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Flips in (bounded degree) triangulations and pseudo-triangulations

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Abstract

In this thesis we consider the flip distance of triangulations and pseudo-triangulations with and without the additional restriction of bounded vertex degrees. We show that the flip distance of a subset of all zigzag triangulations of n points in convex position in the plane can be given as a function of n. Further, we are able to prove the validity of this result in the degree bounded setting where the vertex degree of $k > 6$ must not be exceeded during the transformation of two triangulations.

For the degree bounded setting we present several approaches for an input sensitive upper bound on the flip distance of two triangulations of n points in general position in the plane with maximum vertex degree k. By providing counterexamples, we show that these approaches do not work in general.

We give account of a computer program that, on the one hand, provides heuristics that approximate the flip distance of two (bounded degree) triangulations or pointed pseudo-triangulations taking the vertex degree bound during the transformation into consideration. On the other hand, the construction of the flip graph and related calculations for very small point sets are implemented.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der Flipdistanz von Triangulierungen bzw. Pseudo-Triangulierungen unter Berücksichtigung oder Vernachlässigung begrenzter Knotengrade. Für Zigzag-Triangulierugen einer n-elementigen Punktmenge in konvexer Lage in der Ebene, die bestimmte Anforderungen an die Lage der Kanten erfüllen, wird gezeigt, dass deren Flipdistanz als Funktion von n angegeben werden kann. Außerdem bleibt dieser Zusammenhang gültig, wenn die beiden Triangulierungen maximalen Knotengrad $k > 6$ besitzen und zusätzlich gefordert wird, dass während der Transformation der beiden Triangulierungen ineinander der maximale Knotengrad aller temporären Triangulierungen kleiner oder gleich k ist.

Ansätze zur Abschätzung der Flipdistanz unter der Hinzunahme der Knotengradbeschränkung zweier gegebener Triangulierungen von n Punkten in allgemeiner Lage in der Ebene werden vorgestellt. Durch die Konstruktion von Gegenbeispielen wird gezeigt, dass die Ansätze nicht allgemein gültig sind.

Verschiedene Heuristiken, die die Flipdistanz zweier Trianuglierungen oder pointed Pseudo-Triangulierungen mit maximalen Knotengrad k annähern, wurden implementiert und werden vorgestellt. Durch die Konstruktion des Flipgraphs bietet das Programm zusätzlich die Möglichkeit der Berechnung der exakten Flipdistanz und des Durchmessers des Flipgraphs unter Berücksichtigung der Knotengradbeschränkung.

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The past two years emphasized the importance of friends and family including my boyfriend Max, who all gave me an amazing support and stability for a partially rough time. Thank you... I also thank Max for his understanding and patience in the past two years.

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Contents

1 Introduction

The term "triangulation" is widely used and can be found in many fields reaching from psychology (description of relationships) to land surveying.

In this thesis we turn to triangulations in the field of computational geometry. To be more precise, we consider triangulations of n points P in general position, i.e., subdivisions of the convex hull of P into interior-disjoint triangles where the vertex set of the triangles coincides with P. First introduced by Lawson in [15], the edge flip in a triangulation is a local operation that transforms a triangulation into another. It exchanges a diagonal of a convex quadrilateral built by two triangles of the triangulation. Since the result of the edge flip is again a triangulation, we can say that the operation is closed under the set of triangulations of P. Further, two arbitrary triangulations of P can be transformed into each other with a maximum number of $\mathcal{O}(n^2)$ flips ([15]).

Given two triangulations of P, we are interested in the flip distance, i.e., the minimum number of flips that are needed to transform the triangulations into each other. Since the number of triangulations is exponential in the size of P, the efficient computation of the flip distance is in general an open problem, see [9]. Only a few results for the exact and efficient calculation of the flip distance can be found in the literature in [10] and [16]. In this thesis, we extend the set of triangulations for which the flip distance can be computed efficiently by a subset of the so-called zigzag triangulations of convex point sets. The arising flip sequence is a result of the recursive application of theorems shown in [4], where possibilities of the subdivision of the flip distance into smaller problems are presented. The length of the flip sequence depends on the size of the underlying point set. Thus, we can define the flip distance in this case as a function of the number of points.

The flip graph of triangulations of P contains a vertex for each triangulation and two vertices are connected in the flip graph by an edge if the corresponding triangulations differ by a single edge flip. The diameter of a flip graph is defined by the maximum length of the shortest path in the flip graph. In [3] Aichholzer et al. consider the connectivity of a subgraph of the flip graph that only consists of vertices that correspond to triangulations with maximum vertex degree $k \in \mathbb{N}$. In other words, they raise the question if two given triangulations with maximum vertex degree k can be transformed into each other without exceeding the vertex degree k. For P in convex position they answer the question in the affirmative for any $k > 6$. Further, they show that at most $\mathcal{O}(n^2)$ edge flips are needed for the transformation. In addition, they decrease that bound to $\mathcal{O}(n)$ in case of zigzag-triangulations. With those results in mind, we are able to prove that the above mentioned flip distance does not exceed the vertex degree $k > 6$ and hence, holds for the degree bounded setting, too.

As a contrast for bounded degree triangulations of convex point sets, Aichholzer et al. show that the flip graph of bounded degree triangulations of point sets in general position can be disconnected for any k. Nevertheless, we are interested in an upper bound on the flip distance in the degree bounded setting, assuming that two given triangulations are transformable without exceeding the vertex degree k. We discuss several estimates with the intention to get an input sensitive upper bound on the flip distance similar to the one presented in [11] for triangulations.

Additionally, we study the latest results concerning the flip distance, the flip graph and its diameter of pseudo-triangulations of point sets. Pseudo-triangulations arose in the 1990's as a generalization of triangulations. The subdivision of the convex hull is done by pseudo-triangles, i.e., simple polygons with three vertices, instead of triangles. Particularly, we consider a subset of pseudo-triangulations — pointed pseudo-triangulations. In a pointed pseudo-triangulation each vertex is incident to an angle larger than π. In short, we say that each vertex is pointed in a pointed pseudo-triangulation.

Besides the theoretical considerations, a program is presented that approximates the flip distance of two triangulations or pointed pseudo-triangulations in the degree bounded setting by means of several heuristics. Furthermore, the construction of the flip graph and related calculations such as the flip distance and the flip diameter are implemented and described.

1.1 Thesis overview

In section 2 we start with some fundamental definitions and basic properties relating to triangulations (section 2.1) and pseudo-triangulations (section 2.3). Further, we present important results of research dealing with the flip graph, the flip distance and the flip diameter.

Section 3 addresses input sensitive upper bounds on the flip distance of two (bounded degree) triangulations. We first show the results of Hanke et al. [11], afterwards we elaborate some approaches for a similar estimate for the degree bounded setting.

In section 4 we investigate the flip distance of two zigzag triangulations.

We introduce in section 4.1 some preliminary definitions and observations related to zigzag triangulations and outline important results of Baril et al. in [4], which mainly deal with the subdivision of the flip distance into smaller problems. We prove in section 4.2 that the flip distance of two zigzag triangulations can be defined as function of the number of points, provided that both zigzag triangulations are subject to conditions concerning the position of the edges. Basically, two different flip sequences corresponding to the flip distance arise. Section 4.3 reveals the equality between one of the two flip sequences and the flip sequence used in [3] by means of proving that zigzag triangulations can be transformed into each other without exceeding the vertex degree k in $\mathcal{O}(n)$ time. With an additional extension of this result in [3] we are able to show that the flip distance introduced in section 4.2 can be transferred to the degree bounded setting for $k > 6$ (see Theorem 4.13 on page 65).

In section 5 we give account of the developed computer program. We describe the implemented heuristics and possible calculations related to the flip graph. Finally, we give an overview of the program control.

Section 6 concludes the thesis with a short summary and an outlook on future work.

2 Basic properties

This section is divided into two main parts. One part deals with triangulations (section 2.1), the other one considers pseudo-triangulations (section 2.3) of point sets. Both parts have the same structure: First we give some basic definitions in order to be able to define a (pseudo-)triangulation of a point set. In this thesis only point sets in general position in the plane are considered. Hence, we assume that no three points are collinear.

After some basic properties of a (pseudo-)triangulation, we consider the terms “flip graph”, “flip diameter” and “flip distance” and give a brief overview of some fundamental results related to these terms. Finally, we consider the effect of applying vertex degree restrictions to these results.

Throughout the whole thesis, we denote with P a finite point set in general position in the plane. If P is in convex position, i.e., all points lie on the boundary of the convex hull, we write P_c (except for section 4, where we only consider triangulations of convex point sets). Furthermore, we define n to be the number of points of P and write m for the number of edges of a (pseudo-)triangulation. We define the vertex degree, $deg_T(v)$, $v \in P$, of v in a (pseudo-)triangulation T as number of edges incident to v in T.

2.1 Triangulations

Before we are able to define a triangulation of a point set, we have to consider the terms ”convex set” and “convex hull”. For each of them, there exists a variety of different definitions. To some extent, they are even different in their meaning, which emphasizes the importance of those definitions.

Definition 2.1 ([8])**.** A subset S of the plane is called a **convex set** iff S entirely contains the line segment $\overline{pq}$ for all pairs $p, q \in S$.

Definition 2.2 ([8])**.** The **convex hull** of P, $\mathrm{CH}(P)$, is the smallest convex set that contains that point set.

Contrary to other definitions in the literature, the convex hull is not reduced to the boundary of the smallest convex set containing P. According to Definition 2.2, we observe that if a convex hull has n_h points on its boundary, then the number of edges on the boundary of the convex hull is the same. We refer to those edges as convex hull edges of P. The definition of the convex hull of P enables us to define a triangulation of P:

Definition 2.3 ([12])**.** A **triangulation** of P is a partition of the convex hull of P into a set of interior-disjoint triangles such that the vertices of each triangle are points of P. The points of P are called **vertices** of T. The edges of the triangles are called **edges** of T. An edge is called **inner edge** or **diagonal** if it is not a convex hull edge.

Based on Lawson [15], we alternatively say that a triangulation of P is the result of connecting the vertices of P with non-crossing straight-line segments such that a set of interior-disjoint triangles arises and the set of corners of the triangles coincides with P. A more formal and short equivalent can be found in [1], where a triangulation is characterized by a maximal planar straight-line graph. That definition is useful in order to point out some basic properties of triangulations. The following observations can be found in [1]:

There is a very well known and useful formula for planar graphs, which relates the number of points, edges and faces (denoted by f) to each other. It is Euler's Polyhedron Formula that says $n-m+f=2$. For a triangulation with t triangles we therefore have $t=f-1$. Taking a closer look on t, we see that t is a function of the number of inner edges m_i and the number of convex hull edges m_h, with $m=m_i+m_h$:

- Each triangle consists of exactly three edges.
- Each inner edge has two incident triangles.
- Each convex hull edge has one incident triangle.

Thus, we have $t=\frac{2m_i+m_h}{3}$. Accordingly, we can replace t and m in Euler's formula and get

$$n-(m_h+m_i)+\frac{2m_i+m_h}{3}=1 \Leftrightarrow 3n-m_i-2m_i=3 \Leftrightarrow m=3n-m_h-3.$$

Hence, we see that the number of edges depends on the number of points and the number of convex hull edges. If we include the observation that each triangulation has at least three convex hull edges, we finally get an upper bound on the number of edges depending on n:

$$m=3n-m_h-3\leq 3n-6.$$

We summarize these basic properties in the following proposition:

Proposition 2.1 ([1]). *Let T be a triangulation of P. Furthermore, we denote with $m_h\geq 3$ the number of convex hull edges of P and abbreviate the number of inner edges of T with m_i such that $m=m_h+m_i$. Then T has the following properties:*

1. *m_h corresponds to the number of points in the boundary of* $\mathrm{CH}(P)$.

2. $m = 3n - m_h - 3$. *Note that for* P *is in convex position, we have* $m = 2n - 3$ *and* $m_i = n - 3$.

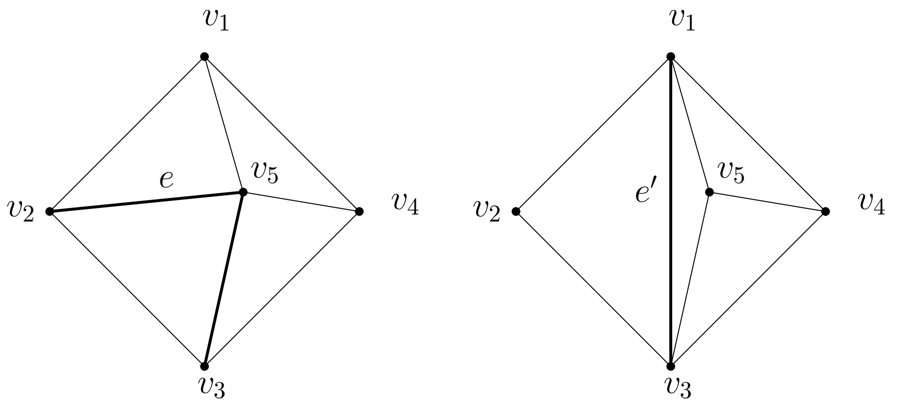
3. *The number of triangles is given by* $\frac{2m_i + m_h}{3}$ *and* $(n-2)$ *for* P *convex, respectively.*

4. $m \leq 3n - 6$.

There are many possible ways to triangulate a point set. Note that each triangulation of the same point set has the same number of edges. We abbreviate the set of all triangulations of P with $\mathcal{T}_P$. If P is in convex position, we write $\mathcal{T}_n$ instead of $\mathcal{T}_P$.

In the following, we will introduce the edge flip in a triangulation, which first appeared in [15] under the name "exchange". It is a local, uniquely reversible operation on $T \in \mathcal{T}_P$ and closed in $\mathcal{T}_P$. Moreover, we will see that it enables us to transform two arbitrary triangulations on the same point set into each other.

Definition 2.4 ([12]). Let t_1, t_2 be two triangles of $T \in \mathcal{T}_P$, that share a common edge e. If $Q = t_1 \cup t_2$ is a convex quadrilateral, then an **edge flip** exchanges e with the other possible diagonal e' in Q. Furthermore, e' is called the **flip target** of e.



(a) T_1 (b) T_2

Figure 1: $T_1, T_2 \in \mathcal{T}_P$, $P = \{v_1, \dots, v_5\}$. The edge flip from e to e' in T_1 results in T_2. Bold edges are flippable.

Figure 1 shows an example of an edge flip. Note that the edge (v_3, v_5) is only flippable in T_1 and thus the number of flippable edges changes from 2 in T_1 to 1 in T_2. Hence, the question on the number of flippable edges in

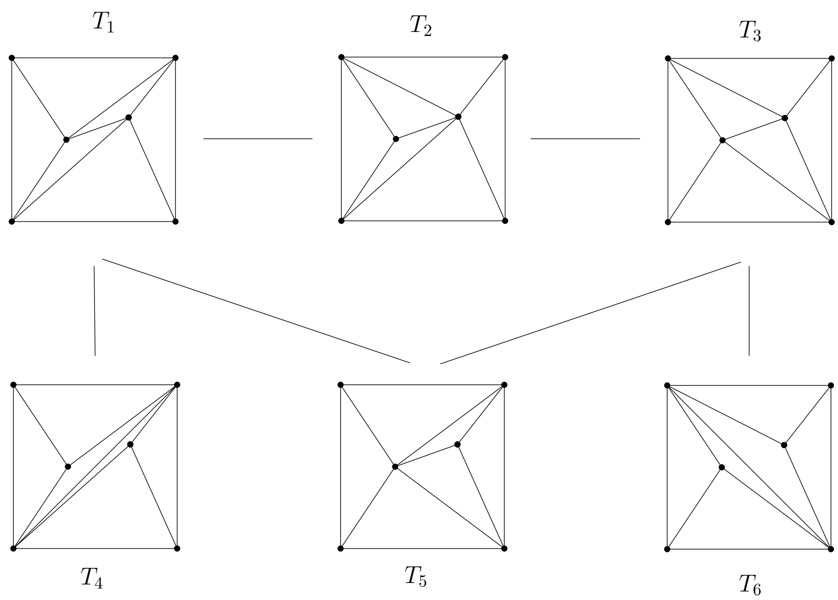


Figure 2: $\mathrm{FG}(\mathcal{T}_P)$ — the flip graph of P.

a triangulation arises. In [12] Hurtado et al. prove that each triangulation has at least $\lceil \frac{n-4}{2} \rceil$ flippable edges. Furthermore, they provide two examples in order to show that this bound is tight. As a consequence, the number of triangulations on a point set in general position is exponential in the number of points as at least a linear number of edges is independently flippable.

Definition 2.5 ([6])**.** A graph $\mathrm{FG}(\mathcal{T}_P) = (V_P, E_P)$ is called the **flip graph** of triangulations of P if all vertices in V_P are triangulations of P and two vertices are connected by an edge if their corresponding triangulations differ by exactly one edge flip.

Figure 2 depicts a flip graph of a point set of size 6. Each triangulation of that point set is transformable into any other of the same point set. Hence, we can say that the depicted flip graph is connected. Additionally, we observe that at most four flips are needed in order to transform two arbitrary triangulations of that point set into each other.

Such observations are also interesting for any $\mathrm{FG}(\mathcal{T}_P)$. Theorem 2.2 reveals the results for the general case. First, we have to define the following terms:

Definition 2.6 ([11])**.** Given $T_1, T_2 \in \mathcal{T}_P$, the **flip distance** of T_1 and T_2, denoted by $\mathrm{fd}(T_1, T_2)$, is defined by the minimum number of edge flips that are needed to transform T_1 into T_2.

Similar to [6] we equivalently say that $\mathrm{fd}(T_1, T_2)$ is given by the length of the shortest path from T_1 to T_2 in $\mathrm{FG}(\mathcal{T}_P)$.

The flip distance of two triangulations can also be seen as a similarity measure in order to "compare" two triangulations. Together with the fact that the number of nodes in $\mathrm{FG}(\mathcal{T}_P)$ is exponential in n, we recognize that the efficient computation of the shortest path of two nodes in $\mathrm{FG}(\mathcal{T}_P)$ turns out to be a real challenge, see [9]. Thus, we are interested in an upper bound on the flip distance, i.e., the maximum length of the shortest path to get information on the maximum number of edge flips that are needed to transform two triangulations into each other. In other words, we are interested in the diameter of the flip graph:

Definition 2.7 ([6])**.** The **diameter** of a flip graph $\mathrm{FG}(\mathcal{T}_P)$ is defined by

$$\max\{\mathrm{fd}(T_1, T_2) | T_1, T_2 \in \mathcal{T}_P\}.$$

The flip graph depicted in Figure 2 has diameter 4, caused by T_4 and T_6. All other triangulations have a flip distance less than 4.

With those definitions in mind we are able to consider the general case of the connectivity and the diameter of a flip graph. Lawson [15] shows that two triangulations can always be transformed into each other via edge flips. The main idea is to find a triangulation T_R — the **reference** or **canonical triangulation** — into which any other triangulation of the same point set can be transformed. In [15], T_R is constructed in the following way: Assume that $(v_1, \ldots, v_n)$ are the vertices in P, lexicographically sorted by their coordinates. Let $\tau_3 = (v_1, v_2, v_3)$ be the initial triangulation. Successively, we add all vertices v_i, $i = 4, \ldots, n$ to the current triangulation τ_{i-1} by connecting v_i with each visible vertex. Note that v_i is called visible to v_j, $j = 0, \ldots, i-1$, if the edge (v_i, v_j) does not cross any other existing edge in τ_{i-1} in the interior. Triangulation T_2 in Figure 1 shows an example of T_R for that point set.

The flip sequence for the conversion of $T_1, T_2 \in \mathcal{T}_P$ can be found by first transforming T_1 into T_R, followed by applying the flip sequence found for the transformation of T_2 to T_R in reverse order on T_R. Lawson describes how to find a flip sequence from $T_1 \in \mathcal{T}_P$ to T_R, $T_1 \neq T_R$. The number of flips needed for that transformation is bounded by $\mathcal{O}(n^2)$. Hence, each triangulation can be converted to any other triangulation of the same point set with $\mathcal{O}(n^2)$ flips. Thus, $\mathrm{FG}(\mathcal{T}_P)$ is connected and its diameter is bounded by $\mathcal{O}(n^2)$.

In [12] Hurtado et al. present an example of two triangulations with flip distance $\Omega(n^2)$ — the so-called "double chain", see Figure 3. Basically, both triangulations consist of a rectangle Q (w.l.o.g. we assume that it is axis-aligned) and two concave chains C_1, C_2 with n vertices each. C_1 and C_2 lie

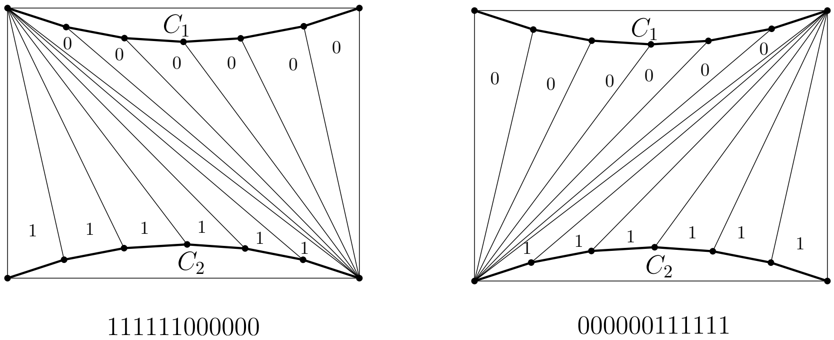


Figure 3: Two triangulations on the point set called the "double chain". The depicted triangulations have a flip distance of $(n-1)^2$ (for the triangulations between the upper and lower chain) + $\Theta(n)$ (for the convex region above and below the chain, respectively) with $|P| = 2n$.

in the interior of Q. Furthermore, C_1 connects the upper two corners of Q whereas C_2 connects the lower two corners of Q.

Hurtado et al. show that there is a bijection between ordered sequences of $2(n-1)$ zeros and ones and the triangulations between C_1 and C_2: Each triangulation between the chains consists of $2(n-1)$ triangles. Such a triangle has either two vertices in C_1 or two vertices in C_2. Dependent on that, we label each triangle with 0 (2 vertices in C_1) or 1 (2 vertices in C_2). Reading the assigned labels from left to right results in an ordered sequence of zeros and ones, that uniquely identifies a triangulation and vice versa. Thus, each triangulation can be encoded with such a sequence. We observe that an edge flip is only possible if the incident triangles have different labels. Additionally, it causes a label-switch of the corresponding triangles and consequently, in the assigned encoding. Hence, if we consider triangulations with encodings $1 \dots 10 \dots 0$ and $0 \dots 01 \dots 1$, we obviously need $(n-1)^2$ edge flips to transform them into each other.

Theorem 2.2 ([15], [12]). *The flip graph of P, $\mathrm{FG}(\mathcal{T}_P)$, is connected. Its diameter is bounded by $\mathcal{O}(n^2)$ and this bound is thight.*

However, for $T_1, T_2 \in \mathcal{T}_P$, $\mathcal{O}(n^2)$ is a rough estimate for the flip distance. Beside some exceptions for point sets in convex position, which will be discussed later in this section, there exist two approaches to specify that bound: In [11] Hanke et. al. present an upper bound on the flip-distance that is sensitive to the input triangulations. They show that the number of edge flips needed to transform T_1 into T_2 is at most the number of intersections between the edges of the triangulations. (See section 3.1). For point sets that include collinear points and do not have any empty pentagons, Eppstein [10]

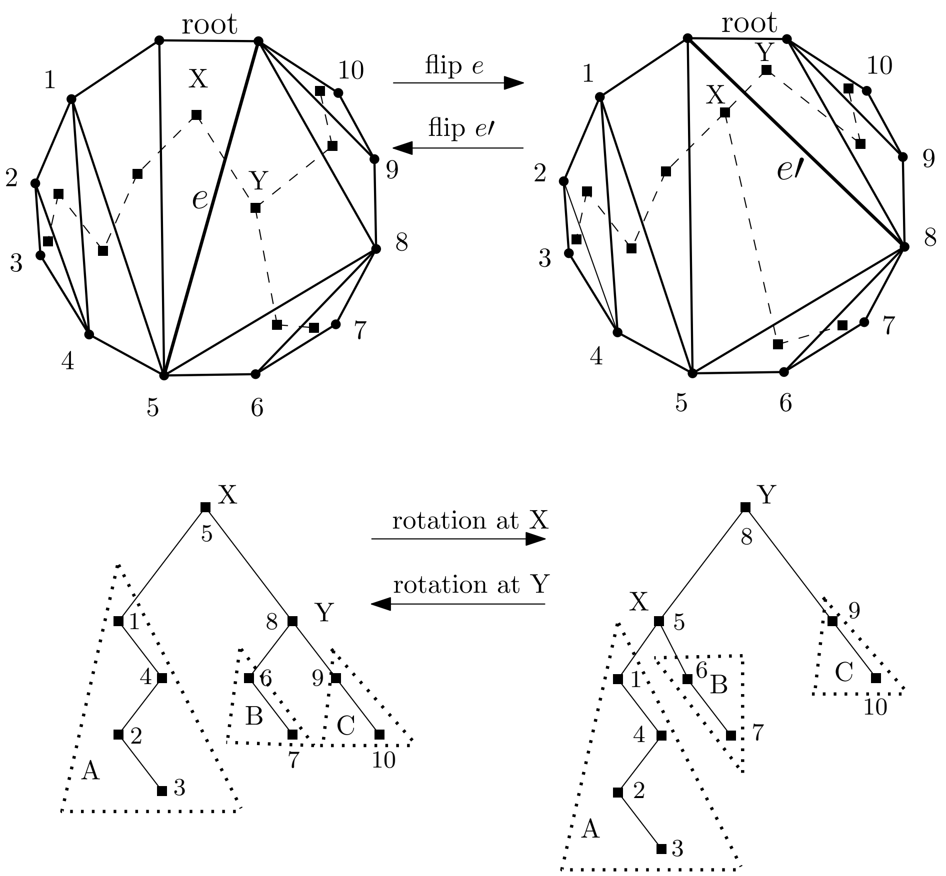


Figure 4: The bijection between triangulations of convex point sets and binary trees. An edge flip corresponds to a rotation of the binary tree. Only interior nodes of the binary trees are depicted.

shows how to compute the flip distance of two triangulations in $\mathcal{O}(n^2)$ time. Nevertheless, the efficient computation of the flip distance generally remains an open problem.

Point sets in convex position A fundamental basis for many achievements concerning triangulations of n points in convex position is the bijection to rooted binary trees with $(n-2)$ internal nodes, shown by Sleator et al. [18]. The edge flip in a triangulation corresponds to a rotation in a binary tree, see Figure 4. Hence, the flip graph $\mathrm{FG}(\mathcal{T}_n)$ is isomorphically to the rotation graph, which has a vertex for each binary tree with $(n-2)$ internal nodes and two vertices are connected by an edge if the corresponding binary trees differ by a single rotation ([18, Lemma 1]).

Each inner edge is a diagonal of a convex quadrilateral. Consequently,

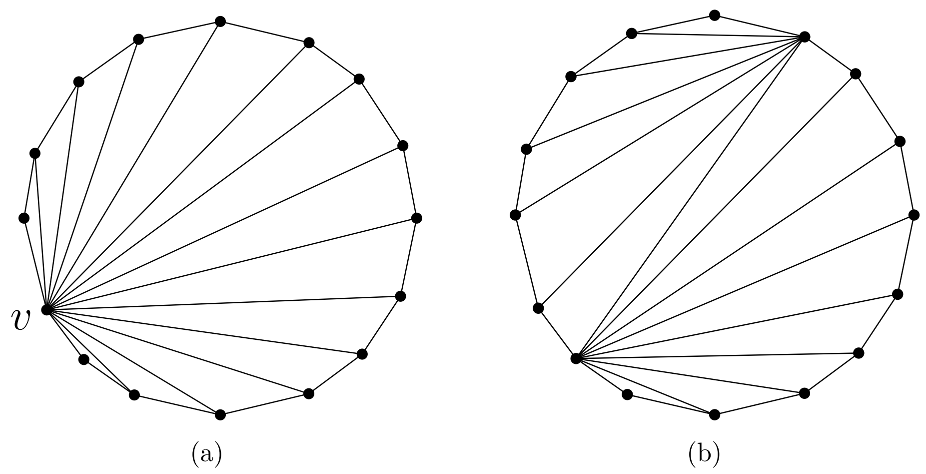


Figure 5: 5(a) A triangulation with a "fan" at v. 5(b) A triangulation with exactly two "fans".

each inner edge is flippable. The number of triangulations of a point set in convex position, P_c, is given by $C_{n-2} = \frac{1}{n-1}\binom{2(n-2)}{n-2}$, the $(n-2)$nd Catalan number. Thus, we still have a flip graph for which the number of vertices is exponential in the size of the underlying point set. However, the diameter of the flip graph decreases: According to Culik and Wood [13, Theorem 2.1], the rotation distance of two binary trees with $(n-2)$ internal nodes is less or equal to $(2(n-2)-2) = (2n-6)$. (The rotation distance is the number of rotations needed to transform two binary trees of the same size into each other.) Consequently, the diameter of the rotation graph is in $\Theta(n)$. Together with the aforementioned isomorphism we can formulate the following theorem:

Theorem 2.3 (Theorem 2.1 in [13], Lemma 1 in [18])**.** *The diameter of the flip graph of a convex point set in the plane is given by* $\Theta(n)$.

A simple proof can be found in [6]: Each triangulation in $\mathcal{T}_n$ has $(n-3)$ diagonals, see Proposition 2.1. Thus, we need at most $(n-3)$ edge flips to transform a triangulation in $\mathcal{T}_n$ into a triangulation consisting of exactly one "fan", i.e., a triangulation where all inner edges have one common vertex, see Figure 5(a). That triangulation acts as canonical triangulation. Consequently, at most $2(n-3)$ edge flips are needed for the transformation of two triangulations in $\mathcal{T}_n$.

A tighter bound for $n > 12$ can be found in [18, Lemma 2]. Sleator et al. show that the diameter of a flip graph is at most $2n-10$. Additionally, there exist many polynomial time algorithms calculating good lower and upper

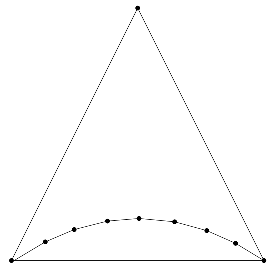


Figure 6: Each triangulation of that point set has a maximum vertex degree of $(n-1)$ [14].

bounds for the flip distance of two triangulations of P_c, see for example [4]. The algorithm is based on the fact that the calculation of the flip distance of T_1 and T_2 can be subdivided, if T_1 and T_2 share a diagonal. It recursively computes upper and lower bounds on the flip distance in $\mathcal{O}(n^3)$ time. We will show more results of that paper in section 4.

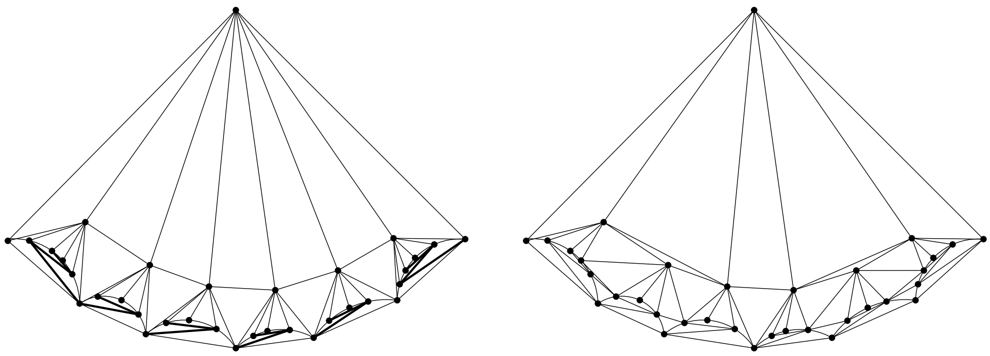
Moreover, Lucas [16] shows how to compute the shortest flip sequence of a restricted kind of triangulations in $\mathcal{O}(n^2)$ time. The presented flip sequence transforms triangulations with no inner triangles into triangulations containing exactly one or two fans, see Figure 5.

However, in general the restriction to convex point sets does not imply a known efficient calculation of the exact flip distance between two triangulations. In section 4.2 we show the flip distance of another restricted kind of triangulations — so-called zigzag triangulations — under certain restrictions on the structure of the edge-positions.

2.2 Bounded degree triangulations

In this section we consider triangulations with bounded vertex degree. We denote by $\mathcal{T}_{P,k}$ the set of triangulations of P where each triangulation has a maximum vertex degree of k. Again, we write $\mathcal{T}_{n,k}$ instead of $\mathcal{T}_{P,k}$ if we know that P is convex.

In [3], Aichholzer et al. consider the connectivity of $\mathrm{FG}(\mathcal{T}_{P,k})$ — a subgraph of $\mathrm{FG}(\mathcal{T}_P)$ — that only contains triangulations with a maximum vertex degree k. Since there exist point sets for which each triangulation has a maximum vertex degree of $(n-1)$ (see Figure 6 for an example), they point out that $\mathrm{FG}(\mathcal{T}_{P,k})$ can be empty for any $k = (n-1)$. Nevertheless, it is worth discussing the question if there exists a flip sequence transforming two arbitrary triangulations with maximum vertex degree k into each other without exceeding that degree bound. Aichholzer et al. show that for each k there exists a point set with two triangulations that can not be transformed into each other without exceeding the vertex degree k. Figure 7(a) and 7(b) give an example for $k = 8$. That example can be modified for any $k \geq 7$ with an underlying point set of size $n = \Theta(k^2)$.



(a) (b)

Figure 7: 7(a) For $k = 8$, only the bold edges are flippable. Thus, that triangulation cannot be transformed into the triangulation depicted in 7(b) without exceeding the vertex degree k. [3]

However, for two given triangulations $T_1, T_2 \in \mathcal{T}_{P,k}$, we are interested in the measure of the flip distance assuming that they are transformable into each other without exceeding the vertex degree bound. In section 3.2 we discuss some estimates. Unfortunately, we could not find an input sensitive upper bound as given in [11].

For point sets in convex position, Aichholzer et al. [3] show that $\mathrm{FG}(\mathcal{T}_{n,k})$ is connected for $k > 6$. The main idea of the transformation for $T_1, T_2 \in \mathcal{T}_{n,k}$ is to convert both triangulations into the "left-most zigzag triangulation" without exceeding the vertex degree bound. For details on zigzag triangulations see section 4. The required number of flips for the conversion is bounded by $\mathcal{O}(n^2)$, i.e., the flip diameter of $\mathrm{FG}(\mathcal{T}_{n,k})$ is in $\mathcal{O}(n^2)$ for $k > 6$.

2.3 Pseudo-triangulations

In the same way as in section 2.1, we start with some preliminary definitions in order to be able to define a pseudo-triangulation. Similar definitions can be found in the literature, see for example [17] and [19].

Definition 2.8. A (simple) **polygon** is a connected subset of the plane, delimited by $n \geq 3$ (non-crossing) line segments such that every end point of a line segment is shared by exactly two line segments. The end points are called vertices and the line segments edges. A polygon with n vertices is called an **n-gon**.

For $n = 3$, we have a triangle, and a 4-gon is commonly known as quadrilateral. The boundary of a polygon is given by the set of edges and the sequence of vertices of the polygon. The interior of the polygon is the finite region delimited by the boundary. It is worth noting that according to that definition, a polygon is the union of the boundary and the interior. Contrary to that, there are many definitions in the literature that reduce a polygon to the boundary.

Since pseudo-triangulations are a generalization of a triangulation, we need to define the equivalent of a triangle. That equivalent is a pseudo-triangle, which is a special case of a pseudo-k-gon.

Definition 2.9. A **pseudo-k-gon** is a simple polygon with exactly k convex vertices, i.e., it has exactly k internal angles less that π. We call these vertices **corners**. The sequence of edges and vertices connecting two consecutive corners of a pseudo-k-gon is called **side chain**. The vertices that are part of a side chain are reflex vertices, i.e., their internal angle is greater than π. A pseudo 3-gon is called **pseudo-triangle**. For any edge e of a side chain s in a pseudo-triangle, we refer to the corner that is not incident to s as **opposite corner** of e (or s). For $k = 4$, we also say **pseudo-quadrilateral**.

Now, we are able to define a pseudo-triangulation analogously to Definition 2.3 in section 2.1:

Definition 2.10. A **pseudo-triangulation** is a partition of the convex hull of P into interior-disjoint pseudo-triangles, whose vertex set is exactly P. An edge is called inner edge if it is not a convex hull edge.

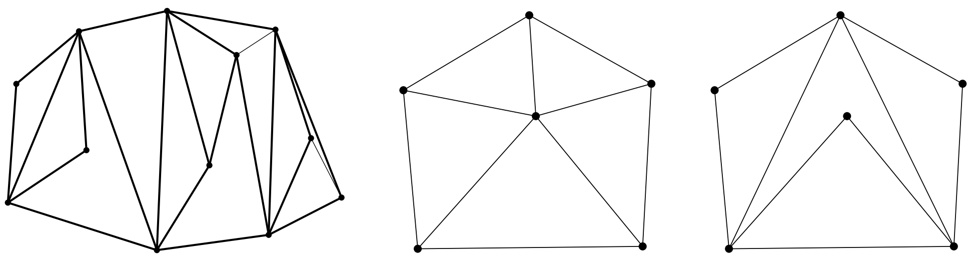
We denote the set of all pseudo-triangulations of P by $\mathcal{PT}_P$. Figure 8 depicts examples of pseudo-triangulations. They reflect a fundamental difference to triangulations: the number of edges of two pseudo-triangulations with a common underlying point set varies. (Recall that each triangulation of the same point set has the same number of edges.) Nevertheless, we know that each triangulation is a pseudo-triangulation because each triangle is a pseudo-triangle. Thus, we have $\mathcal{T}_P \subseteq \mathcal{PT}_P$. Furthermore, the sets even coincide if and only if P is in convex position.

In addition to $\mathcal{T}_P$, we define the following subsets of $\mathcal{PT}_P$:

Definition 2.11 ([14])**.** A pseudo-triangulation $T \in \mathcal{PT}_P$ is a **minimal pseudo-triangulation**, if there is no edge that can be removed such that T

remains a pseudo-triangulation. Contrary to that, we say that T is a **minimum pseudo-triangulation** if no other pseudo-triangulation on P has a smaller number of edges.

According to [14], each minimum pseudo-triangulation also has to be a minimal one, but not necessarily vice-versa. Figure 8(b) and Figure 8(c) demonstrate an example: Neither T_1 nor T_2 result in a pseudo-triangulation if one edge is removed. Thus, according to Definition 2.11, they are both minimal. Considering the number of edges of both depicted pseudo-triangulations, T_1 has five inner edges, whereas T_2 only has four. Additionally, the underlying point set does not have a pseudo-triangulation with a smaller number of inner edges than four. Consequently, T_1 is *not* a minimum, but only a minimal pseudo-triangulation.



(a) (b) T_1 (c) T_2

Figure 8: 8(a) A pseudo-triangulation that contains a minimum pseudo-triangulation (bold edges). 8(b) A minimal pseudo-triangulation, that does not contain a minimum pseudo-triangulation. 8(c) A minimum pseudo-triangulation of point set 8(b). Figure 8(b) and Figure 8(c) are adapted from [14].

Definition 2.12. Given $T \in \mathcal{PT}_P$ and $v \in P$, assume that $E_v = (e_1, e_2, \dots, e_l, e_1)$ is the sequence of all edges incident to v in T, sorted in a cyclic order around v. Then v is called pointed, if there is a pair of consecutive edges in E_v that spans an angle larger than π. Otherwise, v is non-pointed. Furthermore, we say that T is a **pointed pseudo-triangulation**, if each vertex in P is pointed.

We denote the set of pointed pseudo-triangulations of P by $\mathcal{PPT}_P$. Figure 8(a) contains a pointed pseudo-triangulation, indicated by the bold edges. The pseudo-triangulation in Figure 8(c) is pointed, too. Hence, we observe that a pseudo-triangulation can be pointed and minimum at the same time. In fact, we know that each pointed pseudo-triangulation is a minimum pseudo-triangulation and vice versa, formulated in Theorem 2.4.

Theorem 2.4 (Theorem 2.7 in [17], Theorem 2.3 in [19]). *$T \in \mathcal{PT}_P$ is a minimum pseudo-triangulation, if and only if T is pointed.*

Hence, we know that the minimum number of edges is reached if each vertex is pointed. Additionally, it would be interesting to specify the number of edges as a function of n, not only for pointed pseudo-triangulations, but also for pseudo-triangulations. Note that since the number of edges of different pseudo-triangulations of the same point set varies, we cannot reduce m to be a function of n in the general case. The following considerations that can be found in [17] provide the desired information:

Each edge in T is incident to two vertices, i.e., each edge increases the degree of its incident vertices by one. Hence, the total sum of all vertex degrees in T is given by $2m$. Since the degree of a vertex equals the number of incident angles of that vertex, we have $2m$ angles in T. Angles can be categorized into convex and reflex ones. The latter are incident to a pointed vertices. Thus, if the number of non-pointed vertices is given by n_X, we obviously have $n - n_X$ pointed vertices. Furthermore, each pseudo-triangle in T has exactly three convex angles (incident to the corners). Consequently, we have $n_X = 3t$, where t is the number of pseudo-triangles in T. Accordingly, the number of angles is given by

$$2m = 3t + n - n_X. \tag{2.1}$$

Additionally, we know that pseudo-triangulations are planar graphs. Thus, we can use Euler's formula (see section 2.1) in order to eliminate t in equation (2.1) and get

$$2m = 3(m - n + 1) + 3 + n - n_X \Leftrightarrow m = 2n - 3 + n_X$$

In the same way, we can replace m in equation (2.1) and get

$$2(n - t + 1) = 3t + n - n_X \Leftrightarrow t = n - 2 + n_X$$

We formulate those observations in the following proposition:

Proposition 2.5 ([17]). *Given $T \in \mathcal{PT}_P$. Then T has the following properties:*

1. *$m = 2n - 3 + n_X$, where n_X is the number of non-pointed vertices in T.*
2. *$m = 2n - 3$, for T pointed.*
3. *$t = n - 2$, for T pointed, where t is the number of pseudo-triangles in T.*

Analog to section 2.1, we want to define an edge flip in a pseudo-triangulation. Recall that an edge flip in a triangulation is defined by exchanging the diagonals in a convex quadrilateral. That implies that not each edge is flippable in a triangulation. For the edge flip in a pseudo-triangulation we do not have such a restriction. In fact, we will see that each inner edge is flippable even if its incident pseudo-triangles form another pseudo-triangle.

Speaking of diagonals in quadrilaterals, we realize that we have not yet defined a diagonal of a pseudo-quadrilateral. Thus, the generalization of the edge flip requires some preliminary definitions:

Definition 2.13 ([17], [19])**.** A line segment l ending in a vertex $v \in P$ of a pseudo k-gon R is **tangent** to a side-chain s of R if either

- v is a corner of R, s is a side chain incident to v and l lies in the convex angle of v, or
- v is part of s and the two incident edges of v lie on the same side of the supporting line of l.

l is called **bitangent** to two side chains of R if each endpoint of l is tangent to one side chain. A **geodesic path** between two vertices of R is the shortest path inside R connecting these vertices. A **diagonal** of R is a line segment that is

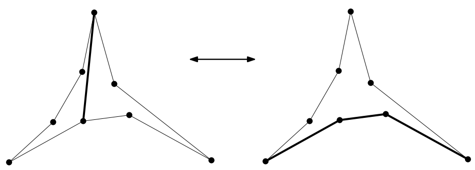
- part of a geodesic path between two corners of R and
- bitangent to two side chains of R.

The bold edges in the right pseudo-triangle of Figure 9(a) build a geodesic path between the lower two corners of the pseudo-triangle. Figure 9(b) depicts two diagonals (bold edges) of a pseudo-quadrilateral. The diagonals are part of a geodesic path between the upper left and the lower right corner and the lower left and the upper right corner, respectively. Note that both diagonals are bitangents in the quadrilateral. Contrary to that, the bold edge in the left pseudo-triangulation of Figure 9(a) is not bitangent to its incident pseudo-triangles and equivalently not part of a geodesic path between two corners.

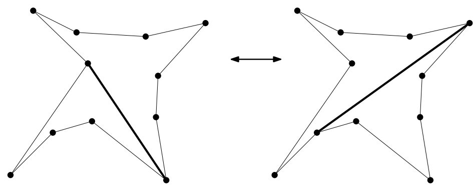
With these terms in mind, we are ready to define an edge flip in a pseudo-triangulation:

Definition 2.14 ([6])**.** Given $T \in \mathcal{PT}_P$ and $e \in T$. Let t_1 and t_2 be the incident pseudo-triangles of e and v_1 and v_2 be the opposite corners of e in t_1 and t_2. Then the removal of e followed by the insertion of the edges of the geodesic path between v_1 and v_2 that are not yet in T or the inversion of the removal operation, is called an **edge flip** in T.

Figure 9 depicts different scenarios of an edge flip. Contrary to the edge flip in Figure 9(b), the edge flip in Figure 9(a) has an influence on the number of edges. Thus, we classify the edge flips into three different types:



(a)



(b)

Figure 9: 9(a) An insertion - deletion flip. 9(b) A diagonal flip.

Definition 2.15 ([17]). Given $T \in \mathcal{PT}_P$. Then the three different types of the edge flip operation in T are defined as follows:

- Assume that T' is the result of removing an existing inner edge e in T.
    1. Then the removal of e is called **deletion**, or **edge-removal flip**, iff $T' \in \mathcal{PT}_P$.
    2. Otherwise, the insertion of a new unique edge $e' \neq e$ leads to a pseudo-triangulation. We call the exchange of e with e' a **diagonal flip**.
- Let T' be the result of inserting a new edge e into T. Then the insertion of e is called **insertion flip**, iff $T' \in \mathcal{PT}_P$.

According to [17] and [19], the diagonal flip is uniquely defined: On the one hand, the removal of an edge e either results in a quadrilateral or in a degenerate quadrilateral, see Figure 10. The degenerate case arises if one incident vertex v of e has degree 2 before the removal. Thus, the deletion of e causes an interior angle of 2π. Nevertheless, it can be seen as a pseudo-quadrilateral: Let $\tilde{e} = (v, \tilde{v})$ be the remaining incident edge of v. Then $\tilde{v}$ can be seen as fourth corner with $\tilde{e}$ as part of an incident side chain. On the other hand, they show that each pseudo-quadrilateral has exactly two diagonals that separate the pseudo-quadrilateral into two pseudo-triangles.

Since the result of an edge flip in a pseudo-triangulation is again a pseudo-triangulation, that operation is closed in $\mathcal{PT}_P$. Furthermore, we know that each edge flip in a pointed pseudo-triangulation is a diagonal flip, shown in [7]. Hence, the diagonal flip is a closed operation in $\mathcal{PPT}_P$.

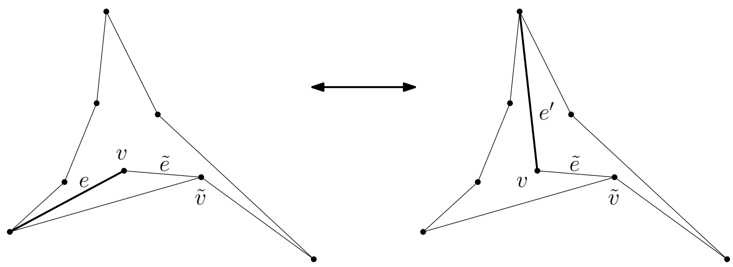


Figure 10: $T_1, T_2 \in \mathcal{T}_P$, $P = \{v_1, \ldots, v_5\}$. Flipping e in T_1 to e' results in T_2.

The definition of a flip graph, its diameter and the flip distance is analog to the definition for triangulations in section 2.1. Thus, the flip graph of all pseudo-triangulations of P is denoted by $\mathrm{FG}(\mathcal{PT}_P)$. According to [17], $\mathrm{FG}(\mathcal{PT}_P)$ as well as its subgraph $\mathrm{FG}(\mathcal{PPT}_P)$ is connected. Furthermore, the diameter of both flip graphs is given by $\mathcal{O}(n \log n)$, shown in [2] for pseudo-triangulations and in [5] for pointed pseudo-triangulations.

All in all, the efficient calculation of the exact flip distance between two pseudo-triangulations remains an open problem. Contrary to triangulations, not even an input sensitive lower nor upper bound on the flip distance is known.

In section 5.1 we present implemented heuristics for the calculation of the flip distance of pointed pseudo-triangulations.

2.4 Bounded degree pointed pseudo-triangulations

In this section, we restrict our considerations to pointed pseudo-triangulations of P with maximum vertex degree k, denoted by $\mathcal{PPT}_{P,k}$. In section 2.2 we saw that for particular point sets each triangulation has a maximum vertex degree $(n-1)$.

Contrary to that, Kettner et al. [14] show that each point set has a pointed pseudo-triangulation with maximum vertex degree five. Furthermore, they prove that this bound is tight by providing an example of a point set for which each pointed pseudo-triangulation has a vertex degree of at least five. Figure 11 depicts that point set. We can observe that each pointed pseudo-triangulation consists of one “real” pseudo-triangle t_p incident to the vertex in the interior of $\mathrm{CH}(P)$. All other pseudo-triangles are triangles, since all other vertices lie on the boundary of the convex hull and have interior angles less that π. Obviously, the corners of t_p lie on the boundary of the convex hull. v_c is part of the side chain s of t_p and has vertex degree 2. W.l.o.g. we assume that (v_t, v_c) is an edge of s. We can add two more edges incident to v_c

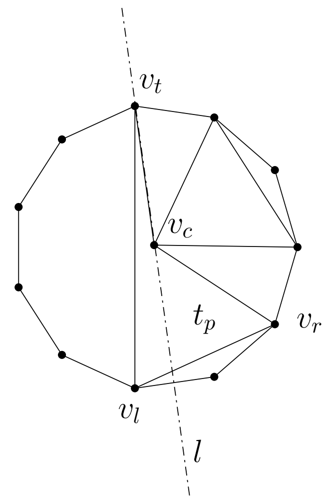


Figure 11: Each pointed pseudo-triangulation of that point set has a vertex degree of at least 5.

or, equivalently, three triangles, each of which has one corner at v_c, without exceeding the vertex degree 4. That implies that at most 3 vertices lie on the boundary of CH(P) between the two corners incident to s. The line l, defined by vertices v_t and v_c, separates $P \setminus \{v_t, v_c\}$ such that on each side of l there are 5 vertices. Consequently, none of the edges of t_p is a convex hull edge. Thus, each corner has vertex degree 4. W.l.o.g. suppose that the 3 vertices between the two corners incident to s lie on the right side of l. Hence, one corner of t_p, v_r, lies on the right side of l, too. Moreover, t_p only consists of inner edges because the opposite corner of s, v_l, has to be on the left side of l. Hence, there are still 4 vertices on the left side, that are not incident to a triangle. In order to complete the pseudo-triangulation, we need to connect one of these vertices with either v_t or v_l. That implies a vertex degree of five.

Accordingly, it would be interesting to study the connectivity of FG($\mathcal{PPT}_{P,k}$) for a $k \geq 5$. Since the set of triangulations equals the set of pseudo-triangulations for P convex, the results presented in section 2.1 for $\mathcal{T}_{n,k}$ hold here as well.

For point sets in general position, Aichholzer et al. [3] show an example of a pointed pseudo-triangulation in which no edge can be flipped without exceeding the vertex degree of 9. Figure 12 depicts that example. Additionally, there exist other pointed pseudo-triangulations of the same point set in $\mathcal{PPT}_{P,9}$. Hence, we can say that FG($\mathcal{PPT}_{P,k}$) can be disconnected for any $k \leq 9$.

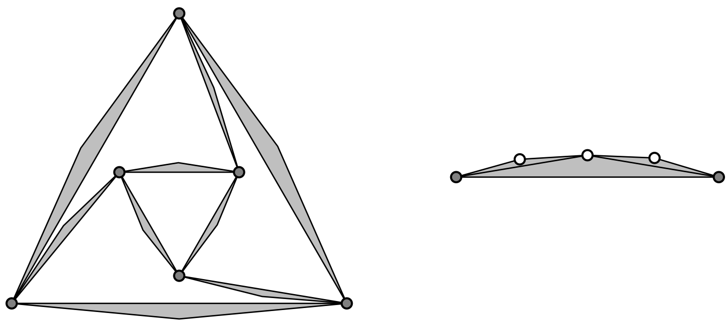


Figure 12: A pointed pseudo-triangulation in which no edge can be flipped without exceeding the vertex degree bound of 9 [3].The shaded parts in the left drawing correspond to the figure depicted on the right side. The dark vertices indicate a vertex degree of 9.

3 Input sensitive upper bounds on the flip distance

In this section we consider input sensitive upper bounds on the flip distance of two given (degree bounded) triangulations on P. On the one hand, we present the main result of Hanke et al. [11], who introduce such a bound for triangulations. On the other hand, we will check the validity of that estimate for the degree bounded setting and discuss several approaches for a similar result for bounded degree triangulations.

In the following, we abbreviate the number of intersections of edge $e \in T_1$ with the edges in T_2 with $\#(e, T_2)$. Furthermore, we denote the number of intersections between the edges of T_1 and T_2 by $\#(T_1, T_2)$. We say that two edges intersect if they intersect in their interior. Note that the expression "$e \in T$" alternatively stands for "T contains e".

3.1 Results for triangulations

As already mentioned in section 2.1, we know that two given triangulations $T_1, T_2 \in \mathcal{T}_P$ can be transformed into each other with at most $\mathcal{O}(n^2)$ edge flips, see Theorem 2.2. Even though there exist pairs of triangulations for which the flip distance is quadratic in the number of points, that estimate is generally not tight. (Just assume that T_1 and T_2 only differ by a constant number of edge flips.)

In [11] Hanke et al. introduce an input sensitive upper bound on the flip distance of T_1 and T_2. That bound depends on the number of intersections between the edges of T_1 and T_2.

Theorem 3.1 (Theorem 1 in [11])**.** *Given $T_1, T_2 \in \mathcal{T}_P$. Then*

$$\mathrm{fd}(T_1, T_2) \leq \#(T_1, T_2) < (m_i)^2, \tag{3.1}$$

where m_i is the number of inner edges of a triangulation in $\mathcal{T}_P$.

The fundamental idea behind the proof of the theorem is formulated in the following lemma:

Lemma 3.2 ([11])**.** *Given $T_1, T_2 \in \mathcal{T}_P$. Then there exists a convex quadrilateral in T_1 given by $v_1, \ldots, v_4 \in P$, with diagonal $e_1 = (v_1, v_3)$ and the property*

$$\#(e_1, T_2) = max\{\#(e, T_2) | e \in T_1\}$$

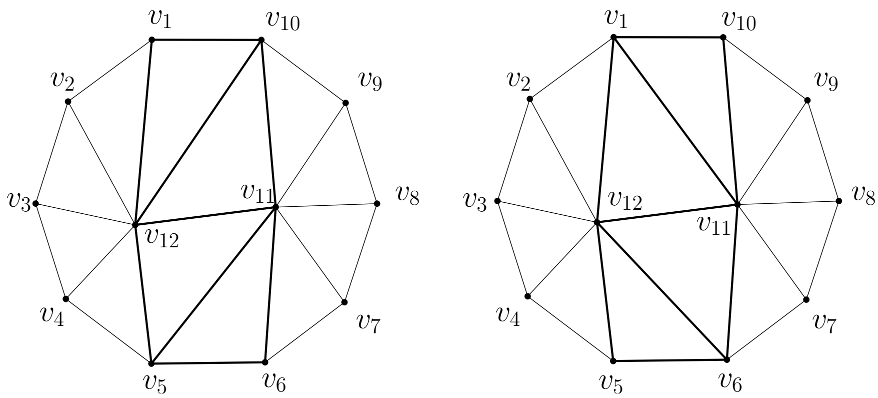
such that the flip of e_1 to (v_2, v_4) reduces the number of intersections.

Thus, we know that as long as $\#(T_1, T_2) > 0$ and equivalently, $T_1 \neq T_2$, there always exists an edge in T_1 that can be flipped such that the number of intersections is reduced by at least one. Therefore, the arising flip sequence, which transforms T_1 into T_2, is bounded by $\#(T_1, T_2)$.

3.2 Adding vertex degree bounds

Generally speaking, we know according to section 2.2 that for each value for k there exists a point set for which the subgraph $FG(\mathcal{T}_{P,k})$ of $FG(\mathcal{T}_P)$, consisting of triangulations with maximum vertex degree k, is not connected. However, that does not mean that $\mathrm{FG}(\mathcal{T}_{P,k})$ is not connected for any point set. Hence, we could ask for an upper bound on the flip distance of two particular triangulations $T_1, T_2 \in \mathcal{T}_{P,k}$, assuming that there is a path in $FG(\mathcal{T}_{P,k})$ between them.

Theorem 3.1 remains valid, as long as the edges with the property of Lemma 3.2 are flippable. Thus, Theorem 3.1 does not hold in general for the degree bounded setting. Figure 13 shows an example. Edges (v_{10}, v_{12}) and $(v_5, v_{11}) \in T_1$ and (v_1, v_{11}) and $(v_6, v_{12}) \in T_2$ satisfy the conditions of Lemma 3.2: they are diagonals of convex quadrilaterals, they have the maximum number of intersections, which is one in this case, and each flip would reduce the number of intersections by one. However, none of the edges is flippable without exceeding the degree bound of 7. It is necessary to flip another incident edge of vertex v_{11} or v_{12} first, which even increases the number of intersections temporarily, before one of the diagonals can be flipped. Thus the flip distance is given by 4.



(a) $T_1 \in \mathcal{T}_P$ (b) $T_2 \in \mathcal{T}_P$

Figure 13: Two triangulations with degree restriction $k = 7$, whose edges with maximal number of intersections cannot be flipped due to degree restriction.

Hence, the input sensitive upper bound on the flip distance of two bounded degree triangulations is not known. Several estimates will be discussed now.

We define $T \cup e$ as

- the result of the insertion of edge e into a triangulation T that does not yet contain e, or
- T if e already is an edge in T.

Consequently, $T_1 \cup T_2$ stands for the result of inserting of each edge of triangulation T_1 that does not exist in T_2 into triangulation T_2. We denote by $\mathcal{V}_k(T_1 \cup T_2) \subset P$ the set of vertices that exceed the vertex degree k in the union of two triangulations T_1, T_2. Additionally, we define $\mathcal{V}_k(e, T)$ as the set of vertices that exceed the vertex degree k in $T \cup e$. We write $\mathcal{V}_{e,k}(T_1 \cup T_2)$ for the set of vertices that are incident to edge e and exceed k in $T_1 \cup T_2$.

An intuitive approach to estimate the overhead of needed flips in order to transform two triangulations into each other in the degree bounded setting is given by the following idea: For each edge e in T_1, we add to the number of intersections with T_2 the number of vertices in T_2 that exceed the vertex degree in $e \cup T_2$. In fact, that estimate is correct for the triangulations depicted in Figure 13. Since T_1 and T_2 only differ in two edges, the value of the estimate depends on

$$\#((v_{10}, v_{12}), T_2) = \#((v_5, v_{11}), T_2) = 1$$

and

$$|\mathcal{V}_k((v_{10}, v_{12}), T_2)| = |\mathcal{V}_k((v_5, v_{11}), T_2)| = 1.$$

Thus, we would get a tight upper bound of 4 for the triangulations shown in Figure 13. (Recall that the flip distance is 4, too.)

Contrary to that, the measure fails for the triangulations depicted in Figure 14: For $k = 5$, their flip distance is given by 14, whereas the value of the estimate is 12. Moreover, we can observe that inserting edges from T_2 into T_1 would increase the value of the estimate to 14. Since we are searching for an upper bound, we have to take both values into consideration. Thus, we have

Measure 1 (**not** valid in general)**.**

$$\operatorname{fd}(T_1, T_2) \leq \max\Big(\sum_{e \in T_1}\big(\#(e, T_2) + |\mathcal{V}_k(e, T_2)|\big), \sum_{e \in T_2}\big(\#(e, T_1) + |\mathcal{V}_k(e, T_1)|\big)\Big)$$

Although Measure 1 is correct for the flip distance of the triangulations in Figure 14, it fails for the triangulations depicted in Figure 15 and 16 (the values of Measure 1 are 4 in both cases, whereas the flip distances of the triangulations are 14 and 6, respectively). Another disadvantage of Measure 1 is the neglect of additional edge flips, needed for the case that we have vertices with the following properties:

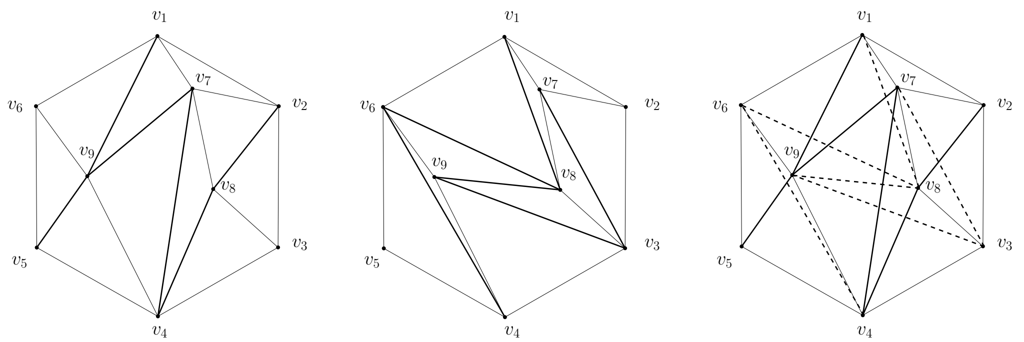
- $v \in P$ has a high vertex degree in T_1 as well as in T_2 but still less than k, and
- the number of common incident edges of v in T_1 and T_2 is small.

We can handle that problem by considering the vertex degree in $T_1 \cup T_2$. Consequently, we try to improve the estimate by the following measure:

Measure 2 (**not** valid in general)**.**

$$\mathrm{fd}(T_1, T_2) \leq \#(T_1, T_2) + |\mathcal{V}_k(T_1 \cup T_2)|$$

Thus, the value of Measure 2 depends on the number of intersections between the edges in T_1 and T_2 and the number of vertices that exceed the vertex degree k in $T_1 \cup T_2$. It is a correct measure of the flip distance for the triangulations depicted in Figure 13 (=2+2) and 14 (=10+4) and remains incorrect for Figure 15 (=2+2) and 16 (=2+2).



(a) T_1 (b) T_2 (c) $T_1 \cup T_2$

Figure 14: For $k = 5$, the flip distance of T_1 and T_2 is given by 14. Furthermore, we have $\#(T_1, T_2) = 10$ and $\mathcal{V}_k(T_1 \cup T_2) = \{v_4, v_7, v_8, v_9\}$. In 14(a) and 14(b) the bold edges only occur in T_1 or T_2. 14(c) Bold edges belong to T_1, dashed ones to T_2.

Though Measure 2 takes the vertex degree of $T_1 \cup T_2$ into consideration, only one additional edge flip for each degree exceeding vertex is included. Of course, it would be more reasonable to consider the number of edges that only occur in one of both triangulations and are incident to a vertex $v \in \mathcal{V}_k(T_1 \cup T_2)$. Measure 3 takes that into account:

Measure 3 (**not** valid in general)**.**

$$\mathrm{fd}(T_1, T_2) \leq \max\Big(\sum_{e \in T_1 \wedge e \notin T_2} \big(\#(e, T_2) + |\mathcal{V}_{e,k}(T_1 \cup T_2)|\big), \sum_{e \in T_2 \wedge e \notin T_1} \big(\#(e, T_1) + |\mathcal{V}_{e,k}(T_1 \cup T_2)|\big)\Big)$$

For the flip distance of the triangulations depicted in Figure 13 and 14, Measure 3 gives a correct upper bound with values (2 + 2) and (10 + 9). Nevertheless, the resulting values for the triangulations in Figure 15 and 16 are still underestimates (given by 4 in both cases).

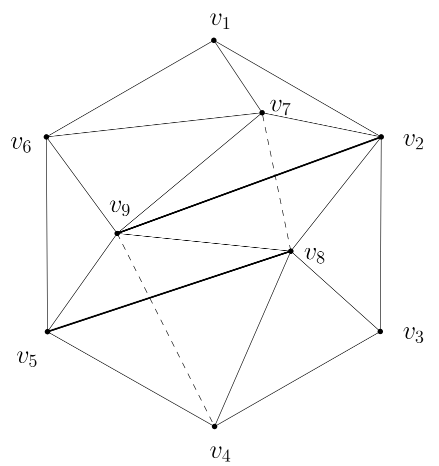


Figure 15: Dashed edges only occur in T_1. The bold edges belong to T_2. For $k = 5$, we have $\#(T_1, T_2) = 2$, $\mathcal{V}_k(T_1 \cup T_2) = \{v_8, v_9\}$ and $\mathrm{fd}(T_1, T_2) = 14$.

Extending Measure 3 by taking into consideration all edges in T_1 and T_2 that are incident to vertices in $\mathcal{V}_k(T_1 \cup T_2)$ results in a correct estimate for the flip distance of the triangulations shown in Figure 15 and 16. Thus we have

Measure 4 (**not** valid in general)**.**

$$\mathrm{fd}(T_1, T_2) \leq \#(T_1, T_2) + \sum_{v \in \mathcal{V}_k(T_1 \cup T_2)} \deg_{(T_1 \cup T_2)}(v)$$

Assuming that the estimate is valid for any two triangulations would imply that the overhead of needed edge flips in order to transform two triangulations in the degree bounded setting depends on the number of edges in T_1

and T_2 that are incident to the vertices that exceed the degree restriction in the union of both triangulations. Generally, that does not seem reasonable. Furthermore, in section 3.3 we can show how to construct two triangulations for which that estimate is wrong.

3.3 Construction of a counterexample

Figure 13 shows an example with flip distance 4, although the number of intersections of T_1 and T_2 is 2. Here, two additional flips are necessary, because the vertices v_{11} and v_{12} have — beside the edges of the two quadrilaterals $Q_1 = (v_1, v_{12}, v_{11}, v_{10})$ and $Q_2 = (v_5, v_6, v_{11}, v_{12})$ — $(k-4)$ additional incident edges. These two flips have to take place in the outer region R_1, defined by

$$R_1 = \{(v_2, v_{12}), \dots, (v_4, v_{12}), (v_7, v_{11}), \dots, (v_9, v_{11})\},$$

i.e., inner edges, which are not part of Q_1 or Q_2. Thus, the corresponding flip sequence starts with an edge in the outer region, followed by the two diagonals of Q_1 and Q_2 and ends by flipping the edge in the outer region back to its original position. (Starting the transformation by flipping edge (v_{11}, v_{12}) would lead to a flip sequence of length 7. Hence, we do not consider that case.)

Extending the point set of Figure 13 leads to the triangulation shown in Figure 16. A second region

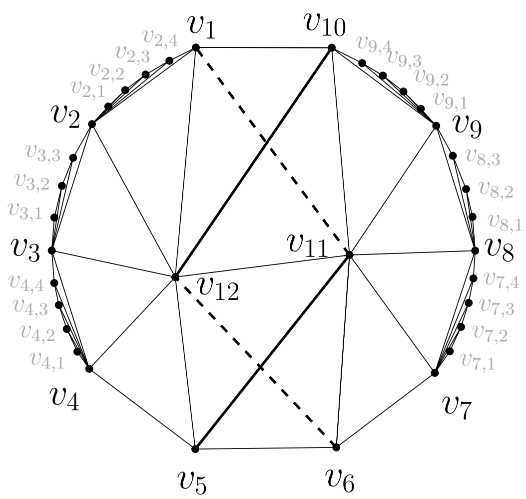


Figure 16: Extension of the pointset shown in Figure 13 with $k = 5$. T_1 has the solid diagonals, T_2 the dashed ones.

$$R_2 = \{(v_2, v_{2,1}), \dots, (v_4, v_{4,4}), (v_7, v_{7,1}), \dots, (v_9, v_{9,4})\}$$

arises. None of the edges in R_1 are flippable anymore. The set of flippable edges equals

$$E_f = \{(v_{11}, v_{12})\} \cup \{\Big(\bigcup_{i \in \{3,8\}} (v_i, v_{i,2}) \Big) \cup \Big(\bigcup_{\substack{i \in \{2,4,7,9\} \\ j \in \{2,3\}}} (v_i, v_{i,j}) \Big) \cup \Big(\bigcup_{i \in \{2,9\}} (v_i, v_{i,4}) \Big).$$

W.l.o.g. we want to flip edge (v_5, v_{11}) to (v_6, v_{12}) in T_1. Again, it is necessary to decrease the vertex degree of v_{12} first. Because none of the incident edges (except (v_{11}, v_{12})) is flippable, we have to start with an edge flip in R_2, say edge $(v_3, v_{3,2}) \in E_f$. Consequently, edge (v_2, v_{12}) is flippable in R_1 and finally, both diagonals (v_5, v_{11}) and $(v_{10}, v_{12}) \in T_1$ can be flipped, too. After two additional edge flips for the edges in R_1, R_2 that have to turn back to their original position, T_1 is converted into T_2 with 6 edge flips.

That flip sequence corresponds to the same pattern as the flip sequence of the triangulations depicted in Figure 13: The first edge that is flipped is part of the outer region R_2. The next one belongs to R_1, which is the region adjacent to R_2. Afterwards, the two diagonals of Q_1 and Q_2 are flipped and finally, the edges in R_1 and R_2 are flipped back to the original position.

In the same way, the point set in Figure 16 can again and again be extended with points and edges creating additional regions. To assure that the shortest flip sequence has the same pattern as in the above described examples of that section, we have to exclude edge (v_{11}, v_{12}) from E_f. Otherwise, it would be possible that at a certain point of extension the shortest flip sequence contains that edge and therefore, differs from the above described pattern of the flip sequence. To avoid that, we can move v_{11} and v_{12} as depicted in Figure 17 such that $(v_{11}, v_{12}) \notin E_f$.

In the same manner, we can create point sets and triangulations for any arbitrary $k > 5$. For each triangulation with that structure, there exists a shortest flip sequence, that equals the aforementioned pattern. In fact, their flip distance depends on the number of regions. Hence, for two triangulations T_1, T_2 of point sets with this structure we can estimate the flip distance between T_1 and T_2 by

$$\mathrm{fd}(T_1, T_2) \leq \#(T_1, T_2) + \mathcal{O}(\log n)$$

In summary, we could not find a correct input sensitive upper bound for the flip distance of two bounded degree triangulations. We only showed a few approaches, that work for some particular point sets, but not in general. Thus, we can not say that for each degree exceeding vertex, or even each edge that is incident to a degree exceeding vertex, we need a constant number of additional flips. Probably, it is necessary to consider the whole structure of the triangulation in order to obtain better upper bounds.

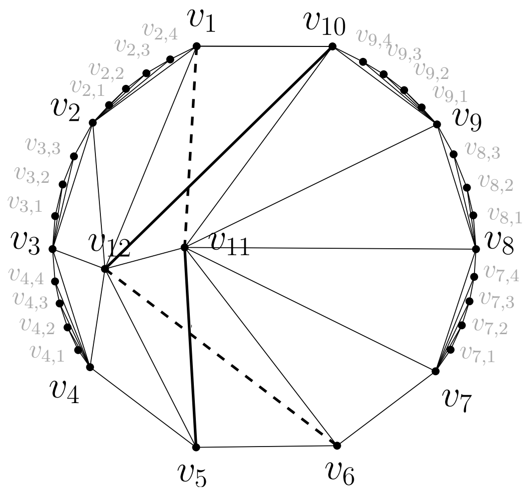


Figure 17: Solid edges occur in $T_1 \cup T_2$, bold ones belong to T_1, dashed ones to T_2. The vertices v_{11}, v_{12} are moved such that (v_{11}, v_{12}) is a diagonal of a concave quadrilateral.

4 Zigzag Triangulations

In general, the efficient calculation of the flip distance of two arbitrary triangulations of a convex point set P in the plane is an open problem. In this section we focus on a special kind of triangulations of P — so-called zigzag triangulations. Section 4.2 shows that the set of triangulations for which the shortest flip sequence is efficiently computable can be extended by zigzag triangulations that have restrictions on the cardinality of points and on the structure of the edge-positions.

Furthermore, that result can be extended for the degree bounded setting of triangulations. In [3, Lemma 4] Aichholzer et al. show that two zigzag triangulations can be transformed into each other in $\mathcal{O}(n)$ time without exceeding a vertex degree of $k > 6$. The corresponding flip sequence presented in the proof of that lemma, is equal to one of the two flip sequences discussed in section 4.3.

4.1 Preliminaries

In this section, P is a set of $n \geq 4$ points in convex position in the plane. For simplicity the vertices are labeled from v_1 to v_n in counterclockwise order, starting at the top (w.l.o.g. assume that the top vertex is unique). We call $v_{i_{succ}} := v_{((i \mod n)+1)}$ the counterclockwise neighbor of $v_i \in P$ and $v_{i_{pred}} := v_{(((i-2) \mod n)+1)}$ the clockwise neighbor of v_i. In other words, if we traverse the boundary of the convex hull in counterclockwise direction, then $v_{i_{pred}}$ is the immediate predecessor of v_i and $v_{i_{succ}}$ the immediate successor of v_i. In case of $2 \leq i \leq (n-1)$ we equivalently write v_{i-1} instead of v_{ipred} and v_{i+1} instead of $v_{i_{succ}}$.

Throughout this section, the distance between two vertices is defined by the minimum number of consecutive edges on the boundary of the convex hull of P connecting these vertices. For brevity, we often write $\text{vdist}_P(v_i, v_j)$ for the distance of $v_i, v_j \in P$, or $\text{vdist}_P(v_i, v_j)$ if the considered point set is clear from the context.

Recall that the set of all possible triangulations of P is denoted by $\mathcal{T}_n$. For $T \in \mathcal{T}_n$ and $v \in P$, $\text{ideg}_T(v)$ stands for the number of diagonals incident to v in T (i.e. the number of inner edges of T that have v as a vertex). Recall that we write $\text{fd}(T_1, T_2)$ as an alternative for the flip distance of $T_1, T_2 \in \mathcal{T}_n$.

For the definition of a zigzag triangulation, we first consider the dual graph D_T of a triangulation $T \in \mathcal{T}_n$. In D_T, each vertex corresponds to a triangle in T, and two vertices are connected in D_T if their corresponding triangles in T share an edge. Due to the convex position of P, D_T is a tree with $(n-2)$ nodes. (Recall that T has $(n-2)$ triangles.) We categorize the triangles of T into three different types:

- **inner triangles**, which consist of three diagonals; the corresponding nodes in D_T of such triangles have degree 3.
- **path triangles**, which have one convex hull edge and two diagonals; in D_T they are equivalent to nodes of degree 2.
- **ears**, which are composed of two convex hull edges and one diagonal; they are dual to the leaves in D_T. The **tip** of an ear is the vertex that is not incident to a diagonal.

A set of path triangles that is dual to a subtree of D_T whose nodes have degree 2 in D_T, is called a **zigzag** if the maximum vertex degree is 4.

Definition 4.1. $T \in \mathcal{T}_n$ is called a **zigzag triangulation**, if and only if T satisfies the following two conditions:

1. D_T is a path.
2. The maximum vertex degree in T is less or equal 4.

In other words, a zigzag triangulation consists of $l = n - 4$ consecutive path triangles, $p_1, \ldots, p_l$, and exactly two ears. Furthermore, the convex hull edge of p_i is adjacent to the convex hull edge of p_{i+2} on the boundary of the convex hull of P, $i \in \{1, \ldots, l-2\}$. Figure 18 shows three zigzag triangulations.

By $\mathcal{Z}_n$, we denote the set of all zigzag triangulations on P. For a $Z \in \mathcal{Z}_n$ that has an ear with the tip v_i, we often abbreviate that v_i is one tip of Z. Taking a closer look to zigzag triangulations that have one tip at v_i, we notice that there exist exactly two possible triangulations in $\mathcal{Z}_n$ for $n > 4$: In one triangulation $v_{i_{succ}}$ has exactly one incident diagonal. For the other triangulation the number of inner edges incident to $v_{i_{succ}}$ is two. For both triangulations, we alternatively call v_i the **starting vertex**. Additionally, we can assign to both triangulations a **starting diagonal** d_{v_i}, which corresponds to the diagonal that is

- incident to the counterclockwise neighbor of $v_{i_{succ}}$, $v_{i_{succ_{succ}}}$, and $v_{i_{pred}}$, in case of $\operatorname{ideg}_{Z(v_i)}(v_{i_{succ}}) = 1$, or
- incident to $v_{i_{succ}}$ and $v_{i_{pred_{pred}}}$, the clockwise neighbor of $v_{i_{pred}}$, in case of $\operatorname{ideg}_{Z(v_i)}(v_{i_{succ}}) = 2$.

For $n = 4$, the number of diagonals of each triangulation in $\mathcal{T}_4 = \mathcal{Z}_4$ is one. Thus, there is exactly one triangulation in $\mathcal{Z}_4$ with one tip at v_i. This leads us to

Definition 4.2. Assume that $Z_1, Z_2 \in \mathcal{Z}_n$ both have one tip at $v_i \in P$ and $Z_1 \neq Z_2$. Then $Z_1(v_i)^I = Z_2$ is called the **inversion** of Z_1 with respect to v_i. For the sake of completeness, we define for $Z \in \mathcal{Z}_4$ with one tip at $v_i \in P$, $Z(v_i)^I := Z$, i.e., the inversion of Z with respect to v_i is again Z.

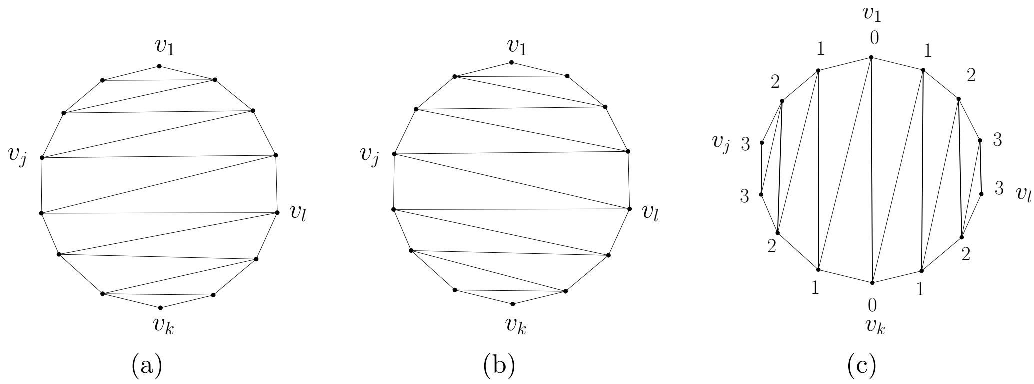


Figure 18: 18(a) A zigzag triangulation $Z \in \mathcal{Z}_{14}$ with tips at v_1 and v_k. 18(b) $Z(v_1)^I = Z(v_k)^I$; the inversion of Z with respect to v_1 equals the inversion of Z with respect to v_k. 18(c) Z^N, the normal zigzag triangulation to Z.

In other words, if we assume that Z has one tip at v_i, then $Z(v_i)^I$ is the result of flipping every second diagonal of Z, beginning with the starting diagonal d_{v_i} of Z. Trivially, the flip distance of Z and $Z(v_i)^I$ is $\lfloor \frac{n-3}{2} \rfloor$, because Z has $(n-3)$ diagonals. Note that $(Z(v_i)^I)^I = Z$. Figure 18(b) shows an example for the inversion of the triangulation shown in Figure 18(a) with respect to v_1. The number of points is even and thus, the tips of the ears remain the same for the inversion.

Observation 4.1. Given $Z \in \mathcal{Z}_n$ with tips at v_i and v_k. If **n is even**, we have

$$\Big(\mathrm{ideg}_Z(v_{i_{succ}}) = 1\Big) \iff \Big(\mathrm{ideg}_Z(v_{k_{succ}}) = 1\Big) \text{ and}$$
$$\Big(\mathrm{ideg}_Z(v_{i_{succ}}) = 2\Big) \iff \Big(\mathrm{ideg}_Z(v_{k_{succ}}) = 2\Big).$$

For the transformation from Z to $Z(v_i)^I$, the starting diagonal of v_k, d_{v_k}, is flipped. Therefore, v_k is the tip of the second ear in $Z(v_i)^I$. Hence, $Z(v_i)^I = Z(v_k)^I$.

If **n is odd**, we observe that

$$\Big(\mathrm{ideg}_Z(v_{i_{succ}}) = 1\Big) \iff \Big(\mathrm{ideg}_Z(v_{k_{succ}}) = 2\Big) \text{ and}$$
$$\Big(\mathrm{ideg}_Z(v_{i_{succ}}) = 2\Big) \iff \Big(\mathrm{ideg}_Z(v_{k_{succ}}) = 1\Big).$$

When flipping Z to $Z(v_i)^I$, the diagonal that is part of the ear with tip v_k is flipped. Hence, the position of the second ear changes and its tip moves to $v_{k_{succ}}$ or $v_{k_{pred}}$. Consequently, we have $Z(v_i)^I \neq Z(v_k)^I$.

Observation 4.2. Given $Z \in \mathcal{Z}_n$ with tips at $v_i, v_k \in P$. Then $v_j, v_l \in P$, $i < j < k < l$, are connected by an edge in Z, if and only if

1. $\text{vdist}(v_i, v_j) = \text{vdist}(v_i, v_l)$ and
2. (a) $\text{vdist}(v_i, v_j) = \Big(\text{vdist}(v_i, v_l) + 1\Big)$ and $\text{ideg}_Z(v_{i_{succ}}) = 1$ or

   (b) $\Big(\text{vdist}(v_i, v_j) + 1\Big) = \text{vdist}(v_i, v_l)$ and $\text{ideg}_Z(v_{i_{succ}}) = 2$.

In general, P has $\Theta(n)$ zigzag triangulations that differ in the position of the tips and the "type" of inversion, meaning the two zigzag triangulations in $\mathcal{Z}_n$ that can be assigned to a tip. We observe that all statements valid for $Z \in \mathcal{Z}_n$ with one tip at v_1 also hold for any $Z' \in \mathcal{Z}_n$: If we rotate P in such a way that v_i has in the rotated point set P' the position of v_1 in P then Z' is a triangulation with one tip at the top vertex v_1' in P', whereas v_1' in P' corresponds to the transformed v_i in P. Thus, Z' with the underlying point set P' corresponds to Z on P. Hence, all definitions and results we show in the following for a zigzag triangulation with one tip at v_1 are equally valid for any other zigzag-triangulation with one tip at $v_i \in P$.

Definition 4.3 ([3])**.** Given $Z_1, Z_2 \in \mathcal{Z}_n$ with tips at v_1, v_k and v_j, v_l, $i < j < k < l$. Label each vertex in P with the minimum distance to the tips of Z_1. Then Z_2 is the **normal zigzag triangulation** of Z_1 if and only if

1. Z_2 contains each edge that connects two vertices with the same label and
2. (a) $\Big(\text{ideg}_{Z_1}(v_2) = 1\Big) \Rightarrow \Big(\text{ideg}_{Z_2}(v_{j+1}) = 1\Big)$ or

   (b) $\Big(\text{ideg}_{Z_1}(v_2) = 2\Big) \Rightarrow \Big(\text{ideg}_{Z_2}(v_{j+1}) = 2\Big)$.

We denote the normal zigzag triangulation of Z_1 by Z_1^N.

Figure 18(c) shows the normal zigzag triangulation of the triangulation depicted in Figure 18(a).

Observation 4.3. Given $Z \in \mathcal{Z}_n$ with tips at v_1 and v_k and let v_j, v_l be the tips of the ears of Z^N. We call $v_2, \ldots, v_{k-1}$ the "left side" of P and $v_{k+1}, \ldots, v_n$ the "right side" of P. Then

1. v_j belongs to the left side of P and v_l to the right side and

2. v_j has the maximum value among all labels of the vertices on the left side and, analogously, v_l has the maximum value of all labels of the vertices on the right side.

With Definition 4.2 and Definition 4.3 we introduced two types of zigzag triangulations related to a $Z \in \mathcal{Z}_n$. This allows a combination of both types. It is worth noting that, dependent on n and the order of building an inversion and a normal zigzag triangulation, respectively, differences in the resulting triangulation can occur:

Remark 4.1. For $(n \mod 4) = 2$, the order of building an inversion of a zigzag triangulation and creating a normal zigzag triangulation is a determining factor for the resulting triangulation, see Figure 19. Otherwise, the results are the same.

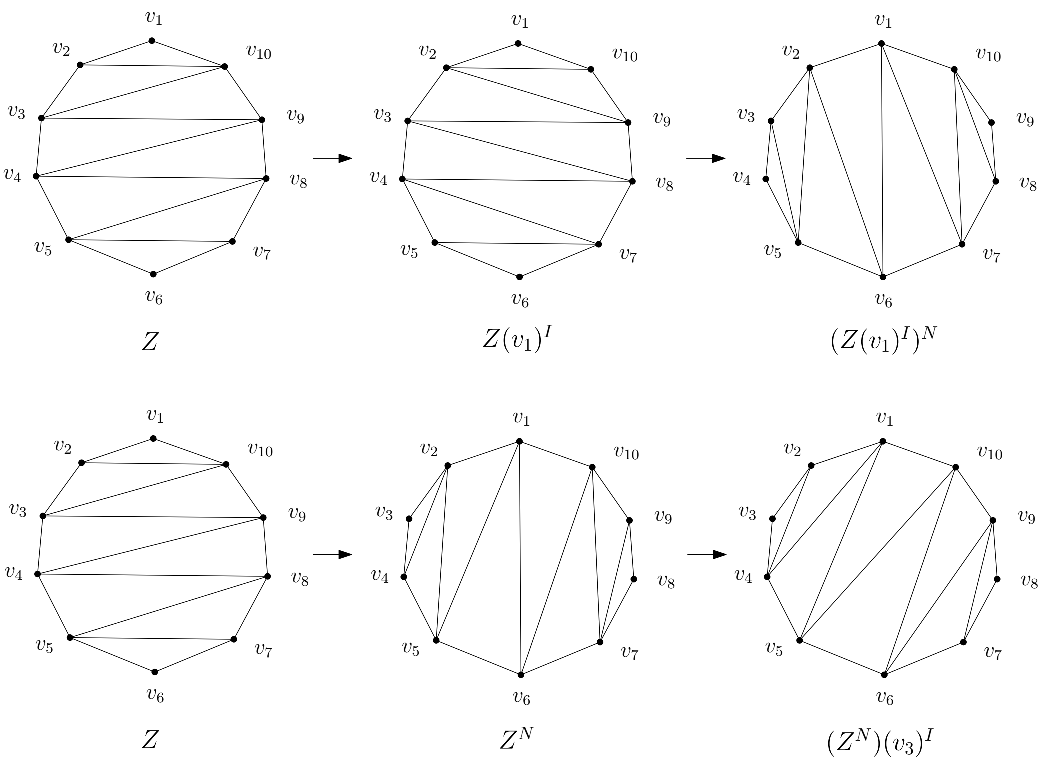


Figure 19: Row 1: First, the initial triangulation Z is inverted with respect to v_1, then the resulting triangulation is the normal zigzag triangulation of $Z(v_1)^I$. Row 2: Starting with the creation of the normal zigzag triangulation of Z leads to Z^N with tips at v_3, v_8, depicted in the second column. Neither $(Z^N)(v_3)^I$ nor $(Z^N)(v_8)^I$ equal $(Z(v_1)^I)^N$.

Turning back to normal zigzag triangulations, we observe that the tips of a normal zigzag triangulation are connected by an edge in the initial zigzag triangulation, if and only if $(n \mod 4) \neq 2$. Thus, the aforementioned irregularity in Remark 4.1 is not the only one for point sets of size $n = (4k_n + 2)$.

Lemma 4.1. *Given $Z_1, Z_2 \in \mathcal{Z}_n$ with $Z_2 = Z_1^N$ and $k_n \in \mathbb{N}$. Furthermore let $v_1, v_k \in P$ be the tips of Z_1 and $v_j, v_l \in P$ be the tips of Z_2, $1 < j < k < l$. Then*

(i) edge (v_j, v_l) is a diagonal in Z_1, if and only if (a) $n = 4k_n$, (b) $n = (4k_n + 1)$, or (c) $n = (4k_n + 3)$, and

*(ii) edge (v_j, v_l) is **not** a diagonal in Z_1, if and only if $n = (4k_n + 2)$.*

Proof. To get an overview of the proof, we first present a rough structure: We consider the cases (i)(a) - (i)(c), with (i)(b) and (i)(c) containing two sub-cases caused by the need of distinction between $\mathrm{ideg}_{Z_1}(v_2) = 1$ (called Case 1) and $\mathrm{ideg}_{Z_1}(v_2) = 2$ (called Case 2). Finally, we study case (ii), which is again subdivided into Case 1 and Case 2.

We define $V_L := \{v_2, \ldots, v_{k-1}\}$, the vertices on the “left side” of Z_1, and $V_R := \{v_{k+1}, \ldots, v_n\}$, the vertices on the “right side” of Z_1. Furthermore, we define $\tilde{n} := 4k_n \in \mathbb{N}$.

We begin the proof with **(i)**:

(a) $n = \tilde{n}$: Independent of $\mathrm{ideg}_{Z_1}(v_2)$, the index of the second tip of Z_1 is $k = \frac{\tilde{n}}{2} + 1$. Furthermore, the number of vertices on the left and right side is odd:

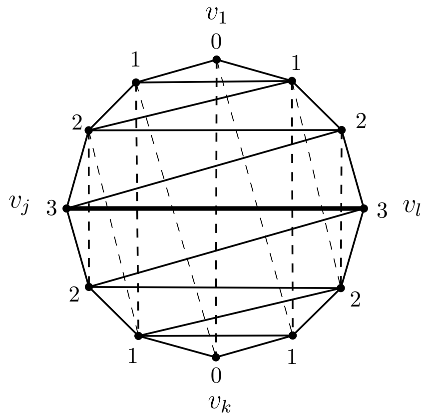
$$|V_L| = |V_R| = \frac{\tilde{n} - 2}{2} = \frac{4k_{\tilde{n}} - 2}{2} = \frac{2(2k_{\tilde{n}} - 1)}{2} = (2k_{\tilde{n}} - 1)$$

Consequently, there exists exactly one vertex for each side with the maximum value of $\frac{\tilde{n}}{4}$ among all labels. According to Observation 4.3 that label belongs to v_j and v_l. Additionally, the labels correspond to $\min\Big(\mathrm{vdist}(v_j, v_1), \mathrm{vdist}(v_j, v_k)\Big)$ and $\min\Big(\mathrm{vdist}(v_l, v_1), \mathrm{vdist}(v_l, v_k)\Big)$. Hence, we have

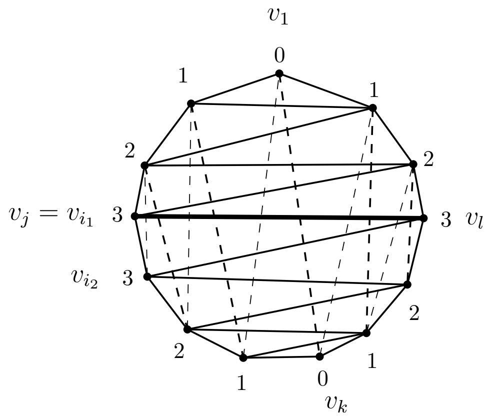
$$\Big(\mathrm{vdist}(v_j, v_1) = \mathrm{vdist}(v_l, v_1)\Big) \xRightarrow{\text{Observation 4.2}} \Big(Z_1 \text{ contains edge } (v_j, v_l)\Big).$$

Figure 20(a) shows an example.

(b) $n = \tilde{n} + 1$: Compared to (a), we have one additional vertex. Depending on $\mathrm{ideg}_{Z_1}(v_2)$, that vertex either belongs to V_L or to V_R. Thus, we have to distinguish between the two zigzag triangulations in $\mathcal{Z}_n$ that can be assigned to v_1 :



(a) $(n \mod 4) = 0$



(b) $(n \mod 4) = 1$. One vertex is added to the "left side" and v_k is shifted counterclockwise.

Figure 20: Solid lines indicate Z_1, dashed lines are part of $Z_2 = Z_1^N$ (the heavier ones are the connection between vertices with the same label). For both cases, we have $(v_j, v_l) \in Z_1$.

Case (i)(b)1: $\mathrm{ideg}_{Z_1}(v_2) = 1$: The index k of v_k increases by one and thus, equals $(\frac{\tilde{n}}{2} + 2)$. Therefore, the additional vertex belongs to V_L. Accordingly, $|V_L| = \frac{\tilde{n}-2}{2} + 1$, which is even. Hence, there are two vertices $v_{i_1}, v_{i_2} \in V_L$, $(i_1 + 1) = i_2$, with the maximum label $\frac{\tilde{n}}{4}$. (The label of v_{i_1} equals $\mathrm{vdist}(v_{i_1}, v_1)$ and the label of v_{i_2} corresponds to $\mathrm{vdist}(v_{i_2}, v_k)$.) Since v_{i_1} and v_{i_2} have the same label, they are connected by an edge in Z_1^N. (v_{i_1}, v_{i_2}) is a convex hull edge and thus, both vertices have an inner degree of 0 before the remaining diagonals are added during the process of building Z_1^N.

If we consider the clockwise neighbor of v_{i_1}, v_{i_1-1} and the counterclockwise neighbor of v_{i_2}, v_{i_2+1}, we know that

$$\mathrm{vdist}(v_1, v_{i_1-1}) = \Big(\mathrm{vdist}(v_1, v_{i_1}) - 1\Big) \text{ and}$$
$$\mathrm{vdist}(v_k, v_{i_2+1}) = \Big(\mathrm{vdist}(v_k, v_{i_2}) - 1\Big).$$

Together with the fact that

$$\Big(\mathrm{vdist}(v_1, v_{i_1}) = \mathrm{vdist}(v_k, v_{i_2})\Big)$$
$$\Rightarrow \Big(\mathrm{vdist}(v_1, v_{i_1-1}) = \mathrm{vdist}(v_k, v_{i_2+1})\Big)$$

we know that Z_1^N contains the edge (v_{i_1-1}, v_{i_2+1}). Thus, before the remaining diagonals are added in the process of building Z_1^N, the inner vertex degree of v_{i_1-1} and v_{i_2+1} is 1. Furthermore, we have the precondition that

$$\Big(\mathrm{ideg}_{Z_1}(v_2) = 1\Big) \xRightarrow{\text{Definition 4.3}} \Big(\mathrm{ideg}_{Z_2}(v_{j+1}) = 1\Big)$$

Adding the edge (v_{i_1}, v_{i_2+1}) during the process of building Z_1^N leads inevitably to one tip at v_{i_2}. Contrary to the aforementioned precondition, the number of diagonals incident to v_{i_2+1} is 2. Due to that contradiction, we exclude v_{i_2} as a tip of Z_1^N.

If we add the edge (v_{i_1-1}, v_{i_2}) for the construction of Z_1^N, then v_{i_1} is one tip of Z_1^N. Since, on the one hand, v_{i_2} is the counterclockwise neighbor of v_{i_1} and on the other hand, v_{i_2} has only one incident diagonal in Z_1^N, we have $v_j = v_{i_1}$.

The number of vertices in V_R is equal to $|V_R|$ in (a). Therefore, v_l is the vertex with the maximum label $\frac{\tilde{n}}{4}$. Hence,

$$\Big(\mathrm{vdist}(v_j, v_1) = \mathrm{vdist}(v_l, v_1) = \frac{\tilde{n}}{4}\Big) \\ \xRightarrow{\text{Observation 4.2}} \Big(Z_1 \text{ contains edge } (v_j, v_l)\Big).$$

Figure 20(b) shows an example of this case.

Case (i)(b)2: $\mathrm{ideg}_{Z_1}(v_2) = 2$: The additional vertex compared to (a) is part of V_R and the index of v_k remains the same. Therefore, we can apply the same arguments used for the left side in (a) in order to prove $\mathrm{vdist}(v_j, v_1) = \frac{\tilde{n}}{4}$.

$|V_R| = \frac{\tilde{n}-2}{2} + 1$, which is even. Analogous to the left side in Case (i)(b)1, there are two vertices $v_{i_3}, v_{i_4} \in V_R$, $(i_3 + 1) = i_4$, with the maximum label $\frac{\tilde{n}}{4}$. According to Definition 4.3:

$$\Big(\mathrm{ideg}_{Z_2}(v_{j+1}) = 2\Big) \xRightarrow{\text{Observation 4.1}} \Big(\mathrm{ideg}_{Z_2}(v_{l+1}) = 1\Big) \Rightarrow \Big(v_l = v_{i_3}\Big).$$

Thus, we have

$$\Big((\mathrm{vdist}(v_j, v_1) + 1) = \mathrm{vdist}(v_l, v_1)\Big) \\ \xRightarrow{\text{Observation 4.2}} \Big(Z_1 \text{ contains edge } (v_j, v_l)\Big).$$

(c) $n = \tilde{n} + 3$: We add three vertices to the point set in (a). Depending on the value of $\mathrm{ideg}_{Z_1}(v_2)$, two vertices are added to V_L and one to V_R, or vice-versa.

Case (i)(c)1: $\mathrm{ideg}_{Z_1}(v_2) = 1$: In this case, we have two additional vertices on the left side. The index of v_k, $k = (\frac{\tilde{n}}{2} + 3)$, is odd. Consequently, exactly one vertex, v_j, in V_L has the maximum value of all labels. The maximum value equals $(\frac{\tilde{n}}{4} + 1)$.

Compared to (a), $|V_R|$ increases by one. Thus, the number of vertices on the right side is even. Again, we have two vertices $v_{i_3}, v_{i_4} \in V_R$, $(i_3 + 1) = i_4$, with label $\frac{\tilde{n}}{4}$.

$$\Big(\mathrm{ideg}_{Z_2}(v_{j+1}) = 1\Big) \xRightarrow{\text{Observation 4.1}} \Big(\mathrm{ideg}_{Z_2}(v_{l+1}) = 2\Big) \Rightarrow \Big(v_l = v_{i_4}\Big).$$

The label of v_{i_4} corresponds to the distance to v_1. All in all, we have

$$\Big(\mathrm{vdist}(v_j, v_1) = \Big(\mathrm{vdist}(v_l, v_1) + 1\Big)\Big) \xRightarrow{\text{Observation 4.2}} \Big(Z_1 \text{ contains edge } (v_j, v_l)\Big).$$

See Figure 21(b) for an example.

Case (i)(c)2: $\mathrm{ideg}_{Z_1}(v_2) = 2$: Compared to (a), the number of vertices on the left side increases by one and V_L has two more vertices. The index of v_k, $k = (\frac{\tilde{n}}{2}+2)$, is even. Consequently, we have $v_{i_1}, v_{i_2} \in V_L$, $i_2 = (i_1+1)$, with the maximum distance $\frac{\tilde{n}}{4}$ to v_1 and v_k, respectively.

$$\Big(\mathrm{ideg}_{Z_2}(v_{j+1}) = 2\Big) \Rightarrow \Big(v_j = v_{i_2}\Big).$$

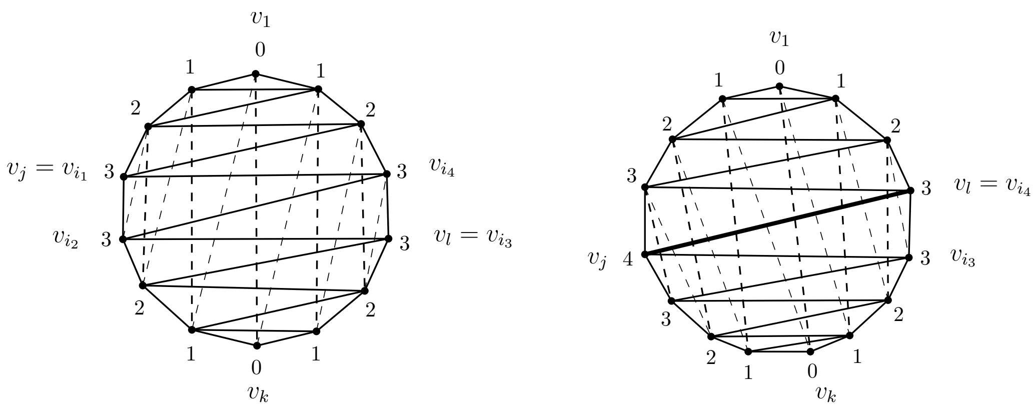
For the same arguments used for the left side in (c), Case 1, v_l has the distance $(\frac{\tilde{n}}{4} + 1)$ to v_1. Thus,

$$\Big(\Big(\mathrm{vdist}(v_j, v_1) + 1\Big) = \Big(\frac{\tilde{n}}{4} + 1\Big) = \mathrm{vdist}(v_l, v_1)\Big) \xRightarrow{\text{Observation 4.2}} \Big(Z_1 \text{ contains edge } (v_j, v_l)\Big).$$

Finally, we turn to **(ii)**:
$n = \tilde{n} + 2$ and $k = \frac{\tilde{n}}{2} + 2$. Compared to case (i)(a), the number of vertices on both sides increases by one and changes from odd to even. Therefore, the vertices $v_{i_1}, v_{i_2} \in V_L$, $i_2 = (i_1 + 1)$, and $v_{i_3}, v_{i_4} \in V_R$, $i_4 = (i_3 + 1)$, have the maximum value of all labels. Again, we distinguish two cases:

Case (ii)1: $\mathrm{ideg}_{Z_1}(v_2) = 1$:

$$\Big(\mathrm{ideg}_{Z_1}(v_2) = 1\Big) \xRightarrow{\text{Definition 4.3}} \Big(\mathrm{ideg}_{Z_2}(v_{j+1}) = 1\Big) \Rightarrow \Big(v_j = v_{i_1}\Big).$$



(a) ($n \mod 4) = 2$ and Z_1 does **not** contain edge (v_j, v_l).

(b) ($n \mod 4) = 3$ and Z_1 contains edge (v_j, v_l).

Figure 21: Solid lines indicate Z_1, dashed lines are part of $Z_2 = Z_1^N$ (the heavier ones are the connection between vertices with the same label).

Furthermore, we have

$$\Big(\operatorname{ideg}_{Z_2}(v_{j+1}) = 1\Big) \xRightarrow{\text{Observation 4.1}} \Big(\operatorname{ideg}_{Z_2}(v_{l+1}) = 1\Big) \Rightarrow \Big(v_l = v_{i_3}\Big).$$

$$\bigg(\operatorname{vdist}(v_{i_3}, v_1) = \Big(\frac{\tilde{n}}{4} + 1\Big) = \Big(\operatorname{vdist}(v_{i_1}, v_1) + 1\Big)\bigg)$$
$$\xRightarrow{\text{Observation 4.2}} \Big(Z_1 \text{ does not contain the edge } (v_j, v_l)\Big).$$

Figure 21(a) depicts an example.

Case (ii)2: $\operatorname{ideg}_{Z_1}(v_2) = 2$: We conclude the proof analog to Case (ii)1. Note that we have $v_j = v_{i_2}$ and $v_l = v_{i_4}$.

□

Contrary to Lemma 4.1, we can observe the same behavior for each value of $n \geq 4$ if one zigzag triangulation is the normal of the inversion of the other given zigzag triangulation. The starting vertex of the inversion depends on the initial triangulation. We specify that observation in the following lemma.

Lemma 4.2. *Given $Z_1, Z_2 \in \mathcal{Z}_n$ with tips at $v_1, v_k \in P$ and $v_j, v_l \in P$, respectively, $1 < j < k < l$. Then edge (v_j, v_l) is a diagonal of Z_1, if one of the following is true:*

(i) n is even and $Z_2 = (Z_1(v_1)^I)^N$,

(ii) $n = (4k_n + 1)$ *and*

(a) $\operatorname{ideg}_{Z_1}(v_2) = 1$ *and* $Z_2 = (Z_1(v_k)^I)^N$*, or*

(b) $\operatorname{ideg}_{Z_1}(v_2) = 2$ *and* $Z_2 = (Z_1(v_1)^I)^N$*,*

(iii) $n = (4k_n + 3)$ *and*

(a) $\operatorname{ideg}_{Z_1}(v_2) = 1$ *and* $Z_2 = (Z_1(v_1)^I)^N$*, or*

(b) $\operatorname{ideg}_{Z_1}(v_2) = 2$ *and* $Z_2 = (Z_1(v_k)^I)^N$*,*

$k_n \in \mathbb{N}$.

Proof. Basically, the proof is analog to the proof of Lemma 4.1. Due to the additional inversion with respect to either v_1 or v_k, we have to replace arguments like

$$\Big(\operatorname{ideg}_{Z_1}(v_2) = 1\Big) \Rightarrow \Big(\operatorname{ideg}_{Z_2}(v_{j+1}) = 1\Big)$$

by

$$\Big(\operatorname{ideg}_{Z_1}(v_2) = 1\Big) \Rightarrow \Big(\operatorname{ideg}_{Z_1(v_1)^I}(v_2) = 2\Big) \Rightarrow \Big(\operatorname{ideg}_{Z_2}(v_{j+1}) = 2\Big).$$

Furthermore, recall that $Z(v_1)^I = Z(v_k)^I$ for n even and $Z(v_1)^I \neq Z(v_k)^I$ for n odd, see Observation 4.1 on page 32. That is the reason why we have to distinguish between the Case 1 and the Case 2 in the cases (ii) and (iii). □

We now turn from properties of two specific zigzag triangulations to possibilities to break down the calculation of the flip distance of two initial triangulations in $\mathcal{T}_n$ into the calculation of the flip distance of sub-triangulations. In this respect, Sleator et al. [18] show the base for all approaches presented in this section.

Lemma 4.3 (Lemma 3 in [18])**.** *Given* $T_1, T_2 \in \mathcal{T}_n$*.*

(a) If it is possible that one edge flip in T_1 *creates a triangulation* T_1' *that has one more diagonal in common with* T_2*, than* T_1 *has then there exists a shortest path from* T_1 *to* T_2 *in which the first edge flip creates* T_1'*.*

(b) If T_1 *and* T_2 *have a diagonal in common, then a shortest path from* T_1 *to* T_2 *never flips this diagonal. In fact, any path that flips this diagonal is at least two flips longer than a shortest path.*

Consequently, if $v \in P$ has one incident diagonal e in $T_1 \in \mathcal{T}_n$ and no incident diagonal in $T_2 \in \mathcal{T}_n$, then flipping e to e' in T_1 creates a triangulation, that is part of one shortest path from T_1 to T_2. Furthermore, that path does not contain a triangulation (except T_1), in which e' does not exist.

Corollary 4.1. *Given $T_1, T_2 \in \mathcal{T}_n$.*

(a) If T_1' is the result of one edge flip in T_1 such that T_1' has one more diagonal in common with T_2 than T_1 has then

$$\mathrm{fd}(T_1, T_2) = \mathrm{fd}(T_1', T_2) + 1.$$

(b) If T_1, T_2 share a common diagonal, which separates T_1 into sub-triangulations T_1' and T_1'', and T_2 into T_2' and T_2'', then the calculation of the flip distance can be subdivided in the following way:

$$\mathrm{fd}(T_1, T_2) = \mathrm{fd}(T_1', T_2') + \mathrm{fd}(T_1'', T_2'').$$

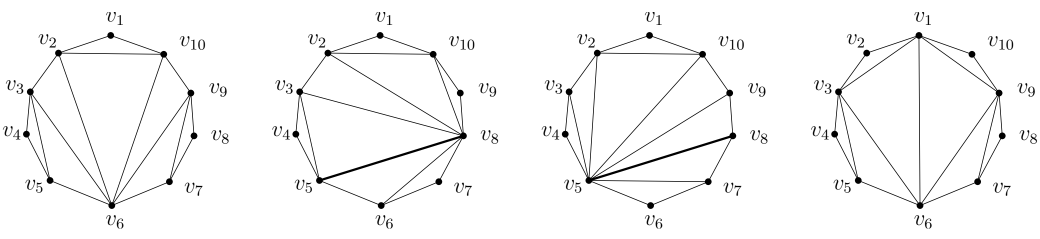
Lucas presents that corollary in [16], where it is formulated for binary trees. Additionally, Corollary 4.1 (b) can be found in [4] for triangulations of convex point sets.

In the following, we will consider further ways that contribute to a possible simplification of the calculation of the flip distance. We introduce the two preliminary definitions of a normalized and a double-normalized triangulation. Note that these definitions have nothing in common with the normal zigzag triangulation.

Definition 4.4 (see Definition 1 in [4] and [18])**.** Given $T \in \mathcal{T}_n$ and $v_i, v_j \in P$. Then we define the **normalized triangulation** $N_T(v_i, v_j)$ (and analogously $N_T'(v_i, v_j)$) of T with respect to the diagonal (v_i, v_j) as follows:

1. $N_T(v_i, v_j)$ and $N_T'(v_i, v_j)$ contain the diagonal (v_i, v_j);
2. $N_T(v_i, v_j)$ and $N_T'(v_i, v_j)$ contain every diagonal of T that does not cross the diagonal (v_i, v_j);
3. if T contains a diagonal (v_k, v_l) that crosses the diagonal (v_i, v_j), then $N_T(v_i, v_j)$ ($N_T'(v_i, v_j)$) contains the diagonals (v_k, v_j) and (v_l, v_j) ((v_k, v_i) and (v_l, v_i)).

If $\mathrm{vdist}(v_i, v_j) = 2$ and v_c is the vertex with distance one to v_i as well as v_j, then we abbreviate $N_T(v_i, v_j)$ as $N_T(v_c)$ (and analogously $N_T'(v_i, v_j)$ as $N_T'(v_c)$). We call $N_T(v_c)$ the normalized triangulation of T with respect to v_c.



(a) T (b) $N_T(v_5, v_8)$ (c) $N'_T(v_5, v_8)$ (d) $N''_T(v_2, v_{10})$

Figure 22: 22(a) $T \in \mathcal{T}_{10}$. 22(b) One normalization of T with respect to v_5, v_8. 22(c) The other normalization of T with respect to v_5, v_8. 22(d) The double-normalized triangulation of T with respect to edge (v_2, v_{10}).

Figure 22(b) and 22(c) depict both normalizations of the triangulation in Figure 22(a) with respect to v_5 and v_8.

Remark 4.2. Given $T \in \mathcal{T}_n$ and $v \in P$. Then

$$\mathrm{ideg}_{N_T(v)}(v) = \mathrm{ideg}_{N'_T(v)}(v) = 0,$$

i.e., each diagonal, which is incident to v in T, is flipped in $N_T(v)$ and $N'_T(v)$.

Remark 4.3. Trivially, if (v_i, v_j) is an edge of T, then

$$T = N_T(v_i, v_j) = N'_T(v_i, v_j),$$

$T \in \mathcal{T}_n$, $v_i, v_j \in P$. Further, if $\mathrm{ideg}_T(v) = 1$, $v \in P$, then

$$N_T(v) = N'_T(v).$$

Furthermore, we observe that if $\mathrm{ideg}_T(v_i) = 2$ for any $T \in \mathcal{T}$ and $v_i \in P$, then we only have to flip the diagonals e_1, e_2 that are incident to v_i in order to transform T into $N_T(v_i)$ and $N'_T(v_i)$. Additionally, we know that $N_T(v_i)$ and $N'_T(v_i)$ differ by exactly one edge. That difference arises due to the varying order of flipping e_1, e_2. In any case, the flip target of e_1 or e_2 is edge $(v_{i_{pred}}, v_{i_{succ}})$. The flip target of the other edge is incident to either $v_{i_{pred}}$ or $v_{i_{succ}}$. We summarize these results in the following observation:

Observation 4.4. Given $T \in \mathcal{T}_n$ and $v_i \in P$ with $\mathrm{ideg}_T(v_i) = 2$. Then the following equations are valid:

(a) $\mathrm{fd}(T_1, N_T(v_i)) = \mathrm{fd}(T_1, N'_T(v_i)) = 2$ and

(b) $\mathrm{fd}(N_T(v_i), N'_T(v_i)) = 1$.

The observation is essential to improve the understanding of the relation between Lemma 4.4 and Theorem 4.5, presented later in this section.

Definition 4.5 (see Definition 3 in [4])**.** Let $e = (v_i, v_j)$ be a diagonal of $T \in \mathcal{T}_n$ with the property that $\text{ideg}_T(v_i) = \text{ideg}_T(v_j) = 2$. The **double-normalized** triangulation of T with respect to e, called $N''_T(v_i, v_j)$, can be obtained by the following edge flips:

1. flip edge e,
2. flip the remaining incident diagonal of vertex v_i,
3. flip the remaining incident diagonal of vertex v_j.

Figure 22(d) shows the double-normalized triangulation of the triangulation depicted in Figure 22(a).

Remark 4.4. According to Definition 4.5, we have

$$\text{ideg}_{N''_T(v_i,v_j)}(v_i) = \text{ideg}_{N''_T(v_i,v_j)}(v_j) = 0.$$

Remark 4.5. Contrary to Definition 4.4, the edge (v_i, v_j) of Definition 4.5 has to be a diagonal of the initial triangulation.

With those definitions and basic properties in mind, we are able to introduce additional possibilities for the subdivision of the calculation of the flip distance between two triangulations.

Lemma 4.4 (see Lemma 5 in [4])**.** *Given $T_1, T_2 \in \mathcal{T}_n$ with $v \in P$ such that $\text{ideg}_{T_1}(v) = 2$ and $\text{ideg}_{T_2}(v) = 0$. Then there exists a shortest path from T_1 to T_2 in which the first two flips create either $N_{T_1}(v)$ or $N'_{T_1}(v)$.*

As a consequence of Lemma 4.4, the calculation of the flip distance of T_1 and T_2 can be subdivided. Since $N_{T_1}(v)$ or $N'_{T_1}(v)$ are part of a shortest path from T_1 to T_2, both, the flip distance between $N_{T_1}(v)$ and T_2, as well as the flip distance of $N'_{T_1}(v)$ and T_2 has to be taken into consideration. The minimum of both is crucial for the resulting flip distance of T_1 and T_2. Together with Observation 4.4(a) we get equation (4.1) in Theorem 4.5.

Let $\mathcal{P} : T_1 = \tau_0, \tau_1, \ldots, \tau_p = T_2,\ \tau_i \in \mathcal{T}_n,\ 0 \le i \le p$ be a shortest path from T_1 to T_2. Additionally, we assume that $\tau_2 = N_{T_1}(v)$, i.e.,

$$\min\Big(\text{fd}\Big(N_{T_1}(v), T_2\Big), \text{fd}\Big(N'_{T_1}(v), T_2\Big)\Big) = \text{fd}\Big(N_{T_1}(v), T_2\Big) = p - 2.$$

As a consequence of $\text{fd}(N_{T_1}(v), N'_{T_1}(v)) = 1$ (see Observation 4.4(b)), there exists a path

$$\mathcal{P}' : T_1 = \tau'_0, \tau'_1, \tau'_2 = N'_{T_1}(v), \ldots, \tau'_q = T_2,\ \tau'_i \in \mathcal{T}_n,\ 0 \le i \le q$$

of a maximum length of $p+1$: If $q=p$, then $N'_{T_1}(v)$ is the second triangulation of another shortest path from T_1 to T_2 and thus, $\mathrm{fd}(N_{T_1}(v), T_2) = \mathrm{fd}(N'_{T_1}(v), T_2)$. Otherwise, we can always flip $N'_{T_1}(v)$ to $N_{T_1}(v)$ and receive a path of length $q = (2+1+(p-2)) = (p+1)$. (2 flips from T_1 to $N'_{T_1}(v)$, 1 flip from $N'_{T_1}(v)$ to $N_{T_1}(v)$ and, according to the assumption, $p-2$ flips from $N_{T_1}(v)$ to T_2.) Generally speaking, the following inequality holds:

$$\begin{aligned}&\max\Big(\mathrm{fd}\left(N_{T_1}(v), T_2\right), \mathrm{fd}\left(N'_{T_1}(v), T_2\right)\Big)\\ &\leq \min\Big(\mathrm{fd}\left(N_{T_1}(v), T_2\right), \mathrm{fd}\left(N'_{T_1}(v), T_2\right)\Big) + 1.\end{aligned}$$

Together with equation (4.1) that inequality leads to equation (4.2).

Theorem 4.5 (Theorem 1 in [4]). *Given $T_1, T_2 \in \mathcal{T}_n$ and $v \in P$ with $\mathrm{ideg}_{T_1}(v) = 2$ and $\mathrm{ideg}_{T_2}(v) = 0$. Then we have*

$$\mathrm{fd}(T_1, T_2) = \min\Big(\mathrm{fd}\left(N_{T_1}(v), T_2\right), \mathrm{fd}\left(N'_{T_1}(v), T_2\right)\Big) + 2 \tag{4.1}$$

and

$$\mathrm{fd}(T_1, T_2) \geq \max\Big(\mathrm{fd}\left(N_{T_1}(v), T_2\right), \mathrm{fd}\left(N'_{T_1}(v), T_2\right)\Big) + 1. \tag{4.2}$$

Corollary 4.2. *Given $T_1, T_2 \in \mathcal{T}_n$ and $v \in P$ with $\mathrm{ideg}_{T_1}(v) = 2$ and $\mathrm{ideg}_{T_2}(v) = 0$. Then we have*

$$\Big(\mathrm{fd}\left(N_{T_1}(v), T_2\right) + 1\Big) \leq \mathrm{fd}(T_1, T_2) \leq \Big(\mathrm{fd}\left(N_{T_1}(v), T_2\right) + 2\Big). \tag{4.3}$$

Moreover, we are able to specify the results of Theorem 4.5, if there exists a so-called $(2,2)$-diagonal:

Theorem 4.6 (see Theorem 2 in [4]). *Given $T_1, T_2 \in \mathcal{T}_n$ and let edge $e = (v_i, v_j)$ of T_1 be a* ***(2,2)-diagonal****, i.e., the incident vertices satisfy the following properties:*

- $\mathrm{ideg}_{T_1}(v_i) = \mathrm{ideg}_{T_1}(v_j) = 2$,
- $\mathrm{ideg}_{T_2}(v_i) = \mathrm{ideg}_{T_2}(v_j) = 0$.

Then the calculation of the flip distance of T_1 and T_2 can be subdivided:

$$\mathrm{fd}(T_1, T_2) = \mathrm{fd}\left(N''_{T_1}(v_i, v_j), T_2\right) + 3.$$

Figure 23 depicts two possible triangulations T_1, T_2 on which Theorem 4.6 can be applied. Thus, the two double-normalized triangulations are part of a shortest path from T_1 to T_2.

Remark 4.6. $N''_{T_1}(v_i, v_j)$ from Theorem 4.6 has two more edges in common with T_2 than T_1 has. Those edges are given by $(v_{i_{pred}}, v_{i_{succ}})$ and $(v_{j_{pred}}, v_{j_{succ}})$.

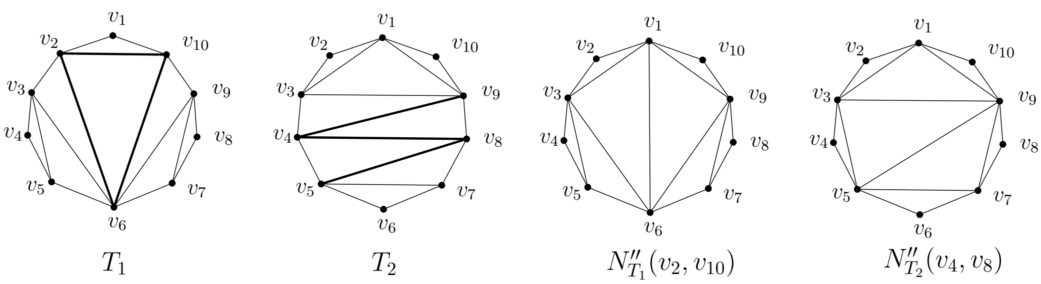


Figure 23: $(v_2, v_{10}) \in T_1$ and $(v_4, v_8) \in T_2$ are (2,2)-diagonals.

4.2 Flip distance of two zigzag triangulations

Now, all basic preconditions are available to extend the subset of triangulations in $\mathcal{T}_n$ for which the flip distance is efficiently computable. In fact, the shown extension is only possible, if the given triangulations are zigzag triangulations and

(a) one is the normal zigzag triangulation of the other one and $x = (n \mod 4) \in \{0, 1, 3\}$, see Theorem 4.9, or

(b) one is the normal zigzag triangulation of the inversion (with respect to the starting vertex) of the other one and $x = (n \mod 4) \in \{1, 2, 3\}$, see Theorem 4.10. The number of points and the inner vertex degree of the neighbors of the tips are the determining factors for the inversions' starting vertex.

For simplicity we assume in the following w.l.o.g. that v_1 and v_k are the tips of $Z \in \mathcal{Z}_n$. Then one tip of the normal zigzag triangulation lies "on the left side", i.e., it is an element of the set $\{v_2, \ldots, v_{k-1}\}$.

In order to be able to prove the above mentioned result, formulated in Theorem 4.9 and 4.10, we first present two fundamental observations in Lemma 4.7 and 4.8.

Lemma 4.7. *Given $Z_1, Z_2 \in \mathcal{Z}_n$ with $Z_2 = Z_1^N$, Furthermore, let $v_1, v_k \in P$ be the tips of Z_1. For (i) $n = 4k_n$, (ii) $n = (4k_n + 1)$ and (iii) $n = (4k_n + 3)$, $k_n \in \mathbb{N}$, $k_n > 1$, the flip distance of Z_1 and Z_2 can be calculated by the following equation:*

$$\mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(Z'_1, Z'_2) + 3$$

with $Z'_1, Z'_2 \in \mathcal{Z}_{n'}$, $n' = n - 2$ and

(i) $Z'_2 = (Z'_1(v_1)^I)^N$

(ii) $\Big(Z_2' = (Z_1'(v_1)^I)^N$ *iff* $\mathrm{ideg}_{Z_1}(v_2) = 1\Big)$ *and*
$\Big(Z_2' = (Z_1'(v_k)^I)^N$ *iff* $\mathrm{ideg}_{Z_1}(v_2) = 2\Big)$

(iii) $\Big(Z_2' = (Z_1'(v_k)^I)^N$ *iff* $\mathrm{ideg}_{Z_1}(v_2) = 1\Big)$ *and*
$\Big(Z_2' = (Z_1'(v_1)^I)^N$ *iff* $\mathrm{ideg}_{Z_1}(v_2) = 2\Big)$

Proof. Since we have to distinguish many cases, we give a short overview of the proof: We start with some definitions and basic observations valid for all cases. Then we prove the cases (i) - (iii) provided that $\mathrm{ideg}_{Z_1}(v_2) = 1$ (Case 1) and conclude the proof for the case $\mathrm{ideg}_{Z_1}(v_2) = 2$ (Case 2) by analogy with Case 1.

During the proof, we sometimes consider triangulations of $P' \subset P$. In order to avoid confusion with expressions like $v_{j-2} \in P'$, we clarify that the index relates to P and not to P'.

In the following, we assume that v_j and v_l are the tips of the ears in Z_2. According to Lemma 4.1, the edge $e = (v_j, v_l)$ is a diagonal of Z_1. Furthermore, we have

$$\mathrm{vdist}(v_1, v_j) \geq \frac{n}{4} \geq 2$$

for $k_n > 1$. The same holds for $\mathrm{vdist}(v_1, v_l)$. Therefore, v_j and v_l are not vertices of an ear in Z_1 and e is a $(2,2)$-diagonal. According to Theorem 4.6, the following equation holds:

$$\mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(N''_{Z_1}(v_j, v_l), Z_2) + 3.$$

Since there are two triangulations in $\mathcal{Z}_n$ that can be assigned to the tip v_1, we have to distinguish between them:

Case 1: $\mathrm{ideg}_{Z_1}(v_2) = 1$
In order to transform Z_1 into $N''_{Z_1}(v_j, v_l)$, we need the following edge flips: For $n = 4k_n$ and $n = (4k_n + 1)$:

1.) (v_j, v_l) to (v_{j+1}, v_{l+1}),

2.) (v_j, v_{l+1}) to (v_{j-1}, v_{j+1}) and

3.) (v_{j+1}, v_l) to (v_{l-1}, v_{l+1}).

Figure 24(a), 24(b) and Figure 25(a), 25(c) show an example. We name those edge flips **Transformation 1a**.

For $n = (4k_n + 3)$, there exists a slight difference to Transformation 1a:

1.) (v_j, v_l) to (v_{j-1}, v_{l-1}),

2.) (v_{j-1}, v_l) to (v_{l-1}, v_{l+1}) and

3.) (v_j, v_{l-1}) to (v_{j-1}, v_{j+1}).

Figure 26(a) and 26(c) depict Z_1 and $N''_{Z_1}(v_j, v_l)$ for that setting. We call those edge flips **Transformation 1b**. Note that both transformations have the same consequence: $N''_{Z_1}(v_j, v_l)$ and Z_2 have two common edges: (v_{j-1}, v_{j+1}) and (v_{l-1}, v_{l+1}). According to Corollary 4.1(b) the calculation of the flip distance can be subdivided:

$$\mathrm{fd}(N''_{Z_1}(v_j, v_l), Z_2) = \mathrm{fd}(Z'_1, Z'_2),$$

where Z'_1 and Z'_2 are two triangulations on the point set $P' = P \setminus \{v_j, v_l\}$. The set of edges is defined by the corresponding edges in $N''_{Z_1}(v_j, v_l)$ and Z_2.

In the following, we define the sets

$$V_{t_1} := \{v_2, \ldots, v_{j-1}\}, V_{t_2} := \{v_{l+1}, \ldots, v_n\} \text{ and } V_t := V_{t_1} \cup V_{t_2} \cup \{v_1\}$$

as well as

$$V_{b_1} := \{v_{j+1}, \ldots, v_{k-1}\}, V_{b_2} := \{v_{k+1}, \ldots, v_{l-1}\} \text{ and } V_b = V_{b_1} \cup V_{b_2} \cup \{v_k\}.$$

Z'_1 consists of two zigzags S_1, S_2 and the diagonal $d_Q = (v_{j+1}, v_{l+1})$ (for $(n \mod 4) \in \{0, 1\}$) or $d_Q = (v_{j-1}, v_{l-1})$ (for $(n \mod 4) = 3$), that separates S_1 and S_2. S_1 is the sub-triangulation of Z_1, that covers the vertices in V_t. The diagonals in S_2 are given by the edges in Z_1 connecting the vertices in V_b. Trivially, the tips of Z'_1 are v_1 and v_k. Since, on the one hand, we have

$$\mathrm{vdist}_{P'}(v_{j+1}, v_1) = \Big(\mathrm{vdist}_{P'}(v_{l+1}, v_1) + 1\Big)$$

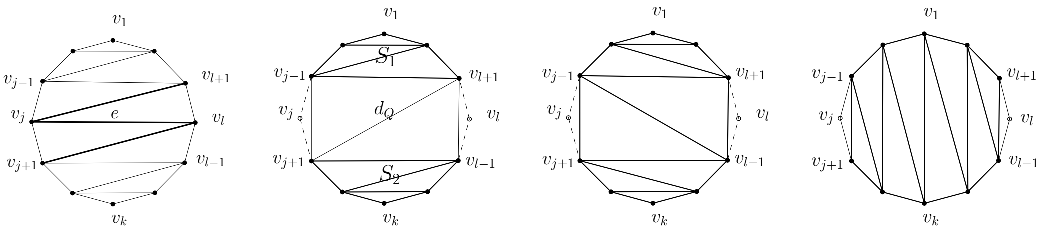
for $(n \mod 4) \in \{0, 1\}$, and

$$\mathrm{vdist}_{P'}(v_{j-1}, v_1) = \mathrm{vdist}_{P'}(v_{l-1}, v_1),$$

for $(n \mod 4) = 3$, and on the other hand, $\mathrm{ideg}_{Z'_1}(v_2) = 1$, we know that Z'_1 is a zigzag triangulation.

Obviously, we have $Z'_2 \in \mathcal{Z}_{n'}$ because Z'_2 is a sub-triangulation of Z_2. The tip of the ear on the "left side" of Z'_2 is v_{j+1}: Due to Definition 4.3, the number of diagonals incident to v_{j+1} in Z_2 is one. The corresponding edge is (v_{j-1}, v_{j+1}). In Z'_2, that edge is a convex hull edge. Therefore, we have $\mathrm{ideg}_{Z'_2}(v_{j+1}) = 0$. Moreover, $\mathrm{ideg}_{Z'_2}(v_{j+2}) = 2$, whereas $\mathrm{ideg}_{Z'_1}(v_2) = 1$.

For further properties of Z'_1 and Z'_2, we have to distinguish according to (i) - (iii) from the lemma:



(a) $Z_1 \in \mathcal{Z}_{4k_n}$ (b) $Z_1' \subset N''_{Z_1}(e)$ (c) $Z_1'(v_1)^I$ (d) $Z_2' \subset Z_2 \in \mathcal{Z}_{4k_n}$

Figure 24: $n = 4k_n$: 24(a) Z_1 with $(2,2)$-diagonal $e = (v_j, v_l)$. 24(b) Z_1' consisting of edges S_1, S_2 and $d_Q = (v_{j+1}, v_{l+1})$. It is a sub-triangulation of $N''_{Z_1}(v_j, v_l)$. 24(c) $Z_1'(v_1)^I = Z_1'(v_k)^I$. Therefore the labels remain the same. 24(d) $Z_2 = Z_1^N$. Bold edges indicate Z_2'.

(i) Figure 24 shows an example of this case ($n = 4k_n$). In order to prove that v_{l+1} is the second tip of Z_2', we can use the same arguments as for v_{j+1} above. Recall that, according to Observation 4.1, we have $\text{ideg}_{Z_2}(v_{l+1}) = 1$, too.

Finally, we have to show that $Z_2' = (Z_1'(v_1)^I)^N$: Compared to Z_1', the tips of $(Z_1'(v_1))^I$ remain the same, see Observation 4.1 on page 32. Consequently, the labels (defined in Definition 4.3) of the vertices in $Z_1'(v_1)^I$ are equal to the labels in Z_1' and Z_1. Hence, all edges connecting the vertices with the same label are diagonals in Z_2'.

As already described in the proof of Lemma 4.1, the tips of the ears of $(Z_1'(v_1)^I)^N$ are given by v_{j+1}, v_{l+1}. Furthermore, we have

$$\text{ideg}_{(Z_1'(v_1)^I)^N}(v_{j+2}) = \text{ideg}_{(Z_1'(v_1)^I)^N)}(v_{l+2}) = 2.$$

The same holds for the counterclockwise neighbors of the tips in Z_2'. Together with the fact, that $Z_2' \in \mathcal{Z}_{n'}$, we have $Z_2' = (Z_1'(v_1)^I)^N$.

(ii) According to Observation 4.1, the number of diagonals incident to v_{l+1} in Z_2 is 2. Consequently, v_{l-1} has an inner vertex degree of one in Z_2. The corresponding incident diagonal in Z_2 is (v_{l-1}, v_{l+1}). As already mentioned before, that edge is a convex hull edge in Z_2'. Thus, we have $\text{ideg}_{Z_2'}(v_{l-1}) = 0$ and v_{l-1} is the tip of the second ear of Z_2', see Figure 25(a), 25(b) and 25(c).

The proof of $Z_2' = (Z_1'(v_1)^I)^N$ in this case is more challenging than it was for case (i): the inversion of Z_1' with respect to v_1 causes a clockwise shift of the second tip, see Figure 25(d). In short, v_{k-1} is the second tip of $Z_1'(v_1)^I$. Hence, all labels that correspond to the distance to v_1 remain the same. Since the label of each vertex in V_t equals the distance to v_1, all labels

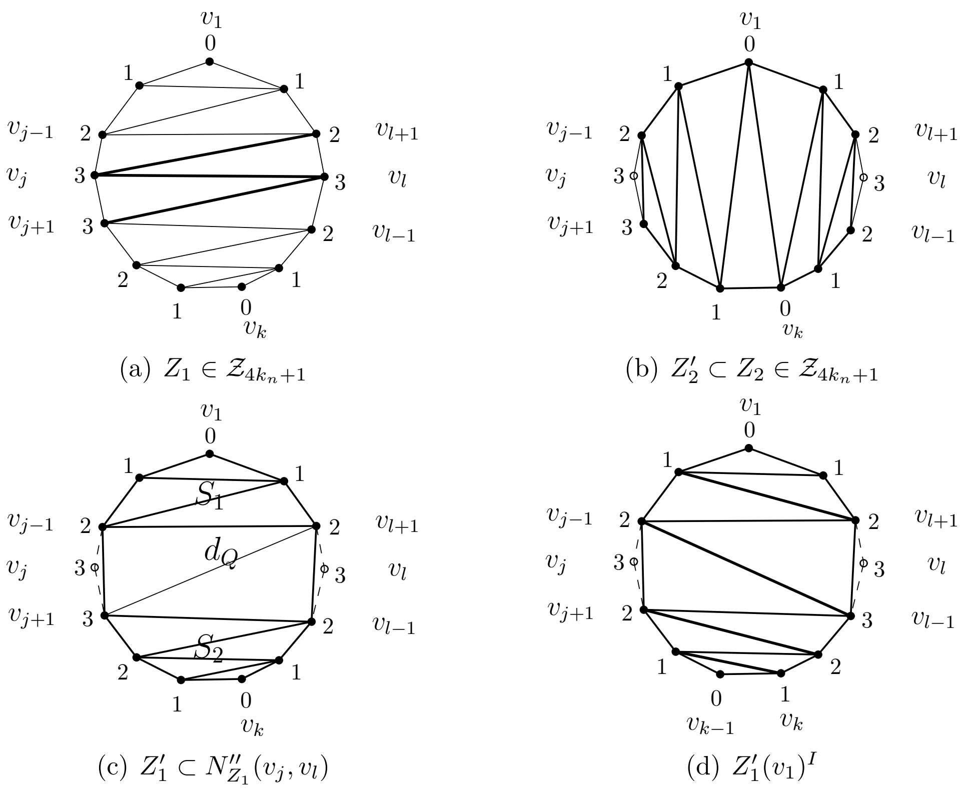


Figure 25: $n = (4k_n + 1)$: 25(a) Z_1 with labels for building Z_1^N. 25(b) $Z_2 = Z_1^N$ with the same labels. Bold edges indicate Z_2'. 25(c) Z_1' - a sub-triangulation of $N''_{Z_1}(v_j, v_l)$. The labels for $(Z_1')^N$ or $(N''_{Z_1}(v_j, v_l))^N$, respectively, would not change. 25(d) $Z_1'(v_1)^I$ with the second tip v_{k-1} and the new labels for the vertices in V_b.

in V_t are unchanged. The labels of vertices in V_{b_1} decrease by one and the labels of vertices $\{v_k\} \cup V_{b_2}$ increase by one. The tips of $(Z_1'(v_1)^I)^N$ are given by v_{j+1}, v_{l-1}. We can observe that

$$\mathrm{ideg}_{(Z_1'(v_1)^I)^N}(v_{j+2}) = \mathrm{ideg}_{Z_2'}(v_{j+2}) = 2$$

and

$$\mathrm{ideg}_{(Z_1'(v_1)^I)^N}(v_{l+1}) = \mathrm{ideg}_{Z_2'}(v_{l+1}) = 1.$$

Additionally, we consider the labels of Z_2 that arose in building the normal zigzag triangulation of Z_1. Figure 25(b) depicts an example. We observe that edge (v_{i_1}, v_{i_2}) is a diagonal of Z_2, if $v_{i_2} \in V_t$ and

(1) $v_{i_1} \in V_b$ and the label of v_{i_1} is equal the label of v_{i_2} , or

(2) $v_{i_1} \in V_{b_1}$ and the label of v_{i_1} equals the label of v_{i_2} plus one , or

(3) $v_{i_1} \in V_{b_2} \cup \{v_k\}$ and the label of v_{i_1} plus one equals the label of v_{i_2}.

Hence, the edges in $(Z_1'(v_1))^I$, that connect vertices with equal labels, correspond to the edges described in case (2) and (3). Therefore, they are edges in Z_2'. In summary, we have $Z_2' = (Z_1'(v_1)^I)^N$.

(iii) Analog to (ii), v_{l-1} is the tip of the second ear of Z_2'. Moreover, the fundamental idea behind the proof of $Z_2' = (Z_1'(v_k)^I)^N$ is the same as in case (ii): Due to the inversion with respect to v_k, the tip of $Z_1'(v_k)^I$ moves to v_2. Hence, each label that corresponds to the distance to v_1 changes its value in $Z_1'(v_k)^I$: The label of each vertex in V_{t_1} decreases by one, whereas the labels of the vertices in V_{t_2} increase by one. Since the labels that arose for Z_1^N of each vertex in V_b equal the distance to v_k, the labels in $Z_1'(v_k)^I$ remain the same. In short, we only have changing labels in V_t. Figure 26 depicts an example. Note, that $(Z_1'(v_1)^I)^N \neq Z_2'$.

Case 2: $\mathrm{ideg}_{Z_1}(v_2) = 2$
For cases (i) and (ii), Transformation 1b converts Z_1 into $N''_{Z_1}(v_j, v_l)$. Otherwise, we need Transformation 1a for case (iii). Hence, the assignment of Case 1 switches. The proof concludes mutatis mutandis. □

Remark 4.7. According to the proof of Lemma 4.7, the tips of Z_2' are given by

- v_{j+1}, v_{l+1}, for $n = 4k_n$ and
- v_{j+1}, v_{j-1}, for n odd,

if $\mathrm{ideg}_{Z_1}(v_2) = 1$.
Otherwise, if $\mathrm{ideg}_{Z_1}(v_2) = 2$, the tips of Z_2' are

- v_{j-1}, v_{l-1}, for $n = 4k_n$ and
- v_{j-1}, v_{l+1}, for n odd.

Remark 4.8. Recall that for $n = (4k_n + 2)$ the edge (v_j, v_l) is not a diagonal of $Z_1 \in \mathcal{Z}_n$, $Z_2 = Z_1^N$ with tips at $v_j, v_l \in P$, see Lemma 4.1 on page 35.

We are able to expand the results of Lemma 4.7 even for $n = (4k_n + 2)$ if one zigzag triangulation is the normal zigzag triangulation of the inversion of the other given zigzag triangulation. The starting vertex of the inversion depends on the initial triangulation.

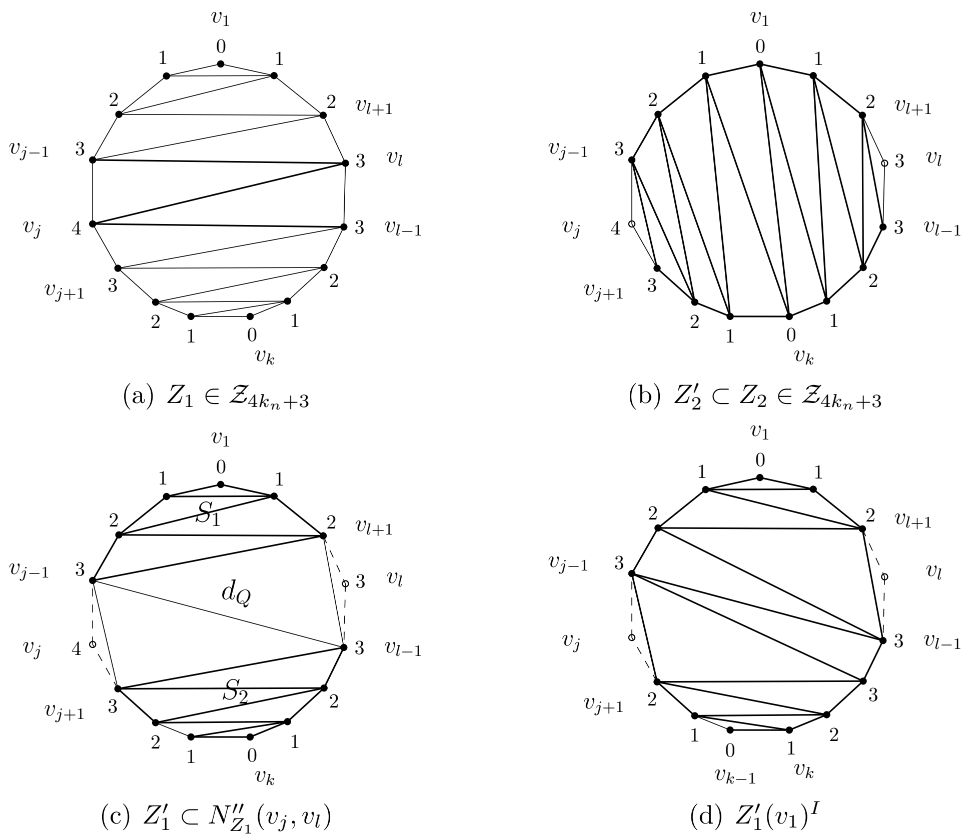


Figure 26: $n = (4k_n + 3)$: 26(a) Z_1 with labels for building Z_1^N. 26(b) $Z_2 = Z_1^N$ with the same labels. Bold edges indicate Z_2'. 26(c) Z_1' - a sub-triangulation of $N''_{Z_1}(v_j, v_l)$. The labels for $(Z_1')^N$ or $(N''_{Z_1}(v_j, v_l))^N$, respectively, would not change. 26(d) $Z_1'(v_1)^I$ with the second tip v_{k-1} and the new labels for the vertices in V_b. All edges, connecting vertices with the same label in $(Z_1'(v_1)^I)^N$ are not in Z_2.

Lemma 4.8. *Given $Z_1, Z_2 \in \mathcal{Z}_n$. For*

(i) $n = 4k_n$ *and* $Z_2 = (Z_1(v_1)^I)^N$,

(ii) $n = (4k_n + 1)$ *and*

- *(a)* $\text{ideg}_{Z_1}(v_2) = 1$ *and* $Z_2 = (Z_1(v_k)^I)^N$, *or*
- *(b)* $\text{ideg}_{Z_1}(v_2) = 2$ *and* $Z_2 = (Z_1(v_1)^I)^N$,

(iii) $n = (4k_n + 2)$ *and* $Z_2 = (Z_1(v_1)^I)^N$,

(iv) $n = (4k_n + 3)$ *and*

- *(a)* $\text{ideg}_{Z_1}(v_2) = 1$ *and* $Z_2 = (Z_1(v_1)^I)^N$, *or* *(b)* $\mathrm{ideg}_{Z_1}(v_2) = 2$ *and* $Z_2 = (Z_1(v_k)^I)^N$*,*

$k_n \in \mathbb{N}$*,* $k_n > 1$*, the flip distance of* Z_1 *and* Z_2 *can be calculated by*

$$\mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(Z_1', Z_2') + 3,$$

with $Z_1', Z_2' \in \mathcal{Z}_{n'}$*,* $n' = n - 2$ *and* $Z_2' = (Z_1')^N$*.*

Proof. Again, we need the same transformations defined in the proof of Lemma 4.7. We have the following assignments:

$$\text{For } \mathrm{ideg}_{Z_1}(v_2) = 1, \text{ we need } \begin{cases} \text{Transformation 1a, for } (i) \text{ and } (iv), \\ \text{Transformation 1b, for } (ii) \text{ and } (iii). \end{cases}$$

In case of $\mathrm{ideg}_{Z_1}(v_2) = 2$, the assignments switch.

Since case (iii) does not occur in the proof of Lemma 4.7, we give a proof sketch for this case:

1. According to Lemma 4.2 edge (v_j, v_l) is a diagonal in Z_1. Since

$$\mathrm{vdist}(v_1, v_j) \geq \frac{n}{4} \geq 2$$

   for $k_n > 1$, (v_j, v_l) is a (2,2)-diagonal. As a consequence of the application of Transformation 1b and Corallary 4.1(b), we have

$$\mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(Z_1', Z_2') + 3$$

2. We have to show that $Z_1' \in \mathcal{Z}_{n-2}$ with an underlying point set $P' = P \setminus \{v_j, v_l\}$ with tips at v_1, v_k: The structure of Z_1' consists of two zigzags S_1, S_2 (see proof of Lemma 4.7 for details) and a diagonal d_Q separating S_1 and S_2. In this case, $d_Q = (v_{j-1}, v_{l-1})$. According to the proof of case (ii) Case 2 in Lemma 4.2, the labels of the tips of the ears of Z_2 correspond to $\mathrm{vdist}_P(v_j, v_k)$ and $\mathrm{vdist}_P(v_l, v_1)$. Thus,

$$\mathrm{vdist}_P(v_j, v_1) = \Big(\mathrm{vdist}_P(v_l, v_1) + 1\Big)$$

   and consequently,

$$\Big(\mathrm{vdist}_P(v_{j-1}, v_1) + 1\Big) = \mathrm{vdist}_P(v_{l-1}, v_1).$$

   Furthermore,

$$\mathrm{vdist}_P(v_{j-1}, v_1) = \mathrm{vdist}_{P'}(v_{j-1}, v_1).$$

Since v_l contributes to $\mathrm{vdist}_P(v_{l-1}, v_1)$ and v_l is not an element of P', we have

$$\mathrm{vdist}_{P'}(v_{l-1}, v_1) = \mathrm{vdist}_{P'}(v_{j-1}, v_1).$$

According to Observation 4.2, we know that d_Q is part of a zigzag triangulation with underlying point set P'. Together with S_1 and S_2, we have $Z_1' \in \mathcal{Z}_{n-2}$.

3. Finally, we have to prove that $Z_2' = (Z_1')^N$. We refer to case (i), because the behavior of an inversion of zigzag triangulation is the same for n even.

Thus, we conclude the proof analog to Lemma 4.7. $\square$

Remark 4.9. For $\mathrm{ideg}_{Z_1}(v_2) = 1$ the tips of Z_2' correspond to the vertices listed in Remark 4.7, case $\mathrm{ideg}_{Z_1}(v_2) = 2$. Otherwise, the tips are given by the vertices, listed for case $\mathrm{ideg}_{Z_1}(v_2) = 1$ (in Remark 4.7).

Each combination of Z_1' and Z_2', listed in the cases (i) - (iii) in Lemma 4.7, occurs as Z_1, Z_2 in Lemma 4.8. Further, the combinations of Z_1' and Z_2' in Lemma 4.8 occur as Z_1 and Z_2 in Lemma 4.7. Thus, we obviously can apply both lemmata successively. The result of starting with Lemma 4.7 is formulated in Theorem 4.9.

Theorem 4.9. *Given $Z_1, Z_2 \in \mathcal{Z}_n$ with $Z_2 = {Z_1}^N$, i.e., Z_2 is the normal zigzag triangulation of Z_1. Then their flip distance equals one of the following equations:*

(i)

$$\mathrm{fd}(Z_1, Z_2) = 6k_n - 5 \tag{4.4}$$

iff $n = 4k_n$,

(ii)

$$\mathrm{fd}(Z_1, Z_2) = 6k_n - 4 \tag{4.5}$$

iff $n = (4k_n + 1)$,

(iii)

$$\mathrm{fd}(Z_1, Z_2) = 6k_n - 1 \tag{4.6}$$

iff $n = (4k_n + 3)$,

where $k_n \in \mathbb{N}$. For $k_n > 1$, the equations can be reformulated to their recursive equivalent:

$$\mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(\widetilde{Z_1}, \widetilde{Z_2}) + 6 \tag{4.7}$$

where $\widetilde{Z_1}, \widetilde{Z_2} \in \mathcal{Z}_{\tilde{n}}$ and $\widetilde{Z_2} = \widetilde{Z_1}^N$. The underlying point set $\widetilde{P}$ is a set of $\tilde{n} = n - 4 =$ (i) $4(k_n - 1)$, (ii) $\big(4(k_n - 1) + 1\big)$, (iii) $\big(4(k_n - 1) + 3\big)$ points in convex position.

Proof. For simplicity, we first prove equation (4.7). Obviously, it is a consequence of applying Lemma 4.7 followed by Lemma 4.8. Both lemmata reduce the calculation of the flip distance of two given zigzag triangulations to two zigzag triangulations for which the size of the underlying point set is decreased by two. On the one hand, both lemmata have the precondition $k_n > 1$ and thus, $n > 7$. On the other hand, equation (4.7) has to be valid for $k_n > 1$ and the reduction of the calculation of the flip distance from Z_i to $\widetilde{Z}_i$, $i = 1, 2$, needs the application of both lemmata. Consequently, we have to distinguish between $k_n = 2$ and $k_n > 2$:

$\mathbf{k_n = 2}$:

- $n = 4 * 2 = 8$: At first, we use Lemma 4.7 in order to get

$$\mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(Z_1', Z_2') + 3,$$

  with $Z_1', Z_2' \in \mathcal{Z}_6$ and $Z_2' = (Z_1'(v_1)^I)^N$. Since $n' = 6$, we cannot continue with the application of Lemma 4.8. Hence, we take a closer look to Z_1', Z_2':

  Independent on $\mathrm{ideg}_{Z_1'}(v_2)$, three additional edge flips reduce the calculation of $\mathrm{fd}(Z_1', Z_2')$ to the calculation of the flip distance of two quadrilaterals with different diagonals. See Figure 27. Therefore, equation (4.7) holds for this case.

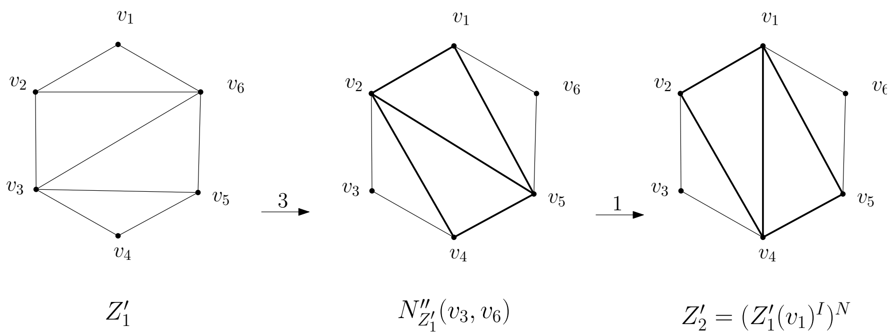


Figure 27: The edge (v_3, v_6) of Z_1' is a (2,2)-diagonal. Three edge flips create triangulation $N''_{Z_1'}(v_3, v_6)$, that has two edges in common with Z_2'. Thus, we have $\mathrm{fd}(Z_1', Z_2') = \mathrm{fd}(Q_1, Q_2) + 3$. Q_1, Q_2 are indicated by bold edges.

- $n = (4 * 2 + 1) = 9$: This case is analog to $n = 8$. Instead of quadrilaterals, we have pentagons. See Figure 28 for the case $\mathrm{ideg}_{Z_1'}(v_2) = 1$.

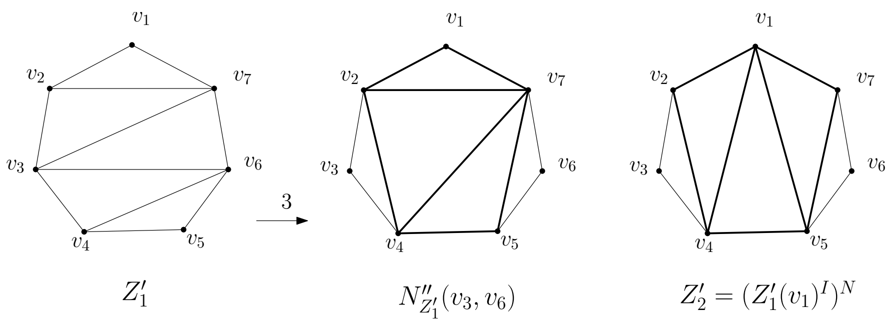


Figure 28: The edge (v_3, v_6) of Z_1' is a (2,2)-diagonal. Three edge flips create triangulation $N''_{Z_1'}(v_3, v_6)$, that has two edges in common with Z_2'. Thus, we have $\mathrm{fd}(Z_1', Z_2') = \mathrm{fd}(P_1, P_2) + 3$. P_1, P_2 are indicated by bold edges.

- $n = (4 * 2 + 3) = 11$: Since Lemma 4.7 and Lemma 4.8 can be applied once without violating the precondition $n > 7$, we refer to case $k_n > 2$.

$\mathbf{k_n > 2}$:

It follows that $n > 11$. We apply Lemma 4.7 followed by Lemma 4.8. The results can be described by following equation:

$$\mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(Z_1', Z_2') + 3 = (\mathrm{fd}(\widetilde{Z_1}, \widetilde{Z_2}) + 3) + 3 = \mathrm{fd}(\widetilde{Z_1}, \widetilde{Z_2}) + 6,$$

with Z_1', Z_2' as described in Lemma 4.7. Note, that the underlying point set of Z_1', Z_2' still has size $n' = (n-2) > 7$. As already mentioned above, each combination of Z_1' and Z_2', listed in that lemma, occurs as Z_1 an Z_2 in Lemma 4.8. Thus, we can again subdivide the expression $\mathrm{fd}(Z_1', Z_2')$ and get two zigzag triangulations $\widetilde{Z_1}, \widetilde{Z_2} \in \mathcal{Z}_{\tilde{n}}$, $\tilde{n} = (n-4)$ with $\widetilde{Z_2} = \widetilde{Z_1}$. Taken together, equation (4.7) is valid for $k_n > 1$.

We prove equations (4.4) - (4.6) by recursively resolving equation (4.7). Assume, that B_1 and B_2 are two zigzag triangulations with $B_2 = B_1^N$. Additionally, let $n_b < 8$ be the size of the underlying point set. Thus, we have:

$$\begin{aligned} \mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(\widetilde{Z_1}, \widetilde{Z_2}) + 1 * 6 = \mathrm{fd}(\widetilde{Z_1}', \widetilde{Z_2}') + 2 * 6 = \dots \\ \dots = \mathrm{fd}(B_1, B_2) + (k_n - 1) * 6, \end{aligned} \tag{4.8}$$

with $\widetilde{Z_1}', \widetilde{Z_2}' \in \mathcal{Z}_{\tilde{n}'}$, $\tilde{n}' = (\tilde{n} - 4) = (n-8) > 8$ and $\widetilde{Z_2}' = \widetilde{Z_2}^N$. In order to complete the resolution of equation (4.8), we have to consider the flip distance of B_1 and B_2. This includes the following base cases:

(i) $n = 1 * 4$. B_1 and B_2 equal to two quadrilaterals with different diagonals. Obviously their flip distance is 1.

(ii) $n = (1 * 4 + 1) = 5$. B_1 and B_2 are two zigzag triangulations of a pentagon. See Figure 29. Independent on $\text{ideg}_{B_1}(v_2)$, their flip distance is 2.

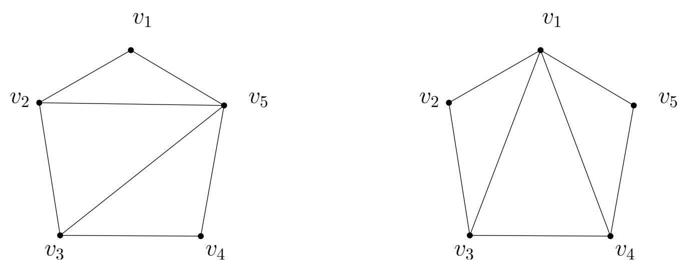


Figure 29: Two normal zigzag triangulations of a pentagon.

(iii) $n = (1*4+3) = 7$. W.l.o.g. let v_1 and v_5 be the tips of the ears of B_1. For $\text{ideg}_{B_1}(v_2) = 1$, v_3 and v_7 are the tips of the ears of B_2, see Figure 30. The edge (v_3, v_7) in B_1 is a (2,2)-diagonal. Thus, the flip distance is 3 plus the flip distance of $N''_{B_1}(v_3, v_7)$ and B_2. They in turn have two common edges: (v_1, v_6) and (v_2, v_4). According to Corollary 4.1(b), the calculation of the flip distance can be reduced to the corresponding sub-triangulations B'_1, B'_2 of $N''_{B_1}(v_3, v_7)$ and B_2 of the point set $P \setminus \{v_3, v_7\}$. Finally, applying Lemma 4.3, the edge flips (v_4, v_6) and (v_2, v_6) create B_2. The total number of flips is $(3 + 2) = 5$.

For $\text{ideg}_{B_1}(v_2) = 2$, the same arguments can be applied.

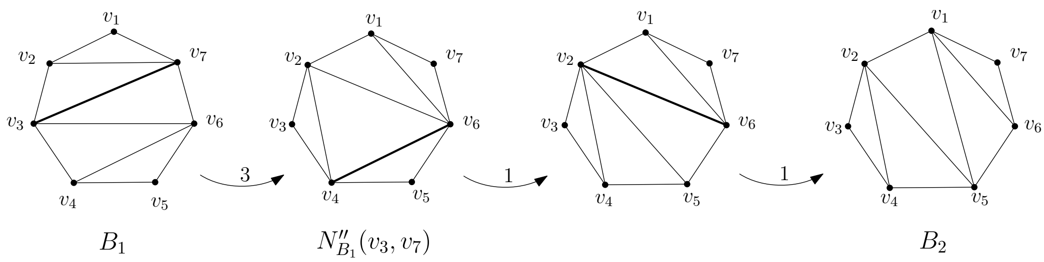


Figure 30: Triangulations on the shortest path from B_1 to B_2.

In summary, we have

$$\mathrm{fd}(Z_1, Z_2) = \dots$$

$$\dots = \mathrm{fd}(B_1, B_2) + (k_n - 1) * 6 = \begin{cases} (i) & \big(1 + (k_n - 1) * 6\big) = 6k_n - 5 \\ (ii) & \big(2 + (k_n - 1) * 6\big) = 6k_n - 4 \\ (iii) & \big(5 + (k_n - 1) * 6\big) = 6k_n - 1 \end{cases}$$

□

Alternatively, we can assume that the initial triangulations Z_1, Z_2 satisfy the preconditions of Lemma 4.8, cases (ii) - (iv). Thus, we can apply Lemma 4.8 on Z_1, Z_2, followed by Lemma 4.7. The results are formulated in Theorem 4.10.

Remark 4.10. We have to exclude case (i) of Lemma 4.8 because the reduced triangulations Z_1', Z_2' do not exist as initial triangulations Z_1, Z_2 in Lemma 4.7. (In fact, we have $Z_1', Z_2' \in \mathcal{Z}_{4k_n+2}$ and $Z_2' = (Z_1')^N$, see Lemma 4.8.)

Theorem 4.10. *Given $Z_1, Z_2 \in \mathcal{Z}_n$ with*

(i) $n = (4k_n + 1)$ *and*

- *(a)* $\mathrm{ideg}_{Z_1}(v_2) = 1$ *and* $Z_2 = (Z_1(v_k)^I)^N$, *or*
- *(b)* $\mathrm{ideg}_{Z_1}(v_2) = 2$ *and* $Z_2 = (Z_1(v_1)^I)^N$,

(ii) $n = (4k_n + 2)$ *and* $Z_2 = (Z_1(v_1)^I)^N$,

(iii) $n = (4k_n + 3)$ *and*

- *(a)* $\mathrm{ideg}_{Z_1}(v_2) = 1$ *and* $Z_2 = (Z_1(v_1)^I)^N$, *or*
- *(b)* $\mathrm{ideg}_{Z_1}(v_2) = 2$ *and* $Z_2 = (Z_1(v_k)^I)^N$,

$k_n \in \mathbb{N}$. *Then the flip distance of Z_1 and Z_2 equals one of the following equations:*

(i)

$$\mathrm{fd}(Z_1, Z_2) = 6k_n - 4 \tag{4.9}$$

(ii)

$$\mathrm{fd}(Z_1, Z_2) = 6k_n - 2 \tag{4.10}$$

(iii)

$$\mathrm{fd}(Z_1, Z_2) = 6k_n - 1 \tag{4.11}$$

Proof. For $k_n > 1$, we apply Lemma 4.8, followed by Theorem 4.9. Thus, we have

$$\mathrm{fd}(Z_1, Z_2) = \mathrm{fd}(Z_1', Z_2') + 3 = \begin{cases} (i) & (6(k_n - 1) - 1) + 3 = 6k_n - 4 \\ (ii) & (6k_n - 5) + 3 = 6k_n - 2 \\ (iii) & (6k_n - 4) + 3 = 6k_n - 1 \end{cases}$$

with $Z_1', Z_2' \in \mathcal{Z}_{n'}$, $Z_2' = Z_1'^N$ and $n' = (n-2)$. Note, that k_n decreases by one for case (i):

$$n' = (n-2) = \Big((4k_n + 1) - 2\Big) = \Big(4(k_n - 1) + 3\Big)$$

Hence, Theorem 4.10 holds for $k_n > 1$. We conclude the proof with the base case $k_n = 1$ for the cases (ii) - (iii):

(i) $n = (4 * 1 + 1) = 5$:
For this case, Z_1 and Z_2 are two triangulations of a pentagon, see Figure 31. The flip distance is 2: Applying Lemma 4.3 on both diagonals of Z_1 results in Z_2. Thus, we have:

$$\mathrm{fd}(Z_1, Z_2) = 2 = 4 = (6 * 1 - 4).$$

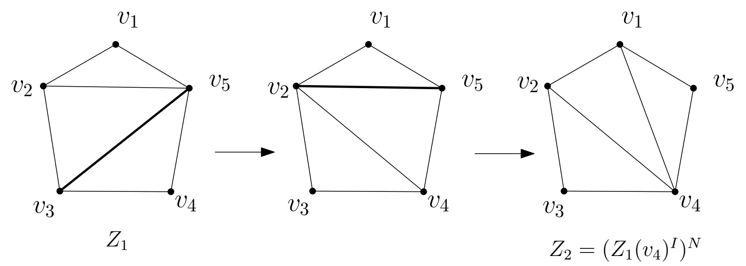


Figure 31: A flip sequence with minimum length from Z_1 to Z_2.

(ii) $n = (4 * 1 + 2) = 6$:
According to the proof of equation (4.7) in Theorem 4.9, we have

$$\mathrm{fd}(Z_1, Z_2) = \Big(\mathrm{fd}(Q_1, Q_2) + 3\Big) = 4 = (6 * 1 - 2),$$

where Q_1, Q_2 are two quadrilaterals with different diagonals. See Figure 27.

(iii) $n = (4 * 1 + 3) = 7$:
Again, we refer to the proof of equation (4.7) in Theorem 4.9. Consequently, we have

$$\mathrm{fd}(Z_1, Z_2) = \Big(\mathrm{fd}(P_1, P_2) + 3\Big) = 5 = (6 * 1 - 1),$$

where P_1, P_2 are two zigzag triangulations of a pentagon with $P_2 = P_1^N$, see Figure 28 and 29. According to the proof of the base case of equation (4.6) in Theorem 4.9, we have $\mathrm{fd}(P_1, P_2) = 2$. □

Finally, we consider the flip distance of $Z_1, Z_2 \in \mathcal{Z}_n$ with $Z_2 = Z_1^N$ and $n = (4k_n + 2)$, $k_n \in \mathbb{N}$. Recall that according to Lemma 4.1 on page 35, the edge connecting the tips of Z_2 is not a diagonal of Z_1. Thus, Theorem 4.6 cannot be applied. Nevertheless, v_j and v_l satisfy the preconditions of Theorem 4.5 and Corollary 4.2 for $k_n > 1$. That leads us to the following Theorem:

Theorem 4.11. *Given $Z_1, Z_2 \in \mathcal{Z}_n$ with $Z_2 = Z_1^N$ and $n = (4k_n+2)$, $k_n > 1$, $k_n \in \mathbb{N}$. Then the flip distance of Z_1 and Z_2 is bounded by*

$$(5k_n - 2) \leq \mathrm{fd}(Z_1, Z_2) \leq (7k_n - 4). \tag{4.12}$$

Proof. For $k_n > 1$, v_j and v_l have at least distance $\frac{n-2}{4} = 2$ to the tips of Z_1 (see proof of Lemma 4.1). Consequently, we have

$$\mathrm{ideg}_{Z_1}(v_j) = \mathrm{ideg}_{Z_1}(v_l) = 2.$$

Thus, both vertices fulfill the preconditions of Corollary 4.2. Assume that $T_1 = N_{Z_1}(v_j)$. Then Corollary 4.2 applied on vertex v_j leads to the following estimate:

$$\Big(\mathrm{fd}(T_1, Z_2) + 1\Big) \leq \mathrm{fd}(Z_1, Z_2) \leq \Big(\mathrm{fd}(T_1, Z_2) + 2\Big).$$

Figure 32 depicts an example of the triangulations. Because v_j and v_l are not connected by an edge in Z_1, v_l still has two incident diagonals in T_1. According to Corollary 4.2, we get

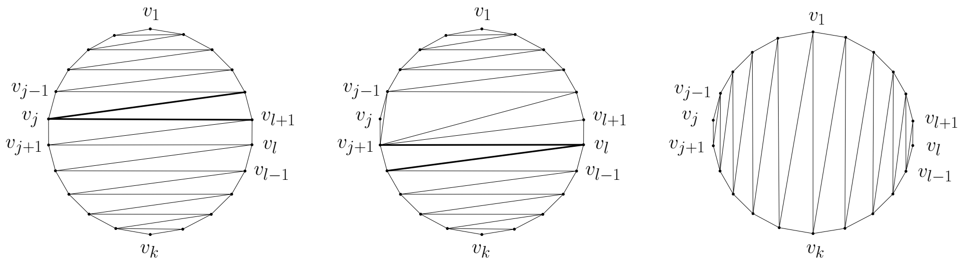
$$\Big(\mathrm{fd}(T_2, Z_2) + 1\Big) \leq \mathrm{fd}(T_1, Z_2) \leq \Big(\mathrm{fd}(T_2, Z_2) + 2\Big)$$

with $T_2 = N_{T_1}(v_l)$ and consequently,

$$\Big(\mathrm{fd}(T_2, Z_2) + 2\Big) \leq \mathrm{fd}(Z_1, Z_2) \leq \Big(\mathrm{fd}(T_2, Z_2) + 4\Big).$$

T_2 and Z_2 share two edges: (v_{j-1}, v_{j+1}) and (v_{l-1}, v_{l+1}). Thus, we can subdivide the calculation of the flip distance of T_2 and Z_2:

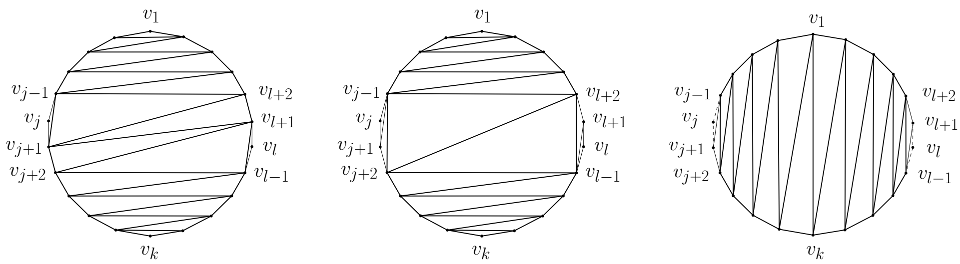
$$\mathrm{fd}(T_2, Z_2) = \mathrm{fd}(T_2', Z_2'),$$



(a) Z_1 (b) $T_1 = N_{Z_1}(v_j)$ (c) $Z_2 = Z_1^N$

Figure 32: 32(a) $Z_1 \in \mathcal{Z}_{4k_n+2}$. 32(b) The normalized triangulation of Z_1 with respect to v_j. 32(c) Z_2, the normal zigzag triangulation of Z_1.

where T_2' and Z_2' are the sub-triangulations of T_2 and Z_2 on the point set $P' = P \setminus \{v_j, v_l\}$, $|P'| = (n-2)$. Furthermore, $Z_2' = (T_2'(v_1)^I)^N$, see Figure 33.



(a) $T_2' \subset T_2 = N_{T_1}(v_l)$ (b) $\widetilde{Z_1} \subset N''_{T_2}(v_{j+1}, v_{l+1})$ (c) $Z_2' \subset \widetilde{Z_2} \subset Z_2 = Z_1^N$

Figure 33: 33(a) T_2' is a sub-triangulation of T_2. Bold edges belong to T_2'. Edge (v_{j+1}, v_{l+1}) is a $(2,2)$-diagonal. 33(b) Bold edges are part of $\widetilde{Z_1}$, a sub-triangulation of $N''_{T_2}(v_{l+1}, v_{j+1})$. 33(c) $\widetilde{Z_2}$, indicated by the bold edges, is a sub-triangulation of Z_2. Z_2' consists of Z_2 less the dashed edges.

According to Lemma 4.7, we have

$$\mathrm{fd}(T_2', Z_2') = \Big(\mathrm{fd}(\widetilde{Z_1}, \widetilde{Z_2}) + 3\Big)$$

with $\widetilde{Z_1}, \widetilde{Z_2} \in \mathcal{Z}_{n-4}$ and $\widetilde{Z_2} = \widetilde{Z_1}^N$. Hence, we have again the initial problem on the reduced point set $\widetilde{P} = P \setminus \{v_j, v_{j+1}, v_l, v_{l+1}\}$ with size $\tilde{n} = (n-4) = \Big(4(k_n - 1) + 2\Big)$:

$$\Big(\mathrm{fd}(\widetilde{Z_1}, \widetilde{Z_2}) + 5\Big) \le \mathrm{fd}(Z_1, Z_2) \le \Big(\mathrm{fd}(\widetilde{Z_1}, \widetilde{Z_2}) + 7\Big).$$

We resolve that inequation recursively to

$$\Big(\mathrm{fd}(B_1, B_2) + (k_n - 1) * 5\Big) \leq \mathrm{fd}(Z_1, Z_2) \leq \Big(\mathrm{fd}(B_1, B_2) + (k_n - 1) * 7\Big)$$

with $B_1, B_2 \in \mathcal{Z}_6$ and $B_2 = B_1^N$. Finally, we conclude the proof with $\mathrm{fd}(\widetilde{Z}_1, \widetilde{Z}_2) = 3$ (see Figure 34).

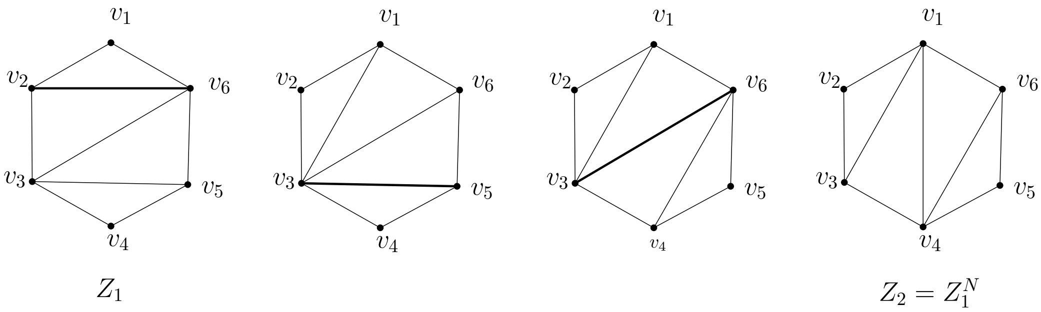


Figure 34: A shortest flip sequence from Z_1 to Z_2 .

□

Remark 4.11. For $Z_1, Z_2 \in \mathcal{Z}_{4k_n+2}$, $Z_2 = Z_1^N$ and $k_n \in \{1, \ldots, 4\}$, calculations showed that the flip distance is given by $3, 10, 17$ and 24. Those values correspond to the upper bound of Theorem 4.11.

4.3 From the shortest flip sequence to the degree bounded setting

In this section we analyze the flip sequences that arise in the proofs of Theorem 4.9 and 4.10. Assume, that Z_1 andZ_2 satisfy the preconditions of one of the two aforementioned theorems. Then we can observe that the flip sequence corresponds to the repetitive and recursive application of Transformation 1a and/or 1b, defined in the proof of Lemma 4.7. Each time two transformations are applied, a triangulation that has 4 edges in common with the other one is created. Since the number of diagonals of a triangulation with an underlying convex point set is given by $n - 3$, the transformations can be applied $\lfloor \frac{n-3}{4} \rfloor$ times. The final edge flips equal the base cases listed in the proof of Theorem 4.9.

If we consider that flip sequence (see Figure 35) for $\mathbf{n = 4k_n}$ and $\mathbf{Z_2 = (Z_1)^N}$ for the entire point set and its corresponding temporary triangulations, which are all part of the shortest path from Z_1 to Z_2, the equality to the flip sequence described in the proof of Lemma 4 by Aichholzer et al., [3], becomes visible. In their paper that flip sequence is used in the context of proving that two normal zigzag triangulations can be transformed into each other in $\mathcal{O}(n)$ time without exceeding a vertex degree of $k > 6$. The following arguments are used:

Applying Transformation 1a and Transformation 1b, respectively, on Z_1 creates a triangulation Z_1' that can be divided into five sub-triangulations: a quadrilateral Q with diagonal d_Q, which delimits four zigzags: two shrinking ones at the tip of the ears of Z_1 (consisting of edges of Z_1) and two growing zigzags at the tip of the ears of Z_2 (containing edges equal to Z_2), see Figure 35.

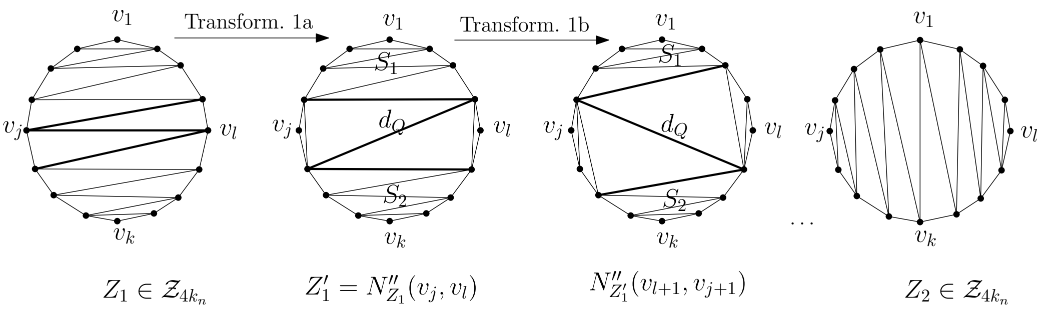


Figure 35: Transformation 1a and 1b applied recursively.

Regarding the vertex degree bound, the following invariant is established: Each vertex $v \in P$ is incident to two convex hull edges, at most two of four zigzags and maybe the diagonal d_Q. That results in a maximum degree of 7. Each subsequent application of Transformation 1a or 1b (except the last one) can alternatively be described by the following steps:

Step 1. Flip diagonal d_Q.

Step 2. Flip the two edges that are common to Q and the shrinking zigzags. Note that this step includes exactly two edge flips, before we can apply Step 1 again. (Obviously, this case is only needed, if the shrinking zigzags are not empty. Otherwise the target triangulation has already been created by Step 1.)

All the resulting triangulations of those flips have the same structure as Z_1'. Therefore, the invariant holds and the degree restriction is never violated.

In case of $\mathbf{Z_2} = (\mathbf{Z_1}(\mathbf{v_1})^{\mathbf{I}})^{\mathbf{N}}$ and $\mathbf{n} = (\mathbf{4k_n} + \mathbf{2})$ the invariant holds, too: Let $\hat{Z}_1, \hat{Z}_2 \in \mathcal{Z}_{4(k_n+1)}$ be two zigzag triangulations with $\hat{Z}_2 = \hat{Z}_1^N$ and say that $\hat{Z}_1'$ is the result of Transformation 1a or Transformation 1b, respectively. It arises from the proof of equation (4.7) in Theorem 4.9 that Z_1 and Z_2 appear as sub-triangulations of triangulation $\hat{Z}_1'$ and $\hat{Z}_2$. Since the vertex degree of a sub-triangulation can only decrease, the invariant is true for Z_1 and Z_2.

For $Z_1, Z_2 \in \mathcal{Z}_{\mathbf{4k_n+1}}$ and $\mathbf{Z_2 = Z_1^N}$, the flip sequence that arises from the proof of Theorem 4.9 corresponds to the repetitive and recursive application of Transformation 1a (or Transformation 1b in case of $\mathrm{ideg}_{Z_1}(v_2) = 2$), see Figure 36. Alternatively, it can be presented as follows:

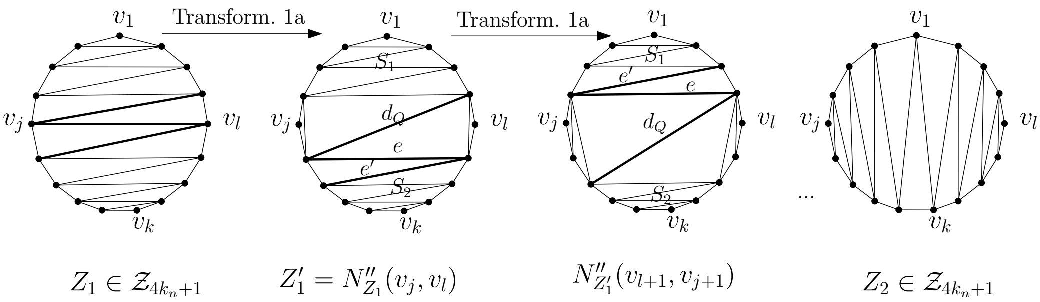


Figure 36: The repetitive recursive application of Transformation 1a leads to Z_2.

1. The first application of Transformation 1a (or 1b, in case of $\mathrm{ideg}_{Z_1}(v_2) = 2$) on Z_1 creates a triangulation with the same structure as Z_1' (case $n = 4k_n$ above): A quadrilateral Q with diagonal d_Q delimits four zigzags: two shrinking ones with tips at v_1 and v_k and two growing ones with tips at v_j and v_l.

2. Alternately, we perform the edge flips described in the following by 2a and 2b in a loop, until only one edge of one shrinking zigzag remains:

   (a)
   - flip the edge e that is common to Q and the shrinking zigzag at v_k. A new structure appears: A path triangle t_p consisting of d_Q and the flip target of edge e, d_e. t_p is adjacent to two inner triangles t_1, t_2, where each delimits one shrinking zigzag and one growing one.
   - flip edge d_Q. The structure created by the flip before is replaced by a quadrilateral Q', which has the same properties as Q.
   - flip edge e', which is common to Q' and the shrinking zigzag at v_k. A new quadrilateral Q'' is created. $Q := Q''$.

   (b) Replace v_k in 2a by v_1.

For $\mathrm{ideg}_{Z_1}(v_2) = 1$, start the loop with 2a. (Otherwise begin with the edge flips in 2b.) As soon as only one edge of the shrinking zigzags remains, the flip sequence can be completed by the corresponding edge flips mentioned in the base cases.

The new structure that appears after the first edge flip described in 2a and 2b above claims for an extension of the before defined **invariant**: Each vertex $v \in P$ is incident to two convex hull edges, at most two zigzags and at most two diagonals d_Q and d_e. In order to assure that the maximum vertex degree remains less or equal 7 during the whole transformation from Z_1 to Z_2, the following lemma is needed:

Lemma 4.12. *Given $Z_1, Z_2 \in$ (a) $\mathcal{Z}_{4k_n+1}$, or (b) $\mathcal{Z}_{4k_n+3}$ with $Z_2 = Z_1^N$ and $T_i \in \mathcal{P}$, where $\mathcal{P}$ is the shortest path from Z_1 to Z_2 that arises from the above presented flip sequence. Assume that vertex $v \in P$ is incident to two convex hull edges, two zigzags and two diagonals d_Q and d_e in T_i. Furthermore, the zigzags incident to v are called z_s (the shrinking one) and z_g (the growing one). Then z_g has the property, that $\operatorname{ideg}_{z_g}(v) = 1$, i.e., v has only one incident edge that belongs to z_g.*

Proof. Assume that $\operatorname{ideg}_{z_g}(v) = 2$. Then the edge flips e and e' described in 2a and 2b above would stop the growth of the zigzag z_s, which leads to a contradiction. Figure 37 shows an example. $\square$

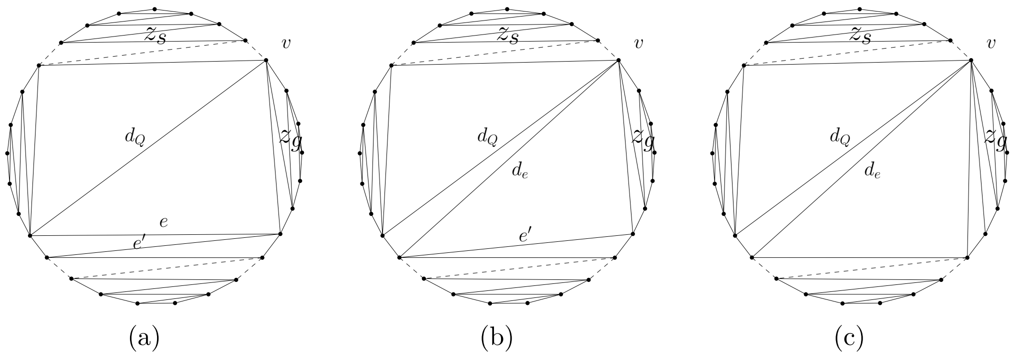


Figure 37: 37(a)Triangulation T, which has the same structure as Z_1'. 37(b)edge e is flipped to diagonal d_e, 37(c)edge e' flipped, too. The flip target is incident to v

For $\mathcal{Z}_{\mathbf{4k_n+3}}$ and $\mathbf{Z_2 = Z_1^N}$, we can use the same arguments as for $\mathcal{Z}_{4k_n+1}$. Note, that for $\operatorname{ideg}_{Z_1}(v_2) = 1$, we have to apply Transformation 1b in a repetitive manner, instead of Transformation 1a before. For $\operatorname{ideg}_{Z_1}(v_2) = 2$, we use Transformation 1a. Figure 38 gives an example.

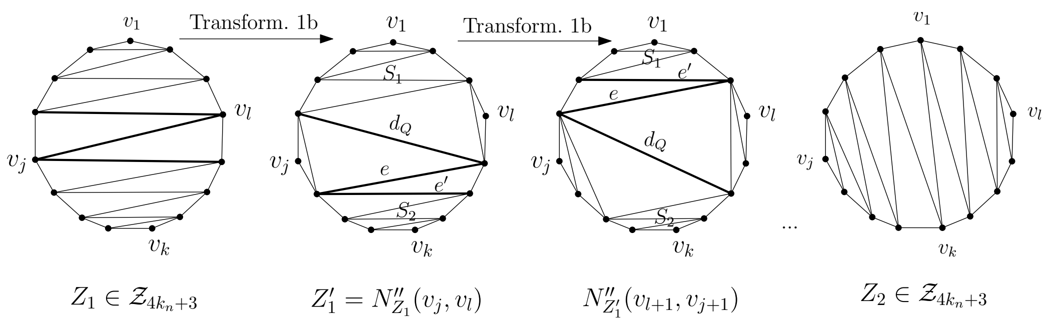


Figure 38: Transformation 1b applied recursively.

Finally, we consider the remaining cases

$$\begin{aligned} \mathrm{ideg}_{Z_1}(v_2) = 1,\ & \mathbf{n = 4k_n + 1} \text{ and } \mathbf{Z_2 = (Z_1(v_k)^I)^N} \\ & \mathbf{n = 4k_n + 3}, \text{ and } \mathbf{Z_2 = (Z_1(v_1)^I)^N} \\ \mathrm{ideg}_{Z_1}(v_2) = 2\ & \mathbf{n = 4k_n + 1} \text{ and } \mathbf{Z_2 = (Z_1(v_1)^I)^N} \\ & \mathbf{n = 4k_n + 3}, \text{ and } \mathbf{Z_2 = (Z_1(v_k)^I)^N}. \end{aligned}$$

We can use the same arguments that are used in the case $Z_2 = (Z_1(v_1)^I)^N$, $n = (4k_n + 2)$. Hence, the extended invariant holds and the maximum vertex degree of 7 is not exceeded during the whole transformation.

Taken together, we can extend Theorem 4.9 and 4.10 to

Theorem 4.13. *Given $Z_1, Z_2 \in \mathcal{Z}_n$. According to Theorem 4.9 and 4.10, the flip distance of Z_1 to Z_2 is given by*

(a) $(6k_n - 5)$ *flips, if* $n = 4k_n$ *and* $Z_2 = Z_1^N$,

(b) $(6k_n - 4)$ *flips, if* $n = (4k_n + 1)$ *and*

- *(1)* $Z_2 = Z_1^N$,
- *(2)* $Z_2 = (Z_1(v_k)^I)^N$ *and* $\mathrm{ideg}_{Z_1}(v_2) = 1$,
- *(3)* $Z_2 = (Z_1(v_1)^I)^N$ *and* $\mathrm{ideg}_{Z_1}(v_2) = 2$,

(c) $(6k_n - 2)$ *flips, if* $n = (4k_n + 2)$ *and* $Z_2 = (Z_1(v_1)^I)^N$,

(d) $(6k_n - 1)$ *flips, if* $n = (4k_n + 3)$ *and*

- *(1)* $Z_2 = Z_1^N$,
- *(2)* $Z_2 = (Z_1(v_1)^I)^N$ *and* $\mathrm{ideg}_{Z_1}(v_2) = 1$,

(3) $Z_2 = (Z_1(v_k)^I)^N$ *and* $\mathrm{ideg}_{Z_1}(v_2) = 2$,

where v_1 *and* v_k *are the tips of* Z_1. *There exists a corresponding shortest path* $\mathcal{P} : T_0 = Z_1, T_1, \ldots, T_{p-1}, T_p = Z_2$ *with the following property:*

$$\forall T_i \in \mathcal{P} : T_i \in \mathcal{T}_{n,k},$$

where $\mathcal{T}_{n,k}$ *is the set of all triangulations of* P *with a maximum vertex degree of* $k > 6$.

Proof. Follows from the above considerations. $\square$

5 Software

This section describes a computer program that approximates the flip distance of two (pointed pseudo-)triangulations in the degree bounded setting by several heuristics. In other words, the program computes a flip sequence that transforms two given bounded degree (pointed pseudo-)triangulations without exceeding the vertex degree bound k. Additionally, the construction of the flip graph of a given point set is implemented. That enables the user to calculate the flip distance of two (pointed pseudo-)triangulations in the degree bounded setting for very small point sets. In both cases, the resulting flip sequence can be traced step by step. The bounded degree (pointed pseudo-)triangulations have to be defined by the user. Since many implemented functions are the same for triangulations and pointed pseudo-triangulations, we denote the initial user-defined triangulations or pointed pseudo-triangulations by T_1 and T_2. As soon as differences occur, we will specify the type. Additionally, we reduce our considerations concerning pseudo-triangulations to pointed pseudo-triangulations in this section. Therefore, we will omit the word "pointed" in order to improve the readability.

Recall that the expression $e \in T$ stands for a (pseudo-)triangulation T that contains the edge e. Furthermore, we denote the result of removing the edge e of T by $T \setminus e$. For brevity, we write $\#(e, T_j)$ for the number of interior intersections between $e \in T_i$ and the edges of T_j, $i, j \in \{1, 2\}$ and $i \neq j$. $T_1 \cup T_2$ stands for the result of adding all edges of T_2 to T_1.

5.1 Description of the implemented heuristics

Basically, all implemented heuristics have the same principle: First, different types of edge weights as described in section 5.1.1 are assigned to each edge of T_1 and T_2. Then the edges are enqueued in a priority queue Q. The prioritization of the edges depends on the Comparator that is assigned to the heuristic. (A Comparator compares the weights of two edges. All implemented Comparators are presented in section 5.1.2.) Afterwards, the two following steps are repeated until either the flip sequence that transforms T_1 and T_2 into each other is found or an infinite loop appears:

- Flip the edge that is the first element in Q.
- Update the edge weights that are affected by the flip as well as Q.

5.1.1 Edge weights

For triangulations two kinds of edge weights are defined: the edge crossing weight and the node degree weight. For pseudo-triangulations a third weight that deals with the pointedness of the vertices incident to the considered edge is added: the non-pointed weight. For each kind of edge weight we can say that the higher the value of the weight the higher the priority to flip that edge.

In the following, we assume that T_i and T_j ($T_i \neq T_j$) are two arbitrary (pseudo-)triangulations on the same point set that have a maximum vertex degree k. Further, we let $e' = (v'_k, v'_l)$ be the flip target of $e = (v_k, v_l) \in T_i$ in case e is flippable with $\{v_k, v'_k, v_l, v'_l\} \subseteq P$. Additionally, we call T'_i the result of flipping e to e' in T_i.

Edge crossing weights There exist two different types of edge crossing weights:

1. The edge crossing weight $ecw_1(e)$ depends on the number of interior intersections of e with the edges in the other given (pseudo-)triangulation. Thus, we have
$$ecw_1(e) := \#(e, T_j).$$

2. Since $ecw_1(e)$ disregards $\#(e', T_j)$, we define $ecw_2(e)$ by the difference between $\#(e, T_j)$ and $\#(e', T_j)$. If e is not flippable, we assign to $ecw_2(e)$ the value $-\infty$.

$$ecw_2(e) := \begin{cases} \#(e, T_j) - \#(e', T_j) & \text{if } e \text{ is flippable, or} \\ -\infty & \text{otherwise.} \end{cases}$$

Node degree weights We subdivide the node degree weights into three categories:

1. The node degree weight $ndw_1(e)$ can be seen as an indicator whether e is flippable without exceeding k.

$$ndw_1(e) := \begin{cases} 1 & \begin{array}{l}\text{if } e \text{ is flippable and} \\ \deg_{T_i \setminus e}(v'_k) < k \text{ and } \deg_{T_i \setminus e}(v'_l) < k,\end{array} \\ -\infty & \text{otherwise.} \end{cases}$$

2. In case e is flippable, $ndw_2(e)$ equals the difference of the vertex degree bound k and the maximum among the vertex degrees of the vertices incident to e'. If e is not flippable, we assign to $ndw_2(e)$ the value $-\infty$. Thus, the higher the value of $ndw_2(e)$ the smaller the maximum vertex degree of v'_k and v'_l.

$$ndw_2(e) := \begin{cases} k - \max\Big(\deg_{T'_i}(v'_k), \deg_{T'_i}(v'_l)\Big) & \begin{array}{l}\text{if } e \text{ is flippable and}\\ \deg_{T_i \setminus e}(v'_k) < k \text{ and}\\ \deg_{T_i \setminus e}(v'_l) < k,\end{array} \\ -\infty & \text{otherwise.} \end{cases}$$

3. The third type of node degree weight takes the maximum vertex degree of v_k and v_l as well as the maximum vertex degree of v'_k and v'_l into account:

$$ndw_3(e) := \begin{cases} \begin{array}{l}\max\Big(\deg_{T_i}(v_k), \deg_{T_i}(v_l)\Big) - \\ \max\Big(\deg_{T'_i}(v'_k), \deg_{T'_i}(v'_l)\Big)\end{array} & \begin{array}{l}\text{if } e \text{ is flippable and}\\ \deg_{T \setminus e}(v'_i) < k \text{ and}\\ \deg_{T \setminus e}(v'_j) < k,\end{array} \\ -\infty & \text{otherwise.} \end{cases}$$

Non-pointed weights This kind of edge weight is intended for an additional weight for edges of pseudo-triangulations. Two different types are implemented:

1. $npw_1(e)$ takes the number of pointed vertices in $T_i \cup T_j$ incident to e into consideration:

$$npw_1(e) := \begin{cases} 2 & \text{if } v_k \text{ and } v_l \text{ are non-pointed in } T_i \cup T_j \\ 1 & \text{if either } v_k \text{ or } v_l \text{ are non-pointed in } T_i \cup T_j \\ 0 & \text{if } v_k \text{ and } v_l \text{ are pointed in } T_i \cup T_j. \end{cases}$$

2. For the second type of non-pointed weight the pointedness of the vertices incident to e' contributes to the value of the weight:

$$npw_2(e) := \begin{cases} npw_1(e) & \text{if } v'_k \text{ and } v'_l \text{ are pointed in } T_i \cup T_j \\ npw_1(e) - 1 & \text{if either } v'_k \text{ or } v'_l \text{ are non-pointed in } T_i \cup T_j \\ npw_1(e) - 2 & \text{if } v'_k \text{ and } v'_l \text{ are non-pointed in } T_i \cup T_j. \end{cases}$$

5.1.2 Comparators

Each edge weight contributes to the prioritization in Q. The extent of each weight depends on the Comparator. Since there exists an additional edge weight for $T_1, T_2 \in \mathcal{PPT}_P$, we distinguish Comparators for triangulations and Comparators for pseudo-triangulations. The following Comparators are implemented for $\mathbf{T_1}, \mathbf{T_2} \in \mathcal{T}_{\mathbf{P,k}}$:

1. The prioritization of the first Comparator is based on the product of the edge crossing weight and node degree weight.

2. The second Comparator sorts Q according to the edge crossing weight. If two edges have the same value of the edge crossing weight, this Comparator takes the node degree weight into consideration.

3. The third Comparator priorizes the edges according to the product of the edge crossing weight and the node degree weight. In case of equality, the node degree weight is used.

4. The prioritization of the fourth Comparator is based on the node degree weight.

For $\mathbf{T_1}, \mathbf{T_2} \in \mathcal{PPT}_{\mathbf{P,k}}$, the following Comparators exist:

1. The first Comparator sorts the edges according to the edge crossing weight. If two edges have the same value of the edge crossing weight, the non-pointed weight is used. If the non-pointed weights are also equal, the node degree weight is taken into consideration.

2. The prioritization of the second Comparator is based on the node degree weight. In case of equality the non-pointed weight is used.

5.2 Description of the flip graph construction and related calculations

Since the number of (pseudo-)triangulations of a given point set is exponential in the size of the underlying point set, the construction of the flip graph and hence all related calculations are time-consuming processes. It is recommended to use them only in the case of very small point sets.

5.2.1 Construction of the (subgraph of the) flip graph

Independent of the vertex degree bound, at first the complete flip graph $FG(\mathcal{T}_P)$ (for triangulations) or $FG(\mathcal{PPT}_P)$ (for pseudo-triangulations) is constructed. Since both flip graphs are connected, breadth-first search can be used for the construction. The initial node corresponds to T_1. The subgraphs $FG(\mathcal{T}_{P,k})$ (for triangulations) and $FG(\mathcal{PPT}_{P_k})$ (for pseudo-triangulations) are built based on the complete flip graph.

5.2.2 Calculation of the flip distance

There are two possible ways to calculate the flip distance of T_1 and T_2.

1. *Flip Graph → Calculate Flip Distance*: If the complete flip graph has already been constructed before, the subgraph $FG(\mathcal{T}_{P,k})$ or $FG(\mathcal{PPT}_{P,k})$ is created. The shortest path between T_1 and T_2 in that subgraph is calculated with Dijkstra's algorithm. In fact, Dijkstra's algorithm can be replaced by using again breadth-first search, since we have no edge weights in the subgraph.

2. Under *Flip Graph → Calculate Parital Flip Graph* a connected component $\mathcal{C}$ of $FG(\mathcal{T}_{P,k})$ or $FG(\mathcal{PPT}_{P,k})$ is created by breadth-first search. The initial node corresponds to T_1. In this case, the vertex degree bound is taken into consideration for the construction. Thus, $\mathcal{C}$ only consists of nodes that correspond to (pseudo-)triangulations with maximum vertex degree k and are connected to T_1. The construction stops as soon as

   - either T_2 is an element of $\mathcal{C}$ or
   - the maximum depth defined by the user is reached or
   - T_1 and T_2 are not in the same connected component of $FG(\mathcal{T}_{P,k})$ or $FG(\mathcal{PPT}_{P,k})$.

5.2.3 Calculation of the diameter of the flip graph

For each $T \in FG(\mathcal{T}_{P,k})$ or $FG(\mathcal{PPT}_{P,k})$ the node at the maximum length of the shortest path to T is calculated by Dijkstra's algorithm. The calculation of the diameter in this way is definitely the most time-consuming process among the calculations presented in this section. Hence, the implementation of a heuristic would be a more efficient approach for that purpose.

5.3 Program control

The user interface of the program contains two areas that are intended to display the initial (pseudo-)triangulations T_1 and T_2. As soon as they are set, the user has the possibility to trace the resulting flip sequences of different implemented heuristics or even the shortest flip sequence. Moreover, the diameter of the flip graph for the given point set can be calculated. As already mentioned before, the calculation of the exact flip distance as well as the calculation of the diameter of the flip graph is only recommended in the case of very small point sets.

5.3.1 Defining T_1 and T_2

The definition of T_1 and T_2 consists of two steps: At first, the underlying point set has to be defined. Afterwards, the edges can be set. For both steps there are different ways that we will present now:

Defining the underlying point set In the *Edit* menu, different functions to define a point set can be found. Independent of the chosen function, a point editor appears. A provided area in that editor enables the user to set points via left mouse click in that area and delete points by marking a point (via left mouse click on this point) followed by pressing the *Delete Point* button.

1. The selection of *New Point Set* enables the user to define a completely new point set.
2. The *Convex Point Set* function creates a point set in convex position of the user defined size. Note that the size is limited by at least 3 and at most 100.
3. Under *Random Point Set* a set of random points is created. The number of points depends on the user input and has to lie between 3 and 300.
4. The *Load Point Set* function allows the user to load the point set from a file that must not violate the format specified in section 5.3.3.

Independent of the chosen item, there is the possibility to edit the points. If the point set is loaded from a file, a subsequent modification is not stored automatically in the file. This only can be done afterwards by using the *Save Points As...* function under the *File* menu.

After pressing the *OK* button, the editor disappears and the defined point set is displayed in the two areas of the main frame.

Defining the edge set of T_1 and T_2 and the vertex degree bound Before the user has the opportunity to define the edge set of T_1 and T_2, a number-input for the vertex degree bound is required. Afterwards, an editor for (pseudo-)triangulations allows the user to flip, delete and set edges. It contains an area, where at least the predefined point set appears. A left mouse click on two points in this area results in either a new edge or marks an existing edge. The latter allows the user to delete that edge via the *Delete Edge* button or to flip it by pressing the *Flip Edge* button. A right mouse click on an edge is the alternative way to flip an edge. There are three possibilities to set the edges.

1. The selection of *Edit → New (Pseudo-)Triangulation 1/2* allows the user to create a new (pseudo-)triangulation. Only the convex hull edges of the underlying point set are already set.

2. Under *Edit → Edit (Pseudo-)Triangulation 1/2* the (pseudo-)triangulations, which are displayed in the main frame, can be modified by the (pseudo-)triangulations editor.

3. Under *Edit → Load (Pseudo-)Triangulation 1/2* a (pseudo-)triangulation can be loaded from a file. The corresponding (pseudo-)triangulation appears in the (pseudo-)triangulation editor. The demanded file format is described in section 5.3.3.

The edge sets of T_1 and T_2 can be stored by selecting the item *Save (Pseudo-)Triangulation 1 /2* in the *File* menu.

For the calculation of the diameter of the flip graph at least one (pseudo-)triangulation has to be defined. This function can be found under *Flip Graph → Calculate Diameter of Flip Graph*. As soon as the calculation is completed, a dialog with the information of the diameter pops up, which additionally gives the user the possibility to store all pairs of (pseudo-)triangulations that have a flip distance equal to the diameter. If we assume that there are p different (pseudo-)triangulations $T_{i,j}$, $0 \leq j < p$, for a (pseudo-)triangulation T_i that are all elements of $\mathcal{T}_{p,k}$ or $\mathcal{PPT}_{P,k}$, respectively, and have the property that $\mathrm{fd}(T_i, T_{i,j})$ equals the diameter of the flip graph, then the files corresponding to these (pseudo-)triangulations have the following names: "t[i]_diameter.txt" and "t[i]_[j]_diameter.txt".

Further, the flip graph and its corresponding (pseudo-)triangulations can be saved optionally. The interpretation of the flip graph file is described in section 5.3.3. This function has been developed to get an overview of the flip graph. Loading the flip graph from a file has not been implemented.

As soon as T_1 and T_2 are displayed in the predefined areas in the main frame and the vertex degree bound of them is equal, the user has the possibility to

- start the calculation of a heuristic by pressing the *Start Heuristic* button.
- calculate the flip distance. This function can be found under *Flip Graph* $\rightarrow$ *Calculate Flip Distance*.
- check if the flip distance is less or equal a number defined by the user. Choose *Flip Graph* $\rightarrow$ *Calculate Partial Flip Graph*.

If the calculation of the heuristic does not create a circulation, the resulting flip sequence can be traced step by step by means of pressing the *Next* and *Prev* button. In the same way the flip sequence that corresponds to the flip distance can be traced. Additionally, the flip graph, its corresponding nodes and the (pseudo-)triangulations that arise due to the flip sequence can optionally be saved.

5.3.2 Program settings

Under the *Settings* item following preferences can be set:

- The selection of one of four heuristics that is applied in case of pressing the *Start Heuristic* button.
- If the item *Log file* is selected, a file named "flipdistanceheuristic.log" is created in the program directory and reports the progress of the program.
- In order to display the point id's in the (pseudo-)triangulation areas in the main frame as well as in the editors, the item *Show Point Id's* has to be selected.
- The assigned weights and the used Comparator for Heuristic 1 – 4 can be set under *Heuristics' Settings*. The default settings are depicted in Table 1 and Table 2. The numbers refer to the enumeration in section 5.1.1.
- The vertex degree bound for T_1 and T_2 can be reset if the item *Set Degree Restriction* is selected. Note that this is only possible if none of the vertices in T_1 and T_2 exceeds the new vertex degree bound.

	Edge crossing weight	Node degree weight	Comparator
Heuristic 1	1	1	1
Heuristic 2	2	1	1
Heuristic 3	2	2	2
Heuristic 4	1	3	3

Table 1: Default edge weights and Comparators of Heuristic 1-4 for triangulations.

	Non-pointed w.	Edge crossing w.	Node degree w.	Comp.
Heuristic 1	1	1	1	1
Heuristic 2	2	1	1	1
Heuristic 3	1	2	1	1
Heuristic 4	2	2	1	1

Table 2: Default edge weights and Comparators of Heuristic 1-4 for pointed pseudo-triangulations.

- In order to switch between calculations for pseudo-triangulations and triangulations the *pseudo-triangulation mode* has to be selected or deselected.

If the program is terminated via *File* → *Exit*, the current settings concerning the log file, the display of point id's, the pseudo-triangulation mode and the last used path are stored in a file called "flipdistanceheuristic.ini", which is located in the program directory. If the application remains in the same directory and is started again, those preferences are loaded from the file.

5.3.3 File format

Point sets and the corresponding sets of edges creating (pseudo-)triangulations are stored in and loaded from text files with the following format:

Point set files The i^{th} line contains two real numbers that are interpreted as coordinates of the point with the label i. The numbers are separated by a blank. The first one corresponds to the x-value of the point, whereas the second number equals the y-value. Note that $1 \leq i \leq n$.

(Pseudo-)Triangulation files Each line contains two integers separated by a blank. The integers stand for the labels of two points. Hence, each line can be interpreted as an edge that is incident to the vertices with the label of the given integers. The labels refer to the corresponding point set defined before.

Flip graph files Files containing information about the flip graph can be used to get an overview of the flip graph. The program provides only the possibility to store flip graphs, but not to load them. Each line of a flip graph file has the following structure:

[label of the node]: [label of neighbor 1]; ...; [label of neighbor j]

with the labels begin integers again. Note that they are unrelated to the aforementioned labels of the point set. Here, we speak of labels of nodes in the flip graph. Furthermore, we say that a node is a neighbor of another node in the flip graph if they are connected by an edge.

Since the nodes of the flip graph correspond to (pseudo-)triangulations, they can optionally be stored in the format of a (pseudo-)triangulation file. These files have the name “t_n[label of the node].txt”.

6 Conclusion and future work

In this thesis we showed that the flip distance of two zigzag triangulations Z_1, Z_2 can be given as a function of n if

1. Z_2 is the normal zigzag triangulation of Z_1 and $(n \mod 4) \in \{0, 1, 3\}$, or

2. Z_2 is the normal zigzag triangulation of the inversion of Z_1 and $(n \mod 4) \in \{1, 2, 3\}$. The starting vertex of the inversion depends on n and the inner vertex degree of the counterclockwise neighbors of the tips of Z_1.

Moreover, we proved that the flip distance remains the same if we add a maximum vertex degree bound $k > 6$ for Z_1, Z_2 and all intermediate triangulations that arise during the transformation from Z_1 into Z_2.

Disregarding the maximum vertex degree, we established a tight lower and upper bound on the flip distance in the case of:

1. $(n \mod 4) = 2$ and Z_2 is the normal zigzag triangulation of Z_1, or

2. $(n \mod 4) = 0$ and Z_2 is the normal zigzag triangulation of the inversion of Z_1. (The starting vertex of the inversion is one of the two tips of Z_1.)

In this context the question arises if these bounds can be replaced by the exact flip distance as described at the beginning of this section. Furthermore, it would be interesting to specify other pairs of zigzag triangulations for which the exact flip distance or even a tight estimate of the flip distance can be shown. What is more, it would be worth investigating the exact flip distance or tight bounds on the flip distance of any two zigzag triangulations.

For point sets in general position we presented approaches for an input sensitive upper bound on the flip distance in the degree bounded setting of two triangulations. We have introduced several measures that do not only take the number of interior intersections between the triangulations' edges into account but also the number of vertices that exceed the degree restriction under different circumstances. By providing counterexamples we were able to show that the discussed measures are not valid input sensitive upper bounds for any two degree bounded triangulations of a point set in general position.

Besides those theoretical considerations, a computer program was developed. On the one hand, it approximates the flip distance of two degree bounded triangulations and pointed pseudo-triangulations by several heuristics. (Note that the resulting flip sequences do not exceed the vertex degree bound.) On the other hand, the implemented construction of the flip graph and related calculations are described. For future work it would be interesting to investigate the results of the heuristics shown in this thesis in order to be able to establish structural characteristics of point sets for which the heuristics give exact bounds.

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