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Limit Behaviors for Random Walks and Branching Random Walks on some Products of Groups

PHD THESIS

written to obtain the academic degree of a Doctor of Engineering

Doctoral studies of Engineering at the doctoral school "Mathematics and Scientific Computing"

Graz University of Technology

Graz University of Technology

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Institut für Mathematische Strukturtheorie

Graz, in May 2012

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Introduction

In this work we analyze the behavior of random walks and branching random walks in different settings. The first structure that we consider is a free product of groups, where we investigate the evolution of transient random walks and branching random walks. The other setting is a Cartesian product of groups, where we answer different questions about branching random walks.

This work is split into three parts.

In the first one we investigate the long-term behavior of a random walk on free products of groups, more precisely we study the possible asymptotic behaviors of its return probabilities, in dependence of the properties of the groups and of the measure governing the random walk itself.

In the second part we study branching random walks on free products of groups. Here the main goal of our work is to understand how "big" (in the sense of the Hausdorff dimension) the limit set of accumulation points of the process can be, in relation to the boundary of the underlying structure.

The third part deals with critical branching random walks on Cartesian products. Our investigation aims at answering the following questions posed to Matthew Roberts and myself by Itai Benjamini: denoting by T_3 the binary tree, does the trace (i.e., the subgraph of sites visited by particles of the branching random walk) of a critical process on $T_3 \times \mathbb{Z}$ have infinitely many ends, or only finitely many? What happens if we consider a critical branching random walk on $T_3 \times T_3$?

Part One: Random Walks on Free Products

The main results that we present in Part One appeared in [6].

Consider Γ_1 and Γ_2 finitely generated groups with identity elements e_1 and e_2 respectively. The free product of these two groups is defined as

$$\Gamma := \Gamma_1 * \Gamma_2 := \big\{ x_1 x_2 \dots x_n \ : \ x_j \in (\Gamma_1 \setminus \{e_1\}) \cup (\Gamma_2 \setminus \{e_2\}) \, , \ j \in \{1, \dots, n\}, \\ \text{and} \ \ x_j \in \Gamma_i \Rightarrow x_{j+1} \notin \Gamma_i \big\} \cup \{e\}.$$

In other words, the group Γ consists of all finite words whose letters (the "blocks" x_j) are elements of one of the two starting groups. The condition

$$x_j \in \Gamma_i \Rightarrow x_{j+1} \notin \Gamma_i$$

means that all words are *reduced*, i.e., two consecutive blocks do not belong to the same group.

In order to define a random walk on Γ we start with random walks defined on each of the two starting groups. Let μ_1 and μ_2 be probability measures defined on the generators of Γ_1 and Γ_2 respectively. Then we can define a probability measure μ on Γ as follows:

$$\mu := \alpha_1\mu_1 + (1-\alpha_1)\mu_2,$$

where $\alpha_1 \in (0,1)$.

The goal is to estimate the asymptotic behavior of the quantities $\mu^{(n)}(x)$ (by this we denote the n-th convolution power of μ): $\mu^{(n)}(x)$ is the probability that a random walk starting at x returns to x in n steps. In most situations, it is of the form

$$\mu^{(n)}(x) \sim C_x \rho^{n\delta} n^{-\lambda}, \tag{$*$}$$

where $\rho < 1$ is the spectral radius of the random walk governed by μ, δ is its period and λ a positive parameter. C_x is a positive constant dependent only on x.

Gerl (see [22]) conjectured that if μ is symmetric and the asymptotic behavior of $\mu^{(n)}(x)$ is of the form $(*)$, then the parameter λ is a group invariant.

This conjecture was disproved by Cartwright (see [10]), who showed that on the free product $\mathbb{Z}^d * \mathbb{Z}^d$ (with $d \geq 5$) there are at least two random walks, governed by symmetric probability measures, that yield different values for λ. In one case $\lambda = 3/2$ and in the other $\lambda = d/2$, being d the dimension of the lattices.

A natural question (see [13]) is whether there are other possible types of asymptotic behaviors. In our work we give a positive answer to this question. In particular, we prove that on a free product of the form $\mathbb{Z}^{d_1} * \cdots * \mathbb{Z}^{d_r}$ (where all d_j's are integers strictly larger than 4) we get up to $(r+1)$ different possible behaviors. Moreover, we study the case $\Gamma = \Gamma_1 * \Gamma_2$ for finitely generated groups and we give precise phase transitions in dependence of the parameter α_1.

Our investigation starts from the works by Cartwright (see [10]) and Woess (see [64]), but work in this direction has been done also by Gerl and Woess (see [23]), Sawyer (see [54]), Woess (see e.g. [63]), Cartwright and Soardi (see e.g. [11]) and Lalley (see e.g. [35]).

For finite range random walks on free groups it is known (see [63] and [35]) that

$$\mu^{(n)}(x) \sim C_x \rho^{n\delta} n^{-3/2}.$$

The same estimate holds for random walks on free products of *finite* groups: results in this direction can be found in [22], [61] and [63].

In order to achieve more general results, Cartwright and Soardi (see [11]), Woess (see [63]), Voiculescu (see [58]) and McLaughlin (in his PhD thesis, see [43]), found a method to express the Green function defined on Γ in terms of a functional equation of the Green functions defined on each factor.

We generalize their methods to a much wider set of free products, and then apply the *method of Darboux* (described in Appendix A) to extrapolate the asymptotic behavior of $\mu^{(n)}$ from the singular expansion of the Green function.

Organization of Part One : After two introductory chapters (namely 1 and 2, contained in "Part 0"), where we recall the fundamental results and definitions useful for our discussions, we split Part One into four chapters.

The aim of Part One is to find the asymptotic behavior of the n-step return probabilities of a random walk defined on the free product of $r \geq 2$ groups (denoted by Γ). These behaviors depend on the structure of the free factors and on the chosen measure defined on Γ. In the particular case of $r = 2$ we find explicit phase transitions.

In Chapter 3 we explain how to define a random walk on a free product of groups, given probability measures on each free factor. We recall the most important generating functions, in particular the Green function, because their properties play a fundamental role all throughout the first part of the work. Following the structure of [64, Section II.9] we introduce a functional equation concerning the Green function. Different properties of this functional equation can lead to different behaviors for the random walk.

In Chapter 4 we consider the case of a product of two free factors. We make a case distinction under some assumption on the Green functions associated with the random walks defined on each factor. In each situation we find the explicit singular expansion of the Green function associated with the random walk on Γ, and by Darboux's method (see Appendix A) we get the asymptotic behavior of its return probabilities.

In Chapter 5 we investigate the general case $r > 2$, and present a few concrete examples where the asymptotic behavior of the return probabilities can be easily computed.

We conclude Part One with Chapter 6, where we find explicit phase transitions with respect to the measure that governs the random walk on Γ.

Part Two: Branching Random Walks on Free Products

The main results that we present in Part Two appeared in [7].

A branching random walk (BRW for short) is a stochastic process characterized by two different kinds of randomness. It starts with one particle at a vertex, and can be defined inductively as follows. At each unit of time, the alive particles split (independently of each other) into a *random amount* of offspring, according to a probability measure ν defined on the non-negative integers. Afterwards, the newly-born particles make one step independently of each other, according to an underlying *random walk*.

The study of branching processes started around 1874 to answer a problem about survival of surnames. In their work (see [60]) Galton and Watson investigate the survival of the surname of a family that reproduces according to a probability measure ν defined on the non-negative integers.
Denote by ν_k the probability that an individual has exactly k descendants, and let $\mathbb{E}\nu := \sum_{k\geq 0} k\nu_k$ be the expected value of ν. Galton and Watson showed

that

$$\begin{aligned} &\text{if } \ \mathbb{E}\nu \leq 1 \implies \mathbb{P}(\text{process dies out within finite time}) = 1; \\ &\text{if } \ \mathbb{E}\nu > 1 \implies \mathbb{P}(\text{process dies out within finite time}) < 1. \end{aligned}$$

A BRW on a graph is called *recurrent* if every vertex of the graph is visited infinitely often by the particles of the BRW. On the other side, it is said to be *transient* if every finite set of vertices is eventually free of particles.

In 1994 Benjamini and Peres (see [4]) showed that:

$$\begin{aligned} &\text{if } \ \mathbb{E}\nu < \rho^{-1} \implies \text{BRW is transient;} \\ &\text{if } \ \mathbb{E}\nu > \rho^{-1} \implies \text{BRW is recurrent,} \end{aligned}$$

being ρ, once again, the spectral radius of the underlying random walk.

Some years later Gantert and Müller (see [20]) proved that at criticality, i.e., if $\mathbb{E}\nu = \rho^{-1}$, the BRW is still *transient*.

The main "source of inspiration" for our investigation is a work by Hueter and Lalley (see [31]): they prove that the limit set of a supercritical branching random walk on a homogeneous tree presents a phase transition in the dimension. More precisely: if the process is transient, its Hausdorff dimension can reach at most 1/2 the Hausdorff dimension of the boundary of the tree; if it is recurrent, then the two dimensions coincide.

Lalley and Sellke (see [36]) studied this type of phase transitions for branching Brownian motion on the hyperbolic disc.
Karpelevich, Pechersky, and Suhov (see [32]) generalized these results to higher dimensional Lobachevsky spaces, while Grigor'yan and Kelbert (in [26]) studied recurrence and transience for branching diffusion processes on Riemannian manifolds.
Cammarota and Orsingher (see [5]) investigated a "linear" growing system of particles on the hyperbolic disc.

What we prove in the setting of free products of groups, is a more general version of the main result in [31]. In addition, we show that there are two possible types of accumulation points in the limit set of the process. One type will be called "typical", and the other one "atypical". This is due to the fact that the first one is always present, while for the second one we need some extra condition (which we state precisely).

Our motivation to analyze the behavior of branching random walks on free products came from the results of the first part: selecting different measures or different groups to build the structure, we can get different asymptotic behaviors for a random walk. Does a similar phenomenon happen in the case of BRW's as well?

Intuitively speaking, by considering a BRW, the phenomena we see in a single random walk should be "amplified" in some sense: how does this fact affect the limit set of the process?

Organization of Part Two : The aim of Part Two is to study BRW's on free products of groups: we consider a transient BRW conditioned on survival.

It turns out that the Hausdorff dimension of the limit set of the process is a function only of the expected value of the offspring distribution governing the branching phenomenon. Moreover, this function is continuous up to the critical value, that determines a phase transition for the process from transience to recurrence.
At this point, the Hausdorff dimension of the limit set can be at most 1/2 the Hausdorff dimension of the whole boundary of Γ. For every larger value of the mean of the offspring distribution, the two Hausdorff dimensions coincide.

Another important result that we get in Part Two is the following: if at least one of the free factors is infinite, we can get different types of accumulation points for the process. We state precise conditions for this to happen, and we see that the "atypical" accumulation points do not contribute to the Hausdorff dimension of the limit set of the BRW.

More precisely, Part Two is organized as follows.

In Chapter 7 we explain how to define the BRW on Γ and recall some useful results, moreover we introduce the definition of the two types of accumulation points of the process. With the help of an auxiliary Galton-Watson process, we find when the "atypical" accumulation points can come in play: we show that when this new Galton-Watson process is supercritical, the ends of the infinite free factor turn out to be inside the limit set of the process.

In Chapter 8 we introduce the *growth functions*, which are the tools we need in order to find the growth rate of the BRW.
We also show that the box-counting dimension and the Hausdorff dimension of the limit set of the BRW coincide, and we determine these values explicitly.

Chapter 9 is essentially devoted to finding the Hausdorff dimension of the boundary of Γ, while in Chapter 10 we present some simplified formulas that we obtain in case all free factors are finite. We end Part Two with a short description (see Section 10.2) of BRW's on free products with amalgamation.

Part Three: Branching Random Walks on Cartesian Products

In this part of the work we present some preliminary results obtained by Matthew Roberts and myself ([8]).

Given $d \geq 2$ finitely generated groups $\Gamma_1, \ldots, \Gamma_d$, we can construct their *Cartesian product* $\Gamma_1 \times \cdots \times \Gamma_d$ in the following way:

$$\Gamma_1 \times \cdots \times \Gamma_d := \{(a_1, \ldots, a_d) \mid a_i \in \Gamma_i \text{ for all } i \in \{1, \ldots, d\}\}.$$

In 1921 Pólya (see [52]) showed that on a Cartesian product where all the factors $\Gamma_i \equiv \mathbb{Z}$, a symmetric nearest neighbor random walk presents different behaviors depending on the value d. More precisely, he showed that for all $d \geq 3$ the walk is *transient*, otherwise it is *recurrent*.

Pólya's proof relies on finding explicit asymptotics for the return probabilities, i.e., denoting by $\mathbf{0} := (0, \ldots, 0)$ the origin of $\mathbb{Z}^d$, we can summarize his result as follows:

$$\mu^{(2n)}(\mathbf{0}) \sim n^{-d/2}.$$

Here $\mu^{(n)}(\mathbf{0})$ denotes the probability that the symmetric random walk comes back to the origin after n steps, and the symbol $\sim$ in this context means that the estimate is accurate up to a constant.

Another result in this direction is due to Cartwright and Soardi (see [12]). They consider a Cartesian product of the form $\Gamma_1 \times \cdots \times \Gamma_d$, where every Γ_i is finitely generated and equipped with a probability measure μ_i defined on its generators. Given positive values $\alpha_1, \ldots, \alpha_d$ such that $\sum_{i=1}^d \alpha_i = 1$, the measure on the Cartesian product is defined as follows:

$$\mu := \alpha_1\mu_1 + \ldots + \alpha_d\mu_d.$$

Denote by $(X_n)_n$ the random walk governed by μ. One of the main results of [12] can be summarized as follows (see Theorem 12.2.2). Suppose that for every element $y_j \in \Gamma_j$ we have

$$\mathbb{P}(X_n^j = y_j) \sim C_j \rho_j^n \,/ n^{a_j}, \qquad \text{for all } j \in \{1, \ldots, d\},$$

where $(X_n^j)_n$ is the random walk on Γ_j governed by μ_j; ρ_j its the spectral radius, and $a_j > 0$ numbers independent of n. Then the random walk $(X_n)_n$ on Γ satisfies:

$$\mathbb{P}(X_n = y) \sim \frac{C(\alpha_1\rho_1 + \ldots + \alpha_d\rho_d)^n}{n^{a_1+\ldots+a_d}},$$

for all $y = (y_1, \ldots, y_d)$. For more details and further explanations we refer the interested reader to [64, Sections I.4.B and III.18].

In this part of our work, we investigate *critical branching random walks* on *some Cartesian products*. In particular we consider two settings:

1. the Cartesian product of a *homogeneous tree* with the d-dimensional grid $\mathbb{Z}^d$;

2. the Cartesian product of two *homogeneous trees*.

Our aim is to understand some properties of the trace of a critical BRW: does it have *finitely* or *infinitely* many ends?

The two approaches presented in these two settings are quite different, because the methods that we can use to solve the first case, drastically fail in the second case.

The idea came up in a very nice environment: during the 41st Probability Summer school in St. Flour, a meeting with Itai Benjamini brought my collaborator Matthew and myself to work together on the following problems. Given the Cartesian product of a homogeneous tree T and the set $\mathbb{Z}$ (this product is non-amenable, has exponential growth and has only *one end* in the graph topology), we can consider a critical BRW on it. The question looks very simple: does the trace of this process have finitely many, or infinitely many ends?

We could find that in the isotropic case (we can replace the simple random walk defined on T by any nearest neighbor random walk on T) it has infinitely many ends, but if the underlying random walk defined on $\mathbb{Z}$ has a bias towards one direction, then the BRW on $T \times \mathbb{Z}$ has only one end.

The next problem, looked very similar to the previous one: does the trace of a critical BRW on $T \times T$ have finitely many, or infinitely many ends? The expectation to get an answer quickly exploiting similar techniques used to solve the first problem was high, but as it happens quite often, these methods could not be applied to this new structure.

We investigate the model under two different points of view: we consider the process with respect to the Martin topology and with respect to the graph topology. We show that there are two types of accumulation points for the process in the Martin topology, but just one in the graph topology.

At the end we manage to show that the limit set of the trace of an *isotropic* BRW on $T_3 \times T_3$ has infinitely many ends almost surely, even in the critical case.

Organization of Part Three : In Chapter 11 we start by introducing a BRW on a Cartesian product of two groups. In Section 11.1 we show that the isotropic, critical BRW on $T_d \times A$ (where T_d is a homogeneous tree of degree d and A any finitely generated amenable group) has infinitely many ends almost surely. On the contrary, in Section 11.2 we show that a critical BRW on $T_q \times \mathbb{Z}^d$, whose underlying random walk has a bias in one direction, has only one end.
In Section 11.3 we present a generalization of the results to the case $T \times \mathbb{Z}^d$, where T is a Galton-Watson tree satisfying some additional conditions.

Chapter 12 is organized as follows: in Section 12.1 we give a small introduction on the Martin compactification of the Cartesian product. We show that despite the fact that every element of the Martin boundary is an accumulation point of the process, we can distinguish two cases: there are elements which are "attractive" for the BRW (we call them *stable*) and others that are "repulsive" (we call them *unstable*).
In Section 12.2 we show that the trace of an isotropic BRW on $T_3 \times T_3$ has infinitely many ends almost surely.
We conclude Chapter 12 with Section 12.3, where we present an example of a critical BRW whose underlying random walk has a bias but, in contrast with what happens in the case of $T_3 \times \mathbb{Z}$, its trace has infinitely many ends almost surely.

Acknowledgements

First of all, I would like to thank my supervisor, Prof. Wolfgang Woess, for his support and especially for giving me the great opportunity to work on such interesting topics like random walks and branching random walks.

Then I would like to thank my coauthors, Lorenz Gilch and Sebastian Müller for the support and help, that led to the results presented in Parts I and II of this thesis.

I would like to thank my collaborator Matthew Roberts, for his help and enthusiasm in working with me on the problems described in Part III of this work. About this topic, I would like to thank Itai Benjamini for stating the question to us.

I am grateful to Prof. Yuval Peres for inviting me at Microsoft Research, where I could meet and have very interesting and useful discussions with some of the members of the *Theory Group*.

Personally I would like to thank all the members of the *Math C* department, as well as Indira Chatterjee and Steven Lalley for the support and for the fruitful discussions about random walks and branching random walks.

Thanks to all my friends, who always manage to improve my mood and give me incentive to go on.

A special thank goes to my family: I would like to deeply thank my parents and my sister for believing in me and for supporting me... always.

The author is grateful to the following entities for the finantial support: Research and Technology Office of the TU Graz; NAWI Graz (grant number F-NW501-GASS); Austrian Academy of Science (DOC-fFORTE fellowship, project number D-1503000014); FWF, Austrian Science Fund (project number D-1502S09606); Microsoft Research.

Moreover, I have been an “associated PhD student” within the project *DK plus*, funded by FWF, Austrian Science Fund (project number E-1503W01230).

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Part 0

Preliminaries

Chapter 1

Background: a flavor of Algebra and Geometry

1.1 Groups and Cayley Graphs

In this section we recall the basic concepts and fix the notation that will be used in the rest of the work.

1.1.1 Groups

A *group* is a set G endowed with a binary operation $(\cdot) : G \times G \longrightarrow G$ that satisfies the following properties:

- associativity: for every $a, b, c \in G$ it holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
- existence of the identity: there is an element $e \in G$ such that for all $a \in G$ it holds $a \cdot e = e \cdot a = a$;
- existence of an inverse: for every $a \in G$ there is an element $b \in G$ such that $a \cdot b = b \cdot a = e$. This element is unique and usually denoted by a^{-1}.

Given a group G and a set X, a *left group-action* of G on X is a map from $G \times X \longrightarrow X$ such that:

- for all $x \in X$ it holds $e \cdot x = x$;
- for all $g, h \in G$ and $x \in X$ it holds $(g \cdot h) \cdot x = g \cdot (h \cdot x)$.

Analogously, a *right group-action* of G on X is a map from $G \times X \longrightarrow X$ such that:

- for all $x \in X$ we have $x \cdot e = x$;
- for all $g, h \in G$ and $x \in X$ we have $x \cdot (g \cdot h) = (x \cdot g) \cdot h$.

All throughout this work, the considered groups always act from the right.

1.1.2 Graphs

By a *graph* Γ we mean a set $V(\Gamma)$ of *vertices* and a set $E(\Gamma)$ of *edges*, each edge being associated to an unordered pair of vertices.

Every vertex $x \in \Gamma$ is associated to (or "contained in") a certain number of edges. Denote by $d_x \geq 0$ this value: d_x is called the *degree* of x. A graph is *locally finite* if each vertex has finite degree. In this work we always consider locally finite graphs, in which for every $x \in \Gamma$ we have $d_x \geq 1$.

A graph is said to be *connected* if every two vertices can be joined by a sequence of edges, which are called *paths*. A cycle (or *loop*) is a non-trivial path connecting a vertex with itself, without repetition of other vertices.

A *tree* is a connected graph with no loops. A d-regular tree is a tree where every vertex has degree d. A *symmetry* of a graph Γ is a bijection $\alpha : \Gamma \longrightarrow \Gamma$ taking vertices to vertices and edges to edges, such that

$$\alpha(\{v, w\}) = \{\alpha(v), \alpha(w)\}$$

for every $\{v, w\} \in E(\Gamma)$, $v, w \in V(\Gamma)$. The set of all symmetries is a group, called the *symmetry group*.

If for any two vertices $v, w \in V(\Gamma)$ there is a symmetry α such that $\alpha(v) = w$, we say that Γ is *vertex transitive*.
Analogously, if for any two edges $\{v, w\}$ and $\{v', w'\}$ there is a symmetry α such that $\alpha(\{v, w\}) = \{v', w'\}$, we say that Γ is *edge transitive*.

Consider a group G with a subset S. We write that $G = \langle S \rangle$ meaning that S *generates* G if every element $g \in G$ can be expressed as a product of elements of S. The group is *finitely generated* if the cardinality of S is finite.

At this point we can state the theorem that allows us to work with graphs rather than directly with groups:

Theorem 1.1.1 (Cayley's Theorem)**.** *Every finitely generated group can be represented as a symmetry group of a connected, directed (every edge is an ordered pair of vertices), locally finite graph.*

For a proof of Cayley's Theorem we refer to [44, Section 1.5.2].

The graph we can associate to a group using Cayley's Theorem is called the *Cayley graph*.

Roughly speaking, the Cayley graph of a group G generated by S, is a graph $\Gamma(G, S)$ whose vertices are the elements of G and whose edges are labeled by some $s \in S$.

Throughout this work we assume that S is symmetric, i.e. if an element is contained in S then also its inverse is in S.

For more details and examples we refer once more to [44].

1.2 Introduction to Free Products of Groups

In this section we would like to give an overview on how free products arise in literature as purely algebraic objects, as well as fundamental groups of geometric structures.

For details we refer e.g. to [44, Chapter 3] or [15, Chapter 2].

1.2.1 Free Groups

Consider a group G generated by a finite set S.

A word $\omega \in G$ is said to be *freely reduced* if it does not contain two consecutive letters (elements of the generating set S) that are one the inverse of the other. For example, for $s, t \in S$, the word $\omega = sts^{-1}$ is freely reduced but $\omega' = ss^{-1}t$ is not.

The group G generated by S is a *free group* if all freely reduced words that are equivalent to the identity are trivial.

The cardinality of the set S is called *rank*, and the free group of rank $2n$ is denoted by $\mathbb{F}_n$.

The following fundamental criterion holds (see [44, Theorem 3.20]):

Theorem 1.2.1. *A group is free if and only if it acts freely on a tree.*

Remark 1.2.2. *Here is a reason why free groups are fundamental objects in algebra: if G is a group generated by n elements, then G is a quotient of $\mathbb{F}_n$ (see [44, Corollary 3.17] and [15, Chapter 2, Corollary 6]).*

1.2.2 Free Products

Let us consider $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ finitely generated groups, and denote their identity elements by $e_1, e_2, \ldots, e_m$ respectively. Their free product (we will denote it by Γ) is defined as the set of all finite words whose letters are elements of one of the groups, and two consecutive letters do not belong to the same group. In formulas we can write:

$$\begin{aligned}\Gamma := \Gamma_1 * \ldots * \Gamma_m = \big\{ & x_1 x_2 \ldots x_n \ : \ x_j \in \bigcup_{i=1}^{m} \Gamma_i \setminus \{e_i\}, \ j \in \{1, \ldots, n\}, \\ & \text{and } \ x_j \in \Gamma_i \Rightarrow x_{j+1} \notin \Gamma_i \big\} \cup \{e\}.\end{aligned} \tag{1.1}$$

The element e denotes the *empty word*. The group operation on Γ can be described as follows: if $u = u_1 \ldots u_m, v = v_1 \ldots v_n \in \Gamma$ then uv stands for their concatenation as words with possible contractions and cancellations in the middle.

It is clear that Γ_i embeds naturally into Γ, while e_i is identified with the empty word e in Γ.

We can look at these object also from a topological point of view: recall that given two topological spaces X (with a base point x_0) and Y (with a base point y_0), their *wedge sum* is defined as the "point union". This means it is the quotient of their disjoint union modulo the identification of $x_0 \in X$ with $y_0 \in Y$. Roughly speaking, we "attach" X to Y through one point.

In order to understand the connection between topology and free products of groups, we need another tool known as the *fundamental group*. Without going into technical details of its formal definition, we can think of it as a group associated to a topological space X and a base point $x \in X$. It gives a precise description of whether two paths in X can be continuously deformed into each other. The fundamental group is the first and simplest of the *homotopy groups*.

Example 1.2.3. *The wedge sum of two circles is shaped like the number "8", and analogously the wedge sum of k circles is shaped like a flower with k petals.*

This construction is related to free products, because the fundamental group of a wedge sum of k circles is a free group of degree k. Indeed, every petal is generated by only one element and there are no relations between the petals.

Example 1.2.4. *The fundamental group of the wedge sum of two circles is the free group $\mathbb{F}_2$.*

More in general, it is true that the fundamental group of any connected graph is free. (For more details and an idea of the proof, the interested reader can refer to e.g. [29, Section 1.2].)

Example 1.2.5. *For two positive, relatively prime integers m and n, we can consider the so called* torus knot *$K_{m,n}$, which is the image in $\mathbb{R}^3$ of the following map*

$$\begin{aligned} f \ &: S^1 \longrightarrow S^1 \times S^1 \\ z \ &\longmapsto (z^m, z^n), \end{aligned}$$

where S^1 denotes the unit circle in $\mathbb{R}^2$ and therefore $S^1 \times S^1$ denotes the torus in $\mathbb{R}^3$. It turns out (see e.g. [29, Example 1.24]) that the fundamental group of $\mathbb{R}^3 \setminus K_{m,n}$ is isomorphic to the free product $\mathbb{Z}_m * \mathbb{Z}_n$, being $\mathbb{Z}_j$ the cyclic group of order j.

For more detailed explanations and a better understanding of free products of groups arising as a consequence of the *Seifert – van Kampen Theorem*, the reader can refer e.g. to [42, Chapter 4]. For a more algebraic approach we refer to [14].

For completeness, we give a more abstract definition of a free product of groups in terms of a *universal property* (see e.g. [15, Chapter II]):

Universal Property : given $m \geq 2$ groups $\Gamma_1, \ldots, \Gamma_m$, and a family of homomorphisms $(h_j : \Gamma_j \to \Gamma)_{j=1,\ldots,m}$, where Γ is itself a group, then there exists a *unique* homomorphism $h : *_{j=1,\ldots,m}\Gamma_j \to \Gamma$ that extends $h_j : \Gamma_j \to \Gamma$.

1.2.3 Free Products with Amalgamation

Free products of groups are a particular case of the well-known *amalgamated products*: consider two groups Γ_1 and Γ_2 that have a common subgroup H. There are two homomorphisms $h_1 : H \to \Gamma_1$ and $h_2 : H \to \Gamma_2$. The amalgamated product is then defined as the free product $\Gamma_1 * \Gamma_2$, modulo the relation $h_1(a) = h_2(a)$, for all $a \in H$, and this product is denoted by

$$\Gamma_1 *_H \Gamma_2. \tag{1.2}$$

The universal property that characterizes the free product with amalgamation is as follows: for a group Γ and two homomorphisms $h_1 : H \to \Gamma_1$ and $h_2 : H \to \Gamma_2$ such that $h_1(a) = h_2(a)$, for all $a \in H$, there is a *unique* homomorphism $h : \Gamma_1 *_H \Gamma_2 \to \Gamma$ that extends h_1 and h_2. More details can be found in [15, Section III.14].

1.3 Introduction to Amenability

Amenability is a widely used concept, defined in several (equivalent) ways: we will give a brief description of a few different definitions.

1.3.1 Definition by von Neumann

The original definition was given by von Neumann. In his work [59], he was referring to amenable structures calling them *measurable*, here we quote his words:

> “Sei $\mathfrak{M}$ eine beliebige Menge, $\mathfrak{W}$ eine Teilmenge von $\mathfrak{M}$ und $\mathfrak{G}$ eine Gruppe eineindeutiger Abbildungen von $\mathfrak{M}$ auf sich selbst.
>
> Von einem allgemeinen nichtnegativen additiven und gegen alle Abbildungen aus $\mathfrak{G}$ invarianten Maß in $\mathfrak{M}$ das durch $\mathfrak{W}$ normiert ist kurz: einem $[\mathfrak{M}, \mathfrak{W}, \mathfrak{G}]$-Maß verlangen wir:
>
> Jeder Teilmenge $\mathfrak{N}$ von $\mathfrak{M}$ sei eine Zahl $\mu(\mathfrak{M}) \geq 0$ zugeordnet, derart dass
>
> α'. Wenn $\mathfrak{N}$ und $\mathfrak{P}$ elementfremd sind, so ist
> $$\mu(\mathfrak{N} + \mathfrak{P}) = \mu(\mathfrak{N}) + \mu(\mathfrak{P}).$$
> β'. Wenn σ zu $\mathfrak{G}$ gehört, so ist
> $$\mu(\mathfrak{N}) = \mu(\sigma\mathfrak{N}).$$
> γ'. Es ist
> $$\mu(\mathfrak{W}) = 1.$$
> Die Frage ist nunmehr offenbar: wie müssen $\mathfrak{M}, \mathfrak{W}$ und $\mathfrak{G}$ beschaffen sein, damit ein $[\mathfrak{M}, \mathfrak{W}, \mathfrak{G}]$-Maß existiert?”

Roughly speaking, he defined a discrete group $\mathfrak{M}$ to be amenable (here “measurable”) if there is a finitely additive probability measure defined on all subsets of $\mathfrak{M}$, which is invariant under left multiplication by elements of $\mathfrak{M}$.

1.3.2 Growth Functions and Følner Sequences

Consider a group Γ, generated by a finite symmetric set S. The *growth function* $\beta(\Gamma, S, k)$ is the number of vertices in Γ such that their distance from the origin is at most k. In other words, denoting by B_k the set of all these elements:

$$\beta(\Gamma, S; k) = \mathrm{Card}(B_k).$$

The corresponding *growth series* is defined as

$$B(\Gamma, S \mid z) := \sum_{k\geq 0} \beta(\Gamma, S; k) z^k.$$

Another fundamental tool is the so-called *spherical growth function*, defined as follows:

$$\sigma(\Gamma, S; k) := \beta(\Gamma, S; k) - \beta(\Gamma, S; k-1),$$

where of course $\sigma(\Gamma, S; 0) = \beta(\Gamma, S; 0) = 1$. Analogously, we can define the *spherical growth series*

$$\Sigma(\Gamma, S \mid z) := \sum_{x\in\Gamma} z^{l(x)} = \sum_{k\geq 0} \sigma(\Gamma, S; k) z^k.$$

For more details, we refer the reader to [15, Chapter VI].

Define the *S-boundary* of a subset $A \subset \Gamma$ as:

$$\partial_S A := \{y \notin A \text{ and } ys \in A \text{ for some } s \in S\}.$$

At this point we can define the *Følner sequence*: it is a sequence of finite subsets $(F_k)_{k\geq 1}$ s.t.

$$\lim_{k\to\infty} \frac{\mathrm{Card}(F_k \cup \partial_S F_k)}{\mathrm{Card}(F_k)} = 1.$$

A group is said to be *amenable* if it has a Følner sequence.

In [59], von Neumann showed that if a group contains a copy of $\mathbb{F}_2$, then it cannot be "measurable" (amenable). It follows that every free product of groups (except for $\mathbb{Z}_2 * \mathbb{Z}_2$) is non-amenable.

1.3.3 Isoperimetric Inequalities

Another definition of *amenability* comes from the following concept: the *isoperimetric number* of a group G, denoted by $\iota(G)$, is defined as

$$\iota(G) := \inf_{A\subset G} \frac{\mathrm{Card}(\partial_S A)}{\mathrm{Card}(A)},$$

where A denotes a finite set, and $\partial_S A$ denotes its boundary.

If $\iota(G) = 0$, then the group is amenable, while if $\iota(G) > 0$ the group is non-amenable.

In the setting described in [64, Section I.4], the *isoperimetric inequality* is presented on a *network*, i.e. a reversible (and irreducible) Markov chain. We denote the network by $\mathcal{N} = (X, E, r(\cdot))$, where X is a countable set, E the set of its edges, and $r(\cdot)$ a real function defined on E. We remark that in a more physical context $r(\{x, y\})$ is called the *resistance* of the edge $\{x, y\} \in E$.

Consider a function $f : X \to \mathbb{R}$, finitely supported. Its *Sobolev norm* is defined as follows:

$$S(f) := \frac{1}{2} \sum_{x,y\in X} \frac{|f(x) - f(y)|}{r(\{x, y\})}.$$

If $r(\cdot)$ is a symmetric function (i.e. $r(\{x,y\}) = r(\{y,x\})$), we can define

$$m(x) := \sum_{y\in X} \frac{1}{r(\{x,y\})},$$

moreover, if this function is positive and finite for every $x \in X$, then

$$p(x,y) := \frac{1}{r(\{x,y\})m(x)}$$

defines a *reversible* Markov chain on X, i.e. $m(x)p(x,y) = m(y)p(y,x)$ for all $x, y \in X$.

Remark 1.3.1. *The matrix* $P := \big(p(x,y)\big)_{x,y\in X}$ *is often called* transition matrix *of the Markov chain.*

Another norm can be considered for f, i.e. the norm in $\ell^p(X,m)$:

$$\|f\|_p := \left(\sum_{x\in X} |f(x)|^p m(x)\right)^{1/p},$$

whenever this sum converges.

At this point, fix a value $1 \le d \le \infty$. We can define the d-dimensional isoperimetric inequality (IS_d): let P denote the transition matrix described in Remark 1.3.1, then we say that (X,P) satisfies IS_d if and only if there exists a constant $\kappa > 0$ such that

$$\|f\|_{\frac{d}{d-1}} \le \kappa S(f).$$

It holds (see e.g. [64, Section II.10]) that

$$\mathrm{IS}_\infty \text{ is satisfied} \iff X \text{ is non-amenable.}$$

1.3.4 Spectral Radius and Amenability

Here we present a criterion for amenability due to Kesten (see [33]). This result will be used all throughout this work. At this point we just explain the main idea, for more details we refer to e.g. [64, Section II.12].

Let P denote the transition matrix of a (symmetric) reversible Markov chain (as mentioned in Remark 1.3.1), and let P^n be its n-th power. Define

$$\rho := \lim_{n\to\infty} \left(p^{(n)}(x,y)\right)^{\frac{1}{n}},$$

where $\left(p^{(n)}(x,y)\right)_{x,y\in X}$ are the entries of P^n. The quantity $\rho \le 1$ is called *spectral radius* of the Markov chain associated to P.

In [33] Kesten showed that

$$\rho < 1 \iff \mathrm{IS}_\infty \text{ is satisfied},$$

giving a criterion connecting a geometric property of a group with a symmetric random walk (Markov chain).

Chapter 2

Boundaries and Compactifications

In this chapter we recall the definitions of the *end compactification* and the *Martin compactification*.

2.1 End Compactification

Let Γ be a finitely generated group, then we recall the fundamental definitions, for further details we refer the reader to [64, Section 21].

(i) A *ray* is a semi-infinite, non-backtracking path $[x_0, x_1, x_2, \dots]$, i.e., $x_i \neq x_j$ if $i \neq j$. At this point, we would like to be able to distinguish rays ending up into different "zones" at infinity, therefore we introduce an equivalence relation. Two rays η_1 and η_2 are *equivalent* if and only if there is a third ray which shares infinitely many vertices with η_1 and η_2.

(ii) An equivalence class of rays is called *end*.

(iii) The set of equivalence classes of rays is called the *end boundary* of Γ, denoted by $\partial\Gamma$.

We would like to remark that from a wider (topological) point of view, the *end-compactification* is a particular case of the so-called *ℓ-TOP*, which was introduced and studied mainly by A. Georgakopoulos (see e.g. [21] for an introduction on the topic).

2.1.1 End Compactification of the Free Product

The graph $\mathcal{X}$ of Γ (free product of groups or free product with amalgamation) is a countable, connected, locally finite graph with a distinguished vertex e which we will refer to as *the root*. Γ is a finitely generated group: denote by S its generating set.

A *path* in Γ is a finite sequence of vertices $[x_0, x_1, \dots, x_n]$ such that there is an edge from x_{i-1} to x_i for each $i \in \{1, \dots, n\}$.

At this point we can naturally define two types of metrics on $\mathcal{X}$: the so-called *Cayley graph distance* and the *block length*.

W.r.t. the first, we denote by $l(u)$ the *length* of $u \in \Gamma$ relatively to e. In particular for every element $u \in \Gamma$ we have

$$l(u) := \min\{n \in \mathbb{N} : u = s_1^{k_1} s_2^{k_2} \dots s_j^{k_j},\ s_i \in S\ (\forall\, i = 1, \dots, j) \text{ and } \sum_{i=1}^{j} k_j = n\}.$$

We say that a *geodesic* of u is a shortest path from e to u. W.r.t. this metric, the geodesic is not necessarily unique (this fact will play an important role in Chapter 7).

Naïvely, we can think of $l(u)$ as the minimum amount of edges that we need to cross in order to connect e to u.

The second definition of distance comes naturally by looking at Equation (1.1): the block length of a word $u = u_1 \dots u_n \in \Gamma$, is given by

$$\|u\| = \|u_1 \dots u_n\| := n.$$

Since e represents the empty word, we define $\|e\| = 0$.

Later on we will investigate situations in which the graph length of an element differs drastically from its block length.

There are different types of ends occurring in the Cayley graph $\mathcal{X}$ of Γ: denote by

$$\Omega_i^{(0)} := \text{set of ends arising from } \mathcal{X}_i,$$

and by Ω_∞ the set of ends we will refer to as "infinite words", more precisely

$$\Omega_\infty := \left\{ x_1 x_2 x_3 \dots \in \left(\bigcup_{i \in \mathcal{I}} \Gamma_i \backslash \{e_i\} \right)^{\mathbb{N}} \mid x_j \in \Gamma_k \backslash \{e_k\} \Rightarrow x_{j+1} \notin \Gamma_k \setminus \{e_k\} \right\}.$$

For $\omega_i \in \Omega_i^{(0)}$, let $\eta = [e_i, y_1, y_2, \dots]$ be an element of the equivalence class ω_i and choose a geodesic $x := [x_0, x_1, \dots, x_n]$ from x_0 to x_n. Then, the ray $x\eta := [x_0, x_1, \dots, x_n, y_1, y_2, \dots]$ describes an end in Γ, denoted by $x\omega_i$.

For simplicity of notation, we set $\Omega_i := \{x\omega_i \mid x \in \Gamma, \omega_i \in \Omega_i^{(0)}\}$.

At this point, it is easy to see that Ω, i.e. the set of ends of $\mathcal{X}$, can be decomposed in the following way:

$$\Omega = \Omega_\infty \sqcup \Omega_1 \sqcup \Omega_2 \sqcup \dots \sqcup \Omega_m,$$

where $\sqcup$ denotes the disjoint union.

Observe that Ω_i is empty if and only if Γ_i is finite. Thus, if all factors Γ_i are finite, then $\Omega = \Omega_\infty$.

2.1.2 How to measure the Boundary

In order to estimate the size of Ω we need to define a metric on it. We say that an end $\omega_1 \in \Omega$ is *contained* in a connected component of $\mathcal{X}$ if all its representatives have all but finitely many vertices there.

By removing any finite subset $F \subseteq \mathcal{X}$ (including the edges connected to vertices in F), there is a unique connected component in the reduced graph $\mathcal{X} \setminus F$, containing a fixed end ω_1. We call this component the ω_1-component and say that ω_1 ends up in this component.

Denote by $B_m := \{x \in \Gamma \mid l(x) \leq m\}$ the ball centered at e with radius (w.r.t. the Cayley graph distance) $m \geq 0$.

Let us consider two different (i.e. non-equivalent) ends $\omega_1, \omega_2 \in \Omega$. Since they are not equivalent, there is a maximal $m \in \mathbb{N}_0$ such that ω_1 and ω_2 end up in the same connected component of $\mathcal{X} \setminus B_{m-1}$. We denote by $c(\omega_1, \omega_2)$ this maximal integer m. The metric on Ω that we will use is defined by

$$d_\Omega(\omega_1, \omega_2) := \alpha^{c(\omega_1, \omega_2)}, \tag{2.1}$$

where $\alpha \in (0, 1)$ is arbitrary, but fixed. Additionally, we set $d_\Omega(\omega_1, \omega_1) := 0$.

The ball $B(\omega, \varepsilon)$ centered at $\omega \in \Omega$ with radius $\varepsilon > 0$ is given by all elements $\hat{\omega} \in \Omega$ such that $d_\Omega(\omega, \hat{\omega}) \leq \varepsilon$. In other words, if $\varepsilon = \alpha^m$ then $\hat{\omega} \in B(\omega, \varepsilon)$ if and only if ω and $\hat{\omega}$ end up in the same component of $\mathcal{X} \setminus B_{m-1}$.

A *cover* of a subset $\Omega' \subseteq \Omega$ is a finite or countable set of balls of the form $B(\omega, \varepsilon)$ with $\omega \in \Omega'$ and $\varepsilon > 0$ such that the union of these balls contains Ω'.

For every $\omega \in \Omega'$ and $\varepsilon > 0$ let $N_\varepsilon(\Omega')$ be the minimal amount of balls of the form $B(\omega, \varepsilon)$ needed to cover Ω'. It is easy to see that $N_\varepsilon(\Omega')$ is bounded from above by the number of elements in Γ at graph distance $m = \lceil \log(\varepsilon)/\log(\alpha) \rceil$.

At this point it is natural to introduce the *lower* and *upper box-counting dimension* (also known as *Minkowski dimension*) of Ω', defined as

$$\underline{\mathrm{BD}}(\Omega') := \liminf_{\varepsilon \downarrow 0} \frac{\log N_\varepsilon(\Omega')}{-\log \varepsilon} \quad \text{and} \quad \overline{\mathrm{BD}}(\Omega') := \limsup_{\varepsilon \downarrow 0} \frac{\log N_\varepsilon(\Omega')}{-\log \varepsilon}. \tag{2.2}$$

If the two limits are equal, the common value is called *box-counting dimension* of Ω', denoted by $\mathrm{BD}(\Omega')$.

Another well-known tool to estimate the size of Ω' is given by the Hausdorff dimension, defined as a function of the Hausdorff measure. For $\delta > 0$, the *δ-dimensional Hausdorff measure* of Ω' is defined by

$$\mathcal{H}_\delta(\Omega') := \liminf_{\varepsilon \downarrow 0} \Big\{ \sum_i \varepsilon_i^\delta \;\Big|\; \big\{B(\cdot, \varepsilon_i)\big\}_i \text{ is the smallest cover of } \Omega' \text{ s.t. } \varepsilon_i < \varepsilon \Big\}.$$

Then the *Hausdorff dimension* of Ω' is defined as

$$\mathrm{HD}(\Omega') := \inf\big\{\delta \geq 0 \;\big|\; \mathcal{H}_\delta(\Omega') = 0\big\} = \sup\big\{\delta \geq 0 \;\big|\; \mathcal{H}_\delta(\Omega') = \infty\big\}. \tag{2.3}$$

Since $\mathcal{X}$ has bounded vertex degree, we have $\mathrm{HD}(\Omega') < \infty$. It is well-known that, for all $\Omega' \subseteq \Omega$,

$$\mathrm{HD}(\Omega') \leq \underline{\mathrm{BD}}(\Omega').$$

2.2 Martin Boundary

The aim of this section is to give a short introduction to the Martin Boundary: the interested reader can find more details and references in [64, Section 24].

The Martin boundary is an analytical object associated to a structure with a random walk defined on it.

Throughout the work, we consider different structures: here we give a general introduction, and we will recall these concepts later on, when needed.

Let us consider a finitely generated, non-amenable group Γ equipped with a random walk with n-step transition probabilities denoted by $p^{(n)}(x,y)$, for all $x, y \in \Gamma$. In the next sections it will be explained and made rigorous what we mean by this.

Since Γ is non-amenable, the power series

$$G(x,y|z) := \sum_{n\geq 0} p^{(n)}(x,y)z^n \tag{2.4}$$

has radius of convergence $R > 1$ (this is a consequence of Kesten's result, see Section 1.3.4), and this holds for every $x, y \in \Gamma$.

A function $h : \Gamma \to \mathbb{R}$ is said to be z-harmonic if

$$h(x) = z \sum_{y\in\Gamma} p(x,y)h(y)$$

for all $x \in \Gamma$. We work in the space of all positive z-harmonic functions, and in order to do this we need to assume $0 < z \leq R$ (see e.g. [64, Section 24] and references therein). This abstract space is completely described by the Martin boundary, which we introduce in the following.

For $0 < z \leq R$ we set $t := 1/z$ and we define the *Martin kernel* as:

$$K(x,y|t) := \frac{G(x,y|z)}{G(e,y|z)}, \tag{2.5}$$

where $G(x,y|z)$ is defined in (2.4) and e denotes the identity element of the group Γ.

The *Martin Compactification* $\hat{\Gamma}_z$ of Γ (this depends not only on Γ, but also on z and on μ), is the unique smallest compactification to which all kernels $K(x,\cdot|t)$ extend continuously.

The *Martin Boundary* is $\mathcal{M}_z := \hat{\Gamma}_z \setminus \Gamma$.

Remark 2.2.1. *As explained in [64, Section 24], the term "smallest" refers to the partial order of compactifications, where* id_Γ *extends to a continuous surjection from the larger to the smaller compactification. Equality means in this contest that the two compactifications are homeomorphic.*

For more details the interested reader is referred to [62], [64, Section 24] and references therein.

For various interesting results (that go beyond the aims of this work) about the Martin boundary of nearest neighbor random walks on trees and non-amenable graphs, we refer to [48] and [49].

For more detailed explanations and direct computations for Martin boundaries of Cartesian products, we refer to [51] and [50]. In particular, we would like to mention that in [51, Corollary 4.3] the Martin boundary of $T_a \times T_b$ (the Cartesian product of two homogeneous trees of degrees a and b respectively) is computed explicitely.

Part I

Random Walks on Free Products

Chapter 3

Construction of a Random Walk on a Free Product

In this chapter we describe how to define a random walk on a free product, and this concept will be used as well in the context of branching random walks (Chapter 7).

3.1 Main Definitions and Visualization of a Free Product

Let us introduce some notation that will be used all throughout the first two parts of our work; for more details we refer to [64]. Let Γ be a finitely generated group with identity e (the group operation is written multiplicatively) and generating set S, and fix a probability measure μ such that $\mathrm{supp}(\mu) = S$.

The random walk on Γ governed by μ is the Markov chain with state space Γ and transition probabilities given by $p(x, y) = \mu(x^{-1}y)$ for $x, y \in \Gamma$. Therefore the random walk starting at $x \in \Gamma$ can be written as

$$X_n = x\eta_1 \cdots \eta_n, \quad n \geq 0,$$

where η_j is a sequence of iid random variables with common distribution μ. The law of X_n is the n-th convolution power $\mu^{(n)}$ of μ, and if not mentioned otherwise the random walk starts at the group identity e. For every two elements x and y of Γ, we denote by

$$p^{(n)}(x, y) := \mathbb{P}[X_n = y | X_0 = x] = \mu^{(n)}(x^{-1}y)$$

the probability to go from x to y in n steps. Furthermore, we always assume the random walk to be irreducible, i.e., for all x, y there exists a $k \in \mathbb{N}$ such that $p^{(k)}(x, y) > 0$.

We say that μ is *symmetric* if $\mu(x) = \mu(x^{-1})$ for all $x \in \Gamma$.

Given a finite set of integers $\mathcal{I} := \{1, 2, \ldots, r\}$, where $r \geq 2$, consider r finitely generated groups $\Gamma_1, \ldots, \Gamma_r$. Each of these groups has a presentation of the form $\Gamma_i = \langle S_i \mid R_i \rangle$, where S_i is a symmetric generating set, and R_i is the set of relations. The free product (refer to Equation 1.1) can be written as

$$\Gamma := \Gamma_1 * \ldots * \Gamma_r = \langle S_1, \ldots, S_r \mid R_1, \ldots, R_r \rangle.$$

We exclude the cases where Γ_i is the trivial group and the case $r = 2$ with $\mathrm{Card}(\Gamma_1) = \mathrm{Card}(\Gamma_2) = 2$.

From now on, in order to simplify the notation, we define $\Gamma_i^{\times} := \Gamma_i \setminus \{e_i\}$, for every $i \in \mathcal{I}$. Hence, the free product Γ defined in Equation (1.1) can be written as

$$\Gamma = \left\{ x_1 x_2 \ldots x_n \;\middle|\; n \in \mathbb{N}, x_j \in \bigcup_{i \in \mathcal{I}} \Gamma_i^{\times}, x_j \in \Gamma_k^{\times} \Rightarrow x_{j+1} \notin \Gamma_k^{\times} \right\} \cup \{e\}.$$

We associate to each group its Cayley graph with respect to the finite generating set S. In this context we will be more precise than in Section 1.1.2.

The Cayley graph $\mathcal{X} = \mathcal{X}(\Gamma, S)$ has vertex set $V(\mathcal{X}) = \Gamma$, and the edge $\{x, y\}$ is an element of $E(\mathcal{X})$ if and only if $x^{-1}y \in S$.

From now on, $\mathcal{X}$ will denote the Cayley graph of the free product. Its construction is as follows: consider the Cayley graphs $\mathcal{X}_1, \ldots, \mathcal{X}_r$ of the factors $\Gamma_1, \ldots, \Gamma_r$ respectively, w.r.t. the (finite) symmetric generating sets $S_1, \ldots, S_r$. Take copies of $\mathcal{X}_1, \ldots, \mathcal{X}_r$ and glue them together at their identities to one single common vertex, which becomes the representation of the empty word e. Inductively, at each vertex $v = v_1 \ldots v_k$ with $v_k \in \Gamma_i$ attach a copy of every $\mathcal{X}_j$, $j \neq i$, identifying v with the identity e_j of the new copy of $\mathcal{X}_j$.

Example 3.1.1. *According to the previous construction, the Cayley graph of the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ looks like the one in Figure 3.1.*

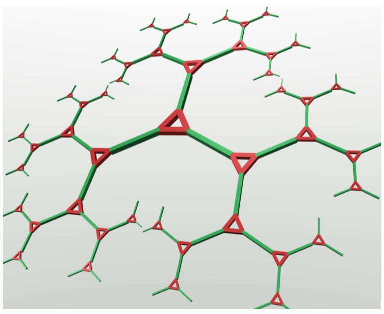

Figure 3.1: Cayley graph of the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$.

3.2 Definition of a Random Walk on Γ

Assume that on every free factor Γ_i, $i \in \mathcal{I}$, we are given a symmetric probability measure μ_i defined on the (finite) set of generators S_i. In other words, we can consider a random walk defined on each factor Γ_i governed by μ_i.

The most natural way to construct a random walk on Γ, is to start from the ones defined on its free factors, and to make a convex combination out of them. Each of these random walks is irreducible, and for sake of simplicity, we assume $\mu_i(e_i) = 0$ for every $i \in \mathcal{I}$.

In order to fix the notation, the single-step transition probability of the RW defined on Γ_i is denoted by $p_i(x,y) := \mu_i(x^{-1}y)$, (for all $x, y \in \Gamma_i$), and the n-step transition probability is denoted by $p_i^{(n)}(x,y) := \mu_i^{(n)}(x^{-1}y)$.

We lift μ_i to a probability measure $\bar{\mu}_i$ on Γ by defining:

$$\bar{\mu}_i(x) := \begin{cases} \mu_i(x) & \text{if } x \in \Gamma_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now let us fix positive real numbers $\alpha_1, \ldots, \alpha_r$ such that $\sum_{i \in \mathcal{I}} \alpha_i = 1$. We construct a probability measure defined on $S = S_1 \cup \ldots \cup S_r$ (which is the generating set of Γ) by a convex combination of the $\bar{\mu}_i$'s, i.e.

$$\mu := \sum_{i \in \mathcal{I}} \alpha_i \bar{\mu}_i. \tag{3.1}$$

The random walk on Γ starting at e and governed by μ, is hence defined as follows: for $x, y \in \Gamma$, the associated single and n-step transition probabilities are given by $p(x,y) := \mu(x^{-1}y)$ and $p^{(n)}(x,y) := \mu^{(n)}(x^{-1}y)$ respectively.

Remark 3.2.1. *Intuitively, the coefficients α_i's can be thought as "weights": α_i is the weight of the measure μ_i relatively to μ. If a α_j is very large (very close to 1), then it seems plausible that the random walk governed by μ will behave like the one defined on Γ_j. Later on (see Chapter 6) we will see that this heuristic explanation lies at the basis of one of our main results.*

3.3 Main Tool: Green Function

The aim of the first part of this work is to find the asymptotic behavior of the non-exponential part of $\mu^{(n)}(e)$. In order to achieve our results, we investigate the so-called *Green Functions*, which are defined through series: the type of their singularity contains fundamental information that we can exploit for our purposes.

First of all, we should introduce some notation.
For a function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ such that $f(z_0) = 0$, for $z_0 \in D$, $0 < q \in \mathbb{R}$ and $k \in \mathbb{N}_0$, we write:

$$f(z) = \mathbf{o}\big((z_0 - z)^q \log^k(z_0 - z)\big) \quad \text{if} \quad \lim_{z \to z_0} \frac{f(z)}{(z_0 - z)^q \log^k(z_0 - z)} = 0;$$

$$f(z) = \mathbf{O}_c\big((z_0 - z)^q \log^k(z_0 - z)\big) \quad \text{if} \quad 0 < \lim_{z \to z_0} \frac{f(z)}{(z_0 - z)^q \log^k(z_0 - z)} < \infty;$$

$$f(z) = \mathbf{O}\big((z_0 - z)^q \log^k(z_0 - z)\big) \quad \text{if} \quad \limsup_{z \to z_0} \frac{f(z)}{(z_0 - z)^q \log^k(z_0 - z)} < \infty.$$

Furthermore, we introduce the *order relation* $\preceq$, we write

$$(z_0 - z)^{q_1} \log^{k_1}(z_0 - z) \preceq (z_0 - z)^{q_2} \log^{k_2}(z_0 - z)$$

if $(z_0-z)^{q_2}\log^{k_2}(z_0-z)=\mathbf{O}\big((z_0-z)^{q_1}\log^{k_1}(z_0-z)\big)$.
What the value z_0 represents, will be clear from the context.

For $z\in\mathbb{C}$, the *Green functions* associated to the random walks on Γ_i and Γ are defined as

$$G_i(z):=\sum_{n=0}^{\infty}\mu_i^{(n)}(e_i)z^n \quad \text{and} \quad G(z):=\sum_{n=0}^{\infty}\mu^{(n)}(e)z^n,$$

respectively.
The corresponding radii of convergence are denoted by $\mathbf{R}_i$ and $\mathbf{R}$ respectively. According to Pringsheim's Theorem, these values are the smallest singularities of the above defined functions.

We would like to point out (recall Section 1.3.4) that $\mathbf{R}>1$, since Γ is non-amenable (see e.g. [64, Corollary 12.5], recalling that $\mathbf{R}$ is the inverse of the spectral radius of the random walk). In the following we assume that $G_i(z)$ is exactly d_i-times differentiable at $z=\mathbf{R}_i$, for some non-negative integer d_i.

The next assumption will be fundamental:

Assumption 3.3.1. *Whenever $G_i'(\mathbf{R}_i)<\infty$, the expansions of the Green functions $G_i(z)$ in a neighborhood of $z=\mathbf{R}_i$ have the form*

$$G_i(z)=\sum_{k=0}^{d_i}g_k^{(i)}(\mathbf{R}_i-z)^k+\sum_{(q,k)\in\mathcal{T}_i}g_{(q,k)}^{(i)}(\mathbf{R}_i-z)^q\log^k(\mathbf{R}_i-z)+\mathbf{O}\big((\mathbf{R}_i-z)^{d_i+2}\big),$$

where $\mathcal{T}_i$ is a finite subset of $\{(q,k)\in\mathbb{R}\times\mathbb{N}_0 \mid d_i<q\leq d_i+2\}$. In other words, up to order $(\mathbf{R}_i-z)^{d_i+2}$, the only "admissible" singular terms are of logarithmic and algebraic type.

Remark 3.3.2. *Higher order terms are not necessary for the computation of the non-exponential type of the n-step return probabilities of the random walk on Γ.*

Remark 3.3.3. *Let us also emphasize that, in the case $G_i'(\mathbf{R}_i)=\infty$, we do not need any assumptions on the singularity type.*

In the following we want to motivate this assumption on $G_i(z)$: this property is satisfied in several well-known cases, e.g., the Green functions of nearest neighbor random walks on the d-dimensional lattice $\mathbb{Z}^d$ have such an expansion, see Proposition 5.3.1.

With some effort, it can be proved that also $\mathbb{Z}^d\times(\mathbb{Z}/n\mathbb{Z})$ satisfies Assumption (3.3.1) by the same methods used for $\mathbb{Z}^d$. Moreover we will prove our main result by induction on the number r of free factors of Γ: we will show that Assumption (3.3.1) is stable under free products, except for some degenerate cases (see Chapter 5).

3.4 More Generating Functions

In the following we look at free products of the form $\Gamma_1*\Gamma_2$, while free products with more than two factors are discussed in Section 5.2.

Let us consider $z \in \mathbb{C}$, $i \in \{1,2\}$ and $s_i \in \mathrm{supp}(\mu_i)$. For all $s \in \mathrm{supp}(\mu) = \mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_2)$ we can define the *first visit generating functions* as follows:

$$\begin{aligned} F_i\big(s_i\big|z\big) &:= \sum_{n\geq 0} \mathbb{P}\big[X_n^{(i)} = e_i, \forall m < n : X_m^{(i)} \neq e_i \,\big|\, X_0^{(i)} = s_i\big]\, z^n, \\ F\big(s\big|z\big) &:= \sum_{n\geq 0} \mathbb{P}\big[X_n = e, \forall m < n : X_m \neq e \,\big|\, X_0 = s\big]\, z^n, \end{aligned} \tag{3.2}$$

where $\big(X_n^{(i)}\big)_{n\in\mathbb{N}_0}$ denotes the random walk on Γ_i governed by μ_i. By conditioning on the number of visits of e_i the functions $F_i\big(s_i\big|z\big)$ are directly linked with $G_i(z)$ via

$$G_i(z) = \frac{1}{1 - \sum_{s_i \in \mathrm{supp}(\mu_i)} \mu_i(s_i)\, z\, F_i\big(s_i\big|z\big)}. \tag{3.3}$$

In the following we summarize some important basic facts, we will refer to Woess [64] for further details. We will make a wide use of the following functions:

$$\begin{aligned} \zeta_1(z) &:= \frac{\alpha_1 z}{1 - \alpha_2 z \sum_{s_2 \in \mathrm{supp}(\mu_2)} \mu_2(s_2) F(s_2|z)}, \\ \zeta_2(z) &:= \frac{\alpha_2 z}{1 - \alpha_1 z \sum_{s_1 \in \mathrm{supp}(\mu_1)} \mu_1(s_1) F(s_1|z)}. \end{aligned} \tag{3.4}$$

Remark 3.4.1. *$\zeta_i(1)$ is the probability that the process (starting at e) makes a step from e into Γ_i within a finite time.*

Remark 3.4.2. *For $s_i \in \mathrm{supp}(\mu_i)$ we have $F\big(s_i\big|z\big) = F_i\big(s_i\big|\zeta_i(z)\big)$. The interested reader is referred to [64, Proposition 9.18c)] for more details.*

By [64, Equation (9.20)] and (3.3), the functions $F_i\big(s_i\big|\zeta_i(z)\big)$, $G_i(z)$ and $G(z)$ satisfy the following relations:

$$\alpha_i\, zG(z) = \zeta_i(z) G_i\big(\zeta_i(z)\big) = \frac{\zeta_i(z)}{1 - \sum_{s_i \in \mathrm{supp}(\mu_i)} \mu_i(s_i)\, \zeta_i(z)\, F_i\big(s_i\big|\zeta_i(z)\big)}. \tag{3.5}$$

Hence, in order to get a singular expansion for $G(z)$ in a neighborhood of $z = \mathbf{R}$, we need to expand $\zeta_i(z)$.

We recall that (see [64, Proposition 9.10]) there are functions Φ_i, $i \in \{1,2\}$, and Φ with the following properties:

$$G_i(z) = \Phi_i\big(zG_i(z)\big) \quad\text{and}\quad G(z) = \Phi\big(zG(z)\big) \tag{3.6}$$

for all $z \in \mathbb{C}$ in an open neighborhood of the intervals $[0, \mathbf{R}_i)$ and $[0, \mathbf{R})$ respectively. In particular, denoting by

$$\theta_i := \mathbf{R}_i G_i(\mathbf{R}_i), \qquad \text{and} \qquad \theta := \mathbf{R} G(\mathbf{R}),$$

the functions Φ_i and Φ are analytic in an open neighborhood of the intervals $[0, \theta_i)$ and $[0, \theta)$, strictly increasing and strictly convex in $[0, \theta_i)$ and $[0, \theta)$ respectively.

In order to proceed with our discussion, we need to define a few further functions, which will play a fundamental role in the next part of the work:

$$\Psi_i(t) := \Phi_i(t) - t\Phi_i'(t) \quad \text{and} \quad \Psi(t) := \Phi(t) - t\Phi'(t). \tag{3.7}$$

It turns out that (a proof of this fact can be found in [64, Theorem 9.19]) the following relations hold:

$$\Phi(t) = \Phi_1(\alpha_1 t) + \Phi_2(\alpha_2 t) - 1 \quad \text{and} \quad \Psi(t) = \Psi_1(\alpha_1 t) + \Psi_2(\alpha_2 t) - 1. \tag{3.8}$$

For simplicity of notation, we write $\Psi_i(\theta_i) := \lim_{t\to\theta_i-} \Psi_i(t)$. Moreover, define

$$\bar{\theta} := \min\left\{\frac{\theta_1}{\alpha_1}, \frac{\theta_2}{\alpha_2}\right\},$$

then we will write $\Psi(\bar{\theta}) := \lim_{t\to\bar{\theta}-} \Psi(t)$ as well.

Despite the fact that we should be very careful not to create confusion between the quantities $\bar{\theta}$ and θ, we will see in the next chapter that, in the situations we are interested in, they coincide (see e.g. [64, Theorem 9.22]).

Chapter 4

Case Distinction

In the next sections, we will make a case distinction according to the finiteness of $G_i(\mathbf{R}_i)$ and $G_i'(\mathbf{R}_i)$, as well as to the sign of $\Psi(\bar{\theta})$. We will prove our results separately for each situation.

4.1 Case $\Psi(\bar{\theta}) < 0$

Let δ denote the period of the random walk. In this case it is known that (see e.g. [64, Theorem 17.3]) the n-step return probabilities of the random walk on Γ behave asymptotically like

$$\mu^{(n\delta)}(e) \sim C \cdot \mathbf{R}^{-n\delta} \cdot n^{-3/2}.$$

Moreover the Green function of the random walk on Γ has the form (see e.g. [64, Proposition 17.4] or [19, Section VI.7]).

$$G(z) = A(z) + \sqrt{\mathbf{R} - z}\, B(z), \tag{4.1}$$

where $A(z)$ and $B(z)$ are analytic functions in a neighborhood of $z = \mathbf{R}$, and moreover $B(\mathbf{R}) \neq 0$.

As usual, let us denote by S_i a finite, symmetric set of generators for Γ_i, $i \in \{1, 2\}$. If each S_i contains at least one element of order larger than 2, then μ_1, μ_2 and α_1 can always be chosen in a suitable way in order to obtain $\Psi(\bar{\theta}) < 0$, provided that $\mathrm{supp}(\mu_i) = S_i$. A proof of this fact can be found in [10] and [64, Corollary 17.10].

Example 4.1.1. *A particular case where the Green function is of the form (4.1) is the free product $\Gamma_1 * \Gamma_2$ for finite groups Γ_1 and Γ_2, see [63], but the easiest example we can think of, is the homogeneous tree.*

Motivated by Example (4.1.1), we assume from now on that at least one out of Γ_1 and Γ_2 is infinite, and we may restrict our investigation to the cases $\Psi(\bar{\theta}) > 0$ and $\Psi(\bar{\theta}) = 0$.

4.2 Case $\Psi(\bar{\theta}) > 0$

We start by remarking some important facts about the case $\Psi(\bar{\theta}) \geq 0$.

We have that $\theta = \bar{\theta}$ and $G(\mathbf{R}) < \infty$, see [64, Theorem 9.22]. Furthermore, by [64, Lemma 17.1.a)] it holds that $\zeta_i(\mathbf{R}) \leq \mathbf{R}_i$ for $i \in \{1, 2\}$, with equality if and only if $\theta = \theta_i/\alpha_i$.

Throughout this chapter we assume that $G_1(z)$ and $G_2(z)$ are differentiable $d_i \geq 1$ times at their radii of convergence, and they satisfy Assumption (3.3.1).

In this setting we can apply the well-known *method of Darboux* (the reader who is not familiar with this method, can get a flavor of the idea by reading Appendix A, Section A.1). This yields that the n-step return probabilities of the random walk on Γ_i behave asymptotically like the coefficients of the Taylor expansion of the leading singular term in (3.3.1) in a neighborhood of $z = \mathbf{R}$.

Denote by $\mathscr{S}_i(z) := (\mathbf{R}_i - z)^{q_i} \log^{k_i}(\mathbf{R}_i - z)$ the *leading* singular term (i.e. the smallest term of the singular expansion w.r.t. $\preceq$) i.e., $q > q_i$ or $(q = q_i \wedge k < k_i)$ for all $(q, k) \in \mathcal{T}_i \setminus \{(q_i, k_i)\}$, then the coefficients of $\mathscr{S}_i(z)$ in a neighborhood of $z = \mathbf{R}$ behave asymptotically like the n-step return probabilities on Γ_i. More precisely, their behavior is asymptotically of type $\hat{C}_i \mathbf{R}_i^{-n\delta_i} n^{-\lambda_i} \log^{\kappa_i}(n)$, where

$$\delta_i := \gcd\big\{n \in \mathbb{N} \mid \mu_i^{(n)}(e_i) > 0\big\}$$

is the period of the random walk on Γ_i and

$$\lambda_i := q_i + 1 \quad \text{and} \quad \kappa_i := \begin{cases} k_i, & \text{if } q_i \notin \mathbb{N}, \\ k_i - 1 & \text{if } q_i \in \mathbb{N}. \end{cases} \tag{4.2}$$

Analogously, $\delta := \gcd\big\{n \in \mathbb{N} \mid \mu^{(n)}(e) > 0\big\} = \gcd\{\delta_1, \ldots, \delta_m\}$ is the period of the random walk on Γ. For more details on the asymptotic behavior of the coefficients for the expansion of $(\mathbf{R}_i - z)^{q_i} \log^{k_i}(\mathbf{R}_i - z)$ in a neighborhood of $z = \mathbf{R}$, see e.g. Flajolet and Sedgewick [19, Chapter VI.2].

The method of Darboux needs some differentiability assumptions at $z = \mathbf{R}_i$; therefore, we need the expansions of $G_i(z)$ up to terms of order $(\mathbf{R}_i - z)^{d_i+2}$.

We point out that another – modern – tool to handle singular expansions as in (3.3.1) is *Singularity Analysis*, introduced by Flajolet and Odlyzko in [18]. For a brief explanation on this method, we invite the reader to take a look at Appendix A, Section A.2. However, in our context it turns out that the verification of the specific requirements of this method is quite cumbersome as one can also see in the work by Lalley [34].

The aim of this section is to prove the following:

Theorem 4.2.1. *Assume that $G_1(z)$ and $G_2(z)$ are differentiable at $z = \mathbf{R}_1$ and $z = \mathbf{R}_2$ respectively, and satisfy Assumption (3.3.1). If $\mathscr{S}_1(z) \preceq \mathscr{S}_2(z)$ and $\Psi(\bar{\theta}) > 0$ then:*

$$\mu^{(n\delta)}(e) \sim \begin{cases} C_1 \cdot \mathbf{R}^{-n\delta} \cdot n^{-\lambda_1} \cdot \log^{\kappa_1}(n), & \text{if } \alpha_1 \geq \frac{\theta_1}{\theta_1+\theta_2}, \\ C_2 \cdot \mathbf{R}^{-n\delta} \cdot n^{-\lambda_2} \cdot \log^{\kappa_2}(n), & \text{if } \alpha_1 < \frac{\theta_1}{\theta_1+\theta_2}, \end{cases}$$

for some constants $C_1, C_2 > 0$.

Recall from Remark 3.4.2 that $F(s_i|z) = F_i(s_i|\zeta_i(z))$ for all $s_i \in \mathrm{supp}(\mu_i)$. Then we rewrite (3.4) as follows:

$$\alpha_1 z = \zeta_1(z)\Big(1 - \alpha_2 z \sum_{s_2 \in \mathrm{supp}(\mu_2)} \mu_2(s_2) F_2\big(s_2|\zeta_2(z)\big)\Big), \tag{4.3}$$

$$\alpha_2 z = \zeta_2(z)\Big(1 - \alpha_1 z \sum_{s_1 \in \mathrm{supp}(\mu_1)} \mu_1(s_1) F_1\big(s_1|\zeta_1(z)\big)\Big). \tag{4.4}$$

In the following we assume w.l.o.g. that $\theta = \bar{\theta} = \theta_1/\alpha_1$, therefore according to what mentioned at the beginning of the section, $\zeta_1(\mathbf{R}) = \mathbf{R}_1$ and $\zeta_2(\mathbf{R}) \leq \mathbf{R}_2$, with equality if and only if $\theta = \theta_1/\alpha_1 = \theta_2/\alpha_2$.

Remark 4.2.2. $\Psi(\bar{\theta}) > 0$ *implies $G'(\mathbf{R}) < \infty$: since $\Phi'(\bar{\theta}) < \Phi(\bar{\theta})/\bar{\theta} = 1/\mathbf{R}$, by differentiating (3.6) we get*

$$G'(\mathbf{R}) = \lim_{z \to \mathbf{R}} \frac{\Phi'\big(zG(z)\big)\, G(z)}{1 - z\,\Phi'\big(zG(z)\big)} = \frac{\Phi'(\bar{\theta})\, G(\mathbf{R})}{1 - \mathbf{R}\,\Phi'(\bar{\theta})} < \infty.$$

To make the notation more clear, define

$$D := \begin{cases} d_1, & \text{if } \bar{\theta} < \theta_2/\alpha_2, \\ \min\{d_1, d_2\}, & \text{if } \bar{\theta} = \theta_1/\alpha_1 = \theta_2/\alpha_2. \end{cases} \tag{4.5}$$

Denoting by $\mathscr{S}(z)$ the *main leading singular term*, we have

$$\mathscr{S}(z) = \begin{cases} \mathscr{S}_1(z), & \text{if } \bar{\theta} < \theta_2/\alpha_2, \\ \min\big\{\mathscr{S}_1(z), \mathscr{S}_2(z)\big\}, & \text{if } \bar{\theta} = \theta_2/\alpha_2. \end{cases}$$

The next lemma shows that if $G(z)$ is differentiable at its radius of convergence, then the functions $\zeta_1(z)$ and $\zeta_2(z)$ are as well.

Lemma 4.2.3. $0 < \zeta_1'(\mathbf{R}) < \infty$ *and* $0 < \zeta_2'(\mathbf{R}) < \infty$.

Proof. We prove the result only for $\zeta_1'(\mathbf{R})$, since the proof for $\zeta_2'(\mathbf{R})$ is completely analogous. We write

$$H_2(z) := \alpha_2 z \sum_{s_2 \in \mathrm{supp}(\mu_2)} \mu_2(s_2) F_2\big(s_2|\zeta_2(z)\big).$$

Since $\zeta_1(\mathbf{R}) = \mathbf{R}_1$, we have $H_2(\mathbf{R}) < 1$; compare with the definition of $\zeta_1(z)$. Furthermore, the coefficient of z^n in $H_2(z)$ is just the probability that the random walk on Γ (starting at e) makes the first step w.r.t. μ_2 and returns for the first time to e at time n. Thus, this probability is bounded from above by $\mu^{(n)}(e)$, and consequently $H_2'(\mathbf{R}) < G'(\mathbf{R}) < \infty$. Computing the derivative of $\zeta_1(z)$ in a neighborhood of $z = \mathbf{R}$ gives

$$\zeta_1'(z) = \frac{\alpha_1\big(1 - H_2(z)\big) + \alpha_1 z H_2'(z)}{\big(1 - H_2(z)\big)^2} > 0.$$

Finiteness of $\zeta_1'(\mathbf{R})$ follows directly from the remarks above. $\square$

Under Assumption (3.3.1), the functions $F_i(s_i|z)$, for $i \in \{1,2\}$ and $s_i \in \mathrm{supp}(\mu_i)$, are at least d_i-times differentiable at $z = \mathbf{R}_i$, therefore we can compare the n-th coefficients of $F_i(s_i|z)$ and $G_i(z)$ as follows:

$$\mu_i^{(n)}(e_i) \geq \mu_i(s_i) \cdot \mathbb{P}\big[X_n^{(i)} = e_i, \forall m < n : X_m^{(i)} \neq e_i \,\big|\, X_0^{(i)} = s_i\big].$$

Thus, we can rewrite these functions in the form

$$F_i(s_i|z) = \sum_{n=0}^{d_i} f_n(s_i)(\mathbf{R}_i - z)^n + E^{(i)}(s_i|z), \tag{4.6}$$

where the coefficients $f_n(s_i)$ are real numbers, and $E^{(i)}(s_i|z) = \mathbf{o}\big((\mathbf{R}_i - z)^{d_i}\big)$. If we have $\zeta_2(\mathbf{R}) < \mathbf{R}_2$, then $F_2(s_2|z)$ is analytic at $z = \zeta_2(\mathbf{R})$ for all $s_2 \in \mathrm{supp}(\mu_2)$ and we can even write

$$F_2(s_2|z) = \sum_{n \geq 0} f_n(s_2)\big(\zeta_2(\mathbf{R}) - z\big)^n.$$

Our first aim is to find out the right order of $E^{(i)}(s_i|z)$. An intermediate result that will be useful to get to this goal, is Lemma 4.2.4. It shows that between order d_i and $d_i + 2$ in the expansion of $\sum_{s_i \in \mathrm{supp}(\mu_i)} \mu_i(s_i) z E^{(i)}(s_i|z)$, we can have only finitely many singular terms.

Lemma 4.2.4. *For $z \in \mathbb{C}$ in a neighborhood of $\mathbf{R}_i$,*

$$\begin{aligned} \sum_{s_i \in \mathrm{supp}(\mu_i)} \mu_i(s_i)\, z\, E^{(i)}(s_i|z) &= e^{(i)}_{(q_i,k_i)}(\mathbf{R}_i - z)^{q_i} \log^{k_i}(\mathbf{R}_i - z) + \\ &\quad + \sum_{(q,k) \in \widehat{\mathcal{T}}_i} e^{(i)}_{(q,k)}(\mathbf{R}_i - z)^q \log^k(\mathbf{R}_i - z) + \mathbf{O}\big((\mathbf{R}_i - z)^{d_i+2}\big), \end{aligned}$$

where $e^{(i)}_{(q_i,k_i)} \neq 0$ and $\widehat{\mathcal{T}}_i$ is a finite subset of

$$\big\{(q,k) \in \mathbb{R} \times \mathbb{N}_0 \mid d_i < q \leq d_i + 2, q > q_i \text{ or } (q = q_i \Rightarrow k < k_i)\big\}.$$

Proof. Define the *first return generating function*

$$U_i(z) := \sum_{s_i \in \mathrm{supp}(\mu_i)} \mu_i(s_i)\, z\, F_i(s_i|z). \tag{4.7}$$

The expansions of $U_i(z)$ and $G_i(z)$ have the same leading singular term since both functions are d_i-times differentiable in a neighborhood of $z = \mathbf{R}_i$. This can be seen very clearly with the help of the well-known (see [64, Lemma 1.13]) relation

$$G_i(z) = 1/\big(1 - U_i(z)\big).$$

Therefore, we have expansions

$$G_i(z) = \sum_{k=0}^{d_i} g_k^{(i)}(\mathbf{R}_i - z)^k + R_{G_i}(z) \quad \text{and} \quad U_i(z) = \sum_{k=0}^{d_i} u_k^{(i)}(\mathbf{R}_i - z)^k + R_{U_i}(z),$$

where $R_{G_i}(z) = \mathbf{O}_c\big(S_i(z)\big)$ and $R_{U_i}(z) = \mathbf{o}\big((\mathbf{R}_i - z)^{d_i}\big)$. Plugging these expansions into $G_i(z)(1 - U_i(z)) = 1$, and taking all polynomial terms to one side, we get

$$\big(1 - U_i(\mathbf{R}_i)\big)\, R_{G_i}(z) - G_i(\mathbf{R}_i)\, R_{U_i(z)} = p(z) + \mathbf{o}\big((\mathbf{R}_i - z)^{d_i+1}\big),$$

where $p(z)$ is a polynomial. This equation implies that the right hand side is of order $\mathbf{O}\big((\mathbf{R}_i - z)^{d_i+1}\big)$, i.e. $R_{U_i(z)} = \mathbf{O}_c\big(\mathscr{S}_i(z)\big)$ and we can write

$$U_i(z) = \sum_{k=0}^{d_i} u_k^{(i)} (\mathbf{R}_i - z)^k + u^{(i)}_{(q_i,k_i)} S_i(z) + \widehat{R}_{U_i}(z),$$

where $\widehat{R}_{U_i}(z) = \mathbf{o}\big(S_i(z)\big)$. Plugging this expansion into $G_i(z)(1 - U_i(z)) = 1$, comparing the error terms and iterating the last steps, together with using (4.6) in (4.7), yields the claim. $\square$

Recall the definition of D from Equation 4.5. The next goal is to show that both $\zeta_1(z)$ and $\zeta_2(z)$ are D times differentiable in a neighborhood of $z = \mathbf{R}$.

Proposition 4.2.5. *There are real numbers $x_0, x_1, \dots, x_D$ and $y_0, y_1, \dots, y_D$ such that*

$$\zeta_1(z) = \sum_{k=0}^{D} x_k\, (\mathbf{R} - z)^k + X_D^{(1)}(z) \quad \textit{and} \quad \zeta_2(z) = \sum_{k=0}^{D} y_k\, (\mathbf{R} - z)^k + X_D^{(2)}(z),$$

where $X_D^{(1)}(z) = \mathbf{o}\big((\mathbf{R} - z)^D\big)$ and $X_D^{(2)}(z) = \mathbf{o}\big((\mathbf{R} - z)^D\big)$.

Proof. Our strategy is to determine $x_0, x_1, \dots, x_D$ and $y_0, y_1, \dots, y_D$ inductively. By Lemma 4.2.3 we can rewrite $\zeta_1(z)$ and $\zeta_2(z)$ as follows:

$$\begin{aligned} \zeta_1(z) &= \mathbf{R}_1 - \zeta_1'(\mathbf{R})\,(\mathbf{R} - z) + X_1^{(1)}(z), \text{ where } X_1^{(1)}(z) = \mathbf{o}(\mathbf{R} - z), \\ \zeta_2(z) &= \zeta_2(\mathbf{R}) - \zeta_2'(\mathbf{R})\,(\mathbf{R} - z) + X_1^{(2)}(z), \text{ where } X_1^{(2)}(z) = \mathbf{o}(\mathbf{R} - z). \end{aligned} \tag{4.8}$$

Thus, we have determined x_0, x_1 and y_0, y_1. Now assume that for some $t < D$ we can write

$$\zeta_1(z) = \sum_{k=0}^{t} x_k\, (\mathbf{R}-z)^k + X_t^{(1)}(z) \;\text{ and }\; \zeta_2(z) = \sum_{k=0}^{t} y_k\, (\mathbf{R}-z)^k + X_t^{(2)}(z), \tag{4.9}$$

where $X_t^{(1)}(z) = \mathbf{o}\big((\mathbf{R} - z)^t\big)$ and $X_t^{(2)}(z) = \mathbf{o}\big((\mathbf{R} - z)^t\big)$.

By (4.6) we have

$$\begin{aligned} F_1(s_1|z) &= \textstyle\sum_{n=0}^{D} a_n(s_1)(\mathbf{R}_1 - z)^n + E^{(1)}(s_1|z) \;\text{ and} \\ F_2(s_2|z) &= \textstyle\sum_{n=0}^{D} b_n(s_2)\big(\zeta_2(\mathbf{R}) - z\big)^n + E^{(2)}(s_2|z), \end{aligned} \tag{4.10}$$

where $E^{(i)}(s_i|z) = \mathbf{o}\big((\zeta_i(\mathbf{R}) - z)^D\big)$. We use the expansions (4.9) and (4.10) and plug them into Equations (4.3) and (4.4), and obtain the following system:

$$\begin{aligned}
\alpha_1 z &= \Big(\sum_{k=0}^{t} x_k\,(\mathbf{R}-z)^k + X_t^{(1)}(z)\Big)\Bigg[1-\alpha_2\big(\mathbf{R}-(\mathbf{R}-z)\big)\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)\cdot\\
&\quad\cdot\Big[\sum_{n=0}^{D} b_n(s_2)\Big(-\sum_{k=1}^{t} y_k\,(\mathbf{R}-z)^k - X_t^{(2)}(z)\Big)^n + E^{(2)}\big(s_2\big|\zeta_2(z)\big)\Big]\Bigg],\\
\alpha_2 z &= \Big(\sum_{k=0}^{t} y_k\,(\mathbf{R}-z)^k + X_t^{(2)}(z)\Big)\Bigg[1-\alpha_1\big(\mathbf{R}-(\mathbf{R}-z)\big)\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)\cdot\\
&\quad\cdot\Big[\sum_{n=0}^{D} a_n(s_1)\Big(-\sum_{k=1}^{t} x_k\,(\mathbf{R}-z)^k - X_t^{(1)}(z)\Big)^n + E^{(1)}\big(s_1\big|\zeta_1(z)\big)\Big]\Bigg].
\end{aligned} \tag{4.11}$$

We bring all polynomial and higher order terms to one hand side: by comparison, we see that a convex sum of $X_t^{(1)}(z)$ and $X_t^{(2)}(z)$ is of the desired order $\mathbf{O}\big((\mathbf{R}-z)^{t+1}\big)$:

$$\begin{aligned}
P_t^{(1)}(z) + \mathbf{o}\big((\mathbf{R}-z)^{t+1}\big) &= \Big[1-\alpha_2\mathbf{R}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_0(s_2)\Big]X_t^{(1)}(z)\\
&\quad+\Big[\alpha_2\mathbf{R}_1\mathbf{R}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2)\Big]X_t^{(2)}(z),\\
P_t^{(2)}(z) + \mathbf{o}\big((\mathbf{R}-z)^{t+1}\big) &= \Big[\alpha_1\zeta_2(\mathbf{R})\mathbf{R}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_1(s_1)\Big]X_t^{(1)}(z)\\
&\quad+\Big[1-\alpha_1\mathbf{R}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_0(s_1)\Big]X_t^{(2)}(z),
\end{aligned} \tag{4.12}$$

where $P_t^{(1)}(z)$ and $P_t^{(2)}(z)$ are polynomials in the variable z. By assumption on $X_t^{(1)}(z)$ and $X_t^{(2)}(z)$, the right hand sides of (4.12) are of order $\mathbf{o}\big((\mathbf{R}-z)^t\big)$. Therefore, the left hand sides have to be of order $\mathbf{O}_c\big((\mathbf{R}-z)^{t+1}\big)$, and consequently the right hand sides are also of order $\mathbf{O}_c\big((\mathbf{R}-z)^{t+1}\big)$.

It still remains to show that both $X_t^{(1)}(z)$ and $X_t^{(2)}(z)$ are $\mathbf{O}_c\big((\mathbf{R}-z)^{t+1}\big)$. For this purpose, we show that the coefficients of the convex sum are linearly independent proving that the matrix of the coefficients has non-zero determinant.

Define the matrix $M = (m_{ij})_{1\le i,j\le 2}$ by

$$\begin{aligned}
m_{11} &:= 1-\alpha_2\mathbf{R}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_0(s_2),\\
m_{12} &:= \alpha_2\mathbf{R}_1\mathbf{R}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2),\\
m_{21} &:= \alpha_1\zeta_2(\mathbf{R})\mathbf{R}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_1(s_1),\\
m_{22} &:= 1-\alpha_1\mathbf{R}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_0(s_1).
\end{aligned}$$

Then the system (4.12) is equivalent to

$$M \cdot \begin{pmatrix} X_t^{(1)}(z) \\ X_t^{(2)}(z) \end{pmatrix} = \begin{pmatrix} Q_t^{(1)}(z) \\ Q_t^{(2)}(z) \end{pmatrix},$$

where $Q_t^{(1)}(z) = \mathbf{O}_c\big((\mathbf{R}-z)^{t+1}\big)$ and $Q_t^{(2)}(z) = \mathbf{O}_c\big((\mathbf{R}-z)^{t+1}\big)$. If the matrix M is invertible, then obviously $X_t^{(1)}(z) = \mathbf{O}_c\big((\mathbf{R}-z)^{t+1}\big)$ and $X_t^{(2)}(z) = \mathbf{O}_c\big((\mathbf{R}-z)^{t+1}\big)$.

Therefore we prove that M is indeed invertible: the last part of the proof is devoted to show that $\det(M) \neq 0$.

We start by differentiating Equations (4.3) and (4.4) in the variable z:

$$\begin{aligned}\alpha_1 = \Big(&-\alpha_2 \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2)F_2\big(s_2|\zeta_2(z)\big) - \alpha_2 z \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2)F_2'\big(s_2|\zeta_2(z)\big)\zeta_2'(z)\Big)\zeta_1(z)\\ &+ \zeta_1'(z)\Big(1-\alpha_2 z \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2)F_2\big(s_2|\zeta_2(z)\big)\Big),\end{aligned}$$

$$\begin{aligned}\alpha_2 = \Big(&-\alpha_1 \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1)F_1\big(s_1|\zeta_1(z)\big) - \alpha_1 z \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1)F_1'\big(s_1|\zeta_1(z)\big)\zeta_1'(z)\Big)\zeta_2(z)\\ &+ \zeta_2'(z)\Big(1-\alpha_1 z \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1)F_1\big(s_1|\zeta_1(z)\big)\Big).\end{aligned}$$

We would like to point out that in the expansions given by relations (4.10) we have:

$$\begin{aligned} &a_0(s_1) = F_1(s_1|\mathbf{R}_1); \quad a_1(s_1) = -F_1'(s_1|\mathbf{R}_1)\\ &b_0(s_2) = F_2\big(s_2|\zeta_2(\mathbf{R})\big); \quad b_1(s_2) = -F_2'\big(s_2|\zeta_2(\mathbf{R})\big).\end{aligned}$$

Substituting these values in the above system and letting $z \to \mathbf{R}$ yields

$$\begin{aligned}\alpha_1 =&\Big(-\alpha_2 \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2)b_0(s_2) + \alpha_2\mathbf{R} \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2)b_1(s_2)\zeta_2'(\mathbf{R})\Big)\mathbf{R}_1\\ &+ \zeta_1'(\mathbf{R})m_{11},\end{aligned}$$

$$\begin{aligned}\alpha_2 =&\Big(-\alpha_1 \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1)a_0(s_1) + \alpha_1\mathbf{R} \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1)a_1(s_1)\zeta_1'(\mathbf{R})\Big)\zeta_2(\mathbf{R})\\ &+ \zeta_2'(\mathbf{R})m_{22}.\end{aligned}$$

Since $\zeta_1(\mathbf{R}), \zeta_2(\mathbf{R}) > 0$ and $a_1(s_1), b_1(s_2) < 0$, the last equations imply that both $m_{11}, m_{22} > 0$. We proceed by rewriting the last system:

$$\begin{aligned} \alpha_2\mathbf{R}_1\mathbf{R}\textstyle\sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2)b_1(s_2)\zeta_2'(\mathbf{R}) &= A - \zeta_1'(\mathbf{R})m_{11},\\ \alpha_1\zeta_2(\mathbf{R})\mathbf{R}\textstyle\sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1)a_1(s_1)\zeta_1'(\mathbf{R}) &= B - \zeta_2'(\mathbf{R})m_{22},\end{aligned} \tag{4.13}$$

where we set

$$\begin{aligned} A &:= \alpha_1 + \alpha_2\mathbf{R}_1 \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2)b_0(s_2)\\ B &:= \alpha_2 + \alpha_1\zeta_2(\mathbf{R}) \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1)a_0(s_1).\end{aligned}$$

Multiplying side by side equations in (4.13) yields

$$\zeta_1'(\mathbf{R})\zeta_2'(\mathbf{R})m_{12}m_{21} = AB - \zeta_1'(\mathbf{R})m_{11}B - \zeta_2'(\mathbf{R})m_{22}A + \zeta_1'(\mathbf{R})\zeta_2'(\mathbf{R})m_{11}m_{22}.$$

The condition $\det(M) = 0$ would imply

$$\zeta_1'(\mathbf{R})m_{11}B + \zeta_2'(\mathbf{R})m_{22}A = AB,$$

or equivalently,

$$\zeta_2'(\mathbf{R}) = \frac{AB - \zeta_1'(\mathbf{R})m_{11}B}{m_{22}A}. \tag{4.14}$$

Furthermore, (4.13) implies

$$\zeta_1'(\mathbf{R}) = \big(A - C\zeta_2'(\mathbf{R})\big)/m_{11},$$

where $C := \alpha_2\mathbf{R}_1\mathbf{R}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2) < 0$. Plugging the last relation into (4.14) leads to

$$\zeta_2'(\mathbf{R}) = \frac{BC}{m_{22}A}\zeta_2'(\mathbf{R}).$$

Observe now that $A, B, m_{22} > 0$ and $C < 0$. This yields a contradiction in the last equation, since $\zeta_2'(\mathbf{R}) > 0$. Thus, $\det(M) \neq 0$.

At this point we proved that both $X_t^{(1)}(z)$ and $X_t^{(2)}(z)$ are $\mathbf{O}_c\big((\mathbf{R}-z)^{t+1}\big)$: in this way we obtain inductively the values $x_0, x_1, \dots, x_D$ and $y_0, y_1, \dots, y_D$. $\square$

The next aim is to show that at least one of the functions $X_D^{(1)}(z)$ and $X_D^{(2)}(z)$ has order $\mathbf{O}_c\big((\mathbf{R}-z)^{q_i}\log^{k_i}(\mathbf{R}-z)\big)$. To this end, we look at the final step of the induction in the proof of Proposition 4.2.5. For $t = D$, the system (4.11) becomes

$$\begin{aligned}
&\Big[1 - \alpha_2\mathbf{R}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_0(s_2)\Big]\cdot X_D^{(1)}(z) + \Big[\alpha_2\mathbf{R}_1\mathbf{R}\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)b_1(s_2)\Big]\cdot X_D^{(2)}(z)\\
&\quad - \alpha_2\mathbf{R}_1\sum_{s_2\in\mathrm{supp}(\mu_2)}\mu_2(s_2)\,z\,E^{(2)}\big(s_2|\zeta_2(z)\big) = P_D^{(1)}(z) + \mathbf{o}\big((\mathbf{R}-z)^{D+1}\big),\\
&\Big[\alpha_1\zeta_2(\mathbf{R})\mathbf{R}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_1(s_1)\Big]\cdot X_D^{(1)}(z) + \Big[1 - \alpha_1\mathbf{R}\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)a_0(s_1)\Big]\cdot X_D^{(2)}(z)\\
&\quad - \alpha_1\zeta_2(\mathbf{R})\sum_{s_1\in\mathrm{supp}(\mu_1)}\mu_1(s_1)\,z\,E^{(1)}\big(s_1|\zeta_1(z)\big) = P_D^{(2)}(z) + \mathbf{o}\big((\mathbf{R}-z)^{D+1}\big),
\end{aligned}$$

where $P_D^{(1)}(z)$ and $P_D^{(2)}(z)$ are polynomials in the variable z. By (4.8), we may conclude that $\big(\zeta_i(\mathbf{R}) - \zeta_i(z)\big) = \mathbf{O}_c(\mathbf{R}-z)$. Since $\zeta_i'(\mathbf{R}_i) < \infty$ by Lemma 4.2.3, we have for $1 < p \in \mathbb{R}$

$$\begin{aligned}
\big(\zeta_i(\mathbf{R}) - \zeta_i(z)\big)^p &= \big(\zeta_i'(\mathbf{R}_i)(\mathbf{R}-z) + \mathbf{o}(\mathbf{R}-z)\big)^p\\
&= \zeta_i'(\mathbf{R}_i)^p\,(\mathbf{R}-z)^p\,\big(1+\mathbf{o}(1)\big)^p\\
&= \mathbf{O}_c\big((\mathbf{R}-z)^p\big)
\end{aligned}$$

and

$$\begin{aligned}
\log\big(\zeta_i(\mathbf{R}) - \zeta_i(z)\big) &= \log\big(\zeta_i'(\mathbf{R}_i)(\mathbf{R}-z) + \mathbf{o}(\mathbf{R}-z)\big)\\
&= \log\big(\zeta_i'(\mathbf{R}_i)\big) + \log(\mathbf{R}-z) + \log\big(1+\mathbf{o}(1)\big)\\
&= \log\big(\zeta_i'(\mathbf{R}_i)\big) + \log(\mathbf{R}-z) + \mathbf{o}(1).
\end{aligned}$$

We remark that $(1+z)^p$ and $\log(1+z)$ are analytic functions in a neighborhood of $z=0$.

In the following we denote by $\iota \in \{1,2\}$ the index such that $\mathscr{S}(z) = \mathscr{S}_\iota(z)$. Then the computations above, together with Lemma 4.2.4, imply that

$$\sum_{s_\iota \in \mathrm{supp}(\mu_\iota)} \mu(s_\iota)\, z\, E^{(\iota)}\big(s_\iota | \zeta_\iota(z)\big) = \mathbf{O}_c\big((\mathbf{R}-z)^{q_\iota} \log^{k_\iota}(\mathbf{R}-z)\big).$$

With an analogous reasoning to the one that allowed us to finish the proof of Proposition 4.2.5, we can conclude that

$$\begin{aligned} X_D^{(1)}(z) &= \mathbf{O}_c\big((\mathbf{R}-z)^{q_\iota} \log^{k_\iota}(\mathbf{R}-z)\big), \\ X_D^{(2)}(z) &= \mathbf{O}_c\big((\mathbf{R}-z)^{q_\iota} \log^{k_\iota}(\mathbf{R}-z)\big). \end{aligned}$$

Thus, what we have obtained, can be summarized as follows: the leading singular term of $\zeta_\iota(z)$ has the same order as the leading singular term in the expansion of $G_\iota(z)$ if $\mathscr{S}(z) = \mathscr{S}_\iota(z)$. Using (3.5), we conclude that the leading singular term in the expansion of $G(z)$ at $z=\mathbf{R}$ has the same form as the leading singular term in the expansion of $G_\iota(z)$ at $z=\mathbf{R}_\iota$, namely $(\mathbf{R}-z)^{q_\iota} \log^{k_\iota}(\mathbf{R}-z)$.

Recall that we assumed throughout this section that $G_i(z)$ is d_i times differentiable at $z=\mathbf{R}_i$. For an application of *Darboux's method* we need in a first step the expansion of $G(z)$ in a neighborhood of $z=\mathbf{R}$ up to terms of order $(\mathbf{R}-z)^{D+2}$. Thus, by (3.5), we have to extend the expansions of $\zeta_1(z)$ and $\zeta_2(z)$ up to terms of order $(\mathbf{R}-z)^{D+2}$.
We present a result that is analogous to Lemma 4.2.4: the next lemma ensures that there are only finitely many singular terms in the expansions of $\zeta_1(z)$ and $\zeta_2(z)$ up to order $D+2$.

Lemma 4.2.6. *For $i \in \{1,2\}$, $\zeta_i(z)$ has an expansion of the form*

$$\sum_{k=0}^{D} x_k (\mathbf{R}-z)^k + \sum_{(q,k)\in\mathcal{T}} x_{(q,k)} (\mathbf{R}-z)^q \log^k(\mathbf{R}-z) + \mathbf{o}\big((\mathbf{R}-z)^{D+2}\big),$$

where $x_k, x_{(q,k)} \in \mathbb{R}$, $\mathcal{T}$ is a finite subset of

$$\widehat{\mathcal{T}} := \big\{(q,k) \in \mathbb{R}\times\mathbb{N}_0 \mid D < q \le D+2\big\}.$$

In particular, if $(q_i,k_i) \in \mathcal{T}$ with $x_{(q_i,k_i)} \neq 0$ and $(q,k)\in\mathcal{T}$, then we have $(q_i,k_i) \preceq (q,k)$.

Proof. Recall the expansion of $\sum_{s_i\in\mathrm{supp}(\mu_i)} \mu_i(s_i)\, z\, E^{(i)}(s_i|z)$ from Lemma 4.2.4. Assume that $\zeta_i(z)$ has already an expansion of the form

$$\sum_{k=0}^{D} x_k (\mathbf{R}-z)^k + \sum_{(q,k)\in\mathcal{T}'} x_{(q,k)} (\mathbf{R}-z)^q \log^k(\mathbf{R}-z) + \mathbf{o}(\max\mathcal{T}'), \tag{4.15}$$

where $\mathcal{T}'$ is a finite subset of $\widehat{\mathcal{T}}$ and

$$\max\mathcal{T}' := \max_{\preceq}\big\{(\mathbf{R}-z)^q \log^k(\mathbf{R}-z) \mid (q,k)\in\mathcal{T}'\big\}.$$

In particular, $x_{(q_i,k_i)} \in \mathcal{T}'$, and $x_{(q_i,k_i)} \neq 0$. We proceed with expanding the next terms of $\zeta_i(z)$ analogously to the proof of Proposition 4.2.5. For this purpose, observe that for $p > 1$

$$\big(\zeta_i(\mathbf{R}) - \zeta_i(z)\big)^p = (-x_1)^p\,(\mathbf{R}-z)^p\,\Bigg(1 + \sum_{k=2}^{D} \frac{x_k}{x_1}(\mathbf{R}-z)^{k-1} + \\ + \sum_{(q,k)\in\mathcal{T}'} \frac{x_{(q,k)}}{x_1}(\mathbf{R}-z)^{q-1}\log^k(\mathbf{R}-z) + \mathbf{o}\Big(\frac{\max\mathcal{T}'}{\mathbf{R}-z}\Big)\Bigg)^p. \tag{4.16}$$

Analogously we have that

$$\log\big(\zeta_i(\mathbf{R}) - \zeta_i(z)\big) = C + \log(\mathbf{R}-z) + \log\Bigg(1 + \sum_{k=2}^{D} \frac{x_k}{x_1}(\mathbf{R}-z)^{k-1} + \\ + \sum_{(q,k)\in\mathcal{T}'} \frac{x_{(q,k)}}{x_1}(\mathbf{R}-z)^{q-1}\log^k(\mathbf{R}-z) + \mathbf{o}\Big(\frac{\max\mathcal{T}'}{\mathbf{R}-z}\Big)\Bigg). \tag{4.17}$$

We plug relations (4.15), (4.16) and (4.17) into Equation (4.6) and then everything into Equations (4.3) and (4.4). Finally we compare again the error terms (we will repeat this procedure in each of the following steps). Therefore, if $\max\mathcal{T}' = (\mathbf{R}-z)^{\hat{q}}\log^{\hat{k}}(\mathbf{R}-z)$ then the next possible terms up to order $(\mathbf{R}-z)^{\hat{q}}$ in the expansion may only be

$$(\mathbf{R}-z)^{\hat{q}}\log^{\hat{k}-1}(\mathbf{R}-z), (\mathbf{R}-z)^{\hat{q}}\log^{\hat{k}-2}(\mathbf{R}-z), \dots, (\mathbf{R}-z)^{\hat{q}}.$$

Analogously to the proof of Proposition 4.2.5 we determine step by step the corresponding coefficients of these terms. The next term in the expansion of $\zeta_i(z)$ has now the form $(\mathbf{R}-z)^{\check{q}}\log^{\check{k}}(\mathbf{R}-z)$, where $\check{q} > \hat{q}$ is a sum of elements from the finite set

$$\big\{1, q, q-1 \mid (q,\cdot) \in \mathcal{T}_1 \cup \mathcal{T}_2\big\}$$

where $\mathcal{T}_i$ has the properties described in Assumption (3.3.1). The value of $\check{q}$ is minimal such that $\check{q} > \hat{q}$. By relations (4.16) and (4.17) there is a maximal $\check{k} \in \mathbb{N}_0$ such that $(\mathbf{R}-z)^{\check{q}}\log^{\check{k}}(\mathbf{R}-z)$ may be a non-vanishing next term in the expansion of $\zeta_i(z)$. Thus, we may iterate the last few steps again. Since there are only finitely many possible values for q such that a term of the form $(\mathbf{R}-z)^q\log^k(\mathbf{R}-z)$ may appear in the expansion up to order $(D+2)$, we have shown the claim. $\square$

With the help of the last lemma we are now able to prove Theorem 4.2.1:

Proof of Theorem 4.2.1. We start by expanding $\zeta_1(z)$ and $\zeta_2(z)$ as in Proposition 4.2.5. We have three possibilities:

1. $\alpha_1 > \theta_1/(\theta_1+\theta_2)$, implying $\bar{\theta} = \theta_1/\alpha_1 < \theta_2/\alpha_2$. Moreover $\zeta_1(\mathbf{R}) = \mathbf{R}_1$ and $\zeta_2(\mathbf{R}) < \mathbf{R}_2$.
   Consequently the leading singular term in the expansion of $\zeta_1(z)$ (and $\zeta_2(z)$) is of the same type as $\mathscr{S}_1(z) = (\mathbf{R}-z)^{q_1}\log^{k_1}(\mathbf{R}-z)$.

2. $\bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1$, implying $\zeta_2(\mathbf{R}) = \mathbf{R}_2$ and $\zeta_1(\mathbf{R}) < \mathbf{R}_1$.
   Therefore the leading singular term is $\mathscr{S}_2(z) = (\mathbf{R}-z)^{q_2}\log^{k_2}(\mathbf{R}-z)$.

3. $\alpha_1 = \theta_1/(\theta_1+\theta_2)$ implies $\bar{\theta} = \theta_1/\alpha_1 = \theta_2/\alpha_2$, therefore $\zeta_1(\mathbf{R}) = \mathbf{R}_1$ and $\zeta_2(\mathbf{R}) = \mathbf{R}_2$.
   In this case the leading singular term in the expansions of $\zeta_1(z)$ and $\zeta_2(z)$ is $\mathscr{S}_j(z) = (\mathbf{R}-z)^{q_j}\log^{k_j}(\mathbf{R}-z)$, where $j=1$, if $\mathscr{S}_1(z) \preceq \mathscr{S}_2(z)$, and $j=2$, if $\mathscr{S}_2(z) \prec \mathscr{S}_1(z)$.

Like we did in Proposition 4.2.5, for the rest of the proof we denote by $\iota \in \{1,2\}$ the index such that $\mathscr{S}(z) = \mathscr{S}_\iota(z)$. Therefore, the expansion of the common leading singular term of $\zeta_1(z)$ and $\zeta_2(z)$, namely $\mathscr{S}_\iota(z)$, in a neighborhood of $z=\mathbf{R}$ has coefficients of asymptotic order proportional to $\mathbf{R}^{-n}n^{-\lambda_\iota}\log^{\kappa_\iota} n$.

We will use *Darboux's method*, described in Appendix A, Section A.1. Briefly, the key of the method is the *Riemann-Lebesgue-Lemma*. It states that if a function $H(z) = \sum_{n\geq 0} h_n z^n$ has radius of convergence $\mathbf{R}_H$ and if H is k times continuously differentiable on its circle of convergence, then $h_n \mathbf{R}_H^n n^k \to 0$ as $n \to \infty$. Thus, one identifies all singularities into one (which is $z=\mathbf{R}$) and subtracts parts of the expansion in a neighborhood of $z=\mathbf{R}$, such that the remaining part is sufficiently often differentiable on the circle of convergence. Therefore the asymptotics of the coefficients h_n arise from the main leading singular term of the singular expansion. We refer e.g. to Olver [46, Chapter 8, §9.2] for more details.

Lemma 4.2.6 assures that we have a singular expansion of $\zeta_1(z)$ up to terms of order $\lceil\lambda_\iota\rceil = \lceil q_\iota\rceil + 1 = D+2$, which allows us to apply Darboux's method: we get the asymptotic behavior of $\mu^{(n\delta)}(e)$ by plugging $\zeta_1(z)$ into Equation (3.5). Thus, the expansions of $G(z)$ and of $\zeta_1(z)$ have leading singular term of the same type, namely $(\mathbf{R}-z)^{q_\iota}\log^{k_\iota}(\mathbf{R}-z)$.

We still need to show that the expansion of $G(z)$ at every singular point on the disc of convergence has the same form. The singularities are exactly the points $\mathbf{R}\exp(\mathrm{i}2\pi j/\delta)$ with $0 \leq j < \delta$; see e.g. [64, Theorem 9.4]. Writing $z = \lambda\mathbf{R}\omega_j$, where $\omega_j = \exp(\mathrm{i}2\pi j/\delta)$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$,

$$\begin{aligned} G(z) = G(\lambda\mathbf{R}\omega_j) &= \sum_{n\geq 0} \mu^{(n\delta)}(e)(\lambda\mathbf{R}\omega_j)^{n\delta} \\ &= \sum_{n\geq 0} \mu^{(n\delta)}(e)(\lambda\mathbf{R})^{n\delta} = G(\lambda\mathbf{R}) = G(z/\omega_j). \end{aligned}$$

Thus, for every $j \in \{0,1,\dots,\delta-1\}$, we can expand $G(z)$ in a neighborhood of $z = \mathbf{R}\omega_j$ as follows:

$$\begin{aligned} G(z) = &\sum_{k=0}^{D} g_k(\mathbf{R}-z/\omega_j)^k + \\ &+ \sum_{(q,k)\in\widehat{\mathcal{T}}_\iota} g_{(q,k)}(\mathbf{R}-z/\omega_j)^q \log^k(\mathbf{R}-z/\omega_j) + \mathbf{O}\big((\mathbf{R}\omega_j - z)^{D+2}\big), \end{aligned}$$

where again (like in Lemma 4.2.6) the set $\widehat{\mathcal{T}}_\iota$ is a finite subset of

$$\{(q,k)\in\mathbb{R}\times\mathbb{N} \mid D < q \leq D+2, q > q_\iota \vee (q = q_\iota \Rightarrow k < k_\iota)\},$$

$g_{(q_\iota,k_\iota)} \in \widehat{\mathcal{T}}_\iota$ where $g_{(q_\iota,k_\iota)} \neq 0$ and $(q,k) \in \widehat{\mathcal{T}}_\iota$ implies $(q_\iota,k_\iota) \preceq (q,k)$. Therefore, the difference

$$G(z) - \sum_{j=0}^{\delta-1} \sum_{(q,k)\in\widehat{\mathcal{T}}_\iota} g_{(q,k)}(\mathbf{R} - z/\omega_j)^q \log^k(\mathbf{R} - z/\omega_j)$$

is $(D+2)$-times differentiable on the circle of convergence. Observe now that the coefficients of the expansion of $(\mathbf{R} - z/\omega_j)^{q_\iota} \log^{k_\iota}(\mathbf{R} - z/\omega_j)$ in a neighborhood of 0 behave asymptotically like $C\,(\mathbf{R}\omega_j)^{-n}\, n^{-\lambda_\iota} \log^{\kappa_\iota}(n)$. We can drop higher order terms in the above difference because the corresponding coefficients have higher asymptotic order. Since $G(z) = \sum_{n\geq 0} \mu^{(n)} z^n$, we can conclude that

$$\mu^{(n)}(e) \sim \sum_{j=0}^{\delta-1} C\, n^{-\lambda_\iota} \log^{\kappa_\iota}(n)\, \mathbf{R}^{-n}\, \omega_j^{-n}.$$

Observe that $\sum_{j=0}^{\delta-1} \omega_j^{-n} = \delta$ if δ divides n, and this sum is zero otherwise.

We remark that the asymptotic behavior of the coefficients in the singular expansion of $s^{q_\iota} \log^{k_\iota}(s)$ near $s = 0$ are well-known; for more details we refer to e.g. Flajolet and Sedgewick [19]. □

4.3 The Case $\Psi(\bar\theta) = 0$

In this section (and *only* in this section) we work under the following assumption:

Assumption 4.3.1. *for $i = \{1,2\}$ the quantities $G_i(\zeta_i(\mathbf{R}))$ and $G_i'(\zeta_i(\mathbf{R}))$ are finite.*

The cases that do not satisfy Assumption 4.3.1 will be treated separately in Chapter 5.

Analogously to what we did in Section 4.2, we can assume $\theta = \bar\theta = \theta_1/\alpha_1$. The aim of this section is to prove the following:

Theorem 4.3.2. *Under Assumption 4.3.1, if $\Psi(\bar\theta) = 0$ we have*

$$\mu^{(n\delta)}(e) \sim C \cdot \mathbf{R}^{-n\delta} \cdot n^{-3/2}.$$

In the following we will derive expansions of $\zeta_i(z)$ and $G(z)$ in a neighborhood of $z = \mathbf{R}$ in order to prove Theorem 4.3.2. Recall from (3.7) that $\Psi(\bar\theta) = 0$ implies

$$\Phi'(\bar\theta) = \frac{\Phi(\bar\theta)}{\bar\theta} = \frac{\Phi(\theta)}{\theta} = \frac{\Phi\big(\mathbf{R}\,G(\mathbf{R})\big)}{\mathbf{R}\,G(\mathbf{R})} = \frac{G(\mathbf{R})}{\mathbf{R}\,G(\mathbf{R})} = \frac{1}{\mathbf{R}}.$$

Differentiating (3.6) yields

$$G'(z) = \frac{G(z)\Phi'\big(zG(z)\big)}{1 - z\Phi'\big(zG(z)\big)}. \tag{4.18}$$

Therefore, in this setting we have

$$G'(\mathbf{R}) = \infty, \tag{4.19}$$

and consequently we have to proceed differently from the previous section in order to find the expansion of $G(z)$.

A few important facts that we want to point out are the following. We can rewrite the functions $\zeta_1(z)$ and $\zeta_2(z)$ as

$$\zeta_1(z) = \mathbf{R}_1 + X_1(z), \quad \text{and} \quad \zeta_2(z) = \zeta_2(\mathbf{R}) + X_2(z), \tag{4.20}$$

where $X_1(\mathbf{R}) = X_2(\mathbf{R}) = 0$. This, together with Equation 4.19 tells us that

$$X_1(x), X_2(z) \neq \mathbf{O}((\mathbf{R} - z)). \tag{4.21}$$

Moreover, for $i \in \{1, 2\}$,

$$G_i\big(\zeta_i(z)\big) = G_i\big(\zeta_i(\mathbf{R})\big) - G_i'\big(\zeta_i(\mathbf{R})\big)\big(-X_i(z)\big) + \mathbf{o}\big(X_i(z)\big). \tag{4.22}$$

Substituting Equations (4.20) and (4.22) into (3.5) we get:

$$G(z) = \frac{1}{\alpha_i z}\left(\zeta_i(\mathbf{R}) + X_i(z)\right)\left(G_i\big(\zeta_i(\mathbf{R})\big) - G_i'\big(\zeta_i(\mathbf{R})\big)\big(-X_i(z)\big) + \mathbf{o}\big(X_i(z)\big)\right).$$

Remark 4.3.3. *By Assumption 4.3.1 and the last equality, we get that in a neighborhood of $z = \mathbf{R}$ the expansions of $G(z)$ and $\zeta_i(z)$ must be of the same type.*

In order to direct the reader on the right way, we can summarize the reasoning that concludes this section in the following steps:

1. in Lemma 4.3.4 we show that $\Phi''(\bar{\theta}) > 0$;
2. in Lemma 4.3.5 we show that $\Phi''(\bar{\theta}) < \infty$;
3. in Proposition 4.3.6 we use Step 1 and Step 2 to find the first singular term of the expansion of $G(z)$ in a neighborhood of $z = \mathbf{R}$.

Lemma 4.3.4. *Under Assumption 4.3.1, if $\Psi(\bar{\theta}) = 0$ then $\Phi''(\bar{\theta}) > 0$.*

Proof. Differentiating (3.8) twice yields

$$\Phi''(\bar{\theta}) = \alpha_1^2 \Phi_1''(\alpha_1\bar{\theta}) + \alpha_2^2 \Phi_2''(\alpha_2\bar{\theta}). \tag{4.23}$$

Since $\Phi_1(t)$ and $\Phi_2(t)$ are strictly convex for $t \in [0, \theta_1)$ and $t \in [0, \theta_2)$ respectively, we get $\Phi''(\bar{\theta}) > 0$ whenever $\theta_1/\alpha_1 \neq \theta_2/\alpha_2$.
If $\bar{\theta} = \theta_1/\alpha_1 < \theta_2/\alpha_2$ then $\alpha_2\bar{\theta} < \theta_2$, i.e., $\Phi_2''(\alpha_2\bar{\theta}) > 0$.

Consider the case $\theta_1/\alpha_1 = \theta_2/\alpha_2$, i.e., $\zeta_2(\mathbf{R}) = \mathbf{R}_2$, and assume (by contradiction) that $\Phi''(\bar{\theta}) = 0$.

By this assumption together with Equation (4.23) it follows that we must have $\Phi_1''(\theta_1) = \lim_{t\to\theta_1-} \Phi_1''(t) = 0$ and $\Phi_2''(\theta_2) = \lim_{t\to\theta_2-} \Phi_2''(t) = 0$. For $i \in \{1, 2\}$, differentiating (3.6) yields

$$G_i'(\mathbf{R}_i) = \lim_{z\to\mathbf{R}_i} \frac{G_i(z)\Phi_i'\big(zG_i(z)\big)}{1 - z\Phi_i'\big(zG_i(z)\big)},$$

or equivalently

$$\Phi_i'(\theta_i) = \lim_{z\to\mathbf{R}_i} \frac{G_i'(z)}{zG_i'(z)+G_i(z)} = \frac{G_i'(\mathbf{R}_i)}{\mathbf{R}_iG_i'(\mathbf{R}_i)+G_i(\mathbf{R}_i)} < \infty.$$

In particular, we have $\Phi_i'(\theta_i) < 1/\mathbf{R}_i$ since $G_i'(\mathbf{R}_i) < \infty$ by Assumption 4.3.1. If $\Phi_i''(\theta_i) = 0$, differentiating (3.6) twice yields

$$\begin{aligned} G_i''(\mathbf{R}_i) &= \lim_{z\to\mathbf{R}_i} \frac{\Phi_i''\big(zG_i(z)\big)\big(G_i(z)+zG_i'(z)\big)^2 + 2\Phi_i'\big(zG_i(z)\big)G_i(z)}{1-z\Phi_i'\big(zG_i(z)\big)} \\ &= \frac{2\Phi_i'(\theta_i)G_i(\mathbf{R}_i)}{1-\mathbf{R}_i\Phi_i'(\theta_i)} < \infty. \end{aligned}$$

At this point we consider again the *first return generating function*. It is defined through a different formula than in Equation (4.7), but they are of course equivalent:

$$U_i(z) := \sum_{n\geq 1} \mathbb{P}\big[X_n^{(i)} = e_i, \forall m \in \{1,\dots,n\} : X_m^{(i)} \neq e_i \mid X_0^{(i)} = e_i\big]\, z^n.$$

It satisfies the well-known equation $G_i(z) = 1/\big(1-U_i(z)\big)$ and is strictly convex. Since $G_i''(\mathbf{R}_i) < \infty$ we get immediately that $U_i''(\mathbf{R}_i) < \infty$. We can use this result to compute $\Phi_i''(\theta_i)$:

$$\Phi_i''(\theta_i) = \lim_{z\to\mathbf{R}_i} \frac{G_i(z)^3U_i''(z)}{\big(G_i(z)+zG_i'(z)\big)^3} = \frac{G_i(\mathbf{R}_i)^3U_i''(\mathbf{R}_i)}{\big(G_i(\mathbf{R}_i)+\mathbf{R}_iG_i'(\mathbf{R}_i)\big)^3} > 0.$$

This is a contradiction with our assumption that $\Phi_i''(\theta_i) = 0$. Consequently Equation (4.23) lead us to $\Phi''(\bar\theta) > 0$. ☐

Lemma 4.3.5. *Under Assumption 4.3.1, if $\Psi(\bar\theta) = 0$ then $\Phi''(\bar\theta) < \infty$.*

Proof. We will prove by contradiction that the situation $\Phi''(\bar\theta) = \infty$ cannot be compatible with our hypotheses.

Consider the auxiliary function $H(z) := \big(G(z) - G(\mathbf{R})\big)^2$ and its first derivative $H'(z) = 2G'(z)\big(G(z)-G(\mathbf{R})\big)$. Using Equation (4.18), we get

$$H'(z) = 2\frac{G(z)\Phi'\big(zG(z)\big)}{1-z\Phi'\big(zG(z)\big)}\big(G(z)-G(\mathbf{R})\big).$$

We want to compute the first derivative of $H(z)$ at $z = \mathbf{R}$. For this purpose, we consider the following limit:

$$\lim_{z\to\mathbf{R}} H'(z) = \lim_{z\to\mathbf{R}} 2G(z)\Phi'\big(zG(z)\big)\frac{G(z)-G(\mathbf{R})}{1-z\Phi'\big(zG(z)\big)}.$$

Since $2G(z)\Phi'\big(zG(z)\big)$ tends to $A := 2G(\mathbf{R})/\mathbf{R} < \infty$, we just look at:

$$\begin{aligned} \lim_{z\to\mathbf{R}} \frac{G(z)-G(\mathbf{R})}{1-z\Phi'\big(zG(z)\big)} &= \lim_{z\to\mathbf{R}} \frac{\Phi\big(zG(z)\big)-G(\mathbf{R})}{1-z\Phi'\big(zG(z)\big)} \\ &= \lim_{z\to\mathbf{R}} \frac{\Phi'\big(zG(z)\big)\big(G(z)+zG'(z)\big)}{-\Phi'\big(zG(z)\big)-z\Phi''\big(zG(z)\big)\big(G(z)+zG'(z)\big)}. \end{aligned} \tag{4.24}$$

In the last equality we applied De L'Hôpital's rule. In order to simplify the notation we can write $\mathcal{G}(z) := G(z)+zG'(z)$, which tends to infinity for $z \to \mathbf{R}$. Recall that $\bar{\theta} = \theta = \mathbf{R}G(\mathbf{R})$ if $\Psi(\bar{\theta}) = 0$. Therefore, Equation (4.24) yields

$$\begin{aligned} H'(\mathbf{R}) &= \lim_{z\to\mathbf{R}} \frac{A\Phi'(\theta)\mathcal{G}(z)}{-\Phi'(\theta) - \mathbf{R}\Phi''(\theta)\mathcal{G}(z)} \\ &= \lim_{x\to\infty} \frac{A\Phi'(\theta)x}{-\Phi'(\theta) - \mathbf{R}\Phi''(\theta)x} = \frac{A}{-\mathbf{R}^2\Phi''(\theta)}. \end{aligned}$$

If $\Phi''(\bar{\theta}) = \infty$ we get $H'(\mathbf{R}) = 0$, and consequently $X_1(z), X_2(z) = \mathbf{o}(\sqrt{\mathbf{R}-z})$ where $X_1(z)$ and $X_2(z)$ are defined in Equation (4.20).

For $i \in \{1,2\}$ and $s_i \in \mathrm{supp}(\mu_i)$, we will write $F_i(s_i|z) = \sum_{n\geq 1} f_n^{(i)}(s_i) z^n$ for some suitable coefficients $f_n^{(i)}(s_i) \in \mathbb{R}$. Our next aim is to find real numbers $C_1^{(i)}$ and $C_2^{(i)}$ such that

$$C_1^{(i)} X_1(z) + C_2^{(i)} X_2(z) + \mathbf{o}(\mathbf{R}-z) = \mathrm{LP}_i, \tag{4.25}$$

where LP_i is a linear polynomial. For this purpose, we rewrite Equations (4.3) and (4.4) with the help of (4.20). In the following denote by j the element of $\{1,2\}$ which is different from i. We get:

$$\begin{aligned} &\Bigl(1 - \alpha_j(\mathbf{R}-(\mathbf{R}-z)) \sum_{s_j\in\mathrm{supp}(\mu_j)} \mu_j(s_j) \sum_{n\geq 1} f_n^{(j)}(s_j) \bigl(\zeta_j(\mathbf{R}) + X_j(z)\bigr)^n\Bigr) \times \\ &\times \bigl(\zeta_i(\mathbf{R}) + X_i(z)\bigr) = \alpha_i z. \end{aligned} \tag{4.26}$$

Therefore the coefficients $C_1^{(i)}$ and $C_2^{(i)}$ of $X_1(z)$ and $X_2(z)$ respectively, are

$$\begin{aligned} C_1^{(1)} &:= 1 - \alpha_2 \mathbf{R} \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2) \sum_{n\geq 1} f_n^{(2)}(s_2)\, \zeta_2(\mathbf{R})^n \\ &= 1 - \alpha_2 \mathbf{R} \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2) F_2\bigl(s_2|\zeta_2(\mathbf{R})\bigr), \\ C_2^{(1)} &:= -\alpha_2 \mathbf{R}_1 \mathbf{R} \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2) \sum_{n\geq 1} f_n^{(2)}(s_2) n\, \zeta_2(\mathbf{R})^{n-1} \\ &= -\alpha_2 \mathbf{R}_1 \mathbf{R} \sum_{s_2\in\mathrm{supp}(\mu_2)} \mu_2(s_2) F_2'\bigl(s_2|\zeta_2(\mathbf{R})\bigr), \\ C_1^{(2)} &:= -\alpha_1 \zeta_2(\mathbf{R}) \mathbf{R} \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1) \sum_{n\geq 1} f_n^{(1)}(s_1) n \mathbf{R}_1^{n-1} \\ &= -\alpha_1 \zeta_2(\mathbf{R}) \mathbf{R} \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1) F_1'\bigl(s_1|\mathbf{R}_1\bigr), \\ C_2^{(2)} &:= 1 - \alpha_1 \mathbf{R} \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1) \sum_{n\geq 1} f_n^{(1)}(s_1) \mathbf{R}_1^n \\ &= 1 - \alpha_1 \mathbf{R} \sum_{s_1\in\mathrm{supp}(\mu_1)} \mu_1(s_1) F_1(s_1|\mathbf{R}_1). \end{aligned}$$

For $i = 1$, the linear polynomial term on the left hand side of (4.26) is

$$\mathbf{R}_1\Big(1 - \alpha_2 z \sum_{s_2 \in \operatorname{supp}(\mu_2)} \mu_2(s_2) F_2\big(s_2|\zeta_2(\mathbf{R})\big)\Big),$$

while on the right hand side it is $\alpha_1 z$.
For $i = 2$, we have on the left hand side of (4.26)

$$\zeta_2(\mathbf{R})\Big(1 - \alpha_1 z \sum_{s_1 \in \operatorname{supp}(\mu_1)} \mu_1(s_1) F_1(s_1|\mathbf{R}_1)\Big),$$

and on the right hand side $\alpha_2 z$. Therefore, (4.25) holds with

$$\begin{aligned}
\mathrm{LP}_1 &:= \alpha_1 z - \mathbf{R}_1\Big(1 - \alpha_2 z \sum_{s_2 \in \operatorname{supp}(\mu_2)} \mu_2(s_2) F_2\big(s_2|\zeta_2(\mathbf{R})\big)\Big) \quad \text{and} \\
\mathrm{LP}_2 &:= \alpha_2 z - \zeta_2(\mathbf{R})\Big(1 - \alpha_1 z \sum_{s_1 \in \operatorname{supp}(\mu_1)} \mu_1(s_1) F_1(s_1|\mathbf{R}_1)\Big).
\end{aligned}$$

The coefficients $C_1^{(i)}, C_2^{(i)}$ satisfy

$$C_1^{(1)} C_2^{(2)} - C_1^{(2)} C_2^{(1)} = 0. \tag{4.27}$$

Indeed, assume that $C_1^{(1)} C_2^{(2)} - C_1^{(2)} C_2^{(1)} \neq 0$. Then the following linear system

$$\begin{aligned}
C_1^{(1)} X_1(z) + C_2^{(1)} X_2(z) + \mathbf{o}(\mathbf{R} - z) &= \mathrm{LP}_1, \\
C_1^{(2)} X_1(z) + C_2^{(2)} X_2(z) + \mathbf{o}(\mathbf{R} - z) &= \mathrm{LP}_2
\end{aligned}$$

would have a unique solution for $X_1(z)$ and $X_2(z)$, but this would mean that both of them are of order $\mathbf{O}(\mathbf{R} - z)$. This is in contradiction with (4.21).

Equation (4.27) yields

$$\mathrm{LP}_1 - \frac{C_2^{(1)}}{C_2^{(2)}} \mathrm{LP}_2 = 0. \tag{4.28}$$

Evaluating the last equation at $z = 0$ yields

$$-\mathbf{R}_1 + \frac{C_2^{(1)}}{C_2^{(2)}} \cdot \zeta_2(\mathbf{R}) = 0. \tag{4.29}$$

Since $C_2^{(1)} < 0$ and $C_2^{(2)} > 0$ (this follows by evaluating Equation (4.26) at $z = \mathbf{R}$ with $i = 2$), Equation (4.29) gives us a contradiction, therefore $\Phi''(\bar{\theta}) = \infty$ cannot hold when $\Psi(\bar{\theta}) = 0$. □

Proposition 4.3.6. *Under Assumption 4.3.1, if $\Psi(\bar{\theta}) = 0$ then we can expand $G(z)$ in a neighborhood of $z = \mathbf{R}$ as follows:*

$$G(z) = g_0 + g_1\sqrt{\mathbf{R} - z} + \mathbf{o}\big(\sqrt{\mathbf{R} - z}\big),$$

where $g_0, g_1 \in \mathbb{R}$ with $g_1 \neq 0$.

Proof. By Lemma 4.3.5 it follows that $\Phi''(\bar{\theta}) < \infty$. Therefore we get that the limit (4.24) tends to a non-zero constant. In particular $H'(\mathbf{R}) < 0$, thus we have:

$$\lim_{z\to\mathbf{R}} \frac{G(\mathbf{R}) - G(z)}{\sqrt{\mathbf{R}-z}} = \lim_{z\to\mathbf{R}} \sqrt{\frac{\big(G(z) - G(\mathbf{R})\big)^2}{\mathbf{R}-z}} = \sqrt{-H'(\mathbf{R})} \in (0,\infty).$$

This leads to the proposed expansion, namely

$$G(z) = G(\mathbf{R}) - \sqrt{-H'(\mathbf{R})}\sqrt{\mathbf{R}-z} + \mathbf{o}(\sqrt{\mathbf{R}-z}),$$

where $\sqrt{-H'(\mathbf{R})} \neq 0$. □

Now we proceed analogously to the previous section: we substitute the expansions found in Lemma 4.3.5 into Equations (4.3) and (4.4). Afterwards we determine step by step the following terms of the expansions of $\zeta_1(z)$ and $\zeta_2(z)$. By the argument explained in Remark 4.3.3, we get the expansion of $G(z)$.

The next lemma shows that we get only a finite number of singular terms up to order $(\mathbf{R}-z)^2$:

Lemma 4.3.7. *Let $i \in \{1,2\}$. If $\Psi(\bar{\theta}) = 0$, we can expand $\zeta_i(z)$ in a neighborhood of $z = \mathbf{R}$ in the following way:*

$$\zeta_i(z) = \zeta_i(r) + c_0\sqrt{\mathbf{R}-z} + \sum_{(q,k)\in\mathcal{T}} c_{(q,k)}(\mathbf{R}-z)^q \log^k(\mathbf{R}-z) + \mathbf{O}\big((\mathbf{R}-z)^2\big),$$

where $\mathcal{T}$ is a finite subset of $\widehat{\mathcal{T}} := \big\{(q,k) \in \mathbb{R}\times\mathbb{N}_0 \mid 1/2 < q \leq 2\big\}$ and $c_0, c_{(q,k)} \in \mathbb{R}$ with $c_0 \neq 0$.

Proof. We start by plugging

$$\zeta_i(z) = \zeta_i(\mathbf{R}) + c_0\sqrt{\mathbf{R}-z} + X_0^{(i)}(z), \quad \text{where} \quad X_0^{(i)}(z) = \mathbf{o}(\sqrt{\mathbf{R}-z})$$

into Equations (4.3) and (4.4) and determine step by step the next terms inductively analogously to the proof of Lemma 4.2.6. Assume now that $\zeta_i(z)$ has an expansion of the form

$$\zeta_i(\mathbf{R}) + c_0\sqrt{\mathbf{R}-z} + \sum_{(q,k)\in\mathcal{T}'} c_{(q,k)}(\mathbf{R}-z)^q \log^k(\mathbf{R}-z) + \mathbf{o}(\max\mathcal{T}'),$$

with $\mathcal{T}' \subseteq \widehat{\mathcal{T}}$ finite. For $p > 1$, $\big(\zeta_i(\mathbf{R}) - \zeta_i(z)\big)^p$ can be rewritten as

$$(-c_0)^p\,(\mathbf{R}-z)^{p/2}\left(1 + \sum_{(q,k)\in\mathcal{T}'} \frac{c_{(q,k)}}{c_0}(\mathbf{R}-z)^{q-1/2}\log^k(\mathbf{R}-z) + \mathbf{o}\Big(\frac{\max\mathcal{T}'}{\sqrt{\mathbf{R}-z}}\Big)\right)^p \tag{4.30}$$

and $\log\big(\zeta_i(\mathbf{R}) - \zeta_i(z)\big)$ as

$$C + \frac{1}{2}\log(\mathbf{R}-z) + \log\left(1 + \sum_{(q,k)\in\mathcal{T}'} \frac{c_{(q,k)}}{c_0}(\mathbf{R}-z)^{q-1/2}\log^k(\mathbf{R}-z) + \mathbf{o}\Big(\frac{\max\mathcal{T}'}{\sqrt{\mathbf{R}-z}}\Big)\right). \tag{4.31}$$

Once again, if $\max \mathcal{T}' = (\mathbf{R}-z)^{\hat{q}} \log^{\hat{k}}(\mathbf{R}-z)$ then the next possible terms up to order $(\mathbf{R}-z)^{\hat{q}}$ in the expansion may only be

$$(\mathbf{R}-z)^{\hat{q}} \log^{\hat{k}-1}(\mathbf{R}-z), (\mathbf{R}-z)^{\hat{q}} \log^{\hat{k}-2}(\mathbf{R}-z), \dots, (\mathbf{R}-z)^{\hat{q}}.$$

We determine step by step the corresponding coefficients of these terms by plugging the expansions of $\zeta_i(z)$, given by (4.30) and (4.31) into Equations (4.3) and (4.4), afterwards we compare the error terms.

The following term has the form $(\mathbf{R}-z)^{\check{q}} \log^{\check{k}}(\mathbf{R}-z)$, where $\check{q} \leq 2$ is now a sum of elements from the finite set $\{1/2, q/2, q/2-1/2 \mid (q,\cdot) \in \mathcal{T}_1 \cup \mathcal{T}_2\}$ such that $\check{q} > \hat{q}$ (recall the definitions of $\mathcal{T}_i$ from (3.3.1)).

By (4.30) and (4.31) there is a maximal $\check{k} \in \mathbb{N}_0$ such that $(\mathbf{R}-z)^{\check{q}} \log^{\check{k}}(\mathbf{R}-z)$ may be a non-vanishing next term in the expansion of $\zeta_i(z)$. Iterating the last steps yields the claim of the lemma, since there are only finitely many possible values for q such that the term $(\mathbf{R}-z)^{q} \log^{k}(\mathbf{R}-z)$ can appear in the expansion of $\zeta_i(z)$. $\square$

Substituting the obtained expansion of $\zeta_i(z)$ into Equation (3.5) yields the proposed claim of Theorem 4.3.2.

Remark 4.3.8. *The result could also be obtained by singularity analysis (see [19, Section VI.7]), but one still has to distinguish different cases according to positivity and finiteness of* $\Phi''(\bar{\theta})$.

Chapter 5

Remaining Cases and Examples

In this chapter we finish the classification of the free products of the form $\Gamma_1 * \Gamma_2$ (see Section 5.1); afterwards (in Section 5.2) we extend our results to the more general free products $\Gamma_1 * \ldots * \Gamma_m$ with $m > 2$. Finally, in Section 5.3 we show a few examples with the relative computations.

5.1 The remaining Cases

In this section we look at all remaining cases of free products of type $\Gamma_1 * \Gamma_2$ not covered by Chapter 4.

5.1.1 Case $G_1(\mathbf{R}_1) < \infty$ and $G_1'(\mathbf{R}_1) = \infty$

Theorem 5.1.1. *Consider the free product of the form $\Gamma_1 * \Gamma_2$, such that $G_1(\mathbf{R}_1) < \infty$, $G_1'(\mathbf{R}_1) = \infty$ and $G_2'(\mathbf{R}_2) < \infty$. Then:*

$$\mu^{(n\delta)}(e) \sim \begin{cases} C_1 \cdot \mathbf{R}^{-n\delta} n^{-3/2}, & \text{if } \bar{\theta} = \theta_1/\alpha_1 \quad \text{or} \quad \Psi(\bar{\theta}) \leq 0, \\ C_2 \cdot \mathbf{R}^{-n\delta} n^{-\lambda_2} \cdot \log^{\kappa_2}(n), & \text{if } \bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1 \text{ and } \Psi(\bar{\theta}) > 0. \end{cases}$$

Proof. We will divide the proof into two parts, and we will show that under the given hypotheses both behaviors are possible.

For the first part let us assume that $\bar{\theta} = \theta_1/\alpha_1$. We use again the first return generating function, defined by Equation (4.7):

$$U_1(z) = \sum_{g \in \Gamma_1} \mu_1(g)\, z\, F_1(g^{-1}|z).$$

As previously said (see the proof of Lemma 4.2.4), we have the well-known equation $G_1(z) = 1/\big(1 - U_1(z)\big)$. Therefore, $G_1'(\mathbf{R}_1) = \infty$ implies $U_1'(\mathbf{R}_1) = \infty$, and by [64, Equation (9.14)] we get:

$$\Psi_1(\alpha_1\bar{\theta}) = \Psi_1(\theta_1) = \lim_{z \to \mathbf{R}_1} \Psi_1\big(zG(z)\big) = \lim_{z \to \mathbf{R}_1} \frac{1}{zU_1'(z) + 1 - U_1(z)} = 0. \quad (5.1)$$

Thus,

$$\Psi(\bar{\theta}) = \Psi_1(\alpha_1\bar{\theta}) + \Psi_2(\alpha_2\bar{\theta}) - 1 = \Psi_1(\theta_1) + \Psi_2(\alpha_2\bar{\theta}) - 1 = \Psi_2(\alpha_2\bar{\theta}) - 1.$$

Recall that $\Psi(t)$ is strictly decreasing and $\Psi_2(0) = 1$. Therefore $\Psi(\bar{\theta}) < 0$, and consequently we obtain the asymptotic behavior $\mu^{(n\delta)}(e) \sim C_1 \mathbf{R}^{-n\delta} n^{-3/2}$; see [64, Theorem 17.3].

For the case $\bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1$ and $\Psi(\bar{\theta}) = 0$, we perform the same computations as explained in Section 4.3.

In the case $\bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1$ and $\Psi(\bar{\theta}) > 0$ the Green function $G_1(z)$ is analytic at $z = \zeta_1(\mathbf{R}) < \mathbf{R}_1$ and thus we may apply the techniques described in Section 4.2 to obtain the proposed asymptotic behavior. □

At this point, let us remark that the formula for $\Psi(t)$ used in Equation (5.1) always implies $\Psi_i(\theta_i) = 0$ whenever $G_i'(\mathbf{R}_i) = \infty$. Moreover:

Corollary 5.1.2. *If $G_1'(\mathbf{R}_1) = G_2'(\mathbf{R}_2) = \infty$, then $\mu^{(n\delta)}(e) \sim C \cdot \mathbf{R}^{-n\delta} \cdot n^{-3/2}$.*

Proof. Since $U_1'(\mathbf{R}_1) = U_2'(\mathbf{R}_2) = \infty$, Equation (5.1) implies that at least one of $\Psi_1(\alpha_1\bar{\theta})$ and $\Psi_2(\alpha_2\bar{\theta})$ equals zero, yielding $\Psi(\bar{\theta}) < 0$. □

5.1.2 Case $G_1(\mathbf{R}_1) = \infty$

As mentioned earlier (see Section 4.1), for finite groups Γ_1 and Γ_2, Woess [63] proved that the asymptotic behavior of the n-step return probabilities is of the form

$$\mu^{(n\delta)}(e) \sim C \cdot \mathbf{R}^{-n\delta} \cdot n^{-3/2}.$$

Moreover, we get the following asymptotic behaviors:

Theorem 5.1.3. *Consider the free product $\Gamma_1 * \Gamma_2$ with $G_1(\mathbf{R}_1) = \infty$. Then:*

$$\mu^{(n\delta)}(e) \sim \begin{cases} C_1 \cdot \mathbf{R}^{-n\delta} \cdot n^{-3/2}, & \text{if } \Psi(\bar{\theta}) \leq 0, \\ C_2 \cdot \mathbf{R}^{-n\delta} \cdot n^{-\lambda_2} \cdot \log^{\kappa_2}(n), & \text{if } \Psi(\bar{\theta}) > 0. \end{cases}$$

Proof. We have three possibilities:

- If $G_2'(\mathbf{R}_2) = \infty$, we have $\Psi(\bar{\theta}) < 0$; see proof of Corollary 5.1.2.
- If $G_2(\mathbf{R}_2) < \infty$ and $G_2'(\mathbf{R}_2) = \infty$ then $\bar{\theta} = \theta_2/\alpha_2$, and $U_2'(\mathbf{R}_2) = \infty$. This implies once again $\Psi(\alpha_2\bar{\theta}) = 0$, and thus $\Psi(\bar{\theta}) < 0$.
- If $G_2'(\mathbf{R}_2) < \infty$ then $\bar{\theta} = \theta_2/\alpha_2$ and $\zeta_1(\mathbf{R}) < \mathbf{R}_1$. Therefore, we can argue in the same way as in Sections 4.2 and 4.3 to prove the proposed claim.

□

5.2 Free Products with more than two Factors

Let $r \in \mathbb{N}$ with $r \geq 3$. Suppose we are given finitely generated groups $\Gamma_1, \ldots, \Gamma_r$, and consider the free product $\Gamma := \Gamma_1 * \ldots * \Gamma_r$, on which the random walk is governed by the measure μ defined as $\mu := \sum_{j=1}^{r} \alpha_j \bar{\mu}_j$; see Section 3.2. We get the following result:

Theorem 5.2.1. *Consider the free product $\Gamma := \Gamma_1 * \ldots * \Gamma_r$ ($r \geq 3$) equipped with a random walk governed by μ. Assume that the Green functions $G_i(z)$ on the free factors Γ_i satisfy Assumption 3.3.1, whenever $G_i'(\mathbf{R}) < \infty$. Then the asymptotic behavior of the corresponding n-step transition probabilities must obey one of the following laws: $C\,\mathbf{R}^{-n\delta}\, n^{-\lambda_i} \log^{\kappa_i}(n)$, where λ_i and κ_i are inherited from one of the μ_i's, or $C\,\mathbf{R}^{-n\delta}\, n^{-3/2}$ with some constant $C = C_\mu$ depending on μ.*

Proof. The proof is based on the induction on the number of free factors: we consider the Green function associated to the random walk on $\Gamma^* := \Gamma_1 * \ldots * \Gamma_{r-1}$ governed by $\mu^* := \sum_{j=1}^{r-1} \frac{\alpha_j}{\alpha_1 + \ldots + \alpha_{r-1}} \bar{\mu}_j$ (denote by $\mathbf{R}^*$ its radius of convergence). It has an expansion either of the form (5.2) or of the form (5.3):

$$\begin{aligned} G^*(z) = \sum_{k=0}^{D} g_k (\mathbf{R}^* - z)^k + \sum_{(q,k)\in\mathcal{T}} g_{(q,k)} (\mathbf{R}^* - z)^q \log^k(\mathbf{R}^* - z) \\ + \mathbf{O}\big((\mathbf{R}^* - z)^{D+2}\big), \end{aligned} \tag{5.2}$$

where $\mathcal{T}$ is a finite subset of $\{(q,k) \in \mathbb{R}\times\mathbb{N}_0 \mid D < q \leq D{+}2\}$ and $g_k, g_{(q,k)} \in \mathbb{R}$, and:

$$\begin{aligned} G^*(z) = g_0 + g_1\sqrt{\mathbf{R}^* - z} + \sum_{(q,k)\in\mathcal{T}} g_{(q,k)} (\mathbf{R}^* - z)^q \log^k(\mathbf{R}^* - z) \\ + \mathbf{O}\big((\mathbf{R}^* - z)^{2}\big), \end{aligned} \tag{5.3}$$

where $\mathcal{T}$ is a finite subset of $\{(q,k) \in \mathbb{R} \times \mathbb{N}_0 \mid 1/2 < q \leq 2\}$ and $g_0, g_1, g_{(q,k)}$ are real values. Thus, we may apply the results from Chapter 4 to the free product $\Gamma^* * \Gamma_r$ equipped with $\mu = (\alpha_1 + \ldots + \alpha_{r-1})\mu^* + \alpha_r \bar{\mu}_r$ and obtain the proposed claim. □

5.3 Examples

In this section we would like to present some examples to clarify the fundamental concepts and to show to the reader that in a few concrete cases these behaviors can be explicitly computed.

5.3.1 Free Products of Lattices

Let us take $d_1, \ldots, d_r \in \mathbb{N}$. In this subsection we consider free products of the form $\Gamma := \mathbb{Z}^{d_1} * \ldots * \mathbb{Z}^{d_r}$, equipped with a nearest neighbor random walk, that is, we always assume $\mathrm{supp}(\mu_i) = \{\pm e_j^{(i)} \mid 1 \leq j \leq d_i\}$, where $e_j^{(i)}$ is the j-th unit vector in $\mathbb{Z}^{d_i}$. In the following subsection we show that the Green functions of nearest neighbor random walks on $\mathbb{Z}^d$ satisfy Assumption 3.3.1.

Expansion of the Green Function on $\mathbb{Z}^d$

Since the factors of the free product are d-dimensional lattices, we compute explicitly the Green function on each factor, depending on $d \in \mathbb{N}$.

Given a probability measure π supported by $\{\pm e_1, \dots, \pm e_d\}$, the set of natural generators of $\mathbb{Z}^d$. Then π defines a random walk on $\mathbb{Z}^d$, and we denote by $\pi^{(n)}$ its n-fold convolution power. We write for $1 \leq i \leq d$

$$\beta_i := \pi(e_i) + \pi(-e_i) \quad \text{and} \quad p_i := \frac{\pi(e_i)}{\pi(e_i) + \pi(-e_i)}.$$

Denote by $\mathbf{0}$ the zero vector in $\mathbb{Z}^d$. Once again $G_d(z) := \sum_{n\geq 0} \pi^{(n)}(\mathbf{0}) z^n$ denotes the associated Green function, which has radius of convergence $\mathbf{R}_d$. The crucial point for our later discussion is the following:

Proposition 5.3.1. *The Green function of the random walk on $\mathbb{Z}^d$ has an expansion of the form*

$$G_d(z) = \begin{cases} f(z) + g(z)(\mathbf{R}_d - z)^{(d-2)/2}, & \text{if } d \text{ is odd,} \\ f(z) + g(z)(\mathbf{R}_d - z)^{(d-2)/2} \log(\mathbf{R}_d - z), & \text{if } d \text{ is even,} \end{cases}$$

where the functions $f(z), g(z)$ are analytic in a neighborhood of $z = \mathbf{R}_d$ and moreover $g(\mathbf{R}_d) \neq 0$.

Remark 5.3.2. *For simple random walks on $\mathbb{Z}^d$, i.e. $\pi(\pm e_i) = 1/(2d)$, a proof of this proposition can be found in [64, Proposition 17.16]. Here we generalize the statement to arbitrary nearest neighbor random walks on $\mathbb{Z}^d$, but we will only give a sketch of the proof and refer once again to [64].*

Proof. First, note that the spectral radius of the random walk on $\mathbb{Z}^d$ is given by

$$\rho = \sum_{i=1}^{d} \beta_i \sqrt{4p_i(1-p_i)} = \frac{1}{\mathbf{R}_d};$$

compare with [64, Theorem 8.23]. For every $i \in \{1, \dots, d\}$, we define a random walk on the i-th factor $\mathbb{Z}$, governed by a probability measure π_i such that $\pi_i(1) := p_i$ and $\pi_i(-1) := 1 - p_i$. A standard tool that comes into play when dealing with random walks on Cartesian products is the *exponential generating function*: for $z \in \mathbb{C}$

$$E(z) := \sum_{n=0}^{\infty} \pi^{(n)}(\mathbf{0}) \frac{z^n}{n!},$$

defined on $\mathbb{Z}^d$. Analogously we can define it coordinate-wise as follows: on the i-th factor the exponential generating function is given by

$$E_i(z) := \sum_{n\geq 0} \pi_i^{(n)}(0) \frac{z^n}{n!} = \int_{-1}^{1} e^{\sqrt{4p_i(1-p_i)}tz} \frac{1}{\pi\sqrt{1-t^2}} dt.$$

In the last equation we applied the following relation:

$$\pi_i^{(n)}(0) = \int_{-1}^{1} \sqrt{4p_i(1-p_i)}^n t^n \frac{1}{\pi\sqrt{1-t^2}} dt.$$

Furthermore, we get $E(z) = \prod_{i=1}^{d} E_i(\beta_i z) = \int_{-\rho}^{\rho} e^{tz}\big(\hat{f}_1 * \ldots * \hat{f}_d\big)(t)dt$, where

$$\hat{f}_i(t) := \frac{1}{\beta_i\sqrt{4p_i(1-p_i)}} f_0\Big(\frac{t}{\beta_i\sqrt{4p_i(1-p_i)}}\Big)$$

with

$$f_0(t) := \begin{cases} \frac{1}{\pi\sqrt{1-t^2}}, & \text{if } t \in (-1,1),\\ 0, & \text{otherwise.}\end{cases}$$

This allows us to rewrite the Green function in the following way:

$$G_d(z) = \int_{-\rho}^{\rho} \frac{1}{1-zt}\big(\hat{f}_1 * \ldots * \hat{f}_d\big)(t)\,dt. \tag{5.4}$$

Our next aim is to prove that there is a function $g_d(t)$, which is analytic in a neighborhood of $t=\rho$ and satisfies $g_d(\rho) \neq 0$ such that

$$\big(\hat{f}_1 * \ldots * \hat{f}_d\big)(t) = (\rho - t)^{(d-2)/2} g_d(t). \tag{5.5}$$

To prove this, we define $\bar{f}_i(t) := \hat{f}_i\big(\beta_i\sqrt{4p_i(1-p_i)} - t\big)$ and show inductively that we can write

$$(\bar{f}_1 * \ldots * \bar{f}_d)(t) = t^{(d-2)/2}\bar{g}_d(t),$$

where the function $\bar{g}_d(t)$ is analytic in a neighborhood of $t=0$ and $\bar{g}_d(0) \neq 0$. Analogously to the proof of [64, Proposition 17.16], we may conclude together with (5.4) and (5.5) that $G_d(z)$ has the proposed expansion. □

Remark 5.3.3. *With the help of Darboux's method it follows that the asymptotic behavior of $\pi^{(2n)}(\mathbf{0})$ is of the form $C\,\mathbf{R}_d^{-2n}\,n^{-d/2}$. This asymptotic behavior can also be deduced by Cartwright and Soardi [11].*

Complete Classification of the Asymptotic Behavior

Observe that a nearest neighbor random walk on $\mathbb{Z}^d$ has period 2 since it can return to the origin only in an even number of steps. Therefore, the period of a nearest neighbor random walk on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ is $\delta = 2$. Now we can give a complete classification of the asymptotic behavior of the n-step return probabilities of nearest neighbor random walks on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$:

Theorem 5.3.4. *Consider irreducible nearest neighbor random walks on the lattices $\mathbb{Z}^{d_1}$ and $\mathbb{Z}^{d_2}$ with $d_1 \leq d_2$. Then the n-step return probabilities of the associated random walk on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ obey one the following laws:*

$$\mu^{(2n)}(e) \sim \begin{cases} C_1 \cdot \mathbf{R}^{-2n} \cdot n^{-d_1/2}, & \text{if } \Psi(\bar{\theta}) > 0 \text{ and } \bar{\theta} = \theta_1/\alpha_1,\\ C_2 \cdot \mathbf{R}^{-2n} \cdot n^{-d_2/2}, & \text{if } \Psi(\bar{\theta}) > 0 \text{ and } \bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1,\\ C_3 \cdot \mathbf{R}^{-2n} \cdot n^{-3/2}, & \text{otherwise.}\end{cases}$$

Remark 5.3.5. *For seek of completeness, even though the reader might have already noticed it, we point out the following fact: if both d_1 and d_2 are smaller than 5, the function $\Psi(\bar{\theta})$ cannot be positive. Therefore the first two behaviors described by Theorem 5.3.4 can show up only if at least one of the factors has dimension at least 5.*

Consider now the multi-factor free product $\mathbb{Z}^{d_1} * \ldots * \mathbb{Z}^{d_r}$. Let μ_i be the simple random walk on $\mathbb{Z}^{d_i}$ for each $i \in \{1, \ldots, r\}$, i.e., $\mu_i(\pm e_j^{(i)}) = 1/(2d_i)$, where $e_j^{(i)}$ is the j-th unit vector in $\mathbb{Z}^{d_i}$.

As described in Section 3.2, choose $\alpha_1, \ldots, \alpha_r > 0$ s.t. $\sum_{j=1}^r \alpha_j = 1$ and denote by $G_i(z)$ the Green function of the simple random walk on $\mathbb{Z}^{d_i}$. In this case $\mathbf{R}_i = 1$, since each factor is amenable.

Cartwright [9] computed numerically some of the values of $\Psi_i\big(G_i(1)\big)$ (where $\Psi_i(t)$ is defined by Equation (3.7)) and showed that $\Psi_i\big(G_i(1)\big) \to 1$ when d_i grows to infinity. Thus, for large d_i we have $\Psi_i\big(G_i(1)\big) > 1 - 1/r$. Recall also that $\Psi_i(t)$ is decreasing. By [64, Equation 9.21] we know that

$$\Psi(\bar{\theta}) = 1 + \sum_{j=1}^{r} \big(\Psi_i(\alpha_i \bar{\theta}) - 1\big),$$

where $\bar{\theta} = \min_{1 \le i \le r} \theta_i/\alpha_i$. If all exponents $d_i \ge 5$ are large enough, we get $\Psi(\bar{\theta}) > 0$. Furthermore, if α_i is chosen large enough, we get an asymptotic behavior of the form $C_i\, \mathbf{R}^{-2n}\, n^{-d_i/2}$.

On the other hand, one can define (symmetric) measures $\mu_1, \ldots, \mu_r$ supported on the natural generators to obtain a $C_0\, \mathbf{R}^{-2n}\, n^{-3/2}$-law: it suffices to choose μ_1 and μ_2 such that $\Psi_1(\theta_1), \Psi_2(\theta_2) < 1/2$, and α_1 and α_2 such that $\bar{\theta} = \theta_1/\alpha_1 = \theta_2/\alpha_2$, yielding

$$\Psi(\bar{\theta}) = 1 + \underbrace{\big(\Psi_1(\theta_1) - 1\big)}_{<-1/2} + \underbrace{\big(\Psi_2(\theta_2) - 1\big)}_{<-1/2} + \underbrace{\sum_{k=3}^{r} \big(\Psi_k(\alpha_k \bar{\theta}) - 1\big)}_{\le 0} < 0.$$

That is, we can have $r+1$ different asymptotic behaviors. This finally proves the following

Theorem 5.3.6. *Let $r \in \mathbb{N}$, $r \ge 2$ and $d_1, \ldots, d_r \in \mathbb{N}$. For $i \in \{1, \ldots, r\}$, consider a probability measure μ_i supported on the natural set of generators of $\mathbb{Z}^{d_i}$. For any $\alpha_1, \ldots, \alpha_r > 0$ with $\sum_{i=1}^r \alpha_i = 1$, let $\mu := \sum_{i=1}^r \alpha_i \mu_i$ govern a (irreducible) random walk on the free product $\mathbb{Z}^{d_1} * \cdots * \mathbb{Z}^{d_r}$ starting at its identity e.*

Then the return probabilities $\mu^{(2n)}(e)$ have an asymptotic behavior either of the form $C \cdot \rho^{2n} \cdot n^{-d_i/2}$ for $i \in \{1, \ldots, r\}$ or of the form $C \cdot \rho^{2n} \cdot n^{-3/2}$ for some constant $C = C_\mu$ depending on μ.

Moreover, if all exponents d_i are different and $\min\{d_1, \ldots, d_r\} \ge 5$ then exactly $r+1$ different asymptotic behaviors may occur by choosing the random walk adequately.

For instance, consider $\Gamma = \mathbb{Z}^5 * \mathbb{Z}^6 * \mathbb{Z}^7$ equipped with simple random walks μ_1, μ_2 and μ_3 on each free factor. For $i \in \{1, 2, 3\}$, we define $\Psi_i(t)$ according to Equation (3.7). In [9] Cartwright computed the values of $\Psi_1\big(G_1(1)\big)$, $\Psi_2\big(G_2(1)\big)$ and $\Psi_3\big(G_3(1)\big)$, which are 0.691, 0.824 and 0.876 respectively. Thus, the random walk on $\mathbb{Z}^5 * \mathbb{Z}^6$ governed by $\mu_{12} := \alpha_1^* \bar{\mu}_1 + \alpha_2^* \bar{\mu}_2$, where $\alpha_1^* = \alpha_1/(\alpha_1 + \alpha_2)$ and $\alpha_2^* = \alpha_2/(\alpha_1 + \alpha_2)$, satisfies $\Psi(M) \ge 0.515$ with

$M := \min\{\theta_1/\alpha_1^*, \theta_2/\alpha_2^*\}$. That is, $M = \mathbf{R}_{1,2} G_{1,2}(\mathbf{R}_{1,2})$, where $G_{1,2}(z)$ is the Green function of the random walk on $\mathbb{Z}^5 * \mathbb{Z}^6$ with radius of convergence $\mathbf{R}_{1,2}$. Since all Ψ_i-functions are strictly decreasing, we obtain for the random walk on $\Gamma = \Gamma_1 * \Gamma_2$ with $\Gamma_1 = \mathbb{Z}^5 * \mathbb{Z}^6$ and $\Gamma_2 = \mathbb{Z}^7$:

$$\Psi(\bar{\theta}) \quad = \quad \Psi_1\bigl((\alpha_1 + \alpha_2)\bar{\theta}\bigr) + \Psi_2(\alpha_3\bar{\theta}) - 1 \geq 0.515 + 0.876 - 1 > 0.$$

For the simple random walk on Γ, we have then the asymptotic non-exponential type $n^{-7/2}$, if $\alpha_1 + \alpha_2 < M/\bigl(M + G_3(1)\bigr)$. Otherwise, if $M = \theta_1/\alpha_1^*$ we have the asymptotic behavior $n^{-5/2}$, and if $M = \theta_2/\alpha_2^* \neq \theta_1/\alpha_1^*$ we have n^{-3}.

5.3.2 $(\mathbb{Z}/m\mathbb{Z}) * \mathbb{Z}^d$

Consider the groups $\Gamma_1 = \mathbb{Z}/m\mathbb{Z}$ and $\Gamma_2 = \mathbb{Z}^d$ for any $m, d \in \mathbb{N}$ with $m \geq 2$. Suppose we are given a probability measure μ_1 on Γ_1 and a probability measure μ_2 on $\mathbb{Z}^d$, which is supported on the natural generators. Then $G_1(1) = \infty$, and thus we get the following classification:

$$\mu^{(n\delta)}(e) \sim \begin{cases} C_1 \cdot \mathbf{R}^{-n\delta} \cdot n^{-d/2}, & \text{if } \Psi(\bar{\theta}) > 0, \\ C_2 \cdot \mathbf{R}^{-n\delta} \cdot n^{-3/2}, & \text{otherwise.} \end{cases}$$

Once again let us remark that if $d \leq 4$ we have $\Psi(\bar{\theta}) < 0$: this follows from the fact that $G_2'(\mathbf{R}_2) = \infty$ (see Proposition 5.3.1) and Corollary 5.1.2.

5.3.3 $\Pi_q * \mathbb{Z}^d$

Consider the groups $\Gamma_1 = \Pi_q := *_{i=1}^q (\mathbb{Z}/2\mathbb{Z})$ and $\Gamma_2 = \mathbb{Z}^d$ for any $q, d \in \mathbb{N}$ with $q \geq 2$. Observe that the Cayley graph of Γ_1 is the homogeneous tree of degree q. Suppose we are given probability measures μ_1 on Γ_1 and μ_2 on $\mathbb{Z}^d$, which are both supported on the natural generators. If $q = 2$ then $G_1(1) = \infty$, and thus we get the same classification as in the case $(\mathbb{Z}/m\mathbb{Z}) * \mathbb{Z}^d$. If $q \geq 3$, then it is well-known (see e.g. Section 4.1, [64, Proposition 17.4] and [65, Equation (4.5)]) that $G_1(z)$ can be written as

$$G_1(z) = A(z) + \sqrt{\mathbf{R}_1 - z}\, B(z),$$

where $A(z), B(z)$ are analytic in a neighborhood of $z = \mathbf{R}_1$ and $B(\mathbf{R}_1) \neq 0$. Therefore, we get the following classification for the associated random walk on the free product $\Gamma_1 * \Gamma_2$:

$$\mu^{(2n)}(e) \sim \begin{cases} C_1 \cdot \mathbf{R}^{-2n} \cdot n^{-d/2}, & \text{if } \bar{\theta} = \theta_2/\alpha_2 < \theta_1/\alpha_1 \text{ and } \Psi(\bar{\theta}) > 0, \\ C_2 \cdot \mathbf{R}^{-2n} \cdot n^{-3/2}, & \text{otherwise.} \end{cases}$$

Analogously to the previous example, observe that $d \leq 4$ implies $\Psi(\bar{\theta}) < 0$.

Chapter 6

Phase Transitions

6.1 Classification of Phase Transitions

At this point we would like to change the point of view of our investigation: up to now we have been interested in whether it was possible to find different measures that give us certain behaviors.
Now we fix the measures μ_1 and μ_2 on the free factors Γ_1 and Γ_2, and see what happens on the free product $\Gamma = \Gamma_1 * \Gamma_2$ when we make the parameter α_1 vary. More precisely we look at the variation of $\Psi(\bar{\theta})$ as a function of α_1.

We start with the following result:

Lemma 6.1.1. *Assume $\bar{\theta} < \infty$. Then the function $\Upsilon : (0,1) \mapsto \mathbb{R}$ defined by*

$$\Upsilon(\alpha_1) := \Psi_1(\alpha_1\bar{\theta}) + \Psi_2\big((1-\alpha_1)\bar{\theta}\big) - 1$$

is continuous, strictly decreasing on the interval $\big(0, \frac{\theta_1}{\theta_1+\theta_2}\big]$ and strictly increasing on the interval $\big[\frac{\theta_1}{\theta_1+\theta_2}, 1\big)$.

Proof. Continuity of Υ is immediate, since Ψ_1 and Ψ_2 are analytic in an open neighborhood of the intervals $[0, \theta_1)$ and $[0, \theta_2)$ respectively.

Note that $\Upsilon(\alpha_1)$ coincides with $\Psi(\bar{\theta})$, but seen as a function of α_1.

We divide the proof of this lemma into three parts, according to finiteness of θ_1 and θ_2.

<u>Case $\theta_1, \theta_2 < \infty$</u>. If $0 < \alpha_1 < \frac{\theta_1}{\theta_1+\theta_2}$ then $\bar{\theta} = \theta_2/\alpha_2$, therefore

$$\Upsilon(\alpha_1) = \Psi_1\Big(\frac{\alpha_1}{1-\alpha_1}\theta_2\Big) + \Psi_2(\theta_2) - 1.$$

Since the function $\frac{\alpha_1}{1-\alpha_1}$ is strictly increasing, it follows that $\Psi_1(\frac{\alpha_1}{1-\alpha_1}\theta_2)$ is strictly decreasing, implying $\Upsilon(\alpha_1)$ strictly decreasing.
If $\alpha_1 = \frac{\theta_1}{\theta_1+\theta_2}$ we obtain $\bar{\theta} = \theta_1/\alpha_1 = \theta_2/\alpha_2$, i.e.,

$$\Upsilon(\alpha_1) = \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1.$$

If $\frac{\theta_1}{\theta_1+\theta_2} < \alpha_1 < 1$ we have

$$\Psi(\bar{\theta}) = \Psi_1(\theta_1) + \Psi_2\left(\frac{1-\alpha_1}{\alpha_1}\theta_1\right) - 1.$$

Since $\frac{1-\alpha_1}{\alpha_1}$ is strictly decreasing, $\Upsilon(\alpha_1)$ is a strictly increasing function in the above-mentioned interval.

Case $\theta_1 = \infty$. In this case $\bar{\theta} = \frac{\theta_2}{1-\alpha_1}$. The same reasoning as before shows that $\Upsilon(\alpha_1)$ is strictly decreasing in the interval $(0,1)$.

Case $\theta_2 = \infty$. In this case $\bar{\theta} = \frac{\theta_1}{\alpha_1}$. Analogously, $\Upsilon(\alpha_1)$ is strictly increasing in the interval $(0,1)$. □

Let us remark that $\bar{\theta} = \infty$ implies $\Psi(\bar{\theta}) < 0$ (see [64, Theorem 9.22]); otherwise we would have a contradiction to transience of the process.

Now we can give a complete picture of the *phase transition* of the asymptotic behavior of the return probabilities in dependence of the parameter α_1, afterwards we present some examples.

In the following we discuss the different possible behaviors of the function $\Upsilon(\alpha_1) = \Psi(\bar{\theta})$. In Figures 6.1–6.6, the dashed line represents *approximately* the qualitative behavior of $\Upsilon(\alpha_1)$; we denote its zeros (if they exist) by α_{low} and α_{high} (with $\alpha_{\mathrm{low}} \leq \alpha_{\mathrm{high}}$). Moreover, we denote

$$\alpha_{\mathrm{c}} := \theta_1/(\theta_1+\theta_2).$$

Our approach is the following: we decompose the interval $(0,1)$ into subintervals such that every choice of α_1 in a fixed subinterval leads to the same non-exponential type. With the help of Figures 6.1–6.6 (at the end of this chapter) we discuss case by case the different behaviors of $\Upsilon(\alpha_1)$, and for each situation we give an example of a nearest neighbor random walk on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$. Recall that $\Psi(0) = \Psi_i(0) = 1$.

A: See Figure 6.1. Example: $\Gamma = \mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ with $d_1, d_2 \geq 5$ and μ_1, μ_2 such that $\Psi_1(\theta_1) < 1/2$ and $\Psi_2(\theta_2) < 1/2$ (by the argument at the end of Section 4.1 and by [64, Lemma 17.9], such measures exist). Since $\Psi_i(\theta_i) = 0$ implies $\Phi_i'(\theta_i) = 1/\mathbf{R}_i$, by differentiating (3.6) yields a contradiction to Proposition 5.3.1 (according to which $G_i'(\mathbf{R}_i)$ must be finite due to $d_i \geq 5$), therefore we get $\Psi_i(\theta_i) > 0$. Hence:

- If α_1 is small (close to zero), then $\bar{\theta} = \theta_2/(1-\alpha_1)$ and therefore on the limit for $\alpha_1 \to 0$ we have $\Psi_1\left(\alpha_1 \frac{\theta_2}{1-\alpha_1}\right) \to 1$. Then

$$\Psi(\bar{\theta}) = \Psi_1\left(\alpha_1 \frac{\theta_2}{1-\alpha_1}\right) + \Psi_2(\theta_2) - 1, \tag{6.1}$$

  that is, $\Psi(\bar{\theta}) > 0$ if α_1 is sufficiently small. This yields a $n^{-d_2/2}$-law for small values of α_1.

- With a similar reasoning to the one above, we can see that if α_1 is close to 1 then $\bar{\theta} = \theta_1/\alpha_1$, and we get a $n^{-d_1/2}$-law.

- For $\alpha_1 = \alpha_c$, we get $\Psi(\bar{\theta}) = \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1 < 0$, therefore in this case we have a $n^{-3/2}$-law.

B: See Figure 6.2. Example: $\Gamma = \mathbb{Z}^2 * \mathbb{Z}^7$. By Lemma 6.1.1 the function $\Upsilon(\alpha_1)$ is strictly decreasing and $\bar{\theta} = \theta_2/\alpha_2$. As in Case A we can divide the reasoning according to the following different situations:

- If α_1 is small (close to zero) then the same reasoning as in (6.1) leads to $\Psi(\bar{\theta}) > 0$, that is, we have a $n^{-d_2/2}$-law for small α_1.
- If α_1 is close to 1 then $\Psi_1(\alpha_1 \frac{\theta_2}{1-\alpha_1}) \to 0$ (recall that by Equation (5.1) we have $\lim_{t\to\infty} \Psi_1(t) = 0$), and since $\Psi_2(\theta_2) < 1$, we get

$$\Psi(\bar{\theta}) = \Psi_1\left(\alpha_1 \frac{\theta_2}{1-\alpha_1}\right) + \Psi_2(\theta_2) - 1 < 0.$$

  Therefore we have a $n^{-3/2}$-law for α_1 close to 1.

C: See Figure 6.3. Example: $\Gamma = \mathbb{Z}^7 * \mathbb{Z}^2$, we have the symmetric situation as in Case B. This gives an example for this case by exchanging the roles of $\mathbb{Z}^2$ and $\mathbb{Z}^7$.

D: See Figure 6.4. Example: $\Gamma = \mathbb{Z}^5 * \mathbb{Z}^6$ with μ_1 and μ_2 simple random walks. By a result of Cartwright [9], we have $\Psi_1(\theta_1) = 0.691$ and $\Psi_2(\theta_2) = 0.824$. Since $\Psi_1(t)$ and $\Psi_2(t)$ are strictly decreasing, we have

$$\Upsilon(\alpha_1) \geq \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1 > 0.$$

Thus, we obtain a $n^{-5/2}$-law for $\alpha_1 \geq \alpha_c$, and a n^{-3}-law for $\alpha_1 < \alpha_c$.

E: See Figure 6.5. Example: $\Gamma = \mathbb{Z}^3 * \mathbb{Z}^4$. By Equation (5.1) it follows that $\Psi_1(\alpha_1\bar{\theta}) = 0$ or $\Psi_2(\alpha_2\bar{\theta}) = 0$, that is, we have $\Upsilon(\alpha_1) < 0$ for all $\alpha_1 \in (0,1)$. This yields a $n^{-3/2}$-law for all $\alpha_1 \in (0,1)$.

At this point we would like to give an example (see Figure 6.6) where the $n^{-3/2}$-interval described in Case A collapses to a singleton. For this purpose, we need to prove the following:

Lemma 6.1.2. *Consider $\Gamma = \mathbb{Z}^5 * \mathbb{Z}^6$. Then there are probability measures μ_1 and μ_2 supported on the natural generators of $\mathbb{Z}^5$ and $\mathbb{Z}^6$ respectively, such that $\Psi_1(\theta_1) = \Psi_2(\theta_2) = \frac{1}{2}$.*

Proof. Let $i \in \{1,2\}$. In this example we have $d_1 = 5, d_2 = 6$. Choose any $\delta \in (0,1)$ and define

$$\nu_\delta^{(i)}(x) := \begin{cases} (1-\delta)/2, & \text{if } x = (\pm 1, 0, \ldots, 0) \in \mathbb{Z}^{d_i}, \\ \frac{\delta}{2d_i - 2}, & \text{if } x = (0, \ldots, 0, \pm 1, 0, \ldots, 0) \in \mathbb{Z}^{d_i} \setminus \{(\pm 1, 0, \ldots, 0)\}. \end{cases} \tag{6.2}$$

The Green function associated with the random walk on $\mathbb{Z}^{d_i}$ governed by the symmetric measure $\nu_\delta^{(i)}$ has radius of convergence $\mathbf{R}_i = 1$; see [64, Cor. 8.15]. Moreover, if $\delta = 1 - 1/d_i$ then $\Psi_1(\theta_1) = 0.691 > 1/2$ and $\Psi_2(\theta_2) = 0.824 > 1/2$; see Cartwright [9].

On the other hand, choosing δ small enough, then $\Psi_1(\theta_1) < 1/2$ and $\Psi_2(\theta_2) < 1/2$. A proof of this fact can be seen in [64, Lemma 17.9]. It remains to show that $\Psi_i(\theta_i)$ varies continuously in dependence of δ, which implies that there is some value $\delta_0^{(i)}$ such that $\Psi_i(\theta_i) = 1/2$.

Denote by $U_i(\delta|z)$ and $\Psi_i(\delta|t)$ the generating functions $U_i(z)$ and $\Psi_i(t)$ relative to the measure defined in (6.2).

Recall that

$$\Psi_i(\delta|\theta_i) = \frac{1}{U_i'(\delta|1) + 1 - U_i(\delta|1)}.$$

Since $U_i(\delta|1)$ can be rewritten as a power series in the variable δ, the function $\delta \mapsto \Psi_i(\delta|\theta_i)$ is continuous in δ. This finishes the proof. □

Therefore, an example for Figure 6.6 is $\Gamma = \mathbb{Z}^5 * \mathbb{Z}^6$ with measures μ_1 and μ_2 chosen in such a way that $\Psi_1(\theta_1) = \Psi_2(\theta_2) = 1/2$. Obviously, we have $\Upsilon(\alpha_c) = \Psi_1(\theta_1) + \Psi_2(\theta_2) - 1 = 0$, implying the following possible asymptotic behaviors:

$$\mu^{(2n)}(e) \sim \begin{cases} C_1 \cdot \mathbf{R}^{-2n} \cdot n^{-5/2}, & \text{if } \alpha_1 > \alpha_c, \\ C_2 \cdot \mathbf{R}^{-2n} \cdot n^{-3/2}, & \text{if } \alpha_1 = \alpha_c, \\ C_3 \cdot \mathbf{R}^{-2n} \cdot n^{-3}, & \text{if } \alpha_1 < \alpha_c. \end{cases}$$

As a final remark let us explain that if $\Upsilon(\alpha_1) > 0$, it is not possible that $\Upsilon(\alpha_1)$ is strictly increasing or decreasing for all $\alpha_1 \in (0,1)$. In order to show this assume (by contradiction) that $\Upsilon(\alpha_1)$ is strictly increasing. Then, by Lemma 6.1.1, we have that $\theta_2 = \infty$ must hold, that is, $G_2(\mathbf{R}_2) = \infty$.

The same reasoning as in Equation (5.1) leads to $\lim_{z\to\mathbf{R}_2} \Psi_2\bigl(zG(z)\bigr) = \lim_{t\to\infty} \Psi_2(t) = 0$. Therefore, for α_1 small enough we obtain

$$\Psi(\bar{\theta}) = \Psi_1(\theta_1) + \Psi_2\left((1-\alpha_1)\frac{\theta_1}{\alpha_1}\right) - 1 < 0.$$

Analogously, under the assumption $\Upsilon(\alpha_1)$ strictly decreasing, we find out that $\Psi(t)$ must have a zero.

6.2 Higher Asymptotic Orders

The techniques we used for determining the asymptotic behavior give us not only the leading term $n^{-\lambda}\log^{\kappa} n$, but also the following *higher order* terms, according to the singular terms in the expansion after the leading one. For instance, consider a nearest neighbor random walk on $\mathbb{Z}^7 * \mathbb{Z}^8$ with $\alpha_1 = \theta_1/(\theta_1+\theta_2)$. Then the associated Green function has the following expansion:

$$\begin{aligned}\sum_{k=0}^{4} g_k(\mathbf{R}-z)^4 + \hat{g}_1(\mathbf{R}-z)^{5/2} + \check{g}_1(\mathbf{R}-z)^3\log(\mathbf{R}-z) \\ + \hat{g}_2(\mathbf{R}-z)^{7/2} + \check{g}_2(\mathbf{R}-z)^4\log(\mathbf{R}-z) + \mathbf{o}\bigl((\mathbf{R}-z)^4\bigr),\end{aligned}$$

where $\hat{g}_1 \neq 0$. That is,

$$\mu^{(2n)}(e) \sim \mathbf{R}^{-2n} \cdot \bigl(C_1\, n^{-7/2} + C_2\, n^{-4} + C_3\, n^{-9/2} + C_4\, n^{-5} + \mathbf{o}(n^{-5})\bigr),$$

where $C_1 \neq 0$.

Pictures of the different possible Behaviors

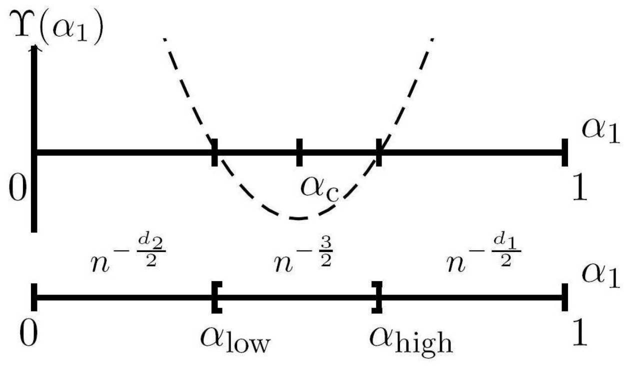

Figure 6.1: Type A

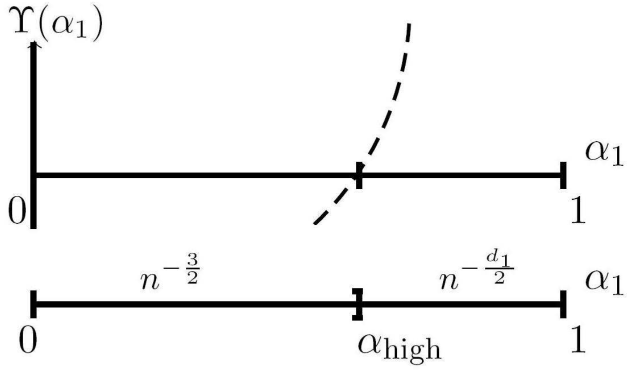

Figure 6.2: Type B

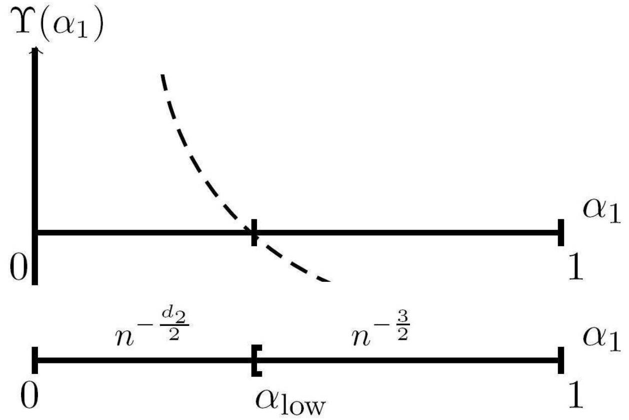

Figure 6.3: Type C

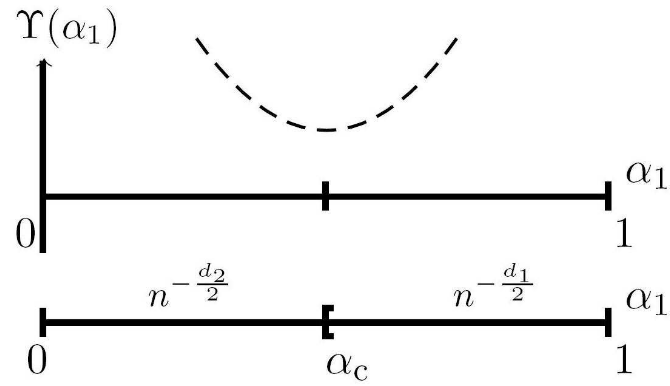

Figure 6.4: Type D

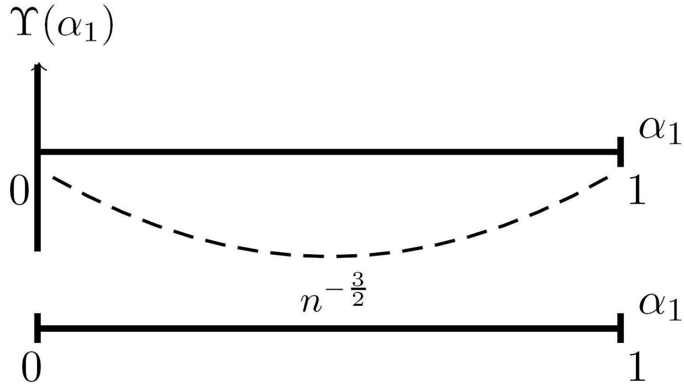

Figure 6.5: Type E

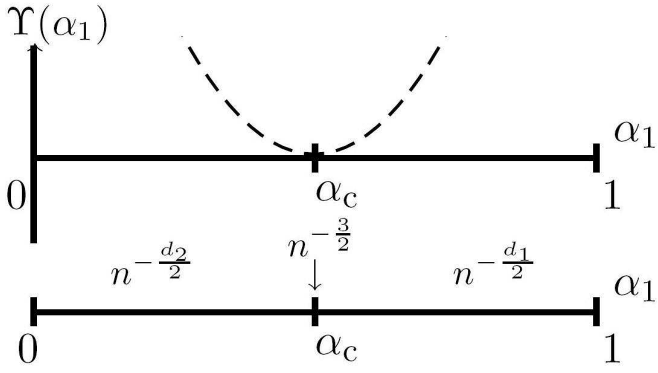

Figure 6.6: Type A - singleton

Part II

Branching Random Walks on Free Products

Chapter 7

Branching Random Walks on Free Products

A Branching Random Walk (BRW for short) is a stochastic process, discrete in space and time, which we can visualize as a cloud of particles, growing and moving. It is characterized by two kinds of randomness, that we can describe as follows: at each unit of time every alive particle has a random amount of offspring, afterwards the old particles die and in the meanwhile the new particles move independently of each other, performing one step according to a underlying random walk.

We investigate BRW's defined on free products of groups, more precisely we look at their *trace* in the so-called *weak-survival* case.

7.1 Definition of the BRW

Let us fix two probability measures $\nu : \mathbb{N} \cup \{0\} \to [0,1]$ and $\mu : \Gamma \to [0,1]$. The former is the *offspring distribution*, while the latter is the distribution of the *underlying random walk*.
The law ν is defined such that for all $k \in \mathbb{N} \cup \{0\}$

$$\nu_k := \text{probability that a particle has exactly } k \text{ offspring.}$$

We have already described in Chapter 3 the features of the law μ: recall Equation 3.1. From now on we assume that μ is *symmetric*.

As it can easily be seen, ν and μ are completely independent of each other: they are fixed at the beginning and do not change throughout the whole time of the process.

A BRW is a process that can be defined inductively, and this description can be better followed with the help of Figures 7.1–7.5:

Step 1: We start with one particle at the root (a distinguished vertex), we will always assume that the root is the element e; see Figure 7.1.

Step 2: The particle splits into a random number of offspring, according to the law ν, afterwards it dies; see Figure 7.2.

Step 3: The new particles perform (independently of each other) one step according to the law μ; see Figure 7.3.

Step 4: As soon as they reach the new vertex, they split again (independently) into a random amount of particles, according to ν, and so on; see Figures 7.4 and 7.5.

Among the most referred books about BRW's there are the work of Athreya and Ney (see [2]) and the pioneer book by Harris (see [28]), whose contribution has been collected in [1]. To have quite a good understanding of this concept, the reader is invited to consult these references.

Let $\mathbb{E}\nu := \sum_{k\geq 0} k\nu_k$ denote the expected value of the offspring distribution. We mention here some of the fundamental results about BRW's.

Long ago, in 1873, Francis Galton asked the question regarding the probability of survival of surnames. The following year, Reverend Henri Watson gave him a solution, and they published the well-known paper [60]. The main result is the following:

$$\begin{aligned} \mathbb{E}\nu \leq 1 \quad &\Longrightarrow \quad \mathbb{P}(\text{the process dies out within a finite time}) = 1, \\ \mathbb{E}\nu > 1 \quad &\Longrightarrow \quad \mathbb{P}(\text{the process survives for infinite time}) > 0. \end{aligned}$$

A BRW on a Cayley graph is called *recurrent* if each vertex is visited infinitely many times and *transient* if any finite subset is eventually free of particles. The interested reader can find useful explanations about Galton–Watson processes and Branching Markov Chains in [66, Chapter 5].

In the nineties, Benjamini and Peres showed another fundamental result (see [4]): if the given structure is non-amenable, there is a value $R > 1$ such that

$$\begin{aligned} \mathbb{E}\nu < R \quad &\Longrightarrow \quad \text{the BRW is transient}, \\ \mathbb{E}\nu > R \quad &\Longrightarrow \quad \text{the BRW is recurrent}. \end{aligned}$$

Moreover they find that R coincides with the inverse of the spectral radius of the corresponding underlying random walk. By analogy to the previous part of the work, we replace the notation R by $\mathbf{R}$.

Gantert and Müller (see [20]) showed that at the critical value $\mathbb{E}\nu = \mathbf{R}$ the process is transient.

Whenever we consider the situation $1 < \mathbb{E}\nu \leq \mathbf{R}$, we say that the process is in the *weak survival case*, while if $\mathbb{E}\nu > \mathbf{R}$ we are speaking about the *strong survival case*.

We also refer to [26] for the corresponding result in the continuous setting.

In view of Section 2.1.2, a natural question that can be asked is "*how big is the random set of accumulation points of the BRW, in relation to the whole boundary of the free product?*"

Denote by Λ the random set of accumulation points of the process, and by Ω the boundary of Γ.

In the specific case of homogeneous trees, a precise answer comes from the work of Hueter and Lalley [31], where they prove that in the weak survival case (the only non-trivial case):

$$\dim(\Lambda) \leq \frac{1}{2}\dim(\Omega),$$

where $\dim(\cdot)$ can denote the Box-counting dimension or the Hausdorff dimension. We will prove (see Theorems 8.1.1 and 9.1.1) that in our setting they coincide.

Recall from Section 2.1.2 that for every set $\Omega' \subset \Omega$ we denote by $\mathrm{HD}(\Omega')$ its Hausdorff dimension, and by $\mathrm{BD}(\Omega')$ its Box-counting dimension. In order not to create confusion, we will be consistent with this notation.

An interesting phenomenon that we find on free products is the following: we can divide the set Λ into two subsets: one consists of the *typical ends*, and the other of the *atypical ends*. We speak of typical accumulation points when we consider ends arising as limit of infinite words, while the atypical accumulation points arise from parts of the process remaining infinitely often in a copy of one of the factors. This is of course possible in case of transiency of the process, but the necessary condition is that at least one factor is infinite. We will give a characterization of this phenomenon (see Theorem 7.4.1) and we prove that the amount of the *atypical accumulation points* does not affect the Hausdorff (Box-counting) dimension of Λ, see Corollary 8.1.2.

Similar results are proved in the deterministic case where we show that the atypical ends of the boundary of the free product do not give any contribution to the Hausdorff (or Box-counting) dimension of Ω (see Corollary 9.1.2).

In case Γ is a free product of finite groups, we present a simpler version of our results (see Chapter 10). In the same chapter we show that everything holds as well if we consider free products with amalgamation.

At this point we can define a very useful concept, i.e. the *type* of a word. For every $u = u_1 u_2 \cdots u_m \in \Gamma$ define the type of u as follows:

$$\tau(u) = i \tag{7.1}$$

if its last "block" $u_m \in \Gamma_i^{\times}$. Moreover we set $\tau(e) := 0$.

Remark 7.1.1. *Recall that in this second part of the work we always assume that the random walk on Γ is symmetric, unless otherwise explicitly stated. This assumption can be dropped for free products of finite groups. In this case the crucial property $F(e, x|\mathbf{R}) < 1$ holds for all $x \in \Gamma \setminus \{e\}$. More precisely, let us point out that in the symmetric case we have*

$$G(e, e|z) > F(e, x|z)G(x, x|z)F(x, e|z),$$

implying $F(e, x|\mathbf{R}) < 1$. But if the groups are finite we get, using [64, Lemma 17.1 and Proposition 9.8] for any $x = x_1 \dots x_m \in \Gamma \setminus \{e\}$

$$F(e, x_1 \dots x_m|\mathbf{R}) = \prod_{j=1}^{m} F_{\tau(x_j)}\big(e_{\tau(x_j)}, x_j \mid \zeta_{\tau(x_j)}(\mathbf{R})\big) < 1,$$

because $\zeta_i(\mathbf{R}) < 1$, since $G_i(e_i, e_i \mid 1) = \infty$ on a finite group Γ_i.

7.2 Pictures of the inductive Steps of a BRW

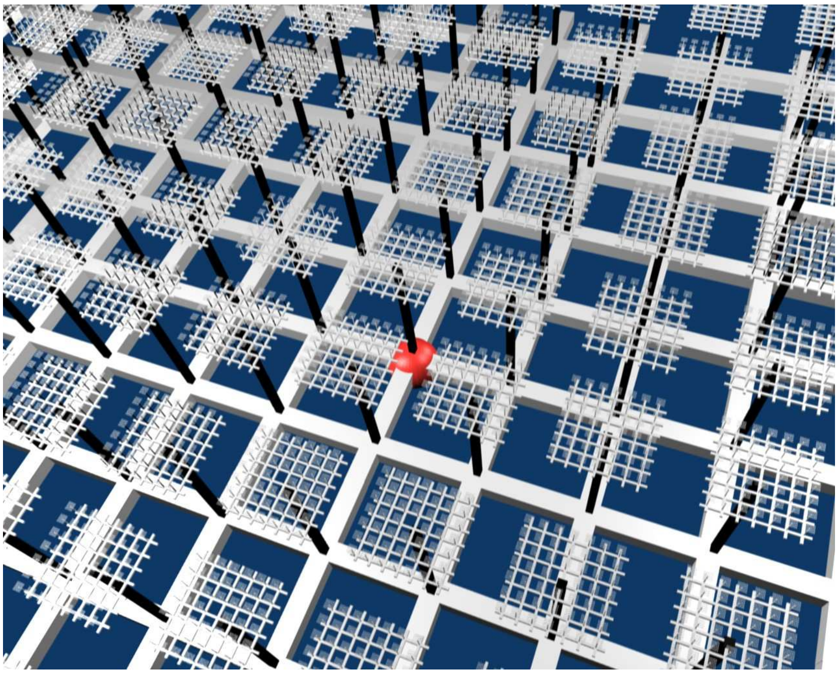

Figure 7.1: Step 1, one particle at e

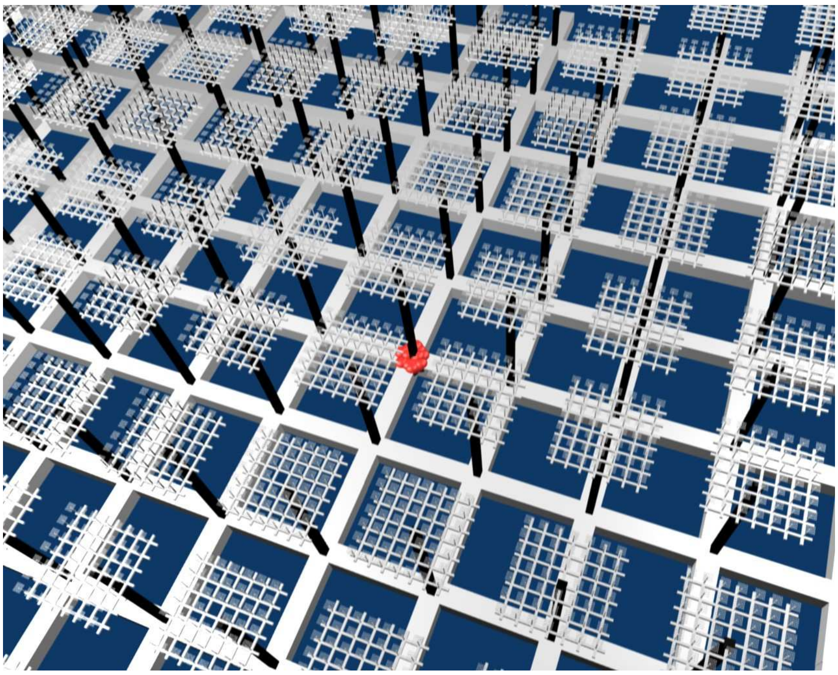

Figure 7.2: Step 2, offspring at e

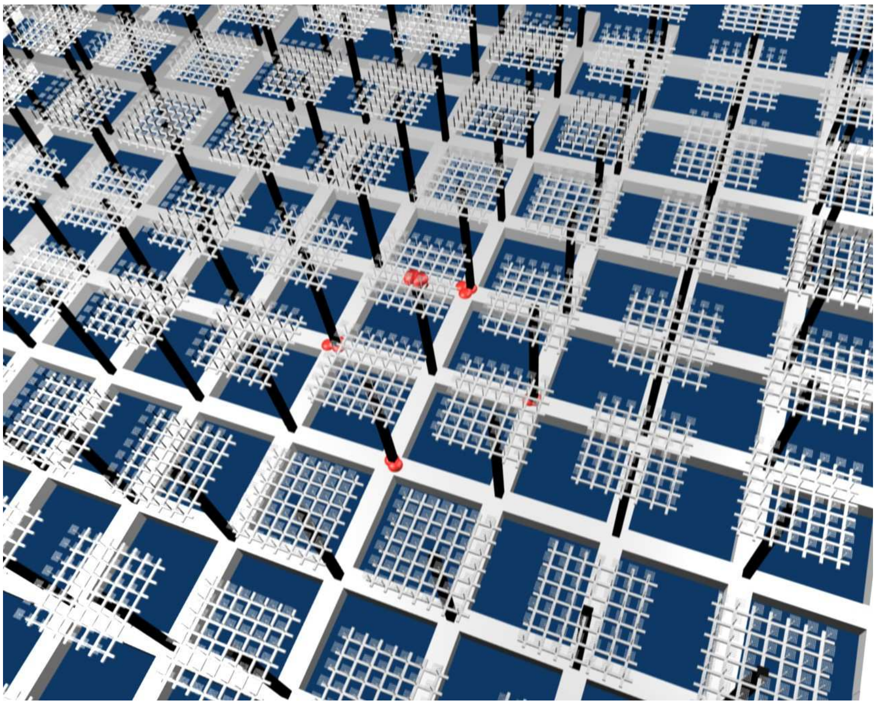

Figure 7.3: Step 3, the offspring move

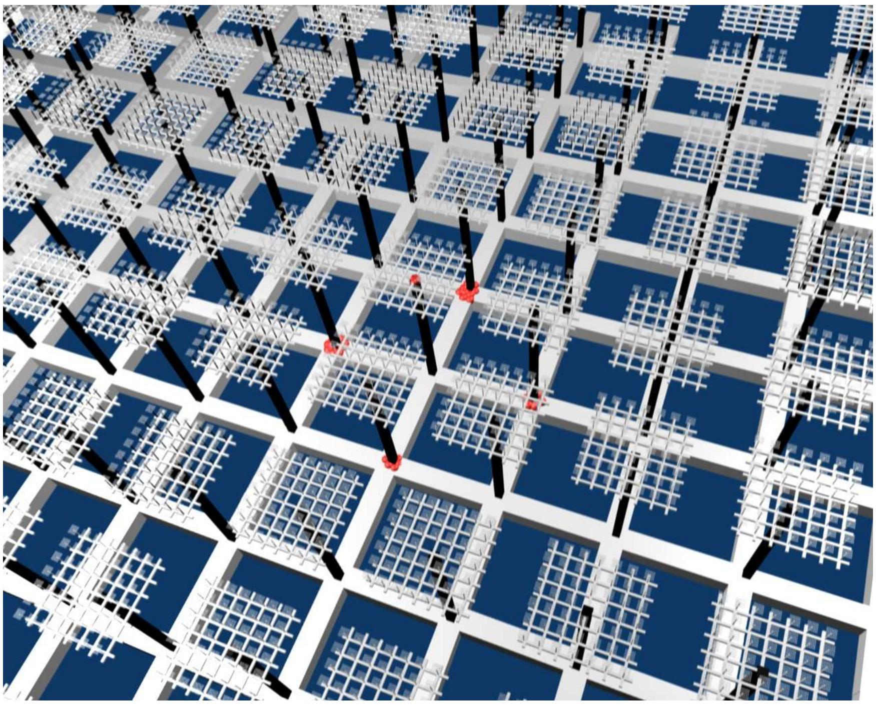

Figure 7.4: Step 4, the offspring have descendants

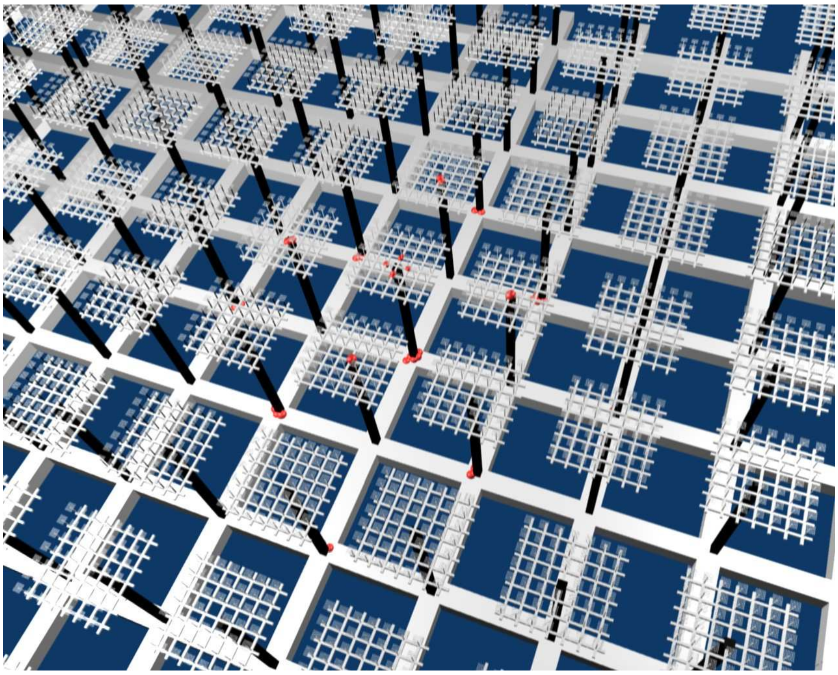

Figure 7.5: Repetition, the descendants move

7.3 Tree-indexed BRW's

For sake of clarity, we would like to summarize the main argument described in [4].

Let $\mathscr{T}$ be a rooted infinite tree. The root is denoted by o and let v be any vertex of $\mathscr{T}$. Then we denote by $l(v)$ the (graph) distance of v from o.

The random walk on Γ indexed by $\mathscr{T}$ is the collection of Γ-valued random variables $(S_v)_{v\in\mathscr{T}}$ defined as follows: label the edges of $\mathscr{T}$ with i.i.d. random variables η_v with distribution μ, i.e. the random variable η_v is the label of the edge (v^-, v).
Define $S_v := e \cdot \prod_{i=1}^{l(v)} \eta_{v_i}$ where $\langle v_0 = o, v_1, \ldots, v_n = v\rangle$ is the path, up to level n, of the unique geodesic connecting the root o to v.

A tree-indexed random walk becomes a BRW if the underlying tree is a Galton–Watson tree induced by the offspring distribution ν (for a detailed description about Galton–Watson trees the reader is referred to [38, Section 1 and Chapter 5]).

We will refer to $\mathscr{T}$ as the *family tree*. More precisely, a vertex $v \in \mathscr{T}$ is a particle of the BRW, and the vertices at level n of $\mathscr{T}$ represent the particles alive at generation n.

7.3.1 Colored BRW

A variation of the BRW is the *colored BRW*, see [31]. This process behaves like a standard BRW where in addition each particle is either blue or red. In order to define this colored version we choose a subset M of Γ that plays the role of a "paint bucket". We start the BRW with one blue particle at e. Blue particles located outside M produce blue offspring. A blue particle that hits M is frozen (makes no further movement and has no offspring anymore) and is replaced by a red particle. This new particle starts an ordinary (red-colored) branching random walk. As a consequence, every red particle has exactly one "frozen" ancestor in M.

We denote by $Z_\infty(M) \in \mathbb{N} \cup \{\infty\}$ the random number of (blue) frozen particles in M during the whole branching process. If $M = \{x\}$ then we just write $Z_\infty(x)$.

Depending on the different purpose, we will be freely switching between the different versions of the BRW; nevertheless it will always be clear from the context which description we are using.

7.4 Typical vs. Atypical Accumulation Points

Analogously to [31], we have that the Hausdorff (or Box-counting) dimension of the set Λ depends only on the expected value $\mathbb{E}\nu$. To simplify the notation, we denote it by $\lambda := \mathbb{E}\nu$. Recall that $\mathbf{R}$ is the inverse of the spectral radius of the random walk governed by μ, i.e. the critical value described by [4].

Recall the notation introduced in Section 2.1 and Equation 3.4.

The aim of this section is to prove the following result:

Theorem 7.4.1. *Consider* $\lambda \in (1, \mathbf{R}]$. *Then* $\mathbb{P}\big[\Lambda \cap \Omega_i \neq \emptyset\big]$ *is either* 0 *or* 1, *and* $\mathbb{P}\big[\Lambda \cap \Omega_i \neq \emptyset\big] = 1$ *if and only if* $\zeta_i(\lambda) > 1$. *More precisely:*

1. *If* $\zeta_i(\lambda) \leq 1$ *then* $\emptyset \subsetneq \Lambda \subseteq \Omega_\infty$.
2. *If* $\zeta_i(\lambda) > 1$ *then* $\emptyset \subsetneq (\Omega_\infty \cap \Lambda) \subset \Lambda$ *with* $\Lambda \cap \Omega_i \neq \emptyset$ *and we have that* $\mathrm{Card}(\Lambda \cap \Omega_i) = \infty$.

The last possibility can only show up if Γ_i is infinite, for some index $i \in \mathcal{I}$.

Remark 7.4.2. *If one of the free factors is an infinite amenable group, its ends do not appear in* Λ. *In other words, if* $\mathbf{R}_i = 1$ *is the radius of convergence of* $G_i(e_i, e_i|z)$ *then* $\zeta_i(\lambda) \leq 1$ *for all* $\lambda \in (1, \mathbf{R}]$; *see [64, Lemma 17.1a]. Consequently, none of the ends belonging to* Ω_i *contribute to* Λ, *that is,* $\Lambda \cap \Omega_i = \emptyset$ *almost surely.*

Before proving the theorem, we illustrate the above described behavior by two examples:

Example 7.4.3. *Consider* $\Gamma = \mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ *and let* μ_1 *and* μ_2 *be two symmetric probability measures on* $\mathbb{Z}^{d_1}$ *and* $\mathbb{Z}^{d_2}$ *respectively. Due to Kesten's amenability criterion (see Section 1.3.4) we have* $\mathbf{R}_1 = \mathbf{R}_2 = 1$. *Consequently,* $\Lambda \subseteq \Omega_\infty$ *almost surely for all* $\lambda \leq \mathbf{R}$.

Example 7.4.4. *Consider* $\Gamma = \Gamma_1 * \Gamma_2$, *where* Γ_1 *and* Γ_2 *are non-amenable groups, and let* μ_i *define a symmetric random walk on* Γ_i *for* $i \in \{1, 2\}$.

Non-amenability implies $\mathbf{R}_1, \mathbf{R}_2 > 1$ *and* $G_i(\mathbf{R}_i) < \infty$.
Choosing

$$\alpha_1 = \frac{\mathbf{R}_1\, G_1(\mathbf{R}_1)}{\mathbf{R}_1\, G_1(\mathbf{R}_1) + \mathbf{R}_2\, G_2(\mathbf{R}_2)} = \frac{\theta_1}{\theta_1 + \theta_2}$$

we obtain by [64, Lemma 17.1] that $\zeta_1(\mathbf{R}), \zeta_2(\mathbf{R}) > 1$. *Therefore, there are values* $\lambda_1, \lambda_2 \in (1, \mathbf{R})$ *such that* $\zeta_1(\lambda_1) = \zeta_2(\lambda_2) = 1$. *This determines a change of behavior at* λ_1 *and* λ_2.

7.4.1 Proof of Theorem 7.4.1

Using the description of a tree-indexed random walk (see [4]) it is easy to see that the distribution of the location of a particle at the n-th generation has the same distribution as the location of a (non-branching) random walk on Γ after n steps.

In other words:

Fact 7.4.5. *Let* $v \in \mathscr{T}$ *with* $l(v) = n$ *for some* $n \geq 1$. *Then*

$$\mathbb{P}[S_v = y] = P[X_n = y] = \mu^{(n)}(y).$$

Consider the colored BRW: the next Lemma gives us a formula for the expected number of elements frozen in a set $M \subseteq \Gamma$. A proof of this fact can be found for example in [45] or [31, Lemma 1].

Lemma 7.4.6. *For any* $M \subseteq \Gamma$, *we have* $\mathbb{E}\big[Z_\infty(M)\big] = F(e, M|\lambda)$.

For seek of clarity, we split the proof of Theorem 7.4.1 into the proof of Propositions 7.4.7, 7.4.8 and 7.4.9.

Recall from Section 2.1 that $\Omega_i^{(0)} \subseteq \Omega_i \subseteq \Omega$.

Proposition 7.4.7. *Ends of* $\Omega_i^{(0)}$ *belong to* Λ *with positive probability if and only if* $\zeta_i(\lambda) > 1$, *i.e.,* $\mathbb{P}\big[\Lambda \cap \Omega_i^{(0)} \neq \emptyset\big] > 0$ *if and only if* $\zeta_i(\lambda) > 1$.

Proof. In this context it is convenient to work with the colored BRW. In fact, the idea of the proof is to define an embedded Galton–Watson process with mean value $\zeta_i(\lambda)$, that counts the number of particles that hit Γ_i.

The BRW starts with one particle at e. Recall from Section 1.2.2 that by construction we have that every identity element e_i of Γ_i is identified with e. The first generation of the embedded Galton–Watson process consists of all particles of the BRW frozen at $\Gamma_i^\times$.

Since μ has finite support, every particle visiting $\Gamma_i^\times$ has to pass through $\mathrm{supp}(\mu_i)$. Hence, $Z_\infty(\Gamma_i^\times) = Z_\infty(\mathrm{supp}(\mu_i))$, which is almost surely finite, because the BRW is transient. Therefore the amount of particles of the first generation is almost surely finite.

Now let us fix a vertex $x \in \Gamma_i^\times$. For each particle frozen at x we start a new BRW where particles freeze when reaching $\Gamma_i \setminus \{x\}$. The second generation of the embedded Galton–Watson process consists of all these newly-frozen particles.

Further generations are constructed inductively in the same way. Let ψ_n denote the amount of particles of this process at generation n, and let $F(e,x|z)$ be as defined in Equation (3.2). Then $(\psi_n)_{n\geq 0}$ is a Galton–Watson process with mean

$$m_i = \mathbb{E}\big[Z_\infty\big(\mathrm{supp}(\mu_i)\big)\big] = F\big(e, \mathrm{supp}(\mu_i)|\lambda\big) = \zeta_i(\lambda).$$

Hence, this Galton–Watson process survives with positive probability if and only if $\zeta_i(\lambda) > 1$; see e.g. [28, Theorem 6.1] or the original paper by Galton and Watson [60]. Therefore, we have that

$$\zeta_i(\lambda) > 1 \quad \Longleftrightarrow \quad \Gamma_i \text{ is visited infinitely often by the process.}$$

The previous statement can be written as

$$\zeta_i(\lambda) > 1 \quad \Longleftrightarrow \quad \mathbb{P}\big[\Lambda \cap \Omega_i^{(0)} \neq \emptyset\big] > 0,$$

which concludes the proof. □

Recall from (7.1) the definition of *type* $\tau(\cdot)$ of a word $u = u_1 u_2 \cdots u_m$, i.e. the value $i \in \mathcal{I}$ such that $u_m = i$.

In the next proposition we show that

$$\zeta_i(\lambda) > 1 \quad \Longrightarrow \quad \mathrm{Card}(\Lambda \cap \Omega_i) = \infty.$$

Proposition 7.4.8. *If $\zeta_i(\lambda) > 1$ then there are almost surely infinitely many cosets $x\Gamma_i$, where the BRW accumulates. That is, the set*

$$\Big\{x \in \Gamma \,\Big|\, \tau(x) \neq i, x\Omega_i^{(0)} \cap \Lambda \neq \emptyset\Big\}$$

is almost surely infinite.

Proof. We construct the family tree $\mathscr{T}$ of the BRW with branching distribution ν in the following way. We start with one geodesic $v_\infty = \langle o, v_1, v_2, \ldots\rangle$ and attach to each of the vertices independent copies of Galton–Watson trees where the distribution of the first generation is $\tilde{\nu}(k) = \nu(k+1)$ for $k \geq 0$ and ν for the other generations.

As already argued in Fact 7.4.5, the trajectory along v_∞ has the same distribution of a non-branching random walk. Hence, S_{v_n} converges almost surely to a random infinite word $g_\infty = g_1 g_2 \ldots \in \Omega_\infty$ as $n \to \infty$ (here we mean convergence in the sense that the block length of the common prefix of the location of S_{v_n} and g_∞ tends to infinity).

Moreover, we define the random indices $n_1 := \min\{m \in \mathbb{N} \mid g_m \in \Gamma_i\}$, and recursively $n_k := \min\{m \in \mathbb{N} \mid m > n_{k-1}, g_m \in \Gamma_i\}$. Note that these indices are almost surely finite; see e.g. [24, Section 7.I].

Denote by $\hat{v}_k$ the first vertex in v_∞ such that $\hat{v}_k = g_1 \ldots g_{n_k}$, and by Λ_v the set of accumulation points of the descendants of any element $v \in \mathscr{T}$. Moreover, let B_k be the set of offspring of $\hat{v}_k$ different from the one on the geodesic connecting the root to v_∞.

Let A_k denote the following event:

$$A_k := \{\Lambda_v \cap S_v \Omega_i^{(0)} \neq \emptyset \text{ for some } v \in B_k \text{ s.t. } \tau(v) = i\}.$$

We point out that the events A_k are i.i.d. because by transitivity we have

$$\mathbb{P}[\Lambda_v \cap S_v \Omega_i^{(0)} \neq \emptyset] = \mathbb{P}[\Lambda \cap \Omega_i^{(0)} \neq \emptyset], \qquad \text{for every } v \in \mathscr{T}.$$

Now, due to Proposition 7.4.7 and the fact that

$$\begin{aligned}\mathbb{P}[B_k \neq \emptyset, \exists v \in B_k : \tau(S_v) = i] &= \big(1 - \nu(1)\big) \cdot \mathbb{P}[v \in B_k : \tau(S_v) = i \mid B_k \neq \emptyset] \\ &\geq \big(1 - \nu(1)\big) \cdot \alpha_i > 0\end{aligned}$$

we have $\mathbb{P}[A_k] \geq c$ for all k and some $c > 0$. The Borel–Cantelli Lemma finishes the proof. □

In order to complete the proof of Theorem 7.4.1, we still need to look at the critical and subcritical cases $\zeta_i(\lambda) \leq 1$, and this is done in the next

Proposition 7.4.9. *If $\zeta_i(\lambda) \leq 1$ then $\mathbb{P}[\Lambda \cap \Omega_i \neq \emptyset] = 0$.*

Proof. From Proposition 7.4.7 it follows that $\mathbb{P}\left[\Lambda \cap x\Omega_i^{(0)} \neq \emptyset\right] = 0$ for all $x \in \Gamma$: indeed, each $x \in \Gamma$ is visited only a finite amount of times almost surely. Each particle that hits x, starts its own BRW there and each of these BRW's hits $x\Omega_i^{(0)}$ only finitely many times with probability one.

Since we can write $\Lambda \cap \Omega_i$ as a disjoint union, i.e.

$$\Lambda \cap \Omega_i = \bigsqcup_{x \in \Gamma : \tau(x) \neq i} (\Lambda \cap x\Omega_i^{(0)}),$$

we have

$$\mathbb{P}\left[\Lambda \cap \Omega_i \neq \emptyset\right] = \sum_{x \in \Gamma : \tau(x) \neq i} \mathbb{P}\left[\Lambda \cap x\Omega_i^{(0)} \neq \emptyset\right] = 0,$$

which concludes the proof. □

Chapter 8

Dimension of Λ

The aim of this chapter is to show how to measure the size of Λ. i.e. the random set of accumulation points of the BRW.

The most important tool that we are going to exploit is a variation of the *growth function* defined in Section 1.3:

$$\mathcal{F}(\lambda|z) := \sum_{x\in\Gamma} F(e,x|\lambda)\, z^{l(x)}, \tag{8.1}$$

where we recall that $l(u)$ is the graph distance of u from the root.

For $i \in \mathcal{I}$ we define

$$\mathcal{F}_i^+(\lambda|z) := \sum_{x\in\Gamma_i^\times} F(e,x|\lambda)\, z^{l(x)} = \sum_{x\in\Gamma_i^\times} F_i\big(e_i,x\big|\zeta_i(\lambda)\big)\, z^{l(x)}, \tag{8.2}$$

and

$$\mathcal{F}_i(\lambda|z) := \sum_{n\geq 1} \sum_{\substack{x=x_1\dots x_n\in\Gamma:\\ x_1\in\Gamma_i^\times}} F(e,x|\lambda)\, z^{l(x)} = \mathcal{F}_i^+(\lambda|z)\Big(1+\sum_{j\in\mathcal{I}\setminus\{i\}}\mathcal{F}_j(\lambda|z)\Big). \tag{8.3}$$

By definition it follows:

$$\mathcal{F}(\lambda|z) = 1+\sum_{i\in\mathcal{I}}\mathcal{F}_i(\lambda|z). \tag{8.4}$$

We denote by $R(\mathcal{F})$ and by $R(\mathcal{F}_i^+)$ the radii of convergence of 8.1 and 8.2 respectively.

By relations (8.3) and (8.4) we get

$$\mathcal{F}_i(\lambda|z) = \mathcal{F}_i^+(\lambda|z)\big(\mathcal{F}(\lambda|z)-\mathcal{F}_i(\lambda|z)\big),$$

or equivalently

$$\mathcal{F}_i(\lambda|z) = \mathcal{F}(\lambda|z)\frac{\mathcal{F}_i^+(\lambda|z)}{1+\mathcal{F}_i^+(\lambda|z)}. \tag{8.5}$$

Hence we have

$$\mathcal{F}(\lambda|z) = 1+\sum_{i\in\mathcal{I}}\mathcal{F}_i(\lambda|z) = 1+\mathcal{F}(\lambda|z)\sum_{i\in\mathcal{I}}\frac{\mathcal{F}_i^+(\lambda|z)}{1+\mathcal{F}_i^+(\lambda|z)},$$

which leads to

$$\mathcal{F}(\lambda|z) = \frac{1}{1 - \sum_{i\in\mathcal{I}} \frac{\mathcal{F}_i^+(\lambda|z)}{1+\mathcal{F}_i^+(\lambda|z)}}. \tag{8.6}$$

This relation holds for every $z \in \mathbb{C}$ with $|z| < R(\mathcal{F})$.

8.1 Box-counting Dimension & Hausdorff Dimension

The the main result that we prove is the following:

Theorem 8.1.1. *Suppose that ν has finite second moment. Then the box-counting dimensions of Λ and $\Lambda\cap\Omega_\infty$ exist and equal the Hausdorff dimensions of Λ and $\Lambda\cap\Omega_\infty$ respectively. Furthermore:*

$$\mathrm{BD}(\Lambda) = \mathrm{BD}(\Lambda\cap\Omega_\infty) = \mathrm{HD}(\Lambda) = \mathrm{HD}(\Lambda\cap\Omega_\infty) = \frac{\log z^*}{\log\alpha},$$

where z^* is the smallest real positive number such that

$$\sum_{i\in\mathcal{I}} \frac{\mathcal{F}_i^+(\lambda|z^*)}{1+\mathcal{F}_i^+(\lambda|z^*)} = 1. \tag{8.7}$$

A first consequence of Theorem 8.1.1 that we obtain, is that only the set of infinite words contributes to the dimension of Λ:

Corollary 8.1.2. *For $i \in \mathcal{I}$, $\mathrm{HD}(\Lambda\cap\Omega_i) < \mathrm{HD}(\Lambda\cap\Omega_\infty)$.*

The proof of Theorem 8.1.1 is split into two sections. In Section 8.1.1 we find the upper bound for the Box-counting dimension, that is the same as the lower bound computed in Section 8.1.2.

8.1.1 Upper Bounds

The aim of this section is to show that $\log z^*/\log\alpha$ is an upper bound for $\overline{\mathrm{BD}}(\Lambda)$. For this purpose we estimate the growth rate of the set of visited sites. Therefore denote by

$$\mathcal{H}_n := \left\{x\in\Gamma \,\middle|\, l(x) = n,\ x \text{ is visited by the BRW}\right\}. \tag{8.8}$$

Since the random walk governing the considered BRW is of nearest-neighbor type, we are sure that there are no jumps along the paths. Recall that on the boundary we are working w.r.t. the metric defined by Equation (2.1). Therefore, for every $m \in \mathbb{N}$, the set of accumulation points of the process can be covered by a *finite* amount of balls of radius α^m.

Finiteness follows from the fact that the free product itself is locally finite, and that the BRW has finite support.

From now on, B_m denotes the ball of radius m (in the Cayley-graph distance) centered at the origin. In formulas we have

$$\Lambda \subseteq \bigcup_{x\in\mathcal{H}_m} \{\omega\in\Omega \mid x \text{ lies in the } \omega\text{-component of } \mathcal{X}\setminus B_{m-1}\}.$$

The growth rate of the set of occupied vertices is given by

$$\lim_{m\to\infty} (\mathrm{Card}(\mathcal{H}_m))^{1/m},$$

which is what we are going to estimate.

Recall from Section 7.3.1 the definition of $Z_\infty(x)$. We remark that $x \in \mathcal{H}_m$ if and only if $Z_\infty(x) \geq 1$. Therefore, by Lemma 7.4.6,

$$1 \leq \mathbb{E}\,\mathrm{Card}(\mathcal{H}_m) \leq \sum_{x\in\Gamma:l(x)=m} \mathbb{E}Z_\infty(x) = \sum_{x\in\Gamma:l(x)=m} F(e,x|\lambda) =: H_m.$$

Since each vertex can be reached more than once, we have that $H_{m+n} \geq H_m H_n$ and hence Fekete's lemma (see [16, Satz II] and [56, Lemma 11.6]) implies that $\lim_{m\to\infty} H_m^{1/m}$ exists.

We can rewrite Equation (8.1) as $\mathcal{F}(\lambda|z) = \sum_{m\geq 0} H_m\, z^m$. With this notation, Equation (8.6) yields

$$1 \leq \lim_{m\to\infty} H_m^{1/m} = 1/R(\mathcal{F}), \tag{8.9}$$

therefore

$$R(\mathcal{F}) \leq 1. \tag{8.10}$$

By Pringsheim's Theorem $R(\mathcal{F})$ corresponds to the smallest singularity on the positive x-axis of $\mathcal{F}(\lambda|z)$. This value is either one of the $R(\mathcal{F}_i^+)$'s, or the smallest positive number z^* such that

$$\sum_{i\in\mathcal{I}} \frac{\mathcal{F}_i^+(\lambda|z^*)}{1+\mathcal{F}_i^+(\lambda|z^*)} = 1.$$

Before proving that in fact $R(\mathcal{F}) = z^*$, we still need to introduce a few definitions: we will need the so-called *last visit generating functions*. We define:

$$L_i(x_i,y_i|z) := \sum_{n\geq 0} \mathbb{P}\big[Y_n^{(i)} = y_i, \forall 1\leq m\leq n : Y_m^{(i)} \neq x_i \mid Y_0^{(i)} = x_i\big] z^n \quad \text{and}$$
$$L(x,y|z) := \sum_{n\geq 0} \mathbb{P}\big[X_n = y, \forall 1\leq m\leq n : X_m \neq x \mid X_0 = x\big] z^n.$$

By conditioning the random walk on its *first* visit to y_i (on Γ_i) or to y (on Γ), and its *last* visit to x_i (on Γ_i) or to x (on Γ) we obtain:

$$\begin{aligned} G_i(x_i,y_i|z) &= F_i(x_i,y_i|z)\cdot G_i(y_i,y_i|z) = G_i(x_i,x_i|z)\cdot L_i(x_i,y_i|z), \\ G(x,y|z) &= F(x,y|z)\cdot G(y,y|z) = G(x,x|z)\cdot L(x,y|z). \end{aligned} \tag{8.11}$$

Thus, by transitivity we obtain

$$F(x,y|z) = L(x,y|z) \text{ for any } x,y\in\Gamma \text{ and } |z| \leq \mathbf{R}. \tag{8.12}$$

Let $x,y,w \in \Gamma$ be such that all paths of the random walk going from x to w must pass through y. Then

$$F(x,w|z) = F(x,y|z)\cdot F(y,w|z), \quad L(x,w|z) = L(x,y|z)\cdot L(y,w|z); \tag{8.13}$$

this can be checked by conditioning the paths from x to w on the first/last visit to y.

Now we can show that $R(\mathcal{F}) = z^*$. We split this proof into two lemmas: first we show that $R(\mathcal{F})$ must be positive and strictly smaller than 1. Afterwards we show that $R(\mathcal{F}) < R(\mathcal{F}_i^+)$, giving the proposed result.

Lemma 8.1.3. $R(\mathcal{F}) \in (0, 1)$.

Proof. The Cayley graph of Γ grows at most at exponential rate, therefore $R(\mathcal{F}) > 0$.

To see that $R(\mathcal{F}) < 1$ recall from Equation (8.11) that $F(e, x|\lambda)$ and $G(e, x|\lambda)$ are comparable, i.e., $G(e, x|\lambda) = F(e, x|\lambda)G(e, e|\lambda)$.

Hence, for some $C > 0$ we have that for all $m \in \mathbb{N}$

$$\sum_{x:l(x)\le m} F(e, x|\lambda) \ge C \sum_{x:l(x)\le m} G(e, x|\lambda).$$

The sum on the right hand side is the expected number of times (i.e. the total expected occupation time) that the BRW visits the ball B_m. Since the underlying random walk is of nearest-neighbor type, all particles up to generation m must be contained in the ball B_m. Moreover we know that the expected population size at time m is λ^m (recall that $\lambda > 1$). Therefore we have

$$\sum_{x:l(x)\le m} F(e, x|\lambda) = \sum_{k\le m} H_k \ge C \sum_{x:l(x)\le m} G(e, x|\lambda) \ge \lambda^m.$$

Taking the limit on $m \to \infty$ we obtain:

$$\lim_m \left(\sum_{k\le m} H_k \right)^{1/m} \ge \lim_m (\lambda^m)^{1/m} > 1,$$

Therefore H_m grows exponentially. By relation (8.9) we have the statement. $\square$

In the next lemma we show that $R(\mathcal{F})$ must be the solution of (8.6).

Lemma 8.1.4. *For all $i \in \mathcal{I}$, $R(\mathcal{F}) = z^* < R(\mathcal{F}_i^+)$.*

Proof. From an intuitive point of view, this must be true: the value $R(\mathcal{F})^{-1}$ represents the growth rate of the process on Γ, while $R(\mathcal{F}_i^+)^{-1}$ is the growth rate of its projection on the i-th factor. The proof is based on the following considerations: if the growth of the process on Γ_i is less than exponential, then in view of Lemma 8.1.3 this property is trivially true. While, if the process grows exponentially on each Γ_i, since the amount of Γ_i's at each level increases at exponential rate as well, it is becomes natural to guess that

$$R(\mathcal{F}_i^+)^{-1} < R(\mathcal{F})^{-1}.$$

We split the proof into two parts: first we investigate the situation $\zeta_i(\lambda) \le 1$, and then the case $\zeta_i(\lambda) > 1$.

Case $\zeta_i(\lambda) \le 1$: In view of Proposition 7.4.7 we know that in this situation the expected value of the process projected on Γ_i is at most 1. This implies that on each copy of Γ_i it will eventually die out almost surely, which means its growth is less than exponential. Therefore from Lemma 8.1.3 the statement follows.

Case $\zeta_i(\lambda) > 1$: In this case the growth of the projected process is at least exponential. Therefore we consider the following:

$$H_n = \sum_{\substack{x\in\Gamma:\\ l(x)=n}} F(e,x|\lambda) = \sum_{k=1}^{n} \sum_{\substack{x=x_1\dots x_k\in\Gamma:\\ l(x)=n}} \prod_{j=1}^{k} F(e,x_j|\lambda).$$

Another fact that we have to keep in mind is: $\sum_{x\in\Gamma_{i'}:l(x)=1} F(e,x|\lambda) \ge \zeta_{i'}(\lambda)$.

We can minorate H_n by conditioning the BRW on performing $\lfloor n/2\rfloor$ consecutive steps on Γ_i, and then alternating between Γ_i and another factor $\Gamma_{i'}$. This leads to

$$\begin{aligned} H_n &\ge \left(\frac{1}{R(\mathcal{F}_i^+)}\right)^{\lfloor n/2\rfloor} \sum_{k=1}^{\lceil n/2\rceil} \sum_{\substack{x=x_1\dots x_k\in\Gamma:\\ l(x)=\lceil n/2\rceil}} \prod_{j=1}^{k} F(e,x_j|\lambda) \\ &\ge \left(\frac{1}{R(\mathcal{F}_i^+)}\right)^{\lfloor n/2\rfloor} \left(\sum_{k=1}^{\lceil n/2\rceil} \binom{\lceil n/2\rceil}{k} \left(\frac{1}{R(\mathcal{F}_i^+)}\right)^{\lfloor n/2\rfloor - k} (\zeta_{i'}(\lambda))^k \right). \end{aligned}$$

The binomial coefficients come from the fact that we are counting all different possibilities that satisfy our assumption. This is the same as counting in how many ways we can place $(n - \lfloor n/2\rfloor - k)$ indistinguishable balls into k urns.

Applying the binomial theorem (a more general version that can also be used here is explained in [11]) we get

$$\begin{aligned} \lim_n (H_n)^{1/n} &\ge \sqrt{\frac{1}{R(\mathcal{F}_i^+)}} \sqrt{\frac{1}{R(\mathcal{F}_i^+)} + \zeta_{i'}(\lambda)} \\ &= \frac{1}{R(\mathcal{F}_i^+)} \sqrt{1 + R(\mathcal{F}_i^+)\zeta_{i'}(\lambda)} > \frac{1}{R(\mathcal{F}_i^+)}. \end{aligned} \tag{8.14}$$

□

The next lemma gives an almost sure upper bound for $|\mathcal{H}_m|^{1/m}$ as $m \to \infty$. Its proof is based on Markov's Inequality and the Borel–Cantelli Lemma.

Lemma 8.1.5. *Recall from (8.8) the definition of $\mathcal{H}_m$. We have*

$$\limsup_{m\to\infty} (\mathrm{Card}(\mathcal{H}_m))^{1/m} \le \frac{1}{z^*} \quad \text{almost surely.}$$

Proof. Choose a value $\varepsilon > 0$ and define the event

$$A_m := \left[\mathrm{Card}(\mathcal{H}_m)^{1/m} \ge \frac{1+\varepsilon}{z^*}\right].$$

Since

$$\limsup_{m\to\infty}\big(\mathbb{E}\left(\mathrm{Card}(\mathcal{H}_m)\right)\big)^{1/m} \le \limsup_{m\to\infty} H_m^{1/m} = \frac{1}{z^*},$$

there is a value $m_0 \in \mathbb{N}$ such that $\mathbb{E}\left(\mathrm{Card}(\mathcal{H}_m)\right) \le \left(\frac{1}{z^*} + \frac{\varepsilon}{2}\right)^m$ for all $m \ge m_0$.

Therefore, for large m, using Markov's inequality

$$\mathbb{P}[A_m] = \mathbb{P}\Big[\mathrm{Card}(\mathcal{H}_m) \ge \frac{(1+\varepsilon)^m}{(z^*)^m}\Big] \le \frac{(z^*)^m \mathbb{E}\left(\mathrm{Card}(\mathcal{H}_m)\right)}{(1+\varepsilon)^m} \le \frac{\big(1+z^*\varepsilon/2\big)^m}{(1+\varepsilon)^m}.$$

By Lemma 8.1.3 we know that $z^* < 1$, therefore the Borel–Cantelli Lemma yields that A_m occurs only finitely many times almost surely. In other words,

$$\limsup_{m\to\infty} \mathrm{Card}(\mathcal{H}_m)^{1/m} \le (1+\varepsilon)/z^*$$

almost surely. Since this inequality holds for every $\varepsilon > 0$, we get the proposed claim. □

Finally, the desired upper box-counting dimension is obtained:

Proposition 8.1.6.

$$\overline{\mathrm{BD}}(\Lambda) \le \frac{\log z^*}{\log \alpha}$$

Proof. Denote by $N(\alpha^m)$ the number of balls of radius at most α^m needed to cover the random set Λ. Then, for every $\varepsilon > 0$, we have that

$$N(\alpha^m) \le \mathrm{Card}(\mathcal{H}_m) \le \left(\frac{1}{z^*} + \varepsilon\right)^m$$

almost surely for sufficiently large m. Therefore,

$$\overline{\mathrm{BD}}(\Lambda) = \limsup_{m\to\infty} \frac{\log N(\alpha^m)}{-\log \alpha^m} \le \limsup_{m\to\infty} \frac{\log\big(\frac{1}{z^*}+\varepsilon\big)^m}{-\log \alpha^m} = \frac{\log\big(\frac{1}{z^*}+\varepsilon\big)}{-\log \alpha}.$$

Letting $\varepsilon \to 0$ proves the claim. □

8.1.2 Lower Bounds

In this section we show that $\log z^*/\log \alpha$ is also the lower bound for the Hausdorff dimension of Λ. From this fact we can conclude that the box-counting dimension exists, indeed $\mathrm{HD}(\Lambda) \le \underline{\mathrm{BD}}(\Lambda) \le \overline{\mathrm{BD}}(\Lambda)$.

The "skeleton" of the proof recalls the main ideas used to prove a similar result in [31, Section 6.3]: in order to help the reader follow, we use the same notation as [31].

The main idea is to construct a sequence of Galton–Watson trees τ_r to embed in the BRW, in such a way that the limit sets of the τ_r's are subsets of the limit set Λ.

Remark 8.1.7. *In this context, r denotes the parameter of the Galton–Watson trees, like in [31]: the offspring of a vertex x in τ_r are the vertices y at distance r from x, such that a particle of the BRW located at x has at least one descendant entering the level containing y, for the first time at y.*

Let us denote by Λ_{τ_r} the limit set of τ_r.

As r goes to infinity we have $\mathrm{HD}(\Lambda_{\tau_r}) \to \mathrm{HD}(\lambda)$. This approximation relies mainly on the following facts:

- the particles travel essentially along the geodesics;
- the limit sets of multy-type Galton–Watson trees are well understood.

These two facts are still true in the case of free products of finite groups, therefore the proof of the lower bound is similar to the one for homogeneous trees (see [31]).

The case with at least one infinite factor, needs some extra care. In this situation particles do not necessarily travel along geodesics, and infinite-type Galton–Watson processes are not so easy to handle. To overcome these difficulties we *approximate* the infinite factors by an increasing sequence of finite subgraphs. These, denoted by $\mathcal{X}_i^{(d)}$, are the ones induced by balls

$$B_i(d) := \{y \in \Gamma_i \mid l(y) \le d\}, \quad d \ge 1.$$

Letting $d \to \infty$ we get the optimal bound $\log z^* / \log \alpha$.

Fix a value $d \ge 1$. At this point we add an auxiliary vertex $\dagger_i$ to $\mathcal{X}_i^{(d)}$, which we call "the tomb". All edges in $\mathcal{X}_i$ exiting $B_i(d)$ now lead to $\dagger_i$.

The random walk $\big(Y_n^{(i,d)}\big)_{n \in \mathbb{N}_0}$ on $\mathcal{X}_i^{(d)}$ behaves like the random walk on Γ_i, with the exception that each particle that leaves $B_i(d)$ is sent to $\dagger_i$ (i.e. it dies).

The next step is the construction of the free product $\mathcal{X}^{(d)}$ whose free factors are the $\mathcal{X}_i^{(d)}$'s: analogously to Equation (1.1) we obtain

$$\begin{aligned}\mathcal{X}^{(d)} := \Big\{ x_1 \dots x_n \in \Gamma \ : \ n \in \mathbb{N}, x_j \in \bigcup_{i \in \mathcal{I}} \mathcal{X}_i^{(d)} \setminus \{e_i, \dagger_i\}, \\ \text{and } x_j \in \mathcal{X}_i^{(d)} \Rightarrow x_{j+1} \notin \mathcal{X}_i^{(d)} \Big\} \cup \{e, \dagger\},\end{aligned}$$

where $\dagger$ symbolizes the tomb on $\mathcal{X}^{(d)}$. Roughly speaking, we identify all tombs $\dagger_i$ into a single vertex $\dagger \in \mathcal{X}^{(d)}$.

We identify every $x \in \mathcal{X}^{(d)}$ with the corresponding element in Γ. Analogously to Section 3.2, we lift the random walks defined on $\mathcal{X}_i^{(d)}$ to a random walk $\big(X_n^{(d)}\big)_{n \in \mathbb{N}_0}$ defined on $\mathcal{X}^{(d)}$. This new measure is the one governing the associated BRW.

In order to avoid confusions, we write $G^{(d)}(x, y|z)$ for the Green function of the random walk on $\mathcal{X}^{(d)}$. In the same way, we denote the generating functions on $\mathcal{X}^{(d)}$ like the ones on Γ, but with an index "(d)" to distinguish the different settings.

Remark 8.1.8. *The comparison between the BRW defined on Γ and the one defined on $\mathcal{X}^{(d)}$ is in some sense very easy. There are particles of the first process which can come back to the origin e after exiting the ball $B_i(d)$, while in the second case, by definition of $\mathcal{X}_i^{(d)}$ they would be killed. Therefore the return probability in the first situation is (from the exponential point of view) at* *least as large as the second one. It follows easily that the radius of convergence of $G^{(d)}(x,y|z)$ is at least* $\mathbf{R}$.

From now on, unless otherwise stated, we refer to *geodesics* in the sense of the Cayley-graph distance.

For every $x, y \in \Gamma$, we define $\overline{x:y}$ to be the set of vertices $w \in \Gamma$ such that there is a geodesic from x to y which passes through w. For $u \in \Gamma$, we denote by $d(u, \overline{x:y})$ the minimal distance (w.r.t. the Cayley-graph metric) of u to any element of $\overline{x:y}$.

Now let us reason in terms of the colored BRW on $\mathcal{X}^{(d)}$. Let $Z_\infty^{(d)}(y|x)$ denote the total amount of blue particles arriving and freezing at $y \in \mathcal{X}^{(d)}$, under the assumption that the BRW starts with only one blue particle at x. For $r \in \mathbb{N}$, we denote by $Z_{\infty,r}^{(d)}(y|x)$ the total amount of particles counted in $Z_\infty^{(d)}(y|x)$ whose trails remain within distance r from a geodesic connecting x to y.

In other words, at all sites u such that $d(u, \overline{x:y}) > r$ every blue particle turns into a red one.

In the following we set $x_0 := x_1^{-1}$ for any $x = x_1 \dots x_m \in \mathcal{X}^{(d)}$. The proofs of the next two Lemmas follow step by step to the ones of [31, Lemma 4] and [31, Proposition 7]. Nevertheless we present the main ideas for completeness.

Lemma 8.1.9.

$$\lim_{r\to\infty} \inf_{x=x_1\dots x_m\in\mathcal{X}^{(d)}} \left(\frac{\prod_{j=1}^m \mathbb{E}Z_{\infty,r}^{(d)}(x_1\dots x_j|x_1\dots x_{j-1})}{\mathbb{E}Z_\infty^{(d)}(x|e)} \right)^{1/l(x)} = 1.$$

Sketch of the Proof. First of all we see that for every $x \in \mathcal{X}^{(d)} \setminus \{e\}$

$$\prod_{j=1}^m \mathbb{E}Z_{\infty,r}^{(d)}(x_1\dots x_j|x_1\dots x_{j-1}) \le \mathbb{E}Z_{\infty,r}^{(d)}(x|e) \le \mathbb{E}Z_\infty^{(d)}(x|e).$$

This implies that the seeked limit is at most 1.

In order to prove the other direction, consider an arbitrary element in $\mathcal{X}^{(d)}$, say $x = x_1 \dots x_m$, and apply Lemma 7.4.6 in this setting: we have $\mathbb{E}Z_\infty^{(d)}(x|e) = F^{(d)}(e,x|\lambda)$. Now observe that using Equations (8.11) and (8.13) we get

$$\mathbb{E}Z_\infty^{(d)}(x|e) = F^{(d)}(e,x|\lambda) = \frac{G^{(d)}(e,e|\lambda)}{G^{(d)}(x,x|\lambda)} \prod_{j=1}^m L_{\tau(x_j)}^{(d)}\big(e_{\tau(x_j)}, x_j \,\big|\, \zeta_{\tau(x_j)}^{(d)}(\lambda)\big).$$

For every $r \ge 1$, we can denote by $G^{(d,r)}(x,y|z)$ the Green function associated to the random walk on $\mathcal{X}^{(d)}$ which remains within distance r from the geodesics $\overline{x:y}$. In the same way we can define the first-visit and last-visit generating functions $F^{(d,r)}(x,y|z)$, and $L^{(d,r)}(x,y|z)$.

We would like to remark that for all $z \in \mathbb{C}$ such that $|z| \le \mathbf{R}$

$$\lim_{r\to\infty} G^{(d,r)}(x,x|z) = G^{(d)}(x,x|z), \ \lim_{r\to\infty} L_i^{(d,r)}(e_i,x_i|z) = L_i^{(d)}(e_i,x_i|z).$$

If we fix any value $\varepsilon > 0$, there is some r such that

$$L_i^{(d,r)}\big(e_i, x_i \,\big|\, \zeta_i^{(d,r)}(\lambda)\big) \geq (1-\varepsilon) L_i^{(d)}\big(e_i, x_i \,\big|\, \zeta_i^{(d)}(\lambda)\big)$$

for all $i \in \mathcal{I}$ and $x_i \in \mathcal{X}_i^{(d)} \setminus \{e_i, \dagger\}$.

At this point we can conclude the proof using the tree-like structure of the free product, analogously to [31, Lemma 4], and since ε can be chosen arbitrarily small, we obtain the claim. □

The proof of [31, Lemma 5] can be easily adapted to our setting, giving a more general result:

Corollary 8.1.10. *For all $x, y \in \mathcal{X}^{(d)}$ and $r \geq 1$ we have* $\mathrm{Var} Z_{\infty,r}^{(d)}(y|x) < \infty$.

For $x \in \mathcal{X}^{(d)}$, we define the event $E^{(d)}(x)$ that among all particles counted in $Z_\infty^{(d)}(x|e)$ there is at least one particle whose trail has not entered $\Gamma_1^\times$ yet, and enters the set

$$\big\{y \in \mathcal{X}^{(d)} \mid l(y) = l(x)\big\}$$

first at x. Obviously, $Z_\infty^{(d)}(x|e) \geq 1$ on the event $E^{(d)}(x)$ and hence

$$\mathbb{P}\big[E^{(d)}(x)\big] \leq \mathbb{E} Z_\infty^{(d)}(x|e).$$

Lemma 8.1.11.

$$\lim_{k\to\infty} \Bigg(\min_{\substack{x = x_1 \dots x_m \in \mathcal{X}^{(d)}: \\ m \in \mathbb{N}, x_1 \notin \Gamma_1, l(x) = k}} \frac{\mathbb{P}[E^{(d)}(x)]}{\mathbb{E} Z_\infty^{(d)}(x|e)} \Bigg)^{1/k} = 1.$$

Proof. The proof of this Lemma is completely analogous to the one of [31, Proposition 7]. We would like to point out that in this case we must consider the distance from elements $\overline{x:y}$ (which are sets of paths) instead of single geodesics. For the rest, we apply the same reasoning explained in [31, Proposition 7]. □

Analogously to (8.2) and (8.3), we define for $i \in \mathcal{I}$ and $d \in \mathbb{N}$

$$\mathcal{L}_i^{(d)+}(\lambda|z) := \sum_{x \in \Gamma_i^\times} L^{(d)}(e, x|\lambda)\, z^{l(x)} = \sum_{x \in \Gamma_i^\times} L_i^{(d)}\big(e_i, x \big| \zeta_i^{(d)}(\lambda)\big)\, z^{l(x)},$$

$$\mathcal{L}_i^{(d)}(\lambda|z) := \sum_{n \geq 1} \sum_{\substack{x = x_1 \dots x_n \in \mathcal{X}^{(d)} \\ \tau(x_1) = i}} L^{(d)}(e, x|\lambda)\, z^{l(x)}.$$

Therefore we obtain

$$\mathcal{L}_i^{(d)}(\lambda|z) = \mathcal{L}_i^{(d)+}(\lambda|z) \Big(1 + \sum_{j \in \mathcal{I}\setminus\{i\}} \mathcal{L}_j^{(d)}(\lambda|z) \Big). \tag{8.15}$$

Like in the case of Equation (8.6) we have $\mathcal{L}^{(d)}(\lambda|z) = 1 + \sum_{i\in\mathcal{I}} \mathcal{L}_i^{(d)}(\lambda|z)$ and therefore:

$$\mathcal{L}^{(d)}(\lambda|z) = \frac{1}{1 - \sum_{i \in \mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z)}{1 + \mathcal{L}_i^{(d)+}(\lambda|z)}}.$$

Since every $\mathcal{L}_i^{(d)+}(\lambda|z)$ is convergent and strictly increasing for all $0 \leq z < R(\mathcal{L}_i^{(d)+})$ there is some unique $z^*_{d,\mathcal{L}} > 0$ such that

$$\sum_{i\in\mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z^*_{d,\mathcal{L}})}{1+\mathcal{L}_i^{(d)+}(\lambda|z^*_{d,\mathcal{L}})} = 1.$$

Hence the radius of convergence of $\mathcal{L}^{(d)}(\lambda|z)$ is given by $z^*_{d,\mathcal{L}}$.

We define for $k \in \mathbb{N}$

$$\sigma_k^* := \Bigl\{x_1 \dots x_s \in \mathcal{X}^{(d)} \,\Big|\, s \in \mathbb{N}, l(x) = k, x_1 \notin \Gamma_1, x_s \in \Gamma_1\Bigr\}.$$

Since we excluded the case $\mathrm{Card}(\mathcal{I}) = 2 = \mathrm{Card}(\Gamma_1) = \mathrm{Card}(\Gamma_2)$ we have that $\sigma_2^* \neq \emptyset$ and $\sigma_3^* \neq \emptyset$. Therefore, $\sigma_k^* \neq \emptyset$ for all $2 \leq k \in \mathbb{N}$.

Lemma 8.1.12.

$$\limsup_{k\to\infty} \Bigl(\sum_{x\in\sigma_k^*} \mathbb{P}[E^{(d)}(x)]\Bigr)^{1/k} = \frac{1}{z^*_{d,\mathcal{L}}}.$$

Proof. By Lemma 8.1.11, for all k large enough we have

$$\mathbb{P}[E^{(d)}(x)] \geq (1-\varepsilon)^k \mathbb{E}Z_\infty^{(d)}(x|e),$$

uniformly for all x such that $l(x) = k$.

Recall also that $\mathbb{P}[E^{(d)}(x)] \leq \mathbb{E}Z_\infty^{(d)}(x|e)$. Thus, it is sufficient to prove

$$\limsup_{k\to\infty} \Bigl(\sum_{x\in\sigma_k^*} \mathbb{E}Z_\infty^{(d)}(x|e)\Bigr)^{1/k} = \frac{1}{z^*_{d,\mathcal{L}}}.$$

Using again Equations (8.11) and (8.13) we obtain

$$\sum_{x\in\sigma_k^*} \mathbb{E}Z_\infty^{(d)}(x|e) = \sum_{x\in\sigma_k^*} F^{(d)}(e,x|\lambda) = \sum_{x\in\sigma_k^*} \frac{G^{(d)}(e,e|\lambda)}{G^{(d)}(x,x|\lambda)} L^{(d)}(e,x|\lambda).$$

Moreover $1 \leq G^{(d)}(x,x|\lambda) \leq G(x,x|\lambda) = G(e,e|\lambda) < \infty$, therefore

$$\limsup_{k\to\infty} \Bigl(\sum_{x\in\sigma_k^*} L^{(d)}(e,x|\lambda)\Bigr)^{1/k} = \limsup_{k\to\infty} \Bigl(\sum_{x\in\sigma_k^*} \mathbb{E}Z_\infty^{(d)}(x|e)\Bigr)^{1/k}. \tag{8.16}$$

To determine the left-hand side of (8.16) we need some more tools. Since we are considering elements starting with all possible words which are not in Γ_1, we define the following generating function:

$$\mathcal{L}_{\neg 1,1}^{(d)}(\lambda|z) := \sum_{n\geq 2} \sum_{\substack{x=x_1\dots x_n\in\mathcal{X}^{(d)}:\\ x_1\notin\Gamma_1^\times, x_n\in\Gamma_1^\times}} L^{(d)}(e,x|\lambda)\, z^{l(x)},$$

whose z^k-coefficient is just $\sum_{x\in\sigma_k^*} L^{(d)}(e,x|\lambda)$. Equation (8.15) tells us

$$\mathcal{L}_1^{(d)}(\lambda|z) = \mathcal{L}_1^{(d)+}(\lambda|z)\cdot\Bigl(1 + \sum_{i\in\mathcal{I}\setminus\{1\}} \mathcal{L}_i^{(d)}(\lambda|z)\Bigr),$$

and hence the function $1 + \sum_{i\in\mathcal{I}\setminus\{1\}} \mathcal{L}_i^{(d)}(\lambda|z)$ must have the same radius of convergence of $\mathcal{L}^{(d)}(\lambda|z)$, which is $z^*_{d,\mathcal{L}}$. Since

$$0 < \mathcal{L}^{(d)}_{\neg 1,1}(\lambda|z) \leq 1 + \sum_{i\in\mathcal{I}\setminus\{1\}} \mathcal{L}_i^{(d)}(\lambda|z),$$

for all z such that the right hand side converges, the function $\mathcal{L}^{(d)}_{\neg 1,1}$ has also radius of convergence $z^*_{d,\mathcal{L}}$. This yields the claim. $\square$

Our next aim is to show that $z^*_{d,\mathcal{L}}$ tends to z^* as $d \to \infty$.
Since $z^*_{d,\mathcal{L}}$ is strictly decreasing in d, and since

$$\lim_{d\to\infty} L^{(d)}(e,x|\lambda) = L(e,x|\lambda) = F(e,x|\lambda) \tag{8.17}$$

we have $z_\infty = \lim_{d\to\infty} z^*_{d,\mathcal{L}} \geq z^*$. Now we prove that in fact equality holds: assume $z^* < z_\infty$. In this case we obtain

$$\begin{aligned} 1 &= \lim_{d\to\infty} \sum_{i\in\mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z^*_{d,\mathcal{L}})}{1+\mathcal{L}_i^{(d)+}(\lambda|z^*_{d,\mathcal{L}})} \\ &\geq \limsup_{d\to\infty} \sum_{i\in\mathcal{I}} \frac{\mathcal{L}_i^{(d)+}(\lambda|z_\infty)}{1+\mathcal{L}_i^{(d)+}(\lambda|z_\infty)} = \sum_{i\in\mathcal{I}} \frac{\mathcal{F}_i^{+}(\lambda|z_\infty)}{1+\mathcal{F}_i^{+}(\lambda|z_\infty)} > 1, \end{aligned}$$

which is obviously a contradiction. Thus,

$$\lim_{d\to\infty} z^*_{d,\mathcal{L}} = z^*. \tag{8.18}$$

Choose a number $2 \leq k \in \mathbb{N}$ arbitrarily. Similarly to [31], we embed a Galton–Watson process in the BRW defined on the free product $\mathcal{X}^{(d)}$.

For $n \in \mathbb{N}_0$, we define the Galton–Watson process as follows: its generations are denoted by $\mathrm{gen}(n)$ and the level of generation n is denoted by σ^*_{nk}. We denote by ξ_x a distinguished particle associated to a vertex $x \in \mathrm{gen}(n)$. The process is defined inductively as follows:

1. $\mathrm{gen}(0) := \{e\}$ consists of only one particle ξ_e located at e.
2. $y \in \sigma^*_{(n+1)k}$ belongs to $\mathrm{gen}(n+1)$ if and only if there exists a distinguished particle ξ_x in $\mathrm{gen}(n)$ such that some of its offspring counted in $Z^{(d)}_\infty(y|x)$ has a trail which
   (a) remains within the set
   $$\Gamma(x) := \{y \in \Gamma \mid y \text{ has the form } xw_1\dots w_s \text{ with } w_1 \notin \Gamma_1, s \geq 1\} \cup \{x\},$$
   (b) hits the set $\{w \in \mathcal{X}^{(d)} \mid l(w) = (n+1)k\}$ first at y.
3. The first particle hitting $y \in \sigma^*_{(n+1)k}$ becomes the distinguished particle ξ_y.

Let ϕ_n denote the number of particles alive at generation n. Since we have the same offspring distribution at every $x \in \sigma^*_{nk}$, the sequence $(\phi_n)_{n\geq 0}$ defines a Galton–Watson process. We denote its mean value by $M_{d,k}$. At this point we can state the following result:

Proposition 8.1.13.

$$\limsup_{k\to\infty} M_{d,k}^{1/k} = \frac{1}{z^*_{d,\mathcal{L}}}.$$

Proof. The claim follows directly from Lemma 8.1.12 because per definition we have $M_{d,k} = \sum_{x\in\sigma^*_k} \mathbb{P}[E^{(d)}(x)]$. □

A crucial tool in the following is *Hawkes' Theorem*: in his work (see [30]) he finds a way to measure the boundary of a Galton–Watson tree $\mathscr{T}$. This result is extremely interesting and useful because in a Galton–Watson tree (which Hawkes calls a *simple branching process*) each generation has a random amount of elements. Denote by $\partial\mathscr{T}$ the limit set of the tree (i.e. its boundary).

Denote by p_k the probability that a vertex has exactly k descendants (to avoid trivialities assume $p_0, p_1 < 1$), and by $\overline{m} := \sum_{k\geq 0} kp_k$ the expected value of this offspring distribution.

His main result states as follows:

Theorem 8.1.14 (Hawkes' Theorem)**.** *If the offspring distribution has mean $\overline{m} > 1$ and finite second moment, then, in the event of non-extinction, the limit set of the Galton–Watson tree $\mathscr{T}$ has Hausdorff dimension*

$$\mathrm{HD}(\partial\mathscr{T}) = \frac{\log \overline{m}}{-\log\alpha} \qquad \text{a.s.}$$

Remark 8.1.15. *This result was proved with other techniques by Russell Lyons (see [39]), using the so-called* branching number *(which corresponds to $\overline{m}$).*

A sharper version of [30] can be found in [37].

Applying Hawkes' Theorem as in [31, Corollary 7], together with Equation (8.18) we get the following statement:

Proposition 8.1.16. *With probability one,*

$$\mathrm{HD}(\Lambda \cap \Omega_\infty) \geq \frac{\log z^*}{\log\alpha}.$$

8.1.3 Proofs of the Main Theorems

At this point we have all the tools we need in order to prove Theorem 8.1.1:

Proof of Theorem 8.1.1. The following chains of inequalities summarize the previous results and finish the proof of the theorem. For the first part, Propositions 8.1.6 and 8.1.13 give us

$$\frac{\log z^*}{\log\alpha} \leq \mathrm{HD}(\Lambda) \leq \underline{\mathrm{BD}}(\Lambda) \leq \overline{\mathrm{BD}}(\Lambda) \leq \frac{\log z^*}{\log\alpha}.$$

For the second part we use Proposition 8.1.6 again, together with Proposition 8.1.16, obtaining

$$\frac{\log z^*}{\log \alpha} \le \mathrm{HD}(\Lambda \cap \Omega_\infty) \le \underline{\mathrm{BD}}(\Lambda \cap \Omega_\infty) \le \overline{\mathrm{BD}}(\Lambda \cap \Omega_\infty) \le \overline{\mathrm{BD}}(\Lambda) \le \frac{\log z^*}{\log \alpha}.$$

$\square$

Proposition 8.1.16 states that

$$\mathrm{HD}(\Lambda \cap \Omega_i) \le \mathrm{HD}(\Lambda \cap \Omega_\infty),$$

but in the following we prove that in fact strict inequality holds.

Now we prove Corollary 8.1.2, i.e., we show that the amount of non-typical ends does not give any contribution to the Hausdorff dimension of Λ.

Proof of Corollary 8.1.2. A well-known property of the Hausdorff dimension is the following: the dimension of a countable union $\bigcup_i B_i$ of sets $B_i \subseteq \Omega$ is given by the supremum of the dimensions of the single sets B_i. Thus,

$$\mathrm{HD}(\Lambda \cap \Omega_i) = \sup_{x \in \Gamma : \tau(x) \neq i} \mathrm{HD}(\Lambda \cap x\Omega_i^{(0)}) \le \sup_{x \in \Gamma : \tau(x) \neq i} \overline{\mathrm{BD}}(\Lambda \cap x\Omega_i^{(0)}).$$

For any fixed $x \in \Gamma$ with $\tau(x) \neq i$, denote by $\mathcal{H}_m^{(x)}$ the vertices y in the coset $x\Gamma_i$ such that $l(y) = l(x) + m$ and y has been visited by the BRW. Therefore, by the property of *tree-like structure* endowed by the free product, we get

$$\mathbb{E}|\mathcal{H}_m^{(x)}| \le \sum_{y \in \Gamma_i : l(y) = m} F(e, xy|\lambda) = F(e, x|\lambda) \sum_{y \in \Gamma_i : l(y) = m} F(e, y|\lambda).$$

Now we can observe that the function

$$F(e, x|\lambda) \sum_{m \ge 1} \sum_{y \in \Gamma_i : l(y) = m} F(e, y|\lambda)\, z^m$$

has radius of convergence equal to $R(\mathcal{F}_i^+)$.

Therefore, Lemma 8.1.4 yields

$$\limsup_{m \to \infty} \big(\mathbb{E}|\mathcal{H}_m^{(x)}|\big)^{1/m} \le 1/R(\mathcal{F}_i^+) < 1/z^*. \tag{8.19}$$

Applying a similar reasoning to Lemma 8.1.5 and Proposition 8.1.6, we show that

$$\mathrm{HD}(\Lambda \cap \Omega_i) \le \frac{\log(R(\mathcal{F}_i^+))}{\log \alpha},$$

which by (8.19) leads to the statement. $\square$

Chapter 9

Dimension of Ω

In this chapter we find the Hausdorff dimension of the boundary of the free product: we will use the definitions and the tools described in Section 2.1. In this case the proofs of the theorems are slightly easier than in the case of Λ. This is due to the fact that Ω is a *deterministic* set, while Λ is a *random* subset of Ω.

The main methods used to evaluate the dimension of Ω are roughly the ones described in Chapter 8. For completeness, we present them as well.

We show an analogue of Theorem 8.1.1: we prove the existence of the box-counting dimension of Ω and express it as the solution of a functional equation.

In order to do it, we need to introduce some new tools. Recall the definition of the *growth functions* from Section 1.3. To simplify the notation we set:

$$\begin{aligned} \mathcal{S}(z) &:= \Sigma(\Gamma, S \mid z); & \mathcal{S}_i(z) &:= \Sigma(\Gamma_i, S_i \mid z); \\ \sigma(k) &:= \sigma(\Gamma, S\,;\, k); & \sigma_i(k) &:= \sigma(\Gamma_i, S_i\,;\, k). \end{aligned}$$

We denote by $R(\mathcal{S})$ and by $R(\mathcal{S}_i)$ the radii of convergence of $\mathcal{S}(z)$ and $\mathcal{S}_i(z)$ respectively. With our notation, we can also write

$$\sigma(k) = \#\{x \in \Gamma \mid l(x) = k\}, \quad \sigma_i(k) = \#\{x \in \Gamma_i \mid l(x) = k\}.$$

Exactly in the same way as in Chapter 8, we obtain the deterministic correspondent of Equations (8.1)–(8.6): the function corresponding to (8.1) is

$$\mathcal{S}(z) := \sum_{m \geq 0} \sigma(m)\, z^m.$$

Then, for all $i \in \mathcal{I}$ we define

$$\mathcal{S}_i^+(z) := \sum_{m \geq 1} \sigma_i(m)\, z^m,$$

$$\mathcal{S}_i(z) := \sum_{n \geq 1} \sum_{m \geq 1} \sum_{\substack{x = x_1 \ldots x_n \in \Gamma: \\ l(x) = m,\, x_1 \in \Gamma_i^\times}} \sigma(m)\, z^m = \mathcal{S}_i^+(z) \Big(1 + \sum_{j \in \mathcal{I} \setminus \{i\}} \mathcal{S}_j(z) \Big).$$

By definition it follows:

$$\mathcal{S}(z) = 1 + \sum_{i\in\mathcal{I}} \mathcal{S}_i(z).$$

Therefore we get

$$\mathcal{S}_i(z) = \mathcal{S}_i^+(z)\big(\mathcal{S}(z) - \mathcal{S}_i(z)\big),$$

hence we have

$$\mathcal{S}(z) = 1 + \sum_{i\in\mathcal{I}} \mathcal{S}_i(z) = 1 + \mathcal{S}(z)\sum_{i\in\mathcal{I}} \frac{\mathcal{S}_i^+(z)}{1+\mathcal{S}_i^+(z)},$$

which leads to

$$\mathcal{S}(z) = \frac{1}{1 - \sum_{i\in\mathcal{I}} \frac{\mathcal{S}_i^+(z)}{1+\mathcal{S}_i^+(z)}}. \tag{9.1}$$

To cover Ω with balls of radius α^m we need at least $\sigma(m-1)$ balls, and at most $\sigma(m)$. Therefore we are interested in the asymptotic behavior of $\sigma(m)^{1/m}$ on the limit $m \to \infty$.

9.1 Main Results

The main result we would like to present is the following:

Theorem 9.1.1. *The box-counting dimensions of Ω and Ω_∞ exist and satisfy*

$$\mathrm{BD}(\Omega) = \mathrm{BD}(\Omega_\infty) = \mathrm{HD}(\Omega) = \mathrm{HD}(\Omega_\infty) = \frac{\log z_{\mathcal{S}}^*}{\log \alpha},$$

where $z_{\mathcal{S}}^*$ is the smallest real positive number such that

$$\sum_{i\in\mathcal{I}} \frac{\mathcal{S}_i^+(z_{\mathcal{S}}^*)}{1+\mathcal{S}_i^+(z_{\mathcal{S}}^*)} = 1. \tag{9.2}$$

Analogously to Corollary 8.1.2 we obtain that the Hausdorff dimension of Ω arises only from the ends in Ω_∞.

Corollary 9.1.2. *For all $i \in \mathcal{I}$, $\mathrm{HD}(\Omega_i) < \mathrm{HD}(\Omega_\infty)$.*

9.1.1 Proofs of the Statements

The following lemma shows that the value $1/z_{\mathcal{S}}^*$ corresponds to the growth factor of the free product.

Lemma 9.1.3.

$$\lim_{m\to\infty} \sigma(m)^{1/m} = \frac{1}{z_{\mathcal{S}}^*} < 1.$$

Proof. Obviously, $R(\mathcal{S}) \leq R(\mathcal{F}) < 1$ since $F(e, x|\lambda) < 1$ for all $x \in \Gamma \setminus \{e\}$. With the same reasoning done for Lemma 8.1.4, we get $R(\mathcal{S}) = z_{\mathcal{S}}^*$. Therefore, if the sequence $\sigma(m)^{1/m}$ converges, then we have

$$\limsup_{m\to\infty} \sigma(m)^{1/m} = \frac{1}{z_{\mathcal{S}}^*} = \frac{1}{R(\mathcal{S})} > 1.$$

Now we still have to prove that the sequence $\sigma(m)^{1/m}$ converges for $m \to \infty$.

By transitivity of Γ, we have $\sigma(m)\sigma(n) \geq \sigma(m+n)$ for all $m, n \in \mathbb{N}$, i.e. $(\sigma(m))_{m\in\mathbb{N}}$ is a submultiplicative sequence. Fekete's Lemma implies the statement. $\square$

Remark 9.1.4. *One can show analogously to Lemma 8.1.4 that $z^*_{\mathcal{S}} < R(\mathcal{S}_i^+)$, being $R(\mathcal{S}_i^+)$ the radius of convergence of $\mathcal{S}_i^+(z)$.*

The next proposition shows that the box-counting dimension of Ω equals the dimension of Ω_∞.

Proposition 9.1.5.

$$\mathrm{BD}(\Omega) = \mathrm{BD}(\Omega_\infty) = \frac{\log z^*_{\mathcal{S}}}{\log \alpha}.$$

Proof. If we try to give a rough estimate of the number of balls of radius α^m that we need to cover Ω_∞ we find:

$$\begin{aligned}\underline{\mathrm{BD}}(\Omega) \geq \underline{\mathrm{BD}}(\Omega_\infty) &\geq \liminf_{m\to\infty}\left(-\frac{\log \sigma(m-1)}{\log \alpha^m}\right)\\ &= \liminf_{m\to\infty}\left(-\frac{\log \sigma(m-1)^{1/(m-1)}}{\log \alpha}\frac{m-1}{m}\right) = \frac{\log z^*_{\mathcal{S}}}{\log \alpha}.\end{aligned}$$

Analogously,

$$\begin{aligned}\overline{\mathrm{BD}}(\Omega_\infty) \leq \overline{\mathrm{BD}}(\Omega) &\leq \limsup_{m\to\infty}\left(-\frac{\log \sigma(m)}{\log \alpha^m}\right)\\ &= \limsup_{m\to\infty}\left(-\frac{\log \sigma(m)^{1/m}}{\log \alpha}\right) = \frac{\log z^*_{\mathcal{S}}}{\log \alpha}.\end{aligned}$$

These inequalities lead to the statement. $\square$

Finally, we can prove the formula for the Hausdorff dimensions of Ω and Ω_∞.

Proof of Theorem 9.1.1. It is sufficient to show that $\mathrm{HD}(\Omega_\infty) \geq \frac{\log z^*_{\mathcal{S}}}{\log \alpha}$, therefore we adapt the tools described in Chapter 8 to our new setting.

We approximate the free product Γ by a sequence of "truncated" free products $\mathcal{X}^{(d)}$, and we do the same with the growth functions as well.

Since we are in the deterministic case, Lemma 8.1.11 is trivially true, since here we do not count particles, but just possible trails. Reasoning in the same way as done for the first part of the proof of Lemma 8.1.12, we get that the amount of words of (Cayley graph) length k such that $x_1 \notin \Gamma_1$ and $x_k \in \Gamma_1$ (denote it by $\sigma^{(d)}_{\neg 1,1}(k)$) is such that

$$\left(\sigma^{(d)}_{\neg 1,1}(k)\right)^{1/k} \xrightarrow{k\to\infty} \frac{1}{z^*_{d,\mathcal{S}}} \xrightarrow{d\to\infty} \frac{1}{z^*_{\mathcal{S}}}.$$

We can proceed following the proof of Lemma 8.1.12 and embed a "deterministic" Galton–Watson tree into the free product analogously to what done in Subsection 8.1.2. In this case each generation has exactly $\sigma^{(d)}_{\neg 1,1}(k)$ descendants.

By Hawkes' Theorem, the Hausdorff dimension of the boundary of the embedded tree is bounded from below by $\log z^*_{d,\mathcal{S}}/\log\alpha$, and therefore, considering the limit on d going to infinity we get:

$$\mathrm{HD}(\Omega_\infty) \geq \log z^*_{\mathcal{S}}/\log\alpha.$$

$\square$

At this point we prove the last result of this section:

Proof of Corollary 9.1.2. Analogously to the proof of Corollary 8.1.2 and by Remark 9.1.4, we can use the property $\mathrm{HD}(\cup_i B_i) = \sup_i \mathrm{HD}(B_i)$ for all countable unions of sets $B_i \subseteq \Omega$. In this way we can show that

$$\mathrm{HD}(\Omega_i) = \sup_{x\in\Gamma:\tau(x)\neq i} \mathrm{HD}(x\Omega_i^{(0)}) \leq \overline{\mathrm{BD}}(\Omega_i^{(0)}) < \mathrm{BD}(\Omega_\infty) = \mathrm{HD}(\Omega_\infty).$$

$\square$

9.2 Continuity of the Hausdorff Dimension

The aim of this section is to investigate regularity properties of the function "Hausdorff dimension" in dependence of the parameter λ. For a free product Γ, let us consider the function

$$\begin{aligned} \Phi : [1,\infty) &\to \mathbb{R} \\ \lambda &\mapsto \mathrm{HD}(\Lambda), \end{aligned}$$

which assigns to every λ the Hausdorff dimension of the limit set of a BRW with growth parameter λ. The limit case $\lambda = 1$ corresponds to the degenerate case of a non-branching random walk. In this case the Hausdorff dimension is zero.

We can summarize the main properties of Φ in the following statement:

Theorem 9.2.1. *The function $\Phi(\lambda)$ has the following properties:*

(i) $\Phi(\lambda)$ *is strictly increasing on* $[1,\mathbf{R}]$, $\Phi(1) = 0$ *and* $\Phi(\lambda) = \mathrm{HD}(\Omega)$ *for all* $\lambda > \mathbf{R}$.

(ii) $\Phi(\lambda)$ *is continuous on* $[1,\infty)\backslash\{\mathbf{R}\}$ *and continuous from the left at* $\lambda = \mathbf{R}$. *Moreover*

$$\Phi(\mathbf{R}) \leq \frac{1}{2}\mathrm{HD}(\Omega).$$

Proof. *Statement (i)*: This part follows from the next observations:

- the solution of Equation (8.7) must be strictly decreasing in λ;
- the BRW at $\lambda = 1$ dies out almost surely (see [60]);
- if $\lambda > \mathbf{R}$ the BRW is recurrent (see [4]) and therefore every point in $\Gamma \cup \Omega$ is accumulation point for the process, implying $\mathrm{HD}(\Lambda) = \mathrm{HD}(\Omega)$.

Statement (ii): This part can be split into two steps: in the first one we show that Φ is continuous in $[1,\infty)\setminus\{\mathbf{R}\}$ and continuous from the left at $\lambda=\mathbf{R}$. Afterwards we show that for all $\lambda\le\mathbf{R}$ we have $\mathrm{HD}(\Lambda)\le\frac{1}{2}\mathrm{HD}(\Omega)$.

Step 1.: In order to prove continuity of Φ, it is sufficient to prove continuity of the map $\lambda\mapsto z^*=z^*(\lambda)$.

First, we prove continuity from the left at every $\lambda_0\in(1,\infty)$.
For this purpose, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of strictly increasing real numbers such that $\lambda_n<\lambda_0$ and $\lim_{n\to\infty}\lambda_n=\lambda_0$. By domination arguments, $z^*(\lambda_n)$ can not be smaller than $z^*(\lambda_0)$, therefore assume $z_0:=\lim_{n\to\infty}z^*(\lambda_n)>z^*(\lambda_0)$. We have that $z^*(\lambda_n)$ is strictly decreasing.

Since $f(x)/\big(1+f(x)\big)$ is strictly increasing in $[1,\infty)$ if $f(x)$ is a strictly increasing function on $[1,\infty)$, we get the following contradiction:

$$\begin{aligned}1&=\lim_{n\to\infty}\sum_{i\in\mathcal{I}}\frac{\mathcal{F}_i^+\big(\lambda_n\big|z^*(\lambda_n)\big)}{1+\mathcal{F}_i^+\big(\lambda_n\big|z^*(\lambda_n)\big)}=\sum_{i\in\mathcal{I}}\frac{\mathcal{F}_i^+\big(\lambda_0\big|z_0\big)}{1+\mathcal{F}_i^+\big(\lambda_0\big|z_0\big)}\\&>\sum_{i\in\mathcal{I}}\frac{\mathcal{F}_i^+\big(\lambda_0\big|z^*(\lambda_0)\big)}{1+\mathcal{F}_i^+\big(\lambda_0\big|z^*(\lambda_0)\big)}=1.\end{aligned}$$

Thus, $\lim_{n\to\infty}z^*(\lambda_n)=z^*(\lambda_0)$.

Since $\mathrm{HD}(\Lambda)=\mathrm{HD}(\Omega)$ for all $\lambda>\mathbf{R}$, it remains to prove continuity from the right for $\lambda_0\in[1,\mathbf{R})$. First of all we consider the case $\lambda_0\in(1,\mathbf{R})$ and afterwards we prove continuity at $\lambda_0=1$.

Consider a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of strictly decreasing real numbers such that $\lambda_0<\lambda_n<\mathbf{R}$, and $\lim_{n\to\infty}\lambda_n=\lambda_0$.

We want to show that under the assumption $z_0:=\lim_{n\to\infty}z^*(\lambda_n)<z^*(\lambda_0)$ (by domination arguments, $z^*(\lambda_n)$ can not be larger than $z^*(\lambda_0)$), we get a contradiction. Observe that $z^*(\lambda_n)$ is strictly increasing.

Therefore,

$$\begin{aligned}1&=\lim_{n\to\infty}\sum_{i\in\mathcal{I}}\frac{\mathcal{F}_i^+\big(\lambda_n\big|z^*(\lambda_n)\big)}{1+\mathcal{F}_i^+\big(\lambda_n\big|z^*(\lambda_n)\big)}=\sum_{i\in\mathcal{I}}\frac{\mathcal{F}_i^+\big(\lambda_0\big|z_0\big)}{1+\mathcal{F}_i^+\big(\lambda_0\big|z_0\big)}\\&<\sum_{i\in\mathcal{I}}\frac{\mathcal{F}_i^+\big(\lambda_0\big|z^*(\lambda_0)\big)}{1+\mathcal{F}_i^+\big(\lambda_0\big|z^*(\lambda_0)\big)}=1,\end{aligned}$$

which is a contradiction. Consequently, $\lim_{n\to\infty}z^*(\lambda_n)=z^*(\lambda_0)$.

It remains to prove continuity from the right at $\lambda_0=1$. In this case we have that $\zeta_i(1)<1$ (for a proof of this result, see e.g. [62, Section 6]). It follows that for every $\delta>0$ such that $\zeta_i(\lambda_0+\delta)<1$, we have

$$\begin{aligned}\mathcal{F}_i^+(\lambda_0+\delta|1)&=\sum_{x\in\Gamma_i^\times}F_i(e_i,x\mid\zeta_i(\lambda_0+\delta))=\frac{\sum_{x\in\Gamma_i}G_i(e_i,x\mid\zeta_i(\lambda_0+\delta))}{G_i(\zeta_i(\lambda_0+\delta))}-1\\&=\frac{1}{G_i(\zeta_i(\lambda_0+\delta))\big(1-\zeta_i(\lambda_0+\delta)\big)}-1<\infty.\end{aligned}\tag{9.3}$$

Here we used the fact that a random walk on Γ is transient, therefore it has to pass through all intermediate levels at least once:

$$\sum_{x\in\Gamma_i} G_i(e_i, x \mid \zeta_i(\lambda_0+\delta)) = \sum_{x\in\Gamma_i} \zeta_i(\lambda_0+\delta)^{l(x)} = \frac{1}{1-\zeta_i(\lambda_0+\delta)}.$$

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of real numbers with limit $\lambda_0 = 1$. We write $z_0 = \lim_{n\to\infty} z^*(\lambda_n) \le 1$. Then, for n large enough,

$$1 = \lim_{n\to\infty} \sum_{i\in\mathcal{I}} \frac{\mathcal{F}_i^+\big(\lambda_n \big| z^*(\lambda_n)\big)}{1+\mathcal{F}_i^+\big(\lambda_n \big| z^*(\lambda_n)\big)} \le \sum_{i\in\mathcal{I}} \frac{\mathcal{F}_i^+\big(1\big|z_0\big)}{1+\mathcal{F}_i^+\big(1\big|z_0\big)}. \tag{9.4}$$

In order to finish the proof we verify that $z^*(1) = 1$, from which $z_0 = z^*(1) = 1$ follows.

By Equation (9.3) we get

$$\sum_{i\in\mathcal{I}} \frac{\mathcal{F}_i^+(1|1)}{1+\mathcal{F}_i^+(1|1)} = \sum_{i\in\mathcal{I}} \Big(1 - G_i\big(\zeta_i(1)\big)\big(1-\zeta_i(1)\big)\Big).$$

From [24, Lemma 5.1] it follows that the quantity

$$1 - G_i\big(\zeta_i(1)\big)\big(1-\zeta_i(1)\big)$$

is nothing else but the probability that a non-branching random walk on Γ tends to an infinite word of the form $x_1x_2\cdots \in \Omega_\infty$ with $x_1 \in \Gamma_i^\times$. In other words, the above sum equals 1. By Equation (9.4) the statement follows.

The next result completes the proof of statement (ii):

Step 2.: For all $\lambda \in [1, \mathbf{R}]$, $\mathrm{HD}(\Lambda) \le \frac{1}{2}\mathrm{HD}(\Omega)$.

Following a similar procedure as in [31], define the function

$$\mathcal{F}^{(2)}(\lambda|z) := \sum_{x\in\Gamma} F(e,x|\lambda)^2 \, z^{l(x)},$$

whose radius of convergence is denoted by z_2^*. The Cauchy-Schwarz Inequality gives then

$$\begin{aligned}
\frac{1}{z^*} &= \limsup_{m\to\infty} \Big(\sum_{x\in\Gamma: l(x)=m} F(e,x|\lambda)\Big)^{1/m} \\
&\le \limsup_{m\to\infty} \sqrt{\Big(\sum_{x\in\Gamma: l(x)=m} F(e,x|\lambda)^2\Big)^{1/m}} \cdot \limsup_{m\to\infty} \sqrt{\Big(\sum_{x\in\Gamma: l(x)=m} 1^2\Big)^{1/m}} \\
&= \sqrt{\frac{1}{z_2^*}} \cdot \sqrt{\frac{1}{z_{\mathcal{S}}^*}}.
\end{aligned}$$

At this point it suffices (by the formulas given in Theorems 8.1.1 and 9.1.1) to show that $z_2^* \ge 1$. First,

$$\begin{aligned}
\mathcal{F}^{(2)}(\lambda|1) = \sum_{x\in\Gamma} F(e,x|\lambda)^2 &= \frac{1}{G(e,e|\lambda)^2} \sum_{x\in\Gamma} G(e,x|\lambda)^2 \\
&= \frac{1}{G(e,e|\lambda)^2} \sum_{x\in\Gamma} \Big(\sum_{n\ge 0} p^{(n)}(e,x)\lambda^n\Big)^2.
\end{aligned}$$

We can rewrite the squared sum as

$$\Big(\sum_{n\geq 0} p^{(n)}(e,x)\lambda^n\Big)\Big(\sum_{m\geq 0} p^{(m)}(e,x)\lambda^m\Big) = \sum_{n\geq 0}\sum_{m\geq 0} p^{(n)}(e,x)p^{(m)}(e,x)\lambda^{n+m}.$$

By symmetry we can expand the previous as

$$\sum_{k\geq 0}\sum_{m=0}^{k} p^{(k-m)}(e,x)p^{(m)}(x,e)\lambda^k.$$

Therefore, for every fixed $x \in \Gamma$, the coefficient of λ^k in the inner squared sum can (by symmetry) be rewritten as

$$\frac{1}{G(e,e|\lambda)^2}\sum_{m=0}^{k} p^{(k-m)}(e,x)p^{(m)}(x,e). \tag{9.5}$$

Thus, every path $[x_0 = e, x_1, \ldots, x_k = e]$ of length k (consisting of $k+1$ vertices) from e to e is counted $k+1$ times, because every x_i can play the role of x in Equation (9.5). That is,

$$\mathcal{F}^{(2)}(\lambda|1) = \frac{1}{G(e,e|\lambda)^2}\sum_{k\geq 0} p^{(k)}(e,e)\cdot(k+1)\cdot\lambda^k = \frac{\lambda\, G'(e,e|\lambda)}{G(e,e|\lambda)^2} + \frac{1}{G(e,e|\lambda)}.$$

From this follows $z_2^* \geq 1$ whenever $\lambda < \mathbf{R}$ or $G'(e,e|\mathbf{R}) < \infty$, and therefore we get $\mathrm{HD}(\Lambda) \leq \frac{1}{2}\mathrm{HD}(\Omega)$ for $\lambda < \mathbf{R}$.

By *Step 1* (continuity from the left), we have the result for $\lambda = \mathbf{R}$ as well. □

For some examples the reader is referred to [7, Section 3.1].

Chapter 10

Finite Case

10.1 Free Products of Finite Groups

In this section we give a more explicit formula for the box-counting dimension with respect to a slightly changed metric on the boundary in the case of free products of *finite* groups. In this case we have $\Omega = \Omega_\infty$.

Throughout the whole chapter we do not need the assumption that the μ_i's are symmetric.

For any $\omega_1 = x_1 x_2 \dots, \omega_2 = y_1 y_2 \dots \in \Omega_\infty$ with $\omega_1 \neq \omega_2$, we define the *confluent* $\omega_1 \wedge \omega_2$ of ω_1 and ω_2 to be the word $x_1 \dots x_k$ of maximal length (see below) such that $x_i = y_i$ for all $1 \leq i \leq k$. If $x_1 \neq y_1$, then $\omega_1 \wedge \omega_2 := e$.

Recall from Section 2.1 that by $\|v\|$ we denote the *block length* of the word v. The metric on the boundary Ω_∞ is defined by

$$d_\Omega^{\mathrm{fin}}(\omega_1, \omega_2) := \alpha^{\|\omega_1 \wedge \omega_2\|}$$

for any arbitrary but fixed $\alpha \in (0,1)$.

With respect to this metric on Ω_∞ we can define analogously to (2.2) and (2.3) the box-counting dimension $\mathrm{BD}^{\mathrm{fin}}(\Omega')$ and the Hausdorff dimension $\mathrm{HD}^{\mathrm{fin}}(\Omega')$ for any $\Omega' \subseteq \Omega_\infty$.

Now we set

$$\mathcal{F}_i^+(\lambda) := \mathcal{F}_i^+(\lambda|1),$$

and define the matrix $M = \bigl(m(i,j)\bigr)_{i,j\in\mathcal{I}}$ by

$$m(i,j) := \begin{cases} \mathcal{F}_j^+(\lambda), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Since M is irreducible and has non-negative entries, the Perron–Frobenius eigenvalue exists (see e.g. [55]). We denote it by θ.

Now let us define the matrix $D = \bigl(d(i,j)\bigr)_{i,j\in\mathcal{I}}$ by

$$d(i,j) := \begin{cases} |\Gamma_j| - 1, & \text{if } i \neq j \\ 0, & \text{otherwise}, \end{cases}$$

and denote by ϱ its Perron–Frobenius eigenvalue. With this notation we get:

Corollary 10.1.1.

$$\mathrm{BD}^{\mathrm{fin}}(\Lambda) = \mathrm{HD}^{\mathrm{fin}}(\Lambda) = -\frac{\log\theta}{\log\alpha} \quad \textit{and} \quad \mathrm{BD}^{\mathrm{fin}}(\Omega) = \mathrm{HD}^{\mathrm{fin}}(\Omega) = -\frac{\log\varrho}{\log\alpha}.$$

In order to prove the corollary, we show the following intermediate result:

Lemma 10.1.2.

$$\overline{\mathrm{BD}^{\mathrm{fin}}}(\Lambda) \leq -\frac{\log\theta}{\log\alpha} \quad \textit{and} \quad \mathrm{BD}^{\mathrm{fin}}(\Omega) = -\frac{\log\varrho}{\log\alpha}.$$

Proof. First of all, we define the matrices $M_0 = \big(m_0(i,j)\big)_{i,j\in\mathcal{I}}$ and $D_0 = \big(d_0(i,j)\big)_{i,j\in\mathcal{I}}$ by

$$m_0(i,j) := \begin{cases} \mathcal{F}_i^+(\lambda), & \text{if } i=j,\\ 0, & \text{otherwise,}\end{cases} \qquad d_0(i,j) := \begin{cases} |\Gamma_i|-1, & \text{if } i=j,\\ 0, & \text{otherwise.}\end{cases}$$

For $m\in\mathbb{N}$, denote by $\mathcal{H}_m^{\mathrm{fin}}$ the random number of words of (block) length m visited by the BRW, and by $\mathbf{1}$ the vector of length $r=\mathrm{Card}(\mathcal{I})$ with all entries equal to 1. Then

$$\mathbb{E}\,\mathrm{Card}(\mathcal{H}_m^{\mathrm{fin}}) \leq \sum_{x\in\Gamma:\|x\|=m} F(e,x|\lambda) = \mathbf{1}^T M_0 M^{m-1}\mathbf{1},$$
$$\hat{\sigma}(m) = \mathrm{Card}\big(\{x\in\Gamma \mid \|x\|=m\}\big) = \mathbf{1}^T D_0 D^{m-1}\mathbf{1}.$$

Let $\mathbf{u}\in\mathbb{R}^r$ be an eigenvector w.r.t. the eigenvalue θ such that $\mathbf{u}\geq\mathbf{1}$ (component-wise). Then:

$$\mathbb{E}\,\mathrm{Card}(\mathcal{H}_m^{\mathrm{fin}}) \leq \begin{pmatrix}\mathcal{F}_1(\lambda)\\ \vdots \\ \mathcal{F}_r(\lambda)\end{pmatrix}^T M^{m-1}\mathbf{u} \leq \begin{pmatrix}\mathcal{F}_1(\lambda)\\ \vdots \\ \mathcal{F}_r(\lambda)\end{pmatrix}^T \theta^{m-1}\mathbf{u}.$$

Thus, $\limsup_{m\to\infty}\big(\mathbb{E}\mathcal{H}_m^{\mathrm{fin}}\big)^{1/m}\leq\theta$. Similarly, one can show that

$$\lim_{m\to\infty}\hat{\sigma}(m)^{1/m} = \varrho,$$

obtaining the two inequalities by taking eigenvectors $\mathbf{v}_1\geq\mathbf{1}$ and $\mathbf{v}_2\leq\mathbf{1}$.

Analogously to the proofs of Lemma 8.1.5 and Propositions 8.1.6, 9.1.5 we obtain the claim. □

Now we can prove the stated corollary:

Proof of Corollary 10.1.1. First, we remark that we dropped the assumption on symmetry of the μ_i's, because we are working with finite groups. In the present setting we have already that $F(e,x|\lambda)<1$ (see Remark 7.1.1).

Let us recall Equation 3.5:

$$\alpha_i z G(z) = \zeta_i(z) G_i\big(\zeta_i(z)\big).$$

Since $G(\mathbf{R}) < \infty$ and $G_i(1) = \infty$, we must have $\zeta_i(\mathbf{R}) < 1$. Consequently,

$$\begin{aligned} F(e, x_1 \dots x_k|\lambda) &= \prod_{j=1}^{k} F_{\tau(x_j)}\big(e_{\tau(x_j)}, x_j|\zeta_{\tau(x_j)}(\lambda)\big) \\ &< \prod_{j=1}^{k} F_{\tau(x_j)}\big(e_{\tau(x_j)}, x_j|1\big) \leq 1. \end{aligned}$$

In order to show that $(-\log\theta/\log\alpha)$ is a lower bound for $\mathrm{HD}^{\mathrm{fin}}(\Lambda)$, we can follow the same reasoning explained in [31, Section 6] and in Section 8.1.2.

Analogously to the proof of Theorem 9.1.1 we obtain that $\mathrm{HD}^{\mathrm{fin}}(\Omega) = \mathrm{BD}^{\mathrm{fin}}(\Omega)$. $\square$

As a particular case we can see that when $\Gamma = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| = |\Gamma_2| < \infty$, we get the following explicit formulas for the dimensions:

$$\begin{aligned} \mathrm{BD}^{\mathrm{fin}}(\Lambda) = \mathrm{HD}^{\mathrm{fin}}(\Lambda) &= -\frac{\log\sqrt{\mathcal{F}_1^+(\lambda)\mathcal{F}_2^+(\lambda)}}{\log\alpha} \\ \mathrm{BD}^{\mathrm{fin}}(\Omega) = \mathrm{HD}^{\mathrm{fin}}(\Omega) &= -\frac{\log\sqrt{\big(|\Gamma_1|-1\big)\big(|\Gamma_2|-1\big)}}{\log\alpha}. \end{aligned}$$

10.2 Free Products with Amalgamation

An important generalization of free products are amalgamated products (recall Section 1.2.3 for the definitions).

Take $\Gamma_1, \dots, \Gamma_r, H$ to be finite groups such that each group Γ_i contains a subgroup H_i isomorphic to H and denote by $\phi_i : H_i \to H$ this isomorphism, for each $i \in \mathcal{I}$.

Moreover, we denote by S_i the generating set of Γ_i and by R_i its relations.

In general, we can define the *free product with amalgamation* with respect to the subgroup H by

$$\begin{aligned} \Gamma_H &:= \Gamma_1 *_H \Gamma_2 *_H \cdots *_H \Gamma_r \\ &:= \langle S_1, \dots, S_r \mid R_1, \dots, R_n, (\phi_i(a)) = (\phi_j(a)),\ \forall a \in H_i\ \forall i, j \in \mathcal{I}\rangle. \end{aligned}$$

For $i \in \mathcal{I}$, the quotient Γ_i/H_i consists of all left co-sets of the form

$$x_i H_i = \{x_i h \mid h \in H_i\},$$

where $x_i \in \Gamma_i$.

Now we fix a set of representatives $\mathcal{R}_i := \{g_{i,1} = e_i, g_{i,2}, \dots, g_{i,n_i}\}$ for the elements of Γ_i/H_i, i.e., for each $y_i \in \Gamma_i$ there is a unique $g_{i,k} \in \mathcal{R}_i$ such that $y_i \in g_{i,k}H_i$. We write $\hat{\tau}(x) = i$ if $x \in \mathcal{R}_i \setminus \{e_i\}$.

The amalgam Γ_H consists of all finite words of the form

$$x_1 x_2 \dots x_n h \tag{10.1}$$

with $n \in \mathbb{N}_0$, $x_i \in \bigcup_{j\in\mathcal{I}} \mathcal{R}_j \setminus \{e_j\}$ and $h \in H$. Here we need that $\hat{\tau}(x_i) \neq \hat{\tau}(x_{i+1})$. W.l.o.g. we may identify h with $\phi_1^{-1}(h)$.

Let Ω be the set of all ends of Γ_H, which consists of all infinite words of the form $w_1w_2\ldots \in \left(\bigcup_{i\in\mathcal{I}} \mathcal{R}_i\setminus\{e_i\}\right)^{\mathbb{N}}$ such that $\hat{\tau}(w_i) \neq \hat{\tau}(w_{i+1})$ for all $i \in \mathbb{N}$.

For any two different ends $\omega_1 = x_1x_2\ldots, \omega_2 = y_1y_2\ldots \in \Omega$, we can define the *confluent* $\omega_1 \wedge \omega_2$ of ω_1 and ω_2 to be the word $x_1\ldots x_k$ of maximal length with $x_i = y_i$ for all $1 \leq i \leq k$. Analogously to the previous section, if $x_1 \neq y_1$, then $\omega_1 \wedge \omega_2 := e$.

This definition allows us to define a metric on the boundary Ω:

$$d_\Omega^{(H)}(\omega_1,\omega_2) := \alpha^{\|\omega_1\wedge\omega_2\|} \tag{10.2}$$

for any fixed parameter $\alpha \in (0,1)$.

With respect to this metric, we can define analogously to (2.2) and (2.3) the box-counting dimension $\mathrm{BD}^{(H)}(\Omega')$ and Hausdorff dimension $\mathrm{HD}^{(H)}(\Omega')$ for any $\Omega' \subseteq \Omega$.

Suppose every group Γ_i is equipped with a symmetric probability measure μ_i, and a value $\alpha_i > 0$ such that $\sum_{i\in\mathcal{I}} \alpha_i = 1$.

The random walk on Γ_H is then governed by

$$\mu(x) := \begin{cases} \alpha_i\mu_i(x), & \text{if } x \in \Gamma_i \setminus H_i,\\ \sum_{i\in\mathcal{I}} \alpha_i\mu_i\big(\phi_i^{-1}(\phi_1(x))\big), & \text{if } x \in H_1,\\ 0, & \text{otherwise.}\end{cases}$$

For $g_i \in \mathcal{R}_i$, denote by T_{g_iH} the stopping time of the first visit to the set g_iH_i. We introduce the following generating functions:

$$F_H(gh|z) := \sum_{n\geq 0} \mathbb{P}\big[T_{gH} = n, X_n = gh \mid X_0 = e\big]\, z^n,$$

where $g \in \bigcup_{i\in\mathcal{I}} \mathcal{R}_i\setminus\{e_i\}$, $z \in \mathbb{C}$ and h belongs to one of the H_i's. By symmetry of the μ_i's, we have $F_H(gh|z) \leq F(e,gh|z) < 1$.

Conditioning on the first step of the random walk, we get

$$\begin{aligned} F_H(gh|z) = \mu(gh)z &+ \sum_{g_0\in\Gamma_{\tau(g)}\setminus gH_{\tau(g)}} \mu(g_0)zF_H(g_0^{-1}gh|z)\\ &+ \sum_{i\in\mathcal{I}\setminus\{\tau(g)\}}\sum_{g_0\in\Gamma_i}\mu(g_0)z\sum_{h_0\in H_i} F_H(g_0^{-1}h_0|z)F_H(h_0^{-1}gh|z). \end{aligned} \tag{10.3}$$

Since there are only *finitely* many functions $F_H(\cdot|z)$, one can compute $F_H(\cdot|z)$ by solving the finite system of quadratic equations (10.3). We define also

$$\mathcal{F}_i^{(H)}(\lambda) := \sum_{\substack{g\in\mathcal{R}_i\setminus\{e_i\},\\ h\in H_i}} F_H(gh|\lambda)$$

and the matrix $N = \big(n(i,j)\big)_{i,j\in\mathcal{I}}$ with entries

$$n(i,j) := \begin{cases} \mathcal{F}_j^{(H)}(\lambda), & \text{if } i \neq j,\\ 0, & \text{if } i = j.\end{cases}$$

We denote by θ_H the Perron–Frobenius eigenvalue of N and by ϱ_H the one of the matrix $D_H = \big(d_H(i,j)\big)_{i,j\in\mathcal{I}}$, defined by

$$d_H(i,j) := \begin{cases} [\Gamma_j : H_j] - 1, & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Finally, we can state the following formulas for the dimensions:

Corollary 10.2.1. *With probability one,*

$$\mathrm{BD}^{(H)}(\Lambda) = \mathrm{HD}^{(H)}(\Lambda) = -\frac{\log \theta_H}{\log \alpha}$$

and

$$\mathrm{BD}^{(H)}(\Omega) = \mathrm{HD}^{(H)}(\Omega) = -\frac{\log \varrho_H}{\log \alpha}.$$

To prove this statement, we start by showing the following intermediate result:

Lemma 10.2.2.

$$\overline{\mathrm{BD}^{(H)}}(\Lambda) \leq -\frac{\log \theta_H}{\log \alpha} \quad \textit{and} \quad \mathrm{BD}^{(H)}(\Omega) = -\frac{\log \varrho_H}{\log \alpha}.$$

Proof. We proceed similarly to the proof of Lemma 10.1.2. We define the matrices $N_0 = \big(n_0(i,j)\big)_{i,j\in\mathcal{I}}$ and $D_{0,H} = \big(d_{0,H}(i,j)\big)_{i,j\in\mathcal{I}}$ by

$$n_0(i,j) := \begin{cases} \mathcal{F}_i^{(H)}(\lambda), & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \qquad d_{0,H}(i,j) := \begin{cases} [\Gamma_i : H_i] - 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For $m \in \mathbb{N}$, let $\mathcal{H}_m^{(H)}$ denote the set of words of the form $g_1 \dots g_m h \in \Gamma$ in the sense of (10.1). Since every path from e to $g_1 \dots g_m h \in \Gamma$ has to go through the vertices $g_1 \dots g_j h_j \in \Gamma$, where $h_j \in H$ and $h_m = h$, we have

$$\begin{aligned} \sum_{g_1 \dots g_m h \in \Gamma} F_H(g_1 \dots g_m h | z) &= \sum_{g_1 \dots g_m h \in \Gamma} \sum_{h_1,\dots,h_{m-1} \in H} \prod_{i=1}^{m} F_H(g_i h_i | z) \\ &= \mathbf{1}^T N_0 N^{m-1} \mathbf{1}. \end{aligned}$$

Choose now an eigenvector $\mathbf{v}$ of N w.r.t. the eigenvalue θ_H such that componentwise $\mathbf{v} = (v_1, \dots, v_r)^T \geq \mathbf{1}$. Then

$$\mathbb{E}\,\mathrm{Card}(\mathcal{H}_m^{(H)}) \leq \mathbf{1}^T N_0 N^{m-1} \mathbf{1} \leq \mathbf{1}^T N_0 N^{m-1} \mathbf{v} = \theta_H^{m-1} \cdot \Big(\sum_{i \in \mathcal{I}} v_i \mathcal{F}_i^{(H)}(\lambda) \Big).$$

Therefore, $\limsup_{m\to\infty} \mathbb{E}\,\mathrm{Card}(\mathcal{H}_m^{(H)})^{1/m} \leq \theta_H$.

Furthermore, we remark that

$$\hat{\sigma}_H(m) = \mathrm{Card}\big(\{x_1 \dots x_m \mid x_i \in \bigcup_{j\in\mathcal{I}} \mathcal{R}_j \setminus \{e_j\}, x_i \in \mathcal{R}_j \Rightarrow x_{i+1} \notin \mathcal{R}_j\}\big)$$

can be written as

$$\hat{\sigma}_H(m) = \mathbf{1}^T D_{0,H} D_H^{m-1} \mathbf{1}.$$

Taking eigenvectors $\mathbf{v}_1 \geq \mathbf{1}$ and $\mathbf{v}_2 \leq \mathbf{1}$ w.r.t. ϱ_H leads to

$$\lim_{m\to\infty} \mathrm{Card}(\hat{\sigma}_H(m))^{1/m} = \varrho_H.$$

Like for Lemma 10.1.2, one concludes the proof applying the same reasoning used in the proofs of Lemma 8.1.5 and Propositions 8.1.6, 9.1.5. □

Finally we can prove the advertised corollary:

Proof of Corollary 10.2.1. It is sufficient to show that $-\log\theta_H/\log\alpha$ is also a lower bound for $\mathrm{HD}^{(H)}(\Lambda)$. First, we remark that for $m \in \mathbb{N}$

$$\begin{aligned}&\sum_{g_1\dots g_m h\in\Gamma: g_1\notin\mathcal{R}_1} F(e, g_1\dots g_m h|\lambda)\\ &\quad= \sum_{g_1\dots g_m h\in\Gamma: g_1\notin\mathcal{R}_1}\ \sum_{h_0\in H} F_H(g_1\dots g_m h_0|\lambda)\, F(e, h_0^{-1}h|\lambda).\end{aligned}$$

Since $\mathrm{Card}(H) < \infty$, there are real constants $d, D > 0$ such that for all $h \in H$ it holds $d \leq F(e,h|\lambda) \leq D$. Now we write $\mathbf{1}_0 := (0,1,\dots,1)^T \in \mathbb{R}^r$ and therefore we get:

$$\begin{aligned}\Big(\sum_{g_1\dots g_m h\in\Gamma: g_1\notin\mathcal{R}_1} \mathbb{E}Z_\infty(g_1\dots g_m h)\Big)^{1/m} &\leq \Big(D\cdot \mathbf{1}_0^T N_0 N^{m-1}\mathbf{1}\Big)^{1/m} \xrightarrow{m\to\infty} \theta_H,\\ \Big(\sum_{g_1\dots g_m h\in\Gamma: g_1\notin\mathcal{R}_1} \mathbb{E}Z_\infty(g_1\dots g_m h)\Big)^{1/m} &\geq \Big(d\cdot \mathbf{1}_0^T N_0 N^{m-1}\mathbf{1}\Big)^{1/m} \xrightarrow{m\to\infty} \theta_H.\end{aligned}$$

This can be verified by substituting $\mathbf{1}$ in the first case by an eigenvector (w.r.t. θ_H) $\mathbf{v}_1 \geq \mathbf{1}$, and in the second case by an eigenvector (w.r.t. θ_H) $\mathbf{v}_2 \leq \mathbf{1}$.

With the help of this convergence, we can prove that the upper bounds in Lemma 10.2.2 equal the Hausdorff and the Box-Counting dimensions. This can be done once again following the procedure described in Section 8.1.2 (as well as in [31, Section 6]).

Analogously to the proof of Theorem 9.1.1 we can finally conclude that $\mathrm{HD}^{(H)}(\Omega) = \mathrm{BD}^{(H)}(\Omega)$. □

Example 10.2.3. *Consider the amalgam $(\mathbb{Z}/6\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$ and write $\Gamma_1 = \langle a \mid a^6 = e_1\rangle$, $\Gamma_2 = \langle b \mid b^6 = e_2\rangle$, and $H = \langle c \mid c^2 = e_H\rangle$, being e_H the identity of H.*

The isomorphisms are defined by $\phi_1(a^3) = c = \phi_2(b^3)$. Therefore,

$$(\mathbb{Z}/6\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z}) = \langle a, b \mid a^6 = b^6 = e, a^3 = b^3\rangle.$$

We set $\mu_1(a) = \mu_1(a^5) = \mu_2(b) = \mu_2(b^5) = 1/2$, $\alpha_1 = \alpha_2 = 1/2$ and consider the distance defined by (10.2) with base $\alpha = 1/2$. The system (10.3) becomes then

$$\begin{aligned}F_H(a|z) &= \frac{z}{4} + \frac{z}{4}F_H(a^2|z) + \frac{z}{2}\big(F_H(a|z)^2 + F_H(a^2|z)^2\big),\\ F_H(a^2|z) &= \frac{z}{4}F_H(a|z) + \frac{z}{2}\big(F_H(a|z)F_H(a^2|z) + F_H(a^2|z)F_H(a|z)\big).\end{aligned}$$

Observe that $F_H(a|z) = F_H(a^5|z)$ *and* $F_H(a^2|z) = F_H(a^4|z)$. *The Hausdorff dimension of the limit set of the BRW is then given by*

$$\mathrm{HD}^{(H)}(\Lambda) = \frac{\log\bigl(2F_H(a|\lambda) + 2F_H(a^2|\lambda)\bigr)}{\log 2},$$

while $\mathrm{HD}^{(H)}(\Omega) = 1$.

Part III

Branching Random Walks on Cartesian Products

Chapter 11

Critical BRW's on $T_3 \times \mathbb{Z}$

In this third part of our work, we investigate *critical* BRW's on some Cartesian products of groups. In particular we consider two settings: $T_q \times \mathbb{Z}^d$ and $T_q \times T_q$, being T_q a homogeneous tree of degree q.

Our aim is to understand the limit set of a critical BRW on these structures: does it have *finitely* or *infinitely* many ends?

All computations presented in this chapter are done for the Cartesian product of the *binary tree* (denoted by T_3) with the set of integers $\mathbb{Z}$, but they can easily be generalized to a product of a homogeneous tree (with bounded degree) with $\mathbb{Z}^d$ (for every $d \geq 1$). The reason will become clear by looking at the computations.

Analogously to [12] we define the probability measure on $T_3 \times \mathbb{Z}$ as

$$\mu := \alpha_1 \mu_1 + \alpha_2 \mu_2, \tag{11.1}$$

where $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$, while μ_1 and μ_2 are the measures defined on the generators of the first and second factor, respectively.

By analogy to the previous parts of the work, we denote by ρ the spectral radius of the Markov chain governed by μ and by $\mathbf{R}$ its inverse. By [4, Section 4], the value $\mathbf{R}$ is the critical mean value for the offspring distribution of the BRW, and by [20] we know that at this point the BRW is still transient, afterwards it becomes recurrent.

By [12] we have:

$$\lim_{n\to\infty} \left(\mu^{(n)}(\mathbf{0})\right)^{1/n} = \alpha_1 \rho_1 + \alpha_2 \rho_2, \tag{11.2}$$

where ρ_i denotes, as usual, the spectral radius of the random walk governed by μ_i.

11.1 Isotropic Case

Let us start with the isotropic situation, i.e., μ_1 and μ_2 govern *simple random walks.*

Under the condition of survival, we have the following result:

Proposition 11.1.1. *The limit set of the critical isotropic BRW on $T_3 \times \mathbb{Z}$ has infinitely many ends almost surely.*

Proof. The crucial fact is the following: the critical BRW defined on $T_3 \times \mathbb{Z}$ can be projected on each of the two factors, and such projections are again BRW's.

Let us consider the projection of the process on the tree, in which case every edge $\{x, y\} \in \mathbb{Z}$ reduces to a single vertex in T_3. What we see is a BRW governed by a probability measure (denote it by μ^*), such that every particle moves with transition probabilities

$$\mu^*(x^{-1}y) := \begin{cases} \alpha_1\mu_1(x^{-1}y) & \text{if the edge } \{x,y\} \in T_3, \\ \alpha_2 & \text{if the edge } \{x,y\} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

By standard computations, we can evaluate the spectral radius ρ^* of the Markov chain governed by μ^*: this new Markov chain is nothing else but a *lazy random walk* on T_3. We find that

$$\rho^* = \alpha_1\rho_1 + \alpha_2. \tag{11.3}$$

Using relation (11.2), we can evaluate the spectral radius of μ:

$$\rho = \lim_{n\to\infty} \left(\mu^{(n)}(\mathbf{0})\right)^{1/n} = \alpha_1\rho_1 + \alpha_2\rho_2 = \alpha_1 \frac{2\sqrt{2}}{3} + \alpha_2,$$

where in the last equality we used [64, Lemma 1.24] to find $\rho_1 = 2\sqrt{2}/3$, and the fact that $\rho_2 = 1$, being $\mathbb{Z}$ amenable and μ_2 symmetric (we refer again to Section 1.3.4).

Therefore, since $\rho = \rho^*$, we can immediately deduce that the projection of the entire BRW on the tree is transient (transience is assured by [20]), which means that every copy of $\mathbb{Z}$ is visited by only finitely many particles almost surely.

Consequently, the accumulation set of the considered BRW coincides a.s. (on the event of survival) with a *proper, non-trivial* random subset of the union of the boundaries of the trees. This union has infinitely many ends. ☐

Remark 11.1.2. *The previous computations can be repeated* 1 *to* 1 *if we replace T_3 by any homogeneous tree of degree $q \geq 3$ and $\mathbb{Z}$ by any finitely generated* ***amenable*** *group.*

11.2 Anisotropic Case

In this second situation, we consider a BRW on $T_3 \times \mathbb{Z}$ such that its underlying walk has a drift on the second factor. More precisely, denote by e_1 and e_{-1} the two natural generators of $\mathbb{Z}$ and fix a parameter $0 < \varepsilon < 1$ arbitrarily small. Now we choose μ_1 to be a simple random walk on T_3, and μ_2 such that

$$\mu_2(t) = \begin{cases} \frac{1+\varepsilon}{2} & \text{if } t = e_1 \\ \frac{1-\varepsilon}{2} & \text{if } t = e_{-1}. \end{cases} \tag{11.4}$$

Our aim is to show that in this situation, the limit set of the BRW on $T_3 \times \mathbb{Z}$ is one-ended.

For this purpose, we start by investigating what happens on the second factor, and then how this information can be used to understand the behavior of the BRW on the Cartesian product.

Choosing μ_2 as in (11.4), using (11.2) we get that for every $\varepsilon > 0$:

$$\rho := \lim_n \left(\mu^{(n)}(\mathbf{0})\right)^{1/n} < \rho^*. \tag{11.5}$$

By [4] this means that the critical value of the offspring distribution in this case is larger than the critical value of the projected process on the tree (for every arbitrary choice of α_1 and $\alpha_2 := 1 - \alpha_1$).

From this result it follows immediately that every copy of $\mathbb{Z}$ is visited infinitely often by the particles of the process: hence there are infinitely many connections between different copies of $\mathbb{Z}$.

On each copy of $\mathbb{Z}$ the drift in the direction e_1 "pushes" the random walk, making it transient at its critical value, which can be computed explicitly using [66, Proposition 9.3]. Obviously, it is a function of ε, in fact it turns out to be $\rho_2^{-1} = (\sqrt{1-\varepsilon^2})^{-1}$.

Now we need to prove that all the particles of the critical BRW defined on $\mathbb{Z}$ accumulate in the same direction: we prove this in the next lemma.

Lemma 11.2.1. *For every $\varepsilon > 0$ the critical BRW on $\mathbb{Z}$, whose underlying random walk is governed by (11.4), is almost surely one-ended.*

Proof. By [66, Proposition 9.3], we can easily compute the first arrival generating functions relatively to the second factor

$$\begin{aligned} F_2(0, e_1 \mid z) &= \frac{1 \pm \sqrt{1-(1-\varepsilon^2)z^2}}{(1-\varepsilon)z} \\ F_2(0, e_{-1} \mid z) &= \frac{1 \pm \sqrt{1-(1-\varepsilon^2)z^2}}{(1+\varepsilon)z}. \end{aligned}$$

Exploiting the natural tree-structure of $\mathbb{Z}$ we know that every element of $\mathbb{Z}$ is either of the form $x = (e_1)^n$, or of the form $y = (e_{-1})^n$, for some $n \in \mathbb{N}$.

In the first case we have:

$$F_2(0, x \mid z) = (F_2(0, e_1 \mid z))^n,$$

and for every $y = (e_{-1})^n$

$$F_2(0, y \mid z) = (F_2(0, e_{-1} \mid z))^n.$$

Since the ratio $\sqrt{1-\varepsilon^2}/(1-\varepsilon)$ is larger than one, while $\sqrt{1-\varepsilon^2}/(1+\varepsilon)$ is strictly smaller, for every word $x = (e_1)^n$ and $y = (e_{-1})^n$ we get:

$$\begin{aligned} \lim_n F_2(0, x \mid \rho_2^{-1}) &= \lim_n \big(F_2(0, e_1 \mid \rho_2^{-1})\big)^n = \lim_n \left(\frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon}\right)^n = \infty, \\ \lim_n F_2(0, y \mid \rho_2^{-1}) &= \lim_n \big(F_2(0, e_{-1} \mid \rho_2^{-1})\big)^n = \lim_n \left(\frac{\sqrt{1-\varepsilon^2}}{1+\varepsilon}\right)^n = 0. \end{aligned}$$

Applying Lemma 7.4.6 (or, equivalently [31, Lemma 1]) and then Markov inequality, we get the claim: the end determined by the direction $\lim_n (e_{-1})^n$ is *not* an accumulation point for the process. The critical BRW on $\mathbb{Z}$ is almost surely one-ended. $\square$

At this point we know that the BRW on $\mathbb{Z}$ will accumulate at the end identified by direction e_1.

Proposition 11.2.2. *Choose μ_1 to be a simple random walk on T_3, and μ_2 as in (11.4). Then the limit set of the critical BRW on $T_3 \times \mathbb{Z}$ is one-ended.*

Proof. By relation (11.5) and Lemma 11.2.1, it follows that there are infinitely many connections between the infinitely many one-ended sets of accumulation points, i.e. the one ended-sets are all equivalent. Hence the BRW defined on the Cartesian product accumulates on a one-ended limit set. $\square$

We would like to add that by the previous computations can be repeated 1 to 1 if we replace T_3 by any homogeneous tree of degree $q > 2$ and $\mathbb{Z}$ by any Cartesian product of the form $\mathbb{Z}^d$. In this case an analogue of Lemma 11.2.1 still holds in the following form:

Lemma 11.2.3. *Denote by $e_{\pm 1}, \dots, e_{\pm d}$ the natural generators of $\mathbb{Z}^d$ and for every $\varepsilon > 0$ fix the measure on $\mathbb{Z}^d$ defined by*

$$\mu_2(t) := \begin{cases} \beta_1 \frac{(1+\varepsilon)}{2} & \text{if } t = e_1 \\ \beta_1 \frac{(1-\varepsilon)}{2} & \text{if } t = e_{-1} \\ \frac{\beta_i}{2} & \text{if } t = e_{\pm i}, \ \text{for } i \in \{2, \dots, d\}, \end{cases}$$

where $\beta_1, \dots, \beta_d > 0$ and $\sum_{j=1}^d \beta_j = 1$.

Then the limit set of the critical BRW on $\mathbb{Z}^d$ is almost surely one-ended.

Proof. By projecting the process defined on $\mathbb{Z}^d$ onto the first factor $\mathbb{Z}$, we see a BRW governed by the following probability measure:

$$\mu_1^*(t) = \begin{cases} \beta_1 \frac{(1+\varepsilon)}{2} & \text{if } t = e_1 \\ \beta_1 \frac{(1-\varepsilon)}{2} & \text{if } t = e_{-1} \\ \sum_{i=2}^d \beta_i & \text{otherwise.} \end{cases}$$

This is the probability measure governing a biased lazy random walk on $\mathbb{Z}$: it stays in place with probability $\sum_{i=2}^d \beta_i$ and it moves on the considered factor otherwise.

Using again [12] and [66, Proposition 9.3], we get that the spectral radius of μ_1^* is given by

$$\rho_1^* = \beta_1 \sqrt{1 - \varepsilon^2} + \sum_{i=2}^d \beta_i.$$

In this case $\rho_1^* = \rho_2$ (where as usual ρ_2 denotes the spectral radius of μ_2), giving us a transient BRW on $\mathbb{Z}$. By reasoning on the first factor $\mathbb{Z}$, we can repeat the same steps done in the proof of Lemma 11.2.1, obtaining the claim. In fact we know that the projection of the BRW on any factor other than the first one would give us a recurrent lazy random walk (since the random walk on the other factors is symmetric).

It follows that there are infinitely many connections between the infinitely many one-ended sets, i.e., the limit set of the BRW on $\mathbb{Z}^d$ is one-ended. □

Now we can state the main result of this chapter:

Theorem 11.2.4. *Denote by A any finitely generated amenable group. For every probability measure μ_1 and μ_2, and every value of α_1 and $\alpha_2 := 1 - \alpha_1$ we have the following characterization (phase transition):*

(i) if μ_2 is a symmetric measure, then the limit set of the critical BRW on $T_q \times A$ has infinitely many ends almost surely;

(ii) if μ_2 is non-symmetric (i.e. the random walk on $\mathbb{Z}^d$ is biased towards one direction), then the limit set of the critical BRW on $T_q \times \mathbb{Z}^d$ is one-ended almost surely.

Proof. *Statement (i):* if μ_2 is symmetric (i.e. it governs a simple random walk on $\mathbb{Z}^d$), then a straightforward generalization of Proposition 11.1.1 holds when μ_1 is not symmetric. This implies that the limit set of the critical BRW on $T_q \times A$ has infinitely many ends almost surely. This happens independently of the measure μ_1 and of the values of α_1, α_2.

Statement (ii): If μ_2 is a non-symmetric measure (i.e. the random walk on $\mathbb{Z}^d$ is biased towards one direction), then a straightforward generalization of Proposition 11.2.2 holds. Using Lemma 11.2.3 we obtain that the limit set of the critical BRW on $T_q \times \mathbb{Z}^d$ has only one end almost surely. This is true independently of the measure μ_1 and of the values of α_1, α_2. □

At this point it is natural to conjecture the following (more general) statement:

Conjecture 11.2.5. *Choose any one-ended, finitely generated amenable group A with a probability measure μ_2 on A such that the random walk governed by μ_2 is biased towards one direction. If μ_1 governs a simple random walk on T_3, then the limit set of the critical BRW on $T_3 \times A$ (whose underlying walk is governed by μ defined in (11.1)) is one-ended.*

11.3 Generalization

The results found so far with the help of [4], [20] and [12], can be pushed further to investigate more general situations.

We can, for example, consider the Cartesian product $T \times \mathbb{Z}^d$, where T is a Galton-Watson tree (conditioned on survival).

In order to approach this topic, we need to recall the most important results about BRW's on Galton-Watson trees: the main work we refer to, is the one by Pemantle and Stacey [47]. In their paper, they find precise conditions for a BRW defined on a Galton-Watson tree to survive and have two phase transitions.

Denote by T a Galton-Watson tree whose evolution is determined by a probability measure η, i.e.,

$$\eta_k := \mathbb{P}(\text{a vertex has } k \text{ descendants}).$$

We are interested in the case when T is infinite, then we need to assume $m := \sum_k k\eta_k > 1$.

Remark 11.3.1. *The topic "random walks on Galton-Watson trees" has been studied by many authors, and the reader can find a summary of the known results in [38, Chapter 16].*

Grimmett and Kesten (see [27]) proved that the simple random walk on infinite Galton-Watson trees is a.s. transient. This result was deepened by Lyons, Pemantle and Peres: they investigated the rate of escape of the simple random walk (in [40]) and of the biased random walk (in [41]).

Denote by $\deg(x)$ the degree of a vertex $x \in T$, then the simple random walk on T is governed by the measure

$$\mu_1(x^{-1}y) = \begin{cases} 1/\deg(x) & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases} \tag{11.6}$$

Using the same techniques described in [12], we get that the simple random walk on $T \times A$ (being A any finitely generated amenable group) has spectral radius

$$\rho = \frac{1}{2}(\rho_1 + 1),$$

where ρ_1 is the spectral radius of the simple random walk on T.

In [47], Pemantle and Stacey consider the development of BRW's and contact processes on Galton-Watson (as well as non-homogeneous) trees. They define the BRW in a slightly different way, which we recall here for seek of simplicity.

Denote by $n(x,t)$ the amount of particles alive at vertex $x \in T$ at time t. According to their definition, at every unit of time, each vertex v gives particles away at rate $n(v,t)$, while it receives particles from its neighbors at rate:

$$\beta \sum_{u:u\sim v} n(u,t),$$

being $\beta > 0$ a fixed parameter determining the evolution of the model. They prove that under some conditions (see below) on the underlying Galton-Watson tree, the BRW has two *distinct* critical values for β: one (denoted by β_w) above which the process survives weakly, and the other (denoted by β_s) above which every vertex is visited infinitely often by the BRW.

In particular, they show that (see [47, Propositions 2.5 and 2.6])

$$\beta_s = 1/(2\sqrt{d}),$$

where $(d+1) < \infty$ is the maximum degree of the underlying Galton-Watson tree T. They find a bound for β_w as well:

$$\beta_w \leq 1/(m+1),$$

where $m > 1$ is the mean value of η, the offspring distribution of T. In addition, this inequality becomes strict if η is not concentrated on a single value.

It follows (see [47, Theorem 2.1]) that if we have $(m+1) \geq 2\sqrt{d}$, then $\beta_w < \beta_s$ almost surely, on the event of nonextinction.

Therefore, from now on we assume the following condition:

Assumption 11.3.2. $(m+1) \geq 2\sqrt{d}$.

In our setting, we can choose the tree T appropriately, i.e., in such a way that Assumption 11.3.2 is satisfied and T has no leaves (i.e., $\eta_0 = 0$). At this point we define a BRW on T with transition probabilities governed by the measure μ_1 on T such that

$$\mathbb{P}(\text{a particle of the BRW moves from } u \text{ to } v \text{ in one step}) = \mu_1(u^{-1}v),$$

where μ_1 is defined by (11.6), i.e., it governs a simple random walk on T.

Denote by ν the probability distribution governing the offspring distribution of the BRW, and once again denote by

$$\lambda := \sum_k k\nu_k,$$

where $\nu_k := \mathbb{P}(\text{a particle of the BRW defined on } T \text{ has } k \text{ offspring})$.

For each couple of neighbors $u, v \in T$ denote by $\xi_{u,v}$ the amount of particles of the BRW that start at u and reach v. Then, for every $\beta_w < \beta < \beta_s$ we have

$$\mathbb{E}\left(\beta \sum_{t \geq 1} n(u,t)\right) = \mathbb{E}(\xi_{u,v}) = \sum_{t \geq 1} \lambda^t \mu_1^{(t)}(u^{-1}v) = G_1(u, v \mid \lambda), \qquad (11.7)$$

where $G_1(u, v \mid \lambda)$ is the Green function associated to the simple random walk on T evaluated at $z = \lambda$. In the last equality we used a more general version of [31, Lemma 1], considering all the particles going from u to v, and not only the ones stopped on arriving at v for the first time. The proof of this equality is based on the same techniques as the one of [31, Lemma 1], with some extra care because the random walk is defined on a random structure.

Relation (11.7) shows us the connection between the paramenter β and the mean value λ. It is clear that, in order to preserve equality in (11.7), if we change the value of β then we need to modify λ accordingly. In particular it is easy to see that λ not only varies continuously in β, but it can be seen as a monotone increasing function of β.

In other words, there are two values λ_w (determining a phase transition between death and weak survival) and λ_s (determining a transition between weak and strong survival) such that under Assumption 11.3.2 we have

$$\lambda_w < \lambda_s.$$

The necessary condition for survival of the BRW is $\lambda > 1$, then we get

$$1 < \lambda_s.$$

Since T has bounded maximum degree $(d+1)$, it is a *quasi-transitive* graph. We can therefore apply the methods used in [20] to find that at $\lambda = \lambda_s$ the BRW is transient.

Given all these facts, we can perform the same steps as in Sections 11.1 and 11.2 in order to prove the next two statements:

Proposition 11.3.3. *Consider a finitely generated, amenable group A and a Galton-Watson tree T (without leaves) such that Assumption 11.3.2 is satisfied. Then the limit set of the critical isotropic BRW on $T \times A$ has infinitely many ends almost surely.*

Proposition 11.3.4. *Consider a Galton-Watson tree T (without leaves) such that Assumption 11.3.2 is satisfied. Let the measure μ_2 have a bias in one direction, then the limit set of the critical BRW on $T \times \mathbb{Z}^d$ is almost surely one-ended.*

Chapter 12

BRW's on $T_3 \times T_3$

In this chapter, all computations are made for the case $\Gamma := T_3 \times T_3$, but they can easily be generalized to the Cartesian product of two homogeneous trees of finite degrees $a \geq 3$ and $b \geq 3$ respectively.

Our aim is to investigate the critical BRW on $T_3 \times T_3$ whose underlying random walk is a simple random walk. This means that the underlying random walk is governed by

$$\mu := \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2, \tag{12.1}$$

with μ_1 and μ_2 governing simple random walks on T_3. Denote once again by ρ the spectral radius of the simple random walk governed by μ.

In order to distinguish the two factors of the Cartesian product we will denote by T_3^h the *horizontal* factor T_3, and by T_3^v the *vertical* factor T_3.

We can consider two compactifications of the Cartesian product: namely the *end compactification* (recall Section 2.1) and the *Martin compactification* (recall Section 2.2). As the reader can easily check, the former consists of only one element. In order to deal with the latter we will need to be more careful.

Denote by ∂T_3^h and ∂T_3^v the end compactifications of T_3^h and T_3^v respectively.

Here we are dealing with critical BRW's, therefore all generating functions are evaluated at $z = \mathbf{R}$, which means that the *Martin Kernel* (recall Equation (2.5)) is evaluated at $t = \mathbf{R}^{-1} = \rho$.

We would like to anticipate that the Martin boundary is the tool that gives us a more formal idea of what we mean by "infinitely ended (random) set", since Γ itself is one-ended.

By [51] we know that the Martin boundary $\mathcal{M}$ of $T_3^h \times T_3^v$ (when $z = \mathbf{R}$) is given by

$$\mathcal{M} = (\partial T_3^h \times T_3^v) \cup (\partial T_3^h \times \partial T_3^v) \cup (T_3^h \times \partial T_3^v). \tag{12.2}$$

Roughly speaking, $\mathcal{M}$ is the set of all *possible directions* that the process can take. Since the BRW is transient, all particles will eventually move away from any finite set: this means that they will follow a path going to some element of $\mathcal{M}$.

12.1 Martin Topology

Denote by o_1 and o_2 the roots of T_3^h and T_3^v respectively, and by o_1o_2 the origin of the Cartesian product. By $(Z_1^{(n)})_{n\geq 0}$ and $(Z_2^{(n)})_{n\geq 0}$ we denote any sequence of (non necessarily distinct) vertices in T_3^h and T_3^v respectively. In this way each sequence on the Cartesian product Γ can be written as $(Z_1^{(n)}Z_2^{(n)})_{n\geq 0}$, for some suitable sequences. For simplicity, we set $Z_1^{(0)}Z_2^{(0)} := o_1o_2$

Recall that $l(u)$ is the Cayley graph distance of an element from the origin (see Section 2.1). If at least one, out of $(Z_1^{(n)})_{n\geq 0}$ and $(Z_2^{(n)})_{n\geq 0}$ (denote it by $(Z_\iota^{(n)})_{n\geq 0}$), is such that

$$l(Z_\iota^{(n+1)}) = 1 + l(Z_\iota^{(n)}) \quad \forall\, n \geq 0,$$

then we say that $(Z_1^{(n)}Z_2^{(n)})_{n\geq 0}$ is an *increasing sequence*. We point out that it is possible to describe any element $\omega \in \mathcal{M}$ as limit of an *increasing* sequence of vertices $(Z_1^{(n)}Z_2^{(n)})_{n\geq 0} \in T_3^h \times T_3^v$.

Define $T_3^h(a,b)$ (resp. $T_3^v(a,b)$) to be the horizontal (resp. vertical) binary tree rooted at b not including a.

At this point we can state how the neighborhoods of each element of $\mathcal{M}$ look like. By Relation (12.2), we have three possible cases.
For suitable increasing sequences $(Z_1^{(n)})_{n\geq 0}$ and $(Z_2^{(n)})_{n\geq 0}$, every element ω belonging to $(\partial T_3^h \times T_3^v)$ or $(T_3^h \times \partial T_3^v)$ has neighborhoods of type

$$U^{(n)} := (T_3^h(o_1, Z_1^{(n)}) \times y), \tag{12.3}$$

or

$$U^{(n)} := (x \times T_3^v(o_2, Z_2^{(n)})), \tag{12.4}$$

respectively. Here $x \in T_3^h$ and $y \in T_3^v$.

In case $\omega \in (\partial T_3^h \times \partial T_3^v)$, its neighborhoods are:

$$U^{(n)} := (T_3^h(o_1, Z_1^{(n)}) \times T_3^v(o_2, Z_2^{(n)})). \tag{12.5}$$

Now we can define what an *accumulation point* of the BRW is: it is an element ω of $\mathcal{M}$ such that any arbitrarily small neighborhood $U^{(n)}$ of ω contains trails of the BRW.

The goals of this section are the following:

(i) to understand which elements of $\mathcal{M}$ are accumulation points;

(ii) to distinguish different types of accumulation points.

Now we can show our first result.

Lemma 12.1.1. *Every element of $\mathcal{M}$ is an accumulation point.*

Proof. Given Relation (12.2) we should consider the three cases given by Equations (12.3)–(12.5) separately but, exploiting the symmetry of (12.3) and (12.4), we can reduce our investigation to only two situations.

To show that in neighborhoods $U^{(n)}$ described by Equations (12.3) and (12.4) there are trails for every n, it suffices to show that $U^{(n)}$ is touched infinitely many times by the BRW.

This is a simple consequence of the following fact. Denote by ρ the spectral radius of the simple random walk on $T_3^h \times T_3^v$. By [4], the critical mean value of the BRW is ρ^{-1}, which (by [64, Lemma 1.24]) equals $3/(2\sqrt{2})$.

If we project the BRW on T_3^h (or, equivalently, on T_3^v), we see a BRW governed by the lazy random walk, whose spectral radius is

$$\rho_{\text{lazy}} = \frac{1}{2}\left(\frac{2\sqrt{2}}{3} + 1\right).$$

Since $\rho < \rho_{\text{lazy}}$, we have that the BRW on each projection is recurrent (see [4, Proposition 4.5]). Therefore elements ω whose neighborhoods are of type (12.3) and (12.4) are accumulation points (in the Martin topology).

In the situation where ω has neighborhoods of type (12.5) we can analyze things in a similar way: now we know that every copy of T_3^h and of T_3^v is touched infinitely often (i.o. for short) by the process, i.e.

$$\mathbb{P}(T_3^h \text{ touched i.o. by BRW}) = \mathbb{P}(T_3^v \text{ touched i.o. by BRW}) = 1.$$

By symmetry we can argue that if we split T_3^h (or T_3^v) into three equal subtrees (denote them by $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$), the following holds:

$$\begin{aligned}&\mathbb{P}(T^{(1)} \text{ touched i.o. by BRW}) + \\ &+ \mathbb{P}(T^{(2)} \text{ touched i.o. by BRW}) + \mathbb{P}(T^{(3)} \text{ touched i.o. by BRW}) \geq 1.\end{aligned}$$

This implies

$$\begin{aligned}\mathbb{P}(T^{(1)} \text{ touched i.o. by BRW}) &\geq 1/3,\\ \mathbb{P}(T^{(2)} \text{ touched i.o. by BRW}) &\geq 1/3,\\ \mathbb{P}(T^{(3)} \text{ touched i.o. by BRW}) &\geq 1/3.\end{aligned}$$

Since for $j \in \{1,2,3\}$ the event $\{T^{(j)} \text{ touched i.o. by BRW}\}$ is a tail event, its probability must be 1. By (12.5) we see that the probability that $U^{(n)}$ is touched infinitely often by the BRW is at least the product of the probabilities of $T_3^h(o_1, Z_1^{(n)})$ and $T_3^v(o_2, Z_2^{(n)})$ being touched infinitely often.

It is clear that $T_3^h(o_1, Z_1^{(n)})$ and $T_3^v(o_2, Z_2^{(n)})$ are isomorphic to one of the $T^{(j)}$'s. By this observation we get that $U^{(n)}$ is touched infinitely often by the BRW as well:

$$\begin{aligned}&\mathbb{P}(U^{(n)} \text{ touched i.o. by BRW})\\ &\geq \mathbb{P}(T^{(i)} \text{ touched i.o. by BRW})\mathbb{P}(T^{(j)} \text{ touched i.o. by BRW}) = 1,\end{aligned}$$

for some appropriate $i, j \in \{1,2,3\}$.

We would like to emphasize that the calculations do *not* depend on the starting point of the process. More precisely, every neighborhood $U^{(n)}$ of every element $\omega \in \mathcal{M}$ is reached infinitely often by the BRW, independently of the initial location of the particles. $\square$

Remark 12.1.2. *In particular, we can start the process as far away from the chosen $U^{(n)}$ as we wish, and Lemma 12.1.1 assures that (with probability one) there are particles of the BRW that eventually reach $U^{(n)}$.*

Now we can distinguish two different types of accumulation points, that we will call *stable* and *unstable*.

We say that $\omega \in \mathcal{M}$ is stable if it is "attractive" for the process, i.e., for every neighborhood $U^{(n)}$ of ω we have that

$$\mathbb{P}(\exists \text{ particles of BRW that enter } U^{(n)} \text{ and eventually stay in } U^{(n)}) = 1.$$

Likewise, we say that $\omega \in \mathcal{M}$ is unstable if

$$\mathbb{P}(\exists \text{ particles of BRW that enter } U^{(n)} \text{ and eventually stay in } U^{(n)}) = 0.$$

Our next result is:

Proposition 12.1.3. *Every $\omega \in (\partial T_3^h \times T_3^v) \cup (T_3^h \times \partial T_3^v)$ is an unstable accumulation point.*

Proof. Markov inequality tells us that for every random variable A

$$\mathbb{P}(|A| \geq a) \leq \frac{\mathbb{E}|A|}{a}. \tag{12.6}$$

Let us consider the following event:

$$A_{n,k} := \{\exists \text{ particles that stay in } T_3^h \text{ between time } n \text{ and } n+k\}.$$

We get:

$$\begin{aligned}\mathbb{E}(|A_{n,k}|) &= \rho^{-(n+k)}\mathbb{P}(X_{n+i} \in T_3^h,\ \forall i = 1, \ldots, k)\\ &= \rho^{-(n+k)}\mathbb{P}(X_n \in T_3^h)\left(\frac{1}{2}\right)^k \leq \left(\frac{3}{2\sqrt{2}}\right)^n \left(\frac{3}{4\sqrt{2}}\right)^k.\end{aligned}$$

This last quantity goes to zero as $k \to \infty$. Therefore

$$\begin{aligned}&\mathbb{P}(\exists \text{ particles that stay in } T_3^h \text{ eventually})\\ &= \mathbb{P}(\cup_{n=0}^{\infty} \exists \text{ particles that stay in } T_3^h \text{ after time } n)\\ &\leq \sum_{n\geq 0} \mathbb{P}(\exists \text{ particles that stay in } T_3^h \text{ after time } n)\\ &= \sum_{n\geq 0} \lim_{k\to\infty} \mathbb{P}(|A_{n,k}| \geq 1) \leq \sum_{n\geq 0} \lim_{k\to\infty} \mathbb{E}(|A_{n,k}|) = 0.\end{aligned}$$

The last inequality is just (12.6), evaluated at $a = 1$.

This proves that there are no infinite trails of particles connecting a vertex of T_3^h (resp. T_3^v) to an element of ∂T_3^h (resp. ∂T_3^v). □

The meaning of Proposition 12.1.3 is the following: there are infinitely many particles going through all neighborhoods $U^{(n)}$ of type (12.3) or (12.4), but with probability one they are spending a very short time there. They are leaving to reach some other accumulation point, i.e. a stable one.

On the other hand we have:

Proposition 12.1.4. *Every $\omega \in (\partial T_3^h \times \partial T_3^v)$ is a stable accumulation point.*

Proof. Let us split T_3 into three equal subtrees, and let us analyze what a single random walk $(X_n)_{n\in\mathbb{N}}$ governed by μ (defined by (12.1)) does.

We know that for every fixed n, $U^{(n)}$ (of type (12.5)) is touched infinitely often by the BRW with probability one. We claim that the following holds:

$$\mathbb{P}(\exists j \geq 1 \text{ s.t. } X_k \in U^{(n)},\ \forall k \geq j) \geq \frac{4}{9}. \tag{12.7}$$

This comes from the fact that the random walk must move in the right subtree of T_3^h and of T_3^v containing the projections of $U^{(n)}$. In other words, once the random walk X_j reaches $U^{(n)}$, it has to choose between three possible directions on T_3^h and three possible directions on T_3^v. Therefore, X_j has probability 2/3 of (eventually) ending up in $T_3^h(o_1, Z_1^{(n)})$, as well as of (eventually) ending up in $T_3^v(o_2, Z_2^{(n)})$ on T_3^v. Thus:

$$\begin{aligned}
&\mathbb{P}(\exists j \geq 1 \text{ s.t. } X_k \in U^{(n)}\ \forall k \geq j)\\
&\geq \mathbb{P}\big(X_j \in T_3^h(o_1, Z_1^{(n)}) \text{ eventually}\big)\mathbb{P}\big(X_j \in T_3^v(o_2, Z_2^{(n)}) \text{ eventually}\big)\\
&\geq \left(\frac{2}{3}\right)\left(\frac{2}{3}\right),
\end{aligned}$$

which is relation (12.7).

Define the following event:

$$\begin{aligned}
A_k := \{&\text{all descendants of a particle that hits } U^{(n)}\\
&\text{for the } k\text{-th time, eventually exit } U^{(n)}\}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\mathbb{P}(A_k) &= \mathbb{P}(A_k \cap \{U^{(n)} \text{ is visited infinitely often}\})\\
&\leq \lim_{m\to\infty} \mathbb{P}\left(\cup_{k=1}^m A_k \cap \{U^{(n)} \text{ is visited infinitely often}\}\right)\\
&\leq \lim_{m\to\infty}\left(1 - \frac{4}{9}\right)^m = 0.
\end{aligned}$$

This means that the complement event has probability one, i.e. the elements in $(\partial T_3^h \times \partial T_3^v)$ are stable. ◻

Remark 12.1.5. *We would like to point out that by these results we know that the directions $\omega \in (\partial T_3^h \times T_3^v) \cup (T_3^h \times \partial T_3^v)$ are not limit points (in the graph topology) of the trace of the BRW, since there are no infinite trails connecting the starting point to any of these directions.*
Therefore the limit set of the BRW on $T_3^h \times T_3^v$ is contained in the set of stable accumulation points.

12.2 The isotropic BRW has infinitely many Ends

So far we have considered BRW's with their Martin topology. In this section we state another result, that tells us what happens in the *graph topology.*

Consider an isotropic BRW (with mean value $\mathbf{m}$ smaller or equal to ρ^{-1}) on $T_3^h \times T_3^v$. In the event of non-extinction we have the following result:

Proposition 12.2.1. *If the BRW is isotropic, then its limit set has infinitely many ends (in the graph topology) almost surely.*

Proof. At o_1o_2 start two identical but independent BRW's (with mean value $\mathbf{m} \le \rho^{-1}$) and say that one is the *blue* BRW (denote it by $\mathbf{B}$), and the other one is the *red* BRW (denote it by $\mathbf{R}$).

By $\mathbf{B}_k$ we mean the blue BRW that ran for k steps, analogously we write $\mathbf{R}_k$ for the red one. Moreover, we write ξ_k for any particle of $\mathbf{B}_k$ alive at time k, and ξ'_n for any particle of $\mathbf{R}_n$ alive at time n.

By x_u we denote the position of the particle u.

Let us say that a vertex is *purple* if it is visited by a red and a blue particle. Then:

$$\begin{aligned}\mathbb{E}\left[\#\{\text{purple vertices}\}\right] &\le \mathbb{E}\left[\sum_{k\ge 0}\sum_{n\ge 0}\#\{u\in\mathbf{B}_k, v\in\mathbf{R}_n \;:\; x_u = x_v\}\right]\\ &= \sum_{k\ge 0}\sum_{n\ge 0}\mathbb{E}\left[\#\{u\in\mathbf{B}_k, v\in\mathbf{R}_n \;:\; x_u = x_v\}\right] = \sum_{k\ge 0}\sum_{n\ge 0}\mathbf{m}^{k+n}\mathbb{P}[\xi_k = \xi'_n].\end{aligned}$$

Since the process is isotropic, we know the following facts:

(i) ξ_k and ξ'_n are two particles that performed simple random walks;

(ii) $\mathbb{P}[\xi_k = \xi'_n]$ is the probability that two simple random walks meet. Then:

$$\mathbb{P}[\xi_k = \xi'_n] = \mathbb{P}[\xi_{k+n} = o_1o_2],$$

by invariance and reversibility of the process.

Here we recall the fundamental result by Cartwright and Soardi:

Theorem 12.2.2 ([12])**.** *Consider a Cartesian product G of d discrete groups $G_1,\ldots,G_d$, such that every G_j is equipped with a probability measure μ_j governing a random walk $(X_n^j)_{n\ge 1}$. Suppose that for every element $y\in G$ of the form $y=(y_1,\ldots,y_d)\in G$ we have*

$$\mathbb{P}(X_n^j = y_j) \sim C_j\rho_j^n\,/n^{a_j}, \qquad \text{for all } j\in\{1,\ldots,d\},$$

where ρ_j is the spectral radius of the random walk $(X_n^j)_n$ and $a_j > 0$ are numbers independent of n. Then the random walk $(X_n)_n$ on G governed by

$$\mu := \alpha_1\mu_1 + \ldots + \alpha_d\mu_d, \qquad (\text{for } \alpha_j > 0,\ \sum_{j=1}^d \alpha_j = 1)$$

satisfies

$$\mathbb{P}(X_n = y) \sim \frac{C(\alpha_1\rho_1 + \ldots + \alpha_d\rho_d)^n}{n^{a_1+\ldots+a_d}}.$$

Remark 12.2.3. *In this case the random walk has period* 2*, then we should consider only even values of* n*.*

On every homogeneous tree it is known (see e.g. [23], [63] and [11] compare also with Section 4.1) that

$$\mathbb{P}(X_n^j = y_j) \sim C_j \rho_j^n / n^{3/2},$$

therefore, applying Theorem 12.2.2 we get that

$$\mathbb{P}(X_n = y) \sim \frac{C\rho^n}{n^3}.$$

Since

$$\mathbb{P}(X_n = o_1 o_2) \sim \mathbb{P}(X_n = y) \quad \text{for any } y \in T_3^h \times T_3^v,$$

we have

$$\mathbb{P}[\xi_k = \xi_n'] = \mathbb{P}[\xi_{k+n} = o_1 o_2] \sim \frac{C\rho^n}{n^3}.$$

From this reasoning it follows that

$$\mathbb{E}\left[\#\{\text{purple vertices}\}\right] \leq \sum_{k\geq 0}\sum_{n\geq 0} \mathbf{m}^{k+n}\mathbb{P}[\xi_{k+n} = o_1 o_2]$$

$$\leq \sum_{k\geq 0}\sum_{n\geq 0} \mathbf{m}^{k+n}\frac{C'\rho^{n+k}}{(n+k)^3}.$$

The last sums are finite for all $\mathbf{m} \leq \rho^{-1}$. This means that the expected amount of blue particles that touch the red ones is finite, i.e., by definition of the expected value it follows that

$$\mathbb{P}\left[\mathrm{Card}(\mathbf{B} \cap \mathbf{R}) < \infty\right] = 1,$$

which implies that there are only finitely many connections between trails of blue and red particles. Eventually the trails of particles will separate, which means the limit set of the BRW has infinitely many ends almost surely. □

12.3 A small Note on a BRW with a special Bias

The measure μ defined by (12.1) is an instance of a more general one:

$$\mu := \alpha_1\mu_1 + (1-\alpha_1)\mu_2,$$

where $\alpha_1 \in (0,1)$, while μ_1 and μ_2 are the two measures governing random walks on T_3^h and T_3^v respectively. By [11] we have that the spectral radius ρ of the random walk governed by μ is given by

$$\rho = \alpha_1\rho_1 + (1-\alpha_1)\rho_2,$$

being ρ_1 and ρ_2 the spectral radii of the random walks governed by μ_1 and μ_2 respectively.

We consider μ_2 defined as

$$\mu_2 := \begin{cases} \mathbb{P}(o, a_1) = 1/4; \\ \mathbb{P}(o, a_2) = 1/4; \\ \mathbb{P}(o, a_3) = 1/2; \end{cases} \tag{12.8}$$

being a_1, a_2, a_3 the generators of T_3^v. In this case it is easy to verify that $\rho_2 = 1$.

Proposition 12.3.1. *For every measure μ_1 and every value α_1, choose μ_2 as in Equation (12.8). Then the limit set of the critical BRW on $T_3^h \times T_3^v$ has infinitely many ends (in the graph topology) almost surely.*

Proof. By projecting the considered BRW on T_3^h we see a BRW whose underlying random walk is a lazy random walk, it stays in place with probability $(1 - \alpha_1)$ and moves on the tree with probability α_1. The spectral radius of this random walk is given by

$$\rho_{\text{lazy}} = \alpha_1 \rho_1 + (1 - \alpha_1),$$

which obviously coincides with ρ in this case.

The lazy random walk on the tree is transient at its critical value then, with probability one, every copy of T_3^v is visited only *finitely many times* by the BRW.

The proof ends in the same way as the one of Proposition 11.1.1: the particles of the BRW accumulate on a (proper, non-trivial) random subset of the union of the boundaries of all copies of T_3^h, giving the statement. $\square$

Given these results, it is natural to conjecture the following statement:

Conjecture 12.3.2. *For every biased underlying measure μ, the limit set of a critical BRW on $T_3^h \times T_3^v$ has infinitely many ends, almost surely.*

Appendix A

Darboux's Method vs. Singularity Analysis

As mentioned in Chapter 3 there are mainly two methods to deal with singular expansions: one is Darboux's Method, and the other is known by the name of Singularity Analysis.

The authors who work in this setting are sympathizers either of one, or of the other method. In our work we used mostly the first, this is why we are going to describe it more in details.

There are a few references which can be looked at, e.g. [63], [65], [9] and [10], where the Method of Darboux has been described and applied. Let us explain how this method works.

A.1 Darboux's Method

First of all we find the singular expansion of the function $G(z)$, which means that we have to understand how fast the quantity $G(\mathbf{R}) - G(z)$ tends to zero in a neighborhood of $z = \mathbf{R}$. Therefore, in the considered cases, we were able to find an explicit function $a(\cdot)$ such that

$$G(z) = G(\mathbf{R}) + a(\mathbf{R} - z) + \mathbf{o}(a(\mathbf{R} - z)),$$

with $a(\mathbf{R} - z)$ going to zero as z tends to $\mathbf{R}$. This function is the "leading singular term" of the expansion. Under our assumption –see Equation (3.3.1)– $a(s)$ has a well known Taylor expansion in a neighborhood of $s = 0$, which means that the asymptotic behavior of its coefficients a_n is known. In our case it is of type

$$a_n \sim \mathbf{R}^{-n} f(n),$$

where "$\sim$" stands for "asymptotically equivalent up to a constant", and the function $f(\cdot) : \mathbb{N} \to \mathbb{R}$ is of the form $f(n) = n^{-\lambda} \log^{\kappa} n$ for some real non-negative (non both zero) λ and κ. This can be deduced from the book of Flajolet and Sedgewick [19, Chapter VI.2].

It is clear that if $\mathbf{R} > 1$ the sequence a_n converges to zero.

The main point of this method is to compare a_n with $\mu^{(n)}(e)$, and to find out that

$$\mu^{(n)}(e) \sim a_n,$$

whoch means to show that their difference tends to zero faster than a_n:

$$|\mu^{(n)}(e) - a_n| \to 0 \quad \text{as} \quad n \to \infty.$$

In order to achieve this asymptotic estimate, the resource that we need to use is the so-called *Riemann-Lebesgue Lemma*. Its statement can be found in many works (see e.g. [57, Section 2.2]), but will be stated here for completeness:

Lemma A.1.1 (Riemann-Lebesgue Lemma)**.** *If a function $H(z) = h_n z^n$ is analytic for $|z| < 1$, continuous for $|z| \leq 1$ and d times continuously differentiable over $|z| = 1$, then*

$$h_n = \mathbf{o}(n^{-d}).$$

Remark A.1.2. *By an easy normalization, we can see that the Riemann-Lebesgue Lemma can be generalized to the case when the radius of convergence of $H(z)$ is $r \geq 1$, obtaining that*

$$h_n = \mathbf{o}(r^{-n} n^{-d}).$$

In order to be able to apply Lemma A.1.1 we need a function that is enough times continuously differentiable on $|z| = \mathbf{R}$. The first idea is to consider

$$H(z) := G(\mathbf{R}) - G(z) - a(\mathbf{R} - z) = \mathbf{o}(a(\mathbf{R} - z)),$$

in order to obtain that the difference of the coefficients $|\mu^{(n)}(e) - a_n|$ tends to zero *faster* than a_n.

At this point (we can consider this as the first step) this is normally not the case yet, which means that we need to expand the function further, to find the next leading singular term. Let us denote it by $b(\mathbf{R} - z)$. Therefore we find

$$G(\mathbf{R}) - G(z) - a(\mathbf{R} - z) = b(\mathbf{R} - z) + \mathbf{o}(b(\mathbf{R} - z))$$

We can consider (this is the second step)

$$G(\mathbf{R}) - G(z) - a(\mathbf{R} - z) - b(\mathbf{R} - z) = \mathbf{o}(b(\mathbf{R} - z)),$$

and hope that we have achieved already the right value of d that we need, in order to exploit Lemma A.1.1 to obtain the desired approximation

$$|\mu^{(n)}(e) - a_n| = \mathbf{o}(a_n) \quad \text{as} \quad n \to \infty.$$

If this is not the case yet, we must keep on expanding the considered function, until we have enough differentiability on its circle of convergence.

A.2 Singularity Analysis

This method has been introduced by Flajolet and Odlyzko (see [18]) and further developed by Flajolet and Sedgewick (see [19]).

Like the method of Darboux, it is used to extract the asymptotic behavior of the coefficients of a power series.

Apparently it is easier to use and faster to verify than the method of Darboux, but in our case we found out that there was no advantage. The problems that we had to face using Darboux's method, arose as well when we tried to verify the hypotheses of sufficient regularity of the function, required to apply this second method.

The different situations described in Chapters 4 and 5, arising according to different expansions of the Green functions, must be treated differently with both methods.

A few references that can give an idea about the difference between these two different approaches are the following: [17], [3].

We would like to describe shortly the basic facts that characterize this method. For deep explanations and details we refer to [19, Part B].

The theory of singularity analysis essentially relies on two objects: Gamma functions and Cauchy integrals. In order to make a proper use of these powerful tools, there is a "price" to pay: we need some *regularity assumptions* on the considered function.

This method relies on the contour integration by means of Hankel-type paths, therefore the fundamental assumption that we need, is:

Assumption A.2.1. *The considered function has an analytic continuation to a small neighborhood outside its circle of convergence, except close to its singularity.*

Essentially, this means that the function must be analytic in a "pacman-shaped" region containing the full disc of convergence, see Figure A.1.

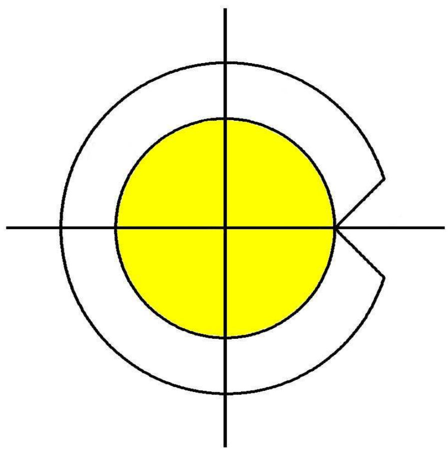

Figure A.1: Pacman-shaped region (in yellow the disc of convergence).

Consider a function $f(z)$ that can be written as a formal power series $f(z) = \sum_{n\geq 0} f_n z^n$. We denote by $[z^n]f(z)$ the coefficient f_n.

Using the same notation as [19, Chapter VI], we define the following set:

$$\mathcal{S} := \{(1-z)^\alpha \lambda(z)^\beta \mid \alpha, \beta \in \mathbb{C}\}, \quad \lambda(z) := \frac{1}{z}\log\frac{1}{1-z}.$$

Here we would like to recall the following fundamental result: for its proof we refer to [19, Theorem VI.4].

Theorem A.2.2. *Let $f(z)$ be an analytic function at 0 and such that satisfies Assumption A.2.1; let us denote by R its singularity. Assume that there are two functions σ and τ, where σ is a finite combination of elements of $\mathcal{S}$ and $\tau \in \mathcal{S}$ such that*

$$f(z) = \sigma(z/R) + \mathbf{O}(\tau(z/R)), \quad \text{as} \quad z \to R.$$

Then one has

$$[z^n]f(z) = R^{-n}\sigma_n + \mathbf{O}(R^{-n}\tau_n^\star),$$

where

$$\sigma_n := [z^n]\sigma(z) = n^{\alpha-1}(\log n)^\beta \left((\Gamma(\alpha))^{-1} + \mathbf{O}\left((\log n)^{-1}\right)\right),$$

and $\tau_n^\star = \mathbf{o}(\sigma_n)$.

We conclude this appendix by recalling the *Tauberian theorem* (see e.g. [53] for a detailed exposition of the Tauberian theory), which is another powerful tool to get the asymptotic behavior of the coefficients of the power series $f(z)$, provided to know very little about $f(z)$. For more details and more references we refer to [19, Section VI.11].

For this method we only need to know the growth of $f(z)$ on the positive real line. It is very convenient to use this theorem when, for example, $f(z)$ has a very irregular behavior on its circle of convergence (a very interesting example is described in [25]).

In the following, a function $\Lambda(x)$ is said to be *slowly varying* at infinity if, for any $c > 0$ we have

$$\Lambda(cx)/\Lambda(x) \to 1 \quad \text{as} \quad x \to \infty.$$

Finally, we have all tools to state the Tauberian Theorem:

Theorem A.2.3 (Tauberian Theorem)**.** *Let $f(z)$ be a power series with radius of convergence equal to 1, satisfying (for $z \to 1$)*

$$f(z) \sim \frac{1}{(1-z)^\alpha}\Lambda\left(\frac{1}{1-z}\right),$$

for some value $\alpha \geq 0$ and $\Lambda(x)$ slowly varying function. If the coefficients $f_n = [z^n]f(z)$ are all non-negative, then

$$\sum_{k=0}^{n} f_k \sim \frac{n^\alpha}{\Gamma(\alpha+1)}\Lambda(n).$$

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