derivative is (realised by) the Martin kernel:

$$\frac{d\nu_x}{d\lambda_{\partial\mathbb{T}}}(\xi) = K(x,\xi)\,. \tag{1.21}$$

We have a perfect analogy with Theorem 1.2.9.

Theorem 1.2.10. (a) *For every $\xi \in \partial\mathbb{T}$, the function $x \mapsto K(x,\xi)$ is harmonic on $\mathbb{T}$.*

(b) *For every positive harmonic function h on $\mathbb{T}$, there is a unique Borel measure ν^h on $\partial\mathbb{T}$ such that*

$$h(x) = \int_{\partial\mathbb{T}} K(x,\cdot)\,d\nu^h\,.$$

(c) [Solution of the Dirichlet problem] *For every continuous function φ on $\partial\mathbb{T}$,*

$$h(x) = \int_{\partial\mathbb{T}} \varphi\,d\nu_x = \int_{\partial\mathbb{T}} K(x,\cdot)\,\varphi\,d\lambda_{\partial\mathbb{T}}\,.$$

is the unique harmonic function h on $\mathbb{T}$ such that

$$\lim_{x\to\xi} h(x) = \varphi(\xi) \quad \text{for every } \xi \in \partial\mathbb{T}\,.$$

There are many further analogies between analysis, probability, group actions, etc. on $\mathbb{D}$ and $\mathbb{T}$. The present introduction is not intended to cover all those aspects. For further tips of the iceberg, see e.g. M. Rigoli and Vignati, 2005, Cohen, Colonna, and Singman, 2008, Atanasi and M. A. Picardello, 2008 or Casadio Tarabusi and Figá-Talamanca, 2010, and the references given there.

2 Moments of Riesz measures

The study of boundary properties of analytic functions, began in the second half of the 19th century with the Yu.V. Sokhotskii theorem and the E. Picard theorem regarding the behavior of analytic functions in neighborhoods of isolated singular points. The first systematic study of the certain boundary properties of analytic functions was provided in the dissertation of P. Fatou (1906). The development of the theory of boundary properties is closely related with various fields of mathematical analysis and mathematics in general, first and foremost with the probability theory, the theory of harmonic functions, the theory of conformal mapping, boundary value problems of analytic function theory, the potential theory, the value-distribution theory, Riemann surfaces, subharmonic functions. The theory of boundary properties made considerable advances in the first third of the 20th century.

Let us distinguish the following two major approaches in the theory of boundary properties of analytic functions.

- The study of the behavior in a neighborhood of an isolated boundary point. The most important case here is the case of an essential singular point, which was studied by Yu.V. Sokhotskii, E. Picard, G. Julia, and F. Iversen (see Sokhotskii, 1868, Iversen, 1914).
- The study of the behavior in the case when the boundary is an everywhere-discontinuous set provided by V.V. Golubev (see Golubev, 1961, Carleson, 1967).

In this section we provide some results concerning properties of the Riesz measure of subharmonic functions defined on the unit disk and on the homogeneous tree.

2.1 Motivation

We start with a well-known result about zeros of bounded analytic functions in the unit disc. This is a so-called *Blaschke theorem* (see Blaschke, 1915).

Theorem 2.1.1 (W. Blaschke)**.** *A sequence $\{z_n\}$ (with possible repetitions) of points on the unit disk $\mathbb{D}$ defines a bounded analytic function on the unit disk with zero set consisting precisely of the z_n-s, counted according to their multiplicities if and only if*

$$\sum_{z_n \in \mathbb{D}} 1 - |z_n| < \infty.$$

If the series $\sum_{z_n \in \mathbb{D}} 1 - |z_n|$ converges, we say that the *Blaschke condition* holds.

Example 2.1.2. Let z_n be the following sequence of points in $\mathbb{D}$

$$z_n = 1 - \frac{1}{2^n}.$$

This sequence satisfies the Blaschke condition, indeed

$$\sum_{z_n \in \mathbb{D}} 1 - \left(1 - \frac{1}{2^n}\right) = \sum_{z_n \in \mathbb{D}} \frac{1}{2^n} < \infty.$$

Using the Blaschke product one can construct a bounded analytic function in $\mathbb{D}$ with zeros at z_n.

Definition 22. The *Blaschke product* is defined as

$$B(z) = \prod_i B(z_n, z),$$

where $B(z_n, z)$ are factors

$$B(z_n, z) = \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}$$

with $z_n \neq 0$. If $z_n = 0$ we put $B(0, z) = z$.

Numerous different results followed the work of Blaschke of 1915 in which he proved his famous theorem stated above. For instance, in early 20s came out a series of works by Golubev, 1961. His book entitled ”The study on the theory of singular points of single valued analytic functions” (which was published in 1961, after the author's death) contains this results as the second part. He was by far the first one to show that a Blaschke-type condition

$$\sum_{z_i}(1-|z_n|)^{\alpha+1+\varepsilon}$$

holds for a function f of a finite order at most α, i.e., for a functions f satisfying

$$|f(z)| \leq h(1-|z|), \quad z \in \mathbb{D}, \quad h(x) = \exp\left(\frac{1}{x}\right)^{\alpha}, \quad \alpha > 0.$$

Later this result was extended to a wide class of weight functions by Shamoyan, 1983. For further information we refer the interested reader to Djrbashian, 1975, Matsaev and Mogulskii, 1976, Hayman and Korenblum, 1980, Jerbashian, 2005.

In a recent paper by Borichev, Golinskii, and Kupin, 2009 the authors introduced a class of analytic functions in the unit disk $\mathbb{D}$ of finite order at most q having a *finite* set E as the set of singular points, i.e.,

$$|f(z)| \leq exp\Big(\frac{D}{\rho^q_{\mathbb{D}}(z,E)}\Big)$$

with D, $q \geq 0$, $\rho_{\mathbb{D}}$ Euclidean distance and proved that the Blaschke-type condition for such functions is

$$\sum_{z\in Z_f}(1-|z|)\rho_{\mathbb{D}}(z,E)_+^{(q-1+\varepsilon)} \leq C(\varepsilon,q,E)D,$$

where Z_f is the zero set of f.

In parer by Favorov and Golinskii, 2009 authors expanded this result to the case of an *arbitrary closed set* E. They showed that for some $\alpha \in \mathbb{R}$

$$I(\alpha,E) = \int_0^2 \frac{|\zeta \in \partial\mathbb{D} : \rho_{\mathbb{D}}(\zeta,E) < t|}{t^{\alpha+1}}dt < \infty,$$

(here $|\ |$ is normalized Lebesgue measure) and for an analytic function f such that

$$|f(z)| \leq exp\Big(\frac{D}{\rho^q_{\mathbb{D}}(z,E)}\Big)$$

holds

$$\sum_{z \in Z_f} (1 - |z|)\rho_{\mathbb{D}}(z, E)_+^{(q-\alpha)} \leq C(\alpha, q, E)D.$$

Remark 2.1.3. The classical Blaschke condition one gets if case if $E = \partial\mathbb{D}$ and $\alpha = -q < \infty$, where $q \to 0$.

This is a consequence of two theorems, which we formulate now.

Theorem 2.1.4. *Let E be a closed subset of the unit circle $\partial\mathbb{D}$ and $q > 0$. Suppose that a subharmonic function v in $\mathbb{D}$ satisfies an inequality for some q*

$$v(z) \leq \frac{1}{\rho^q(z, E)}, \quad z \in \mathbb{D}.$$

- *If*
$$I(q, E) = \int_0^2 \frac{|\zeta \in \partial\mathbb{D} : \rho_{\mathbb{D}}(\zeta, E) < t|}{t^{q+1}} dt < \infty,$$
then the Riesz measure satisfies the Blaschke-type condition
$$\int_{\mathbb{D}} (1 - |\lambda|) d\mu_v(\lambda) < \infty.$$
- *If $I(q, E) = \infty$ and $I(\alpha, E) < \infty$ for some $\alpha < q$, then*
$$\int_{\mathbb{D}} (1 - |\lambda|)\rho_{\mathbb{D}}(\lambda, E)^{q-\alpha} d\mu_v(\lambda) < \infty.$$

The second theorem establishes sharpness of the previous result.

Theorem 2.1.5. *Let E be a closed subset of $\partial\mathbb{D}$ such that $I(\alpha, E) = \infty$ for some $\alpha \geq 0$. Consider the subharmonic function*

$$u_0 = \frac{1}{\rho_{\mathbb{D}}^q(z, E)}$$

with its Riesz measure μ_0. Suppose that $q \geq \alpha$. Then it holds

$$\int_{\mathbb{D}} (1 - |\lambda|)\rho_{\mathbb{D}}(\lambda, E)^{q-\alpha} d\mu_{u_0}(\lambda) = \infty.$$

Our goal is to find an analog of the last two results for the case of an arbitrary decreasing function $\psi(\rho_{\mathbb{D}}(z, E)) : [0, 2] \to \mathbb{R}$ instead of $\frac{1}{\rho_{\mathbb{D}}^q(z,E)}$. Also we generalize these results for the case of subharmonic functions on the homogeneous tree. For the case of the unit disk similar results were obtained by Favorov and Radchenko, 2013 in their recent paper.

2.2 Moment conditions and harmonic majorants

Let $\mathbb{X} = \mathbb{D}$ or $\mathbb{X} = \mathbb{T}$, with respective boundary $\partial\mathbb{X}$ and compactification $\widehat{\mathbb{X}} = \mathbb{X} \bigcup \partial\mathbb{X}$. The boundary carries the metric dist and measure λ, where $\mathsf{dist} = \mathsf{d}_{\mathbb{D}}$ and $\lambda = \lambda_{\mathbb{S}}$ in case of the disk, while $\mathsf{dist} = \rho_{\mathbb{T}}$ and $\lambda = \lambda_{\mathbb{T}}$ in case of the tree. Given a subharmonic function u on $\mathbb{X}$ and its Riesz measure μ_u, we are interested in finiteness of its *first (boundary) moment*

$$\int_{\mathbb{X}} \mathsf{dist}(x, \partial\mathbb{X})\, d\mu_u(x) \tag{2.1}$$

and variants thereof. One principal tool is the following lemma.

Lemma 2.2.1. *The subharmonic function u has a harmonic majorant on $\mathbb{X}$ if and only if μ_u has finite first moment* (2.1).

Proof. Our function u has a harmonic majorant if and only if $G_{\mathbb{X}}\mu_u(x) < \infty$ for all $x \in \mathbb{X}$.
If $\mathbb{X} = \mathbb{D}$ and there is a harmonic majorant then we choose $x = 0$ and get

$$\infty > G_{\mathbb{D}}\mu_u(0) = -\int_{\mathbb{D}} \log|z|\, d\mu_u(z) \geq \int_{\mathbb{D}} (1 - |z|)\, d\mu_u(z).$$

Conversely, if the first moment is finite, then $G_{\mathbb{D}}\mu_u$ is finite on $\mathbb{D}$ by Armitage and Gardiner, 2001 [Thm. 4.2.5].

If $\mathbb{X} = \mathbb{T}$ then by (1.7), $G_{\mathbb{T}}(x, y) \leq q^{|x|} G_{\mathbb{T}}(o, y)$ for all x, y, so that $G_{\mathbb{T}}\mu_u$ is finite on $\mathbb{T}$ if and only if $G_{\mathbb{T}}\mu_u(o) < \infty$. Now

$$G_{\mathbb{T}}\mu_u(o) = \sum_{x \in \mathbb{T}} \frac{q}{q-1} q^{-|x|} \mu_u(x) = \frac{q}{q-1} \int_{\mathbb{T}} \rho_{\mathbb{T}}(x, \partial\mathbb{T})\, d\mu_u(x)\,. \qquad \square$$

So in fact what we are going to do is to exhibit a sufficient condition for a subharmonic function on $\mathbb{X} = \mathbb{D}$, resp. $\mathbb{X} = \mathbb{T}$, to possess a (global or restricted) harmonic majorant, even if it is not bounded above.

Theorem 2.2.2. *Let u be a subharmonic function on $\mathbb{X}$ and consider the closed set*

$$E = \Big\{ \xi \in \partial\mathbb{X} : \limsup_{\mathbb{X} \ni x \to \xi} u(x) = \infty \Big\}.$$

Suppose that $\Psi : [0, \mathrm{diam}(\mathbb{X})] \to [0, \infty]$ *is a continuous, decreasing function with*

$$\Psi(t) = \infty \iff t = 0 \quad \textit{and} \quad \lim_{t \to 0} \Psi(t) = \infty,$$

and that

$$u(x) \le \Psi\big(\mathrm{dist}(x, E)\big) \quad \textit{for all } x \in \mathbb{X}.$$

If

$$\int_{\partial\mathbb{X}} \Psi\big(\mathrm{dist}(\xi, E)\big)\, d\lambda(\xi) < \infty, \tag{2.2}$$

then u has a finite harmonic majorant, and the Riesz measure μ_u has finite first boundary moment.

We note that for condition (2.2) it is necessary that $\lambda(E) = 0$. For the proof of the theorem, we shall work with the function

$$h = \int_{\partial\mathbb{X}} K_{\mathbb{X}}(\cdot, \xi)\, \Psi\big(\mathrm{dist}(\xi, E)\big)\, d\lambda(\xi), \tag{2.3}$$

where $K_{\mathbb{X}}$ is the Poisson kernel

$$P(z, \xi) = \frac{1 - |z|^2}{|\xi - z|^2} = \lim_{w \to \xi} \frac{G_{\mathbb{H}}(z, w)}{G_{\mathbb{H}}(0, w)} = e^{-\mathfrak{h}_{\mathbb{H}}(z, \xi)}, \tag{2.4}$$

when $\mathbb{X} = \mathbb{D}$, and the Martin kernel

$$K(x, \xi) = \lim_{y \to \xi} \frac{G_{\mathbb{T}}(x, y)}{G_{\mathbb{T}}(o, y)} = \mathsf{q}^{-\mathfrak{h}_{\mathbb{T}}(x, \xi)}, \tag{2.5}$$

when $\mathbb{X} = \mathbb{T}$. Since for fixed $x \in \mathbb{X}$, the function $\xi \mapsto K_{\mathbb{X}}(x, \xi)$ is continuous on $\partial\mathbb{X}$ (whence bounded), the function h is finite and harmonic on $\mathbb{X}$ under condition (2.2).

We need some preparations. We let $0 < t \le \max\{\mathrm{dist}(x, E) : x \in \mathbb{X}$ and consider the sets

$$E^{(t)} = \{\xi \in \partial\mathbb{X} : \mathrm{dist}(\xi, E) \le t\} \quad \text{and} \quad E_*^{(t)} = \{\xi \in \partial\mathbb{X} : \mathrm{dist}(\xi, E) > t\},$$

and, for $0 < t < 1$, the set $\mathbb{X}^{(t)}$ which is the component of the origin of the set $\{x \in \mathbb{X} : \mathrm{dist}(x, E) > t\}$.

Disk case: $\mathbb{D}^{(t)}$ (denoted Ω_t in Favorov and Golinskii, 2009) is an open domain, and its boundary is

$$\partial\mathbb{D}^{(t)} = \partial_\infty\mathbb{D}^{(t)} \cup \Gamma^{(t)}, \quad \text{where } \partial_\infty\mathbb{D}^{(t)} \subset \overline{E_*^{(t)}} \text{ and } \Gamma^{(t)} = \Gamma_{\mathbb{D}}^{(t)} = \{z \in \mathbb{D} : \mathsf{d}_{\mathbb{D}}(z,E) = t\}.$$

The sets $E^{(t)}$ and $\partial_\infty\mathbb{D}^{(t)}$ are both unions of finitely many closed arcs on $\mathbb{S}$ and meet at finitely many endpoints of those arcs. $\partial_\infty\mathbb{D}^{(t)}$ may be a strict subset of the closure of $E_*^{(t)}$, because some arcs of the latter set can be the boundary of a different component of $\{z \in \mathbb{D} : \mathsf{d}_{\mathbb{D}}(z,E) > t\}$. (The latter can arise as "triangular" regions bounded by an arc of $\mathbb{S}$ and of arcs of two intersecting circles $\{z : |z - \zeta_j| = t\}$, where $\zeta_j \in E$, $j = 1, 2$.)

Tree case: The origin is of course the root vertex of $\mathbb{T}$. The metric $\mathsf{dist} = \rho_{\mathbb{T}}$ takes only the countably many values q^{-k}, $k \geq 0$ (integer). For $0 < t < 1$ let $k \geq 1$ be the integer such that

$$\mathsf{q}^{-k} \leq t < \mathsf{q}^{-(k-1)}, \quad k = k(t). \tag{2.6}$$

For any vertex $y \in \mathbb{T}$, we consider the *branch* of $\mathbb{T}$ at y. This is the subtree (induced by)

$$T_y = \{u \in \mathbb{T} : y \in \pi(o,u)\}.$$

Its boundary $\partial T_y \subset \partial\mathbb{T}$ consists of those ends which are represented by geodesics that lie entirely within T_y. Note that the open-compact sets ∂T_y, $y \in \mathbb{T}$, are a basis of the topology of $\partial\mathbb{T}$. Given t, let $k = k(t)$ and consider the set

$$\Gamma^{(t)} = \Gamma_T^{(t)} = \{y \in \mathbb{T} : |y| = k,\ \partial T_y \cap E \neq \emptyset\}.$$

We have

$$E^{(t)} = E_{\mathbb{T}}^{(t)} = \bigcup_{y \in \Gamma^{(t)}} \partial T_y .$$

For small $t \equiv$ large $k = k(t)$, only few vertices y with $|y| = k$ belong to $\Gamma^{(t)}$: as $t \to 0 \equiv k \to \infty$, we have

$$\frac{|\Gamma^{(t)}|}{|\{y \in \mathbb{T} : |y| = k(t)\}|} = \lambda_{\partial\mathbb{T}}(E^{(t)}) \to \lambda_{\partial\mathbb{T}}(E) = 0.$$

When $\mathbb{X} = \mathbb{T}$, the set $\mathbb{T}^{(t)}$ is the subtree of $\mathbb{T}$ obtained by chopping off each branch T_y, $y \in \Gamma^{(t)}$, that is,

$$\mathbb{T}^{(t)} = \mathbb{T} \setminus \bigcup\nolimits_{y \in \Gamma^{(t)}} T_y .$$

The *boundary* of this truncated tree is

$$\partial\mathbb{T}^{(t)} = \partial_\infty\mathbb{T}^{(t)} \cup \Gamma^{(t)}_{\mathbb{T}}, \quad \text{where} \quad \partial_\infty\mathbb{T}^{(t)} = E^{(t)}_*,$$

while $\Gamma^{(t)}$ is the outer vertex boundary of $\mathbb{T}^{(t)}$: it consists of those vertices in the complement that have a neighbour (here: precisely one neighbour) in $\mathbb{T}^{(t)}$. In the topology of $\widehat{\mathbb{T}}$, we have the compact subspaces $\widehat{\mathbb{T}}^{(t)} = \mathbb{T}^{(t)} \cup \partial\mathbb{T}^{(t)}$ and the boundary $\partial\mathbb{T}^{(t)}$.

We shall need the following simple estimate.

Lemma 2.2.3. *For $x \in \mathbb{X}$, consider the harmonic measure ν_x on $\partial\mathbb{X}$. Then*

$$\textit{for } \; y \in \Gamma^{(t)}, \quad \nu_y(E^{(t)}) \geq 1/c_{\mathbb{X}} \; = \begin{cases} 1/3\,, \textit{if } \; \mathbb{X} = \mathbb{D}\,, \\ \mathsf{q}/(\mathsf{q}+1)\,, \textit{if } \; \mathbb{X} = \mathbb{T}\,. \end{cases}$$

Proof. **A. Disk case.** For $y \in \Gamma^{(t)}_{\mathbb{D}}$ there is $\zeta = \zeta_y \in E$ such that $|y - \zeta_y| = \mathsf{d}(y, E) = t$. Consider the arc $\gamma_\zeta = \{\xi \in \mathbb{S} : |\xi - \zeta| \leq t\} \subset E^{(t)}$, as well as the circle $\{z \in \mathbb{C} : |z - \zeta| = t\}$. At any of the two intersection points of that circle with $\mathbb{S}$, the angle α between the tangents to the two circles is such that $\pi/2 > \alpha > \pi/3$, as $0 < t < 1$. By Garnett, 2007[p. 13, Fig.1.1], $\nu_y(\gamma_\zeta) = \alpha/\pi > 1/3$. (In Favorov and Golinskii, 2012, the lower estimate $1/6$ is used, but apparently also $1/3$ works.)

B. Tree case. For $y \in \Gamma^{(t)}_{\mathbb{T}}$, we have that $\partial\mathrm{T}_y \subset E^{(t)}$. We note that ν_y gives equal mass to the boundaries of each of the $\mathsf{q}+1$ branches of $\mathbb{T}$ that are emanating from y. Among those, q branches are part of T_y, that is, $\nu_y(\partial\mathrm{T}_y) = \mathsf{q}/(\mathsf{q}+1)$, providing the lower bound. □

Proof of Theorem 2.2.2. Consider the continuous function

$$\psi^{(t)}(\xi) = \min\{\Psi(t), \Psi\big(\mathsf{dist}(\xi, E)\big)\}$$

on $\partial\mathbb{X}$ and the harmonic function

$$h^{(t)}(x) = \int_{\partial\mathbb{X}} K(x,\cdot)\,\psi^{(t)}\,d\lambda = \int_{\partial\mathbb{X}} \psi^{(t)}\,d\nu_x\,.$$

It is well known that it is the solution of the Dirichlet problem on $\mathbb{X}$ with boundary function $\psi^{(t)}$. We have $\psi^{(t)}(\xi) = \Psi(t)$ on $E^{(t)}$, while $\psi^{(t)}(\xi) \le \Psi(t)$ on $E_*^{(t)} \supset \partial_\infty \mathbb{X}^{(t)}$. Thus,

$$h^{(t)}(x) = \int_{E_*^{(t)}} \Psi\big(\mathsf{dist}(\cdot, E)\big)\, d\nu_x + \Psi(t)\, \nu_x(E^{(t)})\,. \tag{2.7}$$

Taking boundary limits for points x within $\widehat{\mathbb{X}}^{(t)}$, and using Lemma 2.2.3,

$$\begin{aligned} \lim_{x\to\xi} h^{(t)}(x) &= \Psi\big(\mathsf{dist}(\xi, E)\big)\,, \quad \text{for } \xi \in \partial_\infty \mathbb{X}^{(t)}\,, \quad \text{and} \\ \lim_{x\to y} h^{(t)}(x) = h^{(t)}(y) &\ge \Psi(t)\, \nu_y(E^{(t)}) \ge \Psi(t)/c_{\mathbb{X}} \quad \text{for } y \in \Gamma^{(t)}. \end{aligned} \tag{2.8}$$

(In the tree case, since y is an isolated point, the last limit just means stabilisation at y.) On the other hand, by assumption our subharmonic function u satisfies

$$\begin{aligned} \limsup_{x\to\xi} u(x) \le \Psi\big(\mathsf{dist}(\xi, E)\big) \quad &\text{for } \xi \in \partial_\infty \mathbb{X}^{(t)}\,, \quad \text{and} \\ \limsup_{x\to y} u(y) \le \Psi(t) \quad &\text{for } y \in \Gamma^{(t)}\,. \end{aligned} \tag{2.9}$$

Therefore, again taking boundary limits within $\widehat{\mathbb{X}}^{(t)}$,

$$\limsup_{x\to\eta} \Big(u(x) - c_{\mathbb{X}}\, h^{(t)}(x)\Big) \le 0 \quad \text{for every } \eta \in \partial \mathbb{X}^{(k)}.$$

Thus, by the maximum principle (which also holds on the tree because $\mathbb{T}^{(t)}$ is a connected graph),

$$u(x) \le c_{\mathbb{X}}\, h^{(t)}(x) \quad \text{for every } x \in \mathbb{X}^{(t)}. \tag{2.10}$$

Having this, we obtain the proposed first moment: let

$$h(x) = \int_{\partial\mathbb{X}} K(x,\cdot)\, \Psi\big(\mathsf{dist}(\cdot, E)\big)\, d\lambda$$

be the harmonic function proposed in (2.3). Then $h^{(t)} \le h$ on $\mathbb{X}^{(t)}$ for any t. Given any $x \in \mathbb{X}$, we can choose $t < \mathsf{dist}(x, E)$ to see that $c_{\mathbb{X}} \cdot h$ is a (finite) harmonic majorant for our subharmonic function u. □

There is a simple converse to Theorem 2.2.2.

Proposition 2.2.4. *Let u be a subharmonic function on $\mathbb{X}$, and let E and Ψ be as in Theorem 2.2.2. If*

$$u(x) \geq \Psi\big(\mathsf{dist}(x,E)\big) \quad \textit{for all } x \in \mathbb{X}$$

and

$$\int_{\partial\mathbb{X}} \Psi\big(\mathsf{dist}(\xi,E)\big)\, d\lambda(\xi) = \infty \tag{2.11}$$

then u has no harmonic majorant on $\mathbb{X}$, and the first moment of μ_u is infinite.

Proof. We give a combined proof for $\mathbb{X} = \mathbb{D}$ and $\mathbb{X} = \mathbb{T}$. Suppose that the first moment of μ_u is finite. Then by Lemma 2.2.1, u has a (finite) harmonic majorant h. Consider the continuous function $\Psi_M = \min\{\Psi, M\}$. Then for all $x \in \mathbb{X}$,

$$h(x) \geq u(x) \geq \Psi_M\big(\mathsf{dist}(x,E)\big)$$

The function

$$g_M(x) = \int_{\partial\mathbb{X}} K_{\mathbb{X}}(\cdot,\xi)\, \psi_M\big(\mathsf{dist}(\xi,E)\big)\, d\lambda(\xi)\,,$$

defined analogously to (2.3), provides the solution of the Dirichlet problem on $\mathbb{X}$ with boundary data $\psi_M\big(\mathsf{dist}(\xi,E)\big)$. We have

$$\liminf_{x\to\xi}\big(h(x) - g_M(x)\big) \geq 0 \quad \text{for every} \ \ \xi \in \partial\mathbb{X}\,.$$

By the minimum principle, $h \geq g_M$ on $\mathbb{X}$, and in particular, $h(o) \geq g_M(o)$. Letting $M \to \infty$, monotone convergence yields $h(o) = \infty$, contradicting finiteness of h. $\square$

Next, in a similar spirit to Favorov and Golinskii, 2009, we want to extend Theorem 2.2.2 to a situation where the integral in (2.2) is infinite. For that purpose, we shall need an estimate of the Green function of $\mathbb{X}^{(t)}$ $G_{\mathbb{X}^{(t)}}(x,y) = G_{\mathbb{X}^{(t)}}(y,x)$ On the disk, this function is of course well described in the classical potential theory literature.

On the tree, for $x,y \in \mathbb{T}^{(k)}$, it is the expected number of visits to y of the random walk starting at x before it hits $\Gamma^{(t)}$. It is natural to define $G_{\mathbb{T}^{(t)}}(x,y) = 0$ when one of x,y lies in $\Gamma^{(t)}$ and the other in $\mathbb{T}^{(t)}$. In potential theoretic terms, $f = G_{\mathbb{T}^{(t)}}(\cdot,y)$ is the smallest non-negative function on $\mathbb{T}^{(t)} \cup \Gamma^{(t)}$ satisfying $\Delta_{\mathbb{T}} f(x) = -\delta_y(x)$ for $x \in \mathbb{T}^{(k)}$. This corresponds directly to the disk situation.

Theorem 2.2.5. *Define $r = r_{\mathbb{X}}$, $a = a_{\mathbb{X}}$ and $b = b_{\mathbb{X}}$ and for $\mathbb{X} = \mathbb{D}$ or $= \mathbb{T}$ by*

$$r_{\mathbb{D}} = 7 \quad \text{and} \quad a_{\mathbb{D}} = b_{\mathbb{D}} = 18\,, \quad \text{resp.} \quad r_{\mathbb{T}} = 1\,,\ a_{\mathbb{T}} = \mathsf{q}/(\mathsf{q}-1) \quad \text{and} \quad b_{\mathbb{T}} = 1\,.$$

Let $0 < t < 1/r$. Then for any $x \in \mathbb{X}^{(rt)}$, we have

$$G_{\mathbb{X}}(x,o) \geq G_{\mathbb{X}^{(t)}}(x,o) \geq \frac{1}{a} G_{\mathbb{X}}(x,o) \geq \frac{1}{b}\,\mathsf{dist}(x,\partial X)\,,$$

where o is the origin (root) of $\mathbb{X}$.

Proof. The first inequality is clear in both cases. The third inequality is also clear, and it is an equality in the tree case. We need to prove the second inequality separately for tree and disk, and begin this time with the tree.

A. Tree case. Let $\nu_x^{(t)}$ be the harmonic measure of $\mathbb{T}^{(t)}$ on its boundary. In particular, for $y \in \Gamma^{(t)}$, the probability that the random walk starting at x first hits $\Gamma^{(t)}$ in y is $\nu_x^{(t)}(y)$. The function $g^{(t)}(x) = G_{\mathbb{T}}(x,o) - G_{\mathbb{T}^{(k)}}(x,o)$ is positive harmonic on $\mathbb{T}^{(t)}$. We have $\lim_{x\to\xi} g^{(t)}(x) = 0$ for $\xi \in \partial_\infty \mathbb{T}^{(t)}$ (because this holds for $G_{\mathbb{T}}(x,o)$), while $g^{(t)}(y) = G_{\mathbb{T}}(y,o)$ for $y \in \Gamma^{(t)}$. Since the Dirichlet problem on $\widehat{\mathbb{T}}^{(t)}$ admits solution (a straightforward adaptation of **CSW** including in that argument vertices which are boundary points), we get that

$$g^{(t)}(x) = \sum_{y\in\Gamma^{(t)}} G_{\mathbb{T}}(y,o)\nu_x^{(t)}(y) = \frac{\mathsf{q}}{\mathsf{q}-1}\mathsf{q}^{-k}\,\nu_x^{(t)}(\Gamma^{(t)})\,,$$

where $k = k(t)$, as defined in (2.6). In the last identity (which can of course also be derived probabilistically), (1.7) was used. Now let $x \in \mathbb{T}^{(t)}$ and let x_0 be the last point on the geodesic $\pi(o,x)$ that lies on some $\pi(o,y)$ with $y \in \Gamma^{(t)}$. Note that $|x_0| \leq k-1$. In order to reach $\Gamma^{(t)}$, the random walk starting at x needs to pass through x_0. Unless $x = x_0$, this is unrestricted random walk on $\mathbb{T}$ before the first visit in x_0, because up to that time it evolves on a branch of $\mathbb{T}$ that contains no element of $\Gamma^{(t)}$. It is well known and easy to see that

$$\mathsf{Pr}[\exists n : Z_n = x_0 \mid Z_0 = x] = G(x,x_0)/G(x_0,x_0),$$

see e.g. Woess, 2009 [Thm.1.38]. Thus (compare with Woess, 2009[Prop.9.23]),

$$\nu_x^{(t)}(\Gamma^{(t)}) = \mathsf{Pr}[\exists n : Z_n = x_0 \mid Z_0 = x]\,\underbrace{\nu_{x_0}^{(t)}(\Gamma^{(t)})}_{\leq 1} \leq \mathsf{q}^{-\mathsf{d}_{\mathbb{T}}(x,x_0)} = \mathsf{q}^{|x_0|-|x|} \leq \mathsf{q}^{k-1-|x|}\,.$$

We infer that

$$g_k(x) \le \frac{\mathsf{q}}{\mathsf{q}-1}\mathsf{q}^{-k}\mathsf{q}^{k-1-|x|} = \frac{1}{\mathsf{q}-1}\mathsf{q}^{-|x|}$$

Consequently,

$$G_{\mathbb{T}^{(k)}}(x,o) = G_{\mathbb{T}}(x,o) - g_k(x) = \frac{\mathsf{q}}{\mathsf{q}-1}\mathsf{q}^{-|x|} - g_k(x) \ge \mathsf{q}^{-|x|},$$

and in view of (1.7), the proposed estimate is proved for the tree.

B. Disk case. The proof follows Favorov and Golinskii, 2009, but we re-elaborate it to get the constant $a_{\mathbb{D}} = 7$ and to have $G_{\mathbb{D}}(z,0)$ in the lower bound. As before, we prefer to write z instead of x for the elements of $\mathbb{D}$. We start in the same way as for the tree. We know that $G_{\mathbb{D}}(z,0) = \log\frac{1}{|z|}$, and we can decompose

$$G_{\mathbb{D}^{(t)}}(z,0) = G_{\mathbb{D}}(z,0) - g^{(t)}(z), \quad z \in \mathbb{D}^{(t)},$$

where $g^{(t)}$ is harmonic on $\mathbb{D}^{(t)}$ with boundary values 0 at $\partial_\infty \mathbb{D}^{(t)}$. For $z \in \Gamma^{(t)}$, there is $\zeta \in E$ with $|z-\zeta| = t$, whence $|z| \ge 1-t$. Thus, using (2.2.3),

$$g^{(t)}(z) = G_{\mathbb{D}^{(t)}}(z,0) \le \log\frac{1}{1-t} \le 3\,\log\frac{1}{1-t}\,\nu_z(E^{(t)}). \tag{2.12}$$

The right hand side is a harmonic function of z on the whole of $\mathbb{D}$. By the maximum principle, (2.12) holds on all of $\mathbb{D}^{(t)}$.

We now choose real parameters $r > s > 1$ with $r - s > 1$. We assume that $t < 1/r$. Let $z \in \mathbb{D}$.

Case 1. Let $|z| < (1-t)^s$. Then $g^{(t)}(z) \le \log\frac{1}{1-t} \le \frac{1}{s}\log\frac{1}{|z|}$, and

$$G_{\mathbb{D}^{(t)}}(z,0) \ge \frac{s-1}{s} G_{\mathbb{D}}(z,0).$$

Case 2. Let $z \in \mathbb{D}^{(rt)}$ with $|z| \ge (1-t)^s$. By the Bernoulli inequality, $|z| \ge 1 - st$. Following Favorov and Golinskii, 2009, we write $z = |z|e^{i\theta}$ and

$$\nu_z(E^{(t)}) = \int_{E^{(t)}} P(z,\xi)\, d\lambda_{\mathbb{D}}(\xi) = (1-|z|^2)\frac{1}{2\pi}\int_{\{\varphi: e^{i\varphi}\in E^{(t)}\}} \frac{d\varphi}{(1-|z|)^2 + 4|z|\sin^2\frac{\varphi-\theta}{2}}.$$

Then for $\varphi \in (-\pi, \pi]$ with $e^{i\varphi} \in E^{(t)}$, using $rt \le \mathsf{dist}(z,E) \le 1 - |z| + \mathsf{dist}(e^{i\theta}, E)$,

$$\pi \ge |\phi - \theta| \ge 2|\sin\tfrac{\varphi-\theta}{2}| = |e^{i\theta} - e^{i\varphi}| \ge \mathsf{dist}(e^{i\theta}, E) - t \ge rt - (1-|z|) - t \ge \tau\, t,$$

where $\tau = r - s - 1$. Combining these estimates with (2.12),

$$\begin{aligned} g^{(t)}(z) &\le 3\Big(\log\frac{1}{1-t}\Big)\,(1-|z|^2)\,\frac{1}{2\pi}\int_{\{\varphi\,:\,\tau t\le|\varphi-\theta|\le\pi\}}\frac{d\varphi}{(1-|z|)^2+4|z|\sin^2\frac{\varphi-\theta}{2}} \\ &= \frac{6}{\pi}\Big(\log\frac{1}{1-t}\Big)\,(1-|z|^2)\int_{\tau t/2}^{\pi/2}\frac{d\varphi}{(1-|z|)^2+4|z|\sin^2\varphi} \\ &= \frac{6}{\pi}\Big(\log\frac{1}{1-t}\Big)\arctan\Big(\frac{1-|z|}{1+|z|}\cot\Big(\frac{\tau t}{2}\Big)\Big) \\ &\le \frac{6}{\pi}\Big(\log\frac{1}{1-t}\Big)\Big(\cot\frac{\tau t}{2}\Big)\,(1-|z|)\,. \end{aligned}$$

Since $rt < 1 < \pi/3$, we have $\tau t/2 < \pi/6$, whence $\cot(\tau t/2) \le 2\pi/(3\tau t)$. Also, for $0 < t < 1/r$, we have $\log 1/(1-t) \le r\,t/(r-1)$. Therefore

$$g^{(t)}(z) \le \frac{4}{\tau t}\Big(\log\frac{1}{1-t}\Big)\,(1-|z|) \le \frac{4r}{(r-1)(r-s-1)}\log\frac{1}{|z|}$$

Thus, in Case 2,

$$G_{\mathbb{D}^{(t)}}(z,0) \ge \Big(1-\frac{4r}{(r-1)(r-s-1)}\Big)G_{\mathbb{D}}(z,0)\,.$$

Choosing $r = 7$ and $s = 18/17$, we get the proposed estimate. □

At the cost of increasing r, one can get a better (bigger) lower bound on the disk. For our purpose, smaller $r_{\mathbb{D}}$ will be better. The proof allows to take any number $r > (7+\sqrt{41})/2$.

With u and Ψ as in Theorem 2.2.2, we would like to have a more general type of boundary moment to be finite, even when the integral in (2.2) is infinite. To this end, we consider a continuous, increasing function $\Phi : \big[0\,,\,\mathsf{diam}(\mathbb{X})\big] \to [0\,,\,\infty)$ with $\Phi(0) = 0$. With Φ as well as with Ψ, we associate the continuous, non-negative measures $d\Phi$ and $d\Psi$ on $\big(0\,,\,\mathsf{diam}(\mathbb{X})\big]$ which give mass $\Phi(b) - \Phi(a)$, resp. $\Psi(a) - \Psi(b)$ to any interval $(a\,,\,b] \subset \big(0\,,\,\mathsf{diam}(\mathbb{X})\big]$. Furthermore, we consider the decreasing, continuous function

$$\Upsilon : \big[0\,,\,\mathsf{diam}(\mathbb{X})\big] \to [0\,,\,\infty]\,,\quad \Upsilon(t) = \int_t^{\mathsf{diam}(\mathbb{X})} \Phi(s)\,d\Psi(s)\,. \tag{2.13}$$

It will (typically) occur that $\Upsilon(0) = \infty$. We should consider Υ as a downscaling of Ψ; indeed, $\Upsilon(t) \le \|\Phi\|_\infty\,\Psi(t)$. If Ψ is differentiable on $\big(0\,,\,\mathsf{diam}(\mathbb{X})\big)$, then $d\Psi(t) =$

$-\Psi'(t)\,dt$, and $\Upsilon'(t) = \Phi(t)\,\Psi'(t)$. The case considered in Favorov and Golinskii, 2009 is the one where $\Psi(t) = t^{-q}$ and $\Phi(t) = t^{\alpha}$, where $0 < \alpha < q$, so that $\Upsilon(t) \asymp t^{\alpha-q}$.

Theorem 2.2.6. *Let the subharmonic function u on $\mathbb{X}$, the "singular" set $E \subset \partial\mathbb{X}$ and the function Ψ be as in Theorem 2.2.2, but with infinite integral in (2.2). For continuous, increasing $\Phi : \big[0\,,\,\mathrm{diam}(\mathbb{X})\big] \to [0\,,\,\infty)$ with $\Phi(0) = 0$ and the associated function $\Upsilon(t)$ according to (2.13), suppose that*

$$\int_{\partial\mathbb{X}} \Upsilon\big(\mathrm{dist}(\xi, E)\big)\, d\lambda(\xi) < \infty\,.$$

Then the Riesz measure μ_u satisfies the extended boundary moment condition

$$\int_{\mathbb{X}} \mathrm{dist}(x, \partial\mathbb{X})\,\Phi\big(\mathrm{dist}(x,E)/R\big)\,d\mu_u(x) < \infty\,, \tag{2.14}$$

where $R = R_{\mathbb{X}}$ is given by $R_{\mathbb{D}} = 14$, resp. $R_{\mathbb{T}} = 1$.

For the disk case, when $\Psi(t) = t^{-q}$ and $\Phi(t) = t^{\alpha}$ $(0 < \alpha < q)$, this boils down to Theorem 1-(ii)-(7) of Favorov and Golinskii, 2009.

In typical instances, Φ will have the *doubling property* $\Phi(t/2) \geq C \cdot \Phi(t)$ for a fixed $C > 0$. In this case, division by R can be omitted in (2.14) even on the disk.

Corollary 2.2.7. *Consider the disk. Under the assumptions of Theorem 2.2.6, if $1/\Psi$ is doubling and*

$$\int_{\mathbb{S}} \Psi\big(\mathrm{d}_{\mathbb{D}}(\xi, E)\big)^{1-\varepsilon}\, d\lambda_{\mathbb{S}}(\xi) < \infty\,,$$

then

$$\int_{\mathbb{D}} \mathrm{d}_{\mathbb{D}}(x, \mathbb{S})\,\Psi\big(\mathrm{d}_{\mathbb{D}}(x,E)\big)^{-\varepsilon}\,d\mu_u(x) < \infty\,.$$

Proof of Theorem 2.2.6. Once again, the proof works in similar ways on disk and tree. We should keep in mind that on the tree, integrals with respect to the Riesz measure are infinite sums.

For most of the proof, we assume that $u(o)$ is finite. On the tree, this is always required, but on the disk, one may have $u(z) = -\infty$ on a set of measure 0. We shall briefly explain at the end how to handle the case $u(0) = -\infty$.

We take up the thread from the end of the proof of Theorem 2.2.2, in particular (2.10). That inequality tells us that u has $c_{\mathbb{X}}\, h^{(t)}$ as a harmonic majorant on $\mathbb{X}^{(t)}$. Thus, it has its least harmonic majorant $v^{(t)}$ on that set, and we have the Riesz decomposition

$$u(x) = v^{(t)}(x) - G_{\mathbb{X}^{(t)}}\mu_u(x)\,, \quad x \in \mathbb{X}^{(t)}.$$

We have $G_{\mathbb{D}}(z,0) \geq 1 - |z| = \mathsf{d}_{\mathbb{D}}(z,\mathbb{S})$ on the disk, and $G_{\mathbb{T}}(x,o) = b_{\mathbb{T}}\, \rho_{\mathbb{T}}(x,\partial\mathbb{T})$. Using Theorem 2.2.5, we get for $0 < t < 1/r$ ($r = r_{\mathbb{X}}$)

$$\begin{aligned}
\int_{\mathbb{X}^{(rt)}} \mathsf{dist}(x,\partial\mathbb{X})\, d\mu_u(x) &\leq b_{\mathbb{X}}\, G_{\mathbb{X}^{(t)}}\mu_u(o) \\
= b_{\mathbb{X}}\, \big(v^{(t)}(o) - u(o)\big) &\leq b_{\mathbb{X}}\, c_{\mathbb{X}}\, h^{(t)}(o) - b_{\mathbb{X}}\, u(o) \\
= b_{\mathbb{X}}\, c_{\mathbb{X}} \int_{E_*^{(t)}} \Psi\big(\mathsf{dist}(\cdot,E)\big)\, d\lambda + b_{\mathbb{X}}\, c_{\mathbb{X}}\, &\Psi(t)\, \lambda(E^{(t)}) - b_{\mathbb{X}}\, u(o).
\end{aligned}$$

(In the disk case, o stands once more for the origin.) For the next computation, we note that $\max\{\mathsf{dist}(x,E) : x \in \mathbb{X}\}$ has value 1 for the tree, but may be between 1 and 2 for the disk. Tacitly using continuity of the involved measures, and using monotonicity of Ψ, for $0 < t < 1$

$$\begin{aligned}
\int_{E_*^{(t)}} \Psi\big(\mathsf{dist}(\xi,E)\big)\, d\lambda(\xi) &= \int_{E^{(1)}\cap E_*^{(t)}} \Psi\big(\mathsf{dist}(\xi,E)\big)\, d\lambda(\xi) + \int_{E_*^{(1)}} \Psi\big(\mathsf{dist}(\xi,E)\big)\, d\lambda(\xi) \\
&\leq \int_{\mathsf{dist}(\xi,E)}^{1} d\Psi(s)\, d\lambda(\xi) \;\; + \;\; \Psi(1)\, \lambda(E^{(1)} \cap E_*^{(t)}) + \Psi(1)\, \lambda(E_*^{(1)}) \\
&= \int_t^1 \lambda\big(\{\xi \in \partial\mathbb{D} : t < \mathsf{dist}(\xi,E) \leq s\}\big)\, d\Psi(s) \;\; + \;\; \Psi(1)\, \lambda(E_*^{(t)})) \\
&= \int_t^1 \lambda(E^{(s)})\, d\Psi(s) \;\; - \;\; \lambda(E^{(t)})\, \Psi(t) \;\; + \;\; \Psi(1)\,.
\end{aligned}$$

Combining this with the previous inequality, we get for $0 < t < 1$

$$\int_{x\in\mathbb{X}^{(t)}} \mathsf{dist}(x,\partial\mathbb{X})\, d\mu_u(x) \leq b_{\mathbb{X}}\, c_{\mathbb{X}} \int_{t/r}^{1} \lambda(E^{(s)})\, d\Psi(s) + C_1\,, \tag{2.15}$$

where $C_1 = b_{\mathbb{X}}\, c_{\mathbb{X}}\, \Psi(1) - b_{\mathbb{X}}\, u(o)$. Because of several smaller subtleties, we now conclude the proofs separately.

A. Tree case. Recalling that $b_{\mathbb{T}} = r_{\mathbb{T}} = R_{\mathbb{T}} = 1$,

$$\begin{aligned}
\sum_{x\in\mathbb{T}} \rho_{\mathbb{T}}(x,\partial\mathbb{T})\,\Phi\big(\rho_{\mathbb{T}}(x,E)\big)\,\mu_u(x) &= \sum_{x\in\mathbb{T}} \rho_{\mathbb{T}}(x,\partial\mathbb{T}) \int_0^{\rho_{\mathbb{T}}(x,E)} d\Phi(t)\,\mu_u(x) \\
&= \int_0^1 \Bigg(\sum_{x\in\mathbb{T}^{(t)}} \rho_{\mathbb{T}}(x,\partial\mathbb{T})\,\mu_u(x) \Bigg) d\Phi(t) \\
\text{[by (2.15)]} \qquad &\le c_{\mathbb{T}} \int_0^1 \int_t^1 \lambda_{\mathbb{T}}(E^{(s)})\,d\Psi(s)\,d\Phi(t) + C_1\,\Phi(1) \\
\text{[Fubini]} \qquad = c_{\mathbb{T}} \int_0^1 \lambda_{\mathbb{T}}(E^{(s)})\,\Phi(s)\,d\Psi(s) + C_2 &= c_{\mathbb{T}} \int_{\partial\mathbb{T}} \Upsilon\big(\rho_{\mathbb{T}}(\xi,E)\big)\,d\lambda_{\mathbb{T}}(\xi) + C_2\,,
\end{aligned}$$

which is finite by assumption.

B. Disk case. Note that the maximum possible value of $\mathsf{d}_{\mathbb{D}}(z,E)$ is 2. We refer to a simple observation of Favorov and Golinskii, 2009: if $0 < t < 2$ then for every $z \in \mathbb{D}$ and $\alpha \in [0\,,1]$, we have $\mathsf{d}_{\mathbb{D}}(z,E) \le 2\mathsf{d}_{\mathbb{D}}(\alpha\, z,E)$. In particular, if $\mathsf{d}_{\mathbb{D}}(z,E) > t$ then $\mathsf{d}_{\mathbb{D}}(\alpha\, z,E) > t/2$, so that z lies in the component of 0 of the set $\{w \in \mathbb{D} : \mathsf{d}_{\mathbb{D}}(w,E) > t/2\}$. This means that

$$\{z \in \mathbb{D} : \mathsf{d}_{\mathbb{D}}(z,E) > t\} \subset \mathbb{D}^{(t/2)}\,. \tag{2.16}$$

Using this, we now compute

$$\begin{aligned}
\int_{\mathbb{D}} \mathsf{d}_{\mathbb{D}}(z,\mathbb{S})\,\Phi\big(\mathsf{d}_{\mathbb{D}}(z,E)/14\big)\,d\mu_u(z) &= \int_{\mathbb{D}} \int_0^{\mathsf{d}_{\mathbb{D}}(z,E)/14} \mathsf{d}_{\mathbb{D}}(z,\mathbb{S})\,d\Phi(t)\,d\mu_u(z) \\
\text{[since } \mathsf{d}(z,E) < 2] \qquad &= \int_0^{1/7} \int_{\{z\in\mathbb{D}\,:\,\mathsf{d}_{\mathbb{D}}(z,E)>14t\}} \mathsf{d}_{\mathbb{D}}(z,\mathbb{S})\,d\mu_u(z)\,d\Phi(t) \\
\text{[by (2.16)]} \qquad &\le \int_0^{1/7} \int_{\mathbb{D}^{(7t)}} \mathsf{d}_{\mathbb{D}}(z,\mathbb{S})\,d\mu_u(z)\,d\Phi(t) \\
\text{[by (2.15)]} \qquad &\le b_{\mathbb{X}}\,c_{\mathbb{X}} \int_0^1 \int_t^1 \lambda(E^{(s)})\,d\Psi(s) + C_1\,\Phi(1)\,,
\end{aligned}$$

which is seen to be finite by the same calculation as in the tree case.

The case when $u(0) = -\infty$ can be treated exactly as in Favorov and Golinskii, 2009 [p.43] (where the subharmonic function is denoted v) and is omitted here. $\square$

Finally, we want to prove a converse to Theorem 2.2.6 analogous to Proposition 2.2.4.

Theorem 2.2.8. *Let the set $E \subset \partial\mathbb{X}$ and the function Ψ be as in Theorem 2.2.2, but with infinite integral in (2.2). Let $\Phi : [0,1] \to [0\,,\,\infty)$ be continuous and increasing with $\Phi(0) = 0$ and $\Phi(t) > 0$ for $t > 0$. For the associated function $\Upsilon(t)$ according to (2.13), suppose that*

$$\int_{\partial\mathbb{X}} \Upsilon\big(\mathsf{dist}(\xi, E)\big)\, d\lambda(\xi) = \infty\,.$$

If u is a subharmonic function on $\mathbb{X}$ such that

$$u(x) \geq \Psi\big(\mathsf{dist}(x, E)\big)$$

then the Riesz measure μ_u is such that

$$\int_{\mathbb{X}} \mathsf{dist}(x, \partial\mathbb{X})\, \Phi\big(\mathsf{dist}(x, E)\big)\, d\mu_u(x) = \infty\,. \tag{2.17}$$

Proof. First of all, we note that (2.17) hold if and only if

$$\int_{\mathbb{X}} G(x, o)\, \Phi\big(\mathsf{dist}(x, E)\big)\, d\mu_u(x) = \infty\,. \tag{2.18}$$

On the tree, this is obvious, because $G_{\mathbb{T}}(x, o) = \frac{\mathsf{q}}{\mathsf{q}-1}\, \rho_{\mathbb{T}}(x, \partial\mathbb{T})$. On the disk, it is clear that (2.17) implies (2.18). Conversely,

$$\int_{|z|<1/2} G(z, 0)\, \Phi\big({}_{\mathbb{D}}(\mathsf{z}, \mathsf{E})\big)\, \mathsf{d}^{-}{}_{\mathsf{u}}(\mathsf{z}) \leq \|\Phi\|_\infty \int_{|z|<1/2} \mathsf{G}(\mathsf{z}, 0)\, \mathsf{d}^{-}{}_{\mathsf{u}}(\mathsf{z}) < \infty\,,$$

while for $|z| \geq 1/2$, we have $G(z, 0) = \log \frac{1}{|z|} \leq (2\log 2)(1 - |z|)$, so that (2.18) implies

$$2\log 2 \int_{|z|\geq 1/2} (1 - |z|)\, \Phi\big(\mathsf{d}_{\mathbb{D}}(z, E)\big)\, d\mu_u(z) \geq \int_{|z|\geq 1/2} G(z, 0)\, \Phi\big({}_{\mathbb{D}}(\mathsf{z}, \mathsf{E})\big)\, \mathsf{d}^{-}{}_{\mathsf{u}}(\mathsf{z}) = \infty\,.$$

Case 1. Suppose that there is $t \in (0\,,\,1)$ such that u has no harmonic majorant on the set $\mathbb{X}^{(t)}$. Then $G_{\mathbb{X}^{(t)}}\mu_u$ is infinite on that set. Thus,

$$\begin{aligned}\int_{\mathbb{X}} G(x, o)\, \Phi\big(\mathsf{dist}(x, E)\big)\, d\mu_u(x) &\geq \int_{\mathbb{X}^{(t)}} G_{\mathbb{X}^{(t)}}(x, o)\, \Phi\big(\mathsf{dist}(x, E)\big)\, d\mu_u(x) \\ &\geq \Phi(t)\, G_{\mathbb{X}^{(t)}}\mu_u(o) = \infty\,,\end{aligned}$$

and the equivalence of (2.17) with (2.18) implies the result.

Case 2. We are left with the case when for each $t \in (0\,,\,1)$ there is the (finite) least harmonic majorant $v^{(t)}$ of u on $\mathbb{X}^{(t)}$. Recall the function $h^{(t)}$ of (2.7). Then for every $\eta \in \partial\mathbb{X}^{(t)}$,

$$\limsup_{x\to\eta} v^{(t)}(x) \geq \limsup_{x\to\eta} v^{(t)}(x) \geq \Psi\big(\mathsf{dist}(\eta, E)\big) = \lim_{x\to\eta} h^{(t)}(x)\,.$$

By the minimum principle, applied to the harmonic function $v^{(t)} - h^{(t)}$, we have $v^{(t)} \geq h^{(t)}$ on $\mathbb{X}^{(t)}$. Now we can replace the computations of the proof of Theorem 2.2.6 with similar inequalities in the reverse direction.

$$\begin{aligned}
\int_{\mathbb{X}^{(t)}} G(x,o)\,d\mu_u(x) &\geq \int_{\mathbb{X}^{(t)}} G_{\mathbb{X}^{(t)}}\mu_u(o) = v^{(t)}(o) - u(o) \geq h^{(t)}(o) - u(o)\\
&= \int_{E^{(1)}\cap E_*^{(t)}} \Psi\big(\mathsf{dist}(\cdot, E)\big)\,d\lambda + \int_{E_*^{(1)}} \Psi\big(\mathsf{dist}(\cdot, E)\big)\,d\lambda + \Psi(t)\,\lambda(E^{(t)}) - u(o)\\
&\geq \int_t^1 \lambda(E^{(s)} \setminus E^{(t)})\,d\Psi(s) + \Psi(1)\,\lambda(E^{(1)} \setminus E^{(t)}) + \Psi(1)\,\lambda(E_*^{(1)}) + \Psi(t)\,\lambda(E^{(t)}) - u(o)\\
&= \int_t^1 \lambda(E^{(s)})\,d\Psi(s) + C_3\,, \qquad\qquad \text{where} \quad C_3 = \Psi(1) - u(o)\,.
\end{aligned}$$

Now let $0 < \varepsilon < 1$. Let $\Phi_\varepsilon(s) = \max\{\Phi(s) - \Phi(\varepsilon)\,,\,0\}$. Since u has a harmonic majorant on $\mathbb{X}^{(\varepsilon)}$, the first integral in the following computation is finite. The above estimate is used in the third line.

$$\begin{aligned}
\int_{\mathbb{X}^{(\varepsilon)}} G(x,o)\,\Phi\big(\mathsf{dist}(x,E)\big)\,d\mu_u(x) &\geq \int_{\mathbb{X}^{(\varepsilon)}} G(x,o) \int_\varepsilon^{\mathsf{dist}(x,E)} d\Phi(t)\,d\mu_u(x)\\
&\geq \int_\varepsilon^1 \int_{\mathbb{X}^{(t)}} G(x,o)\,d\mu_u(x)\,d\Phi(t)\\
&\geq \int_\varepsilon^1 \int_t^1 \lambda(E^{(s)})\,d\Psi(s)\,d\Phi(t) + (1-\varepsilon)C_3\\
&= \int_\varepsilon^1 \lambda(E^{(s)}) \int_\varepsilon^s d\Phi(t)\,d\Psi(s) + (1-\varepsilon)C_3\\
&= \int_0^1 \bigg(\int_{\{\xi\in\partial\mathbb{X}:\mathsf{dist}(\xi,E)\leq s} d\lambda(\xi)\bigg)\Phi_\varepsilon(s)\,d\Psi(s) + (1-\varepsilon)C_3\\
&= \int_{E^{(1)}} \int_{\mathsf{dist}(\xi,E)}^1 \Phi_\varepsilon(s)\,d\Psi(s)\,d\lambda(\xi) + (1-\varepsilon)C_3
\end{aligned}$$

As $\varepsilon \to 0$, by monotone convergence, the double integral in the last line tends to

$$\int_{E^{(1)}} \Big(\Upsilon\big(\mathsf{dist}(\xi, E)\big) - \Upsilon(1)\Big)\,d\lambda(\xi)\,,$$

which is infinite by assumption. □

Remark 2.2.9. (a) *[Hyperbolic versus Euclidean.]* In the introduction we insisted on a hyperbolic "spirit" inherent in the material presented here. After all, this was not dominant in most of our computations. Not only on the disk, we always used the Euclidean metric $d_{\mathbb{D}}$, but also on the tree, the dominant role was played by the metric $\rho_{\mathbb{T}}$ which is the tree-analogue of the Euclidean metric. One point is that to see the latter analogy, one should first understand that the graph metric on the tree corresponds to the hyperbolic one on the disk.

One result where hyperbolicity is strongly present is Theorem 2.2.5. The proof in the tree case relies directly on the fact that the tree with its graph metric is δ-hyperbolic in the sense of Gromov, 1987, with $\delta = 0$: every vertex is a cut-point (it disconnects the tree). Analogously, one might try to prove that theorem in the disk case using δ-hyperbolicity with $\delta = \log(1 + \sqrt{2})$. Indeed, this is related with the inequalities of Ancona, 1987 which say that the Green kernel of the open disk is almost submultiplicative along hyperbolic geodesics. (For the disk, this can be seen by direct inspection via the explicit formulas for the Green kernel.) Now, for points $z \in \mathbb{D}^{(rt)}$ and $\xi \in E^{(t)}$, the hyperbolic geodesic from z to ξ must be at bounded hyperbolic distance from the origin (depending on r and t), similarly to the (simpler) tree case. However, this idea is more vague than the down-to-earth proof following Favorov and Golinskii, 2009.

(b) In view of the equivalence (2.17) $\iff$ (2.18), in all the results presented here, one can replace the distance to the boundary $\mathsf{dist}(x, \partial\mathbb{X})$ with the Green kernel $G(x, o)$.

(c) Among the common features of disk and tree which allowed us to formulate and prove the results in very similar ways, the key facts are

- comparability of $G(x, o)$ with $\mathsf{dist}(x, \partial\mathbb{X})$ (the metric is "intrinsic" in this sense),
- solvability of the Dirichlet problem for continuous functions on $\partial\mathbb{X}$, and in particular, vanishing of the Green kernel at the boundary, and
- the Green kernel estimate of Theorem 2.2.5.

2.19. An extension for trees. Instead of the homogeneous tree, we can take an arbitrary locally finite tree $\mathbb{T}$ and equip its edges with *conductances* $a(x, y) = a(y, x) > 0 \iff x \sim y$. Letting $m(x) = \sum_y a(x, y)$, the transition probabilities $p(x, y) = a(x, y)/m(y)$

give rise to a nearest neighbour random walk $(Z_n)_{n\geq 0}$ and to the associated Laplacian

$$\Delta_{\mathbb{T}} f(x) = \sum_{y\sim x} p(x,y)\big(f(y)-f(x)\big).$$

We assume the following.

(i) Strong irreducibility: $0 < m_0 \leq m(x) \leq M_0 < \infty$ and $a(x,y) \geq a_0 > 0$ for all x and all $y \sim x$.
(ii) Strong transience: $F(x,y) \leq \delta < 1$ for all x and all $y \sim x$, where for arbitrary $x,y \in \mathbb{T}$,

$$F(x,y) = \Pr[\exists n \geq 0 : Z_n = y \mid Z_0 = x]$$

The associated Green kernel

$$G(x,y) = \sum_{n=0}^{\infty} p^{(n)}(x,y)\,, \quad \text{where} \quad p^{(n)}(x,y) = \Pr[Z_n = y \mid Z_0 = x]\,, \quad x,y \in X$$

is finite and tends to 0 at infinity by assumption (ii). Note that in our notation, $G(x,y) = F(x,y)G(y,y)$.

We can adapt all the above results regarding the homogenous tree to this more general situation. The main issue is to define a suitable metric on the compactification $\widehat{\mathbb{T}}$ in the right way: for $z,w \in \widehat{\mathbb{T}}$,

$$\rho_{\mathbb{T}}(w,z) = \begin{cases} F(w \wedge z, o)\,, \text{if } z \neq w\,, \\ 0\,, \text{if } z = w\,. \end{cases}$$

[For simple random walk on the homogeneous tree, as considered above, this is just the metric of (1.18).]

In this setting, the tree-versions of theorems 2.2.2, 2.2.6 and 2.2.8 remain true. This applies, in particular, to arbitrary symmetric nearest neighbour random walks on the free group ($\equiv$ homogeneous tree with even degree).

In conclusion, we remark that the very recent note by Favorov and Radchenko, 2013 was written in parallel to the present article without mutual knowledge. The results of

Favorov and Radchenko, 2013 concern the disk case and are a bit less general than ours. We want to point out that here, our main focus has been on elaborating some aspects of the very strong analogies of the potential theory on disk and tree, respectively, via focussing on properties of Riesz measures.

3 Mean value property for nonharmonic functions

3.1 Generalized mean value property in $\mathbb{R}^n$

In this section we show how to generalize the mean value property in $\mathbb{R}^n$ to the case of nonharmonic functions. Without loss of generality we study the value at the origin and take the integrals over the spheres centered at the origin. In Subsections 3.1.1 and 3.1.2 we introduce some preliminary general notions and definitions. Further in Subsection 3.1.3 we formulate and prove the main results concerning the mean value property in $\mathbb{R}^n$.

3.1.1 Operator on $\mathbb{R}^1$ associated to the d-dimensional Laplace operator

Consider the Laplace operator $\triangle$ on $\mathbb{R}^d$. In polar coordinates one can write

$$\triangle f = \triangle_r f + \frac{1}{r^2}\triangle_{S^{d-1}} f, \quad \text{where} \quad \triangle_r f = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\Big(r^{d-1}\frac{\partial f}{\partial r}\Big),$$

the radial part, and $\triangle_{S^{d-1}}$ is the Laplace–Beltrami operator on the $(d-1)$-sphere. Let us associate to the Laplace operator $\triangle$ the following operator on the real line:

$$\tilde{\triangle}_d g(x) = \frac{\partial}{\partial x}\Big(x^{d-1}\frac{\partial}{\partial x}\Big(\frac{g(x)}{x^{d-1}}\Big)\Big), \qquad x \in \mathbb{R}.$$

Proposition 3.1.1. *For an analytic function f it holds*

$$\int\limits_{S^{d-1}(x)} \triangle f(v)d\mu = \tilde{\triangle}_d \left(\int_{S^{d-1}(x)} f(v)d\mu \right).$$

Remark 3.1.2. For the convenience of the reader we consider only the functions that are analytic in some neighborhood of the ball of radius x centered at zero. We should mention that similar statement can be considered for analytic functions in the interior of the balls.

Proof. First, notice that for Laplace–Beltrami operator $\triangle_{S^{d-1}}$ it holds

$$\int\limits_{S^{d-1}(x)} h(v)\triangle_{S^{d-1}} f(v)d\mu = -\int\limits_{S^{d-1}(x)} \langle \operatorname{grad} h(v), \operatorname{grad} f(v)\rangle d\mu,$$

where the function grad is the gradient operator on the tangent space to the sphere $S^{d-1}(x)$, and $\langle v, w\rangle$ is the scalar product of v and w. Therefore, substituting $h = 1$ we have

$$\int\limits_{S^{d-1}(x)} \triangle_{S^{d-1}} f(v)d\mu = 0.$$

Second, we make the following transformations.

$$\begin{aligned}
\tilde{\triangle}_d\Big(\int_{S^{d-1}(x)} f(v)d\mu\Big) &= \tilde{\triangle}_d\Big(x^{d-1}\int_{S^{d-1}(1)} f(xv)d\mu\Big) \\
&= \frac{\partial}{\partial x}\left(x^{d-1}\frac{\partial}{\partial x}\Big(\int_{S^{d-1}(1)} f(xv)d\mu\Big)\right) \\
&= \int_{S^{d-1}(1)} \frac{\partial}{\partial x}\Big(x^{d-1}\frac{\partial}{\partial x} f(xv)\Big)d\mu \\
&= \int_{S^{d-1}(x)} \frac{1}{x^d}\frac{\partial}{\partial x}\Big(x^{d-1}\frac{\partial}{\partial x} f(xv)\Big)d\mu \\
&= \int_{S^{d-1}(x)} \triangle_r f(v)d\mu = \int_{S^{d-1}(x)} \triangle f(v)d\mu.
\end{aligned}$$

This concludes the proof of Proposition 3.1.1. □

Iteratively applying Proposition 3.1.1 we get the following corollary.

Corollary 3.1.3. *For an analytic function f on $\mathbb{R}^d$ and a nonnegative integer n we have*

$$\int_{S^{d-1}(x)} \triangle^n f(v) d\mu = \tilde{\triangle}_d^n \left(\int_{S^{d-1}(x)} f(v) d\mu \right).$$

□

3.1.2 Bessel functions and some important generating functions

Let J_p denote Bessel functions of the first kind. Recall that the power series decomposition of J_p at $x = 0$ is written as

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{p+2k}}{k!\Gamma(p+k+1)}.$$

Let us define two collections of coefficients $\alpha_{i,d}$ and $\beta_{i,d}$. Recall that

$$\sum_{i=0}^{\infty} \alpha_{i,d} x^{2i} = \frac{(Ix/2)^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2}) J_{\frac{d-2}{2}}(Ix)}.$$

Remark 3.1.4. In case if $d = 1$ and $d = 3$ we have the following

$$\operatorname{sech} x = \sum_{i=0}^{\infty} \alpha_{i,1} x^{2i} \qquad \text{and} \qquad x \operatorname{csch} x = \sum_{i=0}^{\infty} \alpha_{i,3} x^{2i}.$$

Set the coefficients $\beta_{i,d}$ as follows

$$\sum_{i=0}^{\infty} \beta_{i,d} x^{2i} = \frac{J_{\frac{d-2}{2}}(Ix)}{(Ix/2)^{\frac{d-2}{2}}},$$

Proposition 3.1.5. *Let k be a nonnegative integer and d be a positive integer. Then it holds*

(i) $\beta_{k,d} = \dfrac{1}{4^k k! \Gamma(p+k+1)}$;

(ii) $\sum_{i=0}^{k} \alpha_{i,d} \beta_{k-i,d} = \begin{cases} \frac{1}{\Gamma(d/2)}, & \text{if } k = 0; \\ 0, & \text{if } k \geq 1. \end{cases}$

Proof. The first statement follows directly from the power series decomposition for the function $J_{\frac{d-2}{2}}(Ix)$. The second statement holds, since by the definition of generating functions

$$\sum_{i=0}^{\infty} \alpha_{i,1} x^{2i} \sum_{i=0}^{\infty} \beta_{i,1} x^{2i} = \frac{(Ix/2)^{\frac{d-2}{2}}}{\Gamma(\frac{d}{2}) J_{\frac{d-2}{2}}(Ix)} \cdot \frac{J_{\frac{d-2}{2}}(Ix)}{(Ix/2)^{\frac{d-2}{2}}} = \frac{1}{\Gamma(d/2)}.$$

□

3.1.3 Generalized mean value property

We start with several definitions.

Definition 23. For an arbitrary dimension d, a smooth function f on $\mathbb{R}^n$, and a smooth function g on $\mathbb{R}^1$, set

$$T_d(f,r)(v) = \sum_{i=0}^{\infty} \alpha_{i,d} r^{2i} \triangle^i f(v),$$
$$\tilde{T}_d g(x) = \sum_{i=0}^{\infty} \alpha_{i,d} x^{2i} \tilde{\triangle}_d^i g(x),$$

where the generating function for the coefficients $\alpha_{i,d}$ is as above.

For an arbitrary analytic function $f : \mathbb{R}^d \to \mathbb{R}$ we denote by $\tilde{f} : \mathbb{R} \to \mathbb{R}$ the function defined as follows. For positive x we set

$$\tilde{f}(x) = \frac{1}{\mathrm{Vol}(S^{d-1}(x))} \int_{S^{d-1}(x)} f(v) d\mu.$$

For negative x we put $\tilde{f}(x) = \tilde{f}(-x)$. Finally we define $\tilde{f}(0)$ by continuity as f(o):

$$\tilde{f}(0) \lim_{x\to 0} \tilde{f}(x) = \lim_{x\to 0} \left(\frac{1}{\mathrm{Vol}(S^{d-1}(x))} \int_{S^{d-1}(x)} f(v) d\mu \right) = f(0).$$

Definition 24. Let $a > 0$. We say that a function $f : \mathbb{R}^d \to \mathbb{R}$ is *spherically a-analytic at* 0 for some $a > 0$ if the Taylor series for $\tilde{f}$ at the origin converges to $\tilde{f}$ on the segment $[-a, a]$.

Theorem A. *Consider $0 < r < a$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function that is spherically a-analytic at 0. Then we have*

$$f(0) = \frac{1}{\mathrm{Vol}(S^{d-1}(r))} \int\limits_{S^{d-1}(r)} T_d(f,r) d\mu.$$

Example 3.1.6. Let a function φ on $\mathbb{R}^3$ satisfy the Poisson equation

$$\triangle \varphi = f$$

for some harmonic function f. Then

$$\varphi(0) = \frac{1}{4\pi} \int\limits_{S^2(1)} \Big(\varphi(x) - \frac{1}{6}\triangle\varphi(x)\Big) d\mu.$$

We start the proof of Theorem A with the following lemma.

Lemma 3.1.7. *Let k be a nonnegative integer. Then*

$$\tilde{T}_d(x^{2k+d-1}) = \begin{cases} x^{d-1}, & \text{if } k = 0; \\ 0, & \text{if } n \geq d. \end{cases}$$

Proof. First, observe the following

$$\tilde{\triangle}_d x^n = (n-d+1)(n-1)x^{n-2}.$$

Therefore,

$$\tilde{\triangle}_d^i x^{2k+d-1} = 4^i \frac{k!}{(k-i)!} \frac{\Gamma(k+\frac{d}{2})}{\Gamma(k-i+\frac{d}{2})} x^{n-2i}.$$

In particular, this means that for $i > k$ we have $\tilde{\triangle}_d^i(x^{2k+d-1}) = 0$. Hence we get

$$\begin{aligned}
\tilde{T}_d(x^{2k+d-1}) &= \sum_{i=0}^{\infty} \alpha_{i,d} x^{2i} 4^i \frac{k!}{(k-i)!} \frac{\Gamma(k+\frac{d}{2})}{\Gamma(k-i+\frac{d}{2})} x^{n-2i} \\
&= 4^k k! \Gamma\Big(k+\frac{d}{2}\Big) x^{2k+d-1} \sum_{i=0}^{k} \alpha_{i,d} \frac{1}{4^{k-i}(k-i)!\Gamma(k-i+\frac{d}{2})} \\
&= 4^k k! \Gamma\Big(k+\frac{d}{2}\Big) x^{2k+d-1} \sum_{i=0}^{k} \alpha_{i,d} \beta_{k-i,d} \\
&= \begin{cases} x^{d-1}, & \text{if } k = 0; \\ 0, & \text{if } k \geq 0. \end{cases}
\end{aligned}$$

The last two equalities follows from Proposition 3.1.5(i) and Proposition 3.1.5(ii) respectively. ☐

Corollary 3.1.8. *Consider an even analytic function g whose Taylor series taken at 0 converges on the segment $[-a,a]$. Let also x satisfy $0 < x < a$. Then*

$$\frac{\tilde{T}_d\big(x^{d-1}g(x)\big)}{x^{d-1}} = g(0).$$

Remark 3.1.9. In fact, if g is not even then a more general statement holds

$$g(0) = \frac{\tilde{T}_d\big(x^{d-1}g(x)\big) + \tilde{T}_d\big(x^{d-1}g(-x)\big)}{2x^{d-1}}.$$

Proof. Let g be an even function, i.e.,

$$g(x) = \sum_{i=0}^{\infty} c_i x^{2i}.$$

Then from Lemma 3.1.7 we have

$$\begin{aligned}\frac{\tilde{T}_d\big(x^{d-1}g(x)\big)}{x^{d-1}} &= \frac{\tilde{T}_d\big(\sum_{i=0}^{\infty} c_i x^{2i+d-1}\big)}{x^{d-1}} = \frac{\big(\sum_{i=0}^{\infty} c_i \tilde{T}_d(x^{2i+d-1})\big)}{x^{d-1}}\\ &= \frac{c_0 x^{d-1}}{x^{d-1}} = c_0 = g(0).\end{aligned}$$

We demand the convergence of Taylor series in order to exchange the sum operation with $\tilde{T}_d$ in the second equality. ☐

Proof of Theorem A. By Corollary 3.1.3 and by the definition of $\tilde{f}$ we have

$$\int\limits_{S^{d-1}(x)} \triangle^n f(v)d\mu = \tilde{\triangle}_d^n \left(\int_{S^{d-1}(x)} f(v)d\mu\right) = \mathrm{Vol}(S^{d-1}(1))\tilde{\triangle}_d^n\big(x^{d-1}\tilde{f}(x)\big).$$

Since f is spherically a-analytic, the function $\tilde{f}$ satisfies all the conditions of Corollary 3.1.8. Applying Corollary 3.1.8 we get

$$\begin{aligned}f(0) = \tilde{f}(0) &= \frac{\tilde{T}_d\big(x^{d-1}\tilde{f}(x)\big)}{x^{d-1}} = \frac{1}{x^{d-1}\,\mathrm{Vol}(S^{d-1}(1))} \int\limits_{S^{d-1}(x)} T_d(f,r)d\mu\\ &= \frac{1}{\mathrm{Vol}(S^{d-1}(x))} \int\limits_{S^{d-1}(x)} T_d(f,r)d\mu.\end{aligned}$$

This concludes the proof of Theorem A. $\square$

3.1.4 Laplace-Dirichlet series

Definition of Laplace-Dirichlet series, $\triangle$-analyticity

Notice that the inverse Laplace operator is uniquely identified up to addition of a harmonic function. Let us distinguish one special operator among them.

Let f be an arbitrary continuous function. Denote by $harm(f)$ a harmonic function coinciding with f at the boundary (i.e., the solution of the Dirichlet problem for f).

Definition 25. We define the *principal inverse Laplace operator* $\triangle^{-1}$ as follows. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an analytic function in the disc, and let a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfy the Poisson equation $\triangle\varphi = f$. Set

$$\triangle^{-1}(f) = \varphi - harm(\varphi).$$

Remark 3.1.10. Notice that the principal inverse Laplace operator does not depend on the choice of φ. Note also that

$$\triangle(\triangle^{-1}(f)) = f, \quad \text{and} \quad \triangle^{-1}(\triangle(u)) - u = -harm(u).$$

There is a natural procedure to generate an infinite sequence of harmonic functions starting from a given analytic function f. Namely, we consider the functions $harm(\triangle^n(f))$ for $n \geq 0$. These functions are similar to the coefficients of Taylor series. We call such sequence the *Laplace-Dirichlet sequence*.

Definition 26. For a sequence of harmonic functions h_n we say that the sum

$$\sum_{n=0}^{\infty} \triangle^{-n}(h_n),$$

is *Laplace-Dirichlet series*.

Suppose that the harmonic functions h_n are defined by some function f in the disk as $h_n = \triangle^n(f)$, then we say that the corresponding sum is the *Laplace-Dirichlet series* for f.

So Laplace-Dirichlet series represents a function (in case of convergency) that have a prescribed Laplace-Dirichlet sequence. In this context it is natural to say that a function f is $\triangle$-*analytic* if it coincides with its Laplace-Dirichlet series.

Discrete analytic functions for the case of the square grid were introduced in the 40's by Ferrand, 1944 and studied quite extensively in the 50's by Duffin, 1956. In the case of a general map, the notion of discrete analytic functions is implicit in paper of R.L. Brooks, C.A Smith, A.H. Stone and W.T. Tutte (see Brooks et al., 1940) and more recent work by I. Benjamini and O. Schramm (Benjamini and Schramm, 1996). They were formally introduced later by C. Mercat Mercat, 2001.

Example 3.1.11. Let us show that not every smooth function is $\triangle$-analytic. For instance, consider a nonzero function f on B_1 whose support is contained in $B_{1-\varepsilon}$ for some small positive ε. Then all the elements of the Laplace-Dirichlet series are zero functions, and therefore the corresponding Laplace-Dirichlet series is a zero function not equivalent to f.

Laplace-Dirichlet series of a segment

Let us consider a particular example of the segment $[-1, 1]$. The set of harmonic functions on the segment coincides with the set of linear functions, which is generated by the functions 1 and x.

Direct calculations proves the following proposition

Proposition 3.1.12.

$$\triangle^{-n}(1) = \sum_{i=0}^{n} \nu_{2i} \frac{x^{2n-2i}}{(2n-2i)!};$$
$$\triangle^{-n}(x) = \sum_{i=0}^{n} \mu_{2i+1} \frac{x^{2n-2i+1}}{(2n-2i+1)!}.$$

where the coefficients ν_k and μ_k are generated as follows:

$$\frac{1}{\cos(It)} = \sum_{i=0}^{\infty} \nu_i t^i.$$
$$\frac{It^2}{\sin(It)} = \sum_{i=0}^{\infty} \nu_i t^i.$$

□

Remark 3.1.13. Notice that the polynomials of the previous proposition are interesting because of the following interesting property. Their i-th Laplacian powers for $i < n$ have roots ± 1, and their n-th Laplacian powers either equal to 1 or to x.

Example 3.1.14. Let us consider the exponent: $f(x) = e^x$. We have $\triangle(e^x) = e^x$. Therefore, all the elements of the Dirichlet-Laplace series coincide and equal to

$$harm(\triangle^i(e^x)) = harm(e^x) = \sinh(1)x + \cosh(1).$$

Then the corresponding Dirichlet-Laplace series are as follows:

$$\sum_{n=0}^{\infty} \left(\Big(\sum_{i=0}^{n} \mu_{2i+1} \frac{x^{2n-2i+1}}{(2n-2i+1)!} \Big) \sinh 1 + \Big(\sum_{i=0}^{n} \nu_{2i} \frac{x^{2n-2i}}{(2n-2i)!} \Big) \cosh 1 \right).$$

Let us show that e^x is $\triangle$-analytic. We rearrange the above expression as follows:

$$\begin{aligned} &\Big(\sum_{n=0}^{\infty} \mu_n \Big) \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \sinh 1 + \Big(\sum_{n=0}^{\infty} \nu_n \Big) \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \cosh 1 \\ &= \frac{I}{\sin(I)} \sinh x \sinh 1 + \frac{1}{\cos(I)} \cosh x \cosh 1 \\ &= \sinh x + \cosh x = e^x. \end{aligned}$$

Therefore, the function e^x is $\triangle$-analytic.

3.2 Maximal proper arc formula for homogeneous trees

In this section we study the situation in the discrete case of homogeneous trees. We start in Subsection 3.2.1 with necessary notions and definitions. Further in Subsection 3.2.3 we formulate the statements regarding the generalization of the Dirichlet problem at infinity. In Subsection 3.2.4 we study some necessary tools that are further used in the proofs of the main result. We conclude the proofs in Subsection 3.2.5.

3.2.1 Notions and definitions

Consider a homogeneous tree T_q with its boundary $\partial\mathbb{T}_q$. WE recall, that if v and w are connected by an edge we write $v \sim w$.

Laplace operator

In this section we consider the standard Laplace operator on the space of all functions on $\mathbb{T}_q$, which is defined as

$$\triangle f(v) = \frac{\sum\limits_{w \sim v} f(w)}{q+1} - f(v).$$

The compositions $\triangle^i$ are defined inductively in i. Set $\triangle^0$ the identity operator.

Remark 3.2.1. The statements of this section have a straightforward generalization to arbitrary locally finite graphs. For simplicity reasons we restrict ourselves entirely to homogeneous trees.

3.2.2 Weighted Laplace operator on $\mathbb{Z}$.

Denote by $\mathbb{R}^{\mathbb{Z}}$ the set of all real valued functions on the set of integers. In this section we briefly investigate some properties of operators on $\mathbb{R}^{\mathbb{Z}}$ that are usually considered as discrete versions of second order differential operators.

For simplicity we identify the set of all functions with the set of real-valued sequences that are infinite on both side. From now on a function in $\mathbb{R}^{\mathbb{Z}}$ is defined as a sequence of real numbers, i.e., as $(a_i)_{-\infty}^{+\infty}$.

We begin with the following definition.

Definition 27. Let $p \in [0,1]$. The operator $\Delta_p : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ defined as

$$\Delta_p(a_n) = pa_{n+1} + (1-p)a_{n-1} - a_n.$$

is called the *weighted Laplace operator* on the set of all real-valued functions on $\mathbb{Z}$.

Now we are interested in the expressions for the coefficients of the powers of weighted Laplace operators. It turns out that these coefficients coincide with the coefficients of the polynomial

$$h_k(x) = \left(px^2 - x + (1-p)\right)^k.$$

It is clear that the polynomial h_k is of degree $2k$ and, therefore, it is written in the form

$$C_{2k}(k)x^{2k} + C_{2k-1}(k)x^{2k-1} + \ldots + C_1(k)x + C_0(k)$$

for some real numbers $C_0(k), \ldots, C_{2k}(k)$. We use this coefficients in the following proposition.

Proposition 3.2.2. *Let k be a positive integer. The k-th degree of the Laplace operator on $\mathbb{Z}$ has the following form*

$$\triangle_p^k(a_n) = C_{2n}(k)a_{n+k} + C_{2n-1}(k)a_{n+k-1} + \ldots + C_0(k)a_{n-k}.$$

Proof. We prove this proposition by induction on k.

Base of induction. In the case of $k = 1$ we have a weighted Laplace operator itself. Consider

$$\Delta_p(a_0) = pa_1 + (1-p)a_{-1} - a_0 = pa_1 - a_0 + (1-p)a_{-1}.$$

Now the polynomial $h_1(x)$ has the following form

$$h_1(x) = \left(px^2 - x + (1-p)\right)^1 = px^2 - x + (1-p).$$

Therefore the corresponding coefficients coincide. Hence for $k = 1$ the statement is true.

Step of induction. Let the statement hold for k for $k \geq 1$. Let

$$\triangle_p^k(a_0) = C_{2k}a_{n+k} + C_{2k-1}a_{n+k-1} + \ldots + C_0a_{n-k}.$$

and, respectively,

$$h_k(x) = C_{2k}x^{2k} + C_{2k-1}x^{2k-1} + \ldots + C_1 x + C_0$$

(for simplicity in the proof of the induction step we write C_i for $C_i(k)$). Let us prove the statement for $k+1$.

$$\begin{aligned}
\triangle(\triangle_p^k(a_0)) &= C_{2k}(pa_{n+k+1} - a_{n+k} + (1-p)a_{n+k-1}) \\
&\quad + C_{2k-1}(pa_{n+k} - a_{n+k-1} + (1-p)a_{n+k-2}) + \ldots \\
&\quad + C_0(pa_{n-k+1} - a_{n-k} + (1-p)a_{n-k-1}) \\
&= (pC_{2k})a_{n+k+1} + (-C_{2k} + pC_{2k-1})a_{n+k} + \\
&\quad + \sum_{i=1}^{2k-1}(pC_{2k-i-1} - C_{2k-i} + (1-p)C_{2k-i+1})a_{n+k-i} + \\
&\quad + ((1-p)C_1 - C_0)a_{n-k} + ((1-p)C_0)a_{n-k-1}.
\end{aligned}$$

Let us write h_{k+1}

$$\begin{aligned}
h_{k+1}(x) &= (px^2 - x + (1-p))(C_{2k}x^{2k} + C_{2k-1}x^{2k-1} + \ldots + C_1 x + C_0) \\
&= (pC_{2k})x^{2k+2} + (-C_{2k} + pC_{2k-1})x^{2k+1} + \\
&\quad + \sum_{i=1}^{2k-1}(pC_{2k-i-1} - C_{2k-i} + (1-p)C_{2k-i+1})x^{2k-i+1} + \\
&\quad + ((1-p)C_1 - C_0)x^1 + ((1-p)C_0).
\end{aligned}$$

Hence the corresponding coefficients of $\triangle_p^{k+1}(a_0)$ and $h_{k+1}(x)$ are equal. This concludes the proof of induction step.

Therefore, the statement holds for all positive integers. □

Remark 3.2.3. Let us mention explicit formulas for the coefficients of the polynomial h^k. Each polynomial $h^k(x)$ satisfies

$$h^k(x) = \sum_{0 \le i+j \le k} \frac{k!}{i!j!(k-i-j)!}(px^2)^i(-x)^j(1-p)^{k-i-j}.$$

Hence its coefficients are expressed as follows

$$C_n(k) = \sum_{l=0}^{[\frac{n}{2}]} \frac{k!}{l!(n-2l)!(k-l-(n-2l))!} p^l(-1)^{n-2l}(1-p)^{k-l-(n-2l)}.$$

(Here we substitute $i = l$, $j = n - 2l$ to the previous expression and collect only the coefficients at x^n.)

In particular we have

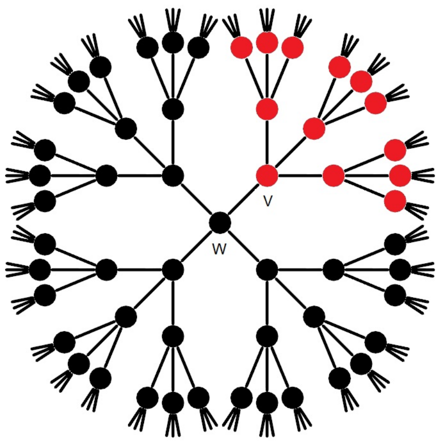
$$\begin{aligned} &C_0(k) = (1-p)^k, \\ &C_1(k) = -k(1-p)^{k-1}, \\ &C_2(k) = \tfrac{k(k-1)}{2}(1-p)^{k-2} + kp(1-p)^k, \\ &\dots \end{aligned}$$

Maximal cones and MP-arcs

We start with the definition of maximal proper cones.

Definition 28. Consider two vertices $v, w \in T_q$ connected by an edge e. The maximal connected component of $T_q \setminus e$ containing v is called the *maximal proper cone with vertex at v (with respect to w)*. We denote it by C^{v-w}.

Here is the maximal proper cone centered at v with respect to w.



C^{v-w}

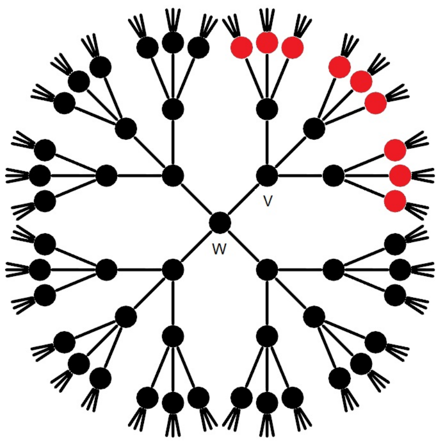
Here *distance* between two vertices $v, w \in T_q$ is the length of the path between v and w, that is the minimal number of edges needed to reach the vertex w starting from the vertex v. For an arbitrary nonnegative integer r and an arbitrary vertex v we denote by $S_r(v)$ the set of all vertices at distance r to v, we call such set the *circle of radius r with center v*. Note that $S_r(v)$ contains exactly $(q+1)q^{r-1}$ points.

Definition 29. Let C^{v-w} be a maximal proper cone of $\mathbb{T}_q$ and let n be a nonnegative integer. The set

$$C_n^{v-w} = C^{v-w} \cap S_n(v)$$

is called the *maximal proper arc of radius n with center at v with respect to w* (or, the *MP-arc*, for short).

We illustrate the last definition by the following picture. Here is the maximal proper arc of radius 2 centered at v with respect to w.



C_2^{v-w}

Integral series

For an arbitrary function $f : \mathbb{T}_q \to \mathbb{R}$ we write

$$f(C_n^{v-w}) = \frac{1}{q^n} \sum_{u \in C_n^{v-w}} f(u).$$

Definition 30. In what follows we consider the *maximal proper cone integrals* (or, *MPC-integrals*, for short) defined by the following formal expression (in the left hand side):

$$\int_{\partial C^{v-w}} \left[\sum_{i=0}^{\infty} \lambda_i(\infty) \triangle^i f(t) \right] dt = \lim_{n\to\infty} \left(\sum_{i=0}^{n} \lambda_i(n) \triangle^i f(C_n^{v-w}) \right),$$

where f is a function on the tree, λ_i are arbitrary functions on the set of positive integers. Respectively we write

$$\int_{\partial \mathbb{T}_q} \left[\sum_{i=0}^{\infty} \lambda_i(\infty) \triangle^i f(t) \right]_v dt = \lim_{n\to\infty} \left(\sum_{i=0}^{n} \lambda_i(n) \sum_{u \in S^n(v)} \frac{\triangle^i f(u)}{q^n} \right).$$

Here we specify by an index v that the series are taken with respect to the vertex v, since in such settings v is defined by from the integration domain.

For instance,

$$\begin{aligned} \int_{\partial C^{v-w}} [2^{\infty}(1+(-\infty)^3) f(t)] dt &= \lim_{n\to\infty} \left(2^n (1-n^3) f(C_{n+1}^{v-w}) \right) \\ &= \lim_{n\to\infty} \left(2^n (1-n^3) \sum_{u \in C_n^{v-w}} \frac{f(u)}{q^n} \right). \end{aligned}$$

Remark 3.2.4. Notice that the limit operation is not always commute with the sum operation. To illustrate this we mention, that the expression from the limit exists for every harmonic function even if the integral at boundary diverges (see Theorem 3.2.7). So the notion of integral series extends the notion of integration of functions at boundary.

In the context of Theorem B it remains the following open questions: *What are the discrete analogs of Bessel functions generated by the coefficients λ_i? What properties do they have?*

3.2.3 Maximal proper cone integral formula

In this subsection we formulate the mean value property for certain nonharmonic functions.

MPC-integrals for C^{v-w}-summable functions

We start with the following definition.

Definition 31. We say that a functions f is C^{v-w}-*summable* if

$$\lim_{n\to\infty}\left(q^n f(C_{2n}^{v-w})\right) = \lim_{n\to\infty}\Big(\sum_{u\in C_{2n}^{v-w}}\frac{f(u)}{q^n}\Big) = 0.$$

Theorem B. *Consider two vertices $v, w \in \mathbb{T}_q$ connected by an edge, and let f be a C^{v-w}-summable function. Then*

$$f(v) = \int_{\partial C^{v-w}}\left[\sum_{i=0}^{\infty}\left((q+1)^i\Big(\gamma_i(\infty)+q^{\infty}\gamma_i(-\infty)\Big)\triangle^i f(t)\right)\right]dt,$$

where

$$\gamma_i(n) = c_{i,i}n^i + \ldots + c_{i,1}n + c_{i,0}, \tag{3.1}$$

whose collection of coefficients $c_{i,j}$ (for a fixed i) is the solution of the following linear system

$$A\begin{pmatrix} c_{i,i} \\ \vdots \\ c_{i,1} \\ c_{i,0}\end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1\end{pmatrix} \tag{3.2}$$

where

$$A = \begin{pmatrix} 0 & \dots & 0 & 2 \\ 1^i(1+(-1)^i q^1) & \dots & 1^1(1+(-1)^1 q^1) & 1^0(1+(-1)^0 q^1) \\ 2^i(1+(-1)^i q^2) & \dots & 2^1(1+(-1)^1 q^2) & 2^0(1+(-1)^0 q^2) \\ \vdots & \ddots & \vdots & \vdots \\ (i-1)^i(1+(-1)^i q^{i-1}) & \dots & (i-1)^1(1+(-1)^1 q^{i-1}) & (i-1)^0(1+(-1)^0 q^{i-1}) \\ i^i(1+(-1)^i q^i) & \dots & i^1(1+(-1)^1 q^i) & i^0(1+(-1)^0 q^i) \end{pmatrix}.$$

In addition, the existence of the integral in the right part of the equation is equivalent to the condition that f is C^{v-w}-summable.

Here we consider the integration in the following sense

$$\int\limits_{\partial T_q} \left[\sum_{i=0}^{\infty} \lambda_i(\infty) \triangle^i f(t) \right]_v dt = \lim_{n\to\infty} \left(\sum_{i=0}^{n} \lambda_i(n) \sum_{w \in S_n(v)} \triangle^i f(w) \right),$$

where $S_n(v)$ is the set of all vertices at distance n to the vertex v.

We prove this theorem later in Subsection 3.2.5.

Note that it would be interesting to relate the coefficients at terms $\triangle^i(f)$ with discretizations of Bessel functions.

Example 3.2.5. Let us check Theorem B for the function χ_v that is zero everywhere except for the point v and $\chi_v(v) = 1$. We have

$$\triangle^i \chi_v(C_n^{v-w}) = \frac{1}{q^n} \sum_{u \in C_n^{v-w}} f(u) = \begin{cases} 0, & \text{if } i < n; \\ \frac{1}{(q+1)^n}, & \text{if } i = n. \end{cases}$$

(notice that C_n^{v-w} contains exactly q^n vertices). Therefore,

$$\int\limits_{\partial C^{v-w}} \left[\sum_{i=0}^{\infty} \left((q+1)^i \Big(\gamma_i(\infty) + q^\infty \gamma_i(-\infty) \Big) \triangle^i \chi_v(t) \right) \right] dt$$
$$= \lim_{n\to\infty} a_{n,n} \triangle^n \chi_v(C_n^{v-w}) = \lim_{n\to\infty} (q+1)^n \frac{1}{(q+1)^n} = 1 = \chi_v(v).$$

It is clear from this example that it is not always possible to exchange the sum operator and the limit operator. For the function χ_v we have

$$\sum_{i=0}^{\infty} \lim_{n\to\infty} \left((q+1)^i \Big(\gamma_i(n) + q^n \gamma_i(-n) \Big) \triangle^i \chi_v(C_n^{v-w}) \right) = \sum_{i=0}^{\infty} 0 = 0 \neq 1 = \chi_v(v).$$

Let us write a weaker version of Theorem B for the integration over all the boundary.

Corollary 3.2.6. *Consider a vertex $v \in \mathbb{T}_q$, and let f be a C^{v-w}-summable function for all vertices w adjacent to v. Then*

$$f(v) = \frac{q}{q+1} \int\limits_{\partial \mathbb{T}_q} \left[\sum_{i=0}^{\infty} \left((q+1)^i \Big(\gamma_i(\infty) + q^{\infty} \gamma_i(-\infty) \Big) \triangle^i f(t) \right) \right]_v dt.$$

Proof. Let us sum up the expression obtained in Theorem B for all maximal proper cones with vertex at v. From one hand there are exactly $q+1$ such cones so the sum equals to $(q+1)f(v)$. From the other hand each point of the boundary was integrated q times. Therefore, we get the constant $\frac{q}{q+1}$ in the statement of the corollary. □

Remark. Note that it is possible to write similar series for arbitrary locally-finite trees, although the formulas for the coefficients would be more complicated.

MPC-integral formula for harmonic functions

We conclude this subsection with the following more general statement for harmonic functions.

Corollary 3.2.7. *Consider an arbitrary harmonic function h on a homogeneous tree $\mathbb{T}_q$. Let v be a vertex of $\mathbb{T}_q$ and C^{v-w} be one of its proper maximal cones. Then the following holds:*

$$h(v) = \int\limits_{\partial C^{v-w}} [h(t)]dt + \int\limits_{\partial C^{v-w}} \left[q^{\infty} \Big(h(t) - \int_{\partial C^{v-w}} [h(t)]dt \Big) \right] dt.$$

□

Remark 3.2.8. Suppose that h is integrable on ∂C^{v-w} with respect to the standard probability measure $d\mu$ on the boundary. Then

$$\int\limits_{\partial C^{v-w}} h d\mu = \int\limits_{\partial C^{v-w}} [h(t)] dt.$$

In case if h is not integrable with respect to probability measure, the MPC-integral nevertheless exists. In some sense MPC-integrability is an improper integrability with respect to integration over probability measure. MPC-integral exists for every harmonic function h and for every cone C^{v-w}.

We illustrate the last theorem by example.

Example 3.2.9. Let h be the harmonic function with the following values at the boundary

$$\begin{aligned} h(t) &\to |t|, & &\text{for } t \in \left[-\tfrac{2\pi}{3}, \tfrac{2\pi}{3}\right] \subset \partial\mathbb{T}_2; \\ 2^n\left(h - \int\limits_{\frac{-2\pi}{3}}^{\frac{2\pi}{3}} h(u) du\right) &\to t^2, & &\text{for } t \in \left[-\tfrac{2\pi}{3}, \tfrac{2\pi}{3}\right] \subset \partial\mathbb{T}_2. \end{aligned}$$

Let us calculate $h(v)$. Applying the last theorem to h (on C^{v-w}) we get

$$h(v) = \int\limits_{\frac{-2\pi}{3}}^{\frac{2\pi}{3}} |t| dt + \int\limits_{\partial C^{v-w}} \left[2^{\infty} \left(h(t) - \int\limits_{\frac{-2\pi}{3}}^{\frac{2\pi}{3}} h(u) d\, u \right) \right] dt.$$

Notice that

$$\int\limits_{\frac{-2\pi}{3}}^{\frac{2\pi}{3}} |t| dt = \frac{4\pi^2}{9} \quad \text{and} \quad \int\limits_{\frac{-2\pi}{3}}^{\frac{2\pi}{3}} t^2 dt = \frac{16\pi^3}{81}.$$

Hence, we have

$$h(v) = \frac{4\pi^2}{9} + \int\limits_{\partial C^{v-w}} \left[2^{\infty} \left(h(t) - \int\limits_{\frac{-2\pi}{3}}^{\frac{2\pi}{3}} h(u) d\, u \right) \right] dt = \frac{4\pi^2}{9} + \frac{16\pi^3}{81}.$$

So, we have calculated the value of the harmonic function h in vertex v

$$h(v) = \frac{4\pi^2}{9} + \frac{16\pi^3}{81}.$$

3.2.4 Relations on special Laurent polynomial

In this subsection we prove some supplementary statements. For every integer n we denote

$$D_n(x) = x^n + \frac{q^n}{x^n}.$$

Note that $D_0(x) = x^0 + \frac{q^0}{x^0}$.

For every nonnegative integer we set

$$S_n(x) = \frac{(x-1)^n(x-q)^n}{(q+1)^n x^n}.$$

We have the following recurrent relation for the defined above Laurent polynomials.

Proposition 3.2.10. *For every integer n we have*

$$S_1 D_n = \frac{D_{n+1} - (q+1)D_n + qD_{n-1}}{q+1}.$$

Proof. For every integer n (including $n = -1, 0, 1$) it holds

$$\begin{aligned}
S_1 D_n &= \left(\frac{(x-1)(x-q)}{(q+1)x}\right)\left(x^n + \frac{q^n}{x^n}\right) \\
&= \frac{x^{n+1}}{q+1} - x^n + \frac{q}{q+1}x^{n-1} + \frac{q^n}{(q+1)x^{n-1}} - \frac{q^n}{x^n} + \frac{q^{n+1}}{(q+1)x^{n+1}} \\
&= \frac{1}{q+1}\left(x^{n+1} + \frac{q^{n+1}}{x^{n+1}}\right) - \left(x^n + \frac{q^n}{x^n}\right) + \frac{q}{q+1}\left(x^{n-1} + \frac{q^{n-1}}{x^{n-1}}\right) \\
&= \frac{D_{n+1} - (q+1)D_n + qD_{n-1}}{q+1}.
\end{aligned}$$

□

The following proposition is straightforward.

Proposition 3.2.11. *For every integer n there exists a unique decomposition*

$$D_n = \sum_{i=0}^{n} a_{n,i} S_i.$$

□

Now we are interested in the coefficients $a_{n,i}$. The next statement follows directly from Proposition 3.2.10.

Corollary 3.2.12. *For every positive integer i and every integer n it holds*

$$a_{n,i-1} = \frac{a_{n+1,i} - (q+1)a_{n,i} + qa_{n-1,i}}{q+1}.$$

Additionally in the case $i = 0$ it holds

$$0 = a_{n+1,0} - (q+1)a_{n,0} + qa_{n-1,0}.$$

Proof. By the definition we have

$$S_1 S_k = S_{k+1}.$$

Propositions 3.2.10 and 3.2.11 imply

$$\begin{aligned}\sum_{i=1}^{n+1} a_{n,i-1} S_i = S_1 D_n &= \frac{D_{n+1} - (q+1)D_n + qD_{n-1}}{q+1} \\ &= \frac{1}{q+1}\Big(\sum_{i=0}^{n+1} a_{n+1,i} S_i - (q+1)\sum_{i=0}^{n} a_{n,i} S_i + q\sum_{i=0}^{n-1} a_{n-1,i} S_i\Big).\end{aligned}$$

Collecting the coefficients at S_i we get the recurrence relations of the corollary. □

Definition 32. For a positive integer k we define the linear form L_k in $2k+1$ variables as follows

$$L_k(y_1, \dots, y_{2k+1}) = \sum_{i=-n}^{n} c_{i,n} y_i,$$

where $c_{i,n}$ are defined as the coefficients of S_n, i.e., from the expression

$$S_n(x) = \frac{(x-1)^n (x-q)^n}{(q+1)^n x^n} = \sum_{i=-n}^{n} c_{i,n} x^i.$$

Proposition 3.2.13. *For every nonnegative integer i and every integer n we have*

$$L_i(a_{n-i,i}, a_{n-i+1,i}, \dots, a_{n+i,i}) = 0.$$

Proof. We prove the proposition by induction in i.

Base of induction. For the case $i = 0$ the statement holds by Corollary 3.2.12.

Step of induction. Suppose that the statement holds for $i - 1$. Let us prove it for i. We have

$$L_i(a_{n-i,i}, \dots, a_{n+i,i}) = 0.$$

By Corollary 3.2.12 and linearity of L_i we have

$$\begin{aligned}
&L_i(a_{n-i,i}, \dots, a_{n+i,i}) \\
&\quad = L_i\big(\tfrac{a_{n-i+1,i+1}-(q+1)a_{n-i,i+1}+qa_{n-i-1,i+1}}{q+1}, \dots, \\
&\qquad\qquad \tfrac{a_{n+i+1,i+1}-(q+1)a_{n-i,i+1}+qa_{n-i-1,i+1}}{q+1}\big) \\
&\quad = \frac{1}{q+1}\Big(L_i(a_{n-i+1,i+1}, \dots, a_{n+i+1,i+1}) - (q+1)L_i(a_{n-i,i+1}, \dots, a_{n+i,i+1}) \\
&\qquad + qL_i(a_{n-i-1,i+1}, \dots, a_{n+i-1,i+1})\Big) \\
&\quad = L_{i+1}(a_{n-i-1,i+1}, a_{n-i,i+1}, \dots, a_{n+i,i+1}, a_{n+i+1,i+1}).
\end{aligned}$$

Therefore, by induction assumption we have

$$\begin{aligned}
L_{i+1}(a_{n-i-1,i+1}, a_{n-i,i+1}, \dots, a_{n+i,i+1}, a_{n+i+1,i+1}) &= L_i(a_{n-i,i}, \dots, a_{n+i,i}) \\
&= 0.
\end{aligned}$$

This concludes the proof of the induction step. $\square$

Corollary 3.2.14. *For every fixed nonnegative integer k we have the*

$$a_{n,k} = P_k(n) + q^n \hat{P}_k(n),$$

where $P_k(n)$ and $\hat{P}_k(n)$ are polynomials of degree at most k. $\square$

We skip the proof here. This is a general statement about linear recursive sequences whose characteristic polynomial has roots 1 and q both of multiplicity n.

Example 3.2.15. Direct calculations show that in case $q = 2$ we have

$$\begin{aligned}
a_{n,0} &= 1 + 2^n, \\
a_{n,1} &= \tfrac{3^1}{1!}(-n + 2^n n), \\
a_{n,2} &= \tfrac{3^2}{2!}(n^2 + 3n + 2^n(n^2 - 3n)), \\
a_{n,3} &= \tfrac{3^3}{3!}\big(-n^3 - 9n^2 - 26n + 2^n(n^3 - 9n^2 + 26n)\big), \\
&\dots
\end{aligned}$$

Let us prove a general theorem on numbers $a_{n,i}$.

Theorem 3.2.16. *For every admissible k and n it holds*

$$a_{n,k} = (q+1)^k \big(\gamma_k(n) + q^n \gamma_k(-n)\big),$$

where the coefficients of γ_k are defined by System (3.2).

We start the proof of Theorem 3.2.16 with the following two lemmas.

Lemma 3.2.17. *For every nonnegative integer k and every n we have*

$$\hat{P}_k(-n) = P_k(n).$$

Proof. For every integer x we have

$$D_{-n} = x^{-n} + \frac{q^{-n}}{x^{-n}} = \frac{1}{q^n}\Big(\frac{q^n}{x^n} + x^n\Big) = \frac{D_n}{q^n}.$$

By Proposition 3.2.11 the coefficients $a_{n,i}$ and $a_{-n,i}$ are uniquely defined, therefore,

$$a_{n,k} = q^n a_{-n,k}.$$

Let us rewrite this equality in terms of polynomials P_k and $\hat{P}_k$:

$$P_k(n) + q^n \hat{P}_k(n) = q^n (P_k(-n) + q^{-n} \hat{P}_k(-n)),$$

and hence

$$P_k(n) + q^n \hat{P}_k(n) = \hat{P}_k(-n) + q^n P_k(-n).$$

Since this equality is fulfilled for every n we have $\hat{P}_k(-n) = P_k(n)$. This concludes the proof. $\square$

Lemma 3.2.18. *For every nonnegative k it holds*

$$P_k(k) + q^k P_k(-k) = (q+1)^k$$

Proof. We prove the proposition by induction in k.

Base of induction. For the case $k = 0, 1$ we have

$$P_0(0) + q^0 P_0(0) = a_{0,0} = 1 \qquad \text{and} \qquad P_1(1) + qP_1(-1) = a_{1,1} = q + 1.$$

Step of induction. Let $P_k(k) + q^k P_k(-k) = (q+1)^k$. Then

$$\begin{aligned}(q+1)^k = P_k(k) + q^k P_k(-k) = a_{k,k} &= \frac{a_{k+1,k+1} - (q+1)a_{k,k+1} + qa_{k-1,k+1}}{q+1} \\ = \frac{a_{k+1,k+1}}{q+1} &= \frac{P_{k+1}(k{+}1) + q^{k+1}P_{k+1}(-k{-}1)}{q+1}.\end{aligned}$$

The third equality follows from the recursive formula of Corollary 3.2.12. Hence

$$P_{k+1}(k{+}1) + q^{k+1}P_{k+1}(-k{-}1) = (q+1)^{k+1}.$$

This concludes the step of induction. □

Proof of Theorem 3.2.16. From Lemma 3.2.17 we know that $\hat{P}_k(-n) = P_k(n)$. In addition, by Corollary 3.2.14 the degree of P_k equals to k, and hence it has $k+1$ coefficient. The coefficients of the polynomial P_k are uniquely defined by the conditions for $a_{j,k}$ for $j = 0, \ldots, k$:

$$P_k(j) + q^j P_k(-j) = \begin{cases} 0, & \text{for } j = 0, \ldots, k-1, \\ (q+1)^k, & \text{for } j = k. \end{cases}$$

The expression for k follows from Lemma 3.2.18. We consider these equalities as linear conditions on the coefficients of the polynomial $\frac{P_k}{(q+1)^k}$. These conditions form a linear system, which coincides with System (3.2) (substituting k to i).

We should also show that the determinant of the matrix in System (3.2) is nonzero. We prove this by reductio ad absurdum. Suppose the determinant of the matrix is zero. Thus, it has a nonzero kernel. Therefore, there exists an expression

$$R(n) = r(n) + r(-n)q^n,$$

where $r(n)$ is a polynomial of degree k having at least one nonzero coefficient, satisfying

$$R(-k) = R(-k+1) = \ldots = R(k) = 0.$$

Let $R(k+1) = a$. Let us find the value $R(-k-1)$. From one hand, our sequence satisfy the linear recursion condition determined by the coefficients of the polynomial $(x-1)^k(x-q)^k$, and hence

$$R(-k-1) = -\frac{a}{q^{k+1}}.$$

From another hand,

$$R(-k-1) = r(-k-1) + r(k+1)q^{-k-1} = \frac{r(k+1) + r(-k-1)q^{k+1}}{q^{k+1}} = \frac{a}{q^{k+1}}.$$

This implies that $a = 0$, and hence $R(k+1) = R(-k-1) = 0$.

Therefore, the linear recursive sequence $R(n)$ determined by the coefficients of the polynomial of degree $2k+3$ has $2k+3$ consequent elements equal zero. Hence $R(n) = 0$ for any integer n, which implies that all the coefficients of $r(n)$ equal zero. We come to the contradiction. Hence the determinant of the matrix in System (3.2) is nonzero.

So both the coefficients of $\frac{P_k}{(q+1)^k}$ and the coefficients of γ_k are solutions of System (3.2). Since System (3.2) has a unique solution, the polynomials P_k and $(q+1)^k\gamma_k$ coincide. Therefore, by Lemma 3.2.17 it holds

$$a_{n,k} = P_k(n) + q^n\hat{P}_k(n) = P_k(n) + q^n P_k(-n) = (q+1)^k\big(\gamma_k(n) + q^n\gamma_k(-n)\big).$$

This concludes the proof of Theorem 3.2.16. □

Observe the following corollary.

Corollary 3.2.19. *For every integer $k > 0$ we have $P_k(0) = 0$, and $P_0(1) = 1$.* □

3.2.5 Proof of Theorem B

Finally we have all necessary tools to prove of Theorem B. We start with the following lemma.

Lemma 3.2.20. *Let f be a function on $\mathbb{T}_q$ and v, w be two vertices of $\mathbb{T}_q$ connected by an edge. Then for every nonnegative n it holds*

$$f(v) + q^n f(C_{2n}^{v-w}) = \sum_{k=0}^{n} \Big((q+1)^k \big(\gamma_k(n) + q^n \gamma_k(-n)\big) \triangle^k f(C_n^{v-w}) \Big).$$

Proof. For $0 < k \leq n$ set

$$\hat{D}_{k,n} = f(C_{n-k}^{v-w}) + q^k f(C_{n+k}^{v-w}),$$
$$\hat{S}_{k,n} = \sum_{i=-k}^{k} c_{i,k} f(C_{n+i}^{v-w}),$$

where the coefficients $c_{i,k}$ are generated by

$$S_k = \frac{\big((x-1)(x-q)\big)^k}{(q+1)^k x^k} = \sum_{i=-k}^{k} c_{i,k} x^i.$$

Notice that all linear expressions over S_k and D_k are identically translated to the linear expressions over $\hat{S}_{k,n}$ and $\hat{D}_{k,n}$. Then from Proposition 3.2.11 it follows

$$f(v) + q^n f(C_{2n}^{v-w}) = \hat{D}_{n,n} = \sum_{k=0}^{n} a_{n,k} \hat{S}_{k,n},$$

where the coefficients $a_{n,k}$ as in Theorem 3.2.16, i.e.,

$$a_{n,k} = (q+1)^k (\gamma_k(n) + q^n \gamma_k(-n)),$$

where the coefficients of γ_k are defined by System (3.2). In addition note that

$$\hat{S}_{k,n} = \triangle^k (C_n^{v-w}).$$

Therefore, we obtain

$$f(v) + q^n f(C_{2n}^{v-w}) = \sum_{k=0}^{n} \Big((q+1)^k \big(\gamma_k(n) + q^n \gamma_k(-n)\big) \triangle^k f(C_n^{v-w}) \Big).$$

This concludes the proof. $\square$

Proof of Theorem B. From Lemma 3.2.20 we have

$$f(v) + q^n f(C_{2n}^{v-w}) = \sum_{i=0}^{n} \Big((q+1)^i \big(\gamma_i(n) + q^n \gamma_i(-n)\big) \triangle^i f(C_n^{v-w}) \Big).$$

Hence,

$$\begin{aligned}
\int_{\partial C^{v-w}} & \left[\sum_{i=0}^{\infty}\left((q+1)^i\Big(\gamma_i(\infty)+q^{\infty}\gamma_i(-\infty)\Big)\triangle^i f(t)\right)\right] dt \\
&= \lim_{n\to\infty}\sum_{i=0}^{n}\left((q+1)^i\Big(\gamma_i(n)+q^n\gamma_i(-n)\Big)\triangle^i f(C_n^{v-w})\right) \\
&= \lim_{n\to\infty}\left(f(v)+q^n f(C_{2n}^{v-w})\right) = f(v)+\lim_{n\to\infty}\left(q^n f(C_{2n}^{v-w})\right) \\
&= f(v).
\end{aligned}$$

Therefore, the integral converges to the value $f(v)$ if and only if the sequence $\left(q^n f(C_{2n}^{v-w})\right)$ converges to zero as n tends to infinity. This means that f is C^{v-w}-summable. This concludes the proof. $\square$

3.2.6 Laplace-Dirichlet series in discrete settings

Finally, we introduce the notions of discrete Laplace-Dirichlet series. We do it very briefly, since the definitions almost literally repeat the corresponding ones in the Euclidean case. For an arbitrary function f we denote by $harm(f)$ a harmonic function coinciding with f at the boundary (in case of existence).

Definition 33. The discrete *principal inverse Laplace operator* $\triangle^{-1}$ is as follows. Consider a function f on $\mathbb{T}_q$ such that $harm(f)$ exists. Let a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfy the Poisson equation $\triangle\varphi = f$. Set

$$\triangle^{-1}(f) = \varphi - harm(\varphi).$$

Remark 3.2.21. As in Euclidean case $\triangle^{-1}(f)$ does not depend on the choice of φ in the definition. Similarly we have

$$\triangle(\triangle^{-1}(f)) = f, \quad \text{and} \quad \triangle^{-1}(\triangle(u)) - u = -harm(u).$$

Definition 34. For a sequence of harmonic functions h_n we say that the sum

$$\sum_{n=0}^{\infty}\triangle^{-n}(h_n),$$

is *Laplace-Dirichlet series*.
Suppose that the harmonic functions h_n are defined by some function f as $h_n = \triangle^n(f)$, then we say that the corresponding sum is the *Laplace-Dirichlet series* for f.
We say that a function f is $\triangle$*-analytic* if it coincides with its Laplace-Dirichlet series.

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Index of Notation

Unit disk

$\mathbb{D}$	unit disk
$\partial\mathbb{D}$	unit circle
$\triangle$	Laplace operator
z, w	point on $\mathbb{D}$
ζ, ξ	point on $\partial\mathbb{D}$
$P(z, \zeta)$	Poisson kernel with $\lvert z\rvert < 1$ and $\lvert\zeta\rvert = 1$
$P_{\mathbb{D}}\phi$	Poisson integral of ϕ on $\mathbb{D}$
p_μ	potential of the measure μ
$K(z - \zeta)$	Newtonian kernel
μ_u	Riesz measure associated with a subharmonic function u
$G_{\mathbb{D}}$	Green function

Homogeneous tree

G	locally finite, connected graph
$\mathbb{T}_q$	homogeneous tree of degree q
o	origin of $\mathbb{T}_q$
$\partial\mathbb{T}_q$	boundary of $\mathbb{T}_q$
ν_o	measure on $\partial\mathbb{T}_q$
μ_u	Riesz measure associated with a subharmonic function u
$G(v,w\|x)$	Green kernel associated with the simple random walk on $\mathbb{T}_q$
$K(v,w\|x)$	Martin Kernel

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