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Titchmarsh–Weyl Theory and Inverse Problems for Elliptic Differential Operators

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Introduction

Partial differential equations and corresponding boundary value problems appear in the modeling of numerous processes in science and engineering. Many mathematical models such as, e.g., the Schrödinger equation with electric or magnetic potentials, lead to second order, formally symmetric, uniformly elliptic differential expressions of the form

$$\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{a_j} \right) + a \tag{0.1}$$

with variable coefficients on bounded or unbounded Lipschitz domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$. To such a differential expression $\mathcal{L}$ one relates the Dirichlet-to-Neumann map, which acts on the boundary $\partial\Omega$ and is defined by

$$M(\lambda) u_\lambda|_{\partial\Omega} = -\frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega}.$$

Here u_λ is a solution of the differential equation $\mathcal{L} u_\lambda = \lambda u_\lambda$, $u_\lambda|_{\partial\Omega}$ is the trace of u_λ at $\partial\Omega$, and $\frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega}$ is the trace of the conormal derivative with respect to $\mathcal{L}$; cf. Chapter 1 for further details. The mapping $M(\lambda)$ is well-defined for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and can be regarded as a bounded linear operator between appropriate Sobolev spaces on $\partial\Omega$. The Dirichlet-to-Neumann map plays a major role in, e.g., electrical impedance tomography. It can be interpreted as an operator which assigns to a given voltage on the surface of an inhomogeneous body the corresponding current flux.

The main objective of the present thesis is to investigate the connection between the selfadjoint operators associated with $\mathcal{L}$ in the Hilbert space $L^2(\Omega)$ and the Dirichlet-to-Neumann map $M(\lambda)$. On the one hand we solve a Calderón type inverse problem. We prove that the selfadjoint Dirichlet operator

$$A_{\mathrm{D}} u = \mathcal{L} u, \quad \operatorname{dom} A_{\mathrm{D}} = \left\{ u \in H^1(\Omega) : \mathcal{L} u \in L^2(\Omega), u|_{\partial\Omega} = 0 \right\}, \tag{0.2}$$

in $L^2(\Omega)$ is uniquely determined up to unitary equivalence by the knowledge of $M(\lambda)$ on any nonempty, open subset of $\partial\Omega$ for a proper set of points λ; here $H^1(\Omega)$ is the L^2-based Sobolev space of order one. On the other hand, we give a complete characterization of the eigenvalues and corresponding eigenfunctions as well as the continuous and absolutely continuous spectrum of A_{D} in terms of the limiting behavior of the operator function $\lambda \mapsto M(\lambda)$. In addition, we provide analogous results for the operator realizations of $\mathcal{L}$ with Neumann and generalized Robin boundary conditions. Our results require comparatively weak regularity conditions on the differential expression $\mathcal{L}$. We assume that the coefficients a_{jk}

and a_j, $1 \leq j,k \leq n$, are bounded, Lipschitz continuous functions, and that a is bounded and measurable.

In the following we discuss the objectives of this thesis in more detail. The first main objective is the solution of an inverse problem. We show that the partial knowledge of the Dirichlet-to-Neumann map $M(\lambda)$ determines the Dirichlet operator A_{D} in (0.2) on the (possibly unbounded) Lipschitz domain Ω uniquely up to unitary equivalence. The result is the following; cf. Theorem 3.7 below.

Theorem 0.1. *Let Ω be a connected, bounded or unbounded Lipschitz domain and let $\omega \subset \partial\Omega$ be a nonempty, open set. Let $\mathcal{L}_1, \mathcal{L}_2$ be two formally symmetric, uniformly elliptic differential expressions of the form* (0.1)*, and let $A_{\mathrm{D},1}$, $A_{\mathrm{D},2}$ and $M_1(\lambda)$, $M_2(\lambda)$ be the corresponding Dirichlet operators and Dirichlet-to-Neumann maps, respectively. If*

$$M_1(\lambda)g = M_2(\lambda)g \quad \textit{on } \omega$$

holds for all g with support in ω and all λ in a set with an accumulation point outside the spectra of $A_{\mathrm{D},1}$ and $A_{\mathrm{D},2}$ then $A_{\mathrm{D},1}$ and $A_{\mathrm{D},2}$ are unitarily equivalent.

The result of Theorem 0.1 is closely related to Calderón's inverse conductivity problem in electrical impedance tomography: In the special case of the elliptic differential expression $\mathcal{L} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\gamma\frac{\partial}{\partial x_j}$ on a bounded, sufficiently smooth domain Ω the coefficient $\gamma : \overline{\Omega} \to \mathbb{R}$ corresponds to an isotropic conductivity and it is known that the knowledge of $M(\lambda)$ for, e.g., $\lambda = 0$ on all of $\partial\Omega$ does even determine the coefficient γ itself uniquely, see [40, 101, 104, 118] for the space dimension $n \geq 3$ and [16, 102] for the two-dimensional case; in recent publications this was also shown for the case of partial data, that is, $M(0)$ is known only on certain, special subsets of $\partial\Omega$, see [39, 77, 82, 83, 103] and [53] for a magnetic Schrödinger operator. For unbounded Ω such results exist, to the best of our knowledge, only under the much more restrictive assumption that the coefficient γ is constant outside some compact subset of Ω; cf. [76, 87, 94]. For general $\mathcal{L}$ of the form (0.1) the single coefficients are not uniquely determined by the knowledge of $M(\lambda)$; cf. [80]. In the anisotropic case $\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k}$ on a bounded domain Ω uniqueness up to diffeomorphisms by the knowledge of $M(0)$ was shown for smooth coefficients in [93, 117, 119]; more general cases were treated in [15, 52, 116], see also [78, 91, 92] for results with partial boundary data. For closely related problems such as, e.g., the multidimensional Gelfand inverse spectral problem and inverse problems for the wave equation we refer the reader to [28–30, 79, 80, 88]. For a detailed review on Calderón's problem and further references see also [122].

In addition to Theorem 0.1 we show how the Dirichlet operator can be recovered from the knowledge of $M(\lambda)$ on ω under the assumption that Ω is bounded. In this case the spectrum of A_{D} consists of isolated eigenvalues with finite multiplicities and, hence, A_{D} is completely determined by its eigenvalues and the corresponding eigenfunctions. We indicate how the eigenvalues and eigenfunctions can be recovered from the poles and the corresponding residues of the operator-valued meromorphic function $\lambda \mapsto M(\lambda)$. The results of this part of the thesis are complemented by an additional section which treats the case of selfadjoint elliptic differential operators A_Θ in $L^2(\Omega)$ of the form

$$A_\Theta u = \mathcal{L}u, \quad \operatorname{dom} A_\Theta = \left\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} + \Theta u|_{\partial\Omega} = 0 \right\}, \tag{0.3}$$

where the parameter Θ in the boundary condition can be chosen as a (nonlocal) bounded operator between certain Sobolev spaces on $\partial\Omega$; see Chapter 2 for further details. We show that the operator A_Θ is uniquely determined up to unitary equivalence by the knowledge of the operator Θ in the boundary condition and of the Dirichlet-to-Neumann map on $\partial\Omega$. Moreover, we show that uniqueness can even be guaranteed when the boundary data is only known on an open part ω of $\partial\Omega$ in case Θ gives rise to a local (classical Robin) boundary condition, that is, Θ is the operator of multiplication with a bounded, real-valued function on the boundary. For further information and recent results on elliptic differential operators with (generalized) Robin boundary conditions we refer the reader to [10, 11, 43, 57, 58, 90, 105, 124] and the references therein.

The second main objective of the present thesis is a complete description of the spectrum $\sigma(A_{\mathrm{D}})$ of the Dirichlet operator A_{D} in terms of the limiting behavior of the analytic operator function $\lambda \mapsto M(\lambda)$ when λ approaches the real axis. One of the main results is the following theorem, which characterizes all eigenvalues and the complete continuous spectrum of A_{D}; cf. Theorem 4.2 below. Here s-lim denotes the strong limit of an operator-valued function.

Theorem 0.2. *Let Ω be a bounded or unbounded Lipschitz domain and let A_{D} be the selfadjoint Dirichlet operator in* (0.2)*. Then for $\lambda \in \mathbb{R}$ the following assertions hold.*

(i) *$\lambda \notin \sigma(A_{\mathrm{D}})$ if and only if $M(\cdot)$ can be continued analytically into λ.*

(ii) *λ is an eigenvalue of A_{D} if and only if* $\operatorname{s-lim}_{\eta \searrow 0} \eta M(\lambda + i\eta) \neq 0$.

(iii) *λ is an isolated eigenvalue of A_{D} if and only if λ is a pole of $M(\cdot)$.*

(iv) λ *belongs to the continuous spectrum of* A_{D} *if and only if* $M(\cdot)$ *cannot be continued analytically into* λ *and* $\operatorname{s-lim}_{\eta\searrow 0}\eta M(\lambda+i\eta)=0$.

We remark that in the case of a bounded domain Ω the spectrum of A_{D} consists only of isolated eigenvalues, see Chapter 3. Thus in this case only item (iii) in the above theorem is of interest.

In addition to Theorem 0.2, we provide a characterization of all eigenspaces of A_{D}. Moreover, we prove that the absolutely continuous spectrum of the Dirichlet operator A_{D} can also be detected with the help of the function $M(\cdot)$. In the following theorem $\mathrm{cl}_{\mathrm{ac}}(\chi)=\{x\in\mathbb{R}:|(x-\varepsilon,x+\varepsilon)\cap\chi|>0 \text{ for all } \varepsilon>0\}$ is the absolutely continuous closure (or essential closure) of a Borel set χ and $(\cdot,\cdot)$ is an extension of the inner product in $L^2(\partial\Omega)$; for the details see Theorem 4.4 below.

Theorem 0.3. *The absolutely continuous spectrum of* A_{D} *is given by*

$$\sigma_{\mathrm{ac}}(A_{\mathrm{D}})=\overline{\bigcup_g \mathrm{cl}_{\mathrm{ac}}\left(\left\{x\in\mathbb{R}:0<\mathrm{Im}(M(x+i0)g,g)<+\infty\right\}\right)}.$$

We complement these spectral characterizations for the Dirichlet operator by a sufficient condition for the absence of singular continuous spectrum and sufficient conditions for the spectrum of A_{D} to be purely absolutely continuous or purely singular continuous, respectively, in some interval.

Theorem 0.2 and Theorem 0.3 are multidimensional analogs of well-known facts from the Titchmarsh–Weyl theory for ordinary differential operators. The classical Titchmarsh–Weyl m-function associated with a singular Sturm–Liouville differential expression goes back to the work [125] by H. Weyl and plays a fundamental role in the direct and inverse spectral theory of the corresponding ordinary differential operators. For, e.g., a one-dimensional Schrödinger differential expression $-\frac{d^2}{dx^2}+q$ on the half-axis $(0,\infty)$ with a bounded, real-valued potential q the corresponding Titchmarsh–Weyl coefficient $m(\lambda)\in\mathbb{C}$ may be defined as

$$m(\lambda)f_\lambda(0)=f_\lambda'(0),\quad \lambda\in\mathbb{C}\setminus\mathbb{R},$$

where f_λ is the unique solution in $L^2(0,\infty)$ of the equation $-f''+qf=\lambda f$. It is due to E. C. Titchmarsh that the function $\lambda\mapsto m(\lambda)$ is analytic and is closely related to the spectrum. The limiting behavior of the function $m(\cdot)$ towards the real axis is in one-to-one correspondence to the spectra of the selfadjoint realizations of $-\frac{d^2}{dx^2}+q$ in $L^2(0,\infty)$ in the same way as in the multidimensional theorems Theorem 0.2 and Theorem 0.3 above. For instance, λ is an eigenvalue of the selfadjoint realization T_{D} subject to a Dirichlet boundary condition $f(0)=0$ if and only if $\lim_{\eta\searrow 0}\eta m(\lambda+i\eta)\neq 0$, and the absolutely continuous spectrum of T_{D} can be represented as

$$\sigma_{\mathrm{ac}}(T_{\mathrm{D}})=\mathrm{cl}_{\mathrm{ac}}\left\{\lambda\in\mathbb{R}:0<\mathrm{Im}\, m(\lambda+i0)<+\infty\right\}.$$

Analogously the spectra of realizations with other boundary conditions can be characterized in terms of the function $m(\cdot)$ and the boundary condition; cf. [41, 121]. Because of this one-to-one correspondence the Titchmarsh–Weyl m-function became an indispensable tool in the spectral analysis of Sturm–Liouville differential operators as well as more general Hamiltonian and canonical systems; for a small selection from the vast number of contributions during the last decades see, e.g., [13, 20, 35, 45, 60, 64, 73, 85, 113, 114] for direct spectral problems and [31, 33, 38, 61–63, 89, 99, 115] for inverse problems. We point out that in the recent past various attempts were made to carry over several elements of the classical Titchmarsh–Weyl theory to partial differential operators; for contributions to this field we refer the reader to [4, 5, 21, 36, 37, 58, 111]. However, to the best of our knowledge no generalizations of the classical spectral characterization via the Titshmarsh–Weyl m-function to partial differential operators were obtained so far.

Besides Theorem 0.2 and Theorem 0.3 we provide extensions and generalizations of these results. We show that the spectrum of A_{D} can even be recovered from the partial knowledge of $M(\lambda)$ on any nonempty, open subset of $\partial\Omega$. Furthermore, we provide characterizations of the spectra of the operators A_Θ in (0.3) in terms of the Dirichlet-to-Neumann map $M(\lambda)$ and the boundary operator Θ.

The methods which serve us to prove the main results of the present thesis are strongly inspired by modern approaches to the extension theory of symmetric operators. In the abstract framework of a boundary triple for the adjoint S^* of a closed, densely defined, symmetric operator S with equal defect numbers in a Hilbert space $\mathcal{H}$ one fixes two boundary mappings $\Gamma_0, \Gamma_1 : \operatorname{dom} S^* \to \mathcal{G}$, where $\mathcal{G}$ is an auxiliary Hilbert space. It is assumed that the pair $\{\Gamma_0, \Gamma_1\}$ is surjective onto $\mathcal{G} \times \mathcal{G}$ and satisfies the abstract Green identity

$$(S^*u, v) - (u, S^*v) = (\Gamma_1 u, \Gamma_0 v) - (\Gamma_0 u, \Gamma_1 v), \quad u, v \in \operatorname{dom} S^*, \tag{0.4}$$

where $(\cdot, \cdot)$ stands for both the inner products in $\mathcal{H}$ and in $\mathcal{G}$. One defines an abstract Weyl function via the relation

$$M(\lambda)\Gamma_0 u_\lambda = \Gamma_1 u_\lambda, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{0.5}$$

where u_λ is the unique solution in $\operatorname{dom} S^*$ of the equation $S^* u_\lambda = \lambda u_\lambda$. The restriction of S^* to the kernel of Γ_0 defines a selfadjoint operator A_0 in $\mathcal{H}$. It can be shown that the function $M(\cdot)$ in this abstract setting determines the operator A_0 uniquely up to unitary equivalence and contains the complete spectral information of A_0—if and only if the underlying symmetric operator S is simple or completely non-selfadjoint, that is, S does not possess any nontrivial reducing subspace in which it defines a selfadjoint operator; cf. Appendix A.2 for more details. In order to treat elliptic differential operators on Lipschitz domains in a similar way it is natural to consider the symmetric operator

$$Su = \mathcal{L}u, \quad \operatorname{dom} S = \Big\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), u|_{\partial\Omega} = \frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} = 0 \Big\}, \tag{0.6}$$

in $L^2(\Omega)$ and the boundary mappings

$$\Gamma_0 u = u|_{\partial\Omega} \quad \text{and} \quad \Gamma_1 u = \frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega}.$$

The mappings Γ_0 and Γ_1 are well-defined on the space $\{u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega)\}$, which is not the domain of S^* but is dense in $\operatorname{dom} S^*$ with respect to the graph norm of S^*. Furthermore, Γ_0 and Γ_1 are not surjective onto a joint Hilbert space $\mathcal{G}$ but their ranges coincide with the Sobolev spaces $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. Nevertheless, many of their properties are similar to those of the boundary mappings in an abstract boundary triple. For instance, the relation (0.4) is satisfied due to the classical second Green identity and the Dirichlet-to-Neumann map $M(\lambda)$ satisfies the identity (0.5). In particular, the relation between the operator $A_{\mathrm{D}} = \mathcal{L} \upharpoonright \ker \Gamma_0$ and the Dirichlet-to-Neumann map is similar to that between A_0 and $M(\lambda)$ in the framework of a boundary triple; cf., e.g., the formulas in Lemma 3.1 below. However, in order to obtain a complete picture of the Dirichlet operator and its spectrum from the knowledge of the Dirichlet-to-Neumann map it is necessary to ensure the simplicity of the symmetric operator S in (0.6). This problem is solved in the present thesis for a large class of domains and elliptic differential expressions; cf. Proposition 3.4 and Appendix A.2. This generalizes results on the simplicity of symmetric ordinary differential operators from [65].

A more detailed discussion of modern methods in the extension theory of symmetric operators as boundary triples and their generalizations can be found in [21, 46–50, 66, 84, 107, 108, 111]. The treatment of elliptic differential operators with the help of extension theory goes back to the fundamental works [18, 32, 67, 68, 95, 123]. More recent results on this topic can be found in, e.g., [1, 21, 36, 37, 59, 96, 106]. For further recent publications in the field of direct and inverse spectral theory for elliptic differential operators see [6, 9, 14, 22–24, 58, 70–72, 98].

Let us give a brief outline of the thesis. In the first chapter we shortly provide some basic facts on bounded and unbounded linear operators and, especially, on the spectra of selfadjoint operators in Hilbert spaces. Moreover, we recall the definitions and some of the most important facts concerning Sobolev spaces on Lipschitz domains and on their boundaries. In Chapter 2 we introduce operator realizations of elliptic differential expressions with Dirichlet, Neumann, and generalized Robin boundary conditions. We prove their selfadjointness and investigate the solvability of related boundary value problems, which finally allows us to define Dirichlet-to-Neumann and Robin-to-Dirichlet maps. The remaining two chapters contain the main results of this thesis. Chapter 3 is devoted to the uniqueness and reconstruction results of Calderón type and in Chapter 4 we develop spectral theory for selfadjoint elliptic differential operators via the Dirichlet-to-Neumann map as a multidimensional Titchmarsh–Weyl m-function. The thesis closes with two appendices. The first one provides facts on the spectra of finite Borel measures that are used in Chapter 4 for the description of the absolutely continuous and singular continuous spectrum. In the second one we discuss the notion of simplicity of a symmetric operator and point out its connection to the present work.

1 Preliminaries

In this preliminary chapter we provide basic facts and definitions which play a role in the main part of the present thesis. We are concerned with linear operators in Banach spaces, particularly with selfadjoint operators in Hilbert spaces and with their spectra. Moreover, we recall some elements of the representation theory for semibounded sesquilinear forms and discuss some of the most important statements concerning Sobolev spaces on Lipschitz domains and on their boundaries.

1.1 Linear operators and analytic operator functions

In this section we discuss basics on bounded and unbounded linear operators in Banach spaces and on analytic functions whose values are bounded linear operators. For a more detailed exposition we refer the reader to the standard works [3,54,109].

Let X and Y be complex Banach spaces. For a linear operator T from X to Y we denote by $\operatorname{dom} T$, $\ker T$, and $\operatorname{ran} T$ the domain, kernel, and range, respectively, of T. The restriction of T to a subspace D of $\operatorname{dom} T$ is denoted by $T \upharpoonright D$. For a closed operator T from X to X we denote by

$$\rho(T) = \big\{\lambda \in \mathbb{C} : (T - \lambda)^{-1} \text{ is bounded and everywhere defined in } X\big\}$$

the resolvent set of T and by $\sigma(T) = \mathbb{C} \setminus \rho(T)$ the spectrum of T. Recall that $\rho(T)$ is an open subset of $\mathbb{C}$ and that, hence, $\sigma(T)$ is closed.

A *conjugation* on a complex Banach space X is a continuous, antilinear mapping $X \ni u \mapsto \overline{u} \in X$ with $\overline{(\overline{u})} = u$; the reader may think of the complex conjugation on a function space. Assume that X is equipped with a conjugation and let X' denote the dual space of X, which consists of all bounded, linear functionals $v : X \to \mathbb{C}$. We define the *dual pairings* between X and X' by

$$(v, u)_{X',X} := \overline{(u, v)_{X,X'}} := v(\overline{u}), \quad u \in X, v \in X'.$$

Note that each of the mappings $(\cdot,\cdot)_{X',X} : X' \times X \to \mathbb{C}$ and $(\cdot,\cdot)_{X,X'} : X \times X' \to \mathbb{C}$ is linear in the first and antilinear in the second entry.

Assume that the Banach spaces X and Y are equipped with conjugations and let $T : X \to Y$ be a bounded, everywhere defined linear operator. The adjoint operator $T^* : Y' \to X'$ of T is defined by the identity

$$(Tu, v)_{Y,Y'} = (u, T^*v)_{X,X'}, \quad u \in X, v \in Y'.$$

It follows from the closed graph theorem that T^* is bounded. Moreover, it is clear from the definition that $(T^*)^* = T$ holds.

Let $G \subset \mathbb{C}$ be an open, nonempty set and let $M(z) : X \to Y$ be a bounded linear operator for each $z \in G$. The operator function $z \mapsto M(z)$ is called *holomorphic* on G if for each $z_0 \in G$ the limit

$$\lim_{z \to z_0} \frac{M(z) - M(z_0)}{z - z_0}$$

exists in the space of bounded linear operators from X to Y with respect to the usual operator topology. Recall that the operator function $M(\cdot)$ is holomorphic on G if and only if it is analytic on G, that is, $M(\cdot)$ can be represented locally by a power series which converges with respect to the operator topology.

Let the operator function $z \mapsto M(z)$ be analytic on G and assume that $\lambda \in \mathbb{C}$ is a pole of $M(\cdot)$ of order n, that is, there exists an open ball B centered at λ such that $\overline{B} \setminus \{\lambda\} \subset G$,

$$\lim_{z \to \lambda} (z - \lambda)^n M(z) \text{ exists and is nontrivial,} \quad \text{and} \quad \lim_{z \to \lambda} (z - \lambda)^{n+1} M(z) = 0$$

in the operator topology. Then the residue $\operatorname{Res}_\lambda M$ of $M(\cdot)$ at λ is given by

$$\operatorname{Res}_\lambda M = \frac{1}{2\pi i} \int_\Gamma M(z) dz, \tag{1.1}$$

where Γ is the boundary of the ball B. If the order of the pole is one then the relation

$$\operatorname{Res}_\lambda M = \lim_{z \to \lambda} (z - \lambda) M(z)$$

holds.

1.2 Selfadjoint linear operators and their spectra

In this section we shortly recall some well-known facts on (unbounded) selfadjoint operators and on their spectra. In particular, we discuss the notions of the absolutely continuous and the singular continuous spectrum. This and more can be found in the text books [3, 81, 109].

Let $\mathcal{H}$ be a complex Hilbert space with scalar product $(\cdot, \cdot)$ and corresponding norm $\|\cdot\|$, where we comply with the convention that the scalar product $(\cdot, \cdot)$ is linear in the first and antilinear in the second entry. Let A be a densely defined linear operator in $\mathcal{H}$. Then the adjoint A^* of A in $\mathcal{H}$ is defined by $A^* v = w$, $v \in \operatorname{dom} A^*$, where

$$\operatorname{dom} A^* = \big\{ v \in \mathcal{H} : \text{exists } w \in \mathcal{H} \text{ such that } (Au, v) = (u, w) \text{ for all } u \in \operatorname{dom} A \big\}.$$

Note that in the case $\operatorname{dom} A = \mathcal{H}$ this coincides with the above definition of A^*. Furthermore, note that A^* always is a closed operator in $\mathcal{H}$. The operator A is called *symmetric* if $(Au, v) = (u, Av)$ holds for all $u, v \in \operatorname{dom} A$ or, equivalently, if (Au, u) is real for all $u \in \operatorname{dom} A$. Moreover, A is called *selfadjoint* if $A = A^*$ holds. Clearly each selfadjoint operator is symmetric, but the converse does not hold.

Let A be a selfadjoint linear operator in $\mathcal{H}$. Then its spectrum $\sigma(A)$ is contained in $\mathbb{R}$ and it is the union of the disjoint sets $\sigma_{\mathrm{p}}(A)$ and $\sigma_{\mathrm{c}}(A)$, where the set of eigenvalues

$$\sigma_{\mathrm{p}}(A) = \big\{\lambda \in \mathbb{R} : \ker(A - \lambda) \neq \{0\}\big\}$$

of A is called the *point spectrum* and

$$\sigma_{\mathrm{c}}(A) = \big\{\lambda \in \mathbb{R} : (A - \lambda)^{-1} \text{ exists and is unbounded}\big\}$$

is called the *continuous spectrum* of A. Recall that the spectrum of A is said to be *purely discrete* if $\sigma(A)$ consists of isolated eigenvalues with finite multiplicities. For instance, $\sigma(A)$ is purely discrete if the operator $(A - \lambda)^{-1}$ is compact for one (and, hence, all) $\lambda \in \rho(A)$.

A selfadjoint operator A is called *semibounded from below* by $\mu \in \mathbb{R}$ if and only if

$$(Au, u) \geq \mu \|u\|^2, \quad u \in \operatorname{dom} A.$$

In this case the spectrum of A is bounded from below by μ, i.e., $\sigma(A) \subset [\mu, +\infty)$.

Each selfadjoint operator A gives rise to an operator-valued measure $E(\cdot)$ on the Borel σ-algebra in $\mathbb{R}$, whose values are orthogonal projections in $\mathcal{H}$, such that

$$A = \int_{\sigma(A)} t dE(t)$$

holds, where the integral on the right-hand side converges in the strong sense. The measure $E(\cdot)$ is called the *spectral measure of* A. For each measurable function $f : \sigma(A) \to \mathbb{R}$ the operator $f(A)$ is defined as

$$f(A) = \int_{\sigma(A)} f(t) dE(t), \quad \operatorname{dom} f(A) = \Big\{u \in \mathcal{H} : \int_{\sigma(A)} |f(t)|^2 d(E(t)u, u) < \infty\Big\}.$$

Recall that $\lambda \in \sigma_{\mathrm{p}}(A)$ if and only if $E(\{\lambda\}) \neq 0$ and that in this case $\operatorname{ran} E(\{\lambda\}) = \ker(A - \lambda)$ holds, that is, $E(\{\lambda\})$ is the orthogonal projection in $\mathcal{H}$ onto the eigenspace of A corresponding to λ. If the eigenvalue λ is isolated in $\sigma(A)$ then λ is a pole of order one of the analytic operator function $\rho(A) \ni \lambda \mapsto R(\lambda) = (A-\lambda)^{-1}$ and the formula

$$E(\{\lambda\}) = -\operatorname{Res}_\lambda R = -\frac{1}{2\pi i}\int_\Gamma (A-z)^{-1}dz \tag{1.2}$$

holds, where Γ is the boundary of an open ball B centered at λ with $\overline{B}\setminus\{\lambda\} \subset \rho(A)$; cf. (1.1). Moreover, $\lambda \in \sigma(A)$ if and only if $E((\lambda-\varepsilon, \lambda+\varepsilon)) \neq 0$ holds for each $\varepsilon > 0$. Note that, in particular, each isolated point in $\sigma(A)$ is an eigenvalue. For $a, b \notin \sigma_\mathrm{p}(A)$ the spectral projection $E((a,b))$ of the interval (a,b) with respect to A can be expressed via the Stone formula

$$E((a,b))u = \lim_{\varepsilon\searrow 0}\frac{1}{2\pi i}\int_a^b \Big(\big(A-(t+i\varepsilon)\big)^{-1}u - \big(A-(t-i\varepsilon)\big)^{-1}u\Big)dt, \quad u \in \mathcal{H}, \tag{1.3}$$

where the integral and the limit have to be taken with respect to the topology in $\mathcal{H}$.

With the help of the spectral measure the continuous spectrum of a selfadjoint operator A can be decomposed into an absolutely continuous and a singular continuous part. For each $u \in \mathcal{H}$ let us introduce the scalar measure $\mu_u = (E(\cdot)u, u)$ on the Borel σ-algebra of $\mathbb{R}$. Recall that the measure μ_u is called *absolutely continuous* (with respect to the Lebesgue measure $|\cdot|$) if $\mu_u(\chi) = 0$ for each Borel set $\chi \subset \mathbb{R}$ with $|\chi| = 0$ and *singular* if there exists a Borel set χ with $|\chi| = 0$ and $\mu_u(\mathbb{R}\setminus\chi) = 0$. We define the *absolutely continuous part* and the *singular part* of $\mathcal{H}$ with respect to A by

$$\mathcal{H}_\mathrm{ac} = \mathcal{H}_\mathrm{ac}(A) = \{u \in \mathcal{H} : \mu_u \text{ is absolutely continuous}\}$$

and

$$\mathcal{H}_\mathrm{s} = \mathcal{H}_\mathrm{s}(A) = \{u \in \mathcal{H} : \mu_u \text{ is singular}\},$$

respectively. Recall that $\mathcal{H} = \mathcal{H}_\mathrm{ac} \oplus \mathcal{H}_\mathrm{s}$ holds. Furthermore, let us denote by $\mathcal{H}_\mathrm{p} = \mathcal{H}_\mathrm{p}(A)$ the closed span of all eigenvectors of A. Then $\mathcal{H}_\mathrm{p} \subset \mathcal{H}_\mathrm{s}$ and we call $\mathcal{H}_\mathrm{sc} = \mathcal{H}_\mathrm{sc}(A) = \mathcal{H}_\mathrm{s} \ominus \mathcal{H}_\mathrm{p}$ the *singular continuous part* of $\mathcal{H}$ with respect to A. It turns out that the Hilbert spaces $\mathcal{H}_\mathrm{p}, \mathcal{H}_\mathrm{ac}, \mathcal{H}_\mathrm{sc}$, and $\mathcal{H}_\mathrm{s}$ are reducing subspaces for the operator A. Let

$$A_i u = Au, \quad \operatorname{dom} A_i = \operatorname{dom} A \cap \mathcal{H}_i$$

in $\mathcal{H}_i$ denote the restriction of A to $\mathcal{H}_i$, $i = \mathrm{ac}, \mathrm{sc}$. Then the *absolutely continuous spectrum* $\sigma_\mathrm{ac}(A)$ and the *singular continuous spectrum* $\sigma_\mathrm{sc}(A)$ of A are defined by

$$\sigma_\mathrm{ac}(A) = \sigma(A_\mathrm{ac}) \quad \text{and} \quad \sigma_\mathrm{sc}(A) = \sigma(A_\mathrm{sc}),$$

respectively.

1.3 Sesquilinear forms and representation theorems

In this section we shortly recall some basic facts on semibounded sesquilinear forms and their representations via selfadjoint operators, as they will be used in Chapter 2 below. The text of this section is based on [81, Chapter VI].

Let us first introduce the basic notions. In this section $\mathcal{H}$ is a complex Hilbert space with scalar product $(\cdot,\cdot)$ and corresponding norm $\|\cdot\|$.

Definition 1.1. Let $D \subset \mathcal{H}$ be a linear subspace of $\mathcal{H}$. A mapping $\mathfrak{a} = \mathfrak{a}[\cdot,\cdot] : D \times D \to \mathbb{C}$ is called a *sesquilinear form* (in short: a form) in $\mathcal{H}$ if $\mathfrak{a}[\cdot,\cdot]$ is linear in the first and antilinear in the second entry. For the *domain* D of $\mathfrak{a}$ we usually write $\operatorname{dom}\mathfrak{a}$. The form $\mathfrak{a}$ is called *densely defined* if $\operatorname{dom}\mathfrak{a}$ is dense in $\mathcal{H}$. It is called *symmetric* if

$$\mathfrak{a}[u,v] = \overline{\mathfrak{a}[v,u]}, \quad u,v \in \operatorname{dom}\mathfrak{a}.$$

Moreover, we say that $\mathfrak{a}$ is *semibounded from below* if there exists $\mu \in \mathbb{R}$ with

$$\mathfrak{a}[u,u] \geq \mu\|u\|^2, \quad u \in \operatorname{dom}\mathfrak{a};$$

in this case we shortly write $\mathfrak{a} \geq \mu$.

Note that a form $\mathfrak{a}$ in $\mathcal{H}$ is symmetric if and only if $\mathfrak{a}[u,u]$ is real for all $u \in \operatorname{dom}\mathfrak{a}$. In particular, if $\mathfrak{a} \geq \mu$ for some $\mu \in \mathbb{R}$ then $\mathfrak{a}$ is symmetric and generates a scalar product $(\cdot,\cdot)_\mathfrak{a}$ on the linear space $\operatorname{dom}\mathfrak{a}$ via

$$(u,v)_\mathfrak{a} = a[u,v] + (1-\mu)(u,v), \quad u,v \in \operatorname{dom}\mathfrak{a}; \tag{1.4}$$

the norm which is induced by $(\cdot,\cdot)_\mathfrak{a}$ on $\operatorname{dom}\mathfrak{a}$ we denote by $\|\cdot\|_\mathfrak{a}$. Obviously the choice of μ is not unique, since $\mathfrak{a} \geq \mu$ implies $\mathfrak{a} \geq \widetilde{\mu}$ for each $\widetilde{\mu} < \mu$. Nevertheless, if we replace μ in (1.4) by $\widetilde{\mu}$, we obtain a norm on $\operatorname{dom}\mathfrak{a}$ which is equivalent to $\|\cdot\|_\mathfrak{a}$. Therefore in the following we do not care about the precise choice of μ.

Furthermore we need the important notion of a closed semibounded form.

Definition 1.2. A semibounded sesquilinear form $\mathfrak{a}$ in $\mathcal{H}$ is called *closed* if $(\operatorname{dom}\mathfrak{a}, (\cdot,\cdot)_\mathfrak{a})$ is a Hilbert space. Moreover, we say that a subspace D' of $\operatorname{dom}\mathfrak{a}$ is a *core of* $\mathfrak{a}$ if D' is dense in $(\operatorname{dom}\mathfrak{a}, (\cdot,\cdot)_\mathfrak{a})$.

Alternatively, the notions of a closed form and of a core can be defined via form convergence, see [81, Section VI.1.3], but we will not make use of this concept in the following.

One of the main statements on closed semibounded forms is the following famous representation theorem. We remark that it is provided in [81, Chapter VI, Theorem 2.1] in the more general framework of sectorial forms.

Theorem 1.3. *Let $\mathfrak{a}$ be a densely defined, closed, symmetric sesquilinear form which is semibounded from below by some $\mu \in \mathbb{R}$. Then there exists a unique selfadjoint operator A in $\mathcal{H}$ with* $\operatorname{dom} A \subset \operatorname{dom} \mathfrak{a}$ *and*

$$\mathfrak{a}[u, v] = (Au, v), \quad u \in \operatorname{dom} A, v \in \operatorname{dom} \mathfrak{a}.$$

The operator A is semibounded from below by μ. Moreover, if $u \in$ $\operatorname{dom} \mathfrak{a}$ *and $w \in \mathcal{H}$ satisfy*

$$\mathfrak{a}[u, v] = (w, v)$$

for all v belonging to a core of $\mathfrak{a}$ then $u \in$ $\operatorname{dom} A$ *and $Au = w$.*

1.4 Sobolev spaces and trace maps

In this section we define Sobolev spaces on Lipschitz domains and on their boundaries and provide some basic facts which are connected with them. We mainly follow the account in [100]; see also [2, 55, 56, 95] for more details.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open set. As usual, we denote by $L^2(\Omega)$ the Hilbert space of (equivalence classes of) square-integrable, complex-valued functions on Ω, equipped with the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u\overline{v} dx, \quad u, v \in L^2(\Omega),$$

and the associated norm $\| \cdot \|_{L^2(\Omega)}$. In the following we will usually just write $(\cdot, \cdot)$ and $\| \cdot \|$ instead of $(\cdot, \cdot)_{L^2(\Omega)}$ and $\| \cdot \|_{L^2(\Omega)}$, respectively, when no confusion can arise. By $C_0^\infty(\Omega)$ we denote the space of all functions from Ω to $\mathbb{C}$ which are arbitrarily often differentiable and have a compact support in Ω. Recall that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ and $\varphi \in C_0^\infty(\Omega)$ we set

$$D^\alpha \varphi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}} \varphi.$$

Moreover, we write $\operatorname{supp} \varphi$ for the support of $\varphi \in C_0^\infty(\Omega)$. We say that a sequence $(\varphi_k)_{k \in \mathbb{N}} \subset C_0^\infty(\Omega)$ converges to $\varphi \in C_0^\infty(\Omega)$ in $C_0^\infty(\Omega)$ if there exists a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_k, \operatorname{supp} \varphi \subset K$ for all $k \in \mathbb{N}$ and $D^\alpha \varphi_k$ converges to $D^\alpha \varphi$ uniformly on K for each $\alpha \in \mathbb{N}_0^n$. A *distribution* is a linear mapping from $C_0^\infty(\Omega)$ to $\mathbb{C}$ which is sequentially continuous with respect to this convergence in $C_0^\infty(\Omega)$. We say that a distribution T *belongs to* $L^2(\Omega)$ and write $T \in L^2(\Omega)$ if there exists $u \in L^2(\Omega)$ with

$$T\varphi = \int_\Omega u \varphi dx, \quad \varphi \in C_0^\infty(\Omega). \tag{1.5}$$

In this sense we can identify each element of $L^2(\Omega)$ with a distribution. Moreover, we define the derivative $D^\alpha T$ of a distribution T by $(D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi)$, where $|\alpha| = \sum_{i=1}^n \alpha_i$. Clearly, $D^\alpha T$ is again a distribution.

We are now able to introduce the Sobolev spaces of integer order. For every integer $k \geq 0$ we set

$$H^k(\Omega) = \big\{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k \big\}.$$

Equipped with the scalar product

$$(u,v)_{H^k(\Omega)} = \sum_{0 \leq |\alpha| \leq k} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}, \quad u, v \in H^k(\Omega),$$

$H^k(\Omega)$ is a separable Hilbert space; the corresponding norm is denoted by $\|\cdot\|_{H^k(\Omega)}$. We write $H_0^k(\Omega)$ for the closure of $C_0^\infty(\Omega)$ in the Hilbert space $H^k(\Omega)$. Moreover, we denote by $(\cdot,\cdot)_{-k,k}$ the sesquilinear duality between $H_0^k(\Omega)$ and its dual $H^{-k}(\Omega) = (H_0^k(\Omega))'$ (with respect to the usual complex conjugation; cf. Section 1.1). It satisfies

$$(u,v)_{-k,k} = \int_\Omega u\bar{v}dx = (u,v)_{L^2(\Omega)}, \quad u \in L^2(\Omega), v \in H_0^k(\Omega).$$

Additionally, we define the local Sobolev space $H^k_{\mathrm{loc}}(\Omega)$ by

$$H^k_{\mathrm{loc}}(\Omega) = \big\{ u \in L^2_{\mathrm{loc}}(\Omega) : \varphi u \in H^k(\Omega) \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^n) \text{ with } \operatorname{supp}\varphi \subset \Omega \big\},$$

$k \geq 1$, where we write $u \in L^2_{\mathrm{loc}}(\Omega)$ if and only if $u|_{\mathcal{O}} \in L^2(\mathcal{O})$ holds for each open, bounded set $\mathcal{O}$ with $\overline{\mathcal{O}} \subset \Omega$.

Recall that the Fourier transformation $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the unique unitary operator in $L^2(\mathbb{R}^n)$ which satisfies

$$(\mathcal{F}u)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(y)dy, \quad x \in \mathbb{R}^n, \quad u \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$

where $x \cdot y$ denotes the scalar product of x and y in $\mathbb{R}^n$. We define the Sobolev space $H^s(\mathbb{R}^n)$ of real order $s \geq 0$ on $\mathbb{R}^n$ by

$$H^s(\mathbb{R}^n) = \big\{ u \in L^2(\mathbb{R}^n) : (1 + |\cdot|^2)^{\frac{s}{2}} \mathcal{F}u \in L^2(\mathbb{R}^n) \big\}$$

and equip it with the scalar product

$$(u,v)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|x|^2)^s u(x)\overline{v(x)}dx, \quad u, v \in H^s(\mathbb{R}^n). \tag{1.6}$$

Then $H^s(\mathbb{R}^n)$ is a separable Hilbert space. We denote the norm associated with (1.6) by $\|\cdot\|_{H^s(\mathbb{R}^n)}$. For each positive integer $s = k$ this new definition of $H^s(\mathbb{R}^n)$ coincides with the above one for $\Omega = \mathbb{R}^n$ with equivalent norms so that we will not distinguish them.

In order to define Sobolev spaces on the boundary of an open set we need additional assumptions. Following the lines of [100, Chapter 3] we say that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a *Lipschitz hypograph* if there exists a Lipschitz continuous function $\zeta : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega = \big\{(x_1, \dots, x_n)^T \in \mathbb{R}^n : x_n < \zeta(x_1, \dots, x_{n-1})\big\}. \tag{1.7}$$

Using this notion a Lipschitz domain is defined in the following way.

Definition 1.4. An open set $\Omega \subset \mathbb{R}^n$ is called a *Lipschitz domain* if its boundary $\partial\Omega$ is compact and there exist sets $W_1, \dots, W_k, \Omega_1, \dots, \Omega_k \subset \mathbb{R}^n$ with the following properties.

(i) W_j is open, $1 \leq j \leq k$, and $\partial\Omega \subset \bigcup_{1\leq j\leq k} W_j$.

(ii) Ω_j can be transformed by a rotation and a translation into a Lipschitz hypograph, $1 \leq j \leq k$.

(iii) $W_j \cap \Omega = W_j \cap \Omega_j$ for $1 \leq j \leq k$.

We remark that a Lipschitz domain does not have to be bounded; only its boundary is compact. For instance, if Ω is a bounded Lipschitz domain then also $\mathbb{R}^n \setminus \overline{\Omega}$ is a Lipschitz domain. Moreover, we remark that with this definition a Lipschitz domain does not need to be connected.

On the boundary of a Lipschitz domain a surface measure and an outer unit normal field can be defined in the following way. Let first Ω be a Lipschitz hypograph and let the Lipschitz continuous function ζ be as above. By Rademacher's theorem ζ is differentiable almost everywhere on $\mathbb{R}^{n-1}$ and its gradient $\nabla\zeta$ is bounded. Thus the integral of a function $g : \partial\Omega \to \mathbb{C}$ with respect to the *surface measure* σ may be defined as

$$\int_{\partial\Omega} g d\sigma := \int_{\mathbb{R}^{n-1}} g(x, \zeta(x))\sqrt{1 + |\nabla\zeta(x)|^2} dx$$

(if the integral on the right-hand side exists). Moreover, the outer unit normal vector field is given by

$$\nu(s) = (\nu_1(s), \dots, \nu_n(s))^T := \frac{(-\nabla\zeta(x), 1)^T}{\sqrt{1 + |\nabla\zeta(x)|^2}}, \quad s = (x, \zeta(x))^T, x \in \mathbb{R}^{n-1}.$$

If Ω is a Lipschitz domain as in Definition 1.4 we can choose a partition of unity with respect to the open cover $\{W_j\}$ of $\partial\Omega$, that is, functions $\varphi_j \in C_0^\infty(W_j)$, $1 \le j \le k$, with $\sum_{j=1}^k \varphi_j(x) = 1$ for all $x \in \partial\Omega$. Then we define

$$\int_{\partial\Omega} g d\sigma := \sum_{j=1}^k \int_{\mathbb{R}^{n-1}} \varphi_j(x, \zeta_j(x)) g(x, \zeta_j(x)) \sqrt{1 + |\nabla \zeta_j(x)|^2} dx,$$

where the ζ_j correspond to the Ω_j as in (1.7). As usual we denote by $L^2(\partial\Omega)$ the space of (equivalence classes of) complex-valued functions on $\partial\Omega$ which are square-integrable with respect to σ.

In order to define Sobolev spaces on $\partial\Omega$ let us first assume that Ω is a Lipschitz hypograph with ζ as above. For each $g \in L^2(\partial\Omega)$ we define a function $g_\zeta : \mathbb{R}^{n-1} \to \mathbb{C}$ by

$$g_\zeta(x) = g(x, \zeta(x)), \quad x \in \mathbb{R}^{n-1}.$$

With this notation for real $s \ge 0$ we put

$$H^s(\partial\Omega) = \left\{ g \in L^2(\partial\Omega) : g_\zeta \in H^s(\mathbb{R}^{n-1}) \right\}$$

and

$$(g, h)_{H^s(\partial\Omega)} = (g_\zeta, h_\zeta)_{H^s(\mathbb{R}^{n-1})}, \quad g, h \in H^s(\partial\Omega).$$

With this scalar product $H^s(\partial\Omega)$ is a Hilbert space.

Let now Ω be a Lipschitz domain as in Definition 1.4 and let φ_j, $1 \le j \le k$, form a partition of unity as above. For real $s \ge 0$ we define

$$H^s(\partial\Omega) = \left\{ g \in L^2(\partial\Omega) : \varphi_j g \in H^s(\partial\Omega_j), 1 \le j \le k \right\}$$

and

$$(g, h)_{H^s(\partial\Omega)} = \sum_{j=1}^k (\varphi_j g, \varphi_j h)_{H^s(\partial\Omega_j)}, \quad g, h \in H^s(\partial\Omega),$$

so that $H^s(\partial\Omega)$ becomes a Hilbert space. Finally we denote by $H^{-s}(\partial\Omega)$ the dual of $H^s(\partial\Omega)$. The sesquilinear duality between $H^s(\partial\Omega)$ and $H^{-s}(\partial\Omega)$ is denoted by $(\cdot, \cdot)_{-s,s}$ and the norm on $H^{-s}(\partial\Omega)$ is given by

$$\|g\|_{H^{-s}(\partial\Omega)} = \sup_{\substack{h \in H^s(\partial\Omega) \\ \|h\|_{H^s(\partial\Omega)} = 1}} |(g, h)_{-s,s}|, \quad g \in H^{-s}(\partial\Omega).$$

In the present thesis we will mainly deal with the case $s = \frac{1}{2}$. We will write $(\cdot,\cdot)_{\partial\Omega}$ for both $(\cdot,\cdot)_{-1/2,1/2}$ and $(\cdot,\cdot)_{1/2,-1/2}$ when no confusion can arise.

In our further considerations trace maps will play an important role. We use the notation

$$C_0^\infty(\overline{\Omega}) = \{u|_{\overline{\Omega}} : u \in C_0^\infty(\mathbb{R}^n)\}\,.$$

The following proposition allows us to consider boundary values of functions in $H^1(\Omega)$, see, e.g., [100, Theorem 3.37].

Proposition 1.5. *Let Ω be a Lipschitz domain. Then the trace map $C_0^\infty(\overline{\Omega}) \ni u \mapsto u|_{\partial\Omega}$ has a unique extension to a bounded linear operator $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ and the operator γ has a bounded right inverse. In particular, γ is surjective.*

In the following we will always write $u|_{\partial\Omega}$ instead of γu also for $u \in H^1(\Omega)$. We remark that on each Lipschitz domain the space $H_0^1(\Omega)$ defined as above coincides with the kernel of the trace operator γ, that is,

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\right\}. \tag{1.8}$$

This fact we will use extensively.

Besides the trace we will also make use of the trace of the conormal derivative of $u \in H^1(\Omega)$ with respect to the differential expression

$$\mathcal{L} = -\sum_{j,k=1}^n \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^n \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{a_j} \right) + a$$

on Ω, where $a_{jk}, a_j : \overline{\Omega} \to \mathbb{C}$ are bounded Lipschitz functions and $a : \overline{\Omega} \to \mathbb{R}$ is measurable and bounded. In order to make $\mathcal{L}$ formally symmetric we additionally assume $a_{jk} = \overline{a_{kj}}$, $1 \le j,k \le n$. Note that under these assumptions for each $u \in H^1(\Omega)$ one can calculate $\mathcal{L}u$ in the sense of distributional derivatives, see above, and the distribution $\mathcal{L}u$ is always bounded with respect to the norm $\|\cdot\|_{H^1(\Omega)}$. Indeed, if M denotes a joint upper bound of all the functions $|a_{jk}|, |a_j|$, and $|a|$, $1 \le j,k \le n$, then for $v \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} |(\mathcal{L}u)v| &= \left| \int_\Omega \left(\sum_{j,k=1}^n a_{jk} \frac{\partial u}{\partial x_k} \cdot \frac{\partial v}{\partial x_j} + \sum_{j=1}^n \big(a_j \frac{\partial u}{\partial x_j} \cdot v + a_j u \cdot \frac{\partial v}{\partial x_j} \big) + auv \right) dx \right| \\ &\le M \sum_{j,k=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_k} \right| \left| \frac{\partial v}{\partial x_j} \right| dx + M \sum_{j=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_j} \right| |v| dx \\ &\quad + M \sum_{j=1}^n \int_\Omega \left| \frac{\partial v}{\partial x_j} \right| |u| dx + M \int_\Omega |u||v| dx \end{aligned}$$

$$\leq M \sum_{j,k=1}^{n} \Big\| \frac{\partial u}{\partial x_k} \Big\|_{L^2(\Omega)} \Big\| \frac{\partial v}{\partial x_j} \Big\|_{L^2(\Omega)} + M\|v\|_{L^2(\Omega)} \sum_{j=1}^{n} \Big\| \frac{\partial u}{\partial x_j} \Big\|_{L^2(\Omega)}$$
$$+ M\|u\|_{L^2(\Omega)} \sum_{j=1}^{n} \Big\| \frac{\partial v}{\partial x_j} \Big\|_{L^2(\Omega)} + M\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)};$$

cf. (1.5). Thus $\mathcal{L}u$ can be regarded as an element of $H^{-1}(\Omega)$ and we will just write $\mathcal{L}u \in H^{-1}(\Omega)$. Corresponding to the differential expression $\mathcal{L}$ we consider the sesquilinear form which is given by

$$\mathfrak{a}[u,v] = \int_\Omega \Big(\sum_{j,k=1}^{n} a_{jk} \frac{\partial u}{\partial x_k} \cdot \overline{\frac{\partial v}{\partial x_j}} + \sum_{j=1}^{n} \big(a_j \frac{\partial u}{\partial x_j} \cdot \overline{v} + \overline{a_j} u \cdot \overline{\frac{\partial v}{\partial x_j}} \big) + au\overline{v} \Big) dx \quad (1.9)$$

for $u, v \in H^1(\Omega)$. We will study the properties of this form later on; cf. Lemma 2.2 below. It can be verified that the following definition makes sense, see, e.g., [100, Lemma 4.3].

Definition 1.6. Let $u \in H^1(\Omega)$ with $\mathcal{L}u \in L^2(\Omega)$. Then the unique $g \in H^{-1/2}(\partial\Omega)$ with

$$\mathfrak{a}[u,v] = (\mathcal{L}u, v) + (g, v|_{\partial\Omega})_{\partial\Omega}, \quad v \in H^1(\Omega), \quad (1.10)$$

is called the *conormal derivative of u with respect to $\mathcal{L}$*; we write $g = \frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega}$.

Note that for $u, v \in C_0^\infty(\overline{\Omega})$ the duality on the right-hand side of (1.10) turns into an integral and (1.10) then is simply the first Green identity with

$$g = \sum_{j,k=1}^{n} a_{jk} \nu_j \frac{\partial u}{\partial x_k}\Big|_{\partial\Omega} + \sum_{j=1}^{n} \overline{a_j} \nu_j u|_{\partial\Omega}.$$

Moreover, from (1.10) we immediately conclude the second Green identity

$$(\mathcal{L}u, v) - (u, \mathcal{L}v) = \Big(u|_{\partial\Omega}, \frac{\partial v}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} \Big)_{\partial\Omega} - \Big(\frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, v|_{\partial\Omega} \Big)_{\partial\Omega} \quad (1.11)$$

for $u, v \in H^1(\Omega)$ with $\mathcal{L}u, \mathcal{L}v \in L^2(\Omega)$.

We conclude this section with a collection of embedding statements; the first of them is known as the criterion of Rellich. For proofs we refer the reader to, e.g., [55, 100].

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then the following assertions hold.*

(i) *If Ω is bounded then the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact.*

(ii) *If Ω is a bounded Lipschitz domain then the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact.*

(iii) *If Ω is a Lipschitz domain then the embedding of $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$ is compact.*

2 Selfadjoint elliptic differential operators and boundary mappings

The aim of the present chapter is to introduce the central objects which appear in the main results of this thesis and to provide some of their basic properties. We consider a second order, formally symmetric, uniformly elliptic differential expression $\mathcal{L}$ on a (bounded or unbounded) Lipschitz domain $\Omega \subset \mathbb{R}^n$. We establish a wide class of selfadjoint realizations of $\mathcal{L}$ in the Hilbert space $L^2(\Omega)$ subject to Dirichlet and nonlocal generalized Robin boundary conditions; this will be done with the help of the classical theory of representing operators for semibounded sesquilinear forms in Hilbert spaces; cf. Section 1.3. Moreover, we introduce the Dirichlet-to-Neumann map and Robin-to-Dirichlet maps on the boundary $\partial\Omega$ corresponding to the differential expression $\mathcal{L} - \lambda$ with a spectral parameter $\lambda \in \mathbb{C}$.

For more details on selfadjoint elliptic differential operators on bounded and unbounded domains we refer the reader to the classical works [55, 67, 95, 123] and to the recent publications [19, 21, 24, 57, 59, 69, 70, 98]. For recent studies of the corresponding boundary operators and related questions the reader may consult [4, 7, 8, 36, 58].

In this and in the following chapters we will make the following assumptions.

Assumption 2.1. *The set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a Lipschitz domain, see Definition 1.4. Moreover, $\mathcal{L}$ is a second order partial differential expression on Ω of the form*

$$\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{a_j} \right) + a$$

with bounded Lipschitz coefficients $a_{jk} = \overline{a_{kj}}, a_j : \overline{\Omega} \to \mathbb{C}$, $1 \leq j, k \leq n$, and a bounded, measurable coefficient $a : \Omega \to \mathbb{R}$. The expression $\mathcal{L}$ is uniformly elliptic *on $\overline{\Omega}$, that is, there exists $E > 0$ such that*

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq E \sum_{k=1}^{n} \xi_k^2, \quad x \in \overline{\Omega}, \quad \xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n. \tag{2.1}$$

We remark that the condition (2.1) already implies

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\overline{\xi_k} \geq E \sum_{k=1}^{n} |\xi_k|^2, \quad x \in \overline{\Omega}, \quad \xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{C}^n. \tag{2.2}$$

We first focus on the selfadjoint realization of $\mathcal{L}$ with a Dirichlet boundary condition.

2.1 The Dirichlet operator and the Dirichlet-to-Neumann map

In this section we define the selfadjoint Dirichlet operator corresponding to the differential expression $\mathcal{L}$ in $L^2(\Omega)$ and the Dirichlet-to-Neumann map associated with $\mathcal{L}-\lambda$ for λ outside the spectrum of the Dirichlet operator. The results of this section are essentially known but it is difficult to find references under our precise assumptions. Therefore, for the convenience of the reader we provide proofs. We make use of the following lemma.

Lemma 2.2. *Let Assumption 2.1 be satisfied and let the sesquilinear form $\mathfrak{a}$ in $L^2(\Omega)$ be defined by*

$$\mathfrak{a}[u,v] = \int_\Omega \left(\sum_{j,k=1}^n a_{jk} \frac{\partial u}{\partial x_k} \cdot \overline{\frac{\partial v}{\partial x_j}} + \sum_{j=1}^n \left(a_j \frac{\partial u}{\partial x_j} \cdot \overline{v} + \overline{a_j} u \cdot \overline{\frac{\partial v}{\partial x_j}} \right) + au\overline{v} \right) dx,$$
$$\operatorname{dom}\mathfrak{a} = H^1(\Omega). \tag{2.3}$$

Then $\mathfrak{a}$ is densely defined, symmetric, and semibounded from below by some $\mu \in \mathbb{R}$. Moreover, $\mathfrak{a}$ is bounded in $H^1(\Omega)$, that is, there exists $C > 0$ such that

$$|\mathfrak{a}[u,v]| \le C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \quad u,v \in H^1(\Omega).$$

The expression $\|u\|_\mathfrak{a}^2 = (\mathfrak{a} - \mu + 1)[u,u]$, $u \in H^1(\Omega)$, defines a norm on $H^1(\Omega)$ which is equivalent to $\|\cdot\|_{H^1(\Omega)}$; in particular, $\mathfrak{a}$ is closed.

Proof. Clearly, $\mathfrak{a}$ is densely defined in $L^2(\Omega)$. For $u, v \in H^1(\Omega)$ we have

$$\begin{aligned}\overline{\mathfrak{a}[v,u]} &= \int_\Omega \left(\sum_{j,k=1}^n \overline{a_{jk}} \frac{\partial u}{\partial x_j} \cdot \overline{\frac{\partial v}{\partial x_k}} + \sum_{j=1}^n \left(a_j \frac{\partial u}{\partial x_j} \cdot \overline{v} + \overline{a_j} u \cdot \overline{\frac{\partial v}{\partial x_j}} \right) + \overline{a}u\overline{v} \right) dx \\ &= \mathfrak{a}[u,v],\end{aligned}$$

see Assumption 2.1; hence $\mathfrak{a}$ is symmetric. Let us observe next that $\mathfrak{a}$ is semibounded from below. Indeed, for $u \in \operatorname{dom}\mathfrak{a} = H^1(\Omega)$ we obtain with the help of (2.2)

$$\begin{aligned}\mathfrak{a}[u,u] &\ge \int_\Omega \left(\sum_{j=1}^n \left(E\left|\frac{\partial u}{\partial x_j}\right|^2 - 2\|a_j\|_\infty |u| \left|\frac{\partial u}{\partial x_j}\right| \right) + (\inf a)|u|^2 \right) dx \\ &= \frac{E}{2} \int_\Omega |\nabla u|^2 dx \\ &\quad + \int_\Omega \left(\sum_{j=1}^n \left(\left(\sqrt{E/2}\left|\frac{\partial u}{\partial x_j}\right| - \frac{\|a_j\|_\infty}{\sqrt{E/2}}|u| \right)^2 - \frac{2\|a_j\|_\infty^2}{E}|u|^2 \right) + (\inf a)|u|^2 \right) dx\end{aligned}$$

$$\geq \frac{E}{2}\int_\Omega |\nabla u|^2 dx + \Big(-\frac{2}{E}\sum_{j=1}^n \|a_j\|_\infty^2 + \inf a\Big)\|u\|^2_{L^2(\Omega)} \geq \mu \|u\|^2_{L^2(\Omega)}$$

with $\mu := -\frac{2}{E}\sum_{j=1}^n \|a_j\|_\infty^2 + \inf a$, where $\|a_j\|_\infty = \sup_{x\in\Omega}|a_j(x)|$. Hence $\mathfrak{a}$ is semibounded from below. Note that for each $u \in H^1(\Omega)$ the above estimate also yields

$$\|u\|_{\mathfrak{a}}^2 = (\mathfrak{a} - \mu + 1)[u,u] \geq \frac{E}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^2 dx \geq \min\Big\{\frac{E}{2}, 1\Big\}\|u\|^2_{H^1(\Omega)}. \tag{2.4}$$

Moreover, for $u, v \in H^1(\Omega)$ we have

$$\begin{aligned} |\mathfrak{a}[u,v]| &\leq M \sum_{j,k=1}^n \int_\Omega \Big|\frac{\partial u}{\partial x_k}\Big|\Big|\frac{\partial v}{\partial x_j}\Big| dx + M\sum_{j=1}^n \int_\Omega \Big|\frac{\partial u}{\partial x_j}\Big||v| dx \\ &\quad + M\sum_{j=1}^n \int_\Omega \Big|\frac{\partial v}{\partial x_j}\Big||u| dx + M\int_\Omega |u||v| dx \\ &\leq M \sum_{j,k=1}^n \Big\|\frac{\partial u}{\partial x_k}\Big\|_{L^2(\Omega)}\Big\|\frac{\partial v}{\partial x_j}\Big\|_{L^2(\Omega)} + M\|v\|_{L^2(\Omega)}\sum_{j=1}^n \Big\|\frac{\partial u}{\partial x_j}\Big\|_{L^2(\Omega)} \\ &\quad + M\|u\|_{L^2(\Omega)}\sum_{j=1}^n \Big\|\frac{\partial v}{\partial x_j}\Big\|_{L^2(\Omega)} + M\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}, \end{aligned}$$

where M denotes a joint upper bound of all the functions $|a_{jk}|, |a_j|$, and $|a|$, $1 \leq j, k \leq n$. Hence there exists $C > 0$ such that

$$|\mathfrak{a}[u,v]| \leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \quad u, v \in H^1(\Omega).$$

Hence it follows together with (2.4) that the norms $\|\cdot\|_{\mathfrak{a}}$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $H^1(\Omega)$. In particular, $(H^1(\Omega), \|\cdot\|_{\mathfrak{a}})$ is complete, i.e., $\mathfrak{a}$ is closed. This completes the proof of the lemma. □

The *Dirichlet operator* associated with $\mathcal{L}$ in $L^2(\Omega)$ is defined by

$$A_{\mathrm{D}} u = \mathcal{L}u, \quad \operatorname{dom} A_{\mathrm{D}} = \big\{u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), u|_{\partial\Omega} = 0\big\}. \tag{2.5}$$

We prove that it is selfadjoint and summarize some of its properties in the following theorem; cf. also the monographs [44, 55].

Theorem 2.3. *Let Assumption 2.1 be satisfied. Then the Dirichlet operator A_{D} in (2.5) is selfadjoint and semibounded from below in $L^2(\Omega)$. Moreover, if Ω is bounded then the spectrum of A_{D} is purely discrete and accumulates to $+\infty$.*

Proof. Let us define a sesquilinear form $\mathfrak{a}_\mathrm{D}$ in $L^2(\Omega)$ by

$$\mathfrak{a}_\mathrm{D}[u,v] := \mathfrak{a}[u,v], \quad \operatorname{dom}\mathfrak{a}_\mathrm{D} = H^1_0(\Omega),$$

where $\mathfrak{a}$ is given by (2.3). Clearly, $\mathfrak{a}_\mathrm{D}$ is densely defined, and it follows from Lemma 2.2 that $\mathfrak{a}_\mathrm{D}$ is symmetric and semibounded from below by some $\mu \in \mathbb{R}$ and that the norm induced by $\|\cdot\|^2_{\mathfrak{a}_\mathrm{D}} := (\mathfrak{a}_\mathrm{D} - \mu + 1)[\cdot,\cdot]$ on $H^1_0(\Omega)$ is equivalent to the H^1-norm on $H^1_0(\Omega)$. In particular, $(\operatorname{dom}\mathfrak{a}_\mathrm{D}, \|\cdot\|_{\mathfrak{a}_\mathrm{D}})$ is a Hilbert space, that is, $\mathfrak{a}_\mathrm{D}$ is closed. By Theorem 1.3 there exists a unique selfadjoint operator A in $L^2(\Omega)$ with $\operatorname{dom} A \subset \operatorname{dom}\mathfrak{a}_\mathrm{D} = H^1_0(\Omega)$ and

$$\mathfrak{a}_\mathrm{D}[u,v] = (Au, v), \quad u \in \operatorname{dom} A, v \in \operatorname{dom}\mathfrak{a}_\mathrm{D}.$$

We will prove next that $A = A_\mathrm{D}$. Let $u \in \operatorname{dom} A$. Then for each $v \in C^\infty_0(\Omega) \subset \operatorname{dom}\mathfrak{a}_\mathrm{D}$ we have

$$\begin{aligned}(Au, v) &= \mathfrak{a}_\mathrm{D}[u,v] \\ &= \int_\Omega \Bigg(\sum_{j,k=1}^n a_{jk} \frac{\partial u}{\partial x_k} \cdot \overline{\frac{\partial v}{\partial x_j}} + \sum_{j=1}^n \Big(a_j \frac{\partial u}{\partial x_j} \cdot \overline{v} + \overline{a_j} u \cdot \overline{\frac{\partial v}{\partial x_j}} \Big) + a u \overline{v} \Bigg) dx \\ &= \int_\Omega \Bigg(-\sum_{j,k=1}^n \frac{\partial}{\partial x_j} \Big(a_{jk} \frac{\partial u}{\partial x_k} \Big) \overline{v} + \sum_{j=1}^n \Big(a_j \frac{\partial u}{\partial x_j} \cdot \overline{v} - \frac{\partial}{\partial x_j} (\overline{a_j} u) \cdot \overline{v} \Big) + a u \overline{v} \Bigg) dx \\ &= (\mathcal{L}u, v)_{-1,1} \end{aligned} \tag{2.6}$$

by the definition of the distributional derivative. In particular, $Au = \mathcal{L}u$ in the distributional sense. Since A is an operator in $L^2(\Omega)$, it turns out that $\mathcal{L}u \in L^2(\Omega)$, that is, $u \in H^1_0(\Omega)$ with $\mathcal{L}u \in L^2(\Omega)$. Hence u belongs to $\operatorname{dom} A_\mathrm{D}$, see (1.8), and satisfies $A_\mathrm{D} u = \mathcal{L}u = Au$. Let, conversely, u belong to $\operatorname{dom} A_\mathrm{D}$, that is, $u \in H^1_0(\Omega)$ with $\mathcal{L}u \in L^2(\Omega)$. Then for each $v \in C^\infty_0(\Omega)$ we obtain

$$(A_\mathrm{D} u, v) = (\mathcal{L}u, v) = \mathfrak{a}_\mathrm{D}[u,v] \tag{2.7}$$

by the definition of the distributional derivative; cf. (2.6). Note that it follows from the equivalence of the norms $\|\cdot\|_\mathfrak{a}$ and $\|\cdot\|_{H^1(\Omega)}$, see Lemma 2.2, that $C^\infty_0(\Omega)$ is a core of $\mathfrak{a}_\mathrm{D}$. From (2.7) and Theorem 1.3 we obtain $u \in \operatorname{dom} A$ and $Au = A_\mathrm{D} u$. Thus A_D coincides with A. In particular, A_D is selfadjoint. Moreover, from Theorem 1.3 we obtain that A_D is semibounded from below.

Let now Ω be bounded. Clearly, for each $\lambda \in \rho(A_\mathrm{D})$ the operator $(A_\mathrm{D} - \lambda)^{-1}$ is bounded and everywhere defined in $L^2(\Omega)$ with $\operatorname{ran}(A_\mathrm{D} - \lambda)^{-1} = \operatorname{dom} A_\mathrm{D} \subset H^1_0(\Omega)$, see (1.8). By the closed graph theorem the operator $R_\lambda : L^2(\Omega) \to H^1_0(\Omega), u \mapsto (A_\mathrm{D} - \lambda)^{-1} u$, is also bounded. Moreover, by Theorem 1.7 the embedding ι of $H^1_0(\Omega)$ into $L^2(\Omega)$ is compact. Consequently, also $(A_\mathrm{D} - \lambda)^{-1} = \iota R_\lambda$ is compact in $L^2(\Omega)$. Therefore the spectrum of A_{D} only consists of isolated eigenvalues with finite multiplicities. Since, as a selfadjoint operator which is only densely defined, A_{D} is unbounded, the eigenvalues accumulate to $+\infty$. □

In order to define the Dirichlet-to-Neumann map corresponding to the differential expression $\mathcal{L}-\lambda$ for λ in the resolvent set $\rho(A_{\mathrm{D}})$ of A_{D} we need the following lemma.

Lemma 2.4. *Let Assumption 2.1 be satisfied and let A_{D} be the selfadjoint Dirichlet operator in* (2.5). *Then for each $\lambda \in \rho(A_{\mathrm{D}})$ and each $g \in H^{1/2}(\partial\Omega)$ the boundary value problem*

$$\mathcal{L}u = \lambda u, \quad u|_{\partial\Omega} = g \tag{2.8}$$

has a unique solution $u_\lambda \in H^1(\Omega)$.

Proof. Let $g \in H^{1/2}(\partial\Omega)$ and $\lambda \in \rho(A_{\mathrm{D}})$. By Proposition 1.5 there exists $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = g$. Let $\mathfrak{a}$ be the sesquilinear form in (2.3) with $\operatorname{dom}\mathfrak{a} = H^1(\Omega)$. By Lemma 2.2 $\mathfrak{a}$ is bounded in $H^1(\Omega)$; in particular, the mapping

$$F_\zeta : H_0^1(\Omega) \to \mathbb{C}, \quad v \mapsto -(\mathfrak{a} - \zeta + 1)[u, v]$$

is bounded in $H^1(\Omega)$ and antilinear for each $\zeta \in \mathbb{R}$; hence F_ζ belongs to the antidual of $H_0^1(\Omega)$. Moreover, it follows from Lemma 2.2 that the sesquilinear form

$$\mathfrak{a}_{\mathrm{D}}[u, v] = \mathfrak{a}[u, v], \quad \operatorname{dom}\mathfrak{a}_{\mathrm{D}} = H_0^1(\Omega),$$

is semibounded by some $\mu \in \mathbb{R}$ and closed; cf. the proof of Theorem 2.3. In particular, $H_0^1(\Omega)$ is a Hilbert space when it is equipped with the scalar product $\mathfrak{a}_{\mathrm{D}} - \mu + 1$. By the Fréchet–Riesz theorem there exists a unique $u_0 \in H_0^1(\Omega)$ with

$$(\mathfrak{a}_{\mathrm{D}} - \mu + 1)[u_0, v] = F_\mu(v) = -(\mathfrak{a} - \mu + 1)[u, v], \quad v \in H_0^1(\Omega).$$

Consequently, $(\mathfrak{a} - \mu + 1)[u_0 + u, v] = 0$ for all $v \in H_0^1(\Omega)$, which implies $(\mathcal{L} - \mu + 1)(u_0 + u) = 0$; in particular, $(\mathcal{L} - \lambda)(u_0 + u) = (\mu - 1 - \lambda)(u_0 + u) \in L^2(\Omega)$. Let us set

$$u_\lambda = u_0 + u - (A_{\mathrm{D}} - \lambda)^{-1}(\mathcal{L} - \lambda)(u_0 + u) \in H^1(\Omega).$$

Then $u_\lambda|_{\partial\Omega} = u|_{\partial\Omega} = g$ and $(\mathcal{L} - \lambda)u_\lambda = 0$. Thus u_λ is a solution of (2.8).

In order to prove the uniqueness let $v_\lambda \in H^1(\Omega)$ be a further solution of (2.8). Then we have

$$\mathcal{L}(u_\lambda - v_\lambda) = \lambda(u_\lambda - v_\lambda) \quad \text{and} \quad (u_\lambda - v_\lambda)|_{\partial\Omega} = 0,$$

that is, $(u_\lambda - v_\lambda) \in \ker(A_{\mathrm{D}} - \lambda)$. Since $\lambda \in \rho(A_{\mathrm{D}})$, it follows $u_\lambda = v_\lambda$. □

We use the observation of Lemma 2.4 in order to define the Dirichlet-to-Neumann map.

Definition 2.5. Let Assumption 2.1 be satisfied and let A_{D} be the selfadjoint Dirichlet operator in (2.5). Then for each $\lambda \in \rho(A_{\mathrm{D}})$ the *Dirichlet-to-Neumann map* is defined by

$$M(\lambda) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega), \quad g = u_\lambda|_{\partial\Omega} \mapsto -\frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \tag{2.9}$$

where u_λ is the unique solution in $H^1(\Omega)$ of (2.8).

We remark that the minus sign in the definition of $M(\lambda)$ is not essential for the validity of our main results. The sign was chosen in order to obtain an operator-valued Nevanlinna function in analogy to the Titchmarsh–Weyl m-function for ordinary differential equations; cf. Remark 3.2 below.

2.2 Generalized Robin operators and Robin-to-Dirichlet maps

In this section we introduce a class of selfadjoint realizations of the differential expression $\mathcal{L}$ with nonlocal boundary conditions of Robin type. We consider the operators A_Θ in $L^2(\Omega)$ given by

$$A_\Theta u = \mathcal{L}u, \quad \operatorname{dom} A_\Theta = \Big\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} + \Theta u|_{\partial\Omega} = 0 \Big\}, \tag{2.10}$$

where we make the following assumption on the operator Θ; cf. [57].

Assumption 2.6. *The operator* $\Theta : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ *is bounded and satisfies the symmetry condition*

$$(\Theta g, h)_{\partial\Omega} = (g, \Theta h)_{\partial\Omega}, \quad g, h \in H^{1/2}(\partial\Omega).$$

Moreover, $\Theta = \Theta_1 + \Theta_2$ *holds, where* $\Theta_i : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ *are bounded operators,* $i = 1, 2$, Θ_1 *is* L^2*-semibounded from below, i.e., there exists* $c_{\Theta_1} \in \mathbb{R}$ *with*

$$(\Theta_1 g, g)_{\partial\Omega} \geq c_{\Theta_1} \|g\|^2_{L^2(\partial\Omega)}, \quad g \in H^{1/2}(\partial\Omega), \tag{2.11}$$

and Θ_2 *is compact.*

Note that, as a special case, Θ may be chosen to be the operator of multiplication with a bounded, measurable function $\vartheta : \partial\Omega \to \mathbb{R}$. In this case the functions in the domain of A_Θ satisfy the classical Robin boundary condition

$$\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} + \vartheta u|_{\partial\Omega} = 0.$$

Moreover, for $\Theta = 0$ we obtain the well-known *Neumann operator*. We are going to prove that A_Θ is selfadjoint. We remark that Assumption 2.6 on Θ is inspired by the recent publication [57], where selfadjointness of A_Θ is shown for $\mathcal{L} = -\Delta$ on a bounded Lipschitz domain. Our assumptions on Θ may be slightly weakened when one follows the ideas of [57]. In order to keep the situation simple we restrict ourselves to the above conditions.

We use the following lemma which can basically be found in [8, Lemma 2.3] and [57, Lemma 4.2–Lemma 4.3] in more general versions. We give a short proof in our precise setting. Recall that $(H^1(\Omega))'$ denotes the dual space of $H^1(\Omega)$.

Lemma 2.7. *Let $K : H^1(\Omega) \to (H^1(\Omega))'$ be a compact linear operator. Then for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with*

$$|(Ku, u)_{1',1}| \le \varepsilon \|u\|^2_{H^1(\Omega)} + C_\varepsilon \|u\|^2_{L^2(\Omega)}, \quad u \in H^1(\Omega),$$

where $(\cdot,\cdot)_{1',1}$ denotes the duality between $H^1(\Omega)$ and $(H^1(\Omega))'$.

Proof. Assume the converse, that is, there exist $\varepsilon > 0$ and $(u_j)_{j\in\mathbb{N}} \subset H^1(\Omega)$ with $\|u_j\|_{H^1(\Omega)} = 1$, $j \in \mathbb{N}$, and

$$|(Ku_j, u_j)_{1',1}| \ge \varepsilon + j\|u_j\|^2_{L^2(\Omega)}, \quad j \in \mathbb{N}. \tag{2.12}$$

Since $H^1(\Omega)$ is reflexive, it is no restriction to assume that there exists $u \in H^1(\Omega)$ with $u_j \to u$ weakly in $H^1(\Omega)$. Moreover, since K is compact, it follows $Ku_j \to Ku$ (strongly) in $(H^1(\Omega))'$ and, hence,

$$(Ku_j, u_j)_{1',1} \to (Ku, u)_{1',1} \quad \text{as} \quad j \to \infty. \tag{2.13}$$

Thus (2.12) yields $\|u_j\|^2_{L^2(\Omega)} \to 0$ as $j \to \infty$, hence $u = 0$. Then (2.13) yields $(Ku_j, u_j)_{1',1} \to 0$ as $j \to \infty$ and (2.12) leads to the contradiction $\varepsilon \le 0$. □

We are now able to prove the selfadjointness of A_Θ; we remark that for an arbitrary bounded, symmetric operator $\Theta : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ the operator A_Θ in (2.10) in general is not selfadjoint, cf. [21,57,67]. Nevertheless, if Θ satisfies the above assumption then selfadjointness can be guaranteed.

Theorem 2.8. *Let Assumption 2.1 and Assumption 2.6 be satisfied. Then A_Θ in* (2.10) *is selfadjoint and semibounded from below in $L^2(\Omega)$. Moreover, if Ω is bounded then the spectrum of A_Θ is purely discrete and accumulates to $+\infty$.*

Proof. Let us define a sesquilinear form $\mathfrak{a}_\Theta$ in $L^2(\Omega)$ by

$$\mathfrak{a}_\Theta[u,v] = \mathfrak{a}[u,v] + (\Theta u|_{\partial\Omega}, v|_{\partial\Omega})_{\partial\Omega}, \quad \operatorname{dom}\mathfrak{a}_\Theta = H^1(\Omega),$$

where $\mathfrak{a}$ is given in (2.3). It is clear that $\mathfrak{a}_\Theta$ is symmetric and densely defined in $L^2(\Omega)$; cf. Lemma 2.2. Let us show that $\mathfrak{a}_\Theta$ is closed and semibounded from below. For this let us observe two consequences of Lemma 2.7. On the one hand, if $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator in Proposition 1.5, we may choose $K = \gamma^*\iota\gamma$ in Lemma 2.7, where ι is the compact embedding of $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$; cf. Theorem 1.7 (iii). Then K is compact and we obtain that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with

$$\begin{aligned}\|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)} &= (\iota\gamma u, \gamma u)_{\partial\Omega} = (Ku,u)_{1',1}\\ &\le \varepsilon\|u\|^2_{H^1(\Omega)} + C_\varepsilon\|u\|^2_{L^2(\Omega)}, \quad u \in H^1(\Omega).\end{aligned} \tag{2.14}$$

On the other hand we may choose $K = \gamma^*\Theta_2\gamma$. Then by Lemma 2.7 for each $\varepsilon > 0$ there exists a number $\widetilde{C}_\varepsilon > 0$ with

$$\begin{aligned}|(\Theta_2 u|_{\partial\Omega}, u|_{\partial\Omega})_{\partial\Omega}| &= |(\gamma^*\Theta_2\gamma u, u)_{1',1}| = |(Ku,u)_{1',1}|\\ &\le \varepsilon\|u\|^2_{H^1(\Omega)} + \widetilde{C}_\varepsilon\|u\|^2_{L^2(\Omega)}, \quad u \in H^1(\Omega).\end{aligned} \tag{2.15}$$

Recall from Lemma 2.2 that the norm $\|\cdot\|_\mathfrak{a}$ induced by the scalar product $\mathfrak{a}-\mu+1$ on $H^1(\Omega)$ for appropriate $\mu \in \mathbb{R}$ is equivalent to $\|\cdot\|_{H^1(\Omega)}$, that is, there exist $c, C > 0$ with

$$c\|u\|^2_{H^1(\Omega)} \le \|u\|^2_\mathfrak{a} \le C\|u\|^2_{H^1(\Omega)}, \quad u \in H^1(\Omega). \tag{2.16}$$

It is no restriction to assume that c_{Θ_1} in (2.11) is negative. Let $\varepsilon > 0$ be such that $c + c_{\Theta_1}\varepsilon - \varepsilon > 0$. Then we obtain from (2.14), (2.15), and (2.16)

$$\begin{aligned}\mathfrak{a}_\Theta[u,u] &= (\mathfrak{a}-\mu+1)[u,u] + (\mu-1)\|u\|^2_{L^2(\Omega)}\\ &\quad + (\Theta_1 u|_{\partial\Omega}, u|_{\partial\Omega})_{\partial\Omega} + (\Theta_2 u|_{\partial\Omega}, u|_{\partial\Omega})_{\partial\Omega}\\ &\ge c\|u\|^2_{H^1(\Omega)} + (\mu-1)\|u\|^2_{L^2(\Omega)} + c_{\Theta_1}\varepsilon\|u\|^2_{H^1(\Omega)} + c_{\Theta_1}C_\varepsilon\|u\|^2_{L^2(\Omega)}\\ &\quad - \varepsilon\|u\|^2_{H^1(\Omega)} - \widetilde{C}_\varepsilon\|u\|^2_{L^2(\Omega)}\\ &= (c + c_{\Theta_1}\varepsilon - \varepsilon)\|u\|^2_{H^1(\Omega)} + \widetilde{\mu}\|u\|^2_{L^2(\Omega)}, \qquad u \in H^1(\Omega),\end{aligned}$$

with $\widetilde{\mu} = \mu - 1 + c_{\Theta_1}C_\varepsilon - \widetilde{C}_\varepsilon$. It follows

$$\mathfrak{a}_\Theta[u,u] \ge \widetilde{\mu}\|u\|^2_{L^2(\Omega)}, \quad u \in H^1(\Omega),$$

that is, a_Θ is bounded from below, and

$$(\mathfrak{a}_\Theta - \widetilde{\mu} + 1)[u,u] \ge (c + c_{\Theta_1}\varepsilon - \varepsilon)\|u\|^2_{H^1(\Omega)}, \quad u \in H^1(\Omega). \tag{2.17}$$

On the other hand, since $\Theta : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ as well as the trace map $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ are bounded, there exists $M > 0$ such that

$$(\mathfrak{a}_\Theta - \widetilde{\mu} + 1)[u, u] \leq C\|u\|^2_{H^1(\Omega)} + M\|u\|^2_{H^1(\Omega)} + |c_{\Theta_1} C_\varepsilon - \widetilde{C}_\varepsilon - 1|\|u\|^2_{L^2(\Omega)} \quad (2.18)$$

holds for all $u \in H^1(\Omega)$. From (2.17) and (2.18) it follows that $\mathfrak{a}_\Theta - \widetilde{\mu} + 1$ induces a norm on $H^1(\Omega)$ which is equivalent to $\| \cdot \|_{H^1(\Omega)}$. In particular, $\mathfrak{a}_\Theta$ is closed. By Theorem 1.3 there exists a selfadjoint operator A in $L^2(\Omega)$ with $\operatorname{dom} A \subset \operatorname{dom} \mathfrak{a}_\Theta = H^1(\Omega)$ and

$$(Au, v) = \mathfrak{a}_\Theta[u, v], \quad u \in \operatorname{dom} A, v \in \operatorname{dom} \mathfrak{a}_\Theta.$$

We are going to show $A = A_\Theta$. Let first $u \in \operatorname{dom} A$. Then for each $v \in C_0^\infty(\Omega)$ we have

$$(Au, v) = \mathfrak{a}_\Theta[u, v] = \mathfrak{a}[u, v] = (\mathcal{L}u, v)_{-1,1};$$

cf. the proof of Theorem 2.3. Hence $\mathcal{L}u = Au$ and, in particular, $\mathcal{L}u \in L^2(\Omega)$. Moreover, for arbitrary $v \in H^1(\Omega)$ we have

$$\begin{aligned} \Big(\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} + \Theta u|_{\partial\Omega}, v|_{\partial\Omega}\Big)_{\partial\Omega} &= \mathfrak{a}[u, v] - (\mathcal{L}u, v) + (\Theta u|_{\partial\Omega}, v|_{\partial\Omega})_{\partial\Omega} \\ &= \mathfrak{a}_\Theta[u, v] - (Au, v) = 0 \end{aligned}$$

by the first Green identity (1.10). Since $v|_{\partial\Omega}$ runs through all of $H^{1/2}(\partial\Omega)$ as v runs through $H^1(\Omega)$, it follows $\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} + \Theta u|_{\partial\Omega} = 0$, hence $u \in \operatorname{dom} A_\Theta$ and $A_\Theta u = \mathcal{L}u = Au$. Conversely, if $u \in \operatorname{dom} A_\Theta$ then

$$(A_\Theta u, v) = (\mathcal{L}u, v) = \mathfrak{a}_\Theta[u, v] - \Big(\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} + \Theta u|_{\partial\Omega}, v|_{\partial\Omega}\Big)_{\partial\Omega} = \mathfrak{a}_\Theta[u, v]$$

holds for all $v \in H^1(\Omega)$ and Theorem 1.3 implies $u \in \operatorname{dom} A$; thus $A_\Theta = A$. In particular, A_Θ is selfadjoint. Since $\mathfrak{a}_\Theta$ is semibounded from below, by Theorem 1.3 the same holds for A_Θ.

Let now Ω be bounded. Then the embedding ι of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, see Theorem 1.7. Moreover, by the closed graph theorem the operator $R_\lambda : L^2(\Omega) \to H^1(\Omega), u \mapsto (A_\Theta - \lambda)^{-1} u$ is bounded for all $\lambda \in \rho(A_\Theta)$. Therefore $(A_\Theta - \lambda)^{-1} = \iota R_\lambda$ is compact. From this it follows that the spectrum of A_Θ is purely discrete. Since A_Θ is selfadjoint but not everywhere defined, A_Θ is unbounded. Thus the eigenvalues accumulate to $+\infty$. □

In order to define a Robin-to-Dirichlet map we make use of the following lemma.

Lemma 2.9. *Let Assumption 2.1 and Assumption 2.6 hold. Then for each $\lambda \in \rho(A_\Theta)$ and each $g \in H^{-1/2}(\partial\Omega)$ the boundary value problem*

$$\mathcal{L}u = \lambda u, \quad \frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u|_{\partial\Omega} = g \tag{2.19}$$

has a unique solution $u_\lambda \in H^1(\Omega)$.

Proof. As we have seen in the proof of Theorem 2.8 the form

$$\mathfrak{a}_\Theta[u,v] = \mathfrak{a}[u,v] + (\Theta u|_{\partial\Omega}, v|_{\partial\Omega})_{\partial\Omega}, \quad \operatorname{dom}\mathfrak{a}_\Theta \in H^1(\Omega),$$

in $L^2(\Omega)$ is semibounded from below by some $\mu \in \mathbb{R}$ and $(H^1(\Omega), \mathfrak{a}_\Theta - \mu + 1)$ is a Hilbert space. Let us first prove that for each $g \in H^{-1/2}(\partial\Omega)$ the boundary value problem

$$(\mathcal{L} - \mu + 1)u = 0, \quad \frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u|_{\partial\Omega} = g$$

has a solution u in $H^1(\Omega)$. Let $g \in H^{-1/2}(\partial\Omega)$. Indeed, by the continuity of the trace $H^1(\Omega) \ni u \mapsto u|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ the mapping $H^1(\Omega) \ni v \mapsto (g, v|_{\partial\Omega})_{\partial\Omega}$ is bounded and, hence, belongs to the antidual of $H^1(\Omega)$. By the Fréchet–Riesz theorem there exists a unique $u \in H^1(\Omega)$ with

$$(\mathfrak{a}_\Theta - \mu + 1)[u,v] = (g, v|_{\partial\Omega})_{\partial\Omega}, \quad v \in H^1(\Omega). \tag{2.20}$$

In particular, $(\mathfrak{a}-\mu+1)[u,v] = 0$ for all $v \in C_0^\infty(\Omega)$, which implies $(\mathcal{L}-\mu+1)u = 0$; in particular, $\mathcal{L}u = (\mu-1)u \in L^2(\Omega)$. Then (2.20) yields

$$\mathfrak{a}[u,v] - (\mathcal{L}u, v) = (g - \Theta u|_{\partial\Omega}, v|_{\partial\Omega})_{\partial\Omega}, \quad v \in H^1(\Omega),$$

hence $\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} + \Theta u|_{\partial\Omega} = g$; cf. Definition 1.6.

Let now $\lambda \in \rho(A_\Theta)$ and u as above. Then $(\mathcal{L}-\lambda)u = (\mu-1-\lambda)u \in L^2(\Omega)$ and there exists $u_\Theta \in \operatorname{dom} A_\Theta$ with $(A_\Theta - \lambda)u_\Theta = (\mathcal{L}-\lambda)u$. Let us set $u_\lambda = u - u_\Theta$. Then

$$\frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u_\lambda|_{\partial\Omega} = \frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u|_{\partial\Omega} = g$$

and, clearly, $(\mathcal{L}-\lambda)u_\lambda = 0$, that is, $u_\lambda \in H^1(\Omega)$ solves (2.19).

In order to prove uniqueness, let $v_\lambda \in H^1(\Omega)$ be a further solution of (2.19). Then $u_\lambda - v_\lambda$ satisfies

$$\mathcal{L}(u_\lambda - v_\lambda) = \lambda(u_\lambda - v_\lambda), \quad \frac{\partial(u_\lambda - v_\lambda)}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta(u_\lambda - v_\lambda)|_{\partial\Omega} = 0,$$

in particular, $u_\lambda - v_\lambda \in \operatorname{dom} A_\Theta$ and $(A_\Theta - \lambda)(u_\lambda - v_\lambda) = 0$. From $\lambda \in \rho(A_\Theta)$ it follows $u_\lambda - v_\lambda = 0$. Thus we have proved the uniqueness of the solution. □

Lemma 2.9 allows us to make the following definition.

Definition 2.10. Let Assumption 2.1 and Assumption 2.6 hold. For $\lambda \in \rho(A_\Theta)$ we define the *Robin-to-Dirichlet map*

$$M_\Theta(\lambda) : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega), \quad g = \frac{\partial u_\lambda}{\partial \nu_\mathcal{L}}\big|_{\partial\Omega} + \Theta u_\lambda|_{\partial\Omega} \mapsto u_\lambda|_{\partial\Omega} \tag{2.21}$$

where $u_\lambda \in H^1(\Omega)$ is the unique solution of (2.19).

We remark that for $\lambda \in \rho(A_\Theta) \cap \rho(A_\mathrm{D})$ the Robin-to-Dirichlet map can be written more explicitly as

$$M_\Theta(\lambda) = (\Theta - M(\lambda))^{-1}, \tag{2.22}$$

where $M(\lambda)$ is the Dirichlet-to-Neumann map in (2.9). Indeed, let $\lambda \in \rho(A_\Theta) \cap \rho(A_\mathrm{D})$ and let $g \in H^{1/2}(\partial\Omega)$ with $(\Theta - M(\lambda))g = 0$. By Lemma 2.4 there exists a unique $u_\lambda \in H^1(\Omega)$ with $\mathcal{L}u_\lambda = \lambda u_\lambda$ and $u_\lambda|_{\partial\Omega} = g$. Then $\Theta u_\lambda|_{\partial\Omega} + \frac{\partial u_\lambda}{\partial \nu_\mathcal{L}}|_{\partial\Omega} = 0$, thus $u_\lambda \in \operatorname{dom} A_\Theta$ and $A_\Theta u_\lambda - \lambda u_\lambda = 0$. Now $\lambda \in \rho(A_\Theta)$ implies $u_\lambda = 0$ and, hence, $g = u_\lambda|_{\partial\Omega} = 0$. Therefore $\Theta - M(\lambda)$ is injective. Moreover, if $u_\lambda \in H^1(\Omega)$ satisfies $\mathcal{L}u_\lambda = \lambda u_\lambda$, then

$$(\Theta - M(\lambda))u_\lambda|_{\partial\Omega} = \Theta u_\lambda|_{\partial\Omega} + \frac{\partial u_\lambda}{\partial \nu_\mathcal{L}}\big|_{\partial\Omega},$$

which leads to the representation (2.22).

3 Inverse problems of Calderón type

The present chapter contains some of the main results of this thesis. We are concerned with inverse problems of Calderón type with partial data for a uniformly elliptic differential expression

$$\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{a_j} \right) + a$$

on a connected (not necessarily bounded) Lipschitz domain Ω. We prove that the knowledge of the Dirichlet-to-Neumann map

$$M(\lambda) u_\lambda|_{\partial\Omega} = -\frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \qquad \mathcal{L} u_\lambda = \lambda u_\lambda,$$

see (2.9), on an arbitrarily small nonempty, open subset ω of the boundary $\partial\Omega$ for a certain collection of points λ determines the selfadjoint Dirichlet operator

$$A_{\mathrm{D}} u = \mathcal{L} u, \quad \operatorname{dom} A_{\mathrm{D}} = \left\{ u \in H^1(\Omega) : \mathcal{L} u \in L^2(\Omega), u|_{\partial\Omega} = 0 \right\}$$

associated with $\mathcal{L}$ in $L^2(\Omega)$, see (2.5), uniquely up to unitary equivalence. In addition, we prove a reconstruction formula for A_{D} from the knowledge of $M(\lambda)$ on ω in the case that the domain Ω is bounded. Moreover, we provide analogous results for selfadjoint elliptic differential operators with Robin boundary conditions. The results of this chapter were partly published in [26].

In the whole chapter we assume that the domain Ω and the differential expression $\mathcal{L}$ on Ω satisfy Assumption 2.1 above, that is, Ω is a Lipschitz domain and the differential expression $\mathcal{L}$ is uniformly elliptic on Ω with bounded Lipschitz coefficients $a_{jk} = \overline{a_{kj}}, a_j : \overline{\Omega} \to \mathbb{C}$, $1 \leq j,k \leq n$, and a bounded, measurable coefficient $a : \Omega \to \mathbb{R}$. Moreover, $\omega \subset \partial\Omega$ is assumed to be a nonempty, relatively open set.

3.1 Preliminaries

In this section we provide some preliminary material. As an important tool in the proofs of our main results we introduce the *Poisson operator*

$$\gamma(\lambda) : H^{1/2}(\partial\Omega) \to L^2(\Omega), \quad g \mapsto u_\lambda \tag{3.1}$$

for $\lambda \in \rho(A_{\mathrm{D}})$, where $u_\lambda \in H^1(\Omega)$ is the unique solution of the boundary value problem

$$\mathcal{L} u = \lambda u, \quad u|_{\partial\Omega} = g$$

for a given $g \in H^{1/2}(\partial\Omega)$; cf. Lemma 2.4 in Chapter 2 above. We will make use of a couple of statements and formulas for the Poisson operator and the Dirichlet-to-Neumann map, which are collected in the following lemma. Its proof is based on the second Green identity (1.11). Similar statements in an abstract setting of extension theory of symmetric operators in Hilbert spaces and associated Weyl functions were proved in, e.g., [21, 49].

Lemma 3.1. *Let Ω and $\mathcal{L}$ be as in Assumption 2.1 and let A_{D} be the Dirichlet operator associated with $\mathcal{L}$ in $L^2(\Omega)$ in* (2.5). *Then for $\lambda, \mu \in \rho(A_{\mathrm{D}})$ the Dirichlet-to-Neumann maps $M(\lambda), M(\mu)$ in* (2.9) *and the Poisson operators $\gamma(\lambda), \gamma(\mu)$ in* (3.1) *satisfy the following assertions.*

(i) *$\gamma(\lambda)$ is a bounded operator and its adjoint $\gamma(\lambda)^* : L^2(\Omega) \to H^{-1/2}(\partial\Omega)$ is given by*

$$\gamma(\lambda)^* u = -\frac{\partial}{\partial \nu_{\mathcal{L}}}\big((A_{\mathrm{D}} - \overline{\lambda})^{-1} u\big)\big|_{\partial\Omega}, \quad u \in L^2(\Omega).$$

(ii) *The identity*

$$\gamma(\lambda) = \big(I + (\lambda - \mu)(A_{\mathrm{D}} - \lambda)^{-1}\big)\, \gamma(\mu)$$

holds.

(iii) *The Poisson operators and the Dirichlet-to-Neumann maps satisfy*

$$(\lambda - \overline{\mu})\gamma(\mu)^* \gamma(\lambda) = M(\lambda) - M(\overline{\mu}),$$

and $(M(\lambda)g, h)_{\partial\Omega} = (g, M(\overline{\lambda})h)_{\partial\Omega}$ holds for all $g, h \in H^{1/2}(\partial\Omega)$.

(iv) *$M(\lambda)$ is a bounded operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, which satisfies*

$$M(\lambda) = M(\overline{\mu}) + (\lambda - \overline{\mu})\gamma(\mu)^* \big(I + (\lambda - \mu)(A_{\mathrm{D}} - \lambda)^{-1}\big)\, \gamma(\mu). \tag{3.2}$$

In particular, $\lambda \mapsto M(\lambda)$ is analytic on $\rho(A_{\mathrm{D}})$.

Proof. (i) Let us fix $\lambda \in \rho(A_{\mathrm{D}})$. In order to calculate $\gamma(\lambda)^*$ we choose $g \in H^{1/2}(\partial\Omega)$ and $u \in L^2(\Omega)$. Moreover, we set $v = (A_{\mathrm{D}} - \overline{\lambda})^{-1} u$ and $u_\lambda = \gamma(\lambda) g$, that is, $\mathcal{L} u_\lambda = \lambda u_\lambda$ and $u_\lambda|_{\partial\Omega} = g$. Then the second Green identity (1.11) yields

$$\begin{aligned} (\gamma(\lambda)g, u) &= \big(u_\lambda, (A_{\mathrm{D}} - \overline{\lambda})v\big) = (u_\lambda, \mathcal{L}v) - (\mathcal{L}u_\lambda, v) \\ &= \Big(u_\lambda|_{\partial\Omega}, -\frac{\partial v}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega}\Big)_{\partial\Omega} - \Big(-\frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega}, v|_{\partial\Omega}\Big)_{\partial\Omega}. \end{aligned}$$

Since $v = (A_D - \overline{\lambda})^{-1}u \in \operatorname{dom} A_D$ implies $v|_{\partial\Omega} = 0$, it follows

$$(\gamma(\lambda)g, u) = \Big(g, -\frac{\partial}{\partial \nu_{\mathcal{L}}}\big((A_D - \overline{\lambda})^{-1}u\big)\big|_{\partial\Omega}\Big)_{\partial\Omega},$$

from which we conclude with the help of the closed graph theorem that $\gamma(\lambda)$ is bounded and that $\gamma(\lambda)^* u = -\frac{\partial}{\partial \nu_{\mathcal{L}}}((A_D - \overline{\lambda})^{-1}u)|_{\partial\Omega}$ holds.

(ii) For $\lambda, \mu \in \rho(A_D)$, $g \in H^{1/2}(\partial\Omega)$, and $u \in L^2(\Omega)$ we obtain from (i)

$$\begin{aligned}
&(\gamma(\lambda)g, u) - (\gamma(\mu)g, u) \\
&\quad = \Big(g, -\frac{\partial}{\partial \nu_{\mathcal{L}}}\big((A_D - \overline{\lambda})^{-1}u\big)|_{\partial\Omega}\Big)_{\partial\Omega} - \Big(g, -\frac{\partial}{\partial \nu_{\mathcal{L}}}\big((A_D - \overline{\mu})^{-1}u\big)|_{\partial\Omega}\Big)_{\partial\Omega} \\
&\quad = \Big(g, -\frac{\partial}{\partial \nu_{\mathcal{L}}}\big((A_D - \overline{\mu})^{-1}(\overline{\lambda} - \overline{\mu})(A_D - \overline{\lambda})^{-1}u\big)\big|_{\partial\Omega}\Big)_{\partial\Omega} \\
&\quad = \big(\gamma(\mu)g, (\overline{\lambda} - \overline{\mu})(A_D - \overline{\lambda})^{-1}u\big) = (\lambda - \mu)\left((A_D - \lambda)^{-1}\gamma(\mu)g, u\right),
\end{aligned}$$

which implies $\gamma(\lambda) - \gamma(\mu) = (\lambda - \mu)(A_D - \lambda)^{-1}\gamma(\mu)$ and leads to the assertion.

(iii) Let $\lambda, \mu \in \rho(A_D)$ and choose $g, h \in H^{1/2}(\partial\Omega)$. Moreover, define $u_\lambda = \gamma(\lambda)g$ and $v_\mu = \gamma(\mu)h$. Then the second Green identity (1.11) yields

$$\begin{aligned}
(\lambda - \overline{\mu})\,(\gamma(\lambda)g, \gamma(\mu)h) &= (\mathcal{L}u_\lambda, v_\mu) - (u_\lambda, \mathcal{L}v_\mu) \\
&= (M(\lambda)g, h)_{\partial\Omega} - (g, M(\mu)h)_{\partial\Omega}
\end{aligned} \tag{3.3}$$

and the special choice $\mu = \overline{\lambda}$ implies the second statement in (iii). The first statement now follows immediately from (3.3).

(iv) From $(M(\lambda)g, h)_{\partial\Omega} = (g, M(\overline{\lambda})h)_{\partial\Omega}$ for $\lambda \in \rho(A_D)$ and $g, h \in H^{1/2}(\partial\Omega)$ it follows with the closed graph theorem that $M(\lambda) : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is bounded. Furthermore, by (ii) and (iii) we have

$$\begin{aligned}
M(\lambda) &= M(\overline{\mu}) + (\lambda - \overline{\mu})\gamma(\mu)^*\gamma(\lambda) \\
&= M(\overline{\mu}) + (\lambda - \overline{\mu})\gamma(\mu)^*\left(I + (\lambda - \mu)(A_D - \lambda)^{-1}\right)\gamma(\mu).
\end{aligned}$$

Since $\lambda \mapsto (A_D - \lambda)^{-1}$ is an analytic mapping on $\rho(A_D)$, it follows from (3.2) that $\lambda \mapsto M(\lambda)$ is also analytic. □

Remark 3.2. It follows from Lemma 3.1 (iii) and (iv) that the mapping $\lambda \mapsto M(\lambda)$ can be viewed as an operator-valued Nevanlinna function since $M(\cdot)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$, $M(\lambda)^* = M(\overline{\lambda})$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (after an identification of $H^{1/2}(\partial\Omega)$ with the dual space of $H^{-1/2}(\partial\Omega)$), and

$$\frac{\operatorname{Im}(M(\lambda)g, g)_{\partial\Omega}}{\operatorname{Im}\lambda} = (\gamma(\lambda)g, \gamma(\lambda)g) \geq 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, g \in H^{1/2}(\partial\Omega).$$

Recall that ω is an open, nonempty subset of $\partial\Omega$. For $\lambda \in \rho(A_{\mathrm{D}})$ we define

$$\begin{aligned}\mathcal{N}_\lambda &= \left\{u \in H^1(\Omega) : \mathcal{L}u = \lambda u,\ \operatorname{supp}(u|_{\partial\Omega}) \subset \omega\right\}\\ &= \left\{\gamma(\lambda)g : g \in H^{1/2}(\partial\Omega),\ \operatorname{supp} g \subset \omega\right\},\end{aligned} \tag{3.4}$$

the space of solutions of the differential equation $\mathcal{L}u = \lambda u$ whose trace is supported in ω, where we define $\operatorname{supp} g$ to be the smallest closed set such that g vanishes almost everywhere on its complement.

The following proposition serves as a further preparation and will be crucial for the proofs of our main results. Its proof is partially inspired by an idea from [17]: We extend a certain L^2-function on $\Omega \subset \mathbb{R}^n$ to a function in $n+1$ variables via introducing a semigroup and, afterwards, apply a unique continuation theorem to this function. Unique continuation theorems for second order elliptic differential operators are due to [12, 42, 74, 75] and others. In the following formulation such a theorem can be found in [126].

Theorem 3.3. *Let $G \subset \mathbb{R}^N$, $N \geq 2$, be an open, connected set and let $\alpha_{jk} : \overline{G} \to \mathbb{C}$ be bounded Lipschitz functions, $1 \leq j,k \leq N$, such that*

$$\sum_{j,k=1}^{N} \alpha_{jk}(x)\xi_j\xi_k \geq E\sum_{k=1}^{N}\xi_k^2, \quad x \in \overline{G}, \quad \xi = (\xi_1,\dots,\xi_N)^T \in \mathbb{R}^N,$$

for some $E > 0$. Let $f \in H^2_{\mathrm{loc}}(G)$ and assume that there exist $A, B \in \mathbb{R}$ with

$$\Big|\sum_{j,k=1}^{N} \alpha_{jk}\frac{\partial^2 f}{\partial x_j \partial x_k}\Big| \leq A|f| + B\sum_{j=1}^{N}\Big|\frac{\partial f}{\partial x_j}\Big|$$

almost everywhere on G. If f vanishes almost everywhere in an open, nonempty subset of G then $f = 0$ identically on G.

Proposition 3.4. *Let Assumption 2.1 be satisfied, let Ω be connected, and let $\omega \subset \partial\Omega$ be open and nonempty. Then*

$$\operatorname{span}\{\mathcal{N}_\lambda : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$$

is dense in $L^2(\Omega)$.

Proof. Let $\widetilde{\Omega}$ be a Lipschitz domain such that $\widetilde{\Omega} \supset \Omega$, $\partial\Omega \setminus \omega \subset \partial\widetilde{\Omega}$, and there exists an open ball $\mathcal{O} \subset \widetilde{\Omega} \setminus \Omega$. Let $\widetilde{a}_{jk}, \widetilde{a}_j$ be bounded Lipschitz functions on $\widetilde{\Omega}$ which extend a_{jk} and a_j, respectively, $1 \leq j,k \leq n$, and let $\widetilde{a} : \widetilde{\Omega} \to \mathbb{R}$ be a bounded, measurable extension of a to $\widetilde{\Omega}$ such that the differential expression

$$\widetilde{\mathcal{L}} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\widetilde{a}_{jk}\frac{\partial}{\partial x_k} + \sum_{j=1}^{n}\left(\widetilde{a}_j\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\overline{\widetilde{a}_j}\right) + \widetilde{a}$$

is uniformly elliptic on $\widetilde{\Omega}$; cf. Assumption 2.1. Let $\widetilde{A}_{\mathrm{D}}$ denote the selfadjoint Dirichlet operator associated with $\widetilde{\mathcal{L}}$ in $L^2(\widetilde{\Omega})$, i.e.,

$$\widetilde{A}_{\mathrm{D}}\widetilde{u} = \widetilde{\mathcal{L}}\widetilde{u}, \quad \operatorname{dom}\widetilde{A}_{\mathrm{D}} = \left\{\widetilde{u} \in H^1(\widetilde{\Omega}) : \widetilde{\mathcal{L}}\widetilde{u} \in L^2(\widetilde{\Omega}), \widetilde{u}|_{\partial\widetilde{\Omega}} = 0\right\}.$$

Since $\widetilde{A}_{\mathrm{D}}$ is semibounded from below, see Theorem 2.3, it is no restriction to assume that this operator has a lower bound $\mu > 0$. Let $\widetilde{v} \in L^2(\widetilde{\Omega})$ be such that $\widetilde{v}$ vanishes on Ω, and define

$$\widetilde{u}_{\lambda,\widetilde{v}} = (\widetilde{A}_{\mathrm{D}} - \lambda)^{-1}\widetilde{v}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, denote by $u_{\lambda,\widetilde{v}}$ the restriction of $\widetilde{u}_{\lambda,\widetilde{v}}$ to Ω. Then $\mathcal{L}u_{\lambda,\widetilde{v}} = \lambda u_{\lambda,\widetilde{v}}$ and $\operatorname{supp}(u_{\lambda,\widetilde{v}}|_{\partial\Omega}) \subset \omega$, since $\partial\Omega \setminus \omega \subset \partial\widetilde{\Omega}$ and $\widetilde{u}_{\lambda,\widetilde{v}}|_{\partial\widetilde{\Omega}} = 0$. Hence $u_{\lambda,\widetilde{v}} \in \mathcal{N}_\lambda$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and all $\widetilde{v} \in L^2(\widetilde{\Omega})$ with $\widetilde{v}|_\Omega = 0$.

Let us choose $u \in L^2(\Omega)$ such that u is orthogonal to $\mathcal{N}_\lambda$ for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then, in particular,

$$0 = (u, u_{\overline{\lambda},\widetilde{v}}) = \left(\widetilde{u}, (\widetilde{A}_{\mathrm{D}} - \overline{\lambda})^{-1}\widetilde{v}\right)_{L^2(\widetilde{\Omega})} = \left((\widetilde{A}_{\mathrm{D}} - \lambda)^{-1}\widetilde{u}, \widetilde{v}\right)_{L^2(\widetilde{\Omega})}$$

for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$, where $\widetilde{u}$ denotes the extension of u by zero to $\widetilde{\Omega}$. Since $\widetilde{v} \in L^2(\widetilde{\Omega})$ was chosen arbitrarily such that $\widetilde{v}|_\Omega = 0$, it follows

$$\left((\widetilde{A}_{\mathrm{D}} - \lambda)^{-1}\widetilde{u}\right)\Big|_{\widetilde{\Omega}\setminus\Omega} = 0, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.5}$$

Following an idea from [17, Section 3] we consider the semigroup $T(t) = e^{-t\sqrt{\widetilde{A}_{\mathrm{D}}}}$, $t \geq 0$, which is generated by the square root of the uniformly positive operator $\widetilde{A}_{\mathrm{D}}$. Then $t \mapsto T(t)\widetilde{u} \in L^2(\widetilde{\Omega})$ is twice differentiable with

$$\frac{d^2}{dt^2}T(t)\widetilde{u} = \widetilde{A}_{\mathrm{D}}T(t)\widetilde{u}, \quad t > 0,$$

which implies

$$\left(-\frac{\partial^2}{\partial t^2} + \mathcal{L}\right)T(t)\widetilde{u}(x) = 0, \quad x \in \widetilde{\Omega},\ t > 0, \tag{3.6}$$

in the distributional sense. Indeed, if we choose $\eta_1 \in C_0^\infty(\widetilde{\Omega})$ and $\eta_2 \in C_0^\infty((0,\infty))$ and denote by η the tensor product of both, i.e., $\eta(x,t) = \eta_1(x)\eta_2(t)$ for $x \in \widetilde{\Omega}$, $t \in (0,\infty)$, then $\eta \in C_0^\infty(\widetilde{\Omega} \times (0,\infty))$ and

$$\left(\frac{\partial^2}{\partial t^2}e^{-t\sqrt{\widetilde{A}_{\mathrm{D}}}}\widetilde{u}\right)(\eta) = \int_0^\infty \left(e^{-t\sqrt{\widetilde{A}_{\mathrm{D}}}}\widetilde{u}, \overline{\eta_1}\right)_{L^2(\widetilde{\Omega})}\frac{\partial^2}{\partial t^2}\eta_2(t)dt$$

$$
\begin{aligned}
&= \int_0^\infty \frac{\partial^2}{\partial t^2}\Big(e^{-t\sqrt{\widetilde{A}_{\mathrm{D}}}}\widetilde{u}, \overline{\eta_1}\Big)_{L^2(\widetilde{\Omega})}\eta_2(t)dt \\
&= \int_0^\infty \Big(\widetilde{A}_{\mathrm{D}}e^{-t\sqrt{\widetilde{A}_{\mathrm{D}}}}\widetilde{u}, \overline{\eta_1}\Big)_{L^2(\widetilde{\Omega})}\eta_2(t)dt \\
&= \Big(\widetilde{\mathcal{L}}e^{-t\sqrt{\widetilde{A}_{\mathrm{D}}}}\widetilde{u}\Big)(\eta).
\end{aligned}
$$

Now the density of the tensor product space $C_0^\infty(\widetilde{\Omega}) \otimes C_0^\infty((0,\infty))$ in $C_0^\infty(\widetilde{\Omega} \times (0,\infty))$ with respect to the convergence in $C_0^\infty(\widetilde{\Omega} \times (0,\infty))$ introduced in Section 1.4 implies (3.6). Furthermore, from (3.6) it follows

$$
\begin{aligned}
\Big(-\frac{\partial^2}{\partial t^2} - \sum_{j,k=1}^n \widetilde{a}_{jk}\frac{\partial^2}{\partial x_j \partial x_k}\Big)T(t)\widetilde{u}(x) &= \Big(\sum_{j=1}^n \frac{\partial}{\partial x_j}\overline{\widetilde{a}}_j - \widetilde{a}\Big)T(t)\widetilde{u}(x) \\
&\quad + \sum_{k=1}^n \Big(\sum_{j=1}^n \frac{\partial}{\partial x_j}\widetilde{a}_{jk} - \widetilde{a}_k + \overline{\widetilde{a}}_k\Big)\Big(\frac{\partial}{\partial x_k}T(t)\widetilde{u}(x)\Big).
\end{aligned}
$$

Since the functions $\widetilde{a}_{jk}$ and $\widetilde{a}_j$ and their derivatives of first order as well as $\widetilde{a}$ are bounded, there exist $A, B \in \mathbb{R}$ with

$$
\begin{aligned}
&\Big|\Big(-\frac{\partial^2}{\partial t^2} - \sum_{j,k=1}^n \widetilde{a}_{jk}\frac{\partial^2}{\partial x_j \partial x_k}\Big)T(t)\widetilde{u}(x)\Big| \\
&\qquad \le A|T(t)\widetilde{u}(x)| + B\Big(\Big|\frac{\partial}{\partial t}T(t)\widetilde{u}(x)\Big| + \sum_{j=1}^n \Big|\frac{\partial}{\partial x_j}T(t)\widetilde{u}(x)\Big|\Big). \qquad (3.7)
\end{aligned}
$$

Note that $(x,t) \mapsto T(t)\widetilde{u}(x)$ belongs to $L^2(\widetilde{\Omega} \times (0,\infty))$, since

$$
\begin{aligned}
\int_0^\infty \int_{\widetilde{\Omega}} \Big|\Big(e^{-t\sqrt{\widetilde{A}_{\mathrm{D}}}}\widetilde{u}\Big)(x)\Big|^2 dxdt &= \int_0^\infty \Big\|e^{-t\sqrt{\widetilde{A}_{\mathrm{D}}}}\widetilde{u}\Big\|^2 dt \\
&= \int_0^\infty \int_\mu^\infty \Big|e^{-t\sqrt{\lambda}}\Big|^2 d(E_\lambda \widetilde{u}, \widetilde{u})dt = \int_\mu^\infty \frac{1}{2\sqrt{\lambda}} d(E_\lambda \widetilde{u}, \widetilde{u}) < \infty.
\end{aligned}
$$

Now the uniform ellipticity of the differential expression $-\frac{\partial^2}{\partial t^2} + \widetilde{\mathcal{L}}$ and standard regularity theory imply that $(x,t) \mapsto T(t)\widetilde{u}(x)$ is locally in H^2 on $\widetilde{\Omega} \times (0,\infty)$, see, e.g., [100, Theorem 4.16]. Moreover, for any real numbers $a, b \notin \sigma_{\mathrm{p}}(\widetilde{A}_{\mathrm{D}})$, $a < b$, the Stone formula

$$
E((a,b))\widetilde{u} = \lim_{\varepsilon \searrow 0} \frac{1}{2\pi i}\int_a^b \Big(\big(\widetilde{A}_{\mathrm{D}} - (y + i\varepsilon)\big)^{-1}\widetilde{u} - \big(\widetilde{A}_{\mathrm{D}} - (y - i\varepsilon)\big)^{-1}\widetilde{u}\Big)dy
$$

for the spectral measure $E(\cdot)$ of $\widetilde{A}_\mathrm{D}$ together with (3.5) implies $(E((a,b))\widetilde{u})|_{\widetilde{\Omega}\setminus\Omega} = 0$. Thus, in particular, for each $t \geq 0$ we have

$$\big(T(t)\widetilde{u}\big)\big|_{\widetilde{\Omega}\setminus\Omega} = \Big(\int_\mu^\infty e^{-t\sqrt{\lambda}} dE(\lambda)\widetilde{u}\Big)\Big|_{\widetilde{\Omega}\setminus\Omega} = 0,$$

that is, $(x,t) \mapsto T(t)\widetilde{u}(x)$ vanishes on the nonempty, open subset $\mathcal{O} \times (0,\infty)$ of $\widetilde{\Omega} \times (0,\infty)$. Now (3.7) and Theorem 3.3 yield $T(t)\widetilde{u}(x) = 0$ for all $x \in \widetilde{\Omega}$, $t \in (0,\infty)$, i.e., $T(t)\widetilde{u}$ vanishes identically on $\widetilde{\Omega}$ for all $t > 0$. Thus, taking the limit $t \searrow 0$, we find $\widetilde{u} = 0$ and, hence, $u = 0$. This completes the proof. □

Remark 3.5. We point out that the statement of Proposition 3.4 can be improved in the following way. With the help of the identity theorem for holomorphic functions one can deduce that

$$\operatorname{span}\{\mathcal{N}_\lambda : \lambda \in D\}$$

is dense in $L^2(\Omega)$ for any subset D of $\rho(A_\mathrm{D})$ which has both an accumulation point in the upper and the lower open complex half-plane. We do not elaborate on the details, since we will not make use of this fact in the following.

Remark 3.6. The statement of Proposition 3.4 is equivalent to the fact that the symmetric restriction

$$Su = \mathcal{L}u, \quad \operatorname{dom} S = \Big\{u \in \operatorname{dom} A_\mathrm{D}, \frac{\partial u}{\partial \nu_\mathcal{L}}\big|_\omega = 0\Big\},$$

of the Dirichlet operator in $L^2(\Omega)$ is *simple* or *completely non-selfadjoint*; cf. [3, Chapter VII-81] and [86]. A more detailed discussion of this can be found in Appendix A.2.

3.2 An inverse problem for the Dirichlet operator with partial data

Let us now turn to the main results of this chapter. In order to consider the Dirichlet-to-Neumann map only on an arbitrary open, nonempty subset ω of $\partial\Omega$ we set

$$H^{1/2}_\omega = \big\{g \in H^{1/2}(\partial\Omega) : \operatorname{supp} g \subset \omega\big\}.$$

We first prove that the partial knowledge of the Dirichlet-to-Neumann map on ω determines the Dirichlet operator A_D associated with the elliptic differential expression $\mathcal{L}$ in (2.5) on a bounded or unbounded Lipschitz domain Ω uniquely up to unitary equivalence.

Theorem 3.7. *Let Ω be a connected Lipschitz domain, let $\omega \subset \partial\Omega$ be open and nonempty, and let $\mathcal{L}_1, \mathcal{L}_2$ be two differential expressions as in Assumption 2.1. Moreover, let $M_1(\lambda), M_2(\lambda)$ be the corresponding Dirichlet-to-Neumann maps and let $A^1_{\mathrm{D}}, A^2_{\mathrm{D}}$ be the corresponding Dirichlet operators as in* (2.5)*. Assume that $\mathcal{D} \subset \rho(A^1_{\mathrm{D}}) \cap \rho(A^2_{\mathrm{D}})$ is a set with an accumulation point in $\rho(A^1_{\mathrm{D}}) \cap \rho(A^2_{\mathrm{D}})$ and that*

$$(M_1(\lambda)g, h)_{\partial\Omega} = (M_2(\lambda)g, h)_{\partial\Omega}, \qquad g, h \in H^{1/2}_\omega,$$

holds for all $\lambda \in \mathcal{D}$. Then A^1_{D} and A^2_{D} are unitarily equivalent.

Proof. Note first that by Lemma 3.1 (iv) the functions

$$\rho(A^i_{\mathrm{D}}) \ni \lambda \mapsto (M_i(\lambda)g, h)_{\partial\Omega}, \quad i = 1, 2,$$

are holomorphic for all $g, h \in H^{1/2}_\omega$. Thus, it follows from the assumption of the theorem, that these functions do not only coincide on the set $\mathcal{D}$ but on the whole set $\rho(A^1_{\mathrm{D}}) \cap \rho(A^2_{\mathrm{D}})$, i.e.,

$$(M_1(\lambda)g, h)_{\partial\Omega} = (M_2(\lambda)g, h)_{\partial\Omega}, \qquad g, h \in H^{1/2}_\omega,$$

holds for all $\lambda \in \rho(A^1_{\mathrm{D}}) \cap \rho(A^2_{\mathrm{D}})$ and, in particular, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ be the Poisson operators associated with $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, as in (3.1). For $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \lambda \neq \overline{\mu}$, it follows from Lemma 3.1 (iii) that

$$\begin{aligned}(\gamma_1(\lambda)g, \gamma_1(\mu)h) &= \frac{(M_1(\lambda)g, h)_{\partial\Omega} - (M_1(\overline{\mu})g, h)_{\partial\Omega}}{\lambda - \overline{\mu}} \\ &= \frac{(M_2(\lambda)g, h)_{\partial\Omega} - (M_2(\overline{\mu})g, h)_{\partial\Omega}}{\lambda - \overline{\mu}} = (\gamma_2(\lambda)g, \gamma_2(\mu)h) \qquad (3.8)\end{aligned}$$

holds for all $g, h \in H^{1/2}_\omega$. Let us define a linear mapping V in $L^2(\Omega)$ on

$$\operatorname{dom} V = \operatorname{span}\left\{\gamma_1(\lambda)g : g \in H^{1/2}_\omega, \lambda \in \mathbb{C} \setminus \mathbb{R}\right\}$$

by

$$V\Big(\sum_{j=1}^k \gamma_1(\lambda_j)g_j\Big) := \sum_{j=1}^k \gamma_2(\lambda_j)g_j, \quad \lambda_j \in \mathbb{C} \setminus \mathbb{R},\ g_j \in H^{1/2}_\omega,\ 1 \leq j \leq k.$$

It follows from the identity (3.8) that V is a well-defined, isometric operator in $L^2(\Omega)$. Moreover, by Proposition 3.4 the set

$$\operatorname{span}\left\{\gamma_i(\lambda)g : \lambda \in \mathbb{C} \setminus \mathbb{R}, g \in H^{1/2}_\omega\right\} \qquad (3.9)$$

is dense in $L^2(\Omega)$, $i = 1, 2$, that is, V is densely defined and has a dense range in $L^2(\Omega)$. Hence V extends by continuity to a unitary operator U in $L^2(\Omega)$, which, clearly, satisfies $U\gamma_1(\lambda)g = \gamma_2(\lambda)g$ for all $g \in H_\omega^{1/2}$ and all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Let $\mu \in \mathbb{R} \cap \rho(A_\mathrm{D}^1) \cap \rho(A_\mathrm{D}^2)$. Then Lemma 3.1 (ii) yields

$$\begin{aligned} U(A_\mathrm{D}^1 - \mu)^{-1}\gamma_1(\lambda)g &= \frac{U\gamma_1(\mu)g - U\gamma_1(\lambda)g}{\mu - \lambda} \\ &= \frac{\gamma_2(\mu)g - \gamma_2(\lambda)g}{\mu - \lambda} = (A_\mathrm{D}^2 - \mu)^{-1}U\gamma_1(\lambda)g \end{aligned}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and all $g \in H_\omega^{1/2}$. Using again (3.9) we conclude

$$U(A_\mathrm{D}^1 - \mu)^{-1} = (A_\mathrm{D}^2 - \mu)^{-1}U,$$

thus $U(\operatorname{dom} A_\mathrm{D}^1) = \operatorname{dom} A_\mathrm{D}^2$ and $UA_\mathrm{D}^1 u = A_\mathrm{D}^2 U u$ for all $u \in \operatorname{dom} A_\mathrm{D}^1$. Therefore A_D^1 and A_D^2 are unitarily equivalent. □

Let us now discuss how the Dirichlet operator A_D can be recovered from the knowledge of the corresponding Dirichlet-to-Neumann map $M(\lambda)$ on $\omega \subset \partial\Omega$. Here we will assume that Ω is bounded. Recall that in this case the spectrum of A_D consists of isolated eigenvalues with finite multiplicities only, see Theorem 2.3; in particular, the resolvent of A_D is a meromorphic operator-valued function, whose poles are of order one, and it follows from Lemma 3.1 (iv) that the same holds for the function $M(\cdot)$. We define the residue $\operatorname{Res}_\lambda^\omega M : H_\omega^{1/2} \to (H_\omega^{1/2})'$ of $M(\cdot)$ *on* ω at some $\lambda \in \mathbb{R}$ by

$$(\operatorname{Res}_\lambda^\omega Mg, h)_\omega := (\operatorname{Res}_\lambda Mg, h)_{\partial\Omega}, \quad g, h \in H_\omega^{1/2},$$

where $\operatorname{Res}_\lambda M : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is the usual residue of the operator-valued function $M(\cdot)$ at λ and $(\cdot,\cdot)_\omega$ denotes the duality between $H_\omega^{1/2}$ and its dual space $(H_\omega^{1/2})'$; for more details on the residue of a meromorphic operator function see Section 1.1. Moreover, we say that $M(\cdot)$ *has a pole on* ω *at* λ if the operator $\operatorname{Res}_\lambda^\omega M$ is nontrivial. Let us finally define the restriction of some $h \in H^{-1/2}(\partial\Omega)$ to ω as an element of $(H_\omega^{1/2})'$ by

$$(h|_\omega, g)_\omega := (h, g)_{\partial\Omega}, \quad g \in H_\omega^{1/2}.$$

The Dirichlet operator A_D can be recovered from the partial knowledge of $M(\lambda)$ on ω as follows.

Theorem 3.8. *Let Assumption 2.1 be satisfied, let Ω be bounded and connected, and let $\omega \subset \partial\Omega$ be an open, nonempty set. Moreover, let A_{D} be the Dirichlet operator in* (2.5) *and let $M(\lambda)$ be the Dirichlet-to-Neumann map in* (2.9)*, $\lambda \in \rho(A_{\mathrm{D}})$. Then the eigenvalues of A_{D} coincide with the poles of $M(\cdot)$ on ω. For each eigenvalue λ_k of A_{D} the mapping*

$$\tau_k : \ker(A_{\mathrm{D}} - \lambda_k) \to \operatorname{ran} \operatorname{Res}^{\omega}_{\lambda_k} M, \quad u \mapsto \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\omega}$$

is an isomorphism. In particular, there exist $g_1^{(k)}, \dots, g_{n(k)}^{(k)} \in H^{1/2}_{\omega}$ such that

$$e_i^{(k)} := \tau_k^{-1} \left(\operatorname{Res}^{\omega}_{\lambda_k} M\right) g_i^{(k)}, \quad i = 1, \dots, n(k),$$

form an orthonormal basis of $\ker(A_{\mathrm{D}} - \lambda_k)$ *and the identity*

$$A_{\mathrm{D}} u = \sum_{k=1}^{\infty} \lambda_k \sum_{i=1}^{n(k)} (u, e_i^{(k)}) e_i^{(k)}, \quad u \in \operatorname{dom} A_{\mathrm{D}},$$

holds.

Proof. **Step 1.** Let λ_k, $k \in \mathbb{N}$, be the distinct eigenvalues of A_{D} as in the theorem. In this first step of the proof we show that for each $k \in \mathbb{N}$ the mapping

$$\tau_k : \ker(A_{\mathrm{D}} - \lambda_k) \to \operatorname{ran} \operatorname{Res}^{\omega}_{\lambda_k} M, \quad u \mapsto \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\omega},$$

is an isomorphism. First we observe that τ_k is injective. Indeed, assume $u \in \ker(A_{\mathrm{D}} - \lambda_k)$ satisfies $\tau_k u = 0$, that is $\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\omega} = 0$. Moreover, let $\mu \in \mathbb{C} \setminus \mathbb{R}$, and let $v_\mu \in \mathcal{N}_\mu$, i.e., $\mathcal{L} v_\mu = \mu v_\mu$ and $\operatorname{supp}(v_\mu|_{\partial\Omega}) \subset \omega$; cf. (3.4). Then the second Green identity (1.11) yields

$$\begin{aligned}(\lambda_k - \overline{\mu})(u, v_\mu) &= (A_{\mathrm{D}} u, v_\mu) - (u, \mathcal{L} v_\mu) \\ &= \Big(u|_{\partial\Omega}, \frac{\partial v_\mu}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}\Big)_{\partial\Omega} - \Big(\frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, v_\mu|_{\partial\Omega}\Big)_{\partial\Omega} = 0,\end{aligned}$$

since $u|_{\partial\Omega} = 0$, $\operatorname{supp}(v_\mu|_{\partial\Omega}) \subset \omega$, and $\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\omega} = 0$. Thus with the help of Proposition 3.4 it follows $u = 0$, that is, τ_k is injective.

In order to prove the surjectivity let us fix some $\mu \in \mathbb{R} \cap \rho(A_{\mathrm{D}})$ and note that for $u \in \ker(A_{\mathrm{D}} - \lambda_k)$ Lemma 3.1 (i) yields

$$\begin{aligned}\tau_k u &= \tau_k \left((A_{\mathrm{D}} - \mu)^{-1} A_{\mathrm{D}} u - (A_{\mathrm{D}} - \mu)^{-1} \mu u\right) \\ &= (\lambda_k - \mu) \frac{\partial}{\partial \nu_{\mathcal{L}}} \left((A_{\mathrm{D}} - \mu)^{-1} u\right)\Big|_{\omega} = (\mu - \lambda_k)(\gamma(\mu)^* u)|_{\omega},\end{aligned}$$

where $\gamma(\mu)$ is the Poisson operator in (3.1). Consequently, for the surjectivity of τ_k it is sufficient to ensure

$$\operatorname{ran}\operatorname{Res}^{\omega}_{\lambda_k} M = \left\{ \big(\gamma(\mu)^* u\big)|_\omega : u \in \ker(A_{\mathrm{D}} - \lambda_k) \right\}. \tag{3.10}$$

For the inclusion $\subset$ in (3.10) denote by P the orthogonal projection in $L^2(\Omega)$ onto $\ker(A_{\mathrm{D}} - \lambda_k)$. Let us choose $\eta \in \mathbb{C} \setminus \mathbb{R}$ and an open ball $\mathcal{O}$ centered in λ_k such that η and μ do not belong to $\overline{\mathcal{O}}$ and such that $\sigma(A_{\mathrm{D}}) \cap \mathcal{O} = \{\lambda_k\}$. If Γ denotes the boundary of $\mathcal{O}$ then with the help of the identity (1.2) and of Lemma 3.1 (ii) and (iii) we obtain

$$\begin{aligned}
(P\gamma(\eta)g, \gamma(\mu)h) &= -\frac{1}{2\pi i}\int_\Gamma \big((A_{\mathrm{D}} - \zeta)^{-1}\gamma(\eta)g, \gamma(\mu)h\big)\, d\zeta \\
&= -\frac{1}{2\pi i}\int_\Gamma \left(\frac{1}{\zeta - \eta}(\gamma(\zeta)g, \gamma(\mu)h) - \frac{1}{\zeta - \eta}(\gamma(\eta)g, \gamma(\mu)h) \right) d\zeta \\
&= \frac{1}{2\pi i}\int_\Gamma \left(\frac{(M(\zeta)g, h)_{\partial\Omega}}{(\eta - \zeta)(\zeta - \mu)} + \frac{(g, M(\mu)h)_{\partial\Omega}}{(\mu - \zeta)(\eta - \mu)} + \frac{(M(\eta)g, h)_{\partial\Omega}}{(\zeta - \eta)(\eta - \mu)} \right) d\zeta
\end{aligned}$$

for $g, h \in H^{1/2}_\omega$; cf. the formulas in [51, §I.1]. The second and third fraction under the integral on the right-hand side are holomorphic in a neighborhood of $\mathcal{O}$ as functions of ζ and, hence, their integrals vanish. Note that by Lemma 3.1 (iv) the function $M(\cdot)$ is either analytic in $\mathcal{O}$ or has a pole of order one at λ_k. Together with the fact that $\zeta \mapsto \frac{1}{(\eta-\zeta)(\zeta-\mu)}$ is holomorphic in $\mathcal{O}$ we obtain

$$(P\gamma(\eta)g, \gamma(\mu)h) = \frac{(\operatorname{Res}_{\lambda_k} Mg, h)_{\partial\Omega}}{(\eta - \lambda_k)(\lambda_k - \mu)};$$

cf. (1.1). It follows

$$(\operatorname{Res}_{\lambda_k} Mg, h)_{\partial\Omega} = (\eta - \lambda_k)(\lambda_k - \mu)(P\gamma(\eta)g, \gamma(\mu)h), \quad g, h \in H^{1/2}_\omega,$$

and, in particular,

$$\operatorname{Res}^{\omega}_{\lambda_k} Mg = (\eta - \lambda_k)(\lambda_k - \mu)\big(\gamma(\mu)^* P\gamma(\eta)g\big)|_\omega, \quad g \in H^{1/2}_\omega. \tag{3.11}$$

This implies the inclusion $\subset$ in (3.10).

For the proof of the second inclusion in (3.10) let $u \in \ker(A_{\mathrm{D}} - \lambda_k)$ and let $\varepsilon > 0$. Since $\gamma(\mu)^* P$ is a bounded operator from $L^2(\Omega)$ to $H^{-1/2}(\partial\Omega)$, there exists $\delta > 0$ such that $\|u - v\|_{L^2(\Omega)} < \delta$ implies $\|\gamma(\mu)^* Pu - \gamma(\mu)^* Pv\|_{H^{-1/2}(\partial\Omega)} < \varepsilon$. According to Proposition 3.4 the set

$$\operatorname{span}\left\{ \gamma(\eta)g : \eta \in \mathbb{C} \setminus \mathbb{R}, g \in H^{1/2}_\omega \right\}$$

is dense in $L^2(\Omega)$, hence there exist $l \in \mathbb{N}$, $\eta_j \in \mathbb{C} \setminus \mathbb{R}$ and $g_j \in H_\omega^{1/2}$, $1 \le j \le l$, such that

$$\Big\| u - \sum_{j=1}^{l} \gamma(\eta_j) g_j \Big\|_{L^2(\Omega)} < \delta$$

and, consequently,

$$\Big\| \gamma(\mu)^* P u - \gamma(\mu)^* P \sum_{j=1}^{l} \gamma(\eta_j) g_j \Big\|_{H^{-1/2}(\partial\Omega)} < \varepsilon.$$

Since $u \in \ker(A_\mathrm{D} - \lambda_k)$, we have $Pu = u$. Moreover, the mapping $H^{-1/2}(\partial\Omega) \ni h \mapsto h|_\omega \in (H_\omega^{1/2})'$ is continuous with norm less than one, and with the help of the identity (3.11) it follows

$$\Big\| \big(\gamma(\mu)^* u\big)|_\omega - \sum_{j=1}^{l} \frac{\operatorname{Res}^\omega_{\lambda_k} M g_j}{(\eta_j - \lambda_k)(\lambda_k - \mu)} \Big\|_{(H_\omega^{1/2})'} < \varepsilon,$$

hence, $(\gamma(\mu)^* u)|_\omega$ belongs to the closure of $\operatorname{ran} \operatorname{Res}^\omega_{\lambda_k} M$. Since $\ker(A_\mathrm{D} - \lambda_k)$ is finite-dimensional, the inclusion $\subset$ in (3.10) implies that also the dimension of $\operatorname{ran} \operatorname{Res}^\omega_{\lambda_k} M$ is finite. Thus

$$(\gamma(\mu)^* u)|_\omega \in \operatorname{ran} \operatorname{Res}^\omega_{\lambda_k} M$$

and we have proved the equality (3.10). Therefore τ_k is a bijective linear mapping between finite-dimensional spaces, and, hence, an isomorphism. From this it follows immediately that each eigenvalue of A_D is a pole of $M(\cdot)$ on ω. On the other hand it follows from Lemma 3.1 (iv) that $M(\cdot)$ is holomorphic on $\rho(A_\mathrm{D})$. Hence $\sigma(A_\mathrm{D})$ coincides with the set of poles of $M(\cdot)$ on ω.

Step 2. In this step we prove the statement on the representation of A_D. Since τ_k is an isomorphism for each $k \in \mathbb{N}$, there exist $g_1^{(k)}, \dots, g_{n(k)}^{(k)} \in H_\omega^{1/2}$, $n(k) = \dim\ker(A_\mathrm{D} - \lambda_k) < \infty$, such that the functions $e_i^{(k)}$, $i = 1, \dots, n(k)$, defined as in the theorem, form an orthonormal basis of $\ker(A_\mathrm{D} - \lambda_k)$. Since the spectrum of A_D consist only of the isolated eigenvalues λ_k, it follows that

$$A_\mathrm{D} u = \sum_{k=1}^{\infty} \lambda_k \sum_{i=1}^{n(k)} (u, e_i^{(k)}) e_i^{(k)}, \quad u \in \operatorname{dom} A_\mathrm{D},$$

holds. This completes the proof of the theorem. □

Remark 3.9. The connectedness assumption on Ω in the theorems of this section can be slightly relaxed. The same proof shows that it suffices to require that $\omega \cap \partial\mathcal{O}$ is nonempty for each connected component $\mathcal{O}$ of Ω.

3.3 Inverse problems for generalized Robin operators

In this section we carry over the results of the previous section to realizations A_Θ in $L^2(\Omega)$ of the uniformly elliptic differential expression $\mathcal{L}$ subject to generalized Robin boundary conditions,

$$A_\Theta u = \mathcal{L}u, \quad \operatorname{dom} A_\Theta = \left\{ u \in H^1(\Omega), \mathcal{L}u \in L^2(\Omega), \frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u|_{\partial\Omega} = 0 \right\},$$

see (2.10). Here $\Theta : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is an operator which satisfies Assumption 2.6 from Chapter 2 above, that is, $\Theta = \Theta_1 + \Theta_2$, where $\Theta_i : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ are bounded operators with

$$(\Theta_i g, h)_{\partial\Omega} = (g, \Theta_i h)_{\partial\Omega}, \quad g, h \in H^{1/2}(\partial\Omega),$$

$i = 1, 2$, such that Θ_1 is L^2-semibounded, i.e.,

$$(\Theta_1 g, g)_{\partial\Omega} \geq c_{\Theta_1} \|g\|^2_{L^2(\partial\Omega)}, \quad g \in H^{1/2}(\partial\Omega),$$

for some $c_{\Theta_1} \in \mathbb{R}$, and Θ_2 is compact. It follows from Theorem 2.8 that A_Θ is a selfadjoint operator in $L^2(\Omega)$. By Lemma 2.9 the Robin-to-Dirichlet map

$$M_\Theta \left(\frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u_\lambda|_{\partial\Omega} \right) = u_\lambda|_{\partial\Omega}, \quad \mathcal{L}u_\lambda = \lambda u_\lambda,$$

is well-defined for each $\lambda \in \rho(A_\Theta)$, and it can alternatively be expressed as

$$M_\Theta(\lambda) = (\Theta - M(\lambda))^{-1}, \quad \lambda \in \rho(A_\Theta) \cap \rho(A_{\mathrm{D}}),$$

see (2.22). In this section we show that the knowledge of the mapping $M_\Theta(\lambda)$ for an appropriate set of points λ determines the operator A_Θ uniquely up to unitary equivalence. Additionally we provide a reconstruction result in the case that the domain Ω is bounded. We first restrict ourselves to the case that the Robin-to-Dirichlet map $M_\Theta(\lambda)$ is given on the whole boundary $\partial\Omega$. Afterwards we show that under additional conditions on Θ this assumption can be relaxed. We provide a uniqueness result under local knowledge of the Robin-to-Dirichlet map in the case that Θ is a multiplication operator, i.e., the functions in the domain of A_Θ satisfy a local, classical Robin boundary condition.

In order to develop an analog of Lemma 3.1 for A_Θ and $M_\Theta(\lambda)$ instead of A_{D} and $M(\lambda)$, respectively, we introduce the *Poisson operator for the Robin problem*

$$\gamma_\Theta(\lambda) : H^{-1/2}(\partial\Omega) \to L^2(\Omega), \quad g \mapsto u_\lambda, \tag{3.12}$$

where u_λ is the unique solution of the boundary value problem

$$\mathcal{L}u = \lambda u, \quad \frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u|_{\partial\Omega} = g$$

for a given $g \in H^{-1/2}(\partial\Omega)$; cf. Lemma 2.9. Then the following holds.

Lemma 3.10. *Let Assumption 2.1 and Assumption 2.6 be satisfied. Moreover, let* $\lambda, \mu \in \rho(A_\Theta)$*, let* $M_\Theta(\lambda), M_\Theta(\mu)$ *be the Robin-to-Dirichlet maps in* (2.21) *and let* $\gamma_\Theta(\lambda), \gamma_\Theta(\mu)$ *be given in* (3.12)*. Then the following assertions hold.*

(i) $\gamma_\Theta(\lambda)$ *is a bounded operator and its adjoint* $\gamma_\Theta(\lambda)^* : L^2(\Omega) \to H^{1/2}(\partial\Omega)$ *is given by*

$$\gamma_\Theta(\lambda)^* u = \big((A_\Theta - \overline{\lambda})^{-1} u\big)\,|_{\partial\Omega}, \quad u \in L^2(\Omega).$$

(ii) *The identity*

$$\gamma_\Theta(\lambda) = \big(I + (\lambda - \mu)(A_\Theta - \lambda)^{-1}\big)\,\gamma_\Theta(\mu)$$

holds.

(iii) *We have*

$$(\lambda - \overline{\mu})\gamma_\Theta(\mu)^* \gamma_\Theta(\lambda) = M_\Theta(\lambda) - M_\Theta(\overline{\mu}),$$

and $(M_\Theta(\lambda)g, h)_{\partial\Omega} = (g, M_\Theta(\overline{\lambda})h)_{\partial\Omega}$ *holds for all* $g, h \in H^{-1/2}(\partial\Omega)$*.*

(iv) $M_\Theta(\lambda)$ *is a bounded operator from* $H^{-1/2}(\partial\Omega)$ *to* $H^{1/2}(\partial\Omega)$*, which satisfies*

$$M_\Theta(\lambda) = M_\Theta(\overline{\mu}) + (\lambda - \overline{\mu})\gamma_\Theta(\mu)^* \big(I + (\lambda - \mu)(A_\Theta - \lambda)^{-1}\big)\,\gamma_\Theta(\mu). \quad (3.13)$$

In particular, $\lambda \mapsto M_\Theta(\lambda)$ *is analytic on* $\rho(A_\Theta)$*.*

Proof. (i) Let $\lambda \in \rho(A_\Theta)$, let $g \in H^{-1/2}(\partial\Omega)$, and let $u \in L^2(\Omega)$. Moreover, let $u_\lambda = \gamma_\Theta(\lambda)g$, that is, $\mathcal{L}u_\lambda = \lambda u_\lambda$ and $\frac{\partial u_\lambda}{\partial \nu_\mathcal{L}}|_{\partial\Omega} + \Theta u_\lambda|_{\partial\Omega} = g$, and let $v = (A_\Theta - \overline{\lambda})^{-1}u$. Then the second Green identity (1.11) yields

$$\begin{aligned}
(\gamma_\Theta(\lambda)g, u) &= (u_\lambda, (A_\Theta - \overline{\lambda})v) = (u_\lambda, \mathcal{L}v) - (\mathcal{L}u_\lambda, v) \\
&= \Big(\frac{\partial u_\lambda}{\partial \nu_\mathcal{L}}|_{\partial\Omega}, v|_{\partial\Omega}\Big)_{\partial\Omega} - \Big(u_\lambda|_{\partial\Omega}, \frac{\partial v}{\partial \nu_\mathcal{L}}|_{\partial\Omega}\Big)_{\partial\Omega} \\
&= \Big(\frac{\partial u_\lambda}{\partial \nu_\mathcal{L}}|_{\partial\Omega} + \Theta u_\lambda|_{\partial\Omega}, v|_{\partial\Omega}\Big)_{\partial\Omega} - \Big(u_\lambda|_{\partial\Omega}, \frac{\partial v}{\partial \nu_\mathcal{L}}|_{\partial\Omega} + \Theta v|_{\partial\Omega}\Big)_{\partial\Omega} \\
&= (g, v|_{\partial\Omega})_{\partial\Omega} = \big(g, \big((A_\Theta - \overline{\lambda})^{-1}u\big)|_{\partial\Omega}\big)_{\partial\Omega}.
\end{aligned}$$

From this it follows with the closed graph theorem that $\gamma_\Theta(\lambda)$ is bounded and satisfies $\gamma_\Theta(\lambda)^* u = ((A_\Theta - \overline{\lambda})^{-1}u)|_{\partial\Omega}$.

The proof of item (ii) is analogous to the proof of Lemma 3.1 (ii) and will be omitted.

(iii) Let $\lambda,\mu \in \rho(A_\Theta)$, let $g,h \in H^{-1/2}(\partial\Omega)$, and let $u_\lambda = \gamma_\Theta(\lambda)g$ and $v_\mu = \gamma_\Theta(\mu)h$. Then we have

$$\begin{aligned}(\lambda-\overline{\mu})(\gamma_\Theta(\lambda)g,\gamma_\Theta(\mu)h) &= (\mathcal{L}u_\lambda, v_\mu) - (u_\lambda, \mathcal{L}v_\mu)\\ &= \Big(u_\lambda|_{\partial\Omega}, \frac{\partial v_\mu}{\partial\nu_{\mathcal{L}}}|_{\partial\Omega}\Big)_{\partial\Omega} - \Big(\frac{\partial u_\lambda}{\partial\nu_{\mathcal{L}}}|_{\partial\Omega}, v_\mu|_{\partial\Omega}\Big)_{\partial\Omega}\\ &= \Big(u_\lambda|_{\partial\Omega}, \frac{\partial v_\mu}{\partial\nu_{\mathcal{L}}}|_{\partial\Omega} + \Theta v_\mu|_{\partial\Omega}\Big)_{\partial\Omega} - \Big(\frac{\partial u_\lambda}{\partial\nu_{\mathcal{L}}}|_{\partial\Omega} + \Theta u_\lambda|_{\partial\Omega}, v_\mu|_{\partial\Omega}\Big)_{\partial\Omega}\\ &= (u_\lambda|_{\partial\Omega}, h)_{\partial\Omega} - (g, v_\mu|_{\partial\Omega})_{\partial\Omega}\\ &= (M_\Theta(\lambda)g,h)_{\partial\Omega} - (g, M_\Theta(\mu)h)_{\partial\Omega}. \end{aligned}\tag{3.14}$$

With $\mu = \overline{\lambda}$ it follows $(M_\Theta(\lambda)g,h)_{\partial\Omega} = (g, M_\Theta(\overline{\lambda})h)_{\partial\Omega}$. From this and (3.14) we obtain the remaining statement of (iii).

The assertions of item (iv) follow from (ii) and (iii) analogously to the proof of Lemma 3.1 (iv). □

The following two theorems show that the knowledge of the Robin-to-Dirichlet map determines the operator A_Θ in (2.10) uniquely up to unitary equivalence and that A_Θ can be recovered from the knowledge of $M_\Theta(\lambda)$ in the case that Ω is bounded. We do not carry out the proofs of these theorems. They are analogous to the proofs of Theorem 3.7 and Theorem 3.8 in the previous section in the case $\omega = \partial\Omega$, where one has to replace the use of Lemma 3.1 by the application of Lemma 3.10. Moreover, in order to admit Lipschitz domains Ω which are not necessarily connected we replace Proposition 3.4 (with $\omega = \partial\Omega$) by the following slightly generalized variant.

Proposition 3.11. *Let Assumption 2.1 be satisfied and let*

$$\mathcal{N}_\lambda = \left\{u \in H^1(\Omega) : \mathcal{L}u = \lambda u\right\}, \quad \lambda \in \rho(A_{\mathrm{D}}).\tag{3.15}$$

Then the space

$$\operatorname{span}\{\mathcal{N}_\lambda : \lambda \in \mathbb{C}\setminus\mathbb{R}\}$$

is dense in $L^2(\Omega)$.

Proof. Let $u \in L^2(\Omega)$ be orthogonal to $\mathcal{N}_\lambda$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and let $\mathcal{O}$ be a connected component of Ω. Let $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and set

$$\mathcal{N}_\lambda^{\mathcal{O}} = \left\{v \in H^1(\mathcal{O}) : \mathcal{L}v = \lambda v\right\}.$$

For each $v_\lambda \in \mathcal{N}_\lambda^{\mathcal{O}}$ we define

$$\widetilde{v}_\lambda = \begin{cases} v_\lambda & \text{on } \mathcal{O},\\ 0 & \text{on } \Omega\setminus\mathcal{O}.\end{cases}$$

Then $\widetilde{v}_\lambda$ belongs to $H^1(\Omega)$ and satisfies $\mathcal{L}\widetilde{v}_\lambda = \lambda\widetilde{v}_\lambda$, that is, $\widetilde{v}_\lambda \in \mathcal{N}_\lambda$. In particular,

$$0 = (u, \widetilde{v}_\lambda) = (u|_{\mathcal{O}}, v_\lambda)_{L^2(\mathcal{O})},$$

hence $u|_{\mathcal{O}}$ is orthogonal to $\mathcal{N}_\lambda^{\mathcal{O}}$ for all $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and Proposition 3.4 yields $u|_{\mathcal{O}} = 0$. Since the connected component $\mathcal{O}$ was chosen arbitrarily, it follows $u = 0$. □

The uniqueness result for A_Θ is the following.

Theorem 3.12. *Let Ω be a Lipschitz domain, let $\mathcal{L}_1, \mathcal{L}_2$ be two differential expressions as in Assumption 2.1, and let Θ satisfy Assumption 2.6. Moreover, let $M_\Theta^1(\lambda), M_\Theta^2(\lambda)$ be the corresponding Robin-to-Dirichlet maps as in* (2.21) *and let A_Θ^1, A_Θ^2 be the corresponding Robin operators as in* (2.10). *Assume that $\mathcal{D} \subset \rho(A_\Theta^1) \cap \rho(A_\Theta^2)$ is a set with an accumulation point in $\rho(A_\Theta^1) \cap \rho(A_\Theta^2)$ and that*

$$(M_\Theta^1(\lambda)g, h)_{\partial\Omega} = (M_\Theta^2(\lambda)g, h)_{\partial\Omega}, \qquad g, h \in H^{-1/2}(\partial\Omega),$$

holds for all $\lambda \in \mathcal{D}$. Then A_Θ^1 and A_Θ^2 are unitarily equivalent.

In case Ω is bounded, A_Θ can be recovered from the knowledge of $M_\Theta(\lambda)$ as follows.

Theorem 3.13. *Let Ω and $\mathcal{L}$ be as in Assumption 2.1 and let, additionally, Ω be bounded. Moreover, let Θ satisfy Assumption 2.6, let A_Θ be the selfadjoint operator in* (2.5), *and let $M_\Theta(\lambda)$ be the Robin-to-Dirichlet map in* (2.21), *$\lambda \in \rho(A_\Theta)$. Then the eigenvalues of A_Θ coincide with the poles of $M_\Theta(\cdot)$. For each eigenvalue λ_k of A_Θ the mapping*

$$\tau_k : \ker(A_\Theta - \lambda_k) \to \operatorname{ran} \operatorname{Res}_{\lambda_k} M_\Theta, \quad u \mapsto u|_{\partial\Omega}$$

is an isomorphism. In particular, there exist $g_1^{(k)}, \dots, g_{n(k)}^{(k)} \in H^{-1/2}(\partial\Omega)$ such that

$$e_i^{(k)} := \tau_k^{-1} \left(\operatorname{Res}_{\lambda_k} M_\Theta\right) g_i^{(k)}, \quad i = 1, \dots, n(k),$$

form an orthonormal basis of $\ker(A_\Theta - \lambda_k)$ *and the identity*

$$A_\Theta u = \sum_{k=1}^{\infty} \lambda_k \sum_{i=1}^{n(k)} (u, e_i^{(k)}) e_i^{(k)}, \quad u \in \operatorname{dom} A_\Theta,$$

holds.

Let us finally come to the case of partial data. We now assume additionally that the operator Θ in the boundary condition has the form

$$\Theta g = \vartheta g, \quad g \in H^{1/2}(\partial\Omega), \tag{3.16}$$

where $\vartheta : \partial\Omega \to \mathbb{R}$ is a bounded, measurable function. Then $\Theta : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is a bounded operator with

$$(\Theta g, g)_{\partial\Omega} \geq \inf \vartheta \|g\|^2_{L^2(\partial\Omega)}, \quad g \in H^{1/2}(\partial\Omega),$$

and, hence, satisfies Assumption 2.6. Therefore A_Θ in (2.10) is selfadjoint by Theorem 2.8, the Robin-to-Dirichlet-map $M_\Theta(\lambda)$ in (2.21) is well-defined, and Lemma 3.10 is applicable. In order to prove that A_Θ is determined uniquely by the partial knowledge of $M_\Theta(\lambda)$ we need the following analog of Proposition 3.4.

Let again $\omega \subset \partial\Omega$ be a nonempty, relatively open set. For $\lambda \in \rho(A_\Theta)$ let

$$\mathcal{N}_\lambda^\Theta = \Big\{ u \in H^1(\Omega) : \mathcal{L}u = \lambda u, \ \operatorname{supp} \Big(\frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u|_{\partial\Omega} \Big) \subset \omega \Big\},$$

where the support of some $h \in H^{-1/2}(\partial\Omega)$ is the smallest closed set $\widetilde{\omega}$ such that $(h, g)_{\partial\Omega} = 0$ for all $g \in H^{1/2}(\partial\Omega)$ with $\operatorname{supp} g \subset \partial\Omega \setminus \widetilde{\omega}$.

Proposition 3.14. *Let Assumption 2.1 be satisfied and let Θ be given in* (3.16). *Then*

$$\operatorname{span} \big\{ \mathcal{N}_\lambda^\Theta : \lambda \in \mathbb{C} \setminus \mathbb{R} \big\}$$

is dense in $L^2(\Omega)$.

Proof. Let $\widetilde{\Omega}$ and $\widetilde{\mathcal{L}}$ be defined as in the proof of Proposition 3.4 above. Let us define a function $\widetilde{\vartheta} : \partial\widetilde{\Omega} \to \mathbb{R}$ by

$$\widetilde{\vartheta} = \begin{cases} \vartheta & \text{on } \partial\Omega \setminus \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widetilde{\vartheta}$ is measurable and bounded. If we set $\widetilde{\Theta} g = \widetilde{\vartheta} g$, $g \in H^{1/2}(\partial\widetilde{\Omega})$ then by Theorem 2.8 the operator

$$\widetilde{A}_{\widetilde{\Theta}} \widetilde{u} = \widetilde{\mathcal{L}} \widetilde{u}, \quad \operatorname{dom} \widetilde{A}_{\widetilde{\Theta}} = \left\{ \widetilde{u} \in H^1(\widetilde{\Omega}) : \widetilde{\mathcal{L}} \widetilde{u} \in L^2(\widetilde{\Omega}), \ \frac{\partial \widetilde{u}}{\partial \nu_{\widetilde{\mathcal{L}}}}\big|_{\partial\widetilde{\Omega}} + \widetilde{\Theta} \widetilde{u}|_{\partial\widetilde{\Omega}} = 0 \right\},$$

in $L^2(\widetilde{\Omega})$ is selfadjoint and semibounded from below. Let $\widetilde{v} \in L^2(\widetilde{\Omega})$ be such that $\widetilde{v}$ vanishes on Ω, and define

$$\widetilde{u}_{\lambda,\widetilde{v}} = (\widetilde{A}_{\widetilde{\Theta}} - \lambda)^{-1} \widetilde{v}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Moreover, denote by $u_{\lambda,\widetilde{v}}$ the restriction of $\widetilde{u}_{\lambda,\widetilde{v}}$ to Ω. Then, clearly, $\mathcal{L}u_{\lambda,\widetilde{v}} = \lambda u_{\lambda,\widetilde{v}}$. We check that the distribution $\frac{\partial u_{\lambda,\widetilde{v}}}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u_{\lambda,\widetilde{v}}|_{\partial\Omega}$ vanishes on $\partial\Omega \setminus \omega$. Indeed, let $g \in H^{1/2}(\partial\Omega)$ with $\operatorname{supp} g \subset \partial\Omega \setminus \omega$. By Proposition 1.5 there exists $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$. Let $\widetilde{v}$ denote the extension of v by zero to $\widetilde{\Omega}$. Since $v|_{\partial\Omega}$ is identically zero on ω, we have $\widetilde{v} \in H^1(\widetilde{\Omega})$ and $\operatorname{supp}(\widetilde{v}|_{\partial\widetilde{\Omega}}) \subset \partial\Omega \setminus \omega$. Moreover, by the definition (1.10) of the conormal derivative we have

$$\begin{aligned}
&\Big(\frac{\partial u_{\lambda,\widetilde{v}}}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u_{\lambda,\widetilde{v}}|_{\partial\Omega}, g\Big)_{\partial\Omega} \\
&\qquad = -(\mathcal{L}u_{\lambda,\widetilde{v}}, v)_{L^2(\Omega)} + \mathfrak{a}[u_{\lambda,\widetilde{v}}, v] + (\vartheta u_{\lambda,\widetilde{v}}|_{\partial\Omega}, v|_{\partial\Omega})_{L^2(\partial\Omega)} \\
&\qquad = -(\widetilde{\mathcal{L}}\widetilde{u}_{\lambda,\widetilde{v}}, \widetilde{v})_{L^2(\widetilde{\Omega})} + \widetilde{\mathfrak{a}}[\widetilde{u}_{\lambda,\widetilde{v}}, \widetilde{v}] + \int_{\partial\Omega\setminus\omega} \vartheta u_{\lambda,\widetilde{v}}|_{\partial\Omega}\overline{v|_{\partial\Omega}}\, d\sigma \\
&\qquad = \Big(\frac{\partial \widetilde{u}_{\lambda,\widetilde{v}}}{\partial \nu_{\widetilde{\mathcal{L}}}}\big|_{\partial\widetilde{\Omega}} + \widetilde{\Theta}\widetilde{u}_{\lambda,\widetilde{v}}|_{\partial\widetilde{\Omega}}, \widetilde{v}|_{\partial\widetilde{\Omega}}\Big)_{\partial\widetilde{\Omega}} = 0,
\end{aligned}$$

where we denoted by $\widetilde{\mathfrak{a}}$ the sesquilinear form corresponding to the differential expression $\widetilde{\mathcal{L}}$ on $\widetilde{\Omega}$ as in (1.9); hence $\operatorname{supp}(\frac{\partial u_{\lambda,\widetilde{v}}}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u_{\lambda,\widetilde{v}}|_{\partial\Omega}) \subset \omega$, that is, $u_{\lambda,\widetilde{v}} \in \mathcal{N}_\lambda^\Theta$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and all $\widetilde{v} \in L^2(\widetilde{\Omega})$ with $\widetilde{v}|_\Omega = 0$.

If we choose $u \in L^2(\Omega)$ being orthogonal to $\mathcal{N}_\lambda^\Theta$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and denote by $\widetilde{u}$ the extension of u by zero to $\widetilde{\Omega}$ then we obtain

$$0 = (u, u_{\overline{\lambda},\widetilde{v}})_{L^2(\Omega)} = \big(\widetilde{u}, (\widetilde{A}_{\widetilde{\Theta}} - \overline{\lambda})^{-1}\widetilde{v}\big)_{L^2(\widetilde{\Omega})} = \big((\widetilde{A}_{\widetilde{\Theta}} - \lambda)^{-1}\widetilde{u}, \widetilde{v}\big)_{L^2(\widetilde{\Omega})}$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and all $\widetilde{v} \in L^2(\widetilde{\Omega})$ which vanish on Ω, that is,

$$\big((\widetilde{A}_{\widetilde{\Theta}} - \lambda)^{-1}\widetilde{u}\big)|_{\widetilde{\Omega}\setminus\Omega} = 0$$

for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Proceeding further as in the proof of Proposition 3.4 we conclude $u = 0$, which leads to the statement of the proposition. □

The following theorem can be proved analogously to the proof of Theorem 3.7 with Proposition 3.14 and Lemma 3.10 instead of Proposition 3.4 and Lemma 3.1, respectively.

Theorem 3.15. *Let Ω be a connected Lipschitz domain, let $\omega \subset \partial\Omega$ be an open, nonempty set, let $\mathcal{L}_1, \mathcal{L}_2$ be two differential expressions as in Assumption 2.1, and let Θ be given in* (3.16)*. Moreover, let $M_\Theta^1(\lambda), M_\Theta^2(\lambda)$ be the corresponding Robin-to-Dirichlet maps and let A_Θ^1, A_Θ^2 be the corresponding Robin operators as in* (2.10)*. Assume that $\mathcal{D} \subset \rho(A_\Theta^1) \cap \rho(A_\Theta^2)$ is a set with an accumulation point in $\rho(A_\Theta^1) \cap \rho(A_\Theta^2)$ and that*

$$(M_\Theta^1(\lambda)g, h)_{\partial\Omega} = (M_\Theta^2(\lambda)g, h)_{\partial\Omega}, \qquad g, h \in H^{-1/2}(\partial\Omega), \quad \operatorname{supp} g, h \subset \omega,$$

holds for all $\lambda \in \mathcal{D}$. Then A_Θ^1 and A_Θ^2 are unitarily equivalent.

Note that Remark 3.9 also applies to Theorem 3.15.

4 Titchmarsh–Weyl theory for elliptic differential operators

In this chapter we turn to the second main objective of the present thesis. We develop an approach to the spectral theory of selfadjoint elliptic differential operators which generalizes results of the classical Titchmarsh–Weyl theory for selfadjoint ordinary differential operators. It is a well-known fact, see [41, 121], that the spectra of the selfadjoint realizations of singular Sturm–Liouville differential expressions can be recovered from the limiting behavior of the Titchmarsh–Weyl m-function towards the real axis. In the present chapter we generalize these results to selfadjoint partial, elliptic differential operators. We consider a uniformly elliptic, formally symmetric differential expression

$$\mathcal{L} = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial}{\partial x_k} + \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{a_j} \right) + a$$

on a (bounded or unbounded) Lipschitz domain Ω as in Assumption 2.1. The function $\lambda \mapsto M(\lambda)$, where $M(\lambda)$ is the Dirichlet-to-Neumann map

$$M(\lambda) u_\lambda|_{\partial\Omega} = -\frac{\partial u_\lambda}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \quad \mathcal{L} u_\lambda = \lambda u_\lambda,$$

in (2.9), is the natural multidimensional analog of the Titchmarsh–Weyl m-function. In the main theorems of this section we prove that the whole spectral data of the selfadjoint Dirichlet operator

$$A_{\mathrm{D}} u = \mathcal{L} u, \quad \operatorname{dom} A_{\mathrm{D}} = \left\{ u \in H^1(\Omega) : \mathcal{L} u \in L^2(\Omega), u|_{\partial\Omega} = 0 \right\},$$

in $L^2(\Omega)$, see (2.5), is encoded in the function $M(\cdot)$. Particularly, we give a complete description of all isolated and embedded eigenvalues and of the absolutely continuous spectrum of A_{D} in terms of the limiting behavior of $M(\cdot)$ towards the real line and we prove a sufficient criterion for the absence of singular continuous spectrum. In the second part of this chapter we provide generalizations of these results to the case of a partial knowledge of the Dirichlet-to-Neumann map and to further selfadjoint realizations of $\mathcal{L}$ with (in general) nonlocal boundary conditions of Robin type. Parts of the results of the present chapter were published in [27].

4.1 A characterization of the Dirichlet spectrum

In this section we describe the complete spectrum of the Dirichlet operator A_{D} by means of the behavior of the Dirichlet-to-Neumann map $M(\lambda)$ for λ close to the real axis. In particular, we characterize the isolated and embedded eigenvalues of A_{D} together with the corresponding eigenspaces and the absolutely continuous spectrum and give a sufficient condition for the absence of singular continuous spectrum within some interval. In order to keep the results and proofs simple we first consider the case that the Dirichlet-to-Neumann map $M(\lambda)$ is known on the whole boundary $\partial\Omega$.

In view of the characterization of the eigenvalues of the Dirichlet operator in the first theorem of this section we state the following simple lemma.

Lemma 4.1. *Let Assumption 2.1 be satisfied and let $M(\lambda + i\eta)$ be the Dirichlet-to-Neumann map in* (2.9)*. Then for all $\lambda \in \mathbb{R}$ the strong limit*

$$\operatorname*{s-lim}_{\eta \searrow 0} \eta M(\lambda + i\eta) \tag{4.1}$$

exists, that is, $\lim_{\eta\searrow 0} \eta M(\lambda + i\eta)g$ exists in $H^{-1/2}(\partial\Omega)$ for all $g \in H^{1/2}(\partial\Omega)$.

Proof. Let $\lambda \in \mathbb{R}$ and $g \in H^{1/2}(\partial\Omega)$. For an arbitrary $\mu \in \rho(A_{\mathrm{D}})$ Lemma 3.1 (iii) and (iv) lead to

$$\begin{aligned} M(\lambda+i\eta)g &= M(\overline{\mu})g \\ &\quad + (\lambda + i\eta - \overline{\mu})\gamma(\mu)^*\big(I + (\lambda + i\eta - \mu)(A_{\mathrm{D}} - (\lambda + i\eta))^{-1}\big)\gamma(\mu)g \end{aligned} \tag{4.2}$$

for all $\eta > 0$. Moreover, when $E(\cdot)$ denotes the spectral measure of A_{D}, we have

$$\begin{aligned} &\big\|\eta(A_{\mathrm{D}}-(\lambda + i\eta))^{-1}\gamma(\mu)g - iE(\{\lambda\})\gamma(\mu)g\big\|^2 \\ &\qquad = \int_{\mathbb{R}} \left| \frac{\eta}{t - \lambda - i\eta} - i\mathbb{1}_{\{\lambda\}} \right|^2 d\,(E(t)\gamma(\mu)g, \gamma(\mu)g) \to 0 \quad \text{as} \quad \eta \searrow 0 \end{aligned} \tag{4.3}$$

by the dominated convergence theorem, that is, $\eta(A_{\mathrm{D}} - (\lambda + i\eta))^{-1}\gamma(\mu)g$ converges to $iE(\{\lambda\})\gamma(\mu)g$ as $\eta \searrow 0$. From this and (4.2) we obtain

$$\begin{aligned} \lim_{\eta\searrow 0} \eta M(\lambda + i\eta)g &= \lim_{\eta\searrow 0} \eta(\lambda + i\eta - \overline{\mu})\gamma(\mu)^*(\lambda + i\eta - \mu)(A_{\mathrm{D}} - (\lambda + i\eta))^{-1}\gamma(\mu)g \\ &= (\lambda - \overline{\mu})\gamma(\mu)^*(\lambda - \mu)iE(\{\lambda\})\gamma(\mu)g; \end{aligned}$$

in particular, the strong limit (4.1) exists. □

The following theorem is one of the main results of this thesis. It shows that the whole spectral data of A_{D} can be recovered from the knowledge of the function $M(\cdot)$. Particularly, we provide a complete characterization of all eigenvalues and the corresponding eigenspaces. We point out that this result is the multidimensional analog of the main theorem in [41], where the spectra of selfadjoint singular Sturm–Liouville operators were characterized by means of the limiting behavior of the associated Titchmarsh–Weyl m-function; cf. also [73] for analogous statements for Hamiltonian systems. For similar results in the abstract framework of Q-functions associated with selfadjoint operators in Hilbert spaces see [25,97]. For the idea of the proof of item (i) we refer the reader to [51]. Recall that $\sigma_{\mathrm{p}}(A_{\mathrm{D}})$ and $\sigma_{\mathrm{c}}(A_{\mathrm{D}})$ denote the point spectrum and the continuous spectrum, respectively, of A_{D} and that $\operatorname{Res}_\lambda M$ is the residue of the function $M(\cdot)$ at λ.

Theorem 4.2. *Let Assumption 2.1 be satisfied, let A_{D} be the selfadjoint Dirichlet operator in* (2.5) *and let $M(\lambda)$ be the Dirichlet-to-Neumann map in* (2.9)*. For $\lambda \in \mathbb{R}$ the following assertions hold.*

(i) *$\lambda \in \rho(A_{\mathrm{D}})$ if and only if $M(\cdot)$ can be continued analytically to λ.*

(ii) *$\lambda \in \sigma_{\mathrm{p}}(A_{\mathrm{D}})$ if and only if $\operatorname{s-lim}_{\eta\searrow 0} \eta M(\lambda + i\eta) \neq 0$. If λ is an eigenvalue with finite multiplicity then the mapping*

$$\begin{aligned} \tau : \ker(A_{\mathrm{D}} - \lambda) &\to \Big\{ \lim_{\eta\searrow 0} \eta M(\lambda + i\eta) g : g \in H^{1/2}(\partial\Omega) \Big\}, \\ u &\mapsto \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \end{aligned} \tag{4.4}$$

is bijective; if λ is an eigenvalue with infinite multiplicity then the mapping

$$\begin{aligned} \tau : \ker(A_{\mathrm{D}} - \lambda) &\to \mathrm{cl}_\tau \Big\{ \lim_{\eta\searrow 0} \eta M(\lambda + i\eta) g : g \in H^{1/2}(\partial\Omega) \Big\}, \\ u &\mapsto \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \end{aligned} \tag{4.5}$$

is bijective, where cl_τ denotes the closure in the normed space $\operatorname{ran} \tau$.

(iii) *λ is an isolated eigenvalue of A_{D} if and only if λ is a pole of $M(\cdot)$. If λ is an isolated eigenvalue with finite multiplicity then the mapping*

$$\tau : \ker(A_{\mathrm{D}} - \lambda) \to \operatorname{ran} \operatorname{Res}_\lambda M, \quad u \mapsto \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \tag{4.6}$$

is bijective; if λ is an isolated eigenvalue with infinite multiplicity then the mapping

$$\tau : \ker(A_{\mathrm{D}} - \lambda) \to \mathrm{cl}_\tau(\operatorname{ran} \operatorname{Res}_\lambda M), \quad u \mapsto \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega}, \tag{4.7}$$

is bijective with cl_τ as in (ii).

(iv) *$\lambda \in \sigma_{\mathrm{c}}(A_{\mathrm{D}})$ if and only if $\operatorname{s-lim}_{\eta\searrow 0} \eta M(\lambda + i\eta) = 0$ and $M(\cdot)$ cannot be continued analytically to λ.*

Proof. (i) It follows from Lemma 3.1 (iv) that $M(\cdot)$ is analytic on $\rho(A_\mathrm{D})$. In order to verify the other implication, let us assume that $M(\cdot)$ can be continued analytically to some $\lambda \in \mathbb{R}$. Let us choose $a, b \in \mathbb{R} \setminus \sigma_\mathrm{p}(A_\mathrm{D})$ with $a < b$ such that $\lambda \in (a, b)$ and such that $[a, b]$ is contained in the maximal domain of analyticity of the function $M(\cdot)$. The spectral projection $E((a, b))$ of A_D corresponding to the interval (a, b) is given by Stone's formula

$$E((a,b)) = \lim_{\delta \searrow 0} \frac{1}{2\pi i} \int_a^b \big((A_\mathrm{D} - (t + i\delta))^{-1} - (A_\mathrm{D} - (t - i\delta))^{-1} \big)\, dt, \tag{4.8}$$

see (1.3), where the integral on the right-hand side converges in the strong sense. Let $\gamma(\mu)$ denote the Poisson operator in (3.1). Combining (4.8) with the identity (3.2) in Lemma 3.1 we obtain

$$\begin{aligned}\gamma(\mu)^* E((a,b))\gamma(\mu) = \lim_{\delta \searrow 0} \frac{1}{2\pi i} \int_a^b \bigg(& \frac{M(t+i\delta) - M(\overline{\mu})}{(t + i\delta - \mu)(t + i\delta - \overline{\mu})} - \frac{\gamma(\mu)^*\gamma(\mu)}{t + i\delta - \mu} \\ & + \frac{\gamma(\mu)^*\gamma(\mu)}{t - i\delta - \mu} - \frac{M(t-i\delta) - M(\overline{\mu})}{(t - i\delta - \mu)(t - i\delta - \overline{\mu})} \bigg) dt = 0\end{aligned}$$

for each $\mu \in \mathbb{C} \setminus \mathbb{R}$, since $M(\cdot)$ is holomorphic in an open neighborhood of the interval $[a, b]$ in $\mathbb{C}$. In particular,

$$\big(E((a,b))\gamma(\mu)g, \gamma(\mu)g\big) = 0, \quad g \in H^{1/2}(\partial\Omega),\ \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{4.9}$$

Recall next that by Proposition 3.11

$$\operatorname{span}\big\{\gamma(\mu)g : \mu \in \mathbb{C} \setminus \mathbb{R}, g \in H^{1/2}(\partial\Omega)\big\}$$

is dense in $L^2(\Omega)$. Hence (4.9) yields $E((a, b)) = 0$. Now $\lambda \in (a, b)$ implies $\lambda \in \rho(A_\mathrm{D})$.

(ii) We prove that the operator τ in (4.5) is bijective for all $\lambda \in \mathbb{R}$; from this it follows immediately that λ is an eigenvalue of A_D if and only if $\text{s-lim}_{\eta \searrow 0}\, \eta M(\lambda + i\eta) \neq 0$. Let $\lambda \in \mathbb{R}$. We verify first that the operator τ is injective. Indeed, assume $u \in \ker(A_\mathrm{D} - \lambda)$ satisfies $\tau u = 0$, let $\mu \in \mathbb{C} \setminus \mathbb{R}$, and let $v_\mu \in \mathcal{N}_\mu$, see (3.15), that is, $v_\mu \in H^1(\Omega)$ and $\mathcal{L} v_\mu = \mu v_\mu$. Then the second Green identity (1.11) yields

$$\begin{aligned}(\lambda - \mu)(u, v_\mu) &= (A_\mathrm{D} u, v_\mu) - (u, \mathcal{L} v_\mu) \\ &= \Big(u|_{\partial\Omega}, \frac{\partial v_\mu}{\partial \nu_\mathcal{L}}\big|_{\partial\Omega}\Big)_{\partial\Omega} - \Big(\frac{\partial u}{\partial \nu_\mathcal{L}}\big|_{\partial\Omega}, v_\mu|_{\partial\Omega}\Big)_{\partial\Omega} = 0,\end{aligned}$$

since $u|_{\partial\Omega} = 0$ and $\frac{\partial u}{\partial \nu_\mathcal{L}}|_{\partial\Omega} = \tau u = 0$. Since

$$\operatorname{span}\{\mathcal{N}_\lambda : \lambda \in \mathbb{C} \setminus \mathbb{R}\}$$

is dense in $L^2(\Omega)$ by Proposition 3.11, it follows $u = 0$, that is, τ is injective.

Let us set

$$\mathcal{F}_\lambda = \Big\{ \lim_{\eta \searrow 0} \eta M(\lambda + i\eta) g : g \in H^{1/2}(\partial\Omega) \Big\}.$$

In order to prove the surjectivity of τ we will verify the identity

$$\mathcal{F}_\lambda \subset \operatorname{ran} \tau \subset \overline{\mathcal{F}_\lambda}. \tag{4.10}$$

Since A_{D} is semibounded from below, see Theorem 2.3, we can fix some $\mu \in \mathbb{R} \cap \rho(A_{\mathrm{D}})$. Note that for each $u \in \ker(A_{\mathrm{D}} - \lambda)$ the identity

$$\begin{aligned} \tau u &= \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\partial\Omega} = \frac{\partial}{\partial \nu_{\mathcal{L}}}\big((A_{\mathrm{D}} - \mu)^{-1}(A_{\mathrm{D}} - \mu)u\big)\big|_{\partial\Omega} \\ &= (\lambda - \mu)\frac{\partial}{\partial \nu_{\mathcal{L}}}\big((A_{\mathrm{D}} - \mu)^{-1}u\big)\big|_{\partial\Omega} = (\mu - \lambda)\gamma(\mu)^* u \end{aligned}$$

holds by Lemma 3.1 (i), where $\gamma(\mu)$ is the Poisson operator in (3.1); in particular,

$$\operatorname{ran} \tau = \operatorname{ran} \left(\gamma(\mu)^* \restriction \ker(A_{\mathrm{D}} - \lambda)\right).$$

Thus, in order to verify (4.10) it is sufficient to show

$$\mathcal{F}_\lambda \subset \operatorname{ran} \big(\gamma(\mu)^* \restriction \ker(A_{\mathrm{D}} - \lambda)\big) \subset \overline{\mathcal{F}_\lambda}. \tag{4.11}$$

Indeed, if we denote by $E(\cdot)$ the spectral measure of A_{D} and by $P = E(\{\lambda\})$ the orthogonal projection in $L^2(\Omega)$ onto $\ker(A_{\mathrm{D}} - \lambda)$ then

$$\lim_{\eta \searrow 0} \eta (A_{\mathrm{D}} - (\lambda + i\eta))^{-1} \gamma(\nu) g = i P \gamma(\nu) g \tag{4.12}$$

holds for all $g \in H^{1/2}(\partial\Omega)$ and all $\nu \in \mathbb{C} \setminus \mathbb{R}$, see (4.3). Furthermore, note that the identity

$$\begin{aligned} &\gamma(\mu)^*(A_{\mathrm{D}} - z)^{-1}\gamma(\nu) \\ &\qquad = \frac{M(z)}{(z-\nu)(z-\mu)} + \frac{M(\mu)}{(z-\mu)(\nu-\mu)} - \frac{M(\nu)}{(z-\nu)(\nu-\mu)} \end{aligned} \tag{4.13}$$

holds for $\nu, z \in \mathbb{C} \setminus \mathbb{R}$ satisfying $z \neq \nu$. Indeed, by Lemma 3.1 (ii) and the first statement in Lemma 3.1 (iii) we have

$$\begin{aligned} \gamma(\mu)^*(A_{\mathrm{D}} - z)^{-1}\gamma(\nu) &= \gamma(\mu)^* \left(\frac{\gamma(z) - \gamma(\nu)}{z - \nu} \right) \\ &= \frac{1}{z-\nu}\left(\frac{M(z) - M(\mu)}{z - \mu} - \frac{M(\nu) - M(\mu)}{\nu - \mu} \right) \end{aligned}$$

and an easy computation yields (4.13). The formulas (4.12) and (4.13) and the continuity of $\gamma(\mu)^*$ imply

$$\begin{aligned}\frac{\lim_{\eta\searrow 0}\eta M(\lambda+i\eta)g}{(\lambda-\nu)(\lambda-\mu)} &= \lim_{\eta\searrow 0}\eta\gamma(\mu)^*(A_{\mathrm{D}}-(\lambda+i\eta))^{-1}\gamma(\nu)g\\ &= i\gamma(\mu)^*P\gamma(\nu)g \end{aligned}\tag{4.14}$$

for all $g\in H^{1/2}(\partial\Omega)$ and all $\nu\in\mathbb{C}\setminus\mathbb{R}$. From this we obtain the first inclusion in (4.11). Moreover, it follows from Proposition 3.11 that

$$\operatorname{span}\left\{P\gamma(\nu)g : \nu\in\mathbb{C}\setminus\mathbb{R}, g\in H^{1/2}(\partial\Omega)\right\}$$

is dense in $\ker(A-\lambda)$. Thus (4.14) also leads to the second inclusion in (4.11) and, consequently, we obtain (4.10). In particular, we have $\mathcal{F}_\lambda\subset\operatorname{ran}\tau$ and $\mathrm{cl}_\tau(\mathcal{F}_\lambda)=\overline{\mathcal{F}}_\lambda\cap\operatorname{ran}\tau=\operatorname{ran}\tau$. Therefore τ in (4.5) is surjective and, hence, bijective. Clearly, if $\dim\ker(A_{\mathrm{D}}-\lambda)$ is finite then equality holds in (4.10), which leads to the bijectivity of (4.4) and completes the proof of (ii).

(iii) Let λ be an isolated eigenvalue of A_{D}. Then there exists an open neighborhood $\mathcal{O}$ of λ such that $z\mapsto(A_{\mathrm{D}}-z)^{-1}$ is holomorphic on $\mathcal{O}\setminus\{\lambda\}$. Thus, by (i), $M(\cdot)$ is holomorphic on $\mathcal{O}\setminus\{\lambda\}$. Moreover, by (ii), there exists $g\in H^{1/2}(\partial\Omega)$ such that $\lim_{\eta\searrow 0}i\eta M(\lambda+i\eta)g\neq 0$. Hence λ is a pole of $M(\cdot)$ and it follows from the formula (3.2) in Lemma 3.1 that the order of the pole is one. Moreover, the limit

$$\lim_{z\to\lambda}(z-\lambda)M(z)g=\operatorname{Res}_\lambda M(\cdot)g$$

exists for all $g\in H^{1/2}(\partial\Omega)$ and, clearly, it coincides with $\lim_{\eta\searrow 0}i\eta M(\lambda+i\eta)g$. Therefore (4.7) is a consequence of (4.5). Analogously, the identity (4.6) follows immediately from (4.4). If, conversely, λ is a pole of $M(\cdot)$ then clearly there exists $g\in H^{1/2}(\partial\Omega)$ such that $\lim_{\eta\searrow 0}\eta M(\lambda+i\eta)g\neq 0$ and it follows from (ii) that λ is an eigenvalue of A_{D}. Since $M(\cdot)$ is holomorphic on a punctured neighborhood of λ, by (i) the same holds for the function $z\mapsto(A_{\mathrm{D}}-z)^{-1}$. Therefore λ is isolated in $\sigma(A_{\mathrm{D}})$ and, hence, λ is an isolated eigenvalue of A_{D}.

(iv) Since $\sigma_{\mathrm{c}}(A_{\mathrm{D}})=\mathbb{R}\setminus(\rho(A_{\mathrm{D}})\cup\sigma_{\mathrm{p}}(A_{\mathrm{D}}))$, the statement of (iv) follows immediately from (i) and (ii). □

In the special case that Ω is bounded the spectrum of A_{D} is purely discrete, see Theorem 2.3 above, that is, $\sigma(A_{\mathrm{D}})$ consists of isolated eigenvalues with finite multiplicities. In this case Theorem 4.2 can be regarded as a special case of Theorem 3.8 in Chapter 3 above with $\omega=\partial\Omega$.

In our next main result we characterize the absolutely continuous spectrum of A_{D} by means of the limits of $M(\lambda)$ when λ approaches the real axis. In order to do so we define the *absolutely continuous closure* of a Borel set $\chi \subset \mathbb{R}$ by

$$\mathrm{cl}_{\mathrm{ac}}(\chi) := \big\{x \in \mathbb{R} : |(x-\varepsilon, x+\varepsilon) \cap \chi| > 0 \,\text{for all}\, \varepsilon > 0\big\}, \tag{4.15}$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$; sometimes $\mathrm{cl}_{\mathrm{ac}}(\chi)$ is also called the *essential closure* of χ. The proofs of the next two theorems require the following lemma from measure theory. For a proof of the lemma see Lemma A.1 in Appendix A.1. Recall that each σ-finite Borel measure μ on $\mathbb{R}$ admits a unique decomposition $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}}$, where μ_{ac} is absolutely continuous (with respect to the Lebesgue measure) and μ_{s} is singular, and that the singular part μ_{s} can be decomposed further into the singular continuous part μ_{sc} and the pure point part; cf. Appendix A.1. For a Borel measure μ on $\mathbb{R}$ we define

$$\operatorname{supp} \mu := \{x \in \mathbb{R} : \mu((x-\varepsilon, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\},$$

the set of all growth points of μ.

Lemma 4.3. *Let μ be a finite Borel measure on $\mathbb{R}$ and denote by $F(\cdot)$ its Borel transform, i.e.,*

$$F(\lambda) = \int_{\mathbb{R}} \frac{1}{t-\lambda} d\mu(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Then the limit $\operatorname{Im} F(x+i0) = \lim_{y \searrow 0} \operatorname{Im} F(x+iy)$ exists and is finite for Lebesgue almost all $x \in \mathbb{R}$. Moreover, for the absolutely continuous part μ_{ac} and the singular continuous part μ_{sc} of μ the following assertions hold.

(i) $\operatorname{supp} \mu_{\mathrm{ac}} = \mathrm{cl}_{\mathrm{ac}}(\{x \in \mathbb{R} : 0 < \operatorname{Im} F(x+i0) < +\infty\})$.

(ii) *The set $M_{\mathrm{sc}} = \{x \in \mathbb{R} : \operatorname{Im} F(x+i0) = +\infty, \lim_{y \searrow 0} yF(x+iy) = 0\}$ is a support for μ_{sc}, that is, $\mu_{\mathrm{sc}}(\mathbb{R} \setminus M_{\mathrm{sc}}) = 0$.*

The following result is the multidimensional analog of a well-known and widely used statement from singular Sturm–Liouville theory; cf., e.g., [13, 64, 121]. It states that the absolutely continuous spectrum of A_{D} can be detected by the limits of the imaginary part of the Dirichlet-to-Neumann map towards the real line. For similar results in an abstract framework see, e.g., [34].

Theorem 4.4. *Let Assumption 2.1 be satisfied, let A_{D} be the selfadjoint Dirichlet operator in (2.5), and let $M(\lambda)$ be the Dirichlet-to-Neumann map in (2.9). Then the absolutely continuous spectrum of A_{D} is given by*

$$\sigma_{\mathrm{ac}}(A_{\mathrm{D}}) = \overline{\bigcup_{g \in H^{1/2}(\partial\Omega)} \mathrm{cl}_{\mathrm{ac}}\big(\big\{x \in \mathbb{R} : 0 < \operatorname{Im}(M(x+i0)g, g)_{\partial\Omega} < +\infty\big\}\big)}. \tag{4.16}$$

In particular, if $a, b \in \mathbb{R}$ with $a < b$ then $(a,b) \cap \sigma_{\mathrm{ac}}(A_{\mathrm{D}}) = \emptyset$ if and only if $\operatorname{Im}(M(x+i0)g, g)_{\partial\Omega} = 0$ holds for all $g \in H^{1/2}(\partial\Omega)$ and for almost all $x \in (a,b)$.

Proof. **Step 1.** Recall that $\gamma(\zeta)$ denotes the Poisson operator for $\zeta \in \rho(A_{\mathrm{D}})$, see (3.1). In this first step our aim is to verify that the absolutely continuous spectrum of A_{D} is given by

$$\sigma_{\mathrm{ac}}(A_{\mathrm{D}}) = \overline{\bigcup_{\substack{\zeta\in\mathbb{C}\setminus\mathbb{R},\\ g\in H^{1/2}(\partial\Omega)}} \operatorname{supp}\mu_{\gamma(\zeta)g,\mathrm{ac}}}, \tag{4.17}$$

where $\mu_u = (E(\cdot)u,u)$, $u \in L^2(\Omega)$, and $E(\cdot)$ denotes the spectral measure of A_{D}. Let $\mathcal{H}_{\mathrm{ac}}$ denote the absolutely continuous subspace of $L^2(\Omega)$ with respect to A_{D} and let $A_{\mathrm{D,ac}}$ be the absolutely continuous part of A_{D}. Let $\lambda \notin \sigma_{\mathrm{ac}}(A_{\mathrm{D}}) = \sigma(A_{\mathrm{D,ac}})$. Then there exists $\varepsilon > 0$ such that $E((\lambda-\varepsilon,\lambda+\varepsilon)) \upharpoonright \mathcal{H}_{\mathrm{ac}} = 0$, in particular, $\mu_u((\lambda-\varepsilon,\lambda+\varepsilon)) = 0$ for all $u \in \mathcal{H}_{\mathrm{ac}}$. For arbitrary $\zeta \in \mathbb{C}\setminus\mathbb{R}$ and $g \in H^{1/2}(\partial\Omega)$ we have

$$\mu_{\gamma(\zeta)g,\mathrm{ac}}((\lambda-\varepsilon,\lambda+\varepsilon)) = \mu_{P_{\mathrm{ac}}\gamma(\zeta)g}((\lambda-\varepsilon,\lambda+\varepsilon)) = 0,$$

where P_{ac} denotes the orthogonal projection in $L^2(\Omega)$ onto $\mathcal{H}_{\mathrm{ac}}$. Hence we have

$$\lambda \notin \bigcup_{\substack{\zeta\in\mathbb{C}\setminus\mathbb{R},\\ g\in H^{1/2}(\partial\Omega)}} \operatorname{supp}\mu_{\gamma(\zeta)g,\mathrm{ac}}.$$

Since $\sigma_{\mathrm{ac}}(A_{\mathrm{D}})$ is closed, we have proved the inclusion $\supset$ in (4.17). In order to verify the converse inclusion assume that λ does not belong to the right-hand side of (4.17). Then there exists $\varepsilon > 0$ such that $(\lambda-\varepsilon,\lambda+\varepsilon) \subset \mathbb{R}\setminus\operatorname{supp}\mu_{\gamma(\zeta)g,\mathrm{ac}} = \mathbb{R}\setminus\operatorname{supp}\mu_{P_{\mathrm{ac}}\gamma(\zeta)g}$ for all $\zeta \in \mathbb{C}\setminus\mathbb{R}$, $g \in H^{1/2}(\partial\Omega)$, that is,

$$\|E((\lambda-\varepsilon,\lambda+\varepsilon))P_{\mathrm{ac}}\gamma(\zeta)g\|^2 = (E((\lambda-\varepsilon,\lambda+\varepsilon))P_{\mathrm{ac}}\gamma(\zeta)g, P_{\mathrm{ac}}\gamma(\zeta)g) = 0$$

for all $\zeta \in \mathbb{C}\setminus\mathbb{R}, g \in H^{1/2}(\partial\Omega)$. Since it follows from Proposition 3.11 that

$$\operatorname{span}\left\{P_{\mathrm{ac}}\gamma(\zeta)g : \zeta \in \mathbb{C}\setminus\mathbb{R}, g \in H^{1/2}(\partial\Omega)\right\}$$

is dense in $\mathcal{H}_{\mathrm{ac}}$, we obtain $E((\lambda-\varepsilon,\lambda+\varepsilon)) \upharpoonright \mathcal{H}_{\mathrm{ac}} = 0$, that is, $\lambda \notin \sigma(A_{\mathrm{D,ac}}) = \sigma_{\mathrm{ac}}(A_{\mathrm{D}})$. Thus we have proved (4.17).

Step 2. In this step we observe that for each $\zeta \in \mathbb{C}\setminus\mathbb{R}$ and each $g \in H^{1/2}(\partial\Omega)$ we have

$$\begin{aligned}&\operatorname{supp}\mu_{\gamma(\zeta)g,\mathrm{ac}}\\ &\qquad = \mathrm{cl}_{\mathrm{ac}}\left(\left\{x \in \mathbb{R} : 0 < \operatorname{Im}\left((A_{\mathrm{D}}-(x+i0))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\right) < +\infty\right\}\right).\end{aligned} \tag{4.18}$$

Indeed, for each $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and each $g \in H^{1/2}(\partial\Omega)$ the Borel transform of the finite Borel measure $\mu_{\gamma(\zeta)g} = (E(\cdot)\gamma(\zeta)g, \gamma(\zeta)g)$ is given by

$$\begin{aligned} F_{\gamma(\zeta)g}(x+iy) &= \int_{\mathbb{R}} \frac{1}{t-(x+iy)} d(E(t)\gamma(\zeta)g, \gamma(\zeta)g) \\ &= \big((A_{\mathrm{D}} - (x+iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\big), \qquad x \in \mathbb{R},\ y > 0. \end{aligned}$$

Hence Lemma 4.3 (i) implies (4.18).

Step 3. In this third step we verify that

$$\begin{aligned} 0 < \operatorname{Im} \big((A_{\mathrm{D}} - (x+i0))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\big) < +\infty \\ \Longleftrightarrow 0 < \operatorname{Im}(M(x+i0)g, g)_{\partial\Omega} < +\infty \end{aligned} \tag{4.19}$$

is true for all $x \in \mathbb{R}$, all $g \in H^{1/2}(\partial\Omega)$, and all $\zeta \in \mathbb{C} \setminus \mathbb{R}$. We make use of the formula (3.2) and obtain for $y > 0$, $\zeta \in \mathbb{C} \setminus \mathbb{R}$, and $g \in H^{1/2}(\partial\Omega)$

$$\begin{aligned} &\operatorname{Im}(M(x+iy)g, g)_{\partial\Omega} \\ &\quad = y\|\gamma(\zeta)g\|^2_{L^2(\Omega)} + \big(|x-\zeta|^2 - y^2\big) \operatorname{Im} \big((A_{\mathrm{D}} - (x+iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\big) \\ &\qquad + 2(x - \operatorname{Re}\zeta) y \operatorname{Re} \big((A_{\mathrm{D}} - (x+iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\big). \end{aligned}$$

Moreover, for $y > 0$ we have

$$y \operatorname{Re} \big((A_{\mathrm{D}} - (x+iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\big) = \int_{\mathbb{R}} \frac{y(t-x)}{(t-x)^2 + y^2} d(E(t)\gamma(\zeta)g, \gamma(\zeta)g),$$

which converges to zero for $y \searrow 0$ by the dominated convergence theorem as the integrand is bounded by $1/2$. Hence

$$\operatorname{Im}(M(x+i0)g, g)_{\partial\Omega} = |x-\zeta|^2 \operatorname{Im} \big((A_{\mathrm{D}} - (x+i0))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\big). \tag{4.20}$$

Since $|x-\zeta|^2 > 0$, (4.20) yields (4.19).

From Step 1–Step 3 the representation (4.16) follows.

Step 4. In this last step we prove the remaining assertion of the theorem. Let $a < b$ and assume $(a, b) \cap \sigma_{\mathrm{ac}}(A_{\mathrm{D}}) = \emptyset$. Then (4.16) implies

$$\emptyset = \mathrm{cl}_{\mathrm{ac}} \left(\{x \in \mathbb{R} : 0 < \operatorname{Im}(M(x+i0)g, g)_{\partial\Omega} < +\infty\}\right) \cap (a, b)$$

for each $g \in H^{1/2}(\partial\Omega)$. Consequently, for each $g \in H^{1/2}(\partial\Omega)$ and each $x \in (a, b)$ there exists $\varepsilon > 0$ with

$$\big|(x-\varepsilon, x+\varepsilon) \cap \{x \in \mathbb{R} : 0 < \operatorname{Im}(M(x+i0)g, g)_{\partial\Omega} < +\infty\}\big| = 0. \tag{4.21}$$

Since by (4.20) and Lemma 4.3

$$\operatorname{Im}(M(x+i0)g,g)_{\partial\Omega} = |x-\zeta| \operatorname{Im} F_{\gamma(\zeta)g}(x+i0), \quad \zeta \in \mathbb{C}\setminus\mathbb{R},$$

exists and is finite for Lebesgue almost all $x \in \mathbb{R}$, it follows from (4.21) that $\operatorname{Im}(M(x+i0)g,g)_{\partial\Omega} = 0$ for all $g \in H^{1/2}(\partial\Omega)$ and almost all $x \in (a,b)$. If, conversely, $\operatorname{Im}(M(x+i0)g,g)_{\partial\Omega} = 0$ holds for all $g \in H^{1/2}(\partial\Omega)$ and almost all $x \in (a,b)$, then clearly the intersection of (a,b) with the right-hand side in (4.16) is empty and, hence, we obtain $\sigma_{\mathrm{ac}}(A_{\mathrm{D}}) \cap (a,b) = \emptyset$. □

In the next theorem a sufficient criterion for the absence of singular continuous spectrum of A_{D} within some interval by means of the limiting behavior of $M(\cdot)$ is given; cf. [34] for an abstract approach.

Theorem 4.5. *Let Assumption 2.1 be satisfied, let A_{D} be the Dirichlet operator in* (2.5)*, and let $M(\lambda)$ be the Dirichlet-to-Neumann map in* (2.9)*. Moreover, let $a,b \in \mathbb{R}$ with $a<b$. If for each $g \in H^{1/2}(\partial\Omega)$ there exist at most countably many $x \in (a,b)$ such that*

$$\operatorname{Im}(M(x+iy)g,g)_{\partial\Omega} \to +\infty \quad \text{and} \quad y(M(x+iy)g,g)_{\partial\Omega} \to 0, \qquad y \searrow 0, \tag{4.22}$$

then $(a,b) \cap \sigma_{\mathrm{sc}}(A_{\mathrm{D}}) = \emptyset$.

Proof. Analogously to Step 1 in the proof of Theorem 4.4 it can be seen that the singular continuous spectrum of A_{D} is given by

$$\sigma_{\mathrm{sc}}(A_{\mathrm{D}}) = \overline{\bigcup_{\substack{\zeta\in\mathbb{C}\setminus\mathbb{R},\\ g\in H^{1/2}(\partial\Omega)}} \operatorname{supp} \mu_{\gamma(\zeta)g,\mathrm{sc}}}. \tag{4.23}$$

From (4.22) it follows with the help of (4.20) and (4.14) that for each $g \in H^{1/2}(\partial\Omega)$ and each fixed $\zeta \in \mathbb{C}\setminus\mathbb{R}$ there exist at most countably many $x \in (a,b)$ such that

$$\operatorname{Im}\big((A_{\mathrm{D}} - (x+iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\big) \to +\infty$$

and

$$y\big((A_{\mathrm{D}} - (x+iy))^{-1}\gamma(\zeta)g, \gamma(\zeta)g\big) \to 0$$

as $y \searrow 0$. With Lemma 4.3 (ii) it follows that $\mu_{\gamma(\zeta)g,\mathrm{sc}}$ has a countable support within the interval (a,b) for each $g \in H^{1/2}(\partial\Omega)$ and each $\zeta \in \mathbb{C}\setminus\mathbb{R}$. Since a singular continuous measure does not possess any point masses, we conclude that $\mu_{\gamma(\zeta)g,\mathrm{sc}}$ is trivial on (a,b) for all $\zeta \in \mathbb{C}\setminus\mathbb{R}$ and all $g \in H^{1/2}(\partial\Omega)$. Finally, from (4.23) it follows $\sigma_{\mathrm{sc}}(A_{\mathrm{D}}) \cap (a,b) = \emptyset$. □

Finally, we state the following corollary of the theorems of this section. It provides sufficient criteria for the spectrum of A_{D} to be purely absolutely continuous or purely singular continuous, respectively, in some interval.

Corollary 4.6. *Assume that Assumption 2.1 is satisfied. Let A_{D} be the Dirichlet operator in (2.5), let $M(\lambda)$ be the Dirichlet-to-Neumann map in (2.9), and let $a, b \in \mathbb{R}$ with $a < b$. Moreover, for all $g \in H^{1/2}(\partial\Omega)$ and all $x \in (a, b)$ let*

$$\lim_{y \searrow 0} yM(x + iy)g = 0.$$

Then the following assertions hold.

(i) *If $\operatorname{Im}(M(x + i0)g, g)_{\partial\Omega} = 0$ holds for all $g \in H^{1/2}(\partial\Omega)$ and almost all $x \in (a, b)$ then $\sigma(A_{\mathrm{D}}) \cap (a, b) = \sigma_{\mathrm{sc}}(A_{\mathrm{D}}) \cap (a, b)$.*

(ii) *If for each $g \in H^{1/2}(\partial\Omega)$ there exist at most countably many $x \in (a, b)$ such that $\operatorname{Im}(M(x + i0)g, g)_{\partial\Omega} = +\infty$ then $\sigma(A_{\mathrm{D}}) \cap (a, b) = \sigma_{\mathrm{ac}}(A_{\mathrm{D}}) \cap (a, b)$.*

4.2 Generalizations and extensions

In this section we provide extensions and generalizations of the results of the previous section. On the one hand we show that the spectrum of the selfadjoint Dirichlet operator can be described completely from knowledge of the function $M(\cdot)$ on an open subset ω of $\partial\Omega$ instead of the whole boundary; this complements the results of Chapter 3 with partial boundary data. On the other hand we provide a spectral characterization for operators with generalized Robin boundary conditions as in (2.10). We show that the results of the previous section remain valid for such generalized Robin operators when the Dirichlet-to-Neumann map is replaced by an associated Robin-to-Dirichlet map.

4.2.1 A characterization of the Dirichlet spectrum from partial data

Our aim in this subsection is to characterize the spectrum of the Dirichlet operator A_{D} by the partial knowledge of the Dirichlet-to-Neumann map. The following theorem can be considered to be a local variant of Theorem 4.2. For the sake of completeness we provide a short proof, which is of a similar nature as the proof of Theorem 4.2. Let $\omega \subset \partial\Omega$ be a nonempty, relatively open set. Recall from Chapter 3 that the space of functions in $H^{1/2}(\partial\Omega)$ with support in ω is called $H^{1/2}_{\omega}$, that is,

$$H^{1/2}_{\omega} = \left\{ g \in H^{1/2}(\partial\Omega) : \operatorname{supp} g \subset \omega \right\}.$$

We say that $M(\cdot)$ *can be continued analytically to* $\lambda \in \mathbb{R}$ *on* ω if there exists an open neighborhood $\mathcal{O}$ of λ in $\mathbb{C}$ such that the function $(M(\cdot)g, g)_{\partial\Omega}$ can be continued analytically to $\mathcal{O}$ for all $g \in H^{1/2}_\omega$.

Theorem 4.7. *Let Assumption 2.1 be satisfied, let* Ω *be connected, and let* $\omega \subset \partial\Omega$ *be open and nonempty. Moreover, let* A_{D} *be the selfadjoint Dirichlet operator in* (2.5) *and let* $M(\lambda)$ *be the Dirichlet-to-Neumann map in* (2.9)*. Then for* $\lambda \in \mathbb{R}$ *the following assertions hold.*

(i) $\lambda \in \rho(A_{\mathrm{D}})$ *if and only if* $M(\cdot)$ *can be continued analytically to* λ *on* ω*.*

(ii) $\lambda \in \sigma_{\mathrm{p}}(A_{\mathrm{D}})$ *if and only if* $\lim_{\eta\searrow 0} \eta(M(\lambda+i\eta)g, g)_{\partial\Omega} \neq 0$ *for some* $g \in H^{1/2}_\omega$*.*

(iii) λ *is an isolated eigenvalue of* A_{D} *if and only if* λ *is a pole of* $(M(\cdot)g, g)_{\partial\Omega}$ *for some* $g \in H^{1/2}_\omega$*.*

(iv) $\lambda \in \sigma_{\mathrm{c}}(A_{\mathrm{D}})$ *if and only if* $M(\cdot)$ *cannot be continued analytically to* λ *on* ω *and* $\lim_{\eta\searrow 0} \eta(M(\lambda+i\eta)g, g)_{\partial\Omega} = 0$ *for all* $g \in H^{1/2}_\omega$*.*

Proof. (i) The proof of (i) follows precisely the lines of the proof of Theorem 4.2 (i), where one has to replace $H^{1/2}(\partial\Omega)$ by $H^{1/2}_\omega$ and use Proposition 3.4 instead of Proposition 3.11.

(ii) Let $E(\cdot)$ denote the spectral measure of A_{D}. Making use of Lemma 3.1 (iv) and the calculation (4.3) we obtain

$$\begin{aligned}\lim_{\eta\searrow 0} &\eta(M(\lambda+i\eta)g, g)_{\partial\Omega} \\ &= \lim_{\eta\searrow 0} \eta(\lambda+i\eta-\mu)(\lambda+i\eta-\overline{\mu})\big((A_{\mathrm{D}}-(\lambda+i\eta))^{-1}\gamma(\mu)g, \gamma(\mu)g\big) \\ &= (\lambda-\mu)(\lambda-\overline{\mu})\|E(\{\lambda\})\gamma(\mu)g\|^2 \end{aligned} \tag{4.24}$$

for all $\mu \in \mathbb{C}\setminus\mathbb{R}$ and all $g \in H^{1/2}_\omega$. If $\lim_{\eta\searrow 0} \eta(M(\lambda+i\eta)g, g)_{\partial\Omega} \neq 0$ for some $g \in H^{1/2}_\omega$ then (4.24) implies $E(\{\lambda\})\gamma(\mu)g \neq 0$, that is, λ is an eigenvalue of A_{D}. For the converse implication note that, as a consequence of Proposition 3.4, the linear space

$$\operatorname{span}\big\{E(\{\lambda\})\gamma(\mu)g : \mu \in \mathbb{C}\setminus\mathbb{R}, g \in H^{1/2}_\omega\big\}$$

is dense in $\ker(A_{\mathrm{D}}-\lambda)$. Thus, if λ belongs to $\sigma_{\mathrm{p}}(A_{\mathrm{D}})$ then there exist $\mu \in \mathbb{C}\setminus\mathbb{R}$ and $g \in H^{1/2}_\omega$ with $E(\{\lambda\})\gamma(\mu)g \neq 0$. From this and (4.24) we conclude $\lim_{\eta\searrow 0} \eta(M(\lambda+i\eta)g, g)_{\partial\Omega} \neq 0$.

(iii) This statement is an easy consequence of (i) and (ii); cf. the proof of Theorem 4.2 (iii).

(iv) Since $\sigma_{\mathrm{c}}(A_{\mathrm{D}}) = \mathbb{C}\setminus(\rho(A_{\mathrm{D}})\cup\sigma_{\mathrm{p}}(A_{\mathrm{D}}))$, the claim follows immediately from (i) and (ii). □

In the following two theorems we indicate how the absolutely continuous and singular continuous spectrum of the selfadjoint Dirichlet operator can be detected from the partial knowledge of the Dirichlet-to-Neumann map. Their proofs follow the lines of Theorem 4.4 and Theorem 4.5 with $H^{1/2}_\omega$ instead of $H^{1/2}(\partial\Omega)$ and Proposition 3.4 instead of Proposition 3.11.

A characterization of the absolutely continuous spectrum of A_{D} by means of the limits of the partial Dirichlet-to-Neumann map on ω looks as follows. This is a local version of Theorem 4.4 above.

Theorem 4.8. *Let Assumption 2.1 be satisfied, let Ω be connected, and let $\omega \subset \partial\Omega$ be open and nonempty. Moreover, let A_{D} be the selfadjoint Dirichlet operator in (2.5), and let $M(\lambda)$ be the Dirichlet-to-Neumann map in (2.9). Then the absolutely continuous spectrum of A_{D} is given by*

$$\sigma_{\mathrm{ac}}(A_{\mathrm{D}}) = \overline{\bigcup_{g \in H^{1/2}_\omega} \mathrm{cl}_{\mathrm{ac}}\left(\left\{x \in \mathbb{R} : 0 < \mathrm{Im}(M(x+i0)g, g)_{\partial\Omega} < +\infty\right\}\right)}.$$

In particular, if $a, b \in \mathbb{R}$ with $a < b$ then $(a,b) \cap \sigma_{\mathrm{ac}}(A_{\mathrm{D}}) = \emptyset$ if and only if $\mathrm{Im}(M(x+i0)g, g)_{\partial\Omega} = 0$ holds for all $g \in H^{1/2}_\omega$ and for almost all $x \in (a,b)$.

Furthermore, the following criterion for the absence of singular continuous spectrum of A_{D} in some interval can be proved. It is the local variant of Theorem 4.5 above.

Theorem 4.9. *Let Assumption 2.1 be satisfied, let Ω be connected, and let $\omega \subset \partial\Omega$ be open and nonempty. Moreover, let A_{D} be the selfadjoint Dirichlet operator in (2.5), let $M(\lambda)$ be the Dirichlet-to-Neumann map in (2.9), and let $a, b \in \mathbb{R}$ with $a < b$. If for each $g \in H^{1/2}_\omega$ there exist at most countably many $x \in (a,b)$ such that*

$$\mathrm{Im}(M(x+iy)g, g)_{\partial\Omega} \to \infty \quad \text{and} \quad y(M(x+iy)g, g)_{\partial\Omega} \to 0, \qquad y \searrow 0,$$

then $(a,b) \cap \sigma_{\mathrm{sc}}(A_{\mathrm{D}}) = \emptyset$.

As an immediate consequence of the theorems of this section we obtain the following corollary. It contains sufficient criteria for the spectrum of A_{D} to be purely absolutely continuous or purely singular continuous, respectively, in some interval, and is the counterpart of Corollary 4.6 for partial boundary data.

Corollary 4.10. *Assume that Assumption 2.1 is satisfied, that Ω is connected, and that ω is an open, nonempty subset of $\partial\Omega$. Let A_{D} be the selfadjoint Dirichlet operator in (2.5), let $M(\lambda)$ be the Dirichlet-to-Neumann map in (2.9), and let $a, b \in \mathbb{R}$ with $a < b$. Moreover, for all $g \in H^{1/2}_\omega$ and all $x \in (a, b)$ let*

$$\lim_{y \searrow 0} y(M(x+iy)g, g)_{\partial\Omega} = 0.$$

Then the following assertions hold.

(i) *If $\operatorname{Im}(M(x+i0)g, g)_{\partial\Omega} = 0$ holds for all $g \in H^{1/2}_\omega$ and almost all $x \in (a, b)$ then $\sigma(A_{\mathrm{D}}) \cap (a, b) = \sigma_{\mathrm{sc}}(A_{\mathrm{D}}) \cap (a, b)$.*

(ii) *If for each $g \in H^{1/2}_\omega$ there exist at most countably many $x \in (a, b)$ such that $\operatorname{Im}(M(x+i0)g, g)_{\partial\Omega} = +\infty$ then $\sigma(A_{\mathrm{D}}) \cap (a, b) = \sigma_{\mathrm{ac}}(A_{\mathrm{D}}) \cap (a, b)$.*

Remark 4.11. In all results of this subsection the assumption that Ω is connected can be weakened. It suffices to require that $\omega \cap \partial\mathcal{O}$ is nonempty for each connected component $\mathcal{O}$ of Ω, and the proofs remain the same.

4.2.2 A characterization of the spectra of generalized Robin operators

In this section we focus on the operator

$$A_\Theta u = \mathcal{L}u, \quad \operatorname{dom} A_\Theta = \Big\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), \frac{\partial u}{\partial \nu_{\mathcal{L}}}\big|_{\partial\Omega} + \Theta u|_{\partial\Omega} = 0 \Big\},$$

in $L^2(\Omega)$, cf. (2.10), where $\Theta : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is an operator which satisfies Assumption 2.6 from Chapter 2 above, that is, $\Theta = \Theta_1 + \Theta_2$, where $\Theta_i : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ are bounded operators with

$$(\Theta_i g, h)_{\partial\Omega} = (g, \Theta_i h)_{\partial\Omega}, \quad g, h \in H^{1/2}(\partial\Omega),$$

$i = 1, 2$, such that Θ_1 is L^2-semibounded, i.e.,

$$(\Theta_1 g, g)_{\partial\Omega} \geq c_{\Theta_1} \|g\|^2_{L^2(\partial\Omega)}, \quad g \in H^{1/2}(\partial\Omega),$$

for some $c_{\Theta_1} \in \mathbb{R}$, and Θ_2 is compact. We provide analogs of the theorems in the previous section, where the Dirichlet operator A_{D} is replaced by the operator A_Θ and the Dirichlet-to-Neumann map $M(\lambda)$ is replaced by the Robin-to-Dirichlet map $M_\Theta(\lambda)$ which is given by

$$M_\Theta(\lambda) = (\Theta - M(\lambda))^{-1}, \quad \lambda \in \rho(A_\Theta) \cap \rho(A_{\mathrm{D}}),$$

see (2.21) and (2.22).

The following lemma is a consequence of the formula (3.13) in Lemma 3.10; cf. the proof of Lemma 4.1.

Lemma 4.12. *Let Assumption 2.1 and Assumption 2.6 be satisfied and let $M_\Theta(\lambda + i\eta)$ be the Robin-to-Dirichlet map in* (2.21). *Then for all $\lambda \in \mathbb{R}$ the strong limit*

$$\operatorname*{s-lim}_{\eta \searrow 0} \eta M_\Theta(\lambda + i\eta)$$

exists, that is, $\lim_{\eta\searrow 0} \eta M_\Theta(\lambda + i\eta) g$ exists in $H^{1/2}(\partial\Omega)$ for all $g \in H^{-1/2}(\partial\Omega)$.

The following theorem is an analog of Theorem 4.2 for the operator A_Θ instead of the Dirichlet operator. We denote by $\operatorname{Res}_\lambda M_\Theta$ the residue of the analytic function $M_\Theta(\cdot)$ at some pole λ.

Theorem 4.13. *Let Assumption 2.1 hold and let Θ satisfy Assumption 2.6. Moreover, let A_Θ be the selfadjoint operator given in* (2.10) *and let $M_\Theta(\lambda)$ be the Robin-to-Dirichlet map in* (2.21). *For $\lambda \in \mathbb{R}$ the following assertions hold.*

(i) *$\lambda \in \rho(A_\Theta)$ if and only if $M_\Theta(\cdot)$ can be continued analytically to λ.*

(ii) *$\lambda \in \sigma_{\mathrm{p}}(A_\Theta)$ if and only if $\operatorname{s-lim}_{\eta\searrow 0} \eta M_\Theta(\lambda + i\eta) \neq 0$. If λ is an eigenvalue with finite multiplicity then the mapping*

$$\tau : \ker(A_\Theta - \lambda) \to \Big\{ \lim_{\eta \searrow 0} \eta M_\Theta(\lambda + i\eta) g : g \in H^{-1/2}(\partial\Omega) \Big\}, \quad u \mapsto u|_{\partial\Omega}$$

is bijective; if λ is an eigenvalue with infinite multiplicity then the mapping

$$\tau : \ker(A_\Theta - \lambda) \to \mathrm{cl}_\tau \Big\{ \lim_{\eta \searrow 0} \eta M_\Theta(\lambda + i\eta) g : g \in H^{-1/2}(\partial\Omega) \Big\}, \quad u \mapsto u|_{\partial\Omega}$$

is bijective, where cl_τ denotes the closure in the normed space $\operatorname{ran} \tau$.

(iii) *λ is an isolated eigenvalue of A_Θ if and only if λ is a pole of $M_\Theta(\cdot)$. If λ is an isolated eigenvalue with finite multiplicity then the mapping*

$$\tau : \ker(A_\Theta - \lambda) \to \operatorname{ran} \operatorname{Res}_\lambda M_\Theta, \quad u \mapsto u|_{\partial\Omega}$$

is bijective; if λ is an isolated eigenvalue with infinite multiplicity then the mapping

$$\tau : \ker(A_\Theta - \lambda) \to \mathrm{cl}_\tau(\operatorname{ran} \operatorname{Res}_\lambda M_\Theta), \quad u \mapsto u|_{\partial\Omega}$$

is bijective with cl_τ as in (ii).

(iv) *$\lambda \in \sigma_{\mathrm{c}}(A_\Theta)$ if and only if $M_\Theta(\cdot)$ cannot be continued analytically to λ and $\operatorname{s-lim}_{\eta\searrow 0} \eta M_\Theta(\lambda + i\eta) = 0$.*

The proof of Theorem 4.13 will not be carried out. It is analogous to the proof of Theorem 4.2, where Lemma 3.1 must be replaced by Lemma 3.10.

If Ω is bounded then the spectrum of A_Θ is purely discrete, see Chapter 2, that is, $\sigma(A_\Theta)$ consists of isolated eigenvalues with finite multiplicities. In this case Theorem 4.13 reduces to Theorem 3.13.

The next theorem shows how the absolutely continuous spectrum of the operator A_Θ in (2.10) can be expressed in terms of the limits of the Robin-to-Dirichlet map at real points. Recall the definition of the absolutely continuous closure $\mathrm{cl}_{\mathrm{ac}}$ in (4.15).

Theorem 4.14. *Let Assumption 2.1 hold and let Θ satisfy Assumption 2.6. Moreover, let A_Θ be the selfadjoint operator given in* (2.10) *and let $M_\Theta(\lambda)$ be the Robin-to-Dirichlet map in* (2.21). *Then the absolutely continuous spectrum of A_Θ is given by*

$$\sigma_{\mathrm{ac}}(A_\Theta) = \overline{\bigcup_{g\in H^{-1/2}(\partial\Omega)} \mathrm{cl}_{\mathrm{ac}}\left(\left\{x\in\mathbb{R} : 0 < \mathrm{Im}(M_\Theta(x+i0)g,g)_{\partial\Omega} < +\infty\right\}\right)}.$$

In particular, if $a,b\in\mathbb{R}$ with $a<b$ then $(a,b)\cap\sigma_{\mathrm{ac}}(A_\Theta)=\emptyset$ if and only if $\mathrm{Im}(M_\Theta(x+i0)g,g)_{\partial\Omega}=0$ *holds for all $g\in H^{-1/2}(\partial\Omega)$ and for almost all $x\in(a,b)$.*

A sufficient criterion for the absence of singular continuous spectrum within some interval in terms of the limiting behavior of the function $M_\Theta(\cdot)$ can be formulated as follows.

Theorem 4.15. *Let Assumption 2.1 hold and let Θ satisfy Assumption 2.6. Moreover, let A_Θ be the selfadjoint operator given in* (2.10), *let $M_\Theta(\lambda)$ be the Robin-to-Dirichlet map in* (2.21), *and let $a,b\in\mathbb{R}$ with $a<b$. If for each $g\in H^{-1/2}(\partial\Omega)$ there exist at most countably many $x\in(a,b)$ such that*

$$\mathrm{Im}(M_\Theta(x+iy)g,g)_{\partial\Omega}\to+\infty \quad \textit{and} \quad y(M_\Theta(x+iy)g,g)_{\partial\Omega}\to 0, \quad y\searrow 0,$$

then $(a,b)\cap\sigma_{\mathrm{sc}}(A_\Theta)=\emptyset$.

The proofs of Theorem 4.14 and Theorem 4.15 will be omitted. They are analogs of the proofs of Theorem 4.4 and Theorem 4.5, respectively, with a use of Lemma 3.10 instead of Lemma 3.1.

We conclude this chapter with the following immediate corollary of the theorems of this section. It provides sufficient criteria for the spectrum of A_Θ to be purely absolutely continuous or purely singular continuous in terms of the limiting behavior of the Robin-to-Dirichlet map $M_\Theta(\lambda)$ when λ approaches the real line.

Corollary 4.16. *Let Assumption 2.1 be satisfied, let* Θ *satisfy Assumption 2.6, let* A_Θ *be the selfadjoint operator in* (2.10)*, and let* $M_\Theta(\lambda)$ *be the Robin-to-Dirichlet map in* (2.21)*. Let* $a, b \in \mathbb{R}$ *with* $a < b$*. Moreover, for all* $g \in H^{-1/2}(\partial\Omega)$ *and all* $x \in (a, b)$ *let*

$$\lim_{y \searrow 0} y M_\Theta(x + iy)g = 0.$$

Then the following assertions hold.

(i) *If* $\operatorname{Im}(M_\Theta(x + i0)g, g)_{\partial\Omega} = 0$ *holds for all* $g \in H^{-1/2}(\partial\Omega)$ *and almost all* $x \in (a, b)$ *then* $\sigma(A_\Theta) \cap (a, b) = \sigma_{\rm sc}(A_\Theta) \cap (a, b)$*.*

(ii) *If for each* $g \in H^{-1/2}(\partial\Omega)$ *there exist at most countably many* $x \in (a, b)$ *such that* $\operatorname{Im}(M_\Theta(x + i0)g, g)_{\partial\Omega} = +\infty$ *then* $\sigma(A_\Theta) \cap (a, b) = \sigma_{\rm ac}(A_\Theta) \cap (a, b)$*.*

A Appendix

A.1 Spectral properties of Borel measures

In this appendix we provide some basic statements on the Borel transform of a finite Borel measure. We point out its connection to the absolutely continuous and singular continuous parts of the measure as they are used in the main part of this thesis in order to describe the spectral parts of selfadjoint elliptic differential operators. The results presented in this appendix are known; our presentation is mainly based on [120, Chapter 3 and Appendix A.8] and [110, Chapter 8]; cf. also [112, Chapter IV] for the derivatives of measures.

Let μ be a finite Borel measure on $\mathbb{R}$. Recall that μ admits a unique decomposition $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{s}}$, where μ_{ac} is absolutely continuous and μ_{s} is singular (both with respect to the Lebesgue measure), and that, moreover, the singular part μ_{s} of μ can be decomposed uniquely into $\mu_{\mathrm{s}} = \mu_{\mathrm{pp}} + \mu_{\mathrm{sc}}$, where μ_{pp} is supported on a countable set and μ_{sc} does not possess any point masses, that is, $\mu_{\mathrm{sc}}(\{x\}) = 0$ for all $x \in \mathbb{R}$. Let us denote the set of all growth points of μ by $\operatorname{supp} \mu$, that is,

$$\operatorname{supp} \mu = \{x \in \mathbb{R} : \mu((x - \varepsilon, x + \varepsilon)) > 0 \text{ for all } \varepsilon > 0\} ;$$

sometimes this set is also called the *spectrum of* μ. Note that $\operatorname{supp} \mu$ is a support of μ, that is, $\mu(\mathbb{R} \setminus \operatorname{supp} \mu) = 0$. In order to characterize $\operatorname{supp} \mu_{\mathrm{ac}}$ we define the *absolutely continuous closure* (or *essential closure*) of a Borel set $\chi \subset \mathbb{R}$ by

$$\mathrm{cl}_{\mathrm{ac}}(\chi) = \big\{x \in \mathbb{R} : |(x - \varepsilon, x + \varepsilon) \cap \chi| > 0 \text{ for all } \varepsilon > 0\big\},$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$. It is the aim of this appendix to provide a proof of the following lemma.

Lemma A.1. *Let μ be a finite Borel measure on $\mathbb{R}$ and denote by $F(\cdot)$ its Borel transform, i.e.,*

$$F(\lambda) = \int_{\mathbb{R}} \frac{1}{t - \lambda} d\mu(t), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Then the limit $\operatorname{Im} F(x + i0) = \lim_{y \searrow 0} \operatorname{Im} F(x + iy)$ exists and is finite for Lebesgue almost all $x \in \mathbb{R}$. Moreover, for the absolutely continuous part μ_{ac} and the singular continuous part μ_{sc} of μ the following assertions hold.

(i) $\operatorname{supp} \mu_{\mathrm{ac}} = \mathrm{cl}_{\mathrm{ac}}(\{x \in \mathbb{R} : 0 < \operatorname{Im} F(x + i0) < +\infty\})$.

(ii) *The set $M_{\mathrm{sc}} = \{x \in \mathbb{R} : \operatorname{Im} F(x + i0) = +\infty, \lim_{y \searrow 0} yF(x + iy) = 0\}$ is a support for μ_{sc}, that is, $\mu_{\mathrm{sc}}(\mathbb{R} \setminus M_{\mathrm{sc}}) = 0$.*

Proof. Let us introduce the derivative

$$(D\mu)(x) := \lim_{\varepsilon \searrow 0} \frac{\mu((x-\varepsilon, x+\varepsilon))}{2\varepsilon}$$

of μ at x for all $x \in \mathbb{R}$ where the limit exists in $\mathbb{R} \cup \{+\infty\}$. It is well known that $(D\mu)(x)$ exists in $\mathbb{R} \cup \{+\infty\}$ for Lebesgue almost all $x \in \mathbb{R}$ and coincides Lebesgue almost everywhere with the Radon–Nikodym derivative of μ_{ac}, i.e., for each Borel set $\chi \subset \mathbb{R}$ we have

$$\mu_{\text{ac}}(\chi) = \int_\chi (D\mu)(x)dx, \tag{A.1}$$

see, e.g., [120, Theorem A.37]. In particular, $(D\mu)(x)$ is finite for almost all $x \in \mathbb{R}$. In order to prove the items (i) and (ii) we will verify the following

Claim. If $(D\mu)(x)$ exists in $\mathbb{R} \cup \{+\infty\}$ then the limit $\operatorname{Im} F(x + i0)$ exists in $\mathbb{R} \cup \{+\infty\}$ and coincides with $\pi(D\mu)(x)$.

Proof of the claim. Assume first that $x \in \mathbb{R}$ is chosen such that $(D\mu)(x)$ exists in $\mathbb{R}$. We observe that

$$\operatorname{Im} F(x+iy) = \int_{\mathbb{R}} K_y(t-x)d\mu(t)$$

holds for $y > 0$ with $K_y(s) := \frac{y}{s^2+y^2}$, $s \in \mathbb{R}$. We have to show that $\lim_{y\searrow 0} \operatorname{Im} F(x+iy)$ exists and equals $\pi(D\mu)(x)$. Let us choose $c, C \in \mathbb{R}$ with $c < (D\mu)(x) < C$. Then there exists $\delta > 0$ such that

$$c \le \frac{\mu((x-s, x+s))}{2s} \le C \tag{A.2}$$

holds for all $s \in (0, \delta]$. As an abbreviation we write $I_\delta := (x-\delta, x+\delta)$. Then, clearly,

$$\operatorname{Im} F(x+iy) = \int_{I_\delta} K_y(t-x)d\mu(t) + \int_{\mathbb{R}\setminus I_\delta} K_y(t-x)d\mu(t) \tag{A.3}$$

for all $y > 0$ and the second integral on the right-hand side satisfies

$$0 \le \int_{\mathbb{R}\setminus I_\delta} K_y(t-x)d\mu(t) \le K_y(\delta)\mu(\mathbb{R}) \to 0, \quad y \searrow 0. \tag{A.4}$$

In order to estimate the first integral in (A.3) we integrate $K_y'(s)$ with respect to $dsd\mu(t)$ over the triangle

$$\{(s,t) : x-s < t < x+s, 0 < s < \delta\}$$

$$= \{(s,t) : x-\delta < t < x, -t+x < s < \delta\}$$
$$\cup \{(s,t) : x \le t < x+\delta, t-x < s < \delta\}\,.$$

This yields

$$\int_0^\delta \int_{x-s}^{x+s} K_y'(s) d\mu(t) ds = \int_0^\delta K_y'(s)\mu(I_s) ds$$

and

$$\int_{x-\delta}^{x} \int_{-t+x}^{\delta} K_y'(s) ds d\mu(t) + \int_{x}^{x+\delta} \int_{t-x}^{\delta} K_y'(s) ds d\mu(t)$$
$$= \int_{x-\delta}^{x} \big(K_y(\delta) - K_y(-t+x)\big) d\mu(t) + \int_{x}^{x+\delta} \big(K_y(\delta) - K_y(t-x)\big) d\mu(t)$$
$$= \mu(I_\delta) K_y(\delta) - \int_{x-\delta}^{x+\delta} K_y(t-x) d\mu(t),$$

hence

$$\int_0^\delta K_y'(s)\mu(I_s) ds = \mu(I_\delta)K_y(\delta) - \int_{I_\delta} K_y(t-x) d\mu(t). \tag{A.5}$$

Note further that

$$\delta K_y(\delta) + \int_0^\delta (-sK_y'(s)) ds = \arctan(\delta/y).$$

From this together with (A.2) and (A.5) it follows

$$2c \arctan \delta/y = 2c\delta K_y(\delta) + 2c \int_0^\delta s(-K_y'(s)) ds$$
$$\le \mu(I_\delta)K_y(\delta) - \int_0^\delta \mu(I_s) K_y'(s) ds$$
$$= \int_{I_\delta} K_y(t-x) d\mu(t)$$

and analogously

$$\int_{I_\delta} K_y(t-x) d\mu(t) \le 2C \arctan(\delta/y).$$

Thus we have proved

$$2c \arctan(\delta/y) \le \int_{I_\delta} K_y(t-x) d\mu(t) \le 2C \arctan(\delta/y);$$

passing over to the limit $y \searrow 0$ and taking (A.3) and (A.4) into account we obtain

$$\pi c \leq \liminf_{y \searrow 0} \operatorname{Im} F(x+iy) \leq \limsup_{y \searrow 0} \operatorname{Im} F(x+iy) \leq \pi C. \tag{A.6}$$

Since c and C were chosen arbitrarily with $c < (D\mu)(x) < C$, it follows that $\lim_{y \searrow 0} \operatorname{Im} F(x+iy)$ exists and equals $\pi (D\mu)(x)$. If $(D\mu)(x) = +\infty$ then for an arbitrary $c \in \mathbb{R}$ the above reasoning yields

$$\pi c \leq \liminf_{y \searrow 0} \operatorname{Im} F(x+iy)$$

instead of (A.6), which implies $\lim_{y \searrow 0} \operatorname{Im} F(x+iy) = +\infty$ and completes the proof of the claim. ■

Now we are able to verify the assertions (i) and (ii) of the lemma.

(i) With the definition

$$M_{\mathrm{ac}} := \{x \in \mathbb{R} : 0 < \operatorname{Im} F(x+i0) < +\infty\}$$

we have to prove

$$\operatorname{supp} \mu_{\mathrm{ac}} = \operatorname{cl}_{\mathrm{ac}}(M_{\mathrm{ac}}). \tag{A.7}$$

Assume first that $x \notin \operatorname{cl}_{\mathrm{ac}}(M_{\mathrm{ac}})$, that is, there exists $\varepsilon > 0$ such that we have $|(x-\varepsilon, x+\varepsilon) \cap M_{\mathrm{ac}}| = 0$ and thus $\mu_{\mathrm{ac}}((x-\varepsilon, x+\varepsilon) \cap M_{\mathrm{ac}}) = 0$. With

$$\widetilde{M}_{\mathrm{ac}} := \{x \in \mathbb{R} : 0 < (D\mu)(x) < +\infty\}$$

we have $\widetilde{M}_{\mathrm{ac}} \subset M_{\mathrm{ac}}$ by the above claim and it follows

$$\begin{aligned}\mu_{\mathrm{ac}}((x-\varepsilon, x+\varepsilon)) &= \mu_{\mathrm{ac}}((x-\varepsilon, x+\varepsilon) \setminus M_{\mathrm{ac}}) \leq \mu_{\mathrm{ac}}\big((x-\varepsilon, x+\varepsilon) \setminus \widetilde{M}_{\mathrm{ac}}\big) \\ &= \int_{(x-\varepsilon, x+\varepsilon) \setminus \widetilde{M}_{\mathrm{ac}}} (D\mu)(x) dx = 0,\end{aligned}$$

see (A.1); hence $x \notin \operatorname{supp} \mu_{\mathrm{ac}}$. Let now $x \notin \operatorname{supp} \mu_{\mathrm{ac}}$. Then there exists $\varepsilon > 0$ with

$$\begin{aligned}0 = \mu_{\mathrm{ac}}((x-\varepsilon, x+\varepsilon)) &= \int_{(x-\varepsilon, x+\varepsilon)} (D\mu)(x) dx \\ &= \frac{1}{\pi} \int_{(x-\varepsilon, x+\varepsilon)} \operatorname{Im} F(x+i0) dx = \frac{1}{\pi} \int_{(x-\varepsilon, x+\varepsilon) \cap M_{\mathrm{ac}}} \operatorname{Im} F(x+i0) dx\end{aligned}$$

by the claim. This implies $|(x-\varepsilon, x+\varepsilon) \cap M_{\mathrm{ac}}| = 0$, that is, $x \notin \operatorname{cl}_{\mathrm{ac}}(M_{\mathrm{ac}})$. Thus we have shown (A.7).

(ii) In order to verify the statement of item (ii) we first prove that the singular part μ_{s} of μ does not act on the set of points x with $(D\mu)(x) < +\infty$, that is,

$$\mu_{\mathrm{s}}\big(\{x \in \mathbb{R} : (D\mu)(x) < +\infty\}\big) = 0. \tag{A.8}$$

Let us assume the converse; then there exists a Borel set E with $(D\mu)(x) < +\infty$ for all $x \in E$, $|E| = 0$, and $\mu(E) > 0$. Let us set

$$E_n = \{x \in E : (D\mu)(x) < n\}, \quad n \in \mathbb{N}.$$

Then, clearly, $E = \bigcup_{n\in\mathbb{N}} E_n$; in particular, there exists $n \in \mathbb{N}$, which we fix, with $\mu(E_n) > 0$. Let us further define

$$A_j = \Big\{x \in E_n : \mu((x-\delta, x+\delta)) < n2\delta \text{ for all } \delta < \frac{1}{j}\Big\}, \quad j \in \mathbb{N}. \tag{A.9}$$

We have $E_n = \bigcup_{j\in\mathbb{N}} A_j$, hence there exists some $j \in \mathbb{N}$ with $\mu(A_j) > 0$. By the regularity of μ there exists a compact set $K \subset A_j$ with $\mu(K) > 0$. Moreover, as a subset of E, K satisfies $|K| = 0$. Thus for each $\varepsilon > 0$ there exists an open set $V \supset K$ with $|V| < \frac{\varepsilon}{3n}$. With $r := \operatorname{dist}(\partial V, K) > 0$ we choose a sequence of disjoint, non-degenerate intervals $I_l \subset V$, $l = 1, \dots, N \le \infty$, with $|I_l| < \min\{\frac{r}{3}, \frac{2}{3j}\}$ for all l and $\bigcup_l I_l = V$. Let I_{l_m} be precisely those of these intervals which have a nonempty intersection with K. Moreover, let us extend each I_{l_m} to an open interval $\widetilde{I}_{l_m}$ centered in K with $|\widetilde{I}_{l_m}| \le 3|I_{l_m}|$. Then the $\widetilde{I}_{l_m}$ satisfy

$$\widetilde{I}_{l_m} = (x_m - \delta_m, x_m + \delta_m)$$

for appropriate $x_m \in K$ and $\delta_m > 0$ with $\delta_m < \frac{1}{j}$. In particular, $\mu(\widetilde{I}_{l_m}) < n2\delta_m = n|\widetilde{I}_{l_m}|$ for all m by (A.9). From $K \subset \bigcup_m \widetilde{I}_{l_m}$ we obtain

$$\mu(K) \le \sum_m \mu(\widetilde{I}_{l_m}) < n \sum_m |\widetilde{I}_{l_m}| \le 3n \sum_m |I_{l_m}| \le 3n|V| < \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, it follows $\mu(K) = 0$, a contradiction. Thus we have proved (A.8). From this and the fact that $(D\mu)(x)$ exists in $\mathbb{R} \cup \{+\infty\}$ for μ-almost every $x \in \mathbb{R}$, see [112, Chapter IV-(9.6)], it follows that

$$\{x \in \mathbb{R} : (D\mu)(x) = +\infty\}$$

is a support for μ_{s}. Moreover, $(D\mu)(x) = +\infty$ implies $\operatorname{Im} F(x + i0) = +\infty$, see the above claim. Thus also $\{x \in \mathbb{R} : \operatorname{Im} F(x + i0) = +\infty\}$ is a support for μ_{s}. Furthermore,

$$|yF(x+iy) - i\mu(\{x\})| \le \int_{\mathbb{R}} \left| \frac{y}{t-(x+iy)} - \chi_{\{x\}}(t) \right| d\mu(t) \to 0, \quad y \searrow 0,$$

by the dominated convergence theorem; in particular, $\mu(\{x\}) = 0$ if and only if $\lim_{y\searrow 0} yF(x+iy) = 0$. This yields that

$$M_{\rm sc} = \Big\{ x \in \mathbb{R} : \operatorname{Im} F(x+i0) = +\infty, \lim_{y\searrow 0} yF(x+iy) = 0 \Big\}$$

is a support for $\mu_{\rm sc}$, which completes the proof of the lemma. □

A.2 Simplicity of symmetric elliptic differential operators

In this short appendix we point out that the result of Proposition 3.4 in the main part of this thesis is equivalent to the fact that the symmetric differential operator

$$Su = \mathcal{L}u, \quad \operatorname{dom} S = \left\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega), u|_{\partial\Omega} = 0, \frac{\partial u}{\partial \nu_{\mathcal{L}}}\Big|_{\omega} = 0 \right\}, \quad \text{(A.10)}$$

in $L^2(\Omega)$ is simple (or completely non-selfadjoint), see Definition A.2 below. Here $\mathcal{L}$ is a uniformly elliptic differential expression as in Assumption 2.1 on a connected (bounded or unbounded) Lipschitz domain Ω and $\omega \subset \partial\Omega$ is a nonempty, relatively open set. Theorem A.3 below generalizes the main result in [65], where R. Gilbert proved the simplicity of certain symmetric ordinary differential operators which are in the limit-point case at one endpoint. We remark that in the special case $\omega = \partial\Omega$ the operator S is called the *minimal* symmetric operator associated with $\mathcal{L}$ in $L^2(\Omega)$.

Let us first recall the definition of a simple symmetric operator as it can be found in, e.g., [3, Chapter VII].

Definition A.2. Let S be a closed, densely defined, symmetric operator in a Hilbert space $\mathcal{H}$. Assume that there does not exists a nontrivial, S-invariant, closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ such that the restriction of S to $\mathcal{H}_1$ defines a selfadjoint operator in $\mathcal{H}_1$. Then S is called *simple*.

Sometimes such an operator is also called *completely non-selfadjoint*.

The proof of the following theorem uses arguments similar to the proof of [26, Lemma 2.6]. The idea of Step 2 is due to M. G. Krein, see [86].

Theorem A.3. *Let the differential expression $\mathcal{L}$ satisfy Assumption 2.1, let Ω be a connected Lipschitz domain, and let ω be a nonempty, open subset of $\partial\Omega$. Then the operator S in* (A.10) *is closed, densely defined, symmetric, and simple.*

Proof. **Step 1.** As a restriction of the selfadjoint Dirichlet operator the operator S is symmetric. Moreover, $\operatorname{dom} S$ contains $C_0^\infty(\Omega)$, hence S is densely defined in $L^2(\Omega)$. We verify next that S coincides with the adjoint of the operator T in $L^2(\Omega)$ which is defined as

$$Tu = \mathcal{L}u, \quad \operatorname{dom} T = \left\{ u \in H^1(\Omega) : \mathcal{L}u \in L^2(\Omega),\ \operatorname{supp}(u|_{\partial\Omega}) \subset \omega \right\}.$$

Let first $u \in \operatorname{dom} S$. Then for all $v \in \operatorname{dom} T$ the second Green identity (1.11) yields

$$(Tv, u) = (v, \mathcal{L}u) + \Big(v|_{\partial\Omega}, \frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} \Big)_{\partial\Omega} - \Big(\frac{\partial v}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega}, u|_{\partial\Omega} \Big)_{\partial\Omega} = (v, Su),$$

since $u|_{\partial\Omega} = 0$, $\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\omega} = 0$, and $\operatorname{supp}(v|_{\partial\Omega}) \subset \omega$. Hence $u \in \operatorname{dom} T^*$ and $T^*u = Su$. Let, conversely, $u \in \operatorname{dom} T^*$. Since the Dirichlet operator A_{D} is a restriction of T, we have $T^* \subset A_{\mathrm{D}}$, hence $u \in \operatorname{dom} A_{\mathrm{D}}$ and $T^*u = A_{\mathrm{D}}u = \mathcal{L}u$. It remains to show $\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\omega} = 0$. Indeed, let $g \in H^{1/2}(\partial\Omega)$ with $\operatorname{supp} g \subset \omega$. It follows from Lemma 2.9 that there exists $v \in H^1(\Omega)$ with $\mathcal{L}v \in L^2(\Omega)$ and $v|_{\partial\Omega} = g$; in particular, $v \in \operatorname{dom} T$. From the second Green identity (1.11) we obtain

$$\begin{aligned} \Big(\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega}, g \Big)_{\partial\Omega} &= \Big(\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega}, v|_{\partial\Omega} \Big)_{\partial\Omega} \\ &= (u, Tv) - (T^*u, v) + \Big(u|_{\partial\Omega}, \frac{\partial v}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega} \Big)_{\partial\Omega} = 0, \end{aligned}$$

since $u|_{\partial\Omega} = 0$. Hence $\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\omega} = 0$, that is, $u \in \operatorname{dom} S$. Therefore we have $T^* = S$ and, in particular, S is closed.

Step 2. Let us prove that the subspace

$$\mathcal{M} := \bigcap_{\nu \in \mathbb{C} \setminus \mathbb{R}} \operatorname{ran}(S - \nu)$$

is S-invariant and that the restriction $S_{\mathcal{M}}$ of S to $\mathcal{M}$ is selfadjoint in $\mathcal{M}$. Indeed, let $u \in \operatorname{dom} S \cap \mathcal{M}$. Then for each $\nu \in \mathbb{C} \setminus \mathbb{R}$ there exist $u_\nu \in \operatorname{dom} S$ with $(S - \nu)u_\nu = u$. Hence, for each $\nu \in \mathbb{C} \setminus \mathbb{R}$ we have

$$Su = S(S - \nu)u_\nu = (S - \nu)Su_\nu \in \operatorname{ran}(S - \nu),$$

that is, Su belongs to $\mathcal{M}$. As a restriction of S, the operator $S_{\mathcal{M}}$ is symmetric. In order to show that $S_{\mathcal{M}}$ is selfadjoint, we fix $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and prove that

$$\operatorname{ran}(S_{\mathcal{M}} - \lambda) = \mathcal{M} \tag{A.11}$$

holds. Let $u \in \mathcal{M}$ and define $v := (S - \lambda)^{-1}u$. Then, clearly, $v \in \operatorname{dom} S$, and we will verify that even $v \in \mathcal{M}$ holds, that is, $v \in \operatorname{ran}(S - \nu)$ for all $\nu \in \mathbb{C} \setminus \mathbb{R}$. For $\nu \neq \lambda$ one can see immediately that the element

$$v_\nu := \frac{1}{\nu - \lambda} \left((S - \nu)^{-1} - (S - \lambda)^{-1} \right) u$$

in $\operatorname{dom} S$ satisfies $(S-\nu)v_\nu = v$. It remains to show $v \in \operatorname{ran}(S-\lambda)$. Let us choose a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathcal{O}$, $\lambda_k \neq \lambda$, with $\lambda_k \to \lambda$, $k \to \infty$. As above one gets $(S-\lambda_k)^{-1}u \in \operatorname{ran}(S-\lambda)$ for all $k \in \mathbb{N}$. Since S is a closed, symmetric operator, the estimates $\|(S-\lambda)^{-1}\| \leq |\operatorname{Im}\lambda|^{-1}$ and $\|(S-\lambda_k)^{-1}\| \leq |\operatorname{Im}\lambda_k|^{-1}$ hold, which together with

$$v-(S-\lambda_k)^{-1}u = (S-\lambda)^{-1}u-(S-\lambda_k)^{-1}u = (\lambda-\lambda_k)(S-\lambda)^{-1}(S-\lambda_k)^{-1}u$$

imply $(S-\lambda_k)^{-1}u \to v$, $k\to\infty$. Since $(S-\lambda_k)^{-1}u$ belongs to the closed subspace $\operatorname{ran}(S-\lambda)$, it follows $v \in \operatorname{ran}(S-\lambda)$. This proves (A.11) and thus we have shown that $S_{\mathcal{M}}$ is a selfadjoint operator in the Hilbert space $\mathcal{M}$.

Assume now that S is not simple. Then there exists a nontrivial, S-invariant subspace $\mathcal{M}'$ of $L^2(\Omega)$ such that the restriction $S_{\mathcal{M}'}$ of S to $\mathcal{M}'$ is selfadjoint in $\mathcal{M}'$. It follows that for each $\nu \in \mathbb{C}\setminus\mathbb{R}$ we have $\operatorname{ran}(S_{\mathcal{M}'}-\nu) = \mathcal{M}'$, in particular, $\mathcal{M}' \subset \mathcal{M}$, so that $\mathcal{M}$ is nontrivial. On the other hand, since $T^* = S$ the orthogonal complement $\mathcal{M}^\perp$ of $\mathcal{M}$ in $L^2(\Omega)$ coincides with the closure of

$$\operatorname{span}\{\ker(T-\nu) : \nu \in \mathbb{C}\setminus\mathbb{R}\},$$

and it follows from Proposition 3.4 that $\mathcal{M}^\perp = L^2(\Omega)$, so that $\mathcal{M}$ must be trivial, which is a contradiction. Thus S is simple. □

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