Diplomarbeit in Technischer Mathematik
TU Graz

Case Studies and Problems in Experimental Design

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Ich erkläre, dass ich diese Arbeit selbst verfasst, alle verwendeten Quellen zitiert und mich keiner unerlaubten Hilfsmittel bedient habe.

February 16, 2011,

Unterschrift

Acknowledgement

I would like to thank all the people who have helped me during the work. I owe the greatest debt of gratitude to my advisor Dr. Ernst Stadlober for his great and invaluable help and his scholarly pieces of advice. More over, my gratitude to all the professors who have helped me through the courses. Finally, warm thanks to my family, my beloved parents who made it all possible and my sister Ladan, whose love and help were so important during the work on this thesis.

Abstract

The present diploma thesis is about planning and conducting experiments by means of experimental design. It is a collection of well documented problems and case studies employing one way analysis of variance, randomized complete block designs, analysis of covariance, full factorial 2^k designs, fractional factorial 2^{k-p} designs, full factorial 3^k designs and Taguchi designs.
Each study follows the same format: (i) description of the problem, (ii) choosing an adequate statistical model, (iii) statistical modeling, analysis and interpretation, (iv) problem specific summary, (v) listing of the `R` source code.
All analyses has been carried out with the public domain statistical package `R` (Version 2.11.1).

Zusammenfassung

Die vorliegende Diplomarbeit beschäftigt sich mit der Planung und Ausführung von Experimenten mit Hilfe von Methoden der Versuchsplanung. Sie ist ein Sammlung von gut dokumentierten Problemen und Fallstudien bei denen folgende Methoden eingesetzt werden: einfache Varianzanalyse, randomisierte vollständige Blockversuchspläne, Kovarianzanalyse, vollständige faktorielle 2^k Versuchspläne, fraktionierte faktorielle 2^{k-p} Versuchspläne, vollständige faktorielle 3^k Versuchspläne und Taguchi Versuchspläne.
Jede Studie ist nach demselben Schema aufgebaut: (i) Beschreibung des Problems, (ii) Auswahl eines adäquaten statistischen Modells, (iii) statistische Modellierung, Analyse und Interpretation, (iv) problemspezifische Zusammenfassung, (v) Auflistung des `R` Quellcodes.
Sämtliche Analysen werden mit dem statistischen Programmpaket `R` (Version 2.11.1), das lizenzfrei zur Verfügung steht, durchgeführt.

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Chapter 1

Introduction

Experiments are performed by investigators in virtually all fields of inquiry, usually to discover something about particular processes or systems. Literally, an experiment is a test. More formally, we can define an experiment as a test or series of tests in which purposeful changes are made to the input variables of a processor system so that we may observe and identify the response for changes in the output response [Montgomery D. C. (1997)].

The design and analysis of experiments revolves around the understanding of the effects of different variables on other variables. The dependent variable is called response and the independent variables are called factors. Experiments are run at different factor values, called levels. Each run of the experiment involves a combination of the levels of the investigated factors. Each of the combinations are referred to as a treatment. There are different types of experiment: **One factor designs**, **factorial designs**, **response surface methods** and reliability of design of experiments.

One factor designs: These are the designs where only one factor is under investigation, and the objective is to determine whether the response is significantly different at different factor levels; and only the effect of the factor on the response can be determined.

Factorial designs: In factorial designs, multiple factors are investigated simultaneously during the test. As in one factor designs, qualitative and/ or quantitative factors can be considered. The objective of these designs is to identify the factors that have a significant effect on the response, as well as the effect of the interactions.

General full factorial design: In general full factorial designs, each factor can have a different number of levels, and the factors can be quantitative or qualitative or both.

Two level full factorial design: These are factorial designs where the number of the factors is restricted to two. Restricting the levels to two and running a full factorial experiment reduces the number of treatments, and allows for the investigation of all the factors and their interactions. If all factors are quantitative, then the data from such experiments can be used for predictive purposes, provided a linear model is appropriate for modeling the response.

Two level fractional factorial design: This is a special category of two level designs, where not all factor level combinations are considered, and the experimenter can choose which combinations are to be excluded. Based on the excluded combinations, certain combinations can not be determined.

Three level factorial designs and three level fractional factorial designs: These designs are the same as two level factorial and two level fractional factorial designs, where the number of factors is restricted to three.

Taguchi's orthogonal arrays: Taguchi's orthogonal arrays are highly fractional designs, used to estimate main effects based on only a few experimental runs. These designs are not only applicable to two level factorial experiments, but also can investigate main effects when factors have more than two levels. Designs are also available to investigate the main effects of certain mixed level experiments where the factors included do not have the same number of levels.

This thesis explains a summary of conducting experiments and analyze the resulting data so that valid and objective conclusions are obtained. Experiments often involve several factors. Usually, an objective of the person conducting the experiment, called the experimenter, is to determine the influence that these factors have on the output response of the system. The general approach to planning and conducting the experiments called the strategy of experimentation.

The late Sir Ronald A. Fisher was the innovator in the use of statistical methods in experimental design. Fisher developed and first used the analysis of variance as the primary method of statistical analysis in experimental design. Many of early applications of experimental design methods were in the agricultural and biological sciences, and as a result, much of the terminology of the field is derived from this heritage. The first industrial applications of experimental design began to appear in the 1930s, initially in the British textile and wooden industry. After world war II, experimental design methods were introduced to the chemical and process industries in the United States and Western Europe. These industry groups are still very fertile areas for using the experimental design in product and process development [Montgomery D. C. (1997)].

In this thesis we will give a brief summary of the different methods from a simplest one way ANOVA (Chapter 2, [Winer B. J. , Brown D. R., Michels K. M. (1991)], [Stadlober E. (2010)]) up to the most complicated case Taguchi design (Chapter 8, [Dean A., Voss D. (1999)]) used in experimental designs. The main focus is to deliver corresponding test studies which starts with the description of the problem and end up with a problem specific summary and the corresponding R-code. The software R (version 2.11) is used for analyzing the different problems [Crawley M. J. (2005)] and [Dalgaard P. (2002)].

Chapter 2

One Way ANOVA

2.1 Model

One-way analysis of variance is a method of testing differences between more than two groups or treatments. Multiple comparison procedures and orthogonal contrasts are described as methods for identifying specific differences between pairs of treatments.

Analysis of variance (often referred to as ANOVA) is a technique for analyzing the way in which the mean of a variable is affected by different types and combinations of factors. One-way analysis of variance is the simplest form. It is an extension of the independent t-test and can be used to compare any number of groups or treatments. This method can be used, for example, in the analysis of the effect of three different diets on total serum cholesterol or in the investigation into the extent to which severity of illness is related to occurrence of infection.

Let we have a treatments that we wish to compare. The observed response from each of a treatments is a random variable. Our linear statistical model seems to be:

$$y_{ij} \;=\; \mu + \tau_i + \varepsilon_{ij}, \quad i = 1, 2, \ldots, a\,; \quad j = 1, 2, \ldots, n_i\,; \quad n = \sum_{i=1}^{a} n_i,$$

$$\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$$

where y_{ij} is the ij^{th} observation associated to treatment (or group) i, μ is the treatment mean, and ε_{ij} is a random error with normal distribution.

This model is called a One-Way Analysis of Variance, since the model includes only one major source of variation, namely; the treatment effect, and because the standard analysis of data using this model involves a comparison of measures of variation.

In an experiment involving **a** treatments, an obvious question is wether or not the treatments differ at all in terms of their effects and the response variable. The most important hypothesis test in this case is:

$$\begin{cases} H_0 : \tau_1 = \tau_2 = ... = \tau_a = 0 \\ H_1 : \exists i, \tau_i \neq 0 \end{cases}$$

Source	Sum of Squares (SS)	Degrees of Freedom (d.f)	Mean Square	F
Treat ment A	$SS_A = \sum_{i=1}^{a} n_i(\bar{y}_i - \bar{y})^2$	$a-1$	$MS_T = \frac{SS_T}{k-1}$	$F_0 = \frac{MS_T}{MS_E}$
Error	$SS_E = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2$	$n-k$	$MSE = \frac{SS_E}{n-a}$	
Total	$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(y_{ij} - \bar{y})^2$	$n-1$		

We have

$$\overline{y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij}; \quad \overline{y} = \frac{1}{a}\sum_{i=1}^{a} n_i\overline{y}_i = \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}.$$

The total sum of squares is the sum of squares between groups plus the sum of squares of within groups:

$$SS_T = SS_A + SS_E$$

where

$$SS_A = \sum_i \frac{1}{n_i}\left(\sum_j y_{ij}\right)^2 - \frac{1}{n}\left(\sum_i\sum_j y_{ij}\right)^2 = \sum_i n_i(\overline{y}_i - \overline{y})^2 = n\sum_i(\overline{y}_i - \overline{y})^2$$

$$SS_E = \sum_i \sum_j y_{ij}^2 - \sum_i \frac{1}{n_i} \left(\sum_j y_{ij} \right)^2 = \sum_i \sum_j (\overline{y}_{ij} - \overline{y}_i)^2$$

$$SS_T = \sum_i \sum_j y_{ij}^2 - \frac{1}{n} \left(\sum_i \sum_j y_{ij} \right)^2 = \sum_i \sum_j (\overline{y}_{ij} - \overline{y})^2$$

Note that if the null hypothesis H_0 is false, and treatment effects differ, then the SS_E under the full model is considerably smaller than SS_{E_0} under the reduced model ($y_{ij} = \mu + \varepsilon_{ij}$). Otherwise if the null hypothesis is true, then SS_E and SS_{E_0} will be very similar.

F-Test: The hypothesis H_0 is tested by the F-Test, where

$$F = \frac{MS_T}{MS_E} \sim F_{a-1,n-a} \quad \text{if } H_0 \text{ is true}$$

F-distribution with degrees of freedom $a - 1$ and $n - a$.

H_0 is rejected if the probability $P(F > f_0) = P < \alpha$, where f_0 is the realized value of the test statistic F and $\alpha \leq 0.05$.

2.2 Cement

Example 2.1. (From D. C. Montgomery, (1997), P.117, Problem 3-1 and 3-3) The tensile strength y of Portland cement is being studied. Four different mixing techniques can be used economically. The following data have been collected.

Mixing Technique	Tensile strength(lb/in^2)			
1	3129	3000	2865	2890
2	3200	3300	2975	3150
3	2800	2900	2985	3050
4	2600	2700	2600	2765

(a) Test the hypothesis that mixing techniques affect the strength of the cement. Use $\alpha = 0.05$.

(b) Use Tukeys test with $\alpha = 0.05$ to make comparisons between pairs of means.

(c) Construct a graphical display to compare the mean tensile strengths for the four mixing techniques. What are your conclusions?

(d) Construct a normal probability plot of the residuals. What conclusion would you draw about the validity of the normality assumption?

(e) Plot the residuals versus the predicted tensile strength. Comment the plot.

(f) Prepare box plots of the results to aid the interpretation of the results of this experiment.

In this example the four mixing techniques are factors and response y is the strength of Portland cement. This is an example of a single-factor experiment with $a = 4$ levels of the factor and $n_i = 4, i = 1, ..., 4$, replicates. The $n = 16$ runs are made in a random order. The randomized test sequence is necessary to prevent the effects of unknown nuisance variables. Questions: Does changing the mixing techniques (level) change the mean strength? If so, is there a level which results in the maximum mean strength? To answer this question we carry out an ANOVA.

Figure 2.1 on page 9 shows a scatter plot and a series of boxplots, the scatter plot indicates that the answers are "yes" and "2". The boxplots show that the mixing technique 2 have a much higher strength of Portland cement. Furthermore, the boxplot indicates that the strength of Portland cement increases as the mixing techniques differs; up to about

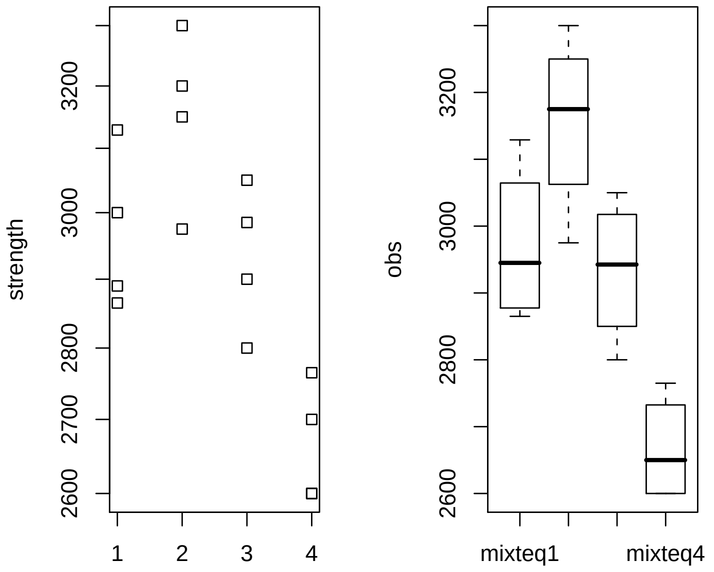


Figure 2.1: Scatter plot and Box plot Series

mixing techniques 2. Beyond mixing technique 2; there is a marked decrease in strength of Portland cement. There is no strong evidence to suggest that the variability in strength of Portland cement around the average depends on the mixing techniques. Based on this simple graphical analysis, we strongly suspect that (1) mixing techniques affects the strength of Portland cement and (2) in mixing technique 2 we have a maximum strength.

```
Analysis of Variance Table
Response: strength
           Df Sum Sq Mean Sq F value    Pr(>F)
mixteq     3  489740  163247  12.728 0.0004887 ***
Residuals  12 153908   12826
```

According to the ANOVA-table; the large value of $f_0 = 12.728 > 3.490295$ indicates that the hypothesis of treatment effects should be rejected. The corresponding P-value $P(F(12, 3) > 12.728) = 0.0004887$ is very small. It appears that (at least one of) the treatment effects are significantly different. Note that the P-value of a test is the smallest choice of α that would allow the null hypothesis to be rejected, reject H_0: if P$< \alpha$. So it means that for any α larger than 0.0004887 the null hypothesis is rejected. Also we can conclude from the above results that the mixing techniques have an effect on the strength of the Portland Cement. The tables of effects $\hat{\tau}_i = \overline{y}_i - \overline{y}$, computes tables of results from a fitted model.

```
Tables of effects
mixteq
  1       2      3       4
39.19  224.44  1.94 -265.56
```

Plot of design gives univariate effects of a design or a model; the supplied function will be called once for each level of each factor in the design and the plot will show these summary values. The levels of a particular factor are shown along a vertical line and the overall mean value for the response is drawn as a horizontal line. Figure 2.2 shows the plot of effects. We see that the mixing technique 4 has the largest negative effect and mixing technique 2 has the largest positive effect on strength of the Portland cement.

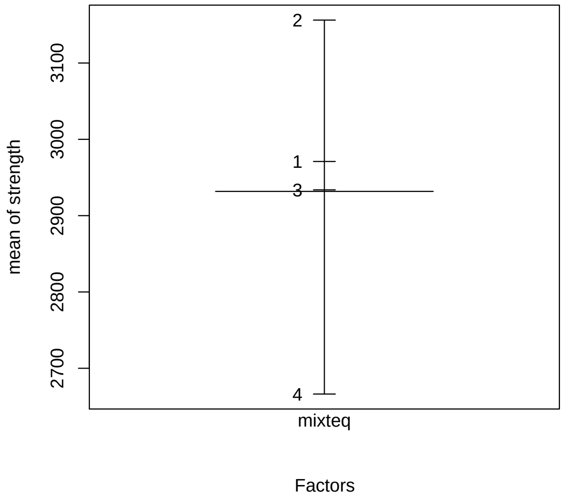


Figure 2.2: Cement, Plot of Effects

Now we are interested in the behavior of the residuals. We are interested to find out if the residuals are normally distributed or not. We see that the mixing technique 4 has the smallest effect and mixing technique 2 has the largest effect on strength of the Portland cement.

An extremely useful procedure is to construct a normal Q-Q-plot of the residuals to check the assumptions of the normality. If the underlying error distribution is normal, this plot will resemble a straight line. If the model is correct and if the assumptions are satisfied, the residuals should be structure less; in particular, they should be unrelated to any other variable including the predicted response. A simple check is to plot the residuals versus the fitted values.

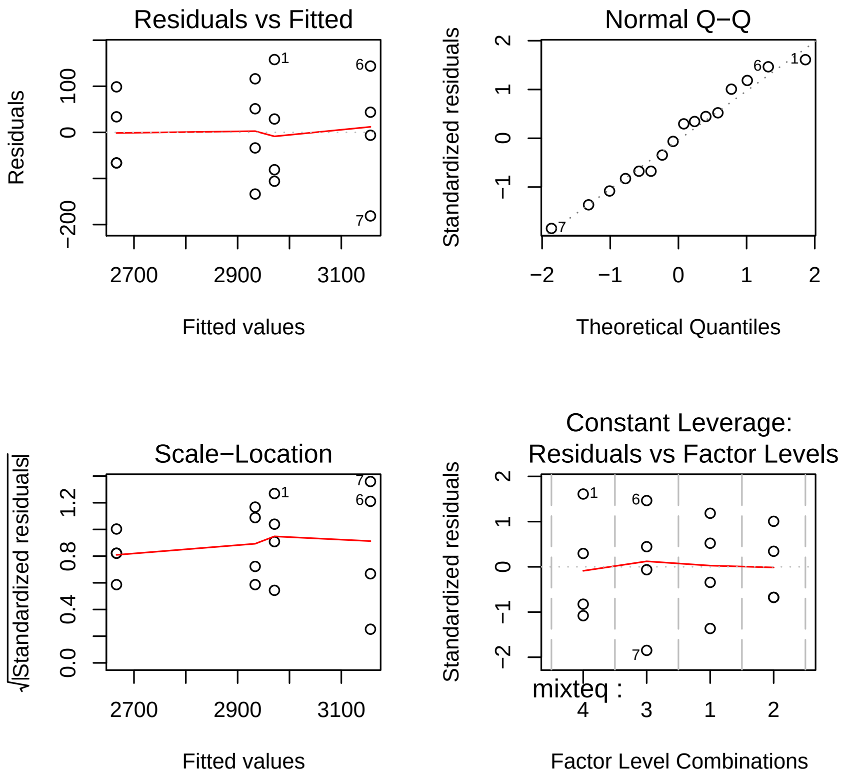


Figure 2.3: Cement, The four diagnostic plots

In figure 2.3 you can see four diagnostic plots. The plot of residuals against the fitted values (row1, column1): from this plot we can inference about the non constancy of an error variance. It shows similar spread for all levels. The Q-Q plot (row1, column2) shows us that the residuals are normally distributed. Now we will do the residuals normality test, one of these non parametric tests is the Shapiro-Wilk test. It tests if the data are from a normal distribution. Here we will use both the Shapiro-Wilk and the Kolmogorov-Smirnov test.

```
1) Shapiro-Wilk normality test
data:  residuals(cement.fit)
W = 0.9705, p-value = 0.846

2) One-sample Kolmogorov-Smirnov test
data:  residuals(cement.fit)
```

```
D = 0.1185, p-value = 0.9783
alternative hypothesis: two-sided
```

As the P-values are large $>> \alpha$ we can say that the null hypothesis that the residuals are normally distributed can not be rejected. In `R` we test the homogeneity of variances by the Fligner-Killeen test. It performs a median test of the null that the variances in each of the groups (samples) are the same. The Fligner-Killeen (median) test has been determined in a simulation study as one of the many tests for homogeneity of variances which is most robust against deviations from the normal distribution.

```
Fligner-Killeen test of homogeneity of variances
data:  strength by mixteq
Fligner-Killeen:med chi-squared = 0.8785, df = 3, p-value = 0.8306
```

We can conclude from the results above that the four variances aren't significant different. The Kruskal-Wallis test is used to test the null hypothesis that the a treatments are identical against the alternative hypothesis that some of the treatments generate observations that are larger than others. The Kruskal-Wallis test is a nonparametric alterantive to the usual analysis of variance. It performs a rank sum test of the null that the location parameters are the same in each group. The alternative is that they differ in at least one.

```
Kruskal-Wallis rank sum test
data:  strength by mixteq
Kruskal-Wallis chi-squared = 10.891, df = 3, p-value = 0.01233
```

What we have learned about these data so far? The F-test and the more robust Kruskal-Wallis test both claimed that the treatment effects are significant, the data are normally distributed and homogeneity of variances can be assumed.

We can now go to answer our second question, which of the treatment means ($\mu_i = \mu + \tau_i$) are significantly different? Can we say with any confidence that a particular one is largest? This is a problem of "multiple comparisons" comparing several means with each other. Hereby we can use TukeyHSD Confidence Intervals. The TukeyHSD creates a set of confidence intervals on the differences between the means of the levels of a factor with the specified family-wise probability of coverage. The intervals are based on the Studentized range statistic and Tukey's "Honest Significant Difference" method. The Hypothesis $\mu_i = \mu_j$, $i = 1, ..., a$; $j = 1, ..., a$; $i \neq j$ at the level of significance α will be rejected, when the confidence intervals for the difference $\mu_i - \mu_j$ do not contain zero.

```
Tukey multiple comparisons of means
    95\% family-wise confidence level
Fit: aov(formula = strength ~ mixteq)
mixteq
```

```
        diff    lwr        upr      p adj
2-1  185.25  -52.50029  423.00029 0.1493561
3-1  -37.25 -275.00029  200.50029 0.9652776
4-1 -304.75 -542.50029  -66.99971 0.0115923
3-2 -222.50 -460.25029   15.25029 0.0693027
4-2 -490.00 -727.75029 -252.24971 0.0002622
4-3 -267.50 -505.25029  -29.74971 0.0261838
```

Figure 2.4 shows simultaneous confidence intervals (level $\alpha = 0.95$) for $\mu_2 - \mu_1$, $\mu_3 - \mu_1$, $\mu_4 - \mu_1$, $\mu_3 - \mu_2$, $\mu_4 - \mu_2$, $\mu_4 - \mu_3$ they are (-52.5,423), (-275,200), (-542.5,-67), (-460,15), (-727.7,-252), (-505,-29.7). While the confidence intervals 4-1, 4-2,4-3 doesn't contain zero, we can conclude that between the mixing technique 4 and the others there is a significant difference at $\alpha = 0.05$.

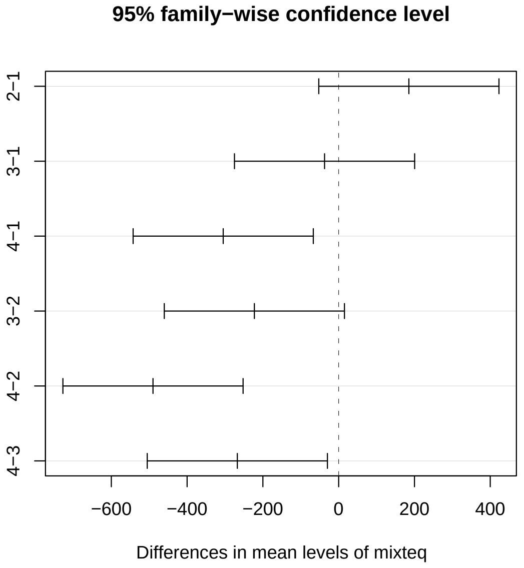


Figure 2.4: Cement, Plot of 95% CI

Cement R-Code

```
cement<-read.table("cement data.txt",header=TRUE); attach(cement)
mixteq<-factor(mixteq)
windows(); par(mfrow=c(1,2))
stripchart(strength ~ mixteq, main = "stripchart(cement)",
vertical = TRUE, log = "y",data = cement)
plot(mixteq,strength,names=c("mixteq1","mixteq2","mixteq3","mixteq4"),ylab="obs")
cement.fit<-lm(strength~mixteq); anova(cement.fit); summary(cement.fit)
model<-aov(strength~mixteq)
model.tables(model); plot.design(strength~mixteq)
par(mfrow=c(2,2)); plot(aov(strength~mixteq))
shapiro.test(residuals(cement.fit))
ks.test(residuals(cement.fit),"pnorm",mean(residuals(cement.fit)),
sd(residuals(cement.fit)))
fligner.test(strength~mixteq); kruskal.test(strength,mixteq)
TukeyHSD(model); plot(TukeyHSD(model))
```

2.3 Radon

Example 2.2. (From D. C. Montgomery (1997), P. 120, Problem 3-9) An article in Environment International (Vol.18, No.4, 1992) describes an experiment in which the amount of radon y released in shower was investigated. Radon enriched water was used in the experiment, and six different orifice diameters were tested in shower heads. The data from the experiment are shown in the following table.

Orifice Diameter	Radon Released in %			
0.37	80	83	83	85
0.51	75	75	79	79
0.71	74	73	76	77
1.02	67	72	74	74
1.40	62	62	67	69
1.99	60	61	64	66

(a) Does the size of the orifice affect the mean percentage of radon released? Use $\alpha = 0.05$.

(b) Find the P-value for the F statistic in part (a).

(c) Analyze the residuals from this experiment.

(d) Find a 95 percent confidence interval on the mean percent of radon released when the orifice diameter is 1.40.

(e) Construct a graphical display to compare the treatment means. What conclusions can you draw?

In this example the factor x is orifice diameter with six levels and response y is the amount of released radon. This is an example of a single-factor experiment with $a = 6$ levels of the factor and $n_i = 4$ replicates resulting in $n = 4 \times 6 = 24$ runs. Here we define the variable orifice diameter as OD factor. In figure 2.5 on page 16 you can see the scatter plot and the series boxplots. The scatter plot shows that with smallest orifice diameter the amount of released radon is higher. The boxplots indicate that by increasing the size of the orifice diameter the amount of released radon will decreased. We are interested in investigating the equality of six means. In other words we want to investigate the treatment effects; the appropriate way of investigation is to establish the ANOVA table. The ANOVA results are as follows:

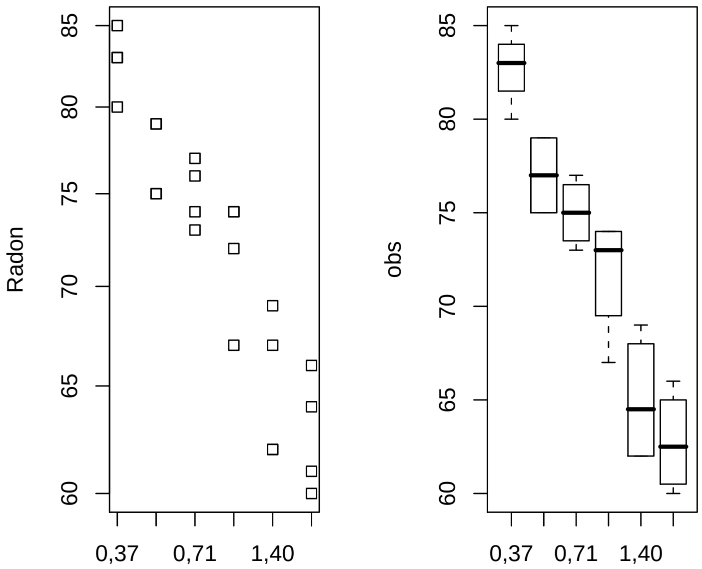


Figure 2.5: Radon, Scatter Plot and Boxplot Series

```
Analysis of Variance Table
Response: Radon
          Df  Sum Sq Mean Sq F value    Pr(>F)
OD         5 1133.38  226.68  30.852 3.160e-08 ***
Residuals 18  132.25    7.35
---
Signif. codes:  0 "***" 0.001 "**" 0.01 "*" 0.05 "." 0.1 " " 1
Residual standard error: 2.711 on 18 degrees of freedom
Multiple R-squared: 0.8955,     Adjusted R-squared: 0.8665
F-statistic: 30.85 on 5 and 18 DF,  p-value: 3.160e-08
```

Does the size of orifice affect the mean percentage of radon released? Yes, there is at least one treatment mean that is different. From the ANOVA table we can see that the F-value is 30.85, which implies that the model is significant. The P-value is small, here the null hypothesis that the treatment means are equal will be rejected. By considering ($R^2_{adj} = 0.8665$) the six orifice diameters determine 86.7% of the variance of the released radon.

The effects of the factors are estimated and the univariate plot of effects is displayed in figure 2.6.

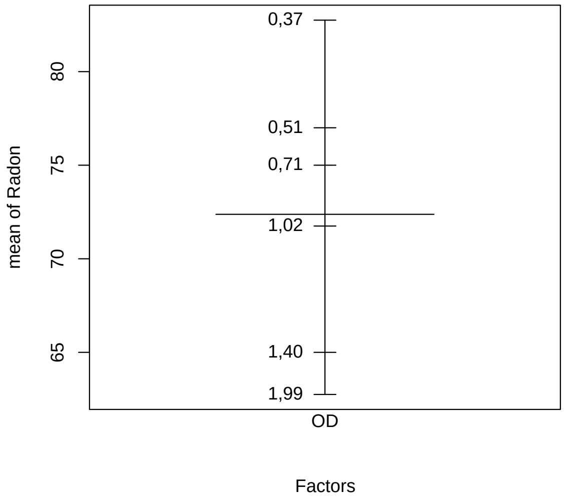


Figure 2.6: Radon, Plot of Effects

For investigation of homogeneity of variances we can use the Fligner-Killeen test:

```
Fligner-Killeen test of homogeneity of variances
data:  Radon by OD
Fligner-Killeen:med chi-squared = 5.6658, df = 5, p-value = 0.3401
```

As we can see from the above results the test statistic is the Fligner-Killeen:med $\chi^2 = 5.6658$ with 5 d.f. and the P-value of the test is $P[\chi^2(5) > 5.6658]$=0.3401. So we can conclude that the six variances are equal, they aren't significant different at level of significant $\alpha = 0.05$.

Now we are interested in the behavior of the residuals. We try to find out if the residuals are normally distributed or not.

The plot of residuals against fitted values shows no structure. The four diagnostic graphs in figure 2.7 on page 18 show that there is nothing unusual about the residuals. The normality tests are as follows:

```
1) Shapiro-Wilk normality test
data:  residuals(radon.fit)
W = 0.9348, p-value = 0.1247
```

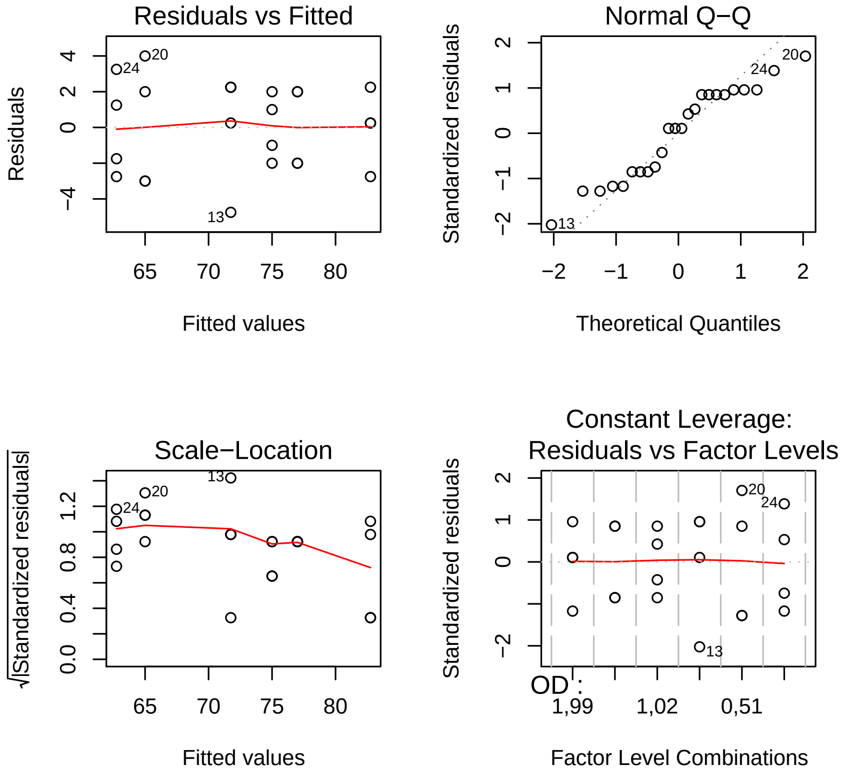


Figure 2.7: Radon, The four diagnostic plots

```
2) One-sample Kolmogorov-Smirnov test
data:  residuals(radon.fit)
D = 0.1729, p-value = 0.47
alternative hypothesis: two-sided
```

The P-values in both tests are larger than $\alpha = 0.05$ so we conclude that the null hypothesis that the residuals are normally distributed has not to be rejected. The Kruskal-Wallis test is written below:

```
Kruskal-Wallis rank sum test
data:  Radon by OD
Kruskal-Wallis chi-squared = 20.6885, df = 5, p-value = 0.0009275
```

While the Kruskal-Wallis chi-squared value is larger than $\chi^2_{0.05,5} = 11.1$ the null hypothesis can be rejected and we conclude that the treatments differ. This is the same conclusion as given by the usual analysis of variance F- test.

The TukeyHSD create a set of confidence intervals on the differences between the means of the levels of a factor with the specified family-wise coverage. The intervals are based on the studentized range statistic, Tukey's 'Honest Significant Difference' method.

```
Tukey multiple comparisons of means
    95% family-wise confidence level
Fit: aov(formula = Radon ~ OD)
OD
             diff   lwr          upr     p adj
0,51-0,37  -5.75 -11.841234   0.3412336 0.0707511
0,71-0,37  -7.75 -13.841234  -1.6587664 0.0084181
1,02-0,37 -11.00 -17.091234  -4.9087664 0.0002404
1,40-0,37 -17.75 -23.841234 -11.6587664 0.0000004
1,99-0,37 -20.00 -26.091234 -13.9087664 0.0000001
0,71-0,51  -2.00  -8.091234   4.0912336 0.8968057
1,02-0,51  -5.25 -11.341234   0.8412336 0.1153360
1,40-0,51 -12.00 -18.091234  -5.9087664 0.0000841
1,99-0,51 -14.25 -20.341234  -8.1587664 0.0000089
1,02-0,71  -3.25  -9.341234   2.8412336 0.5513482
1,40-0,71 -10.00 -16.091234  -3.9087664 0.0007059
1,99-0,71 -12.25 -18.341234  -6.1587664 0.0000650
1,40-1,02  -6.75 -12.841234  -0.6587664 0.0249971
1,99-1,02  -9.00 -15.091234  -2.9087664 0.0021152
1,99-1,40  -2.25  -8.341234   3.8412336 0.8432736
```

It can be observed that 10 of the 15 differences are significant. No significant differences are (2-1, 3-2, 4-2, 4-3, 6-5).

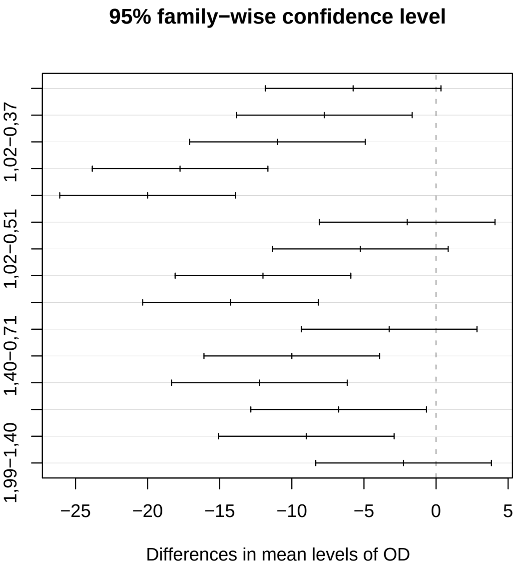


Figure 2.8: Radon, 95% CI

Radon R-Code

```
radon<-read.table("radon data.txt",header=TRUE); attach(radon)
stripchart(Radon ~ OD,main = "stripchart(radon)"
,vertical = TRUE, log = "y", data = radon)
OD<-factor(OD)
plot(OD,Radon,names=c("0,37","0,51","0,71","1,02","1,40","1,99"),ylab="obs")
summary(aov(Radon~OD)); radon.fit<-lm(Radon~OD)
summary(radon.fit); fligner.test(Radon~OD)
model<-aov(Radon~OD); model.tables(model)
plot.design(Radon~OD); TukeyHSD(model)
plot(TukeyHSD(model)); windows(); par(mfrow=c(2,2))
plot(aov(Radon~OD))
shapiro.test(residuals(radon.fit))
ks.test(residuals(radon.fit),"pnorm",mean(residuals(radon.fit))
,sd(residuals(radon.fit)))
kruskal.test(Radon~OD)
```

2.4 Rodding

Example 2.3. (From D. C. Montgomery, 1997, P. 119, Problem 3-8) An article in the ACI Materials Journal (Vol.84, 1987, pp.213-216) describes several experiments investigating the rodding of concrete to remove entrapped air. A 3inch×6inch cylinder was used, and the number of times this rod was used in the design variable. The resulting compressive strength y of the concrete specimen is the response. The data are shown in the following table.

Rodding Level	Compressive strength		
10	1530	1530	1440
15	1610	1650	1500
20	1560	1730	1530
25	1500	1490	1510

(a) Is there any difference in compressive strength due to the rodding level? Use $\alpha = 0.05$.

(b) Find the P-value for the F statistic in part (a).

(c) Analyze the residuals from this experiment. What conclusions can you draw about the underlying model assumptions?

(d) Construct a graphical display to compare the treatment means.

Here we will discuss the four rodding levels as factor to see if they have an effect on the compressive strength of rodding. And to find out if there is a difference in compressive strength due to rodding level. In this problem we have just one factor which is rodding, the rodding level is our factor. The scatter plot below indicates that in level 3 there is the largest strength. The boxplots in figure 2.9 on page 22 show the difference between the medians of the four rodding levels.

We are interested in investigating the equality of the four means, or in other words the treatment effects. To test whether the strengths of the rodding are different due to four rodding levels or not, we construct the ANOVA table.

```
Analysis of Variance Table
Response: CS
          Df  Sum Sq  Mean Sq F value Pr(>F)
RL        3   28633    9544   1.8654 0.2138
Residuals 8   40933    5117
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 71.53 on 8 degrees of freedom
Multiple R-squared: 0.4116,     Adjusted R-squared: 0.1909
F-statistic: 1.865 on 3 and 8 DF,  p-value: 0.2138
```

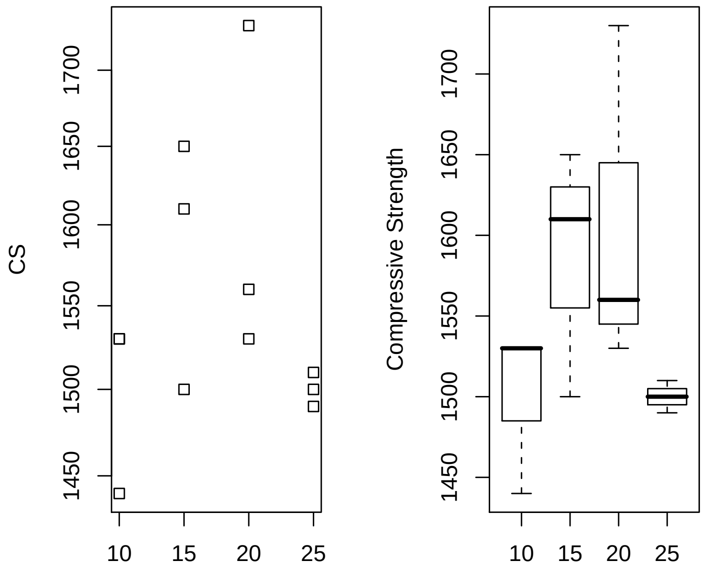


Figure 2.9: Rodding, Scatter plot and Boxplots Series

The model F-value of 1.87 implies that the model is not significant relative due to the noise. The P-value is larger than $\alpha = 0.05$, so we can conclude that the null hypothesis can be hold, i.e. there is no treatment effect, or the treatment means are equal.

The effects are computed and the plot of univariate effects of a design or a model is drawn in figure 2.10 on page 23. The homogeneity of variance will be tested by the Fligner-Killeen test:

```
Fligner-Killeen test of homogeneity of variances
data:  CS by RL
Fligner-Killeen:med chi-squared = 1.6639, df = 3, p-value = 0.6451
```

The P-value is P[$\chi^2(3) > 1.6639$]=0.6452, so there is no doubt against strong evidence of equality of variances.

Now we are interested in the behavior of the residuals. The four diagnostic plots are exhibited in figure 2.11 on page 24. Via these graph we can check the model assumptions. Slight inequality of variance can be observed in the residual plots below, how ever, not enough to be concerned about the assumptions.

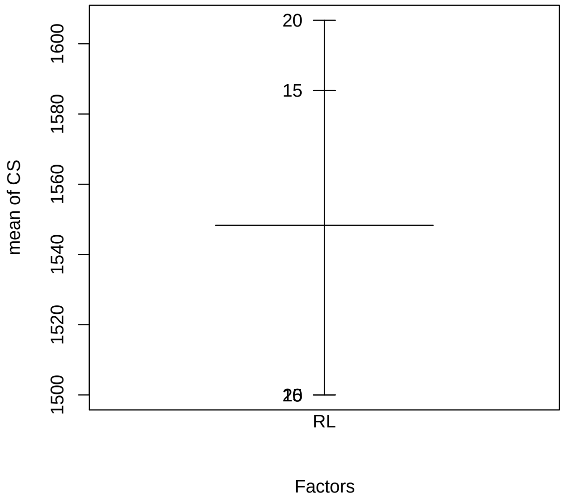


Figure 2.10: Rodding, Plot of Effects

From these computations we find out that the residuals have a normal distribution. The tests of normality show also that the residuals can be assumed as normally distributed.

```
1)  Shapiro-Wilk normality test
data:  residuals(rodding.fit)
W = 0.9596, p-value = 0.7785

2) One-sample Kolmogorov-Smirnov test
data:  residuals(rodding.fit)
D = 0.1448, p-value = 0.9324
alternative hypothesis: two-sided
```

The nonparametric one-way ANOVA can be performed using the Kruskal-Wallis test.

```
Kruskal-Wallis rank sum test
data:  CS by RL
Kruskal-Wallis chi-squared = 5.1803, df = 3, p-value = 0.1591
```

While the Kruskal-Wallis chi-squared is smaller than $\chi^2_{0.05,3} = 7.81$ we can not reject the null hypothesis. This is the same conclusion given by the usual analysis of variance F test. Also the rodding level have no effect on the compressive strength of the rodding.

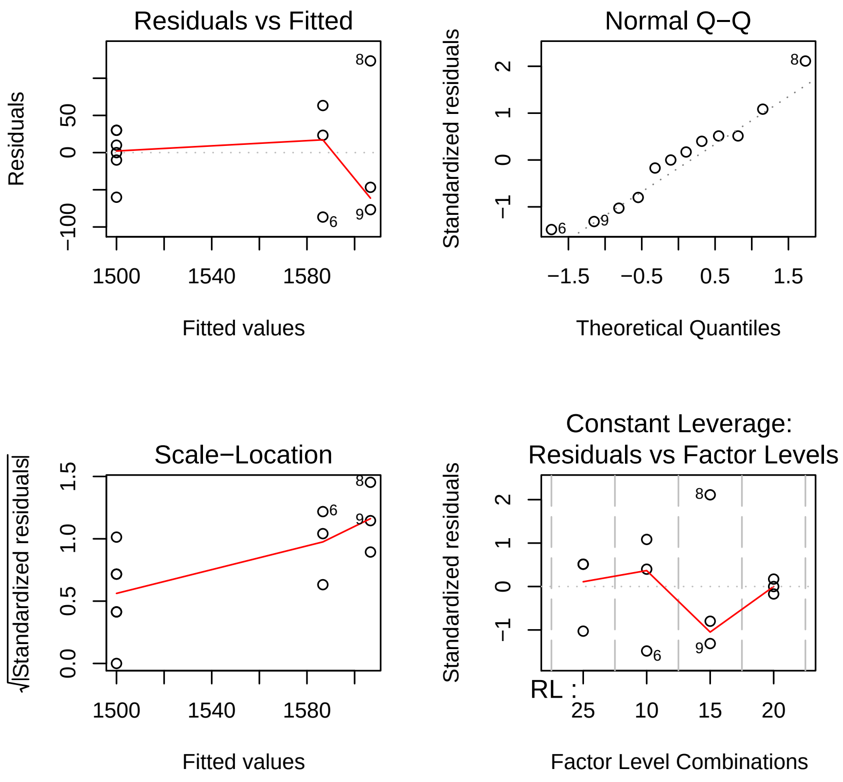


Figure 2.11: Rodding, Four diagnostic plots

In this problem it isn't necessary to conduct a 95% Tukey multiple comparison test, because the F-test isn't significant.

Rodding R-Code

```
rodding<-read.table("rodding data.txt",header=TRUE)
attach(rodding)
stripchart(CS ~ RL,main = "stripchart(rodding)",
vertical = TRUE, log = "y", data = rodding)
windows();  par(mfrow=c(1,2)); RL<-factor(RL)
plot(RL,CS,names=c("10","15","20","25"),ylab="Compressive Strength")
rodding.fit<-lm(CS~RL);summary(rodding.fit)
model<-aov(CS~RL)
model.tables(model); plot.design(CS~RL)
fligner.test(CS~RL); model<-aov(CS~RL)
model.tables(model); TukeyHSD(model)
plot(TukeyHSD(model)); windows(); par(mfrow=c(2,2))
plot(aov(CS~RL)); shapiro.test(residuals(rodding.fit))
ks.test(residuals(rodding.fit),"pnorm",mean(residuals(rodding.fit))
,sd(residuals(rodding.fit))); kruskal.test(CS~RL)
```

Chapter 3

Randomized Complete Block Design

In the previous chapter we have examined the one way ANOVA in which only one factor is considered at a time.

As one-way analysis of variance (ANOVA) measures significant effects of one factor only, a randomized complete block design (RCBD) measures the effects of two factors simultaneously. For example, an experiment might be defined by two parameters, such as treatment and time point. One-way ANOVA tests would be able to assess only the treatment effect or the time effect. On the other hand we can assess both time and treatment in the same test. RCBD test generates two P-values, one for each parameter independently. The model is as follows:

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij}, \quad i = 1, \ldots, a; \quad j = 1, \ldots, b; \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2).$$

Source	Sum of Squares (SS)	Degrees of Freedom (df)	Mean Square	F statistics
A	SS_A	$a-1$	MS_A	$F = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$F = \frac{MS_B}{MS_E}$
Error	SS_E	$(a-1)(b-1)$	MS_E	
Total	SS_T	$ab-1$		

We may interpret factor A as treatment and factor B as block. The different sums of squares are computed as follows:

$$\begin{aligned}
SS_T &= SS_A + SS_B + SS_E \\
SS_A &= a\sum_{j=1}^{b}(\bar{y}_{i.} - \bar{y}_{..})^2 \\
SS_B &= b\sum_{i=1}^{a}(\bar{y}_{.j} - \bar{y}_{..})^2 \\
SS_E &= \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2 \\
SS_T &= \sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij} - \bar{y}_{..})^2
\end{aligned}$$

The means are defined as follows:

$$\begin{aligned}
\overline{y}_{i.} &= \frac{1}{b}\sum_{j=1}^{b} y_{ij} \quad \text{mean of } \; i \\
\overline{y}_{.j} &= \frac{1}{a}\sum_{i=1}^{a} y_{ij} \quad \text{mean of } \; j \\
\overline{y}_{..} &= \frac{1}{a}\sum_{i=1}^{a} \overline{y}_{i.} = \frac{1}{b}\sum_{i=1}^{b} \overline{y}_{.j} = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij} \quad \text{overall mean}
\end{aligned}$$

3.1 Nozzle Design

Example 3.1. (from D. C. Montgomery, 1997, P. 220, Problem 5-5) An article in the Fire Safety Journal ("The Effect of Nozzle Design on the Stability and Performance of Turbulent Water Jets," Vol.4, Aug 1981) describes an experiment in which a shape factor y was determined for five nozzle designs (factor A) at six levels of jet efflux velocity (factor B). Interest focused on potential differences between nozzle designs, the velocity considered as a nuisance variable. The data are shown below:

Nozzle Design	Jet Efflux Velocity ($\frac{m}{s}$) B					
A	11.73	14.37	16.59	20.43	23.46	28.74
1	0.78	0.80	0.81	0.75	0.77	0.78
2	0.85	0.85	0.92	0.86	0.81	0.83
3	0.93	0.92	0.95	0.89	0.89	0.83
4	1.14	0.97	0.98	0.88	0.86	0.83
5	0.97	0.86	0.78	0.76	0.76	0.75

(a) Does nozzle design affect the shape factor? Compare the nozzles with box plots and with an analysis of variance, using $\alpha = 0.05$.

(b) Analyze the residuals from this experiment.

(c) Which nozzle designs are different with respect to shape factor y? Draw a graph of the average shape factor for the nozzle type and compare this to a scaled t distribution. Compare the conclusions that you draw from this plot to those from TukeyHSD test.

In this problem we have two factors one with 5 levels and the other one with 6 levels. It is useful to solve this problem via a completely randomized block design. We can use the 6 Jet Efflux Velocities as **Blocks** and apply each of the 5 nozzles as **treatments** to the Jet Efflux Velocity. The treatments are applied in random order, so each block can be viewed as one CR designed experiment. This is a Randomized Complete Block Design (RCBD). "Complete" means that each block contains all of the treatments.

A basic idea is that the responses y should be less highly varied within a block than between blocks. We first, draw the boxplot in figure 3.1 on page 30. It shows that the 4^{th} nozzle (4^{th} treatment) has the largest effect on the shape factor y.
The ANOVA table results show that both factors are significant due to their P-value which are smaller than the significance level $\alpha = 0.05$.

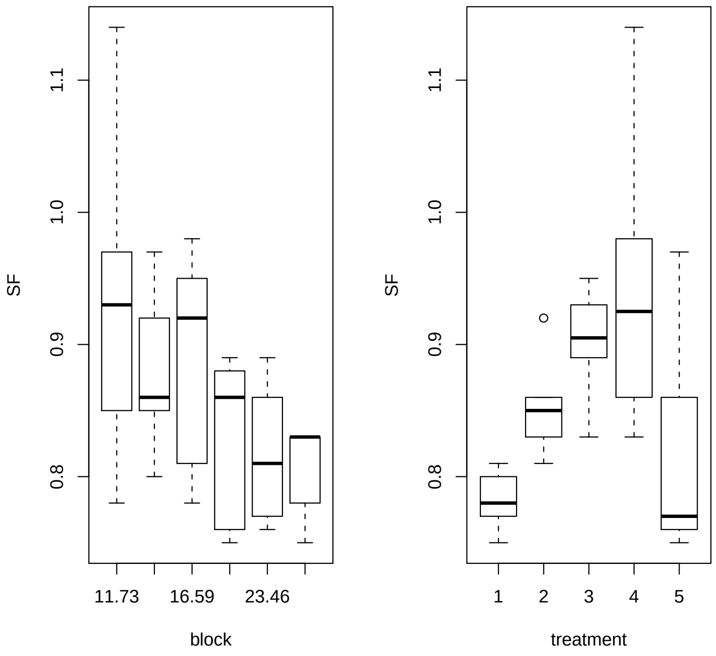


Figure 3.1: Nozzle, Boxplots Series

```
Analysis of Variance Table

Response: SF
          Df   Sum Sq  Mean Sq F value    Pr(>F)    
treatment  4 0.102180 0.025545  8.9162 0.0002655 ***
block      5 0.062867 0.012573  4.3886 0.0073642 ** 
Residuals 20 0.057300 0.002865                      
---
Residual standard error: 0.05353 on 20 degrees of freedom
Multiple R-squared: 0.7423,     Adjusted R-squared: 0.6263
F-statistic: 6.401 on 9 and 20 DF,  p-value: 0.0002787
```

Thus at any level $\alpha > 0.0002655$, we can reject the null hypothesis of no treatment effects ($H_0 : \tau_1 = ... = \tau_a = 0$). It also appears that the blocks have a significant effect. The randomization ensures that the F-test for treatments is approximately valid even if the errors aren't nearly normal. Because of the randomization restriction, the same isn't true for testing the significance of the blocks by $\frac{MS_{Bl}}{MS_E}$. Thus the P-value of 0.00073642 for blocks should be used as a guide, unless we are sure of the normality.

Suppose that we analyze the data as a CRD, what would the ANOVA table be? We have just one factor Nozzle.

```
Analysis of Variance Table

Response: SF
          Df  Sum Sq   Mean Sq F value   Pr(>F)
treatment  4 0.10218 0.0255450  5.3145 0.003079 **
Residuals 25 0.12017 0.0048067
---
```

Since there are no replications (only one observation per combination of treatment of block) we cannot test the additive model against a model with an interaction. However, we may get an idea of the adequacy of the additive model from an interaction plot.

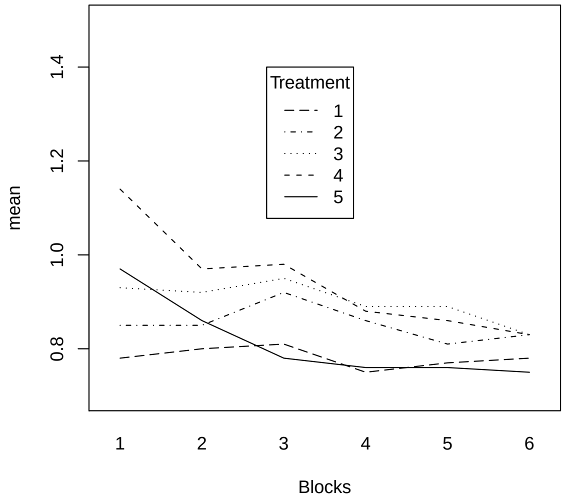


Figure 3.2: Interaction Plot

Figure 3.2 shows that profiles for treatments 1, 2 and 3 are almost parallel, just treatment 5 shows a somewhat different pattern.

Checking the assumptions and viewing the residuals:

1. QQ-plot of the residuals $e_{ij} = y_{ij} - \widehat{y}_{ij}$. Where the fitted values are $\widehat{y}_{ij} = \widehat{\mu} + \widehat{\tau}_i + \widehat{\beta}_j$ with $\hat{\mu} = \overline{y}_{..}$, $\hat{\tau}_i = \overline{y}_{i.} - \overline{y}_{..}$ and $\hat{\beta}_j = \overline{y}_{.j} - \overline{y}_{..}$.

2. residuals vs. treatment labels, block labs, fitted values. Figure 3.3 shows the four diagnostic plots.

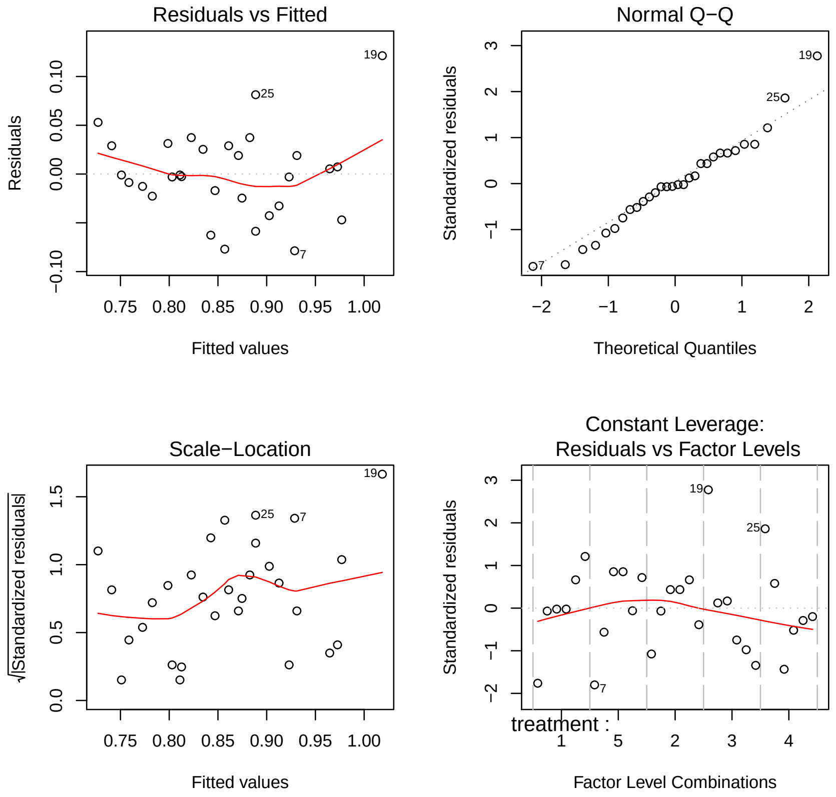


Figure 3.3: Nozzle, The four diagnostic plots

The normality tests are done and the results are as follows:

```
1)      Shapiro-Wilk normality test
data:  residuals(g)
W = 0.9699, p-value = 0.5369

2)        One-sample Kolmogorov-Smirnov test
data:  residuals(g)
D = 0.1005, p-value = 0.8929
alternative hypothesis: two-sided
```

From the results and the plots above we can conclude that the residuals are normally distributed, and the P-value is much larger than $\alpha = 0.05$, so we can't reject H_0 that the residuals are normally distributed.

Does the error variance depend on the treatment, or on the block? To answer this question we will do the variance homogeneity test one for the blocks and one for the treatments:

```
Fligner-Killeen test of homogeneity of variances
        Fligner-Killeen test of homogeneity of variances

data:  SF by treatment
Fligner-Killeen:med chi-squared = 9.9619, df = 4, p-value = 0.04107

        Fligner-Killeen test of homogeneity of variances

data:  SF by block
Fligner-Killeen:med chi-squared = 4.9236, df = 5, p-value = 0.4253
```

While in the first test (shape factor by treatment) the P-value is slightly smaller than α we can conclude that the variances aren't equal and we reject the null hypothesis.

The analogue of the Kruskal-Wallis test, for a RCBD is "Friedman's test". The observations are replaced by their ranks within each block, and the usual ANOVA is run.

```
Friedman rank sum test
data:  y, treatments and blocks
Friedman chi-squared = 17.1034, df = 4, p-value = 0.001846
```

So the assumptions seems to be met, and at least some of the differences in the treatment means, i.e. in the mean readings $\mu_{i.} = \mu + \tau_i$, are significant. The shape factor depends on nozzles used. It's time for looking to a confidence interval by applying the TukeyHSD test. Figure 3.4 on page 34 shows the 95% TukeyHSD confidence intervals.
From the confidence intervals we can conclude that treatment 1 is significant different from treatments 3 and 4, and block 1 is significant different from blocks 5 and 6.

To answer the question of which nozzle designs are different with respect to shape factor y, we will conduct a 95% TukeyHSD confidence Interval

```
  Tukey multiple comparisons of means
    95% family-wise confidence level

Fit: aov(formula = SF ~ treatment)
```

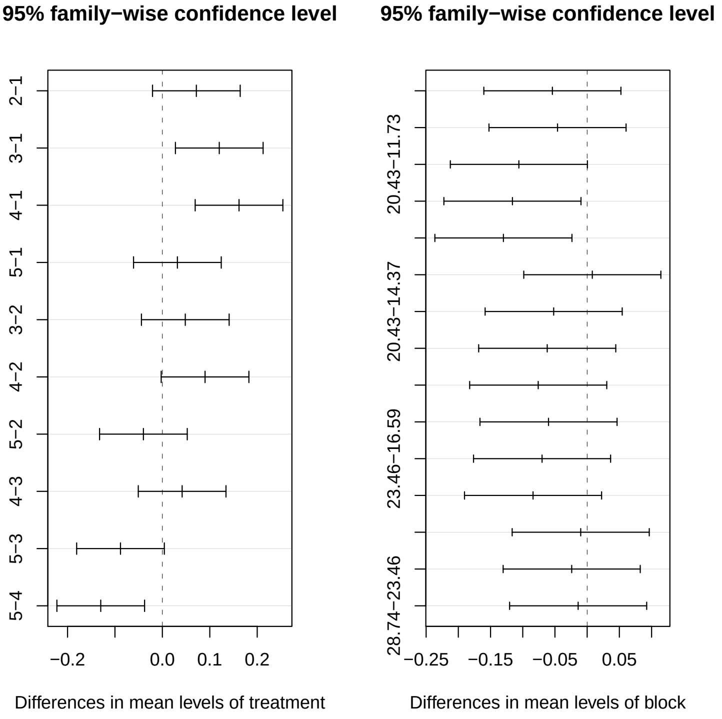


Figure 3.4: 95% CI

```
$treatment
           diff         lwr         upr     p adj
2-1  0.07166667 -0.04588973  0.18922307 0.4009767
3-1  0.12000000  0.00244360  0.23755640 0.0437691
4-1  0.16166667  0.04411027  0.27922307 0.0037468
5-1  0.03166667 -0.08588973  0.14922307 0.9307964
3-2  0.04833333 -0.06922307  0.16588973 0.7471280
4-2  0.09000000 -0.02755640  0.20755640 0.1952713
5-2 -0.04000000 -0.15755640  0.07755640 0.8531635
4-3  0.04166667 -0.07588973  0.15922307 0.8339577
5-3 -0.08833333 -0.20588973  0.02922307 0.2099634
5-4 -0.13000000 -0.24755640 -0.01244360 0.0249866
```

From the results above we conclude that the Nozzle 1 and 3 are significant different from others.

Nozzle R-Code

```
nozzle<-read.table("nozzle.txt", header=TRUE)
attach(nozzle);treatment<-factor(Nozzle)
block<-factor(JEV)
par(mfrow=c(1,2)); plot(SF~blocks,xlab="blocks")
plot(SF~treatments,xlab="treatments")
#get anova table for RCBD; model1<-lm(SF~treatment+block); anova(model1)
model2<-lm(SF~treatment); anova(model2); windows(); par(mfrow=c(2,2))
plot(aov(SF~treatment+block));fligner.test(SF~treatment);
fligner.test(SF~block);
data.matrix<-as.matrix(data); friedman.test(y,treatments,blocks)
model<-aov(SF~treatment+block); TukeyHSD(model)
windows(); par(mfrow=c(1,2)); plot(TukeyHSD(model))
shapiro.test(residuals(g))
ks.test(residuals(g),"pnorm",mean(residuals(g))
,sd(residuals(g)))
interaction.plot(blocks,treatments,y,xlab="Blocks",ylab="mean",
ylim=c(0.7,1.5),fixed=T,legend=F)
legend(2.79,1.4,c("1","2","3","4","5"), title="Treatment",lty=c(5,4,3,2,1))
```

3.2 Ratio Control Algorithm Experiment

Example 3.2. (From D. C. Montgomery, 1997, P. 132) Consider a ratio control algorithm experiment. The experiment was actually conducted as a randomized block design, where six time periods were selected as the blocks (factor A), and all four ratio control algorithms (factor B) were tested in each time period. The voltage y of ratio control algorithm and the standard deviation of voltage are as follows:

Ratio Control Algor.	Time Period					
	1	2	3	4	5	6
1	4.93(.05)	4.86(.04)	4.75(.05)	4.95(.06)	4.79(.03)	4.88(.05)
2	4.85(.04)	4.91(.02)	4.79(.03)	4.85(.05)	4.75(.03)	4.85(.02)
3	4.83(.09)	4.88(.13)	4.90(.11)	4.75(.15)	4.82(.08)	4.90(.12)
4	4.89(.03)	4.77(.04)	4.94(.05)	4.86(.05)	4.79(.03)	4.76(.02)

(a) Analyze the average cell voltage data. (Use $\alpha = 0.05$) Does the choice of the ratio control algorithm affect the average cell voltage?

(b) Perform an appropriate analysis on the standard deviation of voltage. (Recall that this is called "pot noise".) Does the choice of the ratio control algorithm affect the pot noise?

(c) Conduct any residual analysis that seems appropriate.

(d) Which ratio control algorithm would you select if your objective is to reduce both the average cell voltage and the pot noise?

Aluminium is produced by combining alumina with other ingredients in a reaction cell and applying heat by passing electric current through the cell. Alumina is added continuously to different ratio control algorithms while investigated in this experiment. The response variables studied were related to cell voltage.

We have a RCBD problem in which the time period is taken as blocks and the ratio control algorithm is taken as treatments. The treatments are applied in random order to each block. First we draw the boxplots. Figure 3.5 on page 37 shows that the 5^{th} block has the smallest effect on the average cell voltage.

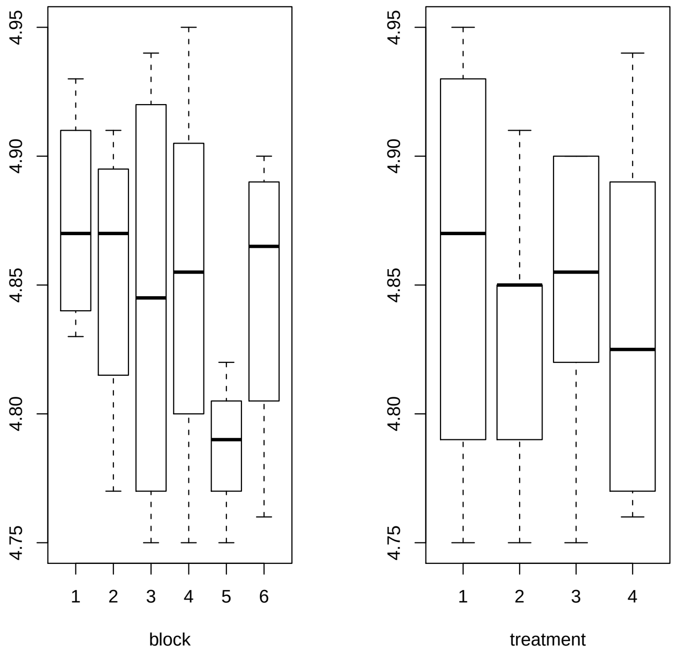


Figure 3.5: RCA, Boxplots Series, Average cell voltage

The ANOVA result for the average cell voltage is as follows:

```
Analysis of Variance Table

Response: Avcv
          Df   Sum Sq   Mean Sq F value Pr(>F)
treatment  3 0.002746 0.0009153  0.1902 0.9014
block      5 0.017438 0.0034875  0.7248 0.6154
Residuals 15 0.072179 0.0048119
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.06937 on 15 degrees of freedom
Multiple R-squared: 0.2185,     Adjusted R-squared: -0.1983
F-statistic: 0.5243 on 8 and 15 DF,  p-value: 0.8205
```

As you can see non of these two factors have an effect on the response variable average cell voltage. So we can not reject the null hypothesis of no treatment effects. We conclude that the choice of the ratio control algorithm doesn't have any effect on the average cell voltage.

Now we are interested in studying the behavior of the residuals. Figure 3.6 shows the four diagnostic plots. It shows that the residuals are normally distributed.

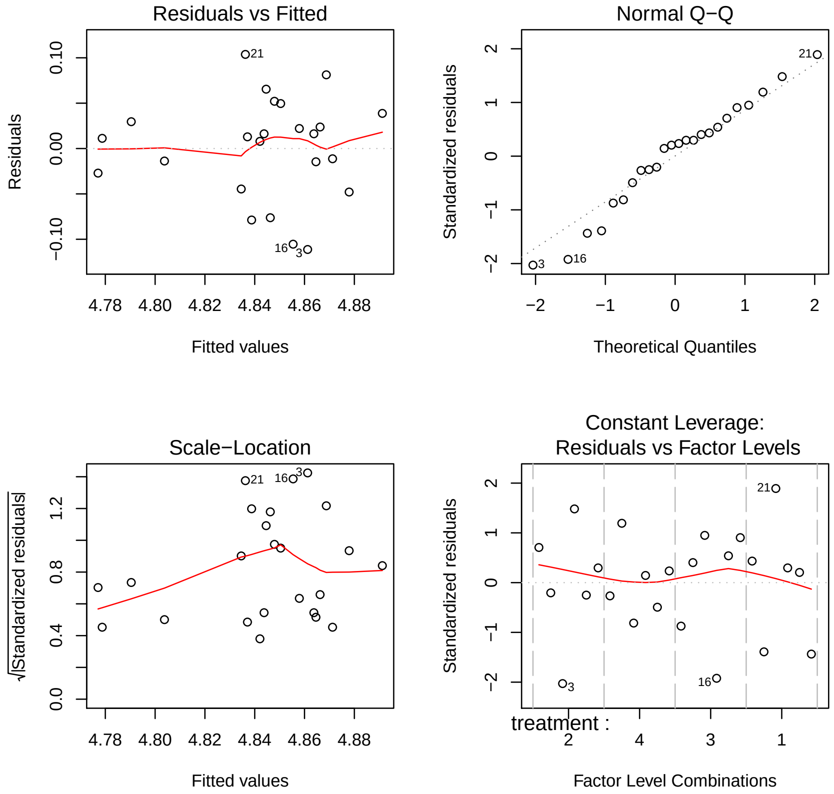


Figure 3.6: RCA, The four Diagnostic plots for average cell voltage

Now we analyze the data for pot noise as response variable. We first draw boxplots, figure 3.7 on page 39 shows that the 4^{th} time period and the third treatment (third ratio control algorithm) have the largest effect on the pot noise.

The ANOVA results for pot noise is as follows:

```
Analysis of Variance Table

Response: Pn
          Df    Sum Sq   Mean Sq F value    Pr(>F)
treatment  3 0.0260125 0.0086708 50.7561 4.345e-08 ***
block      5 0.0027208 0.0005442  3.1854   0.03711 *
Residuals 15 0.0025625 0.0001708
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

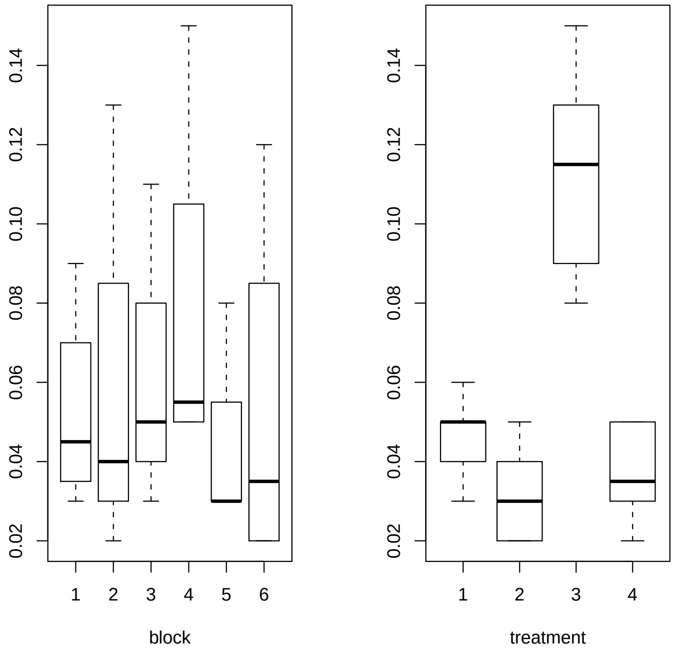


Figure 3.7: RCA, Boxplots Series; Pot noise

```
Residual standard error: 0.01307 on 15 degrees of freedom
Multiple R-squared: 0.9181,     Adjusted R-squared: 0.8745
F-statistic: 21.02 on 8 and 15 DF,  p-value: 7.97e-07
```

As you can see both factors have an effect on the standard deviation (pot noise), and the treatments are much more significant as blocks. That at any level $\alpha > 4.345e^{-8}$, we would reject the null hypothesis of no treatment effects. So we can conclude that the algorithm does affect the pot noise. It also appears that the blocks are significant. Blocking the time period is a good idea for reducing the mean square error. Now we are interested in studying the behavior of the residuals. Figure 3.8 on page 40 shows the four diagnostic plots. It shows that the residuals are normally distributed, and the variability of the residuals seems to be fairly constant across the fitted value of the response.
From the results of the normality tests we can conclude that the residuals are normally distributed while the P-values are larger than the significance level $\alpha = 0.05$.

```
1)  Shapiro-Wilk normality test
data:  residuals(g)
W = 0.9461, p-value = 0.2225
```

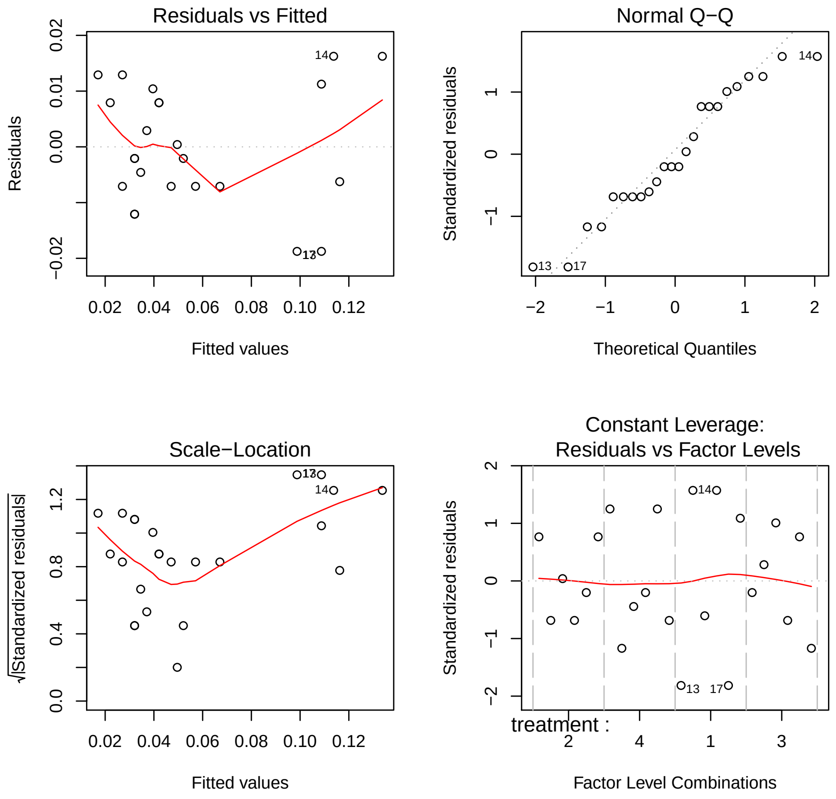


Figure 3.8: RCA, The four Diagnostic plots for Pot noise

```
2)   One-sample Kolmogorov-Smirnov test
data:  residuals(g)
D = 0.1484, p-value = 0.6661
alternative hypothesis: two-sided
```

The normality-based tests can be justified here since we have little evidence of non-normality. It's a good idea to run nonparametric tests too, to reassure ourselves that we reach the same conclusions without assuming normality. The Fligner-Killeen test for equal variances in each block is as follows:

```
        Fligner-Killeen test of homogeneity of variances

data:  Avcv by treatment by block
Fligner-Killeen:med chi-squared = 2.2098, df = 3, p-value = 0.53
```

While the P-value is larger than $\alpha = 0.05$ so we can't reject the null hypothesis that the variances are equal.

The analogue of Kruskal-Wallis test, for a RCBD is "Friedman's test". The observations are replaced by their ranks within each block, and the usual ANOVA is run.

```
Rank:
    X1  X2  X3  X4 X5  X6
r1  3 2.5 2.5 3.0  2 3.0
r2  2 1.0 1.0 1.5  2 1.5
r3  4 4.0 4.0 4.0  4 4.0
r4  1 2.5 2.5 1.5  2 1.5
       Friedman rank sum test
data:  y, treatments and blocks
Friedman chi-squared = 15.4615, df = 3, p-value = 0.001462
```

As you can see from the table above, r3 has the largest rank each time, it indicates that its standard deviation effect is the largest. So the assumptions seems to be met, and at least some of the differences in the treatment means, i.e. in the mean readings $\mu_{i.} = \mu + \alpha_i$, are significant.

It's time for looking for a confidence interval by applying the TukeyHSD test. Figure 3.9 shows the 95% TukeyHSD confidence intervals.
Figure 3.10 on page 42 shows the interaction plot between these two factors , i.e, treatments and blocks. You can see from the confidence interval plots that the treatment 3 is significantly different from other treatments, also we can conclude that blocks 4 and 5 are significantly different. As we see in figure 3.10, the interaction plots show that the treatments 1, 2 and 3 are almost parallel, but the treatment 4 shows a somewhat different pattern.

Lastly, we can conclude that only algorithm 3 results in a larger population standard deviation than the other algorithms. Because the conclusion of part (a) is that the choice of algorithm has no effect on the average cell voltage, it makes no difference which algorithm we use, with respect to average cell voltage. Because of the above results. to reduce the pot noise, we use only algorithms 1, 4 or 2.

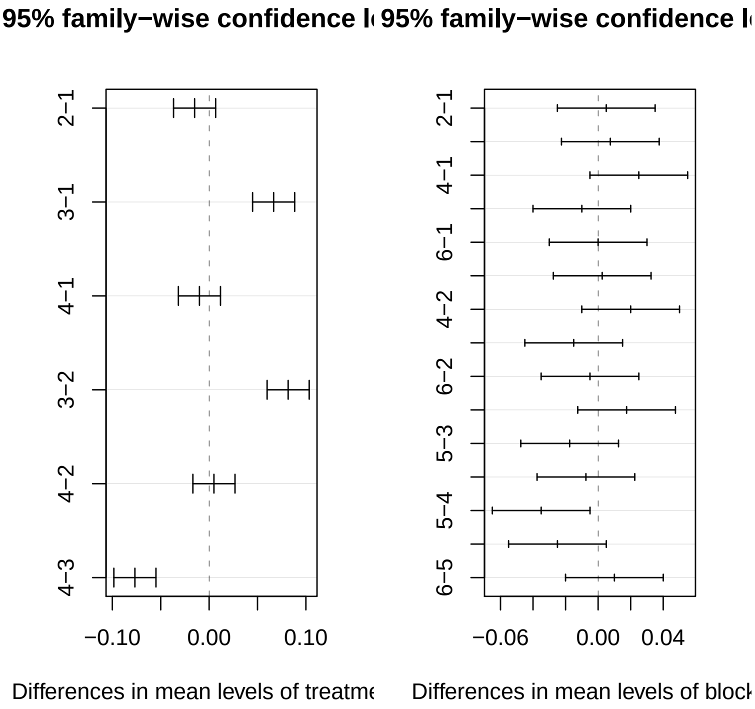


Figure 3.9: RCA, 95% CI

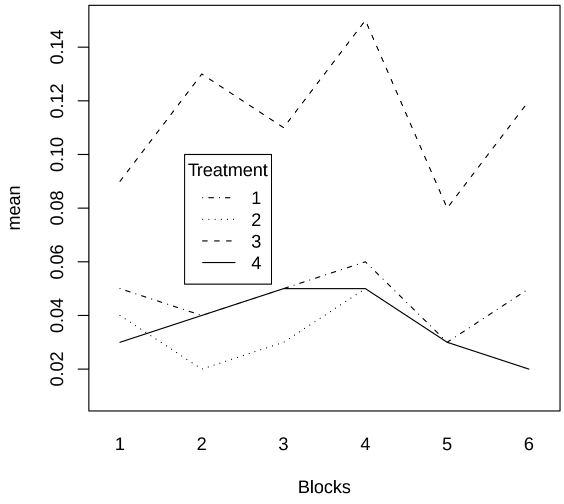


Figure 3.10: RCA, Interaction plot

Ratio Control Algorithm R-Code

```
data<-read.table("RCA.txt", header=TRUE); attach(data)
block<-factor(TP); treatment<-factor(RCA)
par(mfrow=c(1,2)); boxplot(Avcv~block, xlab="block")
boxplot(Avcv~treatment, xlab="treatment")
#get the anova table of RCBD
model1<-lm(Avcv~treatment+block); anova(model1); summary(model1)
#to check the normality assumption and view residuals
windows(); par(mfrow=c(2,2)); plot(aov(model1))
#homogeneity of variance
fligner.test(Avcv~treatment+block)
#friedman's test depends on the ranks within each blocks
data.ranks<-apply(data,2,rank); data.ranks
friedman.test(Avcv,treatment,block)
model<-aov(Avcv~treatment+block); TukeyHSD(model)
windows(); par(mfrow=c(2,2)); plot(TukeyHSD(model))
interaction.plot(block,treatment,Avcv)
shapiro.test(residuals(model1))
ks.test(residuals(model1),"pnorm",mean(residuals(model1))
,sd(residuals(model1)))
par(mfrow=c(1,2)); boxplot(Pn~block, xlab="block")
boxplot(Pn~treatment, xlab="treatment")
model2<-lm(Pn~treatment+block); anova(model2); summary(model2)
#to check the normality assumption and view residuals
windows(); par(mfrow=c(2,2)); plot(aov(model2))
data.ranks<-apply(data,2,rank); data.ranks
friedman.test(Pn,treatment,block)
```

3.3 Aluminium master Alloy

Example 3.3. (From D. C. Montgomery, 1997, P. 222, problem 5-7) An aluminium master alloy manufacturer produces grain refiners in ingot form. The company produces the product in four furnaces. Each furnace is known to have its own unique operating characteristics. So any experiments run in the foundry that involves more than one furnace. We will consider furnaces as a nuisance variable. The process engineers suspect that stirring rate impacts the grain size y. The block design is run for a particular refiner and the resulting grain size data are shown below.

Stirring Rate (rpm)	Furnace			
	1	2	3	4
5	8	4	5	6
10	14	5	6	9
15	14	6	9	2
20	17	9	3	6

(a) Is there any evidence that the stirring rate impacts grain size?

(b) Graph the residuals from this experiment on a normal probability plot. Interpret this plot.

(c) Plot the residuals versus furnace and stirring rate. Does this plot convey any useful information?

(d) What should process engineers recommend concerning the choice of stirring rate and furnace for this particular grain refiner if small grain size is desirable?

We define furnace as blocks and the stirring rate as treatments. Each treatment is applied to the blocks in random way. Figure 3.11 on page 45 shows the boxplot, it indicates that the fourth block (4^{th} furnace) has the largest effect on the grain size.

The ANOVA table below shows that there is no difference in mean grain size due to the different stirring rates, so we can conclude null hypothesis, but there is an evidence that the blocks are significant, and have an effect on grain size. The model F-value of 0.85 implies that the model is not significant relative to the treatments. So the answer to part (a) is no, there is no evidence concerning that the stirring rate impacts grain size.

```
Analysis of Variance Table

Response: Gs
          Df  Sum Sq Mean Sq F value  Pr(>F)
treatment  3  22.187   7.396  0.8527 0.49954
block      3 165.188  55.063  6.3483 0.01334 *
Residuals  9  78.063   8.674
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 2.945 on 9 degrees of freedom
Multiple R-squared: 0.7059,     Adjusted R-squared: 0.5099
F-statistic:   3.6 on 6 and 9 DF,  p-value: 0.04196
```

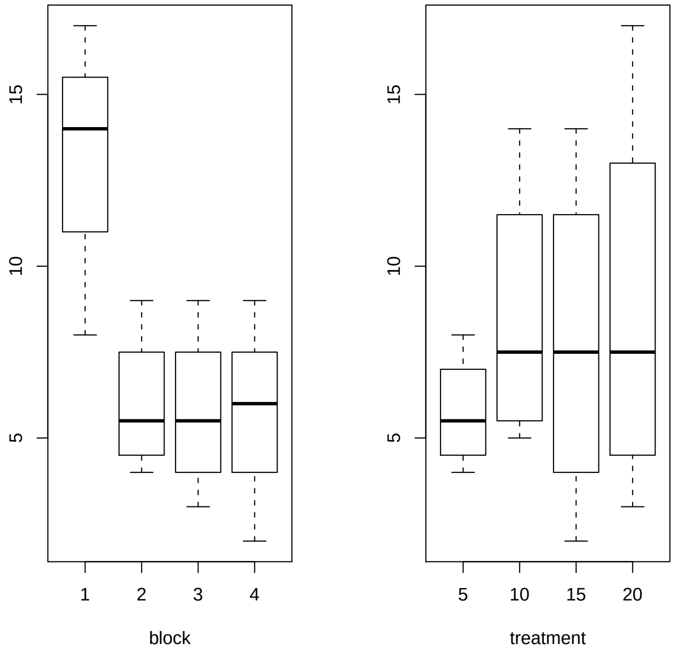


Figure 3.11: AMA, Boxplot

Now we study the behavior of the residuals; figure 3.12 on page 46 shows the four diagnostic plots. It shows that the normality assumption is valid.
The normality tests indicate no significant deviation from normality, here the normality assumption can be made.

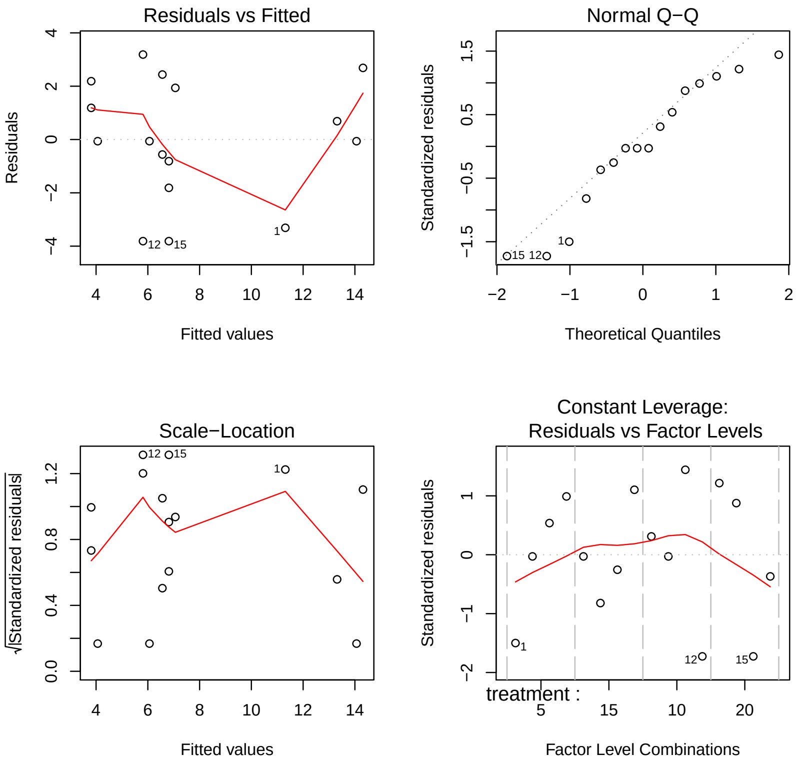


Figure 3.12: AMA, The four diagnostic plots

```
1)   Shapiro-Wilk normality test
data:  residuals(g)
W = 0.9301, p-value = 0.2447
2)   One-sample Kolmogorov-Smirnov test
data:  residuals(g)
D = 0.1146, p-value = 0.9685
alternative hypothesis: two-sided
```

The residuals plots against stirring rate and furnace are shown in figure 3.13 on page 47. The variance is consistent at different stirring rates. Note only that the validity assumption of uniform variance also identifies that the different stirring rates do not affect the variance.

Now there is a question: Does the error variance depend on the treatment, or on the block? The variance homogeneity test confirms that there is no dependency:

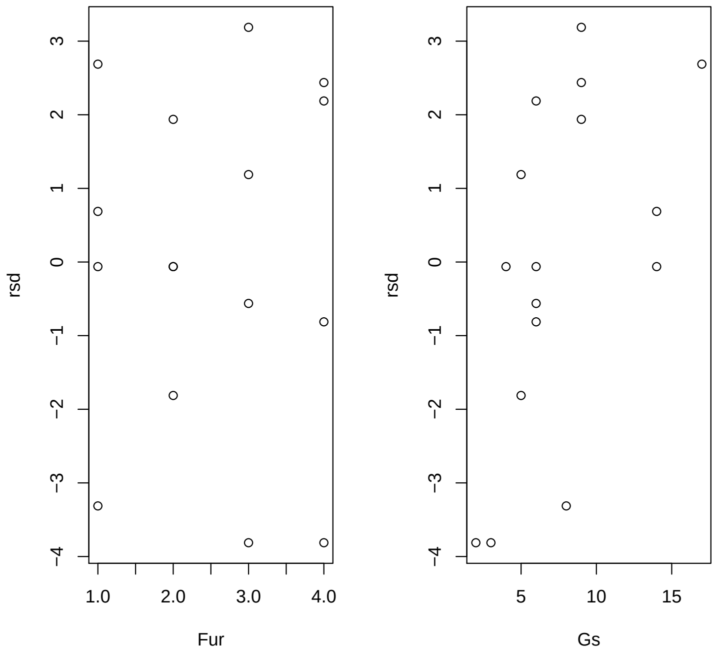


Figure 3.13: AMA, Residual plots Vs. Furnace and Stirring rate

```
        Fligner-Killeen test of homogeneity of variances

data:  Gs by treatment by block
Fligner-Killeen:med chi-squared = 2.9766, df = 3, p-value = 0.3952
```

The analogue of Kruskal-Wallis test, for a RCBD is "Friedman's test". The observations are replaced by their ranks within each block, and the usual ANOVA is run. Here there is also no evidence of a treatment effect (P=0.4429).

```
    X1 X2 X3  X4
s5  1.0  1  2 2.5
s10 2.5  2  3 4.0
s15 2.5  3  4 1.0
s20 4.0  4  1 2.5

        Friedman rank sum test
data:  Gs, treatment and block
Friedman chi-squared = 2.6842, df = 3, p-value = 0.4429
```

It's time for looking for confidence intervals by applying the TukeyHSD test. Figure 3.14 shows the 95% TukeyHSD confidence intervals and the interaction plot is given in figure 3.15 on page 49.

Figure 3.14: AMA, 95% CI

From the confidence interval plots we can conclude that block 1 is significantly different from the other blocks.
The answer to the last part is that there is no effect due to the stirring rate.

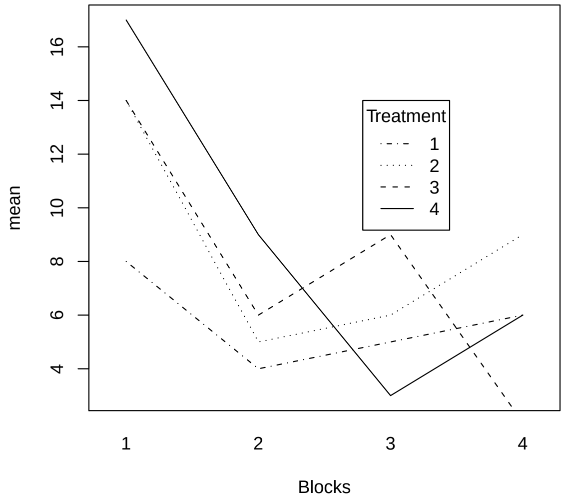


Figure 3.15: AMA, Interaction plot

Aluminium Master Alloy R-Code

```
data<-read.table("AMA.txt",header=TRUE); attach(data)
#generate the block index and the treatment level for response value
block<-factor(Fur); treatment<-factor(SR)
par(mfrow=c(1,2)); boxplot(Gs~block, xlab="block")
boxplot(Gs~treatment, xlab="treatment")
#get the anova table of RCBD
model1<-lm(Gs~treatment+block); anova(model1); summary(model1)
#to check the normality assumption and view residuals
windows(); par(mfrow=c(2,2)); plot(aov(model1))
#residuals against stirring rate and furnace
rsd<-residuals(model1); plot(Fur,rsd); plot(Gs,rsd)
#homogeneity of variance
fligner.test(Gs~treatment+block)
#friedman's test depends on the ranks within each blocks
data.ranks<-apply(data,2,rank); data.ranks
friedman.test(Gs,treatment,block)
model<-aov(Gs~treatment+block); TukeyHSD(model)
windows(); par(mfrow=c(1,2)); plot(TukeyHSD(model))
interaction.plot(block,treatment,Gs,xlab="Blocks",ylab="mean",
ylim=c(3,17),fixed=T,legend=F)
legend(2.79,14,c("1","2","3","4"), title="Treatment",
lty=c(4,3,2,1)); shapiro.test(residuals(model1))
ks.test(residuals(g),"pnorm",mean(residuals(model1))
,sd(residuals(model1)))
```

Chapter 4

Analysis of Covariance

The analysis of covariance (generally known as ANCOVA) combines regression and ANOVA. Here the response variable y is continuous, we have one ore more explanatory factors (treatments) and one or more continuous explanatory variables.

The goal of analysis of covariance is to reduce the error variance. This increases the power of tests and narrows the confidence interval. There may be measurable variables that affect the response but have nothing to do with the factors (treatments) in the experiment. Analysis of covariance adjust for those variables [Montgomery D. C. (1997)].

4.1 Analysis

Along with the usual terms in the effects model for a single factor CRD, we include a term expressing the departure of the covariate from its overall average, and assume that y is linearly related to this departure.

If y_{ij} is the j^{th} observation in the i^{th} treatment group, then the model is

$$y_{ij} = \mu + \tau_i + \beta(x_{ij} - \overline{x}_{..}) + \varepsilon_{ij}, \quad \varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2), \quad i = 1, \ldots, a; \quad j = 1, \ldots, r.$$

Here τ_i is the effect of the i^{th} treatment and we assume $\sum_i \tau_i = 0$, and that the covariate is not affected by the treatments.

- The main hypothesis of interest is $H_0 : \tau_1 = \tau_2 = ... = \tau_a$. This is equivalent to the statement that all $\mu + \tau_i$ are equal, and so is tested by comparing their estimated – adjusted treatment means – with each other.

The analysis of covariance for one linear covariate is as follows:

Source	Sum of Squares (SS)	Degrees of Freedom(d.f.)	Mean Square	F
$T\|\beta$	$SS_{T\|\beta}$	$a-1$	$MS_{T\|\beta} = \frac{SS_T}{a-1}$	$\frac{MS_{T\|\beta}}{MS_E}$
$\beta\|T$	$SS_{\beta\|T}$	1	$MS_{\beta\|T} = \frac{SS_T}{1}$	$\frac{MS_{\beta\|T}}{MS_E}$
Error	SS_E	$n-a-1$	$MS_E = \frac{SS_E}{n-a-1}$	
Total	SS_{yy}	$n-1$		

Where the formulas for computation of this table are:

$$SS_{T|\beta} = \left(ss_{yy} - \frac{(ss_{xy})^2}{ss_{xx}} \right) - \left(ss^*_{yy} - \frac{(ss^*_{xy})^2}{ss^*_{xx}} \right)$$

$$SS_{\beta|T} = \frac{(ss^*_{xy})^2}{ss^*_{xx}}$$

$$SS_E = ss^*_{yy} - \frac{(ss^*_{xy})^2}{ss^*_{xx}}$$

$$ss_{xx} = \sum_i \sum_j (x_{ij} - \overline{x}_{..})^2, \quad ss^*_{xx} = \sum_i \sum_j (x_{ij} - \overline{x}_{i.})^2$$

$$ss_{yy} = \sum_i \sum_j (y_{ij} - \overline{y}_{..})^2, \quad ss^*_{yy} = \sum_i \sum_j (y_{ij} - \overline{y}_{i.})^2$$

$$ss_{xy} = \sum_i \sum_j (x_{ij} - \overline{x}_{..})(y_{ij} - \overline{y}_{..}), \quad ss^*_{xy} = \sum_i \sum_j (x_{ij} - \overline{x}_{i.})(y_{ij} - \overline{y}_{i.})$$

Assumptions in Ancova

We need to verify the following assumptions:

1. The covariate x_{ij} is not affected by the experimental factors.
2. The regression relationship measured by parameter β must be the same for all factor levels.

4.2 General Approach to Ancova

- First we look at the effect of x_{ij}. If it isn't significant, do an Anova.
- Check to see that x_{ij} is not significantly affected by the factor values.
- Test to see that β is not significantly different for all factor levels. This is an interaction between the factors and the covariate.
- Order matters: the covariate comes after the factors in the model because they're less important.
- If both tests pass, do the ANCOVA.

4.3 Monofilament Fiber

Example 4.1. (From D. C. Montgomery, 1997, p. 150) Consider a study performed to determine if there is a difference in the strength y of monofilament fiber produced by three different machines. The data are shown in the table below.

Machine 1		Machine 2		Machine 3	
y	x	y	x	y	x
36	20	40	22	35	21
41	25	48	28	37	23
39	24	39	22	42	26
42	25	45	30	34	21
49	32	44	28	32	15

The process engineer is interested in determining if there is a difference in the breaking strength of the fiber produced by the three machines. Analyze the data and draw conclusions.

In this problem the response variable is y = strength. However the thickness x of the fiber will clearly affect strength. This varies both between and within machines, and can be measured but not controlled. The layout of the design is as for a complete randomized design.

Here we have 3 treatments (machines) and 5 observations made in each treatment group, these are carried out in random order. From the design standpoint the only difference is that the covariate is measured along with the response variable.

The strength of the fiber is related to its diameter, with thicker fibers being generally stronger than thinner ones. Figure 4.1 on page 55 shows the scatter diagram, which shows a strong suggestion of a linear relationship between breaking strength and diameter, and it seems appropriate to remove the effect of diameter on strength by an analysis of covariance.
The Ancova table is shown below (in this case we get the P-value and adjusted sum of squares for machine).

```
Analysis of Variance Table
Response: y
          Df  Sum Sq Mean Sq  F value   Pr(>F)
X          1 305.130 305.130 119.9330 2.96e-07 ***
machine    2  13.284   6.642   2.6106   0.1181
Residuals 11  27.986   2.544
```

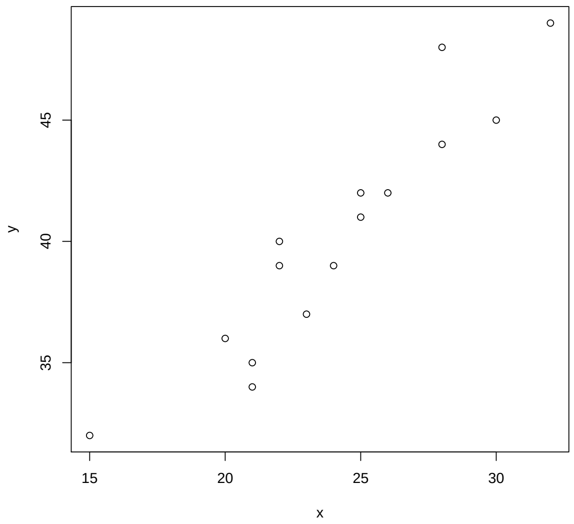


Figure 4.1: Monofilament Fiber, Scatter plot

From this, the P-value for the machine is 0.1181. We conclude that there is no significant difference between the machines, once their output is adjusted for fiber thickness, (ignoring x, gives a P-value of 0.4 and it is an incorrect one-way ANOVA). The 13.284 in the output is referred to as $SS(machine|x)$, the sums of squares for machines adjusted for thickness, it means that the machines have no effect on the monofilament fiber. The Ancova table for thickness is as follows:

```
Analysis of Variance Table
Response: y
          Df  Sum Sq Mean Sq F value    Pr(>F)
machine    2 140.400  70.200  27.593 5.170e-05 ***
X          1 178.014 178.014  69.969 4.264e-06 ***
Residuals 11  27.986   2.544
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
Residual standard error: 1.595 on 11 degrees of freedom
Multiple R-squared: 0.9192,     Adjusted R-squared: 0.8972
F-statistic: 41.72 on 3 and 11 DF,  p-value: 2.665e-06
```

Entering X last shows that the variation in fiber thickness accounts for a significant amount of the variation in strength (P-value$= 4.264 \times 10^{-6}$, $SS(x|machine) = 178.014$). Of course this P-value is also that for the hypothesis $H_0 : \beta = 0$.

Now it's time to do the regression analysis with factor x (diameter), while the machines have no effect on the fiber thickness. So you can see the ANOVA table below:

```
Analysis of Variance Table
Response: y
          Df  Sum Sq Mean Sq F value    Pr(>F)
X          1 305.130 305.130  96.116 2.263e-07 ***
Residuals 13  41.270   3.175
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 1.782 on 13 degrees of freedom
Multiple R-squared: 0.8809,     Adjusted R-squared: 0.8717
F-statistic: 96.12 on 1 and 13 DF,  p-value: 2.263e-07
```

As you can see this model is appropriate with $r^2_{adj} = 87\%$. The regression model is then: $y_{ij} = 14.143 + 1.0797x_i$. From the above computation we conclude that diameter has an effect on fiber thickness.

Now the question is: Does the factor machine have an effect on the variable x? To answer this question we have to apply the one-way ANOVA with response variable x and machine as factor. Below you can see the results:

```
Analysis of Variance Table
Response: X
          Df  Sum Sq Mean Sq F value Pr(>F)
machine    2  66.133  33.067  2.0286 0.1742
Residuals 12 195.600  16.300
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 4.037 on 12 degrees of freedom
Multiple R-squared: 0.2527,     Adjusted R-squared: 0.1281
F-statistic: 2.029 on 2 and 12 DF,  p-value: 0.1742
```

We can conclude from the results above that the factor machine doesn't have any effect on variable x, so we can do our process as already described.
Figure 4.2 on page 57 shows the four diagnostic plots, and the following normality tests show that the residuals assumed to be normally distributed, so the covariance analysis is an appropriate model for thickness of the monofilament fiber.

```
1)  Shapiro-Wilk normality test
data:  residuals(g0)
W = 0.9616, p-value = 0.72
```

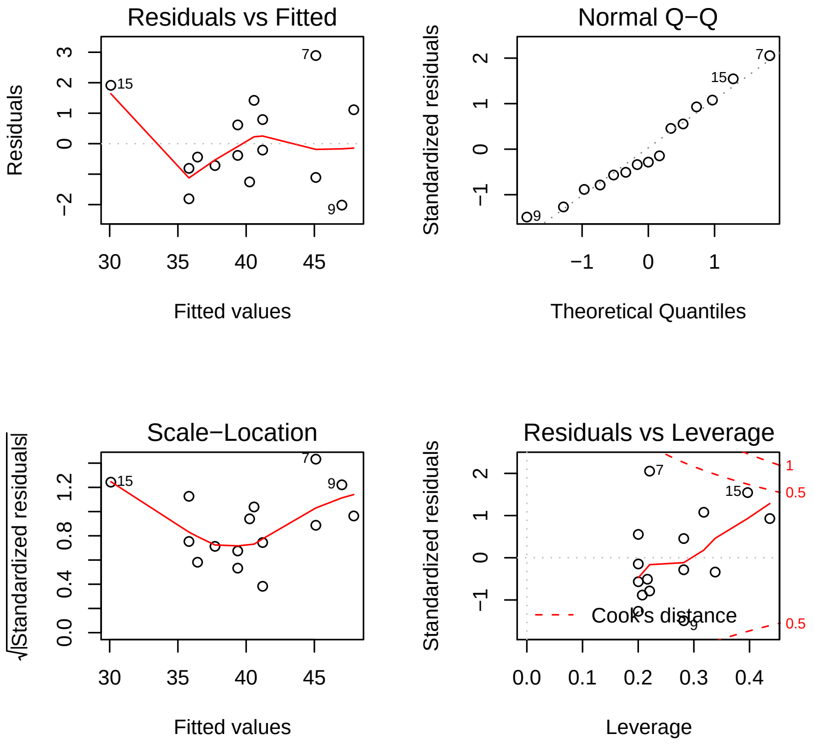


Figure 4.2: Monofilament Fiber, The four diagnostic plots

```
2)  One-sample Kolmogorov-Smirnov test
data:  residuals(g0)
D = 0.1588, p-value = 0.789
alternative hypothesis: two-sided
```

The second important question is: What will happen if the variable X is not as a covariate in the model (i.e. we ignore it from the model)? One way ANOVA leads to the result that machine is significant (hidden by influence of diameter):

```
Analysis of Variance Table
Response: y
          Df  Sum Sq Mean Sq F value  Pr(>F)
machine    2 140.400  70.200  4.0893 0.04423 *
Residuals 12 206.000  17.167
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 4.143 on 12 degrees of freedom
```

```
Multiple R-squared: 0.4053,     Adjusted R-squared: 0.3062
F-statistic: 4.089 on 2 and 12 DF,  p-value: 0.04423
```

The confidence intervals are shown in figure 4.3.

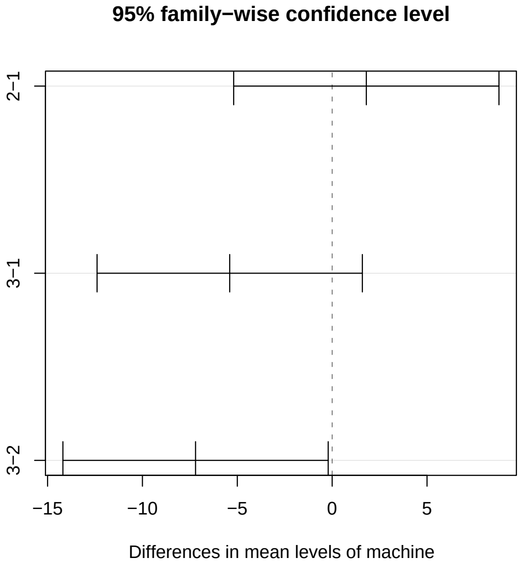


Figure 4.3: Monofilament Fiber, CI

Machines 3 and 2 are significantly different from each other while their confidence interval doesn't contain zero.

Without considering covariate X the machines have an effect on the thickness of the fibers. Figure 4.4 on page 59 shows the boxplot, it indicates the difference in the thickness median of the monofilament fibers. This difference is approved in the Kruskal-Wallis test.

```
Kruskal-Wallis rank sum test
data:  y by machine
Kruskal-Wallis chi-squared = 5.5147, df = 2, p-value = 0.06346
```

While the P-value is slightly larger than $\alpha = 0.05$ so we can not reject the null hypothesis.

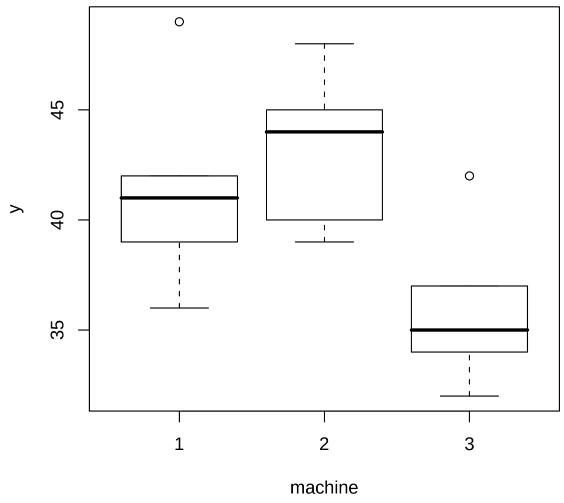


Figure 4.4: Monofilament Fiber, Boxplot

Monofilament Fiber R-Code

```
y<-c(36,41,39,42,49,40,48,39,45,44,35,37,42,34,32)
x<-c(20,25,24,25,32,22,28,22,30,28,21,23,26,21,15)
X<-x-mean(x); machine<-as.factor(rep(1:3,each=5))
data<-data.frame(y,x,machine,X); data; plot(y,x)
#To get p-value and adjusted SS for machine
g<-lm(y~X+machine); anova(g)
#To get p-value and adjusted SS for thickness
g0<-lm(y~machine+X); anova(g0); summary(g)
predict(g, new=data.frame(machine=as.factor(1),X=0), se.fit=T)
g1<-lm(y~X); anova(g1); summary(g1)
#one-way anova
g2<-lm(X~machine); anova(g2); summary(g2)
#residual analysis
windows(); par(mfrow=c(2,2)); plot(aov(y~machine+X))
shapiro.test(residuals(g0))
ks.test(residuals(g0),"pnorm",mean(residuals(g0))
,sd(residuals(g0)))
kruskal.test(y~machine+X)
#one-way anova (x isn't in the model)
g3<-lm(y~machine); anova(g3); summary(g3)
model<-aov(y~machine); TukeyHSD(model)
plot(TukeyHSD(model)); plot(y~machine)
kruskal.test(y~machine)
```

4.4 Glue

Example 4.2. (From D. C. Montgomery, 1997, p. 169, Problem 4-19) Four different formulations of an industrial glue are being tested. The tensile strength of the glue when it is applied to join parts is also related to the application thickness. Five observations on the strength y in pounds and thickness x in 0.01 inches are obtained for each formulation. The data are shown in the following table. Analyze these data and draw appropriate conclusions.

1		2		3		4	
y	x	y	x	y	x	y	x
46.5	13	48.7	12	46.3	15	44.7	16
45.9	14	49.0	10	47.1	14	43.0	15
49.8	12	50.1	11	48.9	11	51.0	10
46.1	12	48.5	12	48.2	11	48.1	12
44.3	14	45.2	14	50.3	10	48.6	11

In this problem we have 4 treatments (glue formulation) and 5 observations for each treatment. The tensile strength of the glue when it is applied to join parts is also related to the application thickness. Figure 4.5 shows the scatter plot, it indicates a negative linear relation between x and y, thats why we can use x as a covariate.

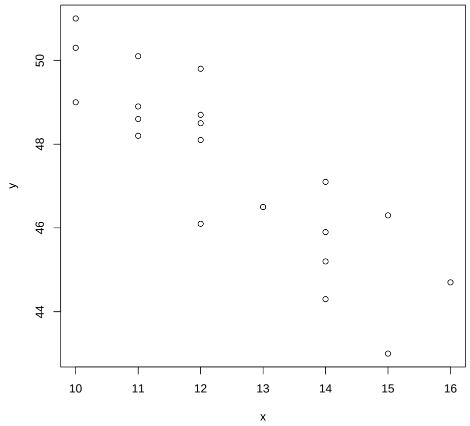


Figure 4.5: Glue, Scatter plot

Question 1: Does the formulation has an effect on the strength of the glue? The Ancova table is shown below (in this case we get the P-value and adjusted sums of squares for glue).

```
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value    Pr(>F)
x          1 68.852  68.852 55.5413 2.038e-06 ***
glue       3  4.138   1.379  1.1128    0.3749
Residuals 15 18.595   1.240
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 1.113 on 15 degrees of freedom
Multiple R-squared: 0.797,      Adjusted R-squared: 0.7428
F-statistic: 14.72 on 4 and 15 DF,  p-value: 4.471e-05
```

In this case, the P-value for the glue is 0.3749. Here there is no significant difference between the glue, once their output is adjusted for glue thickness, (ignoring x, gives a P-value=0.5142 and it is an incorrect one way ANOVA). The 4.138 in the output is referred to as $SS(glue|x)$, the SS for glue adjusted for thickness, it means that the tickness have no effect on the glue. The Ancova table for thickness reads

```
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value    Pr(>F)
glue       3 11.886   3.962  3.1959   0.05399 .
X          1 61.105  61.105 49.2919 4.134e-06 ***
Residuals 15 18.595   1.240
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 1.113 on 15 degrees of freedom
Multiple R-squared: 0.797,      Adjusted R-squared: 0.7428
F-statistic: 14.72 on 4 and 15 DF,  p-value: 4.471e-05
```

From the results above we can conclude that the formulation has no significant effect on glue. While there is no effect of formulation to thickness we will do the regression analysis with factor x.

```
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value   Pr(>F)
x          1 68.852  68.852  54.517 7.53e-07 ***
Residuals 18 22.733   1.263
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 1.124 on 18 degrees of freedom
Multiple R-squared: 0.7518,     Adjusted R-squared: 0.738
F-statistic: 54.52 on 1 and 18 DF,  p-value: 7.53e-07
```

As you see this model is appropriate with $r^2_{adj} = 0.738$, and the regression model is $y_{ij} = 60.5356 - 1.0458x_i$. From the results above it is obvious that the thickness has an affect on the glue.

Question 2: Does the formulation has an effect on the thickness x? In this case we have a one way ANOVA.

```
1) Analysis of Variance Table
Response: X
          Df Sum Sq Mean Sq F value  Pr(>F)
glue       3 20.150   6.717  2.5109 0.09557 .
Residuals 16 42.800   2.675
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
Residual standard error: 1.636 on 16 degrees of freedom
Multiple R-squared: 0.3201,     Adjusted R-squared: 0.1926
F-statistic: 2.511 on 3 and 16 DF,  p-value: 0.09557
```

Since the variable x isn’t affected by the formulation so it can be used as a covariate. Now it’s time to do the residual analysis. Figure 4.6 on page 63 shows the four diagnostic plots. The two normality tests are as follows:

```
1)     Shapiro-Wilk normality test
data:  residuals(g)
W = 0.9489, p-value = 0.3503

2) One-sample Kolmogorov-Smirnov test
data:  residuals(g)
D = 0.1147, p-value = 0.9284
alternative hypothesis: two-sided
```

We conclude that the residuals are normally distributed.

Question 3: What will happen if we eliminate x from the model? Then we have the following one way ANOVA.

```
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value Pr(>F)
glue       3 11.886   3.962  0.7953 0.5142
Residuals 16 79.700   4.981
Residual standard error: 2.232 on 16 degrees of freedom
Multiple R-squared: 0.1298,     Adjusted R-squared: -0.03339
F-statistic: 0.7953 on 3 and 16 DF,  p-value: 0.5142
```

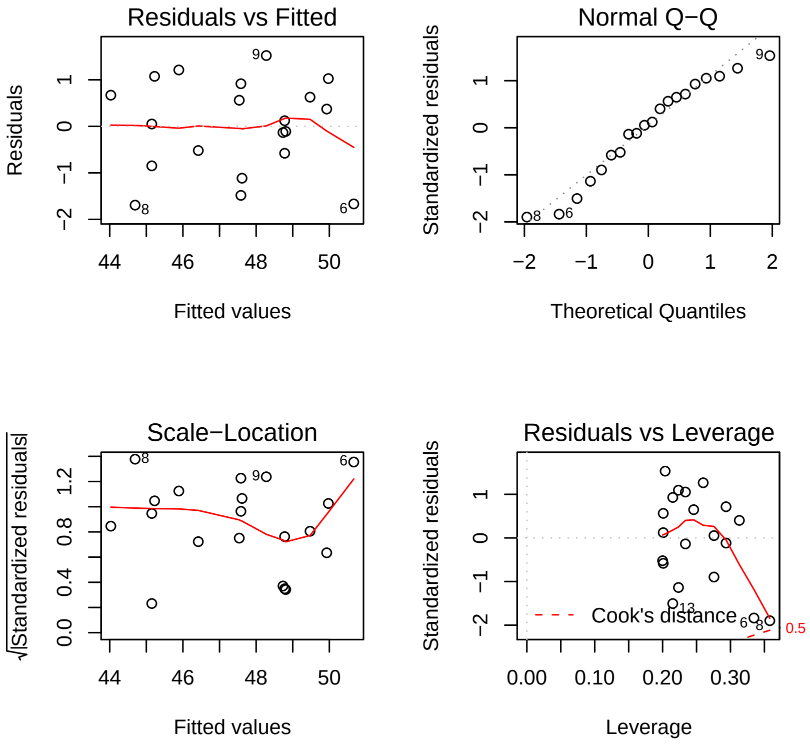


Figure 4.6: Glue, The four diagnostic plots

In this case the "wrong" analysis of variance gives us the correct answer. The boxplot in figure 4.7 on page 64 indicates that there is no difference in the median of the four formulations even though we didn't use covariate x.

```
Kruskal-Wallis rank sum test
data:  y by glue
Kruskal-Wallis chi-squared = 2.6571, df = 3, p-value = 0.4476
```

The Kruskal-Wallis test shows also no significant difference in the mean of the formulations. From the confidence intervals we can conclude that none of these formulations differs from each other so we reject the null hypothesis while all confidence intervals contain the value zero.

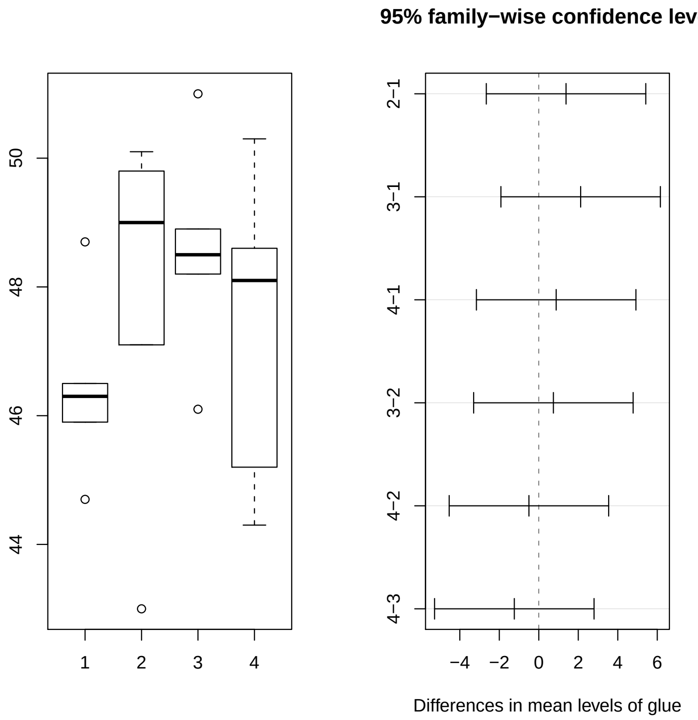


Figure 4.7: Glue, Boxplot and CI

Glue Formulation R-code

```
y<-c(46.5,48.7,46.3,44.7,45.9,49.0,47.1,43.0,49.8,50.1,48.9,51.0,46.1,48.5
,48.2,48.1,44.3,45.2,50.3,48.6)
x<-c(13,12,15,16,14,10,14,15,12,11,11,10,12,12,11,12,14,14,10,11)
X<-x-mean(x); glue<-as.factor(rep(1:4,each=5))
data<-data.frame(y,x,glue,X); data
plot(x,y); g<-lm(y~glue+X); anova(g); summary(g)
g0<-lm(y~x+glue); anova(g0); summary(g0)
#Regression analysis
g1<-lm(y~x); anova(g1); summary(g1)
#one-way Anova
g2<-lm(X~glue); anova(g2); summary(g2)
windows(); par(mfrow=c(2,2)); plot(aov(y~glue+X))
shapiro.test(residuals(g))
ks.test(residuals(g),"pnorm",mean(residuals(g))
,sd(residuals(g)))
#one way anova without X
g3<-lm(y~glue); anova(g3); summary(g3)
windows(); par(mfrow=c(1,2)); plot(glue,y)
model<-aov(y~glue); TukeyHSD(model)
plot(TukeyHSD(model)); kruskal.test(y~glue)
```

Chapter 5

Factorial Designs

We study the effects of two or more factors, each at several levels. A Factorial Design has observations at all combinations of these levels. Factorial ANOVA can be used when factors are crossed with each other, rather than nested. We have to be careful that all the combinations of factors are included in the design. Note that every level of each factor occurs in combination with every level of the other factors [Montgomery D. C. (1997)].

Defining Factorial Design

- Means-

  Cell means: means of all replicates for that combination of treatments.

  Marginal means: means for one factor in analysis.

  Grand mean: overall mean.

- Main effects-

  The effect of a factor is defined to be the change in response produced by a change in the level of the factor.

- Interaction-

  When the difference in response between the levels of one factor is not the same at all levels of the other factors then we have an interaction.

Kinds of Factorial Models

- Model 1- all factors are fixed

  Tests for effect of main effects and interaction effects. Effects tested over mean square errors (MSE).

- Model 2- all factors are random

  Fairly uncommon in biology. Main effects tested over interaction terms, interaction effects over MSE.

- Model 3- Some factors are fixed and the others are random

  This model is called a mixed model.

In the following we discuss only factorial models where all factors are fixed (Model 1).

Note that factorial designs have several advantages. They are more efficient than one-factor- at a time experiments. Furthermore a factorial design is necessary when the interactions may be present to avoid misleading conclusions. Finally, factorial designs allow the effects of a factor to be estimated at several levels of the factors, yielding conclusions that are valid over a range of experimental conditions.

5.1 The Two-Factor Factorial Design

The effects model, including terms for interaction, is that the k^{th} observation at level i of A, j of B is:

$$y_{ijk} \;=\; \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}, \quad \varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$$

$i = 1, \ldots, a$ (levels of factor A); $j = 1, \ldots, b$ (levels of factor B); $k = 1, \ldots, t$.

Constraints are $\sum_i \tau_i = 0$ (average effect of levels of A is 0), $\sum_j \beta_j = 0$ (average effect of levels of B is 0), and average interaction $\sum_i (\tau\beta)_{ij} = 0$ for $j = 1, \ldots, b$; $\sum_j (\tau\beta)_{ij} = 0$ for $i = 1, \ldots, a$.

Reasonable estimates of these effects, obeying these constraints, are

$$\hat{\mu} = \overline{y}_{..}, \quad \widehat{\tau}_i = \overline{y}_{i..} - \overline{y}_{...} \quad \widehat{\beta}_j = \overline{y}_{.j.} - \overline{y}_{...}$$

$$\widehat{(\tau\beta)}_{ij} = \overline{y}_{ij.} - \overline{y}_{...} - \widehat{\tau}_i - \widehat{\beta}_j = \overline{y}_{ij.} - \overline{y}_{i..} - \overline{y}_{.j.} + \overline{y}_{...} \quad \text{(grand mean)}$$

where

$$\begin{aligned}
\overline{y}_{..} &= \frac{1}{abt}\sum_i\sum_j\sum_k y_{ijk} \\
\overline{y}_{i..} &= \frac{1}{bt}\sum_j\sum_k y_{ijk} \quad \text{(mean for A at level i)} \\
\overline{y}_{.j.} &= \frac{1}{at}\sum_i\sum_k y_{ijk} \quad \text{(mean for B at level j)} \\
\overline{y}_{ij.} &= \frac{1}{t}\sum_k y_{ijk} \quad \text{(mean of cell (i,j))}
\end{aligned}$$

The ANOVA table for a two factor factorial experiment with t observations per cell is ($n = abt$):

Source	SS	df	MS	F_0
A	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(t-1)$	$MS_E = \frac{SS_E}{ab(t-1)}$	
Total	SS_T	$abt-1$		

We have

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E = \sum_i\sum_j\sum_k (y_{ijk} - \overline{y}_{...})^2$$

$$SS_A = tb\sum_i \widehat{\tau}_i^2, \quad SS_B = ta\sum_j \widehat{\beta}_j^2, \quad SS_{AB} = t\sum_i\sum_j \widehat{\tau\beta}_{ij}^2,$$

$$SS_E = \sum_i\sum_j\sum_k (y_{ijk} - \overline{y}_{ij.})^2$$

5.2 The General Factorial Design

In the general factorial design more than two factors should be investigated, a levels of factor A, b levels of factor B, c levels of factor C, ..., and t replicates. The total number of observations is *abc...t*.

For example, consider the three-factor analysis of variance model:

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijkl},$$

$$\varepsilon_{ijkl} \overset{iid}{\sim} N(0, \sigma^2); \quad i = 1, \ldots, a; \quad j = 1, \ldots, b; \quad k = 1, \ldots, c; \quad l = 1, \ldots, t.$$

The ANOVA table in this case is as follows:

Source	SS	df	MS	F_0
A	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
C	SS_C	$c-1$	$MS_C = \frac{SS_C}{c-1}$	$F_0 = \frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	$MS_{AC} = \frac{SS_{AC}}{(a-1)(c-1)}$	$F_0 = \frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	$MS_{BC} = \frac{SS_{BC}}{(b-1)(c-1)}$	$F_0 = \frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	$MS_{ABC} = \frac{SS_{ABC}}{(a-1)(b-1)(c-1)}$	$F_0 = \frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(t-1)$	$MS_E = \frac{SS_E}{abc(t-1))}$	
Total	SS_T	$abct-1$		

Where the sum of squares are easily generalized:

$$SS_T = \sum_i \sum_j \sum_k \sum_l (y_{ijkl} - \overline{y}_{....})^2$$

The sum of squares for the interactions are:

$$
\begin{aligned}
SS_A &= bct\sum_i(\overline{y}_{i...} - \overline{y}_{....})^2 \\
SS_B &= act\sum_j(\overline{y}_{.j..} - \overline{y}_{....})^2 \\
SS_C &= abt\sum_k(\overline{y}_{..k.} - \overline{y}_{....})^2 \\
SS_{AB} &= ct\sum_i\sum_j(\overline{y}_{ij..} - \overline{y}_{....})^2 - SS_A - SS_B \\
SS_{AC} &= bt\sum_i\sum_k(\overline{y}_{i.k.} - \overline{y}_{....})^2 - SS_A - SS_C \\
SS_{BC} &= at\sum_j\sum_k(\overline{y}_{.jk.} - \overline{y}_{....})^2 - SS_B - SS_C \\
SS_{ABC} &= t\sum_i\sum_j\sum_k(y_{ijk.} - \overline{y}_{....})^2 - SS_A - SS_B - SS_C - SS_{AC} - SS_{BC} \\
SS_E &= \sum_i\sum_j\sum_k\sum_l(y_{ijkl} - \overline{y}_{ijk.})^2
\end{aligned}
$$

5.3 Soft Drink Bottling Problem

Example 5.1. (From D. C. Montgomery, 1997, p. 259) A soft drink bottler is interested on obtaining more uniform fill heights in the bottles produced by his manufacturing process. The filling machine theoretically fills each bottle to the correct target height, but in practice, there is variation around this target, and the bottler would like to understand better the source of this variability and eventually reduce it.

The process engineer can control three variables during the filling process: the percent carbonation (A), the operating pressure in the filler (B), and the bottles produced per minute or the line speed (C). The pressure and speed are easy to control during actual manufacturing because it varies with product temperature. However, for purpose of an experiment, the engineer can control carbonation at three levels: 10, 12, and 14 percent. He chooses two levels for pressure (25 and 30 psi) and two levels for line speed (200 and 250 bpm). He decides to run two replicates for a factorial design in these three factors, with all 24 runs taken in random order. The response variable y observed is the average deviation from the target fill height observed in a production run of bottles at each set of conditions. The data that resulted from this experiment are shown in the table below:

	Operating Pressure (B)			
	25 psi		30 psi	
Percent	Line Speed (C)			
Carbonation (A)	200	250	200	250
10	-3	-1	-1	1
	-1	0	0	1
12	0	2	2	6
	1	1	3	5
14	5	7	7	10
	4	6	9	11

Analyze the data.

Soft drink bottlers must maintain targets for fill heights, and any variation is a cause for concern. The deviation y from the target is affected by % carbonation (A), pressure in the filter (B) and the line speed (C). In this problem we have 24 runs in random order.

To fit an ANOVA model to this data, we simply generalize the single factor completely randomized design. Figure 7.1 on page 71 shows the interaction plots and the univariate effects of the model.

Figure 5.1 indicates no interaction effects between factors. Notice that all three main variables have positive main effects; that is, increasing the variable moves the average

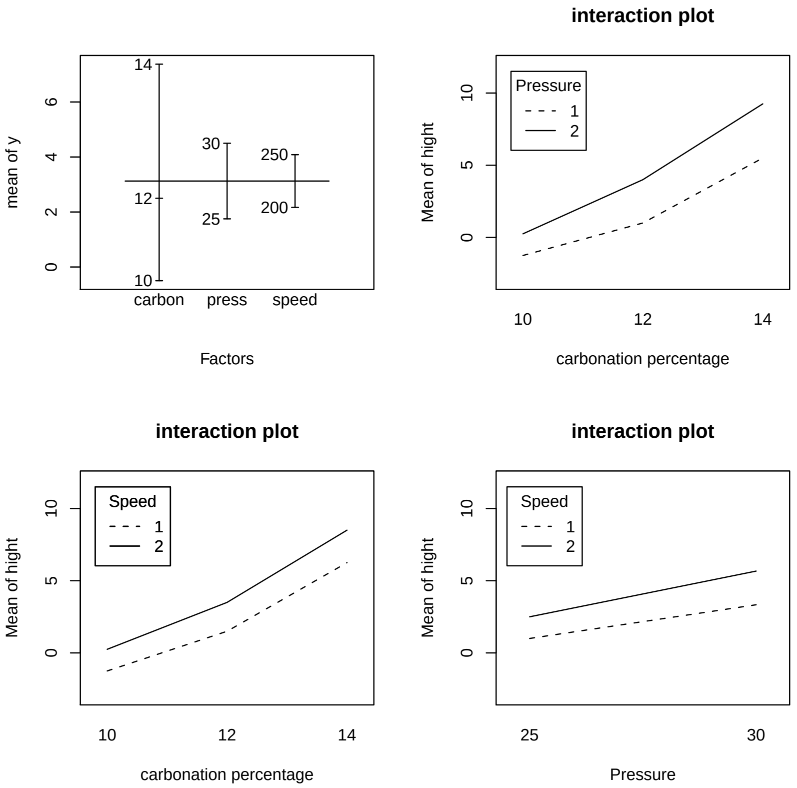


Figure 5.1: Soft Drink, Interaction plots

deviation from the fill target upward. We have a full three-factor model (5.1).

The ANOVA table of this full model is constructed below:

```
Analysis of Variance Table
Response: y
                   Df  Sum Sq Mean Sq  F value    Pr(>F)
carbon              2 252.750 126.375 178.4118 1.186e-09 ***
press               1  45.375  45.375  64.0588 3.742e-06 ***
speed               1  22.042  22.042  31.1176 0.0001202 ***
carbon:press        2   5.250   2.625   3.7059 0.0558081 .
carbon:speed        2   0.583   0.292   0.4118 0.6714939
press:speed         1   1.042   1.042   1.4706 0.2485867
carbon:press:speed  2   1.083   0.542   0.7647 0.4868711
Residuals          12   8.500   0.708
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’0.05 ’.’ 0.1 ’’ 1
```

```
Residual standard error: 0.8416 on 12 degrees of freedom
Multiple R-squared: 0.9747,     Adjusted R-squared: 0.9516
F-statistic: 42.11 on 11 and 12 DF,  p-value: 7.417e-08
```

It seems that interactions are largely absent, and that all three main effects are significant. The R^2_{adj} shows that about 95% of the variability in the fill heights is explained by the % carbonation, pressure in the filter, the line speed and their interactions.

For checking ANOVA model assumption, the best way is analysis of residuals. We take a model with three main effects and an interaction carbon pressure.

```
Response: y
             Df  Sum Sq Mean Sq  F value    Pr(>F)
carbon        2 252.750 126.375 191.6766 2.178e-12 ***
press         1  45.375  45.375  68.8216 2.218e-07 ***
speed         1  22.042  22.042  33.4312 2.210e-05 ***
carbon:press  2   5.250   2.625   3.9814   0.03818 *
Residuals    17  11.208   0.659
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’0.05 ’.’ 0.1 ’’ 1
Residual standard error: 0.812 on 17 degrees of freedom
Multiple R-squared: 0.9667,     Adjusted R-squared: 0.955
F-statistic: 82.26 on 6 and 17 DF,  p-value: 1.294e-11
```

Note that within each of the 12 groups the residuals tend to be of the same sign, with the signs alternating as we move from group to group.

The error would show up in the residual plot against the fitted values. Normal Q-Q plots and residual plots. In factorial designs, though, we should plot residuals against the predicted values (i.e. treatment means) and the levels of each factor 5.3 on page 74. This safe guards against any unstable variance being embedded within the factorial treatment structure.
The Q-Q plot looks fine. The residual plots are showing what appears to be a mild non-constancy of variance. Here there is no need for transformation.

The last step is a multiple comparison test, by computing the TukeyHSD confidence intervals in figure 5.4 on page 75. As you see just three combinations of the carbon:pressure interaction contain zero which means that they are not significant.

Since the company wants the average deviation from the fill target to be close to zero, the engineer decides to recommend the low level of operating pressure (25 psi) B_- and the high level of line speed (250 bpm) C^+ which maximize the production rate.

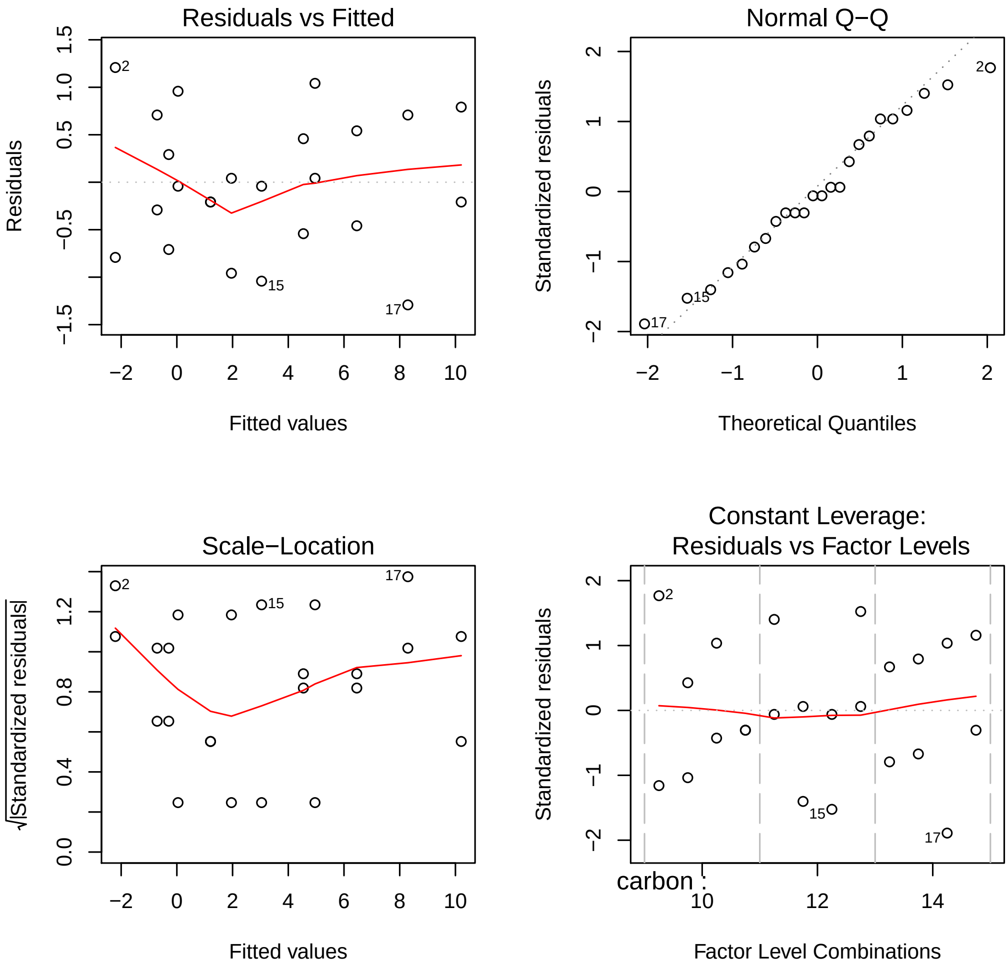


Figure 5.2: Soft Drink, The four diagnostic plots with one interaction

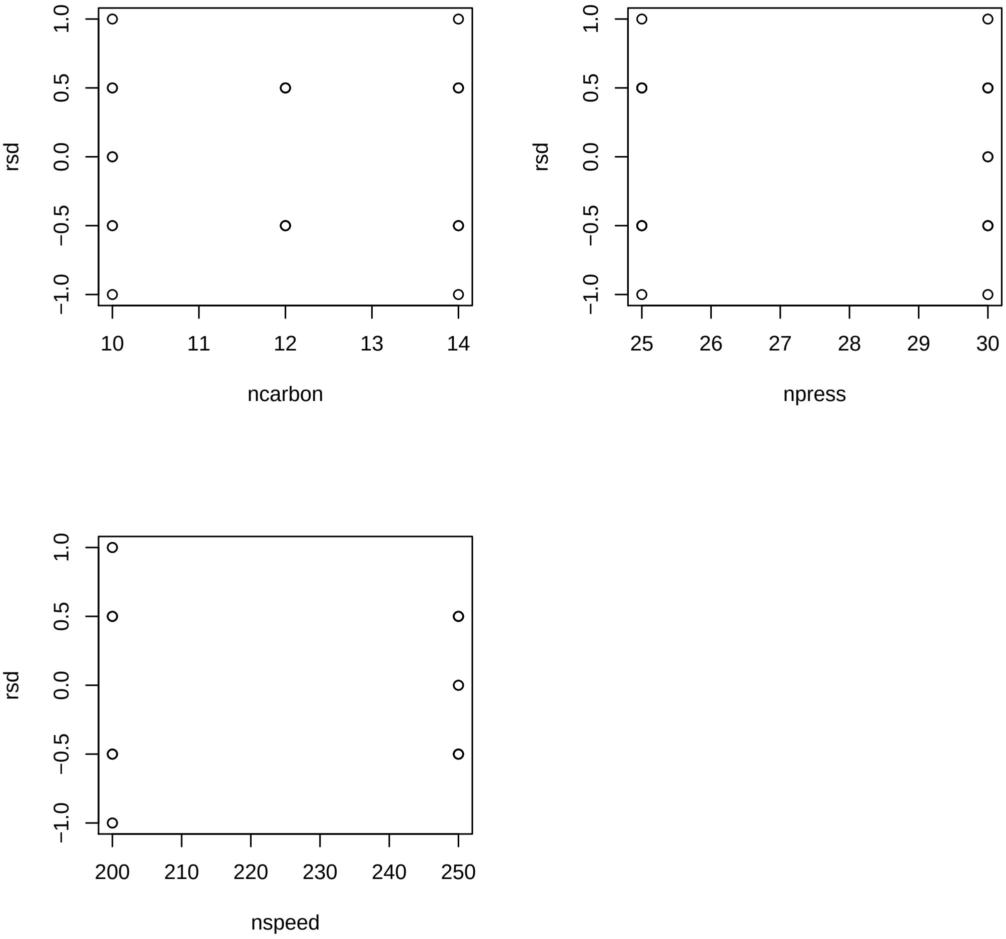


Figure 5.3: Soft Drink, Residuals against level of factors

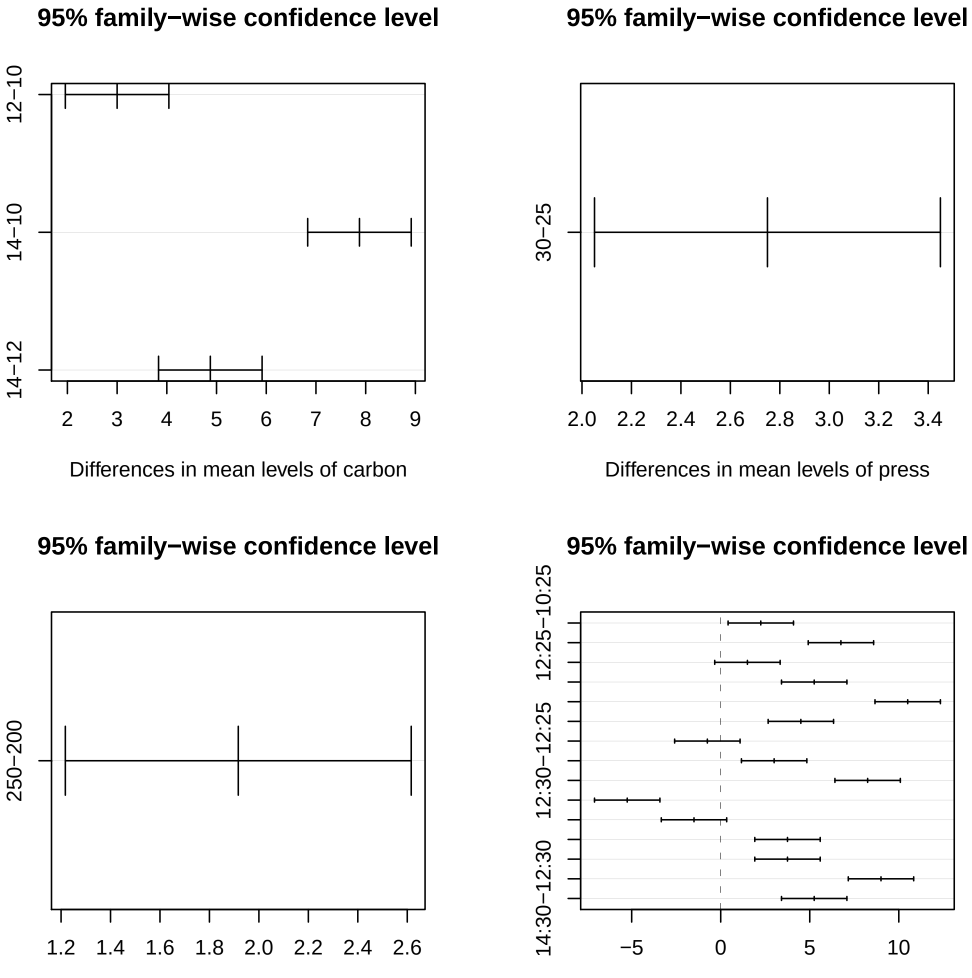


Figure 5.4: Soft Drink, 95% CI

Soft Drink Bottlers- R Code

```
y<-c(-3,-1,0,1,5,4, -1,0,2,1,7,6, -1,0,2,3,7,9, 1,1,6,5,10,11)
carbon<-as.factor(rep(c(10,12,14),each=2, times=4))
press<-as.factor(rep(c(25,30), each=12))
speed<-as.factor(rep(c(200,250), each=6, times=2))
data<-data.frame(y, carbon, press, speed); data
par(mfrow=c(2,2)); plot.design(data)
interaction.plot(carbon,press,y,xlab="carbonation percentage",
ylab="Mean of hight", main="interaction plot",ylim=c(-3,12),
fixed=T,legend=F); legend(0.90,11.50,c("1","2"),
title="Pressure",lty=c(2,1)); interaction.plot(carbon,speed,y,
xlab="carbonation percentage",ylab="Mean of hight",
main="interaction plot",ylim=c(-3,12),fixed=T,legend=F)
legend(0.90,11.50,c("1","2"),title="Speed",lty=c(2,1))
interaction.plot(press,speed,y,xlab="Pressure",
ylab="Mean of hight", main="interaction plot",ylim=c(-3,12),
fixed=T,legend=F)) legend(0.90,11.50,c("1","2"),
title="Speed",lty=c(2,1))
g<-lm(y~carbon+press+speed+carbon*press+carbon*speed+press*speed
+carbon*press*speed)
anova(g); summary(g)
h<-lm(y~carbon+press+speed+carbon+carbon*press); anova(h); par(mfrow=c(2,2))
plot(aov(y~carbon+press+speed+carbon+carbon*press))
model<-aov(y~carbon+press+speed+carbon+carbon*press)
plot(TukeyHSD(model))
ncarbon<-rep(c(10,12,14),each=2, times=4)
npress<-rep(c(25,30), each=12)
nspeed<-rep(c(200,250), each=6, times=2)
windows(); par(mfrow=c(2,2)); plot(ncarbon,y)
plot(npress,y); plot(nspeed,y)
```

5.4 Strength of a paper

Example 5.2. (D. C. Montgomery, 1997, P. 286, Problem 6-16) The percentage of hardwood concentration (A) in row pulp, the vat pressure (C), and the cooking time of the pulp (B) are being investigated for their effects on the strength of paper y. Three levels of hardwood concentration, three levels of pressure, and two cooking times are selected. A factorial experiment with two replicates is conducted ($n = 36$ *runs*), and the following data are obtained:

	Cooking times 3.0 Hours			Cooking times 4.0 Hours		
Percentage of Hardwood Concentration	Pressure			Pressure		
	400	500	650	400	500	650
2	196.6	197.7	199.8	198.4	199.6	200.6
	196.0	196.0	199.4	198.6	200.4	200.9
4	198.5	196.0	198.4	197.5	198.7	199.6
	197.2	196.9	197.6	198.1	198.0	199.0
8	197.5	195.6	197.4	197.6	197.0	198.5
	196.6	196.2	198.1	198.4	197.8	199.8

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

(b) Prepare appropriate residual plots and comment on the models's adequacy.

(c) Under what set of conditions would you operate this process? Why?

For the experimenter the strength of a paper and any variation in it is a cause of concern, the deviation from the target y is affected by % of hardwood concentration (A), the pressure in vat the cooking time of the pulp (B) and (C). In this problem we have $n = 36$ runs in a random order.

We will examine the main effects and the interactions effects by the full factorial model.

```
Analysis of Variance Table
Response: y
                   Df  Sum Sq Mean Sq F value    Pr(>F)
hardwood            2  7.7639  3.8819 10.6193 0.0008996 ***
time                1 20.2500 20.2500 55.3951 6.745e-07 ***
press               2 19.3739  9.6869 26.4992 4.327e-06 ***
hardwood:time       2  2.0817  1.0408  2.8473 0.0842597 .
```

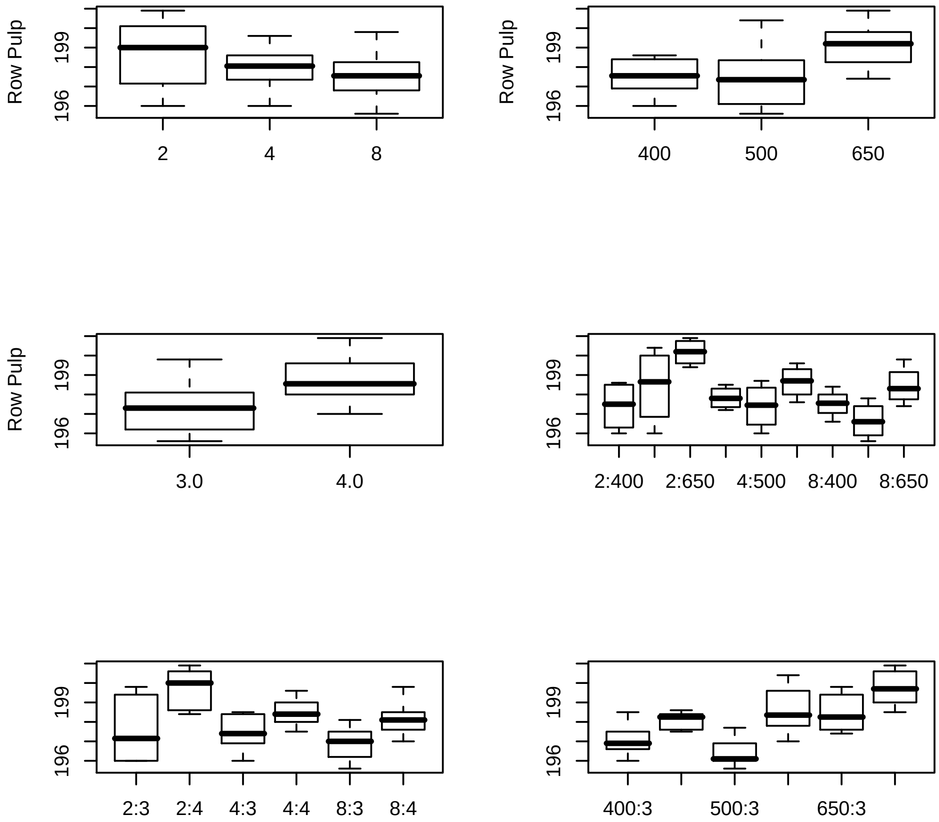


Figure 5.5: Strength of paper dependent on their factos and their combinations, Box plots Series

```
hardwood:press       4  6.0911  1.5228  4.1657 0.0146262 *
time:press           2  2.1950  1.0975  3.0023 0.0749564 .
hardwood:time:press  4  1.9733  0.4933  1.3495 0.2903053
Residuals           18  6.5800  0.3656
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
Residual standard error: 0.6046 on 18 degrees of freedom
Multiple R-squared: 0.9008,     Adjusted R-squared: 0.807
F-statistic: 9.611 on 17 and 18 DF,  p-value: 7.797e-06
```

From the results above we may conclude that the three main effects are highly significant, and the hardwood-time, hardwood-press and time-press interactions with P-values 0.08, 0.01, 0.07 indicate some interactions between these factors. R^2_{adj} indicates that about the 80% of the variability in the strength of the paper is explained by the model.

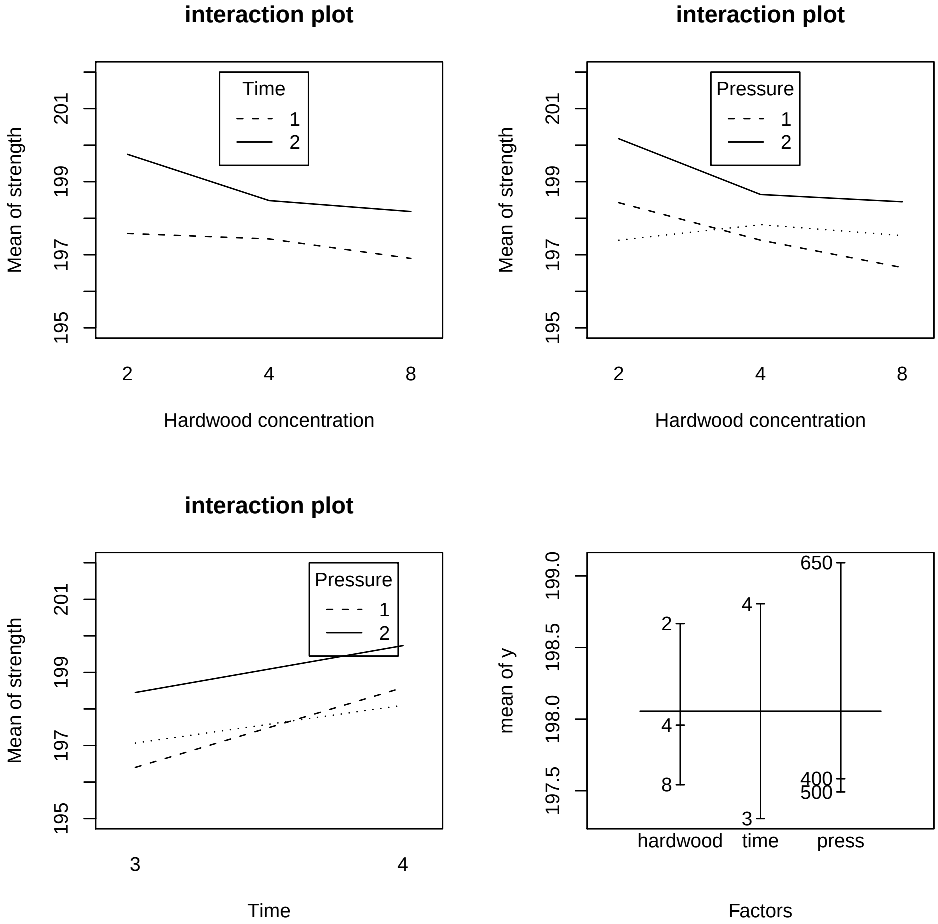


Figure 5.6: Strength of paper, Interaction plots and plot of main effects

Figure 5.6 shows that there is only slight interaction between cooking time and hardwood concentration. At 4 hours cooking time and 2% hardwood concentration we will have the largest strength in the paper. The two other plots show that by choosing the high level of pressure (650), (C_+) and the lowest hardwood percentage (2%) (A_-) and (4 hours) cooking time (B_+) we will have the largest strength in the paper.

We take a model with the three main effects and an interaction factor hardwood-pressure as a new model and we investigate the behavior of the residuals. The four diagnostic plots for the new model are shown in figure 5.7 on page 80.

```
Analysis of Variance Table
Response: y
               Df  Sum Sq Mean Sq F value    Pr(>F)
hardwood        2  7.7639  3.8819  7.8668   0.00213 **
time            1 20.2500 20.2500 41.0366 8.711e-07 ***
press           2 19.3739  9.6869 19.6306 6.370e-06 ***
hardwood:press  4  6.0911  1.5228  3.0859   0.03322 *
Residuals      26 12.8300  0.4935
---
Residual standard error: 0.7025 on 26 degrees of freedom
Multiple R-squared: 0.8065,     Adjusted R-squared: 0.7395
F-statistic: 12.04 on 9 and 26 DF,  p-value: 3.153e-07
```

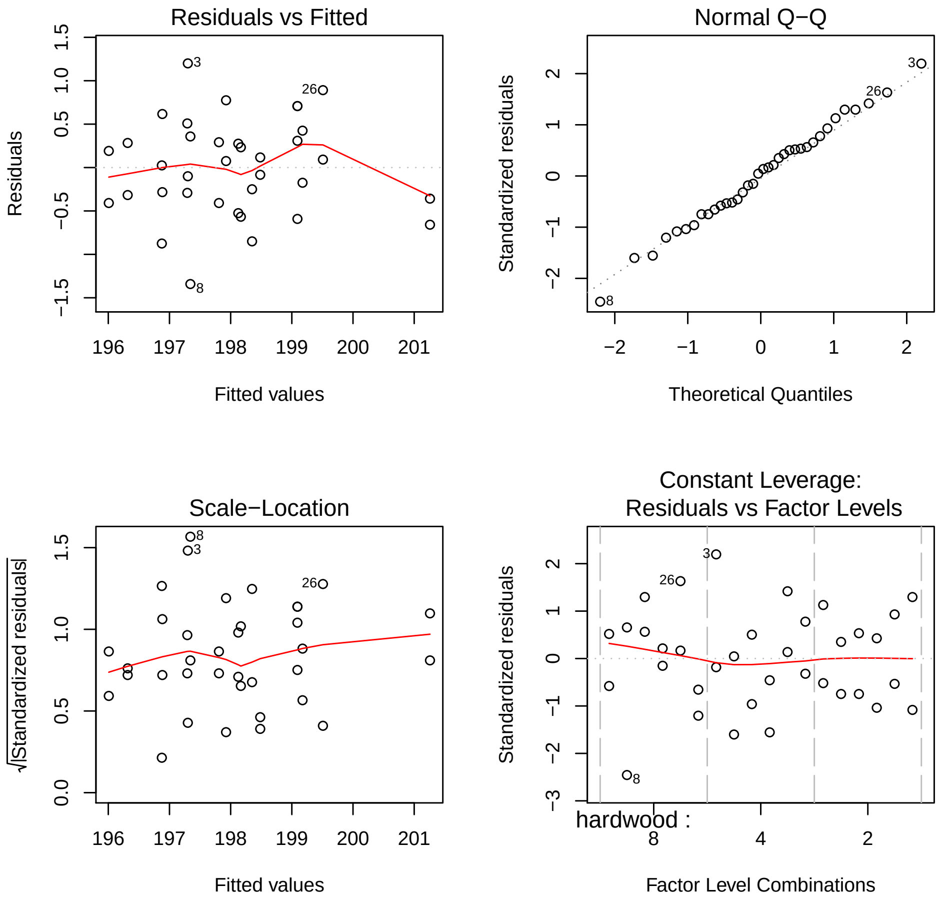


Figure 5.7: Strength of paper,Four diagnostic plots with one interaction

As you can see the Q-Q plots doesn't show any major concerns in the normality of the residuals.

As we see from the above plots in figure 5.7, the normal probability plot and the other plots do not indicate any major concerns. The last thing we can do is the TukeyHSD test, you can see the 95% TukeyHSD confidence intervals in figure 5.8 on page 81.

Figure 5.8 indicates that the hardwood concentration at level 2 is significantly different from the other two levels and the pressure 650 is significantly different from the others.

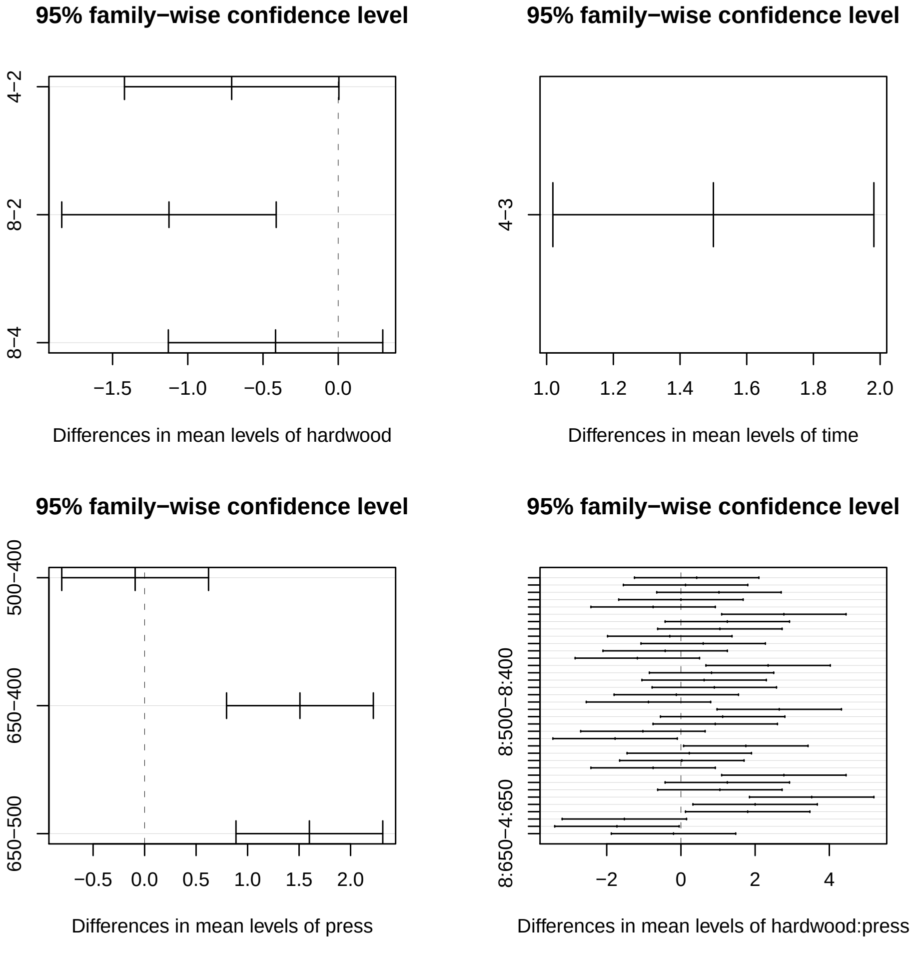


Figure 5.8: Strength of paper, 95% CI

Hardwood R-Code

```
y<-c(196.6,196.0,198.5,197.2,197.5,196.6,197.7,196.0,196.0,196.9,195.6,
196.2,199.8,199.4,198.4,197.6,197.4,198.1,198.4,198.6,197.5,198.1,197.6,
198.4,199.6,200.4,198.7,198.0,197.0,197.8,200.6,200.9,199.6,199.0,198.5,
199.8); hardwood<-as.factor(rep(c(2,4,8),each=2,times=6))
press<-as.factor(rep(c(400,500,650),each=6,times=2))
time<-as.factor(rep(c(3.0,4.0),each=18))
data<-data.frame(y,hardwood,time,press); attach(data); data
windows(); par(mfrow=c(2,2))
interaction.plot(hardwood,time,y,xlab="Hardwood concentration",ylab="Mean
of strength", main="interaction plot",ylim=c(195,202),fixed=T,legend=F)
legend(1.65,202,c("1","2"),title="Time",lty=c(2,1))
interaction.plot(hardwood,press,y,xlab="Hardwood concentration",ylab="Mean
of strength", main="interaction plot",ylim=c(195,202),fixed=T,legend=F)
legend(1.65,202,c("1","2"),title="Pressure",lty=c(2,1))
interaction.plot(time,press,y,xlab="Time",ylab="Mean of strength",
main="interaction plot",ylim=c(195,202),fixed=T,legend=F)
legend(1.65,202,c("1","2"),title="Pressure",lty=c(2,1))
plot.design(data); g<-lm(y˜hardwood+time+press+hardwood*time+hardwood*press
+time*press+hardwood*time*press); anova(g); summary(g)
h<-lm(y˜hardwood+time+press+hardwood*press); anova(h)
windows(); par(mfrow=c(2,2))
plot(aov(y˜hardwood+time+press+hardwood*time+hardwood*press))
windows(); par(mfrow=c(2,2))
model<-aov(y˜hardwood+time+press+hardwood*press)
TukeyHSD(model); plot(TukeyHSD(model))
```

5.5 Metal Part Surface

Example 5.3. (D. C. Montgomery, 1997, p. 282, Exp.6-2) An engineer suspects that the surface finish of a metal part y is influenced by feed rate (A) and the depth of cut (B). She selects three feed rates and four depths of cut. She then conducts a factorial experiment and obtains the following data:

	Depth of Cut (in)			
Feed Rate (in/min)	0.15	0.18	0.20	0.25
0.20	74	79	82	99
	64	68	88	104
	60	73	92	96
0.25	92	98	99	104
	86	104	108	110
	88	88	95	99
0.30	98	99	110	111
	99	104	108	114
	102	95	99	107

(a) Analyze the data and draw conclusions. Use $\alpha = 0.05$.

(b) Prepare appropriate residual plots and comment on the model's adequacy.

(c) Obtain point estimates of the mean surface finish at each feed rate.

(d) Find the P-values for the tests in part (a).

In this problem we have a two factor-factorial design with $n = 36$ runs in a random order. The surface finish y of a metal part is affected by feed rate (A) and the depth of cut (B). The Anova table is shown below:

```
Analysis of Variance Table
Response: y
               Df  Sum Sq Mean Sq F value    Pr(>F)
feedrate        2 3160.50 1580.25 55.0184 1.086e-09 ***
depth           3 2125.11  708.37 24.6628 1.652e-07 ***
feedrate:depth  6  557.06   92.84  3.2324   0.01797 *
Residuals      24  689.33   28.72
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 5.359 on 24 degrees of freedom
Multiple R-squared: 0.8945,     Adjusted R-squared: 0.8461
F-statistic: 18.49 on 11 and 24 DF,  p-value: 4.111e-09
```

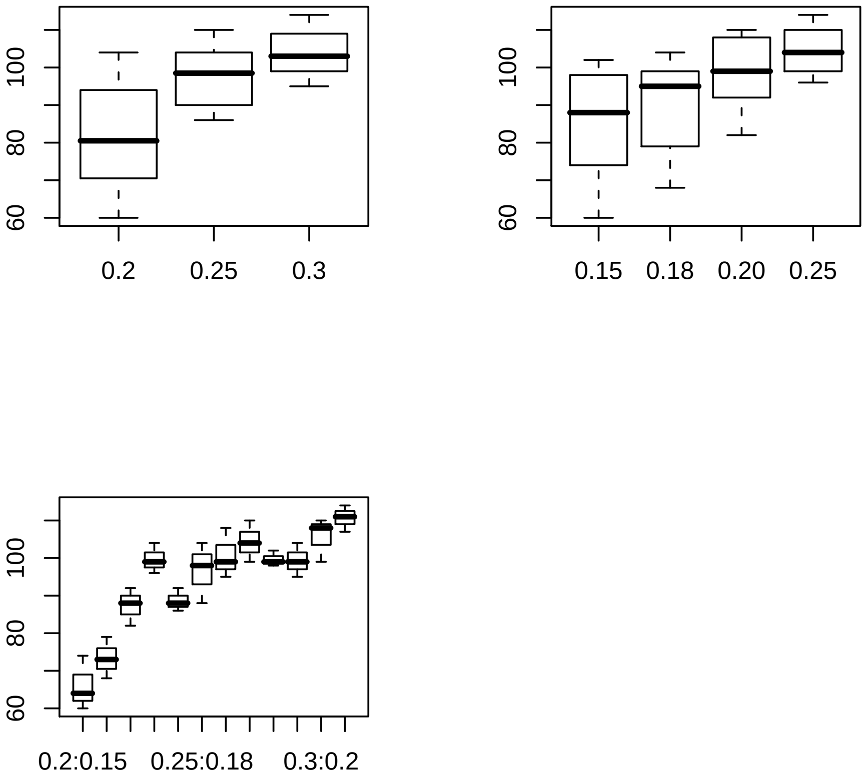


Figure 5.9: Metal Surface, Boxplots Series

The plots in figure 5.9 show that by choosing the feed rate of 0.3 (A_3) and and the depth of 0.25 (B_4) we will have the best surface in a metal part.
From the above results (ANOVA table) we can conclude that the main effects are highly significant and the feed rate-depth interaction with P-value of 0.01797 is also significant. R^2_{adj} indicates that the 85% of the variability in the surface of a metal part is explained by the main effects and their interactions.

Figure 5.10 on page 85 shows the interaction effect and the plot of the design. As we see from figure 5.11 on page 86 the normal probability plot and the other plots do not indicate any major concerns. Figure 5.12 on page 87 shows the 95% TukeyHSD confidence intervals. It indicates that the levels of the feed rate are significantly different from each other while the confidence interval doesn't contain zero. This conclusion can be made also for the depth of cut levels just the two levels (0.18 and 0.15) don't differ from each other.

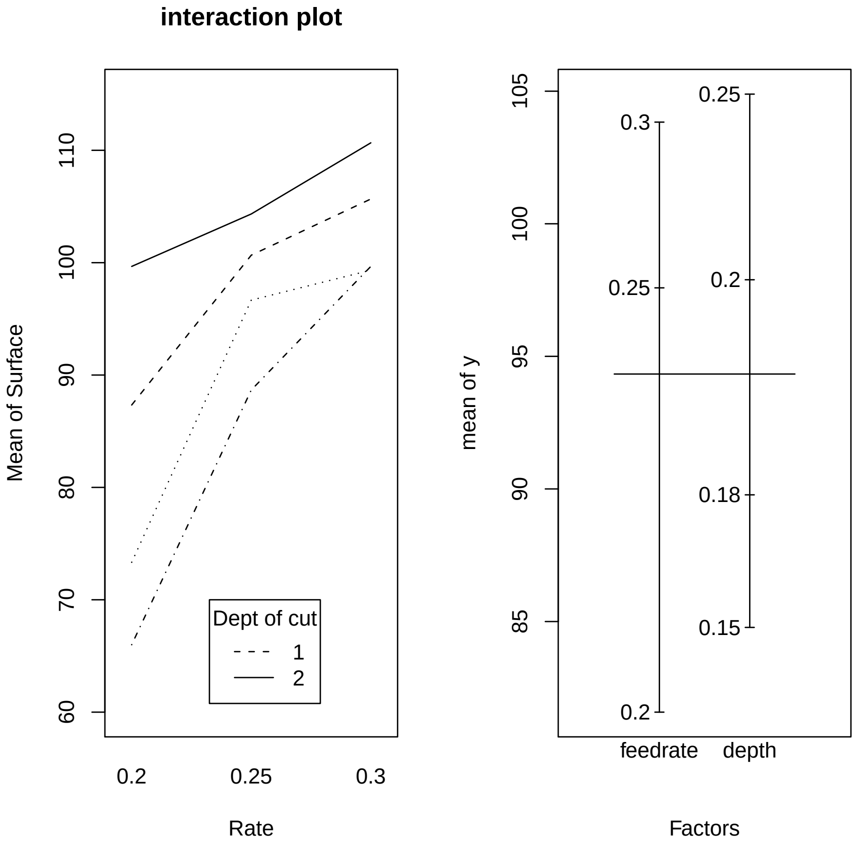


Figure 5.10: Metal Surface, Main effects plots and Interaction plots

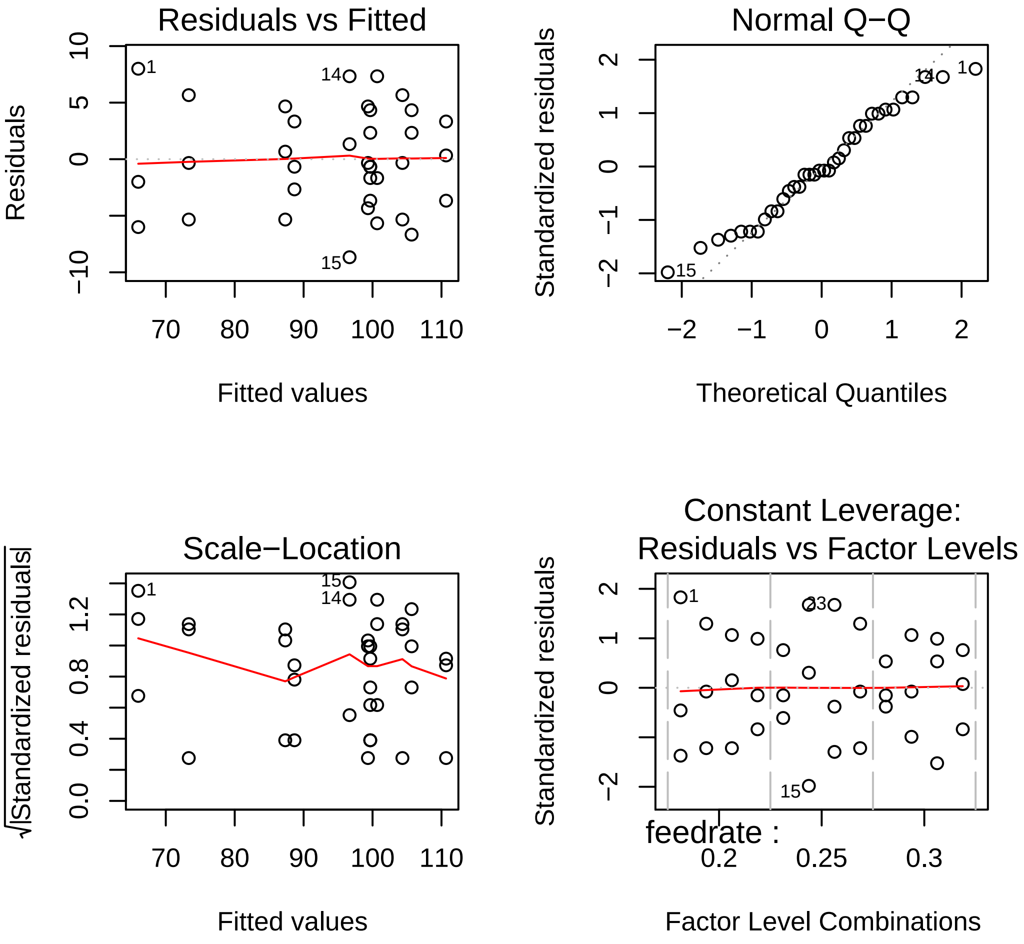


Figure 5.11: Metal Surface, The four diagnostic plots with interaction

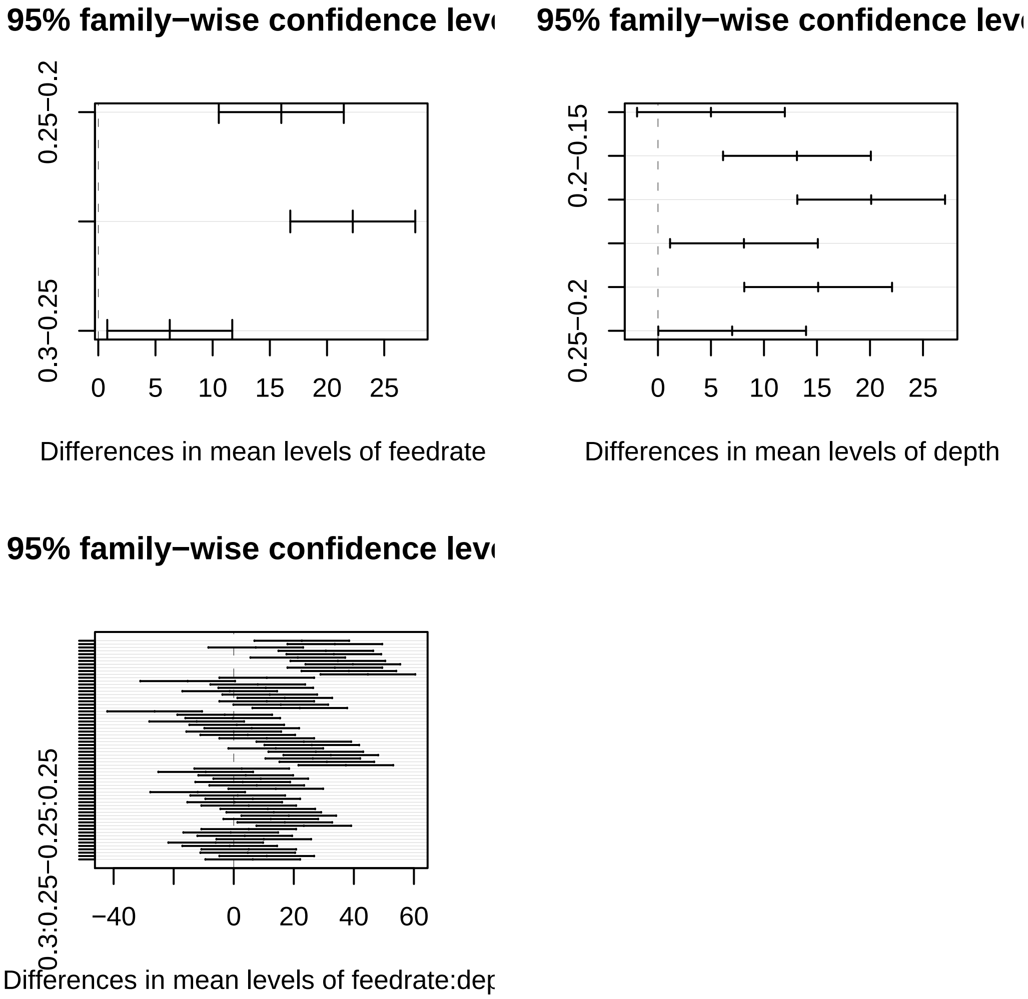


Figure 5.12: Metal Surface, 95% CI

Metal Surface R-code

```
y<- c(74, 64,60,92,86,88,99,98,102,79,68,73,98,104,88,104,99,95,82,88,92,99,
108,95,108,110,99,99,104,96,104,110,99,114,111,107)
feedrate<-as.factor(rep(c(0.20,0.25,0.30),each=3, times=4))
depth<- as.factor(rep(c(0.15,0.18,0.20,0.25),each=9))
data<-data.frame(y,feedrate,depth); attach(data); data
windows(); par(mfrow=c(1,2))
interaction.plot(feedrate,depth,y,xlab="Rate",ylab="Mean of Surface",
main="interaction plot",ylim=c(60,115),fixed=T,legend=F)
legend(1.65,70,c("1","2"),title="Dept of cut",lty=c(2,1))
plot.design(data)
g<-lm(y~feedrate+depth+feedrate:depth); anova(g); summary(g)
h<-lm(y~feedrate+depth); anova(h)
windows(); par(mfrow=c(2,2)); plot(aov(y~feedrate+depth))
windows(); par(mfrow=c(2,2))
plot(aov(y~feedrate+depth+feedrate*depth))
model<-aov(y~feedrate+depth+feedrate*depth)
TukeyHSD(model); windows(); par(mfrow=c(2,2)); plot(TukeyHSD(model))
```

Chapter 6

Two-Level Factorial Design

Two-Level Factorial designs are widely used in experiments involving several factors where it is necessary to study the joint effect of the factors on a response variable. In the previous chapter we presented general methods for the analysis of factorial designs.

The most important special case of factorial design is the case of **k** factors, each at only two levels (2^k design). These levels can be quantitative or qualitative. Throughout this chapter we assume that: (1) the factors are fixed, (2) the designs are completely randomized, and (3) the usual normality assumptions are satisfied.

The 2^k design provides the smallest number of runs on which **k** factors can be studied in a complete factorial design. However, as the number of factors grows, the number of required observations grows very quickly (exponentially). Our goal is to examine via experiment a large number of factors in a small number of runs. To accomplish this goal we shall do the following:

1. Reduce the number of levels of each factor to 2. If we have **k** factors to investigate and each is at 2 levels, the result is a design with 2^k cells; and this is called a 2^k factorial design.

2. Occasionally we shall reduce the number of observations per cell to 1.

6.1 The Simplest Case: 2^2 Factorial Designs

We'll start with a basic 2^2 design, where it is easy to see what is going on. These designs are very widely used in industrial experiments. Assume that we have two factors (A and B) each with two levels low "$-$" and high "$+$", with number of replicates equal to t. In the case of a 2^2 factorial design we have four treatment combinations.

A	B	Yates Notation
$-$	$-$	(1)
$+$	$-$	a
$-$	$+$	b
$+$	$+$	ab

The Yates notation used for denoting the factor combinations is as follows:

- $(1) =$ sum of observations at low levels of both factors,
- $a =$ sum of observations with A high and B low ,
- $b =$ sum of observations with A low and B high ,
- $ab =$ sum of observations at high levels of both factors.

Suppose that our model is as follows:

$$y_{ijk} \quad = \quad \mu + A_i + B_j + (AB)_{ij} + \varepsilon_{ijk}, \quad \varepsilon \overset{iid}{\sim} N(0, \sigma^2); \quad i, j = 1, 2; \quad k = 1, ..., t.$$

What is the effect of A? Since A is the effect of changing factor A from high to low, we expect

$$A = \frac{a + ab}{2t} - \frac{(1) + b}{2t} = \frac{a + ab - (1) - b}{2t}.$$

Similarly,

$$B = \frac{a + ab - (1) - a}{2t}$$

and the interaction effects is denoted by

$$AB = \frac{ab - b}{2t} - \frac{a - (1)}{2t} = \frac{ab - b - a + (1)}{2t}.$$

Contrasts. In the estimates of the effects we have used only the terms ab, a, b and (1), each is the sum of t independent terms.

$$A = \frac{C_A}{2t}, \quad B = \frac{C_B}{2t}, \quad AB = \frac{C_{AB}}{2t}$$

Where C_A, C_B, C_{AB} are orthogonal contrasts in ab, a, b and (1). We can display the contrast coefficients table as follows:

Effects	(1)	a	b	ab
A	$-$	$+$	$-$	$+$
B	$-$	$-$	$+$	$+$
AB	$+$	$-$	$-$	$+$

The estimation of the sum of squares for factors A, B and AB are :

$$\begin{aligned} SS_A &= \frac{[ab + a - b - (1)]^2}{4t} = \frac{C_A^2}{4t} \\ SS_B &= \frac{[ab + b - a - (1)]^2}{4t} = \frac{C_B^2}{4t} \\ SS_{AB} &= \frac{[ab + (1) - a - b]^2}{4t} = \frac{C_{AB}^2}{4t} \end{aligned}$$

The total sum of squares is found in the usual way, that is,

$$SS_T = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{t} y_{ijk}^2 - \frac{y_{...}^2}{4t} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{t}(y_{ijk} - \overline{y}_{...})^2.$$

The error sum of squares, with $4(t-1)$ degrees of freedom, is usually computed by subtraction as

$$SS_E = SS_T - SS_A - SS_B - SS_{AB}.$$

All of this can be generalized to the 2^k factorial design, in which k factors are investigated, each at two levels. To easily write down the estimates of the effects, and the contrasts, we start with a table of ± signs. We will construct the 2^3 ((1), a, b, ab, c,ac, bc, abc) table as follows.

Effects	(1)	a	b	ab	c	ac	bc	abc
I	+	+	+	+	+	+	+	+
A	−	+	−	+	−	+	−	+
B	−	−	+	+	−	−	+	+
AB	+	−	−	+	+	−	−	+
C	−	−	−	−	+	+	+	+
AC	+	−	+	−	−	+	-	+
BC	+	+	−	−	−	−	+	+
ABC	−	+	+	−	+	−	−	+

In the next section we will study the general form of the factorial design.

6.2 The General 2^K Design

The method of analysis that we have presented may be generalized to the case of a 2^K factorial design, that is a design with K factors each at two levels. The statistical model for a 2^K design would include K main effects, $\frac{K(K-1)}{2}$ two factor interactions, $\ldots$, and one K-factor interaction. That is, for a 2^K design the complete model would contain $2^K - 1$ effects. The notation introduced earlier for treatment combinations is also used here. The general approach to the statistical analysis of the 2^K design is summarized in the table below.

Table 6.1: Table of 2^K Design

Sources of Variation	Sum of Squares	Degrees of Freedom
Kmain effects		
A	SS_A	1
B	SS_B	1
.	.	.
.	.	.
.	.	.
K	SS_K	1
two factor interaction		
AB	SS_{AB}	1
AC	SS_{AC}	1
.	.	.
.	.	.
.	.	.
JK	SS_{JK}	1
three factor interaction		
ABC	SS_{ABC}	1
ABD	SS_{ABD}	1
.	.	.
.	.	.
.	.	.
IJK	SS_{IJK}	1
.	.	.
.	.	.
.	.	.
K factor interaction		
ABC....K	$SS_{ABC...K}$	1
Error	SS_E	$2^K(t-1)$
Total	SS_T	$t2^K-1$

6.3 Router

Example 6.1. (From D. C. Montgomery, 1997, P. 342, Problem 7-5) A router is used to cut location notches on printed circuit board. The vibration level y at the surface of the board as it is considered to be a major source of dimensional variation in the notches. Two factors are thought to influence vibration: bit size (A) and cutting speed (B). Two bit sizes ($\frac{1}{16}$ and $\frac{1}{8}$ inch) and two speeds (40 and 90 rpm) are selected, and four boards are cut at each set of conditions shown below. The response variable is vibration measured as the resultant vector of three accelerometers (x, y, and z) on each test circuit board.

A	B	Treatment combination	1	2	3	4
−	−	(1)	18.2	18.9	12.9	14.4
+	−	a	27.2	24.0	22.4	22.5
−	+	b	15.9	14.5	15.1	14.2
+	+	ab	41.0	43.9	36.3	39.9

(a) Analyze the data from this experiment.

(b) Construct a Q-Q-plot of the residuals, and plot the residuals versus predicted vibration level. Interpret these plots.

(c) Draw the AB interaction plot. Interpret this plot. What level of bit size (A) and cutting speed (B) would you recommend for routine operation?

In this problem we have a 2^2 factorial design with four replicates ($n = 16$ runs). The two factors that may influence the response variable (vibration) are bit size and the cutting speed.

We may first conduct the ANOVA table to see which factor is significant and have an effect on the vibration of circuit board. Below you can see that both main effects and the interaction effect are highly significant.

```
            Df  Sum Sq Mean Sq F value    Pr(>F)
A            1 1107.23 1107.23 185.252 1.175e-08 ***
B            1  227.26  227.26  38.023 4.826e-05 ***
A:B          1  303.63  303.63  50.801 1.201e-05 ***
Residuals   12   71.72    5.98
---
Signif. codes:  ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1  ’ ’ 1
Residual standard error: 2.445 on 12 degrees of freedom
Multiple R-squared: 0.9581,     Adjusted R-squared: 0.9476
F-statistic: 91.36 on 3 and 12 DF,  p-value: 1.569e-08
```

The F-statistic of 91.36 implies that the model is significant. The R^2_{adj} indicates that the 94.8% of the vibration at the surface of the circuit board is explained by main effects and their interaction.

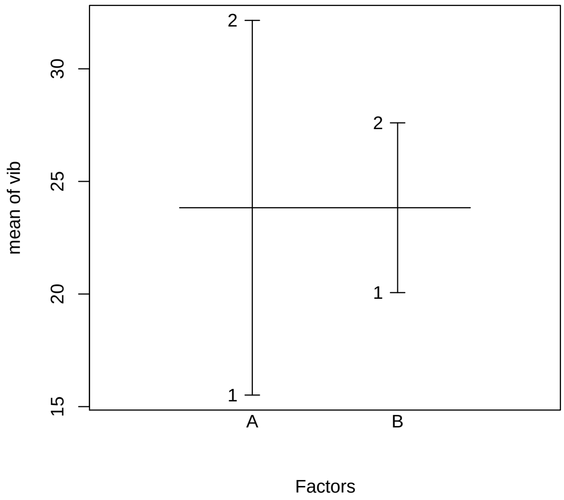


Figure 6.1: Router, Plot of main effects

The effects can be computed as follows:

```
Tables of effects
 A
A
    1      2
-8.319  8.319
effect of A: 8.319-(-8.319)=16.638
 B
B
    1      2
-3.769  3.769\
effect of B: 3.769-(-3.769)=7.538
 A:B
   B\
A   1      2
  1  4.356 -4.356
  2 -4.356  4.356
effect of AB: -4.356-4.356=-8.712
```

As we can see from the plot of main effects in figure 6.1 on page 95 the best case is when both factors A and B are at their low level. In this case we have the smallest vibration mean. Note that effects of AB in R is given with the reversed sign.

To estimate the regression model we first analyze the linear model with interaction:

```
Call:
lm(formula = vib ~ speedA * sizeB, data = vibration.dat)
Residuals:
   Min     1Q Median     3Q    Max
-3.975 -1.550 -0.200  1.256  3.625
Coefficients:
***********  Estimate Std. Error t value Pr(>|t|)
(Intercept)   23.8312     0.6112  38.991 5.22e-14 ***
speedA         8.3187     0.6112  13.611 1.17e-08 ***
sizeB          3.7687     0.6112   6.166 4.83e-05 ***
speedA:sizeB   4.3562     0.6112   7.127 1.20e-05 ***
---
Signif. codes:  '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1  ' ' 1
Residual standard error: 2.445 on 12 degrees of freedom
Multiple R-squared: 0.9581,     Adjusted R-squared: 0.9476
F-statistic: 91.36 on 3 and 12 DF,  p-value: 1.569e-08
```

So the estimated regression surface is

$$\hat{y} = 23.8312 + \frac{16.638}{2}x_1 + \frac{7.538}{2}x_2 + \frac{8.712}{2}x_1x_2.$$

The regression model can be used to generate the response surface plots. Since the model contains interaction, the fitted response surface is a "twisted" plane and as well for the contour, the contour lines of less vibration are curved. From examining the contour plot, we see that the vibration decreases as both factors are in low level. You can see the response surface plot and the contour plot in figure 6.2 on page 97 and in figure 6.3 on page 98.

The interaction plot is shown in figure 6.3 on page 98. It indicates that at the high level of A and high level of B ($\frac{1}{8}$ inch of bit size and with the speed of 90 rpm) we have more vibration at the surface of the board. To reduce the vibration we have to use the smaller bit size.

Now we are interested in the behavior of the residuals. We are interested to find out if the residuals are normally distributed. Figure 6.4 on page 99 shows the four diagnostic plots. The QQ-plot shows that the residuals are normally distributed.

The test of normality are as follows:

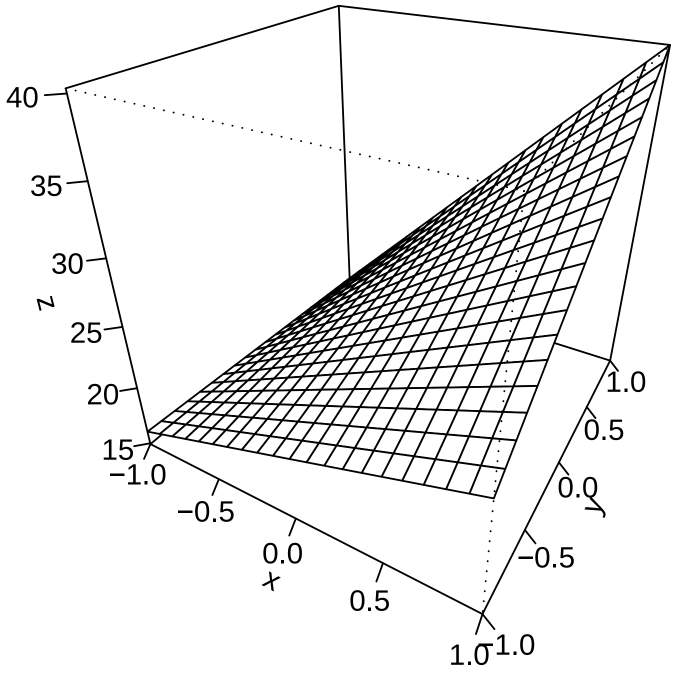


Figure 6.2: Router, Response Surface Plot

```
1)        Shapiro-Wilk normality test
data:  residuals(g)
W = 0.9714, p-value = 0.861

2)        One-sample Kolmogorov-Smirnov test
data:  residuals(g)
D = 0.0934, p-value = 0.9966
alternative hypothesis: two-sided
```

From the results above we can conclude that the residuals can be taken as normal distributed.

The 95% confidence intervals for multiple comparisons will be drawn and the re-

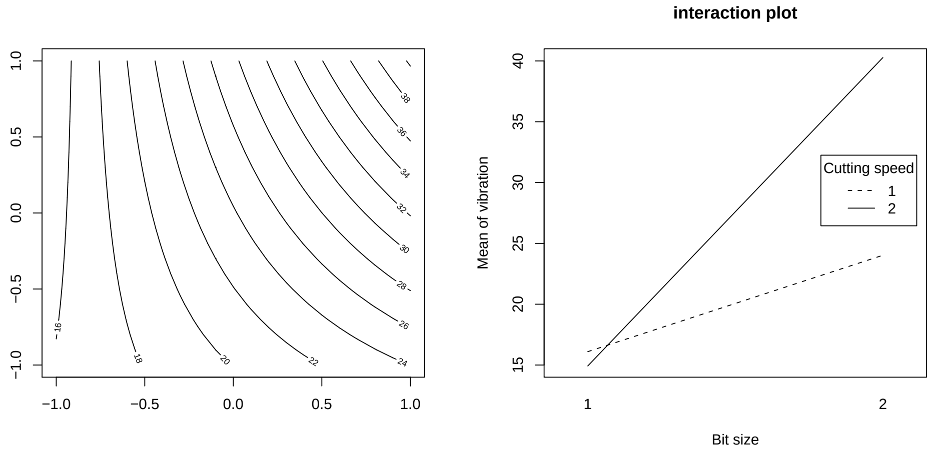


Figure 6.3: Router, Contour Plot and Interaction plot

sult is shown in figure 6.5 on page 100. The confidence interval for the difference of (-1,-1) to (-1,1) contains zero, so the two combinations (-1,1), (-1,-1) aren't significant from each other.

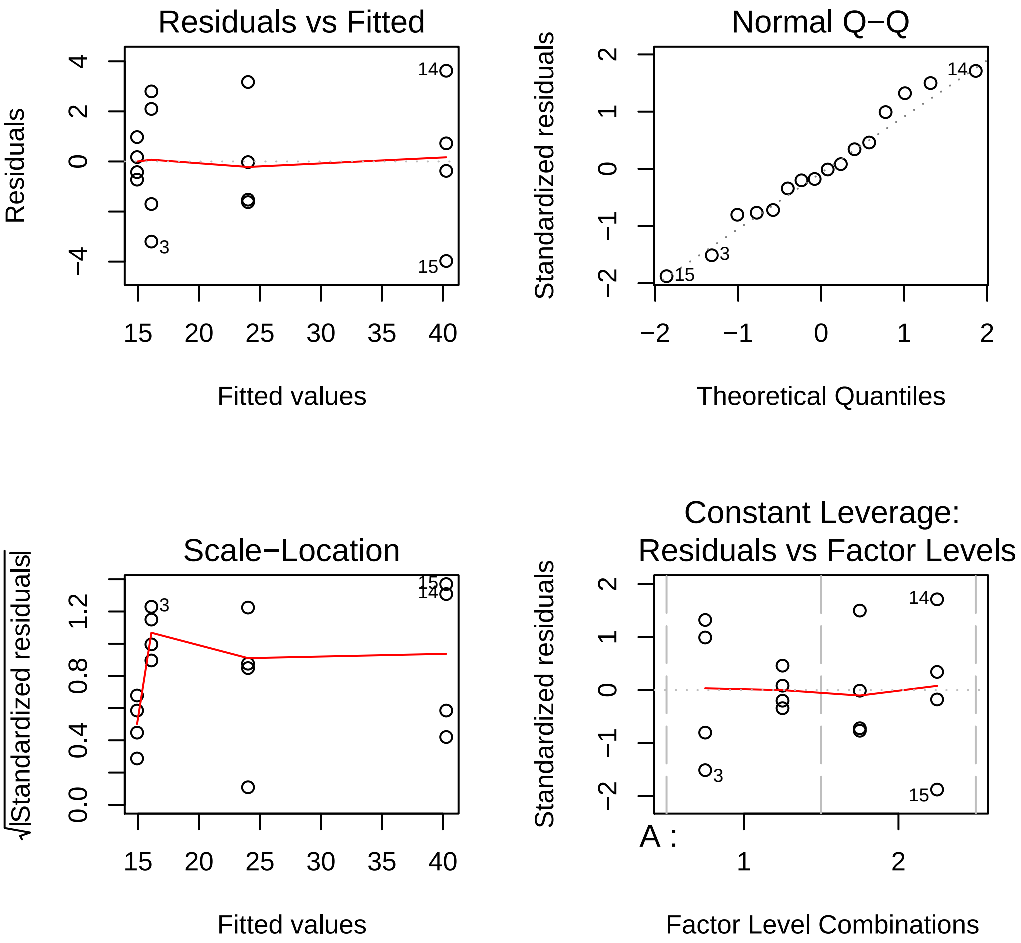


Figure 6.4: Router, The four diagnostic plots

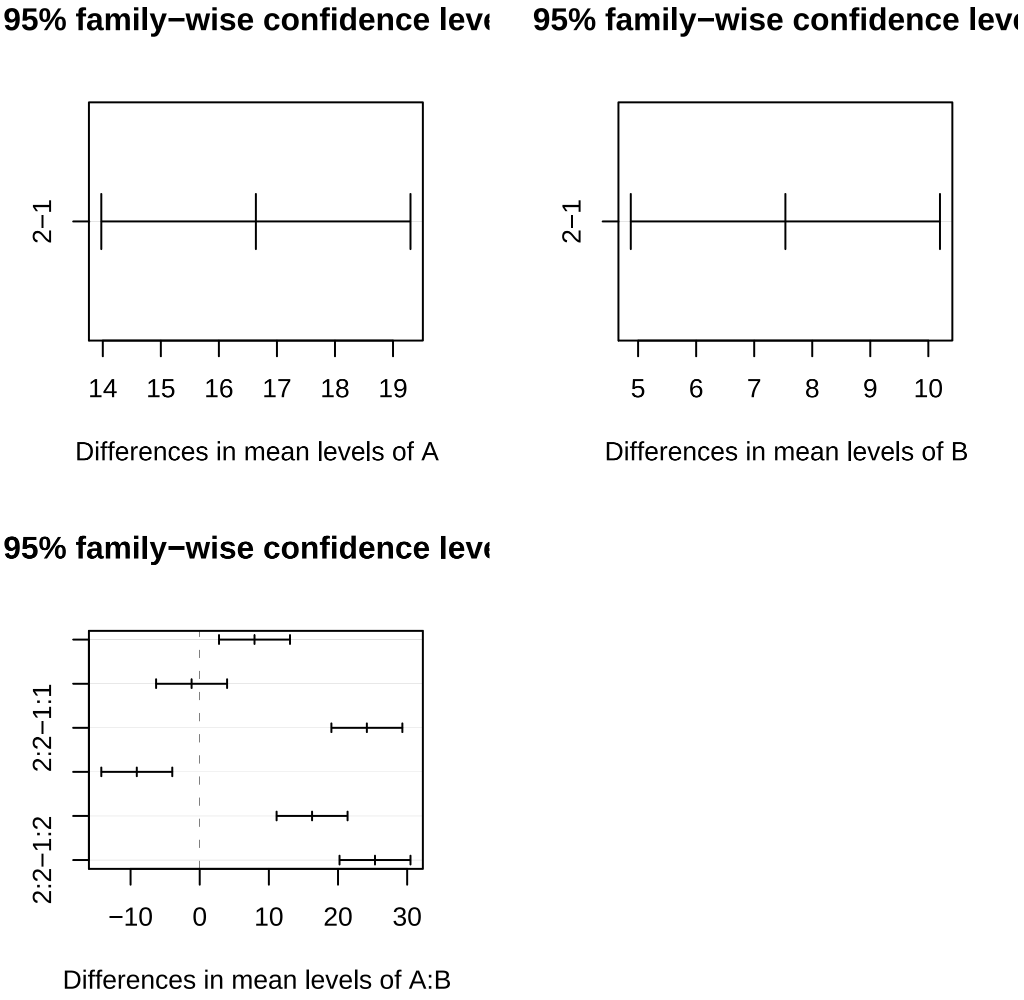


Figure 6.5: Router, 95% CI

Router R-Code

```
vibration.dat<-data.frame(vib=c(18.2,18.9,12.9,14.4,27.2,24.0
,22.4,22.5,15.9,14.5,15.1,14.2,41.0,43.9,36.3,39.9),
A=gl(2,4,16),B=gl(2,8,16)); vibration.dat
#Anova Table:
vib.mod<-aov(vib~A*B, data=vibration.dat)
summary(vib.mod)
#Effects:
model.tables(vib.mod)
#QQ-plot and residual plot:
qqnorm(residuals(vib.mod))
qqline(residuals(vib.mod))
plot(fitted(vib.mod),residuals(vib.mod),main="scatterplot")
#four diagnostic plots:
windows(); par(mfrow=c(2,2))
plot(aov(vib~A*B,data=vibration.dat))
#plot of design
plot.design(vib~A*B,data=vibration.dat)
#Interactionplot:
interaction.plot(vibration.datA,vibration.datB,vibration.datvib,
xlab="Bit size",ylab="Mean of vibration",main="interaction plot",
ylim=c(15,40),fixed=T,legend=F)
legend(1.79,32.25,c("1","2"),title="Cutting speed",lty=c(2,1))
#linear model:
speedA<-rep(c(-1,1),each=4,times=2)
sizeB<-rep(c(-1,1),each=8,times=1)
vib.lm<-lm(vib~speedA*sizeB,data=vibration.dat)
summary(vib.lm)
#Surfaceplot and contour plot:
x<-seq(-1,1,0.1); y<-seq(-1,1,0.1)
model<-function(a,b){23.832+(8.3187)*a+(3.7687)*b+(4.3563)*a*b}
z<-outer(x,y,model);z
persp(x,y,z,theta=30,phi=30,ticktype="detailed")
contour(x,y,z,nlevels=10)
#TukeyHSD CI:
model1<-aov(vib~A*B, data=vibration.dat)
TukeyHSD(model1); windows(); par(mfrow=c(2,2))
plot(TukeyHSD(model1))
```

6.4 Nickel-Titanium allloy

Example 6.2. (From D. C. Montgomery, 1997, P. 345, Problem 7-15) A nickel−titanium alloy is used to make components for jet turbine aircraft engines. Cracking is a potentially serious problem in the final part, as it can lead to nonrecoverable failure. A test is run at the parts procedure to determine the effect of four factors on cracks. The four factors are pouring temperature (A), titanium content (B), heat treatment method (C), and amount of grain refiner used (D). Two replicates of a 2^4 design are run, and the length of crack (in mm) y induced in a sample coupon subjected to a standard test is measured. The data are shown below.

A	B	C	D	Treatment combination	1	2
−	−	−	−	(1)	1.71	2.01
+	−	−	−	a	1.42	1.58
−	+	−	−	b	1.35	1.63
+	+	−	−	ab	1.67	1.65
−	−	+	−	c	1.23	1.48
+	−	+	−	ac	1.25	1.36
−	+	+	−	bc	1.46	1.52
+	+	+	−	abc	1.29	1.37
−	−	−	+	d	2.04	2.29
+	−	−	+	ad	1.86	1.95
−	+	−	+	bd	1.79	2.05
+	+	−	+	abd	1.42	1.69
−	−	+	+	cd	1.81	2.02
+	−	+	+	acd	1.34	1.39
−	+	+	+	bcd	1.46	1.63
+	+	+	+	abcd	0.85	1.00

(a) Estimate the factor effects. Which four effects appear to be large?

(b) Conduct an analysis of variance. Do any of the factors affect cracking? Use $\alpha = 0.05$.

(c) Write down a regression model that can be used to predict crack length as a function of the significant main effects and interactions you have identified in part (b).

(d) Analyze the residuals from this experiment.

(e) Is there an indication that any of the factors affect the variability in cracking?

In this problem we have a 2^4 factorial design with two replicates ($n = 32$ runs). We have four factors and we are interested to find out if these factors have an effect on cracking, so we will do the analysis of variance, where the ANOVA table is shown as follows:

```
Analysis of Variance Table
             Df  Sum Sq Mean Sq F value    Pr(>F)
A             1 0.60225 0.60225 32.7701 3.133e-05 ***
B             1 0.26463 0.26463 14.3991  0.001591 **
C             1 0.99758 0.99758 54.2807 1.587e-06 ***
D             1 0.21288 0.21288 11.5832  0.003634 **
A:B           1 0.00750 0.00750  0.4083  0.531893
A:C           1 0.03990 0.03990  2.1712  0.160015
B:C           1 0.00300 0.00300  0.1634  0.691390
A:D           1 0.24325 0.24325 13.2360  0.002214 **
B:D           1 0.22950 0.22950 12.4878  0.002759 **
C:D           1 0.07315 0.07315  3.9804  0.063351 .
A:B:C         1 0.04575 0.04575  2.4895  0.134169
A:B:D         1 0.04425 0.04425  2.4079  0.140276
A:C:D         1 0.03445 0.03445  1.8747  0.189854
B:C:D         1 0.04278 0.04278  2.3277  0.146615
A:B:C:D       1 0.05695 0.05695  3.0990  0.097441 .
Residuals    16 0.29405 0.01838
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.2868 on 16 degrees of freedom
Multiple R-squared: 0.5878,     Adjusted R-squared: 0.2013
F-statistic: 1.521 on 15 and 16 DF,  p-value: 0.2072
```

From the result above we may conclude that the four main factors A, B, C and D and the two-factor interactions AD and BD are highly significant at the significance level of $\alpha < 0.05$. The two-factor interaction CD is slightly significant at the significant level of $\alpha = 0.063$.

The effects of the model are shown below:

```
A=-0.274374 , B=-0.181874, C=0.353124, D=0.166876
AB=0.030624, AC=-0.070626, BC=-0.009687, AD=-0.174376
BD=-0.169376, CD=-0.095626, ABC=-0.075624, ABD=0.074374
ACD=-0.0656246, BCD=-0.073124, ABCD=0.084374
```

Figure 6.6 on page 104 is the normal plot of effects, it shows the same result as the ANOVA table.
The fitted regression line needs to be estimated for drawing the surface plot and the contour plot. The estimated parameters are as follows. These estimates are for a new model containing four main effects and two interactions:

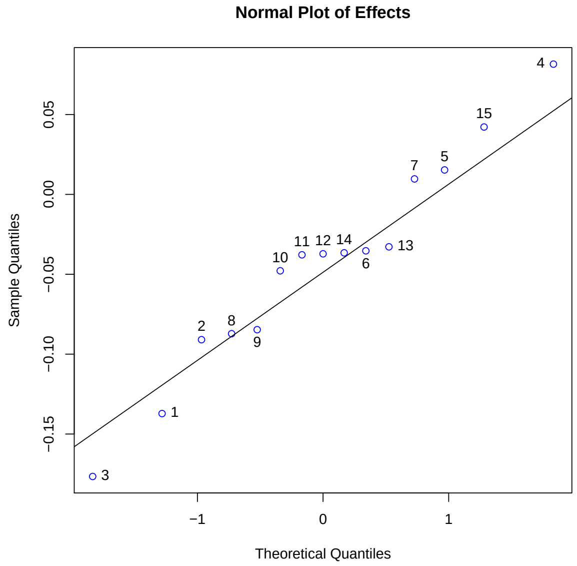


Figure 6.6: Nickel-Titanium, Normal Plot

```
Call:
lm(formula = crack ~ nA + nB + nC + nD + nA * nD + nB * nD, data = NickTit.dat)
Residuals:
      Min        1Q    Median        3Q       Max
-0.369062 -0.086250  0.002812  0.097344  0.278438
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.58031    0.02832  55.794  < 2e-16 ***
nA          -0.13719    0.02832  -4.843 5.59e-05 ***
nB          -0.09094    0.02832  -3.211  0.00362 **
nC          -0.17656    0.02832  -6.234 1.61e-06 ***
nD           0.08156    0.02832   2.880  0.00805 **
nA:nD       -0.08719    0.02832  -3.078  0.00500 **
nB:nD       -0.08469    0.02832  -2.990  0.00619 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.1602 on 25 degrees of freedom
Multiple R-squared: 0.7989,     Adjusted R-squared: 0.7507
F-statistic: 16.56 on 6 and 25 DF,  p-value: 1.27e-07
```

So the regression line will be:

$$\hat{y} = 1.5803 - 0.13719x_1 - 0.0909x_2 - 0.017656x_3 + 0.08156_4 - 0.08719x_1x_4 - 0.08469x_2x_4$$

Let us consider that both factors B and C are at the high level. The corresponding surface plot with $x_2 = 1$ and $x_3 = 1$ and the contour plot will be shown in figure 6.7 and figure 6.8 on page 106.

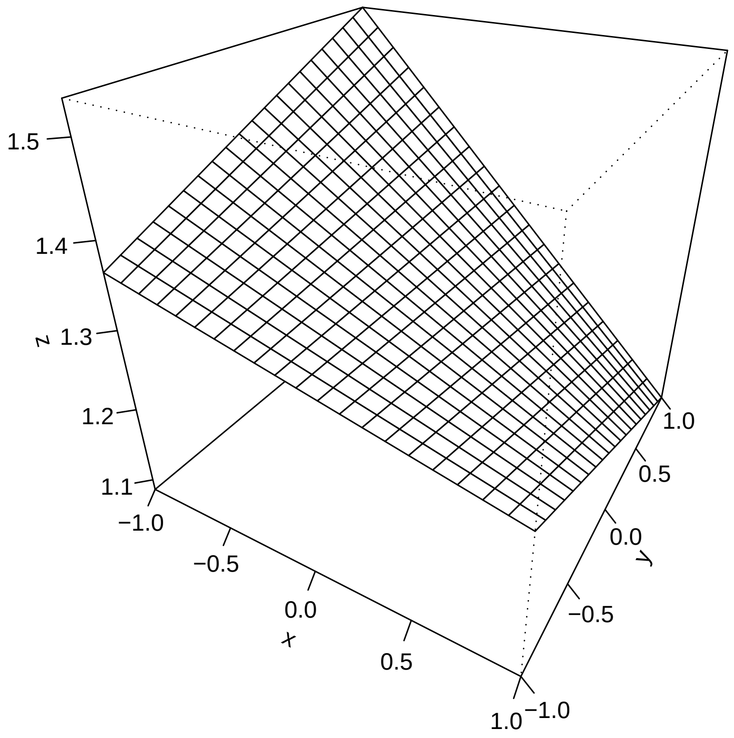


Figure 6.7: Nickel-Titanium, Response Surface Plot

Plot of main effects is drawn in figure 6.9 on page 107, from this plot we may conclude that the best case is when we set factor A, B and C at their high level and factor D at it's low level.

As you can see from figure 6.9 on page 107, when factor A has the highest level and the factors B, C and D have the highest level then we have the smallest cracks. When B is high and C is high we have a smallest crack. In the case of B high and D low we have a smallest crack and when C high and D low we have the smallest cracks.

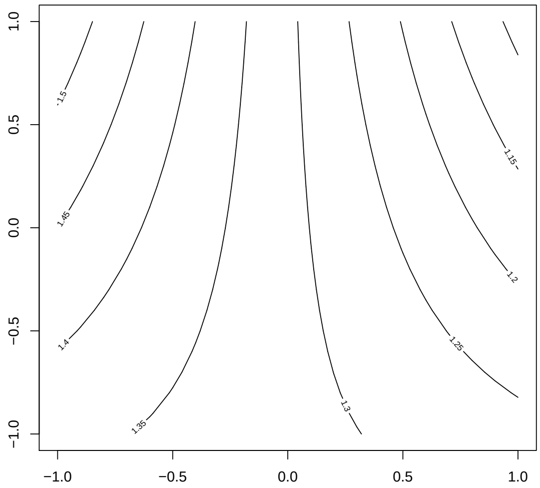


Figure 6.8: Nickel-Titanium, Contour Plot

From these figures we can see that in the following univariate cases we have a smallest crack: A^+, B^+, C^+, D^+ and A^+, B^+, C^+, D^- and by interactions the best cases are: A^+D^+, B^+D^+, C^+D^-.

Now we are interested in the behavior of the residuals, we are interested to find out if the residuals are normally distributed. The normality test result is shown below

```
        Shapiro-Wilk normality test
data:  residuals(mod1)
W = 0.9822, p-value = 0.8604
```

From the above results and the normality test we can conclude that the residuals are normal distributed.

From figure 6.10 on page 108 we can conclude that the residuals are normal distributed.

The 95% Confidence Interval for multiple Comparisons will be drawn in figure 6.11 on page 109.

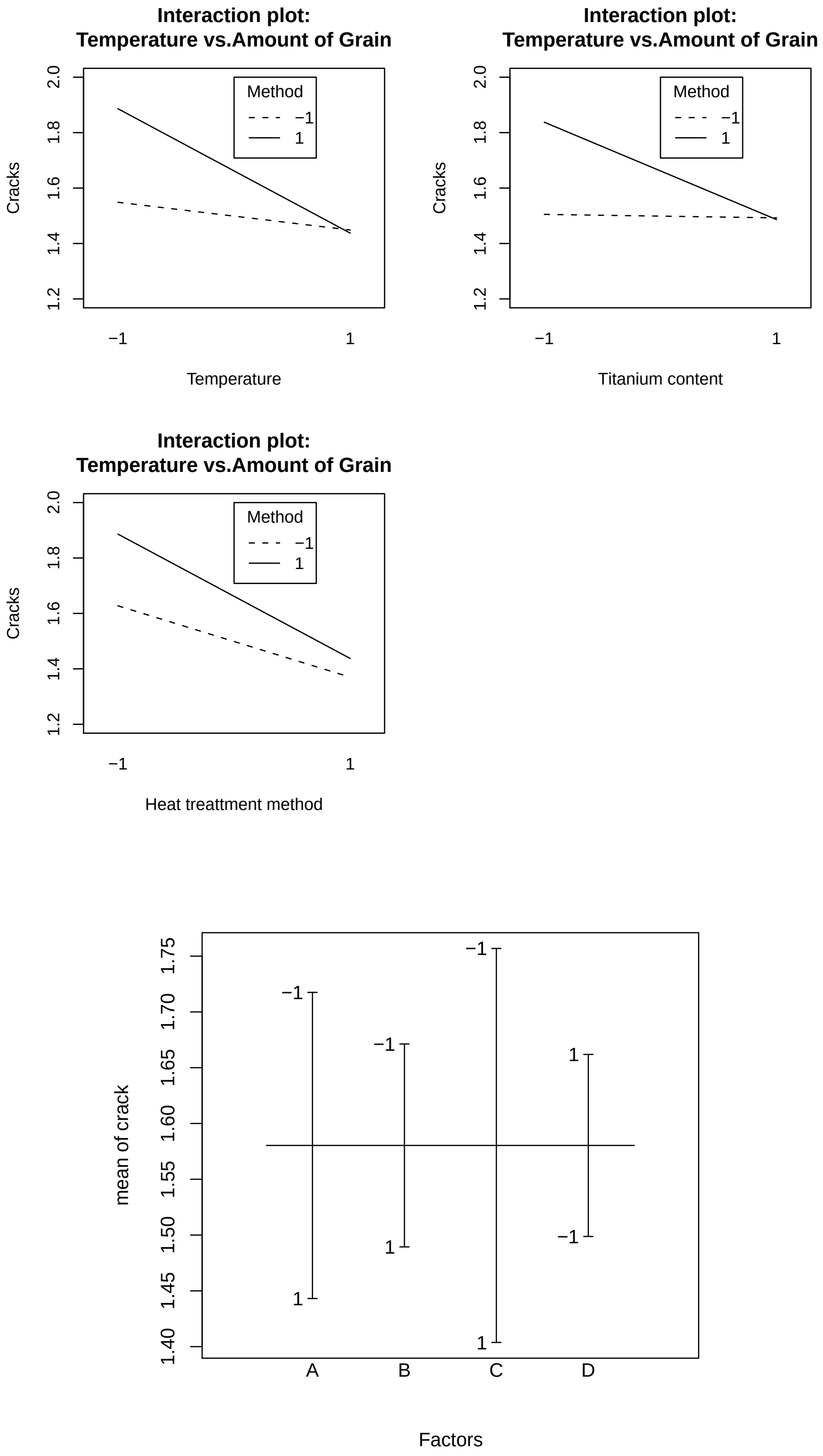


Figure 6.9: Nickel-Titanium, Interaction plots and Plot of Design

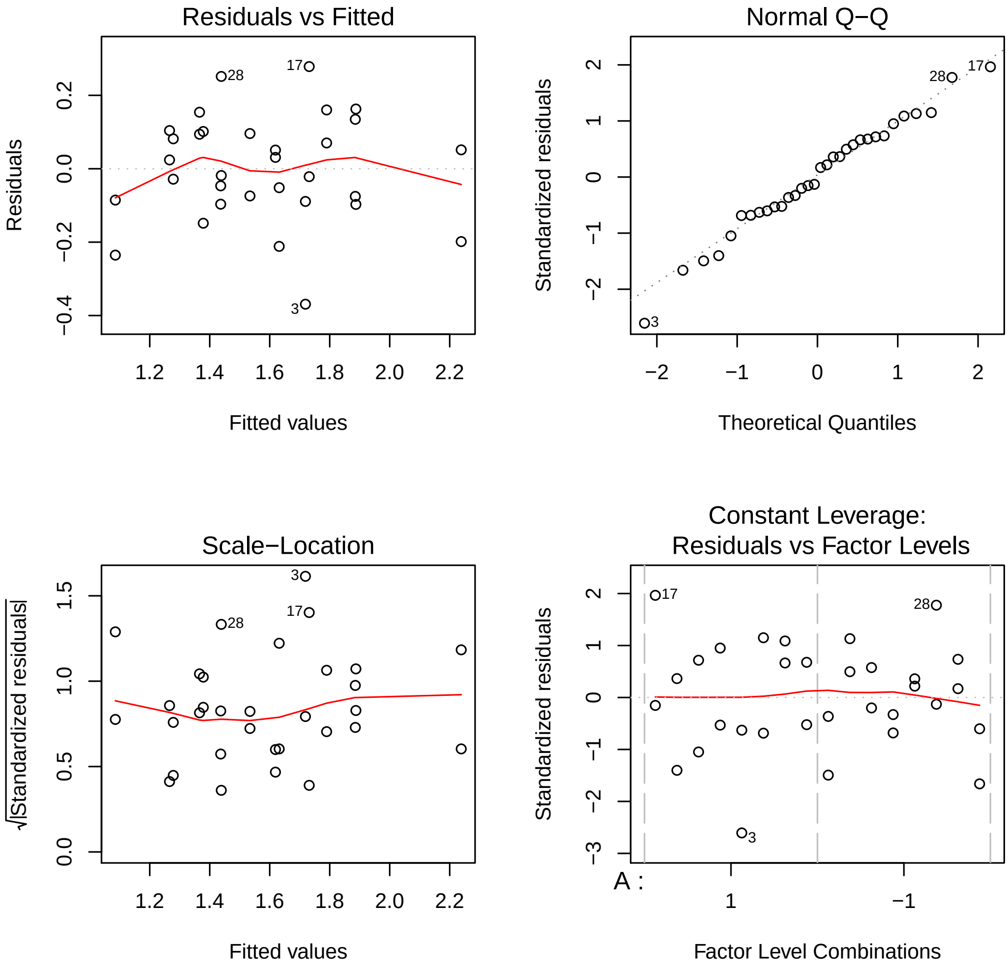


Figure 6.10: Nickel-Titanium, The four diagnostic plots

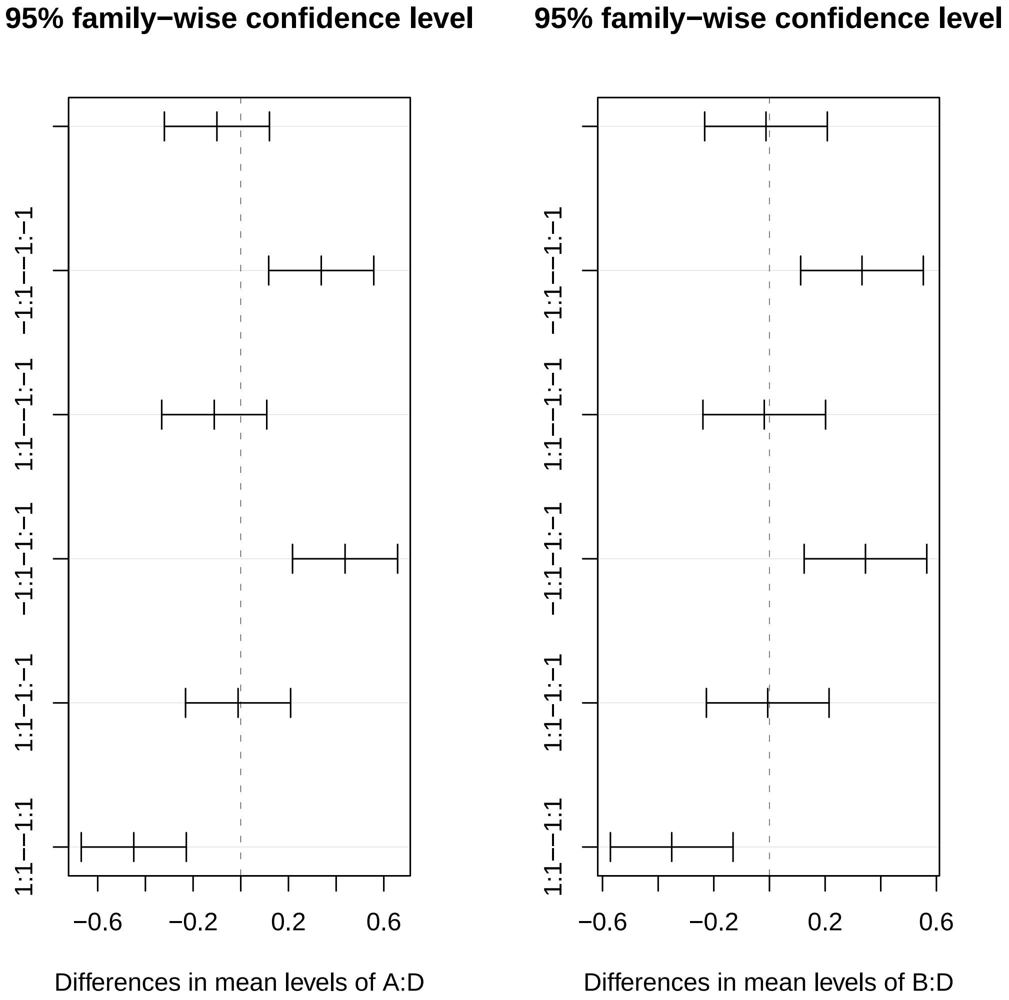


Figure 6.11: Nickel-Titanium, 95% CI

Nickel-Titanium R-Code

```
NickTit.dat<-data.frame(crack=c(1.71,1.42,1.35,1.67,1.23,1.25,1.46,
1.29,2.04,1.86,1.79,1.42,1.81,1.34,1.46,0.85,2.01,1.58,1.63,1.65,
1.48,1.36,1.52,1.37,2.29,1.95,2.05,1.69,2.02,1.39,1.63,1.00),
A=factor(rep(c(-1,1),each=1,times=8)),B=factor(rep(c(-1,1),each=2,
times=4)); C=factor(rep(c(-1,1),each=4,times=2));
D=factor(rep(c(-1,1),each=8,times=1))); NickTit.dat
#Anova Table:
crack.mod<-aov(crack~A*B*C*D, data=NickTit.dat); summary(crack.mod)
#Effects and Linear Model:
model.tables(crack.mod); nA=rep(c(-1,1),each=1,times=16);
nB=rep(c(-1,1),each=2,times=8); nC=rep(c(-1,1),each=4,times=4);
nD=rep(c(-1,1),each=8,times=2)
crack.lm<-lm(crack~nA*nB*nC*nD, data=NickTit.dat)
summary(crack.lm)
#Q-Q plot of Effects:
eff<-summary(crack.lm)$coef[2:16]; eff
qq<-qqnorm(eff,main="Normal Plot of Effects",col="blue")
qqline(eff); identify(qq$x,qq$y,1:length(qq$x))
crack1.lm<-lm(crack~nA+nB+nC+nD+nA*nD+nB*nD, data=NickTit.dat)
summary(crack1.lm)
#Plot of Design and Interactionplots:
plot.design(crack~A+B+C+D+A:D+B:D+C:D, data=NickTit.dat)
windows(); par(mfrow=c(2,2))
interaction.plot(NickTit.datA, NickTit.datD, NickTit.datcrack,
xlab="Temperature", ylab="Cracks", main="Interaction plot:
Temperature vs.Amount of Grain",ylim=c(1.2,2), fixed=T, legend=F)
legend(1.50,2.00,c("-1","1"),title="Method",lty=c(2,1))
interaction.plot(NickTit.datB, NickTit.datD, NickTit.datcrack,
xlab="Titanium content", ylab="Cracks", main="Interaction plot:
Temperature vs.Amount of Grain",ylim=c(1.2,2), fixed=T, legend=F)
legend(1.50,2.00,c("-1","1"),title="Method",lty=c(2,1))
interaction.plot(NickTit.datC, NickTit.datD, NickTit.datcrack,
xlab="Heat treattment method", ylab="Cracks", main="Interaction plot:
Temperature vs.Amount of Grain",ylim=c(1.2,2), fixed=T, legend=F)
legend(1.50,2.00,c("-1","1"),title="Method",lty=c(2,1))
#Surface Plot and contour Plot:
x<-seq(-1,1,0.1);y<-seq(-1,1,0.1)
func<-function(a,b){1.31284-0.1371*a-0.00313*b-0.08719*a*b}
z<-outer(x,y,func);z ; persp(x,y,z,theta=30,phi=30,
ticktype="detailed"); contour(x,y,z,nlevels=10)
crack.mod1<-aov(crack~A+B+C+D+A:D+B:D,data=NickTit.dat)
summary(crack.mod1)
#The four diagnostic plots, Hemogenity of variance Test and 95% CI:
windows();par(mfrow=c(2,2)); plot(crack.mod1)
shapiro.test(residuals(crack.mod1))
windwos();par(mfrow=c(1,2)); plot(TukeyHSD(crack.mod1))
```

6.5 Etch rate for silicon nitride

Example 6.3. (From D. C. Montgomery, 1997, P. 347, Problem 7-18) An article in Solid State Technology ("Orthogonal Design for Process Optimization and its Application in Plasma Etching," May 1987, pp. 127-132) describes the application of factorial designs in developing a nitride etch process on a single wafer plasma etcher. The process uses C_2F_6 as the reactant gas. Four factors are of interest: anode-cathode gap (A), pressure in a reactor chamber (B), C_2F_6 gas flow (C), and power applied to the cathode (D). The response variable y of interest is the etch rate for silicon nitride. A single replicate of a 2^4 design is run, and the data are shown below.

Run Number	Actual Run Order	A	B	C	D	Etch Rate	Factor Levels
1	13	−	−	−	−	550	A(cm) 0.80 1.20
2	8	+	−	−	−	669	B(mTorr)4.50 550
3	12	−	+	−	−	604	C(SCCM) 125 200
4	9	+	+	−	−	650	D (W) 275 325
5	4	−	−	+	−	633	
6	15	+	−	+	−	642	
7	16	−	+	+	−	601	
8	3	+	+	+	−	635	
9	1	−	−	−	+	1037	
10	14	+	−	−	+	749	
11	5	−	+	−	+	1052	
12	10	+	+	−	+	868	
13	11	−	−	+	+	1075	
14	2	+	−	+	+	860	
15	7	−	+	+	+	1063	
16	6	+	+	+	+	729	

(a) Estimate the factor effects. Construct the Q-Q-plot of the factor effects. Which effects appear large?

(b) Conduct an analysis of variance to confirm your findings for part (a).

(c) What is the regression model relating etch rate to the significant process variables?

(d) Analyze the residuals from this experiment. Comment on the model's adequacy.

(e) If not all the factors are important, project the 2^4 design into a 2^k design with $k \leq 3$ and conduct the analysis of variance.

(f) Draw a graph to interpret any significant interactions.

(g) Plot the residuals versus the actual run order. What problems might be revealed by this plot?

In this problem we have a 2^4 factorial design with four factors and one replicate. We are interested to find out if these four factors have an effect on the etch rate for silicon nitride. First we will compute the effects and draw the normal plot to find out which effects appear to be large. We first conduct the ANOVA Table.

```
            Df Sum Sq Mean Sq
A            1  41311   41311
B            1     11      11
C            1    218     218
D            1 374850  374850
A:B          1    248     248
A:C          1   2475    2475
B:C          1   7700    7700
A:D          1  94403   94403
B:D          1      2       2
C:D          1     18      18
A:B:C        1    977     977
A:B:D        1     68      68
A:C:D        1    127     127
B:C:D        1   2576    2576
A:B:C:D      1   6440    6440
```

From the table above we can see that factors A and D and the two factor interaction AD have large effects on the response variable.
From the figure 6.12 on page 113 we can see that the significant terms are A, D and the interaction AD.

The estimated effects are as follows:

```
A=-101.625, B=-1.625, C=7.375, D=306.125
AB= -7.875, AC=-24.875,AD=-153.62,
BC=-43.875, BD=-0.625, CD=-2.125,
ABC=-15.625, BCD=-25.375, ABCD=-40.125
```

Now we eliminate the two factors B and C from our data, and we will conduct the ANOVA table for a 2^2 design with 4 replicates:

```
             Df Sum Sq Mean Sq F value    Pr(>F)
A             1  41311   41311  23.767 0.0003816 ***
D             1 374850  374850 215.661 4.951e-09 ***
A:D           1  94403   94403  54.312 8.621e-06 ***
Residuals    12  20858    1738
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

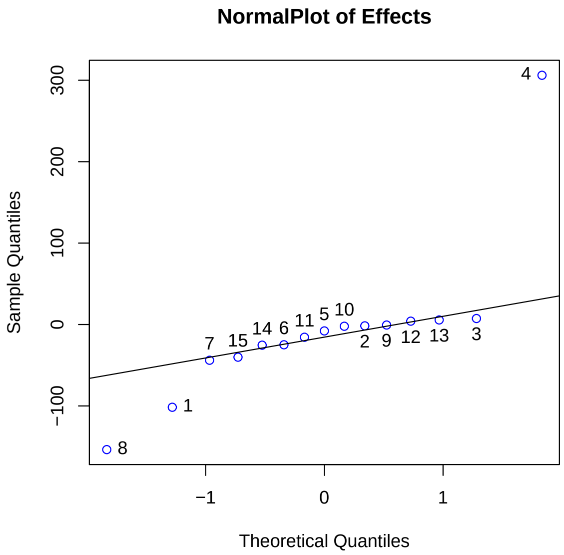


Figure 6.12: Etch rate for silicon nitride, Normal plot

The main factors A and D and their interaction are highly significant at the significance level of $\alpha < 0.0004$.

The estimation of the regression parameters are:

```
Call:
lm(formula = nitride ~ nA * nD, data = plasma1.dat)
Residuals:
   Min      1Q Median      3Q     Max
-72.50 -15.44   2.50  18.69  66.50
Coefficients:
*********** Estimate Std. Error t value Pr(>|t|)
(Intercept)   776.06      10.42  74.458  < 2e-16 ***
nA            -50.81      10.42  -4.875 0.000382 ***
nD            153.06      10.42  14.685 4.95e-09 ***
nA:nD         -76.81      10.42  -7.370 8.62e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 41.69 on 12 degrees of freedom
Multiple R-squared: 0.9608,     Adjusted R-squared: 0.9509
F-statistic: 97.91 on 3 and 12 DF,  p-value: 1.054e-08
```

So the regression equation is as follows:

$\hat{y} = 776.06 - 50.81x_1 + 153.06x_2 - 76.81x_1x_2$

The surface plot and the contour plot are shown in figures 6.13 and 6.14 on page 115.

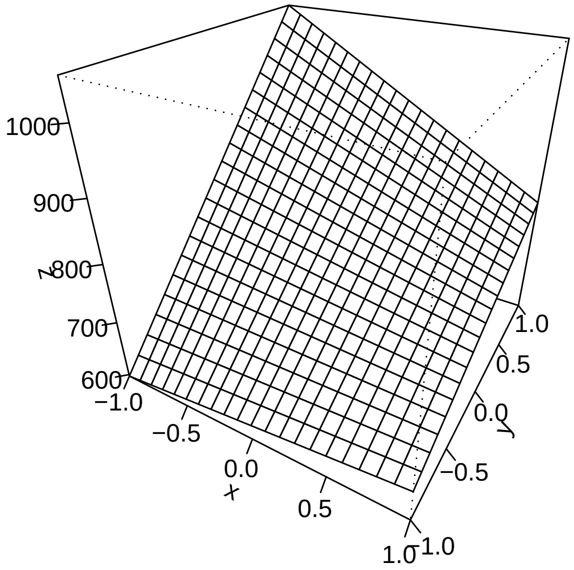


Figure 6.13: Etch rate for silicon nitride, Surface Plot

The plot of residuals against the actual run order is shown in figure 6.14 on page 115. The actual run order, represents the order of execution of each treatment combination, then a plot of the residuals of these observations versus the actual run order will test for any time dependency. With help of this plot we can reveal the problems of variation of observations and finding the outliers.
Notice that the contours are curved while there is an interaction in the model, we can conclude from this plots that when the two factors A and D (i.e. the anode-cathode gap and the power applied to the cathode) increases then the silicon nitride etch rate increases as well.

From the plots of main effects and interaction plot figure 6.15 on page 116 the best case would be to set factor A at its high level and factor D at its high level for observing a better response (A_+D_-).

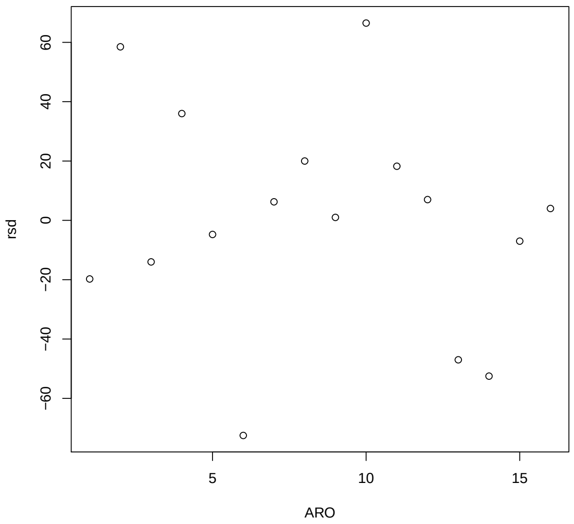
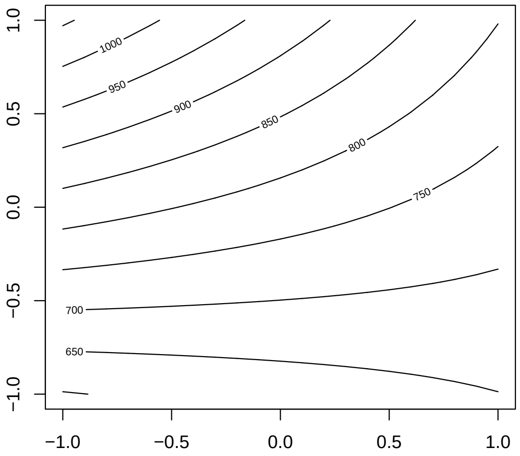
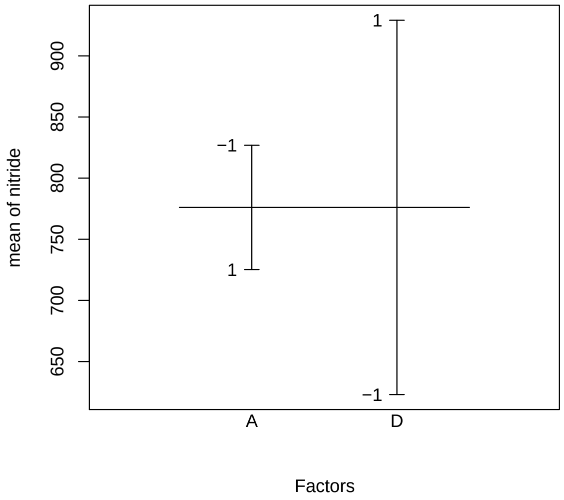


Figure 6.14: Etch rate for silicon nitride, Residuals against Actual Run Order

Figure 6.16 on page 117 shows the four diagnostic plots. From the residual plots we conclude that there is no indication of problems with assumptions of model adequacy, The plot of residuals shows high variability at the levels $(A_{-}D_{+})$.





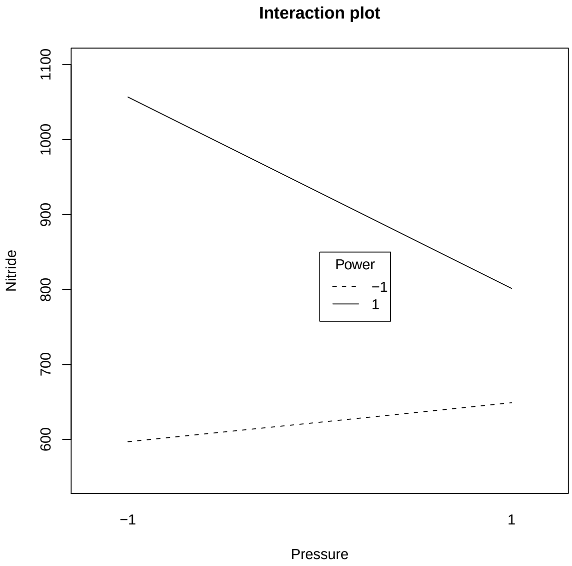


Figure 6.15: Etch rate of silicon nitride. Contour Plot, Plot of main effects and

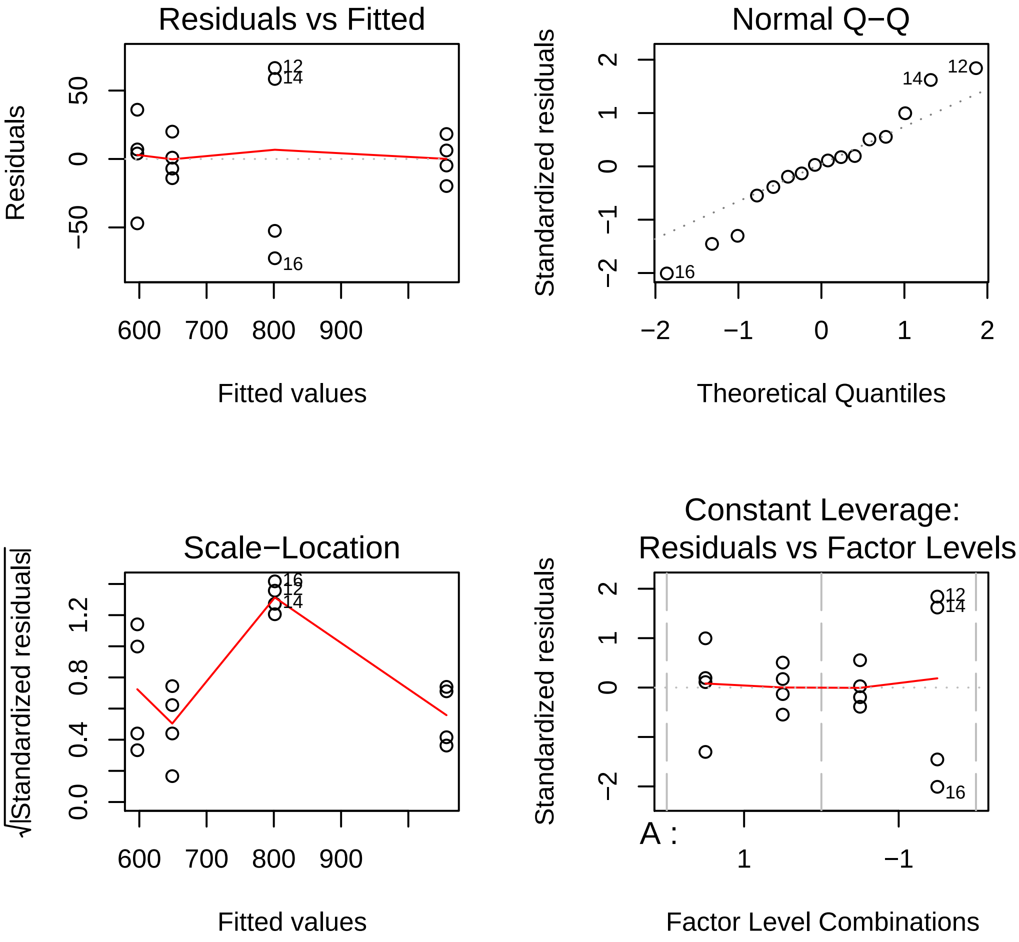


Figure 6.16: Etch rate for silicon nitride, Four diagnostic plots

Etch rate for silicon nitride R-code

```
A<-rep(c(-1,1),times=8);B<-rep(c(-1,1),each=2,times=4)
C<-rep(c(-1,1),each=4,times=2);D<-rep(c(-1,1),each=8)
y<-c(550,669,604,650,633,642,601,635,1037,749,1052,868,1075,860,1063,729)
ARO<-c(13,8,12,9,4,15,16,3,1,14,5,10,11,2,7,6)
A<-as.factor(A);B<-as.factor(B); C<-as.factor(C);D<-as.factor(D)
data<-data.frame(A,B,C,D,y,ARO)
g<-lm(y~A+B+C+D+A*B*A*C+A*D+B*C+B*D+C*D+A*B*C+A*B*D+A*C*D+B*C*D+A*B*C*D)
anova(g);geffects
effects<-abs(geffects[-1]);qq<-qqnorm(effects,type="n")
text(qq$x, qq$y,labels=names(effects))
interaction.plot(A,D,y,xlab="Pressure",
ylab="Nitride",main="Interaction plot"ylim=c(550,1100),fixed=T,legend=F)
legend(1.50,2.00,c("-1","1"),title="Power",lty=c(2,1));
x<-seq(-1,1,0.1);y<-seq(-1,1,0.1)
func<-function(a,b){776.063-99.375*a+303.875*b-154.25*a*b}
z<-outer(x,y,func);z
windows();par(mfrow=c(1,2))
persp(x,y,z,theta=30,phi=30,ticktype="detailed")
contour(x,y,z,nlevels=10); h<-lm(y~A+D+A*D);anova(h)
model<-aov(y~A+D+A*D);model.tables(model)
windows();par(mfrow=c(3,2))
plot(aov(y~A+D+A*D);plot.design(y~A+D+A*D)
windows();par(mfrow=c(2,2)); plot(TukeyHSD(model))
kruskal.test(y~A+D+A*D); rsd<-residuals(h);plot(ARO,rsd)
```

6.6 Chemical process

Example 6.4. (From D. C. Montgomery, 1997, P. 343, Problem 7-7) An experiment was performed to improve the yield y of a chemical process. Four factors were selected, and two replicates of a completely randomized experiment were run. The results are shown in the following table:

Treatment Combination	Replicate 1 2	Treatment Combination	Replicate 1 2
(1)	90 93	d	98 95
a	74 78	ad	72 76
b	81 85	bd	87 83
ab	83 80	abd	85 86
c	77 78	cd	99 90
ac	81 80	acd	79 75
bc	88 82	bcd	87 84
abc	73 70	abcd	80 80

(a) Estimate the factor effects.

(b) Prepare an analysis of variance table, and determine which factors are important in explaining yield.

(c) Write down a regression model for predicting yield, assuming that all four factors were varied over the range from -1 to $+1$ (in coded units).

(d) Plot the residuals versus the predicted yield on a normal probability scale. Does the residual analysis appear satisfactory?

(e) Two Three-factor interactions, ABC and ABD, apparently have large effects. Draw a cube plot in the factors A, B, and C with the average yields shown at each corner. Repeat using the factors A, B, and D. Do these two plots aid in data interpretation? Where would you recommend that the process be run with respect to the four variables?

In this problem we have a 2^4 factorial design with four factors and two replicates ($n = 32$ runs). We want to investigate the effects of these factors on the response variable y. You can see from figure 6.17 on page 120 that the main factors A, C and D and the interactions effects AB, ABC, ABD and ABCD are significant. Since factor B is included in an interaction, it will also be added to the model in the next step. The effects of this model are:

```
A=-9.06     AB=4.06     BD=-0.19     ACD=-0.94
B=-1.31     AC=0.69     CD=1.69      BCD=-0.94
C=-2.69     AD=-2.19    ABC=-5.19    ABCD=2.44
D=3.94      BC=-0.56    ABD=4.69
```

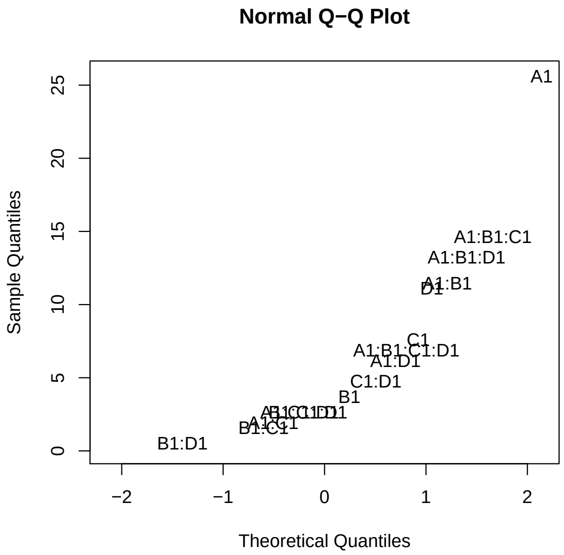


Figure 6.17: Chemical process, The normal plot

The factors A and ABC effects are strongly negative, but the other three are positive. We begin our investigation with the full model, the ANOVA table is shown below.

```
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value    Pr(>F)
A          1 657.03  657.03 85.8163 7.875e-08 ***
B          1  13.78   13.78  1.8000 0.1984451
C          1  57.78   57.78  7.5469 0.0143171 *
D          1 124.03  124.03 16.2000 0.0009794 ***
A:B        1 132.03  132.03 17.2449 0.0007491 ***
A:C        1   3.78    3.78  0.4939 0.4923019
A:D        1  38.28   38.28  5.0000 0.0399447 *
B:C        1   2.53    2.53  0.3306 0.5732962
B:D        1   0.28    0.28  0.0367 0.8504174
C:D        1  22.78   22.78  2.9755 0.1037933
A:B:C      1 215.28  215.28 28.1184 7.146e-05 ***
A:B:D      1 175.78  175.78 22.9592 0.0001997 ***
A:C:D      1   7.03    7.03  0.9184 0.3521621
B:C:D      1   7.03    7.03  0.9184 0.3521621
A:B:C:D    1  47.53   47.53  6.2082 0.0240766 *
Residuals 16 122.50    7.66
---
```

```
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
Residual standard error: 2.767 on 16 degrees of freedom
Multiple R-squared: 0.9247,     Adjusted R-squared: 0.8542
F-statistic:  13.1 on 15 and 16 DF,  p-value: 2.919e-06
```

The main effects A, C and D and the interactions effects AB, AD, ABC and ABD are significant. The ABCD interaction with P-value of 0.024077 is also significant, but we try to omit it from the model. The model F-value of 13.1 indicates that the model is significant.

The best model to fit our data is to have these factors in our model: (A, B, C, D, AB, AC, BC, BD, ABC, ABD), so the fitted regression model will be:

$$\begin{aligned}\hat{y} = 82.78 - 4.531x_1 - 0.656x_2 - 1.344x_3 + 1.969x_4 + 2.031x_1x_2 + 0.344x_1x_3 - 1.094x_1x_4\\ -0.281x_2x_3 - 0.009375x_2x_4 - 2.594x_1x_2x_3 + 2.344x_1x_2x_4\\ 1.22x_1x_2x_3x_4\end{aligned}$$

While the above model contains factor combinations which are not significant. Consider a new model with the factor combination (A, C, D, AB, AD, ABC and ABD).

```
Analysis of Variance Table
Response: y
********* Df Sum Sq Mean Sq F value    Pr(>F)
A          1 657.03  657.03 63.5196 1.235e-07 ***
B          1  13.78   13.78  1.8000 0.1984451
C          1  57.78   57.78  5.5861  0.028340 *
D          1 124.03  124.03 11.9909  0.002458 **
A:B        2 145.81   72.91  7.0483  0.004821 **
A:D        1  38.28   38.28  3.7009  0.068733 .
A:C:B      3 221.59   73.86  7.1410  0.001907 **
A:D:B      2 176.06   88.03  8.5106  0.002117 **
Residuals 20 206.87   10.34
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

In this model all factors are significant so the fitted regression model could be:

$$\begin{aligned}\hat{y} = 82.78 - 4.531x_1 - 1.344x_3 + 1.969x_4 + 2.031x_1x_2 - 1.094x_1x_4\\ -2.594x_1x_2x_3 + 2.344x_1x_2x_4 + 1.22x_1x_2x_3x_4\end{aligned}$$

Consider both factors B and C are at their high level, the response surface plot and the contour plots are shown in figure 6.18 on page 122 and 6.19 on page 123. It displays the

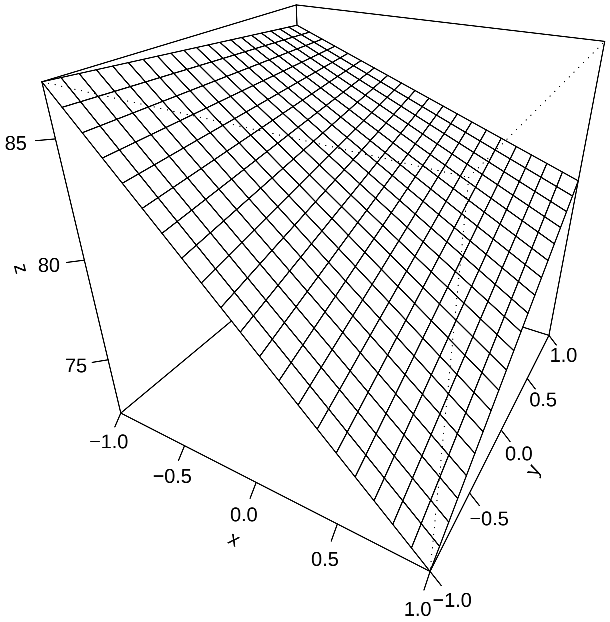


Figure 6.18: Chemical process, Response surface plot

curvature of the response in the design region, possibly indicating where robust operating conditions can be found.

The residual plots ares shown in figure 6.20 on page 124. The Q-Q-plot shows that the residuals are normally distributed. The two normality tests indicate that the residuals are normal distributed.

```
1)      Shapiro-Wilk normality test
data:  residuals(h)
W = 0.9496, p-value = 0.1406
2)    One-sample Kolmogorov-Smirnov test
data:  residuals(h)
D = 0.1568, p-value = 0.4112
alternative hypothesis: two-sided
```

The two-factor interaction plots are shown in figure 6.21 on page 125 and the three-factor interaction plots are shown in figure 6.22 on page 126.

The interaction plot indicates that there is just one two-factor interaction effect which is AB, the others are approximately parallel lines. By the three-factor interactions as we see

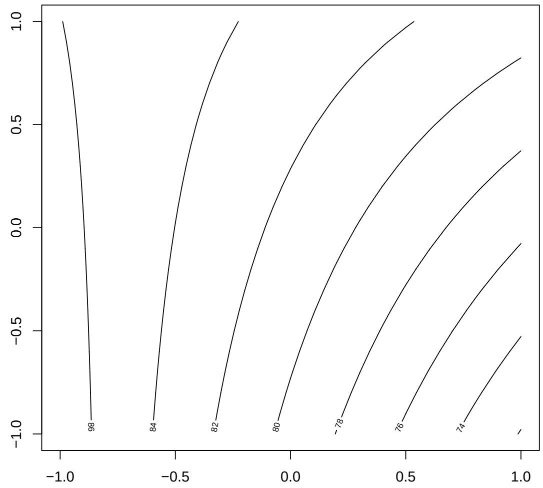


Figure 6.19: Chemical process, Contour plot

from figure 6.22 that one time we have parallel lines and the other one are crossed lines. In these figures the best response which is maximized are: A and B the low level, A and C at the low level, A at low level and D at high level, B and C at their low levels, B at low level and D at high level, when A, B and C are at low levels and A and B at low levels and D at high level. The response is optimized for $A_{-}B_{-}C_{-}D_{+}$.

Figures 6.23 on page 127 shows the cube plots for three-interaction factors ABC and ABD. The recommended setting for maximizing yield is $A_{-}B_{-}C_{-}D_{+}$. This information comes from studying the two cube plots and determining which side of the cube for each factor has the higher average response. The selection is confirmed by examining the regression equation and seeing that it agrees with the results listed above (factors A, B, and C at the low level and factor D at the high level).

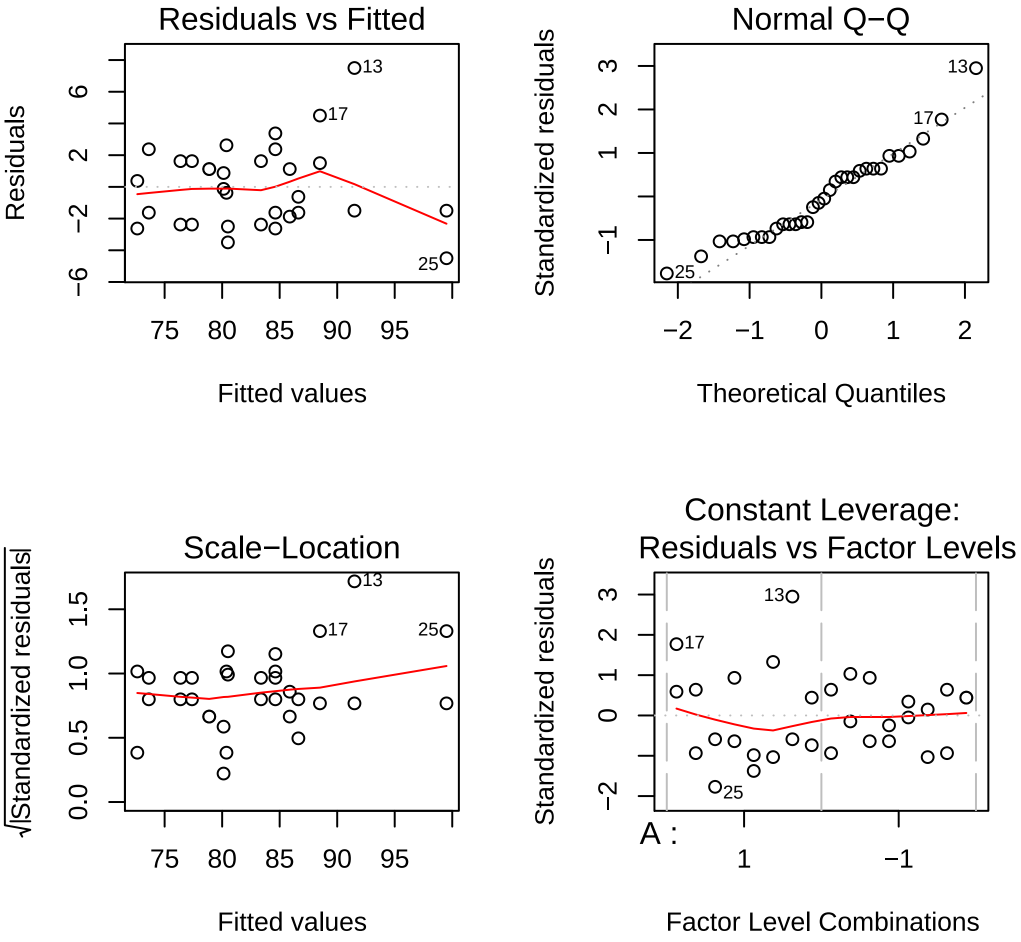


Figure 6.20: Chemical process, The four diagnostic plots

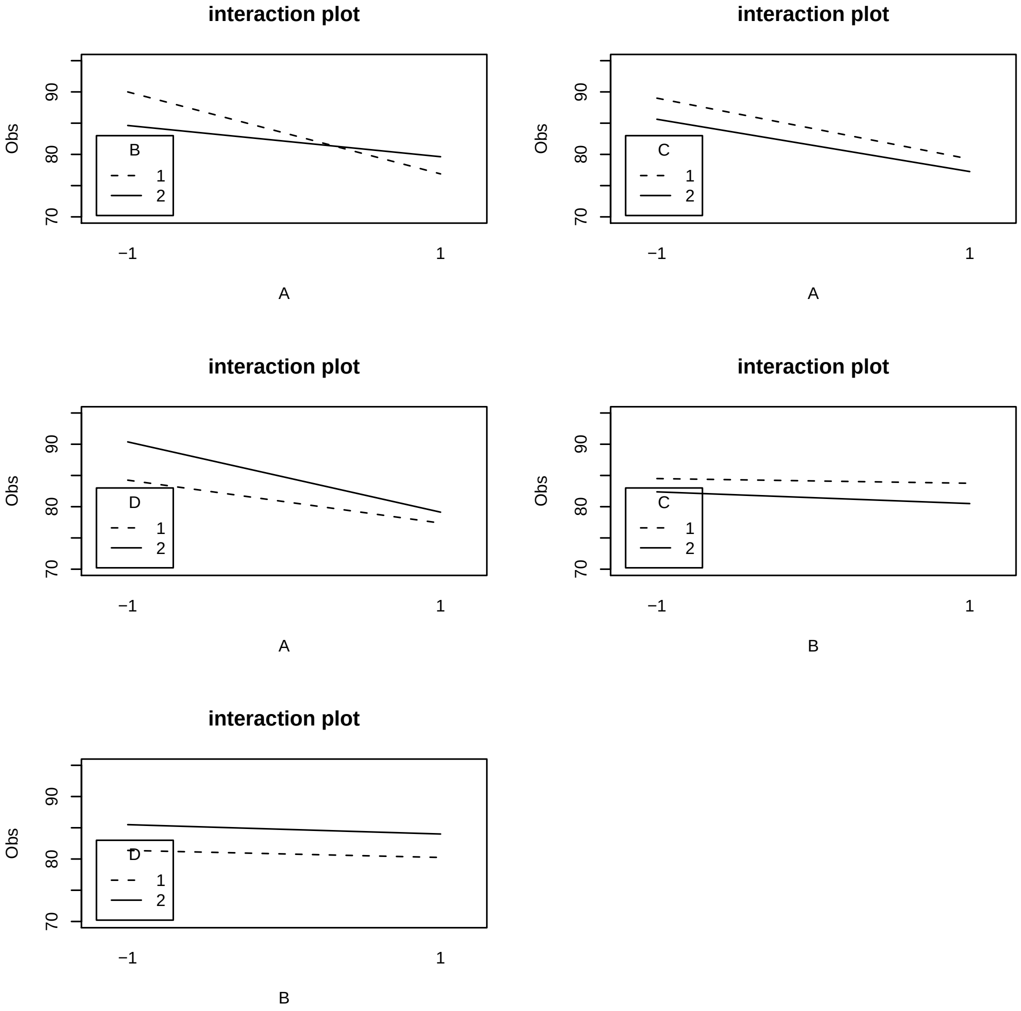


Figure 6.21: Chemical process, 2-factor interaction plot

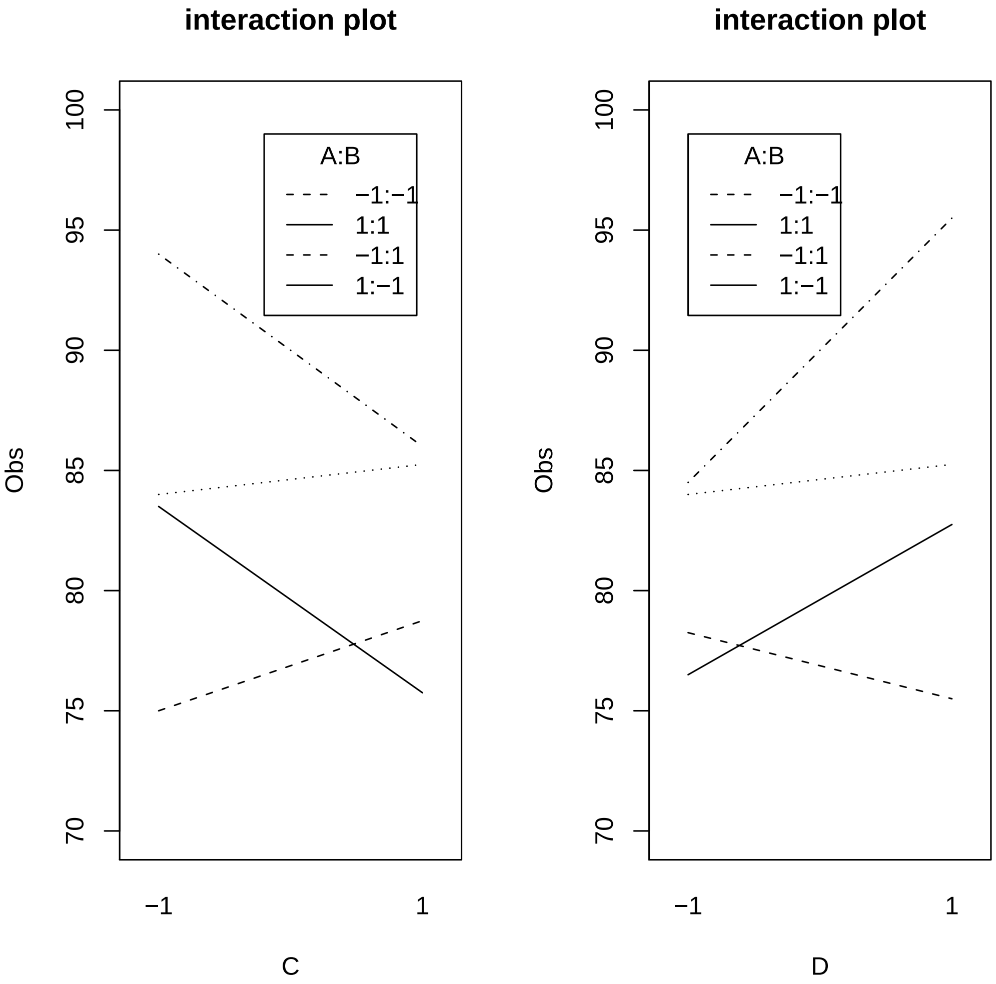


Figure 6.22: Chemical process, 3-factor interaction plot

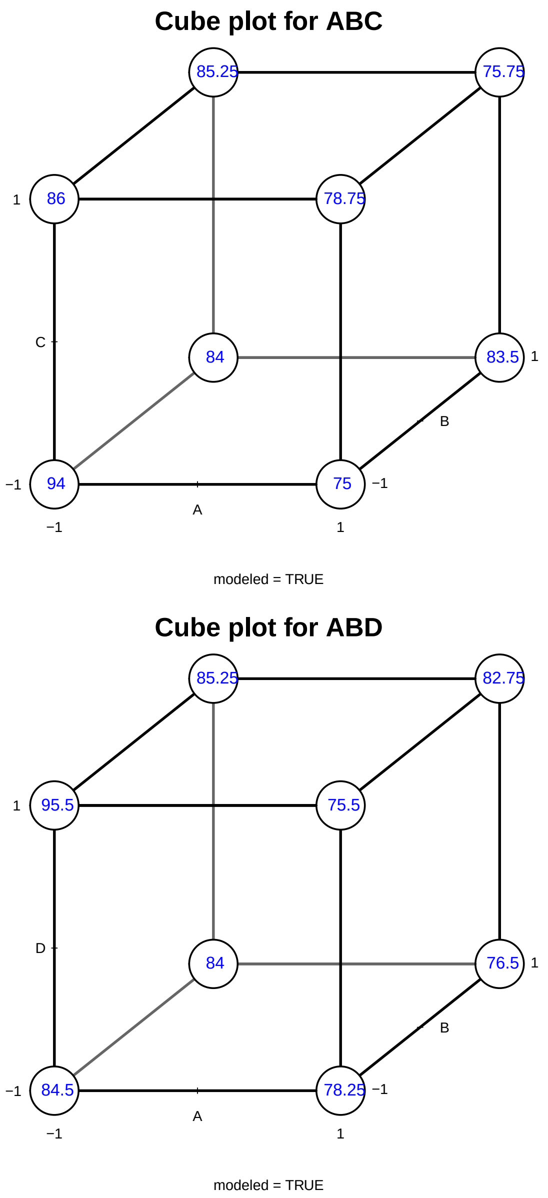


Figure 6.23: Chemical process, Cube Plot for ABC and ABD

Chemical process R-code

```
A<-rep(c(-1,1),times=8); B<-rep(c(-1,1),each=2,times=4)
C<-rep(c(-1,1),each=4,times=2); D<-rep(c(-1,1),each=8)
e<-c(90,74,81,83,77,81,88,73,98,72,87,85,99,79,87,80)
f<-c(93,78,85,80,78,80,82,70,95,76,83,86,90,75,84,80)
data<-data.frame(A,B,C,D,e,f); data
A<-as.factor(rep(A,2)); B<-as.factor(rep(B,2))
C<-as.factor(rep(C,2)); D<-as.factor(rep(D,2))
y<-c(e,f); data<-data.frame(A,B,C,D,y)
g<-lm(y~A+B+C+D+A*B+A*C+A*D+B*C+B*D+A*B*C+A*B*D+B*C*D+A*B*C*D); anova(g)
geffects; effects<-abs(geffects[-1]); qq<-qqnorm(effects,type="n")
text(qq$x,qq$y,labels=names(effects))
model<-aov(y~A+B+C+D+A*B+A*C+A*D+B*C+B*D+A*B*C+A*B*D+B*C*D+A*B*C*D)
model.tables(model); h<-lm(y~A+B+C+D+A:B+A:D+A:B:C+A:B:D); anova(h)
x<-seq(-1,1,0.1); y<-seq(-1,1,0.1)
func<-function(a,b){81.436-5.094*a+1.969*b+2.47*a*b}
z<-outer(x,y,func);z
persp(x,y,z,theta=30,phi=30,ticktype="detailed")
contour(x,y,z,nlevels=10); windows(); par(mfrow=c(2,2))
plot(aov(y~A+C+D+A:B+A:D+A:B:C+A:B:D)); windows(); par(mfrow=c(3,2))
interaction.plot(A,B,y,xlab="A",ylab="Obs",
main="interaction plot",ylim=c(70,95),fixed=T,legend=F)
legend(0.90,83,c("1","2"),title="B",lty=c(2,1))
interaction.plot(A,C,y,xlab="A",ylab="Obs",
main="interaction plot",ylim=c(70,95),fixed=T,legend=F)
legend(0.90,83,c("1","2"),title="C",lty=c(2,1))
interaction.plot(A,D,y,xlab="A",ylab="Obs",
main="interaction plot",ylim=c(70,95),fixed=T,legend=F)
legend(0.90,83,c("1","2"),title="D",lty=c(2,1))
interaction.plot(B,C,y,xlab="B",ylab="Obs",
main="interaction plot",ylim=c(70,95),fixed=T,legend=F)
legend(0.90,83,c("1","2"),title="C",lty=c(2,1))
interaction.plot(B,D,y,xlab="B",ylab="Obs",
main="interaction plot",ylim=c(70,95),fixed=T,legend=F)
legend(0.90,83,c("1","2"),title="D",lty=c(2,1))
windows(); par(mfrow=c(1,2))
interaction.plot(C,A:B,y,xlab="C",ylab="Obs",
main="interaction plot",ylim=c(70,100),fixed=T,legend=F)
legend(1.40,99,c("-1:-1","1:1","-1:1","1:-1"),title="A:B",lty=c(2,1))
interaction.plot(D,A:B,y,xlab="D",ylab="Obs",
main="interaction plot",ylim=c(70,100),fixed=T,legend=F)
legend(1.0,99,c("-1:-1","1:1","-1:1","1:-1"),title="A:B",lty=c(2,1))
cubePlot(g,"A","B","C"); cubePlot(g,"A","B","D")
```

Chapter 7

Fractional Factorial Design

Motivation for a fractional factorial design is obvious; as the number of factors becomes large to be interesting, the size of designs grows very quickly. For example, consider a 2^5 factorial. Even without replicates, there are $2^5 = 32$ observations required to estimate the effects: 5 main effects, 10 two factor interactions, 10 three factor interactions, 5 four factor interactions and 1 five factor interaction.

If three or more factor interactions are not of interest then only 15 effects are left so that (including 1 d.f. for μ) perhaps only one half as many observations are needed.

Question- Why do fractional factorial designs work?

- The sparsity of effects principle
  - There may be lots of factors, but few are important.
  - System is dominated by main effects, low-order interactions.
- The projection property
  - Every fractional factorial contains full factorials in fewer factors.
- Sequential experimentation
  - To resolve difficulties (or ambiguities) on interpretation.

7.1 The One-Half Fraction of the 2^k Design

A 2^{k-1} design or "one-half fraction of the 2^k design", is one in which only half of the 2^k treatment combinations are observed. Let us have K binary factors A, B, C, ..., K with $(-1/+1)$ high/low; and we have 2^k combinations. Note that because the design has $\frac{2^k}{2}$ runs, it's referred to as a 2^{k-1}.

Consider a really simple case, the 2^{3-1}; which $k = 3$. The table of plus and minus signs for the 2^3 is shown as below:

Treatment Combination	Factorial Effect							
	I	A	B	C	AB	AC	BC	ABC
a	+	+	-	-	-	-	+	+
b	+	-	+	-	-	+	-	+
c	+	-	-	+	+	-	-	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	-	+	-	-	-
ac	+	+	-	+	-	+	-	-
bc	+	-	+	+	-	-	+	-
(1)	+	-	-	-	+	+	+	-

In this example, ABC is called the generator of this fraction (only + in ABC column). The identity column I is always plus, so we call $I = ABC$ the defining relation of our design. To estimate the main effects of A, B and C we use the linear combinations of observations:

$$l_A = \tfrac{1}{2}(a - b - c + abc), \quad l_B = \tfrac{1}{2}(-a + b - c + abc), \quad l_C = \tfrac{1}{2}(-a - b + c + abc)$$

The two-factor interactions estimations are as follows:

$$l_{BC} = \tfrac{1}{2}(a - b - c + abc), \quad l_{AC} = \tfrac{1}{2}(-a + b - c + abc), \quad l_{AB} = \tfrac{1}{2}(-a - b + c + abc)$$

$$l_A = l_{BC}, \quad l_B = l_{AC}, \quad l_C = l_{AB}$$

So we may note that the contrast for estimating the main effect A is exactly the same as the contrast we used for estimating the BC interaction. These phenomena are called aliasing and they occur in all fractional designs. Aliases can be found directly from the columns in the table with + and − signs.

Aliasing in the one-half fraction of the 2^3

Aliases can be found from the defining relation $I = ABC$ by multiplication.

$$AI = A(ABC) = A^2BC = BC, \quad BI = B(ABC) = AB^2C = AC$$

$$CI = C(ABC) = ABC^2 = AB$$

Also for estimating the main effects A, B and C we are really estimating A+BC, B+AC and C+AB so the aliases are:

$$l_A \longrightarrow A + BC, \quad l_B \longrightarrow B + AC, \quad l_C \longrightarrow C + AB$$

The alternate fraction of the 2^{3-1}

Let $I = -ABC$ as the defining relation implies slightly different aliases. When we estimate A, B and C using this design, we are really estimating A-BC, B-AC and C-AB, i.e. $l_A^{'} \longrightarrow A - BC$, $l_B^{'} \longrightarrow B - AC$, $l_C^{'} \longrightarrow C - AB$.

Both designs belong to the same family $I = \pm ABC$. Suppose after running the principal fraction, the alternate fraction was also run, the two groups of runs can be combined to form a full factorial. The de-aliased estimates of all effects by analyzing the eight runs as a full 2^3 design in two blocks. Hence

$$\begin{aligned} \frac{1}{2}\left(l_A + l_A^{'}\right) &= \frac{1}{2}(A + BC + A - BC) \longrightarrow A \\ \frac{1}{2}\left(l_A - l_A^{'}\right) &= \frac{1}{2}(A + BC - A + BC) \longrightarrow BC \end{aligned}$$

Design Resolution

1. Resolution III Designs:
   $I = ABC \Longrightarrow 2^{3-1}_{iii}$
   No main effect is with another main effect aliased. Main effects and two-factor interactions are with two-factor interactions aliased.

2. Resolution IV Designs:
   $I = ABC \Longrightarrow 2^{4-1}_{iv}$
   No main effect is with another main effect or with one two-factor interactions aliased. Two-factor interactions are aliased with two-factor interactions. Main effects are aliased only with three-factor interactions.

3. Resolution V Designs:
   $I = ABC \Longrightarrow 2^{5-1}_{v}$
   No main effect or two-factor interactions are with other main effects or two-factor interactions aliased. Two-factor interactions aliased with three factor interactions.

In general, the resolution of a two-level fractional factorial design is the smallest number of letters in any word in the defining relation. The higher the resolution, the less restrictive the assumptions that are required regarding which interactions are negligible to obtain a unique interpretation of the data.

Constructing one-half fraction

- Write down a full 2^{k-1} design
- Add the K^{th} factor by identifying its plus minus levels with the signs of ABC...(K-1)
- $K = ABC...(K-1) \Longrightarrow I = ABC...K$
- Another way is to partition the runs into two blocks with the highest-order interaction ABC...K confounded.

7.2 2^{5-1} factorial design

Example 7.1. (From D. C. Montgomery, 1997, P. 423, Problem 9-6) R.D.Snee ("Experimenting with a Large Number of variables," in Experiments in Industry: Design, Analysis and Interpretation of Results, by R.D.Snee, L.B Hare, and J.B.Trout, Editors, ASQC, 1985) describes an experiment in which a 2^{5-1} design with I=ABCDE was used to investigate the effects of five factors on the y=color of a chemical product. The factors are A=solvent/reactant, B=catalyst/reactant, C=temperature, D= reactant purity, and E= reactant PH. The results obtained were as follows:

```
e=-0.63     d=6.79
a=2.51      ade=5.47
b=-2.68     bde=3.45
abe=1.66    abd=5.68
c=2.06      cde=5.22
ace=1.22    acd=4.38
bce=-2.09   bcd=4.30
abc=1.93    abcde=4.05
```

(a) Prepare a normal probability plot of the effects. Which effect seems active?

(b) Calculate the residuals. Construct the normal probability plot of the residuals and plot the residuals versus the fitted values. Comment on the plots.

(c) If any factors are negligible, collapse the 2^{5-1} design into a full factorial in the active factors. Comment on the resulting design and interpret the results.

We attain resolution V (the best possible choice) with the defining relation I=ABCDE to investigate the effects of five factors on the color of a chemical product. Note that:

1. The defining relationship implies that E=ABCD.
2. The principal (or complementary) half of a 2^k factorial is a full 2^{k-1} factorial for $k-1$ of the factors.

We can get the basic design by writing down a full 2^4 factorial for A, B, C and D, and computing the signs for E from E=ABCD. The resulting 2_V^{5-1} design with simulated data is:

```
*************************************************************
  Basic Design
 ******************                       Treatment
 A   B    C    D         E=ABCD          Combination
*************************************************************
 -   -    -    -          +                 e            0.63
 +   -    -    -          -                 a            2.51
 -   +    -    -          -                 b           -2.68
 +   +    -    -          +                 abe          5.68
 -   -    +    -          -                 c            2.06
 +   -    +    -          +                 ace          1.22
 -   +    +    -          +                 bce         -2.09
 +   +    +    -          -                 abc          1.93
 -   -    -    +          -                 d            6.79
 +   -    -    +          +                 ade          5.47
 -   +    -    +          +                 bde          3.45
 +   +    -    +          -                 abd          5.68
 -   -    +    +          +                 cde          5.22
 +   -    +    +          -                 acd          4.38
 -   +    +    +          -                 bcd          4.30
 +   +    +    +          +                 abcde        4.05
 ************************************************************
```

We note that each main effect is aliased with a four-factor interaction:

$$l_A \longrightarrow A + BCDE, \quad l_B \longrightarrow B + ACDE, \quad l_C \longrightarrow C + ABDE,$$

$$l_D \longrightarrow D + ABCE, \quad l_E \longrightarrow E + ABCD$$

and each two-factor interaction is aliased with a three-factor interaction:

$$l_{AB} \longrightarrow AB + CDE, \;\; l_{BD} \longrightarrow BD + ADE, \;\; l_{AC} \longrightarrow AC + BDE,$$

$$l_{BE} \longrightarrow BE + ACD, \;\; l_{AD} \longrightarrow AD + BCE, \;\; l_{CD} \longrightarrow CD + ABE,$$

$$l_{AE} \longrightarrow AE + BCD, \;\; l_{CE} \longrightarrow CE + ABD,$$

$$l_{BC} \longrightarrow BC + ADE, \;\; l_{DE} \longrightarrow DE + ABC$$

Figure 7.1 on page 135 shows the normal plot and indicates that the factors A, B and D and the interactions effects AB and AD are significant.

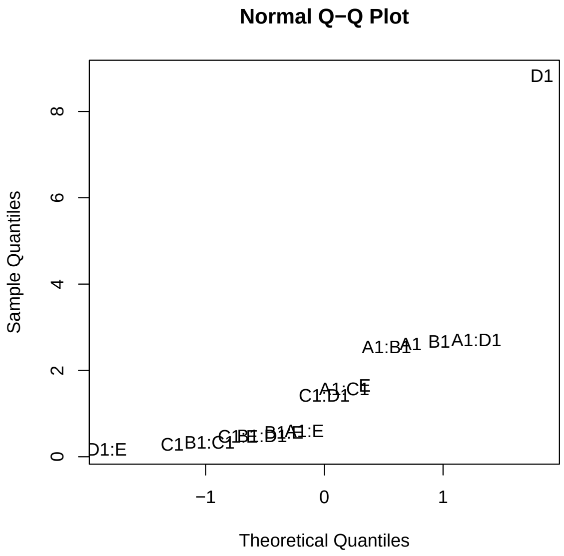


Figure 7.1: 2^{5-1} factorial design, Normal Plot

```
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value Pr(>F)
A          1  6.864   6.864
B          1  7.182   7.182
C          1  0.087   0.087
D          1 78.146  78.146
E          1  2.739   2.739
A:B        1  6.502   6.502
A:C        1  2.481   2.481
A:D        1  7.344   7.344
A:E        1  0.366   0.366
B:C        1  0.112   0.112
B:D        1  0.240   0.240
B:E        1  0.331   0.331
C:D        1  2.031   2.031
C:E        1  0.230   0.230
D:E        1  0.031   0.031
Residuals  0  0.000
```

From the above results we can conclude that the main effects A, B and D and the two two-factor interaction effects AB and AD are significant. Remember because of aliasing, these effects are really A+BCDE, B+ACDE, D+ABCE, AB+CDE, and AD+BCE. We now omit factors C and E from our model, and again we construct an ANOVA for a new model.

```
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value    Pr(>F)
A          1  6.864   6.864  7.3484   0.02397 *
B          1  7.182   7.182  7.6888   0.02165 *
D          1 78.146  78.146 83.6557 7.481e-06 ***
A:B        1  6.502   6.502  6.9610   0.02699 *
A:D        1  7.344   7.344  7.8619   0.02058 *
B:D        1  0.240   0.240  0.2570   0.62436
Residuals  9  8.407   0.934
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 0.9665 on 9 degrees of freedom
Multiple R-squared: 0.9267,     Adjusted R-squared: 0.8778
F-statistic: 18.96 on 6 and 9 DF,  p-value: 0.0001235
```

The model F-value of 18.96 implies that the model is significant. The estimated effects of these factors are as below:

```
Term           Effect
A               1.31
B              -1.34
D               4.42
AB              1.275
AD             -1.355
```

Now we investigate the residuals by drawing the four diagnostic plots in figure 7.2 on page 137. The residual plots are satisfactory, the residuals are normally distributed.
In this case the 2^{5-1} model is reduced to a 2^3 design with two replicates.

```
Analysis of Variance Table
Response: y
          Df Sum Sq Mean Sq F value   Pr(>F)
A          1  6.864   6.864  6.7160  0.03204 *
B          1  7.182   7.182  7.0271  0.02922 *
D          1 78.146  78.146 76.4559 2.29e-05 ***
A:B        1  6.502   6.502  6.3619  0.03568 *
A:D        1  7.344   7.344  7.1853  0.02790 *
B:D        1  0.240   0.240  0.2349  0.64090
A:B:D      1  0.230   0.230  0.2254  0.64763
```

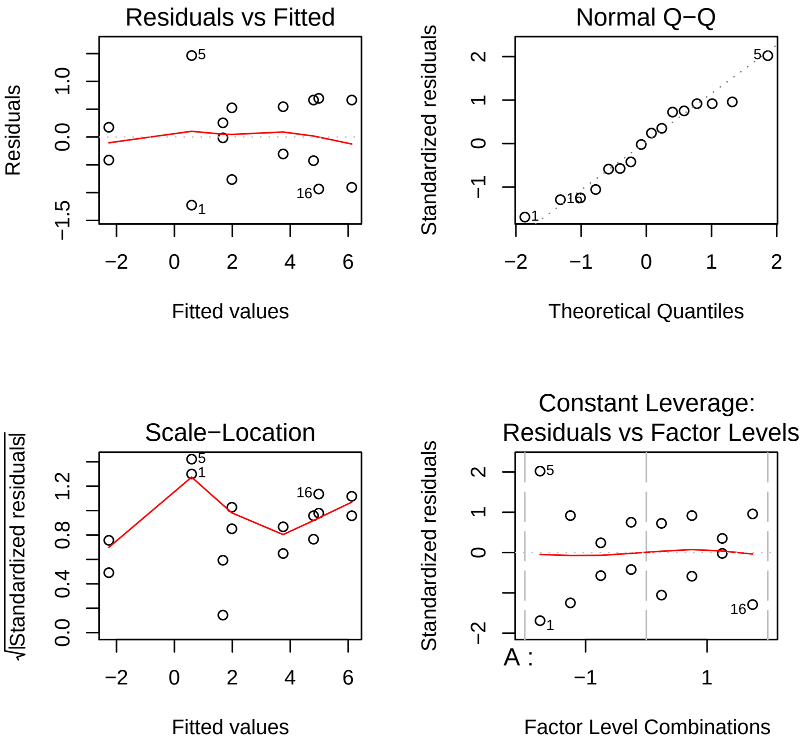


Figure 7.2: 2^{5-1} factorial design, The four diagnostic plots

```
Residuals  8  8.177   1.022
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
Residual standard error: 0.9665 on 9 degrees of freedom
Multiple R-squared: 0.9267,     Adjusted R-squared: 0.8778
F-statistic: 18.96 on 6 and 9 DF,  p-value: 0.0001235
```

This model has 87.8% explanation . The interaction plots show us which factor has an effect on the response variable. In other words what is the role of interactions on the response. Figure 7.3 on page 138 shows the two-factor interactions. As we see for the higher value we have to take the main effects A^+, B^-, D^+. It means we have to take the solvent with high values, low values of catalysts and high reactant purity. We can conclude that by two-factor interaction AB we have always a uniform high value for all combinations just not for A^-B^+. For AD it is just important to have a high value of reactant purity.

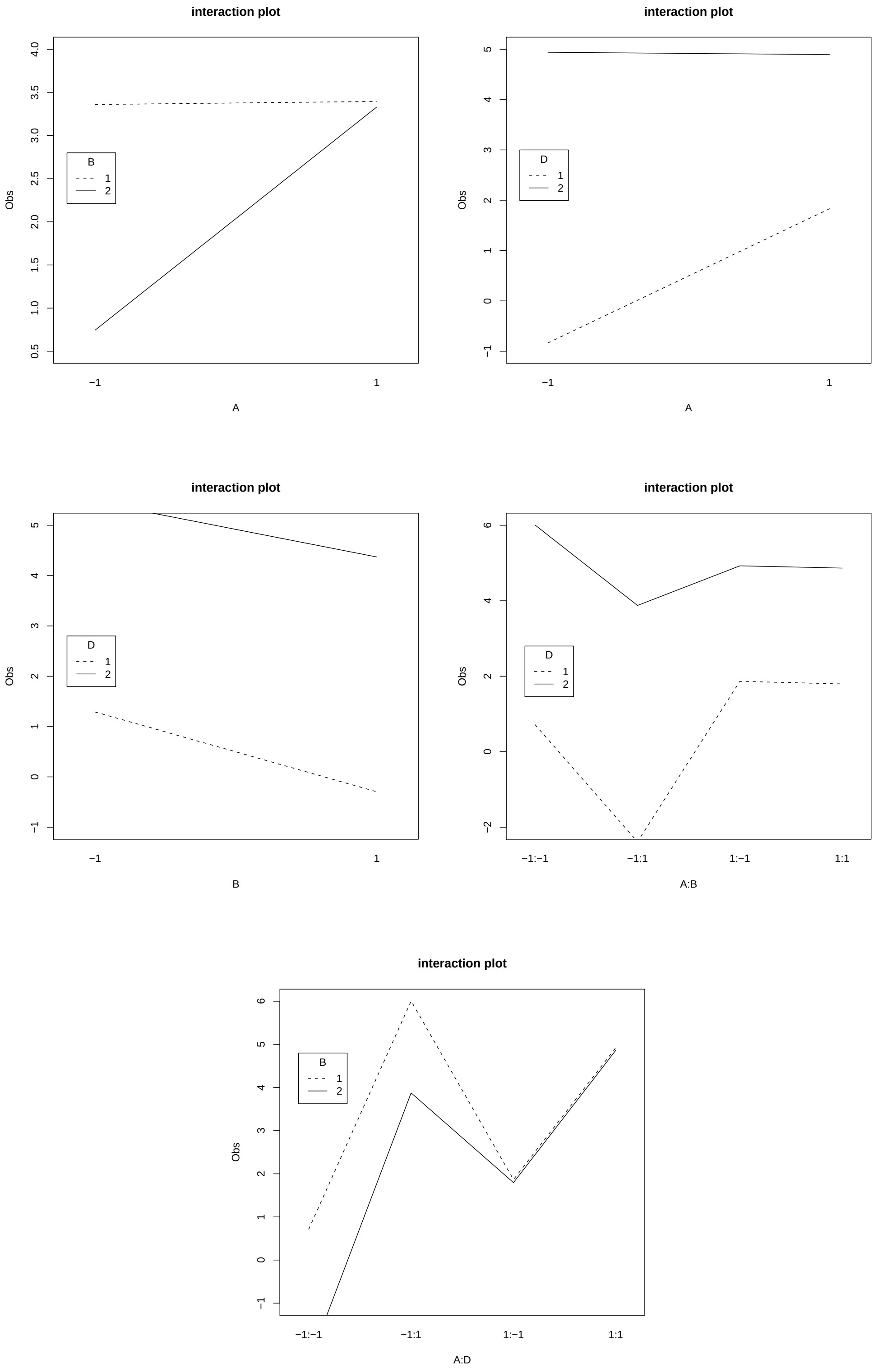


Figure 7.3: 2^{5-1} factorial design, Interaction plots

The fitted regression model in this case is:

$$\hat{y} = 3.0375 + 1.31x_1 - 1.34x_2 + 4.42_4 + 1.275x_1x_2 - 1.355x_1x_4$$

The response surface plot and the contour plot are shown in figures 7.4 on page 139 and 7.4 on page 139 respectively. When we set factor D (the reactant purity) at its high level ($x_4 = 1$). We will conclude the same result as in the interaction plots.

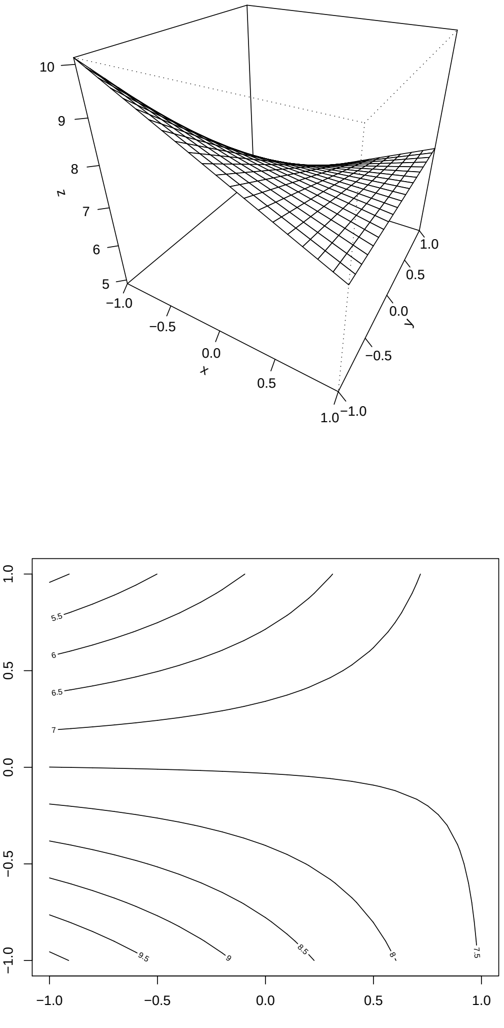


Figure 7.4: 2^{5-1} factorial design, Response surface plot and Contour plot

2^{5-1} factorial design R-code

```
A<-rep(c(-1,1),times=8); B<-rep(c(-1,1),each=2,times=4)
C<-rep(c(-1,1),each=4,times=2); D<-rep(c(-1,1),each=8)
y<-c(-0.63,2.51,-2.68,1.66,2.06,1.22,-2.09,1.93,6.79,5.47,3.45,5.68,
5.22,4.38,4.30,4.05)
E<-A*B*C*D; A<-as.factor(A); B<-as.factor(B)
C<-as.factor(C); D<-as.factor(D)
data<-data.frame(A,B,C,D,E,y); data
g<-lm(y~(A+B+C+D+E)^5); anova(g)
g$effects; effects<-abs(g$effects[-1])
qq<-qqnorm(effects,type="n")
text(qq$x,qq$y,labels=names(effects))
h<-lm(y ~(A+B+D)^2); anova(h); summary(h)
model<-aov(y~(A+B+D)^2); model.tables(model)
windows(); par(mfrow=c(2,2))
plot(aov(y ~(A+B+D)^2))
windows(); par(mfrow=c(3,2))
interaction.plot(A,B,y,xlab="A",ylab="Obs",
main="interaction plot",ylim=c(0.5,4.0),fixed=T,legend=F)
legend(0.90,2.80,c("1","2"),title="B",lty=c(2,1))
interaction.plot(A,D,y,xlab="A",ylab="Obs",
main="interaction plot",ylim=c(-1,5.0),fixed=T,legend=F)
legend(0.90,3.0,c("1","2"),title="D",lty=c(2,1))
interaction.plot(B,D,y,xlab="B",ylab="Obs",
main="interaction plot",ylim=c(-1,5.0),fixed=T,legend=F)
legend(0.90,2.80,c("1","2"),title="D",lty=c(2,1))
interaction.plot(A:B,D,y,xlab="A:B",ylab="Obs",
main="interaction plot",ylim=c(-2,6.0),fixed=T,legend=F)
legend(0.90,2.80,c("1","2"),title="D",lty=c(2,1))
interaction.plot(A:D,B,y,xlab="A:D",ylab="Obs",
main="interaction plot",ylim=c(-1,5.0),fixed=T,legend=F)
legend(0.90,2.80,c("1","2"),title="B",lty=c(2,1))
x<-seq(-1,1,0.1); y<-seq(-1,1,0.1)
func<-function(a,b){7.4575-0.045*a-1.34*b+1.275*a*b}
z<-outer(x,y,func);z
persp(x,y,z,theta=30,phi=30,ticktype="detailed")
contour(x,y,z,nlevels=10)
```

7.3 Monte Carlo Simulation Model

Example 7.2. An industrial engineer is conducting an experiment using a Monte Carlo Simulation Model of an inventory system. The independent variables in her model are the order quantity (A), the reorder point (B), the setup cost (C), the back order cost (D), and the carrying cost rate (E). The response variable y is average annual cost. To conserve computer time, she decides to investigate these factors using a 2^{5-2}_{III} design with I=ABD and I=BCE. The results she obtains are:

`de=95, ae=134, b=158, abd=195, cd=92, ac=187, bce=155,and abcde=185.`

(a) Verify that the treatment combinations given are correct. Estimate the effects, assuming three-factor and higher interactions are negligible.

(b) Suppose that a second fraction is added to the first, for example:

`ade=136, e=93, ab=187, bd=153, acd=139, c=99, abce=191, and bcde=150`

How was this second fraction obtained? Add this data to the original fraction, and estimate the effects.

(c) Suppose that the fraction :

`abc=189, ce=96, bcd=154, acde=135, abe=193, bde=152, ad=137, and (1)=98`

was run. How was the fraction obtained? Add this fraction to the original fraction and estimate the effects.

First we verify the correctness of the treatment combination given in part (a). For this particular design, the experimenter used only 8 runs ($\frac{1}{4}$ fraction) of a 32 run or (2^5) design. For each of these runs D=AB and E=BC. If we multiply both sides of the first equation by D; we obtain:

$$DD = ABD \quad I = ABD$$

Likewise, if we multiply both sides of E=BC by E, we obtain:

$$EE = BCE \quad I = BCE$$

We can say the design is comprised of the 8 runs for which both ABD and BCE are equal to one, I=ABD=BCE.

$L_1 = x_1 + x_2 + x_4$, $L_2 = x_2 + x_3 + x_5$

```
**********************************************************************
Factor Combination *L_{1}=x_{1}+x_{2}+x_{4} *L_{2}=x_{2}+x_{3}+x_{5} *
**********************************************************************
DE                 *1+0=1(mod2)             *0+1=1 (mod2)            *
AE                 *1+0=1 (mod2)            *0+1=1(mod2)             *
B                  *1+0=1 (mod2)            *0+1=1(mod2)             *
ABD                *1+1+1=1(mod2)           *0+1+0=1(mod2)           *
CD                 *0+1=1 (mod2)            *1+0=1(mod2)             *
AC                 *1+0=1(mod2)             *0+1=1 (mod2)            *
BCE                *1+0+0=1(mod2)           *1+1+1=1(m0d2)           *
ABCDE              *1+1+0+1+0=1(mod2)       *0+1+1+0+1=1(mod2)       *
**********************************************************************
```

An experimenter wants to study the design in 8 runs by assigning D=AB and E=BC in the three factor 8 runs signs table:

```
********************************************
A     B    C     D=AB     E=BC     Factor
********************************************
-     -     -       +         +       ABCDE
+     -     -       -         +       BCE
-     +     -       -         -       CD
+     +     -       +         -       AC
-     -     +       +         -       ABD
+     -     +       -         -       B
-     +     +       -         +       AE
+     +     +       +         +       DE
********************************************
```

We see from this table that, for each of the given generators we have $L_1 = L_2 = 1$. So the given factor combination is correct. Now we have to estimate the effects. Assume that the three-factor interactions and the interactions of higher power can be eliminated. The normal plot is shown in figure 7.5.

```
*****************************
Factor Combination *Effects
*****************************
l_{A}->A+BD        *   50.25
l_{B}->B+AD+CE     *   46.25
l_{C}->C+BE        *   9.25
l_{D}->D+AB        *  -16.75
l_{E}->E+BC        *  -15.75
l_{AC}->AC+DE      *   12.25
```

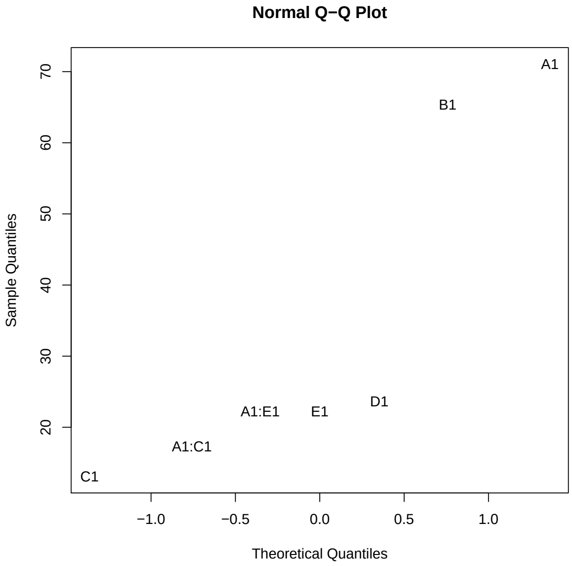


Figure 7.5: Monte Carlo, Normal plot

```
l_{AE}->AE+CD      *  -15.75
*****************************
```

From the above results we conclude that the two first effects are positive, the estimates of A+BD and B+AD+CE, the effect of D is the smallest from others. The ANOVA table indicates that the main two factors A and B have high effect on response variable but they aren't significant, and the F-value of 3.647 implies that the model isn't significant. The 69.4% of variability in the model is caused by these two effects.

```
Analysis of Variance Table

Response: y
          Df Sum Sq Mean Sq F value Pr(>F)
A          1 5050.1  5050.1 10.1791 0.1934
B          1 4278.1  4278.1  8.6231 0.2090
C          1  171.1   171.1  0.3449 0.6619
D          1  561.1   561.1  1.1310 0.4804
E          1  496.1   496.1  1.0000 0.5000
A:C        1  300.1   300.1  0.6049 0.5792
Residuals  1  496.1   496.1
---
Residual standard error: 22.27 on 1 degrees of freedom
Multiple R-squared: 0.9563,     Adjusted R-squared: 0.6941
```

```
F-statistic: 3.647 on 6 and 1 DF,  p-value: 0.3807
```

We have received new data that we have to add to the fractions. There are 3 other equivalent $\frac{1}{4}$ fraction, the experimenter could have used:

```
ABD=1,  BCE=-1, (I=ABD=-BCE)
ABD=-1, BCE=1,  (I=-ABD=BCE)
ABD=-1, BCE=-1, (I=-ABD=-BCE)
```

We construct the following table with factor D=-AB and E=BC and I=BCE:

```
**********************************************
A     B    C     D=-AB     E=BC     Factor
**********************************************
+     -     -       +         +       E
-     -     -       -         +       ADE
+     +     -       -         -       BD
-     +     -       +         -       AB
+     -     +       +         -       C
-     -     +       -         -       ACD
+     +     +       -         +       BCDE
-     +     +       +         +       ABCE
**********************************************
```

Here we have a 2^{5-1}_{IV} with 16 runs. The normal plot is drawn in figure 7.6 on page 145. As we see from the normal plot, we can conclude the same as in part (a) that the two first effects are positive, these are the estimators of A and B+CE. AB is the third largest effect. It's value is obviously smaller. Now we estimate the effects.

```
*****************************
Factor Combination *Effects
*****************************
      A            *   44.88
      B+CE         *   49.88
      C+BE         *   5.88
      D            *  -7.38
      E+BC         *  -8.88
      AB           *  -9.38
      AC           *   6.62
      AD           *   3.63
*****************************
Factor Combination *Effects
*****************************
```

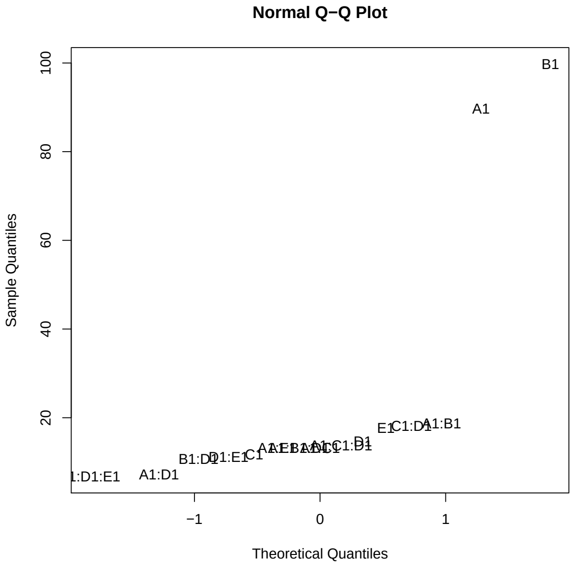


Figure 7.6: Montecarlo, Normal plot

```
      AE            *   -6.62
      BD            *    5.38
      CD            *   -9.12
      DE            *    5.63
      ABD           *    6.62
      ACD           *   -6.88
      ADE           *    3.38
*****************************
```

In the third part (c), we add a new fraction to our data. The table of signs is as follows:

```
*********************************************
A     B    C        D         E    Factor
*********************************************
+     +     +       -         -       E
-     -     +       -         +       ADE
+     -     +       +         -       BD
+     -     +       +         +       AB
+     +     -       -         +       C
-     +     -       +         +       ACD
+     -     -       +         -       BCDE
-     -     -       -         -       ABCE
*********************************************
```

Now we have to combine all the data together so we will have 24 observations. As we have seen the factors A, B, D and AB are the most important. So we can have a full 2^3 factorial design with 3 replicates. We will do the analysis of variance and the ANOVA table shows that just the two factors A and B are significant.

```
            Df  Sum Sq Mean Sq  F value  Pr(>F)
A            1 11745.3 11745.3 104.4025 0.06211 .
B            1 15167.0 15167.0 134.8175 0.05469 .
C            1    95.0    95.0   0.8443 0.52690
D            1   105.3   105.3   0.9361 0.51050
E            1   266.3   266.3   2.3670 0.36692
A:B          1   481.9   481.9   4.2836 0.28654
A:C          1   103.9   103.9   0.9233 0.51270
A:D          1    82.1    82.1   0.7295 0.55000
A:E          1   106.6   106.6   0.9477 0.50854
B:C          1   124.6   124.6   1.1078 0.48371
B:D          1   140.5   140.5   1.2493 0.46465
B:E          1   116.7   116.7   1.0375 0.49414
C:D          1   143.4   143.4   1.2750 0.46143
C:E          1     0.3     0.3   0.0030 0.96501
D:E          1    35.8    35.8   0.3183 0.67298
A:B:C        1   207.1   207.1   1.8409 0.40435
A:B:D        1   222.6   222.6   1.9789 0.39342
A:B:E        1   145.7   145.7   1.2947 0.45900
A:C:D        1   144.9   144.9   1.2881 0.45982
A:C:E        1    14.2    14.2   0.1266 0.78236
A:D:E        1    23.5    23.5   0.2089 0.72706
B:C:E        1     5.6     5.6   0.0501 0.85985
Residuals    1   112.5   112.5
```

We have seen in the first two parts that factors A, B and D and the interaction AB are the most important factors, so we reduce our model in to a model with these three factors. Now we will do the full factorial design, the ANOVA table is as follows:

```
            Df  Sum Sq Mean Sq  F value    Pr(>F)
A            1 11745.3 11745.3 107.0205 1.710e-08 ***
B            1 15167.0 15167.0 138.1982 2.770e-09 ***
D            1   108.2   108.2   0.9856   0.33559
A:B          1   470.1   470.1   4.2839   0.05502 .
A:D          1    58.1    58.1   0.5295   0.47734
B:D          1   124.8   124.8   1.1374   0.30204
```

```
A:B:D        1   161.5   161.5   1.4716   0.24271
Residuals   16  1756.0   109.7
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
Residual standard error: 10.48 on 16 degrees of freedom
Multiple R-squared: 0.9407,     Adjusted R-squared: 0.9147
F-statistic: 36.23 on 7 and 16 DF,  p-value: 1.193e-08
```

As we can see from the following result, factors A and B are highly significant and their interaction is rare significant.So we omit factor D from our model.

```
            Df  Sum Sq Mean Sq  F value    Pr(>F)
A            1 11745.3 11745.3  98.5053 3.590e-09 ***
B            1 15167.0 15167.0 127.2023 4.035e-10 ***
A:B          1   294.0   294.0   2.4658    0.1320
Residuals   20  2384.7   119.2
---
Residual standard error: 10.92 on 20 degrees of freedom
Multiple R-squared: 0.9194,     Adjusted R-squared: 0.9073
F-statistic: 76.06 on 3 and 20 DF,  p-value: 4.116e-11
```

The model F-statistic of 76.06 indicates that the model is significant. The effects of model are:

```
*****************************
Factor Combination *Effects
*****************************
      A            *   44.4
      B            *   50.45
*****************************
```

Monte-Carlo-Simulation R-Code

```
A<-c(-1,1,-1,1,-1,1,-1,1); B<-c(-1,-1,1,1,-1,-1,1,1)
C<-c(-1,-1,-1,-1,1,1,1,1); D<-c(1,-1,-1,1,1,-1,-1,1)
E<-c(1,1,-1,-1,-1,-1,1,1)
y<-c(95,134,158,195,92,187,155,185)
A<-as.factor(A); B<-as.factor(B); C<-as.factor(C)
D<-as.factor(D); E<-as.factor(E)
data<-data.frame(A,B,C,D,E,y); data
mod1<-aov(y~(A+B+C+D+E)^5); summary(mod1)
model.tables(mod1); g<-lm(y~(A+B+C+D+E)^5)
g$effects; effects<-abs(g$effects[-1])
qq<-qqnorm(effects,type="n")
text(qq$x,qq$y,labels=names(effects))
***********************************************************
A<-c(-1,1,-1,1,-1,1,-1,1,1,-1,1,-1,1,-1,1,-1)
B<-c(-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1)
C<-c(-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1)
D<-c(1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1)
E<-c(1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1)
y1<-c(95,134,158,195,92,187,155,185,136,93,187,153,139,99,191,150)
data1<-data.frame(A,B,C,D,E,y1)
A<-as.factor(A); B<-as.factor(B); C<-as.factor(C)
D<-as.factor(D); E<-as.factor(E)
mod2<-aov(y1~(A+B+C+D+E)^4); summary(mod2)
model.tables(mod2); g<-lm(y1~(A+B+C+D+E)^4)
g$effects; effects<-abs(g$effects[-1])
qq<-qqnorm(effects,type="n")
text(qq$x,qq$y,labels=names(effects))
*************************************************************************
A<-c(-1,1,-1,1,-1,1,-1,1,1,-1,1,-1,1,-1,1,-1,1,-1,1,1,1,-1,1,-1)
B<-c(-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,1,-1,-1,-1,1,1,-1,-1)
C<-c(-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,-1,-1,-1,-1)
D<-c(1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,-1,-1,1,1,-1,1,1,-1)
E<-c(1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,-1,1,-1,1,1,1,-1,-1)
y3<-c(95,134,158,195,92,187,155,185,136,93,187,153,139,99,191,150,189,96,
154,135,193,152,137,98); A<-as.factor(A); B<-as.factor(B); D<-as.factor(D)
data3<-data.frame(A,B,C,D,E,y3); data3
mod3<-aov(y3~(A+B+C+D+E)^3); summary(mod3)
mod4<-aov(y3~(A+B+D)^3); summary(mod4); model.tables(mod4)
mod5<-aov(y3~(A+B)^2); summary(mod5); model.tables(mod5)
```

Chapter 8

Three-Level Factorial Design

In the previous chapters we studied two level factorial and fractional factorial designs, which have widely used in industrial research and development. There are some intentions and variations of these designs that are occasionally useful, such as the designs for cases where all the factors are present at three levels. These 3^k designs will be discussed in this chapter [Montgomery D. C. (1997)].

8.1 The 3^k Factorial Design

In this chapter we discuss the 3^k factorial design, that is a factorial arrangement with k factors each at three levels. Each treatment combination in the 3^k design will be denoted by k digits, where the first digit indicates the level of factor A, the second digit the level of factor B and so on.

The 3^k design is certainly a possible choice by an experimenter who is concerned about curvature in the response functions. However, two points must be considered:

1. The 3^k design is not the most efficient way to model a quadratic relationship.
2. The 2^k design augmented with center points, is an excellent way to obtain an indication of curvature.

8.2 The 3^2 design

The simplest design in the 3^k system is the 3^2 design, which has 2 factors, each at three levels. In this model we have three-levels (–1,0,1) for each factor; and we have

$t \geq 2$ replicates for each treatment combination. The ANOVA model for this case is:

$$y_{ijk} = \mu_0 + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$$

where

$$e_{ijk} \overset{iid}{\sim} N(0, \sigma^2), \quad \text{for } i, j = 1, 2 \text{ and } k = 1, 2, \ldots, t$$

and the following conditions may noticed:

$$\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = 0, \quad \gamma_{1j} + \gamma_{2j} + \gamma_{3j} = \gamma_{i1} + \gamma_{i2} + \gamma_{i3} = 0$$

Since there are $3^2 = 9$ treatment combinations, there are 8 degrees of freedom between these treatment combinations. The main effects of A and B each have two degrees of freedom, and the AB interaction has four degrees of freedom.

If there are t replicates, there will be $3^2(t-1)$ degrees of freedom for error.

This model has the following sum of squares:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{t}(y_{ijk} - \overline{y}_{...})^2 = 3t\sum_{i=1}^{3}(\overline{y}_{i..} - \overline{y}_{...})^2 + 3t\sum_{j=1}^{3}(\overline{y}_{.j.} - \overline{y}_{...})^2 +$$
$$t\sum_{i=1}^{3}\sum_{j=1}^{3}(\overline{y}_{ij.} - \overline{y}_{i..} - \overline{y}_{.j.} + \overline{y}_{...}))^2 + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{t}(y_{ijk} - \overline{y}_{ij.})^2$$

In other words:

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

with the following degrees of freedom:

$$df(SS_T) = df(SS_A) + df(SS_B) + df(SS_{AB}) + df(SS_E)$$

$$9t - 1 = 2 + 2 + 4 + 9(t-1)$$

Note that the sum of squares for any main effect may be partitioned in to a linear and a quadratic component, each with one degree of freedom, using the orthogonal contrast constants. This is only meaningful if the factor is quantitative and if the three levels are equally spaced. The two-factor interaction AB maybe partitioned in two-ways:

1. This method consists of subdividing AB into the four single-degree-of-freedom components corresponding to $AB_{L\times L}$, $AB_{L\times Q}$, $AB_{Q\times L}$ and $AB_{Q\times Q}$. The technique for computing the single degree of freedom interaction components requires the call totals (recall the differences between calls define an interaction effect) and the orthogonal contrast coefficients. To illustrate the approach, consider the coefficients for the $AB_{L\times L}$ interaction components as below:

```
******************************************
                          B
A         *******************************
            -1            0           +1
-1          +1            0           -1
 0           0            0            0
+1          -1            0           +1
*******************************************
```

$$\begin{pmatrix} +1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & +1 \end{pmatrix}$$

$$AB_{L\times L} = \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}y_{ij.}, \quad SSAB_{L\times L} = \frac{(AB_{L\times L})^2}{t\sum_i \sum_j c_{ij}^2}$$

```
******************************************
                          B
A      *******************************
             1           -2            1
-1          -1            2           -1
 0           0            0            0
+1           1           -2           +1
*******************************************
```

$$\begin{pmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

$$AB_{L\times Q} = \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij} y_{ij.}, \quad SSAB_{L\times Q} = \frac{(AB_{L\times Q})^2}{t\sum_i \sum_j c_{ij}^2}$$

```
*****************************************
                      B
A       *******************************
          -1           0            1
 1        -1           0            1
-2         2           0           -2
 1        -1           0            1
******************************************
```

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

$$AB_{Q\times L} = \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij} y_{ij.}, \quad SSAB_{Q\times L} = \frac{(AB_{Q\times L})^2}{t\sum_i \sum_j c_{ij}^2}$$

```
*****************************************
                      B
A       *******************************
           1          -2            1
 1         1          -2            1
-2        -2           4           -2
 1         1          -2            1
******************************************
```

$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$AB_{Q\times Q} = \sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij} y_{ij.}, \quad SSAB_{Q\times Q} = \frac{(AB_{Q\times Q})^2}{t\sum_i \sum_j c_{ij}^2}$$

So we can write:

$$SSAB = SSAB_{L\times L} + SSAB_{L\times Q} + SSAB_{Q\times L} + SSAB_{Q\times Q}$$

The polynomial approximation is:

$$y = \alpha_0 + \alpha_1 P_1(x) + \alpha_2 P_2(x) + \epsilon; \quad \epsilon \overset{iid}{\sim} N(0, \sigma^2)$$

Where $P_i(x)$ are the orthogonal polynomials, and $\widehat{\alpha}$ can be estimated as follows:

$$\widehat{\alpha}_1 = \frac{\sum_{i=1}^{3} y_i P_1(x_i)}{\sum_{i=1}^{3} P_1^2(x_i)}, \quad \widehat{\alpha}_2 = \frac{\sum_{i=1}^{3} y_i P_2(x_i)}{\sum_{i=1}^{3} P_2^2(x_i)}$$

2. The second method is based on orthogonal Latin squares. Consider the two factors A and B, they correspond to the rows and columns respectively, of a 3×3 Latin square.

Q	R	S
R	S	Q
S	Q	R

Q	R	S
S	Q	R
R	S	Q

These two Latin squares are orthogonal; that is, if one square is superimposed on the other, each letter in the first square will appear exactly once with each letter in the second square.

The sum of squares computed from the first square is called the *AB* component of interaction, and the sum of squares computed from the second square is called the AB_2 component of interaction. The components AB and AB_2 each has two degrees of freedom. This terminology is used because if we denote the levels $(-1, 0, 1)$ for A and B by X_1 and X_2 respectively, then we find that the letters occupy cells according to the following pattern, for the first square we will have:

$$Q: \quad (-1,-1); \quad (0,1); \quad (1,0)$$

$$Q: X_1 + X_2 \equiv 0 \mod 3$$

$$R: \quad (-1,0); \quad (0,-1); \quad (1,1)$$

$$R: X_1 + X_2 \equiv 1 \mod 3$$

$$S: \quad (-1,1); \quad (0,0); \quad (1,-1)$$

$$S: X_1 + X_2 \equiv 2 \mod 3$$

For the second square we have the following equations:

$$Q: X_1 + 2X_2 \equiv 0 \mod 3$$

$$S: X_1 + 2X_2 \equiv 1 \mod 3$$

$$R: X_1 + 2X_2 \equiv 2 \mod 3$$

When considering expressions of the form A^pB^q, we establish the convention that the only exponent allowed on the first letter is one. If the first letter isn't one, the entire expression is squared and the exponents are reduced modulus 3. For example, A^2B is the same as AB^2 since $A^2B = (A^2B)^2 = A^4B^2 = AB^2$.

Yates called these components of interactions the I and J components of interaction, respectively. We use both notations interchangeably, that is:

$$I(AB) = AB^2, \quad J(AB) = AB$$

$$SS_{AB} = I(AB) + J(AB)$$

$$I(AB) = \frac{1}{3t}\left(SQ^2 + SR^2 + SS^2\right)_a - \frac{1}{9t}\left(SQ + SR + SS\right)_b^2$$

$$I(AB) = \frac{1}{3t}\left(SQ^2 + SR^2 + SS^2\right)_b - \frac{1}{9t}\left(SQ + SR + SS\right)_a^2$$

8.3 The 3^3 design

Now suppose there are three factors A, B and C. Each factor is at three levels arranged in a factorial experiment. This is a 3^3 factorial design. In this case we have 27 treatment combinations having 26 degrees of freedom. Each main effect has two degrees of freedom, each two-factor interaction has four degrees of freedom, and the three-factor interaction has eight degrees of freedom. If there are t replicates, there are $t3^3-1$ total degrees of freedom and $3^3(t-1)$ degrees of freedom for error.

The sum of squares may be calculated using the standard methods for factorial designs. Here we have a same computation as in the 3^2-design. It is also possible to partition the two-factor interactions into their I and J components. These are designated AB, AB^2, AC, AC^2, BC and BC^2 and each component has two degrees of freedom.

The three-factor interaction ABC may be partitioned into four orthogonal two degrees of freedom components, which are usually called the W, X, Y and Z

components of the interaction. They are also referred to as the AB^2C^2, AB^2C, ABC^2, and ABC components of the ABC interaction, respectively. The two notations are used interchangeably, that is:

$$W(ABC) = AB^2C^2, \quad X(ABC) = AB^2C, \quad Y(ABC) = ABC^2, \quad Z(ABC) = ABC$$

Note that the first letters have an exponent equal to 1.

8.4 The General 3^k Design

The concepts utilized in the 3^2 and 3^3 designs can be readily extended to the case of k factors, each at three levels, that is, to a 3^k factorial design. There are 3^k treatment combinations with $3^k - 1$ degrees of freedom between them. These treatment combinations allow sums of squares to be determined for k main effects, each with two degrees of freedom; $\binom{k}{2}$ two-factor interactions, each with four degrees of freedom; etc and one k-factor interaction with 2^k degrees of freedom. If there are t replicates, there are $t3^k - 1$ total degrees of freedom and $3^k(t-1)$ degrees of freedom for error.

For example, the four-factor interaction $ABCD$ has $2^{4-1} = 8$ orthogonal two degrees of freedom components, denoted by $ABCD^2$, ABC^2D, $ABCD$, ABC^2D^2, AB^2C^2D, AB^2CD^2.

In writing these components, note that the only exponent allowed on the first letter is 1. If the exponent on the first letter isn't 1 then the entire expression must be squared and the exponents reduced modulus 3.

8.5 Confounding the 3^k factorial design

Even when a single replicate of the 3^k design is considered, the design requires so many runs that it is unlikely that all 3^k runs can be made under uniform conditions. That's why confounding in blocks is often necessary. The 3^k design may be confounded in 3^p incomplete blocks, where $p < k$. So these designs may be confounded in three blocks, nine blocks and so on.

8.5.1 Confounding the 3^k factorial design in three blocks

Suppose we wish to confound the 3^k design in three incomplete blocks. These three blocks have two degrees of freedom among them; thus there must be two degrees of freedom confounded with blocks.

In the 3^k factorial design each main effect has two degrees of freedom, every two-factor interaction has four degrees of freedom and can be decomposed into two components of interaction for example (AB, AB^2) each with two degrees of freedom; every three-factor interaction has eight degrees of freedom and can be decomposed in to four components of interaction (ABC, ABC^2, AB^2C and AB^2C^2), each with two degrees of freedom, and etc. Therefore, it is convenient to confound a component of interaction with blocks. The general procedure is to construct a defining contrast:

$$L = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_k x_k$$

Where α_i represents the exponent on the i^{th} factor in the effect to be confounded and x_i is the level of the i^{th} factor in a particular treatment combination. For the 3^k factorials, we have $\alpha_i = 0, 1, 2$ with the first non zero α_i being unity, and $x_i = 0$ (low level), 1 (intermediate level), 2 (high level).

The treatment combinations in the 3^k design are assigned to blocks based on the value of L (mod 3). Since L (mod 3) can take only the values 0, 1, 2, three blocks are uniquely defined. The treatment combinations satisfying $L = 0$ (mod 3) constitute the principal block.

In general, for the 3^k design in three blocks, we would always select a component of the highest order interaction to confound with blocks. The remaining unconfounded components of this interaction could be obtained by computing the k-factor interaction in the usual way and subtracting from this quantity the sum of squares for blocks.

8.6 The 3^k Factorial Design in Nine Blocks

In some experimental situations it is useful to confound the 3^k design in nine blocks, so that eight degrees of freedom will be confounded with blocks. For constructing these designs, we choose two components of interaction and, as a result two more will be confounded automatically, yielding the required eight degrees of freedom. In the 3^k system, the generalized interactions of two effects (P, Q) are defined as PQ and PQ^2 (P^2Q). The two components of interaction initially chosen yield two defining contrasts:

$$L_1 = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_k x_k = u \mod 3, \quad u = 0, 1, 2$$

$$L_1 = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k = v\text{mod } 3, \quad v = 0, 1, 2$$

Where $\alpha_i, \beta_i \in \{0, 1, 2\}$ are the exponents in the first and second generalized interactions respectively.

The defining contrasts imply nine simultaneous equations specified by the pair of values for L_1 and L_2. Treatment combinations having the same pair of values for (L_1, L_2) are assigned to the same block. Consider the 3^4 factorial design confounded in nine blocks of nine runs each. Suppose we choose to confound ABC and AB^2D^2. Their generalized interactions

$$(ABC)(AB^2D^2) = A^2B^3CD^2 = (A^2B^3CD^2)^2 = AC^2D$$

$$(ABC)(AB^2D^2)^2 = A^3B^5CD^4 = B^2CD = (B^2CD)^2 = BC^2D^2$$

are also confounded with blocks. The defining contrasts for ABC and AB^2D^2 are:

$$L_1 = X_1 + X_2 + X_3, \quad L_2 = X_1 + 2X_2 + 2X_4$$

The nine blocks may be constructed by using the defining contrasts. For the 3^k design in nine blocks, there will be four components of interaction confounded.

8.7 The 3^k Factorial Design in 3^p Blocks

The 3^k factorial design is confounded in 3^p blocks of 3^{k-p} observations each, where $p \leq k$. The procedure is to select p independent effects to be confounded with blocks. As a result, exactly $\frac{3^p-2p-1}{2}$ other blocks are automatically confounded.

8.8 The one-third fraction of the 3^k factorial design

The largest fraction of the 3^k design is a one-third fraction containing 3^{k-1} fractional factorial designs. For constructing a 3^{k-1} fractional factorial design, first we select a two degrees of freedom component of interaction (the highest-order interaction) and then we partition the full 3^k design in three blocks, we have to define a defining relation

$$I = AB^{\alpha_2}C^{\alpha_3}...K^{\alpha_k}$$

Each main effect of component of interaction estimated from the 3^{k-1} design has two aliases, which may be found by multiplying the effect by both I and I^2 modulus 3.

The treatment combinations in a 3^{k-1} design with the defining relation $I = AB^{\alpha_2}C^{\alpha_3}...K^{\alpha_k}$ can be constructed using a method similar to that employed in the 2^{k-p} series. First, write down the 3^{k-1} runs for a full three-level factorial design in $k-1$ factors, with the unusual 0, 1, 2 notation. This is the basic design terminology, then introduce the k^{th} factor by equating its levels x_k to the appropriate component of the highest-order interaction, say $I = AB^{\alpha_2}C^{\alpha_3}...(K-1)^{\alpha_{k-1}}$, through the relationship

$$x_k = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_{k-1} x_{k-1}$$

where

$$\beta_i = (3-\alpha_k)\alpha_i \mod 3 \quad \text{for} 1 \leq i \leq k-1$$

8.9 Other 3^{k-p} Fractional Factorial Designs

For moderate to large values of k, even further fractioning of the 3^k design is desirable. In general, we may construct a $(\frac{1}{3})^p$ fraction of the 3^k design for $p < k$, where the fraction contains 3^{k-p} runs. Such a design is called a 3^{k-p} fractional factorial design. The defining relation I of any fraction consists of the p effects initially chosen and their $\frac{3^p-2p-1}{2}$ generalized interactions. The alias of any main effect or component of interaction is produced by multiplication modulus 3 of the effect by I and I^2.

8.10 Dye Experiment

Example 8.1. (From Dean/Voss page 467) An experimenter investigates three reactants (the base material and two ignoric materials, called here M and N) in the manufacture of a cotton dye stuff y. The three factors of interest in the experiment were the concentration of M in the water in the reaction mixture (factor A at three equally spaced levels), the volume of the water in the reaction mixture (factor B at three equally spaced levels), and the concentration of N in the free water in the reaction mixture (factor C at three equally spaced levels). The experimenters divided the treatment combinations into three blocks of size nine. The data are as follows:

Block I		Block II		Block III	
Comb.	Volume	Comb.	Volume	Comb.	Volume
000	74	020	69	010	13
021	130	011	46	001	112
012	56	002	71	022	125
110	110	100	211	120	199
101	166	121	220	111	218
122	227	112	216	102	201
220	195	210	147	200	74
211	146	201	47	221	198
202	90	222	164	212	102

(a) What structure is behind this experimental design? Which two contrasts are aliased?

(b) Prepare an analysis of variance for the data, assuming that the three factor interaction is negligible.

(c) Investigate the linear and quadratic trends of main effects.

(d) Draw any plot that help to illustrate any important features of the analysis.

To answer the question in part (a) we select a contrast with high component which gave us the same results for blocks confounded above. So the effect AB^2C^2 is our contrast. For running a 3^3 factorial in 3 blocks we have a defining contrast

$$L = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3.$$

ABC interactions: ABC, ABC^2, AB^2C, AB^2C^2, where AB^2C^2 gives us the same blocks as above. We now would exponentiate these contrasts to get other redundant contrasts which accounts for two d.f:

$$(ABC)^2 \equiv A^2B^2C^2, \quad (ABC^2)^2 \equiv A^2B^2C$$

$$(AB^2C)^2 \equiv A^2BC^2, \quad (AB^2C^2)^2 \equiv A^2BC$$

Now we will illustrate an ANOVA without three-factor interaction effect:

```
            Df Sum Sq Mean Sq F value    Pr(>F)
A            2  64196   32098 34.5947 0.0001154 ***
B            2  16858    8429  9.0844 0.0087343 **
C            2   2335    1167  1.2582 0.3348696
A:B          4  12513    3128  3.3715 0.0674709 .
A:C          4   4045    1011  1.0899 0.4231899
B:C          4   2699     675  0.7272 0.5977302
Residuals    8   7423     928
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

As we see at the level of significance of $\alpha = 0.05$ the main factors A and B are highly significant. We can also investigate to see if both the linear and quadratic component of these two factors have an effect on the response variable or not. The two-factor interaction AB is at level of significance $\alpha = 0.1$ rare significant.

We can also investigate to see if the blocks have an effect on the response variable, the related ANOVA results are shown below:

```
            Df Sum Sq Mean Sq F value   Pr(>F)
A            2  64196   32098 26.5982 0.001041 **
B            2  16858    8429  6.9845 0.027126 *
C            2   2335    1167  0.9674 0.432358
Block        2    182      91  0.0754 0.928230
A:B          4  12513    3128  2.5922 0.142840
A:C          4   4045    1011  0.8380 0.548069
B:C          4   2699     675  0.5591 0.701477
Residuals    6   7241    1207
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

As we see from the results above the two main factors A and B are still significant at $\alpha = 0.05$, furthermore the block as factor isn't significant and there is no difference between the blocks; the other results are the same as in the first table. So the model containing just the two main factors A and B is a good model.

Now we will see if the linear and quadratic component of these factors have an effect on the response variable.

Factor A:

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  134.333      8.414  15.966 2.77e-14 ***
A.L           36.691     14.573   2.518   0.0189 *
A.Q          -76.070     14.573  -5.220 2.38e-05 ***
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

Factor B:

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   134.33      11.99  11.201 5.14e-11 ***
B.L            37.79      20.77   1.819   0.0814 .
B.Q            21.09      20.77   1.015   0.3201
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

Factor C:

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   134.33      12.89  10.418 2.19e-10 ***
C.L            12.57      22.33   0.563    0.579
C.Q           -10.07      22.33  -0.451    0.656
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

Factor A has both linear and quadratic effects, factor B has just a linear effect on the response variable. Factor C isn’t significant in both linear and quadratic.

The effect of these factors can be estimated and here are the results:

```
Call:
lm(formula = reactant ~ nA + nB + nA2, data = reactant.dat)

Residuals:
    Min      1Q  Median      3Q     Max
-86.444 -25.694   8.056  22.472  61.389
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  196.444     12.630  15.554 1.07e-13 ***
nA            25.944      8.931   2.905  0.00798 **
nB            26.722      8.931   2.992  0.00651 **
nA2          -93.167     15.468  -6.023 3.84e-06 ***
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
Residual standard error: 37.89 on 23 degrees of freedom
Multiple R-squared:   0.7,      Adjusted R-squared: 0.6609
F-statistic: 17.89 on 3 and 23 DF,  p-value: 3.258e-06
```

The fitted Regression line is estimated:

$$\widehat{y}_{ijk} = 196.444 + 25.944x_1 - 93.167x_1^2 + 26.722x_2$$

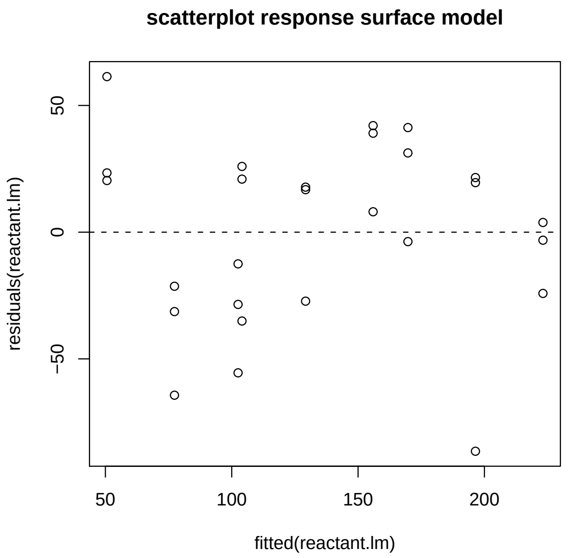


Figure 8.1: Dye Experiment, Scatter plot

The scatter plot 8.1 shows no abnormalities, so the hypothesis of equality of variance can be justified. The QQ-Plot 8.2 on page 164 shows light waves, but we can’t even reject the null hypothesis of non normality because of the Shapiro-Wilks test with significance level of 30%.

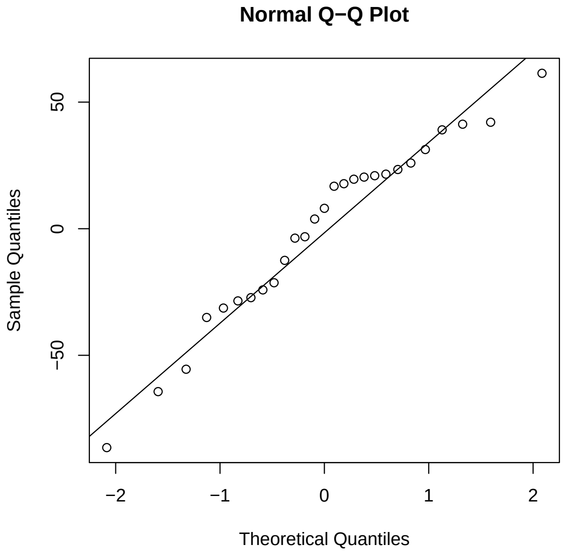


Figure 8.2: Dye Experiment, QQ-plot

```
        Shapiro-Wilk normality test
data:  residuals(reactant.lm)
W = 0.9568, p-value = 0.3118
```

Now we are interested to see the behavior of these factors, so we will construct the profile plots. A and B are main factors that are important. The effect of A on y is not linear (figure 8.3 on page 165). The level 0 differs greatly from the other two levels. From level 0 to level 1 is a decent decrease. A should probably used on the intermediate level to maximize the volume y.

With B (figure 8.3 on page 165) as the second important factor is just linearity interesting. It is obvious that B at its high level has the largest volume. From figure 8.3 on page 165 it is obvious that the factor A is the most important and then factor B. Factor C and Block factors do not have any effect on the response variable.

The interaction plots in figure 8.4 on page 166 show us approximately parallel lines. It means that there is no interaction effect which we have seen in the analysis of variance that they aren't significant.

The best setting for observing an optimized volume y are:

- $A = 0$ and $B = 1$
- $A = 0$ and $C = 0$ or $C = 1$

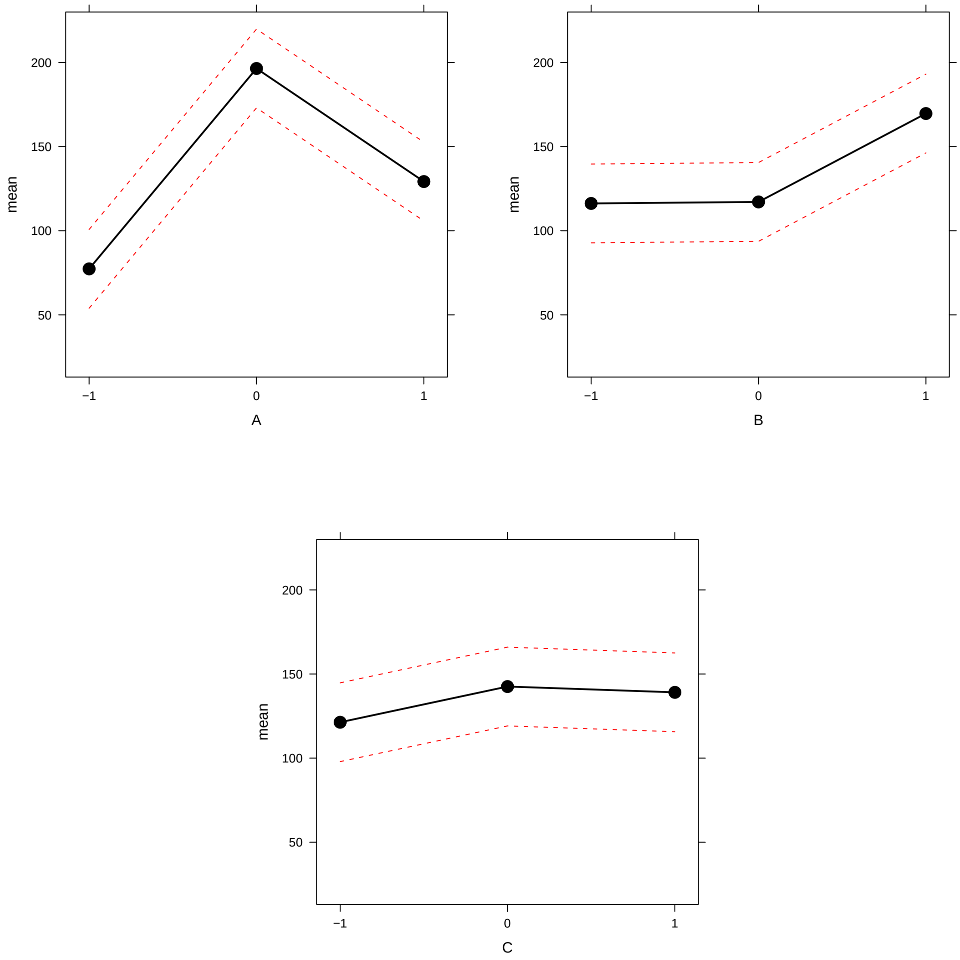


Figure 8.3: Dye Experiment, Profil plot of factors A, B, C

So the best case would be : $A = 0$ and $B = 1$ and $C = 0$ or $C = 1$
Figure 8.4 on page 166 is a plot of design.

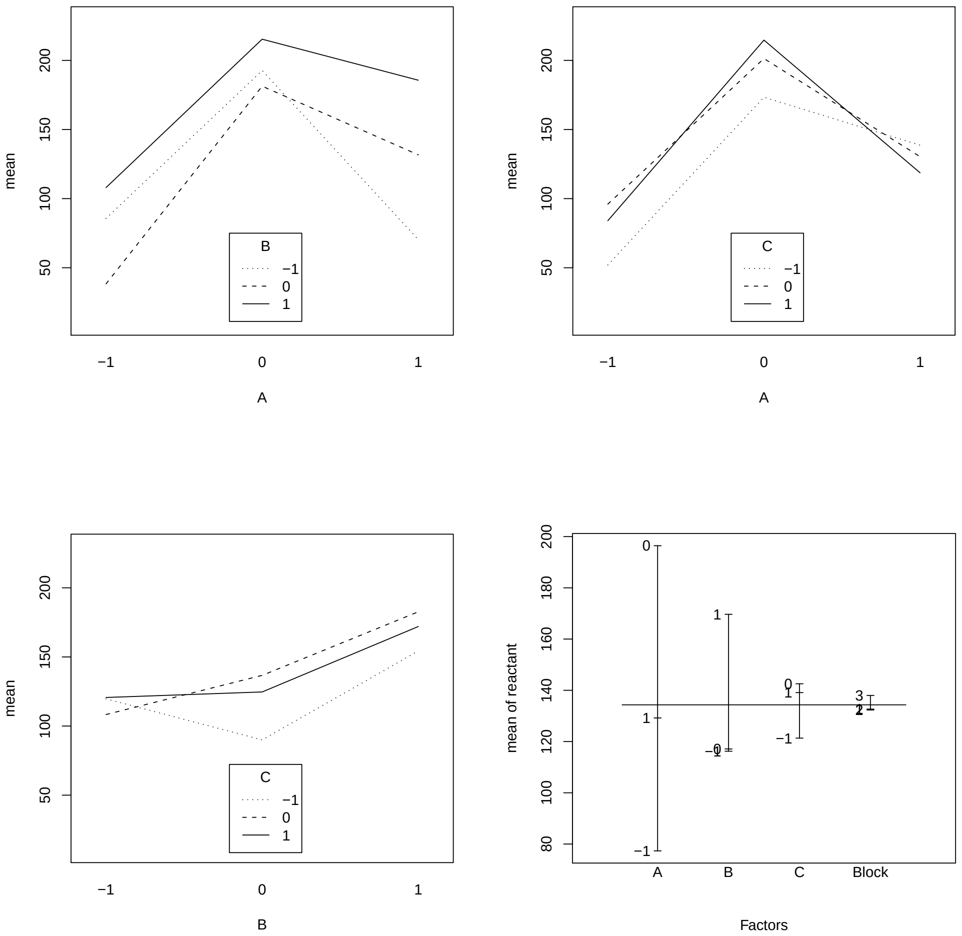


Figure 8.4: Dye Experiment, AB, AC, BC Interaction Plot and the Plot of Design

Dye experiment R-Code

```
reactant.dat<-data.frame(reactant=c(74,112,71,13,46,56,69,130,125,211,166,201,
110,218,216,199,220,227,74,47,90,147,146,102,195,198,164),
A=factor(rep(c(-1,0,1),1,each=9)), B=factor(rep(c(-1,0,1),3,each=3)),
C=factor(rep(c(-1,0,1),9,each=1)), Block=factor(c(1,3,2,3,2,1,2,1,3,2,1,3,1,3,
2,3,2,1,3,2,1,2,1,3,1,3,2)))
reactant.dat
#Analysis of variance without three-factor interaction:
reactant.mod<-aov(reactant~A+B+C+A*B+A*C+B*C, data=reactant.dat)
summary(reactant.mod)
#Analysis of variance with blocks:
reactant.mod1<-aov(reactant~A+B+C+Block+A*B+A*C+B*C, data=reactant.dat)
summary(reactant.mod1)
#Analysis of variance of reduced model:
reactant.mod2<-aov(reactant~A+B, data=reactant.dat)
summary(reactant.mod2)
#Fitting linear model using orthogonal polynomials:
reactpoly<-lm(reactant~A, contrasts=list(A="contr.poly"),
data=reactant.dat); summary(reactpoly)
reactpoly<-lm(reactant~B, contrasts=list(B="contr.poly"),
data=reactant.dat): summary(reactpoly)
reactpoly<-lm(reactant~C, contrasts=list(C="contr.poly"),
data=reactant.dat); summary(reactpoly)
#Effects computing:
nA=rep(c(-1,0,1),1,each=9); nB=rep(c(-1,0,1),3,each=3)
nC=rep(c(-1,0,1),9,each=1)
reactant.lm<-lm(reactant~nA+nB+nC+nA*nB+nA*nC+nB*nC,data=reactant.dat)
summary(reactant.lm)
reactant.lm1<-lm(reactant~nA+nB+nC+nA2+nB2+nC2+Block+nA*nB+nA*nC+nB*nC,
data=reactant.dat); summary(reactant.lm1)
nA2<-nA^2; nB2<-nB^2; nC2<-nC^2
reactant.lm<-lm(reactant~nA+nB+nA2,data=reactant.dat)
summary(reactant.lm)
#Plot of the Design:
plot.design(reactant~A*B*C+Block, data=reactant.dat)
#Plot for regression model:
plot(fitted(reactant.lm), residuals(reactant.lm),
main="scatterplot response surface model"); abline(h=0,lty=2)
qqnorm(residuals(reactant.lm)); qqline(residuals(reactant.lm))
shapiro.test(residuals(reactant.lm))
#Profle Plots (factor diagramm)
library(effects)
plot(effect("A",reactant.mod),main="",xlab="A",ylab="mean", ylim=c(13,230))
plot(effect("B",reactant.mod),main="",xlab="B",ylab="mean", ylim=c(13,230))
plot(effect("C",reactant.mod),main="",xlab="C",ylab="mean", ylim=c(13,230))
#Interaction Plots:
interaction.plot(reactant.dat$A, reactant.dat$B, reactant.dat$reactant,
```

```
xlab="A", ylab="mean", ylim=c(10,230), fixed=T, legend=F)
legend(1.79,75,c("-1","0","1"), title="B", lty=c(3,2,1))
interaction.plot(reactant.dat$A, reactant.dat$C, reactant.dat$reactant,
xlab="A", ylab="mean", ylim=c(10,230), fixed=T, legend=F)
legend(1.79,75,c("-1","0","1"), title="C", lty=c(3,2,1))
interaction.plot(reactant.dat$B, reactant.dat$C, reactant.dat$reactant,
xlab="B", ylab="mean", ylim=c(10,230), fixed=T, legend=F)
legend(1.79,72.25,c("-1","0","1"), title="C", lty=c(3,2,1))
```

8.11 Refinery Experiment

Example 8.2. (From Dean/Voss, page 497) P.W.M. John (1971) describes an experiment of Vance (1962), to find a set of operating conditions to optimize the quality of lube oil treated as a refinery. There were four factors of interest called here A, B, C and D, and three equally spaced levels were selected for each of these, so that quadratic trends could be measured. Since this was a preliminary, experiment, a $\frac{1}{3}$-fraction of resolution IV was thought to be adequate. The data are shown in the table below:

TC	y	Al	Aq	Bl	Bq	Cl	Cq	Dl	Dq
0000	4.2	-1	1	-1	1	-1	1	-1	1
0012	5.9	-1	1	-1	1	0	-2	1	1
0021	8.2	-1	1	-1	1	1	1	0	-2
0102	13.1	-1	1	0	-2	-1	1	1	1
0111	16.4	-1	1	0	-2	0	-2	0	-2
0120	30.7	-1	1	0	-2	1	1	-1	1
0201	9.5	-1	1	1	1	-1	1	0	-2
0210	22.2	-1	1	1	1	0	-2	-1	1
0222	31.0	-1	1	1	1	1	1	1	1
1002	7.7	0	-2	-1	1	-1	1	1	1
1011	16.5	0	-2	-1	1	0	-2	0	-2
1020	14.3	0	-2	-1	1	1	1	-1	1
1101	11.0	0	-2	0	-2	-1	1	0	-2
1110	29.0	0	-2	0	-2	0	-2	-1	1
1122	55.0	0	-2	0	-2	1	1	1	1
1200	8.5	0	-2	1	1	-1	1	-1	1
1212	37.4	0	-2	1	1	0	-2	1	1
1221	66.3	0	-2	1	1	1	1	0	-2
2001	11.4	1	1	-1	1	-1	1	0	-2
2010	21.1	1	1	-1	1	0	-2	-1	1
2022	57.9	1	1	-1	1	1	1	1	1
2100	13.5	1	1	0	-2	-1	1	-1	1
2112	51.6	1	1	0	-2	0	-2	1	1
2121	76.5	1	1	0	-2	1	1	0	-2
2202	31.0	1	1	1	1	-1	1	1	1
2211	74.5	1	1	1	1	0	-2	0	-2
2220	85.1	1	1	1	1	1	1	-1	1

(a) Save the data in a R-file, test if the defining relation $I = ABCD = A^2B^2C^2D^2$ is right for this design.

(b) Find the factor combinations which have the largest effect. Prepare the analysis of variance and see which interaction effects are important. Draw interaction plots.

(c) Investigate the linear and quadratic trends of main effects. Draw any plot that help to illustrate any important features of the analysis.

(d) Prepare the analysis of variance in part (c).

We first study the treatment combinations, where we want to see if the treatment combinations define the same block. Therefore with the defining relation $I = ABCD = A^2B^2C^2D^2$ we will find if the generator $L = x_1 + x_2 + x_3 + x_4$ modulus 3 confirms the same block.

```
*********************************************************
TC          L=x1+x2+x3+x4          TC      L=x1+x2+x3+X4
*********************************************************
0000                0              1122        1+1+2+2=0
0012            1+2=0              1200              1+2=0
0021            2+1=0              1212        1+2+1+2=0
0102            1+2=0              1221        1+2+2+1=0
0111          1+1+1=0              2001              2+1=0
0120            1+2=0              2010              2+1=0
0201            2+1=0              2022          2+2+2=0
0210            2+1=0              2100              2+1=0
0222          2+2+2=0              2112        2+1+1+2=0
1002            1+2=0              2121        2+1+2+1=0
1011          1+1+1=0              2202          2+2+2=0
1020            1+2=0              2211        2+2+1+1=0
1101          1+1+1=0              2220          2+2+2=0
1110          1+1+1=0
********************************************************
```

The value of L for all 27 treatment combinations is 0. Alternatively we can show that relating to the three first factors there is a 3^3 factorial design. Let $I = ABCD$ so we have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ then we have due to the following formulation:

$$\beta_i = (3 - \alpha_k)\alpha_i \pmod 3 \quad for \quad 1 \le i \le k-1$$

$\beta_1 = \beta_2 = \beta_3 = 2$ and therefor :

$$x_4 = 2x_1 + 2x_2 + 2x_3$$

The alias structure of main effects and interactions is produced by multiplying the effects by I and I^2 modulus 3.

Components	Alias of ABCD	Alias of $A^2B^2C^2D^2$
AB	$AB * ABCD = (A^2B^2CD)^2 = ABC^2D^2$	$AB * A^2B^2C^2D^2 = C^2D^2$
AC	$AC * ABCD = (A^2BC^2D)^2 = AB^2CD^2$	$AC * A^2B^2C^2D^2 = B^2D^2$
AD	$AD * ABCD = (A^2BCD^2)^2 = AB^2C^2D$	$AD * A^2B^2C^2D^2 = B^2C^2$
A^2B^2	$A^2B^2 * ABCD = CD$	$A^2B^2 * A^2B^2C^2D^2 = ABC^2D^2$
A^2C^2	$A^2C^2 * ABCD = BD$	$A^2C^2 * A^2B^2C^2D^2 = AB^2CD^2$
A^2D^2	$A^2D^2 * ABCD = BC$	$A^2D^2 * A^2B^2C^2D^2 = AB^2C^2D$

Each component of a two-factor interaction is confounded with two- and four-factor interactions.

We do the analysis of variance test, because of the alias structure we can just analyze three of the two-factor interactions. Each two-factor interaction is almost with another two-factor interaction aliased ($A^2B^2 = CD$). In the complete model we use the four main effects and their four-factor interaction.

We now do an analysis of variance test to see which effect is significant and has an effect on the response variable:

```
Analysis of Variance Table
Response: oil
          Df Sum Sq Mean Sq F value Pr(>F)
Al         2 4496.3 2248.14
Bl         2 2768.7 1384.35
Cl         2 5519.8 2759.89
Dl         2  283.4  141.68
Al:Bl      4  310.8   77.70
Al:Cl      4 1232.9  308.23
Bl:Cl      4  669.7  167.42
Al:Dl      2   83.4   41.71
Bl:Dl      2  151.3   75.66
Cl:Dl      2   28.2   14.10
Residuals  0    0.0
```

From the above results of testing the complete model we see that the three main effects A, B and C are the largest and they could play a role in our model. The two-factor interaction AC has a large value. We now try our new model which contains main effects and two-factor interactions. The ANOVA table of a new model is shown below and it confirms that the main effects A, B and C and the two-factor interaction AC are significant at the level of significance $\alpha = 0.05$.

```
Analysis of Variance Table
Response: oil
          Df Sum Sq Mean Sq F value    Pr(>F)
Al         2 4496.3 2248.14 37.4763 0.0004072 ***
Bl         2 2768.7 1384.35 23.0769 0.0015226 **
Cl         2 5519.8 2759.89 46.0070 0.0002294 ***
Dl         2  283.4  141.68  2.3619 0.1751524
Al:Bl      4  310.8   77.70  1.2953 0.3693413
Al:Cl      4 1232.9  308.23  5.1381 0.0383308 *
Al:Dl      4  572.7  143.17  2.3866 0.1633890
Residuals  6  359.9   59.99
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

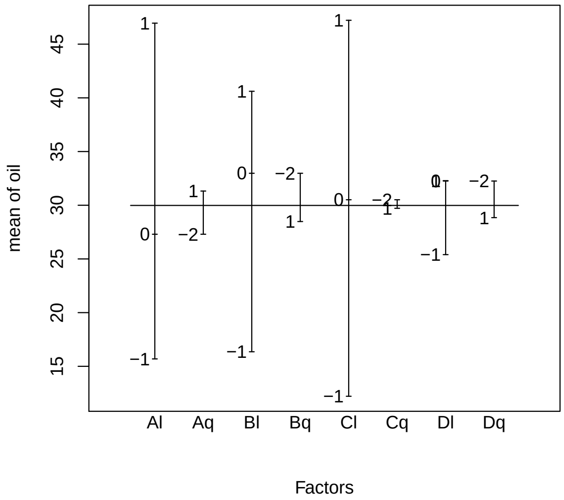


Figure 8.5: Refinery, Plot of Design

From the plot of design, figure 8.5 we see the same result as in the ANOVA table; the linear terms of main effect A, B and C have the largest affect on the response variable and the interaction effect $A_l C_l$ is significant. Now we study the profile plots and the interaction plots of these factors.

As we see from figure 8.6 on page 173 the factors A, B and C have a large linear effect. From the interaction plots, figure 8.7 on page 176 we will see by the interactions between factors A and C three curves that behave almost the same. (Note that AC is significant.)

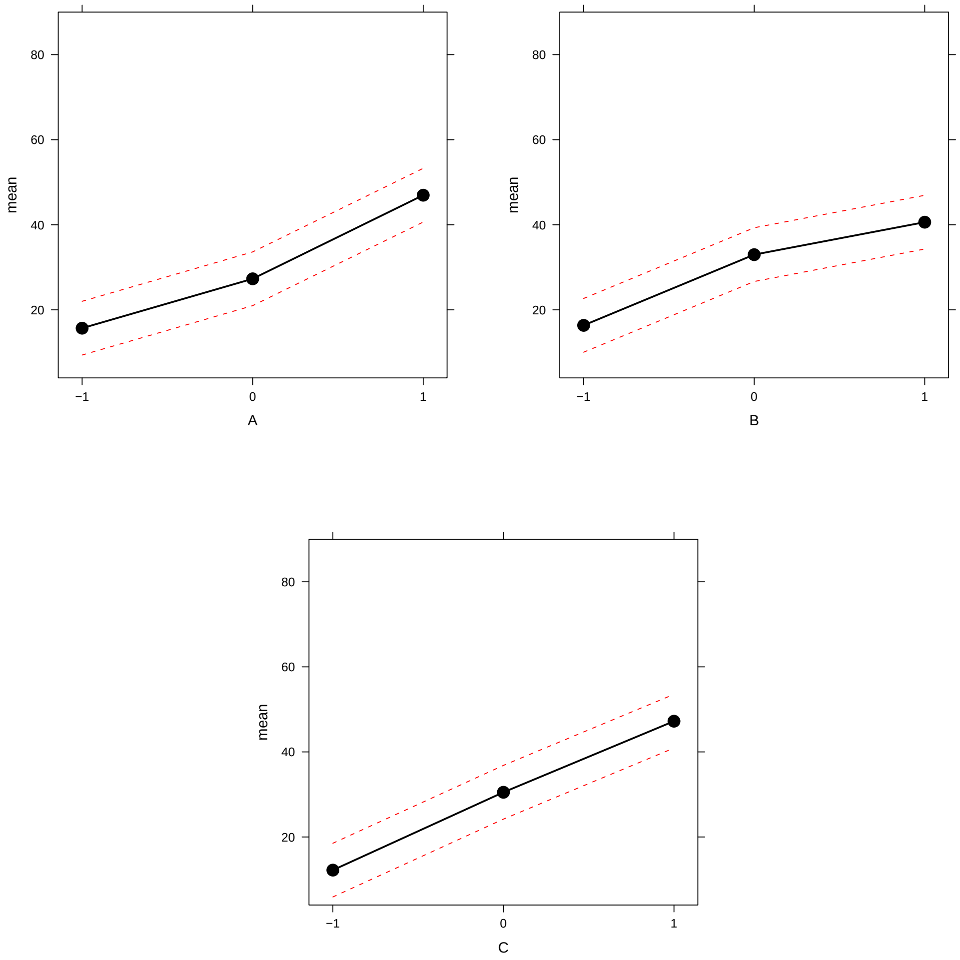


Figure 8.6: Refinery, Profil plot of factors A, B, C

So the reduced model which seems to be a good model is a model containing three main effects A, B and C and the two-factor interaction AC.

```
Analysis of Variance Table
Response: oil
          Df Sum Sq Mean Sq F value    Pr(>F)
Al         2 4496.3 2248.14 23.5597 1.705e-05 ***
Bl         2 2768.7 1384.35 14.5074 0.0002547 ***
Cl         2 5519.8 2759.89 28.9226 4.857e-06 ***
Al:Cl      4 1232.9  308.23  3.2301 0.0401436 *
Residuals 16 1526.8   95.42
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

For computing the linear and quadratic terms of each effect we will use the polynomial contrasts. The following table shows the quadratic and linear terms of each factor and proof their significance.

```
Factor A:
-35.556 -14.478  -2.589  12.978  39.000
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   29.981      4.129   7.261 1.68e-07 ***
Al.L          22.109      7.152   3.091  0.00499 **
Al.Q           3.284      7.152   0.459  0.65022
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Factor B:
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   29.981      4.440   6.752 5.53e-07 ***
Bl.L          17.151      7.691   2.230   0.0354 *
Bl.Q          -3.670      7.691  -0.477   0.6376
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Factor C:
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  29.9815     3.9332   7.623 7.34e-08 ***
Cl.L         24.7566     6.8125   3.634  0.00132 **
Cl.Q         -0.6487     6.8125  -0.095  0.92493
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The effects of the model are shown in table below:

```
   nAl           nAq        nBl          nBq         nCl
 31.2666667    2.6814815   24.2555556   -2.9962963  35.0111111
  nCq                nDl                  nDq
 -0.5296296          6.8888889            -2.2740741
```

We can see from these results that the linear terms of A, B and C have large values of effects and contrasts. We can see the Q-Q-plot of the effects in figure 8.8 on page 177. From this plot it is obvious again that the factors A (1), B (3), C(5) are significant.

```
Analysis of Variance Table
Response: oil
          Df Sum Sq Mean Sq F value    Pr(>F)
Al         2 4496.3 2248.14 23.5597 1.705e-05 ***
Bl         2 2768.7 1384.35 14.5074 0.0002547 ***
Cl         2 5519.8 2759.89 28.9226 4.857e-06 ***
Al:Cl      4 1232.9  308.23  3.2301 0.0401436 *
Residuals 16 1526.8   95.42
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

When we analyze the reduced model with A, B and C we have 2 degrees of freedom for each factor; and it is obvious from the ANOVA table that all three main effects and the AC interaction is significant.
The Q-Q-plot figure 8.9 on page 178 shows no abnormality and the scatter plot shows that the variance is constant, so it shows the homogeneity of variances.

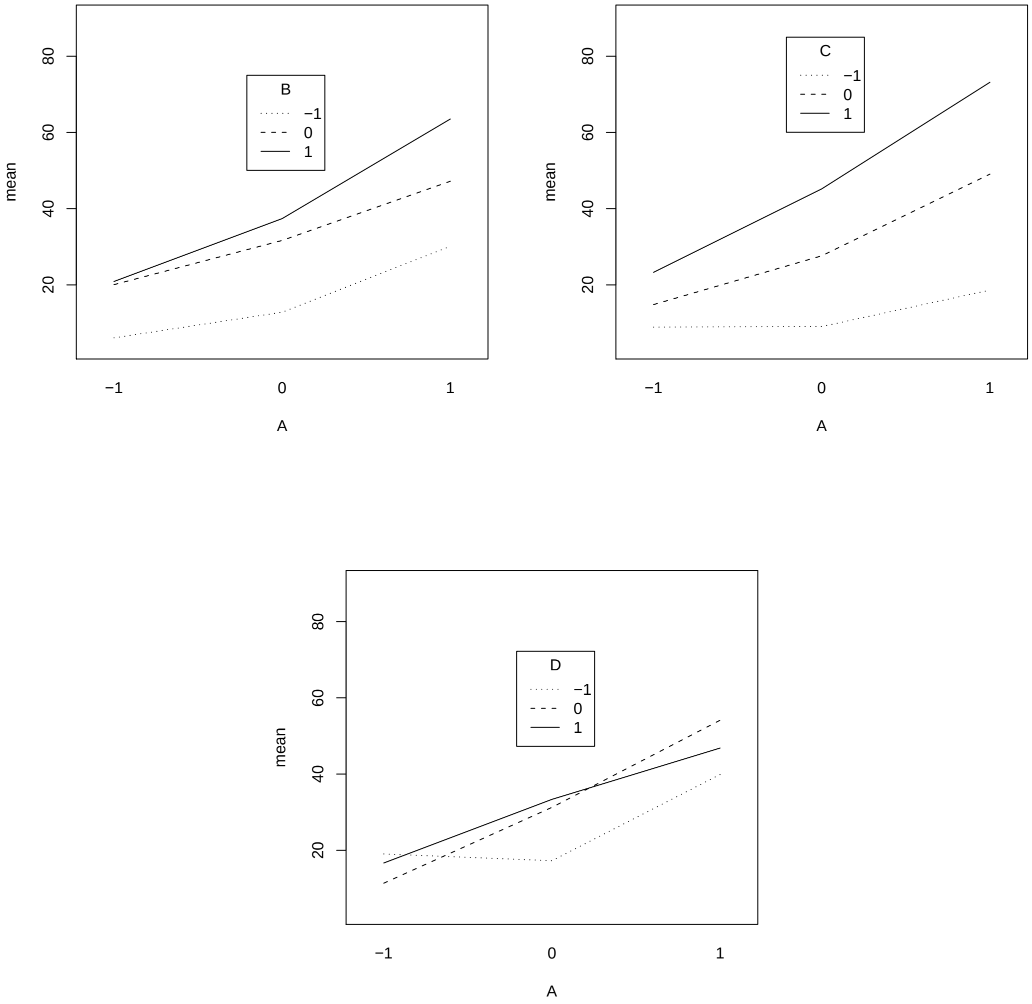


Figure 8.7: Refinery, Interaction plot

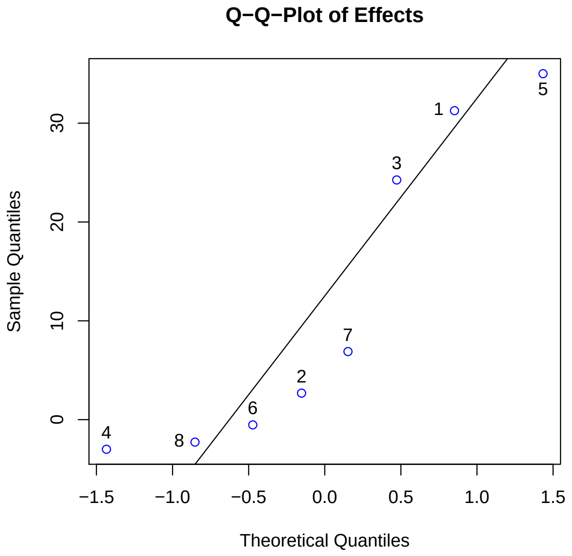


Figure 8.8: Refinery, Q-Q plot

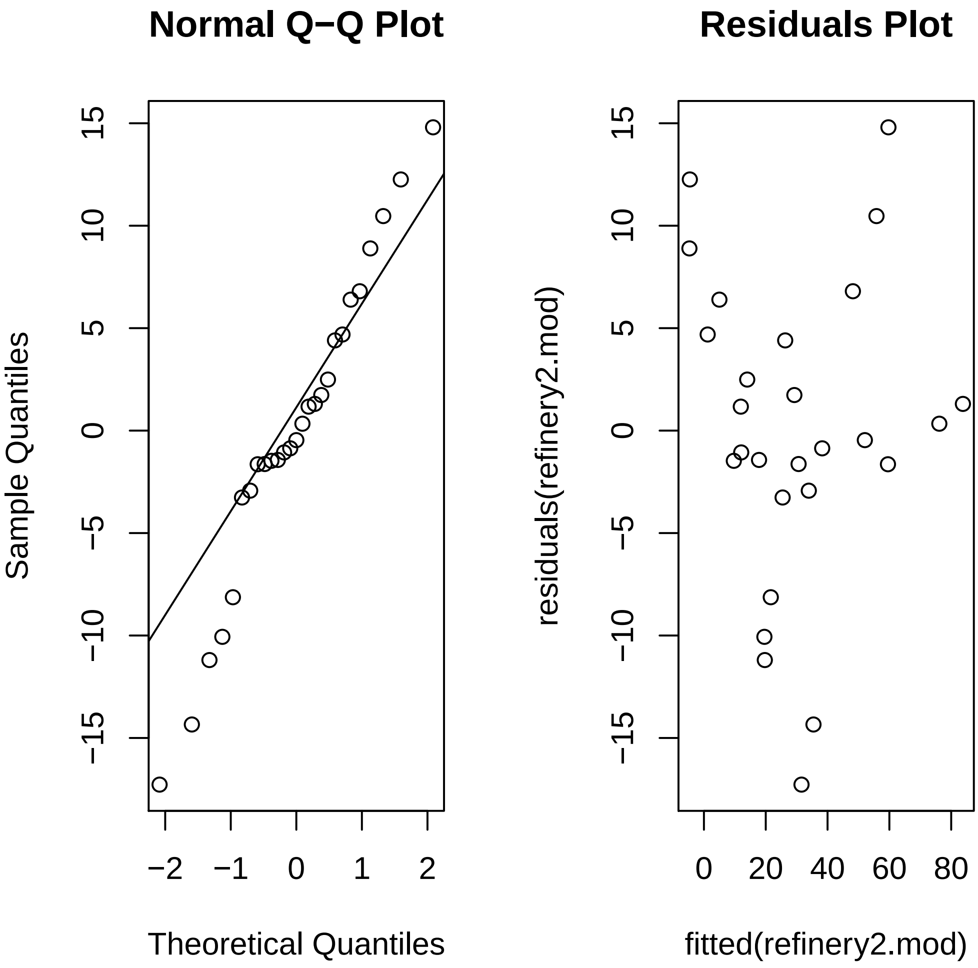


Figure 8.9: Refinery, Residuals plot and Q-Q plot

Refinery R-Code

```
refinery.dat<-data.frame(oil=c(4.2,5.9,8.2,13.1,16.4,30.7,9.5,22.2,31.0,
7.7,16.5,14.3,11.0,29.0,55.0,8.5,37.4,66.3,11.4,21.1,57.9,13.5,51.6,76.5,
31.0,74.5,85.1); Al=factor(rep(c(-1,0,1),each=9)),Aq=factor(rep(c(1,-2,1),
each=9)); Bl=factor(rep(c(-1,0,1),each=3,times=3)),Bq=factor(rep(c(1,-2,1)
,each=3,times=3)); Cl=factor(rep(c(-1,0,1),times=9)),Cq=factor(rep(c(1,-2,1)
,times=9)); Dl=factor(c(-1,1,0,1,0,-1,0,-1,1,1,0,-1,0,-1,1,-1,1,0,0,-1,1,-1,
1,0,1,0,-1)); Dq=factor(c(1,1,-2,1,-2,1,-2,1,1,1,-2,1,-2,1,1,1,1,-2,-2,1,1,
1,1,-2,1,-2,1))); refinery.dat
#Analysis of variance without three-factor interaction:
refinery.mod<-aov(oil~Al*Aq*Bl*Bq*Cl*Cq*Dl*Dq, data=refinery.dat)
summary(refinery.mod)
anova(refinery.mod)
#Effects computing:
nAl=rep(c(-1,0,1),each=9); nAq=rep(c(1,-2,1),each=9)
nBl=rep(c(-1,0,1),each=3,times=3); nBq=rep(c(1,-2,1),each=3,times=3)
nCl=rep(c(-1,0,1),times=9); nCq=rep(c(1,-2,1),times=9)
nDl=c(-1,1,0,1,0,-1,0,-1,1,1,0,-1,0,-1,1,-1,1,0,0,-1,1,-1,1,0,1,0,-1)
nDq=c(1,1,-2,1,-2,1,-2,1,1,1,-2,1,-2,1,1,1,1,-2,-2,1,1,1,1,-2,1,-2,1)
nAl2<-nAl^2; nBl2<-nBl^2; nCl2<-nCl^2;nDl2<-nDl^2
refinery.lm<-lm(oil~nAl*nAq*nBl*nBq*nCl*nCq*nDl*nDq, data=refinery.dat)
anova(refinery.lm)
#Plot of Design
plot.design(oil~Al*Aq*Bl*Bq*Cl*Cq*Dl*Dq,data=refinery.dat)
refinery1.mod<-aov(oil~Al+Bl+Cl+Dl+Al*Bl+Al*Cl+Al*Dl, data=refinery.dat)
summary(refinery1.mod)
anova(refinery1.mod)
#Profle Plots (factor diagramm)
library(effects)
plot(effect("Al",refinery1.mod),main="",xlab="A",ylab="mean", ylim=c(4,90))
plot(effect("Bl",refinery1.mod),main="",xlab="B",ylab="mean", ylim=c(4,90))
plot(effect("Cl",refinery1.mod),main="",xlab="C",ylab="mean", ylim=c(4,90))
#Interaction Plots:
interaction.plot(refinery.dat$Al, refinery.dat$Bl, refinery.dat$oil,
xlab="A", ylab="mean", ylim=c(4,90), fixed=T, legend=F)
legend(1.79,75,c("-1","0","1"), title="B", lty=c(3,2,1))
interaction.plot(refinery.dat$Al, refinery.dat$Cl, refinery.dat$oil,
xlab="A", ylab="mean", ylim=c(4,90), fixed=T, legend=F)
legend(1.79,85,c("-1","0","1"), title="C", lty=c(3,2,1))
interaction.plot(refinery.dat$Al, refinery.dat$Dl, refinery.dat$oil,
xlab="A", ylab="mean", ylim=c(4,90), fixed=T, legend=F)
legend(1.79,72.25,c("-1","0","1"), title="D", lty=c(3,2,1))
#Reduced Model:
refinery2.mod<-aov(oil~Al+Bl+Cl+Al*Cl, data=refinery.dat)
summary(refinery2.mod)
anova(refinery2.mod)
#Fitting linear model using orthogonal polynomials:
```

```
refinpoly<-lm(oil~Al, contrasts=list(Al="contr.poly"),data=refinery.dat)
summary(refinpoly)
refinpoly<-lm(oil~Bl, contrasts=list(Bl="contr.poly"),data=refinery.dat)
summary(refinpoly)
refinpoly<-lm(oil~Cl, contrasts=list(Cl="contr.poly"),data=refinery.dat)
summary(refinpoly)
refinpoly<-lm(oil~Dl, contrasts=list(Dl="contr.poly"),data=refinery.dat)
summary(refinpoly)
#Effects computing:
refinery1.lm<-lm(oil~nAl+nAq+nBl+nBq+nCl+nCq+nDl+nDq,data=refinery.dat)
eff<-2*refinery1.lm$coef[2:9];eff
#Q-Q-Plot of effects :
qq<-qqnorm(eff,main="Q-Q-Plot of Effects",col="blue");qqline(eff)
identify(qq$x,qq$y,1:length(qq$x))
#Reduced model:
refinery2.mod<-aov(oil~Al+Bl+Cl+Al*Cl, refinery.dat)
anova(refinery2.mod)
summary(refinery2.mod)
windows()
par(mfrow=c(1,2))
qqnorm(residuals(refinery2.mod))
qqline(residuals(refinery2.mod))
plot(fitted(refinery2.mod),residuals(refinery2.mod),main="Residuals Plot")
#Linear model of reduced model:
refinery1.lm<-lm(oil~nAl+nBl+nCl+nAl*nCl, data=refinery.dat)
anova(refinery1.lm)
summary(refinery1.lm)
```

Chapter 9

Taguchi-Method

9.1 Design for the control of noise variability

Experiments that involve both design and noise factors are often known colloquially as Taguchi experiments. There are two different types of designs for such an experiment "product arrays" and "mixed arrays". The product arrays are composed of two fractional factorial experiments, one for the design factors and one for the noise factors, and every combination of design factors is observed in conjunction with every combination of noise factors. Mixed arrays, on the other hand, are ordinary fractional factorial designs in which the difference between the design and noise factors is ignored at the design stage except to ensure that the design-by-noise interactions are estimable.[Dean A., Voss D. (1999)]

Product arrays are usually observed in the following way. The order of the design combinations is randomized. For each design combination in turn, observations are taken across all of the noise combinations in a random order.

Such designs are usually analyzed by calculating, for each design combination, the average and log sample variance of the responses obtained under the different noise combinations. The average response and the log variance are taken as separate set of data values for the design factor combinations. The objective of the experiment is to find out which factors most affect the log variance response, and which factor most affect the average response.

9.2 What are Taguchi designs?

Genichi Taguchi (1980), a Japanese engineer, proposed several approaches to experimental designs that are sometimes called "Taguchi Methods". These methods utilize two-, three-, and mixed-level fractional factorial designs. Taguchi refers to experimental design as "off-line quality control" because it is a method of ensuring good performance in the design stage of products or processes. "Taguchi" designs are similar to our familiar fractional factorial designs. However Taguchi has introduced several noteworthy new ways of conceptualizing an experiment that are very valuable, especially in product development and industrial engineering. He has three main ideas, namely System Design, Parameter Design and Tolerance Design.

9.2.1 System Design

This is design at the conceptual level, involving creativity and innovation.

9.2.2 Parameter Design

The aim here is to make a product or process less variable (more robust) in the face of variation over which we have little or no control.

9.2.3 Tolerance Design

This deals with the problem of how, and when, to specify tightened tolerances for a product or a process so that quality and performance/productivity are enhanced. Every product or process has a number - perhaps a large number - of components. It is explained how to identify the critical components to target when the tolerances have to be tightened.

9.3 Inclinometer Experiment

Example 9.1. (From Dean/Voss, P. 516) A collaborative study involving statisticians and mechanical engineers was described. The experiment sought to improve the performance of an inclinometer, which is an instrument that records the angle of an object such as a crane jib. The design of the inclinometer is described in the article as follows.

The basic design of a product is composed in four parts: a bob-weight and flexure, a flanged flywheel and a copper-plated disc (PCB). All are attached to a shaft supported in low-friction bearings. When the object to which the flywheel is attached is tilted, the bob-weight assembly moves to stay perpendicular to the earth, causing the PCB to rotate relative to the casing. The main performance difficulty of the inclinometer is that it doesn't immediately register the true angle of tilt. Spurious swing of the disc is produced by movement of the object.

The purpose of the experiment was to vary the relative sizes of the parts of the inclinometer to find a combination of factors that would reduce the swing. The engineers identified 7 factors that could be altered and that might affect the swing. Three levels were selected for each factor so that linear and quadratic trends could be investigated. The levels of the first six factors were selected to be equally spaced. The factors were:

- A: Flexure length (30.00, 31.25, 32.5)
- B: Flexure thickness (0.05, 0.275, 0.5)
- C: Flexure width (4.0, 5.0, 6.0)
- D: Flange thickness (1.0, 3.5, 6.0)
- E: Flange width (6.0, 10.5, 15.0)
- F: Bob-weight length (12.0, 20.0, 28.0)
- G: Copper plating thickness (0.0175, 0.035, 0.07)

All measurements are in millimeters, and the levels of all factors are coded as 0, 1and 2. A $\frac{1}{9}$ fraction would have been possible except for the fact that there were other considerations that needed to be taken into account. Under designed experimental conditions, it was possible to produce the factor levels exactly as specified, but in mass production variability naturally creeps in. The experimenters decided to build a production variability into the experiment as noise factors as follows:

- H: Flexure length (-0.25,+0.25)
- P: Flexure thickness (-0.005,+0.005)
- J: Flange thickness (-0.025,+0.025)

- K: Flange width (-0.025,+0.025)
- L: Copper plating thickness (-0.005,+0.005)
- M: Tolerance on bob weight mass (-9.0,+9.0)×($\frac{1}{1000}$)g
- N: Maximum horizontal amplitude of vibration (5,25)

The two levels of each noise factor were coded as 0 and 1. The data are shown in the table below:

	Noise factors							
	H : 0	0	0	0	1	1	1	1
	P : 0	0	1	1	0	0	1	1
	J : 0	0	1	1	1	1	0	0
	K : 0	1	0	1	0	1	0	1
	L : 0	1	0	1	1	0	1	0
Design factors	M : 0	1	1	0	0	1	1	0
ABCDEFG	N : 0	1	1	0	1	0	0	1
0000000	0.62	3.54	3.56	0.62	3.09	0.71	0.73	3.20
0011111	0.59	3.11	3.11	0.59	2.98	0.63	0.64	3.02
0022222	0.59	3.01	3.02	0.59	2.97	0.61	0.62	3.00
0101122	0.51	2.65	2.65	0.50	2.53	0.53	0.54	2.56
0112200	0.18	0.96	0.96	0.18	0.89	0.19	0.20	0.90
0202211	0.19	1.03	1.03	0.19	0.97	0.21	0.21	0.93
0210022	1.85	9.46	9.42	1.82	9.19	1.90	1.92	9.35
0221100	0.52	2.73	2.72	0.51	2.61	0.55	0.56	2.64
1001212	0.29	1.56	1.56	0.29	1.45	0.31	0.32	1.47
1012020	0.95	4.98	4.93	0.94	4.79	0.99	1.00	4.82
1020101	1.16	6.09	6.09	1.13	5.70	1.21	1.26	5.93
1102001	0.26	1.45	1.45	0.25	1.30	0.29	0.30	1.30
1110112	1.15	5.99	5.92	1.13	5.69	1.19	1.22	5.84
1121220	0.85	4.31	4.30	0.84	4.23	0.86	0.88	4.28
1200120	1.10	5.74	5.67	1.07	5.43	1.14	1.17	5.57
1211201	0.29	1.55	1.55	0.28	1.45	0.31	0.32	1.47
1222012	0.91	4.64	4.66	0.90	4.56	0.94	0.95	4.57
2002121	0.39	2.05	2.06	0.39	1.96	0.41	0.42	1.97
2010202	0.67	3.61	3.57	0.65	3.27	0.72	0.74	3.41
2021010	1.42	7.31	7.38	1.41	7.14	1.48	1.51	7.24
2100210	0.69	3.66	3.60	0.67	3.37	0.73	0.74	3.47
2111021	1.18	6.04	6.06	1.17	5.90	1.21	1.23	5.95
2122102	0.37	1.95	1.95	0.37	1.87	0.39	0.40	1.88
2201002	0.39	2.15	2.16	0.38	1.94	0.44	0.44	1.96
2212110	0.44	2.29	2.29	0.43	2.21	0.46	0.47	2.22
2220221	1.84	9.35	9.19	1.79	9.06	1.85	1.89	9.28

(a) Analyze the data when response variable is $\ln(s^2)$.

(b) Analyze the data for mean of the data as a response variable.

(c) Which effect is more effective as a control factor and noise factor? Determine the contrasts.

This experiment is a $3^7 \times 2^7$ factorial experiment. The objective of this experiment is to select the combination of the design factors that gave the least amount of swing. In terms of producing a product of consistently high quality, it was also important that the variability of the amount y of swing also remain low across the different noise combinations. The experimenter selects a product array formed from a $(\frac{1}{3})^4$ fraction of the 3^7 design treatment combination and a $(\frac{1}{2})^4$ fraction of the 2^7 noise combinations.

In summary we have 7 control factors each with three levels: A, B, C, D, E, F, G; and 7 noise factors each with 2 levels: H, P, J, K, L, M, N.

The alias structure is as follow:

Design Generators: $D = AB$, $E = AC$, $F = BC$, $G = ABC$

Defining Relation: $I = ABD = ACE = BCDE = BCF = ACDF = ABEF = DEF = ABCG = CDG = BEG = ADEG = AFG = BDFG = CEFG = ABCSEFG$

Aliases: $A = BD = CE = FG$, $B = AD = CF = EG$, $C = AE = BF = DG$, $D = AB = EF = CG$ $E = AC = DF = BG$, $F = BC = DE = AG$, $G = CD = BE = AF$

First we compute for each factor the mean and the variance, and the logarithm of variance will also be computed. In the first analysis of variance our response variable will be ln(var). The ANOVA table is shown below:

```
             Df  Sum Sq Mean Sq  F value    Pr(>F)
A             2  0.6316  0.3158  12.3828  0.001209 **
B             2  0.1358  0.0679   2.6626  0.110411
C             2  9.8448  4.9224 192.9982 7.513e-10 ***
D             2 18.8987  9.4493 370.4922 1.638e-11 ***
E             2  7.0366  3.5183 137.9464 5.244e-09 ***
F             2  9.5150  4.7575 186.5325 9.160e-10 ***
G             2  0.0354  0.0177   0.6942  0.518444
Residuals    12  0.3061  0.0255
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

From the above results we can conclude that the factors A, C, D, E and F are significant at level of significance $\alpha = 0.05$. We can see that the control factors except B and G have a large effect on the variability of the swing. Now we will investigate the factors for linear and quadratic trends.

```
            Df  Sum Sq Mean Sq  F value    Pr(>F)
A            1  0.5798  0.5798  22.7325 0.0004579 ***
A2           1  0.0519  0.0519   2.0331 0.1793949
B            1  0.0581  0.0581   2.2781 0.1570847
B2           1  0.0777  0.0777   3.0471 0.1064062
C            1  9.8241  9.8241 385.1843 1.738e-10 ***
C2           1  0.0207  0.0207   0.8121 0.3852435
D            1 18.3769 18.3769 720.5282 4.391e-12 ***
D2           1  0.5217  0.5217  20.4561 0.0006983 ***
E            1  7.0044  7.0044 274.6287 1.237e-09 ***
E2           1  0.0322  0.0322   1.2641 0.2828660
F            1  9.4043  9.4043 368.7269 2.242e-10 ***
F2           1  0.1106  0.1106   4.3382 0.0593247 .
G            1  0.0244  0.0244   0.9567 0.3473210
G2           1  0.0110  0.0110   0.4317 0.5235370
Residuals   12  0.3061  0.0255
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

As you see the control factors A, C, D, E and F behave linear on the swing, also factors D and F have quadratic effect on swing. So we can say that these factors have an effect on the variability of our process. The alias structure here plays no role in our experiment.

You can see the behavior of the factors in figure 9.1 on page 187.

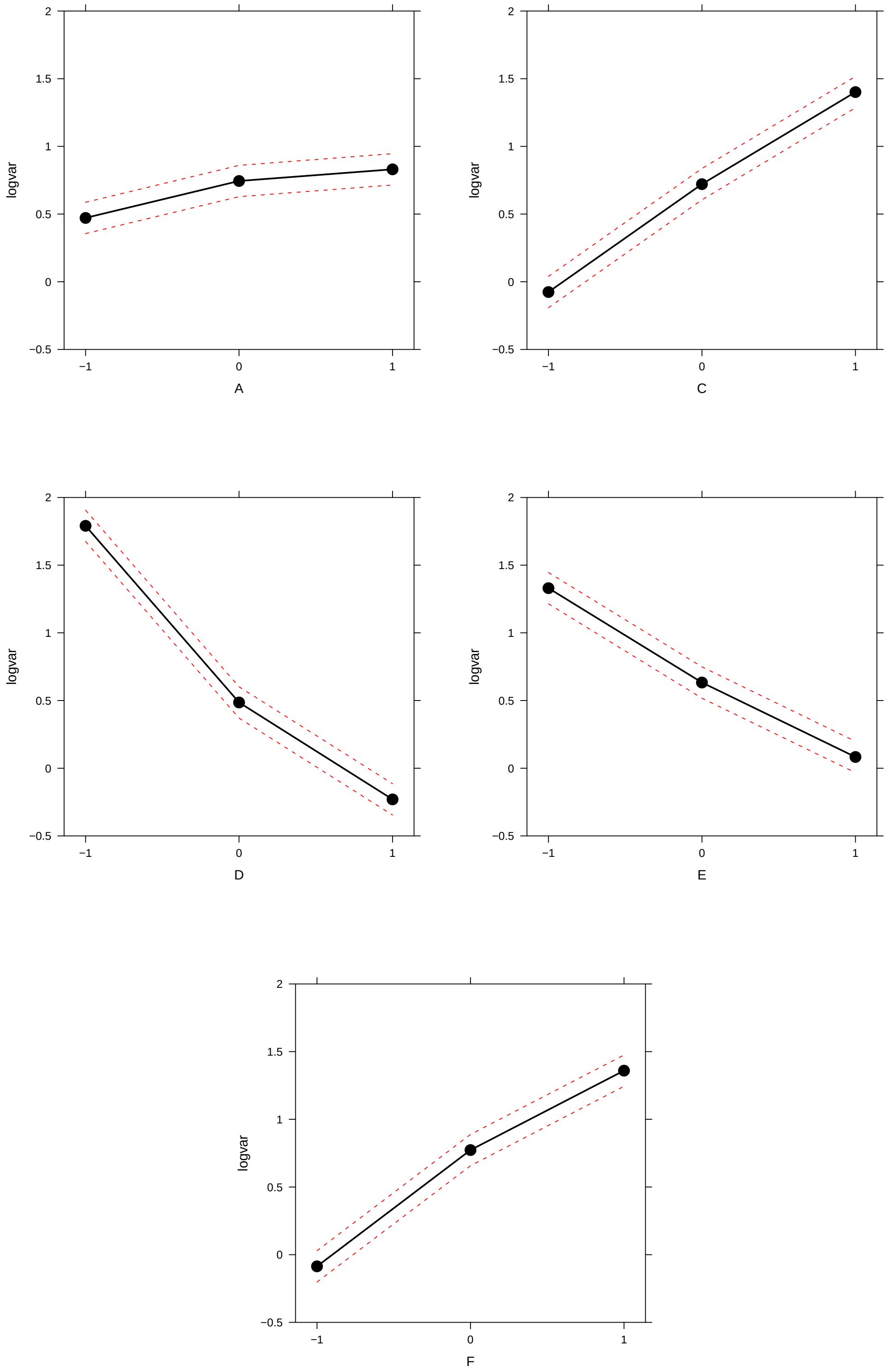


Figure 9.1: Inclinometer, Profile plot of control factors; ln(var) as response variable

Factors A, C and F have linear positive contrasts, it means that by increasing the factors their variability will increase too. As we see from these profile plots we can say that the factors A, C and F at lower level and the factor D and E must be used at high level to keep the variability concerning noise factors as low as possible.

As a result our reduced model contains factors A, C, D, E and F with the ANOVA table as follows:

```
            Df  Sum Sq Mean Sq  F value    Pr(>F)
A            1  0.5798  0.5798  19.4360 0.0004393 ***
A2           1  0.0519  0.0519   1.7383 0.2059161
C            1  9.8241  9.8241 329.3282 4.259e-12 ***
C2           1  0.0207  0.0207   0.6943 0.4169731
D            1 18.3769 18.3769 616.0433 3.350e-14 ***
D2           1  0.5217  0.5217  17.4897 0.0007044 ***
E            1  7.0044  7.0044 234.8044 5.547e-11 ***
E2           1  0.0322  0.0322   1.0808 0.3139786
F            1  9.4043  9.4043 315.2572 5.947e-12 ***
F2           1  0.1106  0.1106   3.7091 0.0720759 .
Residuals   16  0.4773  0.0298
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The table of effects are shown below:

lin A	lin C	lin D	quad D	lin E	lin F	quad F
0.358	1.478	-2.021	0.589	-1.248	1.446	0.272

For the second part of this experiment we take the mean of each factor as a response variable. The ANOVA table is shown below:

```
            Df  Sum Sq Mean Sq F value    Pr(>F)
A            2  0.1288  0.0644  0.3323    0.7237
B            2  0.2899  0.1449  0.7477    0.4943
C            2 12.9528  6.4764 33.4096 1.245e-05 ***
D            2 24.6042 12.3021 63.4626 4.153e-07 ***
E            2  9.0561  4.5281 23.3588 7.286e-05 ***
F            2 11.5710  5.7855 29.8455 2.199e-05 ***
G            2  0.6725  0.3362  1.7345    0.2179
Residuals   12  2.3262  0.1938
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

As we see the factors C, D, E and F are highly significant at the level of significance $\alpha = 0.05$. Now we will estimate the linear and quadratic terms of each factor.

```
 Df  Sum Sq Mean Sq  F value    Pr(>F)
A           1  0.1024  0.1024   0.5281   0.48133
A2          1  0.0264  0.0264   0.1364   0.71830
B           1  0.2850  0.2850   1.4703   0.24863
B2          1  0.0049  0.0049   0.0251   0.87677
C           1 12.8863 12.8863  66.4761 3.094e-06 ***
C2          1  0.0665  0.0665   0.3431   0.56892
D           1 22.9193 22.9193 118.2331 1.439e-07 ***
D2          1  1.6850  1.6850   8.6922   0.01218 *
E           1  7.9385  7.9385  40.9519 3.406e-05 ***
E2          1  1.1177  1.1177   5.7656   0.03345 *
F           1 11.2476 11.2476  58.0229 6.188e-06 ***
F2          1  0.3234  0.3234   1.6682   0.22083
G           1  0.0030  0.0030   0.0155   0.90301
G2          1  0.6694  0.6694   3.4535   0.08781 .
Residuals  12  2.3262  0.1938
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

As we see from the results above, factors C, D, E and F behave linearly and factors D and E have also a quadratic effect on the response variable. Factors C and F are positive and factors D and E are negative. Now we study the behavior of the significant factors.

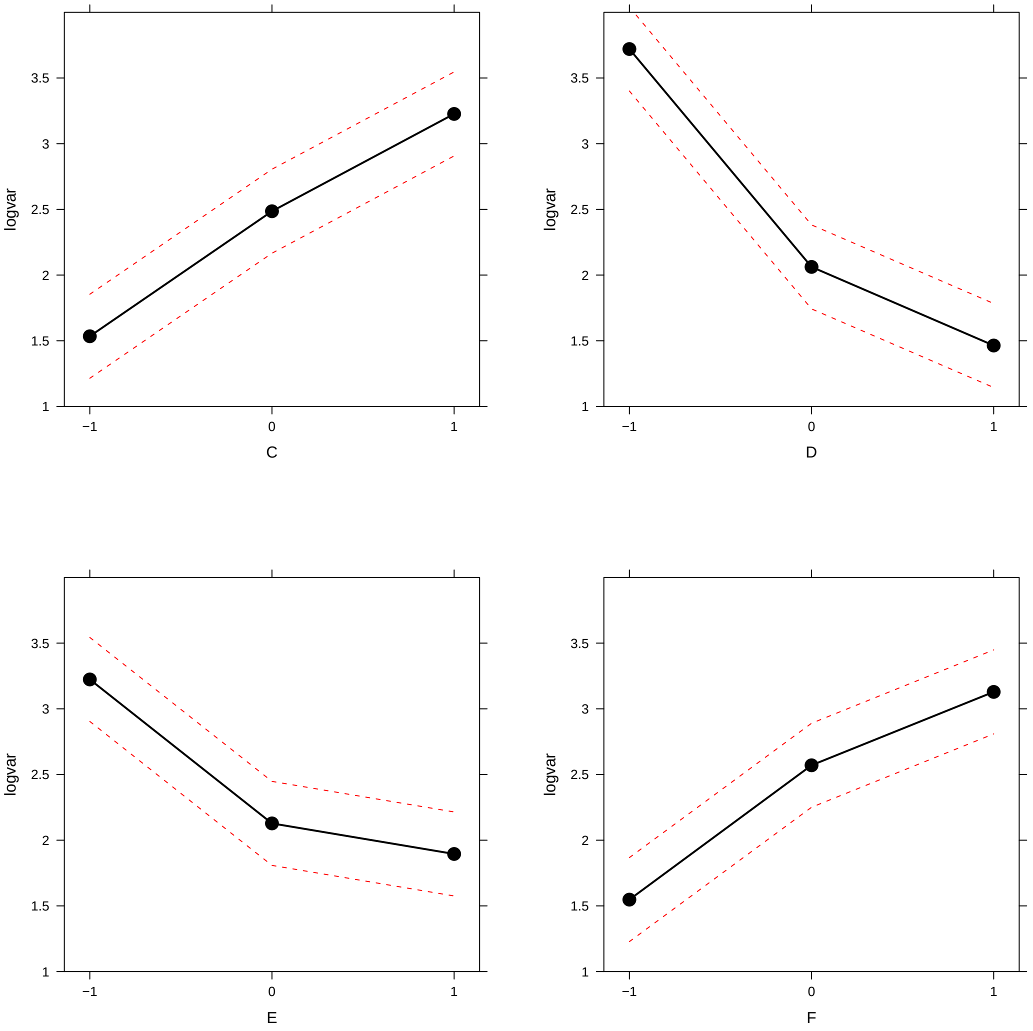


Figure 9.2: Inclonometer, Profile plot of control factors; mean as response variable

As you can see from the figure 9.2 on page 190 the factors D, E and F have a small quadratic effect. To minimize the variability of the swing we will take C and F at the low level, D and E at high level; this would be the best case. The table of effects are shown below:

```
lin C      lin D     quad D       lin E       quad E      lin F
1.692      -2.257    1.059        -1.328      0.863       1.581
```

So from these two parts we can conclude that the most significant factors concerning mean and concerning logarithm of variance are: C (linear), D (linear and quadratic), E (linear) and F (linear). Generally we can say that the best case that the variability of the swing will be as small as possible is $A_{-1}C_{-1}D_1E_1F_{-1}$.
Now we will investigate the complete model, a model containing both control and noise factors. The ANOVA table is as follow:

```
                Df Sum Sq Mean Sq   F value    Pr(>F)
Acontrol         2   1.03    0.52    1.1393   0.32434
Bcontrol         2   2.32    1.16    2.5636   0.08229 .
Ccontrol         2 103.62   51.81  114.5506 < 2.2e-16 ***
Dcontrol         2 196.83   98.42   350.90  < 2.2e-16 ***
Econtrol         2  72.45   36.22   129.16  < 2.2e-16 ***
Fcontrol         2  92.57   46.28   165.02  < 2.2e-16 ***
Gcontrol         2   5.38    2.69    9.59     0.00002 ***
Hnoise           1   0.12    0.12    0.2621   0.60988
Pnoise           1   0.01    0.01    0.0256   0.87333
Jnoise           1   0.04    0.04    0.0934   0.76062
Knoise           1   0.01    0.01    0.0218   0.88288
Lnoise           1   0.01    0.01    0.0195   0.88925
Mnoise           1   0.48    0.48    1.0691   0.30374
Nnoise           1 559.83  559.83 1237.7417 < 2.2e-16 ***
Acontrol:Hnoise  2   0.00    0.00 3.787e-05   0.99996
Acontrol:Pnoise  2   0.00    0.00 1.024e-06   1.00000
Acontrol:Jnoise  2   0.00    0.00    0.0011   0.99890
Acontrol:Knoise  2   0.00    0.00    0.0001   0.99986
Acontrol:Lnoise  2   0.00    0.00 3.992e-05   0.99996
Acontrol:Mnoise  2   0.00    0.00    0.0002   0.99978
Acontrol:Nnoise  2   0.46    0.23    0.5059   0.60458
Bcontrol:Hnoise  2   0.00    0.00    0.0043   0.99566
Bcontrol:Pnoise  2   0.00    0.00    0.0023   0.99766
Bcontrol:Jnoise  2   0.00    0.00    0.0003   0.99973
Bcontrol:Knoise  2   0.00    0.00    0.0001   0.99995
Bcontrol:Lnoise  2   0.00    0.00    0.0007   0.99932
```

```
Bcontrol:Mnoise  2   0.01    0.01    0.0127   0.98735
Bcontrol:Nnoise  2   1.03    0.51    1.1366   0.32519
Ccontrol:Hnoise  2   0.01    0.00    0.0088   0.99124
Ccontrol:Pnoise  2   0.00    0.00    0.0019   0.99813
Ccontrol:Jnoise  2   0.00    0.00    0.0034   0.99664
Ccontrol:Knoise  2   0.00    0.00    0.0009   0.99915
Ccontrol:Lnoise  2   0.00    0.00    0.0014   0.99856
Ccontrol:Mnoise  2   0.01    0.00    0.0073   0.99269
Ccontrol:Nnoise  2  46.05   23.03   50.9089 8.478e-16 ***
Dcontrol:Hnoise  2   0.02    0.01    0.0259   0.97448
Dcontrol:Pnoise  2   0.01    0.00    0.0060   0.99399
Dcontrol:Jnoise  2   0.03    0.02    0.0371   0.96364
Dcontrol:Knoise  2   0.02    0.01    0.0177   0.98246
Dcontrol:Lnoise  2   0.01    0.00    0.0071   0.99294
Dcontrol:Mnoise  2   0.08    0.04    0.0886   0.91525
Dcontrol:Nnoise  2  78.50   39.25   86.7788 < 2.2e-16 ***
Econtrol:Hnoise  2   0.00    0.00    0.0015   0.99852
Econtrol:Pnoise  2   0.00    0.00    0.0003   0.99971
Econtrol:Jnoise  2   0.00    0.00    0.0004   0.99959
Econtrol:Knoise  2   0.00    0.00    0.0002   0.99983
Econtrol:Lnoise  2   0.00    0.00    0.0003   0.99967
Econtrol:Mnoise  2   0.01    0.00    0.0096   0.99050
Econtrol:Nnoise  2  35.72   17.86   39.4874 3.063e-13 ***
Fcontrol:Hnoise  2   0.01    0.00    0.0077   0.99228
```

It is obvious that the control factors B, C, D, E, F and G are significant. From the noise factors just factor N is significant, and the interactions CN, DN, EN, FN and GN are significant too. The interaction plots are shown in figure 9.3 on page 193.

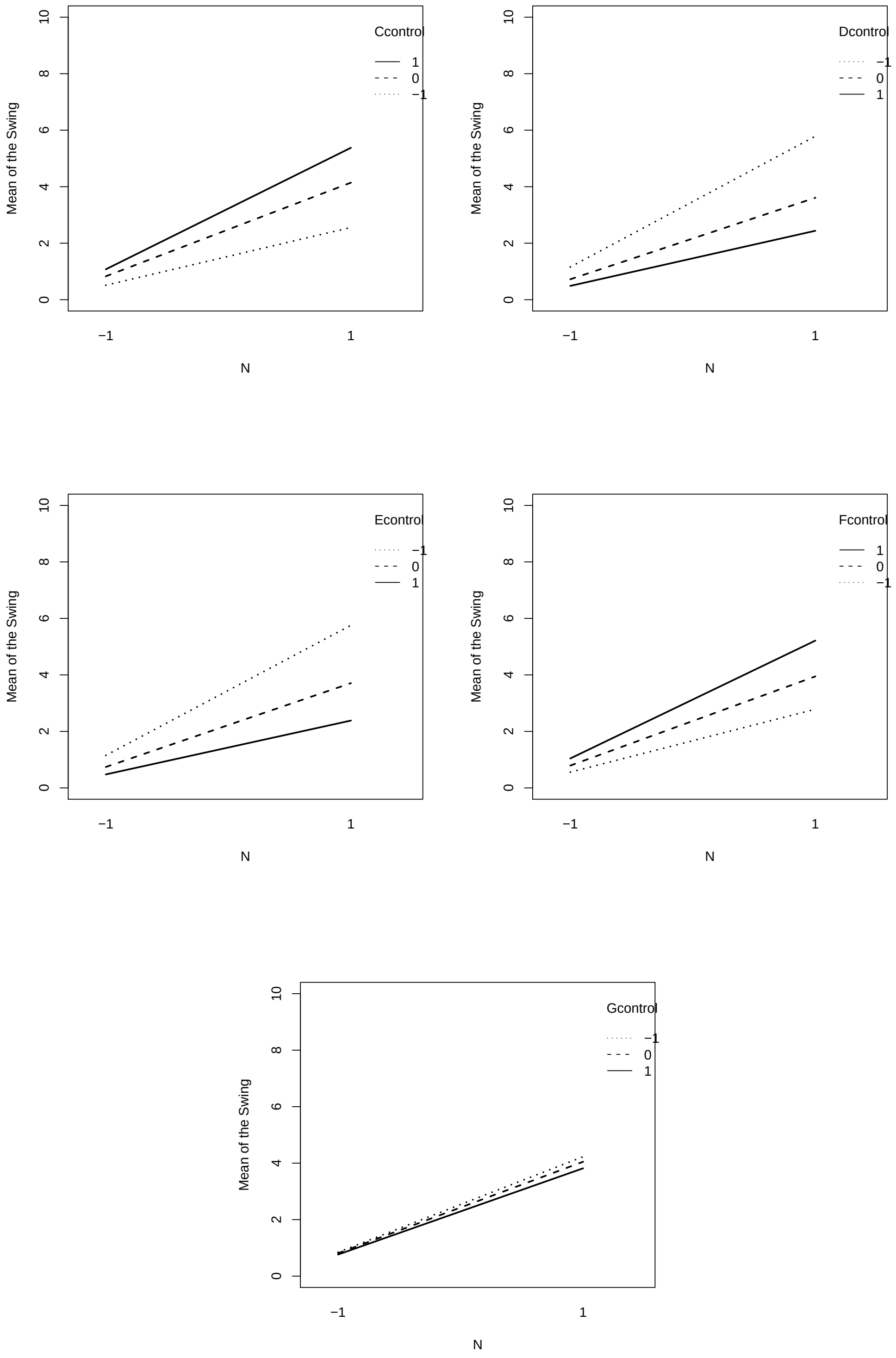


Figure 9.3: Inclinometer, Interaction plots

The interactions of the first four plots show obvious interactions between factors. The last one between control factor G and noise factor N where the lines are almost parallel so there is no interaction between them. If we want to minimize the variability of the swing the best case therefore is $C_{-1}D_1E_1F_{-1}G_{-1}$ or G_1.
The linear and quadratic term of control factor B is as follows:

```
             Df  Sum Sq Mean Sq F value Pr(>F)
Bcontrol      1    2.28  2.28    8.13   0.0053
Bcontrol2     1    0.04  0.04    0.14   0.7104
Residuals   213  270.36  5.9641
```

Inclinometer R-Code

```
menge<-matrix(c(0.62, 3.54, 3.56, 0.62, 3.09, 0.71, 0.73, 3.20,
0.59, 3.11, 3.11, 0.59, 2.98, 0.63, 0.64, 3.02,
0.59, 3.01, 3.02, 0.59, 2.97, 0.61, 0.62, 3.00,
0.51, 2.65, 2.65, 0.50, 2.53, 0.53, 0.54, 2.56,
0.18, 0.96, 0.96, 0.18, 0.89, 0.19, 0.20, 0.90,
1.88, 9.58, 9.55, 1.85, 9.30, 1.92, 1.94, 9.48,
0.19, 1.03, 1.03, 0.19, 0.97, 0.21, 0.21, 0.93,
1.85, 9.46, 9.42, 1.82, 9.19, 1.90, 1.92, 9.35,
0.52, 2.73, 2.72, 0.51, 2.61, 0.55, 0.56, 2.64,
0.29, 1.56, 1.56, 0.29, 1.45, 0.31, 0.32, 1.47,
0.95, 4.98, 4.93, 0.94, 4.79, 0.99, 1.00, 4.82,
1.16, 6.09, 6.09, 1.13, 5.70, 1.21, 1.26, 5.93,
0.26, 1.45, 1.45, 0.25, 1.30, 0.29, 0.30, 1.30,
1.15, 5.99, 5.92, 1.13, 5.69, 1.19, 1.22, 5.84,
0.85, 4.31, 4.30, 0.84, 4.23, 0.86, 0.88, 4.28,
1.10, 5.74, 5.67, 1.07, 5.43, 1.14, 1.17, 5.57,
0.29, 1.55, 1.55, 0.28, 1.45, 0.31, 0.32, 1.47,
0.91, 4.64, 4.66, 0.90, 4.56, 0.94, 0.95, 4.57,
0.39, 2.05, 2.06, 0.39, 1.96, 0.41, 0.42, 1.97,
0.67, 3.61, 3.57, 0.65, 3.27, 0.72, 0.74, 3.41,
1.42, 7.31, 7.38, 1.41, 7.14, 1.48, 1.51, 7.24,
0.69, 3.66, 3.60, 0.67, 3.37, 0.73, 0.74, 3.47,
1.18, 6.04, 6.06, 1.17, 5.90, 1.21, 1.23, 5.95,
0.37, 1.95, 1.95, 0.37, 1.87, 0.39, 0.40, 1.88,
0.39, 2.15, 2.16, 0.38, 1.94, 0.44, 0.44, 1.96,
0.44, 2.29, 2.29, 0.43, 2.21, 0.46, 0.47, 2.22,
1.84, 9.35, 9.19, 1.79, 9.06, 1.85, 1.89, 9.28
),27,8,byrow=T)
#Mean and Variance and logarithm of variance computing:
menge.means<-apply(menge,1,mean)
menge.means
lnvar<-function(x) {log(sum((x -mean(x))^2)/(length(x)-1))}
menge.lnvar<-apply(menge,1,lnvar)
menge.var
SNL<-function(x){-10*log(sum(1/x^2)/length(x),10)}
menge.SNL<-apply(menge,1,SNL)
menge.SNL
#Constructing inner and outer Design:
menge.dat<-data.frame(menge.means,menge.lnvar,menge.SNL,A=factor(rep(c(-1,0,1),
each=9)) ,B=factor(rep(c(-1,0,1),each=3,times=3)),C=factor(rep(c(-1,0,1),9)),
D=factor(c(-1,0,1,0,1,-1,1,-1,0,0,1,-1,1,-1,0,-1,0,1,1,-1,0,-1,0,1,0,1,-1)),
E=factor(c(-1,0,1,0,1,-1,1,-1,0,1,-1,0,-1,0,1,0,1,-1,0,1,-1,1,-1,0,-1,0,1)),
F=factor(c(-1,0,1,1,-1,0,0,1,-1,0,1,-1,-1,0,1,1,-1,0,1,-1,0,0,1,-1,-1,0,1)),
G=factor(c(-1,0,1,1,-1,0,0,1,-1,1,-1,0,0,1,-1,-1,0,1,0,1,-1,-1,0,1,1,-1,0)))
menge.dat
#Analysis of variance for the logarithm of variance:
```

```
menge.mod.lnvar<-aov(menge.lnvar~A+B+C+D+E+F+G, data=menge.dat)
summary(menge.mod.lnvar)
A<-rep(c(-1,0,1),each=9); A2<-A^2
B<-rep(c(-1,0,1),each=3,times=3); B2<-B^2
C<-rep(c(-1,0,1),9); C2<-C^2
D<-c(-1,0,1,0,1,-1,1,-1,0,0,1,-1,1,-1,0,-1,0,1,1,-1,0,-1,0,1,0,1,-1)
E<-c(-1,0,1,0,1,-1,1,-1,0,1,-1,0,-1,0,1,0,1,-1,0,1,-1,1,-1,0,-1,0,1)
F<-c(-1,0,1,1,-1,0,0,1,-1,0,1,-1,-1,0,1,1,-1,0,1,-1,0,0,1,-1,-1,0,1)
G<-c(-1,0,1,1,-1,0,0,1,-1,1,-1,0,0,1,-1,-1,0,1,0,1,-1,-1,0,1,1,-1,0)
D2<-D^2; E2<-E^2; F2<-F^2; G2<-G^2
menge.mod.lnvar<-aov(menge.lnvar~A+A2+B+B2+C+C2+D+D2+E+E2+F+F2+G+G2)
summary(menge.mod.lnvar)
#Profil Plots:
library(effects)
plot(effect("A",menge.mod.lnvar),main="",xlab="A",ylab="logvar", ylim=c(-0.5,2))
plot(effect("C",menge.mod.lnvar),main="",xlab="C",ylab="logvar", ylim=c(-0.5,2))
plot(effect("D",menge.mod.lnvar),main="",xlab="D",ylab="logvar", ylim=c(-0.5,2))
plot(effect("E",menge.mod.lnvar),main="",xlab="E",ylab="logvar", ylim=c(-0.5,2))
plot(effect("F",menge.mod.lnvar),main="",xlab="F",ylab="logvar", ylim=c(-0.5,2))
#Reduced model analysis:
reduce.mod.lnvar<-aov(lnvar~A+C+D+E+F, data=menge.dat)
summary(reduce.mod.lnvar)
menge.mod2.lnvar<-aov(menge.lnvar~A+B+C+D+E+F+G)
model.tables(menge.mod2.lnvar)
#Analysis with mean as a response variable:
menge.mod.means<-aov(menge.means~A+B+C+D+E+F+G, data=menge.dat)
summary(menge.mod.means)
menge.poly<-lm(menge.means~A,contrasts=list(A="contr.poly"),data=menge.dat)
summary(menge.poly)
menge.poly<-lm(menge.means~B,contrasts=list(B="contr.poly"),data=menge.dat)
summary(menge.poly)
menge.poly<-lm(menge.means~C,contrasts=list(C="contr.poly"),data=menge.dat)
summary(menge.poly)
menge.poly<-lm(menge.means~D,contrasts=list(D="contr.poly"),data=menge.dat)
summary(menge.poly)
menge.poly<-lm(menge.means~E,contrasts=list(E="contr.poly"),data=menge.dat)
summary(menge.poly)
menge.poly<-lm(menge.means~F,contrasts=list(F="contr.poly"),data=menge.dat)
summary(menge.poly)
menge.poly<-lm(menge.means~G,contrasts=list(G="contr.poly"),data=menge.dat)
summary(menge.poly)
plot(effect("C",menge.mod.means),main="",xlab="C",ylab="logvar", ylim=c(1,4))
plot(effect("D",menge.mod.means),main="",xlab="D",ylab="logvar", ylim=c(1,4))
plot(effect("E",menge.mod.means),main="",xlab="E",ylab="logvar", ylim=c(1,4))
plot(effect("F",menge.mod.means),main="",xlab="F",ylab="logvar", ylim=c(1,4))
#PART C: Defining new data set:
menge1.dat<-data.frame(menge1=c(0.62,3.54,3.56,0.62,3.09,0.71,0.73,3.20,0.59,
```

```
3.11,3.11,0.59,2.98,0.63,0.64,3.02,0.59,3.01,3.02,0.59,2.97,0.61,0.62,3.00,
0.51,2.65,2.65,0.50,2.53,0.53,0.54,2.56,0.18,0.96,0.96,0.18,0.89,0.19,0.20,
0.90,1.88,9.58,9.55,1.85,9.30,1.92,1.94,9.48,0.19,1.03,1.03,0.19,0.97,0.21,
0.21,0.93,1.85,9.46,9.42,1.82,9.19,1.90,1.92,9.35,0.52,2.73,2.72,0.51,2.61,
0.55,0.56,2.64,0.29,1.56,1.56,0.29,1.45,0.31,0.32,1.47,0.95,4.98,4.93,0.94,
4.79,0.99,1.00,4.82,1.16,6.09,6.09,1.13,5.70,1.21,1.26,5.93,0.26,1.45,1.45,
0.25,1.30,0.29,0.30,1.30,1.15,5.99,5.92,1.13,5.69,1.19,1.22,5.84,0.85,4.31,
4.30,0.84,4.23,0.86,0.88,4.28,1.10,5.74,5.67,1.07,5.43,1.14,1.17,5.57,0.29,
1.55,1.55,0.28,1.45,0.31,0.32,1.47,0.91,4.64,4.66,0.90,4.56,0.94,0.95,4.57,
0.39,2.05,2.06,0.39,1.96,0.41,0.42,1.97,0.67,3.61,3.57,0.65,3.27,0.72,0.74,
3.41,1.42,7.31,7.38,1.41,7.14,1.48,1.51,7.24,0.69,3.66,3.60,0.67,3.37,0.73,
0.74,3.47,1.18,6.04,6.06,1.17,5.90,1.21,1.23,5.95,0.37,1.95,1.95,0.37,1.87,
0.39,0.40,1.88,0.39,2.15,2.16,0.38,1.94,0.44,0.44,1.96,0.44,2.29,2.29,0.43,
2.21,0.46,0.47,2.22,1.84,9.35,9.19,1.79,9.06,1.85,1.89,9.28),
Acontrol=factor(rep(c(-1,0,1),each=72)), Bcontrol=factor(rep(c(-1,0,1),
each=24,times=3)),Ccontrol=factor(rep(c(-1,0,1),each=8,times=9)),
Dcontrol=factor(c(-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,0,0,
0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,-1,-1,-1,
-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,
-1,-1,-1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,-1,-1,-1,-1,
-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,
-1,-1,0,0,0,0,0,0,0,0,-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,
-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1)),
Econtrol=factor(c(-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,
0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,
-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,
0,0,0,0,0,0,0,0,-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,
0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,
1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,
0,0,0,0,0,0,0,0,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1)),
Fcontrol=factor(c(-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,
-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,
-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,
-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,
0,0,0,0,0,1,1,1,1,1,1,1,1)),
Gcontrol=factor(c(-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,
-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,
1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,
0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,
0,0,0,0,0,0,0,0)),Hnoise=factor(rep(c(-1,1),each=4,times=27)),
Pnoise=factor(rep(c(-1,1),each=2,times=54)),Jnoise=factor(c(-1,-1,1,1,1,1,-1,-1,
-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,
```

```
-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,
-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,
-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,
-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,
-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1,
-1,-1,1,1,1,1,-1,-1,-1,-1,1,1,1,1,-1,-1)),Knoise=factor(rep(c(-1,1),each=1,
times=108)),Lnoise=factor(c(-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,
-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,
-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,
-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,
-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,
-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,
-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,
-1,1,-1)),Mnoise=factor(c(-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,
1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,
1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,
1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,
1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,
1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,
1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,1,1,-1,-1,
1,1,-1)),Nnoise=factor(c(-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,
-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,
-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,
-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,
-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,
-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,
-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,-1,1,-1,1,1,-1,1,-1,
-1,1)))
#Analysis of variance:
menge9.mod<-aov(menge1~Acontrol+Bcontrol+Ccontrol+Dcontrol+Econtrol+Fcontrol+
Gcontrol+Hnoise+Pnoise+Jnoise+Knoise+Lnoise+Mnoise+Nnoise+
Acontrol*Hnoise+Acontrol*Pnoise+Acontrol*Jnoise+Acontrol*Knoise+Acontrol*Lnoise
+Acontrol*Mnoise+Acontrol*Nnoise+
Bcontrol*Hnoise+Bcontrol*Pnoise+Bcontrol*Jnoise+Bcontrol*Knoise+Bcontrol*Lnoise
+Bcontrol*Mnoise+Bcontrol*Nnoise+
Ccontrol*Hnoise+Ccontrol*Pnoise+Ccontrol*Jnoise+Ccontrol*Knoise+Ccontrol*Lnoise
+Ccontrol*Mnoise+Ccontrol*Nnoise+
Dcontrol*Hnoise+Dcontrol*Pnoise+Dcontrol*Jnoise+Dcontrol*Knoise+Dcontrol*Lnoise
+Dcontrol*Mnoise+Dcontrol*Nnoise+
Econtrol*Hnoise+Econtrol*Pnoise+Econtrol*Jnoise+Econtrol*Knoise+Econtrol*Lnoise
+Econtrol*Mnoise+Econtrol*Nnoise+
Fcontrol*Hnoise+Fcontrol*Pnoise+Fcontrol*Jnoise+Fcontrol*Knoise+Fcontrol*Lnoise
+Fcontrol*Mnoise+Fcontrol*Nnoise+
Gcontrol*Hnoise+Gcontrol*Pnoise+Gcontrol*Jnoise+Gcontrol*Knoise+Gcontrol*Lnoise
+Gcontrol*Mnoise+Gcontrol*Nnoise, data=menge1.dat)
summary(menge9.mod)
#Interaction effects:
```

```
interaction.plot(Nnoise,Ccontrol,menge1,lwd=2,xlab="N",ylab="Mean of the Swing",
ylim=c(0,10))
interaction.plot(Nnoise,Dcontrol,menge1,lwd=2,xlab="N",ylab="Mean of the Swing",
ylim=c(0,10))
interaction.plot(Nnoise,Econtrol,menge1,lwd=2,xlab="N",ylab="Mean of the Swing",
ylim=c(0,10))
interaction.plot(Nnoise,Fcontrol,menge1,lwd=2,xlab="N",ylab="Mean of the Swing",
ylim=c(0,10))
interaction.plot(Nnoise,Gcontrol,menge1,lwd=2,xlab="N",ylab="Mean of the Swing",
ylim=c(0,10))
#Linaer and quadratic term of factor B:
Bcontrol=rep(c(-1,0,1),each=24,times=3); Bcontrol2<-Bcontrol^2
menge10.mod<-aov(menge1~Bcontrol+Bcontrol2,data=menge1.dat)
summary(menge10.mod)
Gcontrol=c(-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,
-1,-1,-1,-1,-1,-1,-1,-1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,
1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,-1,
0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,
0,0,0,0,0,0,0,0)
```

Chapter 10

Summary

10.1 Content

- Cement (Ch. 2, Sec. 2-2, P. 8)
- Radon (Ch. 2, Sec. 2-3, P. 15)
- Rodding (Ch. 2, Sec. 2-4, P. 21)
- Nozzle Design (Ch. 3, Sec. 3-1, P. 29)
- Ratio Control Algorithm (Ch. 3, Sec. 3-2, P. 36)
- Aluminium Master Alloy (Ch. 3, Sec. 3-3, P. 44)
- Monofilament Fiber (Ch. 4, Sec. 4-3, P. 54)
- Glue (Ch. 4, Sec. 4-4, P. 54)
- Soft Drink Bottlers (Ch. 5, Sec. 5-3, P. 70)
- Strength of a paper (Ch. 5, Sec. 5-4, P. 77)
- Metal Surface Part (Ch. 5, Sec. 5-5, P. 83)
- Router (Ch. 6, Sec. 6-3, P. 94)
- Nickel-Titanium (Ch. 6, Sec. 6-4, P. 102)
- Etch rate for silicon-nitride (Ch. 6, Sec. 6-5, P. 113)
- Chemical Process (Ch. 6, Sec. 6-6, P. 122)
- 2^{5-1} Factorial Design (Ch. 7, Sec. 7-2, P. 137)

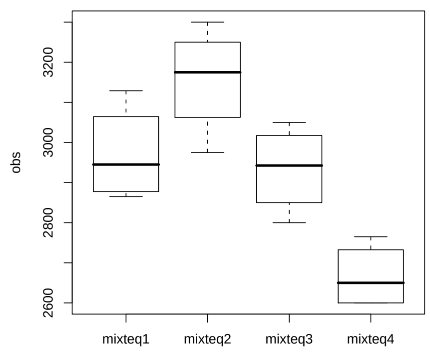
- Monte Carlo Simulation Model (Ch. 7, Sec. 7-3, P. 146)
- Dye Experiment (Ch. 8, Sec. 8-10, P. 164)
- Refinery Experiment (Ch. 8, Sec. 8-11, P. 174)
- Inclinometer (Ch. 9, Sec. 9-3, P. 187)

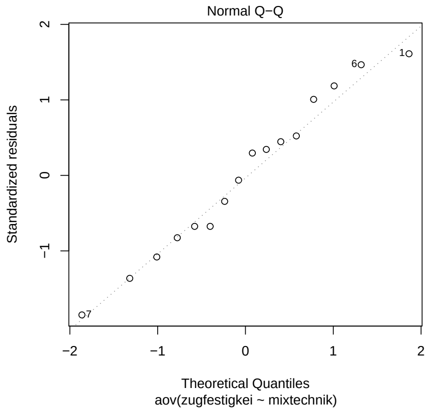
10.2 Summary Cement

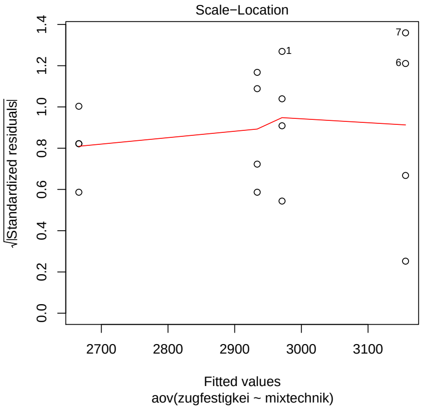
The data are saved in the file `"cement data.txt"`. The Boxplot shows that when by using the mixing technique 2 there is a larger strength in Portland cement.
With one way analysis of variance we find out that mixing techniques have an effect on the strength of Portland cement. By using a multiple comparison test especially Tukey-HSD confidence intervals we find out that the mean 4 is significant different from others.

The residuals are approximately normally distributed, and the ANOVA test indicates that 70% of variability in the model is explained by the mixing techniques.

If we want to have a strongest cement we have to work with mixing technique 2.







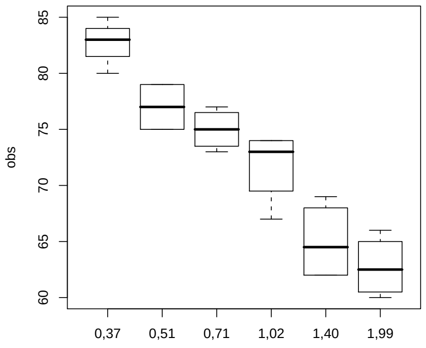
10.3 Summary Radon

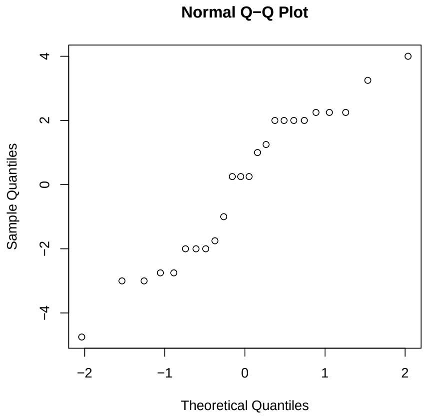
The data are saved in the file `"radon data.txt"`. The Boxplot shows the larger the orifice diameter, the smaller the amount of released radon.

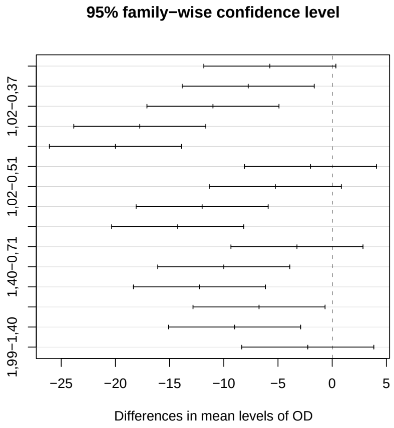
The one way ANOVA indicates that there is at least one treatment mean that is different. The model F-value is 30.85 and it implies that the model is significant.

The QQ-plot shows no abnormality in the residuals. The residual analysis also indicates that the orifice diameter "1.40" has the largest residuals skewness.

The Tukey-HSD confidence intervals give us the result that the treatments 5 and 6 are different from the other means; and the treatments 2, 3 and 4 are also different from other means.





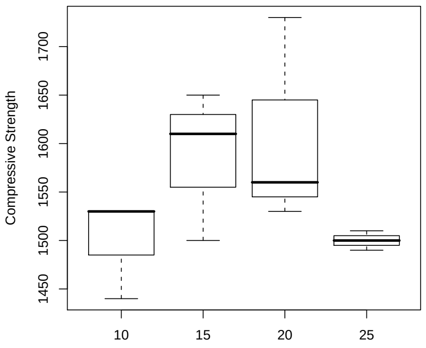


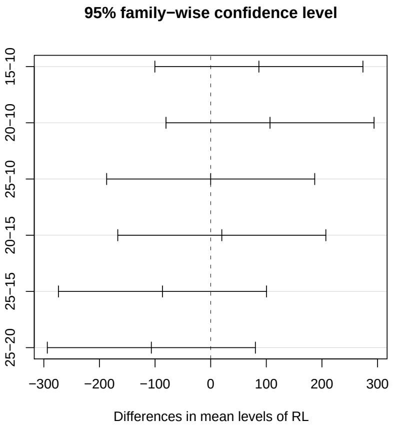
10.4 Summary Rodding

The data are saved in the file `"rodding data.txt"`. The Boxplot shows the difference between the median of the four rodding levels, also by using rod 20 we have a larger compressive strength.

The one way ANOVA indicates there is no effect on the response variable due to rodding level. The model F-value is 1.865 with the P-value 0.2138 indicating that the model is not significant.

From the Tukey-HSD confidence intervals we can say that non of the treatment means are different from others.





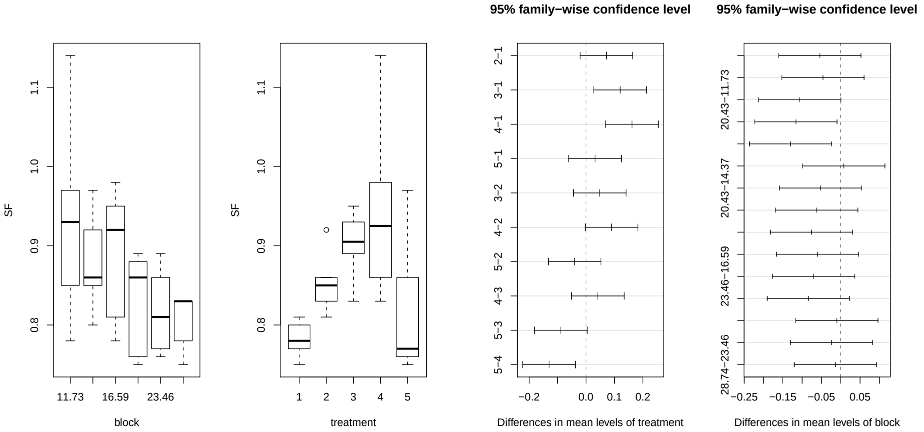
10.5 Summary Nozzle Design

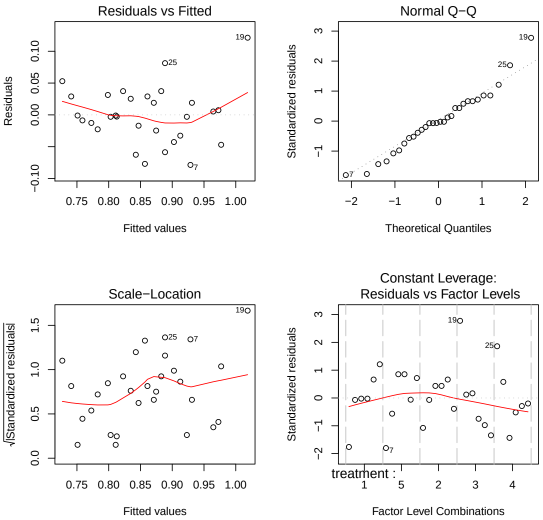
The two way ANOVA gives us the significant effects of the factor nozzle on the response variable shape factor. The model F-value of 6.401 implies that the model is significant, in this case we have a R^2 of 63%.

The ANOVA table indicates that there is no interaction effect between these two factors.

The residual analysis verifies that the residuals are normally distributed. From the Tukey-HSD confidence interval we can conclude that nozzle 1 and 5 are significant different from the nozzle 3 and 4.

For larger shape factor values we have to select nozzles 3 and 4.



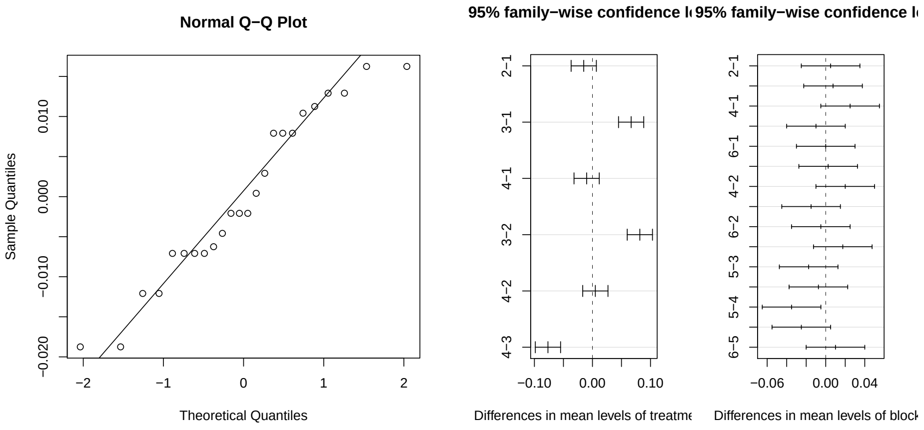


10.6 Summary Ratio Control Algorithm

The two way ANOVA indicates that both factors have a significant effect on the average cell voltage, also there is no interaction effect between these two factors. The model F-value implies that the model is significant.

The QQ-plot indicates the normality of residuals.

The Tukey-HSD confidence intervals indicate that the treatment 3 is significant different from others. If we want to have a better cell voltage we have to select treatment 3.



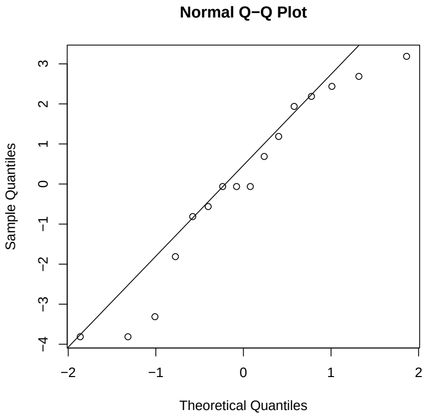
10.7 Summary Aluminium Master Alloy

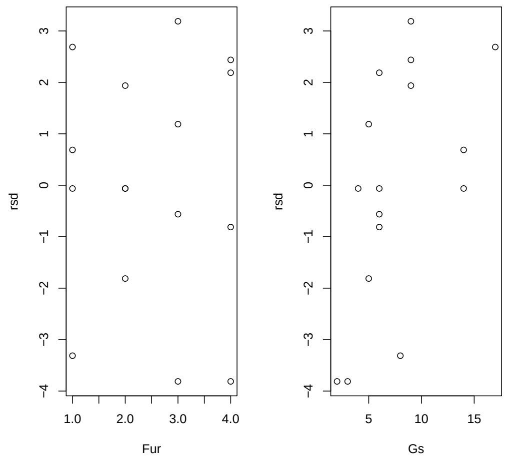
The two way ANOVA indicates that factor furnace (here blocks) is significant, but the stirring rate (here treatments) with the F-value of 0.85 and the P-value of 0.4995 isn't significant. The model F-value of 3.6 implies that the model is significant.

From the ANOVA table we can say there is no difference in mean grain size due to the different stirring rates.

The QQ-plot indicates that the normality assumption is valid. From the plot of residuals versus furnace and stirring rate we can conclude that the variance is consistent at different stirring rates, it also indicates that the different stirring rate does not affect variance.

From the Tukey-HSD confidence intervals we can see that none of the treatments are significant different from others so there is no effect on the grain size due to stirring rate.





10.8 Summary Monofilament Fiber

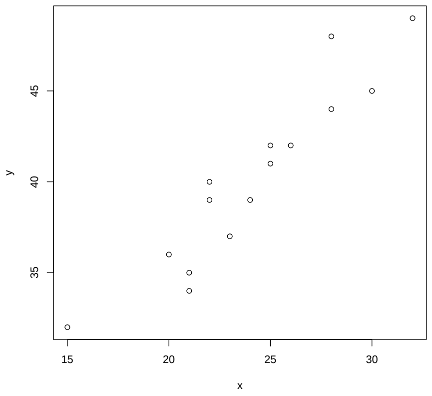
All conditions of monofilament fiber are fulfilled. From the scatter plot we can find out that the variables X and Y are linear dependent.

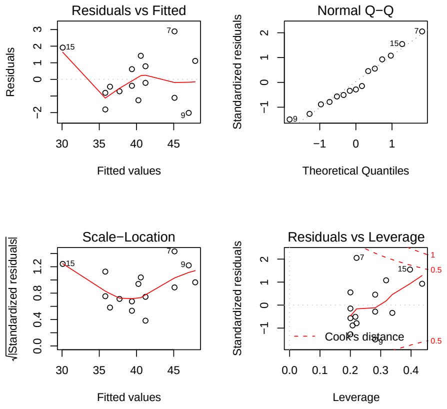
From the ANOVA table we conclude that $H_1 : \beta \neq 0$ at the significance level of < 0.0001

The one-way ANOVA gives us the result that the covariate X isn't independent from the machine with R^2_{adj} of 0.174.

The residual analysis shows that the residuals are normally distributed.

In total the covariance analysis indicates that the factor machine has no effect on the breaking strength of the monofilament fiber.





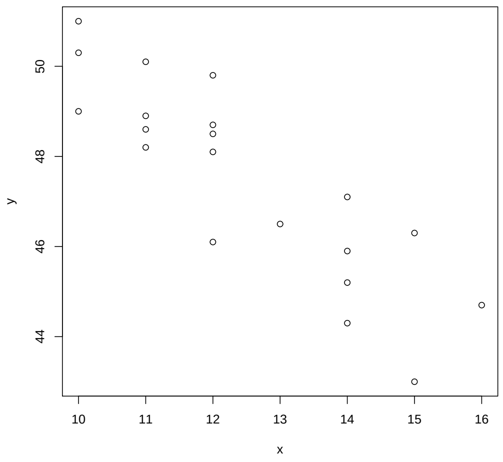
10.9 Summary Glue

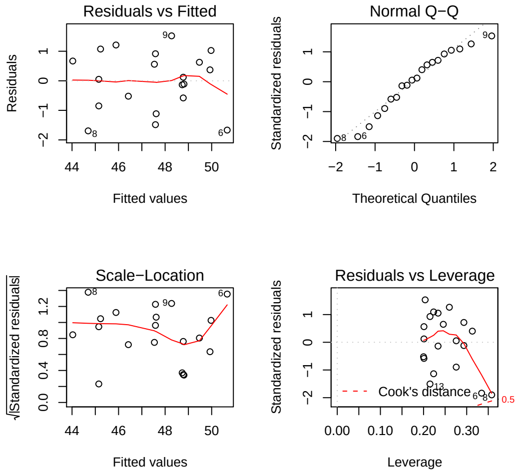
The scatter plot shows that there is a negative linear relationship between X and Y.

The ANOVA table gives us the result that $H_1 : \beta \neq 0$ with significance level of < 0.0001, and the covariate X isn't independent from the type of formulation with R^2_{adj} of 0.744.

The QQ-plot shows that the residuals are normally distributed.

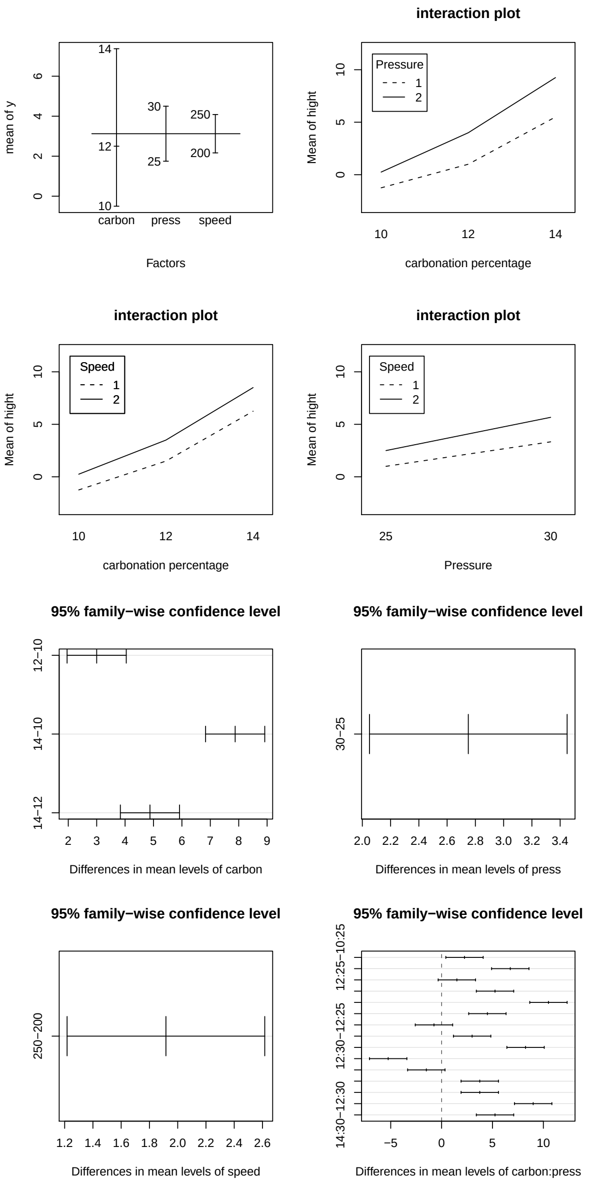
In total, the covariance analysis says that the formulation doesn't have any effect on the strength of the glue.





10.10 Summary Soft Drink Bottlers

As we see from the interaction plots and also from the ANOVA table there is no interaction between three factors, but all three main effects are highly significant. In particular, the low level of pressure results in smaller mean deviance from the target. This can be seen in the confidence intervals.



10.11 Summary Strength of a paper

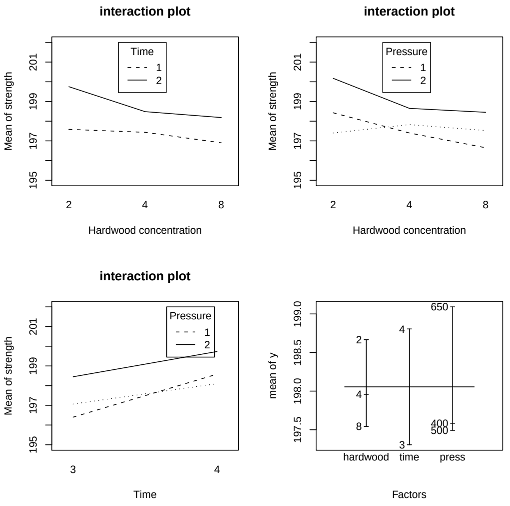
From the boxplots we can see that the factors do have an effect on the strength of a paper. By doing the three way analysis of variance we find out that the three main effects are highly significant.

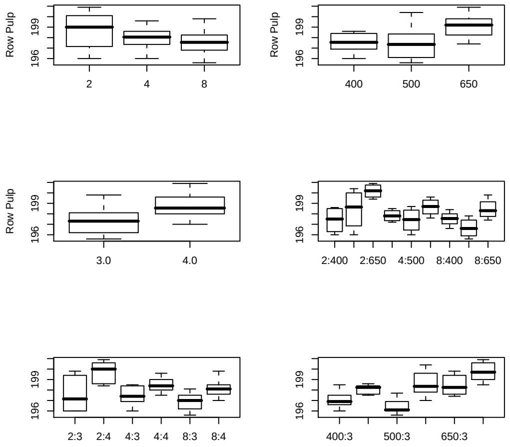
From the interaction plots we see that there is an interaction between hardwood-pressure and rare interaction between hardwood-time and time-pressure.

The F-value of 9.611 implies that the model is significant. The R^2 implies that about the 81% of the variability in the strength of the paper is explained by the percentage of the hardwood, the pressure, the cocking time and their interactions.

The interaction plot shows that the hardwood-time lines are approximately parallel. The hardwood-pressure and pressure-time lines are approximately parallel. Just in two cases, the lines of pressure 400 and 500 aren't parallel, they are crossed.

The residuals are approximately normally distributed. To observe a higher strength of a paper the combination of pressure=650, hardwood percentage=2% and the cooking time=4 will be effective.





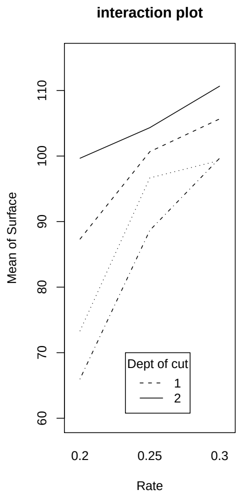
10.12 Summary Metal Part Surface

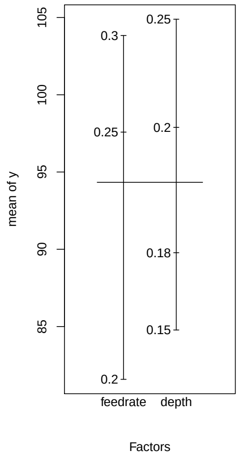
From the boxplots we can see that the factors do have an effect on the surface finish of a metal part. By doing the two-way analysis of variance we find out that the two main effects are highly significant.

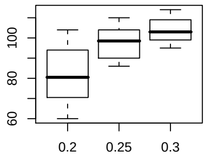
From the interaction plots we see that there is an interaction between feed rate and depth when the cutting depth lines are at level 0.15 and 0.18.

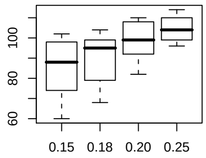
The F-value of 18.49 implies that the model is significant. The R^2 implies that about 85% of the variability in the surface finish of a metal part is explained by the feed rate, the cutting depth and their interactions.

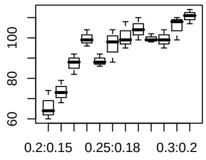
The residuals are approximately normal distributed. To observe a better result the combination of feed rate at high level (0.3) and the cutting depth of (0.25) will be effective.











10.13 Summary Router

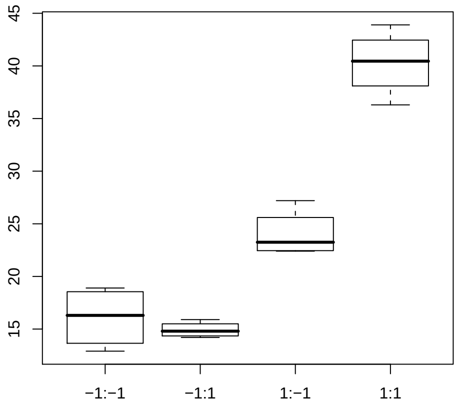
The boxplots show that by the smaller bit size $\frac{1}{16}$ the vibration is almost equal for each cutting speed. The analysis of a 2^2 factorial design gives us the result that the main effect and their interactions are significant and that they have an effect on the vibration of the circuit board.

The F-value of 91.36 implies that the model is significant. The R^2 implies that about 95.8% of the variability in the vibration is explained by the bit size, the cutting speed and their interactions.

The residual analysis verifies that the residuals are normally distributed, and we can see that the larger vibration is caused when we are at the high level of both factors.

To have a smaller vibration during the work, the bit size must be in the low level.

The interaction plot shows that by the combination form we have to select the bit size $\frac{1}{16}$ and the cutting speed of 90 for observing a smaller vibration.

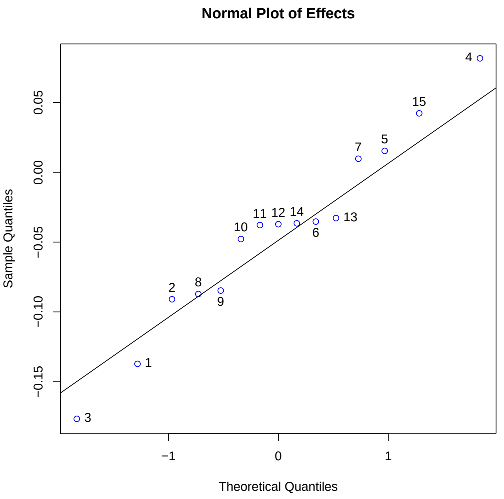


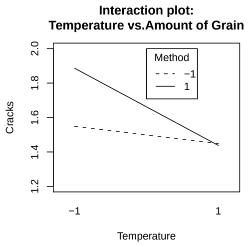
10.14 Summary Nickel-Titanium

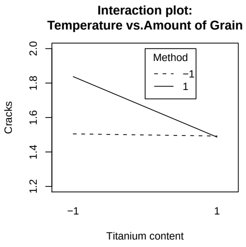
The normal plot indicates that the four main effects and the two interaction effects AD and BD are significant. The F-value of 10.51 implies that the model is significant and 82% of the variability is explained by the significant factors above.

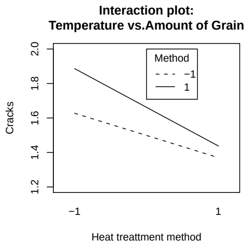
As we see from the interaction plots the following combinations lead to smallest length of the cracks: $A^+B^-D^-$, $A^+D^-C^+$, $B^+D^+A^+$ and $B^+D^+C^+$.

The residual analysis verifies the normality assumption of the model. To observe a smaller cracking length the best solution is to have all four factors at their high levels.





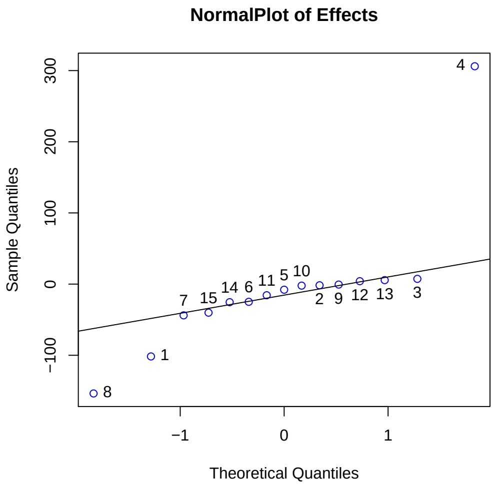


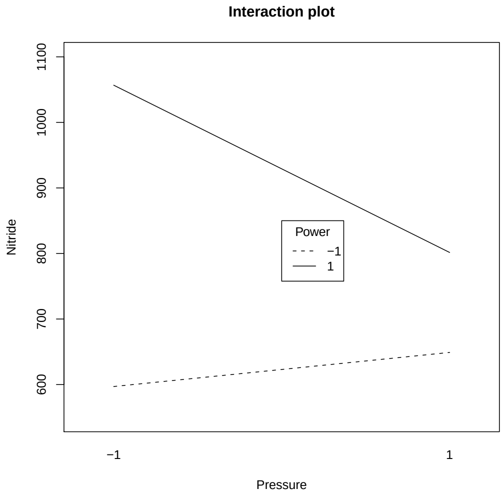


10.15 Summary Etch rate for silicon-nitride

The normal plot indicates that the two main effects A and D and their interaction effects AD are significant. The 2^2 factorial design has a F-value of 97.91 and it implies that the model is significant and 95% of the variability is explained by the significant factors above.

As we see from the interaction plots when both factors A and D are at their high level we will have a larger response. The residual analysis verifies the normality assumption of the model.



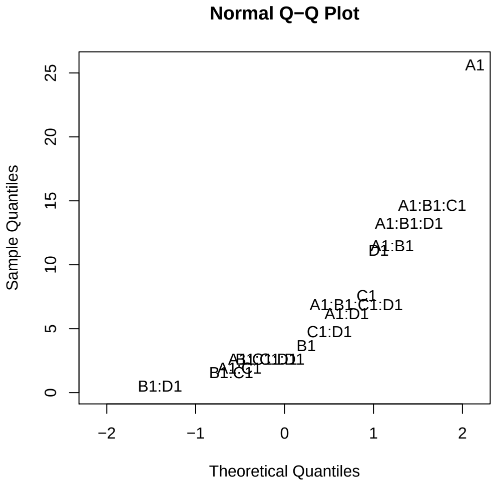


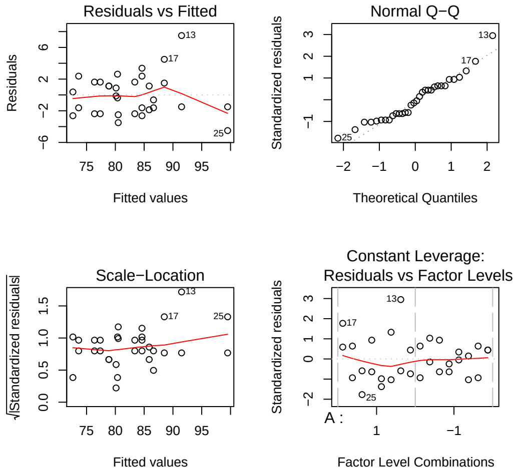
10.16 Summary Chemical Process

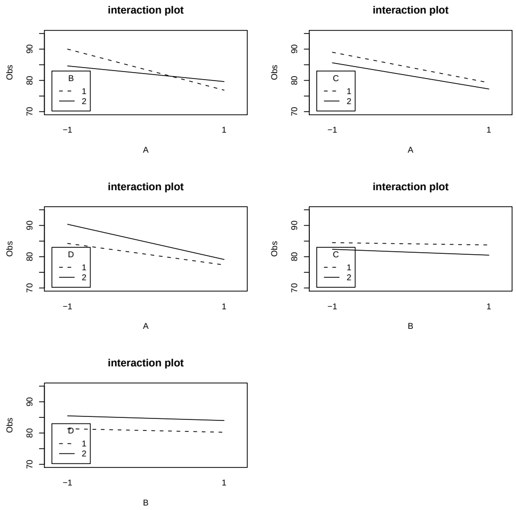
The normal plot indicates that the main factors A, C and D and the interaction effects AB, ABC and ABD are significant. The F-value of 13.1 implies that the model is significant.

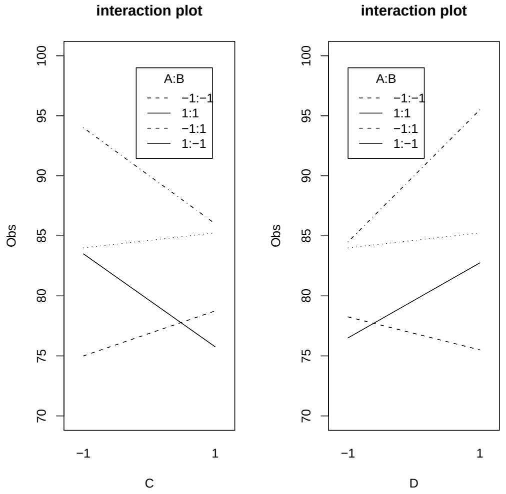
In analyzing the reduced model we see that 86% of the variability is explained by the factors above.

The QQ-plot verifies that the residuals are normally distributed. The interaction plots show that when the factors A, B and C are at their low level and D at high level then we will have a better yield.



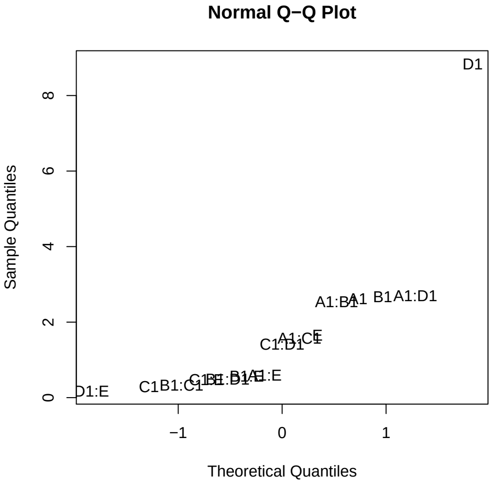


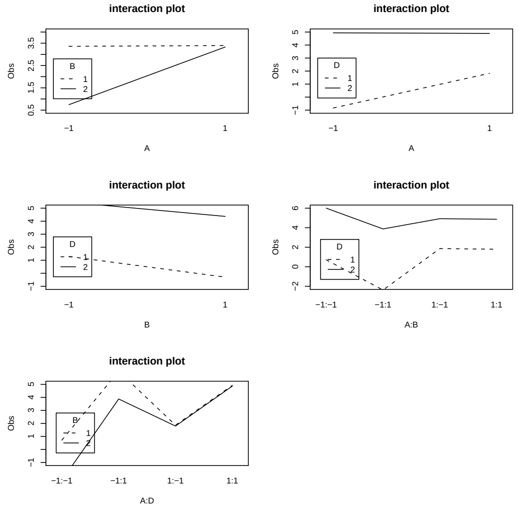




10.17 Summary 2^{5-1} Factorial Design

As we can see from the normal plot the main effects A, B and D and the two factor interaction effects AB and AD are significant. The ANOVA table approves the above results. The F-value of 18.96 implies that the model is significant and 88% of the variability is explained by the above effects. The residual analysis verifies the normality of the residuals.
The analysis of a reduced 2^3 factorial design implies that BD and ABD aren't significant, but the other factors are significant. The interaction plot indicates that for observing higher values we have to set A and D at their high level and B at low level.

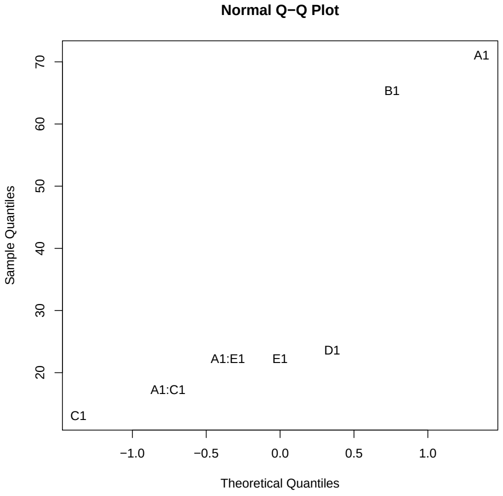


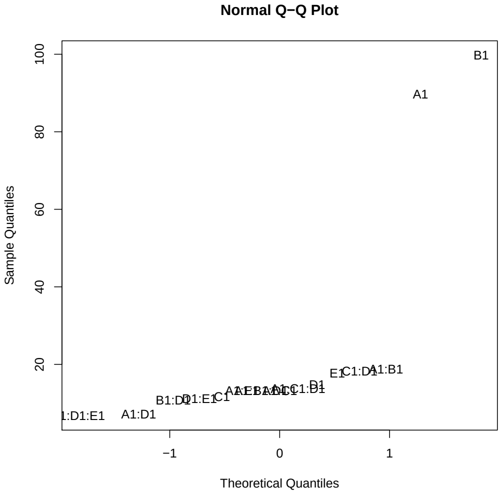


10.18 Summary Monte Carlo Simulation Model

As we see in the first part the two generators are equal, the factors D and E can be aliased with AB and BC respectively. From the normal plot we indicate that the main effects A and B are very large. Recall that we are estimating A+BD and B+AD+CE.

As we have seen from the ANOVA table the two main effects A and B are significant. Here D is the smallest effect and is negative. In the second part D will be estimated as -AB. As we see in this case A and B are still large.



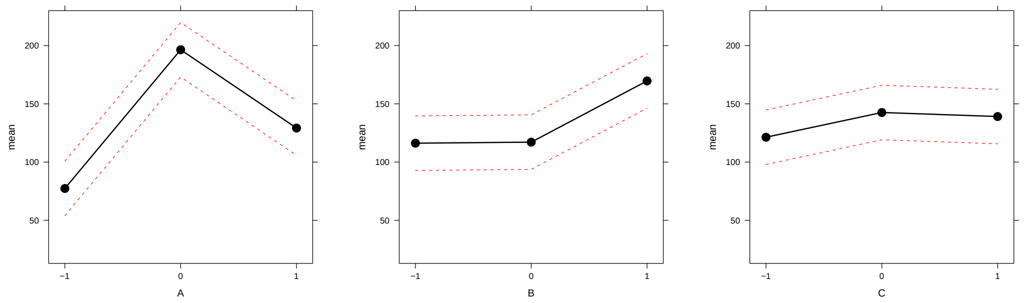


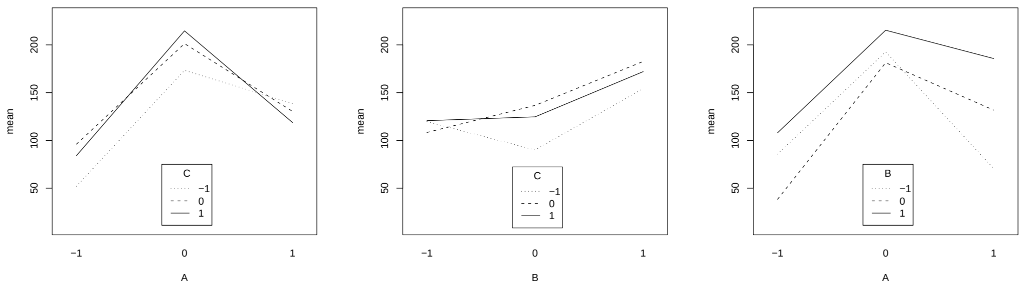
10.19 Summary Dye Experiment

The analysis of variance indicates that the two main factors A and B are significant at significance level of $\alpha = 0.05$ and their interaction is rare significant at $\alpha = 0.1$. Factor Block doesn't have any effect on the response variable and it isn't significant.

As we see from the factor diagram, factor A behaves quadratic; at the intermediate level (0) it has the largest value, but factor C shows no difference, and the value of each level is almost the same, factor B behave the same from level (-1) to (0), from level (0) to (1) it increases and has the largest value at level (1). It behaves linearly. One can also see the interaction plots below.

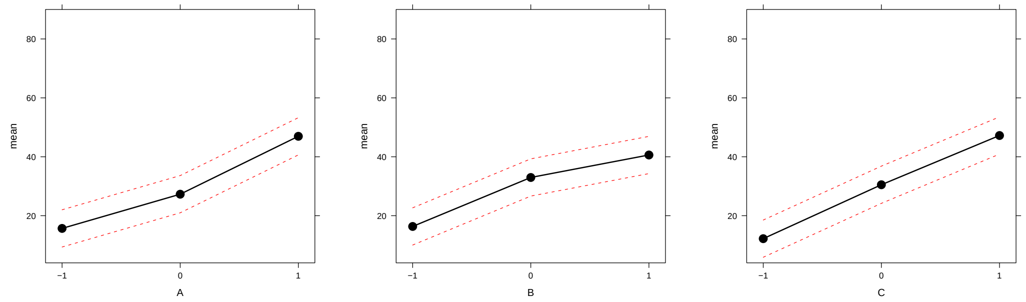
Factor A is the most important followed by factor B, The best setting for observing an optimized volume is $A = 0$, $B = 1$, $C = 0$ or $C = 1$

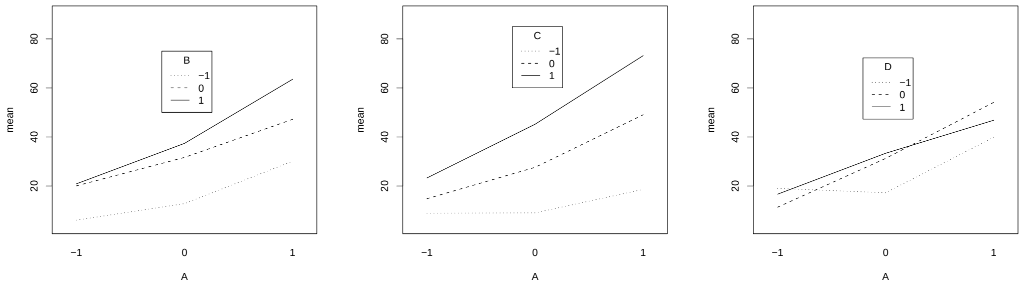




10.20 Summary Refinery Experiment

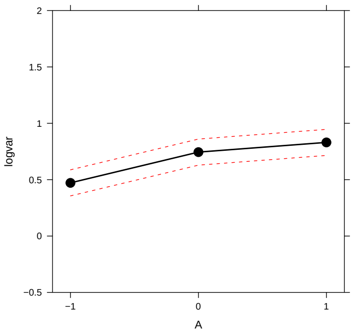
The analysis of variance table gives us the result that factors A, B and C have the largest effect on the response variable and the interaction effect $A_l C_l$ is also significant. From the factor diagram below we can also see that these three factors behave linearly, their linearity increases from level 0 to level 1, and the largest value is observed at their highest level. The plot of the main effects and their interaction effects is shown below.

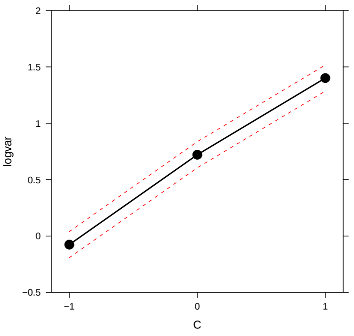


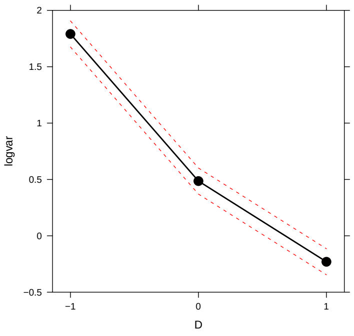


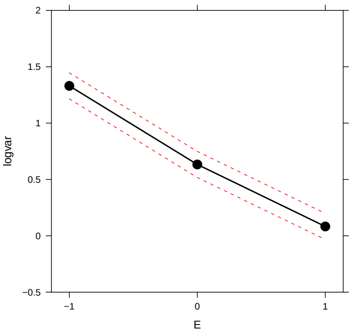
10.21 Summary Inclinometer

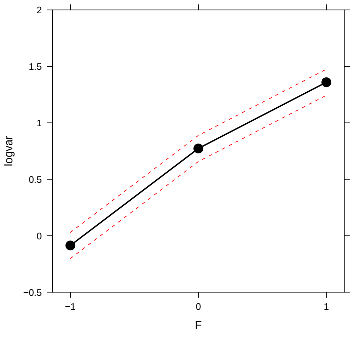
From the analysis of the logarithm of variance ln(var) as response variable we have seen that factors A, C, D, E and F have an effect on the response variable. Factors C, D, E, F behave linearly as one can see.



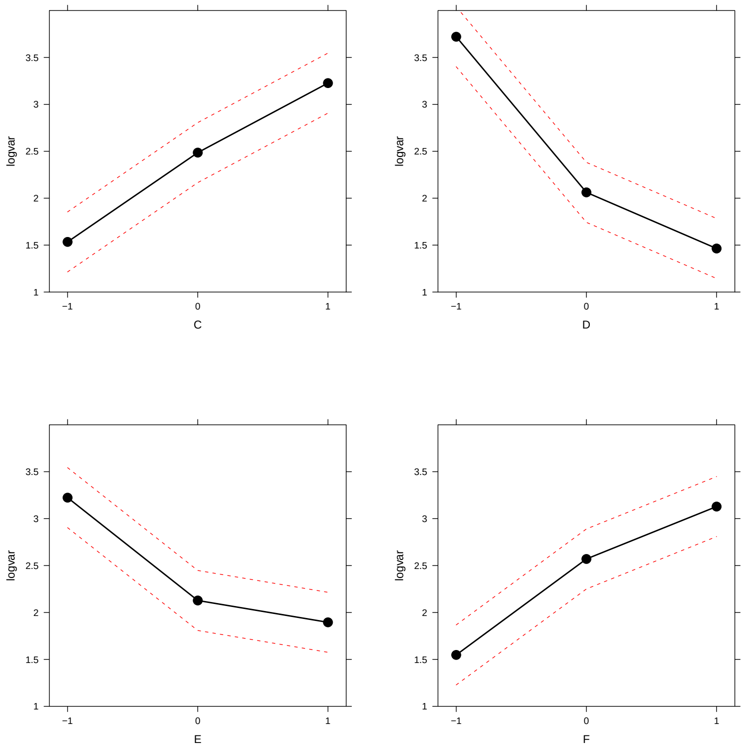








From the analysis of the mean as a response variable factors C, D, E and F are highly significant and have an effect on the response variable. These factors also behave linearly, as you can see from the figures below. The best case in this regard is when factors A, C and F are at their lowest level (-1) and factors D and E are at their high level (1).



By analyzing the model with both control and noise factor we discover that the control factors B, C, D, E, F and G are significant. From the noise factors only factor N is significant. Here interactions CN, DN, EN, FN and GN are also significant.

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