Cong Dinh Doan

Generalized Clifford Analysis

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Technische Universität Graz

Betreuer/in:
Em.Univ.-Prof. Mag.rer.nat. Dr.rer.nat. Wolfgang Tutschke

Institut für Numerische Mathematik (Math D)

Dr.renat.natur.habil.Prof. Le Hung Son

Hanoi University of Science and Technology

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Dedication

To my family

Abstract

The thesis deals with first-order linear elliptic systems for m real-valued functions depending on $n+1$ real variables $x_0,\ x_1,\ \cdots,\ x_n$. Instead of the Cauchy-Riemann operators

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \text{ and } D = \partial_{x_0} + \sum_{i=1}^{n} e_i \partial_{x_i}$$

in complex analysis and Clifford analysis, we consider generalized Cauchy-Riemann operators

$$Mu = \partial_{x_0} u + \sum_{i=1}^{n} E_i(x)\partial_{x_i} u + Q(x)u.$$

The conditions $i^2 = -1$ in the case of complex analysis and $e_i e_j + e_j e_i = -2\delta_{ij}$ in the case of Clifford analysis are replaced by the condition

$$E_i E_j + E_j E_i = -2a_{ij} I_m,$$

where I_m is the $m \times m$ identity matrix and $[a_{ij}]_{i,j=1}^{n}$ is a positive definite matrix. In the thesis we consider the case of an arbitrary number m of real-valued functions, whereas in the case of the Cauchy-Riemann equation in $\mathbb{R}^{n+1}$ the number is $m = 2^n$. This is obtained by a matrix notation of the system.

In the first part of the thesis, we prove that under the above condition there exist fundamental solutions. In the cases of complex analysis and Clifford analysis, the well- known Cauchy kernels are fundamental solutions. In the case under consideration, the fundamental solution can be obtained by a Levi function and a solution of a weakly singular integral equation. The Levi functions are defined by the coefficients of E_i. Then the " Unique Continuation Property" leads to the solvability of the integral equation using the Fredholm alternative.

In the second part, the Dirichlet boundary value problem for monogenic functions of Clifford analysis is solved (solutions of the Cauchy-Riemann system of Clifford analysis). By using the Cauchy kernel of the Clifford analysis, the problem can be reduced step by step to the construction of holomorphic functions in the complex plane. The result is: one half of the components of the desired monogenic function can be prescribed on the whole boundary, while the other components can be prescribed only on lower-dimensional parts of the boundary. The Dirichlet boundary value problem for generalized monogenic functions is also investigated. This problem is reduced to a fixed point problem, then it can be solved by the contraction mapping principle and the second version of the Schauder fixed point theorem (under suitable conditions).

In the last part, initial value problems for first order equations are solved. These problems generalize the classical Cauchy-Kovalevsky theorem. Initial value problems for first order equations can be solved if the initial functions belong to an associated space. In the thesis, new necessary and sufficient conditions for associated pairs are proved by using fundamental solutions. The solution of the initial value problem is a fixed point of an operator whose contractivity is proved by using interior estimates of solutions of elliptic equations.

Zusammenfassung

Die Thesis beschäftigt sich mit linearen elliptischen System erster Ordnung für m gesuchte reellwertige Funktionen, die von $n+1$ reellen Variablen $x_0,\ x_1,\cdots,\ x_n$ abhängen. Anstelle der Cauchy-Riemann Operatoren

$$\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y) \text{ und } D = \partial_{x_0} + \sum_{i=1}^{n} e_i \partial_{x_i}$$

der komplexen Analysis bzw. der Clifford Analysis betrachten wir den verallgemeinerten Cauchy-Riemann Operator

$$Mu = \partial_{x_0} u + \sum_{i=1}^{n} E_i(x)\partial_{x_i} u + Q(x)u.$$

Die Bedingungen $i^2 = -1$ in Fall der komplexen Analysis bzw. $e_i e_j + e_j e_i = -2\delta_{ij}$ im Falle der Clifford Analysis werden ersetzt durch die Bedingung

$$E_i E_j + E_j E_i = -2a_{ij} I_m,$$

wobei I_m die Einheitsmatrix und $[a_{ij}]_{i,j=1}^{n}$ eine positiv definite Matrix ist. In der Dissertation betrachten wir den Fall einer beliebigen Anzahl m gesuchter reellwertiger Funktionen, während im Fall des Cauchy-Riemann-Systems im $\mathbb{R}^{n+1}$ deren Anzahl gleich $m = 2^n$ ist. Dies wird durch eine Matrix-Schreibweise der betrachteter System erreicht.

Erstens wird bewiesen, dass unter den genannten Bedingungen Fundamentallösungen existieren. Im Falle der komplexen Analysis bzw. der Clifford Analysis sind die bekannten Cauchy-Kerne die erforderlichen Fundamentallösungen. In dem hier betrachteten Falle werden die Fundamentallösungen "im grossen" aus einer Levi-Funktion und einer Lösung einer schwach singulären Integralgleichung gewonnen. Die Levi-Funktion erhält aus den Koeffizienten der E_i, während die Lösbarkeit der Integralgleichung aus der Fredholmschen Alternative und der "Eindeutigkeit der analytischen Fortsetzung" folgt.

Zweitens wird das Dirichletsche Randwertproblem für monogene Funktionen der Clifford-Analysis gelöst (Lösung des Cauchy-Riemann-Systems der Clifford-Analysis). Durch Verwendung des Cauchy-Kerns der Clifford Analysis kann das Problem schrittweise auf die Konstruktion von holomorphen Funktionen in der komlexen Ebene reduziert werden. Das Resultat ist: die Hälfte der Komponenten der gesuchten monogene Funktion kann auf dem gesamten Rand vorgeschrieben werden, während die anderen Komponenten nur auf niedriger-dimensionalen Teilen des Randes vorgeschrieben werden können. Ebenfalls wird das Dirichletsche Randwertproblem für verallgemeinerte monogene Funktionen untersucht. Dieses Problem wird auf ein Fixpunkt- Problem zurückgeführt, das sowohl mit Hilfe des Kontraktion Mapping Prinzip als auch mit der zweiten Version des Schauderschen Fixpunktsatzes (unter geeigneten Bedingungen) gelöst werden kann.

Schliesslich werden im letzten Teil der Thesis Anfangswertprobleme für Gleichungen erster Ordnung gelöst. Diese Probleme verallgemeinern den klassischen Satz von Cauchy-Kovalevsky. Anfangswertprobleme für Gleichungen erster Ordnung sind lösbar , falls die Anfangs-Funktion einem assoziierten Raum angehört. In der Thesis werden neue notwendige und hinreichende Bedingungen für assoziierte Paare mit Hilfe von Fundamentallösungen bewiesen. Dann ergibt sich die Lösung des Anfangswertproblems als Fixpunkt eines Operators, dessen Kontraktivität mittels innerer Abschätzungen von Lösungen elliptischer Gleichungen bewiesen wird.

CONTENTS

Introduction

In the theory of generalized analytic functions in the plane, a first order linear elliptic system of two real equations for two desired real-valued functions u and v can be rewritten as a complex equation for $w = u + iv$, which contains first derivatives $\partial_z w$ and $\partial_{\bar{z}} w$. However, carrying out a transformation of the independent variable z by a new variable ζ, which satisfies the Beltrami equation $\partial_{\bar{z}}\zeta = q(z)\partial_z\zeta$ where $|q(z)| \le q_0 < 1$, we can reduce general cases to the Vekua equation [38]

$$\partial_{\bar{z}} w = A(z)w + B(z)\overline{w}.$$

In Clifford analysis, monogenic functions are solutions of the Cauchy-Riemann equation

$$Du = \sum_{j=0}^{n} e_j \frac{\partial u}{\partial x_j} = 0$$

where $e_0 = 1,\ e_i e_j + e_j e_i = -2\delta_{ij}$ (Kronecker symbol), $i, j = 1, \cdots, n,\ u$ is a Clifford-algebra valued function [5]. After I.N. Vekua many authors developed theories of generalized monogenic functions in the space in order to cover more general first-order linear elliptic systems. Namely, the equation $Du + \tilde{u}h = 0$ is investigated by E. Obolashvili [24] , $Du + L(x)u = F(x)$ by B. Goldschmidt [11].

Solving real partial differential equations within the framework of Clifford analysis has some advantages. First, it leads to the unification of statements, second, in some cases one obtains simpler explicit representations. However, using the matrix notation instead of rewriting in the language of Clifford analysis, sometimes more general results can be obtained. The Cauchy-Riemann type system in $\mathbb{R}^{n+1}$ is

$$Du = \sum_{i=0}^{n} E_i \frac{\partial u}{\partial x_i} = 0 \tag{0.1}$$

where $u(x) = [u_1, u_2, \cdots, u_m]^T$ is a real-valued vector function, $E_0 = I_m$ is the $m \times m$ identity matrix, E_i are $m \times m$ constant matrices satisfying the relation

$$E_i E_j + E_j E_i = -2\delta_{ij} I_m \quad \forall i, j = 1, 2, \cdots, n. \tag{0.2}$$

Some authors considered generalizations of the Cauchy-Riemann type system (0.1). For instance, G.N. Hile introduced the system

$$\sum_{i=1}^{n} P_i \frac{\partial u}{\partial x_i} - Qu = 0 \tag{0.3}$$

where P_i, Q and u are matrix-valued functions. The condition (0.2) is replaced by a condition that there exist matrices R_i such that

$$R_iP_j + R_jP_i = 2a_{ij}I_m \quad 1 \le i, j \le n, \tag{0.4}$$

where $A = [a_{ij}]$ is a positive and symmetric matrix. In the case P_i are constant matrices [13], the author gave representations of the solutions of (0.3). In general case a maximum principle of the solutions was proved in [14]. B. Goldschmidt also investigated generalized analytic vectors in matrix form [12].
An overview of development of generalized analytic functions theory in higher dimension can be found in [33].

The thesis investigates a generalization of the Cauchy-Riemann type system

$$Mv(x) = \sum_{i=0}^{n} E_i(x)\frac{\partial v}{\partial x_i} + Q(x)v(x) = 0, \tag{0.5}$$

with $x = (x_0, x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^{n+1}$, $v(x) = [v_1(x), v_2(x), \cdots, v_m(x)]^T \in \mathcal{C}^1(\Omega)$. $E_0 = I_m$, $Q(x)$, $E_i(x), i = 1, 2\cdots n$ are $m \times m$ real matrix functions. The condition (0.2) is replaced by

$$E_i(x)E_j(x) + E_j(x)E_i(x) = -2a_{ij}(x)I_m, (i, j = 1, \cdots, n), \tag{0.6}$$

$$\xi_0^2 + \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge C|\xi|^2, \quad \forall \xi = (\xi_0, \xi_1 \cdots, \xi_n) \in \mathbb{R}^{n+1}, \forall x \in \Omega \tag{0.7}$$

with some constant $C > 0$.

The first chaper is devoted to construct fundamental solutions of the system (0.5). In general, an elliptic system in the sense of Douglis and Nirenberg with Hölder continuous coefficients has a fundamental solution in local, the global existence can be ensured under a condition that the adjoint system has the unique continuation property [30]. In our case, this additional condition is satisfied [40]. Moreover the Levi functions of the operator M can be constructed explicitly. By using the method of Ljubic [20] we can prove the existence of fundamental solutions of the system $Mv = 0$ ”in the large " which is more explicitly constructed and has better properties.

The second chapter is devoted to solve a Dirichlet boundary value problem for monogenic functions in Clifford analysis. It begins from Dirichlet boundary value problem for holomorphic functions in smooth, simply connected, bounded domains [31]. The real part and the imaginary part of a holomorphic function are harmonic functions, hence they are determined uniquely by their boundary values. If one prescribes the real part on the whole

boundary then the imaginary part is determined uniquely up to a constant, so one can prescribe the imaginary part at one point inside the domain. In case the boundary values of the real part are only continuous, the imaginary part may be not continuous upto the boundary. If the boundary values of the real part are Hölder continuous then the solution is Hölder continuous.
A similar situation arises for monogenic functions in Clifford analysis. Using the Cauchy kernel of the Cauchy-Riemann operator the boundary value problem for monogenic functions is reduced to a problem of the same type in a lower dimension. In the end it is reduced to the classical problem for holomorphic functions. The result is: one half of the components of the desired solution can be prescribed on the whole boundary (an n-dimensional manifold), one half of the rest components can be prescribed on a part of the boundary (an $(n-1)$-dimensional manifold) and so on.... In the last step one component is prescribed on a curve on the boundary (1-dimensional manifold) and one component is prescribed at one point inside the domain. If the boundary data are Hölder continuously differentiable functions, then the unique solution is Hölder continuous. An estimate of the solution by its boundary data is proved.
Dirichlet boundary value problem for generalized monogenic functions is also investigated. Using the Cauchy kernel, the problem is reduced to a fixed point problem. Some additional conditions are required for which the contraction mapping principle and the second version of Schauder's fixed point theorem are applicable.

The last chapter discusses about initial value problems

$$\begin{cases} \dfrac{\partial u}{\partial t} & = \mathcal{F}\left(t, x, u, \dfrac{\partial u}{\partial x_j}\right) \\ u(0,x) & = \varphi(x). \end{cases}$$

The initial value problems are investigated in the case the initial functions are generalized analytic functions in complex analysis, monogenic functions in Clifford analysis, or solutions of the equation (0.5) $Mv = 0$. The method of "associated space" is a new tool for this problem [34]. The problem is solvable if the operator $\mathcal{F}$ is associated to the operator M and an interior estimate in the supremum norm for solutions of the equation $Mv = 0$ is available.
Up to now, the problem of finding associated operators has not been solved completely. This chapter shows how to use fundamental solutions of the equation $Mv = 0$ to realize the necessary and sufficient conditions for associated operators. In some special cases, we can construct explicitly a class of operators associated to the Cauchy-Riemann operator in quaternion analysis is constructed, and operators associated to the generalized Cauchy-Riemann operators in the complex analysis. An interior estimate in supremum norm for solutions of the equation $Mv = 0$ is also proved by using the method in [8].

1 FUNDAMENTAL SOLUTIONS OF A CLASS OF FIRST-ORDER LINEAR ELLIPTIC SYSTEMS

1.1 A class of first-order linear elliptic systems

We introduce a generalization of the Cauchy-Riemann type system

$$Mv(x) = \sum_{i=0}^{n} E_i(x)\frac{\partial v}{\partial x_i} + Q(x)v(x) = 0, \tag{1.1}$$

$x = (x_0, x_1, x_2, \dots, x_n) \in \Omega \subset \mathbb{R}^{n+1}$, $v(x) = [v_1(x), v_2(x), \cdots, v_m(x)]^T \in \mathcal{C}^1(\Omega)$ is a real vector function. $E_0 = I_m$ is the indentity matrix, $Q(x),\ E_i(x), i = 1, 2 \cdots n$ are real $m \times m$ matrix functions satisfying the conditions

$$E_i(x)E_j(x) + E_j(x)E_i(x) = -2a_{ij}(x)I_m, (i, j = 1, \cdots, n), \tag{1.2}$$

$$\xi_0^2 + \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ \geq \ C|\xi|^2, \quad \forall \xi = (\xi_0, \xi_1 \cdots, \xi_n) \in \mathbb{R}^{n+1}, \forall x \in \Omega \tag{1.3}$$

with some constant $C > 0$.

Remark 1. *In the special case the coefficients E_i are constant matrices and $Q \equiv 0$, $a_{ij} = \delta_{ij}$ then the condition (1.2) becomes*

$$E_iE_j + E_jE_i = -2\delta_{ij}I_m,$$

the system (1.1) becomes the Cauchy-Riemann system in $\mathbb{R}^{n+1}$.

In this chapter, we prove the existence of fundamental solutions of the equation $Mv = 0$ with a condition on the smoothness of the coefficients.

Theorem 1.1. *Let Ω_0 be a domain in $\mathbb{R}^{n+1}$. We consider the system (1.1) with the coefficients*

$$E_i(x) \in \mathcal{C}^h(\Omega_0),\ \ h > max\left\{\frac{n+3}{2}; 3\right\},\ \ Q(x) \in \mathcal{C}^k(\Omega_0),\ \ k > max\left\{\frac{n+1}{2}; 2\right\}.$$

Let Ω_1 be an open bounded set with $\overline{\Omega}_1 \subset \Omega_0$, then the system (1.1) has a fundamental solution $\Gamma(x,y)$ in Ω_1, moreover $\Gamma(x,y)$ is a Levi function of the operator M.

In the following, we give some systems of type (1.1) satisfying the conditions (1.2), (1.3).

Example 1. *First-order elliptic systems in the plane*

With $n = 1,\ m = 2$, the system is:

$$Mv = \frac{\partial v(x)}{\partial x_0} + E_1(x)\frac{\partial v(x)}{\partial x_1} + Q(x)v(x) = 0,$$

where $E_1(x) = \begin{bmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{bmatrix}$ with the condition $a(x)^2 + b(x)c(x) < 0$.

$$E_1(x)^2 = \begin{bmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{bmatrix}\begin{bmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{bmatrix} = [a(x)^2 + b(x)c(x)]I_2.$$

Example 2. *First-order elliptic systems in* $\mathbb{R}^{n+1}$

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain. Assume that there exist constant matrices A_i of order $m \times m$ such that

$$A_iA_j + A_jA_i = -2\delta_{ij}I_m, (i, j = 1, \ldots, n).$$

Let $\lambda(x) = [\lambda_{i,k}(x)]_{i,k=1}^n$ be a nonsingular matrix, $\lambda_{i,k}(x) \in \mathcal{C}(\overline{\Omega})$. Define

$$E_i(x) = \sum_{k=1}^{n} \lambda_{i,k}(x)A_k \ \ (i = 1, \cdots, n).$$

We will show that the matrices defined as above satisfy the conditions $(1.2), (1.3)$.

$$\begin{aligned} E_iE_j + E_jE_i &= \left(\sum_{k=1}^{n} \lambda_{i,k}(x)A_k\right)\left(\sum_{k=1}^{n} \lambda_{j,k}(x)A_k\right) + \left(\sum_{k=1}^{n} \lambda_{j,k}(x)A_k\right)\left(\sum_{k=1}^{n} \lambda_{i,k}(x)A_k\right) \\ &= -2\sum_{k=1}^{n} \lambda_{i,k}(x)\lambda_{j,k}(x)I_m. \end{aligned}$$

Denote $a_{ij} = \sum\limits_{k=1}^{n} \lambda_{i,k}(x)\lambda_{j,k}(x)$. Let $\xi = (\xi_0, \xi_1 \ldots, \xi_n) \in \mathbb{R}^{n+1},\ \xi \neq 0$, now we check the condition (1.3): $\xi_0^2 + \sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0$. We have

$$T := \left(\xi_0E_0 - \sum_{i=1}^{n} \xi_iE_i(x)\right)\left(\xi_0E_0 + \sum_{i=1}^{n} \xi_iE_i(x)\right) = \left(\xi_0^2 + \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\right)I_m.$$

On the other hand we have

$$\sum_{i=1}^{n} \xi_i E_i(x) = \sum_{i=1}^{n} \xi_i \sum_{k=1}^{n} \lambda_{i,k} A_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \lambda_{i,k} \xi_i \right) A_k.$$

Hence

$$T = \left[\xi_0 E_0 - \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \lambda_{i,k} \xi_i \right) A_k \right] \left[\xi_0 E_0 + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \lambda_{i,k} \xi_i \right) A_k \right]$$
$$= \left[\xi_0^2 + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \lambda_{i,k} \xi_i \right)^2 \right] I_m.$$

From the two representations of T we have

$$\xi_0^2 + \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j = \xi_0^2 + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \lambda_{i,k} \xi_i \right)^2.$$

Because the matrix $\lambda(x) = [\lambda_{i,k}(x)]$ is not singular everywhere, so the condition (1.3) is satisfied.

Example 3. *Case* $n = 3,\ m = 4$

Let $\Omega \subset \mathbb{R}^4$ be a bounded domain.

$$Mv = \frac{\partial v(x)}{\partial x_0} + E_1(x) \frac{\partial v(x)}{\partial x_1} + E_2(x) \frac{\partial v(x)}{\partial x_2} + E_3(x) \frac{\partial v(x)}{\partial x_3} + Q(x) v(x) = 0.$$

Denote

$$A_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

These matrices satisfy the condition $A_i A_j + A_j A_i = -2\delta_{ij} I_4$.
Let $\lambda(x) = \left[\lambda_{i,k}(x)\right]_{i,k=1}^{3}$, $\lambda_{i,k}(x) \in \mathcal{C}(\overline{\Omega})$ with $Det(\lambda(x)) \neq 0\ \forall x \in \overline{\Omega}$. If we choose

$$E_i(x) = \sum_{k=1}^{3} \lambda_{i,k}(x) A_k \ (i = 1, 2, 3),$$

then these matrices satisfy the conditions $(1.2), (1.3)$.

Example 4. *Case* $n = 3,\ m = 4$ *in more general form*

Let $\Omega \subset \mathbb{R}^4$ be a bounded domain.

$$Mv = \frac{\partial v(x)}{\partial x_0} + E_1(x)\frac{\partial v(x)}{\partial x_1} + E_2(x)\frac{\partial v(x)}{\partial x_2} + E_3(x)\frac{\partial v(x)}{\partial x_3} + Q(x)v(x) = 0.$$

Denote

$$A_1(x) = \begin{bmatrix} 0 & -b(x)t(x) & 0 & 0 \\ a(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & -b(x) \\ 0 & 0 & a(x)t(x) & 0 \end{bmatrix}$$

$$A_2(x) = \begin{bmatrix} 0 & 0 & -d(x)t(x) & 0 \\ 0 & 0 & 0 & d(x) \\ c(x) & 0 & 0 & 0 \\ 0 & -c(x)t(x) & 0 & 0 \end{bmatrix}$$

$$A_3(x) = \begin{bmatrix} 0 & 0 & 0 & -b(x)d(x) \\ 0 & 0 & -a(x)d(x) & 0 \\ 0 & b(x)c(x) & 0 & 0 \\ a(x)c(x) & 0 & 0 & 0 \end{bmatrix}$$

where $a(x), b(x), c(x), d(x), t(x)$ are positive functions in $\mathcal{C}(\overline{\Omega})$.
We have

$$\begin{aligned} A_1^2 &= -a(x)b(x)t(x)I_4, & A_2^2 &= -c(x)d(x)t(x)I_4, \\ A_3^2 &= -a(x)b(x)c(x)d(x)I_4, & A_iA_j + A_jA_i &= 0 \text{ with } i \neq j. \end{aligned}$$

Let $\lambda(x) = \left[\lambda_{i,k}(x)\right]_{i,k=1}^{3}$, $\lambda_{i,k}(x) \in \mathcal{C}(\overline{\Omega})$ with $Det(\lambda(x)) \neq 0 \ \forall x \in \overline{\Omega}$. If we choose

$$E_i(x) = \sum_{k=1}^{3} \lambda_{i,k}(x) A_k \ (i = 1, 2, 3),$$

then these matrices satisfy the conditions (1.2), (1.3).

Example 5. *Case* $n = 2, \ m = 4$

$$Mv = \frac{\partial v(x)}{\partial x_0} + E_1(x)\frac{\partial v(x)}{\partial x_1} + E_2(x)\frac{\partial v(x)}{\partial x_2} + Q(x)v(x) = 0.$$

Denote

$$A_1 = \begin{bmatrix} 0 & -h(x) & 0 & 0 \\ g(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & -h(x) \\ 0 & 0 & g(x) & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & -2b(x) & -a(x) & 0 \\ 0 & 0 & 0 & a(x) \\ c(x) & 0 & 0 & -2b(x) \\ 0 & -c(x) & 0 & 0 \end{bmatrix}$$

with the condition $a(x)c(x)h(x) > b^2(x)g(x)\ \ \forall x$.
We have

$$A_1(x)^2 = -g(x)h(x)I_4, \qquad A_2(x)^2 = -a(x)c(x)I_4,$$
$$A_1(x)A_2(x) + A_2(x)A_1(x) = -2b(x)g(x)I_4.$$

Let $\lambda(x) = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \end{bmatrix}$ be a nonsingular matrix. We can choose

$$E_1(x) = \lambda_{1,1}(x)A_1 + \lambda_{1,2}(x)A_2, \qquad E_2(x) = \lambda_{2,1}(x)A_1 + \lambda_{2,2}(x)A_2.$$

Example 6. *Case $n = 2,\ m = 4$ in more general form*

$$Mv = \frac{\partial v(x)}{\partial x_0} + E_1(x)\frac{\partial v(x)}{\partial x_1} + E_2(x)\frac{\partial v(x)}{\partial x_2} + Q(x)v(x) = 0.$$

Denote

$$A_1 = \begin{bmatrix} 0 & -a(x)t(x) & 0 & 0 \\ b(x) & 0 & 0 & 0 \\ c(x) & 0 & 0 & -a(x) \\ 0 & -c(x)t(x) & b(x)t(x) & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & -m(x)t(x) & n(x)t(x) & 0 \\ 0 & 0 & 0 & n(x) \\ p(x) & 0 & 0 & -m(x) \\ 0 & -p(x)t(x) & 0 & 0 \end{bmatrix}$$

where a, b, c, d, m, n, p, t are positive functions satisfying the condition

$$4abnp > (bm + cn)^2\ \ \forall x.$$

We have

$$A_1(x)^2 = -a(x)b(x)t(x)I_4, \qquad A_2(x)^2 = -n(x)p(x)t(x)I_4,$$
$$A_1(x)A_2(x) + A_2(x)A_1(x) = -t(x)[b(x)m(x) + c(x)n(x)]I_4.$$

Let $\lambda(x) = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \end{bmatrix}$ be a nonsingular matrix. We can choose

$$E_1(x) = \lambda_{1,1}(x)A_1 + \lambda_{1,2}(x)A_2, \qquad E_2(x) = \lambda_{2,1}(x)A_1 + \lambda_{2,2}(x)A_2.$$

1.2 Unique continuation property

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with smooth boundary. Suppose that $E_i(x) \in \mathcal{C}^1(\overline{\Omega})$, the adjoint operator of M is defined by

$$Ru(x) = -\sum_{i=0}^{n} \frac{\partial}{\partial x_i}\left[E_i(x)^T u(x)\right] + Q(x)^T u(x) \tag{2.4}$$

where $u(x) = [u_1(x), u_2(x), \cdots, u_m(x)]^T \in \mathcal{C}^1(\Omega)$, $E_i(x)^T$ is the transpose of $E_i(x)$. Suppose that $u(x), v(x) \in \mathcal{C}^1(\overline{\Omega})$, the Green Integral Formula reads

$$\int_{\Omega} \left[(Mv)^T u - v^T Ru\right] dx = \int_{\partial\Omega} \sum_{i=0}^{n} v^T N_i E_i^T u d\mu(x) \tag{2.5}$$

where $N = (N_0, N_1, \cdots, N_n)$ is the outer unit normal of the boundary $\partial\Omega$.

Definition 1. *$u(x) \in L^2(\Omega)$ is called a weak solution of $Ru(x) = 0$ if*

$$\int_{\Omega} (Mv)^T u dx = 0 \quad \forall v(x) \in \mathcal{C}_0^\infty(\Omega).$$

The following theorem gives conditions for a weak solution becoming a classical solution.

Theorem 2.2 ([9]). *If $E_i(x) \in \mathcal{C}^h(\Omega),\ h > \dfrac{n+3}{2},\ Q(x) \in \mathcal{C}^k(\Omega),\ k > \dfrac{n+1}{2}$, then every $L^2(\Omega)$- weak solution of the system $Ru = 0$ belongs to $\mathcal{C}^1(\Omega)$ (classical solution).*

If $E_i(x) \in \mathcal{C}^3(\Omega),\ \ Q(x) \in \mathcal{C}^2(\Omega)$ then applying the result of N. Weck and W. Wendland we have the system $Ru = 0$ has a unique continuation property which is quoted in the following theorem.

Theorem 2.3 ([40]). *Consider the system*

$$P_1 \frac{\partial u}{\partial x_1} + \cdots + P_n \frac{\partial u}{\partial x_n} = f(x, u)$$

in Ω (a domain in $\mathbb{R}^n$), where P_j are $(m \times m) - \mathcal{C}^1$-matrices, $f \in \mathcal{C}^2(\Omega \times \mathbb{C}^m),\ f(x, 0) = 0$, and $u = (u_1, \cdots, u_m)$ is a $\mathcal{C}^1$-vector-function. Under the conditions that
(i) *There exist $(m \times m)$ matrices $Q_1, \cdots, Q_n$ satisfying $A_{jk}^* A_{lm} = A_{lm}^* A_{jk}$ for*

$$A_{jk} = Q_j P_k + Q_k P_j,$$

(ii) *Re $\sum_{j,k=1}^n \langle \xi_j \xi_k A_{jk}(x) u, u\rangle \geq \varepsilon |\xi|^2 |u|^2$ for all $\xi \in \mathbb{R}^n$, $u \in \mathbb{C}^m$ and some $\varepsilon > 0$.*
Then $u = 0$ in Ω if $u = 0$ in some nonempty open subset of Ω.

Combining Theorem 2.3 and Theorem 2.2 we have

Corollary 1. *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be a connected domain, and suppose that* $E_i(x) \in \mathcal{C}^h(\Omega)$, $h > max\left\{\frac{n+3}{2};3\right\}$, $Q(x) \in \mathcal{C}^k(\Omega)$, $k > max\left\{\frac{n+1}{2};2\right\}$. *Then if an* L^2 *-weak solution of* $Ru = 0$ *vanishes in some nonempty open subset of* Ω*, it implies that* $u \equiv 0$ *in* Ω*.*

This result plays an important role in the proof of Theorem 1.1.

1.3 Levi functions

Denote $A = [a_{ij}(x)]_{i,j=0}^n$, $(a_{00} = 1,\ a_{0i} = a_{i0} = 0\ \ \forall\ i = 1,2,\cdots,n)$. From the condition (1.3), the determinant of A is positive in Ω, denote the inverse matrix of A by $A^{-1} = \left[A_{ij}\right]_{i,j=0}^n$. Denote $\rho(x,y) := \sqrt{(x-y)^T A^{-1}(y)(x-y)}$ and

$$G(x,y) := \begin{cases} \dfrac{\rho(x,y)^{1-n}}{(1-n)\omega_{n+1}\sqrt{DetA(y)}} & \text{if} \quad n > 1 \\ \dfrac{\ln\rho(x,y)}{2\pi\sqrt{DetA(y)}} & \text{if} \quad n = 1 \end{cases} \tag{3.6}$$

where ω_{n+1} is the surface measure of the unit sphere in $\mathbb{R}^{n+1}$. The function $G(x,y)$ is known as a Levi function of single second order elliptic equations with the leading part $\sum\limits_{i,j=0}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}$ [22]. Then define

$$H(x,y) := \sum_{j=0}^{n} \frac{\partial G(x,y)}{\partial x_j}\overline{E_j(y)}$$

where $\overline{E_0} := E_0$, $\overline{E_j(y)} := -E_j(y)$ with $j = 1,\ldots,n$.

Lemma 3.1. $H(x,y)$ *has the following properties*

$$a) \qquad H(x,y) = O(|x-y|^{-n}),\ \ \frac{\partial H(x,y)}{\partial x_i} = O(|x-y|^{-n-1}),$$

$$b) \qquad \sum_{i=0}^{n} E_i(y)\frac{\partial}{\partial x_i}H(x,y) = 0.$$

Proof. The statement a) is trivial by the definition of H. We only need to prove the statement b).

$$\sum_{i=0}^{n} E_i(y)\frac{\partial}{\partial x_i}H(x,y) = \sum_{i=0}^{n} E_i(y)\frac{\partial}{\partial x_i}\left[\sum_{j=0}^{n}\frac{\partial G(x,y)}{\partial x_j}\overline{E_j(y)}\right]$$

$$= \sum_{i,j=0}^{n} E_i(y)\overline{E_j(y)}\frac{\partial^2 G(x,y)}{\partial x_i \partial x_j} = \left[\frac{\partial^2 G(x,y)}{\partial x_0 \partial x_0} + \sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 G(x,y)}{\partial x_i \partial x_j}\right] I_m.$$

$$\frac{\partial G(x,y)}{\partial x_i} = \frac{\sum_{r=0}^{n} A_{ir}(y)(x_r - y_r)}{\omega_{n+1}\sqrt{Det(A(y))}\rho(x,y)^{n+1}},$$

$$\frac{\partial^2 G(x,y)}{\partial x_i \partial x_j} = \frac{-(n+1)\sum_{l,r=0}^{n} A_{jl}(y)A_{ir}(y)\frac{(x_l - y_l)(x_r - y_r)}{\rho(x,y)^2} - A_{ij}(y)}{\omega_{n+1}\sqrt{Det(A(y))}\rho(x,y)^{n+1}}.$$

So we see that

$$\sum_{i,j=0}^{n} a_{ij}(y)\frac{\partial^2 G(x,y)}{\partial x_i \partial x_j} = -(n+1)\frac{\sum_{l,r=0}^{n} \frac{A_{lr}(y)(x_l - y_l)(x_r - y_r)}{\rho(x,y)^2} - 1}{\omega_{n+1}\sqrt{Det(A(y))}\rho(x,y)^{n+1}} = 0.$$

Hence Lemma 3.1 is proved. □

Definition 2. *Every $m \times m$ matrix function $L(x,y)$ continuous in the variables x and y for $x,y \in \Omega, x \neq y$, together with its first-order derivatives with respect to x_i, is called a Levi function of the given operator M if there exists a constant $\lambda > 0$ such that*

$$L - H = O(|x-y|^{\lambda - n}), \quad \frac{\partial (L-H)}{\partial x_i} = O(|x-y|^{\lambda - n - 1}) \tag{3.7}$$

in each compact subset of Ω.

We say $f(x,y) = O(|x-y|^p)$ in Ω if there exists a constant $C > 0$ such that $|f(x,y)| < C|x-y|^p \ \forall x,y \in \Omega, \ x \neq y$. We say this bound holding in each compact subset of Ω if for each compact subset K of Ω there exists a constant $C > 0$ such that the inequality is true for all $x,y \in K, \ x \neq y$.

Remark 2. *Suppose that $E_i(x) \in \mathcal{C}^{\alpha}(\Omega)$, $(0 < \alpha < 1)$ (Hölder continuous space) and $Q(x)$ is bounded in Ω then*

$$\begin{aligned} M_x H(x,y) &= \sum_{i=0}^{n} E_i(x)\frac{\partial H(x,y)}{\partial x_i} + Q(x)H(x,y) \\ &= \sum_{i=0}^{n} [E_i(x) - E_i(y)]\frac{\partial H(x,y)}{\partial x_i} + Q(x)H(x,y) = O\left(|x-y|^{\alpha - n - 1}\right). \end{aligned}$$

So if L is a Levi function of M then $M_x L(x,y) = O\left(|x-y|^{\lambda - n - 1}\right)$ for some $\lambda > 0$.

Lemma 3.2 (Stokes's formula)**.** *Let Ω be a smooth, bounded domain in $\mathbb{R}^{n+1}$. Suppose that $E_i(x) \in \mathcal{C}^1(\overline{\Omega})$, $Q(x) \in \mathcal{C}^\alpha(\Omega)(0 < \alpha < 1)$. Let $u(x) \in \mathcal{C}^1(\overline{\Omega})$ then $\forall\, y \in \Omega$*

$$u(y) = \int\limits_{\partial\Omega} H^T(x,y) \sum_{i=0}^{n} N_i E_i^T(x) u(x) d\mu(x) + \int\limits_{\Omega} \left[H^T(x,y) R u(x) - (M_x H(x,y))^T u(x)\right] dx. \tag{3.8}$$

Proof.
For a fixed point $y \in \Omega$, we choose $\varepsilon > 0$ so small that $B_\varepsilon(y) := \{x \in \Omega |\ \ \rho(x,y) \leq \varepsilon\} \subset \Omega$. Denote $\Omega_\varepsilon := \Omega \setminus B_\varepsilon(y)$, apply the Green Integral Formula in Ω_ε with $v(x) = H(x,y)$ and $u(x) \in C^1(\Omega) \bigcap C^0(\overline{\Omega})$, we get

$$\int\limits_{\Omega_\varepsilon} \left[(M_x H)^T u(x) - H^T R u(x)\right] dx = \int\limits_{\partial\Omega_\varepsilon} H^T \sum_{i=0}^{n} N_i E_i^T(x) u(x) d\mu(x) \tag{3.9}$$

$$= \int\limits_{\partial\Omega} H^T \sum_{i=0}^{n} N_i E_i^T(x) u(x) d\mu(x) + \int\limits_{\rho(x,y)=\varepsilon} H^T \sum_{i=0}^{n} N_i E_i^T(x) u(x) d\mu(x).$$

We calculate the limit

$$\begin{aligned} I &= \lim_{\varepsilon \to 0} \int\limits_{\rho(x,y)=\varepsilon} H^T(x,y) \sum_{i=0}^{n} N_i E_i^T(x) u(x) d\mu(x) \\ &= \lim_{\varepsilon \to 0} \int\limits_{\rho(x,y)=\varepsilon} \frac{\sum\limits_{r,s=0}^{n} A_{rs}(y)(x_s - y_s)\overline{E_r(y)}^T \sum\limits_{i=0}^{n} N_i E_i^T(x) u(x) d\mu(x)}{\omega_{n+1}\sqrt{Det(A(y))}\rho(x,y)^{n+1}} \\ &= \lim_{\varepsilon \to 0} \frac{\int\limits_{\rho(x,y)=\varepsilon} \sum\limits_{r,s,i=0}^{n} A_{rs}(y)(x_s - y_s)\overline{E_r(y)}^T N_i E_i^T(x) u(x) d\mu(x)}{\omega_{n+1}\sqrt{Det(A(y))}\varepsilon^{n+1}}. \end{aligned}$$

Because $\dfrac{|x_s - y_s|}{\varepsilon^{n+1}} \leq \dfrac{C}{\varepsilon^n}$,

$$\begin{aligned} I &= \lim_{\varepsilon \to 0} \frac{\int\limits_{\rho(x,y)=\varepsilon} \sum\limits_{r,s,i=0}^{n} A_{rs}(y)(x_s - y_s)\overline{E_r(y)}^T N_i(x) E_i^T(y) d\mu(x) u(y)}{\omega_{n+1}\sqrt{Det(A(y))}\varepsilon^{n+1}} \\ &= -\lim_{\varepsilon \to 0} \frac{1}{\omega_{n+1}\sqrt{Det(A(y))}\varepsilon^{n+1}} \int\limits_{\rho(x,y)\leq\varepsilon} \sum_{r,s=0}^{n} A_{rs}(y)\overline{E_r(y)}^T E_s^T(y) d(x) u(y) \end{aligned}$$

$$= -\lim_{\varepsilon\to 0}\frac{\sum\limits_{r,s=0}^{n} A_{rs}(y)\overline{E_r(y)}^T E_s^T(y)u(y)}{\omega_{n+1}\sqrt{Det(A(y))}\varepsilon^{n+1}}\int\limits_{\rho(x,y)\le\varepsilon} 1d(x)$$
$$= -\lim_{\varepsilon\to 0}\frac{(n+1)u(y)}{\omega_{n+1}\sqrt{Det(A(y))}\varepsilon^{n+1}}\int\limits_{\rho(x,y)\le\varepsilon} 1dx.$$

We have $\rho(x,y)=\sqrt{(x-y)^TA^{-1}(y)(x-y)}$, $A^{-1}(y)$ is symmetric, positive, so there exists a matrix C such that $A^{-1}(y)=C^TC$.

$$\rho(x,y)=\sqrt{(x-y)^TA^{-1}(y)(x-y)}=\sqrt{(x-y)^TC^TC(x-y)}.$$

Changing variable $z=C(x-y)$

$$\int\limits_{\rho(x,y)\le\varepsilon} 1d(x)=\int\limits_{|z|\le\varepsilon}\frac{1}{Det(C)}dz=\sqrt{Det(A(y))}\int\limits_{|z|\le\varepsilon} 1dz=\varepsilon^{n+1}\tau_{n+1}\sqrt{Det(A(y)},$$

so $I=-\dfrac{(n+1)\tau_{n+1}u(y)}{\omega_{n+1}}=-u(y)$ (τ_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$).
By Remark 2 we get $M_xH(x,y)=O(|x-y|^{\alpha-n-1})$.
From (3.9), let $\varepsilon\to 0$ we have

$$u(y)=\int\limits_{\partial\Omega} H^T\sum_{i=0}^{n} N_iE_i^T(x)u(x)d\mu(x)+\int\limits_{\Omega}\left[H^TRu(x)-(M_xH)^Tu(x)\right]dx.$$

So the Stoke's formula is proved. □

Remark 3. *The Stoke's formula is also true for any Levi function of the operator M.*

Definition 3. *A distribution $\Gamma(x,y)$ on $C_0^\infty(\Omega)^m$ is called a fundamental solution of the equation $Mv=0$ if $M\Gamma(x,.)=\delta_x$ (δ_x is the Dirac δ-distribution) [18].*

Remark 4. *By the Stoke's formula (3.8), if $L(x,y)$ is a Levi function in Ω and satisfies the equation $M_xL(x,y)=0$ for all $x,y\in\Omega$, $x\neq y$ then*

$$u(y)=\int\limits_{\Omega} L^T(x,y)Ru(x)dx\ \ \forall\, y\in\Omega,$$

for every $u\in C^1(\Omega)$ with compact support.
Let $f(x)=(f_1(x),f_2(x),\cdots,f_m(x))\in L^2(\Omega)$. Define

$$U(x):=\int\limits_{\Omega} L(x,y)f(y)dy,$$

then U is a solution of the equation $LU=f$. Hence, L becomes a fundamental solution of the equation $Mv=0$ in Ω.

Remark 5. *In the special case the Cauchy-Riemann type system in $\mathbb{R}^{n+1}$, the coefficients E_i are constant matrices and $Q \equiv 0$, $a_{ij} = \delta_{ij}$*

$$E_iE_j + E_jE_i = -2\delta_{ij}I_m,$$

a fundamental solution is given by (the Cauchy kernel)

$$E(x,y) = \frac{1}{\omega_{n+1}|x-y|^{n+1}} \left[(x_0 - y_0) - \sum_{i=1}^{n} (x_i - y_i)E_i \right].$$

1.4 Some auxiliary tools

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain, $D := \{(x,x) \in \Omega \times \Omega | \ \ x \in \Omega\}$. Let $g,h \in \mathcal{C}^0(\overline{\Omega} \times \overline{\Omega} \setminus D)$ with the property:*
There exist numbers $\beta, \gamma \in (0, n+1)$ and $C_1 > 0$ such that

$$|g(x,y)| \leq C_1|x-y|^{\beta-n-1}, \quad |h(x,y)| \leq C_1|x-y|^{\gamma-n-1} \ \ \forall x,y \in \overline{\Omega}, x \neq y.$$

Then the function $f(x,y) := \int_\Omega g(x,z)h(z,y)dz$ is in $\mathcal{C}^0(\overline{\Omega} \times \overline{\Omega} \setminus D)$ and with an appropriate number $C_2 > 0$ we have

$$|f(x,y)| \leq \begin{cases} C_2|x-y|^{\beta+\gamma-n-1} & if \quad \beta+\gamma < n+1 \\ C_2\left(1+|Ln|x-y||\right) & if \quad \beta+\gamma = n+1. \end{cases} \tag{4.10}$$

Moreover, f can be extended to a function in $\mathcal{C}^0(\overline{\Omega} \times \overline{\Omega})$ if $\beta + \gamma > n+1$.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain, $\omega(x,y) \in \mathcal{C}^0(\overline{\Omega} \times \overline{\Omega} \setminus D)$ such that for some constants $\lambda, \alpha \in (0,1)$, $C > 0$*

$$\omega(.,y) \in \mathcal{C}^1(\Omega \setminus \{y\}), \ \ |\omega(x,y)| \leq C|x-y|^{\alpha-n},$$
$$\left| \frac{\partial}{\partial x_i}\omega(x,y) \right| \leq C|x-y|^{\lambda-n-1} \ \ \forall x,y \in \overline{\Omega}, x \neq y.$$

Then the function

$$f(x) := \int_\Omega \omega(x,y)dy \text{ is in } \mathcal{C}^1(\Omega) \text{ and } \frac{\partial}{\partial x_i} f(x) = \int_\Omega \frac{\partial}{\partial x_i}\omega(x,y)dy.$$

Lemma 4.5. *With the hypothesis of Lemma 4.3 in addition: there exist numbers* $\delta \in [0,1]$ *and* $C_3 > 0$ *such that:*

$$|g(x_1,z) - g(x_2,z)| \leq C_3|x_1 - x_2|^{\delta}(|x_1 - z|^{-n-1} + |x_2 - z|^{-n-1})$$

for all $x_1, x_2, z \in \Omega$ *with* $|z - x_1| \geq 2|x_1 - x_2| > 0$. *Let* $\mu := min\{\beta, \gamma\}$, *then the function*

$$f(x,y) := \int_{\Omega} g(x,z)h(z,y)dz \quad x,y \in \overline{\Omega}, x \neq y$$

satisfies

$$|f(x_1,z) - f(x_2,z)| \leq C_4|x_1 - x_2|^{\eta}(|x_1 - z|^{\mu-n-1} + |x_2 - z|^{\mu-n-1})$$

for all $x_1, x_2 \in \Omega$ *and* $z \in \Omega \setminus \{x_1, x_2\}$ *with suitable numbers* $\eta \in (0,\delta)$, $\eta \leq \mu$ *and* $C_4 > 0$.

(The proof of lemmas 4.3, 4.4, 4.5 can be found in [17])

Lemma 4.6. *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be a bounded domain and*

$$\omega : \overline{\Omega} \times \overline{\Omega} \setminus D \to \mathbb{R}, \quad \omega(x,y) = |y - x|^{-n} l\left(x, \frac{y - x}{|y - x|}\right)$$

with some function $l(x,\xi) \in \mathcal{C}^1(\overline{\Omega} \times S^n)$ *(*S^n *denotes the unit sphere in* $\mathbb{R}^{n+1}$*), and* $u \in \mathcal{C}^{\beta}(\Omega)$ *with* $\beta \in (0,1)$. *Then the function*

$$\Phi(x) := \int_{\Omega} \omega(x,y)u(y)dy$$

is in $\mathcal{C}^1(\Omega)$,

$$\frac{\partial}{\partial x_i}\Phi(x) = \lim_{\varepsilon \to 0} \int_{\Omega \setminus \overline{B_{\varepsilon}(x)}} \frac{\partial}{\partial x_i}\omega(x,y)u(y)dy - u(x) \int_{|\xi|=1} \xi_i l(x,\xi)d\mu(\xi) \;\forall x \in \Omega.$$

(The proof of this lemma can be found in [21], pp. 150-152, 162.)

Lemma 4.7. *Suppose that* $E_i(x) \in \mathcal{C}^{\alpha}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$, $(0 < \alpha < 1)$ *then there exists a constant* $C > 0$ *such that* $\forall x, z \in \Omega,\ x \neq z$

$$\begin{aligned} a)\ \|H(x,z) + H(z,x)\| &\leq C|x - z|^{\alpha-n}, \\ b)\ \left\|\frac{\partial[H(z,x) + H(x,z)]}{\partial x_j}\right\| &\leq C|z - x|^{\alpha-n-1}. \end{aligned}$$

Proof. We use the Euclidean norm for matrices.

$$H(x,y) = \sum_{r=0}^{n} \frac{\partial G(x,y)}{\partial x_r} \overline{E_r(y)} = \frac{\sum\limits_{r,s=0}^{n} A_{rs}(y)(x_s - y_s)\overline{E_r(y)}}{d(y)\rho(x,y)^{n+1}}$$

where $\rho(x,y) := \sqrt{(x-y)^T A^{-1}(y)(x-y)}$ and $d(y) := \omega_{n+1}\sqrt{Det A(y)}$.

$$\begin{aligned}
|\rho(z,x)^{n+1} - \rho(x,z)^{n+1}| &= O(|z-x|^n)|\rho(z,x) - \rho(x,z)| \\
&= O(|z-x|^{n-1})|\rho(z,x)^2 - \rho(x,z)^2|) \\
&= O(|z-x|^{n-1})|(z-x)^T(A^{-1}(z) - A^{-1}(x))(z-x)|) \\
&= O(|z-x|^{\alpha+n+1}). \\
\frac{1}{\rho(x,z)^{n+1}} - \frac{1}{\rho(z,x)^{n+1}} &= \frac{\rho(z,x)^{n+1} - \rho(x,z)^{n+1}}{\rho(x,z)^{n+1}\rho(z,x)^{n+1}} \\
&= \frac{O(|z-x|^{\alpha+n+1})}{\rho(x,z)^{n+1}\rho(z,x)^{n+1}} \\
&= O(|z-x|^{\alpha-n-1}).
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
a)\ H(z,x) + H(x,z) &= \sum_{k,l=0}^{n} \left(\frac{A_{kl}(x)(z_l - x_l)\overline{E_r(x)}}{d(x)\rho(z,x)^{n+1}} + \frac{A_{kl}(z)(x_l - z_l)\overline{E_r(z)}}{d(z)\rho(x,z)^{n+1}} \right) \\
&= \sum_{k,l=0}^{n} \left(\frac{A_{kl}(z)\overline{E_r(z)}}{d(z)\rho(x,z)^{n+1}} - \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)\rho(z,x)^{n+1}} \right)(x_l - z_l).
\end{aligned}$$

To prove a) it is sufficent to show that

$$\left\| \frac{A_{kl}(z)\overline{E_r(z)}}{d(z)\rho(x,z)^{n+1}} - \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)\rho(z,x)^{n+1}} \right\| \leq C|x-z|^{\alpha-n-1}. \tag{4.12}$$

Indeed,

$$\begin{aligned}
\frac{A_{kl}(z)\overline{E_r(z)}}{d(z)\rho(x,z)^{n+1}} - \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)\rho(z,x)^{n+1}} &= \frac{A_{kl}(z)\overline{E_r(z)}}{d(z)} \left[\frac{1}{\rho(x,z)^{n+1}} - \frac{1}{\rho(z,x)^{n+1}} \right] \\
&+ \frac{1}{\rho(z,x)^{n+1}} \left[\frac{A_{kl}(z)\overline{E_r(z)}}{d(z)} - \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)} \right].
\end{aligned}$$

The first term, by (4.11)

$$\left| \frac{1}{\rho(x,z)^{n+1}} - \frac{1}{\rho(z,x)^{n+1}} \right| = O(|x-z|^{\alpha-n-1}).$$

The second term is obviously estimated by $|x-z|^{\alpha-n-1}$.

$$b) \qquad \frac{\partial(H(z,x)+H(x,z))}{\partial x_j} = \sum_{k=0}^{n}\left(\frac{A_{kj}(z)\overline{E_r(z)}}{d(z)\rho(x,z)^{n+1}} - \frac{A_{kj}(x)\overline{E_r(x)}}{d(x)\rho(z,x)^{n+1}}\right) + \sum_{k,l=0}^{n}\frac{\partial}{\partial x_j}\left(\frac{A_{kl}(z)\overline{E_r(z)}}{d(z)\rho(x,z)^{n+1}} - \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)\rho(z,x)^{n+1}}\right)(x_l-z_l).$$

To prove b) it is sufficient to show that

$$i)\ I: \ = \frac{A_{kj}(z)\overline{E_r(z)}}{d(z)\rho(x,z)^{n+1}} - \frac{A_{kj}(x)\overline{E_r(x)}}{d(x)\rho(z,x)^{n+1}} = O(|z-x|^{\alpha-n-1}),$$

$$ii)\ J: \ = \frac{\partial}{\partial x_j}\left(\frac{A_{kl}(z)\overline{E_r(z)}}{d(z)\rho(x,z)^{n+1}} - \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)\rho(z,x)^{n+1}}\right) = O\left(|x-z|^{\alpha-n-2}\right).$$

The first term I was estimated in (4.12) and the second term

$$J = -\left[\frac{\partial}{\partial x_j}\frac{A_{kl}(x)\overline{E_r(x)}}{d(x)}\right]\frac{1}{\rho(z,x)^{n+1}} + \left[\frac{A_{kl}(z)\overline{E_r(z)}}{d(z)}\frac{\partial}{\partial x_j}\frac{1}{\rho(x,z)^{n+1}} - \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)}\frac{\partial}{\partial x_j}\frac{1}{\rho(z,x)^{n+1}}\right].$$

The first term J_1 of J is $O\left(|x-z|^{-n-1}\right)$ because that the derivative of $E_i(x)$ is bounded. The second term J_2 of J

$$J_2 = \left[\frac{A_{kl}(z)\overline{E_r(z)}}{d(z)} - \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)}\right]\frac{\partial}{\partial x_j}\frac{1}{\rho(x,z)^{n+1}} + \frac{A_{kl}(x)\overline{E_r(x)}}{d(x)}\frac{\partial}{\partial x_j}\left[\frac{1}{\rho(x,z)^{n+1}} - \frac{1}{\rho(z,x)^{n+1}}\right].$$

It is easy to see the first term of J_2 is $O\left(|x-z|^{\alpha-n-2}\right)$. The second term of J_2

$$\frac{\partial}{\partial x_j}\left[\frac{1}{\rho(x,z)^{n+1}} - \frac{1}{\rho(z,x)^{n+1}}\right] = \sum_{l,k=0}^{n}(n+1)(z_l-x_l)\left[\frac{A_{jl}(z)}{\rho(x,z)^{n+3}} - \frac{A_{jl}(x)}{\rho(z,x)^{n+3}} - \frac{1}{2}\frac{\partial A_{kl}(x)}{\partial x_j}\frac{(x_k-z_k)}{\rho(z,x)^{n+3}}\right].$$

$$\begin{aligned}
\frac{A_{jl}(z)}{\rho(x,z)^{n+3}} - \frac{A_{jl}(x)}{\rho(z,x)^{n+3}} &= \frac{A_{jl}(z) - A_{jl}(x)}{\rho(x,z)^{n+3}} + A_{jl}(x)\left[\frac{1}{\rho(x,z)^{n+3}} - \frac{1}{\rho(z,x)^{n+3}}\right].\\
\frac{A_{jl}(z) - A_{jl}(x)}{\rho(x,z)^{n+3}} &= O(|z-x|^{\alpha-n-3}).\\
\frac{1}{\rho(x,z)^{n+3}} - \frac{1}{\rho(z,x)^{n+3}} &= O(|z-x|^{\alpha-n-3}).\\
\frac{\partial A_{kl}(x)}{\partial x_j}\frac{((x_k - z_k)}{\rho(z,x)^{n+3}} &= O(|z-x|^{-n-2}).
\end{aligned}$$

Hence, the second term of J_2 is estimated by $O(|z-x|^{\alpha-n-2})$. Therefore J is also estimated by $O(|z-x|^{\alpha-n-2})$.
Thus Lemma 4.7 is proved. □

Lemma 4.8. *Suppose that the coefficients $E_i(x)$ and $Q(x)$ of the operator M is in $\mathcal{C}^{\alpha}(\Omega)$ ($0 < \alpha < 1$), $K(x,y) := -M_x H(x,y)$. Then there exists $C > 0$ such that*

$$\begin{aligned}
||K(x,y)|| &\leq C|x-y|^{\alpha-n-1} \quad \forall\, x,y \in \Omega, x \neq y, \ \textit{and}\\
||K(x,z) - K(y,z)|| &\leq C|x-y|^{\alpha}(|x-z|^{-n-1} + |y-z|^{-n-1})
\end{aligned}$$

for all $x,y,z \in \Omega$ with $|z-x| \geq 2|x-y|$.

Proof. The first estimate is already proved in Remark 2. Let $x \neq y$ and z such that $|z-x| \geq 2|x-y|$, $s > 0$. Define a function $f(t)$, $t \in [0,1]$ by

$$f(t) := \rho(y + t(x-y), z)^{-s}.$$

By the mean value theorem, there exists $\theta \in (0,1)$ such that $f(1) - f(0) = f'(\theta)$.

$$f'(\theta) = -s\sum_{j=0}^{n} \rho(\omega,z)^{-s-1}(x_j - y_j)\frac{\partial\rho}{\partial x_j}(\omega,z) \quad \text{with} \quad \omega = y + \theta(x-y).$$

$$\rho(x,z) := \left((x-z)^T A^{-1}(z)(x-z)\right)^{\frac{1}{2}}, \ \frac{\partial}{\partial x_j}\rho(x,z) = \sum_{l=0}^{n} A_{jl}(z)\frac{x_l - z_l}{\rho(x,z)}.$$

$$\begin{aligned}
\rho(x,z)^{-s} - \rho(y,z)^{-s} &= f(1) - f(0)\\
&= -s\sum_{j,l=0}^{n} \rho(\omega,z)^{-s-1}(x_j - y_j)A_{jl}(z)\frac{\omega_l - z_l}{\rho(\omega,z)}\\
&= -s\frac{(\omega - z)^T A^{-1}(z)(x-y)}{\rho(\omega,z)^{s+2}}.
\end{aligned}$$

We have $|\omega - x| \leq |x-y| \leq \frac{1}{2}|z-x|$ and

$$|\omega - z| \geq |z-x| - |x-\omega| \geq |z-x| - \frac{1}{2}|x-z| = \frac{1}{2}|x-z|.$$

Hence

$$|\rho(x,z)^{-s} - \rho(y,z)^{-s}| \leq C|x-y||x-z|^{-s-1}. \tag{4.13}$$

By definition

$$K(x,z) := M_x H(x,z) = \sum_{i=0}^{n} E_i(x) \frac{\partial H(x,z)}{\partial x_i} + Q(x)H(x,z).$$

We will estimate the substraction

$$\begin{aligned} K(x,z) - K(y,z) = \quad & \sum_{i=0}^{n} \left(E_i(x) \frac{\partial H}{\partial x_i}(x,z) - E_i(y) \frac{\partial H}{\partial x_i}(y,z) \right) \\ & + (Q(x)H(x,z) - Q(y)H(y,z)). \end{aligned} \tag{4.14}$$

First, we consider the second term in the expression (4.14)

$$Q(x)H(x,z) - Q(y)H(y,z) \; = \; \sum_{r,s=0}^{n} \frac{A_{rs}(z)}{d(z)} \left(\frac{Q(x)(x_s - z_s)}{\rho(x,z)^{n+1}} - \frac{Q(y)(y_s - z_s)}{\rho(y,z)^{n+1}} \right) \overline{E_r(z)}.$$

We have

$$\begin{aligned} \frac{Q(x)(x_s - z_s)}{\rho(x,z)^{n+1}} - \frac{Q(y)(y_s - z_s)}{\rho(y,z)^{n+1}} \; &= Q(x)(x_s - z_s) \left(\frac{1}{\rho(x,z)^{n+1}} - \frac{1}{\rho(y,z)^{n+1}} \right) \\ &+ \frac{1}{\rho(y,z)^{n+1}} \left(Q(x)(x_s - z_s) - Q(y)(y_s - z_s) \right). \end{aligned}$$

$$\begin{aligned} \left| \frac{1}{\rho(x,z)^{n+1}} - \frac{1}{\rho(y,z)^{n+1}} \right| \quad & \leq \; C|x-y||x-z|^{-n-2} \text{ (by 4.13)}, \\ \|Q(x)(x_s - z_s) - Q(y)(y_s - z_s)\| \quad & \leq \|Q(x)x_s - Q(y)y_s\| + |z_s|.\|Q(x) - Q(y)\|. \end{aligned}$$

Because $Q(x) \in C^{\alpha}(\Omega)$, it is easy to see

$$\|Q(x)H(x,z) - Q(y)H(y,z)\| = O(|x-y|^{\alpha}|z-x|^{-n-1}).$$

Then, the first term of (4.14) is

$$\begin{aligned} & \sum_{i=0}^{n} \left(E_i(x) \frac{\partial H}{\partial x_i}(x,z) - E_i(y) \frac{\partial H}{\partial x_i}(y,z) \right) = \sum_{i=0}^{n} \left(E_i(x) - E_i(y) \right) \frac{\partial H}{\partial x_i}(x,z) \\ & + \sum_{i=0}^{n} \left(E_i(y) - E_i(z) \right) \left(\frac{\partial H}{\partial x_i}(x,z) - \frac{\partial H}{\partial x_i}(y,z) \right). \end{aligned} \tag{4.15}$$

The first term of (4.15) is estimated by

$$\left\|(E_i(x)-E_i(y))\frac{\partial H}{\partial x_i}(x,z)\right\| \;\leq\; C|x-y|^{\alpha}|x-z|^{-n-1}.$$

For the second term of (4.15), it is sufficient to show that

$$\left\|\frac{\partial H}{\partial x_i}(x,z)-\frac{\partial H}{\partial x_i}(y,z)\right\| \;\leq\; C|x-y||x-z|^{-n-2}.$$

Rewriting

$$\begin{aligned}
H(x,z) \;&=\; \sum_{i=0}^{n}\frac{\partial G(x,z)}{\partial x_i}\overline{E_i(z)},\;\; \frac{\partial H}{\partial x_j}(x,z) \;=\; \sum_{i=0}^{n}\frac{\partial^2 G(x,z)}{\partial x_i\partial x_j}\overline{E_i(z)},\\
\frac{\partial G(x,z)}{\partial x_i} \;&=\; \frac{1}{\omega_{n+1}\sqrt{Det(A(z))}\rho(x,z)^{n+1}}\sum_{j=0}^{n}A_{ij}(z)(x_j-z_j),\\
\frac{\partial^2 G(x,z)}{\partial x_i\partial x_j} \;&=\; \frac{(n+1)\sum\limits_{l,r=0}^{n}A_{jl}(z)A_{ir}(z)\dfrac{(x_l-z_l)(x_r-z_r)}{\rho(x,z)^2}-A_{ij}(z)}{\omega_{n+1}\sqrt{Det(A(z))}\rho(x,z)^{n+1}}.
\end{aligned}$$

Because $\frac{1}{2}|x-z|\leq|y-z|\leq 2|x-z|$ and $|x-y|\leq|x-z|$, we need to prove that

$$\left|\frac{\partial^2 G}{\partial x_i\partial x_j}(x,z)-\frac{\partial^2 G}{\partial x_i\partial x_j}(y,z)\right|\leq\; C|x-y||x-z|^{-n-2}.$$

It is sufficient to prove that

$$\begin{aligned}
\left|\frac{1}{\rho(y,z)^{n+1}}-\frac{1}{\rho(x,z)^{n+1}}\right| &\leq\; C|x-y||x-z|^{-n-2}\\
\left|\frac{(y_l-z_l)(y_r-z_r)}{\rho(y,z)^{n+1}\rho(y,z)^2}-\frac{(x_l-z_l)(x_r-z_r)}{\rho(x,z)^{n+1}\rho(x,z)^2}\right| &\leq C|x-y||x-z|^{-n-2}.
\end{aligned}$$

The first estimate comes from (4.13), the left-hand side of the second estimate is

$$\begin{aligned}
&\Bigg|\left[\frac{1}{\rho(y,z)^{n+1}}-\frac{1}{\rho(x,z)^{n+1}}\right]\frac{(y_l-z_l)(y_r-z_r)}{\rho(y,z)^2}\\
&+\frac{1}{\rho(x,z)^{n+1}}\left[\frac{(y_l-z_l)(y_r-z_r)}{\rho(y,z)^2}-\frac{(x_l-z_l)(x_r-z_r)}{\rho(x,z)^2}\right]\Bigg|.
\end{aligned}$$

Because $\left|\frac{(y_l-z_l)(y_r-z_r)}{\rho(y,z)^2}\right|=O(1)$, the first term is estimated. It is sufficient to prove that

$$\left|\frac{(y_l-z_l)(y_r-z_r)}{\rho(y,z)^2}-\frac{(x_l-z_l)(x_r-z_r)}{\rho(x,z)^2}\right|\leq C|x-y||x-z|^{-1}. \tag{4.16}$$

$$\frac{(y_l-z_l)(y_r-z_r)}{\rho(y,z)^2}-\frac{(x_l-z_l)(x_r-z_r)}{\rho(x,z)^2}=\left[(y_l-z_l)(y_r-z_r)-(x_l-z_l)(x_r-z_r)\right]\frac{1}{\rho(y,z)^2}$$
$$-(x_l-z_l)(x_r-z_r)\left[\frac{1}{\rho(y,z)^2}-\frac{1}{\rho(x,z)^2}\right].$$

Since

$$\begin{aligned}&|(y_l-z_l)(y_r-z_r)-(x_l-z_l)(x_r-z_r)|\\&=|(y_l-z_l)\left[(y_r-z_r)-(x_r-z_r)\right]+(x_r-z_r)\left[(y_l-z_l)-(x_l-z_l)\right]|\\&=|(y_l-z_l)\left[y_r-x_r\right]+(x_r-z_r)\left[y_l-x_l\right]|\le 2|x-y||x-z|\end{aligned}$$

and

$$\left|(x_l-z_l)(x_r-z_r)\left[\frac{1}{\rho(y,z)^2}-\frac{1}{\rho(x,z)^2}\right]\right|\le C|x-z|^2|x-y||x-z|^{-3}=C|x-y||x-z|^{-1},$$

the estimate (4.16) is proved. Therefore Lemma 4.8 is proved. □

Lemma 4.9. *Suppose that* $E_i(x)\in\mathcal{C}^{\alpha}(\Omega)\cap\mathcal{C}^1(\overline{\Omega})$, $Q(x)\in\mathcal{C}^{\alpha}(\Omega)$, $0<\alpha<1$. *Let* $\gamma(x,y)$ *be an* $m\times m$ *matrix satisfying the following properties for some constants* $\mu,\beta,\sigma\in(0,1)$, $C>0$

a) $\gamma(x,y)\in\mathcal{C}^0(\overline{\Omega}\cup\overline{\Omega}\setminus D)$
b) $\gamma(x,y)=O(|x-y|^{\mu-n-1})$ *in* Ω
c) $\|\gamma(z_1,y)-\gamma(z_2,y)\|\le C\eta^{\sigma-n-1}|z_1-z_2|^{\beta}\;\;\forall z_1,z_2\in\overline{\Omega}\setminus\{z|\;|z-y|\le\eta\}$.

Define

$$\Phi(x,y):=\int_{\Omega}H(x,z)\gamma(z,y)dz.$$

Then there exists a constant $\lambda\in(0,1)$ *such that*

i) $\Phi(x,y),\;\frac{\partial}{\partial x_i}\Phi(x,y)\in\mathcal{C}^0(\overline{\Omega}\cup\overline{\Omega}\setminus D)$
ii) $\Phi(x,y)=O(|x-y|^{\lambda-n})$ *in* Ω
iii) $\frac{\partial}{\partial x_i}\Phi(x,y)=O(|x-y|^{\lambda-n-1})$ *in each compact subset of* Ω
iv) $M_x\Phi(x,y)=\int_{\Omega}M_xH(x,z)\gamma(z,y)dz+\gamma(x,y)$.

Proof. Here we use the same notation C for any constant.
By Lemma 4.3, $\Phi(x,y)\in C^0(\Omega\times\Omega\setminus D)$ and $||\Phi(x,y)||\le C|x-y|^{\mu-n}$.

Let $x,y \in \Omega,\ \ x \neq y$, without loss of generality we suppose that $|x-y|<2$. Choose η such that $\frac{1}{4}|x-y| < \eta < \frac{1}{2}|x-y|$ and denote $B_\eta(y) := \{x|\ \ |x-y| \leq \eta\}$.

$$\begin{aligned}\Phi(x,y) &= \int\limits_{B_\eta(y)} H(x,z)\gamma(z,y)dz + \int\limits_{\Omega\setminus B_\eta(y)} H(x,z)\gamma(z,y)dz \\ &= \int\limits_{B_\eta(y)} H(x,z)\gamma(z,y)dz + \int\limits_{\Omega\setminus B_\eta(y)} [H(z,x)+H(x,z)]\,\gamma(z,y)dz \\ &- \int\limits_{\Omega\setminus B_\eta(y)} H(z,x)\gamma(z,y)dz = I_1 + I_2 - I_3.\end{aligned}$$

We will estimate $\frac{\partial I_1}{\partial x_j}, \frac{\partial I_2}{\partial x_j}, \frac{\partial I_3}{\partial x_j}$.
The first integral I_1 has only one weak singular at y,

$$\frac{\partial}{\partial x_j} I_1 = \int\limits_{B_\eta(y)} \frac{\partial H}{\partial x_j}(x,z)\gamma(z,y)dz.$$

Since $\left\|\frac{\partial H}{\partial x_j}(x,z)\right\| \leq C|x-z|^{-n-1}$ and $\|\gamma(z,y)\| \leq C|z-y|^{\mu-n-1}$, by Lemma 4.3 we can estimate $\left\|\frac{\partial}{\partial x_j} I_1\right\| \leq C|x-y|^{\mu-n-1}$.

For the second integral I_2, using the Lemma 4.4 and Lemma 4.7, we can differentiate under the integral sign

$$\frac{\partial}{\partial x_j} I_2 = \int\limits_{\Omega\setminus B_\eta(y)} \frac{\partial}{\partial x_j}\,[H(z,x)+H(x,z)]\,\gamma(z,y)dz.$$

Since $\left\|\frac{\partial}{\partial x_j}\,[H(z,x)+H(x,z)]\right\| \leq C|x-z|^{\alpha-n-1}$ (by Lemma 4.7) and $\|\gamma(z,y)\| \leq C|z-y|^{\mu-n-1}$, we can estimate $\left\|\frac{\partial}{\partial x_j} I_2\right\| \leq C|x-y|^{\alpha+\mu-n-1}$.

Consider the last integral $I_3 := \int\limits_{\Omega\setminus B_\eta(y)} H(z,x)\gamma(z,y)dz$.

$$\begin{aligned}H(z,x) &= \sum_{r,s=0}^{n} \frac{1}{\omega_{n+1}\sqrt{DetA(x)}} A_{rs}(x)(z_s-x_s)\overline{E_r(x)}\left[(z-x)^T A^{-1}(x)(z-x)\right]^{-\frac{n+1}{2}} \\ &= \frac{1}{|x-z|^n} \sum_{r,s=0}^{n} \frac{A_{rs}(x)\overline{E_r(x)}}{\omega_{n+1}\sqrt{DetA(x)}} \frac{z_s-x_s}{|x-z|}\left[\left(\frac{z-x}{|z-x|}\right)^T A^{-1}(x)\left(\frac{z-x}{|z-x|}\right)\right]^{-\frac{n+1}{2}}.\end{aligned}$$

We apply Lemma 4.6 for I_3

$$\frac{\partial}{\partial x_j} I_3 = \lim_{\varepsilon\to 0} \int\limits_{(\Omega\setminus B_\eta(y))\setminus B_\varepsilon(x)} \frac{\partial}{\partial x_j} H(z,x)\gamma(z,y)dz$$
$$- \sum_{r,s=0}^{n} \frac{1}{\omega_{n+1}} \frac{A_{rs}(x)\overline{E_r(x)}}{\sqrt{DetA(x)}} \int\limits_{|\xi|=1} \xi_s.\xi_j \left[\xi^T A^{-1}(x)\xi\right]^{-\frac{n+1}{2}} d\mu(\xi).\ \gamma(x,y).$$

We now prove that $\left\|\frac{\partial}{\partial x_j} I_3\right\| \le C|x-y|^{\lambda-n-1}$ for some constant $C>0,\ \lambda\in(0,1)$ and for all $x,y\in\Omega_1,\ x\neq y$ (Ω_1 is a compact subset of Ω) . The second term is $O(|x-y|^{\mu-n-1})$ by the hypothesis b). The first term is estimated as follows

$$\lim_{\varepsilon\to 0} \int\limits_{(\Omega\setminus B_\eta(y))\setminus B_\varepsilon(x)} \frac{\partial}{\partial x_j} H(z,x)\gamma(z,y)dz = \lim_{\varepsilon\to 0} \int\limits_{(\Omega\setminus B_\eta(y))\setminus B_\varepsilon(x)} \frac{\partial}{\partial x_j} H(z,x)dz.\ \gamma(x,y)$$
$$+ \lim_{\varepsilon\to 0} \int\limits_{(\Omega\setminus B_\eta(y))\setminus B_\varepsilon(x)} \frac{\partial}{\partial x_j} H(z,x)\left[\gamma(z,y)-\gamma(x,y)\right]dz.$$

With $z\in\Omega\setminus B_\eta(y)$

$$\left\|\frac{\partial}{\partial x_j} H(z,x)\left[\gamma(z,y)-\gamma(x,y)\right]\right\| \le C\eta^{\sigma-n-1}|x-z|^{-n-1}|x-z|^{\beta}$$
$$= C.4^{n+1-\sigma}.|x-y|^{\sigma-n-1}|z-x|^{\beta-n-1}.$$

By Lemma 4.6, $\lim\limits_{\varepsilon\to 0} \int\limits_{(\Omega\setminus B_\eta(y))\setminus B_\varepsilon(x)} \frac{\partial}{\partial x_j} H(z,x)dz$ is a continuous function of $x,y\in\Omega_1$ and η, hence this function is bounded. Choose $\lambda := min(\mu,\sigma)$, we have

$$\left\|\lim_{\varepsilon\to 0} \int\limits_{(\Omega\setminus B_\eta(y))\setminus B_\varepsilon(x)} \frac{\partial}{\partial x_j} H(z,x)\gamma(z,y)dz\right\| \le C|x-y|^{\lambda-n-1}\ \forall x,y\in\Omega_1,\ x\neq y.$$

In the end $\left\|\frac{\partial}{\partial x_j} I_3\right\| \le C|x-y|^{\lambda-n-1}\ \forall x,y\in\Omega_1,\ x\neq y.$

We have proved $i),\ ii),\ iii)$, we now prove $iv)$. First we have

$$\sum_{j=0}^{n} E_j(x)\frac{\partial}{\partial x_j} I_3 = \lim_{\varepsilon\to 0} \int\limits_{(\Omega\setminus B_\eta(y))\setminus B_\varepsilon(x)} \sum_{j=0}^{n} E_j(x)\frac{\partial}{\partial x_j} H(z,x)\gamma(z,y)dz -$$
$$- \sum_{r,j,s=0}^{n} \frac{1}{\omega_{n+1}} \frac{A_{rs}(x)E_j(x)\overline{E_r(x)}}{\sqrt{DetA(x)}} \int\limits_{|\xi|=1} \xi_s.\xi_j \left[\xi^T A^{-1}(x)\xi\right]^{-\frac{n+1}{2}} d\mu(\xi)\gamma(x,y).$$

The second term can be written

$$I := \int\limits_{|\xi|=1} \sum_{j=0}^{n} E_j(x)N_j(\xi)H(x+\xi,x)d\mu(\xi)\gamma(x,y)$$

where $\overrightarrow{N} = (N_0, N_1, \cdots, N_n)$ is the outer unit normal of the unit sphere. Let $\varepsilon > 0$ be sufficiently small so that $B_{\rho,\varepsilon} := \{\xi \in \mathbb{R}^{n+1} |\ \xi^T A^{-1}(x)\xi < \varepsilon^2\}$ is contained in the unit ball $B_1(0)$. Applying the Green Integral Formula we can replace the unit sphere by the boundary $\partial B_{\rho,\varepsilon}$ in the integral I, in the end we find that $I = \gamma(x,y)$. From this, the property *iv*) will be proved. □

Remark 6. *Lemma* 4.9 *is still true in the case* $\gamma(x,y) = K(x,y)$.

1.5 The existence of fundamental solutions in the large

Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain, $E_i(x) \in C^{\alpha}(\Omega)$, $Q(x) \in \mathcal{C}^{\alpha}(\Omega), 0 < \alpha < 1$. By Remark 2 in section 1.3 : $K(x,y) = -M_x H(x,y) = O(|x-y|^{\alpha-n-1})$. We define an integral operator

$$\Phi : L^2(\Omega) \longrightarrow L^2(\Omega),\ \ (\Phi u)(x) := \int\limits_{\Omega} K(x,z)u(z)dz$$

with $u(x) = [u_1(x), u_2(x), \cdots, u_m(x)]^T \in L^2(\Omega)$. The scalar product and the norm in $L^2(\Omega)$ are given by

$$(u,v)_{L^2(\Omega)} := \int\limits_{\Omega} u^T(x)v(x)dx,\ \ ||u||_{L^2(\Omega)} := \sqrt{(u,u)_{L^2(\Omega)}}.$$

Denote the adjoint operator of Φ by

$$(\Phi^* u)(x) := \int\limits_{\Omega} K^T(z,x)u(z)dz.$$

With n_0 is sufficiently large ($n_0.\alpha > n+1$) the operator $\Phi^{n_0} = \Phi \circ \Phi \circ \cdots \circ \Phi$ has the kernel $K_{n_0}(x,z) \in \mathcal{C}^0(\overline{\Omega} \times \overline{\Omega})$ (by Lemma 4.3). We know that Φ and Φ^* are compact operators because they have weak singular kernels. Apply the Fredholm theory, $Ker(I-\Phi^*) \subset L^2(\Omega)$ has a finite dimension. Let $u \in Ker(I-\Phi^*)$,

$$u = \Phi^* u = \Phi^* \circ \Phi^* u = \cdots = \Phi^* \circ \Phi^* \circ \cdots \circ \Phi^*(u) = \Phi^{*n_0} u \in \mathcal{C}^0(\overline{\Omega}),$$

then $Ker(I-\Phi^*) \subset \mathcal{C}^0(\overline{\Omega})$.
For a given function $f \in \mathcal{C}^0(\overline{\Omega})$, the equation $(I-\Phi)u = f$ is solvable if and only if

$f \perp Ker(I-\Phi^*)$, and the solution is in $\mathcal{C}^0(\overline{\Omega})$. Applying the operator $I+\Phi+\Phi^2+\cdots+\Phi^{n_0}$ to the above equation, we have

$$(I+\Phi+\Phi^2+\cdots+\Phi^{n_0})(I-\Phi)u = (I+\Phi+\Phi^2+\cdots+\Phi^j)f =: g \in \mathcal{C}^0(\overline{\Omega})$$
$$\Rightarrow (I-\Phi^{n_0+1})u = g \quad \Rightarrow u = g+\Phi^{n_0+1}u \in \mathcal{C}^0(\overline{\Omega}).$$

Using the results which we have obtained we can prove Theorem 1.1.

Proof of Theorem 1.1

Proof. Choose a bounded domain Ω such that $\overline{\Omega_1} \subset \Omega \subset \overline{\Omega} \subset \Omega_0$.
Denote $N_0 := Ker(I-\Phi^*) \cap \mathcal{C}_0(\Omega)$, ($\mathcal{C}_0(\Omega)$ is the space of functions whose support are compact sets in Ω). Decompose $Ker(I-\Phi^*) = N_0 \oplus N_0^{\perp}$, let $\{\varphi_1, \varphi_2 \ldots, \varphi_p\}$ ($\{\psi_1, \psi_2 \ldots, \psi_k\}$) be an orthonormal basis of N_0 ($N_0^{\perp}$).
Denote P be the orthogonal projection onto the finite dimensional space N_0 in the Hilbert space $L_2(\Omega)$. Decompose

$$N_0 = PM(\mathcal{C}_0^{\infty}(\Omega)) \oplus [PM(\mathcal{C}_0^{\infty}(\Omega))]^{\perp}.$$

Let $u \in [PM(\mathcal{C}_0^{\infty}(\Omega))]^{\perp}$, and $\varphi \in \mathcal{C}_0^{\infty}(\Omega)$

$$\int_{\Omega} M_x\varphi(x)^T u(x)dx = (u, M\varphi) = (u, PM\varphi) = 0.$$

That means u is a weak solution of $Ru = 0$ and $u \in N_0$ has compact support in Ω. By the unique continuation property of $Ru = 0$ (Corollary 1), then $u = 0$. This implies that $N_0 = PM(\mathcal{C}_0^{\infty}(\Omega))$. We can write the basis $\{\varphi_j\}$ of N_0 in the form $\varphi_j = PM\alpha_j$, for some $\alpha_j \in \mathcal{C}_0^{\infty}(\Omega)$. Choose a domain U such that $\overline{\Omega_1} \subset U \subset \overline{U} \subset \Omega$, and $supp\varphi_j \subset U$, $(j \in \{1,2...,p\})$.

Write the vector function $\psi_r(x)$ in the form

$$\psi_r(x) = [\psi_{1,r}(x), \psi_{2,r}(x), \cdots \psi_{m,r}(x)]^T.$$

Consider the set of row vectors $\left[\psi_{j,1}(x), \psi_{j,2}(x), \ldots, \psi_{j,k}(x)\right] \in \mathbb{R}^k$, $j \in \{1,2..,m\}$, $x \in \Omega \setminus U$. We will prove that the rank of this set is k . Indeed, if the rank of this set less than k then there exists a vector $\overrightarrow{C} = [C_1, C_2 \ldots, C_k] \neq \overrightarrow{0}$ such that $\overrightarrow{C}$ is orthogonal to all vectors in that set. That means

$$C_1\psi_{j,1}(x) + C_2\psi_{j,2}(x) + \ldots + C_k\psi_{j,k}(x) = 0, \ \forall j \in \{1,2..,m\} \text{ and } \forall x \in \Omega \setminus U,$$

then the function $\psi(x) := C_1\psi_1(x) + C_2\psi_2(x) + \ldots + C_k\psi_k(x) = 0$ $\forall x \in \Omega \setminus U$, the support of ψ is compact in Ω, so $\psi \in N_0 \cap N_0^{\perp}$, this implies $\psi = 0$ in Ω. This is imposible because

$\{\psi_1,\psi_2,..,\psi_k\}$ is a basis of $N_0^{\perp}$.
Denote $\Psi=(\psi_{j_s,r}(x_s))$ be a matrix having k linear independent rows,

$$\Psi=\begin{bmatrix}\psi_{j_1,1}(x_1) & \psi_{j_1,2}(x_1) & \cdots & \psi_{j_1,k}(x_1)\\ \psi_{j_2,1}(x_2) & \psi_{j_2,2}(x_2) & \cdots & \psi_{j_2,k}(x_2)\\ \vdots & & & \\ \psi_{j_k,1}(x_k) & \psi_{j_k,2}(x_k) & \cdots & \psi_{j_k,k}(x_k)\end{bmatrix}$$

$x_1,x_2,\cdots,x_k\in\Omega\setminus U,\ j_1,j_2,\ldots,j_k\in\{1,2,\cdots,m\}$
We consider a function

$$R(x,y)=-\sum_{j=1}^{p}\alpha_j(x)\varphi_j^T(y)-\sum_{s=1}^{k}H(x,x_s)B_s(y)$$

with $x,y\in\overline{\Omega},\ x\notin\{y,x_1,x_2,..,x_k\}$. Denote

$$L(x,y):=M_xR(x,y)=-\sum_{j=1}^{p}M_x\alpha_j(x).\varphi_j^T(y)+\sum_{s=1}^{k}K(x,x_s)B_s(y),$$

B_s are $m\times m$ matrix functions (will be determined later) such that

$$u(y)+\int_{\Omega}L^T(z,y)u(z)dz=0\ \ \forall u\in Ker(I-\Phi^*),\forall y\in\overline{\Omega}. \tag{5.17}$$

First, we check the condition (5.17) for φ_i

$$\varphi_i(y)-\sum_{j=1}^{p}\varphi_j(y)\int_{\Omega}M_x\alpha_j(z)^T.\varphi_i(z)dz+\sum_{s=1}^{k}B_s(y)^T\int_{\Omega}K(z,x_s)^T\varphi_i(z)dz=0$$

$$\Leftrightarrow\varphi_i(y)-\sum_{j=1}^{p}\varphi_j(y)\int_{\Omega}M_x\alpha_j(z)^T.\varphi_i(z)dz+\sum_{s=1}^{k}B_s(y)^T\varphi_i(x_s)=0$$

$$\Leftrightarrow\varphi_i(y)-\sum_{j=1}^{p}\varphi_j(y)\int_{\Omega}M_x\alpha_j(z)^T.\varphi_i(z)dz=0.$$

Because $\varphi_i=PM\alpha_i$, the last condition is satisfied.
Second, we check the condition (5.17) for ψ_r

$$\psi_r(y)-\sum_{j=1}^{p}\varphi_j(y)\int_{\Omega}M_x\alpha_j(z)^T.\psi_r(z)dz+\sum_{s=1}^{k}B_s(y)^T\psi_r(x_s)=0$$

$$\Leftrightarrow\qquad\sum_{s=1}^{k}B_s(y)^T\psi_r(x_s)=-\psi_r(y)+\sum_{j=1}^{p}\varphi_j(y)\int_{\Omega}M_x\alpha_j(z)^T.\psi_r(z)dz=:g^r(y)$$

We have to find k functions $B_s(y), s=1,\dots,k$, such that

$$\sum_{s=1}^{k} B_s(y)^T \psi_r(x_s) = g^r(y)\ \forall r=1,2,\cdots,k, \tag{5.18}$$

with $g^r(y) := [g_{1,r}(y), g_{2,r}(y), .., g_{k,r}(y)]^T$.
We find $B_s(y), B(y)$ in the form

$$B_s(y) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ b^s_{j_s,1} & b^s_{j_s,2} & \cdots & b^s_{j_s,k} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},\ B(y) = \begin{bmatrix} b^1_{j_1,1} & b^1_{j_1,2} & \cdots & b^1_{j_1,k} \\ \vdots & \vdots & \cdots & \vdots \\ b^s_{j_s,1} & b^s_{j_s,2} & \cdots & b^s_{j_s,k} \\ \vdots & \vdots & \cdots & \vdots \\ b^k_{j_k,1} & b^k_{j_k,2} & \cdots & b^k_{j_k,k} \end{bmatrix}.$$

The condition (5.18) becomes $\Psi^T(y)B(y) = G^T(y)$, where $G = (g_{i,j}(y))$. The last equation has a solution $B(y) = (\Psi^{-1})^T(y).G^T(y)$, and $B(y)$ is continuous.

The condition (5.17) is equivalent to

$$\int_{\Omega} [K(z,y)+L(z,y)]^T u(z)dz = 0\ \ \forall u \in Ker(I-\Phi^*), \forall y \in \overline{\Omega}.$$

We will find a solution $\gamma(x,y)$ of the equation

$$\gamma(x,y) - \int_{\Omega} K(x,z)\gamma(z,y)dz = K(x,y)+L(x,y). \tag{5.19}$$

Because $E_i(x), Q(x) \in \mathcal{C}^1(\overline{\Omega})$, with $n_0 \in \mathbb{N}$, $n_0 > n+1$, then the kernel $K_{n_0}(x,y)$ of the operator Φ^{n_0} is continuous in $\overline{\Omega} \times \overline{\Omega}$. We consider the integral equation

$$\sigma(x,y) - \int_{\Omega} K(x,z)\sigma(z,y)dz = f(x,y) \tag{5.20}$$

where $f(x,y) := \int_{\Omega} K_{n_0}(x,z)\,[K(z,y)+L(z,y)]\,dz \in \mathcal{C}(\overline{\Omega}\times\overline{\Omega})$. Let $u \in Ker(I-\Phi^*)$ and $y \in \overline{\Omega}$,

$$\begin{aligned}\int_{\Omega} f(x,y)^T u(x)dx &= \int_{\Omega} \left(\int_{\Omega} K_{n_0}(x,z)\,[K(z,y)+L(z,y)]\,dz\right)^T u(x)dx \\ &= \int_{\Omega} [K(z,y)+L(z,y)]^T \int_{\Omega} K_{n_0}(x,z)^T u(x)dxdz \\ &= \int_{\Omega} [K(z,y)+L(z,y)]^T u(z)dz = 0.\end{aligned}$$

Let $y \in \overline{\Omega}$, there is a unique solution $\sigma(.,y) \in \mathcal{C}(\overline{\Omega})$ of the equation (5.20) which is orthogonal to $Ker(I-\Phi)$. We will prove that $\sigma(x,y) \in \mathcal{C}(\overline{\Omega}\times\overline{\Omega})$.
We show that there exists a number $C > 0$ such that

$$\max_{x\in\overline{\Omega}} ||\sigma(x,y)-\sigma(x,z)|| \leq C.\max_{x\in\overline{\Omega}} ||f(x,y)-f(x,z)||,\ \ \forall y,z \in \overline{\Omega}.$$

Assuming that it is not true, we can choose two sequences $(y_j),\ (z_j)$ in $\overline{\Omega}$ such that

$$M_j := \max_{x\in\overline{\Omega}} ||f(x,y_j)-f(x,z_j)|| > 0, \quad N_j := \max_{x\in\overline{\Omega}} ||\sigma(x,y_j)-\sigma(x,z_j)|| > 0,$$

and $\dfrac{N_j}{M_j} \to \infty$ when $j \to \infty$. It follows that

$$f_j(x) := \frac{1}{N_j}\left[f(x,y_j)-f(x,z_j)\right] \to 0 \text{ as } j \to \infty$$

uniformly in $x \in \overline{\Omega}$. Denote

$$u_j(x) := \frac{1}{N_j}\left[\sigma(x,y_j)-\sigma(x,z_j)\right], \quad v_j(x) := \int\limits_{\Omega} K(x,z)u_j(z)dz,\ \forall x \in \overline{\Omega}.$$

Since (u_j) is bounded, the sequence (v_j) is bounded and equi-continuous, by Arzelà-Ascoli theorem there exists a subsequence v_{j_k} of (v_j) which converges uniformly in $\overline{\Omega}$. Therefore $u_{j_k} = v_{j_k} + f_{j_k}$ also converges uniformly to a function u in $\overline{\Omega}$, the limit function u is in $Ker(I-\Phi)$. Because $\sigma(.,y_j)$ is orthogonal to $Ker(I-\Phi)$, we get $(u,u_j) = 0$.
Let $j \to \infty$, $||u||^2_{L^2} = \lim\limits_{j\to\infty}(u,u_j) = 0$. This is a contradiction since $\max_{x\in\overline{\Omega}} ||u_j|| = 1, \forall j \in \mathbb{N}$.
Let a fixed point $(x_0,y_0) \in \overline{\Omega}\times\overline{\Omega}$,

$$\begin{aligned}\|\sigma(x,y)-\sigma(x_0,y_0)\| \leq\ & \|\sigma(x,y)-\sigma(x,y_0)\| + ||\sigma(x,y_0)-\sigma(x_0,y_0)|| \\ \leq\ & C.\max_{x\in\overline{\Omega}} \|f(x,y)-f(x,z)\| + \|\sigma(x,y_0)-\sigma(x_0,y_0)\|.\end{aligned}$$

This implies that $\sigma(x,y) \in \mathcal{C}(\overline{\Omega}\times\overline{\Omega})$. Define

$$\gamma(x,y) := \sigma(x,y) + K(x,y) + L(x,y) + \sum_{j=1}^{n_0-1} \int\limits_{\Omega} K_j(x,z)\left[K(z,y)+L(z,y)\right]dz.$$

It is obvious to see $\|\gamma(x,y)\| \leq C|x-y|^{\alpha-n-1}$ and easy to check that $\gamma(x,y)$ is a solution of the equation (5.19). In the end, we define

$$\Gamma(x,y) := H(x,y) + \int\limits_{\Omega} H(x,z)\gamma(z,y)dz - R(x,y).$$

By Lemma 4.5 and Lemma 4.8 we can apply Lemma 4.9 and its remark for $\gamma(x,y)$, $\Gamma(x,y)$ is a Levi function of the operator M. Using the conclusion $iv)$ of Lemma 4.9 to prove $\Gamma(x,y)$ is a fundamental solution we check now that $M_x\Gamma(x,y) = 0 \;\; \forall x,y \in \Omega_1, x \neq y$:

$$\begin{aligned} M_x\Gamma(x,y) &= M_xH + M_x \int\limits_{\Omega_1} H(x,z)\gamma(z,y)dz + \int\limits_{\Omega\setminus\Omega_1} M_xH(x,z)\gamma(z,y)dz - M_xR \\ &= M_xH + \int\limits_{\Omega_1} M_xH(x,z)\gamma(z,y)dz + \gamma + \int\limits_{\Omega\setminus\Omega_1} M_xH(x,z)\gamma(z,y)dz - M_xR \\ &= -K - \int\limits_{\Omega} K(x,z)\gamma(z,y)dz + \gamma(x,y) - L = 0. \end{aligned}$$

Thus Theorem 1.1 is proved. □

Remark 7 (Local fundamental solutions of the equation $Mv = 0$).
A local fundamental solution of the equation $Mv = 0$ is constructed as following (see [15], [22], [25]):
Let $\Omega' \Subset \Omega$, the measure of Ω' is small enough, and start with an integral equation:

$$\gamma(x,y) - \int\limits_{\Omega'} K(x,z)\gamma(z,y)dz = K(x,y), \quad K(x,y) := -M_xH(x,y), \quad x,y \in \overline{\Omega'}, x \neq y \tag{5.21}$$

Suppose that the coefficients $E_i(x), Q(x) \in \mathcal{C}^\alpha(\Omega),\ (0 < \alpha < 1)$. We know that

$$||K(x,z)|| = ||M_xH(x,z)|| \leq C||x-z||^{\alpha-n-1}.$$

Determine a sequence $K_j(x,y)$ by induction

$$K_1(x,y) := K(x,y), \cdots, K_{j+1}(x,y) := \int\limits_{\Omega'} K(x,z)K_j(z,y)dz, \;\; j \in \mathbb{N}$$

Let $n_0 \in \mathbb{N}$ with $n_o\alpha \leq n+1,\;\; (n_0+1)\alpha > n+1$, we have:

a) *$K_j \in C^0(\overline{\Omega} \times \overline{\Omega}), \quad j \geq n_0 + 1$*

b) *The series $\sum\limits_{j=n_0+1}^{\infty} ||K_j(x,y)||$ converges uniformly and $\gamma(x,y) := \sum\limits_{j=1}^{\infty} K_j(x,y)$ is a solution of equation (5.21)*

c) *The function $\Gamma(x,y) := H(x,y) + \int\limits_{\Omega} H(x,z)\gamma(z,y)dz, \quad x,y \in \overline{\Omega'},\ x \neq y$ is a fundamental solution of the equation $Mu = 0$ in Ω'.*

The results in this chapter were published in [7].

2 DIRICHLET BOUNDARY VALUE PROBLEM FOR MONOGENIC FUNCTIONS

In this chapter we consider a Dirichlet boundary value problem for monogenic functions in Clifford analysis. We begin with the Dirichlet boundary value problem for holomorphic functions in a smooth, simply connected, bounded domain [31]. If one prescribes the real part on the whole boundary then the imaginary part is determined uniquely up to a constant, hence we can prescribe the imaginary part at one point inside the domain. If the boundary values of the real part are Hölder continuous then the solution is Hölder continuous.

Analogously, we set up a Dirichlet boundary value problem for monogenic functions in Clifford analysis. In [41] the problem was solved in the ball using the harmonic conjugate of the Poisson kernel. In this chapter, we develop the problem in more general domain. Using the Cauchy kernel of the Cauchy-Riemann operator the boundary value problem for monogenic functions is reduced to a problem of the same type in a lower dimension. In the end it is reduced to the classical problem for holomorphic functions. The result is: one half of the components of the desired solution can be prescribed on the whole boundary (an n-dimensional manifold), one half of the rest components can be prescribed on a part of the boundary (an $(n-1)$-dimensional manifold) and so on.... In the last step one component is prescribed on a curve on the boundary (1-dimensional manifold) and one component is prescribed at one point inside the domain. If the boundary data are Hölder continuously differentiable functions, then the unique solution is Hölder continuous. An estimate of the solution by its boundary data is proved. This chapter also presents a method of solving Dirichlet boundary value problems for generalized monogenic functions by reducing them to fixed-point problems. Some additional conditions are required such that the contraction mapping principle and the second version of the Schauder's fixed-point theorem are applicable.

2.1 Some notations in Clifford analysis

Clifford algebra $\mathcal{A}_n$ is generated by $e_0 = 1,\ e_1, e_2, \cdots, e_n$ with the relations:

$$e_i e_j + e_j e_i = -2\delta_{ij},\ \ i, j \in \{1, .., n\}.$$

A basis of $\mathcal{A}_n$ is

$$\{1, e_1, e_2, \cdots, e_n, e_{12}, e_{13}, \cdots, e_{1n}, \cdots, e_{12..n}\},$$

where $e_{i_1 i_2 \cdots i_k} = e_{i_1} e_{i_2} \cdots e_{i_k}$, $(1 \le i_1 < i_2 \cdots < i_k \le n)$. An element $u \in \mathcal{A}_n$ is written in the form $u = \sum_A u_A e_A$. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. A function $f \in \mathcal{C}^1(\Omega, \mathcal{A}_n)$, with $x = (x_0, x_1, \cdots, x_n) \in \Omega$, $f(x) \in \mathcal{A}_n$ is written in the form

$$f(x) = \sum_A f_A(x) e_A, \quad (f_A(x) \in \mathcal{C}^1(\Omega, \mathbb{R})).$$

The Cauchy-Riemann operator D and its adjoint $\overline{D}$ are given by

$$D := e_0 \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}, \quad \overline{D} := e_0 \frac{\partial}{\partial x_0} - \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.$$

Definition 4. *An $\mathcal{A}_n$-valued function f is called a left monogenic function if $Df = 0$ and right monogenic function if $fD = 0$. We also call a left monogenic function shortly a monogenic function.*

Since $D\overline{D} = \overline{D}D = \Delta$, all components of a left (right) monogenic function are harmonic functions. The Cauchy kernel is given by

$$E(x,y) = \frac{1}{\omega_{n+1} |x-y|^{n+1}} \left[(x_0 - y_0) - \sum_{i=1}^{n} (x_i - y_i) e_i \right],$$

where ω_{n+1} is the surface measure of the unit sphere in $\mathbb{R}^{n+1}$. A theory on monogenic functions in Clifford analysis could be found in the book [5].

2.2 Regularity of solutions of Poisson equation in Hölder spaces

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set. Denote

$$\beta = (\beta_0, \beta_1, \cdots, \beta_n) \in \mathbb{N}^{n+1}, \ |\beta| := \sum_{i=0}^{n} \beta_i, \ D^\beta = \frac{\partial^{|\beta|}}{\partial x_0^{\beta_0} \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}.$$

$\mathcal{C}^k(\overline{\Omega}, \mathbb{R})$ is the space of real-valued functions in Ω whose $k-th$ order partial derivatives are continuous in $\overline{\Omega}$. $\mathcal{C}^{k,\alpha}(\Omega, \mathbb{R})$ $(0 < \alpha < 1)$ is the space of real-valued functions in Ω whose $k-th$ order partial derivatives are uniformly Hölder continuous with exponent α in Ω. Supplying the norms

$$||u||_{\mathcal{C}^k(\overline{\Omega},\mathbb{R})} : \quad = \sup_{|\beta| \le k} \sup_{x \in \overline{\Omega}} |D^\beta u(x)|$$

$$||u||_{\mathcal{C}^{k,\alpha}(\Omega,\mathbb{R})} : \quad = ||u||_{\mathcal{C}^k(\overline{\Omega},\mathbb{R})} + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \sup_{|\beta| = k} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

both $\mathcal{C}^k(\overline{\Omega})$ and $\mathcal{C}^{k,\alpha}(\Omega)$ are Banach spaces.
A function $f(x) = \sum_A f_A(x)e_A \in \mathcal{C}^{k,\alpha}(\Omega, \mathcal{A}_n)$ if each its component $f_A \in \mathcal{C}^{k,\alpha}(\Omega, \mathbb{R})$, the norm of f is given by

$$\|f\|_{\mathcal{C}^{k,\alpha}(\Omega,\mathcal{A}_n)} := \sum_A \|f_A\|_{\mathcal{C}^{k,\alpha}(\Omega,\mathbb{R})}.$$

Lemma 2.10 ([10]). *Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ with $\mathcal{C}^{k,\alpha}$-boundary, $k \geq 1$, Ω' be an open set containing $\overline{\Omega}$. Suppose that $\varphi \in \mathcal{C}^{k,\alpha}(\partial\Omega, \mathbb{R})$, then there exists a function $\Phi \in \mathcal{C}^{k,\alpha}(\Omega', \mathbb{R}) \cap \mathcal{C}_0(\Omega', \mathbb{R})$ such that $\Phi = \varphi$ on $\partial\Omega$.*

We can define a norm in $\mathcal{C}^{k,\alpha}(\partial\Omega, \mathbb{R})$ by

$$||\varphi||_{\mathcal{C}^{k,\alpha}(\partial\Omega,\mathbb{R})} = \inf_{\Phi|_{\partial\Omega}=\varphi} ||\Phi||_{\mathcal{C}^{k,\alpha}(\Omega,\mathbb{R})} \tag{2.1}$$

where Φ is any $\mathcal{C}^{k,\alpha}(\Omega', \mathbb{R})$ extension of φ to Ω'. The space $\mathcal{C}^{k,\alpha}(\partial\Omega, \mathbb{R})$ becomes a Banach space.

A fundamental solution of the Laplace equation in $\mathbb{R}^{n+1}$

$$\Gamma(x,y) := \begin{cases} \dfrac{1}{(1-n)\omega_{n+1}}|x-y|^{1-n} & \text{if} \quad n > 1 \\ \dfrac{1}{2\pi}\ln|x-y| & \text{if} \quad n = 1, \end{cases}$$

where ω_{n+1} is the surface measure of the unit sphere in $\mathbb{R}^{n+1}$.
The Newtonian potential of a function f in a domain Ω is given by

$$\mathcal{N}_\Omega(f)(x) = \int_\Omega \Gamma(x,y) f(y) dy.$$

Lemma 2.11 ([10]). *Let $B_1 = B_R(x^0)$, $B_2 = B_{2R}(x^0)$ be concentric balls in $\mathbb{R}^{n+1}$. Suppose that $f \in \mathcal{C}^\alpha(B_2, \mathbb{R})$, $(0 < \alpha < 1)$, and let w be the Newtonian potential of f in B_2. Then $w \in \mathcal{C}^{2,\alpha}(B_1, \mathbb{R})$ and*

$$\begin{aligned} &||w||_{\mathcal{C}^2(\overline{B}_1,\mathbb{R})} + R^\alpha \sup_{\substack{x,y\in B_1 \\ x\neq y, |\beta|=2}} \frac{|D^\beta w(x) - D^\beta w(y)|}{|x-y|^\alpha} \\ &\leq C\left(||f||_{\mathcal{C}(\overline{B}_2,\mathbb{R})} + R^\alpha \sup_{\substack{x,y\in B_2 \\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}\right) \end{aligned}$$

where $C = C(n, \alpha)$.

Remark 8. *If* Ω_1, Ω_2 *are domains such that* $\Omega_1 \subset B_1,\ B_2 \subset \Omega_2$ *and* $f \in \mathcal{C}^{\alpha}(\Omega_2)$ *then Lemma* 2.11 *still holds if* B_1, B_2 *are replaced by* Ω_1, Ω_2 *respectively.*

Remark 9. *From the estimate in Lemma* 2.11 *we have*

$$||w||_{\mathcal{C}^{2,\alpha}(B_1,\mathbb{R})} \leq \frac{C.\max\{1,R^{\alpha}\}}{\min\{1,R^{\alpha}\}}||f||_{\mathcal{C}^{\alpha}(B_2,\mathbb{R})}.$$

Lemma 2.12 (Schauder estimate [2])**.** *Let* Ω *be a bounded domain with* $\mathcal{C}^{1,\alpha}$*-boundary,* $\varphi \in \mathcal{C}^{1,\alpha}(\partial\Omega,\mathbb{R})$*. Then the boundary value problem*

$$\begin{cases} \Delta u &= 0 \quad in\ \Omega \\ u &= \varphi \quad on\ \partial\Omega \end{cases}$$

has a unique solution in $\mathcal{C}^{1,\alpha}(\Omega,\mathbb{R})$ *and we have the estimate*

$$||u||_{\mathcal{C}^{1,\alpha}(\Omega,\mathbb{R})} \leq C||\varphi||_{\mathcal{C}^{1,\alpha}(\partial\Omega,\mathbb{R})}.$$

2.3 An extension of Hölder continuous functions

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with smooth boundary ($\partial\Omega \in \mathcal{C}^2$). Then the outer unit normal of the boundary $\overrightarrow{n}(x),\ x \in \partial\Omega,$ is a Lipschitz function, that means there exists a constant $K > 0$ such that

$$|\overrightarrow{n}(x) - \overrightarrow{n}(y)| \leq K|x-y| \ \ \forall x,y \in \partial\Omega.$$

For each $\delta > 0$ denote

$$\Omega^{\delta} := \{x \in \mathbb{R}^n |\ d(x,\Omega) < \delta\},$$

where $d(x,\Omega) = \inf_{y\in\Omega}|x-y|$. Choose $\delta > 0$ so small that for each $x \in \Omega^{\delta}$ with $d(x,\partial\Omega) < \delta$ there exists a unique projection $\tilde{x}$ of x on the boundary $\partial\Omega$. For a later application we also assume that

$$K\delta < \frac{1}{6} \ \text{ and } \ \ \delta < 1. \tag{3.2}$$

Two points $x \in \Omega$ and $x' \in \Omega^{\delta} \setminus \Omega$ with $d(x,\partial\Omega) < \delta,\ d(x',\partial\Omega) < \delta$ are said to be symmetric through the boundary $\partial\Omega$ if $d(x,\partial\Omega) = d(x',\partial\Omega)$ and they have the same projection on the boundary $\partial\Omega$.

Lemma 3.13. *Let* $x,\ y \in \Omega^{\delta} \setminus \Omega$ *and* $x',\ y'$ *be their reflections through the boundary respectively then we have an inequality*

$$|x'-y'| \leq 7|x-y|. \tag{3.3}$$

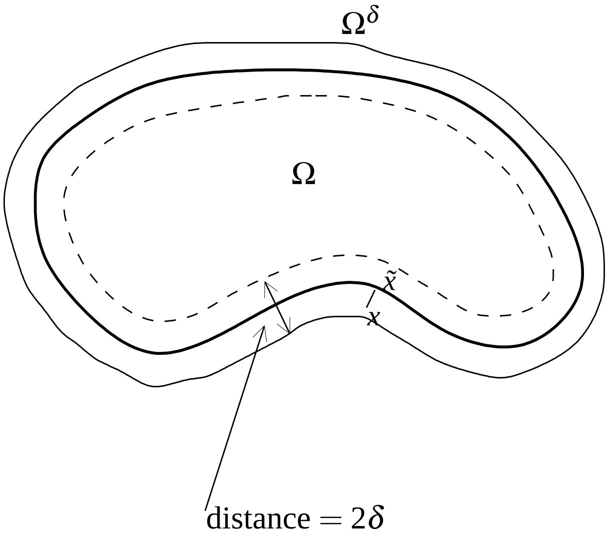

Figure 2.1: Domain extension

Proof. Denote $\tilde{x}$, $\tilde{y}$ be the projections of x, x' and y, y' on the boundary $\partial\Omega$ respectively and

$$t = d(x,\partial\Omega) = d(x',\partial\Omega), \;\; t' = d(y,\partial\Omega) = d(y',\partial\Omega).$$

We can write $x = \tilde{x} + t\overrightarrow{n}(\tilde{x}), \;\; x' = \tilde{x} - t\overrightarrow{n}(\tilde{x}), \;\; y = \tilde{y} + t'\overrightarrow{n}(\tilde{y}), \;\; y' = \tilde{y} - t'\overrightarrow{n}(\tilde{y})$.

$$\begin{aligned}
|x'-y'| &= |\tilde{x} - t\overrightarrow{n}(\tilde{x}) - \tilde{y} + t'\overrightarrow{n}(\tilde{y})| \\
&= |\tilde{x} - \tilde{y} - t\,[\overrightarrow{n}(\tilde{x}) - \overrightarrow{n}(\tilde{y})] + (t'-t)\overrightarrow{n}(\tilde{y})| \\
&\leq |\tilde{x}-\tilde{y}| + t|\overrightarrow{n}(\tilde{x}) - \overrightarrow{n}(\tilde{y})| + |t'-t| \\
&\leq |\tilde{x}-\tilde{y}| + K\delta|\tilde{x}-\tilde{y}| + |t'-t| \\
&\leq (1+K\delta)|\tilde{x}-\tilde{y}| + |t'-t|.
\end{aligned}$$

Because $K\delta < \dfrac{1}{6}$ by (3.2), we have

$$|x'-y'| \leq \frac{7}{6}|\tilde{x}-\tilde{y}| + |t'-t| \leq \frac{7}{6}\left(|\tilde{x}-\tilde{y}| + |t'-t|\right). \tag{3.4}$$

$$\begin{aligned}
|x-y| &= |\tilde{x} + t\overrightarrow{n}(\tilde{x}) - \tilde{y} - t'\overrightarrow{n}(\tilde{y})| \\
&= |\tilde{x} - \tilde{y} + (t-t')\overrightarrow{n}(\tilde{x}) + t'\,[\overrightarrow{n}(\tilde{x}) - \overrightarrow{n}(\tilde{y})]\,| \\
&\geq |\tilde{x} + (t-t')\overrightarrow{n}(\tilde{x}) - \tilde{y}| - t'|\overrightarrow{n}(\tilde{x}) - \overrightarrow{n}(\tilde{y})| \\
&\geq |t-t'| - K\delta|\tilde{x}-\tilde{y}|.
\end{aligned} \tag{3.5}$$

On the other hand

$$\begin{aligned}|x-y| &= |\tilde{x}+t\overrightarrow{n}(\tilde{x})-\tilde{y}-t'\overrightarrow{n}(\tilde{y})| \\ &= |\tilde{x}-\tilde{y}+t\,[\overrightarrow{n}(\tilde{x})-\overrightarrow{n}(\tilde{y})]+(t-t')\overrightarrow{n}(\tilde{y})| \\ &\geq |\tilde{x}-\tilde{y}|-t|\overrightarrow{n}(\tilde{x})-\overrightarrow{n}(\tilde{y})|-|t-t'| \\ &\geq |\tilde{x}-\tilde{y}|-K\delta|\tilde{x}-\tilde{y}|-|t-t'| \\ &\geq (1-K\delta)|\tilde{x}-\tilde{y}|-|t-t'|.\end{aligned} \tag{3.6}$$

Two times of both sides of (3.5) plus (3.6) we get

$$3|x-y| \geq (1-3K\delta)|\tilde{x}-\tilde{y}|+|t'-t|.$$

Since $K\delta < \frac{1}{6}$, we have $3|x-y| \geq \frac{1}{2}|\tilde{x}-\tilde{y}|+|t'-t|$. Hence

$$|x-y| \geq \frac{1}{6}|\tilde{x}-\tilde{y}|+\frac{1}{3}|t'-t| \geq \frac{1}{6}\left(|\tilde{x}-\tilde{y}|+|t'-t|\right). \tag{3.7}$$

From (3.4) and (3.7) we have $|x'-y'| \leq 7|x-y|$. Lemma 3.13 is proved. □

Lemma 3.14. *Let* $f \in \mathcal{C}^{\alpha}(\Omega,\mathbb{R})$ $(0<\alpha<1)$. *Define an extension of* f *to* Ω^{δ} *by*

$$\tilde{f}(x) := \begin{cases} f(x) & if\, x \in \overline{\Omega} \\ f(x') & if\, x \in \Omega^{\delta}\setminus\overline{\Omega}, \end{cases}$$

where x' *is the reflection of* x *through the boundary* $\partial\Omega$. *Then* $\tilde{f} \in \mathcal{C}^{\alpha}(\Omega^{\delta},\mathbb{R})$ *and we have an estimate*

$$\|\tilde{f}\|_{\mathcal{C}^{\alpha}(\Omega^{\delta},\mathbb{R})} \leq 7^{\alpha}\|f\|_{\mathcal{C}^{\alpha}(\Omega,\mathbb{R})}. \tag{3.8}$$

Proof. Denote the Hölder constant of f in Ω by

$$H_f = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}.$$

Let $x,y \in \Omega_{\delta}$. There are three cases:
Case 1: $x,y \in \overline{\Omega}$, then

$$|\tilde{f}(x)-\tilde{f}(y)| = |f(x)-f(y)| \leq H_f|x-y|^{\alpha}. \tag{3.9}$$

Case 2: $x,y \in \Omega_{\delta}\setminus\overline{\Omega}$.
Denote x',y' be the reflections of x,y through the boundary $\partial\Omega$ respectively.

$$|\tilde{f}(x)-\tilde{f}(y)| = |f(x')-f(y')| \leq H_f|x'-y'|^{\alpha}.$$

By Lemma 3.13, we have $|x'-y'| \leq 7|x-y|$, hence

$$|\tilde{f}(x)-\tilde{f}(y)| \leq 7^{\alpha}H_f|x-y|^{\alpha}. \tag{3.10}$$

Case 3: In the last case, we assume that $x \in \Omega^\delta \setminus \overline{\Omega}$ and $y \in \Omega$. Denote x' be the reflection of x through the boundary $\partial\Omega$.

$$|\tilde{f}(x) - \tilde{f}(y)| = |f(x') - f(y)| \leq H_f |x' - y|^\alpha.$$

Since $|x' - y| \leq |x' - x| + |x - y| \leq 3|x - y|$, we obtain

$$|\tilde{f}(x) - \tilde{f}(y)| \leq 3^\alpha H_f |x - y|^\alpha. \tag{3.11}$$

From $(3.9), (3.10)$ and (3.11) we have $\tilde{f} \in \mathcal{C}^\alpha(\Omega^\delta)$ and

$$\frac{|\tilde{f}(x) - \tilde{f}(y)|}{|x - y|^\alpha} \leq 7^\alpha H_f \ \ \forall x, y \in \Omega^\delta, x \neq y.$$

By definition of $\tilde{f}$ we have $\|\tilde{f}\|_{\mathcal{C}(\overline{\Omega^\delta})} = \|f\|_{\mathcal{C}(\overline{\Omega})}$. Hence, we conclude that

$$\|\tilde{f}\|_{\mathcal{C}^\alpha(\Omega^\delta)} = \|\tilde{f}\|_{\mathcal{C}(\overline{\Omega^\delta})} + \sup_{x,y \in \Omega^\delta, x \neq y} \frac{|\tilde{f}(x) - \tilde{f}(y)|}{|x - y|^\alpha} \leq \|\tilde{f}\|_{\mathcal{C}(\overline{\Omega^\delta})} + 7^\alpha H_f \leq 7^\alpha \|f\|_{\mathcal{C}^\alpha(\Omega)}.$$

Lemma 3.14 is proved. ☐

In the following, we define an operator which is similar to the well-known T_Ω in the complex analysis [38].

2.4 $\widetilde{T}_\Omega$ operator on Hölder spaces

Definition 5. *Let $f \in \mathcal{C}^\alpha(\Omega, \mathcal{A}_n)$ and consider its extension $\tilde{f} \in \mathcal{C}^\alpha(\Omega^\delta, \mathcal{A}_n)$ in Lemma 3.14. Define*

$$\widetilde{T}_\Omega f(x) := \int_{\Omega^\delta} E(x,y)\tilde{f}(y)dy \ \forall x \in \Omega.$$

Lemma 4.15. *Let $f \in \mathcal{C}^\alpha(\Omega, \mathcal{A}_n)$, then $\widetilde{T}_\Omega f \in \mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_n)$, $D\widetilde{T}_\Omega f(x) = f(x)$ and*

$$\|\widetilde{T}_\Omega f\|_{\mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_n)} \leq M \|f\|_{\mathcal{C}^\alpha(\Omega, \mathcal{A}_n)},$$

where $M = \dfrac{3(n+1)14^\alpha C(n,\alpha)}{\delta^{2\alpha}}$ and $C(n,\alpha)$ is the constant as in Lemma 2.11.

Proof. Rewrite

$$\widetilde{T}_\Omega f(x) = \int_{\Omega^\delta} \overline{D}_x \Gamma(x,y)\tilde{f}(y)dy = \overline{D}_x \mathcal{N}_{\Omega^\delta}(\tilde{f}),$$

where $\mathcal{N}_{\Omega^\delta}(\tilde{f})$ is the Newtonian potential of $\tilde{f}$ in Ω^δ. From $\tilde{f} \in \mathcal{C}^\alpha(\Omega^\delta, \mathcal{A}_n)$ it implies that $\mathcal{N}_{\Omega^\delta}(\tilde{f}) \in \mathcal{C}^{2,\alpha}(\Omega, \mathcal{A}_n)$ and hence

$$\widetilde{T}_\Omega f = \overline{D}_x \mathcal{N}_{\Omega^\delta}(\tilde{f}) \in \mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_n).$$

$$D\widetilde{T}_\Omega f(x) = D\overline{D}_x \mathcal{N}_{\Omega^\delta}(\tilde{f})(x) = \Delta \mathcal{N}_{\Omega^\delta}(\tilde{f})(x) = \tilde{f}(x) \text{ (see [10]).}$$

In order to estimate $\|\widetilde{T}_\Omega f\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathcal{A}_n)}$, first we consider $f \in \mathcal{C}^\alpha(\Omega, \mathbb{R})$.
Let x be an arbitrary point in Ω. Applying Remark 8 and Remark 9 of Lemma 2.11 for $B_{\delta/2}(x)$ and Ω^δ we obtain

$$\|\mathcal{N}_{\Omega^\delta}(\tilde{f})\|_{\mathcal{C}^{2,\alpha}(B_{\delta/2}(x))} \leq \frac{2^\alpha C(n,\alpha)}{\delta^\alpha} \|\tilde{f}\|_{\mathcal{C}^\alpha(\Omega^\delta)}.$$

By (3.8) it implies that

$$\|\mathcal{N}_{\Omega^\delta}(\tilde{f})\|_{\mathcal{C}^{2,\alpha}(B_{\delta/2}(x))} \leq \frac{2^\alpha 7^\alpha C(n,\alpha)}{\delta^\alpha} \|f\|_{\mathcal{C}^\alpha(\Omega)}. \tag{4.12}$$

Let $x, y \in \Omega, x \neq y$. We consider two cases:
Case 1: $|x-y| \geq \delta$.
For $\beta \in \mathbb{N}^{n+1}$, $|\beta| = 2$

$$\frac{|D^\beta \mathcal{N}_{\Omega^\delta}(\tilde{f})(x) - D^\beta \mathcal{N}_{\Omega^\delta}(\tilde{f})(y)|}{|x-y|^\alpha} \leq \frac{2}{\delta^\alpha} \|D^\beta \mathcal{N}_{\Omega^\delta}(\tilde{f})\|_{\mathcal{C}(\overline{\Omega})} \leq \frac{2^{\alpha+1} 7^\alpha C(n,\alpha)}{\delta^{2\alpha}} \|f\|_{\mathcal{C}^\alpha(\Omega)}.$$

Case 2: $|x-y| < \delta$.
Consider the ball $B_{\delta/2}$ centered at the middle point of x and y and radius $\frac{\delta}{2}$, by (4.12) we have

$$\|\mathcal{N}_{\Omega^\delta}(\tilde{f})\|_{\mathcal{C}^{2,\alpha}(B_{\delta/2})} \leq \frac{2^\alpha 7^\alpha C(n,\alpha)}{\delta^\alpha} \|f\|_{\mathcal{C}^\alpha(\Omega)}.$$

Hence,

$$\frac{|D^\beta \mathcal{N}_{\Omega^\delta}(\tilde{f})(x) - D^\beta \mathcal{N}_{\Omega^\delta}(\tilde{f})(y)|}{|x-y|^\alpha} \leq \frac{2^\alpha 7^\alpha C(n,\alpha)}{\delta^\alpha} \|f\|_{\mathcal{C}^\alpha(\Omega)}.$$

From these two cases we have

$$\frac{|D^\beta \mathcal{N}_{\Omega^\delta}(\tilde{f})(x) - D^\beta \mathcal{N}_{\Omega^\delta}(\tilde{f})(y)|}{|x-y|^\alpha} \leq \frac{2^{\alpha+1} 7^\alpha C(n,\alpha)}{\delta^{2\alpha}} \|f\|_{\mathcal{C}^\alpha(\Omega)} \ \forall x, y \in \Omega, x \neq y.$$

This implies that $\mathcal{N}_{\Omega^\delta}(\tilde{f}) \in \mathcal{C}^{2,\alpha}(\Omega)$ and

$$\|\mathcal{N}_{\Omega^\delta}(\tilde{f})\|_{\mathcal{C}^{2,\alpha}(\Omega)} \leq \frac{3C(n,\alpha)14^\alpha}{\delta^{2\alpha}} \|f\|_{\mathcal{C}^\alpha(\Omega)}.$$

By definition $\widetilde{T}_\Omega f = \overline{D}\mathcal{N}_{\Omega^\delta}(\tilde{f})$, we have $\widetilde{T}_\Omega f \in \mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_n)$.

$$\overline{D}\mathcal{N}_{\Omega^\delta}(\tilde{f}) = \frac{\partial \mathcal{N}_{\Omega^\delta}(\tilde{f})}{\partial x_0} - \sum_{i=1}^{n} e_i \frac{\partial \mathcal{N}_{\Omega^\delta}(\tilde{f})}{\partial x_i}.$$

$$\begin{aligned}\|\widetilde{T}_\Omega f\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathcal{A}_n)} &= \sum_{i=0}^{n} \left\| \frac{\partial \mathcal{N}_{\Omega^\delta}(\tilde{f})}{\partial x_i} \right\|_{\mathcal{C}^{1,\alpha}(\Omega)} \leq (n+1) \left\|\mathcal{N}_{\Omega^\delta}(\tilde{f})\right\|_{\mathcal{C}^{2,\alpha}(\Omega)} \\ &\leq \frac{3(n+1)14^\alpha C(n,\alpha)}{\delta^{2\alpha}} \|f\|_{\mathcal{C}^\alpha(\Omega,\mathbb{R})}.\end{aligned}$$

In general case $f(x) = \sum_A f_A(x) e_A \in \mathcal{C}^\alpha(\Omega, \mathcal{A}_n)$, $\widetilde{T}_\Omega f = \sum_A \widetilde{T}_\Omega f_A(x) e_A$.

$$\begin{aligned}\|\widetilde{T}_\Omega f\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathcal{A}_n)} \leq \sum_A \|\widetilde{T}_\Omega f_A\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathcal{A}_n)} &\leq \frac{3(n+1)14^\alpha C(n,\alpha)}{\delta^{2\alpha}} \sum_A \|f_A\|_{\mathcal{C}^\alpha(\Omega,\mathbb{R})} \\ &= \frac{3(n+1)14^\alpha C(n,\alpha)}{\delta^{2\alpha}} \|f\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)}.\end{aligned}$$

Lemma 4.15 is proved. □

Remark 10. *Because $\mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_n) \subset \mathcal{C}^\alpha(\Omega, \mathcal{A}_n)$, we can find another constant $M > 0$ such that*

$$\|\widetilde{T}_\Omega f\|_{\mathcal{C}^{i,\alpha}(\Omega,\mathcal{A}_n)} \leq M \|f\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)} \ (i = 0,\ 1)$$

(see Lemma 6.23).

2.5 Dirichlet boundary value problem for monogenic functions

2.5.1 Statement of the problem

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with $\mathcal{C}^2$- boundary. Suppose that Ω is a cylinder constructed as follows:

$$\begin{aligned}&\Omega_2 \subset \mathbb{R}^2 \text{ is a simply connected domain,} \\ &\Omega_3 = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \ \mid (x_0, x_1) \in \Omega_2,\ \beta_2(x_0, x_1) < x_2 < \varphi_2(x_0, x_1)\}, \\ &\vdots \\ &\Omega = \Omega_{n+1} = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid x' \in \Omega_n,\ \beta_n(x') < x_n < \varphi_n(x')\},\end{aligned}$$

where $x' := (x_0, x_1, \cdots, x_{n-1}),\ x := (x', x_n)$, $\Omega_k, k = 2, \cdots, n+1$, are bounded domains in $\mathbb{R}^k$ with $\mathcal{C}^2$- boundary.
Let $\psi_k \in \mathcal{C}^{1,\alpha}(\Omega_k, \mathbb{R})$ such that $\beta_k(x) < \psi_k(x) < \varphi_k(x)\ \ \forall x \in \Omega_k$. Define

$$\begin{array}{rccl} \Psi_k : & \Omega_k & \longrightarrow & \Omega_{k+1} \\ & (x_0, x_1, \cdots, x_{k-1}) & \longmapsto & (x_0, x_1, \cdots, x_{k-1}, \psi_k(x_0, x_1, \cdots, x_{k-1})). \end{array}$$

Denote

$$\begin{array}{ll} \Sigma_n : & = \partial\Omega, \\ \Sigma_{n-1} : & = \Psi_n(\partial\Omega_n) \subset \Sigma_n \\ \Sigma_{n-2} : & = \Psi_n \circ \Psi_{n-1}(\partial\Omega_{n-1}) \subset \Sigma_{n-1}, \\ \vdots & \\ \Sigma_k : & = \Psi_n \circ \Psi_{n-1} \circ \Psi_{n-2} \circ \cdots \circ \Psi_{k+1}(\partial\Omega_{k+1}) \subset \Sigma_{k+1}, \\ \vdots & \\ \Sigma_1 : & = \Psi_n \circ \Psi_{n-1} \circ \Psi_{n-2} \circ \cdots \circ \Psi_2(\partial\Omega_2) \subset \Sigma_2, \\ \Sigma_0 : & = \Psi_n \circ \Psi_{n-1} \circ \Psi_{n-2} \circ \cdots \circ \Psi_2(M_0) \in \Omega, \text{ where } M_0 \text{ is a a fixed point in } \Omega_2. \end{array}$$

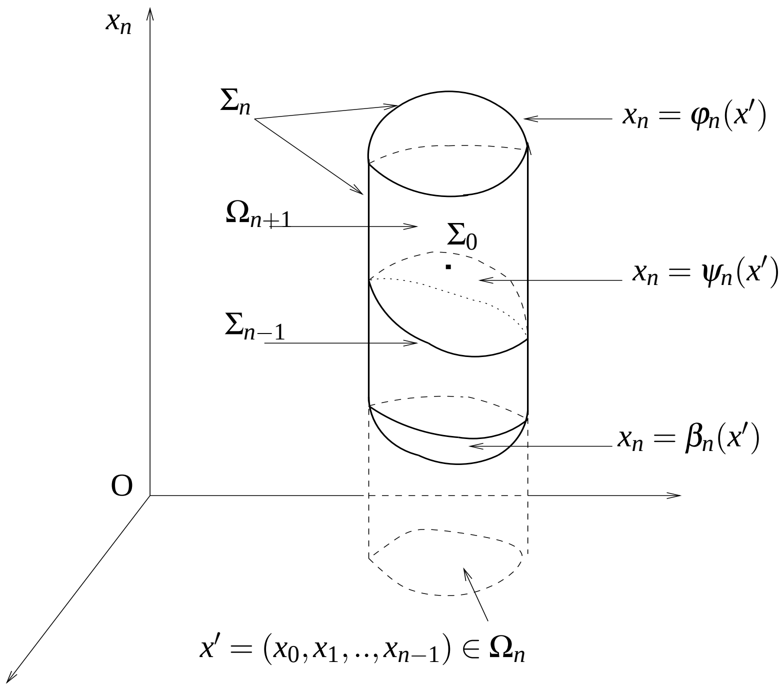

Figure 2.2: Domain and boundary data

Let $g : \Sigma_k \longrightarrow \mathbb{R}$, we write $g \in \mathcal{C}^{1,\alpha}(\Sigma_k, \mathbb{R})$ $(1 \leq k \leq n)$ if

$$g \circ \Psi_n \circ \Psi_{n-1} \circ \Psi_{n-2} \circ \cdots \circ \Psi_{k+1} \in \mathcal{C}^{1,\alpha}(\partial\Omega_{k+1}, \mathbb{R}).$$

Define the norm of g in $\mathcal{C}^{1,\alpha}(\Sigma_k, \mathbb{R})$ by

$$\|g\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} := \|g \circ \Psi_n \circ \Psi_{n-1} \circ \Psi_{n-2} \circ \cdots \circ \Psi_{k+1}\|_{\mathcal{C}^{1,\alpha}(\partial\Omega_{k+1},\mathbb{R})}, \tag{5.13}$$

where the norm in $\mathcal{C}^{1,\alpha}(\partial\Omega_{k+1}, \mathbb{R})$ is defined as in (2.1).

Lemma 5.16. *Let $f \in \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R})$ then the restriction of f on Σ_k $(1 \leq k \leq n)$ is in $\mathcal{C}^{1,\alpha}(\Sigma_k, \mathbb{R})$ and there exists a constant $\lambda > 0$ not depending on f such that*

$$\|f|_{\Sigma_k}\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} \leq \lambda \|f\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathbb{R})}.$$

Proof. Denote $\Phi := \Psi_n \circ \Psi_{n-1} \circ \Psi_{n-2} \circ \cdots \circ \Psi_{k+1}$ in Ω_{k+1}, $(\Phi = (\Phi_0, \Phi_1, \cdots, \Phi_n))$. Because $\psi_k \in \mathcal{C}^{1,\alpha}(\Omega_k, \mathbb{R})$ $(1 \leq k \leq n)$, we have $f \circ \Phi \in \mathcal{C}^{1,\alpha}(\Omega_{k+1}, \mathbb{R})$. By definition (5.13)

$$\|f|_{\Sigma_k}\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} = \|f \circ \Phi\|_{\mathcal{C}^{1,\alpha}(\partial\Omega_{k+1},\mathbb{R})}.$$

By (2.1) it follows

$$\|f|_{\Sigma_k}\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} \leq \|f \circ \Phi\|_{\mathcal{C}^{1,\alpha}(\Omega_{k+1},\mathbb{R})}.$$

$$\frac{\partial(f \circ \Phi)}{\partial x_i} = \sum_{j=0}^{n} \frac{\partial f}{\partial x_j} \circ \Phi \, \frac{\partial \Phi_j}{\partial x_i}.$$

Since $\Phi_j \in \mathcal{C}^{1,\alpha}(\Omega_{k+1}, \mathbb{R})$, there exists a constant $\lambda > 0$ not depending on f such that

$$\|f \circ \Phi\|_{\mathcal{C}^{1,\alpha}(\Omega_{k+1},\mathbb{R})} \leq \lambda \|f\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathbb{R})}.$$

The lemma 5.16 is proved. $\square$

Denote

$$\begin{aligned}
\Lambda_0 :\ & = \{\emptyset\}, \\
\Lambda_1 :\ & = \{(1)\}, \\
\Lambda_2 :\ & = \{(2); (1,2)\}, \\
\vdots\ & \\
\Lambda_{n-1} :\ & = \{(i_1, i_2, \cdots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k = n-1\}, \\
\Lambda_n :\ & = \{(i_1, i_2, \cdots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k = n\}.
\end{aligned}$$

$\mathcal{A}_k$ are subalgebras of $\mathcal{A}_n$ which are generated by $e_0, e_1, \cdots, e_k$ for $1 \leq k \leq n$.
A function $w(x)$ with values in $\mathcal{A}_n$ is written in the form

$$w(x) = \sum_A w_A e_A = \sum_{k=0}^{n} \sum_{A \in \Lambda_k} w_A e_A.$$

Statement of the problem:
Let functions $g_A \in \mathcal{C}^{1,\alpha}(\Sigma_k, \mathbb{R})$ *with* $A \in \Lambda_k,\ (k \geq 1)$, $g_0(x) := g_\emptyset(x) = c$. *Find solutions* w *of the boundary value problem*

$$\begin{cases} Dw(x) & = & 0 & \forall x \in \Omega_{n+1} = \Omega \\ w_A(x) & = & g_A(x) & \forall A \in \Lambda_k \text{ and } x \in \Sigma_k, 0 \leq k \leq n \\ w & = & \sum\limits_A w_A(x)e_A & \in \mathcal{C}^1(\Omega, \mathcal{A}_n) \cap \mathcal{C}^\alpha(\Omega, \mathcal{A}_n). \end{cases} \tag{5.14}$$

This problem is called Dirichlet boundary value problem for monogenic functions.

2.5.2 Reduction of the Dirichlet boundary value problem

A function $w(x)$ with values in $\mathcal{A}_n$ is written in the form [4]

$$\begin{aligned} w(x) = \sum_A w_A(x)e_A &= \sum_{A \notin \Lambda_n} w_A(x)e_A + \sum_{A \in \Lambda_n} w_A(x)e_A \\ &= \sum_{A \notin \Lambda_n} w_A(x)e_A + e_n \sum_{A \in \Lambda_n} (-1)^{|A|-1} w_A(x)e_{A\setminus\{n\}} \\ &= u(x) + e_n v(x), \end{aligned}$$

where $u(x) := \sum\limits_{A \notin \Lambda_n} w_A(x)e_A \in \mathcal{A}_{n-1},\ v(x) := \sum\limits_{A \in \Lambda_n} (-1)^{|A|-1} w_A(x)e_{A\setminus\{n\}} \in \mathcal{A}_{n-1}$.
Suppose that w is a solution of the problem (5.14) then the function v is the solution of the Dirichlet boundary value problem of the Laplace equation

$$\begin{cases} \Delta v(x) & = 0 & \forall x \in \Omega \\ v(x) & = \sum\limits_{A \in \Lambda_n} (-1)^{|A|-1} g_A(x) e_{A\setminus\{n\}} & \forall x \in \Sigma_n = \partial\Omega. \end{cases} \tag{5.15}$$

Using the Schauder estimate in Lemma 2.12, the problem (5.15) has a unique solution $v \in \mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_{n-1})$, and

$$\|v\|_{C^{i,\alpha}(\Omega, \mathcal{A}_{n-1})} \leq C_1 \sum_{A \in \Lambda_n} \|g_A\|_{\mathcal{C}^{1,\alpha}(\partial\Omega, \mathbb{R})}\ (i = 0, 1). \tag{5.16}$$

Denote the Cauchy-Riemann operator and its adjoint operator in Ω_n by

$$D_{x'} = \frac{\partial}{\partial x_0} + \sum_{i=1}^{n-1} e_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \overline{D}_{x'} = \frac{\partial}{\partial x_0} - \sum_{i=1}^{n-1} e_i \frac{\partial}{\partial x_i}.$$

The Cauchy-Riemann equation $D_x w = 0$ is rewritten in the form

$$\begin{aligned}
\left(D_{x'} + e_n \frac{\partial}{\partial x_n}\right)(u + e_n v) = 0 \\
\Leftrightarrow D_{x'}u - \frac{\partial v}{\partial x_n} + e_n\left(\frac{\partial u}{\partial x_n} + \overline{D}_{x'}v\right) = 0 \\
\Leftrightarrow \begin{cases} D_{x'}u - \dfrac{\partial v}{\partial x_n} & = 0 \\ \dfrac{\partial u}{\partial x_n} + \overline{D}_{x'}v & = 0. \end{cases}
\end{aligned} \tag{5.17}$$

The second equation in (5.17) implies that

$$u(x) = -\int_{\psi_n(x')}^{x_n} \overline{D}_{x'}v(x',t)dt + u(x', \psi_n(x'))\ \forall x \in \Omega. \tag{5.18}$$

Then the first equation in (5.17) becomes

$$D_{x'}\left(-\int_{\psi_n(x')}^{x_n} \overline{D}_{x'}v(x',t)dt + u(x', \psi_n(x'))\right) - \frac{\partial v(x)}{\partial x_n} = 0.$$

The above equation is equivalent to

$$D_{x'}[u(x', \psi_n(x'))] = \frac{\partial v(x)}{\partial x_n} + \int_{\psi_n(x')}^{x_n} \Delta_{x'}v(x',t)dt - D_{x'}\psi_n(x')\overline{D}_{x'}v(x', \psi_n(x')).$$

Since $\Delta v = 0$ in Ω, we have $\Delta_{x'}v(x',x_n) + \dfrac{\partial^2 v(x',x_n)}{\partial^2 x_n} = 0$, hence it leads to

$$D_{x'}[u(x', \psi_n(x'))] = \frac{\partial v(x)}{\partial x_n} - \int_{\psi_n(x')}^{x_n} \frac{\partial^2 v}{\partial^2 x_n}(x',t)dt - D_{x'}\psi_n(x')\overline{D}_{x'}v(x', \psi_n(x')).$$

Simplifying the right hand side of the above equation we get

$$D_{x'}(u \circ \Psi_n)(x') = \frac{\partial v}{\partial x_n} \circ \Psi_n(x') - D_{x'}\psi_n(x')\overline{D}_{x'}v \circ \Psi_n(x') =: f(x'), \quad x' \in \Omega_n. \tag{5.19}$$

Since $v \in \mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_{n-1})$ and $\psi_n \in \mathcal{C}^{1,\alpha}(\Omega_n, \mathbb{R})$, the function $f(x') \in \mathcal{C}^{\alpha}(\Omega_n, \mathcal{A}_{n-1})$. Define

$$w' := u \circ \Psi_n - \widetilde{T}_{\Omega_n} f. \tag{5.20}$$

Using Lemma 4.15, we have

$$D_{x'}w' = D_{x'}(u\circ\Psi_n) - D_{x'}(\widetilde{T}_{\Omega_n} f) = D_{x'}(u\circ\Psi_n) - f = 0 \ \text{ in } \Omega_n.$$

Represent $\widetilde{T}_{\Omega_n} f(x')$ by its components

$$\widetilde{T}_{\Omega_n} f(x') = \sum_A (\widetilde{T}_{\Omega_n} f)_A(x')e_A.$$

Define

$$g'_A(x') := g_A\circ\Psi_n(x') - (\widetilde{T}_{\Omega_n} f)_A(x') \ \ \forall A\in\Lambda_k \tag{5.21}$$

with $x'\in\Sigma'_k := \Psi_n^{-1}(\Sigma_k)$, $\ 0\le k\le n-1$.
Lemma 4.15 shows that $(\widetilde{T}_{\Omega_n} f)_A \in \mathcal{C}^{1,\alpha}(\Omega_n,\mathbb{R})$ and hence $g'_A\in\mathcal{C}^{1,\alpha}(\Sigma'_k,\mathbb{R})$ $\forall A\in\Lambda_k$. The function w' becomes a solution of a boundary value problem of type (5.14) in a lower dimensional domain

$$\begin{cases} Dw'(x') &= 0 & \forall x\in\Omega_n \\ w'_A(x') &= g'_A(x') & \forall A\in\Lambda_k,\ \ x'\in\Sigma'_k, 0\le k\le n-1 \\ w' &= \sum\limits_A w'_A(x)e_A & \in \mathcal{C}^1(\Omega_n,\mathcal{A}_{n-1})\cap\mathcal{C}^{\alpha}(\Omega_n,\mathcal{A}_{n-1}). \end{cases} \tag{5.22}$$

Definition 6. *The boundary value problem (5.22) is called the reduced problem of the boundary value problem (5.14).*

Lemma 5.17 (Reduction of the existence and the uniqueness)**.**
Assume that the reduced boundary value problem (5.22) has a unique solution w' for arbitrary boundary data

$$g'_A\in\mathcal{C}^{1,\alpha}(\Sigma'_k,\mathbb{R})\ \forall A\in\Lambda_k\ (0\le k\le n-1).$$

Then the boundary value problem (5.14) has a unique solution.

Proof. We find the solution w of (5.14) in the form $w(x) = u(x) + e_n v(x)$ with $u(x), v(x) \in \mathcal{A}_{n-1}$. The function v becomes the unique solution of the Dirichlet boundary value problem (5.15). Combining (5.18) and (5.20) we get the representation of u

$$u(x) = -\int\limits_{\psi_n(x')}^{x_n} \overline{D}_{x'}v(x',t)dt + \widetilde{T}_{\Omega_n}\left[\frac{\partial v}{\partial x_n}\circ\Psi_n - D_{x'}\psi_n\overline{D}_{x'}v\circ\Psi_n\right](x') + w'(x') \tag{5.23}$$

$\forall x\in\Omega$. ☐

Lemma 5.18 (Reduction of the estimates)**.**
Assume that the solution w' of the reduced boundary value problem (5.22) admits an estimate

$$\|w'\|_{\mathcal{C}^{\alpha}(\Omega_n,\mathcal{A}_{n-1})} \le K_n \sum_{k=0}^{n-1}\sum_{A\in\Lambda_k} \|g'_A\|_{\mathcal{C}^{1,\alpha}(\Sigma'_k,\mathbb{R})} \tag{5.24}$$

for some constant $K_n > 0$. *Then there exists a constant* $K_{n+1} > 0$ *such that for the solution* w *of (5.14)*

$$\|w\|_{\mathcal{C}^{\alpha}(\Omega_{n+1},\mathcal{A}_n)} \leq K_{n+1} \sum_{k=0}^{n} \sum_{A\in\Lambda_k} \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})}\,.$$

Proof. We write the solution w of (5.14) in the form $w(x) = u(x) + e_n v(x)$ with $u(x), v(x) \in \mathcal{A}_{n-1}$. Then v is the unique solution of the Dirichlet boundary value problem (5.15). The function u admits the representation (5.23). We shall estimate the three summands on the right hand side of (5.23) by

$$\sum_{k=0}^{n} \sum_{A\in\Lambda_k} \|g_A\|_{\mathcal{C}^{1,\alpha}(\partial\Omega,\mathbb{R})}$$

.
Because $\psi_n \in \mathcal{C}^{1,\alpha}(\Omega_n,\mathbb{R})$ and $\overline{D}_{x'} v \in \mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)$, there exists a constant $C_2 > 0$ depending on Ω and ψ_n such that the first summand is estimated by

$$\left\| \int_{\psi_n(x')}^{x_n} \overline{D}_{x'} v(x',t)dt \right\|_{\mathcal{C}^{\alpha}(\Omega)} \leq C_2 \|v\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathcal{A}_{n-1})} \leq C_1 C_2 \sum_{A\in\Lambda_n} \|g_A\|_{\mathcal{C}^{1,\alpha}(\partial\Omega,\mathbb{R})}\,. \tag{5.25}$$

By Lemma 4.15 there exists a constant $M > 0$ such that

$$\left\|\widetilde{T}_{\Omega_n} f\right\|_{\mathcal{C}^{i,\alpha}(\Omega_n,\mathcal{A}_{n-1})} \leq M \|f\|_{\mathcal{C}^{\alpha}(\Omega_n,\mathcal{A}_{n-1})} \;\; (i = 0,1).$$

Combining this with the definition of f in (5.19), $\psi_n \in \mathcal{C}^{1,\alpha}(\Omega_n)$, there exists a constant $C_3 > 0$ such that

$$\left\|\widetilde{T}_{\Omega_n} f\right\|_{\mathcal{C}^{i,\alpha}(\Omega_n,\mathcal{A}_{n-1})} \leq C_3 \|v\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathcal{A}_{n-1})} \;\; (i = 0,1).$$

Using the Schauder estimate (5.16), the second summand is estimated as follows

$$\left\|\widetilde{T}_{\Omega_n} f\right\|_{\mathcal{C}^{i,\alpha}(\Omega_n,\mathcal{A}_{n-1})} \leq C_1 C_3 \sum_{A\in\Lambda_n} \|g_A\|_{\mathcal{C}^{1,\alpha}(\partial\Omega,\mathbb{R})}\,. \tag{5.26}$$

The boundary data (5.21) of the reduced boundary value problem are estimated

$$\|g'_A\|_{\mathcal{C}^{1,\alpha}(\Sigma'_k,\mathbb{R})} \leq \|g_A \circ \Psi_n\|_{\mathcal{C}^{1,\alpha}(\Sigma'_k,\mathbb{R})} + \|(\widetilde{T}_{\Omega_n} f)_A\|_{\mathcal{C}^{1,\alpha}(\Sigma'_k,\mathbb{R})}.$$

Since $\psi_n \in \mathcal{C}^{1,\alpha}(\Omega_n,\mathbb{R})$, there exists a constant $C_4 > 0$ such that

$$\|g_A \circ \Psi_n\|_{\mathcal{C}^{1,\alpha}(\Sigma'_k,\mathbb{R})} \leq C_4 \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})}.$$

Combining this with Lemma 5.16, we get

$$\|g'_A\|_{\mathcal{C}^{1,\alpha}(\Sigma'_k,\mathbb{R})} \leq C_4 \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} + \lambda \|(\widetilde{T}_{\Omega_n} f)_A\|_{\mathcal{C}^{1,\alpha}(\Omega_n,\mathbb{R})}.$$

From the assumption (5.24) we estimate the third summand

$$
\begin{aligned}
\|w'\|_{\mathcal{C}^{\alpha}(\Omega_n,\mathcal{A}_{n-1})} &\leq K_n \sum_{k=0}^{n-1} \sum_{A\in\Lambda_k} \left(C_4 \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} + \lambda \|(\widetilde{T}_{\Omega_n} f)_A\|_{\mathcal{C}^{1,\alpha}(\Omega_n,\mathbb{R})} \right) \\
&\leq K_n C_4 \sum_{k=0}^{n-1} \sum_{A\in\Lambda_k} \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} + \lambda K_n \|\widetilde{T}_{\Omega_n} f\|_{\mathcal{C}^{1,\alpha}(\Omega_n,\mathbb{R})} \\
&\leq K_n C_4 \sum_{k=0}^{n-1} \sum_{A\in\Lambda_k} \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} + \lambda K_n C_1 C_3 \sum_{A\in\Lambda_n} \|g_A\|_{\mathcal{C}^{1,\alpha}(\partial\Omega,\mathbb{R})} \\
&\leq K_n \max\{C_4; \lambda C_1 C_3\} \sum_{k=0}^{n} \sum_{A\in\Lambda_k} \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} . \qquad (5.27)
\end{aligned}
$$

Combining the estimates (5.25), (5.26) and (5.27), and denoting

$$K_{n+1} := C_1(C_2 + C_3) + K_n \max\{C_4; \lambda C_1 C_3\},$$

then

$$\|w\|_{\mathcal{C}^{\alpha}(\Omega_{n+1},\mathcal{A}_n)} \leq K_{n+1} \sum_{k=0}^{n} \sum_{A\in\Lambda_k} \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} .$$

Lemma 5.18 is proved. □

With the helps of Lemma 5.17 and Lemma 5.18, the Dirichlet boundary value problem for monogenic functions (5.14) is reduced step by step to the Dirichlet boundary value problem for holomorphic functions. This problem is completely solvable by using results in complex analysis [31].

Lemma 5.19 ([31]). *Let Ω_2 be a simply connected and bounded domain in $\mathbb{C}$ with $\mathcal{C}^2$-boundary. Then the Dirichlet boundary value problem for holomorphic functions*

$$
\begin{cases}
\dfrac{\partial w}{\partial \overline{z}} = 0 & \text{in } \Omega_2 \\
Re(w) = g \in \mathcal{C}^{1,\alpha}(\partial\Omega_2,\mathbb{R}) & \text{on } \partial\Omega_2 \\
Im(w)(M_0) = c & \text{at a fixed point } M_0 \in \Omega_2
\end{cases}
$$

has a unique solution in $\mathcal{C}^{\alpha}(\Omega_2,\mathbb{C})$ and there exist a constant $K_2 > 0$ such that

$$\|w\|_{\mathcal{C}^{\alpha}(\partial\Omega_2,\mathbb{C})} \leq K_2 \left(|c| + \|g\|_{\mathcal{C}^{1,\alpha}(\partial\Omega_2,\mathbb{R})} \right) .$$

Remark 11. *The solution w in Lemma 5.19 indeed belongs to $\mathcal{C}^{1,\alpha}(\Omega)$ (see [31]).*

To summarize this section we give the following theorem.

Theorem 5.4. *The Dirichlet boundary value problem for monogenic functions (5.14) has a unique solution w and we have an estimate*

$$\|w\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)} \le H \sum_{k=0}^{n} \sum_{A\in\Lambda_k} \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})},$$

where H is a positive constant not depending on g_A.

Example

We give an example of the Dirichlet boundary value problem for monogenic functions in $\mathcal{A}_2$.
Let $\Omega := B_1(0)$ be the unit ball in $\mathbb{R}^3$,

$$\Sigma_2 := \partial B_1(0),\ \Sigma_1 := \{(x_0,x_1,0) \in \mathbb{R}^3 |\ x_0^2 + x_1^2 = 1\},\ \Sigma_0 := \{(0,0,0)\}.$$

In this case the Dirichlet boundary value problem (5.14) reads:

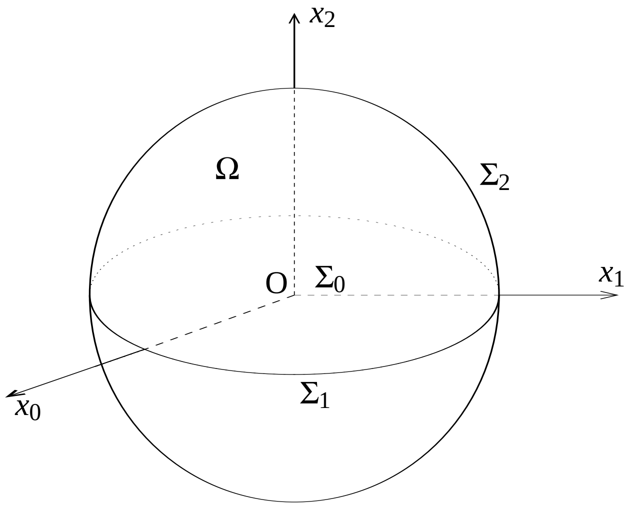

Figure 2.3: Example

Let $g_2, g_{12} \in \mathcal{C}^{1,\alpha}(\partial B_1(0),\mathbb{R}),\ g_1 \in \mathcal{C}^{1,\alpha}(\Sigma_1),\ g_0 = c$ (a real constant), find solutions

$$w(x) = w_0(x) + w_1(x)e_1 + w_2(x)e_2 + w_{12}(x)e_{12} \in \mathcal{C}^1(\Omega,\mathcal{A}_2) \cap \mathcal{C}^\alpha(\Omega,\mathcal{A}_2)$$

of the problem

$$\begin{cases} Dw(x) &= 0 & \forall x \in B_1(0) \\ w_{12}(x) &= g_{12}(x),\ w_2(x) = g_2(x) & \forall x \in \Sigma_2 \\ w_1(x) &= g_1(x) & \forall x \in \Sigma_1 \\ w_0(\Sigma_0) &= g_0. & \end{cases} \tag{5.28}$$

Corollary 2. *There exists a unique solution $w \in \mathcal{C}^1(\Omega,\mathcal{A}_2)\cap\mathcal{C}^\alpha(\Omega,\mathcal{A}_2)$ of the problem (5.28) and we have an estimate*

$$\|w\|_{\mathcal{C}^\alpha(B_1(0),\mathcal{A}_2)} \leq H\left(|g_0| + \|g_1\|_{\mathcal{C}^{1,\alpha}(\Sigma_1,\mathbb{R})} + \|g_2\|_{\mathcal{C}^{1,\alpha}(\Sigma_2,\mathbb{R})} + \|g_{12}\|_{\mathcal{C}^{1,\alpha}(\Sigma_2,\mathbb{R})}\right)$$

for some constant $H > 0$.

2.5.3 Dirichlet boundary value problem for two-sided monogenic functions and Riesz system

In this part we consider the two-sided monogenic functions taking values in $\mathcal{A}_2,\mathcal{A}_3$.

Definition 7. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain, $w \in \mathcal{C}^1(\Omega,\mathcal{A}_n)$ is called two-sided monogenic if*

$$Dw(x) = 0,\ wD(x) = 0\ \forall x \in \Omega.$$

Two-sided monogenic functions with values in $\mathcal{A}_2$

In the case $n = 2$, $\Omega \subset \mathbb{R}^3$ is a connected domain, consider

$$w(x) = w_0(x) + w_1(x)e_1 + w_2(x)e_2 + w_{12}(x)e_{12},\ x = (x_0,x_1,x_2) \in \Omega.$$

w is two-sided monogenic if

$$\begin{cases} \dfrac{\partial w_0}{\partial x_0} - \dfrac{\partial w_1}{\partial x_1} - \dfrac{\partial w_2}{\partial x_2} & = 0 \\ \dfrac{\partial w_0}{\partial x_1} + \dfrac{\partial w_1}{\partial x_0} & = 0 \\ \dfrac{\partial w_0}{\partial x_2} + \dfrac{\partial w_2}{\partial x_0} & = 0 \\ \dfrac{\partial w_1}{\partial x_2} - \dfrac{\partial w_2}{\partial x_1} & = 0 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial w_{12}}{\partial x_0} & = 0 \\ \dfrac{\partial w_{12}}{\partial x_1} & = 0 \\ \dfrac{\partial w_{12}}{\partial x_2} & = 0. \end{cases}$$

We see that w_{12} is a constant in Ω. Define $\overrightarrow{F} := (w_0,-w_1,-w_2)$, then $\overrightarrow{F}$ becomes a potential vector field in Ω

$$\begin{cases} div\overrightarrow{F} & = & 0 \\ \overrightarrow{rot}\,\overrightarrow{F} & = & \overrightarrow{0}. \end{cases} \tag{5.29}$$

The Dirichlet boundary value problem for potential vector field $\overrightarrow{F}$ in Ω reads:
Let $g_2 \in \mathcal{C}^{1,\alpha}(\Sigma_2,\mathbb{R}),\ g_1 \in \mathcal{C}^{1,\alpha}(\Sigma_1),\ g_0 = c_0$ (real constants). To find potential vector fields $\overrightarrow{F} \in \mathcal{C}^1(\Omega,\mathbb{R}^3) \cap \mathcal{C}^\alpha(\Omega,\mathbb{R}^3)$ such that

$$\begin{cases} w_2(x) &= g_2(x) \quad \forall x \in \Sigma_2 \\ w_1(x) &= g_1(x) \quad \forall x \in \Sigma_1 \\ w_0(\Sigma_0) &= g_0. \end{cases} \tag{5.30}$$

(The notations $\Sigma_0, \Sigma_1, \Sigma_2$ and the assumptions of Ω as in Theorem 5.4.)

Corollary 3. *The problem (5.29)-(5.30) has a unique solution $\overrightarrow{F} \in \mathcal{C}^\alpha(\Omega,\mathbb{R}^3) \cap \mathcal{C}^1(\Omega,\mathbb{R}^3)$ and we have an estimate*

$$\|\overrightarrow{F}\|_{\mathcal{C}^\alpha(\Omega,\mathbb{R}^3)} \le H\left(|g_0| + \|g_1\|_{\mathcal{C}^{1,\alpha}(\Sigma_1,\mathbb{R})} + \|g_2\|_{\mathcal{C}^{1,\alpha}(\Sigma_2,\mathbb{R})}\right)$$

for some constant $H > 0$.

Two-sided monogenic functions with values in $\mathcal{A}_3$

In the case $n = 3$, $\Omega \subset \mathbb{R}^4$ is a domain, consider

$$w(x) = w_0(x) + w_1(x)e_1 + w_2(x)e_2 + w_3(x)e_3 + w_{12}(x)e_{12} + w_{13}e_{13} + w_{23}e_{23} + w_{123}e_{123},$$

with $x = (x_0, x_1, x_2, x_3) \in \Omega$.
w is two-sided monogenic if and only if

$$\begin{cases} D(w^*) &= 0 \\ D(w^{**}) &= 0, \end{cases}$$

where

$$\begin{cases} w^* &= w_0(x) &+ w_1(x)e_1 &+ w_2(x)e_2 &+ w_3(x)e_3 \\ w^{**} &= w_{123}(x) &- w_{23}(x)e_1 &+ w_{13}(x)e_2 &- w_{12}(x)e_3. \end{cases}$$

The Dirichlet boundary value problem for two-sided monogenic functions taking values in $\mathcal{A}_3$ reads: Let

$$\begin{cases} g_3,\ g_{12} \in \mathcal{C}^{1,\alpha}(\Sigma_3,\mathbb{R}) \\ g_2,\ g_{13} \in \mathcal{C}^{1,\alpha}(\Sigma_2,\mathbb{R}) \\ g_1,\ g_{23} \in \mathcal{C}^{1,\alpha}(\Sigma_1,\mathbb{R}) \\ g_0 = c_0,\ g_{123} = c_{123}. \end{cases}$$

(The notations $\Sigma_k, k = 0,1,2,3$, and the assumptions of Ω as in Theorem 5.4.)
To find functions $w \in \mathcal{C}^1(\Omega,\mathcal{A}_3) \cap \mathcal{C}^\alpha(\Omega,\mathcal{A}_3)$ such that

$$\begin{cases} Dw(x) &= 0, & wD(x) &= 0 & \forall x \in \Omega \\ w_3(x) &= g_3(x), & w_{12}(x) &= g_{12}(x) & \forall x \in \Sigma_3 \\ w_2(x) &= g_2(x), & w_{13}(x) &= g_{13}(x) & \forall x \in \Sigma_2 \\ w_1(x) &= g_1(x), & w_{23}(x) &= g_{23}(x) & \forall x \in \Sigma_1 \\ w_0(\Sigma_0) &= g_0, & w_{123}(\Sigma_0) &= g_{123}. \end{cases} \tag{5.31}$$

Corollary 4. *The problem (5.31) has a unique solution* $w \in \mathcal{C}^1(\Omega, \mathcal{A}_3) \cap \mathcal{C}^\alpha(\Omega, \mathcal{A}_3)$ *and we have an estimate*

$$\|w\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_3)} \leq H\Big(|g_0| + |g_{123}| + \|g_3\|_{\mathcal{C}^{1,\alpha}(\Sigma_3,\mathbb{R})} + \|g_{12}\|_{\mathcal{C}^{1,\alpha}(\Sigma_3,\mathbb{R})} + \|g_2\|_{\mathcal{C}^{1,\alpha}(\Sigma_2,\mathbb{R})}$$
$$+ \|g_{13}\|_{\mathcal{C}^{1,\alpha}(\Sigma_2,\mathbb{R})} + \|g_1\|_{\mathcal{C}^{1,\alpha}(\Sigma_1,\mathbb{R})} + \|g_{23}\|_{\mathcal{C}^{1,\alpha}(\Sigma_1,\mathbb{R})}\Big)$$

for some constant $H > 0$.

Dirichlet boundary value problem for Riesz system in $\mathbb{R}^{n+1}$

Let $\Omega \subset \mathbb{R}^{n+1}$ with the assumptions as in Section 2.5.1. Consider the Riesz system [26]

$$\begin{cases} \sum_{i=0}^{n} \dfrac{\partial F_i}{\partial x_i} = 0 \\ \dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i} \;\; \forall i,j = 0,\cdots,n, \end{cases} \tag{5.32}$$

where $\overrightarrow{F} = (F_0(x), F_1(x), \cdots, F_n(x)) \in \mathcal{C}^1(\Omega, \mathbb{R}^{n+1})$.
Define

$$F(x) := F_0(x) - F_1(x)e_1 - F_2(x)e_2 - \cdots - F_n(x)e_n \in \mathcal{C}^1(\Omega, \mathcal{A}_n).$$

We see that $\overrightarrow{F}$ is a solution of the system (5.32) if and only if F is a two-sided monogenic function taking values in $\mathcal{A}_n$.
The Dirichlet boundary value problem for Riesz system reads:
Let functions $g_i \in \mathcal{C}^{1,\alpha}(\Sigma_i, \mathbb{R})$, $i = 0, \cdots, n$, $g_0 = c$. To find $\overrightarrow{F} \in \mathcal{C}^1(\Omega, \mathbb{R}^{n+1}) \cap \mathcal{C}^\alpha(\Omega, \mathbb{R}^{n+1})$ such that

$$\begin{cases} \overrightarrow{F} \text{ satisfies the system (5.32)} \\ F_i(x) = g_i(x) \;\forall i = 0, \cdots, n \text{ and } x \in \Sigma_i. \end{cases} \tag{5.33}$$

Corollary 5. *The problem (5.33) has a unique solution* $\overrightarrow{F} \in \mathcal{C}^1(\Omega, \mathbb{R}^{n+1}) \cap \mathcal{C}^\alpha(\Omega, \mathbb{R}^{n+1})$ *and we have an estimate*

$$\|\overrightarrow{F}\|_{\mathcal{C}^\alpha(\Omega,\mathbb{R}^{n+1})} \leq H \sum_{i=0}^{n} \|g_i\|_{\mathcal{C}^{1,\alpha}(\Sigma_i,\mathbb{R})}$$

for some constant $H > 0$.

2.6 Dirichlet boundary value problem for generalized monogenic functions

Let functions $g_A \in \mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})$ with $A \in \Lambda_k,\ (k \geq 1)$, $g_{\emptyset}(x) = g_0(x) = c$. We consider the Dirichlet boundary value problem

$$\begin{cases} Dw(x) &= F(x,w) \quad \forall x \in \Omega_{n+1} = \Omega \\ w_A(x) &= g_A(x) \quad \forall A \in \Lambda_k \text{ and } x \in \Sigma_k, 0 \leq k \leq n \\ w &= \sum\limits_A w_A(x)e_A \in \mathcal{C}^1(\Omega,\mathcal{A}_n) \cap \mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n). \end{cases} \tag{6.34}$$

where $F(x,w)$ is an $\mathcal{A}_n$-valued function of variables $x \in \overline{\Omega}$ and $w \in \mathcal{A}_n$.

2.6.1 Reduction to a fixed- point problem

In this section, we use the method of reduction of boundary value problems to fixed-point problems using fundamental solutions [35].
Denote

$$\mathcal{B}_R(0) := \{w \in \mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n) \big| \ \|w\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)} \leq R\}.$$

We assume that $F(x,w(x)) \in \mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)$ for all $w \in \mathcal{B}_R(0)$ and

$$\|F(x,w(x))\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)} \leq K, \quad \sum_{k=0}^{n} \sum_{A\in\Lambda_k} \|g_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} \leq C.$$

Let w^* be the unique solution of the Dirichlet boundary value problem for monogenic functions

$$\begin{cases} Dw^*(x) &= 0 \quad \forall x \in \Omega \\ w^*{}_A(x) &= g_A(x) \quad \forall A \in \Lambda_k \text{ and } x \in \Sigma_k, 0 \leq k \leq n \\ w^* &= \sum\limits_A w^*{}_A(x)e_A \in \mathcal{C}^1(\Omega,\mathcal{A}_n) \cap \mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n). \end{cases} \tag{6.35}$$

By Theorem 5.4 we have an estimate

$$\|w^*\|_{C^{\alpha}(\Omega,\mathcal{A}_n)} \leq CH, \tag{6.36}$$

where H is a constant not depending on g_A.
Let $w \in \mathcal{B}_R(0)$, define

$$g'(x) = \sum_A g'_A(x)e_A := -\tilde{T}_{\Omega}\left[F(.,w(.))\right](x). \tag{6.37}$$

Since $F[(.,w(.))] \in \mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)$, by Lemma 4.15 we have $\tilde{T}_{\Omega}[F(.,w(.))] \in \mathcal{C}^{1,\alpha}(\Omega,\mathcal{A}_n)$ and there exists a constant $M>0$ such that

$$\left\|\tilde{T}_{\Omega}[F(.,w(.))]\right\|_{\mathcal{C}^{i,\alpha}(\Omega,\mathcal{A}_n)} \leq M\|F(.,w(.))\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)} \leq MK \ (i=0,1). \tag{6.38}$$

We have $g'_A \in \mathcal{C}^{1,\alpha}(\Sigma_k,\mathcal{A}_n)\ \forall A \in \Lambda_k$ and by Lemma 5.16

$$\sum_{k=0}^{n}\sum_{A\in\Lambda_k}\|g'_A\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathcal{A}_n)} \leq \lambda MK.$$

Denote $\tilde{w}$ be the unique solution of the Dirichlet boundary value problem for monogenic functions

$$\begin{cases} D\tilde{w}(x) &= 0 & \forall x \in \Omega \\ \tilde{w}_A(x) &= g'_A(x) & \forall A \in \Lambda_k \text{ and } x \in \Sigma_k, 0 \leq k \leq n \\ \tilde{w} &= \sum\limits_A \tilde{w}_A(x)e_A & \in \mathcal{C}^1(\Omega,\mathcal{A}_n)\cap\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n). \end{cases}$$

Applying the estimate of Theorem 5.4 we have

$$\|\tilde{w}\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)} \leq H\sum_{k=0}^{n}\sum_{A\in\Lambda_k}\left\|g'_A\right\|_{\mathcal{C}^{1,\alpha}(\Sigma_k,\mathbb{R})} \leq \lambda HMK. \tag{6.39}$$

Define an operator

$$\mathcal{U}:\mathcal{B}_R(0) \longrightarrow \mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)$$

$$\mathcal{U}(w) := w^* + \tilde{w} + \tilde{T}_{\Omega}[F(.,w(.))].$$

$$\|\mathcal{U}(w)\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)} \leq \|w^*\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)} + \|\tilde{w}\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)} + \|\tilde{T}_{\Omega}[F(.,w(.))]\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)}.$$

From (6.36), (6.38) and (6.39) we have

$$\|\mathcal{U}(w)\|_{\mathcal{C}^{\alpha}(\Omega,\mathcal{A}_n)} \leq HC + MK + \lambda HMK.$$

This inequality leads to the following lemma.

Lemma 6.20. *The operator $\mathcal{U}$ maps the ball $\mathcal{B}_R(0)$ into itself if*

$$HC + MK + \lambda HMK \leq R. \tag{6.40}$$

Lemma 6.21. *$w \in \mathcal{B}_R(0)$ is a solution of the boundary value problem (6.34) if and only if w is a fixed-point of the operator $\mathcal{U}$.*

Proof.
a) Assume that $w \in \mathcal{B}_R(0)$ is a solution of the problem (6.34). Define

$$\Phi(x) := w - \widetilde{T}_{\Omega}F(.,w(.))(x)\ \forall x \in \Omega.$$

It follows that $\Phi \in \mathcal{C}^1(\Omega,\mathcal{A}_n)\cap\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)$ and

$$D\Phi(x) = Dw(x) - D\left(\widetilde{T}_\Omega F(.,w(.))\right)(x) = Dw(x) - F(x,w(x)) = 0.$$

Define $\tilde{w}$ as in (6.36)

$$w = (\Phi - \tilde{w}) + \tilde{w} + \widetilde{T}_\Omega F(.,w(.)).$$

Because w^* and $\Phi - \tilde{w}$ are monogenic functions with the same boundary data g_A and by Theorem 5.4 it implies that

$$\Phi - \tilde{w} = w^*.$$

We have

$$w = w^* + \tilde{w} + \widetilde{T}_\Omega F(.,w(.)) = \mathcal{U}(w).$$

Hence w becomes a fixed-point of the operator $\mathcal{U}$.
b) Assume that $w \in \mathcal{B}_R(0)$ is a fixed-point of the operator $\mathcal{U}$. It follows that

$$w = w^* + \tilde{w} + \widetilde{T}_\Omega F(.,w(.)) = \mathcal{U}(w).$$

Applying the Cauchy-Riemann operator D we have

$$Dw = Dw^* + D\tilde{w} + D\widetilde{T}_\Omega F(.,w(.)) = F(.,w(.)).$$

Compute the boundary data of w

$$w_A|_{\Sigma_k} = w^*|_{\Sigma_k} + \left[\tilde{w} + \widetilde{T}_\Omega F(.,w(.))\right]\Big|_{\Sigma_k} = w^*|_{\Sigma_k} = g_A \text{ with } A \in \Lambda_k.$$

Hence w turns out to be a solution of the problem (6.34). □

2.6.2 Application of the contraction mapping principle

In this part, we assume that F is a Lipschitz operator with respect to w, that means there exists a constant $L > 0$ such that $\forall w, w' \in \mathcal{B}_R(0)$

$$\|F(.,w(.)) - F(.,w'(.))\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)} \leq L\|w - w'\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)}.$$

We will find conditions under which the operator $\mathcal{U}$ is contractive.
By Lemma 4.15 we have

$$\|\widetilde{T}_\Omega F(.,w(.)) - \widetilde{T}_\Omega F(.,w'(.))\|_{\mathcal{C}^{i,\alpha}(\Omega,\mathcal{A}_n)} \leq M\|F(.,w(.)) - F(.,w'(.))\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)}, i = 0,1.$$

It implies that

$$\|\widetilde{T}_\Omega F(.,w(.)) - \widetilde{T}_\Omega F(.,w'(.))\|_{\mathcal{C}^{i,\alpha}(\Omega,\mathcal{A}_n)} \leq ML\|w - w'\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)}. \tag{6.41}$$

We denote $\tilde{w}'$ be the image of w' in the definition of the operator $\mathcal{U}$. The subtraction $\tilde{w} - \tilde{w}'$ is a monogenic function and it is the unique solution of the Dirichlet boundary value problem with boundary data

$$(\tilde{w} - \tilde{w}')_A|_{\Sigma_k} = -\left[\widetilde{T}_\Omega F(.,w(.)) - \widetilde{T}_\Omega F(.,w'(.))\right]\Big|_{\Sigma_k} \ \forall A \in \Lambda_k.$$

Theorem 5.4 and Lemma 5.16 imply that

$$\|\tilde{w} - \tilde{w}'\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)} \leq \lambda H \|\widetilde{T}_\Omega F(.,w(.)) - \widetilde{T}_\Omega F(.,w'(.))\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathcal{A}_n)}.$$

Combining this with (6.41) we have

$$\|\tilde{w} - \tilde{w}'\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)} \leq \lambda HML \|w - w'\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)}. \tag{6.42}$$

We have

$$\mathcal{U}(w) - \mathcal{U}(w') = (\tilde{w} - \tilde{w}') + \left[\widetilde{T}_\Omega F(.,w(.)) - \widetilde{T}_\Omega F(.,w'(.))\right].$$

From the estimates (6.41) and (6.42) we get

$$\|\mathcal{U}(w) - \mathcal{U}(w')\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)} \leq ML(1+\lambda H)\|w - w'\|_{\mathcal{C}^\alpha(\Omega,\mathcal{A}_n)}.$$

Hence we obtain the following lemma.

Lemma 6.22. *The operator $\mathcal{U}$ is contractive if*

$$ML(1+\lambda H) < 1. \tag{6.43}$$

From lemmas 6.20, 6.21, 6.22 and the contraction mapping principle, we have the following theorem.

Theorem 6.5. *Assume that F is a Lipschitz operator with respect to w and the conditions (6.40), (6.43) are satisfied then the Dirichlet boundary value problem (6.34) has a unique solution in $\mathcal{B}_R(0)$.*

2.6.3 Application of the Schauder's fixed-point theorem

Theorem 6.6 (Second version of the Schauder's fixed-point theorem)**.**
Let M be a closed and convex set of a Banach space, let f be a continuous mapping of M into itself, and suppose that $f(M)$ is relatively compact. Then f has at least one fixed point in M.

Lemma 6.23. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with $\mathcal{C}^2$ boundary. Then the embedding*

$$i : \mathcal{C}^{1,\alpha}(\Omega,\mathbb{R}) \hookrightarrow \mathcal{C}^\alpha(\Omega,\mathbb{R})$$

is a compact operator for all $0 < \alpha < 1$.

Proof. We introduce the Sobolev space $W^{1,p}(\Omega), (p \geq 1)$ be the space of real-valued functions together with their first-order weak derivatives belonging to $L^p(\Omega)$. The norm of $f \in W^{1,p}(\Omega)$ is given by

$$\|f\|_{W^{1,p}(\Omega)} := \left[\int_\Omega \left(\|f\|^p + \sum_{i=0}^n \left(\frac{\partial f}{\partial x_i} \right)^p \right) dx \right]^{1/p}.$$

We have a continuous embedding

$$i_1 : \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R}) \hookrightarrow W^{1,p}(\Omega)$$

with

$$\|f\|_{W^{1,p}(\Omega)} \leq (n+2)^{1/p} (mes\Omega)^{1/p} \|f\|_{\mathcal{C}^{1,\alpha}(\Omega,\mathbb{R})}, \ \forall f \in \mathcal{C}^{1,\alpha}(\Omega, \mathbb{R}).$$

If we choose $p > \dfrac{n+1}{1-\alpha}$ then by Sobolev embedding (see [1]), we have a compact embedding

$$i_2 : W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{\alpha}(\Omega, \mathbb{R}).$$

Hence $i = i_2 \circ i_1$ is a compact embedding. □

By Lemma 6.20 the operator $\mathcal{U}$ maps the ball $\mathcal{B}_R(0)$ into itself if the inequality (6.40) is satisfied. To apply Theorem 6.6 we need to prove that $\mathcal{U}$ is a compact operator.

Lemma 6.24. *With the hypothesis of Section 2.6.1, we assume further that for some $\beta \in (\alpha, 1)$ the operator $F(x, w(x)) \in \mathcal{C}^\beta(\Omega, \mathcal{A}_n)$ for all $w \in \mathcal{B}_R(0)$ and there exists a constant K' such that*

$$\|F(x, w(x))\|_{\mathcal{C}^\beta(\Omega, \mathcal{A}_n)} \leq K'. \tag{6.44}$$

Let a sequence $\{w_n\}_{n\geq 1} \subset \mathcal{B}_R(0)$. Then there exists a subsequence $\{w_{n_k}\}_{k\geq 1}$ such that the sequence $\{\mathcal{U}(w_{n_k})\}_{k\geq 1}$ converges in $\mathcal{C}^\alpha(\Omega, \mathcal{A}_n)$.

Proof. Consider the sequence $\{\widetilde{T}_\Omega F(., w_n(.))\}_{n\geq 1} \subset \mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_n)$ (by Lemma 4.15). We have

$$\left\| \widetilde{T}_\Omega F(., w_n(.)) \right\|_{\mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_n)} \leq M \|F(., w_n(.))\|_{\mathcal{C}^\alpha(\Omega, \mathcal{A}_n)} \leq MK.$$

Hence $\{\widetilde{T}_\Omega F(., w_n(.))\}_{n\geq 1}$ is a bounded sequence in $\mathcal{C}^{1,\alpha}(\Omega, \mathcal{A}_n)$. Applying Lemma 6.23 we have a subsequence $\{\widetilde{T}_\Omega F(., w_{n_k}(.))\}_{k\geq 1}$ which converges in $\mathcal{C}^\alpha(\Omega, \mathcal{A}_n)$.
Consider the function $\tilde{w}_{n_k}$ in the definition of the operator $\mathcal{U}$ corresponding to w_{n_k}. By the estimate in Theorem 5.4 we have

$$\|\tilde{w}_{n_k}\|_{\mathcal{C}^\beta(\Omega, \mathcal{A}_n)} \leq \lambda H \left\| \widetilde{T}_\Omega F(., w_{n_k}(.)) \right\|_{\mathcal{C}^{1,\beta}(\Omega, \mathcal{A}_n)} \leq \lambda HMK'.$$

Because the embedding $i : \mathcal{C}^{\beta}(\Omega, \mathcal{A}_n) \hookrightarrow \mathcal{C}^{\alpha}(\Omega, \mathcal{A}_n)$ is compact, there exists a subsequence $\{\tilde{w}_{n_{k_p}}\}_{p\geq 1}$ of the sequence $\{\tilde{w}_{n_k}\}_{k\geq 1}$ which is convergent in $\mathcal{C}^{\alpha}(\Omega, \mathcal{A}_n)$. In the end the sequence

$$\mathcal{U}(w_{n_{k_p}}) = w^* + \tilde{w}_{n_{k_p}} + \widetilde{T}_{\Omega} F(., w_{n_{k_p}}(.))$$

converges in $\mathcal{C}^{\alpha}(\Omega, \mathcal{A}_n)$ when $p \to +\infty$. Lemma 6.24 is proved. □

By Lemma 6.24, the operator $\mathcal{U}$ is compact and then applying Lemma 6.20, Lemma 6.21 and Theorem 6.6 we have the following theorem.

Theorem 6.7. *If the condition (6.40) and the additional condition (6.44) are satisfied, then the Dirichlet boundary value problem for generalized monogenic functions (6.34) has at least one solution in* $\mathcal{B}_R(0)$.

3 INITIAL VALUE PROBLEMS

This chapter is aimed to solve initial value problems of type

$$\begin{cases} \dfrac{\partial u}{\partial t} & = \mathcal{F}\left(t, x, u, \dfrac{\partial u}{\partial x_j}\right) \\ u(0,x) & = \varphi(x) \end{cases} \tag{0.1}$$

by using the concept of associated operators (see [34]). The initial function φ is defined in a domain $\Omega \subset \mathbb{R}^{n+1}$ and φ is a solution of the equation $M\varphi = 0$. Assume that the operator $\mathcal{F}$ is associated to the operator M in Ω, that means $v = \mathcal{F}\varphi$ satisfies the equation $Mv = 0$ if φ is a solution of the equation $M\varphi = 0$. The initial value problem (0.1) is solvable if the solutions of the equation $M\varphi = 0$ satisfy an interior estimate.
Solutions of the initial value problem (0.1) are fixed points of the operator

$$\mathcal{U}(t,x) = \varphi(x) + \int_0^t \mathcal{F}\left(\tau, x, u(\tau,x), \frac{\partial u}{\partial x_j}(\tau,x)\right) d\tau \tag{0.2}$$

and vice versa. In order to apply a fixed-point theorem (for example contraction mapping principle), the operator (0.2) is estimated in a suitable function space whose elements depend on t and x. The problem is that the integrand in (0.2) depends on $\dfrac{\partial u}{\partial x_j}$, we have to restrict the operator to a space of functions for which $\dfrac{\partial u}{\partial x_j}$ can be estimated by u. This type of estimate is called interior estimate (section 3.2). In general solutions of elliptic systems possess interior estimates (see [8]).

3.1 Differential operators associated to a class of first-order elliptic operators

This section gives a method of constructing operators associated to a class of first-order elliptic operators by using fundamental solutions. This is a new application of fundamental solutions. We will give necessary and sufficient conditions for the associated operators. In the special cases we obtain the criterion for associated operators of the Cauchy-Riemann operators in Quaternion analysis. Some classes of these associated operators are also constructed explicitly. In complex analysis, we give a construction of associated pair (L, D), where D is a generalized Cauchy-Riemann operator. The Lewy example [19] shows that there is a smooth linear partial differential equation without solution, it can happen that there are some differential operators not associated to any elliptic operator.

We consider a class of first-order elliptic operator with variable coefficients

$$Mv(x) = \sum_{i=0}^{n} E_i(x)\frac{\partial v}{\partial x_i} + Q(x)v(x) \tag{1.3}$$

$x \in \Omega \subset \mathbb{R}^{n+1}$, $x = (x_0, x_1, x_2, \dots, x_n)$, $v(x) = [v_1(x), v_2(x), \cdots, v_m(x)]^T \in \mathcal{C}^1(\Omega)$ is a real vector function. $E_0 = I_m$ is the indentity matrix, $Q(x)$, $E_i(x), (i = 1, 2\dots n)$ are $m \times m$ real matrix functions with properties (1.2) and (1.3).

Definition 8.
A first-order differential opeartor L is said to be associated to the operator M if and only if for any solution v of the equation $Mv = 0$ in any subdomain $\Omega' \subset \Omega$, the function $w = Lv$ is a solution of the equation $Mw = 0$ in Ω'.

We consider first-order differential operators L in the form

$$Lv = \sum_{i=0}^{n} B_i(x)\frac{\partial v}{\partial x_i} + C(x)v$$

with $B_i(x), C(x)$ are $m \times m$ matrix functions in $\mathcal{C}^1(\Omega)$. We will find necessary and sufficient conditions such that L is associated to M.

We need the following theorem to make sure that each $\mathcal{C}^1$- solution of the equation $Mv = 0$ becomes its $\mathcal{C}^2$- solution.

Theorem 1.8 ([9]). *If $E_i(x) \in C^h(\Omega)$, $h > \dfrac{n+5}{2}$, $Q(x) \in C^k(\Omega)$, $k > \dfrac{n+3}{2}$, then every $\mathcal{C}^1$- solution of the system $Mv = 0$ belongs to $\mathcal{C}^2$.*

Recall

$$H(x,y) = \sum_{r=0}^{n} \frac{\partial G(x,y)}{\partial x_r}\overline{E_r(y)} = \frac{\sum\limits_{r,s=0}^{n} A_{rs}(y)(x_s - y_s)\overline{E_r(y)}}{\omega_{n+1}\sqrt{DetA(y)}\rho(x,y)^{n+1}}, \tag{1.4}$$

where $\overline{E_0} := E_0$, $\overline{E_r(y)} := -E_r(y)$ for $r \geq 1$, ω_{n+1} is the surface measure of the unit sphere in $\mathbb{R}^{n+1}$, function G is defined as in (3.6), Chapter I.
Assume that $E_i(x) \in C^h(\Omega)$ $\left(h > \dfrac{n+5}{2}\right)$, $Q(x) \in C^k(\Omega)$ $\left(k > \dfrac{n+3}{2}\right)$.

Let $y^0 \in \Omega$ be a fixed point, $R > 0$ such that the closed ball $\overline{B} = \overline{B_R(y^0)} \subset \Omega$. Then the equation $Mv = 0$ has a fundamental solution $\Gamma(x,y)$ in B and we have estimates

$$\Gamma(x,y^0) - H(x,y^0) \leq C|x-y^0|^{1-n} \tag{1.5}$$

$$\frac{\partial}{\partial x_i}\left(\Gamma(x,y^0) - H(x,y^0)\right) \leq C|x-y^0|^{-n}, \qquad i = 0,1,\cdots,n \tag{1.6}$$

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(\Gamma(x,y^0) - H(x,y^0)\right) \leq C|x-y^0|^{-n-1}, \qquad i,j = 0,1,\cdots,n \tag{1.7}$$

for some constant $C > 0$ and for all $x \in B \setminus \{y^0\}$. The fundamental solution $\Gamma(x,y^0)$ and the Levi function $H(x,y^0)$ (and their derivatives) have the same behavior at y^0, these results can be found in [3], [15] or [25].

3.1.1 Necessary and sufficient conditions

Lemma 1.25. *Let the function*

$$G(x) = \begin{cases} |x|^{1-n} & \text{if } n > 1 \\ \ln|x| & \text{if } n = 1 \end{cases} \quad \forall x \in \mathbb{R}^{n+1} \setminus \{0\},$$

then the derivatives $\left\{ \frac{\partial^3 G(x)}{\partial x_0 \partial x_i \partial x_j},\ (1 \leq i \leq j \leq n); \frac{\partial^3 G(x)}{\partial x_i \partial x_j \partial x_k},\ (1 \leq i \leq k \leq j \leq n) \right\}$ *are linear independent.*

Proof. Assume that there exist real numbers α_{ij} and β_{ijk} such that

$$\sum_{1 \leq i \leq j \leq n} \alpha_{ij} \frac{\partial^3 G(x)}{\partial x_0 \partial x_i \partial x_j} + \sum_{1 \leq i \leq j \leq k \leq n} \beta_{ijk} \frac{\partial^3 G(x)}{\partial x_i \partial x_j \partial x_k} = 0$$

for all $0 \neq |x| < \varepsilon$. This implies that

$$P(x) = \sum_{1 \leq i \leq j \leq n} \alpha_{ij} x_0 x_i x_j + \sum_{1 \leq i \leq j \leq k \leq n} \beta_{ijk} x_i x_j x_k + \text{lower order polynomial} = 0,$$

for all $x \in \mathbb{R}^{n+1}$ with $x_0^2 + x_1^2 + \cdots + x_n^2 = 1$. The polynomial $P(x)$ must be divided by the polynomial $x_0^2 + x_1^2 + \cdots + x_n^2 - 1$. But the polynomial $P(x)$ has a degree of 1 with respect to the variable x_0, hence $P(x) \equiv 0$, this implies that $\alpha_{ij} = 0,\ \beta_{ijk} = 0$. ☐

Now we give necessary and sufficient conditions such that the operator L is associated to the operator M.

Theorem 1.9. *Assume that* $E_i(x) \in \mathcal{C}^h(\Omega),\ h > \dfrac{n+5}{2},\ Q(x) \in \mathcal{C}^k(\Omega),\ k > \dfrac{n+3}{2}$*. Let an operator L in the form*

$$Lv = \sum_{i=0}^{n} B_i(x)\frac{\partial v}{\partial x_i} + C(x)v \tag{1.8}$$

with $B_i(x)$, $C(x)$ *are* $m \times m$ *matrices in* $\mathcal{C}^1(\Omega)$*. Denote*

$$\begin{aligned}
P_0 : &= C - B_0 Q,\\
P_j : &= B_j - B_0 E_j,\ j = 1,2,\cdots,n,\\
M_{ij} : &= E_i P_j - P_j E_i,\ \ i,j = 1,2,\cdots,n,\\
M_j : &= \sum_{i=0}^{n}\left(E_i\frac{\partial P_j}{\partial x_i} - P_i\frac{\partial E_j}{\partial x_i}\right) + E_j P_0 - P_0 E_j + QP_j - P_j Q,\ j = 1,\cdots,n\\
M_0 : &= \sum_{i=0}^{n} E_i\frac{\partial P_0}{\partial x_i} - \sum_{i=1}^{n} P_i\frac{\partial Q}{\partial x_i} + QP_0 - P_0 Q.
\end{aligned}$$

The operator L is associated to the operator M in Ω *if and only if*

$$\begin{cases} M_{ij}(x) + M_{ji}(x) & = 0 \quad \forall i,j = 1,2,..,n\\ \qquad M_j(x) & = 0 \quad \forall\ j = 0,1,..,n, \quad \forall x \in \Omega. \end{cases}$$

Proof.
Necessary conditions
Suppose that the operator L in the form (1.8) is associated to the operator M in Ω. Let y^0 be an arbitrary point in Ω. Choose $r > 0$ small enough such that the closed ball $\overline{B_r(y^0)} \subset \Omega$. Denote a fundamental solution of the equation $Mv = 0$ in $B_r(y^0)$ by $\Gamma(x,y)$. Each column of the matrix $\Gamma(x,y^0)$ is a solution of the equation $Mv = 0$ in the set $B_r(y^0)\setminus\{y^0\}$. By the definition of associated operators,

$$M_x\left(L_x\Gamma(x,y^0)\right) = 0\ \forall x \in B_r(y^0)\setminus\{y^0\}. \tag{1.9}$$

$$\begin{aligned}
L_x\Gamma(x,y^0) &= \sum_{j=0}^{n} B_j(x)\frac{\partial\Gamma(x,y^0)}{\partial x_j} + C(x)\Gamma(x,y^0)\\
&= B_0(x)\frac{\partial\Gamma(x,y^0)}{\partial x_0} + \sum_{j=1}^{n} B_j(x)\frac{\partial\Gamma(x,y^0)}{\partial x_j} + C(x)\Gamma(x,y^0)\\
&= -B_0(x)\left(\sum_{j=1}^{n} E_j(x)\frac{\partial\Gamma(x,y^0)}{\partial x_j} + Q(x)\Gamma(x,y^0)\right)\\
&\quad + \sum_{j=1}^{n} B_j(x)\frac{\partial\Gamma(x,y^0)}{\partial x_j} + C(x)\Gamma(x,y^0)
\end{aligned}$$

$$= \sum_{j=1}^{n} \Big(B_j(x) - B_0(x)E_j(x) \Big) \frac{\partial \Gamma(x,y^0)}{\partial x_j} + \Big(C(x) - B_0(x)Q(x) \Big) \Gamma(x,y^0)$$
$$= \sum_{j=1}^{n} P_j(x) \frac{\partial \Gamma(x,y^0)}{\partial x_j} + P_0(x)\Gamma(x,y^0).$$

$$\begin{aligned} M_x\left(L_x\Gamma(x,y^0)\right) &= \sum_{i=0}^{n} E_i(x) \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} P_j(x) \frac{\partial \Gamma(x,y^0)}{\partial x_j} + P_0(x)\Gamma(x,y^0) \right) \\ &\quad + Q(x) \left(\sum_{j=1}^{n} P_j(x) \frac{\partial \Gamma(x,y^0)}{\partial x_j} + P_0(x)\Gamma(x,y^0) \right) \\ &= \sum_{i,j=1}^{n} M_{ij}(x) \frac{\partial^2 \Gamma(x,y^0)}{\partial x_i \partial x_j} + \sum_{j=1}^{n} M_j(x) \frac{\partial \Gamma(x,y^0)}{\partial x_j} + M_0(x)\Gamma(x,y^0). \end{aligned}$$

Multiply (1.9) by $|x-y^0|^{n+2}$ and let $|x-y^0| \to 0$

$$\lim_{|x-y^0|\to 0} |x-y^0|^{n+2} M_x\left(L_x\Gamma(x,y^0)\right) = 0. \tag{1.10}$$

Because $\Gamma(x,y^0) = O(|x-y^0|^{-n})$ and $\dfrac{\partial \Gamma(x,y^0)}{\partial x_i} = O(|x-y^0|^{-n-1})$, the equality (1.10) becomes

$$\lim_{|x-y^0|\to 0} |x-y^0|^{n+2} \sum_{i,j=1}^{n} M_{ij}(x) \frac{\partial^2 \Gamma(x,y^0)}{\partial x_i \partial x_j} = 0. \tag{1.11}$$

From (1.7)

$$\frac{\partial^2}{\partial x_i \partial x_j} \left(\Gamma(x,y) - H(x,y)\right) = O(|x-y|^{-n-1}),$$

(1.11) is equivalent to

$$\begin{aligned} &\lim_{|x-y^0|\to 0} |x-y^0|^{n+2} \sum_{i,j=1}^{n} M_{ij}(x) \frac{\partial^2 H(x,y^0)}{\partial x_i \partial x_j} = 0 \\ \Leftrightarrow &\lim_{|x-y^0|\to 0} |x-y^0|^{n+2} \sum_{i,j=1}^{n} M_{ij}(y^0) \frac{\partial^2 H(x,y^0)}{\partial x_i \partial x_j} \\ &\quad + \lim_{|x-y^0|\to 0} |x-y^0|^{n+2} \sum_{i,j=1}^{n} \left(M_{ij}(x) - M_{ij}(y^0)\right) \frac{\partial^2 H(x,y^0)}{\partial x_i \partial x_j} = 0. \end{aligned} \tag{1.12}$$

Because M_{ij} are Lipschitz functions in $\overline{B_r(y^0)}$, the second limit is zero, so (1.12) is equivalent to

$$\lim_{|x-y^0|\to 0} |x-y^0|^{n+2} \sum_{i,j=1}^{n} M_{ij}(y^0) \frac{\partial^2 H(x,y^0)}{\partial x_i \partial x_j} = 0. \tag{1.13}$$

Since the function G has the property

$$G(y+t(x-y),y) = t^{1-n}G(x,y) \ \ \forall t \in (0,1],$$

we have

$$\begin{aligned} H(y+t(x-y),y) &= t^{-n}H(x,y) \\ \frac{\partial^2 H}{\partial x_i \partial x_j}(y+t(x-y),y) &= t^{-n-2}\frac{\partial^2 H}{\partial x_i \partial x_j}(x,y) \ \forall t \in (0,1]. \end{aligned}$$

Hence the equality (1.13) is equivalent to

$$\sum_{i,j=1}^{n} M_{ij}(y^0)\frac{\partial^2 H(x,y^0)}{\partial x_i \partial x_j} = 0 \ \ \forall x \in B_r(y^0)\setminus\{y^0\}. \tag{1.14}$$

By the same argument we obtain

$$\sum_{j=1}^{n} M_j(y^0)\frac{\partial H(x,y^0)}{\partial x_j} = 0 \ \ \forall x \in B_r(y^0)\setminus\{y^0\}, \tag{1.15}$$

$$M_0(y^0) = 0. \tag{1.16}$$

Substituting $H(x,y)$ from (1.4) into (1.14) we get

$$\begin{aligned} &\sum_{i,j=1,r=0}^{n} M_{ij}(y^0)\overline{E_r(y^0)}\frac{\partial^3 G(x,y^0)}{\partial x_i \partial x_j \partial x_r} = 0 \\ \Leftrightarrow &\sum_{i,j=1}^{n} M_{ij}(y^0)\frac{\partial^3 G(x,y^0)}{\partial x_0 \partial x_i \partial x_j} - \sum_{i,j,r=1}^{n} M_{ij}(y^0)E_r(y^0)\frac{\partial^3 G(x,y^0)}{\partial x_i \partial x_j \partial x_r} = 0. \end{aligned} \tag{1.17}$$

Changing variables $z = C(x-y^0)$, where C is the $(n+1)\times(n+1)$ symmetric matrix such that $A^{-1}(y^0) = C^2$. The function $G(x,y^0)$ reads

$$G(z) = \begin{cases} \dfrac{|z|^{1-n}}{(1-n)\omega_{n+1}\sqrt{DetA(y^0)}} & \text{if} \quad n>1 \\ \dfrac{\ln|z|}{2\pi\sqrt{DetA(y^0)}} & \text{if} \quad n=1 \end{cases}.$$

The equation (1.17) in new variables is

$$\begin{aligned} &\sum_{p,q=1}^{n}\left(\sum_{i,j=1}^{n} M_{ij}(y^0)C_{pi}C_{qj}\right)\frac{\partial^3 G(z)}{\partial z_0 \partial z_p \partial z_q} \\ &- \sum_{p,q,h=1}^{n}\left(\sum_{i,j,r=1}^{n} M_{ij}(y^0)E_r(y^0)C_{pi}C_{qj}C_{hr}\right)\frac{\partial^3 G(z)}{\partial z_p \partial z_q \partial z_h} = 0, \forall\ 0 \neq |z| < \varepsilon. \end{aligned} \tag{1.18}$$

By Lemma 1.25 the system

$$\left\{\frac{\partial^3 G(z)}{\partial z_0 \partial z_p \partial z_q},\ (1 \le p \le q \le n);\frac{\partial^3 G(z)}{\partial z_p \partial z_q \partial z_h},\ (1 \le p \le q \le h \le n)\right\}$$

are linear independent. From (1.18) we have

$$M_{ij}(y^0) + M_{ji}(y^0) = 0\ \ \forall i, j = 1, 2, \cdots, n. \tag{1.19}$$

By the same argument the conditions (1.15), (1.16) lead to

$$M_j(y^0) = 0\ \ \forall j = 0, 1, \cdots, n. \tag{1.20}$$

These conditions are satistied for arbitrary $y^0 \in \Omega$.
Sufficient conditions
Assume that the conditions (1.19), (1.20) are satisfied. Let v be a $\mathcal{C}^1$- solution of the equation $Mv = 0$. By Theorem 1.8 v is a $\mathcal{C}^2$ - solution. We have

$$M\big(Lv\big) = \sum_{i=1}^{n} M_{ii}\frac{\partial^2 v}{\partial x_i^2} + \sum_{1\le i<j\le n}\Big(M_{ij} + M_{ji}\Big)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{j=1}^{n} M_j \frac{\partial v}{\partial x_j} + M_0 v = 0.$$

Theorem 1.9 is proved. □

3.1.2 Operators associated to the Cauchy-Riemann operator in Quaternion analysis

In the case the coefficients of the operator M are constants and satisfied $a_{ij} = \delta_{ij},\ Q \equiv 0$, the operator M becomes the Cauchy- Riemann operator

$$Dv = \sum_{i=0}^{n} E_i \frac{\partial v}{\partial x_i}.$$

We have a corrolary of Theorem 1.9.

Corollary 6. *An operator L in the form (1.8) is associated to the Cauchy-Riemann operator D in Ω if and only if*

$$\begin{cases} E_iP_i - P_iE_i & = & 0 & \quad i = 1, 2, \cdots, n \\ E_iP_j - P_jE_i + E_jP_i - P_iE_j & = & 0 & \quad 1 \le i < j \le n \\ DP_j + E_jC - CE_j & = & 0 & \quad j = 1, 2, \cdots, n \\ DC & = & 0, & \end{cases}$$

where $P_j := B_j - B_0E_j$.

In the following we construct some classes of associated operators in Quaternion analysis. Denote

$$E_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ E_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Then the Cauchy-Riemann operator is given by

$$Du = E_0\frac{\partial u}{\partial x_0} + E_1\frac{\partial u}{\partial x_1} + E_2\frac{\partial u}{\partial x_2}, \tag{1.21}$$

where $u(x) = [u_1(x), u_2(x), u_3(x), u_4(x)]^T \in \mathcal{C}^1(\Omega)$, Ω is a domain in $\mathbb{R}^3$. Consider first-order differential operators in the special form

$$Lv = B_0(x)\frac{\partial v}{\partial x_0} + B_1(x)\frac{\partial v}{\partial x_1} + B_2(x)\frac{\partial v}{\partial x_2} + C(x)v, \tag{1.22}$$

with $B_i(x), C(x)$ are 4×4 matrices in $\mathcal{C}^2(\Omega)$.

Corollary 7. *An operator L of type (1.22) is associated to the Cauchy-Riemann operator D if and only if*

$$E_1B_1 - E_1B_0E_1 - B_1E_1 - B_0 = 0 \tag{1.23}$$
$$E_2B_2 - E_2B_0E_2 - B_2E_2 - B_0 = 0 \tag{1.24}$$
$$E_1B_2 - E_1B_0E_2 - B_2E_1 + E_2B_1 - E_2B_0E_1 - B_1E_2 = 0 \tag{1.25}$$
$$D(B_1 - B_0E_1) + E_1C - CE_1 = 0 \tag{1.26}$$
$$D(B_2 - B_0E_2) + E_2C - CE_2 = 0 \tag{1.27}$$
$$D(C) = 0. \tag{1.28}$$

Remark 12. *The necessary and sufficient conditions for associated operators of the Cauchy-Riemann in Quaternion analysis in other forms were given in [37], (in [23] for Cauchy-Fueter operator and in [6] for potential vector field), but the statement in Corollary 7 is simpler. The sufficient conditions were also given in [16], [27] (for the Dirac operator).*

In the following, we construct a class of solutions of the system (1.23-1.28).

Corollary 8. *Let L be an operator of type (1.22) with the matrices B_i, C in the form*

$$\begin{aligned} B_i(x) &= B_0^i(x)E_0 + B_1^i(x)E_1 + B_2^i(x)E_2 + B_{12}^i(x)E_1E_2,\ (i=0,1,2), \\ C(x) &= C_0(x)E_0 + C_1(x)E_1 + C_2(x)E_2 + C_{12}(x)E_1E_2, \end{aligned}$$

where $B^i_j(x), C_j(x)$ are real-valued functions. The operator L is associated to the Cauchy-Riemann operator D if and only if

$$C(x) = (2nx_0 - 2mx_1 - 2px_2 + p_0)E_0 + (mx_0 + nx_1 + p_1)E_1 \\ + (px_0 + nx_2 + p_2)E_2 + (-px_1 + mx_2 + p_3)E_1E_2,$$

$$B^0_0,\ B^0_1,\ B^0_2,\ B^0_{12} \text{ be arbitrary functions in } \mathcal{C}^2(\Omega), \\ B^1_0 = a - B^0_1,\ B^1_1 = b + B^0_0,\ B^1_2 = B^0_{12},\ B^1_{12} = -B^0_2, \\ B^2_0 = c - B^0_2,\ B^2_1 = B^0_{12},\ B^2_2 = b + B^0_0,\ B^2_{12} = -B^0_1,$$

where

$$a = mx_0^2 - mx_1^2 + mx_2^2 + 2nx_0x_1 - 2px_1x_2 + 2p_1x_0 + p_4x_1 + 2p_3x_2 + p_5, \\ b = -nx_0^2 + nx_1^2 + nx_2^2 + 2mx_0x_1 + 2px_0x_2 - p_4x_0 + 2p_1x_1 + 2p_2x_2 + p_6, \\ c = px_0^2 + px_1^2 - px_2^2 + 2nx_0x_2 - 2mx_1x_2 + 2p_2x_0 - 2p_3x_1 + p_4x_2 + p_7, \\ (m, n, p, p_i \text{ are real constants}).$$

Proof.
From (1.23), (1.24) we have

$$B_0 = E_1B_1 - E_1B_0E_1 - B_1E_1,\ B_0 = E_2B_2 - E_2B_0E_2 - B_2E_2.$$

Substituting these expressions into (1.25) we get

$$E_1B_2E_1 + E_1B_1E_2 + E_2B_1E_1 - E_2B_2E_2 = 0. \tag{1.29}$$

Equality (1.23) is equivalent to

$$B^1_2 - B^0_{12} = 0,\ \ B^1_{12} + B^0_2 = 0, \tag{1.30}$$

so we have $B_1 - B_0E_1 = (B^1_0 + B^0_1) + (B^1_1 - B^0_0)E_1$. Analogously equality (1.24) is equivalent to

$$B^2_1 - B^0_{12} = 0,\ \ B^2_{12} + B^0_1 = 0, \tag{1.31}$$

and hence we have $B_2 - B_0E_2 = (B^2_0 + B^0_2) + (B^2_2 - B^0_0)E_2$. From (1.29) we have the relation

$$B^1_1 - B^2_2 = 0. \tag{1.32}$$

Denoting $a = B^1_0 + B^0_1,\ b = B^1_1 - B^0_0,\ c = B^2_0 + B^0_2$, the system $(1.26 - 1.27)$ becomes

$$\begin{cases} Da + Db.E_1 = 2C_{12}E_2 - 2C_2E_1E_2 \\ Dc + Db.E_2 = -2C_{12}E_1 + 2C_1E_1E_2. \end{cases}$$

This system is equivalent to

$$\begin{cases} \partial_0 a - \partial_1 b &= 0 \\ \partial_1 a + \partial_0 b &= 0 \\ \partial_2 a &= 2C_{12} \\ \partial_2 b &= 2C_2 \\ \partial_0 c - \partial_2 b &= 0 \\ \partial_2 c + \partial_0 b &= 0 \\ \partial_1 c &= -2C_{12} \\ \partial_1 b &= 2C_1. \end{cases} \tag{1.33}$$

Denoting $d = -\partial_0 b$, the system (1.33) is equivalent to three following systems

$$\begin{cases} \partial_0 a &= 2C_1 \\ \partial_1 a &= d \\ \partial_2 a &= 2C_{12} \end{cases} \quad \begin{cases} \partial_0 b &= -d \\ \partial_1 b &= 2C_1 \\ \partial_2 b &= 2C_2 \end{cases} \quad \begin{cases} \partial_0 c &= 2C_2 \\ \partial_1 c &= -2C_{12} \\ \partial_2 c &= d. \end{cases} \tag{1.34}$$

Suppose that $\Omega \subset \mathbb{R}^3$ is a simply connected domain, then the compatibility conditions for (1.34) are

$$\begin{cases} \partial_2 C_1 &= \partial_0 C_{12} \\ \partial_0 d &= 2\partial_1 C_1 \\ \partial_2 d &= 2\partial_1 C_{12}, \end{cases} \quad \begin{cases} \partial_2 C_1 &= \partial_1 C_2 \\ \partial_1 d &= -2\partial_0 C_1 \\ \partial_2 d &= -2\partial_0 C_2, \end{cases} \quad \begin{cases} \partial_1 C_2 &= -\partial_0 C_{12} \\ \partial_1 d &= -2\partial_2 C_{12} \\ \partial_0 d &= 2\partial_2 C_2. \end{cases} \tag{1.35}$$

C is a solution of the equation $DC = 0$, that is,

$$\begin{cases} \partial_0 C_0 - \partial_1 C_1 - \partial_2 C_2 &= 0 \\ \partial_1 C_0 + \partial_0 C_1 + \partial_2 C_{12} &= 0 \\ \partial_2 C_0 + \partial_0 C_2 - \partial_1 C_{12} &= 0 \\ \partial_0 C_{12} + \partial_1 C_2 - \partial_2 C_1 &= 0. \end{cases} \tag{1.36}$$

Combining (1.35) and (1.36) we obtain the system

$$\begin{cases} \partial_2 C_1 &= 0 & \\ \partial_1 C_2 &= 0 & \\ \partial_0 C_{12} &= 0 & \\ \partial_0 d &= 2\partial_1 C_1 &= 2\partial_2 C_2 \\ \partial_1 d &= -2\partial_0 C_1 &= -2\partial_2 C_{12} \\ \partial_2 d &= -2\partial_0 C_2 &= 2\partial_1 C_{12} \\ \partial_0 C_0 - 2\partial_1 C_1 &= 0 & \\ \partial_1 C_0 + 2\partial_0 C_1 &= 0 & \\ \partial_2 C_0 + 2\partial_0 C_2 &= 0. & \end{cases} \tag{1.37}$$

Solving this system we get

$$\begin{aligned} C_1 &= mx_0 + nx_1 + p_1, \\ C_2 &= px_0 + nx_2 + p_2, \\ C_{12} &= -px_1 + mx_2 + p_3, \\ C_0 &= 2nx_0 - 2mx_1 - 2px_2 + p_0, \\ d &= 2nx_0 - 2mx_1 - 2px_2 + p_4, \\ &(m, n, p, p_0, p_1, p_2, p_3 \in \mathbb{R}). \end{aligned}$$

Now the solutions of the system (1.34) are

$$\begin{aligned} a &= mx_0^2 - mx_1^2 + mx_2^2 + 2nx_0x_1 - 2px_1x_2 + 2p_1x_0 + p_4x_1 + 2p_3x_2 + p_5, \\ b &= -nx_0^2 + nx_1^2 + nx_2^2 + 2mx_0x_1 + 2px_0x_2 - p_4x_0 + 2p_1x_1 + 2p_2x_2 + p_6, \\ c &= px_0^2 + px_1^2 - px_2^2 + 2nx_0x_2 - 2mx_1x_2 + 2p_2x_0 - 2p_3x_1 + p_4x_2 + p_7, \end{aligned}$$

with p_4, p_5, p_6, p_7 are arbitrary real constants. We can find the matrices B_i by choosing the coefficients of $B_0^0, B_1^0, B_2^0, B_{12}^0$ arbitrarily in $\mathcal{C}^2(\Omega)$, then the coefficients of matrices B_1, B_2 are given as follows

$$\begin{array}{llll} B_0^1 = a - B_1^0, & B_1^1 = b + B_0^0, & B_2^1 = B_{12}^0, & B_{12}^1 = -B_2^0 \\ B_0^2 = c - B_2^0, & B_1^2 = B_{12}^0, & B_2^2 = b + B_0^0, & B_{12}^2 = -B_1^0. \end{array}$$

□

Remark 13. *Corollary 8 gives a class of operators associated to the Cauchy-Riemann operator. Though they are not the most general associated operators but with this method we can obtain a larger class of associated operators than that in [23], [37], [6], [16] and [27].*

Remark 14. *The class of operators associated to the Cauchy-Riemann operator in Corollary 8 contains operator* $L = \dfrac{\partial}{\partial x_0}$ *satisfying* $Lu \neq 0$ *if* $Du = 0$ *in general. This is an answer to a similar question as the open question in [29].*

3.1.3 Operators associated to the generalized Cauchy-Riemann operators in complex analysis

Denote

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ E_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

A generalized Cauchy-Riemann operator is given by

$$Du = E_0 \frac{\partial u}{\partial x_0} + E_1 \frac{\partial u}{\partial x_1} + Qu, \tag{1.38}$$

where $Q(x) \in \mathcal{C}^2(\Omega)$ is a 2×2 matrix, $u = [u_1(x), u_2(x)]^T \in \mathcal{C}^1(\Omega)$, $\Omega \subset \mathbb{R}^2$. Associated operators to the operator D are found in the form

$$Lu = B_0 \frac{\partial u}{\partial x_0} + B_1 \frac{\partial u}{\partial x_1} + Cu, \tag{1.39}$$

where $B_0(x), B_1(x), C(x) \in \mathcal{C}^1(\Omega)$ are 2×2 matrices.

Corollary 9. *Operator L is associated to the operator D if and only if*

$$\begin{cases} E_1B_1 - B_1E_1 - B_0 - E_1B_0E_1 & = 0 \\ D(C - B_0Q) - (B_1 - B_0E_1)\dfrac{\partial Q}{\partial x_1} - CQ + B_0Q^2 & = 0 \\ D(B_1 - B_0E_1) + E_1C - CE_1 + B_0QE_1 - E_1B_0Q + B_0E_1Q - B_1Q & = 0. \end{cases}$$

Denote $z := x_0 + ix_1$, $w(z) = u_1(x_0,x_1) + iu_2(x_0,x_1)$, the first-order differential operators in matrix form can be transformed into complex form which contains $\partial_{\bar z} w$, $\overline{\partial_{\bar z} w}$, $\partial_z w$, $\overline{\partial_z w}$, w, $\bar w$, and vice versa. We consider the generalized Cauchy-Riemann operator in complex analysis

$$Dw = \partial_{\bar z} w + Aw + B\overline{w}, \;\; A(z), B(z) \in \mathcal{C}^2(\Omega).$$

A general linear first-order differential operator L has the form

$$Lw = M\partial_z w + S\overline{\partial_z w} + N\partial_{\bar z} w + P\overline{\partial_{\bar z} w} + Qw + R\overline{w}, \tag{1.40}$$

where $M(z), S(z), N(z), P(z), Q(z), R(z) \in \mathcal{C}^1(\Omega)$.
If $Dw = 0$ then $\partial_{\bar z} w = -Aw - B\bar w$. Substituting this expression into 1.40 we have

$$Lw = M\partial_z w + S\overline{\partial_z w} + (-NA - P\bar B + Q)\,w + \left(-NB - P\bar A + R\right)\bar w.$$

Therefore it is sufficient to find associated operators L in the form

$$Lw = M\partial_z w + S\overline{\partial_z w} + Nw + P\overline{w}. \tag{1.41}$$

Corollary 10. *An operator L in the form (1.41) is associated to the generalized Cauchy-Riemann operator D if and only if $S = 0$ and*

$$\partial_{\bar z} M \;=\; 0 \tag{1.42}$$

$$P + B\overline{M} \;=\; 0 \tag{1.43}$$

$$M\partial_z A - \partial_{\bar z} N \;=\; 0 \tag{1.44}$$

$$M\partial_z B - MB\overline{A} - \partial_{\bar z} P + NB - AP - B\overline{N} \;=\; 0. \tag{1.45}$$

Proof.
Suppose that the operator L in the form (1.41) is associated to the operator D. Let w be a solution of the equation $Dw = 0$, then

$$\frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y} - 2Aw - 2B\overline{w}. \tag{1.46}$$

Using the expression (1.46) we obtain

$$D(Lw) = \alpha\overline{\frac{\partial^2 w}{\partial y^2}} + \beta\frac{\partial w}{\partial y} + \gamma\overline{\frac{\partial w}{\partial y}} + \chi w + \psi\overline{w},$$

where $\alpha, \beta, \gamma, \chi, \psi$ are coefficients.
By Corollary 9, the conditions are

$$\alpha = 0,\ \beta = 0,\ \gamma = 0,\ \chi = 0,\ \psi = 0. \tag{1.47}$$

On the other hand, instead of using (1.46) we rewrite the condition $Dw = 0$ as

$$\partial_{\overline{z}} w = -Aw - B\overline{w}.$$

Then $D(Lw)$ can be written in other form

$$D(Lw) = \Phi\overline{\partial_{zz} w} + \Psi\partial_z w + \Upsilon\overline{\partial_z w} + \chi w + \psi\overline{w}.$$

The conditions (1.47) are equivalent to $\Phi = 0,\ \Psi = 0,\ \Upsilon = 0,\ \chi = 0,\ \psi = 0$. This is nothing but a system of conditions (1.42-1.45) and $S = 0$.
Corollary 10 is proved. □

Remark 15. *The conditions for associated operators of generalized Cauchy-Riemann operators in [34], [42] are only sufficient conditions. Corollary 9 gives the conditions for such associated operators which are not only sufficient but also necessary conditions.*

Remark 16. *In the case the Cauchy-Riemann operator $\partial_{\overline{z}}$, the conclusion in Corollary 9 coincides with the result in [28].*

From Corollary 10 ($S = 0$) we only need to find associated operators L in the form

$$Lw = M\partial_z w + Nw + P\overline{w}.$$

Solutions of the system $(1.42 - 1.45)$

We assume that the domain $\Omega \subset \mathbb{C}$ is bounded. Assume, further, that A, B, M, N, P are solutions of the system $(1.42 - 1.45)$ with additional conditions

$$A \in C^{1,\alpha}(\Omega),\ M \in C^{0,\alpha}(\Omega) \text{ and } P(z) = e^{X(z)}.$$

The condition (1.42) implies that M is a holomorphic function in Ω.
Denote $\varphi(x,y) := A(x,y)$. From the condition (1.44) : $\partial_{\overline{z}}N = M\partial_z\varphi \in \mathcal{C}^{0,\alpha}(\Omega)$, it implies that

$$N(z) = T_\Omega\left(M\partial_z\varphi\right)(z) + H(z), \tag{1.48}$$

where $H(z)$ is any holomorphic function in Ω and the operator T_Ω [38] is given by

$$T_\Omega f(z) := -\frac{1}{\pi}\iint\limits_{\Omega}\frac{f(\zeta)d\xi d\eta}{\zeta - z}.$$

Substituting $B = -\dfrac{P}{\overline{M}}$ from (1.43) into (1.45) we get

$$M\frac{\partial_z P}{P} + \overline{M}\frac{\partial_{\overline{z}}P}{P} = 2i\mathrm{Im}(M\overline{\varphi} - N). \tag{1.49}$$

Substituting N from (1.48) and $P = e^X$ into (1.49), we obtain

$$M_1\partial_x X + M_2\partial_y X = 2i\mathrm{Im}\left[M\overline{\varphi} - T_\Omega\left(M\partial_z\varphi\right) - H\right], \tag{1.50}$$

where $M(z) = M_1(x,y) + iM_2(x,y)$.
Since solutions of equation (1.50) exist locally, associated pairs (L,D) can be constructed in local. To sum up we have the following corollary.

Corollary 11. *Let $M \in \mathcal{C}^{0,\alpha}(\Omega)$ be any holomorphic function in Ω, ($M \neq 0$). Let $\varphi \in \mathcal{C}^{1,\alpha}(\Omega)$, H be a holomorphic function in Ω. Denote X be a local solution of the equation (1.50). Then coefficients of associated pairs (L,D) are given by*

$$N(z) = T_\Omega\left(M\partial_z\varphi\right)(z) + H(z), \qquad P(z) = e^{X(z)},$$
$$A(z) = \varphi(z), \qquad B(z) = -\frac{P(z)}{\overline{M(z)}}.$$

Example 7. *The case $M \equiv 1$.*

Equation (1.50) reads $\partial_x X = 2i\mathrm{Im}\left[\overline{\varphi} - T_\Omega\left(\partial_z\varphi\right) - H\right]$. The solutions are

$$X = \Phi(y) + 2i\mathrm{Im}\int_0^x\left[\overline{\varphi(t,y)} - T_\Omega\left(\partial_z\varphi\right)(t,y) - H(t,y)\right]dt,$$

where $\Phi \in \mathcal{C}^2$. We can find coefficients as follows:

$$N(z) = T_\Omega\left(\partial_z\varphi\right)(z) + H(z), \qquad \left(\varphi \in \mathcal{C}^{1,\alpha}(\Omega),\ \partial_{\overline{z}}H = 0\right)$$
$$P(z) = \exp\left(\Phi(y) + 2i\mathrm{Im}\int_0^x\left[\overline{\varphi(t,y)} - T_\Omega\left(\partial_z\varphi\right)(t,y) - H(t,y)\right]dt\right)$$
$$A = \varphi, \qquad B = -P.$$

Example 8. *The case $M = z$ in a bounded domain $\Omega \subset \{z = x + iy |\ x > 0\}$.*

Equation (1.50) reads

$$x\partial_x X + y\partial_y X = 2i\mathrm{Im}\left[M\overline{\varphi} - T_\Omega\left(M\partial_z\varphi\right) - H\right].$$

The solutions are given by

$$X = \Phi\left(\frac{y}{x}\right) + 2i\mathrm{Im}\int_1^x \frac{1}{t}\left[\overline{\varphi\left(t, \frac{ty}{x}\right)} - T_\Omega\left(\partial_z\varphi\right)\left(t, \frac{ty}{x}\right) - H\left(t, \frac{ty}{x}\right)\right] dt,$$

where $\Phi \in \mathcal{C}^2(\Omega)$.

Example 9. *The case $M = z^2$ in a bounded domain $\Omega \subset \{z = x + iy |\ y > x > 0\}$.*

Equation (1.50) reads

$$(x^2 - y^2)\partial_x X + 2xy\partial_y X = 2i\mathrm{Im}\left[M\overline{\varphi} - T_\Omega\left(M\partial_z\varphi\right) - H\right] =: g(x,y).$$

The solutions are given by

$$X = \Phi\left(\frac{x^2+y^2}{y}\right) + 2\int_1^x \frac{g\left(t, \frac{1}{2}\frac{x^2+y^2}{y} + \frac{1}{2}\sqrt{\left(\frac{x^2+y^2}{y}\right)^2 - 4t^2}\right)}{4t^2 - \left(\frac{x^2+y^2}{y}\right)^2 - \frac{x^2+y^2}{y}\sqrt{\left(\frac{x^2+y^2}{y}\right)^2 - 4t^2}} dt,$$

where $\Phi \in \mathcal{C}^2(\Omega)$.

3.2 Interior estimates for a class of first-order linear elliptic systems

General interior estimates for solutions of elliptic systems in general case were given by A. Douglis and L. Nirenberg in [8]. The estimates are in weighted Hölder norms. In this section we give an interior estimate for solutions of a class of first-order linear elliptic systems in supremum norm by using the technique in [8] with some modifications.
We consider a system in the form

$$Mu(x) = \sum_{i=0}^{n} E_i(x)\frac{\partial u}{\partial x_i} + Q(x)u(x) = 0, \tag{2.51}$$

where $x=(x_0,x_1,x_2,\cdots,x_n)$ in a bounded domain $\Omega\subset\mathbb{R}^{n+1}$,
$u(x)=[u_1(x),u_2(x),\cdots,u_m(x)]^T\in\mathcal{C}^1(\Omega)$, $Q(x)$, $E_i(x), i=1,2,\cdots,n$ are real $m\times m$ matrix functions with properties (1.2) and (1.3).
Let $\Omega\subset\mathbb{R}^{n+1}$ be a bounded domain. Denote the distance from a point $x\in\Omega$ to the boundary $\partial\Omega$ by d_x. We introduce a weighted function β with some properties:

$$\beta:(0,\frac{1}{2}diam(\Omega)]\to(0,+\infty),\ \left(diam(\Omega):=\max_{x,y\in\Omega}|x-y|\right),$$

β is continuous, increasing, bounded, and $\dfrac{\beta(t)}{\beta(t/2)}$ is also bounded.

Definition 9.
Let a function $u\in\mathcal{C}^1(\Omega)$ we define

$$\|u\|_{\beta,0}=\sup_{x\in\Omega}\beta(d_x)\,|u(x)|,\quad |u|_{\beta,1}=\sup_{x\in\Omega,\ 0\le i\le n}d_x.\beta(d_x)\left|\frac{\partial u}{\partial x_i}(x)\right|,$$
$$\|u\|_{\beta,1}=\|u\|_{\beta,0}+|u|_{\beta,1},\qquad \|u\|_0=\sup_{x\in\Omega}|u(x)|.$$

Let u be a Hölder continuous function with exponent $\alpha\in(0,1)$ in each compact subset of Ω, define

$$H_{x,\Omega}=\sup_{y\in\Omega}\frac{|u(y)-u(x)|}{|y-x|^\alpha}.$$

Lemma 2.26. *Let $\overline{B_R(x^0)}\subset\mathbb{R}^{n+1}$ be a closed ball centered at x^0 with radius R, a function*

$$\omega(x,y)=|y-x|^{-n}l(\xi),\ \ (\xi=\frac{y-x}{|y-x|}),$$

where $l(\xi)\in\mathcal{C}^1(S^n)$, S^n is the unit sphere in $\mathbb{R}^{n+1}$. Let $u\in\mathcal{C}^\alpha(B_R(x^0))$, $(0<\alpha<1)$. Then the function

$$\Phi(x):=\int\limits_{B_R(x^0)}\omega(x,y)u(y)dy$$

is in $\mathcal{C}^1(B_R(x^0))$ and we have

$$\frac{\partial\Phi(x)}{\partial x_i}=\lim_{\varepsilon\to0}\int\limits_{B_R(x^0)\setminus B_\varepsilon(x)}\frac{\partial\omega(x,y)}{\partial x_i}u(y)dy-\int\limits_{|\xi|=1}\xi_i l(\xi)d\mu(\xi)u(x),\tag{2.52}$$

and

$$\left|\frac{\partial\Phi}{\partial x_i}(x^0)\right|\le K_1\big(\|u\|_0+R^\alpha H_{x^0,B_R(x^0)}(u)\big),\tag{2.53}$$

with

$$K_1:=\max\left\{\omega_{n+1}\|l\|_0;\frac{1}{\alpha}w_{n+1}\left(n\|l\|_0+2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right)\right\},$$

Proof.
The statement $\Phi \in \mathcal{C}^1(B_R(x^0))$ and the equality (2.52) were proved by Michlin in [21]. Now we prove the estimate (2.53).
Since $\dfrac{\partial}{\partial x_i}\omega(x,y) = -\dfrac{\partial}{\partial y_i}\omega(x,y)$, we have

$$\frac{\partial \Phi}{\partial x_i}(x^0) = -\lim_{\varepsilon\to 0} \int\limits_{B_R(x^0)\setminus B_\varepsilon(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)u(y)dy - \int\limits_{|\xi|=1} \xi_i l(\xi)d\mu(\xi)u(x^0). \tag{2.54}$$

The first integral in the right-hand side of (2.54) can be estimated

$$I = \left| \int\limits_{|\xi|=1} \xi_i l(\xi)d\mu(\xi)u(x^0) \right| \leq \omega_{n+1}\|l\|_0\|u\|_0. \tag{2.55}$$

In order to estimate the second integral in the right-hand side of (2.54) we need rewrite

$$\begin{aligned}\int\limits_{B_R(x^0)\setminus B_\varepsilon(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)u(y)dy = &\int\limits_{B_R(x^0)\setminus B_\varepsilon(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)\left[u(y)-u(x^0)\right]dy \\ &+ \int\limits_{B_R(x^0)\setminus B_\varepsilon(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)dy.\ u(x^0)\end{aligned}$$

$$\begin{aligned}&= \int\limits_{B_R(x^0)\setminus B_\varepsilon(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)\left[u(y)-u(x^0)\right]dy + \int\limits_{\partial B_R(x^0)} \omega(x^0,y)N_i d\mu(y).\ u(x^0) \\ &- \int\limits_{\partial B_\varepsilon(x^0)} \omega(x^0,y)N_i d\mu(y).\ u(x^0) = \int\limits_{B_R(x^0)\setminus B_\varepsilon(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)\left[u(y)-u(x^0)\right]dy.\end{aligned}$$

$$\Rightarrow \lim_{\varepsilon\to 0}\int\limits_{B_R(x^0)\setminus B_\varepsilon(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)u(y)dy = \int\limits_{B_R(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)\left[u(y)-u(x^0)\right]dy.$$

The second integral in the right-hand side of (2.54) can be estimated

$$II = \left| \lim_{\varepsilon\to 0}\int\limits_{B_R(x^0)\setminus B_\varepsilon(x^0)} \frac{\partial \omega}{\partial y_i}(x^0,y)u(y)dy \right| \leq \int\limits_{B_R(x^0)} \left|\frac{\partial \omega}{\partial y_i}(x^0,y)\right| |y-x^0|^\alpha dy.H_{x^0,B_R(x^0)}(u).$$

We have

$$\frac{\partial \omega}{\partial y_i}(x^0,y) = -n(y_i-x_i^0)|y-x^0|^{-n-2}l\left(\frac{y-x^0}{|y-x^0|}\right) + |y-x^0|^{-n}\frac{\partial}{\partial y_i}l\left(\frac{y-x^0}{|y-x^0|}\right),$$

where $x^0 = (x_0^0, x_1^0, \cdots, x_n^0)$.

$$\left|\frac{\partial \omega}{\partial y_i}(x^0,y)\right| \le n|y-x^0|^{-n-1}\|l\|_0 + 2|y-x^0|^{-n-1}\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0$$
$$= \left(n\|l\|_0 + 2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right)|y-x^0|^{-n-1}.$$

Hence

$$II \le \int_{B_R(x^0)} |y-x^0|^{-n-1+\alpha}dy.\left(n\|l\|_0 + 2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right)H_{x^0,B_R(x^0)}(u)$$
$$= \frac{1}{\alpha}w_{n+1}\left(n\|l\|_0 + 2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right)R^\alpha H_{x^0,B_R(x^0)}(u). \tag{2.56}$$

From the estimates (2.55) and (2.56) if we denote

$$K_1 := \max\left\{\omega_{n+1}\|l\|_0; \frac{1}{\alpha}w_{n+1}\left(n\|l\|_0 + 2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right)\right\},$$

then the inequality (2.53) is proved. □

Lemma 2.27. *Let $B_R(x^0) \subset \mathbb{R}^{n+1}$ be a ball centered at x^0 with radius R, $u = [u_1(x), u_2(x), \cdots, u_m(x)]^T \in \mathcal{C}^1(\overline{B_R(x^0)})$ be a solution of the equation*

$$Mu = \sum_{i=0}^{n} E_i \frac{\partial u}{\partial x_i} = f,$$

where E_i are constant matrices with properties (1.2), (1.3), $f \in \mathcal{C}^\alpha(B_R(x^0))$, $\alpha \in (0,1)$. Then there exists a constant K_2 such that

$$\left|\frac{\partial u}{\partial x_j}(x^0)\right| \le K_2\left[R^{-1}\|u\|_0 + \|f\|_0 + R^\alpha H_{x^0,B_R(x^0)}(f)\right] \ \forall j = 0,1,\cdots,n. \tag{2.57}$$

Proof.
In the following, the Euclidean norm will be used for vectors and matrices. The adjoint operator R of the operator M is given by (2.4). The functions $G(x,y)$, $\rho(x,y)$ are defined in (3.6). The fundamental solution of the system $Rv = 0$ is given by (Chapter I)

$$\Gamma(x,y) = -\sum_{r=0}^{n}\frac{\partial G(x,y)}{\partial x_r}\overline{E_r}^T = -\frac{\sum_{r,s=0}^{n} A_{rs}(x_s-y_s)\overline{E_r}^T}{\omega_{n+1}\sqrt{DetA}\rho(x,y)^{n+1}},$$

where $\overline{E_0} := E_0$, $\overline{E_r} := -E_r$ for $r \geq 1$, ω_{n+1} is the surface measure of the unit sphere in $\mathbb{R}^{n+1}$. Applying the Stoke's formula to the function u we have

$$u(y) = \int\limits_{|x-x^0|=R} \Gamma^T(x,y) \sum_{i=0}^{n} N_i E_i u(x) d\mu(x) + \int\limits_{B_R(x^0)} \Gamma^T(x,y) f(x) dx.$$

$$\frac{\partial u(y)}{\partial y_j} = \int\limits_{|x-x^0|=R} \frac{\partial \Gamma^T(x,y)}{\partial y_j} \sum_{i=0}^{n} N_i E_i u(x) d\mu(x) + \frac{\partial}{\partial y_j} \int\limits_{B_R(x^0)} \Gamma^T(x,y) f(x) dx. \tag{2.58}$$

$$\Gamma^T(x,y) = \frac{\sum\limits_{r,s=0}^{n} A_{rs}(y_s - x_s)\overline{E_r}\rho(x,y)^{-n-1}}{\omega_{n+1}\sqrt{DetA}} = \frac{\sum\limits_{r,s=0}^{n} A_{rs}(y_s - x_s)\overline{E_r}\left((x-y)^T A^{-1}(x-y)\right)^{\frac{-n-1}{2}}}{\omega_{n+1}\sqrt{DetA}}$$

$$= |x-y|^{-n} \frac{1}{\omega_{n+1}\sqrt{DetA}} \sum_{r,s=0}^{n} A_{rs} \frac{y_s - x_s}{|y-x|} \left[\left(\frac{x-y}{|x-y|}\right)^T A^{-1} \left(\frac{x-y}{|x-y|}\right)\right]^{\frac{-n-1}{2}} \overline{E_r}.$$

Denote

$$l(\xi) := \frac{1}{\omega_{n+1}\sqrt{DetA}} \sum_{r,s=0}^{n} A_{rs}\xi_s \left[\xi^T A^{-1} \xi^T\right]^{\frac{-n-1}{2}} \overline{E_r},\ (\xi = \frac{y-x}{|y-x|}),$$

then

$$\Gamma(x,y) = |x-y|^{-n} l\left(\frac{y-x}{|y-x|}\right),\ \left|\frac{\partial \Gamma^T(x,y)}{\partial y_j}\right| \leq \left(n\|l\|_0 + 2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right)|x-y|^{-n-1}.$$

From (2.58) we have

$$\begin{aligned} \left|\frac{\partial u}{\partial y_j}(x^0)\right| \leq \ & \sqrt{m} \int\limits_{|x-x^0|=R} \left|\frac{\partial \Gamma^T}{\partial y_j}(x,x^0)\right| \sum_{i=0}^{n} \|E_i\|_0 \|u\|_0 d\mu(x) \\ & + \left|\frac{\partial}{\partial y_i}\left(\int\limits_{B_R(x^0)} \Gamma^T(x,.) f(x) dx\right)(x^0)\right|. \end{aligned} \tag{2.59}$$

It is easy to see that

$$\int\limits_{|x-x^0|=R} \left|\frac{\partial \Gamma^T}{\partial y_j}(x,x^0)\right| d\mu(x) \leq \omega_{n+1}\left(n\|l\|_0 + 2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right) R^{-1}. \tag{2.60}$$

Applying Lemma 2.26 we have

$$\left|\frac{\partial}{\partial y_j}\left(\int\limits_{B_R(x^0)} \Gamma^T(x,.) f(x) dx\right)(x^0)\right| \leq K_1\big(\|f\|_0 + R^\alpha H_{x^0,B_R(x^0)}(f)\big). \tag{2.61}$$

From (2.60), (2.59) and (2.61) we have

$$\left|\frac{\partial u}{\partial y_j}(x^0)\right| \leq \omega_{n+1}\sqrt{m}\sum_{i=0}^{n}\|E_i\|_0\left(n\|l\|_0+2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right)R^{-1}\|u\|_0 + K_1\left[\|f\|_0+R^{\alpha}H_{x^0,B_R(x^0)}(f)\right].$$

Denote

$$K_2 := max\left\{\omega_{n+1}\sqrt{m}\sum_{i=0}^{n}\|E_i\|_0\left(n\|l\|_0+2\sum_{i=0}^{n}\left\|\frac{\partial l}{\partial y_i}\right\|_0\right);K_1\right\},$$

then the estimate (2.57) is proved. □

Using the above results we can proved the following theorem.

Theorem 2.10. *Suppose that the coefficients* $E_i(x), Q(x) \in \mathcal{C}^{\alpha}(\Omega)(0<\alpha<1)$. *There exists a constant K such that*

$$\|u\|_{\beta,1} \leq K\|u\|_{\beta,0}$$

for every $\mathcal{C}^1-$ *solution u of the system* (2.51) *with finite* $\|u\|_{\beta,0}$.

Proof.
The proof of Theorem 2.10 bases on the method in [8] with some modifications.
We can assume that $|u|_{\beta,1}$ is finite. By considering subdomains $\Omega_\varepsilon := \{x \in \Omega|\ d_x > \varepsilon\}$, $(\varepsilon > 0)$, we work with domains Ω_ε then let ε go to zero. It is sufficient to prove that there exists a constant K such that $|u|_{\beta,1} \leq K\|u\|_{\beta,0}$.
By definition, there exist a point x^0 and a number i such that

$$|u|_{\beta,1} \leq 2d_{x^0}.\beta(d_{x^0})\left|\frac{\partial u}{\partial x_i}(x^0)\right|. \tag{2.62}$$

We consider a ball $B_R(x^0)$ centered at x^0 and radius $R=\lambda d_{x^0}$ with some $\lambda \in (0,\frac{1}{2}]$. Rewrite the equation (2.51) in the form

$$\sum_{i=0}^{n}E_i(x^0)\frac{\partial u}{\partial x_i} = \sum_{i=0}^{n}\left[E_i(x^0)-E_i(x)\right]\frac{\partial u}{\partial x_i} - Q(x)u(x) =: g(x).$$

Applying Lemma 2.27 to the above system in the ball $B_R(x^0)$ we get

$$\begin{aligned}\left|\frac{\partial u}{\partial x_i}(x^0)\right| &\leq K_2\left[R^{-1}\|u\|_0+\|g\|_0+R^{\alpha}H_{x^0,B_R(x^0)}(g)\right]\\ &\leq K_2\left[\frac{\lambda^{-1}\|u\|_{\beta,0}}{d_{x^0}\beta(\frac{d_{x^0}}{2})}+\|g\|_0+\lambda^{\alpha}d_{x^0}^{\alpha}H_{x^0,B_R(x^0)}(g)\right].\end{aligned} \tag{2.63}$$

$$\left|\left[E_i(x^0)-E_i(x)\right]\frac{\partial u}{\partial x_i}\right| \le \frac{1}{d_x\beta(d_x)}\|E_i\|_{C^\alpha}|u|_{\beta,1}|x-x^0|^\alpha \le \frac{\lambda^\alpha(1-\lambda)^{-1}d_{x^0}^\alpha\|E_i\|_{C^\alpha}|u|_{\beta,1}}{d_{x^0}\beta(\frac{d_{x^0}}{2})}.$$

$$|Q(x)u(x)| \le \frac{\|Q\|_0\|u\|_{\beta,0}}{\beta(d_x)} \le \frac{d_{x^0}\|Q\|_0\|u\|_{\beta,0}}{d_{x^0}\beta(\frac{d_{x^0}}{2})}.$$

Hence,

$$\|g\|_0 \le \frac{1}{d_{x^0}\beta(\frac{d_{x^0}}{2})}\left[\lambda^\alpha(1-\lambda)^{-1}d_{x^0}^\alpha\|E_i\|_{C^\alpha}|u|_{\beta,1}+d_{x^0}\|Q\|_0\|u\|_{\beta,0}\right]. \tag{2.64}$$

Now we will estimate $H_{x^0,B_R(x^0)}(g)$

$$H_{x^0,B_R(x^0)}\left\{\left[E_i(x^0)-E_i(.)\right]\frac{\partial u}{\partial x_i}\right\} \le \frac{2\|E_i\|_{C^\alpha}|u|_{\beta,1}}{d_{x^0}\beta(\frac{d_{x^0}}{2})},$$

$$H_{x^0,B_R(x^0)}(Qu) \le \frac{d_{x^0}\|Q\|_{C^\alpha}\|u\|_{\beta,0}+2m(n+1)\|Q\|_{C^\alpha}\lambda^{1-\alpha}d_{x^0}^{1-\alpha}|u|_{\beta,1}}{d_{x^0}\beta(\frac{d_{x^0}}{2})}.$$

So we have

$$H_{x^0,B_R(x^0)}(g) \le \frac{2\|E_i\|_{C^\alpha}|u|_{\beta,1}+d_{x^0}\|Q\|_{C^\alpha}\|u\|_{\beta,0}+2m(n+1)\|Q\|_{C^\alpha}\lambda^{1-\alpha}d_{x^0}^{1-\alpha}|u|_{\beta,1}}{d_{x^0}\beta(\frac{d_{x^0}}{2})}. \tag{2.65}$$

From (2.63), (2.64) and (2.65), there exists a constant K_3 such that

$$\left|\frac{\partial u}{\partial x_i}(x^0)\right| \le \frac{K_3}{d_{x^0}\beta(\frac{d_{x^0}}{2})}\left[(1+\lambda^{-1})\|u\|_{\beta,0}+\lambda^\alpha|u|_{\beta,1}\right].$$

From (2.62) we have

$$|u|_{\beta,1} \le 2K_3\frac{\beta(d_{x^0})}{\beta(\frac{d_{x^0}}{2})}\left[(1+\lambda^{-1})\|u\|_{\beta,0}+\lambda^\alpha|u|_{\beta,1}\right].$$

Since $\dfrac{\beta(t)}{\beta(t/2)}$ is bounded, denote $K_0 := \sup\limits_{0<t<diam(\Omega)/2}\dfrac{\beta(t)}{\beta(t/2)}$. We can find λ so small that $2K_0K_3\lambda^\alpha < \dfrac{1}{2}$, then

$$|u|_{\beta,1} \le 4K_3K_0(1+\lambda^{-1})\|u\|_{\beta,0}.$$

In the end, the constant K in theorem is chosen by $K = 1+4K_3K_0(1+\lambda^{-1})$. □

3.3 Solutions of initial value problems

In this section, we use the standard method in [39], [36] and [32] to solve the initial value prolem.
We consider the initial value problem (0.1), the operator $\mathcal{F}$ in the form

$$\mathcal{F}\left(t,x,u,\frac{\partial u}{\partial x_i}\right)=\sum_{i=0}^{n}B_i(t,x)\frac{\partial u}{\partial x_i}+C(t,x)u,$$

where $B_i(t,x),\ C(t,x)$ are $m\times m$ matrices. The solutions are fixed points of the operator

$$\mathcal{U}(t,x)=\varphi(x)+\int_0^t \mathcal{F}\left(\tau,x,u(\tau,x),\frac{\partial u}{\partial x_j}(\tau,x)\right)d\tau. \tag{3.66}$$

Let $\Omega\subset\mathbb{R}^{n+1}$ be a bounded domain. Denote $d_x=\inf\limits_{y\in\partial\Omega}|x-y|$ and $d(t,x)=d_x-\frac{t}{\eta}$.
Introduce the conical domain

$$M_\eta=\{(t,x):\ x\in\Omega,\ 0\le t<\eta d_x\},$$

where η will be fixed later. Define

$$\|u\|_*=\sup_{M_\eta}|u(t,x)|d^p(t,x),$$

where $p>1$ is fixedly chosen. Denote $\mathcal{B}(M_\eta)$ be the Banach space of functions $u=u(t,x),\ (t,x)\in M_\eta$ with the following properties

$$\begin{aligned}
&(i) && u(.,x),\ \frac{\partial u(.,x)}{\partial x_i}\in\mathcal{C}(0,\eta d_x)\ \forall x\in\Omega,\\
&(ii) && u(t,.)\in\mathcal{C}^1(\Omega_t),\ Mu(t,.)=0,\ \text{where } \Omega_t=\{x\in\Omega|\ d_x>\frac{t}{\eta}\},\\
&(iii) && \|u\|_*<+\infty,
\end{aligned}$$

where the operator M is given by (1.3). Now we assume that

$$\begin{aligned}
&(a) && B_i(.,x),\ C(.,x)\in\mathcal{C}(0,\eta d_x),\ B_i(t,.),\ C(t,.)\in\mathcal{C}^1(\Omega_t),\\
& && |B_i(t,x)|\le c \text{ and } |C(t,x)|\le\frac{c}{d(t,x)},\ \forall(t,x)\in M_\eta \text{ for some constant } c>0\\
&(b) && \mathcal{F} \text{ is associated to } M\\
&(c) && \varphi\in\mathcal{B}(M_\eta).
\end{aligned} \tag{3.67}$$

Lemma 3.28. *Assume that the conditions for the coefficients of the operator M in Theorem 1.9 and Theorem 2.10 are satisfied. If the condition (3.67) is satisfied then the operator $\mathcal{U}$ maps $\mathcal{B}(M_\eta)$ into itself.*

Proof.
Let $u \in \mathcal{B}(M_\eta)$. Consider a point $(t,x) \in M_\eta$. Let x' with $|x'-x| \le r < d(t,x)$ then $d(t,x') = d_{x'} - \dfrac{t}{\eta} \ge d_x - r - \dfrac{t}{\eta} = d(t,x) - r > 0$, and thus $(t,x') \in M_\eta$. We have

$$|u(t,x')| \le \frac{\|u\|_*}{d^p(t,x')} \le \frac{\|u\|_*}{(d(t,x)-r)^p}.$$

Choosing $r = \dfrac{1}{p+1} d(t,x)$, we have

$$|u(t,x')| \le \left(1+\frac{1}{p}\right)^p \frac{\|u\|_*}{d^p(t,x)}.$$

Applying Theorem 2.10 in the ball $\overline{B_r(x)} := \{x' \mid |x'-x| \le r\}$ with the weighted function $\beta \equiv 1$, we have

$$\left|\frac{\partial u}{\partial x_i}(t,x)\right| \le \frac{K}{r} \sup_{|x'-x| \le r} |u(t,x')| \le (p+1)\left(1+\frac{1}{p}\right)^p K \frac{\|u\|_*}{d^{p+1}(t,x)}.$$

$$\begin{aligned}\left|\mathcal{F}\left(t,x,u,\frac{\partial u}{\partial x_i}\right)\right| &\le \sum_{i=0}^{n} |B_i(t,x)| \left|\frac{\partial u}{\partial x_i}\right| + |C(t,x)|\,|u| \\ &\le c\left[(n+1)(p+1)\left(1+\frac{1}{p}\right)^p K + 1\right] \frac{\|u\|_*}{d^{p+1}(t,x)}.\end{aligned}$$

Denote $K' := c\left[(n+1)(p+1)\left(1+\dfrac{1}{p}\right)^p K + 1\right]$. Since

$$\int_0^t \frac{d\tau}{d^{p+1}(\tau,x)} < \frac{\eta}{p} \frac{1}{d^p(t,x)},$$

it follows

$$\left|\int_0^t \mathcal{F}\left(\tau,x,u(\tau,x),\frac{\partial u}{\partial x_j}(\tau,x)\right) d\tau\right| \le \sqrt{m} K' \frac{\eta}{p} \frac{\|u\|_*}{d^p(t,x)}. \tag{3.68}$$

Combining this with $\varphi \in \mathcal{B}(M_\eta)$, we have $\mathcal{U}(t,x) \in \mathcal{B}(M_\eta)$. □

Lemma 3.29. *With the hypothesis of Lemma 3.28, the operator $\mathcal{U}$ is contractive if*

$$\sqrt{m} K' \frac{\eta}{p} < 1.$$

Proof.
Let $\tilde{u}, \hat{u} \in \mathcal{B}(M_\eta)$, denote their images by $\tilde{\mathcal{U}}$, $\hat{\mathcal{U}}$. We have

$$\tilde{\mathcal{U}} - \hat{\mathcal{U}} = \int_0^t \mathcal{F}\left(\tau, x, (\tilde{u} - \hat{u})(\tau, x), \frac{\partial(\tilde{u} - \hat{u})}{\partial x_j}(\tau, x)\right) d\tau.$$

By the estimate (3.68), we have

$$|\tilde{\mathcal{U}}(t,x) - \hat{\mathcal{U}}(t,x)| \leq \sqrt{m} K' \frac{\eta}{p} \frac{\|\tilde{u} - \hat{u}\|_*}{d^p(t,x)}.$$

It follows

$$\|\tilde{\mathcal{U}} - \hat{\mathcal{U}}\|_* \leq \sqrt{m} K' \frac{\eta}{p} \|\tilde{u} - \hat{u}\|_*.$$

Hence, if $\sqrt{m} K' \frac{\eta}{p} < 1$ then the operator $\mathcal{U}$ is contractive. □

Applying the contraction mapping principle for the operator $\mathcal{U}$, we have the following theorem.

Theorem 3.11. *With the hypothesis of Lemma 3.29, if the initial function φ satisfies*

$$\sup_{x \in \Omega} |\varphi(x)| d_x^p < \infty$$

then the initial value problem (0.1) has a unique solution $u(t,x) \in \mathcal{B}(M_\eta)$.

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