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Solutions of Boundary Value Problems for Nonlinear Partial Differential Equations by Fixed Point Methods

DISSERTATION

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Dedicated to my beloved parents

Shamim Iqbal
and
Muhammad Iqbal Asad

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Preface

Comments of the supervisor

The goal of the Thesis is to apply fixed-point methods to partial differential equations of elliptic type. The background of this method is the reduction of the initial value problem

$$y' = f(x,y)$$
$$y(x_0) = y_0$$

for ordinary differential equations to a fixed-point problem for the operator

$$Y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))d\xi$$

(model equation). In case $E(x,\xi)$ is a fundamental solution of the linear differential operator $\mathfrak{L}$, the boundary value problem

$$\begin{aligned} \mathfrak{L}u &= \mathfrak{F}(x,u) \ \text{ in } \ \Omega \\ u &= g \ \text{ on } \ \partial\Omega \end{aligned}$$

can be reduced to a fixed-point problem for the operator

$$U(x) = u_0(x) + \tilde{u}(x) + \int_{\Omega} E(x,\xi)\mathfrak{F}(\xi, u(\xi))d\xi, \qquad (*)$$

where u_0 is a solution of the boundary value problem for the homogeneous equation $\mathfrak{L}u = 0$ and $\tilde{u}$ compensates the boundary values of the domain integral to zero.

In case the right-hand side $\mathfrak{F}(x,u)$ depends only on the desired function u (and not on its derivatives), the corresponding fixed-point problem can be solved in the space of continuous functions. The necessary estimates of the (weakly singular) integral operator are more complicated if the right-hand side $\mathfrak{F}$ depends not only on the function u itself but also on its (first-order) derivatives $\partial_i u$, that is, we consider a partial differential equation of the form

$$\mathfrak{L}u = \mathfrak{F}(x,u,\partial_i u).$$

Of course, in this case a suitable function space is the space of continuously differentiable functions. The auxiliary solutions u_0 and $\tilde{u}$ are to be estimated by Schauder estimates, and therefore the underlying function space is the space of Hölder continuously differentiable functions.

The author of the Thesis has not only to find the corresponding estimates for singular integrals and the necessary estimates of Schauder type in the literature, but also he has to adapt the proofs to the special situation of the operator $(*)$. The author should be in a position to realize complete proofs of all tools which exceed the basic knowledge of Mathematical Analysis. An Appendix of the Thesis should contain at least the sketches of the proofs of all advanced tools which are to be applied. The generality of the basic material should be as high as necessary so that the author is in a position – if desired – to teach a corresponding course in his home university with complete proofs.

The starting point of the thesis is a lecture on "Partial differential equations 2" given by the supervisor. This lecture considers the much simpler case that the right-hand side does not depend on the first-order derivatives, that is, the right-hand side has the form $\mathfrak{F}(x,u)$. The author is allowed to use some arguments of that lecture and of related lectures of supervoisor without quoting those passages.

List of Symbols and Abbreviation

Symbol	Description
Ω	always is an open subset (domain) in Euclidean space $\mathbb{R}^n$, $n \geq 2$
$m\Omega$	the finite measure of the domain
$\mathcal{L}$	is general linear second order elliptic differential operator of divergence type
$\mathcal{L}^*$	Adjoint to the linear second order elliptic differential operator $\mathcal{L}$
$\partial_i = \dfrac{\partial}{\partial x_i}$	first order derivative with respect to the ith component
$E(x, \xi)$	fundamental solution of a linear elliptic partial differential equation with singularity at ξ
C^k	Space of functions of which derivatives up to the order k are continuous
$C^{k,\alpha}$ and $0 < \alpha < 1$	Space of functions of which derivatives up to the order k are Hölder continuous
ℓ	differential operator acting on the boundary
U	Image of u
$\|\cdot\|_{C^{k,\alpha}}$	Hölder norm of the function of which derivative up to order k are Hölder continuous defined in (4.7) Chapter 4
$\|\cdot\|^*_{C^{k,\alpha}}$	Weighted Hölder norm
$\mathfrak{R}$	The radius of the ball (a closed and convex subset of Banach space)

Symbol/Abbr.	**Description**
PDE	Partial differential equation
ODE	Ordinary differential equation
BVP	Boundary value problem
IVP	Initial value problem
ω_n	surface measure of unit ball in $\mathbb{R}^n$
τ_n	volume of unit ball in $\mathbb{R}^n$
ϕ	a twice continuously differentiable function vanishing at neighborhood of the boundary $\partial\Omega$ or simply a *test function*
$\mathcal{F}$	The right hand side of a non-linear partial differential equation mainly a function depending on space like variable x, function $u(x)$ and $\partial_i u(x)$
Schauder (I)	First version of Schauder Fixed Point Theorem
Schauder (II)	Second version of Schauder Fixed Point Theorem

Abstract

We solve the boundary value problems for non-linear second order elliptic partial differential equations when the right hand side depends on the space like variable x, not only on the desired solution u but also on its first order derivatives $\partial_i u$. We show the existence and uniqueness by Schauder Fixed Point Theorem and Contraction Mapping Principle. We consider the $C^{1,\alpha}$ function space for our research work because in our boundary value problems the right hand side involves the first order derivatives generally. First we give the detailed proof of the result by which one can reduce the boundary value problem to a fixed point operator. The corresponding fixed point operator is defined by the fundamental solution of the linear homogeneous equation.

Next we discuss the necessary background material for the existence and uniqueness of the solution. For a special boundary value problem for the Laplace operator in unit disk the Schauder estimates has be proved. Since the corresponding fixed point operators involve the weakly singular kernels so we give also the necessary results on the computation of such integrals. The important result on the estimates of the singular integrals with two weak singularities have been included in Appendix B. Mapping properties of the corresponding fixed point operators are of fundamental importance which we need during the evaluation of $C^{1,\alpha}$-norm. The Chapter 3 deals with these properties of the fixed point operators where one learns to which Banach space the operators belong.

Then the existence and uniqueness of the solution of the boundary value problems have been formulated and various situations and restrictions have been discussed. In order to apply Schauder Fixed Point Theorem, certain restrictions and relatively compactness of the operators is also the part of this thesis. Contraction Mapping Principle put additional restrictions.

At the end we give the optimization results where we state a number of examples dealing with the different situations. We also give the largest possible bound **C** for $C^{1,\alpha}$-norms of the admissible boundary values. Then we determine the radius which leads to the largest **C**.

CONTENTS

Introduction

The main objective of the present thesis is to solve the boundary value problems for the non-linear second order elliptic partial differential equations.

To understand the basics of partial differential equation we refer [26]. Also [12] gives good information on partial differential equations in finite and even in infinite dimensions.

Our goal is confined to find the existence and uniqueness of the solutions of the Dirichlet problem:

$$\mathcal{L}u = \mathcal{F}(\cdot, u, \partial_i u) \quad \text{in} \quad \Omega \tag{0.1}$$
$$u = \varphi \quad \text{on} \quad \partial\Omega \tag{0.2}$$

where $\mathcal{L}$ is linear second order differential operator of the form

$$\mathcal{L}u = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x) \partial_j u \right) + \sum_{i=1}^{n} b_i(x) \partial_i u + cu. \tag{0.3}$$

From now on we shall consider the case of elliptic partial differential operator, i.e,

$$\sum_{i,j=1}^{n} a_{ij}(x) \gamma_i \gamma_j \geq \lambda \|\gamma\|^2 \quad \text{for all} \quad x, \gamma \in \mathbb{R}^n. \tag{0.4}$$

Fixed point methods are great tools to solve the partial differential equations specially the boundary value problem for non-linear partial differential equation (0.1) and (0.2). In the framework of analytical solutions of elliptic partial differential equation, often, the existence and uniqueness is shown by one of fixed point results. Already in the existing literature, there are a number of such fixed point theorems have been proved. Brouwer Fixed Point Theorem is a fundamental result in this regards in finite dimensions. Since we work in infinite dimensional function spaces, we here present the existence and uniqueness results for the solutions of boundary value problems of the form (0.1) and (0.2) by using the fixed point methods like Schauder Fixed Point Theorem and Contraction Mapping Principle. This will be done even for differential equations where right hand sides depend not only on the desired solutions but also on their first order derivatives. Further discussion on fixed point method is given in the chapter four. The explicit proof of Schauder Fixed Point Theorem is presented in the appendix C.

The first chapter of this thesis deals all the necessary aspects of reducing the boundary value problems (0.1) and (0.2) to the corresponding fixed point operator where we necessarily require the fundamental solution of the linear differential equation

$\mathcal{L}u = 0$. That is, we must need enough results on the existence of fundamental solutions of the homogeneous partial differential equations which we consider. In [35] W. Littman, G. Stampaccia and H. F. Weinberger give all necessary details on the existence of fundamental solutions for the (self adjoint) divergence type linear differential operators with measurable coefficients and foremost these fundamental solutions can be estimated by the fundamental solution of the Laplace equation. Similar result on the existence of fundamental solution has been given by C. Miranda in [38] but here the coefficients are required to be Hölder continuously differentiable. Luis Escauriaza [15] extend the results from [35] to the case of elliptic and parabolic partial differential equations when they are in non-divergence form and gives the estimate for the bound of the fundamental solution again with the weaker assumptions on the coefficients, that is, they need to be only measurable. All above results on the existence of fundamental solution are carried out for the domains in $\mathbb{R}^n$ for $n \geq 3$ but for two dimensional case the existence has been shown by C.E. Kenig and W. M. Ni in [29] and is estimated by

$$|E(x,\xi)| \leq \frac{C}{|x-\xi|^{n-2}} \quad \text{for } n \leq 3 \tag{0.5}$$

$$|E(x,\xi)| \leq C(1+\log|x-\xi|) \quad \text{for } n = 2. \tag{0.6}$$

M. Grüter and K.O. Widman [24] has provided the discussion on the fundamental solution of the non-self adjoint elliptic operator.

Chapter two is about the function spaces and other back ground material which is required to solve the boundary value problems by the fixed point techniques. We consider the boundary value problems with right hand side depending on the derivatives of the desired solution. So a natural demand is to consider the Banach space of continuously differentiable functions. But a lot of work has been done on the Schauder type estimates which give the bound of solutions of elliptic partial differential equations by its boundary values in the Hölder spaces. So we consider the $C^{1,\alpha}$ as Banach space. The first major article on Schauder estimates were carried out by S. Agmon, A. Douglis and L. Nirenberg in [2] where a comprehensive discussion is given and the Schauder type estimates for elliptic partial differential equations have been established in Hölder and L_p norms. Very similar estimates are found for the systems of elliptic partial differential equations in [3]. More over the Schauder estimates on both the interior and up to the boundary in the last two articles are given. E.A. Baderko has also wrote down a number of papers on the Schauder type estimates for elliptic and parabolic partial differential equations. Schauder estimates in Hölder spaces for oblique derivative problems have been found in [6]. Baderko uses her previous result [8] on Schauder estimates in $C^{1,\alpha}$-norm for parabolic partial differential equation to show [7] page(22-24) the Schauder estimates for partial differential equation of elliptic type in $C^{1,\alpha}$-norm which is an important estimate. For further information on Schauder type estimates in Hölder and Sobolev spaces we refer to [18] and [38].

Our fixed point operator is defined by an integral having the fundamental solution in its integrand which is singular but has a weak singularity. So an important consideration is to deal with the singular integrals. Since we shall carry out our constructions in the Hölder spaces so we have also to deal the singular integral with two weak singularities. E. M. Stein, in [47] and [46], has given a number of estimates for such integrals. S. G. Mikhlin, [37], [36] also deals with the singular integrals. We too prove the explicit estimates of the integrals with two weak singularities for a domain in $\mathbb{R}^n$. This result has been added in the appendix B. Another very important book of S. G. Mikhlin, N. F.Morozov, and M.V. Paukshto, *The integral equations of the theory of elasticity* , covers a number of properties on the theory of integral equations and also covers the singular integrals. As long as the fixed point operator (1.9) discussed in the Chapter 1 is not differentiated two times, we stay with weak singularity and then the integrals can be estimated comparatively easily. But if we differentiate two times, the order of the singularity becomes equal to the dimentions of the space and then this integral exists only in the sense of Cauchy principal value. Although these Cauchy type integrals are beyond the scope of the present thesis, but [48] is a good article to understand the facts on such integral due to their importance for future work.

Maximum principles are very important tools in the theory of partial differential equations and we are frequently use them to estimate the solution of linear elliptic equations. These maximum principles give the estimates of solutions by its boundary values. In literature there are a number of such maximum principle are already available. A fundamental result is the Hopf Maximum Principle which says the following

Let $u = u(x), x \in \mathbb{R}^n$ be a C^2 solution of $\mathcal{L}u \geq 0$ where $\mathcal{L}$ is elliptic linear differential operator (3) with $c \equiv 0$ in an open domain Ω and coefficients are locally bounded and if u takes maximum value M in Ω then $u \equiv M$.

D. Gilbarg, N. S. Trudinger in [18] and C. Miranda [38] respectively have given various maximum principles for the elliptic type second order partial differential equations which we use in our approach. For non-uniformly elliptic operator with measurable coefficients, maximum principles have been formulated in [49]. K. O. Widman, in [55] gives a quantitative form of maximum principle for elliptic equations. Another important book which covers most of the areas of elliptic partial differential equation is [42], where the authors give various maximum principles.

The third chapter is about the mapping and regularity properties of singular integral operators where we show that our fixed point operator (1.9) belongs to the spaces of continuously differentiable and Hölder continuously differentiable functions which also shows that the solution U (1.9) defined see in Chapter 1, maps these function spaces into themselves. We also prove an important property of the fixed point operator that is we prove its relative compactness in the space of Hölder continuously differentiable functions. This property is essential to show the existence results by Schauder Fixed Point Theorem. At the end of third chapter we show that operator (1.9) of Chapter 1, is contractive under certain conditions. Later on the contractivity of the operator leads to the additional restrictions to function spaces in some sense. For more regularity properties we refer to [40] where author gives the different situations for classical solutions of non-linear elliptic second order partial differential equations in the plane. X. Cabré, and L. A. Caffarelli, [10] use the Krylov-Safonov Harnack inequality and show the situations where the viscosity solution of fully non-linear second order elliptic Partial differential equation are in $C^{1,\alpha}$, $C^{2,\alpha}$. In our consideration due to the application of the Schauder estimates in the Hölder spaces, we need the boundary of the domain to be sufficiently smooth. For the domains with non-smooth boundaries one has to consider the Sobolev spaces to work in. Many cases of non-smooth domains or domains with edges have been discussed by V. A. Kondrat'ev, and O. A. Oleĭnik, [30]. Here authors provide a long survey on the solutions of non-linear elliptic and other types of differential equations mainly in non-smooth domains. Most of the work of O. A. Oleĭnik, covers the non-smooth domains and very useful references are given in her above article.

Fourth chapter is an important one where we present the existence and uniqueness results by Schauder Fixed point Theorem and Contraction Mapping Principle. We also give the explicit condition for mapping the balls (closed and convex subsets) of the Banach space into itself which then gives the conditions on the radii of the balls. That is we have to find the best balls. Moreover the estimates that we obtain help us also to show the maximum norm of the boundary values that can be considered. Only $C^{1,\alpha}$ is a Banach space that we have considered there. We use then the relative compactness to show the existence of solutions by Schauder Theorem. The second part of the Chapter 4 is about the the existence and uniqueness of the solution of the Dirichlet boundary value problems for non-linear partial differential equation. We show the contractivity and additional condition on the radius of the ball discussed in the fifth chapter. W. Tutschke [53], [54] has proved the results for the spaces of continuous function. S. Graubner, in [21], [22] has also proved such results. He replaces balls by poly-cylinders.

The Chapter 5 is the consequence of the chapter four. And in this we establish the optimization results. These optimizations provide the necessary information on the choice of largest possible interval in which we choose the radii of the balls. Moreover in certain cases we give also the largest possible bound for the boundary values that we can consider. We also give the largest possible bound **C** for $C^{1,\alpha}$-norms of the admissible boundary values. Then we determine the radius of the ball which leads to the largest **C**.

At the end we shall give the summary and planned work for the future.

1. REDUCTION OF BOUNDARY VALUE PROBLEMS FOR NON-LINEAR PDES TO FIXED-POINT PROBLEMS

Mainly, the present dissertation revolves around the solutions of the boundary value problems (BVPs) for non-linear partial differential equations (PDEs) by the fixed-point theorems[1]. We shall be considering the boundary value problems in case when the right hand side depends on the desired solution as well as on its first order derivatives. Naturally, it is most important that we must talk about the reduction of the boundary value problems to the fixed-point operator together with the necessary assumptions on the right hand side and coefficients of the differential operator and boundary values. The current chapter covers all the necessary steps and details required for the reduction. First section scales down the boundary value problem for non-linear partial differential equations to fixed point operator. So the goal of the chapter is that the solution of the boundary value problem is nothing but equivalent to find the fixed-point of the corresponding fixed-point operator.

We shall assume that the boundary value problem is invertible by the integral where the integrand contains a fundamental solution of the homogeneous partial differential equations, so it is obvious that we give a brief note on the fundamental solutions and their existence. So the second section deals the fundamental solutions of the homogeneous partial differential equations. Moreover, the fundamental solution of the Laplace equation for domains in $\mathbb{R}^n$ has been presented which gives good idea about the topic.

1.1. Reduction of boundary value problems for non-linear elliptic PDEs to fixed point operator equations

Let $\mathcal{L}$ be a differential operator defined in a domain Ω of $\mathbb{R}^n$, and let ℓ be an operator acting on the boundary $\partial\Omega$ of Ω. We focus on the reduction of the boundary value problems of the type

$$\mathcal{L}u = \mathcal{F}(\cdot, u, \partial_i u) \quad \text{in} \quad \Omega, \; i = 1, 2, 3, \cdots \tag{1.1}$$

$$\ell u = \varphi \quad \text{on} \quad \partial\Omega \tag{1.2}$$

to a fixed-point problem. Where the right hand side $\mathcal{F}$ of (1.1) is a given function depending on a point of domain Ω, the desired solution u and also on the first derivatives of u with respect to any of the components of the arbitrary point of the domain of $\mathbb{R}^n$.

[1] Contraction Mapping Principle and Schauder Fixed Point Theorem

We are going to solve the boundary value problem (1.1),(1.2) under the following assumptions:

(i) Suppose Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary

(ii) Let $\mathcal{L}$ be a linear differential operator of divergence type, that is

$$\begin{aligned}\mathcal{L}u &= \sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+\sum_i b_i\frac{\partial u}{\partial x_i}+cu\\ &= \sum_{i,j}\partial_i\left(a_{ij}\partial_j u\right)+\sum_i b_i\partial_i u+cu.\end{aligned}$$

(iii) Suppose that the homogeneous differential equation $\mathcal{L}u=0$ possesses the fundamental solution represented by $E(\cdot,\xi)$.

(iv) ℓ is a linear operator acting on the boundary $\partial\Omega$ such as the restriction of a certain function defined in $\overline{\Omega}$ to the boundary.

(v) Finally, we assume that the boundary value problem

$$\begin{aligned}\mathcal{L}u &= 0 \quad \text{in} \quad \Omega\\ \ell u &= \varphi \quad \text{on} \quad \partial\Omega\end{aligned}$$

is uniquely solvable.

It is important to note that the last assumption (v) above restricts not only the admissible boundary operator ℓ but also to the admissible domains Ω, i.e, only those domains can be considered which are bounded by sufficiently smooth boundaries.

All above assumptions are satisfied for the following Dirichlet boundary value problem:

$$\begin{aligned}\Delta u &= 0 \quad \text{in} \quad \Omega\\ u &= \varphi \quad \text{on} \quad \partial\Omega\end{aligned}$$

in sufficiently smoothly bounded domains. Here operator ℓ is only a certain restriction of desired solution to the boundary. It is well known that the Dirichlet problem for Laplace equation in the balls in $\mathbb{R}^n$ can be solved explicitly by the Poisson Integral Formula and of course is uniquely solvable.

Moreover, the unique solution of Dirichlet problem for the bi-potential operator Δ^2 requires not only the restriction of the desired solution u on the boundary itself but also its normal derivatives $\partial u/\partial N$ (so-called Neumann condition) on the boundary $\partial\Omega$.

Let u be a given solution of the following differential equation

$$\mathcal{L}u = \mathcal{F}(\cdot,u,\partial_i u) \quad \text{in} \quad \Omega.$$

We define the following operator,

$$V := \int_{\Omega} E(\cdot,\xi)\mathcal{F}(\xi,u(\xi),\partial_i u(\xi))\,d\xi, \tag{1.3}$$

where $\mathcal{F}$ is the given right hand side of differential equation (1.1).

A property of fundamental solutions implies that

$$\mathcal{L}V = \mathcal{F}(\cdot,u,\partial_i u). \tag{1.4}$$

Before proving a lemma on distributional solution of equations of type (1.4), we give the definition of the distributional solutions of the inhomogeneous partial differential equations

Definition (Distributional Solution) Suppose $\mathcal{L}^*$ is adjoint to the linear differential operator $\mathcal{L}$ and u is an integrable function satisfying the relation

$$\int_{\Omega} \left(\phi h + (-1)^{k+1} u\mathcal{L}^*\phi\right) dx = 0 \tag{1.5}$$

for any choice of a test function ϕ. Then u is called a weak solution of the differential equation $\mathcal{L}u = h$ in the distributional sense. Additionally, a weak solution in distributional sense is necessarily a solution in the classical sense provided that u is k-times continuously differentiable.

Weak solution u of the homogeneous equation $\mathcal{L}u = 0$, consequently, are characterized by the relation.

$$\int_{\Omega} u\mathcal{L}^*\phi\,dx = 0$$

Now we come to a very nice result which gives assurance of the existence of distributional solutions of partial differential equation.

Lemma 1.1 *Suppose $E(x,\xi)$ is a fundamental solution of $\mathcal{L}u = 0$ with singularity at ξ. Where $\mathcal{L}$ is k-th order differential operator of divergence type. Then the function u defined by*

$$u(x) = \int_{\Omega} E(x,\xi)h(\xi)\,d\xi$$

turns out to be a distributional solution of the inhomogeneous differential equation $\mathcal{L}u = h$.

Proof Let us denote Ω as domain of $x-$space and the $\xi-$space by Ω_x and Ω_ξ respectively. If φ is a k-times continuously differentiable test function then one has

$$
\begin{aligned}
\int_{\Omega} u\mathcal{L}^{*}\varphi dx &= \int_{\Omega_x}\left(\int_{\Omega_\xi} E(x,\xi)h(\xi)d\xi\right)\mathcal{L}^{*}\varphi dx \\
\text{Fubini theorem} \Rightarrow \quad &= \int_{\Omega_\xi} h(\xi)\left(\int_{\Omega_x} E(x,\xi)\mathcal{L}^{*}\varphi dx\right) d\xi \\
(1.5) \Rightarrow \quad &= (-1)^{k}\int_{\Omega_\xi} h(\xi)\varphi(\xi)d\xi. \quad \blacksquare
\end{aligned}
$$

In order to reduce the boundary value problem (1.1), (1.2) to a fixed-point problem, we first assume that u_* is a given solution of the differential equation (1.1), that is,

$$\mathcal{L}u_* = \mathcal{F}(\cdot, u_*, \partial_i u_*). \tag{1.6}$$

Let

$$V_* = \int_{\Omega} E(\cdot,\xi)\mathcal{F}(\xi, u_*(\xi), \partial_i u_*(\xi))\, d\xi,$$

Here we define a v_* as follows,

$$v_* =: u_* - V_*, \tag{1.7}$$

Then by the lemma (1.1) we have,

$$\mathcal{L}V_* = \mathcal{F}(\cdot, u_*, \partial_i u_*),$$

Consequently,

$$
\begin{aligned}
\mathcal{L}v_* &= \mathcal{L}(u_* - V_*), \\
&= \mathcal{F}(\cdot, u_*, \partial_i u_*) - \mathcal{F}(\cdot, u_*, \partial_i u_*), \\
&= 0,
\end{aligned}
$$

that is, $v_* = u_* - V_*$, turns out to be the solution of the homogeneous equation $\mathcal{L}v_* = 0$. In view of (1.7), we obtain for a given solution u_* of the equation (1.6) the representation

$$u_* = v_* + V_* = v_* + \int_{\Omega} E(\cdot,\xi)\mathcal{F}(\xi, u_*(\xi), \partial_i u_*(\xi))\, d\xi, \tag{1.8}$$

where v_* is a solution of the homogeneous equation $\mathcal{L}v_* = 0$.

Starting from the representation (1.8) of a given solution, we introduce an operator in order to construct solutions. Let u be any function belonging to a subset of the underlying function space. Further, let v be any solution of the homogeneous equation $\mathcal{L}v = 0$ Then define an image of u by

$$U = v + V = v + \int_{\Omega} E(\cdot, \xi)\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))\, d\xi. \tag{1.9}$$

If u is a fixed point of this operator, that is,

$$\mathcal{L}U = \mathcal{L}v + \mathcal{L}V = 0 + \mathcal{F}(\cdot, u, \partial_i u), \tag{1.10}$$

and so each fixed point turns out to be a solution of the non-linear differential equation

$$\mathcal{L}u = \mathcal{F}(\cdot, u, \partial_i u), \tag{1.11}$$

therefore each fixed-point u is a solution of the differential equation (1.1).

Formula (1.8) shows that especially a given solution u_* is a fixed point of the operator (1.9) provided v_* is defined by (1.7). However, formula (1.9) demonstrates that not only this special u_* but also each fixed point of the operator (1.9) is a solution of the differential equation (1.1) where v is any solution of the the homogeneous differential equation $\mathcal{L}v = 0$.

This fact can be used in order to construct a solution of the boundary value problem (1.1), (1.2). This is possible by a suitable choice of the solution v of the homogeneous equation $\mathcal{L}v = 0$. Of course, we assume that u belongs to a given subset of the function space under consideration. Now we choose,

$$v = u_0 + \tilde{u},$$

where u_0 is the solution of the given boundary value problem for the homogeneous partial differential equation while $\tilde{u}$ is a solution of homogeneous partial differential equation also which compensates the boundary values of ℓV to zero. In other words, u_0 is a solution of

$$\begin{aligned} \mathcal{L}u_0 &= 0 \quad \text{in} \quad \Omega \\ \ell u_0 &= \varphi \quad \text{on} \quad \partial\Omega \end{aligned}$$

and $\tilde{u}$ solves the boundary value problem

$$\begin{aligned} \mathcal{L}\tilde{u}_0 &= 0 \quad \text{in} \quad \Omega \\ \ell \tilde{u} &= -\ell V \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Now let u be the fixed point of the operator

$$\begin{aligned} U &= u_0 + \tilde{u} + V \\ &= u_0 + \tilde{u} + \int_{\Omega} E(\cdot, \xi)\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))\, d\xi, \end{aligned}$$

Clearly we obtain

$$\begin{aligned} \mathcal{L}U &= \mathcal{L}u_0 + \mathcal{L}\tilde{u} + \mathcal{L}V \\ &= 0 + 0 + \mathcal{F}(\cdot, u, \partial_i u) \quad \text{in} \quad \Omega \end{aligned}$$

and

$$\begin{aligned} \ell U &= \ell u_0 + \ell\tilde{u} + \ell V \\ &= \varphi - \ell V + \ell V = \varphi \quad \text{on} \quad \partial\Omega \end{aligned}$$

that is the boundary value problem (1.1), (1.2) is completely satisfied, hence the fixed point $U = u$ is a solution of the BVP (1.1), (1.2).

To sum up, we terms the theorem:

Theorem 1.1 *The solution of boundary value problem (1.1), (1.2) is the fixed-point of the operator*

$$U = u_0 + \tilde{u} + \int_\Omega E(\cdot, \xi)\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))\, d\xi, \tag{1.12}$$

and vice versa

where u_0 and $\tilde{u}$ are the solutions of the following boundary value problems

$$\begin{aligned} \mathcal{L}u_0 &= 0 \quad in \quad \Omega \\ \ell u_0 &= \varphi \quad on \quad \partial\Omega \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}\tilde{u} &= 0 \quad in \quad \Omega \\ \ell u &= -\ell\left(\int_\Omega E(\cdot, \xi)\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))\, d\xi\right) \quad on \ \partial\Omega \end{aligned}$$

respectively.

Example 1.1 *Let u_0 and $\tilde{u}$ are the solutions of Dirichlet problems with,*

$$u_0 = g(x) \ \ and \ \ \tilde{u} = \frac{1}{4\pi}\iiint_\Omega \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x - \xi|}\, d\xi \ \ on \ \partial\Omega \tag{1.13}$$

for Laplace equations $\Delta u_0 = 0$ and $\Delta\tilde{u} = 0$ respectively in the bounded domain Ω in $\mathbb{R}^3$. Then each fixed-point of the operator

$$U(x) = u_0(x) + \tilde{u}(x) - \frac{1}{4\pi}\iiint_\Omega \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x - \xi|}\, d\xi \tag{1.14}$$

is a solution of of the Dirichlet problem

$$\Delta u = \mathcal{F}(\cdot, u, \partial_i u) \;\; in \;\; \Omega \tag{1.15}$$

$$u = g \;\; on \;\; \partial\Omega. \tag{1.16}$$

Indeed, since

$$E(x,\xi) = \frac{1}{4\pi} \cdot \frac{1}{|x-\xi|}$$

is a fundamental solution of the Laplace equation.

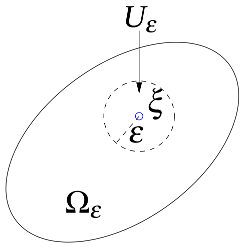
Thus in order to find solutions of the boundary value problem of the form (1.1), (1.2), one has only to apply fixed-point theorems. Of course, a number of such theorems are available in literature but we shall only consider the Schauder Fixed Point Theorem and the Contraction Mapping Principle. In this regards, chapters 4 and 5 of the current manuscript discuss the existence results by fixed point methods. Further, the operator (1.12) must satisfies the necessary conditions. Since the operator (1.12) is defined with the help of fundamental solution $E(x,\xi)$ and since a fundamental solution is weakly singular at ξ, one has to estimate weakly singular integrals. Moreover, if one needs to have considerations in Hölder spaces then integrals with two singularities are of immense importance. The next chapters deal with such integrals in details and their mapping properties.

Before proceeding to next chapter we give a short note on the fundamental solutions. It is important because we shall always assume that the homogeneous equation occurring in our boundary value problem possesses a fundamental solution.

1.2. Fundamental solutions

Solutions u of a partial differential equation $\mathcal{L}u = 0$ with an isolated singularity say ξ are called fundamental solutions in case that u has a special behavior at the isolated singularity ξ. This special behavior can be described with the help of a boundary integral occurring in the Green integral formula.

In order to apply the Green's Integral Formula to functions having a singularity at a point in the interior of a domain Ω, one has to omit a neighborhood of the singular point ξ. Now the domain of integration is then confined to $\Omega_\varepsilon = \Omega - U_\varepsilon$ where U_ε is the ε-neighborhood of ξ. See on next page.



Here the boundary of Ω_ε consists of two parts, first $\partial\Omega$ of the given domain Ω and second the ε-sphere centered at ξ.

Let $E(x,\xi)$ be the solution of $\mathcal{L}u = 0$ having an isolated singularity at ξ, while v is any k-times continuously differentiable function. Then the Green integral formula applied to $u = E(x,\xi)$ and v in Ω_ε implies the relation

$$(-1)^{k+1}\int_{\Omega_\varepsilon} E(x,\xi)\mathcal{L}^* v dx = \int_{\partial\Omega} P[E(x,\xi),v]\,d\mu + \int_{|x-\xi|=\varepsilon} P[E(x,\xi),v]\,d\mu \tag{1.17}$$

where

$$\int_{\partial\Omega} P[u,v]\,d\mu = \int_{\Omega}\left(v\mathcal{L}u + (-1)^{k+1}u\mathcal{L}^* v\right)dx. \tag{1.18}$$

The relation (1.17) leads to the following definition of a fundamental solutions

Definition 1.1 (Fundamental Solution) *The function $u = E(x,\xi)$ is said to be a "fundamental solution" of the equation $\mathcal{L}u = 0$ with singularity at ξ if the following three conditions are satisfied:*

(1) $u = E(x,\xi)$ is the solution of $\mathcal{L}u = 0$ for $x \neq \xi$

(2) The boundary integral over the ε-sphere in (1.17) tends to $-v(\xi)$ as ε tends to zero, that is,

$$\lim_{\varepsilon\to 0}\int_{|x-\xi|=\varepsilon} P[E(x,\xi),v]\,d\mu = -v(\xi) \tag{1.19}$$

where v is any $k-$times continuously differentiable function.

(3) the function $u = E(x,\xi)$ is weakly singular at ξ, i.e, it can be estimated by

$$|E(x,\xi)| \leq \frac{const.}{|x-\xi|^\alpha} \tag{1.20}$$

where $\alpha < n$ (the dimension of the space).

Following lemma gives the proof of the fundamental solution of Laplace equation.

Lemma 1.2 *If ω_n means the surface measure of the unit sphere in $\mathbb{R}^n$, then*

$$-\frac{1}{(n-2)\omega_n|x-\xi|^{n-2}}$$

is a fundamental solution of the Laplace equation in $\mathbb{R}^n$*,* $n \geq 3$*, whose singularity is located at* ξ*.*

Proof . Since the Laplace operator Δ is self adjoint, so by Green's integral formula we have,

$$\int_\Omega (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega} \left(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N} \right) d\mu =: \int_{\partial\Omega} P[u,v]\, d\mu$$

On the ε-sphere centered at ξ, we have,

$$\frac{\partial}{\partial N} = -\frac{\partial}{\partial r}$$

where $r = |x - \xi|$. For the solution of Laplace equation $u = \dfrac{c}{r^{n-2}}$ (c is constant), it follows that

$$P[u,v] = v(x) \cdot \frac{c(n-2)}{\varepsilon^{n-1}} + \frac{c}{\varepsilon^{n-2}} \cdot \frac{\partial v}{\partial r}$$

on the sphere $r = \varepsilon$. This expression can be written in the form

$$(v(x) - v(\xi)) \cdot \frac{c(n-2)}{\varepsilon^{n-1}} + v(\xi)\frac{c(n-2)}{\varepsilon^{n-1}} + \frac{c}{\varepsilon^{n-2}} \cdot \frac{\partial v}{\partial r}. \tag{1.21}$$

Now we have to integrate these three terms over the ε-sphere $r = \varepsilon$. Clearly, the surface measure of ε-sphere is equal to $\varepsilon^{n-1}\omega_n$ where the ω_n is the surface measure of the unit sphere. Therefore absolute value of the integral of the first term can be estimated by

$$\left| \int_{|x-\xi|=\varepsilon} \big(v(x) - v(\xi)\big) \cdot \frac{c(n-2)}{\varepsilon^{n-1}} d\mu \right| \leq \frac{c(n-2)}{\varepsilon^{n-1}} \cdot \sup_{|x-\xi|=\varepsilon} |v(x) - v(\xi)|.$$

Since v is continuous, the supremum tends to zero as ε tends to zero. Consequently, the limit of the integral of the first term equals zero.

The integral of the second term in (1.21) can be estimated by using the Schmidt inequality

$$\begin{aligned} \left| \int_{|x-\xi|=\varepsilon} v(\xi)\frac{c(n-2)}{\varepsilon^{n-1}} d\mu \right| &\leq |v(\xi)|\frac{c(n-2)}{\varepsilon^{n-1}} \cdot \varepsilon^{n-1}\omega_n \\ &\leq c(n-2)\omega_n |v(\xi)|. \end{aligned}$$

Since $\partial v/\partial r$ is bounded (because v is continuous), so the integral over the third term in (1.21) can be estimated as,

$$\left| \int_{|x-\xi|=\varepsilon} \frac{c}{\varepsilon^{n-2}} \cdot \frac{\partial v}{\partial r} dd\mu \right| \leq |c| \cdot const \cdot \omega_n \cdot \varepsilon.$$

So the integral tends to zero as $\varepsilon \to 0$.

To sum up

$$\int_{|x-\xi|=\varepsilon} P[u,v]\,d\mu = c(n-2)\omega_n \cdot v(\xi).$$

Therefore the limit is equal to $-v(\xi)$ if we choose

$$c = -\frac{1}{(n-2)\omega_n}.$$

Which is the desired choice of c that leads to the final result.■

1.3. Existence of fundamental solutions for more general elliptic differential operator

In analysis, an important question is the existence of a fundamental solution in case of a general second or higher order elliptic differential operator of the form,

$$\mathcal{L} \;=\; \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_i b_i \frac{\partial}{\partial x_i} + c,$$

C. Miranda [38] says that for a domain in $\mathbb{R}^n$ and Hölder continuously differentiable coefficients, not only that the fundamental solution $E(x,\xi)$ exists but it also satisfies following estimates:

- $E(x,\xi) \leq \dfrac{\text{const.}}{|x-\xi|^{n-2}}$
- $\partial_i E(x,\xi) \leq \dfrac{\text{const.}}{|x-\xi|^{n-1}}$

2. BACKGROUND MATERIAL

We consider the boundary value problems for non-linear second order elliptic partial differential equations with variable coefficients of the form:

$$\mathcal{L}u = \mathcal{F}(\cdot, u, \partial_i u) \quad \text{in} \quad \Omega \tag{2.1}$$

$$u = \varphi \quad \text{on} \quad \partial\Omega \tag{2.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with finite measure $m\Omega$, the boundary data is assumed to be of class $C^{1,\alpha}$ while the right hand side satisfies the following Lipschitz condition:

$$|\mathcal{F}(\cdot, u_1, \partial_i u_1) - \mathcal{F}(\cdot, u_2, \partial_i u_2)| \leq L_1|u_1 - u_2| + \sum_j L_{2,j}\,|\partial_i u_1 - \partial_i u_2| \tag{2.3}$$

and $\mathcal{F}(\cdot, 0, 0)$ is bounded by M, that is $|\mathcal{F}(\cdot, 0, 0)| \leq M$.

Note : The above Lipschitz condition (2.3) is only required to show the unique existence of solution but for the existence results by Schauder's Fixed Point Theorem, we only require the boundedness of the right hand side.

Where the operator $\mathcal{L}$ is given by

$$\mathcal{L}u = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(\cdot)\partial_j u\right) + \sum_{i=1}^{n} (b_i(\cdot)\partial_i u) + cu. \tag{2.4}$$

We shall require the coefficients $a_{ij}(\cdot)$ and $b_i(\cdot)$ to be continuously differentiable or Hölder continuously differentiable.

Our goal is to show the existence and uniqueness of the solution of the boundary value problem (2.1),(2.2) by the *Schauder* and *Banach* Fixed Point Theorems. Clearly, the right hand side depends on the first order derivative of the desired solution u so we have to choose a suitable Banach space. Due to an availability of the Schauder type estimates, we have the advantage to use the Hölder spaces as required Banach spaces to have investigations in. Also we will need various maximum principles for estimation of solutions u_0 and $\tilde{u}$ of homogeneous problem $\mathcal{L}u_0 = 0$ and $\mathcal{L}\tilde{u} = 0$ respectively.

Before going towards the behavior of the singular integrals, in the first section, we give a short introduction about the function spaces which we shall consider throughout this dissertation.

2.1. Hölder space $\mathbf{C}^{1,\alpha}$

As we have hinted that we shall deal the non-linear second order partial differential equations of elliptic type when the right hand side depends on the desired solution and its derivatives so in general, this leads to the consideration of either space of continuously differentiable functions or space of functions which have Hölder continuous derivatives up to first order.

Since we possess enough literature on the Schauder type estimates so we shall give preference to consider the $C^{1,\alpha}$ function space. In the last section of this chapter, we discuss the Schauder estimates in details.

2.2. Singular integrals

We are going to discuss the two types of singular integrals

i Integral with one weak singularity

ii Integral with two weak singularities

2.2.1. Integral with one weak singularity

It is clear from the first chapter that under certain assumptions the boundary value problem can be reduced to the following fixed point operator

$$U = u_0 + \tilde{u} + \int_\Omega E(\cdot,\xi)\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))\, d\xi \tag{2.5}$$

where we have to deal with the integrals whose kernels have weak singularities. For example with the fundamental solutions of the partial differential equations. Naturally, we must evaluate the behavior of such singular integrals (near the singularity) and their mapping properties see Chapter 3. In [2] Agmon, Douglis and Nirenberg narrate about such kernels with weak singularities very nicely.

The function $\mathcal{F}$ involved in the above integral equation, is assumed to satisfy the Lipschitz condition or is only bounded, so its norm can be taken out of integral and we are then left with a weakly singular integral having one weak singularity of the form,

$$K \cdot \int_\Omega \frac{d\xi}{|x-\xi|^\alpha}, \quad \text{with} \quad \alpha < n. \tag{2.6}$$

To investigate such type of integrals, one need certain results, one of them is well known as Schmidt inequality stated as:

Lemma 2.1 (Schmidt Inequality) *Suppose Ω is a domain in $\mathbb{R}^n$, with finite measure $m\Omega$ not necessarily bounded. Denote the volume of a unit ball in $\mathbb{R}^n$ by τ_n,* *while the measure of surface of unit ball is ω_n then for $0 \leq \alpha < n$*

$$\int_\Omega \frac{d\xi}{|x-\xi|^\alpha} \leq \frac{\omega_n}{n-\alpha} \left(\frac{m\Omega}{\tau_n} \right)^{1-\frac{\alpha}{n}} \tag{2.7}$$

for each x of $\mathbb{R}^n$., [1]

That means, the use of Schmidt inequality will always be an important step to estimate such types of integrals. But importantly, Schmidt inequality provides the result for one weak singularity.

Since our main considerations are to investigate the derivatives of these integrals under the sign of integration because we are going to work in space of Hölder continuously differentiable functions. The first order derivatives leads to the increase in the order of singularity but even then we are staying with weak singularity with $\alpha + 1 < n$ (the dimensions of the space) and estimation of integral can be performed by Schmidt inequality. But if one wants to work in spaces like C^2 or $C^{2,\alpha}$ then the situation is more complicated and tricky because integration involves the strong singularity. Then the integral is understood as a Cauchy principal value in literature. One has to be more careful that case.

2.2.2. Integral with two weak singularities

An important result about the estimates of integrals having integrand with two weak singularities say at $x^{'}$ and $x^{''}$ is the following theorem. This result is even true for an unbounded domain having finite measure $m\Omega$. It is important that the following result is also a counter example and correction to the result used in the book of S. G. Mikhlin see Appendix B.

Theorem 2.1 *Suppose Ω is a domain in $\mathbb{R}^n$, with finite measure $m\Omega$, suppose further that λ and μ are real numbers satisfying the inequalities $0 \leq \lambda < n$ and $0 \leq \mu < n$ then there exist constants C_1, C_2 and C_3 depending only on Ω, λ and μ such that*

$$\int\limits_\Omega \frac{d\xi}{|x^{'}-\xi|^\lambda \cdot |x^{''}-\xi|^\mu} \leq \begin{cases} C_1 |x^{'}-x^{''}|^{n-\lambda-\mu} + C_2, & \text{for} \quad \lambda+\mu \neq n \\ C_3 - 4\pi \ln|x^{'}-x^{''}|, & \text{for} \quad \lambda+\mu = n \end{cases} \tag{2.8}$$

is true for any 2 points $x^{'}$ and $x^{''}$ not necessarily belonging to $\overline{\Omega}$ but having a positive distance less than 2. Here ξ is an element in $\mathbb{R}^n$ and $d\xi$ is the volume

[1]Proof of Schmidt Inequality is available in literature so it is not included here.

element in n-space . Where C_1, C_2 and C_3 are given explicitly by:

$$C_1 = \frac{2\pi}{2^{n-\lambda-\mu}} \cdot M \left\{ \frac{1}{n-\lambda} + \frac{1}{n-\mu} + \frac{2\pi}{n-\lambda-\mu} \right\}$$

$$C_2 = 2\pi M \left\{ \frac{1}{n-\lambda} + \frac{1}{n-\mu} + \frac{2\pi}{n-\lambda-\mu} \right\} + m\Omega$$

$$C_3 = M \left\{ 2\pi \left(\frac{1}{n-\lambda} + \frac{1}{n-\mu} \right) + \ln 2 + \frac{m\Omega}{M} \right\}.$$

Proof : This has been proved in Appendix B.

Remark 2.1 *The estimate is also true if the distances of the two points are even greater than 2 but then we get other constants. This situation is discussed in next chapter under title mapping properties.*

2.2.3. Integral in the sense of Cauchy principal value

When one has to work in different function spaces then various mapping properties of singular integral operators are to be dealt with. In the previous subsections we have discussed the more easy cases concerning the weak singularities which are rather easy to handle. Here we deal with singular integrals having the strong singularities and those unbounded integrals which don't exist as proper or improper integrals. In a broader sense these integrals exist in the sense of Cauchy Principal Value, briefly either CPV. or PV. integrals. A detail note on Cauchy type integrals both in univariate and multivariate cases has been described by A. R. Krommer and C. W. Ueberhuber in [33] p14.

The existence of Cauchy Principal Value Integrals for various classes of integral equations has been discussed by M. A. Golberg [20]. Similarly, these integrals occur in the integral transforms, for instance, in Hilbert and Riesz transforms which have been mentioned by A. J. Jerri in [25].

We now examine the Cauchy type integrals with an example. C. Miranda [38] has dealt such integrals. Here we consider a bounded domain Ω in $\mathbb{R}^n$. Suppose the function $k : \Omega \to \mathbb{R}$ is unbounded in an arbitrary neighborhood of a point ξ of Ω. Let $\mathbb{B}_\varepsilon(\xi)$ be the sphere with respect to the Euclidean norm in $\mathbb{R}^n$ and $\varepsilon \in (0, \infty)$ and let the function k be Riemann integrable over the region $\Omega \backslash \mathbb{B}_\varepsilon(\xi)$. Then the CPV integral is defined by

$$\int_\Omega^* k(x,\xi) d\xi := \lim_{\varepsilon \to 0} \int_{\Omega \backslash \mathbb{B}_\varepsilon(\xi)} k(x,\xi) d\xi \tag{2.9}$$

provided that the limit exists.

G. Monegato [39] stated that if the order of the singularity is at most equal to the dimension of the space of which our domain Ω is taken as subset, the above limit is independent of the particular norm otherwise it depends on the a certain norm. An important example of the CPV integral is given by,

$$J(x) =: \int_{\Omega}^{*} \frac{\mathcal{F}(\xi)}{|x-\xi|^n} d\xi \tag{2.10}$$

We encounter this type of integral during the solutions of boundary value problems for inhomogeneous partial differential equations. A sufficient condition for the existence of above integral defined in (2.10) is that the function $\mathcal{F}$ is Hölder continuous at ξ that is for for $0 < \alpha < 1$ there is a constant C such that

$$|\mathcal{F}(x) - \mathcal{F}(\xi)| \leq C|x-\xi|^{\alpha} \tag{2.11}$$

that means, the the function $\mathcal{F}$ having the property of Hölder continuity leads to the reduction in the order of the singularity of the kernel appearing in the integrand and we stay with a weak singularity and then simplifications can be done similar to the previous sections.

2.3. Schauder estimates

Schauder estimates play a very important role in the theory of elliptic Partial Differential Equations. In view of wikipedia these estimates are based on the existence theory of Juliusz Schauder so they are named after him. Not only that Schauder estimates are of worth importance in existence theory of linear but also for non-linear elliptic PDEs. These estimates guarantee that the Hölder bound for the solutions of PDEs, in general, is controlled by the Hölder norm of the boundary data, i.e, they are very critical in solving the boundary value problems. A lot of literature is already available on these estimates.

In the articles of Agmon, Douglis and Nirenberg [2] and [3], a comprehensive discussion on Schauder estimates is already given. In this regards mostly the $\|\cdot\|_{C^{2,\alpha}}$ bound for the solutions of PDEs is estimated. E. A. Baderko [7] proves the case $\|\cdot\|_{C^{1,\alpha}}$ explicitly which is very important for our considerations, since our main focus is on boundary value problems for non linear second order elliptic PDEs. We will mainly focus on the function space $C^{1,\alpha}$, hence it would be a natural demand to have Schauder estimates for this case at our disposal. D. Gilbarg and N. S. Trudinger [18] has also presented various results on Schauder estimates with applications in the framework of boundary value problems.

Foremost, there are two kinds of Schauder estimates:

- Interior Schauder Estimates
- Estimates near the boundary

The "Interior Estimates", provide us the bounds for the derivatives up to the second order of the solution and their Hölder continuity in any compact subset of the domain that is, for the solution of a boundary value problem (2.1),(2.2), we have the following bound,

$$\|u\|^*_{C^{2,\alpha}(\Omega)} \leq C\left(\|\mathcal{F}\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)}\right) \tag{2.12}$$

Here the norm depends on the behavior of source term and the continuity of the solution u. Sign “*” represents the weighted norm inside the domain that is at a positive distance from the boundary. Constant C depends on the Hölder exponent α, dimension on the space n, ellipticity constant γ and the $C^{0,\alpha}$ bound of the constants appearing in the elliptic operator.

In the later situation we get the same type of estimates as we have in (2.12) but here the Hölder norm depends additionally on the regularity of boundary terms and we have the following Schauder estimates up to the boundary.

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C\left(\|\mathcal{F}\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)} + \|\varphi\|_{C^{2,\alpha}(\partial\Omega)}\right) \tag{2.13}$$

where C additionally depending on domain Ω.

In general setting, for the regularity of $C^{k+2,\alpha}$ of solutions, Gilbarg [19] has proved the lemma as:

Lemma 2.2 *Let Ω be a $C^{k,\alpha}$ domain, $k \geq 0$, and assume $u \in C^{k+2,\alpha}(\overline{\Omega})$, the boundary vales $\varphi \in C^{k+2,\alpha}(\overline{\Omega})$, the right hand side $f \in C^{k,\alpha}(\overline{\Omega})$ and the coefficients of a_{ij}, b_i, c are in $C^{k,\alpha}(\Omega)$. Then the following Schauder estimate is true*

$$\|u\|_{C^{k+2,\alpha}(\overline{\Omega})} \leq C\left(\|f\|_{C^{k,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)} + \|\varphi\|_{C^{k+2,\alpha}(\partial\Omega)}\right) \tag{2.14}$$

The Schauder estimates for the Hölder continuity of the first derivative of the solution up to the boundary is given in [8] and [7]

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\left(\|\mathcal{F}\|_{C^{0,\alpha}(\Omega)} + \|u\|_{C^0(\Omega)} + \|\varphi\|_{C^{1,\alpha}(\partial\Omega)}\right) \tag{2.15}$$

When the solution satisfies a certain maximum principle, the middle term can be dropped or one can estimate it by the Hölder norm of the boundary data.

We also prove the Schauder estimates in $C^{1,\alpha}$-norm in $\mathbb{R}^{\not\models}$ and in this case we evaluate the constants explicitly. For this see Appendix A.

Of course for the differential equations with variable coefficients, additional smoothness properties are required for the variables. Schauder estimates near the boundary for the elliptic partial differential equations of an arbitrary order are presented in [2] where the authors give both the cases of equations with constant and variable coefficients and interior estimates. Moreover, a comprehensive discussion about the boundary conditions and their smoothness is done. Similar estimates for the systems of partial differential equations were given in [3]. Regarding the interior estimates for systems of elliptic PDEs we refer to [14].

2.4. Maximum minimum principles

The maximum principle is an important property of the solutions of certain partial differential equations which can be of elliptic or parabolic type. The maximum principle, in general, says that a maximum value of a function (which is the solution of partial differential equation) in a domain exists on the boundary of the domain. To illustrate more, a solution of elliptic PDE is said to satisfy the strong maximum principle if it attains its maximum value inside the domain then it is uniformly constant in the closure of the domain. Moreover, the strong maximum principle is very useful to find the a priori estimates of the solutions of linear partial differential equations and specially of boundary value problems for non-linear partial differential equations. We investigate boundary value problems for non-linear PDEs where these maximum principles play essential part. C. Miranda [38] p7, gives the maximum principle for the general linear second order PDE of the type $\mathcal{L}u = f$:

Lemma 2.3 *Let Ω be a bounded domain and u be a regular solution of homogeneous equation $\mathcal{L}u = 0$ in Ω. Suppose further that u is non-constant and continuous in $\overline{\Omega}$. If $c \leq 0$ then throughout Ω:*

$$\max_{\Omega} |u| < \max_{\partial\Omega} |u| \tag{2.16}$$

more precisely if $c = 0$ then throughout Ω we have the two sided estimate:

$$\min_{\Omega} u < u < \max_{\partial\Omega} u \tag{2.17}$$

More results on the strong and weak maximum principles are given by D. Gilbarg [19].

The weak maximum principle, on the other hand, says that if we have the maximum on the boundary even then this maximum value may exist inside or interior of the domain, for instance a weak maximum principle has been mentioned in [19] where it says that,

Lemma 2.4 *Let $\mathcal{L}$ be a general linear second order elliptic operator defined in section 1.1, in a bounded domain Ω. Suppose that if*

$$\mathcal{L}u \geq 0 \ \text{ and } \ c = 0 \ \text{ in } \ \Omega$$

with $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ then the maximum of u is given by,

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \tag{2.18}$$

and if

$$\mathcal{L}u \leq 0 \ \text{ and } \ c = 0 \ \text{ in } \ \Omega$$

then minimum of u is given by

$$\inf_{\Omega} u = \inf_{\partial\Omega} u \tag{2.19}$$

The last equation gives the minimum principle.

The maximum principles are the basis of proving the Harnack inequality in the framework of partial differential equations. We shall be frequently using the maximum principle in our existence results for the a priori estimates of the solutions of non-linear partial differential equations.

3. MAPPING PROPERTIES OF RELATED OPERATORS

The chapter one reduces the boundary value problems to the fixed point operator involving the singular integrals. In general these singular integrals are of the form,

$$\int_{\Omega} E(\cdot,\xi)\mathcal{F}(\xi,u(\xi),\partial_i u(\xi))\,d\xi, \tag{3.1}$$

where $E(\cdot,\xi)$ is a fundamental solution of the linear homogeneous elliptic partial differential equation $\mathcal{L}u = 0$ and by definition a fundamental solution is weakly singular.

Our main focus will be on solving the class of the boundary value problems for non-linear partial differential equations with variable coefficients having the principal part as Laplace operator when the right hand side depends not only on the desired solution but also on its first order derivatives. Moreover we shall apply Schauder and Banach Fixed Point Theorems to obtain our existence results. Hence it will be mandatory to choose the suitable function spaces or Banach spaces to work in.

Since the right hand side depends on the derivatives of the solution of the boundary value problem, it is natural to consider the space of continuously differentiable functions C^1. But as mentioned before, we want to utilize the presence of the Schauder estimates which are developed mainly for the Hölder spaces. So we shall work with $C^{1,\alpha}$ function space. The next important chapter gives details on the existence results, where we prove the existence and uniqueness of the solution of BVPs for non-linear PDEs. These existence results demands to have certain mapping properties of the fixed-point operator in our hand first.

Now we shall confine ourselves to the non linear partial differential equations which have the Laplace operator as a principal part and right hand side is a function of the solution u and its first order derivatives. Here we can bring the terms from differential operator with first order derivatives of the solution and the rest lower order terms to the right hand side and we stay with the Laplace operator on the left hand side. The reason is that we can now work with the explicit fundamental solution of the Laplace operator which we have already proved in the Section 1.2 in Chapter 1. We shall consider a bounded domain Ω in $\mathbb{R}^n$ and the corresponding fundamental solution for the Laplace operator is given by

$$E(x,\xi) = -\frac{1}{(n-2)\omega_n|x-\xi|^{n-2}}, \quad n \geq 3.$$

The Dirichlet boundary value problem, considered is

$$\begin{aligned} \Delta u &= \mathcal{F}(\cdot,u,\partial_i u) \quad \text{in} \quad \Omega \\ u &= \varphi \quad \text{on} \quad \partial\Omega \end{aligned}$$

i.e, we have Δ instead of the general second order elliptic differential operator $\mathcal{L}$.

By the Theorem (1.1), the above BVP is reduced to,

$$U = u_0 + \tilde{u} + \int_{\Omega} \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{(n-2)\omega_n |x-\xi|^{n-2}} d\xi,$$

or briefly,

$$U = u_0 + \tilde{u} + V, \tag{3.2}$$

where V is given by

$$V = -\frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x-\xi|^{n-2}} d\xi \tag{3.3}$$

while u_0 and $\tilde{u}$ are the solutions of the Laplace equation with boundary values as φ and $-V$ respectively.

Clearly the singular integral V in (3.3) is a significant component of the fixed point operator (3.2), so we shall be discussing the mapping properties of this integral operator V then we carry these properties to U. It will be important to note that throughout the investigation we shall assume the density function $\mathcal{F}$ to be Lipschitz continuous or bounded.

3.1. Continuity of V

We again consider V

$$V(x) = -\frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x-\xi|^{n-2}} d\xi.$$

The continuity of V can easily be checked as follows

$$\begin{aligned}
V(x) \quad - \quad & V(x') \\
= \quad & \frac{1}{(n-2)\omega_n} \int_{\Omega} \mathcal{F}(\xi, u(\xi), \partial_i u(\xi)) \left(\frac{1}{|x'-\xi|^{n-2}} - \frac{1}{|x-\xi|^{n-2}} \right) d\xi \\
= \quad & \frac{1}{(n-2)\omega_n} \int_{\Omega} \mathcal{F}\big(\xi, u(\xi), \partial_i u(\xi)\big) \left(\frac{|x-\xi|^{n-2} - |x'-\xi|^{n-2}}{|x'-\xi|^{n-2} \cdot |x-\xi|^{n-2}} \right) d\xi \\
= \quad & \frac{1}{(n-2)\omega_n} \int_{\Omega} \mathcal{F}\big(\xi, u(\xi), \partial_i u(\xi)\big) \Big(|x-\xi| - |x'-\xi| \Big) W d\xi \qquad (3.4)
\end{aligned}$$

where

$$W = \frac{|x-\xi|^{n-3} + |x-\xi|^{n-4}|x'-\xi| + |x-\xi|^{n-5}|x'-\xi|^2 + \ldots + |x'-\xi|^{n-3}}{|x'-\xi|^{n-2} \cdot |x-\xi|^{n-2}}.$$

That implies,

$$\left|V(x)-V(x^{'})\right| \leq \frac{\|\mathcal{F}\|}{(n-2)\omega_n}|x-x^{'}|\cdot Y,$$

where

$$Y=\int\limits_{\Omega}\frac{d\xi}{|x^{'}-\xi|^{n-2}\cdot|x-\xi|^{1}}+\int\limits_{\Omega}\frac{d\xi}{|x^{'}-\xi|^{n-3}\cdot|x-\xi|^{2}}+\ldots+\int\limits_{\Omega}\frac{d\xi}{|x^{'}-\xi|^{1}\cdot|x-\xi|^{n-2}} \tag{3.5}$$

In (3.5) each integral with two singularities is bounded by Theorem (2.1) and we have (*n*-2) total terms. Moreover, we have an explicit value for the integral estimate and so the absolute value of left hand side of (3.4) can be made as small as possible when $x \to x^{'}$. But notice that theorem (2.1) is only true for the domain having diameter less than 2. i.e, $|x-x^{'}|<2$. The constants occurring during the estimates of each of above $(n-2)$ number of integrals are the same so one can write down the above inequality:

$$\left|V(x)-V(x^{'})\right| \leq \frac{\|\mathcal{F}\|}{(n-2)\omega_n}|x-x^{'}|\cdot(n-2)\cdot\int\limits_{\Omega}\frac{d\xi}{|x^{'}-\xi|^{n-2}\cdot|x-\xi|^{1}}.$$

For $|x-x^{'}|\geq 1$ we can consider again equation (3.4)

$$\begin{aligned} V(x) \quad &- \quad V(x^{'}) \\ &= \frac{1}{(n-2)\omega_n}\int\limits_{\Omega}\mathcal{F}(\xi,u(\xi),\partial_i u(\xi))\left(\frac{1}{|x^{'}-\xi|^{n-2}}-\frac{1}{|x-\xi|^{n-2}}\right)d\xi \end{aligned} \tag{3.6}$$

$$\begin{aligned} |V(x) \quad &- \quad V(x^{'})| \\ &\leq \frac{\|\mathcal{F}\|}{(n-2)\omega_n}\int\limits_{\Omega}\left(\frac{1}{|x^{'}-\xi|^{n-2}}+\int\limits_{\Omega}\frac{1}{|x-\xi|^{n-2}}\right)d\xi && (3.7)\\ &\leq \frac{\|\mathcal{F}\|}{(n-2)\omega_n}\cdot\left|x-x^{'}\right|\int\limits_{\Omega}\left(\frac{1}{|x^{'}-\xi|^{n-2}}+\int\limits_{\Omega}\frac{1}{|x-\xi|^{n-2}}\right)d\xi && (3.8) \end{aligned}$$

here using the Schmidt inequality two times we get the bound for the integrals that means

$$\left|V(x)-V(x^{'})\right| \leq \frac{\|\mathcal{F}\|}{(n-2)|}\left(\frac{m\Omega}{\tau_n}\right)^{2/3}\cdot|x-x^{'}| \tag{3.9}$$

which shows finally that $V(x)$ is continuous.

3.2. $V(x) \in C^1(\overline{\Omega})$

For $\sup|\partial_i V|$ we differentiate under the sign of integration, so we get

$$\partial_i V = -\frac{1}{(n-2)\omega_n}\int_{\Omega} \partial_i \frac{\mathcal{F}(\xi,u(\xi),\partial_i u(\xi))}{|x-\xi|^{n-2}}\, d\xi \tag{3.10}$$

and

$$\|\partial_i V\| \le \frac{\|\mathcal{F}\|}{(n-2)\omega_n}\int_{\Omega} \frac{1}{|x-\xi|^{n-1}} \cdot \frac{|x_i-\xi_i|}{|x-\xi|} d\xi$$

again Schmidt inequality leads to the following

$$\|\partial_i V\| \le \frac{\|\mathcal{F}\|}{(n-2)} \cdot \left(\frac{m\Omega}{\tau_n}\right)^{\frac{1}{n}} \tag{3.11}$$

$$\begin{aligned}\partial_i V(x^{'}) \quad &- \quad \partial_i V(x^{''}) \\ &= \quad -\frac{1}{(n-2)\omega_n}\int_{\Omega} \mathcal{F}(\xi,u(\xi),\partial_i u(\xi)) \left(\frac{x_i^{'}-\xi_i}{|x^{'}-\xi|^n} - \frac{x_i^{''}-\xi_i}{|x^{''}-\xi|^n}\right) d\xi\end{aligned}$$

this implies that

$$\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right| \le \frac{\|\mathcal{F}\|}{(n-2)\omega_n} \cdot \int_{\Omega} I d\xi \tag{3.12}$$

by the same arguments as previous we have,

$$I \le 2(n-1)|x^{'}-x^{''}| \left(\frac{1}{|x^{'}-\xi|^{n-\frac{1}{2}} \cdot |x^{''}-\xi|^{1/2}}\right),$$

from (3.12) we get,

$$\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right| \le \frac{2(n-1)\|\mathcal{F}\|}{(n-2)\omega_n}|x^{'}-x^{''}| \int_{\Omega} \frac{d\xi}{|x^{'}-\xi|^{n-\frac{1}{2}} \cdot |x^{''}-\xi|^{1/2}},$$

here we apply the following estimate for the integral over a domain with finite measure, say $m\Omega$, in $\mathbb{R}^n$ of the function with two weak singularities lying at $x^{'}$ and $x^{''}$

$$\int_{\Omega} \frac{1}{|x^{'}-\xi|^{\mu} \cdot |x^{''}-\xi|^{\nu}} d\xi \le \frac{8n\pi\tau_n}{2n-1} + \tau_n \ln 2 + m\Omega - 4\pi\left|\ln|x^{'}-x^{''}|\right|$$

where $0 \le \mu < n,\ 0 \le \nu < n$ and $\mu + \nu = n$

we get

$$\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right| \leq \frac{2(n-1)\|\mathcal{F}\|}{(n-2)\omega_n} \cdot |x^{'} - x^{''}| \cdot \left(C - 8\pi \left|\ln|x^{'} - x^{''}|\right|\right)$$

where C is given by

$$C = \frac{8n\pi\tau_n}{2n-1} + \tau_n \ln 2 + m\Omega.$$

Thus we have

$$\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right| \leq \frac{2(n-1)\|\mathcal{F}\|}{(n-2)\omega_n} \left|x^{'} - x^{''}\right|^{\alpha} \left(Ct^{1-\alpha} + 8\pi t^{1-\alpha} \left|\ln|x^{'} - x^{''}|\right|\right) \tag{3.13}$$

where $t := \left|x^{'} - x^{''}\right|$

The last expression shows that the first order derivatives of $V(x)$ are continuous. Hence one can easily find the $\|V\|_{C^1}$ norm. Thus $V(x) \in C^1(\overline{\Omega})$ which is very nice property of $V(x)$.

3.3. $V(x) \in C^{1,\alpha}(\overline{\Omega})$

Now another important mapping property of the singular integral operator $V(x)$ is discussed that is whether it is a member of the Banach space $C^{1,\alpha}(\overline{\Omega})$ or not. Again the last inequality (3.13) implies

$$\sup \frac{\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right|}{|x^{'} - x^{''}|^{\alpha}} \leq \frac{2(n-1)\|\mathcal{F}\|}{(n-2)\omega_n} \cdot \max\left(Ct^{1-\alpha} + 8\pi t^{1-\alpha} \ln t\right) \tag{3.14}$$

suppose $\max\left(Ct^{1-\alpha} + 8\pi t^{1-\alpha} \ln t\right) = m^{'}$

$$\sup \frac{\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right|}{|x^{'} - x^{''}|^{\alpha}} \leq \frac{2(n-1)\|\mathcal{F}\|}{(n-2)\omega_n} \cdot m^{'} \tag{3.15}$$

Now for $t = |x^{'} - x^{''}| \geq 1$, from the previous arguments we have

$$\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right| \leq \frac{\|\mathcal{F}\|}{(n-2)\omega_n} \int\limits_{\Omega} \left\{\frac{1}{|x^{'} - \xi|^{n-1}} + \frac{1}{|x^{''} - \xi|^{n-1}}\right\} d\xi \tag{3.16}$$

using Schmidt inequality 2 times, we get

$$\begin{aligned}
\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right| &\leq \frac{2\|\mathcal{F}\|}{n-2} \cdot \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \\
&\leq \frac{2\|\mathcal{F}\|}{n-2} \cdot \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \cdot 1 \\
&\leq \frac{2\|\mathcal{F}\|}{n-2} \cdot \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \cdot |x^{'} - x^{''}|
\end{aligned}$$

and

$$\sup \frac{|\partial_i V(x') - \partial_i V(x'')|}{|x' - x''|^{\alpha}} \leq \frac{2\|\mathcal{F}\|}{n-2} \cdot \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \cdot \max t^{1-\alpha} = m \cdot \frac{\|\mathcal{F}\|}{n-2}$$

where $t^{1-\alpha}$ is monotonically increasing function and $t \geq 1$ so we have the maximum for each t. Now if $d \geq 1$ is the diameter of the domain then $d^{1-\alpha}$ is the maximum for each $\alpha \in (0,1)$. So ultimately we get

$$\sup \frac{\left|\partial_i V(x^{'}) - \partial_i V(x^{''})\right|}{|x^{'} - x^{''}|^{\alpha}} \leq \left[\frac{2(n-1)}{\omega_n}, \left(\frac{m\Omega}{\tau_n}\right)^{1/n}\right] \frac{\|\mathcal{F}\|}{n-2} = m'(d) \cdot \frac{\|\mathcal{F}\|}{n-2}$$

Clearly the Hölder constant is finite

So now for $m'' = \max\left(m, m'\right)$, $\|V\|_{C^{1,\alpha}}$ is given by

$$\|V\|_{C^{1,\alpha}} \leq \max\left[\left(\frac{m\Omega}{\tau_n}\right)^{2/n}, \left(\frac{m\Omega}{\tau_n}\right)^{1/n}, \quad m''\right] \frac{\|\mathcal{F}\|}{n-2} \tag{3.17}$$

Hence the $V \in C^{1,\alpha}$.

3.4. Relative compactness of the operator $V(x)$

To show that the operator V is relatively compact in $C^{1,\alpha}-$norm, we consider V_k as the image sequence of u_k defined in the ball $\mathfrak{B}_R$

$$\mathfrak{B}_{\mathfrak{R}}(0) := \{u \in C^{1,\alpha} : \|u\|_{C^{1,\alpha}} \leq \mathfrak{R}\}$$

and then later on we will take this result to U_k, where we required to show that the fixed point operator U is relatively compact.

Suppose for u_k, there are V_k the arbitrary sequence in images.

$$V_k(x) = -\frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{\mathcal{F}(\xi, u_k(\xi), \partial_i u_k(\xi))}{|x - \xi|^{n-2}}\, d\xi$$

Let $x^{'}$ be any arbitrary point in the domain Ω, so we have

$$V_k(x)-V_k(x^{'})=\frac{1}{(n-2)\omega_n}\int\limits_{\Omega}\mathcal{F}(\xi,u_k(\xi),\partial_i u_k(\xi))\left\{\frac{1}{|x^{'}-\xi|^{n-2}}-\frac{1}{|x-\xi|^{n-2}}\right\}d\xi$$

$$\left|V_k(x)-V_k(x^{'})\right|\leq\frac{(n-3)\|\mathcal{F}\|}{(n-2)\omega_n}|x-x^{'}|\cdot\int\limits_{\Omega}\frac{d\xi}{|x-\xi||x^{'}-\xi|^{n-2}}.$$

The integral in the right hand side of the last inequality is finite, see for example the result on two weak singularities in appendix or Chapter 2, so the left hand side of the last inequality tends to 0 as $|x-x^{'}|$ is arbitrarily small and the bound is independent of the index k, which shows that the sequence V_k are Lipschitz continuous (consequently equi-continuous too) in the sup-norm. Hence we are in a position of application of the well known Arzelà-Ascoli theorem. Note that in view of Arzelà-Ascoli theorem, we have now a uniformly convergent sub-sequence V_{k_l} of V_k which converges uniformly in the sup-norm.

This is very important to note that up to now, we have found the sub-sequence which converges in the sup norm but it is not sufficient for us because we are in the $C^{1,\alpha}$ function space and we require the convergence in the norm $\|\cdot\|_{C^{1,\alpha}}$.

Since $C^{1,\alpha}$ is a Banach space, so it is enough for us to show that the sequence V_{k_l} or its sub sequence is a fundamental (Cauchy) sequence in the norm $\|\cdot\|_{C^{1,\alpha}}$ and then we can take into consideration the definition of a Banach space to show convergence. Now our main goal in this section, in one sense, is to find a fundamental sub-sequence of V_k in $C^{1,\alpha}-$space. According to the definition of the norm $\|\cdot\|_{C^{1,\alpha}}$, for V_{k_l} to be a fundamental sequence, we have to show for arbitrary two elements of the sequence V_{k_l}, V_{k_m}, $\|V_{k_l}-V_{k_m}\|_{C^{1,\alpha}}$ is arbitrarily small, i.e, $\|V_{k_l}-V_{k_m}\|_{C^{1,\alpha}}<\varepsilon$ for sufficiently large l and m. The crucial step will be to show that the Hölder constant is not only finite but it is arbitrarily small as well.

S. G. Kreĭn, Y. Ī. Petunīn and E. M. Semënov in [31] give a very nice result of such compact embedding. Here authors explain their result for Hölder scale. In essential, the necessary steps for the existence of a fundamental sequence for Hölder (scale) constant are discussed. We use the similar construction for the proof of an important lemma prior to the proof of the existence of a fundamental sequence in $C^{1,\alpha}$ space. An important requirement for the proof of the lemma is the need of a bounded and equi-continuous subsequence in the sup$-$norm.

Now for the derivatives we have already found a result and we have,

$$\begin{aligned}\left|\partial_i V_k(x)-\partial_i V_k(x^{'})\right| &\leq \frac{\|\mathcal{F}\|}{(n-2)\omega_n}\int\limits_{\Omega}\left|\frac{(x_i-\xi_i)|x^{'}-\xi|^n-(x_i^{'}-\xi_i)|x-\xi|^n}{|x-\xi|^n\cdot|x^{'}-\xi|^n}\right|d\xi,\\ &\leq \frac{2(n-1)\|\mathcal{F}\|}{(n-2)\omega_n}\cdot|x-x^{'}|\cdot\left(C-8\pi\left|\ln|x-x^{'}|\right|\right),\end{aligned}$$

which again are equi-continuous, that means, by Arzela-Ascoli[1] theorem, we have found a subsequence $V_{\ell_1}, V_{\ell_2}, V_{\ell_1}, \cdots$ which itself not only is convergent but also the sequence of its first order derivatives with respect to the first argument $\partial_1 V_{\ell_1}, \partial_1 V_{\ell_2}, \partial_1 V_{\ell_3}, \cdots$ are convergent.

Similarly, we can find a subsequence $m_1, m_2, m_3, \cdots$ of $\ell_1, \ell_2, \ell_3, \cdots$ such that $\partial_2 V_{m_1}, \partial_2 V_{m_2}, \partial_2 V_{m_3}, \cdots$ is convergent,and also this sequence, at the same time is convergent for $\partial_1 V_{m_1}, \partial_1 V_{m_2}, \partial_1 V_{m_3}, \cdots$.

By carrying on this procedure we are able to find a sub sequence say $V_{n_1}, V_{n_2}, V_{n_3}, \cdots$ which not only itself is convergent but for which the first order derivatives with respect to all arguments converge too.

Finally, since the $V_{n_1}, V_{n_2}, V_{n_3}, \cdots$ and its first order derivatives with respect to all arguments are bounded we can prove an easy but important result.

Lemma 3.1 *Let $0 < \alpha < \beta \leq 1$, suppose further that f_n be uniformly bounded sequence of functions in $C^{0,\beta}$ and equi-continuous in* sup *norm. Moreover, suppose that $f_{n'}$ be a uniformly convergent subsequence of f_n then the Hölder constant;*

$$\sup \frac{\left| [f_{n'} - f_{m'}] \left(x'\right) - [f_{n'} - f_{m'}] \left(x''\right) \right|}{\left|x' - x''\right|^{\alpha}}$$

is arbitrarily small for sufficiently large n and m for all $\alpha < \beta$.

Proof Of course we can write,

$$\begin{aligned} &\frac{\left| [f_{n'} - f_{m'}] \left(x'\right) - [f_{n'} - f_{m'}] \left(x''\right) \right|}{\left|x' - x''\right|^{\alpha}} \\ &= \frac{\left| \left[f_{n'}\left(x'\right) - f_{n'}\left(x''\right) \right] - \left[f_{m'}\left(x'\right) - f_{m'}\left(x''\right) \right] \right|}{\left|x' - x''\right|^{\alpha}} \\ &= \frac{\left| \left[f_{n'}\left(x'\right) - f_{n'}\left(x''\right) \right] - \left[f_{m'}\left(x'\right) - f_{m'}\left(x''\right) \right] \right|}{\left|x' - x''\right|^{\beta}} \cdot \left|x' - x''\right|^{\beta-\alpha} \\ &\leq \left(\frac{\left| f_{n'}\left(x'\right) - f_{n'}\left(x''\right) \right|}{\left|x' - x''\right|^{\beta}} + \frac{\left| f_{m'}\left(x'\right) - f_{m'}\left(x''\right) \right|}{\left|x' - x''\right|^{\beta}} \right) \cdot \left|x' - x''\right|^{\beta-\alpha} \end{aligned} \tag{3.18}$$

Since, by hypothesis, we know that f_n is uniformly bounded sequence in $C^{0,\beta}$ so we have

$$\frac{\left| f_{n'}\left(x'\right) - f_{n'}\left(x''\right) \right|}{\left|x' - x''\right|^{\beta}} \leq \| f_{n'} \|_{C^{0,\beta}}. \tag{3.19}$$

[1]Arzela-Ascoli theorem guarantees the existence of a uniformly convergent subsequence.

Using (3.19) in (3.18), for sufficiently large m', n', we get

$$\sup \frac{\left|\left[f_{n'}-f_{m'}\right]\left(x^{'}\right)-\left[f_{n'}-f_{m'}\right]\left(x^{''}\right)\right|}{\left|x'-x''\right|^{\alpha}} \leq 2\cdot\max_{n}\left(\|f_n\|_{C^{0,\beta}}\right)\cdot\left|x^{'}-x^{''}\right|^{\beta-\alpha} \quad (3.20)$$

for

$$\left|x^{'}-x^{''}\right| < \underbrace{\left[\frac{\varepsilon}{2\cdot\max_{n}\|f_n\|_{C^{0,\beta}}}\right]^{\frac{1}{\beta-\alpha}}}_{=h}$$

we get

$$\frac{\left|\left[f_{n'}-f_{m'}\right]\left(x^{'}\right)-\left[f_{n'}-f_{m'}\right]\left(x^{''}\right)\right|}{\left|x'-x''\right|^{\alpha}} < \varepsilon \quad (3.21)$$

and now when

$$\left|x^{'}-x^{''}\right| \geq h,$$

due to the fact from hypothesis that $f_{n'}$ is uniformly convergent in sup-norm, so we have

$$\left|f_{n'}(x)-f_{m'}(x)\right| < \frac{\varepsilon^{'}}{2}\cdot h^{\alpha}.$$

for sufficiently large m', n'.

Finally, the inequality (3.18) leads to the desired result,

$$\begin{aligned}\left|\left[f_{n'}-f_{m'}\right]\left(x^{'}\right)-\left[f_{n'}-f_{m'}\right]\left(x^{''}\right)\right| &\leq \left|\left[f_{n'}-f_{m'}\right]\left(x^{'}\right)\right|+\left|\left[f_{n'}-f_{m'}\right]\left(x^{''}\right)\right| \\ &< \frac{\varepsilon^{'}}{2}h^{\alpha}+\frac{\varepsilon^{'}}{2}h^{\alpha} \\ &< \varepsilon\cdot h^{\alpha}\end{aligned}$$

for sufficiently large m', n'.

Now since $\left|x^{'}-x^{''}\right|^{\alpha}$ is monotonically increasing function,

so for

$$\left|x^{'}-x^{''}\right| \geq h$$

we have

$$\left|x^{'}-x^{''}\right|^{\alpha} \geq h^{\alpha}$$

or we can write

$$\frac{1}{\left|x'-x''\right|^{\alpha}} \leq \frac{1}{h^{\alpha}}$$

ultimately we get

$$\sup \frac{\left|\left[f_{n'}-f_{m'}\right]\left(x^{'}\right)-\left[f_{n'}-f_{m'}\right]\left(x^{''}\right)\right|}{\left|x'-x''\right|^{\alpha}} \begin{array}{ll} < & \varepsilon \cdot h^{\alpha} \cdot \dfrac{1}{h^{\alpha}} \\ < & \varepsilon \end{array}$$

Remark Although Lemma (3.1) is an important result for our consideration but it provides us a basis to establish the important embedding results for example Adams [1] p(11-12) has given such embedding results where author also discuss that if the domain is bounded then the embeddings turn out be compact and so on. We have the following embedding result:

If n is nonnegative integer and for $0 < \alpha < \beta \leq 1$ the embedding $C^{n,\beta}(\overline{\Omega}) \to C^{n,\alpha}(\overline{\Omega})$ exists. And moreover, if, Ω is a bounded domain then this embedding is compact. This implies that the Hölder space with larger Hölder exponent is embedded in a Hölder space with a smaller Hölder exponent. Clearly, in our case, the domain is bounded and if we start with a larger Hölder exponent β than compactness in $C^{1,\alpha}$ is obvious by the embedding result we just discussed.

Now we carry on our considerations for the proof that the fixed point operator U defined above is relatively compact in the ball $\mathfrak{B}$ in the Banach space $C^{1,\alpha}$.

Up to now we have prove the existence of a convergent sub-sequence V_{k_l} of V_k in the sup-norm. Since V_{k_l} is a convergent sequence in sup $-$norm this implies that V_{k_l} is fundamental sequence. Consequently, we have

$$\left|V_{k_l}-V_{k_m}\right| < \varepsilon, \qquad \text{for } l \text{ and } m \text{ large enough}$$

so is true for every subsequence.

Next, we are going to show that the V_{k_l} is also convergent in the $\|\cdot\|_{C^{1,\alpha}}$-norm. Clearly the subsequence V_{k_l} can be written as follows

$$\partial_i V_{k_l}(x) = \frac{\|\mathcal{F}\|}{(n-2)\omega_n}\int\limits_{\Omega}\frac{1}{|x-\xi|^{n-1}}\cdot\frac{|x_i-\xi_i|}{|x-\xi|}d\xi$$

For equi-continuity, we use the result, that for the singular integral with two weak singularities we have

$$\left|\partial_i V_{k_l}(x)-\partial_i V_{k_l}(x^{'})\right| \leq \frac{2(n-1)\|\mathcal{F}\|}{(n-2)\omega_n}\max\left(Ct^{1-\alpha}+8\pi t^{1-\alpha}\ln t\right)\left|x-x^{'}\right|.$$

The last inequality shows that the sequence $\partial_i V_{k_l}$ is equi-continuous and in view of Arzelà-Ascoli theorem we have a uniformly convergent subsequence, say $\partial_i V_{k_l'}$. As $\partial_i V_{k_l'}$ is convergent and obviously is a fundamental sequence, so for arbitrary small $\varepsilon > 0$, we get

$$\left|\partial_i V_{k_l'}-\partial_i V_{k_m'}\right| < \varepsilon. \tag{3.22}$$

At the end, we give the final step of the section, i.e, the Hölder constant to be arbitrarily small, and already from the previous sections it is clear that it is finite. To show that the Hölder constant is arbitrarily small we first take into consideration the fact that the sequence $\partial_i V_{k_l'}$ is bounded in the space $C^{0,\beta}$ with larger exponent which can be easily checked with arguments discussed above. Then we apply Lemma (1) proved in this section to get arbitrarily small Hölder constant
i.e,

$$\sup \left| \frac{\left| \left(\partial_i V_{k_l'} - \partial_i V_{k_m'}\right)\left(x'\right) - \left(\partial_i V_{k_l'} - \partial_i V_{k_m'}\right)\left(x''\right) \right|}{\left|x' - x''\right|^{\alpha}} \right| < \varepsilon \tag{3.23}$$

that leads to the desired result concerning the existence of a convergent subsequence of V_k in $C^{1,\alpha}$.

Hence, we have proved the relative compactness of the the operator $V(x)$.

3.5. Contractivity of $V(x)$

To check the that fixed-point operator U is contractive, we shall, first, check that V is contractive in the $\|\cdot\|_{C^{1,\alpha}}$-norm. So, let us consider again the operator V

$$V = -\int_{\Omega} \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x-\xi|^{n-2}}\, d\xi \tag{3.24}$$

where $\mathcal{F}$ satisfies the Lipschitz condition (2.3) in Chapter 2.

Let for arbitrary u_i in the ball defined at the beginning of the Section 3.2 there is an arbitrary image V_i in the same ball then to each u_1 and u_2, images are V_1 and V_2, equation (3.24) leads to the following,

$$V_1 - V_2 = \frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{\mathcal{F}(\xi, u_2(\xi), \partial_i u_2(\xi)) - \mathcal{F}(\xi, u_1(\xi), \partial_i u_1(\xi))}{|x-\xi|^{n-2}}\, d\xi$$

And the absolute value is estimated as

$$\begin{aligned}
|V_1 - V_2| &\leq \frac{1}{(n-2)\omega_n} \cdot \left|\mathcal{F}(x, u_1, p_i^1) - \mathcal{F}(c, u_2, p_i^2)\right| \cdot \int\limits_{\Omega} \frac{1}{|x-\xi|^{n-2}}\, d\xi \\
&\leq \left(L_1 \cdot |u_1 - u_2| + \sum_j L_{2,j} \cdot |\partial_i u_1 - \partial_i u_2|\right) \int\limits_{\Omega} \frac{1}{|x-\xi|^{n-2}}\, d\xi \\
&\leq \frac{1}{(n-2)\omega_n} \left(L_1 \|u_1 - u_2\| + \sum_j L_{2,j} \|u_1 - u_2\|_{C^{1,\alpha}}\right) \frac{\omega_n}{2} \left(\frac{m\Omega}{\tau_n}\right)^{2/n} \\
&\leq \frac{1}{2(n-2)} \cdot \left(L_1 \|u_1 - u_2\|_{C^{1,\alpha}} + \sum_j L_{2,j} \|u_1 - u_2\|_{C^{1,\alpha}}\right) \left(\frac{m\Omega}{\tau_n}\right)^{2/n} \\
&\leq \frac{1}{2(n-2)} \left(\frac{m\Omega}{\tau_n}\right)^{2/n} \left(L_1 + \sum_j L_{2,j}\right) \|u_1 - u_2\|_{C^{1,\alpha}}.
\end{aligned}$$

Hence we get

$$\|V_1 - V_2\| \leq \frac{1}{2(n-2)} \left(\frac{m\Omega}{\tau_n}\right)^{2/n} \left(L_1 + \sum_j L_{2,j}\right) \|u_1 - u_2\|_{C^{1,\alpha}}. \tag{3.25}$$

Next for the derivatives, we have

$$\partial_i V_1 \quad - \quad \partial_i V_2 = \frac{1}{(n-2)\omega_n} \int\limits_{\Omega} \frac{\chi \cdot (x_i - \xi_i)}{|x-\xi|^n}\, d\xi$$

where

$$\chi =: \mathcal{F}\big(\xi, u_1(xi), \partial_i u_1(\xi)\big) - \mathcal{F}\big(\xi, u_2(xi), \partial_i u_2(\xi)\big)$$

Then

$$\begin{aligned}
|\partial_i V_1 - \partial_i V_2| &\leq \frac{1}{(n-2)\omega_n} |\mathcal{F}(x, u_1, \partial_i u_1) - \mathcal{F}(x, u_2, \partial_i u_2)| \cdot \int\limits_{\Omega} \frac{1}{|x-\xi|^{n-1}}\, d\xi \\
&\leq \frac{1}{(n-2)\omega_n} \left(L_1 |u_1 - u_2| + \sum_j L_{2,j} |\partial_i u_1 - \partial_i u_2|\right) \int\limits_{\Omega} \frac{1}{|x-\xi|^{n-1}}\, d\xi \\
&\leq \frac{1}{(n-2)\omega_n} \left(L_1 \|u_1 - u_2\| + \sum_j L_{2,j} \cdot \|u_1 - u_2\|_{C^{1,\alpha}}\right) \cdot 4\pi \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \\
&\leq \frac{1}{(n-2)} \left(L_1 \|u_1 - u_2\|_{C^{1,\alpha}} + \sum_j L_{2,j} \cdot \|u_1 - u_2\|_{C^{1,\alpha}}\right) \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \\
&\leq \frac{1}{(n-2)} \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \left(L_1 + \sum_j L_{2,j}\right) \|u_1 - u_2\|_{C^{1,\alpha}}
\end{aligned}$$

and we get

$$\|\partial_i V_1 - \partial_i V_2\| \leq \frac{1}{(n-2)} \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \left(L_1 + \sum_j L_{2,j}\right) \cdot \|u_1 - u_2\|_{C^{1,\alpha}}. \tag{3.26}$$

Now for Hölder constant we have

$$\partial_i V_1(x^{'}) - \partial_i V_2(x^{'}) - \partial_i V_1)(x^{''}) - \partial_i V_2(x^{''}) = \frac{1}{(n-2)\omega_n} \int_\Omega \chi \left(\frac{x_i^{'} - \xi_i}{|x^{'} - \xi|^n} - \frac{x_i^{''} - \xi_i}{|x^{''} - \xi|^n} \right) d\xi$$

and further

$$\begin{aligned} &\left|\partial_i V_1(x^{'}) - \partial_i V_2(x^{'}) - \partial_i V_1(x^{''}) - \partial_i V_2(x^{''})\right| \\ &\leq \frac{1}{(n-2)\omega_n} \left(L_1 + \sum_j L_{2,j}\right) \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \cdot \int_\Omega \left| \frac{x_i^{'} - \xi_i}{|x^{'} - \xi|^n} - \frac{x_i^{''} - \xi_i}{|x^{''} - \xi|^n} \right| d\xi \\ &\leq \frac{2(n-1)}{(n-2)\omega_n} \left(L_1 + \sum_j L_{2,j}\right) \cdot |x^{'} - x^{''}| \cdot (C - 8\pi \left|\ln |x^{'} - x^{''}|\right|) \cdot \|u_1 - u_2\|_{C^{1,\alpha}}, \end{aligned}$$

For $|x^{'} - x^{''}| = t < 1$ we get

$$\sup \frac{\left|\partial_i V_1(x^{'}) - \partial_i V_2(x^{'}) - \partial_i V_1)(x^{''}) - \partial_i V_2(x^{''})\right|}{|x^{'} - x^{''}|^\alpha} \leq \frac{2(n-1)}{(n-2)\omega_n} \left(L_1 + \sum_j L_{2,j}\right) \cdot \mathbf{O} \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \tag{3.27}$$

where, $\mathbf{O} = \left[\max\left(Ct^{1-\alpha} - 8\pi t^{1-\alpha} |\ln t|\right)\right]$.

and for $|x^{'} - x^{''}| = t \geq 1$ we have

$$\sup \frac{\left|\partial_i V_1(x^{'}) - \partial_i V_2(x^{'}) - \partial_i V_1)(x^{''}) - \partial_i V_2(x^{''})\right|}{|x^{'} - x^{''}|^\alpha} \leq \frac{n-3}{n-2} \left(L_1 + \sum_j L_{2,j}\right) \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \max(t^{1-\alpha}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}}. \tag{3.28}$$

Now $\|V_1 - V_2\|_{C^{1,\alpha}}$ is estimated

$$\|V_1 - V_2\|_{C^{1,\alpha}} \leq m \cdot \left(L_1 + \sum_j L_{2,j}\right) \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \tag{3.29}$$

with

$$m = \max\left[\frac{1}{2}\left(\frac{1}{2}\frac{m\Omega}{\tau_n}\right)^{2/n}, \left(\frac{m\Omega}{\tau_n}\right)^{1/n}, \frac{2(n-1)}{\omega_n}m^{'}\right]$$

where $m^{'}$ is the maximum of two in equations (3.27) and (3.28) for which the construction is done above.
Now for the fixed point operator $U = u_0 + \tilde{u} + V$ by triangle inequality we have

$$\|U_1 - U_2\|_{C^{1,\alpha}} \leq \|\tilde{u_1} - \tilde{u_2}\|_{C^{1,\alpha}} + \|V_1 - V_2\|_{C^{1,\alpha}}. \tag{3.30}$$

Applying the maximum principle for Laplace equation and the Schauder estimate to $\tilde{u}$ having boundary values $-V$, we have

$$\begin{aligned}
\|U_1 - U_2\|_{C^{1,\alpha}} &\leq C^1(\|V_1 - V_2\|_{C^{1,\alpha}} + \|\tilde{u_1} - \tilde{u_2}\|) + \|V_1 - V_2\|_{C^{1,\alpha}} \\
&\leq C^1\left(\|V_1 - V_2\|_{C^{1,\alpha}} + \|\tilde{u_1} - \tilde{u_2}\|_{C(\partial\Omega)}\right) + \|V_1 - V_2\|_{C^{1,\alpha}} \\
&\leq C^1\left(\|V_1 - V_2\|_{C^{1,\alpha}} + \|\tilde{u_1} - \tilde{u_2}\|_{C(\partial\Omega)}\right) + \|V_1 - V_2\|_{C^{1,\alpha}} \\
&\leq C^1(\|V_1 - V_2\|_{C^{1,\alpha}} + \|V_1 - V_2\|_{C^{1,\alpha}}) + \|V_1 - V_2\|_{C^{1,\alpha}} \\
&\leq (2C^1 + 1)\|V_1 - V_2\|_{C^{1,\alpha}}.
\end{aligned}$$

So we get,

$$\|U_1 - U_2\|_{C^{1,\alpha}} \leq (2C^1 + 1) \cdot m \cdot \left(L_1 + \sum_j L_{2,j}\right) \|u_1 - u_2\|_{C^{1,\alpha}}. \tag{3.31}$$

So finally for U to be contractive the following condition is to be satisfied

$$(2C^1 + 1) \cdot m \cdot \left(L_1 + \sum_j L_{2,j}\right) < 1. \tag{3.32}$$

Hence under the conditions (3.32) the operator $U(x)$ is contractive.

All above mapping properties are of worth importance. Since this thesis deals the existence results by Schauder Fixed Point Theorem and Contraction Mapping Principle which we shall carry out in the next chapters.

4. EXISTENCE AND UNIQUENESS THEOREMS

The goal of the current chapter is to solve the boundary value problems for non-linear differential equations where the the right hand side is a function of a point in the domain, the desired solution u and its first order derivative $\partial_i u$, by the Contraction Mapping Principle[1] and Schauder Fixed Point Theorem in the Banach spaces. Here we consider C^0 and $C^{1,\alpha}$ spaces and we solve the boundary value problems for the non-linear Partial differential equations. This chapter is based on the results dealing the existence and uniqueness of solutions in the function spaces by Schauder Fixed Point Theorem and Contraction Mapping Principle respectively.

Mainly, we solve the non-linear partial differential equations by fixed-point techniques. But initially, for motivation, a simple example of an ordinary differential equation is discussed. Here to understand the fundamentals, we consider the space of continuous functions. Then this procedure is extended to the more general PDEs in the Hölder spaces (of course, other Banach spaces can also be considered).

Since the right hand sides of a given non-linear differential equations, generally, are not defined for the whole function space so it will be obvious that we shall apply both fixed point theorems in a ball $\mathfrak{B}$ (a closed and convex subset of the Banach space) with a certain radius $\mathfrak{R}$. More precisely, we will solve our problems in the balls in the Banach space. Optimization of the function space and balls is discussed in the next chapter.

We shall start with the non-linear Poisson equation with the boundary value $\varphi(x) \in C^{1,\alpha}(\partial\Omega)$. Optimality condition in the function space $C^{1,\alpha}$ for the application of Schauder and Banach Fixed-Point Theorems will be investigated. In the nutshell, the current chapter investigates two important results namely:

- Restriction on the radius $\mathfrak{R}$ of the ball when the fixed point operator maps the ball into itself.
- Restriction on the radius $\mathfrak{R}$ of the ball if the contractive condition is satisfied by the image of the ball under the fixed point operator.

The contractive condition will lead to the unique existence of the solution of the non-linear elliptic partial differential equations. The first part of the chapter explains the relative compactness of the operators under consideration for the existence of the solutions by Schauder Fixed Point Theorem.

During the investigations, we apply the well known Schauder estimates which are very important tools to prove our results. Since, these Schauder type estimates give

[1]Banach Fixed Point Theorem

the estimates of the solutions of elliptic PDEs in terms of boundary values in the Hölder spaces, is the reasons to consider the Hölder spaces for our work.

This chapter contains the following sections:

- Section 4.1 is about the basic fixed point theorems which we are considering.
- In Section 4.2, first we give the model case ordinary differential Equation (Application fixed point theorem defined in section(4.1).
- Section 4.3 deals with the condition on the radius of the ball in Banach space for which our fixed-point operator maps the ball $\mathfrak{B}$ into itself. We give the estimate for the restriction to $\mathfrak{R}$ in this case. This section is necessary for the application of both the fixed point theorems.
- In the section 4.4, we prove the relative compactness of the operator which gives rise to state a theorem for the existence of solution for the given problem by Schauder Fixed Point Theorem (II).
- Section 4.5 narrates an additional condition for uniqueness of the solution, that the fixed-point operator is contractive in the Hölder norm and an explicit result for the estimate is proved.

As we have discussed already our main focus is to solve non-linear partial differential equations so the sections (4.3)-(4.5) are all about the PDEs. The consideration of a model problem of an ordinary differential equations in section (4.2) is to carry through the idea of the fixed-point approach to non-linear PDEs.

The whole chapter deals with the various aspects of the existence and uniqueness of solutions by applying the the Schauder Fixed Point Theorem and Contraction Mapping Principle, so it will be better to recall the definitions of the well known fixed point theorems we are going to use.

4.1. Basic fixed-point theorems

In the following definitions, we are recalling the theorems we shall apply for the proofs of the existence and uniqueness of solutions of non-linear PDEs in general and to ODEs in particular.

Definition 4.1 (Contraction Mapping Principle). Let $\mathfrak{A}$ be a closed subset of a Banach space on a domain Ω in $\mathbb{R}^n$ with certain norm. Suppose, further that τ is a contractive operator (under the norm of the Banach space) mapping $\mathfrak{A}$ into itself. Then τ possesses a uniquely determined fixed-point in $\mathfrak{A}$.

Clearly the Contraction Mapping Principle guarantees the unique existence, so it will be an important tool for us to show the uniqueness e of the solution in case of our boundary value problems.

Now we give the definitions of two versions of Schauder Fixed Point Theorems:

Two versions of Schauder Fixed-Point Theorem

Definition 4.2 (Schauder I). Let M be a compact and convex subset of a Banach space, and let f be a continuous mapping of M into itself. Then f has at least one fixed point in M.

And second version of the Schauder theorem is

Definition 4.3 (Schauder II). Let M be a closed and convex subset of a Banach space, let f be a continuous mapping of M into itself, and suppose that $f(M)$ is relatively compact, then f has at least one fixed point in M.

As in our case we will apply the above fixed point theorems to the balls in Banach spaces but these are closed and convex subset. That is why, we will apply only the second version of Schauder Fixed Point Theorem which requires only the subsets to be closed and convex.

Since both the theorems defined above are well known results in literature so we are not including their proofs here. For Contraction Mapping Principle (Banach Fixed Point Theorem) see E. Kreyszig [34] where in detail, its proof is given. To know about the construction of proving the Schauder Theorem, we refer [34] where author gives necessary details of the proof. In appendix C, the second version of Schauder Fixed Point Theorem has been proved.

Also, it is clear from above definitions that we require, for the application of both theorems, the condition that the operator maps the ball (closed and convex subset of Banach space) into itself must have to be satisfied. So the results of Section 4.3 will appear to be necessary for both theorems for their application in the balls. Our main focus will be to solve the boundary problems in general domains for non-linear equations of the following types:

$$\Delta u = \mathcal{F}(\cdot, u, \partial_i u) \text{ in } \Omega \tag{4.1}$$

$$u = \varphi \text{ on } , \partial\Omega \tag{4.2}$$

by fixed point theorems stated above.

Before going to the main Sections 4.3-4.5 which deals with the boundary value problem above, we start with the initial value problem under the following headline:

4.2. The Model case of ordinary differential equation

We have the following initial value problem

$$y' = y^2 \tag{4.3}$$

$$y(0) = 1. \tag{4.4}$$

We are going to solve the above initial value problem by the Contraction Mapping Principle and Schauder Fixed Point Theorem.

For the above initial value problem, we assume:

- The right hand side is Lipschitz continuous in y (at least locally Lipschitz continuous).
- x is point of the real axis.
- We apply the Schauder Fixed Point Theorem and Contraction Mapping Principle to the balls in a Banach space.

Then the above problem is equivalent to the integral equation

$$y(x) = 1 + \int_0^x y^2(\xi)d\xi.$$

In view of Chapter 1, the initial value problem can be reduced to the following fixed point-operator:

$$Y(x) = 1 + \int_0^x y^2(\xi)d\xi \tag{4.5}$$

and the ball B centered at the initial value with radius r in the Banach space $C^0[a,b]$ is given by

$$B_r(1) := \{y \in C^0[0,\rho] : \|y-1\| \leq r\}$$

which clearly is closed and convex subset of C^0

$$\Rightarrow \|y\| \leq 1 + r$$

so we get

$$\begin{aligned} \|Y(x)-1\| &\leq \int_0^x |y^2(\xi)|d\xi \\ &\leq \int_0^x d\xi \cdot \|y^2\| \\ &\leq \rho(1+r)^2. \end{aligned}$$

where ρ is the length of the interval and we want the largest interval on x-axis in which solution is continuous.

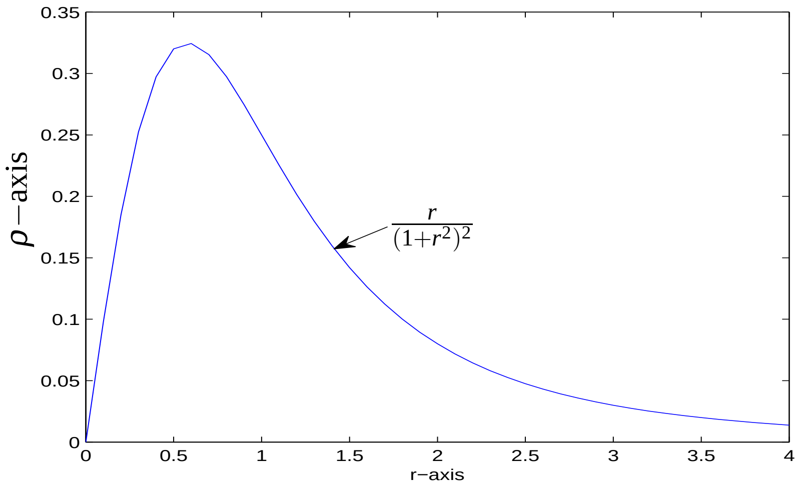
Hence operator Y maps the ball $B_r(1)$ into itself if the following condition is satisfied:

$$\begin{gathered} \|Y(x)-1\| \leq \rho(1+r)^2 \leq r \\ \rho(1+r)^2 \leq r \\ \rho \leq \frac{r}{(1+r)^2} \end{gathered} \tag{4.6}$$

is the first condition for ρ when Y maps the ball into itself.

Here we can apply the Schauder II provided the operator defined in (4.5) is relatively compact in the ball. So we assume that for an arbitrary sequence of the solutions y_k, there exists an arbitrary sequence of images Y_k so we have from (4.5)

$$Y_k(x) = 1 + \int_0^x y_k^2(\xi)d\xi,$$



this implies

$$Y_k(x) - Y_k(x') = \int_{x'}^{x} y_k^2(\xi)\, d\xi,$$

where x' is an arbitrary point in $[0,x]$, we get,

$$\begin{aligned} \left\| Y_k(x) - Y_k(x') \right\| &\leq \|y\|^2 \left| x - x' \right|, \\ &\leq (1+\rho)^2 \left| x - x' \right|, \end{aligned}$$

which shows that the image sequence Y_k is equi-continuous and by Arzelà-Ascoli theorem there exists a uniformly convergent subsequence $Y_{k'}$ of Y_k that converges in sup norm. So the image Y is relatively compact in the ball.

Here we can apply the Schauder Fixed Point Theorem (Schauder II), and the fixed point is the the solution of the above IVP.

Note: To use Schauder Theorem, we do not need the Lipschitz condition on the right hand side necessarily, rather only local boundedness is enough.

Now for the application of contraction mapping principle, we require additionally that the operator Y not only maps the ball into itself but is contractive too.

Contractivity of the fixed point operator Y:

As Y is depending on y, so, for arbitrarily chosen y_1 and y_2 we have the images Y_1 and Y_2 respectively then we can write,

$$\begin{aligned} Y_1(x) - Y_2(x) &= \int_0^x \left(y_1^2(\xi) - y_2^2(\xi)\right) d\xi \\ \|Y_1(x) - Y_2(x)\| &\le \int_0^x |\left(y_1^2(\xi) - y_2^2(\xi)\right)| d\xi \\ &\le \int_0^x |\left(y_1(\xi) - y_2(\xi)\right)\left(y_1(\xi) + y_2(\xi)\right)| d\xi \\ &\le (\|y_1\| + \|y_2\|) \cdot \int_0^x d\xi \cdot \|y_1 - y_2\| \end{aligned}$$

since by the definition of the ball $\|y\| \le 1 + r$

$$\|Y_1 - Y_2\| \le \underbrace{2\rho(1+r)}_{<1} \| y_1 - y_2\|.$$

Of course the operator will be contractive if the following condition holds,

$$2\rho(1+r) < 1$$

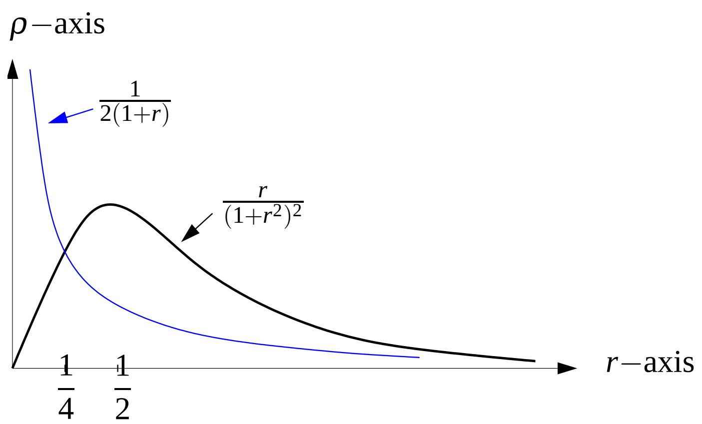
$$\rho < \frac{1}{2(1+r)}$$

which is the 2nd condition for ρ.

Now by Contraction Mapping Principle we have unique fixed point of the operator Y and the above initial value problem has a unique solution that is a fixed point of the above operator. Also now, we are in a position to get an appropriate value of the ρ, i.e the length of the interval in which the function y is continuous. The estimate, for the best ρ is given by

$$\rho < \frac{1}{4}$$

both conditions are shown below



Hence the given initial value problem is solvable by Schauder (II) and the contraction mapping principle.
The above example gives an idea about the way we will develop our constructions for BVPs in the other function (Banach) spaces.

Now we consider the boundary value problems (4.1),(4.2) together with the following assumptions.

From now on we will assume:

1 Ω will be an open bounded domain in $\mathbb{R}^n$ with measure $(m\Omega)$

2 $\varphi(x) \in C^{1,\alpha}(\partial\Omega)$

3 $|\mathcal{F}(\cdot,u_1,\partial_i u_1) - \mathcal{F}(\cdot,u_2,\partial_i u_2)| \leq L_1|u_1 - u_2| + \sum_j L_{2,j}\,|\partial_i u_1 - \partial_i u_2|$
for $i,j = 1,2,3\cdots$ that is Lipschitz condition holds.

4 $|\mathcal{F}(\cdot,0,0)| \leq M$ is given.

We will apply the theorem Schauder (II) instead of Schauder (I) in section 4.3 because in this case we need a weaker condition on the subset of the Banach space, i.e, relatively compactness of the operator. But Section 4.5 deals with the Contraction Mapping Principle as discussed earlier.

Remark to the assumptions on the right hand sides The Lipschitz condition on $\mathcal{F}(x,u,\partial_i u)$ is ncessary for the application of the Contraction Mapping Principle in order to show that the corresonding operator is contractive (provided the Lipschitz constants are small enough). For the application of the second version of the Schauder Fixed-Point Theorem, however, this Lipschitz condition is not necessary. In this case it is enough to assume that the right-hand side $\mathcal{F}(x,u,\partial_i u)$ is continuous (in order to prove the boundedness of $|\mathcal{F}(x,u,\partial_i u)|$ in balls of the underlying function space), see also Remark 4.3 in Subsection 4.4.2.

Since for the application of both theorems, we have to show that the fixed point operator maps the Banach space into itself and then we will show the condition under which the operator maps a ball into itself. Here we consider the $C^{1,\alpha}$ as a Banach space for the solution of boundary value problem (4.1),(4.2) and the norm for any $u \in C^{1,\alpha}$ is defined as follows.

$$\|u\|_{C^{1,\alpha}} := \max\left(\sup|u|, \sup\left|\frac{\partial u}{\partial x_i}\right|, \sup\frac{\left|\partial_i u(x') - \partial_i u(x'')\right|}{|x' - x''|^{\alpha}}\right) \tag{4.7}$$

Since we know from Theorem 1, in Chapter 1 that the solution of the boundary value problem (4.1),(4.2) is a fixed point of the operator equation

$$U(x) = u_0 + \tilde{u} + \int_{\Omega} \frac{\mathcal{F}(\xi,u(\xi),\partial_i u(\xi))}{(n-2)\omega_n|x-\xi|^{n-2}}d\xi \tag{4.8}$$

or

$$U = u_0 + \tilde{u} + V \tag{4.9}$$

where

$$V(x) = \frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x-\xi|^{n-2}} d\xi \tag{4.10}$$

u_0 is the solution of the following problem

$$\begin{aligned} \Delta u_0 &= 0 \quad \text{in}\, \Omega \\ u_0 &= \varphi \quad \text{on}\, \partial\Omega \end{aligned}$$

and $\tilde{u}$ is solution of the homogeneous problem

$$\begin{aligned} \Delta \tilde{u} &= 0 \quad \text{in}\, \Omega \\ \tilde{u} &= -\frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x-\xi|^{n-2}} d\xi = -V(x) \quad \text{on} \quad \partial\Omega \end{aligned} \tag{4.11}$$

4.3. Fixed point operator maps a ball $\mathfrak{B}$ in $\mathbf{C}^{1,\alpha}$, into itself

In this section as mentioned earlier we will construct the balls in which both Contraction Mapping Principle and Schauder Fixed Point Theorem are applicable. Also we are going to show that the operators maps the function space $C^{1,\alpha}$ into itself. In other words, we find the restriction to the radius of the ball to apply the Contraction Mapping Principle and Schauder Fixed Point Theorem. The same Dirichlet problem for a non-linear Poisson equation when the right hand side depends only on the desired solution u has been investigated by W. Tutschke in [53], where C^0 space has been considered and the boundary data is supposed to be only continuous and the right hand side involve only the solution u but not its first order derivatives.

Define to any $u \in C^{1,\alpha}(\overline{\Omega})$ an image U by,

$$U(x) = u_0 + \tilde{u} + \frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x-\xi|^{n-2}} d\xi \tag{4.12}$$

or

$$U = u_0 + \tilde{u} + V \tag{4.13}$$

Now we will show that the operator U maps the space $C^{1,\alpha}$ and a ball in $C^{1,\alpha}$ into itself. First, it is enough to show that $V(x)$ as defined in (4.10) is in $C^{1,\alpha}$. Since the objective of this section is to find the condition under which the fixed point operator (4.12) maps certain balls (closed and convex subsets) of the Banach space $C^{1,\alpha}$ into itself and then to find the explicit formulation in this regards, we define such a ball.

Let the ball $\mathfrak{B}$ in $C^{1,\alpha}$ is defined as,

$$\mathfrak{B}_{\mathfrak{R}}(0) := \{u \in C^{1,\alpha} : \|u\|_{C^{1,\alpha}} \leq \mathfrak{R}\}. \tag{4.14}$$

Again considering V

$$V(x) = \frac{1}{(n-2)\omega_n} \int_{\Omega} \frac{\mathcal{F}(\xi, u(\xi), \partial_i u(\xi))}{|x-\xi|^{n-2}} d\xi \tag{4.15}$$

now for $\sup|V(x)|$, using Schmidt inequality, we have

$$\begin{aligned} \|V(x)\| &\leq \frac{1}{(n-2)\omega_n} \|\mathcal{F}\| \int_{\Omega} \frac{1}{|x-\xi|^{n-2}}\, d\xi \\ &\leq \frac{\|\mathcal{F}\|}{(n-2)\omega_n} \cdot \frac{\omega_n}{2} \left(\frac{m\Omega}{\tau_n}\right)^{\frac{2}{n}} \\ &\leq \frac{\|\mathcal{F}\|}{2(n-2)} \cdot \left(\frac{m\Omega}{\tau_n}\right)^{\frac{2}{n}} \end{aligned} \tag{4.16}$$

and for $\sup|\partial_i V|$ we apply the estimate (3.11) already constructed in Chapter 3, so we get

$$\|\partial_i V\| \leq \frac{\|\mathcal{F}\|}{(n-2)} \cdot \left(\frac{m\Omega}{\tau_n}\right)^{\frac{1}{n}}. \tag{4.17}$$

Now, finally, for the Hölder constant, $\sup \dfrac{\left|\partial_i V(x') - \partial_i V(x'')\right|}{|x'-x''|^{\alpha}}$, we again apply the mapping property of Section 3.3 from to get.

For $t = |x' - x''| \leq 1$,

$$\sup \frac{\left|\partial_i V(x') - \partial_i V(x'')\right|}{|x'-x''|^{\alpha}} \leq \frac{2(n-1)\|\mathcal{F}\|}{(n-2)\omega_n} \cdot m', \tag{4.18}$$

and for $t = |x' - x''| \geq 1$, from the previous arguments we have

$$\sup \frac{\left|\partial_i V(x') - \partial_i V(x'')\right|}{|x'-x''|^{\alpha}} \leq \frac{2\|\mathcal{F}\|}{n-2} \cdot \left(\frac{m\Omega}{\tau_n}\right)^{1/n} \cdot \max t^{1-\alpha}$$

where $t^{1-\alpha}$ is monotonically increasing function and $t \geq 1$ so we have a maximum for each t. Now if $d \geq 1$ is the diameter of the domain then $d^{1-\alpha}$ is the maximum for each $\alpha \in (0,1)$. So ultimately we get

$$\begin{aligned} \sup \frac{\left|\partial_i V(x') - \partial_i V(x'')\right|}{|x'-x''|^{\alpha}} &\leq \max\left[\frac{2(n-1)}{\omega_n} m', \left(\frac{m\Omega}{\tau_n}\right)^{1/n}\right] \frac{\|\mathcal{F}\|}{n-2} \\ &\leq m''(d) \cdot \frac{\|\mathcal{F}\|}{n-2}. \end{aligned} \tag{4.19}$$

where $\max\left(Ct^{1-\alpha}+8\pi t^{1-\alpha}\ln t\right)=m'$

Clearly the Hölder constant is finite.

So now by combining (4.16),(4.17) and (4.19), the norm $\|V(x)\|_{C^{1,\alpha}}$ is given by

$$\|V(x)\|_{C^{1,\alpha}} \leq \max\left[\frac{1}{2}\cdot\left(\frac{m\Omega}{\tau_n}\right)^{2/n},\left(\frac{m\Omega}{\tau_n}\right)^{1/n}, \quad m''\right]\frac{\|\mathcal{F}\|}{n-2} \tag{4.20}$$

Hence $V\in C^{1,\alpha}$.

For an estimation of $\mathcal{F}$ we consider

$$\begin{aligned}
\mathcal{F}\left(x,u,\frac{\partial u}{\partial x_i}\right) &= \mathcal{F}\left(x,u,\frac{\partial u}{\partial x_i}\right)-\mathcal{F}(x,0,0)+\mathcal{F}(x,0,0)\\
\left|\mathcal{F}\left(x,u,\frac{\partial u}{\partial x_i}\right)\right| &\leq \left|\mathcal{F}\left(x,u,\frac{\partial u}{\partial x_i}\right)-\mathcal{F}(x,0,0)\right|+|\mathcal{F}(x,0,0)|\\
\left|\mathcal{F}\left(x,u,\frac{\partial u}{\partial x_i}\right)\right| &\leq L_1|u_1-0|+\sum_j L_{2,j}\cdot\left|\frac{\partial u_1}{\partial x_i}-0\right|+M\\
\left\|\mathcal{F}\left(x,u,\frac{\partial u}{\partial x_i}\right)\right\| &\leq L_1\|u\|_{C^{1,\alpha}}+\sum_j L_{2,j}.\|u\|_{C^{1,\alpha}}+M
\end{aligned}$$

hence

$$\left\|\mathcal{F}\left(x,u,\frac{\partial u}{\partial x_i}\right)\right\| \leq \mathfrak{R}\cdot L_1+\mathfrak{R}\cdot\sum_j L_{2,j}+M \tag{4.21}$$

Now (4.20) takes the form

$$\|V\|_{C^{1,\alpha}} \leq \frac{\max\left[\frac{1}{2}\cdot\left(\frac{m\Omega}{\tau_n}\right)^{2/n},\left(\frac{m\Omega}{\tau_n}\right)^{1/n}, \quad m''\right]}{n-2}\left(\mathfrak{R}\cdot L_1+\mathfrak{R}\cdot\sum_j L_{2,j}+M\right) \tag{4.22}$$

Now we are to show that U in (4.10) belongs to $C^{1,\alpha}$ also.

Again from (4.10)

$$U=u_0+\tilde{u}+V$$

due to the fact that $C^{1,\alpha}$ is a Banach space, the triangle inequality implies,

$$\|U\|_{C^{1,\alpha}(\overline{\Omega})}\leq\|u_0\|_{C^{1,\alpha}(\overline{\Omega})}+\|\tilde{u}\|_{C^{1,\alpha}(\overline{\Omega})}+\|V\|_{C^{1,\alpha}(\overline{\Omega})}.$$

Applying the Schauder estimate from Section 2.3 of Chapter 2.

$$\|\tilde{u}\|_{C^{1,\alpha}(\overline{\Omega})}\leq K\left(\|F\|_{C^\alpha}+\|\varphi\|_{C^{1,\alpha}(\partial\Omega)}+\|\tilde{u}\|\right)$$

where K is constant of the Schauder estimate. Also using the maximum principle for Laplace equation leads to the following calculations:

$$\begin{aligned}
\|U\|_{C^{1,\alpha}(\overline{\Omega})} &\leq K\left(\|\varphi\|_{C^{1,\alpha}(\partial\Omega)} + \|u_0\|\right) + K\left(\|V\|_{C^{1,\alpha}} + \|\tilde{u}\|\right) + \|V\|_{C^{1,\alpha}(\overline{\Omega})} \\
&\leq K\left(\|\varphi\|_{C^{1,\alpha}(\partial\Omega)} + \|\varphi\|\right) + K\left(\|V\|_{C^{1,\alpha}} + \|V\|\right) + \|V\|_{C^{1,\alpha}(\overline{\Omega})} \\
&\leq K\left(\|\varphi\|_{C^{1,\alpha}(\partial\Omega)} + \|\varphi\|_{C^{1,\alpha}}\right) + K\left(\|V\|_{C^{1,\alpha}} + \|V\|\right) + \|V\|_{C^{1,\alpha}(\overline{\Omega})} \\
&\leq K\left(\|\varphi\|_{C^{1,\alpha}(\partial\Omega)} + \|\varphi\|_{C^{1,\alpha}}\right) + K\left(\|V\|_{C^{1,\alpha}} + \|V\|_{C^{1,\alpha}}\right) + \|V\|_{C^{1,\alpha}(\overline{\Omega})} \\
&\leq 2K\|\varphi\|_{C^{1,\alpha}} + 2K\|V\|_{C^{1,\alpha}} + \|V\|_{C^{1,\alpha}(\overline{\Omega})} \\
&\leq 2K\|\varphi\|_{C^{1,\alpha}} + (2K+1)\,\|V\|_{C^{1,\alpha}}
\end{aligned}$$

Using equation (4.22) we have

$$\|U\|_{C^{1,\alpha}(\overline{\Omega})} \leq 2K\|\varphi\|_{C^{1,\alpha}}\,(2K+1))\cdot m\cdot\left(\mathfrak{R}\cdot L_1 + \mathfrak{R}\cdot\sum_j L_{2,j} + M\right)$$

where m is given by,

$$\max\left[\frac{1}{2}\cdot\left(\frac{m\Omega}{\tau_n}\right)^{2/n}, \left(\frac{m\Omega}{\tau_n}\right)^{1/n}, \quad m''\right] = m \tag{4.23}$$

Hence the operator U maps the function space $C^{1,\alpha}$ into itself. And the condition on the radius of ball, under which U maps the ball into itself as follows

$$2K\|\varphi\|_{C^{1,\alpha}} + (2K+1)\cdot m\left(\mathfrak{R}\cdot L_1 + \mathfrak{R}\cdot\sum_j L_{2,j} + M\right) \leq \mathfrak{R} \tag{4.24}$$

or

$$\mathfrak{R} \geq 2K\|\varphi\|_{C^{1,\alpha}} + m\,(2K+1)\left(\mathfrak{R}\cdot L_1 + \mathfrak{R}\cdot\sum_j L_{2,j} + M\right). \tag{4.25}$$

To sum up, we have the following statement:

Theorem 4.1 *Let Ω be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^{1,\alpha}$ then, provided that the right hand side is Lipschitz continuous (at least locally), the boundary value problem (4.1),(4.2) is solvable, in the ball $\mathfrak{B}$ with radius $\mathfrak{R}$ defined in (4.14), by Contraction Mapping Principle and Schauder Fixed Point Theorem, only if the inequality (4.24) is satisfied.*

It is important that the above theorem is not enough for the application of fixed point result but it provides us the necessary condition to be satisfied for the application of the both fixed point results while the additional conditions are yet to be discussed.

From (4.25), following immediate consequence can be drawn.

Corollary 4.1 *An admissible bound for the boundary values is given by*

$$\|\varphi\|_{C^{1,\alpha}} \leq \frac{1}{2K} \max \left[\mathfrak{R} - m\,(2K+1) \left(\mathfrak{R} \cdot L_1(\mathfrak{R}) + \mathfrak{R} \cdot \sum_j L_{2,j}(\mathfrak{R}) + M \right) \right] \quad (4.26)$$

Remark 4.1 Clearly the corollary above gives us the restriction to the range of the boundary values that can be considered to solve the BVPs for non-linear PDEs.

Remark 4.2 The following section deals with the more necessary requirements for Schauder (II) and we are going to discuss an important development for the application of Schauder Fixed Point Theorem, i.e, we will show the relative compactness of the operator.

4.4. Application of Schauder Fixed-Point Theorem

In this section we are going to prove that the operator U defined according to (4.9) is relatively compact in the ball $\mathfrak{B}$ which is a closed and convex subset of the Banach space $C^{1,\alpha}$. Since the main goal of the present section is to apply the Schauder (II). During the construction of relative compactness of the fixed point operator U in the ball in the Banach space, so we will adapt procedure as follows:

i - We assume that for an arbitrary sequence of solution u_k there exists a sequence of images U_k.

ii - We will show that the U_k are equi-continuous in the supremum norm.

iii - In the 3rd step we apply the Arzelà Ascoli theorem that guarantees the existence of convergent subsequence (convergence is again in sup norm).

iv - Finally we will show the existence of subsequence of U_k that converges not only in the sup norm but also in the respective Hölder norm.

v - Using the information from the above steps, we shall apply Schauder Theorem for the existence of the solutions.

Numbers (i) to (iv) will be discussed in subsection (4.4.1), i.e, we will show the relative compactness of our fixed point operator. Subsection (4.4.2) covers the final stage of the section that is we apply the Schauder II.

4.4.1. Relative compactness of the operator

We again consider only V and show that the V_k as the image sequence of the u_k are relatively compact in the ball $\mathfrak{B}$ defined above in $C^{1,\alpha}$, and then later on we will take this result to U_k, which we require.

In Section 3.4 of chapter 3, we have already found the relative compactness of the operator V. There we have found a subsequence of V_k which converges in the $C^{1,\alpha}$ and we are not reproducing it again here.

We start with the assumption of having a subsequence of V_k say $V_{k'_l}$ which is a fundamental (Cauchy) sequence in $C^{1,\alpha}$-norm and hence convergent.

$$\|V_{k'_l} - V_{k'_m}\|_{C^{1,\alpha}} < \varepsilon \quad \text{for large} \quad k'_l, k'_m \tag{4.27}$$

So the triangle inequality confirms the existence of fundamental subsequence of the images U_k.

Since,

$$U = u_0 + \tilde{u} + V$$

implies,

$$U_{k'_l} = u_0 + \tilde{u_{k'_l}} + V_{k'_l}$$

using again the triangle inequality, Schauder estimate and maximum principle

$$\|U_{k'_l} - U_{k'_m}\|_{C^{1,\alpha}} \leq \|\tilde{u}_{k'_l} - \tilde{u}_{k'_m}\|_{C^{1,\alpha}} + \|V_{k'_l} - V_{k'_m}\|_{C^{1,\alpha}}.$$

Since the boundary values of the $\tilde{u}_{k'_l}$ are given by $-V_{k'_l}$, the Schauder estimates and lemma 3.1 show that also the $\tilde{u}_{k'_l}$ are a fundamental sequence in the $C^{1,\alpha}$-norm. Therefore

$$\left\|U_{k'_l} - U_{k'_m}\right\|_{C^{1,\alpha}} \leq \varepsilon \quad \text{for large} \quad k'_l, k'_m. \tag{4.28}$$

4.4.2. Application of Schauder II in the balls

We have given the existence of a fundamental sequence by (4.28), in the Banach space $C^{1,\alpha}$ under the respective norm, so the operator U is relatively compact. Hence by the second version of the Schauder theorem there exist at least one solution of the boundary value problem (4.1),(4.2). To show that U has a fixed point (not necessarily unique) in the ball $\mathfrak{B}$ (the closed and convex subset of Banach space $C^{1,\alpha}$) defined by (4.14), both the conditions that the operator maps the ball into itself by (4.24) and images are relatively compact in the ball by (4.28) are already satisfied. Hence we have the existence of solution in the ball $\mathfrak{B}$.

We prove the following theorem,

Theorem 4.2 *Let assumptions numbered (1)-(4) from page (59) are satisfied, suppose further that the non-linear boundary value problem (4.1),(4.2) is reduced to fixed point operator U in (4.9). Let U maps the ball $\mathfrak{B}$ into itself by (4.25). Moreover, U is relatively compact in the ball then the BVP is solvable by Schauder (II).*

Remark 4.3 For existence by Schauder Fixed Point Theorem, we require right hand side to be only continuous or bounded by a constant $B(\mathfrak{R})$ where $\mathfrak{R}$ is the radius of the ball. If the right hand side is not Lipschitz continuous but is only bounded then $\left(\mathfrak{R}\cdot L_1+\mathfrak{R}\cdot\sum_j L_{2,j}+M\right)$ is replaced by either a constant or by $B(\mathfrak{R})$ accordingly.

For example:

$$F(x,y,u,\partial_x u,\partial_y u)=u^2+|\partial_y u|^{\frac{1}{2}}$$

clearly the above function is bounded locally by $B(\mathfrak{R})=\mathfrak{R}^2+(\mathfrak{R})^{\frac{1}{2}}$ in the ball defined in (4.14). But it is not Lipschitz continuous so only the Schauder Fixed Point Theorem is applicable.

> See also the Remark to the assumptions on the right-hand sides in Section 4.3 (page 59).

4.4.3. Application of Schauder Fixed-Point Theorem in the whole Banach space

Regarding the application of Schauder Fixed Point Theorem in the Whole Banach space, we refer to the results proved in [9], where the author gives the existence results by the Schauder theorem in the whole Banach space as follows:

Theorem 4.3 *Let X be a Banach space. If f is a completely continuous mapping (not necessarily linear) of X into itself and $f(X)$ is bounded then f has a fixed point in X.*

4.5. Application of Contraction Mapping Principle

In the first subsection we shall give the existence and uniqueness of solution by Contraction Mapping Principle in the closed and convex subsets (balls) of a Banach space while in the subsection 4.5.2, we shall give a note on the existence and uniqueness in the whole Banach space, i.e, $C^{1,\alpha}$.

4.5.1. Existence and uniqueness in a Ball

Since we have to apply the Contraction Mapping Principle, so we require that the operator U, defined in (4.9) is contractive. We find the explicit result for the operator to be contractive that leads to the unique existence of the solution of the boundary value problem (4.1),(4.2)

To verify that the fixed point operator U is contractive, we shall, first, check that V is contractive in the $\|\cdot\|_{C^{1,\alpha}}$-norm.

From inequality (3.29) in Chapter 3, we have found the following estimate

$$\|V_1 - V_2\|_{C^{1,\alpha}} \le m \cdot \left(L_1 + \sum_j L_{2,j} \right) \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \tag{4.29}$$

and

$$m = \max\left[\frac{1}{2}\left(\frac{m\Omega}{\tau_n}\right)^{2/n}, \left(\frac{m\Omega}{\tau_n}\right)^{1/n}, \frac{2(n-1)}{\omega_n} m^{'} \right]$$

where m' is explained in Section (3.5).

Now since $U = u_0 + \tilde{u} + V$ so by triangle inequality again we have

$$\|U_1 - U_2\|_{C^{1,\alpha}} \le \|\tilde{u_1} - \tilde{u_2}\|_{C^{1,\alpha}} + \|V_1 - V_2\|_{C^{1,\alpha}}. \tag{4.30}$$

Applying maximum principle for Laplace equation and Schauder estimate to $\tilde{u}$ having boundary values $-V$, we have

$$\begin{aligned}
\|U_1 - U_2\|_{C^{1,\alpha}} &\le K\left(\|V_1 - V_2\|_{C^{1,\alpha}} + \|\tilde{u_1} - \tilde{u_2}\|\right) + \|V_1 - V_2\|_{C^{1,\alpha}} \\
&\le K\left(\|V_1 - V_2\|_{C^{1,\alpha}} + \|\tilde{u_1} - \tilde{u_2}\|_{C(\partial\Omega)}\right) + \|V_1 - V_2\|_{C^{1,\alpha}} \\
&\le K\left(\|V_1 - V_2\|_{C^{1,\alpha}} + \|\tilde{u_1} - \tilde{u_2}\|_{C(\partial\Omega)}\right) + \|V_1 - V_2\|_{C^{1,\alpha}} \\
&\le K\left(\|V_1 - V_2\|_{C^{1,\alpha}} + \|V_1 - V_2\|_{C^{1,\alpha}}\right) + \|V_1 - V_2\|_{C^{1,\alpha}} \\
&\le (2K+1)\|V_1 - V_2\|_{C^{1,\alpha}}.
\end{aligned}$$

Using the results, we get

$$\|U_1 - U_2\|_{C^{1,\alpha}} \le (2K+1) \cdot m \cdot \left(L_1 + \sum_j L_{2,j} \right) \|u_1 - u_2\|_{C^{1,\alpha}} \tag{4.31}$$

so finally for U to be contractive the following condition is to be satisfied

$$(2K+1) \cdot m \cdot \left(L_1 + \sum_j L_{2,j} \right) < 1. \tag{4.32}$$

Hence if the condition in last inequality is satisfied then the $V(x)$ is contractive.

To sum up the following theorem has been proved:

Theorem 4.4 *Let Ω be the bounded domain with finite measure $m\Omega$, suppose further that non-linear boundary value problem (4.1),(4.2) is reduced to the fixed point operator U in (4.9). If U, maps a ball $\mathfrak{B}$ in $C^{1,\alpha}$ into itself with estimate (4.24), moreover, if the operator is contractive with,*

$$(2K+1) \cdot \max\left[\frac{1}{2}\left(\frac{m\Omega}{\tau_n}\right)^{2/n}, \left(\frac{m\Omega}{\tau_n}\right)^{1/n}, \frac{2(n-1)}{\omega_n} m^{'} \right] \cdot \left(L_1 + \sum_j L_{2,j} \right) < 1$$

then the boundary value problem (4.1),(4.2) is uniquely solvable (by Contraction Mapping Principle). Where K is a Constant from the Schauder estimate.

Remark 4.4 The condition for contractivity (4.32) put an additional restriction on $\mathfrak{R}$ together with one we obtained from the self map so that possible bound for the boundary values is restricted additionally.

Remark 4.5 The result which we have in (4.32) for contractivity demands to choose Lipschitz constant sufficiently small for operator to be contractive. And so the boundary value problem in (4.1) and (4.2) is uniquely solvable.

4.5.2. Applications to the whole Banach space

The local Lipschitz condition with respect to the desired solution u and its first order derivatives on the right hand side of (4.1) is necessary for the application of Contraction Mapping Principle in a closed subset of Banach space. But additionally, if the right hand side satisfies the global Lipschitz condition, we can apply the Contraction Theorem in the whole Banach space i.e, the existence and uniqueness can be easily shown in the whole Banach space. For example the right hand side is the function $\frac{1}{2} \cdot \frac{1}{1+u^2}$ which, of course, is global Lipschitz continuous so in this case we can show the existence and uniqueness by Contraction Mapping Principle in the whole Banach space but this does not lead to the explicit calculation. Foremost, in the case of global Lipschitz continuous right hand side we cannot find the solution in the closed and convex subsets of Banach spaces to find the estimates of types (4.25), (4.26) and (4.32).

> In other words, if the constructed solution u belongs to a ball $\mathcal{B}$, then the statement $u \in \mathcal{B}$ can be interpreted as an a-priori estimate of u.

4.6. Solution in the ball centered at the solution of homogeneous equation

Instead of considering the solution of boundary value problem (4.1),(4.2) in the ball defined in (4.33), one can also work in the ball

$$\mathfrak{B}_{\mathfrak{R}}(u_0) := \{u \in C^{1,\alpha} : \|u - u_0\|_{C^{1,\alpha}} \leq \mathfrak{R}\}, \tag{4.33}$$

that is a ball centered at the solution of Laplace equation stated above. In this situation, one can work with a particular boundary value problem but nevertheless, much similar estimates can be found as described in the previous sections of this chapter.

5. OPTIMIZATION OF FIXED-POINT METHODS

This chapter is the consequence of the chapter four and we establish the optimization results. These optimizations provide the necessary information on the choice of largest possible interval in which we choose the radii of the balls. Moreover in certain cases we give also the largest possible bound for the boundary values which we can consider. We also give the largest possible bound **C** for $C^{1,\alpha}$-norms of the admissible boundary values. Then we determine the radius $\mathfrak{R}$ which leads to the largest **C**.

We know from Chapter 4 that the boundary value problem

$$\begin{aligned} \Delta u &= \mathcal{F}(\cdot, u, \partial_i u) \text{ in } \Omega \\ u &= \varphi \text{ on } \partial\Omega \end{aligned}$$

is solvable in the ball,

$$\mathfrak{B}_{\mathfrak{R}}(0) := \{u \in C^{1,\alpha} : \|u\|_{C^{1,\alpha}} \le \mathfrak{R}\}, \tag{5.1}$$

by Schauder Fixed Point Theorem if the following estimate is true,

$$2C^1\|\varphi\|_{C^{1,\alpha}} + (2K+1)\cdot m\left(\mathfrak{R}\cdot L_1 + \mathfrak{R}\cdot\sum_j L_{2,j} + M\right) \le \mathfrak{R}$$

or

$$2K\|\varphi\|_{C^{1,\alpha}} + (2K+1)\cdot m\,(\mathfrak{R}\cdot L(\mathfrak{R}) + M) \le \mathfrak{R} \tag{5.2}$$

where K is constant from the Schauder estimates and $L(\mathfrak{R}) = L_1(\mathfrak{R}) + \sum_j L_{2,j}(\mathfrak{R})$ is the Lipschitz constant revealed from the right hand side of the differential equation, that is we have assumed that it is Lipschitz continuous according to Chapter 4.

For Contraction Mapping Principle we require additionally that

$$(2K+1)\cdot m\cdot\left(L_1(\mathfrak{R}) + \sum_j L_{2,j}(\mathfrak{R})\right) < 1.$$

or

$$(2K+1)\cdot m\cdot(L(\mathfrak{R})) < 1. \tag{5.3}$$

has to be satisfied.

5.1. Schauder Theorem and optimization

Since we already know that, to ensure the existence of solution, we must have to satisfy the condition (5.2), i.e,

$$\|\varphi\|_{C^{1,\alpha}} \leq \frac{1}{2K}\left\{\mathfrak{R} - (2K+1)\cdot m\left(\mathfrak{R}\cdot L(\mathfrak{R}) + M\right)\right\}, \tag{5.4}$$

the above estimate leads to the maximum bound for the boundary value within the ball (5.1). To find the largest possible bound $\|\varphi\|_{C^{1,\alpha}}$ we have to maximize the right hand side of the last inequality, that is, $\max\left\{\mathfrak{R} - \kappa\left(\mathfrak{R}\cdot L(\mathfrak{R}) + M\right)\right\}$ with $\kappa = m(2K+1)$.

Differentiating with respect to $\mathfrak{R}$, we have,

$$\left\{1 - \kappa\left(\mathfrak{R}\cdot L^{'}(\mathfrak{R}) + L(\mathfrak{R})\right)\right\} = 0 \tag{5.5}$$

implies that,

$$\frac{d}{d\mathfrak{R}}\left(\mathfrak{R}\cdot L(\mathfrak{R})\right) = \frac{1}{\kappa}, \tag{5.6}$$

provided $L(\mathfrak{R})$ is differentiable. Suppose, in addition, that $L^{'}(\mathfrak{R})$ is a monotonically increasing, that is, for $\mathfrak{R}_1 \leq \mathfrak{R}_2$ there is $L^{'}(\mathfrak{R}_1) \leq L^{'}(\mathfrak{R}_2)$. So consequently, we have $L^{'}(\mathfrak{R}_1)\cdot\mathfrak{R}_1 \leq L^{'}(\mathfrak{R}_2)\cdot\mathfrak{R}_2$ and hence $L^{'}(\mathfrak{R})\cdot\mathfrak{R}$ is a monotonically increasing function.

Equation (5.6) has a unique solution at $\mathfrak{R}^*$ provided $L(0) < \frac{1}{\kappa}$ and we have,

$$\kappa\cdot L^{'}(\mathfrak{R}^*)\cdot\mathfrak{R}^* + \kappa\cdot L(\mathfrak{R}^*) = 1. \tag{5.7}$$

We have the following result:

Lemma 5.1 *There is at most one solution of the equation (5.6) if* $\frac{d}{d\mathfrak{R}}\left(\mathfrak{R}\cdot L(\mathfrak{R})\right)$ *is monotonically increasing function and* $L(0) < \frac{1}{\kappa}$.

But yet we have to discuss the following cases

5.1.1. Unique existence of optimal radius of the ball

Lemma 5.2 *The equation (5.6) has a unique solution if*

- $\frac{d}{d\mathfrak{R}}\left(\mathfrak{R}\cdot L(\mathfrak{R})\right)$ *is monotonically increasing function.*
- $L(0) < \frac{1}{\kappa}$.
- $\lim\limits_{\mathfrak{R}\to\infty}\frac{d}{d\mathfrak{R}}\left(\mathfrak{R}\cdot L(\mathfrak{R})\right) > \frac{1}{\kappa}$.

That is we have the following two situations when we get the uniquely determined determined radius $\mathfrak{R}$ where the radius of the ball lies in the interval $(0, \mathfrak{R}^*]$. Hence,

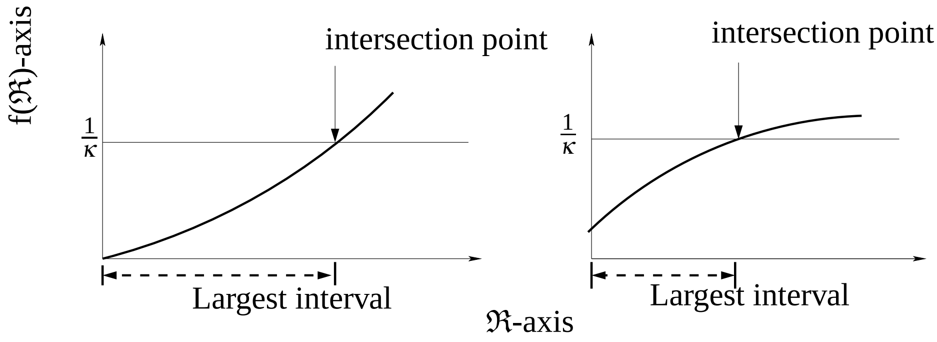


Figure 5.1.: Uniquely determined largest radius $\mathfrak{R}^*$ of the ball.

$(0, \mathfrak{R}^*]$ is the largest interval we get for choosing the radius of the ball.

5.1.2. No solution but application of Schauder Fixed Point theorem in the whole ball

Lemma 5.3 *The equation (5.6) has no solution if*

- $\frac{d}{d\mathfrak{R}}(\mathfrak{R} \cdot L(\mathfrak{R}))$ *is a monotonically increasing function.*
- $L(0) < \frac{1}{\kappa}$.
- $\lim_{\mathfrak{R}\to\infty} \frac{d}{d\mathfrak{R}}(\mathfrak{R} \cdot L(\mathfrak{R})) < \frac{1}{\kappa}$.

In this situation we don't have any solution of equation (5.6) and therefore, no intersection point but even then we are in a position to apply Schauder Fixed Point Theorem in the ball with arbitrary radius. The following figure illustrate this case.

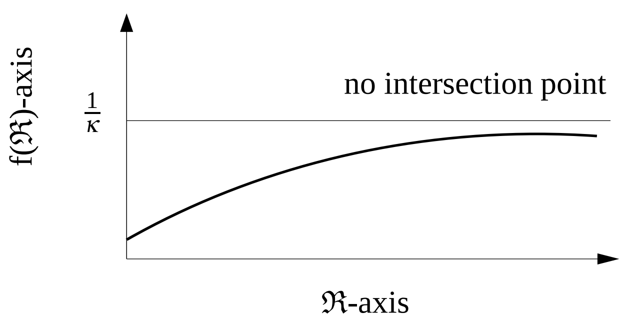


Figure 5.2.: No uniquely determined optimal radius $\mathfrak{R}^*$.

But this is very interesting situation because here, still, we are in a position to apply the Schauder Fixed Point Theorem in the whole ball. For better results, the radius is taken bigger and bigger. To get the explicit results, in this situation, we can fix the radius that leads to the desired results.

5.1.3. No solution and no application of Schauder Fixed Point Theorem

The equation (5.6) has no solution if:

- $\frac{d}{d\mathfrak{R}}(\mathfrak{R}\cdot L(\mathfrak{R}))$ is monotonically increasing function.
- $L(0) > \frac{1}{\kappa}$

In terms of the diagram we have. Here we are in a position where we can't apply

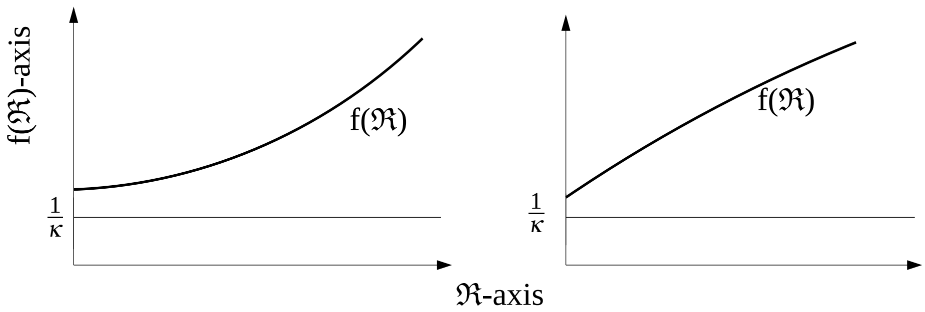


Figure 5.3.: No existence of solution

the Schauder theorem for existence of solution.

5.1.4. Additional condition for Schauder theorem

We again consider equation (5.2)

$$\|\varphi\|_{C^{1,\alpha}} \leq \frac{1}{2K}(\mathfrak{R} - \kappa(\mathfrak{R}\cdot L(\mathfrak{R}) + M)) > 0$$

which imples that

$$\mathfrak{R} - \kappa(\mathfrak{R}\cdot L(\mathfrak{R}) + M) > 0$$

or

$$\mathfrak{R} > \kappa(\mathfrak{R}\cdot L(\mathfrak{R}) + M) \tag{5.8}$$

We have the following figures clarifying the last inequality and choice of the radius.

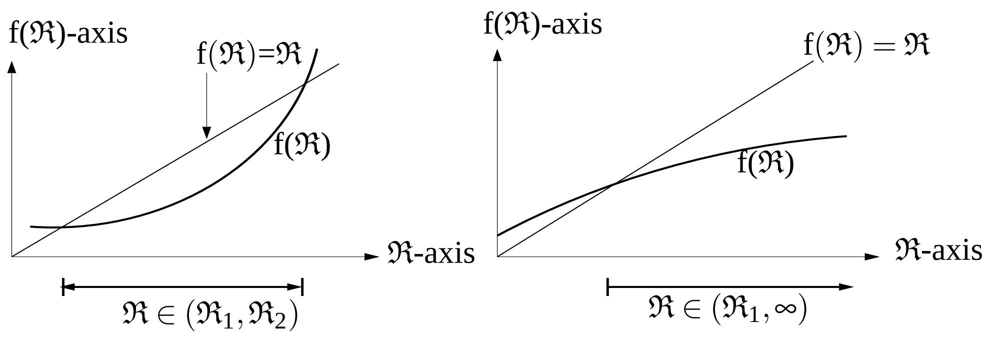


Figure 5.4.: Two different cases for the existence of possible values of the radius of the ball.

The situation in the left side in the figure above demands to choose the radius additionally in the interval [c, d] while the right side interval lead the interval [h, ∞)

5.1.5. Additional condition when ball is centered at the solution of homogeneous equation

$$\mathfrak{B}_{\mathfrak{R}}(u_0) := \{u \in C^{1,\alpha} : \|u - u_0\|_{C^{1,\alpha}} \leq \mathfrak{R}\}, \tag{5.9}$$

by Schauder Fixed Point Theorem if the following estimate is true,

$$\begin{aligned} \|U - u_0\|_{C^{1,\alpha}} &\leq \kappa \left(\mathfrak{R} \cdot L_1 + \mathfrak{R} \cdot \sum_j L_{2,j} + M \right) \leq \mathfrak{R} \\ &\leq \mathfrak{R} - \kappa \left(\mathfrak{R} L(\mathfrak{R}) + M \right) \end{aligned}$$

since $\|U - u_0\|_{C^{1,\alpha}}$ is a norm so we have

$$\mathfrak{R} - \kappa \left(\mathfrak{R} L(\mathfrak{R}) + M \right) > 0 \tag{5.10}$$

$$\left(L(\mathfrak{R}) + \frac{M}{\mathfrak{R}} \right) > \frac{1}{\kappa}$$

We have to ensure that, $\min \left(L(\mathfrak{R}) + \frac{M}{\mathfrak{R}} \right) < \frac{1}{\kappa}$ leads to the interval as in the next figure.

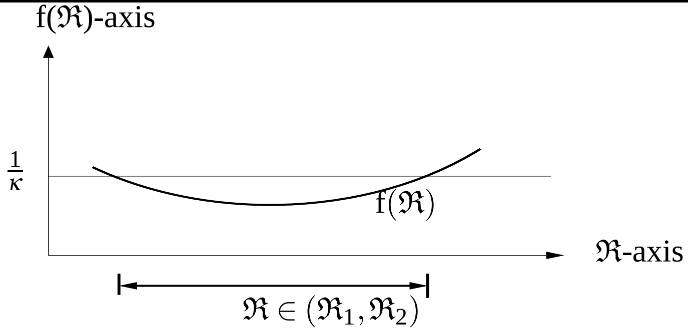


Figure 5.5.: Maximum interval for he radius of the ball for Schauder Fixed Point Theorem.

Note :
For the application of Schauder Fixed Point Theorem we don't require the Lipschitz condition rather we require only the boundedness of the right hand side. If the bound of the right hand side is represented by $B(\mathfrak{R})$ then above all conditions are satisfied for $B(\mathfrak{R})$ instead of $\mathfrak{R}\,(L(\mathfrak{R})+M)$ because all above discussion deals the mapping properties of U mapping the ball into itself for which we don't necessarily require the Lipschitz continuity on the right hand side.

5.2. Contraction Mapping Principle

To apply the contraction mapping principle we have to check the condition that the fixed point operator is not only maps the ball into itself as in condition (5.2) but also that the operator is contractive see (5.3).

Inequality (5.3) leads to the the estimate,

$$L(\mathfrak{R}) < \frac{1}{\kappa} \tag{5.11}$$

To apply the Contraction Mapping Principle we must satisfy the all cases discussed in the previous sections of the current chapter together with the last inequality (5.11) obtained from contractive condition.

Contraction Mapping Principle is applied according to the following figures.

5.2.1. Contraction Mapping Principle for the case of the Ball centered at u_0

In this situation the contractive condition is

$$L(\mathfrak{R}) < \frac{1}{\kappa} \tag{5.12}$$

with of course $L(0) < \dfrac{1}{\kappa}$. Then the following intervals can be considered

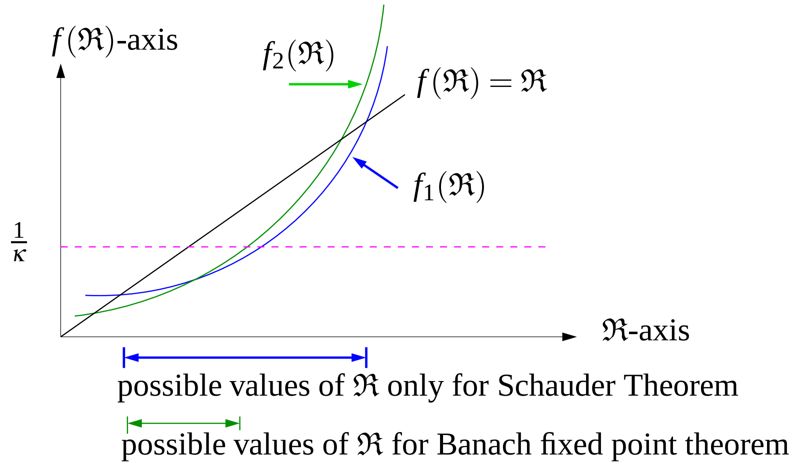


Figure 5.6.: Largest possible interval for the radius for the Contraction Mapping Principle.

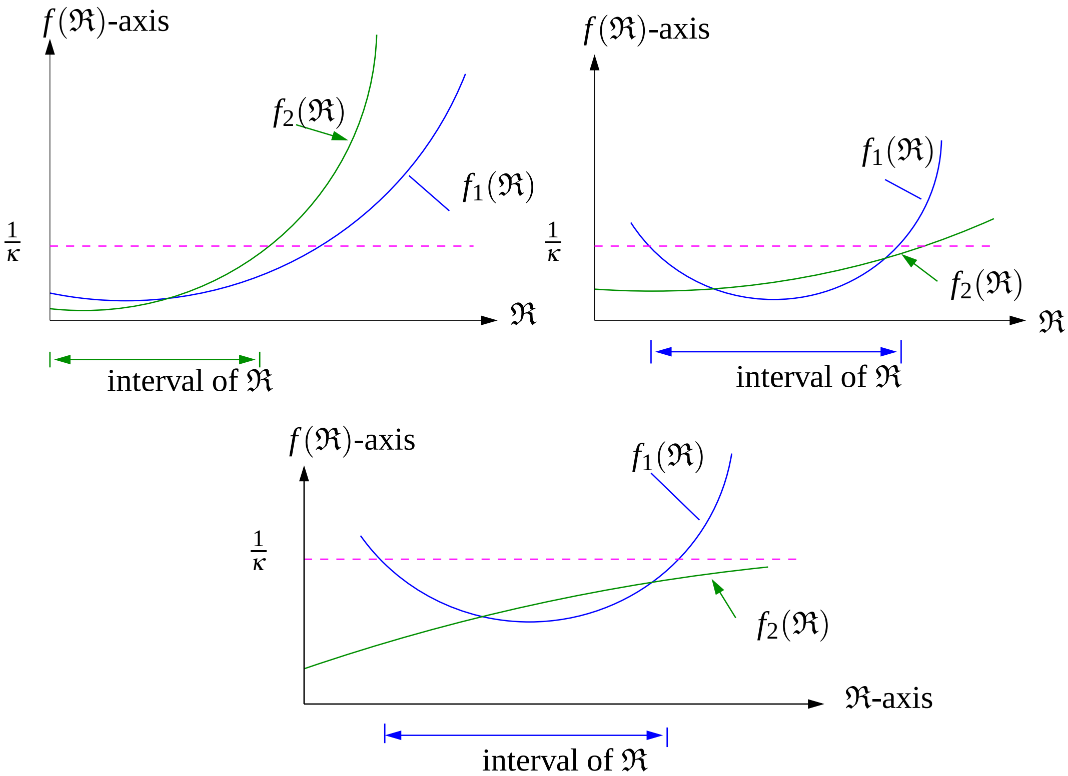


Figure 5.7.: Various cases for Contraction Mapping Principle.

5.3. Solutions of inhomogeneous boundary value problems

We have developed the necessary theory to solve a class of boundary value problems for non-linear elliptic PDEs when the right hand side depends not only on the solution u but also on its first order derivatives. Now we give applications of the results to specific examples. We will consider the boundary value problem for non-linear partial differential equations when the left hand side contains the Laplace operator and the right hand sides always depends on the derivative of the desired solution. We solve the BVPs only in the unit disk in the plane and we will be using the fundamental solution of Laplace equation in two dimensions.

5.3.1. General representation

In this subsection, we estimate the the $C^{1,\alpha}$-norm of the fixed point operator U which is defined by the convolution of the right hand side to the fundamental solution. We also consider the arbitrary boundary values $g(x)$. we shall handle different situations for the $C^{1,\alpha}$-norm of the boundary values to be maximal. We shall also look for the optimal radius of the ball for given fixed boundary values and for a fixed right hand side.

We consider the following general boundary value problem for non-linear elliptic partial differential equation:

$$\begin{aligned} \Delta u &= \mathcal{F}(\cdot, u, \partial_i u) \text{ in } \Omega && (5.13)\\ u &= g(x) \quad \text{on } \partial\Omega && (5.14) \end{aligned}$$

under the following assumptions:

1 - Ω be a unit disk in $\mathbb{R}^2$

2 - $g(x) \in C^{1,\alpha}(\partial\Omega)$

3 - $|\mathcal{F}(\cdot, u_1, \partial_i u_1) - \mathcal{F}(\cdot, u_2, \partial_i u_2)| \leq L_1|u_1 - u_2| + \sum_j L_{2,j}\,|\partial_i u_1 - \partial_i u_2|$

for $i, j = 1, 2, 3 \cdots$ that means that the Lipschitz condition holds or only bounded right hand sides.

4 - $|\mathcal{F}(\cdot, 0, 0)| \leq M$ is given if required.

5 - The homogeneous equation possesses a fundamental solution.

We know that the solution of the above problem is equivalent to finding the fixed-point of the following operator equation

$$U = u_0 + \tilde{u} + V \tag{5.15}$$

where V is given by

$$V =: -\frac{1}{2\pi}\iint\limits_{\Omega} \log|(x,y) - (\xi, \eta)| \cdot \mathcal{F}((\zeta), u(\zeta), \partial_i(\zeta))d\zeta \tag{5.16}$$

while

$$\begin{aligned} \Delta u_0 &= 0 \quad \text{in } \Omega\\ u_0 &= g(x) \text{ on } \partial\Omega \end{aligned}$$

and $\tilde{u}$ is solution of the homogeneous problem:

$$\begin{aligned}
\Delta\tilde{u} &= 0 \quad \text{in } \Omega \\
\tilde{u} &= -\frac{1}{2\pi}\iint\limits_{\Omega} \log|(x,y)-(\xi,\eta)| \cdot \mathcal{F}((\zeta),u(\zeta),\partial_i(\zeta))d\zeta \qquad (5.17) \\
&= -V(x) \qquad \text{on } \partial\Omega. \qquad (5.18)
\end{aligned}$$

We look for the solution in the space $C^{1,\alpha}(\overline{\Omega})$ because the right hand side depends on the derivatives of the desired solution by the fixed point method. So we consider the following closed and convex subset in the the Banach space $C^{1,\alpha}(\overline{\Omega})$ of the Hölder continuously differentiable functions on $C^{1,\alpha}(\overline{\Omega})$,

$$\mathfrak{B}_{\mathfrak{R}}(0) := \{u \in C^{1,\alpha} : \|u\|_{C^{1,\alpha}} \leq \mathfrak{R}\}, \qquad (5.19)$$

5.3.2. General condition on U for mapping the ball into itself

The operator (5.15) will map the ball into itself if,

$$\|U\|_{C^{1,\alpha}(\mathfrak{B})} \leq \|u_0\|_{C^{1,\alpha}(\mathfrak{B})} + \|\tilde{u}\|_{C^{1,\alpha}(\mathfrak{B})} + \|V\|_{C^{1,\alpha}(\mathfrak{B})} \leq (\mathfrak{R}) \qquad (5.20)$$

where $\|u_0\|_{C^{1,\alpha}}$ and $\|\tilde{u}\|_{C^{1,\alpha}}$ will be estimated by Schauder's estimates[1]. To satisfy that (5.15) maps the ball into itself, we first consider only V and check its mapping properties;

$$\begin{aligned}
V &=: -\frac{1}{2\pi}\iint\limits_{\Omega} \log|(x,y)-(\xi,\eta)| \cdot \mathcal{F}((\zeta),u(\zeta),\partial_i(\zeta))d\zeta \qquad (5.21) \\
\Rightarrow |V| &\leq \frac{\|\mathcal{F}\|}{2\pi}\iint\limits_{\Omega} |\log|(x,y)-(\xi,\eta)||d\zeta \qquad (5.22)
\end{aligned}$$

for $0 < |(x,y)-(\xi,\eta)| < 1$, using the Schmidt inequality

$$\begin{aligned}
|V| &\leq \frac{\|\mathcal{F}\|}{2\pi}\iint\limits_{\Omega} \frac{1}{|(x,y)-(\xi,\eta)|}d\zeta \qquad (5.23) \\
\|V\| &\leq \frac{\|\mathcal{F}\|}{2} \qquad (5.24)
\end{aligned}$$

for $|(x,y)-(\xi,\eta)| \geq 1$, we have, $\log|(x,y)-(\xi,\eta)| \leq |(x,y)-(\xi,\eta)|$ and having the fact that the domain is the unit disk in the plane so $|(x,y)-(\xi,\eta)| \leq 2$

$$\begin{aligned}
|V| &\leq \frac{\|\mathcal{F}\|}{2\pi}\iint\limits_{\Omega} |(x,y)-(\xi,\eta)|d\zeta \qquad (5.25) \\
&\leq \frac{\|\mathcal{F}\|}{2\pi} \cdot 1 \cdot \iint\limits_{\Omega} 2d\zeta \qquad (5.26) \\
\|V\| &\leq \|\mathcal{F}\|. \qquad (5.27)
\end{aligned}$$

[1]For a detailed proof of Schauder type estimates in $\mathbb{R}^2$, see the appendix A.

Now for Hölder constant, first we find the norm of the derivative of V

$$\begin{aligned}
\partial_x V &= -\frac{1}{2\pi}\iint\limits_{\Omega}\frac{1}{|(x,y)-(\xi,\eta)|}\cdot\frac{(x,y)-(\xi,\eta)}{|(x,y)-(\xi,\eta)|}\cdot\mathcal{F}((\zeta),u(\zeta),\partial_i(\zeta))d\zeta \\
|\partial_x V| &\leq \frac{\|\mathcal{F}\|}{2\pi}\iint\limits_{\Omega}\frac{1}{|(x,y)-(\xi,\eta)|}d\zeta \\
\|\partial_x V\| &\leq \frac{\|\mathcal{F}\|}{2}
\end{aligned}$$

similarly for $\partial_y V$

$$\|\partial_y V\| \leq \frac{\|\mathcal{F}\|}{2} \tag{5.28}$$

and we have the following estimate

$$\begin{aligned}
\left|V(x',y')-V(x'',y'')\right| &\leq \|\partial_x V\||x'-x''|+\|\partial_y V\||y'-y''| && (5.29)\\
&\leq \frac{\|\mathcal{F}\|}{2}|x'-x''|+\frac{\|\mathcal{F}\|}{2}|y'-y''| && (5.30)\\
&\leq \|\mathcal{F}\|\left|(x',y')-(x'',y'')\right| && (5.31)\\
\frac{|V(x',y')-V(x'',y'')|}{|(x',y')-(x'',y'')|^{\alpha}} &\leq \|\mathcal{F}\|\left|(x',y')-(x'',y'')\right|^{1-\alpha} && (5.32)\\
&\leq \sqrt{2}\|\mathcal{F}\| = H_{\tilde{u}} && (5.33)
\end{aligned}$$

and for the Hölder constant of the derivative, we again have,

$$\begin{aligned}
\partial_x V(x',y')-\partial_x V(x'',y'') &= \frac{1}{2\pi}\iint\limits_{\Omega}\kappa\cdot\mathcal{F}\Big(\zeta,u(\zeta),\partial_x u(\zeta)\Big)(\zeta)d\zeta \\
\left|\partial_x V(x',y')-\partial_x V(x'',y'')\right| &\leq \frac{\|\mathcal{F}\|}{2\pi}\cdot I \\
\max_{(\overline{\Omega})}\frac{\left|\partial_x V(x',y')-\partial_x V(x'',y'')\right|}{|(x',y')-(x'',y'')|^{\alpha}} &\leq 7\|\mathcal{F}\|
\end{aligned}$$

where κ and I are given by,

$$\begin{aligned}
I &= \iint\limits_{\Omega}\frac{d\zeta}{|(x',y')-(\xi,\eta)|\cdot|(x'',y'')-(\xi,\eta))|} && (5.34)\\
\kappa &= \frac{(x'',y'')-(\xi,\eta)}{|(x'',y'')-(\xi,\eta)|^2}-\frac{(x',y')-(\xi,\eta)}{|(x,y)-(\xi,\eta)|^2} && (5.35)\\
\text{if} t &= \left|(x',y')-(x'',y'')\right| && (5.36)
\end{aligned}$$

In our case we have $t \leq 2$ and $\alpha = \frac{1}{2}$, hence we have

$$\max_{(\overline{\Omega})}\frac{|\partial_x V(x',y')-\partial_x V(x'',y'')|}{|(x',y')-(x'',y'')|^{\alpha}} \leq 7\|\mathcal{F}\| \tag{5.37}$$

in view of inequalities (5.28),(5.31),(5.32) and (5.39) we get

$$\|V\|_{C^{1,\alpha}(\overline{\Omega})} = 7\|\mathcal{F}\|. \tag{5.38}$$

Using Schauder estimates, (5.22) in (5.17) we get:

$$\begin{aligned}
\|U\|_{C^{1,\alpha}(\overline{\Omega})} &\leq \|u_0\|_{C^{1,\alpha}(\overline{\Omega})} + \|\tilde{u}\|_{C^{1,\alpha}(\overline{\Omega})} + \|V\|_{C^{1,\alpha}(\overline{\Omega})} \qquad (5.39)\\
&\leq \left[2^{1+\alpha}K_1K_2H + \max_{(\partial\Omega)}|g| + 2K_1K_2\tilde{H} + 4(2^{1-\alpha}+1)K_1^2K_2^2\tilde{H}\right]\\
&\quad + \left[2^{1+\alpha}K_1K_2H_{\tilde{u}} + \max_{(\partial\Omega)}|\tilde{u}| + 2K_1K_2\tilde{H}_{\tilde{u}} + 4(2^{1-\alpha}+1)K_1^2K_2^2\tilde{H}_{\tilde{u}}\right]\\
&\quad +7\|\mathcal{F}\| \qquad (5.40)
\end{aligned}$$

where H and $\tilde{H}$ are the Hölder constants of the given boundary values and their derivatives respectively. Similarly the $H_{\tilde{u}}$ and $\tilde{H}_{\tilde{u}}$ are the Hölder constants of the boundary values of $\tilde{u}$ and its derivatives respectively.

The last inequality can also be written in the following way

$$\begin{aligned}
\|U\|_{C^{1,\alpha}(\overline{\Omega})} &\leq \left[2^{1+\alpha}K_1K_2 + 1 + 2K_1K_2 + 4(2^{1-\alpha}+1)K_1^2K_2^2\right]\|g\|_{C^{1,\alpha}(\partial\Omega)}\\
&\quad + \left[2^{1+\alpha}K_1K_2H_{\tilde{u}} + \max_{(\partial\Omega)}|\tilde{u}| + 2K_1K_2\tilde{H}_{\tilde{u}} + 4(2^{1-\alpha}+1)K_1^2K_2^2\tilde{H}_{\tilde{u}}\right]\\
&\quad +7\|\mathcal{F}\| \qquad (5.41)
\end{aligned}$$

the constants K_1 and K_2,[2] are given

$$K_1 =: \frac{4.2^{\alpha}}{\pi\cos\left(\alpha\frac{\pi}{2}\right)} \tag{5.42}$$

and

$$K_2 =: \left(\frac{2}{\alpha}(1+2^{\alpha})+\pi\right). \tag{5.43}$$

for fixed $\alpha = \frac{1}{2}$, K_1 and K_1 are given by

$$K_1 = \frac{8}{\pi} \tag{5.44}$$

and

$$K_2 = 4(1+\sqrt{2})+\pi. \tag{5.45}$$

Finally, for a unit disk in the plane, the inequality (5.41) can be formulated as follows

$$\|U\|_{C^{1,\alpha}(\overline{\Omega})} \leq 10416\|g\|_{C^{1,\alpha}(\partial\Omega)} + 72395\|\mathcal{F}\|. \tag{5.46}$$

[2]Existence of these constants has been proved in the appendix A

Now if $\|\mathcal{F}\|$ is bounded (by the radius of the ball) then U will map the ball (4.14) into itself when

$$10416\|g\|_{C^{1,\alpha}} + 72395\|\mathcal{F}\| \leq \mathfrak{R}. \tag{5.47}$$

The inequality (5.47) is the condition for U mapping the ball $B_{\mathfrak{R}}(0)$ into itself for each bounded right hand side which also is the sole condition for the application of Schauder Fixed Point Theorem and the relative compactness of U is understood by the discussion that we had in the previous chapters.

5.4. Application of Schauder's Fixed Point Theorem and optimization

Here we consider various examples and then we find the conditions for the best radius of the ball, the maximal bound of the $C^{1,\alpha}$-norm of the boundary values and the restriction on the right hand side. We deal with different situations in the following boundary value problems.

The examples below give the existence of fixed-point solutions of the boundary value problem in the ball centered at the zero element of the function space and then correspondingly, we give the maximum $C^{1,\alpha}$-norm of the boundary values g

5.4.1. Existence of the solution with arbitrary C$^{1,\alpha}$ boundary values

Example 1:

$$\begin{aligned} \Delta u &= k(\cos u + \partial_x u) \quad \text{in } \Omega \\ u &= g(x) \text{ on } \partial\Omega \end{aligned}$$

where k is a given real parameter in general. We assume that the boundary values g are arbitrary in $C^{1,\alpha}(\partial\Omega)$ and Ω is the unit disk in the plane. We know that the given boundary value problem is equivalent to finding the fixed-point of the operator defined in (5.15) if it maps the ball defined in (5.19) into itself.

According to (5.47) a fixed point will be the solution of the boundary value problem if the following condition is satisfied

$$10416\|g\|_{C^{1,\alpha}} + 72395|k|(1+\mathfrak{R}) \leq \mathfrak{R} \tag{5.48}$$

this implies that

$$\|g\|_{C^{1,\alpha}} \leq \frac{1}{10416}(\mathfrak{R} - 72395|k|(1+\mathfrak{R})) \tag{5.49}$$

where $|k|(1+\mathfrak{R})$ is the bound of the right hand side in the ball.

We get the following conditions to be satisfied:

- $\mathfrak{R} > 72395|k|(1+\mathfrak{R})$
- $|k| < \dfrac{\mathfrak{R}}{72395(1+\mathfrak{R})}$ that is for sufficiently small $|k|$ we are able to solve the above boundary value problem by Schauder Fixed Point Theorem.

If for a fixed $|k|$, we consider the following fixed right hand side given by

$$\Delta u = 1.25 \times 10^{-6}\big(cosu + \partial_x(u)\big) \quad \text{in } \Omega$$

and we get the condition

$$\mathfrak{R} > 72395 \times 1.25 \times 10^{-6}(1+\mathfrak{R})$$

so any sufficiently large $\mathfrak{R}_0$ larger than 0.09949 will lead to the larger $\|g\|_{C^{1,\alpha}}$.

Example 2:

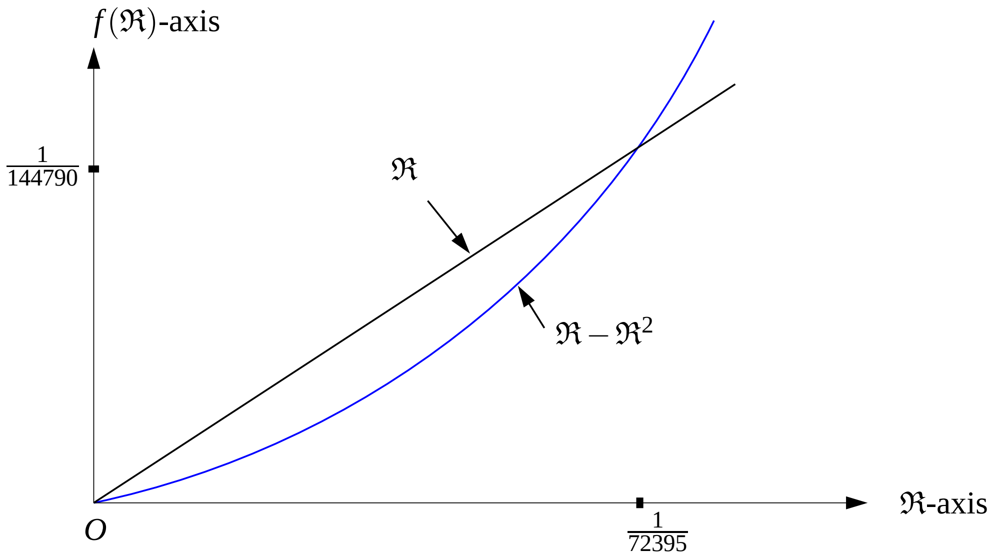
$$\Delta u = \left(\partial_x u\right)^2 \quad \text{in } \Omega \tag{5.50}$$
$$u = g(x) \text{ on } \partial\Omega. \tag{5.51}$$

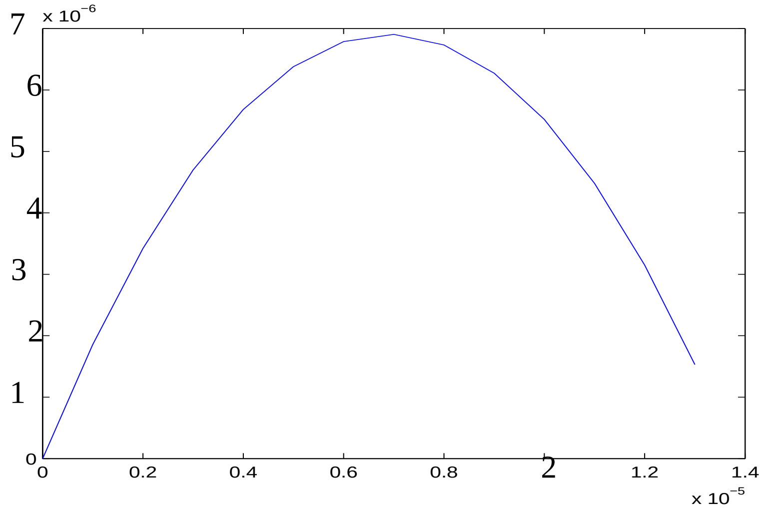
We assume that the boundary value $g(x)$ is Hölder continuously differentiable then the corresponding fixed-point operator maps the ball (4.19) into itself if

$$\|U\|_{C^{1,\alpha}} \leq 10416\|g\|_{C^{1,\alpha}} + 72395\|\mathcal{F}\| \leq \mathfrak{R} \tag{5.52}$$
$$\Rightarrow \|g\|_{C^{1,\alpha}} \leq \frac{1}{10416}\left(\mathfrak{R} - 72395\mathfrak{R}^2\right) \tag{5.53}$$

then the maximal $C^{1,\alpha}$-norm of g leads to the two conditions to be satisfied:

- $\mathfrak{R} > 72395\mathfrak{R}^2$ which implies $\mathfrak{R} < \dfrac{1}{72395}$





- For the maximal value of the $C^{1,\alpha}$-norm of g we have $(1 - 2 \times 72395\mathfrak{R}) = 0$ which leads to $\mathfrak{R}^* = \frac{1}{144790}$. This solves the equation and for this $\mathfrak{R}^*$ we get the maximum value of the $C^{1,\alpha}$-norm of g that is 6.9×10^{-6}.

Th corresponding maximum value of the norm of g is shown in the figure above.

Example 3:

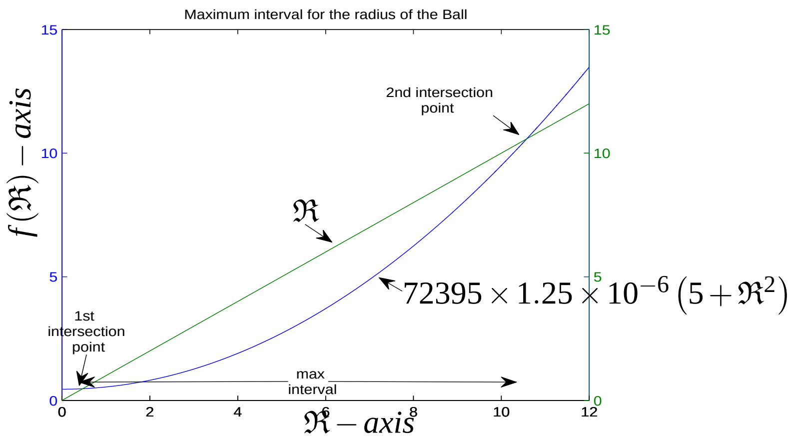
$$\Delta u = 1.25 \times 10^{-6}\left(5 + (\partial_x u)^2\right) \quad \text{in } \Omega \tag{5.54}$$
$$u = g(x) \quad \text{on } \partial\Omega. \tag{5.55}$$

We assume that the boundary value $g(x)$ are Hölder continuously differentiable then the corresponding fixed-point operator maps the ball (4.19) into itself if

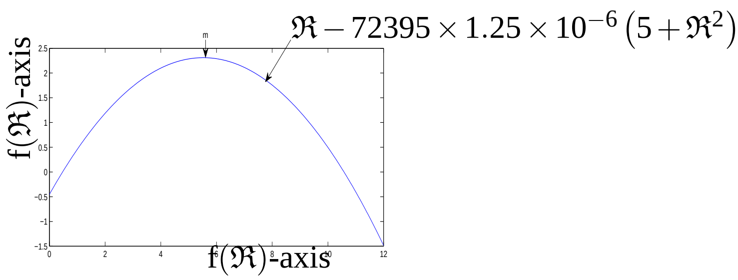
$$\|U\|_{C^{1,\alpha}} \leq 10416\|g\|_{C^{1,\alpha}} + 72395 \times 1.25 \times 10^{-6}\|\mathcal{F}\| \leq \mathfrak{R} \tag{5.56}$$
$$\Rightarrow \|g\|_{C^{1,\alpha}} \leq \frac{1}{10416}\left(\mathfrak{R} - 72395 \times 1.25 \times 10^{-6}\left(5 + \mathfrak{R}^2\right)\right) \tag{5.57}$$

The maximal norm of g leads again to the two conditions to be satisfied,

- $\mathfrak{R} \in (0.4726, 10.5777)$



- for the maximal value of the $C^{1,\alpha}$-norm of g we have $1 - 2 \times 72395 \times 1.25 \times 10^{-6}\mathfrak{R} = 0$ which leads to $\mathfrak{R}^* = \dfrac{10^6}{1.25 \times 144790} \approx 5.5$ which solves the equation and for this $\mathfrak{R}^*$ we get the maximum value of the $C^{1,\alpha}$-norm of g. An easy calculation shows that the maximal value of the $C^{1,\alpha}$-norm of g is equal to 2.3 approximately.



5.4.2. Optimization for a ball centered at the solution of the Laplace equation with given boundary value

Example 4:

$$\begin{aligned} \Delta u &= \frac{1}{1+(\partial_x(u))^2} \quad \text{in } \Omega \\ u &= g(x) \ \text{ on } \partial\Omega. \end{aligned}$$

We assume again that the boundary values are Hölder continuously differentiable on the boundary $\partial\Omega$. And if corresponding fixed point operator U will map the ball,

$$\mathfrak{B}_{\mathfrak{R}}(u_0) := \{u \in C^{1,\alpha} : \|u - u_0\|_{C^{1,\alpha}} \leq \mathfrak{R}\} \tag{5.58}$$

into itself then the boundary value problem with this fixed boundary data g is solvable by Schauder Fixed Point Theorem.

We look for the maximum radius of the ball for which U maps the ball into itself. Here we get the following inequality.

$$\|U - u_0\|_{C^{1,\alpha}} \leq 72395\|\mathcal{F}\|. \tag{5.59}$$

Since here the right hand side is globally bounded by 1 so we get

$$\|U - u_0\|_{C^{1,\alpha}} \leq 72395 \leq \mathfrak{R} \tag{5.60}$$

Result :

- For globally bounded right hand side we solved for given boundary data for an arbitrary parameter k for certain $\mathfrak{R}_0 > 72395k$. For a ball with a small a radius $\mathfrak{R}$, we have to choose k small enough.
- In a similar way, the present boundary value problem with zero boundary value and right hand side globally bounded by k solvable for the same choice of the radius of the ball.

Example 5:

$$\begin{aligned} \Delta u &= \frac{1}{2\left(1+(\partial_x(u))^2\right)} \quad \text{in } \Omega \\ u &= g(x) \ \text{ on } \partial\Omega. \end{aligned}$$

We assume again that the boundary values are Hölder continuously differentiable on the boundary $\partial\Omega$. And if the corresponding fixed-point operator $U = u_0 + \tilde{u} + V$ will map the following ball,

$$\mathfrak{B}_{\mathfrak{R}}(u_0) := \{u \in C^{1,\alpha} : \|u - u_0\|_{C^{1,\alpha}} \leq \mathfrak{R}\} \tag{5.61}$$

into itself then the boundary value problem with this fixed boundary data g is solvable by Schauder Fixed Point Theorem. We look for the maximum radius of the ball for which U maps the ball into itself. Here we get the following inequality.

$$\|U - u_0\|_{C^{1,\alpha}} \leq 72395\|\mathcal{F}\|. \tag{5.62}$$

Since here the right hand side is globally bounded by $\dfrac{1}{2}$. So we get.

$$\|U - u_0\|_{C^{1,\alpha}} \leq 72395 \times \frac{1}{2} \leq \mathfrak{R}. \tag{5.63}$$

Again, the right hand is globally bounded and for each $\mathfrak{R}_0 > 72395 \times \frac{1}{2}$. And the operator U maps the ball into itself.

Example 6:

$$\begin{aligned} \Delta u &= (\partial_x(u))^3 \quad \text{in } \Omega \\ u &= g(x) \text{ on } \partial\Omega. \end{aligned}$$

With the same assumptions the operator $U = u_0 + \tilde{u} + V$ maps the following ball,

$$\mathfrak{B}_{\mathfrak{R}}(u_0) := \{u \in C^{1,\alpha} : \|u - u_0\|_{C^{1,\alpha}} \leq \mathfrak{R}\}, \tag{5.64}$$

into itself. Then the boundary value problem with this fixed boundary data g is solvable by Schauder Fixed Point Theorem. We look for the maximum radius of the ball for which U maps the ball into itself. Here we get the following inequality.

$$\|U - u_0\|_{C^{1,\alpha}} \leq 72395\|\mathcal{F}\|. \tag{5.65}$$

Since the right hand side, here is bounded by $\mathfrak{R}^3$, so we get

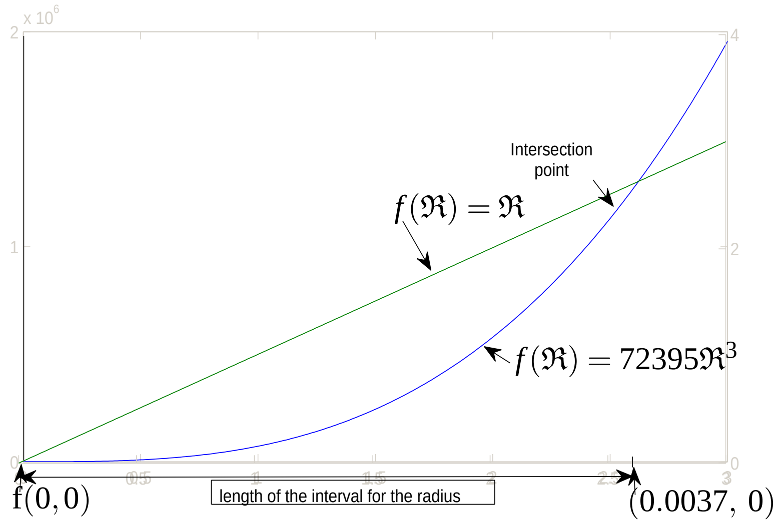
$$\begin{aligned} \|U - u_0\|_{C^{1,\alpha}} &\leq 72395\left(\mathfrak{R}^3\right) \leq \mathfrak{R} && (5.66)\\ &\leq \mathfrak{R} - 72395\left(\mathfrak{R}^3\right) && (5.67) \end{aligned}$$

the last inequality gives the largest possible interval for the radius of the ball, and the largest possible radius $\mathfrak{R}^*$ in this case is 0.0037.

All above examples show the details of various situations for the radius of the ball and we show under circumstances what best ball we can have. That is, we had constructed the best balls.

5.5. Optimal balls for the application of Contraction Mapping Principle

To apply the Contraction Mapping Principle to the boundary value problem (5.13) and (5.14), we have to show;



- The fixed-point operator maps the ball into itself.
- Image of the ball under the fixed point operator is contractive.

The fixed point operator is represented by (5.15).The first condition that U maps a closed and convex set (5.19) in the function space, into itself is already satisfied by (5.47).

The Following calculation will show that the operator defined in (5.15) is contractive. Here, additionally, we assume that,

- The right hand side is Lipschitz continuous, that is $\left|\mathcal{F}(\cdot, u_1, \partial_x u_1) - \mathcal{F}(\cdot, u_2, \partial_x u_2)\right| \le L_1 \left|u_1 - u_2\right| + L_2 \left|\partial u_1 - \partial u_2\right|$.

Where $L(\mathfrak{R}) = L_1(\mathfrak{R}) + L_2(\mathfrak{R})$ is the Lipschitz constant.

5.5.1. General condition on U to be contractive in the ball

The operator (5.15) will be contractive if,

$$\|U_1 - U_2\|_{C^{1,\alpha}} \le \text{const.} \|u_1 - u_2\|_{C^{1,\alpha}} \tag{5.68}$$

and const.< 1.

Again we first consider only V and check its mapping properties;

$$V =: -\frac{1}{2\pi} \iint\limits_{\Omega} \log\left|(x,y) - (\xi, \eta)\right| \cdot \mathcal{F}((\zeta), u(\zeta), \partial_i(\zeta)) d\zeta \tag{5.69}$$

$$|V_1 - V_2| \quad \le \quad \frac{L(\mathfrak{R})}{2\pi} \iint\limits_{\Omega} \left|\log|(x,y) - (\xi, \eta)|\right| d\zeta \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \tag{5.70}$$

for $0 < |(x,y)-(\xi,\eta)| < 1$, using the Schmidt inequality.

$$|V_1 - V_2| \leq \frac{L(\mathfrak{R})}{2\pi} \iint\limits_{\Omega} \frac{1}{|(x,y)-(\xi,\eta)|} d\zeta \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \tag{5.71}$$

$$\|V_1 - V_2\| \leq \frac{L(\mathfrak{R})}{2} \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \tag{5.72}$$

and for $|(x,y)-(\xi,\eta)| \geq 1$, we have, $\log|(x,y)-(\xi,\eta)| \leq |(x,y)-(\xi,\eta)|$, for the unit disk in the plane so $|(x,y)-(\xi,\eta)| \leq 2$

$$|V_1 - V_2| \leq \frac{L(\mathfrak{R})}{2\pi} \iint\limits_{\Omega} |(x,y)-(\xi,\eta)| \frac{|k|}{1+(\partial_x u(\zeta))^2} d\zeta \|u_1 - u_2\|_{C^{1,\alpha}} \tag{5.73}$$

$$\leq \frac{L(\mathfrak{R})}{2\pi} \cdot 1 \cdot \iint\limits_{\Omega} 2 d\zeta \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \tag{5.74}$$

$$\|V_1 - V_2\| \leq L(\mathfrak{R}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}}. \tag{5.75}$$

Now for the Hölder constant, we find the norm of the derivative of $V_1 - V_2$ first. Using the Lipschitz continuity of the right hand side, we have

$$\partial_x V = -\frac{1}{2\pi} \iint\limits_{\Omega} \frac{1}{|(x,y)-(\xi,\eta)|} \cdot \frac{(x,y)-(\xi,\eta)}{|(x,y)-(\xi,\eta)|} \cdot \mathcal{F}((\zeta), u(\zeta), \partial_i(\zeta)) d\zeta$$

$$\Rightarrow |\partial_x V_1 - \partial_x V_2| \leq \frac{L(\mathfrak{R})}{2\pi} \iint\limits_{\Omega} \frac{1}{|(x,y)-(\xi,\eta)|} d\zeta \cdot \|u_1 - u_2\|_{C^{1,\alpha}}$$

$$\|\partial_x V_1 - \partial_x V_2\| \leq \frac{L(\mathfrak{R})}{2} \cdot \|u_1 - u_2\|_{C^{1,\alpha}}$$

similarly for $\partial_y V_1 - \partial_y V_2$

$$\|\partial_y V_1 - \partial_y V_2\| \leq \frac{L(\mathfrak{R})}{2} \cdot \|u_1 - u_2\|_{C^{1,\alpha}} \tag{5.76}$$

Using the above estimates for the derivatives we have the following estimate for $\alpha = \frac{1}{2}$

$$\frac{|(V_1 - V_2)(x',y') - (V_1 - V_2)(x'',y'')|}{|(x',y') - (x'',y'')|^{\alpha}} \leq \sqrt{2} L(\mathfrak{R}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}} := H_{\tilde{u_1} - \tilde{u_2}} \tag{5.77}$$

For the Hölder constant of the derivative, we have again,

$$(V_1 - V_2)(x',y') - (V_1 - V_2)(x'',y'') = \frac{1}{2\pi} \iint\limits_{\Omega} \kappa \cdot \Big(\mathcal{F}(u_1, \partial_i u_1) - \mathcal{F}(u_2, \partial_i u_2) \Big)(\zeta) d\zeta$$

$$\left| \partial_x V(x',y') - \partial_x V(x'',y'') \right| \leq \frac{L(\mathfrak{R}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}}}{2\pi} \cdot I$$

$$\max_{(\overline{\Omega})} \frac{|(V_1 - V_2)(x',y') - (V_1 - V_2)(x'',y'')|}{|(x',y') - (x'',y'')|^{\alpha}} \leq 7 L(\mathfrak{R}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}}$$

where κ and I are given by,

$$I = \iint_{\Omega} \frac{d\zeta}{|(x',y')-(\xi,\eta)||(x'',y'')-(\xi,\eta))|} \quad (5.78)$$

$$\kappa = \frac{(x'',y'')-(\xi,\eta)}{|(x'',y'')-(\xi,\eta)|^2} - \frac{(x',y')-(\xi,\eta)}{|(x,y)-(\xi,\eta)|^2} \quad (5.79)$$

if

$$t = \left|(x',y')-(x'',y'')\right| \quad (5.80)$$

and moreover, in our case $t \leq 2$ and $\alpha = \frac{1}{2}$. Hence we have

$$\max_{(\bar{\Omega})} \frac{|\partial_x V(x',y') - \partial_x V(x'',y'')|}{|(x',y')-(x'',y'')|^{\alpha}} \leq 7L(\mathfrak{R}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}} := \tilde{H}_{\tilde{u}_1-\tilde{u}_2} \quad (5.81)$$

in view of the inequalities (5.75),(5.76),(5.77) and (5.81) we get

$$\|V_1 - V_2\|_{C^{1,\alpha}} = 7L(\mathfrak{R}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}}. \quad (5.82)$$

Using Schauder estimate in (5.41), we have

$$\begin{aligned} \|U_1 - U_2\|_{C^{1,\alpha}} &\leq \|\tilde{u_1} - \tilde{u_2}\|_{C^{1,\alpha}} + \|V_1 - V_2\|_{C^{1,\alpha}} && (5.83)\\ &\leq 2^{1+\alpha} K_1 K_2 H_{\tilde{u}_1-\tilde{u}_2} + \max_{\partial\Omega} |\tilde{u_1} - \tilde{u_2}| + 2K_1K_2\tilde{H}_{\tilde{u}_1-\tilde{u}_2} \\ &\quad +4(2^{1-\alpha}+1)K_1^2K_2^2\tilde{H}_{\tilde{u}_1-\tilde{u}_2} + 7L(\mathfrak{R}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}} && (5.84)\\ &\leq 72395L(\mathfrak{R}) \cdot \|u_1 - u_2\|_{C^{1,\alpha}}. && (5.85) \end{aligned}$$

For U to be contractive we must have to choose $L(\mathfrak{R})$ be sufficiently small, that is, we have to choose $L(\mathfrak{R}) < \frac{1}{72395}$

Result : The above condition for U to be contractive leads to another restriction on the radius of the ball to be chosen subject to the last inequality.

Without further loss of generality we come to some explicit examples

Example 7:

$$\begin{aligned} \Delta u &= u^2 + (\partial_x u)^3 \quad \text{in } \Omega \\ u &= g(x) \text{ on } \partial\Omega. \end{aligned}$$

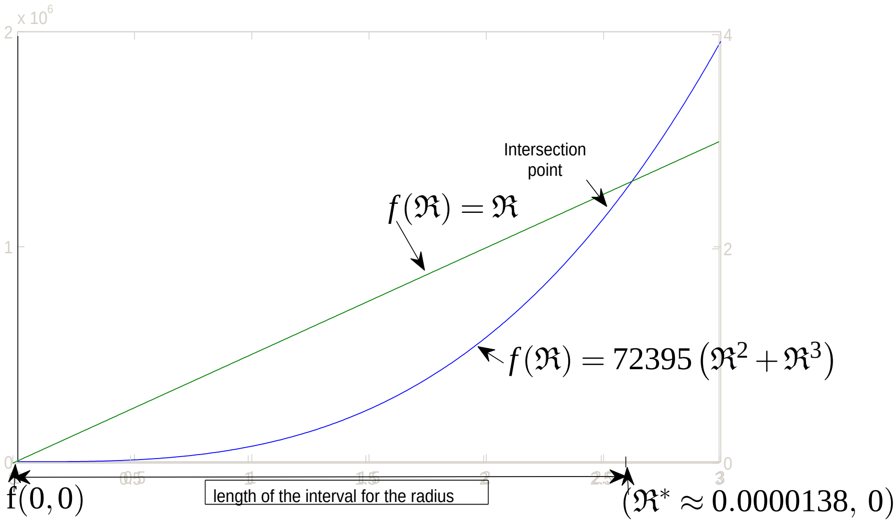
To show the existence and uniqueness by Contraction Mapping Principle, we have to fulfill the following two conditions.

- The fixed-point operator maps the ball into itself.
- The fixed point operator is contractive.

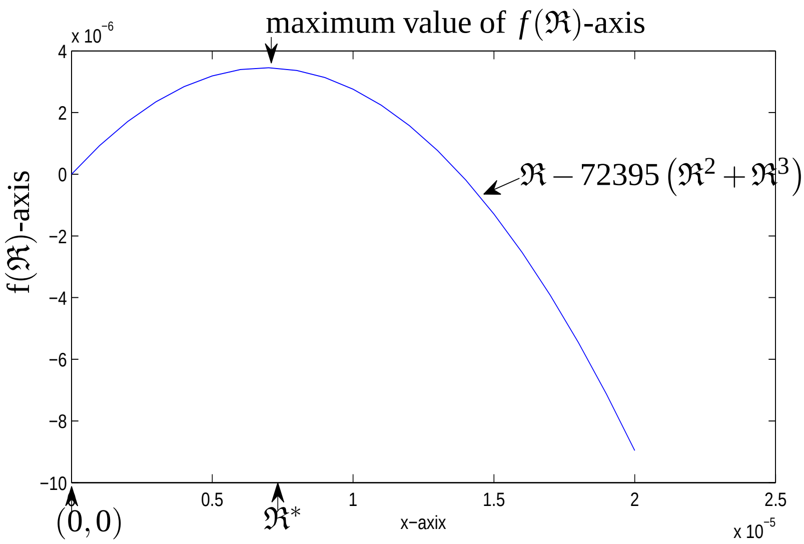
The first condition leads to the following bound for the boundary values

$$\|g\|_{C^{1,\alpha}} \leq \frac{1}{10416}\left(\mathfrak{R} - 72395(\mathfrak{R}^2 + \mathfrak{R}^3)\right) \quad (5.86)$$

Result 1: The above inequality leads to the existence of the largest interval with the radius $\mathfrak{R}$, that is $\mathfrak{R} \in (0, 0.0000138)$.



and the maximal value of the norm of the boundary values is shown below



Now according to condition (5.85) the operator U is contractive in the ball if

$$72395L(\mathfrak{R}) < 1 \tag{5.87}$$

where $L(\mathfrak{R})$ is the Lipschitz constant of the density function and here we can easily find out for the right hand side

$$\begin{aligned}
\left|u_1^2 + (\partial_x(u_1))^3 - u_2^2 - (\partial_x(u_2))^3\right| &\leq |u_1 + u_2|\,|u_1 - u_2| \\
&\quad + |\partial_x u_1 - \partial_x u_2| \left|(\partial_x u_1)^2 + \partial_x u_1 \partial_x u_2 + (\partial_x u_1)^2\right| \\
&\leq \left(2\mathfrak{R} + 3\mathfrak{R}^2\right) \|u_1 - u_2\|_{C^{1,\alpha}}.
\end{aligned}$$

Here the Lipschitz constant is given by,

$$L(\mathfrak{R}) = 2\mathfrak{R} + 3\mathfrak{R}^2$$

(5.88) implies

$$2\mathfrak{R} + 3\mathfrak{R}^2 < \frac{1}{72395} \tag{5.88}$$

$$\Rightarrow \mathfrak{R} < \frac{-2 + \sqrt{\frac{289592}{72395}}}{6} \approx 6.9 \times 10^{-6} \tag{5.89}$$

so we have two conditions on the choice of the radius of the ball and for the the unique existence of solution by Contraction Mapping Principle.

Result : In view of the maximum principle and the Schauder estimates $||u_0||_{C^{1,\alpha}}$, the solution can be estimated by its boundary values $||g||_{C^{1,\alpha}}$ and if $||g||_{C^{1,\alpha}} \leq \mathbf{C}$ then for all possible boundary values we have evaluated the largest **C** that is 0.00000000033.

6. SUMMARY AND OUTLOOK

6.1. Summary

In our research, we have solved the Dirichlet boundary value problems for non-linear second order elliptic partial differential equations, by fixed point methods. BVPs with right hand side depending on the desired solution and its first order derivatives were considered while the left hand sides were general second order elliptic operator with the principal part as Laplace operator.

After reducing the boundary value problems to the corresponding fixed point operator we apply the fixed point theorems. For the existence and uniqueness we applied following fixed point results,

- Schauder Fixed Point Theorem.
- Contraction Mapping Principle.

In order to show the existence of solutions, Schauder Fixed Point Theorem was applied. We have also shown that the corresponding fixed point operator is relatively compact. The solutions both in full Banach space and those in the balls (closed and convex subsets) have been found.

We have uniquely solved the boundary value problems by Contraction Mapping Principal. Here and in the case of Schauder (II), we have given optimal balls in the underlying function spaces. Many other important results and references about the solutions of boundary valued problems for non-linear second order elliptic partial differential equations, are the part of the current manuscript.

6.2. Outlook

We look forward to work in the following directions in future:

- Since it is clear from the summary that in this dissertation, we solved the boundary value problem for non-linear second order elliptic partial differential equations when the linear second order operator has the principal part as the Laplace operator. Now in future, one of our goal will be to consider the more general elliptic operators. Here we have to work with other fundamental solutions instead of the fundamental solution of the Laplace equation.
- We considered only Dirichlet conditions on the boundary. Now we have a plan to work with more general boundary conditions for examples the Neu-

mann boundary conditions. In case BVPs are not uniquely solvable, we shall look for the conditions under which these BVPs are uniquely solvable which in general is not the case.

- Since we need the Schauder estimates to solve the boundary value problems for elliptic equation, we intend to find the Schauder type estimates explicitly. A similar case of Schauder estimates for the Poisson equation in the unit disk.
- We shall consider other function spaces for example Sobolev spaces in stead of Hölder spaces, for our future planed work.

A. APPENDIX

A.1. $C^{1,\alpha}$ bound for the solution of Poisson equation in 2-D

Chapter four deals with the $C^{1,\alpha}$ bound of the solutions of the boundary value problems for non-linear partial differential equations where we use the Schauder estimates. These Schauder type estimates are bounded by certain norms of the boundary values, the right hand side of the differential equation, the solution itself and constants while these constants generally are not explicitly known. Here we give the detail proof of these constants explicitly for the solution of the Poisson equation in the plane. Most of the constructions in this appendix is taken from the the lecture of W. Tutschke [52]. It will be of importance that we will use the concept of the holomorphic functions for our considerations. We will use holomorphic functions because any holomorphic function Φ, in the complex plane, is defined as;

Definition A.1 *A function*

$$\Phi = u(x,y) + iv(x,y) \tag{A.1}$$

is said to be a holomorphic if u and v have continuous first partial derivatives and satisfy the Cauchy-Riemann equation;

$$\partial_x u = \partial_y v \quad \textit{and} \quad \partial_y u = -\partial_x v \tag{A.2}$$

Also both u and v are solutions of the Laplace equation.

Moreover, from (A.1), both u and v can be estimated by Φ.

Since we know that the Laplace equation $\Delta u = 0$ in the balls can be solved by the Poisson integral.

Prior to go with the holomorphic function, we use the Poisson integral to find some necessary estimates for the solution of the Laplace equation and their derivatives, specially when the boundary values are Hölder continuous.

A.1.1. Results from the Poisson integral

We are going to solve the following Laplace equation;

$$\Delta u = 0 \quad \text{in} \quad \Omega. \tag{A.3}$$

where Ω is the domain in the z-plane.

Suppose that the closed disk $|z-z_0| \leq R$ is contained in Ω. Then, we know that at an interior points z of the disk, the function u is represented by the Poisson Integral

$$u(z) = \frac{1}{2\pi R} \int\limits_{|\zeta - z_0| = R} u(\zeta) \frac{R^2 - |z - z_0|^2}{|\zeta - z|^2} ds \tag{A.4}$$

where ζ is the point on the boundary of the disk and ds is the length element of the boundary, i.e, the circle.

The value of u at the center z_0 is given by.

$$u(z_0) = \frac{1}{2\pi R} \int\limits_{|\zeta - z_0| = R} u(\zeta) ds \tag{A.5}$$

that means, the value of u at the center z_0 is the mean value of $u(\zeta)$ of u on the circle $|z - z_0| = R$.

Further we choose any r not greater than R. Applying formula (A.5) to this circle with radius r, we have

$$r \cdot u(z_0) = \frac{1}{2\pi} \int\limits_{|\zeta - z_0| = R} u(\zeta) ds$$

now, integrating over the interval $0 \leq r \leq R$, and taking into consideration that $rdsdr = d\xi d\eta$ is the area element in the z-plane, it follows

$$\frac{1}{2} R^2 \cdot u(z_0) = \frac{1}{2\pi} \iint\limits_{|\zeta - z_0| \leq R} u(\zeta) d\xi d\eta$$

implies

$$u(z_0) = \frac{1}{\pi R^2} \iint\limits_{|\zeta - z_0| \leq R} u(\zeta) d\xi d\eta \tag{A.6}$$

hence the following lemma is proved:

Lemma A.1 *The value $u(z_0)$ of the solution of the Laplace equation at an interior point z_0 is not only the mean value (A.5) of u with respect to a circle centered at z_0. It is the mean value (A.6) of u with respect to a disk centered at z_0 also.*

Since the Laplace equation is a linear differential equation with constant coefficients, the partial derivatives $\partial_x u$ and $\partial_y u$ of the solution u with respect to x and y respectively are also the solutions of the Laplace equation. Lemma (A.1) applied to these derivatives and Green-Gauss Integral Formula leads,

$$\partial_x u(z_0) = \frac{1}{\pi R^2} \int\limits_{|\zeta - z_0| = R} u(\zeta) d\xi. \tag{A.7}$$

and

$$\partial_y u(z_0) = \frac{1}{\pi R^2} \int\limits_{|\zeta - z_0| = R} u(\zeta) d\eta. \tag{A.8}$$

Provided the absolute value of u can be estimated by M everywhere in Ω, we obtain the bound for the above derivatives and as a result we prove:

Lemma A.2 *Suppose u is a solution of the Laplace equation with $|u| \leq M$ everywhere in Ω. Suppose, further, that the closed disk centered at z_0 with radius R is contained in Ω. Then the values $\partial_x(z_0)$ and $\partial_y(z_0)$ of the first order derivative $\partial_x u$ and $\partial_y u$ can be estimated by*

$$|\partial_x u(z_0)| \leq \frac{4M}{\pi R}. \quad \textit{and} \quad |\partial_y u(z_0)| \leq \frac{4M}{\pi R}. \tag{A.9}$$

Up to now, we have estimated the first order derivatives of the solution of the Laplace equation at the center of the disk with radius R. Next we are going to have the result for these derivatives at the boundary.

A.2. Behavior of the first order derivatives of the solution of the Laplace equation near the boundary

For simplicity of the calculations, we will consider that Ω is the unit disk. Suppose also that u is a solution of the Laplace equation in Ω with $|u| \leq M$. Foremost, no assumptions concerning the limits of u at the boundary points has to be made.

Let z be an arbitrary interior point of the unit disk Ω. Then the distance of z from the boundary $|z| = 1$ equals to $1 - |z|$ and thus a closed disk centered at z with radius $\delta < 1 - |z|$ is completely contained in Ω which is shown in the figure below.

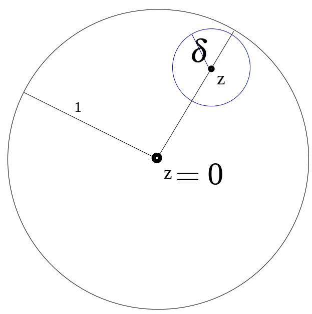


Figure A.1.: Closed disk with radius δ contained in the unit ball

Applying the lemma (A.3) to that disk and carrying out the the limiting process $\delta \to 1 - |z|$, we have the following statement:

Lemma A.3 *Suppose u is a solution of the Laplace equation in the unit disk and $|u| \leq M$ everywhere in Ω. Then the absolute values of the first order derivatives of the solution at z can be estimated by;*

$$|\partial_x u(z)| \leq \frac{4M}{\pi(1-|z|)} \quad \textit{and} \quad |\partial_y u(z)| \leq \frac{4M}{\pi(1-|z|)}. \tag{A.10}$$

A.3. Behavior of the solution when the boundary values are Hölder continuous

Again we consider the same unit disk, and let u be the solution of the Laplace equation in Ω. Suppose now, however, that u now is defined and continuous in the closed unit disk, i.e, for all z with $|z| \leq 1$. We denote the boundary values of u at $z \in \partial\Omega$ by $g(z)$ that means $u(z) = g(z)$ as long as $|z| = 1$.

We suppose additionally that g is Hölder continuous at the boundary point z_0, i.e,

$$|g(z) - g(z_0)| \leq H \cdot |z - z_o|^\alpha \tag{A.11}$$

for every $z \in \partial\Omega$ where $0 < \alpha < 1$. We can write,

$$-H \cdot |z - z_o|^\alpha \leq g(z) - g(z_0) \leq +H \cdot |z - z_o|^\alpha. \tag{A.12}$$

The following construction deals with the estimation of $|z - z_o|^\alpha$ from above by the solution of the Laplace equation. Since we know from complex analysis that,

$$\log(z - z_o) = \ln|z - z_o| + i\arg(z - z_o). \tag{A.13}$$

For $z \in \Omega$, for suitably chosen c, there exists a uniquely defined branch of polar angles $\arg(z - z_o)$ such that

$$c < \arg(z - z_o) < c + \pi \tag{A.14}$$

the last inequality can be re-written as,

$$-\frac{\pi}{2} < \arg(z - z_o) - \left(c + \frac{\pi}{2}\right) < +\frac{\pi}{2} \tag{A.15}$$

subtracting $i\left(c + \frac{\pi}{2}\right)$ on both sides of (A.13) implies

$$\log(z - z_o) - i\left(c + \frac{\pi}{2}\right) = \ln|z - z_o| + i\arg(z - z_o) - i\left(c + \frac{\pi}{2}\right)$$

multiplying α on both sides we have

$$\alpha\left(\log(z - z_o) - i\left(c + \frac{\pi}{2}\right)\right) = \ln|z - z_o|^\alpha + i\alpha\left(\arg(z - z_o) - \left(c + \frac{\pi}{2}\right)\right)$$

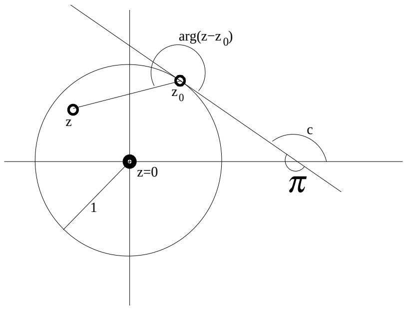


Figure A.2.: $\arg(z - z_o)$

which implies

$$\exp\left[\alpha\left(\log(z-z_o)-i\left(c+\frac{\pi}{2}\right)\right)\right]=|z-z_o|^{\alpha}\cdot\exp\left[i\alpha\left(\arg(z-z_o)-\left(c+\frac{\pi}{2}\right)\right)\right].$$

Since (A.13) is holomorphic in the unit disk, the last expression defines also a holomorphic function. Already we know that $\exp(ix) = \cos(x) + i\sin(x)$, thus real part is given by,

$$U(z) = |z-z_o|^{\alpha}\cdot\cos\left[\alpha\left(\arg(z-z_o)-\left(c+\frac{\pi}{2}\right)\right)\right].$$

Hence by definition (A.1) the $U(z)$ is a positive solution of the Laplace equation. Consequently we get,

$$|z-z_o|^{\alpha} = \frac{U(z)}{\cos\left[\alpha\left(\arg(z-z_o)-\left(c+\frac{\pi}{2}\right)\right)\right]}$$

the inequality (A.15) gives immediately,

$$|z-z_o|^{\alpha} < \frac{U(z)}{\cos\left(\alpha\frac{\pi}{2}\right)}.$$

This is the desired estimate of $|z-z_o|^{\alpha}$ by a solution $U(z)$ of the Laplace equation.

Using the last inequality in (A.12), we get,

$$-\frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}U(z) \le g(z)-g(z_0) \le +\frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}U(z) \qquad \text{(A.16)}$$

Now we return to the solution $u = u(z)$ of the Laplace equation introduced at the beginning of the present section. Since $u(z) = g(z)$ on the boundary, the inequality

(A.16) shows that $u(z) - u(z_0)$ satisfies the inequality,

$$-\frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}U(z) \le u(z) - g(z_0) \le +\frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}U(z) \tag{A.17}$$

on the boundary $\partial\Omega$. It is important to note that the last two-sided estimate is true not only on the boundary $\partial\Omega$, but also everywhere in Ω because for Laplace equation the following statement is true:

Lemma A.4 (Maximum principle) *Suppose u_1 and u_2 are solutions of the Laplace equation in Ω. Suppose further, that the boundary values g_1, g_2 of u_1 and u_2 respectively, satisfy the inequality*

$$g_1 \le g_2 \tag{A.18}$$

everywhere on the boundary $\partial\Omega$. Then one has

$$u_1 \le u_2 \tag{A.19}$$

Indeed, this statement is an immediate consequence of the maximum principle applied to the difference $u_0 = u_1 - u_2$. Since u_0 has the boundary values $g_0 = g_1 - g_2 \le 0$, one has

$$u_0 \le \sup_{\partial\Omega} g_0 \le 0$$

everywhere in Ω. Taking into account the definition of u_0, the inequality (A.19) has thus been proved.

Consequently, (A.16) is true everywhere in Ω. Hence we have

$$|u(z) - g(z_0)| \le \frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}U(z)$$

The definition of $U(z)$ at previous page shows, further, that $U(z) \le |z - z_0|^\alpha$ and, therefore, the last estimate passes into

$$|u(z) - g(z_0)| \le \frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}|z - z_0|^\alpha \tag{A.20}$$

To sum up, the following statement has been proved:

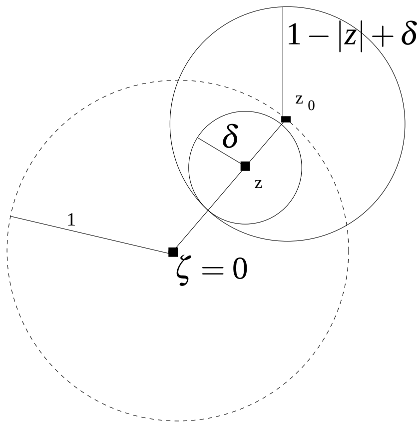
Lemma A.5 *Suppose the boundary function $g = g(z)$ of $u = u(z)$ satisfies a Hölder condition (A.12) with Hölder constant H and Hölder exponent α, $0 < \alpha < 1$, at z_0. Then $u(z)$ satisfies the Hölder condition (A.20) for each $z \in \overline{\Omega}$.*

Note that condition (A.20) is satisfied $z_0 \in \overline{\Omega}$ in case the boundary values $g(z)$ satisfy the Hölder condition (A.12) everywhere on $(\overline{\Omega})$.

A.4. Behavior of the first order derivatives of a solution at the boundary in the case of Hölder continuous boundary values.

In this section, we are going to estimate the derivatives $\partial_x u$ and $\partial_y u$ of a solution of the Laplace equation in the unit disk Ω provided the boundary values g are Hölder-continuous (with Hölder-constant H and Hölder-exponent α, $0 < \alpha < 1$) everywhere on the boundary $\partial\Omega$.

Let $z(\neq 0)$ be an arbitrary point in the unit disk. Let z_0 be the uniquely determined point at which the ray from $\zeta = 0$ through z intersects the boundary $\partial\Omega$. Choose $\delta < 1 - |z|$. Then the (closed) disk with radius δ centered at z is contained in the (closed) disk with radius $1 - |z| + \delta$ centered at z_0:



In view of lemma (A.5), i.e, from (A.20), we have

$$\begin{aligned} |u(\zeta) - g(z_0)| &\leq \frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}|z - z_0|^{\alpha} \\ &\leq \frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}(1 - |z| + \delta)^{\alpha} \end{aligned}$$

everywhere in the disk with radius $1 - |z| + \delta$ centered at z_0, i.e.,

$$M = \frac{H}{\cos\left(\alpha\frac{\pi}{2}\right)}(1 - |z| + \delta)^{\alpha}$$

is a bound of the absolute value of $|u(\zeta) - g(z_0)|$ in that disk. Applying the lemma (A.3) to $u(\zeta) - g(z_0)$ in the (smaller) disk with radius δ centered at z we obtain

$$|\partial_x u(z)| \leq \frac{4M}{\pi R} \leq \frac{4H}{\pi \cos\left(\alpha\frac{\pi}{2}\right)} \cdot \frac{(1 - |z| + \delta)^{\alpha}}{\delta} \tag{A.21}$$

and the same estimate is true for $\partial_y u(z)$.

The limiting process $\delta \to 1-|z|$ yields:

Lemma A.6 *Suppose the boundary values g of a solution u of the Laplace equation in the unit disk are Hölder-continuous with Hölder-constant H and the Hölder-exponent* α, $0<\alpha<1$. *Then at an interior point z the first order derivatives of u can be estimated by*

$$|\partial_x u(z)| \le \frac{K_1 H}{\Big(1-|z|\Big)^{1-\alpha}} \tag{A.22}$$

and

$$|\partial_y u(z)| \le \frac{K_1 H}{\Big(1-|z|\Big)^{1-\alpha}} \tag{A.23}$$

where

$$K_1 = \frac{4.2^{\alpha}}{\pi \cos\Big(\alpha\frac{\pi}{2}\Big)} \tag{A.24}$$

Remark A.1 *The same statements of the last to lemmas are true for* $z=0$, *although we have carried out the construction for* $z \neq 0$ *where we get the results if we have limiting process as* $z \to 0$

A.5. An important criterion for Hölder continuity of the solution of the Laplace equation

Up to now, we have found that if the boundary values are Hölder-continuous then the first order derivatives of the solution of the Laplace equation are bounded and the bounds are known to us by lemma A.6. The following theorem deals with the Hölder continuity of the solution of the Laplace equation and by using lemma A.6, we find the explicit Hölder constant in this case.

Theorem A.1 *Suppose* $u=u(z)$ *is defined and continuously differentiable in the open unit disk* ($|z|<1$). *Suppose, further, that the first order derivatives can be estimated by*

$$|\partial_x u(z)| \le \frac{C}{\Big(1-|z|\Big)^{1-\alpha}} \quad \text{and} \quad |\partial_y u(z)| \le \frac{C}{\Big(1-|z|\Big)^{1-\alpha}} \tag{A.25}$$

where C and α *are given constants,* $0<\alpha<1$. *Then u is Hölder-continuous in the unit disk, and a Hölder constant, in this case, is given by* CK_2 *where*

$$K_2 = \Big(\frac{2}{\alpha}(1+2^{\alpha})+\pi\Big). \tag{A.26}$$

Proof Consider an arbitrary pair of points z_1 and z_2 not coinciding with $z = 0$. Of course, we can write z_j in the following trigonometric form

$$z_j = r_j \exp\left(i\vartheta_j\right), \qquad j = 1,2.$$

Without any loss of generality we may assume that

$$r_2 \geq r_1 \qquad \text{and} \qquad \vartheta_1 \leq \vartheta_2 \leq \vartheta_1 + \pi.$$

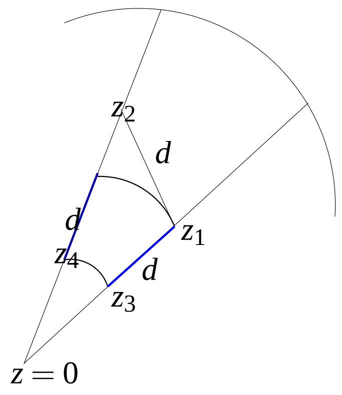
We denote the distance between z_1 and z_2 by d, i.e, we define

$$d = |z_2 - z_1|.$$

Introducing another pair of points z_3 and z_4 defined by

$$z_3 = (r_1 - d)\exp(i\vartheta_1) \qquad \text{and} \qquad z_3 = (r_1 - d)\exp(i\vartheta_2)$$

Define $z_3 = z_4 = 0$ if $r_1 < d$. So we have $|z_1 - z_3| \leq d$ and $|z_2 - z_4| \leq 2d$ because the distance between $r_2 \exp(i\vartheta_2) = z_2$ and $r_1 \exp(i\vartheta_2) = r_2 - r_1$, also, this not larger than as illustrated bellow,



Since,

$$x = r\cos\vartheta \qquad \text{and} \qquad y = r\sin\vartheta$$

Using the chain rule we get

$$\begin{aligned} \partial_r u &= \partial_x u \cdot \cos\vartheta + \partial_y u \cdot \sin\vartheta \\ \partial_\vartheta u &= \partial_x u \cdot (-r\sin\vartheta) + \partial_y u \cdot r\cos\vartheta. \end{aligned}$$

where $z = x + iy$ and $r = |z|$. The assumptions (A.25) are the estimates

$$|\partial_x u| \leq \frac{2C}{\Big(1 - |z|\Big)^{1-\alpha}} \tag{A.27}$$

$$|\partial_y u| \leq \frac{2C}{\Big(1 - |z|\Big)^{1-\alpha}} \tag{A.28}$$

Using (A.27), we get the estimates

$$|u(z_1)-u(z_3)| \leq 2C \int_{\max(r_1-d,0)}^{r_1} (1-r)^{\alpha-1}dr \quad \text{(A.29)}$$

$$|u(z_2)-u(z_4)| \leq 2C \int_{\max(r_1-d,0)}^{r_2} (1-r)^{\alpha-1}dr. \quad \text{(A.30)}$$

The length of the interval of integration is not larger than d in the case of the integral in (A.29), while it is not larger than $2d$ for the integral in (A.30).Therefore, the values of the integrals in (A.29) and (A.30) can only be enlarged if the limits of the integrals are replaced by $\max(1-d,0)$ and 1 and by $\max(1-2d,0)$ and 1 respectively, i.e, the integrals under consideration can be estimated from above by,

$$\frac{2C}{\alpha}\Big[-(1-r)^{\alpha-1}\Big]_{\max(1-d,0)}^{1} \quad \text{(A.31)}$$

and

$$\frac{2C}{\alpha}\Big[-(1-r)^{\alpha-1}\Big]_{\max(1-2d,0)}^{1} \quad \text{(A.32)}$$

respectively.

The expression (A.31) equals $\frac{2C}{\alpha}d^\alpha$ or $\frac{2C}{\alpha}$ according as $1 \geq d$ or $1 < d$ respectively, and, therefore, in any case the expression (A.31) is not smaller than $\frac{2C}{\alpha}d^\alpha$. Similarly, (A.32) is equal to $\frac{2C}{\alpha}2^\alpha d^\alpha$ or $\frac{2C}{\alpha}2^\alpha$ according as $1 \geq 2d$ or $1 < 2d$ respectively. Consequently, in both cases, the expression (A.32) is not larger than $\frac{2C}{\alpha}2^\alpha d^\alpha$. To sum up, we have got the following estimates:

$$|u(z_1)-u(z_3)| \leq \frac{2C}{\alpha}d^\alpha \quad \text{(A.33)}$$

$$|u(z_2)-u(z_4)| \leq \frac{2^{1+\alpha}C}{\alpha}d^\alpha. \quad \text{(A.34)}$$

Integrating (A.28) over the circular arc with radius r_1-d between z_3 and z_4, we obtains

$$|u(z_4)-u(z_3)| \leq \frac{2C(r_1-d)}{(1-r_1+d)^{1-\alpha}}(\vartheta_2-\vartheta_1). \quad \text{(A.35)}$$

Note that

$$2(r_1-d)\sin\frac{\vartheta_2-\vartheta_1}{2} = |z_4-z_3| \leq d. \quad \text{(A.36)}$$

Clearly, for $0 \leq \alpha \leq \frac{\pi}{2}$, $\alpha \leq \frac{\pi}{2}\sin\alpha$. The estimate (A.36) leads, therefore, to

$$4(r_1-d)\frac{1}{2}(\vartheta_2-\vartheta_1) \leq 4(r_1-d)\frac{\pi}{2}\sin\frac{\vartheta_2-\vartheta_1}{2} \leq \pi d$$

Using this in (A.35), and the inequality $1 - r_1 + d > 1$, we have

$$|u(z_4) - u(z_3)| \leq \frac{\pi C d}{d^{1-\alpha}} = \pi d^{\alpha}. \tag{A.37}$$

In view of the triangle inequality, we have

$$|u(z_2) - u(z_1)| \leq |u(z_2) - u(z_4)| + |u(z_4) - u(z_3)| + |u(z_3) - u(z_1)|. \tag{A.38}$$

using (A.33), (A.34) and (A.37) in (A.38), ultimately, we have

$$|u(z_2) - u(z_1)| \leq C\Big(\frac{2}{\alpha}(1 + 2^{\alpha}) + \pi\Big)|z_2 - z_1|^{\alpha}. \tag{A.39}$$

Which is the desired result.

It is important to note that the last theorem guaranties the Hölder-continuity of the solution of the Laplace equation inside the domain when the boundary values are Hölder-continuous.

A.6. Hölder-continuity in the whole domain

Theorem A.2 *Suppose the boundary values g of a solution u of the Laplace equation $\Delta u = 0$ in the unit disk are Hölder-continuous with Hölder-constant H and Hölder-exponent α, $0 < \alpha < 1$. Then u is Hölder-continuous with the same Hölder-exponent α in the closed unit disk, and the Hölder-constant is $K_1 K_2 H$.*

Proof

Proof of this theorem is not included here.

A.7. The Dirichlet boundary value problem for holomorphic functions with Hölder-continuous boundary values

Next we look for a holomorphic solution $\Phi = u(x,y) + iv(x,y)$ in the unit disk Ω the real part of which u has prescribed boundary values g on the unit circle $\partial\Omega$. Again, the boundary values g are supposed to be Hölder-continuous with the Hölder-constant H and the Hölder-exponent α, $0 < \alpha < 1$.

As mentioned in the definition of holomorphic functions at beginning that real part of the desired Φ is the solution of the Laplace equation having the boundary value g on the unit circle. In view of theorem A.2, the real part u is Hölder-continuous in the closed unit disk $\overline{\Omega}$, with a Hölder-constant $K_1 K_2 H$.

Having constructed the real part u of Φ, the Cauchy-Riemann system determines its imaginary part v up to an imaginary constant. By lemma A.6, the first order derivatives of u are estimated by (A.22) and (A.23). And since the real and imaginary part of the Φ are interconnected by equations (A.2), the same estimates, are true for the imaginary part v then. Again by theorem A.2, the imaginary part is v is Hölder-continuous in the closed unit disk $\overline{\Omega}$, and the Hölder-constant is same K_1K_2H.

And we can easily have

$$|\Phi(z_2) - \Phi(z_2)| \leq |u(z_2) - u(z_1)| + |v(z_2) - v(z_1)| \tag{A.40}$$

In view of the Hölder-continuity of u and v, we see the Φ, too, is Hölder-continuous with Hölder-constant $2K_1K_2H$.

Summarizing the above arguments, the following statement has been proved:

Lemma A.7 *Suppose g is Hölder-continuous with the Hölder-constant H and the Hölder-exponent α, $0 < \alpha < 1$. Then each holomorphic function whose real part has the boundary values g turns out to be Hölder-continuous with Hölder-constant $2K_1K_2H$ and the same Hölder-exponent α.*

Now we consider the boundary value problem

$$\partial_{\bar{z}}\Phi = 0 \quad \text{in} \quad \Omega \tag{A.41}$$

$$\text{Re}\Phi = g \quad \text{on} \quad \partial\Omega \tag{A.42}$$

$$\text{Im}\Phi(z_0) = c \tag{A.43}$$

where z_0 is a fixed chosen point (in the unit disk $\overline{\Omega}$). It is important to note that Φ is uniquely determined by up to an imaginary constant and this constant however is uniquely determined by the condition (A.43). Here we can apply the lemma (A.7) to the boundary value problem (A.41)-(A.43).

To have the bound for $|\Phi|$ we can write,

$$\begin{aligned} |\Phi(z)| &= |\Phi(z) - \Phi(z_0) + \Phi(z_0)| \\ &\leq |\Phi(z) - \Phi(z_0)| + |\Phi(z_0)| \\ &\leq 2K_1K_2H|z - z_0|^{\alpha} + |\Phi(z_0)| \end{aligned} \tag{A.44}$$

where K_1, K_2 are explicitly known constants.

Moreover,

$$\begin{aligned} |\Phi(z_0)| &= |u(z_0) + iv(z_0)| \leq |u(z_0)| + |v(z_0)| \\ &\leq |u(z_0)| + |c| \end{aligned}$$

$|u(z_0)|$ can be estimated by the maximum minimum principle of the Laplace equation by its boundary values

$$|\Phi(z_0)| \leq \max_{\partial\Omega} |g| + |c|$$

hence (A.44) implies as follows

$$|\Phi(z)| \leq 2K_1K_2H|z-z_0|^\alpha + \max_{\partial\Omega}|g| + |c| \tag{A.45}$$

since, for unit disk, we have $|z-z_0| \leq 2$, that means, $|z-z_0|^\alpha \leq 2^\alpha$. Hence we get the following result:

Corollary A.1 *The absolute value of the solution of the uniquely determined solution Φ of the boundary value problem (A.41)-(A.43) can be estimated by,*

$$|\Phi(z)| \leq 2^{1+\alpha}K_1K_2H + \max_{\partial\Omega}|g| + |c| \tag{A.46}$$

A.8. Differentiability of boundary values with respect to the polar angle ϑ

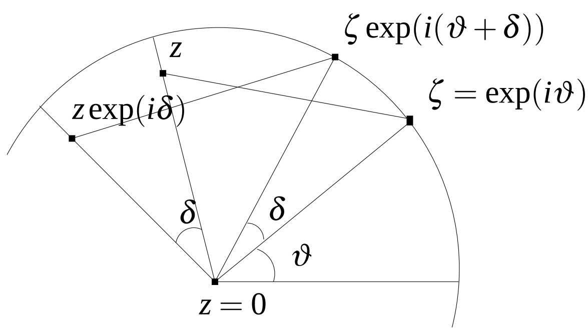
We know that the solution $u(z)$ of the Laplace equation at an interior points of the ball of radius R is represented by Poisson Integral Formula as

$$u(z) = \frac{1}{2\pi R} \int\limits_{|\zeta - z_0| = R} u(\zeta)\frac{R^2 - |z-z_0|^2}{|\zeta - z|^2} ds.$$

For boundary values g, and having the fact that $ds = Rd\vartheta$, we have the following representation for unit disk.

$$u(z) = \frac{1}{2\pi} \int\limits_{\vartheta=0}^{2\pi} g(\vartheta)\frac{1-|z|^2}{|\zeta - z|^2} d\vartheta. \tag{A.47}$$

We suppose now that $\zeta = \exp(i\vartheta)$ be an arbitrary point on the unit circle and z be an arbitrary point in the unit disk.



Here the distance $|z-\zeta|$ of z and ζ is equal to the distance between $z\exp(i\delta)$ and $\zeta\exp(i\delta) = \exp(i(\vartheta+\delta))$, i.e,

formula (A.47) implies

$$\frac{u(z\exp(i\delta))-u(z)}{\delta} = \frac{1}{2\pi}\int_{\vartheta=0}^{2\pi} \frac{g(\vartheta+\delta)-g(\vartheta)}{\delta}\cdot\frac{1-|z|^2}{|\zeta-z|^2}d\vartheta. \tag{A.48}$$

Applying the law of mean to the difference quotient in the integrand and carrying out the limiting process $\delta \to 0$, we have the following lemma proved:

Lemma A.8 *Suppose the boundary values g are continuously differentiable with respect to the polar angle ϑ. Then the solution of the Laplace equation is also continuously differentiable with respect to ϑ, and this derivative can be represented by the Poisson integral with the density $\partial_\vartheta g$:*

$$\partial_\vartheta u(z) = \frac{1}{2\pi}\int_{\vartheta=0}^{2\pi} \partial_\vartheta g(\vartheta)\frac{1-|z|^2}{|\zeta-z|^2}d\vartheta. \tag{A.49}$$

A.9. Hölder-continuously differentiable boundary values

Next we extend the last lemma A.8 in order to investigate the complex derivatives Φ' of the solution Φ of the boundary value problem (A.41)-(A.43), in the case when the boundary values are Hölder-continuously differentiable in ϑ. Let $\tilde{H}$ be the Hölder-constant of $\partial_\vartheta g$. Theorem A.2 says that $\partial_\vartheta u$ is Hölder-continuous in the closed unit disk where a Hölder-constant is given by $K_1K_2\tilde{H}$. Also, since,

$$x = r\cos\vartheta \quad \text{and} \quad y = r\sin\vartheta \tag{A.50}$$

by the chain rule we can write

$$\partial_\vartheta u = \frac{\partial u}{\partial\vartheta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial\vartheta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\vartheta} \tag{A.51}$$

by (A.50) we have,

$$\partial_\vartheta u = \frac{\partial u}{\partial\vartheta} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}. \tag{A.52}$$

Now in view of the C-R equations

$$\Phi' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \tag{A.53}$$

on multiplying $iz = ix - y$ the last equation implies

$$iz\Phi' = \Psi = \Big(-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}\Big) + i\Big(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\Big). \tag{A.54}$$

Comparing (A.52) and (A.54), we obtain

$$\partial_\vartheta u = \mathrm{Re}\Psi \tag{A.55}$$

Theorem A.2 implies that Ψ is Hölder-continuous in the closed unit disk where the Hölder-constant is $2K_1K_2\tilde{H}$. Since $\Psi(0) = 0$ and $|z| \leq 1$ we get, further.

$$|\Psi(z)| = |\Psi(z) - \Psi(0)| \leq 2K_1K_2\tilde{H} \cdot |z-0|^\alpha \leq 2K_1K_2\tilde{H}. \tag{A.56}$$

Since $|\Phi'| = |\Psi(z)|$ for $|z| = 1$, we get

$$|\Phi'| \leq 2K_1K_2\tilde{H}. \tag{A.57}$$

which is true on the boundary $\partial\Omega$. Then the maximum principle for holomorphic functions guarantees that the last inequality is valid everywhere in $\overline{\Omega}$. Further, for points z_1 and z_2 on the unit circle the definition of Ψ implies

$$\begin{aligned}
\Phi'(z_2) - \Phi'(z_1) &= -\frac{i}{z_2}\Psi(z_2) + \frac{i}{z_1}\Psi(z_1) \\
&= -i\Psi(z_2)\Big(\frac{1}{z_2} - \frac{1}{z_1}\Big) - \frac{i}{z_1}\Big(\Psi(z_2) - \Psi(z_1)\Big)
\end{aligned}$$

Thus in view of above inequalities

$$\begin{aligned}
\big|\Phi'(z_2) - \Phi'(z_1)\big| &\leq |\Psi(z_2)| \cdot |z_1 - z_2| + |\Psi(z_2) - \Psi(z_1)| \\
&\leq 2K_1K_2\tilde{H} \cdot |z_1 - z_2| + 2K_1K_2\tilde{H} \cdot |z_1 - z_2|^\alpha \\
&\leq \Big(2K_1K_2\tilde{H} \cdot |z_1 - z_2|^{1-\alpha} + 2K_1K_2\tilde{H}\Big) \cdot |z_1 - z_2|^\alpha \\
&\leq 2\Big(2^{1-\alpha} + 1\Big)K_1K_2\tilde{H} \cdot |z_1 - z_2|^\alpha.
\end{aligned}$$

Consequently, $2\Big(2^{1-\alpha}+1\Big)K_1K_2\tilde{H}$ is a Hölder-constant for the real part of Φ' on the boundary $\partial\Omega$. Once more applying Lemma (A.7), we see that

$$4\Big(2^{1-\alpha}+1\Big)K_1^2K_2^2\tilde{H} \tag{A.58}$$

is a Hölder-constant of Φ' in the closed unit disk.

Summarizing the above arguments, we have proved the following statement:

Theorem A.3 *Suppose that the real part of Φ is Hölder-continuously differentiable on the boundary $\partial\Omega$ where $\tilde{H}$ is a Hölder-constant of the derivative of the boundary values. Then Φ' is Hölder-continuous in $\overline{\Omega}$, and a Hölder-constant of Φ' is given by*

$$4\Big(2^{1-\alpha}+1\Big)K_1^2K_2^2\tilde{H} \tag{A.59}$$

(A.57) proves the following corollary

Corollary A.2 *Provided $\tilde{H}$ is a Hölder-constant of the derivative of the boundary values g of the real part of Φ, the absolute value of Φ' can be estimated by*

$$|\Phi'| \leq 2K_1K_2\tilde{H}. \tag{A.60}$$

Our main goal is to have all necessary constants explicitly for the $C^{1,\alpha}$-norm of the solution of Dirichlet problem where the boundary values are Hölder-continuously differentiable. Here we apply the technique of using the concept of the holomorphic functions, we again consider (A.1)

$$\Phi = u(x,y) + iv(x,y)$$

where u and v are the solutions of the Laplace equation, which leads to

$$u(x,y) = \Phi - iv(x,y)$$

Using the triangle inequality of Banach space we get

$$\begin{aligned}
\|u\|_{C^{1,\alpha}} &\leq \|\Phi\|_{C^{1,\alpha}} + \|v\|_{C^{1,\alpha}} \\
&\leq \left(\|\Phi\| + \|\Phi'\| + \max \frac{|\Phi'(z_2) - \Phi'(z_1)|}{|z_1 - z_2|^\alpha}\right) \\
&\quad + \left(\|v\| + \|\partial_x v\| + \max \frac{|\partial_x v(z_2) - \partial_x v(z_1)|}{|z_1 - z_2|^\alpha}\right) \\
&\leq 2^{1+\alpha}K_1K_2H + \max_{\partial\Omega}|g| + |c| + 2K_1K_2\tilde{H} + 4\left(2^{1-\alpha} + 1\right)K_1^2K_2^2\tilde{H}
\end{aligned} \tag{A.61}$$

A.10. Complete Schauder estimate for the solution of inhomogeneous boundary value problem for $\mathbb{R}^2$

We assume that:

- Ω is a bounded domain in the plane with finite measure $m\Omega$.
- The boundary $\partial\Omega$ is in $C^{1,\alpha}$.
- The right hand side $\mathcal{F}$ is in $C^{1,\alpha}$.

We know that the solution of the inhomogeneous Dirichlet boundary value problem for the Laplace equation

$$\Delta u = \mathcal{F}(\cdot, u, p_i) \quad \text{in} \quad \Omega \quad \text{in the plane} \tag{A.62}$$

$$u = \varphi \quad \text{on}\,\partial\Omega \tag{A.63}$$

is given by the following integral equation

$$U(x) = u_0 - \frac{1}{2\pi}\iint\limits_{\Omega} \mathcal{F}\left((\xi,\eta),u(\xi,\eta),\partial_i u(\xi,\eta)\right)\ln|(x,y)-(\xi,\eta)|d\xi d\eta \tag{A.64}$$

has to estimate for the general bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary where u_0 is the solution of the Laplace equation in the domain in $\mathbb{R}^2$.

For two points in the unit disk we have

$$0 < r < 2 \tag{A.65}$$

for their polar distance r.

If $0 \le r \le 1$, then

$$\frac{1}{e} \le r|\ln r| \le 0 \tag{A.66}$$

If $1 \le r \le 2$, then

$$0 \le r|\ln r| \le 2\ln 2 \tag{A.67}$$

and thus for $0 \le r \le 2$

$$|r\ln r| \le 2\ln 2 \text{ and } |\ln r| \le \frac{2\ln 2}{r}. \tag{A.68}$$

Hence by the triangle inequality.

$$\|U(x)\|_{C^{1,\alpha}} \le \|u_0\|_{C^{1,\alpha}} + \|\mathcal{F}\|_{C^{1,\alpha}}\frac{\ln 2}{2\pi}\iint\limits_{\Omega}\frac{1}{|(x,y)-(\xi,\eta)|}d\xi d\eta \tag{A.69}$$

By using the Schmidt inequality and estimate (A.61), we get

$$\|U(x)\|_{C^{1,\alpha}} \le 2^{1+\alpha}K_1K_2H + \max_{\partial\Omega}|g| + |c| \\ +2K_1K_2\tilde{H} + 4\left(2^{1-\alpha}+1\right)K_1^2K_2^2\tilde{H} + C(\Omega)\|\mathcal{F}\|_{C^{1,\alpha}}.$$

Which is the final form of the Schauder estimates up to the boundary where all constants are explicitly known.

B. APPENDIX

B.1. Estimate for the integral having 2 weak singularities in a domain Ω with finite measure and Ω is domain in $\mathbb{R}^n$

This theorem is about an estimate of an integral having an integrand with two weak singularities at x' and x''. This result is even true for an unbounded domain having finite measure $m\Omega$. Moreover this estimate is true, only when $|x^{'} - x^{''}| < 2$. It is important the that following result is also a counter example and correction to the result used in the book of S. G. Mikhlin *Integral equations and their applications to certain problems in mechanics, mathematical physics and technology* (1964) p(59-62).

Theorem B.1 *Suppose Ω is domain in $\mathbb{R}^n$, with finite measure $m\Omega$, suppose further that λ and μ are real numbers satisfying the inequalities $0 \leq \lambda < n$ and $0 \leq \mu < n$ then there exist constants C_1, C_2 and C_3 depending only on Ω, λ and μ such that*

$$\int\limits_{\Omega} \frac{d\xi}{|x' - \xi|^{\lambda} \cdot |x'' - \xi|^{\mu}} \leq \begin{cases} C_1 |x^{'} - x^{''}|^{n-\lambda-\mu} + C_2, & for \lambda + \mu \neq n \\ C_3 - 4\pi \ln |x^{'} - x^{''}|, & for \lambda + \mu = n \end{cases} \tag{B.1}$$

is true for any 2 points $x^{'}$ and $x^{''}$ not necessarily belonging to $\overline{\Omega}$ but having a positive distance less than 2. Where ξ is an element in $\mathbb{R}^n$ and $d\xi$ is volume element in n-space

Proof The proof is performed by splitting up the domain of integration into five sub-domains then estimating the integral on all sub domains individually and then summing up those integrals. We do this by polar coordinates in n-sphere.
Denote $|x^{'} - x^{''}| = 2\varepsilon$, where, $0 < \varepsilon < 1$. The integral under consideration can be estimated from above by the sum of following integrals I_j , $j = 1, 2, 3, 4, 5$ which are defined as the integrals over the intersection of Ω with the following sets

- I_1 with. $|x^{'} - \xi| \leq \varepsilon$ while $|x^{''} - \xi| \geq \varepsilon$,
- I_2 with. $|x^{''} - \xi| \leq \varepsilon$ while $|x^{'} - \xi| \geq \varepsilon$,
- I_3 with. $\varepsilon \leq |x^{'} - \xi| \leq 1$ while $|x^{'} - \xi| \leq |x^{''} - \xi|$,
- I_4 with. $\varepsilon \leq |x^{''} - \xi| \leq 1$ while $|x^{''} - \xi| \leq |x^{'} - \xi|$,
- I_5 with. $|x^{'} - \xi|$ and $|x^{''} - \xi|$ both are ≥ 1.

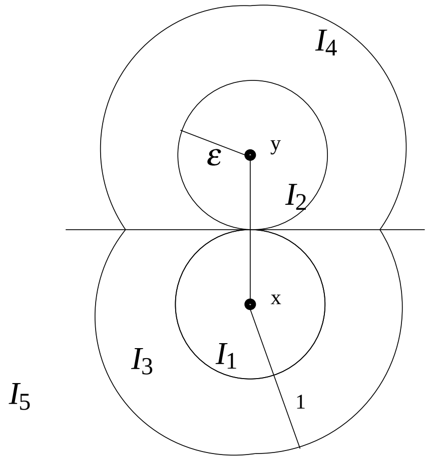


Figure B.1.: 2

In the case, when $j = 1$ one has $|x^{''} - \xi| \geq \varepsilon$ and using polar coordinates, therefore, we have, Finally we obtain

$$I_1 \leq \frac{2\pi}{n-\lambda}\varepsilon^{n-\lambda-\mu} \cdot M. \tag{B.2}$$

Where $2\pi \cdot M$ is surface area of the unit sphere in $\mathbb{R}^n$, and we have the standard result for it

A same estimate we get for I_2

$$I_2 \leq \frac{2\pi}{n-\mu}\varepsilon^{n-\lambda-\mu} \cdot M \tag{B.3}$$

Now for I_3 when $\varepsilon \leq |x^{'} - \xi| \leq 1$ while $|x^{'} - \xi| \leq |x^{''} - \xi|$. Then the integral (B.1) is estimated as follows

for $n \neq \lambda + \mu$, I_3 is given by

$$I_3 \leq \frac{2\pi}{n-\lambda-\mu}(1-\varepsilon^{n-\lambda-\mu}) \cdot M \tag{B.4}$$

for $n = \lambda + \mu$ we get

$$I_3 \leq -2\pi \ln \varepsilon \cdot M, \tag{B.5}$$

and very similar types of estimates for I_4 by a similar calculation
First I_4 for $n \neq \lambda + \mu$

$$I_4 \leq \frac{2\pi}{n-\lambda-\mu}(1-\varepsilon^{n-\lambda-\mu}) \cdot M \tag{B.6}$$

secondly, I_4 for $n = \lambda + \mu$

$$I_4 \leq -2\pi \ln \varepsilon \cdot M \tag{B.7}$$

And finally for I_5 we have $|x^{'} - \xi|$ and $|x^{''} - \xi|$ both are ≥ 1.

$$\begin{aligned} I_5 &\leq \int_{\Omega} d\xi \\ &\leq m\Omega. \end{aligned} \tag{B.8}$$

Now adding all I_i from inequalities (B.2),(B.3),(B.4,B.5),(B.6) and (B.7)

$$\int_{\Omega} \frac{d\xi}{|x' - \xi|^{\lambda}.|x'' - \xi|^{\mu}} \leq \begin{cases} C_1 \left| x^{'} - x^{''} \right|^{n-\lambda-\mu} + C_2, & \text{for} \lambda + \mu \neq n \\ C_3 - 4\pi \ln \left| x^{'} - x^{''} \right|, & \text{for} \lambda + \mu = n \end{cases} \tag{B.9}$$

where C_1, C_2 and C_3 are given explicitly by

$$C_1 = \frac{2\pi}{2^{n-\lambda-\mu}} \cdot M \left\{ \frac{1}{n-\lambda} + \frac{1}{n-\mu} + \frac{2\pi}{n-\lambda-\mu} \right\} \tag{B.10}$$

$$C_2 = 2\pi M \left\{ \frac{1}{n-\lambda} + \frac{1}{n-\mu} + \frac{2\pi}{n-\lambda-\mu} \right\} + m\Omega \tag{B.11}$$

$$C_3 = M \left\{ 2\pi \left(\frac{1}{n-\lambda} + \frac{1}{n-\mu} \right) + \ln 2 + \frac{m\Omega}{M} \right\} \tag{B.12}$$

C. APPENDIX

C.1. Proof of the second version of Schauder's Fixed-Point Theorem

Suppose M is closed and convex subset of a Banach space, f is a continuous mapping of M into itself, and $f(M)$ is relatively compact. Denote the convex hull of $f(M)$ by S. Since M is closed and convex, the set S is a subset of M and thus $f(S) \subset f(M) \subset S$, i.e., f maps S into itself.

Where convex hull is defined as

Definition (Convex hull) *The convex hull of a set S is the smallest closed and convex set containing S.*

To the existence by Schauder(I) , we first define the Mazur's Lemma which state as

Definition (Mazur's Lemma)*The convex hull of a relatively compact set is compact*

Consequently, the first version of Schauder Fixed Point Theorem

Schauder(I) *Let M be a compact and convex subset of a Banach space, and let f be a continuous mapping of M into itself. Then f has at least one fix point in M.*

is applicable to the restriction of f to S, i.e., f has at least one fixed point in S. Since $S \subset M$, the existence of a fixed point of f in M is proved.

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