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Analysis of boundary element methods for wave propagation in porous media

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Abstract

The aim of this work is to analyze a convolution quadrature boundary element approach to simulate wave propagation in porous media. In Laplace domain the model results in an elliptic second order partial differential equation. First, boundary value problems of interest are described and equivalent boundary integral formulations are derived. Unique solvability of all discussed boundary value problems and boundary integral equations is discussed, first in Laplace domain and finally also in time domain. A Galerkin discretization in space and a convolution quadrature discretization in time is applied. Unique solvability of the discrete systems and convergence of the approximate solutions are discussed. Finally, the theoretical results are confirmed by numerical experiments.

Zusammenfassung

Das Ziel dieser Arbeit ist die Analyse eines numerische Näherungsverfahrens zur Simulation von Wellenausbreitung in porösen Medien. Das numerische Näherungsverfahren basiert dabei auf ein Kombination der Randelementmethode mit der Faltungsquadraturmethode. Die Wellenausbreitung in porösen Medien wird mit Hilfe eines elliptischen Differentialoperators zweiter Ordnung und entsprechenden Randwertproblemen im Laplace–Bereich beschrieben. Für die betrachteten Randwertprobleme werden äquivalente Randintegralformulierungen hergeleitet. Die eindeutige Lösbarkeit der Randintegralgleichung wird sowohl im Laplace–Bereich als auch im Zeitbereich diskutiert. Die Randintegralgleichungen werden im Raum durch eine Galerkin Approximation diskretisiert. In der Zeit wird eine Faltungsquadraturmethode verwendet. Im weiteren wird die eindeutige Lösbarkeit der diskretisierten Integralgleichungen und die Konvergenz der näherungsweisen Lösungen diskutiert. Schlussendlich werden die theoretischen Ergebnisse mit Hilfe von numerischen Beispielen bestätigt.

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CONTENTS

1 INTRODUCTION

Wave propagation is a widespread phenomenon within our environment, and porous materials play an important role in many branches of engineering. A porous medium is a solid permeated by an interconnected network of pores filled with fluid. The solid as well as the pores are assumed to be continuous. Natural substances such as rocks, soils, biological tissues, foams, and ceramics can be considered as porous media. Fluid-saturated porous media cannot be modelled satisfyingly with the theory of elastodynamics. Based on the work of Terzaghi, Biot developed a theory to model porous media, see [12, 13]. One of the significant findings was the identification of three waves, two compressional waves and a shear wave. For the numerical simulation, several approaches, both finite element and boundary element, have been developed. An overview on these approaches and on analytical solutions is given in [47].

In this thesis, a formulation based on the solid displacement and the pore pressure as the primary unknowns is chosen. The reduction to these unknowns is only possible in Laplace domain. Boundary integral formulations based on this approach have been developed by Schanz and Messner [39, 40, 46].

Boundary element methods are a popular method to solve boundary value problems. A main advantage of the boundary element method is the reduction of the problem to the boundary. The boundary element method is especially suitable for exterior boundary value problems, since only the boundary of the domains has to be discretized and the radiation condition is already incorporated into the formulation. A comprehensive overview on the topic is given by McLean [38], as well as Hsiao and Wendland [26], Sauter and Schwab [45], and Steinbach [52]. Primarily, elliptic partial differential operators are discussed.

An overview on the application of boundary element methods to parabolic and hyperbolic partial differential equations is given in [17]. Basically two different approaches exist: Space-time integral equation techniques use the fundamental solution in time domain to formulate integral equations. Utilizing a Galerkin discretization by ansatz and test functions with respect to time yields a time stepping procedure. A second approach is based on the Laplace transformation. For fixed frequencies standard boundary element methods for elliptic problems are applied. The transformation back to the time domain employs special methods for the inversion of the Laplace or Fourier transformation. The convolution quadrature method as developed by Lubich [32, 33] falls into this category. This method approximates the convolution by a numerical integration formula, where the integration weights are based on the boundary integral operators in Laplace domain and the underlying multistep method. For poroelasticity this property is essential, since the fundamental solution in time domain is not explicitly known.

The method was first applied to parabolic boundary integral equations by Lubich and Schneider [36], where the authors discussed an indirect single layer approach. The analysis is based on an ellipticity estimate of the single layer boundary integral operator outside a sector of the complex plane with an acute angle to the negative real axis.

For the wave equation a similar approach was studied in [34]. In this case the single layer boundary integral operator is only elliptic in a half-plane. The related estimates for the single layer integral operator and the hyper-singular operator were developed by Bamberger and HaDuong [3,4]. The analysis for the wave equation was recently extended to boundary value problems and transmission problems by Laliena and Sayas [31].

The original convolution quadrature method was developed for multistep methods and has been extended to Runge-Kutta methods in [6, 35]. In recent papers, fast numerical implementations of the convolution quadrature method were investigated [5, 7, 22, 23, 30]. An overview over recent theoretical results is given in [8].

The aim of this thesis is to extend the theoretical results for the wave equation to poroelasticity. It turns out that similar estimates as for the wave equation can be shown. In particular, the theory is applied to the mixed, the Dirichlet and the Neumann boundary value problem. Stability and convergence of the resulting discrete system are obtained and confirmed by numerical examples.

Outline

Starting from constitutive equations, Biot's linear theory of poroelasticity is derived in Chapter 2. The resulting system of partial differential equations is transformed to the Laplace domain, where a simplified system of partial differential equations based on the primary unknowns, the solid displacement and the pore pressure, is derived. Suitable boundary conditions are defined resulting in the statement of the mixed boundary value problem of interest.

In Chapter 3 the analytic preliminaries are introduced. In addition to some basics from functional analysis, some definitions for a simplified notation are introduced. Moreover, Sobolev spaces and the Lamé system are discussed briefly. Furthermore the general framework of strong ellipticity is introduced. In the following, Green's formulae are derived for the operator of poroelasticity and ellipticity and boundedness of the defined sesquilinear form is established. With the help of these theoretical results unique solvability of the mixed boundary value problem is shown. Finally, the conormal derivative of the solution as well as it's adjoint are discussed.

Furthermore, the fundamental solution of poroelasticity as well as some of it's properties are introduced in Chapter 4. In the following, boundary integral operators and their respective mapping properties are discussed. The symmetry relations within these boundary integral operators are investigated, too. Moreover, we show ellipticity of the single layer boundary integral operator and the hyper-singular boundary integral operator. The ellipticity estimates enable us to establish estimates for all boundary integral operators. The dependency of all these estimates on the Laplace parameter s is analyzed and stated explicitly. Moreover, the Steklov–Poincaré as well as the Poincaré–Steklov operator are introduced. Ellipticity estimates are shown for both integral operators.

With the help of the representation formula boundary integral equations are introduced in Chapter 5. Boundary integral equations for the mixed, the Dirichlet and the Neumann boundary value are derived. Unique solvability and estimates for their solutions are presented. Again, the dependency on the Laplace parameter s of all involved constants is stated explicitly.

Moreover, the Galerkin discretization of boundary integral equations is introduced in Chapter 6. The theoretical framework is developed briefly. Estimates for the Galerkin discretization of several boundary integral operators are presented. Furthermore, the discrete boundary integral equations for the mixed, the Dirichlet and the Neumann boundary value problem are presented. Unique solvability and bounds for the solutions are discussed. Error estimates for the unknowns on the boundary as well as for the solution within the domain are given. Additionally, indirect approaches for the Dirichlet and the Neumann boundary value problem are discussed.

The convolution quadrature method is derived in Chapter 7. Error estimates for the approximation of operators are stated. A fast method, developed by Sauter and Banjai [7], for the implementation of the convolution quadrature method is briefly discussed. Finally a Galerkin discretization in space and a convolution quadrature approximation in time are discussed.

In Chapter 8, the analysis done in the Laplace domain is used to obtain statements in time domain. The unique solvability for the continuous system of boundary integral equations as well as for the fully discretized system of boundary integral equations is discussed . Finally, error estimates for the approximate solutions are given.

Numerical examples are discussed in Chapter 8. For this we introduce also a simple collocation approach. In the following sections we compare the Galerkin approach to the collocation approach, and the theoretical convergence orders gained throughout this work. First the error in space and afterwards the error in time are discussed for the mixed, the Dirichlet and the Neumann boundary value problem.

In the last chapter we draw some conclusions and discuss some open questions.

2 BIOT'S THEORY OF POROUS MATERIALS

In case of fluid infiltrated materials like water saturated soil, oil impregnated rocks or air filled foam, the elastic as well as the viscoelastic description of the material shows a rather crude approximation of wave propagation phenomena. Due to an interaction of the solid skeleton with the fluid in between and furthermore the porosity of the material, a different theory is necessary.

In 1941, a theory based on the work of Terzaghi was presented by Biot [12]. In the following years, this theory was extended several times. A collection of Biot's papers on porous materials has been published by Tolstoy [55]. A second theory, the theory of porous media is based on the application of axioms of continuum mechanics. A historical treatment can be found in the review article by de Boer [19]. In this work we will concentrate on the linear Biot theory. A review on linear models, analytic solutions and numerical methods is given in [47].

2.1 Governing equations

In Biot's theory, a fully saturated material is assumed, i.e., an elastic skeleton with a statistical distribution of interconnected pores is considered. Introducing V^f as the volume of the interconnected pores, and V^s as the volume of the solid, the porosity is denoted by $\phi = V^f/V$, where $V = V^s + V^f$. In [13] the balance of momentum in the solid and in the fluid are described as follows: For $i = 1,2,3$ we have

$$\sigma^s_{ij,j} + (1-\phi)\, f^s_i = (1-\phi)\rho_s \ddot{u}^s_i - \rho_a \left(\ddot{u}^f_i - \ddot{u}^s_i\right) - \frac{\phi^2}{\kappa}\left(\dot{u}^f_i - \dot{u}^s_i\right), \tag{2.1}$$

$$\sigma^f_{,i} + \phi\, f^f_i = \phi \rho_f \ddot{u}^f_i + \rho_a \left(\ddot{u}^f_i - \ddot{u}^s_i\right) + \frac{\phi^2}{\kappa}\left(\dot{u}^f_i - \dot{u}^s_i\right), \tag{2.2}$$

where u^s and u^f denote the displacement of the solid and of the fluid respectively. Additionally, f^s_i and f^f_i are the volume forces of the solid and of the interstitial fluid, while ρ_s and ρ_f are the respective densities. Moreover, the apparent mass density ρ_a is introduced to describe the dynamic interaction between the fluid and the skeleton. Note that the Einstein notation is used throughout this work. Finally, κ denotes the permeability. For an isotropic and homogeneous elastic solid and for a viscous interstitial fluid the following partial stress formulations for the stress tensor of the solid σ_{ij}^s and for the stress tensor of the fluid σ_{ij}^f are obtained for $i,j=1,2,3$,

$$\sigma_{ij}^s = \mu(u_{i,j}^s + u_{j,i}^s) + \left(\lambda + \frac{Q^2}{R}\right) u_{k,k}^s \delta_{ij} + Q u_{k,k}^f \delta_{ij}, \tag{2.3}$$

$$\sigma_{ij}^f = -\phi p \delta_{ij} = (Q u_{k,k}^s + R u_{k,k}^f)\delta_{ij}. \tag{2.4}$$

The elastic behaviour of the solid is governed by the Lamé constants λ and μ. The constants Q and R characterize the coupling between the solid and the fluid. The total stress is given as

$$\sigma_{ij} = \sigma_{ij}^s + \sigma_{ij}^f = \mu(u_{i,j}^s + u_{j,i}^s) + \lambda u_{k,k}^s \delta_{ij} - \alpha p \delta_{ij}, \tag{2.5}$$

where

$$\alpha = \phi\left(1 + \frac{Q}{R}\right) \in [0,1]$$

is Biot's effective stress coefficient.

The balance of the mixture is obtained by adding the two partial balances (2.1) and (2.2),

$$\sigma_{ij,j} + f_i = (1-\phi)\rho_s \ddot{u}_i^s + \phi \rho_f \ddot{u}_i^f, \tag{2.6}$$

where

$$f_i = (1-\phi) f_i^s + \phi f_i^f$$

is the bulk body force. Inserting the total stress (2.5) into the balance equation (2.6), and using the density

$$\rho := (1-\phi)\rho_s + \phi\rho_f$$

and the specific flux

$$q_i := \phi\left(\dot{u}_i^f - \dot{u}_i^s\right)$$

results in

$$\mu u_{i,jj}^s + (\lambda+\mu) u_{j,ij}^s - \alpha p_{,i} + f_i = \rho \ddot{u}_i^s + \rho_f \dot{q}_i. \tag{2.7}$$

By using the specific flux q and the fluid stress tensor σ_{ij}^f as given in (2.4), from (2.2) we conclude Darcy's law

$$\rho_f \ddot{u}_i^s + \frac{1}{\phi}\left(\rho_f + \frac{\rho_a}{\phi}\right)\dot{q}_i + \frac{1}{\kappa} q_i + p_{,i} = f_i^f. \tag{2.8}$$

In addition, the variation of fluid volume per unit reference volume ζ is introduced as

$$\zeta = \alpha u_{i,i}^s + \frac{\phi^2}{R} p. \tag{2.9}$$

The variation of the fluid content ζ is governed by the mass balance

$$\dot{\zeta} + q_{i,i} = 0. \tag{2.10}$$

Inserting (2.9) into (2.10) finally yields

$$\alpha \dot{u}^s_{i,i} + \frac{\phi^2}{R}\dot{p} + q_{i,i} = 0. \tag{2.11}$$

Biot's model results in the three coupled partial differential equations (2.7), (2.8) and (2.11), where in addition appropriate initial and boundary conditions have to be formulated. The system describes seven unknowns, namely the solid displacement u^s, the flux q, and the pore pressure p.

2.2 The *u*-*p* model in the Laplace domain

When assuming vanishing initial conditions, the partial differential equations (2.7), (2.8) and (2.11) can be reformulated by using the Laplace transformation

$$\hat{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

with the complex Laplace variable $s \in \mathbb{C}^+$. By convention we have $f(t) = 0$ for $t \leq 0$. The Laplace transformation is a linear transformation and transforms differentiation into multiplication, resulting in the properties

$$\begin{aligned} \mathcal{L}\{af(t)+bg(t)\} &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\} \quad && \text{for all } a,b \in \mathbb{C}, \\ \mathcal{L}\{f^{(n)}(t)\} &= s^n \mathcal{L}\{f(t)\} && \text{for } n \in \mathbb{N}. \end{aligned}$$

The Laplace transformation allows us to eliminate the specific flux from equations (2.7), (2.8) and (2.11). Without the Laplace transformation this elimination is not possible, since in addition to the specific flux the time derivative of the specific flux appears as well.

By using the Laplace transformation we obtain from Darcy's law (2.8)

$$\rho_f s^2 \widehat{u}^s_i + \frac{1}{\phi}\left(\rho_f + \frac{\rho_a}{\phi}\right) s\widehat{q}_i + \frac{1}{\kappa}\widehat{q}_i + \widehat{p}_{,i} = \widehat{f}^f_i,$$

and therefore

$$\widehat{q}_i = \frac{\phi^2\kappa}{s\kappa(\rho_f\phi + \rho_a) + \phi^2}\left(\widehat{f}^f_i - \widehat{p}_{,i} - \rho_f s^2 \widehat{u}^s_i\right) \tag{2.12}$$

follows. For the Laplace transform of (2.7) we conclude

$$\mu\widehat{u}^s_{i,jj}+(\lambda+\mu)\widehat{u}^s_{j,ij}-\alpha\widehat{p}_{,i}+\widehat{f}_i=\rho s^2\widehat{u}^s_i+\beta\left(\widehat{f}^f_i-\widehat{p}_{,i}-\rho_f s^2\widehat{u}^s_i\right),$$

where

$$\beta=\frac{\rho_f s\phi^2\kappa}{s\kappa(\rho_f\phi+\rho_a)+\phi^2}\in\mathbb{C}. \tag{2.13}$$

Analogously, the Laplace transform of (2.11) reads

$$\alpha s\widehat{u}^s_{i,i}+\frac{\phi^2}{R}s\widehat{p}+\frac{\beta}{\rho_f s}\left(\widehat{f}^f_{i,i}-\widehat{p}_{,ii}-\rho_f s^2\widehat{u}^s_{i,i}\right)=0.$$

Hence we consider the coupled system of partial differential equations

$$(\rho-\beta\rho_f)s^2\widehat{u}^s-\mu\Delta\widehat{u}^s-(\lambda+\mu)\operatorname{grad}\operatorname{div}\widehat{u}^s+(\alpha-\beta)\nabla\widehat{p}=\widehat{f}-\beta\widehat{f}^f, \tag{2.14}$$

$$(\alpha-\beta)s\operatorname{div}\widehat{u}^s-\frac{\beta}{\rho_f s}\Delta\widehat{p}+\frac{\phi^2 s}{R}\widehat{p}=-\frac{\beta}{\rho_f s}\operatorname{div}\widehat{f}^f, \tag{2.15}$$

where the related partial differential operator can be written as

$$\mathcal{P}:=\begin{pmatrix}-\mu\Delta-(\lambda+\mu)\operatorname{grad}\operatorname{div}+(\rho-\beta\rho_f)s^2 & (\alpha-\beta)\operatorname{grad}\\ (\alpha-\beta)s\operatorname{div} & -\dfrac{\beta}{\rho_f s}\Delta+\dfrac{\phi^2 s}{R}\end{pmatrix}. \tag{2.16}$$

Note that

$$\mathcal{P}_E:=-\mu\Delta-(\lambda+\mu)\operatorname{grad}\operatorname{div}$$

is related to the system of linear elasticity.

In addition to the partial differential operator (2.16) we need to formulate appropriate boundary conditions. We consider a bounded Lipschitz domain $\Omega\subset\mathbb{R}^3$ with the boundary $\Gamma=\partial\Omega$, where the exterior normal vector n is given almost everywhere. For Dirichlet boundary conditions we prescribe the solid displacement $\widehat{u}^s$ and the pore pressure $\widehat{p}$ on a part of the boundary $\Gamma_D\subset\Gamma$. Neumann boundary conditions describe the traction of the solid displacement $\widehat{u}^s$ and the negative specific flux $\widehat{q}$ in normal direction along the boundary on a part of the boundary $\Gamma_N\subset\Gamma$ with $\Gamma=\Gamma_D\cup\Gamma_N$ and $\Gamma_D\cap\Gamma_N=\varnothing$. The traction is given as

$$\gamma^\mu_1\widehat{u}^s=\boldsymbol{\sigma}\cdot n \tag{2.17}$$

with the total stress tensor $\boldsymbol{\sigma}$, see (2.5). The negative specific flux in normal direction is defined as

$$\gamma^p_1 p=-\widehat{q}\cdot n=s\beta\widehat{u}^s n+\frac{\beta}{s\rho_f}\partial_n p \tag{2.18}$$

see (2.12). For the function $\widehat{U} = (\widehat{u}^s, \widehat{p})^\top$ the mixed boundary value problem of poroelasticity in the Laplace domain is finally given as

$$\begin{aligned} \mathcal{P}\widehat{U} &= f && \text{in } \Omega, \\ \widehat{U} &= g_D && \text{on } \Gamma_D, \\ \gamma_1\widehat{U} &= g_N && \text{on } \Gamma_N, \end{aligned} \tag{2.19}$$

with the Neumann trace operator $\gamma_1\widehat{U} = (\gamma_1^u \widehat{u}^s, \gamma_1^p \widehat{p})^\top$.

A rather similar set of equations can be derived by the linear theory of porous media. The differences between Biot's model and the linear theory of porous media are studied in [48]. There it is shown that the theories for the compressible case contradict each other, due to problems in matching the respective material constants. From a pure mathematical point of view however, both partial differential operators share all the same properties. Therefore, the mathematical theory developed in the subsequent chapters is also applicable to the linear theory of porous media.

3 VARIATIONAL FORMULATIONS AND BOUNDARY VALUE PROBLEMS

3.1 Preliminaries

In this section, some preliminaries from functional analysis are given. The main references are [26, 38, 45, 52]. In particular, we introduce several notations and discuss some basic properties of the Lamé system, see [38, 52].

Definition 3.1. *Let X, Y be Hilbert spaces.*

- *A mapping $a(\cdot,\cdot) : X \times Y \to \mathbb{C}$ is called a sesquilinear form if for all $u_1, u_2 \in X$, all $v_1, v_2 \in Y$ and all $\lambda \in \mathbb{C}$*

$$\begin{aligned} a(u_1 + \lambda u_2, v_1) &= a(u_1, v_1) + \lambda a(u_2, v_1), \\ a(u_1, v_1 + \lambda v_2) &= a(u_1, v_1) + \overline{\lambda} a(u_1, v_2). \end{aligned} \tag{3.1}$$

- *A sesquilinear form is bounded (or continuous) if there exists a constant c_2^a such that*

$$|a(u,v)| \leq c_2^a \|u\|_X \|v\|_Y \tag{3.2}$$

  for all $u \in X$ and $v \in Y$.

- *The sesquilinear form $a(\cdot,\cdot)$ satisfies the inf-sup condition if there exists a constant $\gamma > 0$ such that*

$$\sup_{v \in Y \setminus \{0\}} \frac{|a(u,v)|}{\|v\|_Y} \geq \gamma \|u\|_X \quad \textit{for all } u \in X. \tag{3.3}$$

- *The sesquilinear form $a(\cdot,\cdot)$ is called X-elliptic if there exists a constant $c_1^a > 0$ and a bijective linear operator $\Theta : X \to Y$ such that*

$$\operatorname{Re}\left[a(u, \Theta u)\right] \geq c_1^a \|u\|_X^2 \quad \textit{for all } u \in X. \tag{3.4}$$

From the Riesz representation theorem we deduce that a sesquilinear form induces an operator.

Lemma 3.1. [52] For every sesquilinear form $a(\cdot,\cdot) : X \times Y \to \mathbb{C}$ there exists a unique linear and bounded operator $A : X \to Y^*$ such that

$$a(u,v) = \langle Au, v\rangle \quad \text{for all } u \in X, v \in Y,$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing in $Y * \times Y$. On the other hand, each bounded and linear operator $A : X \to Y^*$ induces a sesquilinear form

$$a(u,v) := \langle Au, v\rangle \quad \text{for all } u \in X, v \in Y.$$

Let X,Y be Hilbert spaces, let $a(\cdot,\cdot) : X \times Y \to \mathbb{C}$ be a continuous sesquilinear form and let $l : Y \to \mathbb{C}$ be a continuous linear functional. We consider the abstract problem:

Find $u \in X$ such that

$$a(u,v) = l(v) \tag{3.5}$$

for all $v \in Y$.

Theorem 3.2. [26,38,45] For every $l \in Y^*$ the abstract problem (3.5) has a unique solution $u \in X$ with

$$\|u\|_X \leq \frac{1}{\gamma} \|l\|_{Y^*}$$

if and only if the sesquilinear form $a(\cdot,\cdot)$ satisfies the inf-sup condition (3.3).

Lemma 3.3. (Lax-Milgram) Let X,Y be Hilbert spaces and additionally let the sesquilinear form $a : X \times Y \to \mathbb{C}$ be X-elliptic. Then the variational problem (3.5) has a unique solution $u \in X$ for all $l \in Y^*$ with

$$\|u\|_X \leq \frac{1}{c_1^a} \|\Theta\|_{X\to Y} \|l\|_{Y^*}.$$

Proof. The X-ellipticity estimate (3.4) can be written as

$$c_1^a \|u\|_X \leq \frac{|a(u,\Theta u)|}{\|u\|_X}.$$

Furthermore we have

$$\|\Theta u\|_Y \leq \|\Theta\|_{X\to Y} \|u\|_X$$

and therefore

$$\frac{c_1^a}{\|\Theta\|_{X\to Y}} \|u\|_X \leq \frac{|a(u,\Theta u)|}{\|\Theta u\|_Y} \leq \sup_{v\in Y\setminus\{0\}} \frac{|a(u,v)|}{\|v\|_Y}.$$

The inf-sup condition (3.3) is consequently fulfilled and hence equation (3.5) is uniquely solvable. The estimate for the solution follows directly from Theorem 3.2. □

Most of the analysis of the partial differential equations of poroelasticity will be done in the Laplace domain. To get estimates in the time domain the dependency on the Laplace parameter s has to be stated explicitly. This analysis is only possible if the Laplace parameter s is an element of the half-space

$$\mathbb{C}_\sigma^+ := \{s \in \mathbb{C} : \operatorname{Re}[s] \geq \sigma > 0\}.$$

This assumption restricts the choice of the time stepping method to A-stable methods, see Chapter 7 or [34].

We will use the following notation throughout the thesis

$$\underline{\sigma} := \min(1, \sigma).$$

An important estimate is

$$\max(1, \operatorname{Re}[s])\underline{\sigma} \leq \operatorname{Re}[s] \quad \text{for all } s \in \mathbb{C}_\sigma^+ \tag{3.6}$$

and similarly

$$\max(1, |s|)\underline{\sigma} \leq |s| \quad \text{for all } s \in \mathbb{C}_\sigma^+. \tag{3.7}$$

To be able to apply the concept of ellipticity, see (3.4), to the system of poroelasticity, we need to introduce an appropriate bijective operator as follows.

Definition 3.2. *Let X_1, X_2, X_3, X_4 be Hilbert spaces over $\mathbb{C}$ and let us consider the product space*

$$X = X_1 \times X_2 \times X_3 \times X_4.$$

The mapping $\Theta_{a,b} : X \to X$ is defined as

$$\Theta_{a,b} := \begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & b \end{pmatrix} \tag{3.8}$$

where $a, b \in \mathbb{C}$. Furthermore we write

$$\Theta_a := \Theta_{a,1}. \tag{3.9}$$

Sobolev spaces

We will consider the setting of a bounded domain $\Omega \subset \mathbb{R}^3$ which is assumed to be Lipschitz. We denote it's boundary with $\Gamma = \partial\Omega$. We will make use of standard results for Sobolev spaces, see, e. g. , [1, 38, 45].

We denote the space of infinitely times differentiable functions with compact support as $\mathcal{D}(\Omega) := \mathcal{C}_0^\infty(\Omega) \subset C^\infty(\Omega)$. The Sobolev spaces are denoted by $H^r(\Omega)$ for $r \in \mathbb{R}$, see [38]. For vector valued functions the Sobolev spaces are taken componentwise.

We denote the norm of a Sobolev space by

$$\|v\|_{r,\Omega} := \|v\|_{[H^r(\Omega)]^d} \quad \text{for } r \geq 0 \text{ and } d \in \mathbb{N}$$

for an element $v \in [H^r(\Omega)]^d$. The dual spaces with respect to the inner product

$$\langle f, v \rangle_\Omega = \int_\Omega f(x) \cdot \overline{v(x)} \, \mathrm{d}x$$

are denoted by $[\widetilde{H}^{-r}(\Omega)]^d$. The norm for $f \in \widetilde{H}^{-r}(\Omega)$ is given by

$$\|f\|_{-r,\Omega} := \sup_{0 \neq v \in [H^r(\Omega)]^d} \frac{\langle f, v \rangle_\Omega}{\|v\|_{r,\Omega}} \quad r > 0.$$

For a Lipschitz domain Ω the Sobolev spaces on the boundary are denoted by $H^r(\Gamma)$ for $r \in (0,1)$, for the definition we refer to [38]. For an element $u \in [H^r(\Gamma)]^d$ we denote the norm by

$$\|u\|_{r,\Gamma} := \|u\|_{[H^r(\Gamma)]^d} .$$

For $r \in (-1,0)$ the space $H^r(\Gamma)$ is defined by duality with respect to the inner product

$$\langle g, v \rangle_\Gamma = \int_\Gamma g(x) \cdot \overline{v(x)} \, \mathrm{d}s_x.$$

The norm is denoted by

$$\|g\|_{r,\Gamma} := \sup_{0 \neq v \in [H^r(\Gamma)]^d} \frac{\langle g, v \rangle_\Gamma}{\|v\|_{-r,\Gamma}} \quad r \in (-1,0)$$

For the definition of Sobolev spaces of higher order, a boundary with a higher regularity is necessary. For Lipschitz domains, Sobolev spaces with higher regularity are defined with respect to piecewise smooth boundaries, see [45, 52].

For an open part $\Gamma_0 \subset \Gamma$ of the boundary Γ, Sobolev spaces of the order $r \in [0,1)$ are defined by

$$\begin{aligned} H^r(\Gamma_0) &:= \{u = \widetilde{u}|_{\Gamma_0} : \widetilde{u} \in H^r(\Gamma)\}, \\ \widetilde{H}^r(\Gamma_0) &:= \{u = \widetilde{u}|_{\Gamma_0} : \widetilde{u} \in H^r(\Gamma) \text{ and } \operatorname{supp} \widetilde{u} \in \Gamma_0\} \end{aligned}$$

with the norm

$$\|u\|_{r,\Gamma_0} := \inf \left\{ \|\widetilde{u}\|_{r,\Gamma} : \widetilde{u} \in H^r(\Gamma) \text{ and } \widetilde{u}|_{\Gamma_0} = u \right\}.$$

The spaces for negative order are defined by duality as

$$H^{-r}(\Gamma_0) := [\widetilde{H}^r(\Gamma_0)]^* \quad \text{for } r > 0$$
$$\widetilde{H}^{-r}(\Gamma_0) := [H^r(\Gamma_0)]^* \quad \text{for } r > 0.$$

Definition 3.3. *The trace operator γ_0 for a function $u \in \mathcal{D}(\overline{\Omega})$ is defined by*

$$\gamma_0 u := u|_\Gamma.$$

Theorem 3.4. [38] If Ω is a Lipschitz domain and if $1/2 < r < 3/2$, then the trace operator γ_0 has a unique extension to a bounded linear operator

$$\gamma_0 : H^r(\Omega) \to H^{r-1/2}(\Gamma),$$

and this extension has a continuous right inverse.

For fixed $s \in C_\sigma^+$ an equivalent norm in $H^1(\Omega)$ is introduced as

$$|||v|||_{|s|,\Omega} := \left(\|\operatorname{grad} v\|_{0,\Omega}^2 + \|sv\|_{0,\Omega}^2 \right)^{\frac{1}{2}}.$$

For a vector valued $U \in [H^1(\Omega)]^d$ the norm is taken component wise. Additionally we introduce the equivalent norm

$$|||U|||_{\widetilde{|s|},\Omega} := \frac{1}{|s|} \, ||| \, \Theta_s U \, |||_{|s|,\Omega} . \tag{3.10}$$

In particular for $U = (u,p)^\top$ with $u \in [H^1(\Omega)]^3$ and $p \in H^1(\Omega)$ we have

$$|||(u,p)|||_{\widetilde{|s|},\Omega}^2 = \|\operatorname{grad} u\|_{0,\Omega}^2 + \|su\|_{0,\Omega}^2 + \left\| \frac{1}{s} \operatorname{grad} p \right\|_{0,\Omega}^2 + \|p\|_{0,\Omega}^2 .$$

From this we conclude the relations

$$\frac{\underline{\sigma}^2}{|s|} \|(u,p)\|_{1,\Omega} \le \frac{\underline{\sigma}}{|s|} \, ||| \, (u,p) |||_{|s|,\Omega} \le |||(u,p)|||_{\widetilde{|s|},\Omega} \le \frac{1}{\underline{\sigma}} \, ||| \, (u,p) |||_{|s|,\Omega} \le \frac{|s|}{\underline{\sigma}^2} \|(u,p)\|_{1,\Omega} . \tag{3.11}$$

Another useful estimate is

$$\|\Theta_s U\|_{1,\Omega} \le \frac{|s|}{\underline{\sigma}} \, ||| \, U \, |||_{\widetilde{|s|},\Omega} \tag{3.12}$$

due to

$$\|\Theta_s U\|_{1,\Omega} \le \frac{1}{\underline{\sigma}} \, ||| \, \Theta_s U |||_{|s|,\Omega} = \frac{|s|}{\underline{\sigma}} \, ||| \, U \, |||_{\widetilde{|s|},\Omega} .$$

In the time domain we apply the spaces

$$H_0^r(0,T,H^k(\Omega)) = \{u(t,\cdot) \in H^k(\Omega) \mid \|u(t,\cdot)\|_{k,\Omega} \in H^r(0,T) \text{ and } u \equiv 0 \text{ for } t < 0\}.$$

and

$$H_0^r(0,T,H^k(\Gamma)) = \{u(t,\cdot) \in H^k(\Gamma) \mid \|u(t,\cdot)\|_{k,\Gamma} \in H^r(0,T) \text{ and } u \equiv 0 \text{ for } t \leq 0\}.$$

An equivalent norm for $H_0^r(0,T;H^k(\Omega))$ is denoted by

$$\|f\|_{r,k,\Omega} := \left(\int_0^T \left| \partial_t^r \, \|f(t,\cdot)\|_{k,\Omega} \right|^2 \, \mathrm{d}t \right)^{\frac{1}{2}}, \tag{3.13}$$

whereas an equivalent norm for $v \in H_0^r(0,T;H^k(\Gamma))$ is denoted by

$$\|g\|_{r,k,\Gamma} := \left(\int_0^T \left| \partial_t^r \, \|g(t,\cdot)\|_{k,\Gamma} \right|^2 \, \mathrm{d}t \right)^{\frac{1}{2}}.$$

Definition 3.4. *For $a,b \in \mathbb{R}$ and $s \in \mathbb{C}$ we abbreviate estimates of the kind*

$$a \leq c_1 c_2(s) b$$

as

$$a \lesssim c_2(s) b \tag{3.14}$$

as long as $c_1 > 0$ does not depend on the Laplace parameter s.

The Lamé system

Some well known results for the Lamé system will be stated in this section. For a more detailed presentation see, e. g. , [38, 52]. The operator of linear elasticity is given by

$$\mathcal{P}_E = -\mu\Delta - (\lambda + \mu) \operatorname{grad}\operatorname{div}.$$

The operator is considered in a Lipschitz domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma = \partial\Omega$, where the outer normal vector n is defined almost everywhere. For the operator $\mathcal{P}_E$ there holds Betti's formula

$$a^E(u,v) = \langle \mathcal{P}_E \, u, v \rangle_\Omega + \langle \mathrm{T_E} \, u, v \rangle_\Gamma$$

with the boundary stress operator

$$\mathrm{T_E} \, u := \lambda \operatorname{div} u n + 2\mu \partial_n u + \mu n \times \operatorname{curl} u \quad \text{on } \Gamma, \tag{3.15}$$

the sesquilinear form

$$a^E(u,v) = \int_\Omega \left[2\mu e_{ij}(u)e_{ij}(\overline{v}) + \lambda \operatorname{div} u \operatorname{div} \overline{v}\right] \, dx,$$

and the strain tensor

$$e_{ij}(u) = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right).$$

A non-trivial result, Korn's second inequality, results in ellipticity and coercivity estimates.

Theorem 3.5 (Korn's second inequality)**.** *Let Ω be a Lipschitz domain, then we have*

$$\int_\Omega e_{ij}(u)e_{ij}(\overline{u}) \, dx + \|u\|_{0,\Omega}^2 \gtrsim \|u\|_{1,\Omega}^2 \quad \text{for all } u \in [H^1(\Omega)]^3.$$

Proof. See [38, 41]. □

By adding the $[L_2(\Omega)]^3$-norm to the sesquilinear form $a^E(\cdot,\cdot)$, we end up with an equivalent norm in $[H^1(\Omega)]^3$.

Theorem 3.6. *For $\mu > 0$, $\lambda \geq 0$ and $s \in \mathbb{C}_\sigma^+$ the following estimates hold for all $u \in [H^1(\Omega)]^3$*

$$a^E(u,u) + \|u\|_{0,\Omega}^2 \gtrsim \|u\|_{1,\Omega}^2, \tag{3.16}$$

$$a^E(u,u) + \|su\|_{0,\Omega}^2 \gtrsim \underline{\sigma}^2 \, |||\, u \,|||_{|s|,\Omega}^2. \tag{3.17}$$

Proof. Inequality (3.16) follows immediately from Korn's second inequality. Furthermore, by applying estimates (3.16) and (3.7) we end up with

$$\begin{aligned}
|||u|||_{|s|,\Omega}^2 &= \|\operatorname{grad} u\|_{0,\Omega}^2 + \|su\|_{0,\Omega}^2 \\
&\leq \|u\|_{1,\Omega}^2 + \|su\|_{0,\Omega}^2 \\
&\leq c_1 a^E(u,u) + \max(1,|s|^2)\,\|u\|_{0,\Omega}^2 \\
&\leq c_2 \left[a^E(u,u) + \frac{|s|^2}{\underline{\sigma}^2}\|u\|_{0,\Omega}^2\right] \\
&\leq \frac{c_2}{\underline{\sigma}^2}\left[a^E(u,u) + \|su\|_{0,\Omega}^2\right].
\end{aligned}$$

□

3.2 Strong ellipticity

Rather general approaches for the analysis of boundary integral equations have been developed, see, e. g. [26,38,45,52]. We follow the approach as given in [38], where strongly elliptic differential operators are considered.

A general partial differential operator P of second order is given by

$$\mathcal{P}u = -\sum_{j=1}^{3}\sum_{k=1}^{3}\partial_j(A_{jk}\partial_k u) + \sum_{j=1}^{3} A_j\partial_j u + Au \quad \text{on } \Omega \subset \mathbb{R}^3$$

where the coefficients

$$A_{jk} = \left[a_{pq}^{jk}\right], \quad A_j = \left[a_{pq}^{j}\right], \quad A = \left[a_{pq}\right] \quad 1 \leq p \leq 3 \text{ and } 1 \leq q \leq 3$$

are functions from Ω into $\mathbb{C}^{3\times 3}$, the space of complex 3×3 matrices. Notice that u is in general vector valued.

Definition 3.5. *A second order partial differential operator $\mathcal{P}$ is called uniform strongly elliptic on Ω if*

$$\mathrm{Re}\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\left[A_{jk}(x)\xi_k\eta\right]^*\xi_j\eta\right] \geq c\,|\xi|^2\,|\eta|^2$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^3$, $\eta \in \mathbb{C}^3$ and $c > 0$.

The operator of poroelasticity (2.16) turns out to be strongly elliptic.

Theorem 3.7. *For $s \in C_\sigma^+$ and $\mu > 0$, $2\mu+\lambda > 0$, $\kappa > 0$, $\phi > 0$, $(\rho_a + \phi\rho_f) \geq 0$ the partial differential operator $\mathcal{P}$ as given in* (2.16) *is strongly elliptic.*

Proof. The Fourier transform of the main part $\mathcal{P}_0$ is $\widehat{\mathcal{P}}_0(\xi)$, where $\widehat{\mathcal{P}}_0(\xi)$ is the homogeneous $\mathbb{C}^{4\times 4}$-valued quadratic polynomial

$$\widehat{\mathcal{P}}_0(\xi) = (2\pi)^2\begin{pmatrix}\mu I_3\,|\xi_1|^2 + (\mu+\lambda)\xi_1\xi_1^* & 0 \\ 0 & \frac{\beta}{s\rho_f}\,|\xi_2|^2\end{pmatrix}$$

with the 3×3 identity I_3 and $\xi = (\xi_1,\xi_2)^\top$. Thus for $\eta = (\eta_1,\eta_2)^\top$,

$$\eta^*\widehat{\mathcal{P}}_0(\xi)\eta = (2\pi)^2\left[\mu\,|\xi_1|^2\,|\eta_1|^2 + (\mu+\lambda)\,|\xi_1\cdot\eta_1|^2 + \frac{\beta}{s\rho_f}\,|\xi_2|^2\,|\eta_2|^2\right]$$

and therefore P is strongly elliptic if and only if

$$\mu > 0, \quad 2\mu + \lambda > 0, \quad \mathrm{Re}\left[\frac{\beta}{s\rho_f}\right] > 0.$$

The last constant is given as

$$\frac{\beta}{s\rho_f} = \frac{\kappa\phi^2}{\phi^2 + s\kappa(\rho_a + \phi\rho_F)} = \frac{\kappa\phi^4 + \bar{s}\kappa^2\phi^2(\rho_a + \phi\rho_F)}{|\phi^2 + s\kappa(\rho_a + \phi\rho_F)|^2},$$

and therefore the real part is strictly positive under the given assumptions. □

3.3 Green's formula in poroelasticity

Let Ω be a Lipschitz domain and let n be the outward unit normal vector on $\Gamma = \partial\Omega$, which is defined almost everywhere. The componentwise multiplication of the partial differential equation (2.14) with the complex adjoint of a test function v_i, integration over Ω, applying integration by parts, and summation gives

$$\begin{aligned}\int_\Omega [\widehat{f}_i - \beta \widehat{f}_i^f]\bar{v}_i \,\mathrm{d}x &= \int_\Omega \left[(\rho - \beta\rho_f)s^2\widehat{u}_i^s - \mu\widehat{u}_{i,jj}^s - (\lambda+\mu)\widehat{u}_{j,ij}^s + (\alpha - \beta)\widehat{p}_{,i}\right]\bar{v}_i \,\mathrm{d}x \\ &= a^E(\widehat{u}^s, v) + (\rho - \beta\rho_f)s^2 \langle \widehat{u}^s, v\rangle_\Omega - \alpha\langle \widehat{p}, \mathrm{div}\, v\rangle_\Omega - \beta\langle \nabla\widehat{p}, v\rangle_\Omega - \langle \mathrm{T_E}\,\widehat{u}^s - \alpha\widehat{p}n, v\rangle_\Gamma .\end{aligned} \tag{3.18}$$

Recall that the L_2-inner products are defined as

$$\langle u, v\rangle_\Omega = \int_\Omega u(x)\cdot\overline{v(x)}\,\mathrm{d}x,$$

and

$$\langle f, g\rangle_\Gamma = \int_\Gamma f(x)\cdot\overline{g(x)}\,\mathrm{d}s_x.$$

When multiplying the partial differential equation (2.15) with the complex adjoint of a test function q we obtain accordingly

$$\begin{aligned}-\int_\Omega \frac{\beta}{\rho_f s}\,\mathrm{div}\,\widehat{f}^f\bar{q}\,\mathrm{d}x &= \int_\Omega \left[(\alpha-\beta)s\,\mathrm{div}\,\widehat{u}^s - \frac{\beta}{\rho_f s}\Delta\widehat{p} + \frac{\phi^2 s}{R}\widehat{p}\right]\bar{q}\,\mathrm{d}x \\ &= \alpha s\langle \mathrm{div}\,\widehat{u}^s, q\rangle_\Omega + \beta s\langle \widehat{u}^s, \nabla q\rangle_\Omega + \frac{\beta}{\rho_f s}\langle \nabla\widehat{p}, \nabla q\rangle_\Omega + \frac{\phi^2 s}{R}\langle \widehat{p}, q\rangle_\Omega \\ &\quad - \beta s\left\langle n^\top\widehat{u}, q\right\rangle_\Gamma - \frac{\beta}{\rho_f s}\langle \partial_n\widehat{p}, q\rangle_\Gamma .\end{aligned} \tag{3.19}$$

Now, by combining (3.18) and (3.19) we conclude Green's first formula in poroelasticity,

$$a_\Omega((\widehat{u}^s,\widehat{p});(v,q)) = \left\langle \mathcal{P}\begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}, \begin{pmatrix}v\\ q\end{pmatrix}\right\rangle_\Omega + \left\langle \gamma_1\begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}, \begin{pmatrix}v\\ q\end{pmatrix}\right\rangle_\Gamma \tag{3.20}$$

with the sesquilinear form

$$\begin{aligned} a_\Omega((\widehat{u}^s,\widehat{p});(v,q)) &= a^E(\widehat{u}^s,v) + (\rho-\beta\rho_f)s^2\langle\widehat{u}^s,v\rangle_\Omega - \alpha\langle\widehat{p},\operatorname{div}v\rangle_\Omega - \beta\langle\nabla\widehat{p},v\rangle_\Omega \\ &\quad + \alpha s\langle\operatorname{div}\widehat{u}^s,q\rangle_\Omega + \beta s\langle\widehat{u}^s,\nabla q\rangle_\Omega + \frac{\beta}{\rho_f s}\langle\nabla\widehat{p},\nabla q\rangle_\Omega + \frac{\phi^2 s}{R}\langle\widehat{p},q\rangle_\Omega \end{aligned} \tag{3.21}$$

and with the boundary stress operator

$$\gamma_1\begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix} = \begin{pmatrix} \mathrm{T_E}\,\widehat{u}^s - \alpha\widehat{p}n \\ \beta s n^\top\widehat{u}^s + \dfrac{\beta}{\rho_f s}\partial_n\widehat{p}\end{pmatrix} = \begin{pmatrix}\mathrm{T_E} & -\alpha n\\ \beta s n^\top & \dfrac{\beta}{\rho_f s}\partial_n\end{pmatrix}\begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}. \tag{3.22}$$

The boundary stress operator (3.22) reflects the dependency of the elastic stress on the pore pressure, while the flux of the pore pressure depends on the displacement. The boundary stress operator can be rewritten by using the stress tensor , see (2.5), the normal vector and the negative specific flux, see (2.18).

In order to deduce Green's second formula for the partial differential equations in poroelasticity we need to introduce the formally adjoint partial differential operator as

$$\widetilde{\mathcal{P}} = \begin{pmatrix} -\mu\Delta-(\lambda+\mu)\operatorname{grad}\operatorname{div}+(\rho-\overline{\beta}\rho_f)\overline{s}^2 & -(\alpha-\overline{\beta})\overline{s}\operatorname{grad}\\ -(\alpha-\overline{\beta})\operatorname{div} & -\dfrac{\overline{\beta}}{\rho_f\overline{s}}\Delta+\dfrac{\phi^2\overline{s}}{R}\end{pmatrix}, \tag{3.23}$$

and the related adjoint boundary stress operator

$$\widetilde{\gamma}_1 = \begin{pmatrix}\mathrm{T_E} & \alpha\overline{s}n\\ -\overline{\beta}n^\top & \dfrac{\overline{\beta}}{\rho_f\overline{s}}\partial_n\end{pmatrix}. \tag{3.24}$$

Then, Green's first formula for the adjoint partial differential operator reads

$$a_\Omega((\widehat{u}^s,\widehat{p});(v,q)) = \left\langle \begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}, \widetilde{\mathcal{P}}\begin{pmatrix}v\\ q\end{pmatrix}\right\rangle_\Omega + \left\langle \begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}, \widetilde{\gamma}_1\begin{pmatrix}v\\ q\end{pmatrix}\right\rangle_\Gamma, \tag{3.25}$$

and therefore by equalizing (3.20) and (3.25), we conclude Green's second formula in poroelasticity,

$$\left\langle \begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}, \widetilde{\mathcal{P}}\begin{pmatrix}v\\ q\end{pmatrix}\right\rangle_\Omega + \left\langle \begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}, \widetilde{\gamma}_1\begin{pmatrix}v\\ q\end{pmatrix}\right\rangle_\Gamma = \left\langle \mathcal{P}\begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}, \begin{pmatrix}v\\ q\end{pmatrix}\right\rangle_\Omega + \left\langle \gamma_1\begin{pmatrix}\widehat{u}^s\\ \widehat{p}\end{pmatrix}, \begin{pmatrix}v\\ q\end{pmatrix}\right\rangle_\Gamma. \tag{3.26}$$

For readability we introduce functions $U = (\widehat{u}^s, \widehat{p})^\top$ and $V = (v, q)^\top$. Green's formulae can therefore be applied to the following setting. We denote an interior domain by Ω^- and the corresponding exterior domain by $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega^-}$. Green's first formulae for both the interior and exterior problems read as

$$a_{\Omega^\pm}(U,V) = \langle \mathcal{P}U, V\rangle_{\Omega^\pm} \mp \left\langle \gamma_1^\pm U, \gamma_0^\pm V\right\rangle_\Gamma \quad \text{for all } U \in [H^2(\Omega^\pm)]^4, V \in [H^1(\Omega^\pm)]^4, \tag{3.27}$$

$$a_{\Omega^\pm}(U,V) = \left\langle U, \widetilde{\mathcal{P}}V\right\rangle_{\Omega^\pm} \mp \left\langle \gamma_0^\pm U, \widetilde{\gamma}_1^\pm V\right\rangle_\Gamma \text{ for all } U \in [H^1(\Omega^\pm)]^4, V \in [H^2(\Omega^\pm)]^4. \tag{3.28}$$

The radiation condition for the exterior problem is embedded into the Sobolev space $[H^1(\Omega^+)]^4$. For poroelasticity the physically relevant solutions show exponential decay as $\|x\| \to \infty$ and so the Sobolev space $[H^1(\Omega^+)]^4$ can be used for the formulation of the relevant variational problem.

The jumps of the traces over Γ of the conormal derivatives and the adjoint conormal derivatives are denoted by

$$[U]_\Gamma = \gamma_0^- U - \gamma_0^+ U, \quad [\gamma_1 U]_\Gamma = \gamma_1^- U - \gamma_1^+ U, \quad [\widetilde{\gamma}_1 U]_\Gamma = \widetilde{\gamma}_1^- U - \widetilde{\gamma}_1^+ U.$$

Additionally, if $\gamma_0^+ U = \gamma_0^- U$ we denote it simply by $\gamma_0 U$. This notation will be used accordingly for $\gamma_1 U$ and $\widetilde{\gamma}_1 U$.

Lemma 3.8. *Let $u \in [L_2(\mathbb{R}^3)]^4$ with $u|_{\Omega^\pm} \in [H^1(\Omega^\pm)]^4$. If*

$$Pu^\pm = 0 \quad \text{on } \Omega^\pm,$$

then

$$\langle \mathcal{P}U, \psi\rangle_\Gamma = \left\langle U, \widetilde{\mathcal{P}}\psi\right\rangle_\Gamma = -\langle [U]_\Gamma, \widetilde{\gamma}_1 \psi\rangle_\Gamma + \langle [\gamma_1]_\Gamma, \gamma_0 \psi\rangle_\Gamma \quad \text{for all } \psi \in [\mathcal{D}(\mathbb{R}^3)]^4, \tag{3.29}$$

and

$$a_{\Omega^+}(U|_{\Omega^+}, V) + a_{\Omega^-}(U|_{\Omega^-}, V) = \left\langle \gamma_1^- U, \gamma_0^- V\right\rangle_\Gamma - \left\langle \gamma_1^+ U, \gamma_0^+ V\right\rangle_\Gamma \quad \text{for all } V \in [H^1(\mathbb{R}^3 \setminus \Gamma)]^4. \tag{3.30}$$

3.4 Boundary value problems

With the help of Green's formulae (3.27) and (3.28) we can analyze related boundary value problems. Unique solvability is proven by the Lemma of Lax-Milgram (Lemma 3.3), which requires boundedness and ellipticity of the sesquilinear form (3.21). Furthermore the dependency of the ellipticity and boundedness constants on the Laplace parameter s will be studied.

For readability we use the abbreviations $U = (u, p)^\top$ and $V = (v, q)^\top$ and the operator Θ_s as introduced in (3.9). The operator Θ_s is bounded by

$$\|\Theta_s\|_{X\to X} \le \max(1, |s|) \le \frac{|s|}{\underline{\sigma}}. \tag{3.31}$$

where X is defined as in Definition 3.2.

Theorem 3.9. *Let the Lamé constants $\lambda \ge 0$ and $\mu > 0$, the permeability $\kappa > 0$, the solid and partial densities $\rho_s > 0$ and $\rho_f > 0$, the coupling constants $Q > 0$ and $R > 0$ and the porosity $\phi \in (0,1)$. Moreover, let $s \in \mathbb{C}^+_\sigma$. Then we have*

$$\mathrm{Re}\,[a_\Omega(U, \Theta_s U)] \gtrsim \underline{\sigma}^5 \sigma \, |||\, U|||^2_{|s|,\Omega}$$

for all $U \in [H^1(\Omega)]^4$.

Proof. The real part of the sesquilinear form (3.21) with the test function (su, p) is given by

$$\mathrm{Re}\,[a_\Omega((u,p);(su,p))] = \mathrm{Re}\,[s]\, a^E(u,u) + |s|^2\, \mathrm{Re}\left[s(\rho - \beta\rho_f)\right] \|u\|^2_{0,\Omega}$$
$$+ \mathrm{Re}\left[(\overline{\beta} - \beta) s \langle u, \nabla p\rangle_\Omega\right] + \frac{1}{\rho_f} \mathrm{Re}\left[\frac{\beta}{s}\right] \|\nabla p\|^2_{0,\Omega} + \frac{\phi^2}{R} \mathrm{Re}\,[s]\, \|p\|^2_{0,\Omega}.$$

Since α is real valued, the corresponding mixed term vanishes.

For $\mathrm{Im}\,[\beta] = 0$ the second mixed part vanishes as well. In this case the remaining parts can be estimated further. We have

$$\mathrm{Re}\left[s(\rho - \beta\rho_f)\right] = \frac{\rho\phi^4\,\mathrm{Re}\,[s] + K_1\,|s|^2\,\mathrm{Re}\,[s] + K_2\,\mathrm{Re}\,[s]^2 + \phi^4\rho_f^2\kappa\,\mathrm{Im}\,[s]^2}{\left|\phi^2 + s\kappa(\rho_a + \phi\rho_f)\right|^2} \tag{3.32}$$

with

$$K_1 = \kappa(\rho\rho_a + \phi\rho_f\rho_s(1-\phi)\kappa(\rho_a + \phi\rho_f)),$$
$$K_2 = \phi^2\kappa\rho\rho_a + \kappa\phi^3\rho_f\rho_s(1-\phi) + \rho\phi^2\kappa(\rho_a + \phi\rho_f).$$

By using (3.7), the denominator can be estimated by

$$\left|\phi^2 + s\kappa(\rho_a + \phi\rho_f)\right|^2 \lesssim \frac{|s|^2}{\underline{\sigma}^2} \tag{3.33}$$

with the notation as introduced in Definition 3.4. Thus we have

$$|s|^2 \operatorname{Re}\left[s(\rho - \beta\rho_f)\right] \gtrsim \underline{\sigma}^2 \operatorname{Re}[s]\,|s|^2\,.$$

Furthermore we have

$$\frac{1}{\rho_f}\operatorname{Re}\left[\frac{\beta}{s}\right] = \frac{\kappa\phi^2(\phi^2 + \kappa^2 \operatorname{Re}[s]\,(\rho_a + \phi\rho_f))}{\left|\phi^2 + s\kappa(\rho_a + \phi\rho_f)\right|^2} \tag{3.34}$$

and therefore

$$\frac{1}{\rho_f}\operatorname{Re}\left[\frac{\beta}{s}\right] \gtrsim \frac{\underline{\sigma}^2 \operatorname{Re}[s]}{|s|^2}.$$

Finally we end up with the estimate

$$\begin{aligned}
&a_\Omega((u,p);(su,p)) \\
&\quad\gtrsim \operatorname{Re}[s]\,a^E(u,u) + \underline{\sigma}^2 \operatorname{Re}[s]\,|s|^2\,\|u\|_{0,\Omega}^2 + \frac{\operatorname{Re}[s]\,\underline{\sigma}^2}{|s|^2}\,\|\operatorname{grad} p\|_{0,\Omega}^2 + \operatorname{Re}[s]\,\|p\|_{0,\Omega}^2\,.
\end{aligned}$$

Korn's second inequality, or more precisely estimate (3.17) yields the desired result. These estimates complete the proof for the case $\operatorname{Im}[\beta] = 0$.

For $\operatorname{Im}[\beta] \neq 0$ we can estimate the mixed part further by

$$\operatorname{Re}\left[(\overline{\beta} - \beta)s\,\langle u, \nabla p\rangle_\Omega\right] \geq -2\,|\operatorname{Im}[\beta]|\,|s|\,\|\nabla p\|_{0,\Omega}\,\|u\|_{0,\Omega}$$

and we end up with

$$\begin{aligned}
\operatorname{Re}[a_\Omega((u,p);(su,p))] &\geq \operatorname{Re}[s]\,a^E(u,u) + |s|^2 \operatorname{Re}\left[s(\rho - \beta\rho_f)\right]\|u\|_{0,\Omega}^2 + \frac{\phi^2}{R}\operatorname{Re}[s]\,\|p\|_{0,\Omega}^2 \\
&\quad - 2\,|\operatorname{Im}[\beta]|\,|s|\,\|\nabla p\|_{0,\Omega}\,\|u\|_{0,\Omega} + \frac{1}{\rho_f}\operatorname{Re}\left[\frac{\beta}{s}\right]\|\nabla p\|_{0,\Omega}^2 \\
&= \operatorname{Re}[s]\,a^E(u,u) + \left(|s|^2 \operatorname{Re}\left[s(\rho - \beta\rho_f)\right] - \frac{1}{\varepsilon^2}\,|\operatorname{Im}[\beta]|\,|s|\right)\|u\|_{0,\Omega}^2 \\
&\quad + \left(\varepsilon\sqrt{|\operatorname{Im}[\beta]|\,|s|}\,\|\nabla p\|_{0,\Omega}^2 - \frac{1}{\varepsilon}\sqrt{|\operatorname{Im}[\beta]|\,|s|}\,\|u\|_{0,\Omega}^2\right)^2 \\
&\quad + \left(\frac{1}{\rho_f}\operatorname{Re}\left[\frac{\beta}{s}\right] - \varepsilon^2\,|\operatorname{Im}[\beta]|\,|s|\right)\|\nabla p\|_{0,\Omega}^2 + \frac{\phi^2}{R}\operatorname{Re}[s]\,\|p\|_{0,\Omega}^2 \\
&\geq \operatorname{Re}[s]\,a^E(u,u) + \left(|s|^2 \operatorname{Re}\left[s(\rho - \beta\rho_f)\right] - \frac{1}{\varepsilon^2}\,|\operatorname{Im}[\beta]|\,|s|\right)\|u\|_{0,\Omega}^2 \\
&\quad + \left(\frac{1}{\rho_f}\operatorname{Re}\left[\frac{\beta}{s}\right] - \varepsilon^2\,|\operatorname{Im}[\beta]|\,|s|\right)\|\nabla p\|_{0,\Omega}^2 + \frac{\phi^2}{R}\operatorname{Re}[s]\,\|p\|_{0,\Omega}^2
\end{aligned}$$

for all $\varepsilon > 0$. Due to $\mathrm{Re}[s] > 0$ it is sufficient to ensure

$$|s|^2 \,\mathrm{Re}\left[s(\rho - \beta\rho_f)\right] - \frac{1}{\varepsilon^2}\left|\mathrm{Im}[\beta]\right||s| > 0, \tag{3.35}$$

$$\frac{1}{\rho_f}\,\mathrm{Re}\left[\frac{\beta}{s}\right] - \varepsilon^2\left|\mathrm{Im}[\beta]\right||s| > 0 \tag{3.36}$$

for an appropriately chosen ε. In particular, ε^2 needs to satisfy the inclusion

$$\frac{\left|\mathrm{Im}[\beta]\right||s|}{|s|^2\,\mathrm{Re}\left[s(\rho - \beta\rho_f)\right]} < \varepsilon^2 < \frac{1}{\rho_f}\frac{\mathrm{Re}\left[\frac{\beta}{s}\right]}{\left|\mathrm{Im}[\beta]\right||s|}. \tag{3.37}$$

Hence we have to ensure

$$\left|\mathrm{Im}[\beta]\right|^2 < \frac{1}{\rho_f}\,\mathrm{Re}\left[\frac{\beta}{s}\right]\mathrm{Re}\left[s(\rho - \beta\rho_f)\right].$$

Indeed, by using

$$\left|\mathrm{Im}[\beta]\right|^2 = \frac{\phi^8\kappa^2\rho_f^2\left|\mathrm{Im}[s]\right|^2}{|\phi^2 + s\kappa(\rho_a + \phi\rho_f)|^4}$$

and (3.32), (3.34) we obtain

$$\begin{aligned}
&\frac{\rho_f\left|\mathrm{Im}[\beta]\right|^2}{\mathrm{Re}\left[s(\rho - \beta\rho_f)\right]\mathrm{Re}\left[\frac{\beta}{s}\right]} \\
&= \frac{\phi^8\kappa^2\rho_f^2\left|\mathrm{Im}[s]\right|^2}{[\rho\phi^4\,\mathrm{Re}[s] + K_1|s|^2\,\mathrm{Re}[s] + K_2\,\mathrm{Re}[s]^2 + \phi^4\rho_f^2\kappa\,\mathrm{Im}[s]^2]\kappa\phi^2(\phi^2 + \kappa^2\,\mathrm{Re}[s](\rho_a + \phi\rho_f))} \\
&< \frac{\phi^8\kappa^2\rho_f^2\left|\mathrm{Im}[s]\right|^2}{(K_1\,\mathrm{Re}[s] + \phi^8\rho_f^2\kappa^2)\left|\mathrm{Im}[s]\right|^2 + \phi^6\rho_f^2\kappa^4\left|\mathrm{Im}[s]\right|^2\mathrm{Re}[s](\rho_a + \phi\rho_f)} \\
&< \frac{\phi^2}{\phi^2 + \kappa^2\,\mathrm{Re}[s](\rho_a + \phi\rho_f)} < 1.
\end{aligned} \tag{3.38}$$

Thus we can chose

$$\varepsilon^2 = \frac{1}{2}\left(\frac{\left|\mathrm{Im}[\beta]\,|s|\right|}{|s|^2\,\mathrm{Re}\left[s(\rho - \beta\rho_f)\right]} + \frac{1}{\rho_f}\frac{\mathrm{Re}\left[\frac{\beta}{s}\right]}{\left|\mathrm{Im}[s]\right||s|}\right),$$

which obviously fulfills the inclusion (3.37). By using the estimate (3.38),

$$\frac{\left|\mathrm{Im}[\beta]\right|^2|s|^2}{|s|^2\,\mathrm{Re}\left[s(\rho - \beta\rho_f)\right]} < \frac{\phi^2}{\phi^2 + \kappa^2\,\mathrm{Re}[s](\rho_a + \phi\rho_f)}\frac{1}{\rho_f}\,\mathrm{Re}\left[\frac{\beta}{s}\right],$$

we end up with an estimate for the term (3.36)

$$\frac{1}{\rho_f}\operatorname{Re}\left[\frac{\beta}{s}\right]-\varepsilon^2\left|\operatorname{Im}[\beta]\right||s|\geq\frac{1}{2}\frac{1}{\rho_f}\operatorname{Re}\left[\frac{\beta}{s}\right]-\frac{1}{2}\frac{1}{\rho_f}\frac{\phi^2}{\phi^2+\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f)}\operatorname{Re}\left[\frac{\beta}{s}\right].$$

This term can be simplified to

$$\begin{aligned}\frac{1}{2}\left(1-\frac{\phi^2}{\phi^2+\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f)}\right)\frac{1}{\rho_f}\operatorname{Re}\left[\frac{\beta}{s}\right]&\\ =\frac{1}{2}\frac{\kappa^2\operatorname{Re}[s]\,(\rho_a-\phi\rho_f)}{\phi^2+\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f)}&\frac{\kappa\phi^2(\phi^2+\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f))}{\left|\phi^2+s\kappa(\rho_a+\phi\rho_f)\right|^2}\\ =\frac{1}{2}\frac{\kappa\phi^2(\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f))}{\left|\phi^2+s\kappa(\rho_a+\phi\rho_f)\right|^2}&.\end{aligned}$$

We have $\kappa^3\phi(\rho_a+\phi\rho_f)>0$ and together with the estimate (3.33) this results in

$$\frac{1}{2}\frac{\kappa\phi^2(\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f))}{\left|\phi^2+s\kappa(\rho_a+\phi\rho_f)\right|^2}\gtrsim\frac{\operatorname{Re}[s]\,\underline{\sigma}^2}{|s|^2}.$$

Again, using (3.38), i. e.

$$\rho_f\frac{\left|\operatorname{Im}[\beta]\right|^2|s|^2}{\operatorname{Re}\left[\frac{\beta}{s}\right]}>\frac{\phi^2}{\phi^2+\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f)}|s|^2\operatorname{Re}\left[s(\rho-\beta\rho_f)\right]$$

yields an estimate for the term (3.35)

$$\begin{aligned}|s|^2\operatorname{Re}\left[s(\rho-\beta\rho_f)\right]-\frac{1}{\varepsilon^2}\left|\operatorname{Im}[\beta]\right||s|&\\ \geq\frac{1}{2}\left(1-\frac{\phi^2}{\phi^2+\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f)}\right)&|s|^2\operatorname{Re}\left[s(\rho-\beta\rho_f)\right].\end{aligned}$$

The right hand side term is given as

$$\begin{aligned}&|s|^2\operatorname{Re}\left[s(\rho-\beta\rho_f)\right]\left(1-\frac{\phi^2}{\phi^2+\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f)}\right)\\ &=\frac{|s|^2\left(\rho\phi^4\operatorname{Re}[s]+K_1|s|^2\operatorname{Re}[s]+K_2\operatorname{Re}[s]^2+\phi^4\rho_f^2\kappa\operatorname{Im}[s]^2\right)}{\left|\phi^2+s\kappa(\rho_a+\phi\rho_f)\right|^2}\frac{\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f)}{\phi^2+\kappa^2\operatorname{Re}[s]\,(\rho_a+\phi\rho_f)}\end{aligned}$$

with

$$\begin{aligned}K_1&=\kappa(\rho\rho_a+\phi\rho_f\rho_s(1-\phi)\kappa(\rho_a+\phi\rho_f)),\\ K_2&=\phi^2\kappa\rho\rho_a+\kappa\phi^3\rho_f\rho_s(1-\phi)+\rho\phi^2\kappa(\rho_a+\phi\rho_f).\end{aligned}$$

Estimate (3.33) and the estimate $\rho\phi^4 \operatorname{Re}[s] + K_2 \operatorname{Re}[s]^2 + \phi^4 \rho_f^2 \kappa \operatorname{Im}[s]^2 > 0$ yield

$$|s|^2 \operatorname{Re}\left[s(\rho - \beta\rho_f)\right] \left(1 - \frac{\phi^2}{\phi^2 + \kappa^2 \operatorname{Re}[s]\,(\rho_a + \phi\rho_f)}\right) \gtrsim \frac{K_1\,|s|^2 \operatorname{Re}[s]^2\, \kappa^2(\rho_a + \phi\rho_f)\underline{\sigma}^2}{(\phi^2 + \kappa^2 \operatorname{Re}[s]\,(\rho_a + \phi\rho_f))}.$$

Moreover we have $K_1 \kappa^2(\rho_a + \phi\rho_f) > 0$ and

$$\phi^2 + \kappa^2 \operatorname{Re}[s]\,(\rho_a + \phi\rho_f) \leq \max(1, \operatorname{Re}[s])(\phi^2 + \kappa^2(\rho_a + \phi\rho_f) \lesssim \frac{\operatorname{Re}[s]}{\underline{\sigma}}.$$

Combining these estimates with estimate (3.33) yields

$$\frac{K_1\,|s|^2 \operatorname{Re}[s]^2\, \kappa^2(\rho_a + \phi\rho_f)\underline{\sigma}^2}{(\phi^2 + \kappa^2 \operatorname{Re}[s]\,(\rho_a + \phi\rho_f))} \gtrsim \operatorname{Re}[s]\,|s|^2\,\underline{\sigma}^3.$$

Hence we end up with the estimate

$$\begin{aligned} &a_\Omega((u,p);(su,p)) \\ &\qquad \gtrsim \operatorname{Re}[s]\, a^E(u,u) + \underline{\sigma}^3 \operatorname{Re}[s]\,|s|^2\, \|u\|_{0,\Omega}^2 + \frac{\operatorname{Re}[s]\,\underline{\sigma}^2}{|s|^2}\, \|\operatorname{grad} p\|_{0,\Omega}^2 + \operatorname{Re}[s]\,\|p\|_{0,\Omega}^2. \end{aligned}$$

Again, Korn's second inequality, or more precisely estimate (3.17), yields the desired result. □

Corollary 3.10. *The sesquilinear form* (3.21) *is bounded, i. e.*

$$a_\Omega(U,V) \lesssim \frac{1}{\underline{\sigma}}\, |||V|||_{|s|,\Omega}\, |||U|||_{\widetilde{|s|},\Omega}$$

for all $U \in [H^1(\Omega)]^4$ *and* $V \in [H^1(\Omega)]^4$.

Proof. The sesquilinear form is given as in (3.21),

$$\begin{aligned} a_\Omega((u,p);(v,q)) = a^E(u,v) &+ (\rho - \beta\rho_f)s^2 \langle u,v\rangle_\Omega - \alpha\,\langle p, \operatorname{div} v\rangle_\Omega - \beta\,\langle \nabla p, v\rangle_\Omega \\ &+ \alpha s\,\langle \operatorname{div} u, q\rangle_\Omega + \beta s\,\langle u, \nabla q\rangle_\Omega + \frac{\beta}{\rho_f s}\,\langle \nabla p, \nabla q\rangle_\Omega + \frac{\phi^2 s}{R}\,\langle p,q\rangle_\Omega. \end{aligned}$$

All constants in the sesquilinear form have to be estimated. We have

$$|\rho - \beta\rho_f| = \left|\frac{\phi^2 + s\kappa[(1-\phi)\rho_s\rho_f\phi + (1-\phi)\rho_s\rho_a + \phi\rho_f\rho_a]}{\phi^2 + s\kappa(\rho_a + \phi\rho_f)}\right|$$

and with the help of estimate (3.7) we conclude

$$\left|\rho - \beta\rho_f\right| \lesssim \frac{1}{\underline{\sigma}}.$$

Moreover we have

$$\beta = \frac{\rho_f s \phi^2 \kappa}{s\kappa(\rho_f\phi + \rho_a) + \phi^2}$$

and therefore

$$\left|\frac{\beta}{s\rho_f}\right| \lesssim \frac{1}{|s|}.$$

Combining all estimates results in

$$\begin{aligned}
|a_\Omega((u,p);(v,q))| &\lesssim \|\nabla v\|_{0,\Omega}\|\nabla u\|_{0,\Omega} + \frac{1}{\underline{\sigma}}\|sv\|_{0,\Omega}\|su\|_{0,\Omega} + \|\nabla v\|_{0,\Omega}\|p\|_{0,\Omega} \\
&\quad + \|sv\|_{0,\Omega}\left\|\frac{1}{s}\nabla p\right\|_{0,\Omega} + \|sq\|_{0,\Omega}\|\nabla u\|_{0,\Omega} + \|\nabla q\|_{0,\Omega}\|su\|_{0,\Omega} \\
&\quad + \|\nabla q\|_{0,\Omega}\left\|\frac{1}{s}\nabla p\right\|_{0,\Omega} + \|sq\|_{0,\Omega}\|p\|_{0,\Omega} \\
&\lesssim \frac{1}{\underline{\sigma}} |||\,(v,q)\,|||_{|s|,\Omega}\left(|||u\,|||_{|s|,\Omega} + |||\,\frac{1}{s}p|||_{|s|,\Omega}\right) \\
&\lesssim \frac{1}{\underline{\sigma}} |||\,(v,q)\,|||_{|s|,\Omega}\,|||(u,p)\,|||_{\widetilde{|s|},\Omega}\,.
\end{aligned}$$

□

A useful estimate as stated in [3] is given by the following lemma.

Lemma 3.11. *For any function $\phi \in H^{1/2}(\Gamma)$ and $s \in \mathbb{C}^+_\sigma$ there exists an extension $u \in H^1(\Omega)$ such that*

$$\begin{aligned}
-\Delta u + su &= 0 \quad \text{in } \Omega, \\
u &= \phi \quad \text{on } \Gamma
\end{aligned}$$

and

$$|||u|||_{|s|,\Omega} \lesssim \max(1,|s|)^{1/2}\,\|\phi\|_{1/2,\Gamma}\,.$$

Let $\mathcal{E}$ denote the continuous right inverse of the trace γ_0 as stated in Theorem 3.4. For the extension $\mathcal{E}\phi$ of the boundary datum ϕ an estimate of the kind

$$|||u|||_{|s|,\Omega} \lesssim |s|\,\|\phi\|_{1/2,\Gamma}$$

is straight forward. However, Lemma 3.11 defines an extension which has an optimal bound with respect to s, see [31].

The ellipticity estimate (Theorem 3.9), the boundedness of the sesquilinear form (3.21) (Corollary 3.10) and the extension operator as defined in Lemma 3.11 give us a bound for the solution of the mixed boundary value problem (2.19).

Theorem 3.12. *The mixed boundary value problem*

$$\begin{aligned} \mathcal{P}U &= f && \text{in } \Omega,\\ \gamma_0 U &= g_D && \text{on } \Gamma_D,\\ \gamma_1 U &= g_N && \text{on } \Gamma_N \end{aligned}$$

has a unique solution $U \in [H^1(\Omega)]^4$ *satisfying*

$$|||U|||_{\widetilde{|s|},\Omega} \lesssim \frac{|s|}{\sigma \underline{\sigma}^6} \left(\|f\|_{-1,\Omega} + \|g_N\|_{-1/2,\Gamma_N} \right) + \frac{|s|^{3/2}}{\sigma \underline{\sigma}^{13/2}} \|g_D\|_{1/2,\Gamma_D} .$$

Proof. The respective variational formulation of the boundary value problem is given as:

Find $U \in [H^1(\Omega)]^4$ with $\gamma_0 U = g_D$ on Γ_D such that

$$a_\Omega(U,V) = \langle f,V\rangle_\Omega + \langle g_N, \gamma_0 V\rangle_{\Gamma_N} \tag{3.39}$$

for all $V \in [H^1_0(\Omega,\Gamma_D)]^4$.

First we extend the function $g_D \in [H^{1/2}(\Gamma_D)]^4$ to a function $\tilde{g}_D \in [H^{1/2}(\Gamma)]^4$ such that $\tilde{g}_D = g_D$ on Γ_D. Furthermore, the extension operator as defined in Lemma 3.11 is used to define the function $U_g \in [H^1(\Omega)]^4$ such that

$$U_g|_\Gamma = \tilde{g}_D$$

Next we split up the solution U into $U = U_0 + U_g$ with $U_0 \in [H^1_0(\Omega,\Gamma_D)]^4$ to be found. Insertion into (3.39) yields a variational problem:

Find $U_0 \in [H^1_0(\Omega,\Gamma_D)]^4$

$$a_\Omega(U_0,V) = \langle f,V\rangle_\Omega + \langle g_N, \gamma_0 V\rangle_\Gamma - a_\Omega(U_g,V)$$

for all V in $[H^1_0(\Omega,\Gamma_D)]^4$.

Utilizing Corollary 3.10 and Theorem 3.9 results in

$$\begin{aligned} \underline{\sigma}^5 \sigma\, |||U_0|||^2_{\widetilde{|s|},\Omega} &\lesssim \operatorname{Re}\left[a_\Omega(U_0,\Theta_s U_0)\right] \\ &\lesssim |\langle f,U_0\rangle_\Omega| + \left|\langle g_N, \Theta_s \gamma_0 U_0\rangle_{\Gamma_N}\right| + \left|a_\Omega(U_g,\Theta_s U_0)\right| \\ &\lesssim \left(\|f\|_{-1,\Omega} + \|g_N\|_{-1/2,\Gamma_N} \right) \|\Theta_s U_0\|_{1,\Omega} + \frac{|s|}{\underline{\sigma}} |||U_0|||_{\widetilde{|s|},\Omega} |||U_g|||_{\widetilde{|s|},\Omega} . \end{aligned}$$

Estimate (3.12) can be applied to $\|\Theta_s U_0\|_{1,\Omega}$ resulting in

$$|||U_0|||_{\widetilde{|s|},\Omega} \lesssim \frac{|s|}{\underline{\sigma}^6\sigma}\left(\|f\|_{-1,\Omega}+\|g_N\|_{-1/2,\Gamma_N}+|||U_g|||_{\widetilde{|s|},\Omega}\right).$$

Applying Lemma 3.11 to estimate the norm of the function U_g results in the given statement. □

It is well known that the conormal derivative, as defined in (3.22), of a solution of a homoegeneous boundary value problem is bounded. An estimate is given in the following lemma.

Lemma 3.13. *Let $U \in [H^1(\Omega)]^4$ such that $\mathcal{P}U = 0$. Then*

$$\|\gamma_1 U\|_{-1/2,\Omega} \lesssim \frac{|s|^{1/2}}{\underline{\sigma}^{3/2}}\,|||\,U\,|||_{\widetilde{|s|},\Omega}.$$

Proof. Applying Green's first formula and Corollary 3.10 results in

$$\begin{aligned}\|\gamma_1 U\|_{-1/2,\Gamma} &= \sup_{0\neq\phi\in[H^{1/2}(\Gamma)]^4} \frac{|\langle\gamma_1 U,\phi\rangle_\Gamma|}{\|\phi\|_{1/2,\Gamma}}\\ &= \sup_{0\neq\phi\in[H^{1/2}(\Gamma)]^4} \frac{|a_\Omega(U,\mathcal{E}\phi)|}{\|\phi\|_{1/2,\Gamma}}\\ &\lesssim \sup_{0\neq\phi\in[H^{1/2}(\Gamma)]^4} \frac{\frac{1}{\underline{\sigma}}\,|||\,U\,|||_{\widetilde{|s|},\Omega}\,|||\mathcal{E}\phi|||_{|s|,\Omega}}{\|\phi\|_{1/2,\Gamma}},\end{aligned}$$

where $\mathcal{E}$ is an extension of ϕ into $[H^1(\Omega)]^4$, as described in Lemma 3.11 componentwise. Thus we have

$$\|\gamma_1 U\|_{-1/2,\Gamma} \lesssim \frac{\max(1,|s|)^{1/2}}{\underline{\sigma}}\,|||\,U\,|||_{\widetilde{|s|},\Omega},$$

which together with the estimate (3.7) concludes the proof. □

The estimate as given in Lemma 3.13 can be extended to the jump of the conormal derivative.

Corollary 3.14. *Let $U \in [H^1(\Omega^- \cup \Omega^+)]^4$ such that $\mathcal{P}U = 0$ in $\Omega^- \cup \Omega^+$. Then*

$$\|[\gamma_1 U]_\Gamma\|_{-1/2,\Gamma} \lesssim \frac{|s|^{1/2}}{\underline{\sigma}^{3/2}}\,|||\,U\,|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma}.$$

The adjoint conormal derivative as given in (3.24) fulfills a similar bound.

Lemma 3.15. *Given* $V \in [H^1(\Omega^- \cup \Omega^+)]^4$ *such that* $\widetilde{\mathcal{P}}V = 0$ *in* $\Omega^- \cup \Omega^+$. *Then*

$$\|[\widetilde{\gamma}_1 V]_\Gamma\|_{-1/2,\Omega} \lesssim \frac{|s|^{1/2}}{\underline{\sigma}^{5/2}} \, ||| V |||_{|s|,\Omega} \, .$$

Proof. Applying Green's first formula for the adjoint problem and Corollary 3.10 results in

$$\begin{aligned}
\|\widetilde{\gamma}_1 V\|_{-1/2,\Gamma} &= \sup_{0 \neq \phi \in [H^{1/2}(\Gamma)]^4} \frac{|\langle \widetilde{\gamma}_1 V, \phi \rangle_\Gamma|}{\|\phi\|_{1/2,\Gamma}} \\
&= \sup_{0 \neq \phi \in [H^{1/2}(\Gamma)]^4} \frac{|a_\Omega(\mathcal{E}\phi, V)|}{\|\phi\|_{1/2,\Gamma}} \\
&\lesssim \sup_{0 \neq \phi \in [H^{1/2}(\Gamma)]^4} \frac{\frac{1}{\underline{\sigma}} \, ||| \mathcal{E}\phi |||_{\widetilde{|s|},\Omega} \, |||V|||_{|s|,\Omega}}{\|\phi\|_{1/2,\Gamma}}.
\end{aligned}$$

The estimate

$$|||\mathcal{E}\phi|||_{\widetilde{|s|},\Omega} \leq \frac{1}{\underline{\sigma}} \, ||| \mathcal{E}\phi |||_{|s|,\Omega}$$

and Lemma 3.11 conclude the proof. □

4 SURFACE POTENTIALS AND BOUNDARY INTEGRAL OPERATORS

To describe solutions of the partial differential equations (2.14) and (2.15) we use appropriate surface potentials which are based on the use of a related fundamental solution. We proceed as in [38], see also [26, 52].

4.1 Fundamental solution

A fundamental solution of the partial differential operator $\mathcal{P}$ as defined in (2.16) is given by, see, e.g., [46],

$$G_s(x,y) = \begin{bmatrix} U_{ij}^E(x,y) & P_i(x,y) \\ U_j(x,y) & P^p(x,y) \end{bmatrix} \in \mathbb{C}^{4\times 4}, \tag{4.1}$$

where

$$U_{ij}^E(x,y) = \frac{1}{4\pi r(\rho-\beta\rho_f)s^2}\left[R_1\frac{\alpha_4^2-\alpha_2^2}{\alpha_1^2-\alpha_2^2}e^{-\alpha_1 r} - R_2\frac{\alpha_4^2-\alpha_1^2}{\alpha_1^2-\alpha_2^2}e^{-\alpha_2 r} + (\delta_{ij}\alpha_3^2 - R_3)e^{-\alpha_3 r}\right]$$

for $i,j=1,2,3$ with

$$\begin{aligned}\alpha_{1,2}^2 = \frac{1}{2}\Bigg[&\frac{\phi^2 s^2\rho_f}{\beta R} + \frac{s^2(\rho-\beta\rho_f)}{\lambda+2\mu} + \frac{s^2\rho_f(\alpha-\beta)^2}{\beta(\lambda+2\mu)} \\ &\pm\sqrt{\left(\frac{\phi^2 s^2\rho_f}{\beta R} + \frac{s^2(\rho-\beta\rho_f)}{\lambda+2\mu} + \frac{s^2\rho_f(\alpha-\beta)^2}{\beta(\lambda+2\mu)}\right)^2 - 4\frac{s^4\phi^2\rho_f(\rho-\beta\rho_f)}{\beta R(\lambda+2\mu)}}\Bigg]\end{aligned} \tag{4.2}$$

and

$$\alpha_3^2 = \frac{s^2(\rho-\beta\rho_f)}{\mu}, \quad \alpha_4^2 = \frac{s^2(\rho-\beta\rho_f)}{\lambda+2\mu},$$

as well as

$$R_k = \frac{3r_{,i}r_{,j}-\delta_{ij}}{r^2} + \alpha_k\frac{3r_{,i}r_{,j}-\delta_{ij}}{r} + \alpha_k^2 r_{,i}r_{,j}, \quad r = |x-y|.$$

Moreover, there holds

$$\begin{aligned}
P_j(x,y) &= \frac{s(\alpha-\beta)\rho_f r_{,j}}{4\pi r\beta(\lambda+2\mu)(\alpha_1^2-\alpha_2^2)}\left[\left(\alpha_1+\frac{1}{r}\right)e^{-\alpha_1 r}-\left(\alpha_2+\frac{1}{r}\right)e^{-\alpha_2 r}\right],\\
U_j(x,y) &= sP_j(x,y),\\
P^p(x,y) &= \frac{s\rho_f}{4\pi r\beta(\alpha_1^2-\alpha_2^2)}\left[(\alpha_1^2-\alpha_4^2)e^{-\alpha_1 r}-(\alpha_2^2-\alpha_4^2)e^{-\alpha_2 r}\right]
\end{aligned}$$

for $j=1,\dots,3$. The parameters represent the three waves occurring in poroelasticity. $\alpha_{1,2}$ represent the fast and slow compressional waves and α_3 the shear wave. If the property

$$\operatorname{Re}[\alpha_i]>0 \quad \text{for } i=1,\dots,3. \tag{4.3}$$

is fulfilled, the fundamental solution $G_s(x,y)$ decays exponentially as $r=|x-y|\to\infty$.

For α_3 this property follows from an appropriate choice of all parameters involved.

Lemma 4.1. *Let $s\in\mathbb{C}_\sigma^+$ and $\phi\in(0,1)$, $\rho>0$, $\rho_f>0$, $\rho_a>0$, $\rho_s>0$ and $\kappa>0$, then*

$$\operatorname{Re}[\alpha_3]>0$$

and $\alpha_3(s)$ is an analytic function of s.

Proof. Since α_3 is defined as a square root of a complex value, we simply take the square root with the real part greater or equal to zero. This approach fails if the real part is equal to zero. However if $\operatorname{Im}\left[\alpha_3^2\right]\neq 0$ we automatically get $\operatorname{Re}[\alpha_3]\neq 0$ and thus $\operatorname{Re}[\alpha_3]>0$. Remember

$$\alpha_3^2=\frac{s^2(\rho-\beta\rho_f)}{\mu}.$$

We have

$$\rho-\beta\rho_F=\frac{c_1+sc_2}{c_3+sc_4}=\frac{c_1c_3+sc_2c_3+\bar{s}c_4c_1+|s|^2c_2c_4}{|c_3+sc_4|^2} \tag{4.4}$$

with

$$\begin{aligned}
c_1&=\rho\phi^2, & c_2&=\kappa\rho\rho_a+\kappa\phi\rho_F\rho_S(1-\phi),\\
c_3&=\phi^2, & c_4&=\kappa(\rho_a+\phi\rho_F).
\end{aligned}$$

Setting $s=a+bi$ we have $\operatorname{Im}[s]=b$, $\operatorname{Im}\left[s^2\right]=2ab$ and $\operatorname{Im}\left[s^3\right]=b(3a^2-b^2)$. Inserting these definitions results in

$$\begin{aligned}
&\operatorname{Im}\left[c_1c_3s^2+s^3c_2c_3+|s|^2sc_4c_1+|s|^2s^2c_2c_4\right]\\
&\qquad=b\left[a^32c_4c_2+b^2a2c_2c_4+a^2(3c_2c_3+c_4c_1)+b^2(c_4c_1-c_2c_3)+a2c_1c_3\right]\\
&\qquad=b\left[b^2(c_4c_1-c_2c_3+a2c_2c_4)+a^32c_4c_4a^2(3c_2c_3+c_4c_1)+a2c_1c_3\right]
\end{aligned}$$

for the enumerator of (4.4). We can further estimate

$$\begin{aligned} c_4c_1 - c_2c_3 &= -(\kappa\rho\rho_a + \kappa\phi\rho_F\rho_S(1-\phi))\phi^2 + \rho\phi^2\kappa(\rho_a + \phi\rho_F) \\ &= -\phi^2\kappa\rho\rho_a - \phi^3\kappa\rho_F\rho_S(1-\phi) + \rho\phi^2\kappa\rho_a + \phi^3\rho_f\kappa(\rho_S(1-\phi) + \rho_F\phi) \\ &= \phi^4\rho_F^2\kappa \end{aligned}$$

and since $c_i > 0$ for $i = 1,\dots,4$ and $a = \mathrm{Re}[s] > 0$ we conclude that the imaginary part of $s^2(\rho - \beta\rho_f)$ can only be zero if the imaginary part of s is zero. If $\mathrm{Im}[s] = 0$, the expression α_3^2 is strictly positive since $\rho - \beta\rho_f$ is strictly positive. Therefore $\mathrm{Re}[\alpha_3]$ can always be chosen strictly positive. □

For the other two parameters we have to postulate the property (4.3). Additionally $\alpha_1 \neq \alpha_2$ has to be satisfied.

Assumption 4.1. *We assume* $\mathrm{Re}[\alpha_1] > 0$ *and* $\mathrm{Re}[\alpha_2] > 0$. *Furthermore we assume*

$$\mathrm{Re}\left[\left(\frac{\phi^2 s^2\rho_f}{\beta R} + \frac{s^2(\rho-\beta\rho_f)}{\lambda+2\mu} + \frac{s^2\rho_f(\alpha-\beta)^2}{\beta(\lambda+2\mu)}\right)^2 - 4\frac{s^4\phi^2\rho_f(\rho-\beta\rho_f)}{\beta R(\lambda+2\mu)}\right] > 0 \qquad (4.5)$$

if

$$\mathrm{Im}\left[\left(\frac{\phi^2 s^2\rho_f}{\beta R} + \frac{s^2(\rho-\beta\rho_f)}{\lambda+2\mu} + \frac{s^2\rho_f(\alpha-\beta)^2}{\beta(\lambda+2\mu)}\right)^2 - 4\frac{s^4\phi^2\rho_f(\rho-\beta\rho_f)}{\beta R(\lambda+2\mu)}\right] = 0$$

Remark 4.1. *Let* $s \in \mathbb{C}_\sigma^+$ *and* $\phi \in (0,1)$, $\rho > 0$, $\rho_f > 0$, $\rho_a > 0$, $\rho_s > 0$ *and* $\kappa > 0$, *then the fundamental solution as defined in* (4.1) *is an analytic function with respect to s.*

Proof. The square root function $\sqrt{s}$ is analytic for $s \in \mathbb{C} \setminus \{s \in \mathbb{R} | s \leq 0\}$. Since $\beta \neq 0$ for $s \in \mathbb{C}_\sigma^+$, assumption (4.5) guarantees that α_1 and α_2 are analytic. With the help of the same argument the proof of Lemma 4.1 guarantees that α_3 and α_4 are analytic.

Furthermore $s^2(\rho - \beta\rho_f) \neq 0$ for $s \in \mathbb{C}_\sigma^+$ and due to assumption (4.5) $\alpha_1^2 \neq \alpha_1^2$ and therefore the fundamental solution itself is analytic with respect to s. □

The singular behaviour of the fundamental solution as given in (4.1) is well known. We have

$$U_{ij}^E(x,y) = \frac{1}{16\pi\mu(1-\nu)}\left\{r_{,i}r_{,j} + (3-5\nu)\delta_{ij}\right\}\frac{1}{r} + \mathcal{O}(1), \quad U_i(x,y) = \mathcal{O}(1),$$

$$P^p(x,y) = \frac{\rho_f s}{4\pi\beta}\frac{1}{r} + \mathcal{O}(1), \quad P_i(x,y) = \mathcal{O}(1).$$

It turns out that the singularity of the block $U_{ij}^{E}(x,y)$ is the singularity of the fundamental solution of linear elastostatics, whereas $P^p(x,y)$ has the same singularity as the fundamental solution of the Laplace operator. In the remainder of the fundamental solution (4.1) no further singularities appear.

We define the operator, see (3.8)

$$\Lambda = \Theta_{\sqrt{-s},\frac{1}{\sqrt{-s}}}.$$

Remark 4.2. *U_{ij}^{E} and P^p are symmetric with respect to x and y, thus $U^E(x,y) = U^E(y,x)$ and $P^p(x,y) = P^p(y,x)$, whereas P_i and U_j are skew symmetric and thus $P_i(x,y) = -P_i(y,x)$ and $U_j(x,y) = -U_j(y,x)$. Finally, the transposed of the fundamental solution can be expressed as*

$$G_s(y,x)^\top = \Lambda G_s(x,y)\Lambda^{-1}.$$

Remark 4.3. *By using the operator Λ one can rewrite the conormal derivative of the adjoint problem as*

$$\overline{\widetilde{\gamma}_1} = \Lambda\gamma_1\Lambda^{-1}.$$

4.2 Boundary integral operators

By using the fundamental solution $G_s(x,y)$ we introduce the Newton potential

$$(N(s)f)(x) := \int_\Omega G_s(x,y) f(y)\, \mathrm{d}y \quad \text{for } x \in \mathbb{R}^3.$$

Since the underlying partial differential operator $\mathcal{P}$ as given in (2.16) is a strongly elliptic operator with constant coefficients, see (3.7), we conclude, see, e.g. [26, 38],

$$N(s) : [H^{r-1}(\mathbb{R}^3)]^4 \to [H^{r+1}(\mathbb{R}^3)]^4 \quad \text{for all } s \in \mathbb{C}, r \in \mathbb{R}.$$

In addition to the Newton potential $N(s)$ we introduce the single and double layer potentials

$$\mathrm{SL}(\mathrm{s})[\psi](x) := \int_\Gamma G_s(x,y)\psi(y)\, \mathrm{d}s_y,$$
$$\mathrm{DL}(\mathrm{s})[\phi](x) := \int_\Gamma [\widetilde{\gamma}_1 G_s^*(x,y)]^*\, \phi(y)\, \mathrm{d}s_y$$

for $x \in \mathbb{R}^3 \setminus \Gamma$, where $G_s^*(x,y)$ is the fundamental solution of the formally adjoint partial differential operator $\widetilde{\mathcal{P}}$, see (3.23). The surface potentials and related traces fulfill the mapping properties, see [38],

$$\begin{aligned} \mathrm{SL}(\mathrm{s}) &: [H^{-1/2}(\Gamma)]^4 \to [H^1(\Omega)]^4, & \mathrm{DL}(\mathrm{s}) &: [H^{1/2}(\Gamma)]^4 \to [H^1(\Omega)]^4, \\ \gamma_0\,\mathrm{SL}(\mathrm{s}) &: [H^{-1/2}(\Gamma)]^4 \to [H^{1/2}(\Gamma)]^4, & \gamma_0\,\mathrm{DL}(\mathrm{s}) &: [H^{1/2}(\Gamma)]^4 \to [H^{1/2}(\Gamma)]^4, \\ \gamma_1\,\mathrm{SL}(\mathrm{s}) &: [H^{-1/2}(\Gamma)]^4 \to [H^{-1/2}(\Gamma)]^4, & \gamma_1\,\mathrm{DL}(\mathrm{s}) &: [H^{1/2}(\Gamma)]^4 \to [H^{-1/2}(\Gamma)]^4 \end{aligned}$$

and satisfy the jump relations

$$[\mathrm{SL}(\mathrm{s})\,\psi]_{|\Gamma} = 0, \quad [\gamma_1\,\mathrm{SL}(\mathrm{s})\,\psi]_{|\Gamma} = -\psi, \quad [\mathrm{DL}(\mathrm{s})\,\phi]_{|\Gamma} = \phi, \quad [\gamma_1\,\mathrm{DL}(\mathrm{s})\,\phi]_{|\Gamma} = 0. \tag{4.6}$$

Since the partial differential operator $\mathcal{P}$ is not self–adjoint, the resulting boundary integral operators are not self–adjoint. For a complete overview on the properties and the different relations of the boundary integral operators and there adjoints in such a general situation, see, e.g., [38]. The boundary integral operators for the adjoint operator $\widetilde{\mathcal{P}}$ are defined by

$$\begin{aligned} \widetilde{\mathrm{SL}}(\mathrm{s})[\phi](x) &:= \int_\Gamma G_s^*(y,x)\phi(y)\,\mathrm{d}y_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Gamma, \\ \widetilde{\mathrm{DL}}(\mathrm{s})[\psi](x) &:= \int_\Gamma [\gamma_1 G_s(y,x)]^*\,\psi(y)\,\mathrm{d}y_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Gamma. \end{aligned}$$

The following duality relations are a direct consequence of the definition of the boundary integral operators.

Theorem 4.2. *For $\phi_1, \phi_2 \in [H^{-1/2}(\Gamma)]^4$ we have*

$$\langle \gamma_0\,\mathrm{SL}(\mathrm{s})\,\phi_1, \phi_2 \rangle_\Gamma = \left\langle \phi_1, \gamma_0\,\widetilde{\mathrm{SL}}(\mathrm{s})\,\phi_2 \right\rangle_\Gamma .$$

In addition, for $\psi_1, \psi_2 \in [H^{1/2}(\Gamma)]^4$ there holds

$$\begin{aligned} \left\langle \phi_1, \gamma_0^\pm\,\mathrm{DL}(\mathrm{s})\,\psi_1 \right\rangle_\Gamma &= \left\langle \widetilde{\gamma}_1^\pm\,\widetilde{\mathrm{SL}}(\mathrm{s})\,\phi_1, \psi_1 \right\rangle_\Gamma , \\ \left\langle \phi_1, \gamma_0^\pm\,\widetilde{\mathrm{DL}}(\mathrm{s})\,\psi_1 \right\rangle_\Gamma &= \left\langle \gamma_1^\pm\,\mathrm{SL}(\mathrm{s})\,\phi_1, \psi_1 \right\rangle_\Gamma , \\ \langle \psi_1, \gamma_1\,\mathrm{DL}(\mathrm{s})\,\psi_2 \rangle_\Gamma &= \left\langle \widetilde{\gamma}_1\,\widetilde{\mathrm{DL}}(\mathrm{s})\,\psi_1, \psi_2 \right\rangle_\Gamma . \end{aligned}$$

Next we introduce the standard boundary integral operators, in particular the single layer integral operator

$$V(s) := \gamma_0\,\mathrm{SL}(\mathrm{s}) : [H^{-1/2}(\Gamma)]^4 \to [H^{1/2}(\Gamma)]^4, \tag{4.7}$$

the single layer integral operator of the adjoint problem

$$\widetilde{V}(s) := \gamma_0 \widetilde{\mathrm{SL}}(\mathrm{s}) : [H^{-1/2}(\Gamma)]^4 \to [H^{1/2}(\Gamma)]^4,$$

the hyper-singular boundary integral operator

$$D(s) := -\gamma_1 \,\mathrm{DL}(\mathrm{s}) : [H^{1/2}(\Gamma)]^4 \to [H^{-1/2}(\Gamma)]^4, \tag{4.8}$$

the double layer integral operators

$$K(s) := \frac{1}{2}\left(\gamma_0^+ \,\mathrm{DL}(\mathrm{s}) + \gamma_0^- \,\mathrm{DL}(\mathrm{s})\right) : [H^{1/2}(\Gamma)]^4 \to [H^{1/2}(\Gamma)]^4, \tag{4.9}$$

$$\widetilde{K}(s) := \frac{1}{2}\left(\gamma_0^+ \,\widetilde{\mathrm{DL}}(\mathrm{s}) + \gamma_0^- \,\widetilde{\mathrm{DL}}(\mathrm{s})\right) : [H^{1/2}(\Gamma)]^4 \to [H^{1/2}(\Gamma)]^4, \tag{4.10}$$

and its adjoint

$$\widetilde{K}(s)^* := \frac{1}{2}\left(\gamma_1^+ \,\mathrm{SL}(\mathrm{s}) + \gamma_1^- \,\mathrm{SL}(\mathrm{s})\right) : [H^{-1/2}(\Gamma)]^4 \to [H^{-1/2}(\Gamma)]^4.$$

Furthermore we conclude the following expressions for the traces and conormal derivatives of the single and double layer potentials, i.e.,

$$\begin{aligned}
\gamma_0 \,\mathrm{SL}(\mathrm{s})\,\psi &= V(s)\psi, & \gamma_1^{\pm} \,\mathrm{SL}(\mathrm{s})\,\psi &= \mp\frac{1}{2}\psi + \widetilde{K}(s)^*\psi,\\
\gamma_0^{\pm} \,\mathrm{DL}(\mathrm{s})\,\phi &= \pm\frac{1}{2}\phi + K(s)\phi, & \gamma_1 \,\mathrm{DL}(\mathrm{s})\,\phi &= -D(s)\phi,\\
\gamma_0 \,\widetilde{\mathrm{SL}}(\mathrm{s})\,\psi &= V(s)^*\psi, & \widetilde{\gamma}_1^{\pm} \,\widetilde{\mathrm{SL}}(\mathrm{s})\,\psi &= \mp\frac{1}{2}\psi + \widetilde{K}(s)^*\psi,\\
\gamma_0^{\pm} \,\widetilde{\mathrm{DL}}(\mathrm{s})\,\phi &= \pm\frac{1}{2}\phi + \widetilde{K}(s)\phi, & \gamma_1 \,\widetilde{\mathrm{DL}}(\mathrm{s})\,\phi &= -D(s)^*\phi,
\end{aligned}$$

for $\psi \in [H^{-1/2}(\Gamma)]^4$ and $\phi \in [H^{1/2}(\Gamma)]^4$ almost everywhere.

Lemma 4.3. [38] For the boundary integral operators one has the following relations

$$\begin{aligned}
V(s)D(s) &= \frac{1}{4}I - K(s)^2, & V(s)\widetilde{K}(s)^* &= K(s)V(s),\\
D(s)K(s) &= \widetilde{K}(s)^*D(s), & D(s)V(s) &= \frac{1}{4}I - (\widetilde{K}(s)^*)^2.
\end{aligned}$$

Moreover, the traces of the Newton potential $N(s)$ imply the volume integral operators

$$\begin{aligned}
N(s)_0 &:= \gamma_0 N(s) : [H^{-1}(\Omega)]^4 \to [H^{1/2}(\Gamma)]^4,\\
N(s)_1 &:= \gamma_1 N(s) : [H^{-1}(\Omega)]^4 \to [H^{-1/2}(\Gamma)]^4.
\end{aligned}$$

4.3 On symmetry and ellipticity

Boundary integral operators related to partial differential equations with complex parameters are not self–adjoint, see for example the boundary integral operators for the Helmholtz equation in [31]. The single layer integral operator and the hyper-singular operator for the Helmholtz equation are however symmetric.

The original partial differential operator (2.16) for poroelasticity is not symmetric, we therefore cannot expect symmetry for those integral operators. On the other hand, the partial differential operator has a block skew-symmetric structure. This structure is preserved by the boundary integral operators.

Lemma 4.4. *For the boundary integral operators of poroelasticity as defined in* (4.7), (4.8) *and* (4.9) *there hold the following relations:*

$$
\begin{aligned}
V(s)^\top &= \Lambda V(s)\Lambda^{-1},\\
K(s)^\top &= \Lambda \widetilde{K}(s)^*\Lambda^{-1},\\
D(s)^\top &= \Lambda^{-1} D(s)\Lambda.
\end{aligned}
$$

Proof. The traces of the two single layer integral operator (4.7) are adjoint to each other, i. e.

$$\langle V(s)\phi,\psi\rangle_\Gamma = \left\langle \phi,\widetilde{V}(s)\psi\right\rangle_\Gamma \quad \text{for all } \phi,\psi\in[H^{-1/2}(\Gamma)]^4.$$

By using Remark 4.2, the single layer potential of the adjoint problem can be written as

$$\widetilde{\mathrm{SL}}(\mathrm{s}) = \overline{\Lambda\,\mathrm{SL}(\mathrm{s})\,\Lambda^{-1}}.$$

When we consider the Dirichlet traces the first relation follows immediately. For the double layer potential we apply Remark 4.2 and Remark 4.3, resulting in

$$
\begin{aligned}
\mathrm{DL}(\mathrm{s})\,\phi &= \int\limits_\Gamma \left[\widetilde{\gamma}_1 \overline{G_s}^\top(x,y)\right]^* \phi(y)\,\mathrm{d}s_y\\
&= \int\limits_\Gamma \left[\Lambda\gamma_1\Lambda^{-1}\Lambda G_s(y,x)\Lambda^{-1}\right]^\top \phi(y)\,\mathrm{d}s_y\\
&= \int\limits_\Gamma \left[\Lambda\gamma_1 G_s(y,x)\Lambda^{-1}\right]^\top \phi(y)\,\mathrm{d}s_y\\
&= \int\limits_\Gamma \Lambda^{-1}\left[\gamma_1 G_s(y,x)\right]^\top \Lambda\phi(y)\,\mathrm{d}s_y\\
&= \Lambda^{-1}\overline{\widetilde{\mathrm{DL}}(\mathrm{s})}\Lambda\phi
\end{aligned}
$$

and thus

$$\langle K(s)\phi,\psi\rangle_\Gamma = \left\langle \Lambda^{-1}\overline{\widetilde{K}(s)}\Lambda\phi,\psi\right\rangle_\Gamma = \left\langle \phi,\overline{\Lambda}\overline{\widetilde{K}(s)}^*\Lambda^{-1}\psi\right\rangle_\Gamma.$$

Furthermore we have

$$D(s) = \gamma_1\,\mathrm{DL}(s) = \gamma_1\Lambda^{-1}\overline{\widetilde{\mathrm{DL}}(s)}\Lambda = \Lambda^{-1}\overline{\widetilde{\gamma}_1\,\widetilde{\mathrm{DL}}(s)} = \Lambda^{-1}\overline{\widetilde{D}(s)}\Lambda$$

and due to $D(s)^* = \widetilde{D}(s)$ the last relation follows immediately. □

Lemma 4.4 can be used to write the single layer integral operator and the hyper-singular operator in the following form.

Corollary 4.5. *The single layer boundary integral operator can be written as*

$$V(s) = \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ -sV_{12}(s)^\top & V_{22}(s) \end{pmatrix}$$

with the symmetric operators

$$\begin{aligned} V_{11}(s) :&[H^{-1/2}(\Gamma)]^3 &\to\quad& [H^{1/2}(\Gamma)]^3, \\ V_{22}(s) :& H^{-1/2}(\Gamma) &\to\quad& H^{1/2}(\Gamma), \end{aligned}$$

and with the operator

$$V_{12}(s) : H^{-1/2}(\Gamma) \quad\to\quad [H^{1/2}(\Gamma)]^3.$$

Proof. If we split the single layer boundary integral operator into four operators

$$V(s) = \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{pmatrix}$$

and apply Lemma 4.4, the transposed of the operator is given as

$$\begin{aligned} \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{pmatrix}^\top &= \begin{pmatrix} \sqrt{-s} & \\ & \sqrt{-1/s} \end{pmatrix} \begin{pmatrix} V_{11}(s) & V_{12}(s) \\ V_{21}(s) & V_{22}(s) \end{pmatrix} \begin{pmatrix} \sqrt{-1/s} & \\ & \sqrt{-s} \end{pmatrix} \\ &= \begin{pmatrix} V_{11}(s) & -s\,V_{12}(s) \\ -1/s\,V_{21}(s) & V_{22}(s) \end{pmatrix}. \end{aligned}$$

We end up with the relations $V_{11}(s) = V_{11}(s)^\top$, $V_{22}(s) = V_{22}(s)^\top$ and $V_{12}(s) = -1/sV_{21}(s)^\top$ as stated. □

Corollary 4.6. *The hyper-singular operator can be written as*

$$D(s) = \begin{pmatrix} D_{11}(s) & D_{12}(s) \\ -\frac{1}{s} D_{12}(s)^\top & D_{22}(s) \end{pmatrix}$$

with the symmetric operators

$$\begin{aligned} D_{11}(s) : [H^{1/2}(\Gamma)]^3 &\rightarrow [H^{-1/2}(\Gamma)]^3, \\ D_{22}(s) : H^{1/2}(\Gamma) &\rightarrow H^{-1/2}(\Gamma), \end{aligned}$$

and with the operator

$$D_{12}(s) : H^{1/2}(\Gamma) \rightarrow [H^{-1/2}(\Gamma)]^3.$$

Proof. The proof is done in the same way as the proof of Corollary 4.5. □

Corollary 4.7. *The double layer integral operator is given as*

$$K(s) = \begin{pmatrix} K_{11}(s) & K_{12}(s) \\ K_{21}(s) & K_{22}(s) \end{pmatrix}$$

with the operators

$$\begin{aligned} K_{11}(s) : [H^{1/2}(\Gamma)]^3 &\rightarrow [H^{1/2}(\Gamma)]^3, \\ K_{22}(s) : H^{1/2}(\Gamma) &\rightarrow H^{1/2}(\Gamma), \\ K_{12}(s) : H^{1/2}(\Gamma) &\rightarrow [H^{1/2}(\Gamma)]^3, \\ K_{21}(s) : [H^{1/2}(\Gamma)]^3 &\rightarrow H^{1/2}(\Gamma). \end{aligned}$$

Then the adjoint of the adjoint double layer integral operator can be written as

$$\widetilde{K}(s)^* = \begin{pmatrix} K_{11}(s)^\top & -sK_{21}(s)^\top \\ -\frac{1}{s} K_{12}(s)^\top & K_{22}(s)^\top \end{pmatrix}.$$

Proof. Repeating the arguments of Corollary 4.5 results in the statement. □

Remark 4.4. *Notice that the sum of the sesquilinear forms* (3.21) *for* Ω^+ *and* Ω^- *can be equi-valently written as*

$$\mathrm{Re}\left[a_{\mathbb{R}^3\setminus\Gamma}(u, \Theta_s u)\right] = \mathrm{Re}\left[a_{\Omega^+}(u, \Theta_s u)\right] + \mathrm{Re}\left[a_{\Omega^+}(u, \Theta_s u)\right].$$

Theorem 4.8. *The single layer integral operator* $V(s) : [H^{-1/2}(\Gamma)]^4 \to [H^{1/2}(\Gamma)]^4$ *as defined in* (4.7) *is* $[H^{-1/2}(\Gamma)]^4$*-elliptic, i. e.*

$$\operatorname{Re}\left[\langle \psi, \Theta_s V(s)\psi\rangle_\Gamma\right] \gtrsim \frac{\sigma\underline{\sigma}^8}{|s|}\,\|\psi\|^2_{-1/2,\Gamma} \quad \text{for all } \psi \in [H^{-1/2}(\Gamma)]^4. \tag{4.11}$$

Therefore $V(s)$ *is invertible with*

$$\left\|V(s)^{-1}\right\|_{[H^{1/2}(\Gamma)]^4 \to [H^{-1/2}(\Gamma)]^4} \lesssim \frac{|s|^2}{\sigma\underline{\sigma}^9}. \tag{4.12}$$

Proof. We define $u = \mathrm{SL(s)}\,\psi$ which fulfills $\mathcal{P}u = 0$ in $\mathbb{R}^3 \setminus \Gamma$ and thus we have

$$\begin{aligned}
\operatorname{Re}\left[\langle \psi, \Theta_s V(s)\psi\rangle_\Gamma\right] &= -\operatorname{Re}\left[\langle [\gamma_1 u], \gamma_0 \Theta_s V(s)\psi\rangle_\Gamma\right] && \text{(Jump conditions (4.6))}\\
&= \operatorname{Re}\left[a_{\mathbb{R}^3\setminus\Gamma}(u, \Theta_s u)\right] && \text{(Green's first formula (3.30))}\\
&\gtrsim \sigma\underline{\sigma}^5\,|||u|||^2_{|s|,\mathbb{R}^3\setminus\Gamma} && \text{(Theorem 3.9)}\\
&\gtrsim \frac{\sigma\underline{\sigma}^8}{|s|}\,\|\psi\|^2_{-1/2,\Gamma}. && \text{(Corollary 3.14)}
\end{aligned}$$

To prove estimate (4.12) we insert $\psi = V(s)^{-1}\phi \in [H^{1/2}(\Gamma)]^4$ into the ellipticity estimate (4.11), which results in

$$\begin{aligned}
\left\|V(s)^{-1}\phi\right\|^2_{-1/2,\Gamma} &\lesssim \frac{|s|}{\sigma\underline{\sigma}^8}\operatorname{Re}\left[\langle \psi, \Theta_s V(s)\psi\rangle_\Gamma\right]\\
&\lesssim \frac{|s|}{\sigma\underline{\sigma}^8}\,\|\Theta_s\phi\|_{1/2,\Gamma}\left\|V(s)^{-1}\phi\right\|_{-1/2,\Gamma} && \text{(Duality estimate)}\\
&\lesssim \frac{|s|^2}{\sigma\underline{\sigma}^9}\,\|\phi\|_{1/2,\Gamma}\left\|V(s)^{-1}\phi\right\|_{-1/2,\Gamma}. && \text{(Estimate (3.31))}
\end{aligned}$$

□

Proposition 4.9. *For* $\psi \in [H^{-1/2}(\Gamma)]^4$, $\phi \in [H^{1/2}(\Gamma)]^4$ *and* $s \in \mathbb{C}^+_\sigma$ *the following estimates hold:*

$$\|\mathrm{SL(s)}\,\psi\|_{1,\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^2}{\underline{\sigma}^8\sigma}\,\|\psi\|_{-1/2,\Gamma}, \tag{4.13}$$

$$\left\|\mathrm{SL(s)}\,V(s)^{-1}\phi\right\|_{1,\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{15/2}\sigma}\,\|\phi\|_{1/2,\Gamma}, \tag{4.14}$$

$$\left\|\gamma_1^{\pm}\,\mathrm{SL(s)}\,\psi\right\|_{-1/2,\Gamma} \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{13/2}\sigma}\,\|\psi\|_{-1/2,\Gamma}. \tag{4.15}$$

Proof. Inserting $u = \mathrm{SL}(s)\,\psi$ into the ellipticity estimate (Theorem 3.9) yields

$$\underline{\sigma}^5 \sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \left| a_{\mathbb{R}^3\setminus\Gamma}(u, \Theta_s u) \right|.$$

We apply the jump condition (4.6) of the single layer potential and Green's first formula (3.18) and end up with

$$\underline{\sigma}^5 \sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim |\langle \psi, \Theta_s V(s)\psi\rangle_\Gamma| \lesssim \|\psi\|_{-1/2,\Gamma} \|\Theta_s V(s)\psi\|_{1/2,\Gamma}. \tag{4.16}$$

With the help of estimate (3.12) we get

$$\|\Theta_s V(s)\psi\|_{1/2,\Gamma} \leq \|\Theta_s u\|_{1,\mathbb{R}^3\setminus\Gamma} \leq \frac{|s|}{\underline{\sigma}}\, |||\, u|||_{\widetilde{|s|},\Omega}$$

and therefore we have

$$\underline{\sigma}^5 \sigma\, |||\, u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|}{\underline{\sigma}} \|\psi\|_{-1/2,\Gamma}. \tag{4.17}$$

The norm equivalence (3.11) yields

$$\|u\|_{1,\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|}{\underline{\sigma}^2}\, |||\, u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^2}{\underline{\sigma}^8 \sigma} \|\psi\|_{-1/2,\Gamma}$$

or estimate (4.13).

To show the estimate (4.14) we start from estimate (4.17) and apply Corollary 3.14 resulting in

$$\left\|\gamma_1^\pm \mathrm{SL}(s)\,\psi\right\|_{-1/2,\Gamma} \lesssim \frac{|s|^{1/2}}{\underline{\sigma}^{3/2}}\, |||\, u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{13/2}\sigma} \|\psi\|_{-1/2,\Gamma}.$$

Finally, to prove estimate (4.14), we reconsider estimate (4.16)

$$\underline{\sigma}^5 \sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \|\psi\|_{-1/2,\Gamma} \|\Theta_s V(s)\psi\|_{1/2,\Gamma}$$

and apply Corollary 3.14, the bound of Θ_s (3.31) and introduce $\psi = V(s)^{-1}\phi \in [H^{1/2}(\Gamma)]^4$, which results in

$$\underline{\sigma}^5 \sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{5/2}}\, |||\, u\, |||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \|\phi\|_{1/2}.$$

The norm equivalence (3.11) yields

$$\left\|\mathrm{SL}(s)\, V(s)^{-1}\phi\right\|_{1,\mathbb{R}^3\setminus\Gamma} = \|u\|_{1,\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{15/2}\sigma} \|\phi\|_{1/2}.$$

□

Remark 4.5. *Due to Remark 4.2 the fundamental solution and thus the operator $V(s)$ can be symmetrised by applying the operator Θ_{-s}. Theorem 4.8 shows that the operator $\Theta_s V(s)$ is $[H^{1/2}(\Gamma)]^4$-elliptic.*

Real valued block skew-symmetric systems can be transformed to a symmetric and positive definite system by a Bramble-Pasciak transformation, see [14]. For complex valued block skew-symmetric systems the theory is however incomplete.

Theorem 4.10. *Let $s \in C^+_\sigma$, then the hyper-singular integral operator $D(s) : [H^{1/2}(\Gamma)]^4 \to [H^{-1/2}(\Gamma)]^4$ as defined in (4.8) is $[H^{1/2}(\Gamma)]^4$-elliptic, i. e.*

$$\operatorname{Re}[\langle D(s)\phi, \Theta\phi\rangle_\Gamma] \gtrsim c_1^D(s)\frac{\underline{\sigma}^7\sigma}{|s|^2}\|\phi\|^2_{1/2,\Gamma} \quad \text{for all } \phi \in [H^{1/2}(\Gamma)]^4. \tag{4.18}$$

Therefore $D(s)$ is invertible satisfying

$$\left\|D(s)^{-1}\right\|_{[H^{-1/2}(\Gamma)]^4 \to [H^{1/2}(\Gamma)]^4} \lesssim \frac{|s|^2}{\underline{\sigma}^8\sigma}. \tag{4.19}$$

Proof. We start with $u = -\mathrm{DL}(s)\,\phi$, which fulfills $\mathcal{P}u = 0$ in $\mathbb{R}^3 \setminus \Gamma$, and which can be estimated in the following way:

$$\begin{aligned}
\operatorname{Re}[\langle D(s)\phi, \Theta_s\phi\rangle_\Gamma] &= \operatorname{Re}[-\langle \gamma_1 u, [u]\rangle_\Gamma] && \text{(Jump conditions (4.6))}\\
&= \operatorname{Re}\left[a_{\mathbb{R}^3\setminus\Gamma}(u, \Theta u)\right] && \text{Green's first formula (3.18)}\\
&\gtrsim \underline{\sigma}^5\sigma\,|||\,u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} && \text{(Ellipticity (Theorem 3.9))}\\
&\gtrsim \frac{\underline{\sigma}^7\sigma}{|s|^2}\,|||\,u\,|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma}. && \text{(Norm equivalence (3.11))}
\end{aligned} \tag{4.20}$$

The trace theorem (Theorem 3.4) and the jump conditions (4.6) can be applied to estimate

$$\|u\|_{1,\mathbb{R}^3\setminus\Gamma} \geq \|[u]|_\Gamma\|_{1/2,\Gamma} = \|\phi\|_{1/2,\Gamma},$$

which results in the ellipticity estimate (4.18) for the hyper-singular operator.

Estimate (4.19) for the norm of the inverse hyper-singular operator can be calculated by using (4.20), which results in

$$\begin{aligned}
\underline{\sigma}^5\sigma\,|||\,u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} &\lesssim \operatorname{Re}[\langle D(s)\phi, \Theta_s\phi\rangle_\Gamma]\\
&\lesssim \|D(s)\phi\|_{-1/2,\Gamma}\,\|\Theta_s\phi\|_{1/2,\Gamma} && \text{(Duality estimate)}\\
&\lesssim \|D(s)\phi\|_{-1/2,\Gamma}\,\|\Theta_s[u]|_\Gamma\|_{1/2,\Gamma} && \text{(Jump conditions (4.6))}\\
&\lesssim \frac{|s|}{\underline{\sigma}}\,\|D(s)\phi\|_{-1/2,\Gamma}\,\|u\|_{1,\mathbb{R}^3\setminus\Gamma} && \text{(Thm. 3.4, estimate (3.31))}\\
&\lesssim \frac{|s|}{\underline{\sigma}}\,\|D(s)\phi\|_{-1/2,\Gamma}\,|||\,u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} && \text{(Norm equivalence (3.11))}
\end{aligned}$$

which yields

$$|||u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|}{\underline{\sigma}^6\sigma} \|D(s)\phi\|_{-1/2,\Gamma}.$$

The norm equivalence (3.11) and the trace theorem (Theorem 3.4) conclude the proof. □

Proposition 4.11. *For $s \in C_\sigma^+$, $\phi \in [H^{1/2}(\Gamma)]^4$ and $\psi \in [H^{-1/2}(\Gamma)]^4$ the following inequalities hold:*

$$\|\mathrm{DL}(s)\,\phi\|_{1,\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^{5/2}}{\underline{\sigma}^{19/2}\sigma} \|\phi\|_{1/2,\Gamma}, \tag{4.21}$$

$$\left\|\mathrm{DL}(s)\,D(s)^{-1}\psi\right\|_{1,\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^2}{\underline{\sigma}^6\sigma} \|\psi\|_{-1/2,\Gamma}, \tag{4.22}$$

$$\|\gamma_1\,\mathrm{DL}(s)\,\phi\|_{-1/2,\Gamma} \lesssim \frac{|s|^2}{\underline{\sigma}^8\sigma} \|\phi\|_{1/2,\Gamma}. \tag{4.23}$$

Proof. Settting $u = \mathrm{DL}(s)\,\phi$ results in

$$\begin{aligned}
\underline{\sigma}^5\sigma\,|||\,u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} &\lesssim \left|a_{\mathbb{R}^3\setminus\Gamma}(u,\Theta_s u)\right| && \text{(Theorem 3.9)}\\
&= |\langle\gamma_1 u, [\Theta_s u]|_\Gamma\rangle_\Gamma| && \text{(Green's first formula (3.18))}\\
&\lesssim \|\Theta_s\phi\|_{1/2,\Gamma}\,\|\gamma_1 u\|_{-1/2,\Gamma} && \text{(Duality, jump conditions (4.6))}\\
&\lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{5/2}} \|\phi\|_{1/2,\Gamma}\,|||\,u\,|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma}. && \text{(Corollary 3.14, Estimate (3.31))}
\end{aligned}$$

Finally, with estimate (3.11) we have

$$\|u\|_{1,\mathbb{R}^3\setminus\Gamma} \lesssim |||u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{19/2}\sigma} \|\phi\|_{1/2,\Gamma},$$

which concludes the estimate (4.21).

To show estimate (4.22) we start with

$$\underline{\sigma}^5\sigma\,|||\,u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \|\Theta_s\phi\|_{1/2,\Gamma}\,\|\gamma_1 u\|_{-1/2,\Gamma} = \|\Theta_s\phi\|_{1/2,\Gamma}\,\|D(s)\phi\|_{-1/2,\Gamma}$$

and introduce $\phi = D(s)^{-1}\psi \in [H^{1/2}(\Gamma)]^4$, which in addition to the trace theorem (Theorem 3.4) and the estimate (3.12) leads to

$$\underline{\sigma}^5\sigma\,|||\,u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|}{\underline{\sigma}}\,|||\,u\,|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma}\,\|\psi\|_{-1/2,\Gamma}.$$

Finally, to prove estimate (4.23) we apply Corollary 3.14

$$\|\gamma_1\,\mathrm{DL}(s)\,\phi\|_{-1/2,\Gamma} \lesssim \frac{|s|^{1/2}}{\underline{\sigma}^{3/2}}\,|||\,u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^2}{\underline{\sigma}^9\sigma} \|\phi\|_{1/2,\Gamma}.$$

□

Proposition 4.12. *For $s \in C_\sigma^+$, $\phi \in [H^{1/2}(\Gamma)]^4$ and $\psi \in [H^{-1/2}(\Gamma)]^4$ the following inequalities hold:*

$$\|K(s)\phi\| \lesssim \frac{|s|^{5/2}}{\underline{\sigma}^{19/2}\sigma} \|\phi\|_{1/2,\Gamma},$$
$$\left\|\widetilde{K}(s)\phi\right\| \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{13/2}\sigma} \|\phi\|_{1/2,\Gamma},$$
$$\|K(s)^*\psi\| \lesssim \frac{|s|^{5/2}}{\underline{\sigma}^{19/2}\sigma} \|\psi\|_{-1/2,\Gamma},$$
$$\left\|\widetilde{K}(s)^*\psi\right\| \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{13/2}\sigma} \|\psi\|_{-1/2,\Gamma}.$$

Proof. The first two estimates for the double layer integral operatos $K(s)$ and $\widetilde{K}(s)^*$ follow immediately from the estimates (4.21) and (4.15). The last two operators are the adjoint operators and thus fulfill the same bounds. □

4.4 The Steklov–Poincaré operator

Additionally we introduce the interior and exterior Steklov–Poincaré operator $\mathrm{S}^{\pm}(s)$,

$$\mathrm{S}^-(s) = V(s)^{-1}\left(\frac{1}{2}I + K(s)\right) = D(s) + \left(\frac{1}{2}I + \widetilde{K}(s)^*\right)V(s)^{-1}\left(\frac{1}{2}I + K(s)\right),$$
$$-\mathrm{S}^+(s) = V(s)^{-1}\left(\frac{1}{2}I - K(s)\right) = D(s) + \left(\frac{1}{2}I - \widetilde{K}(s)^*\right)V(s)^{-1}\left(\frac{1}{2}I - K(s)\right)$$

and its inverse, the Poincaré–Steklov operator $\mathrm{T}^{\pm}(s)$

$$\mathrm{T}^-(s) = D(s)^{-1}\left(\frac{1}{2}I - \widetilde{K}(s)^*\right) = V(s) + \left(\frac{1}{2}I - K(s)\right)D(s)^{-1}\left(\frac{1}{2}I - \widetilde{K}(s)^*\right),$$
$$-\mathrm{T}^+(s) = D(s)^{-1}\left(\frac{1}{2}I + \widetilde{K}(s)^*\right) = V(s) + \left(\frac{1}{2}I + K(s)\right)D(s)^{-1}\left(\frac{1}{2}I + \widetilde{K}(s)^*\right).$$

The Steklov–Poincaré operator is equivalent to the Dirichlet to Neumann map for homogeneous problems. Similarly the Poincaré–Steklov operator is equivalent to the Neumann to Dirichlet map for homogeneous problems. These two operators are very popular in domain decomposition methods, see, e. g. , [51].

Proposition 4.13. *For the operator $S^{\pm}(s)$ and $T^{\pm}(s)$ we have the ellipticity estimates*

$$\operatorname{Re}\left[\left\langle S^{\pm}(s)\phi,\Theta_s\phi\right\rangle_\Gamma\right] \gtrsim \frac{\underline{\sigma}^9\sigma}{|s|^2}\,\|\phi\|^2_{1/2,\Gamma} \quad \textit{for all } \phi\in[H^{1/2}(\Gamma)]^4,$$

$$\operatorname{Re}\left[\left\langle \Theta_{\bar s}\psi, T^{\pm}(s)\psi\right\rangle_\Gamma\right] \gtrsim \frac{\underline{\sigma}^8\sigma}{|s|}\,\|\psi\|^2_{1/2,\Gamma} \quad \textit{for all } \psi\in[H^{-1/2}(\Gamma)]^4,$$

and the bounds

$$\left\|T^{\pm}(s)\right\|_{[H^{-1/2}(\Gamma)]^4\to[H^{1/2}(\Gamma)]^4} \lesssim \frac{|s|^2}{\underline{\sigma}^8\sigma}, \tag{4.24}$$

$$\left\|S^{\pm}(s)\right\|_{[H^{1/2}(\Gamma)]^4\to[H^{-1/2}(\Gamma)]^4} \lesssim \frac{|s|^2}{\underline{\sigma}^9\sigma} \tag{4.25}$$

for all $s\in\mathbb{C}^+_{\sigma}$.

Proof. We define $u^{\pm}\in[H^1(\Omega^{\pm})]^4$ as the solution of $\mathcal{P}u^{\pm}=0$ in $\Omega^{\pm}$, $\gamma_0^{\pm}u=\phi$ on Γ. Inserting this function into Green's first formula (3.18) yields

$$\operatorname{Re}\left[\left\langle \mathrm{S}^{\pm}(s)\phi,\Theta_s\phi\right\rangle_\Gamma\right] = \operatorname{Re}\left[\langle\gamma_1 u,\Theta_s\gamma_0 u\rangle_\Gamma\right] = \operatorname{Re}\left[a_{\Omega^{\pm}}(u,\Theta_s u)\right].$$

The ellipticity estimate in Theorem 3.9 for the sesquilinear form, estimate (3.11) and the trace theorem (Theorem 3.4) result in

$$\operatorname{Re}\left[\left\langle \mathrm{S}^{\pm}(s)\phi,\Theta_s\phi\right\rangle_\Gamma\right] \gtrsim \underline{\sigma}^5\sigma\,|||\,u\,|||^2_{\widetilde{|s|},\Omega^{\pm}} \gtrsim \frac{\underline{\sigma}^9\sigma}{|s|^2}\,\|\phi\|^2_{1/2,\Gamma}$$

and thus the ellipticity estimate for the Steklov–Poincaré operator is obtained.

Furthermore we have

$$\underline{\sigma}^5\sigma\,|||\,u\,|||^2_{\widetilde{|s|},\Omega^{\pm}} \lesssim \left|\left\langle \mathrm{S}^{\pm}(s)\phi,\Theta_s\phi\right\rangle_\Gamma\right| \lesssim \left\|\mathrm{S}^{\pm}(s)\phi\right\|_{-1/2,\Gamma}\|\Theta_s u\|_{1,\mathbb{R}^3\setminus\Gamma}$$

and using the estimate (3.12) we end up with

$$|||u|||_{\widetilde{|s|},\Omega^{\pm}} \lesssim \frac{|s|}{\underline{\sigma}^6\sigma}\left\|\mathrm{S}^{\pm}(s)\phi\right\|_{-1/2,\Gamma}.$$

The Dirichlet trace on the boundary can be estimated by

$$\|\phi\|_{1/2,\Gamma} \le \|u\|_{1,\Omega^{\pm}} \le \frac{|s|}{\underline{\sigma}^2}\,|||\,u\,|||_{\widetilde{|s|},\Omega^{\pm}}.$$

Introducing $\phi=\mathrm{T}^{\pm}(s)\psi\in[H^{1/2}(\Gamma)]^4$ concludes the proof for estimate (4.24).

For the ellipticity estimate of the Poincaré–Steklov operators $\mathrm{T}^{\pm}(s)$, we define v as the solution of $\mathcal{P}v^{\pm} = 0$ in $\Omega^{\pm}, \gamma_1^{\pm}v = \psi$ on Γ. We have

$$\operatorname{Re}\left[\left\langle \psi, \Theta_s \mathrm{T}^{\pm}(s)\psi\right\rangle_\Gamma\right] = \operatorname{Re}\left[a_{\Omega^\pm}(u,\Theta_s u)\right] \gtrsim \underline{\sigma}^5 \sigma \, |||\, u|||^2_{|s|,\Omega^\pm} \gtrsim \frac{\underline{\sigma}^8\sigma}{|s|} \,\|\psi\|^2_{-1/2,\Gamma}$$

and replacing $\psi = \mathrm{S}^{\pm}(s)\phi$ results in

$$\left\|\mathrm{S}^{\pm}(s)\phi\right\|^2_{-1/2,\Gamma} \lesssim \frac{|s|^2}{\underline{\sigma}^9\sigma} \,\|\phi\|_{1/2,\Gamma} \left\|\mathrm{S}^{\pm}(s)\phi\right\|_{-1/2,\Gamma},$$

and we obtain the bound for the Poincaré–Steklov operator. □

Remark 4.6. *The bounds for the Steklov–Poincaré operators $S^{\pm}(s)$ and the Poincaré–Steklov operator $T^{\pm}(s)$ as given in Proposition 4.13 give an alternative proof for the bound of the inverse of the single layer boundary integral operator $V(s)$ and the hyper-singular boundary integral operator $D(s)$ since*

$$D(s)^{-1} = T^-(s) - T^+(s) \quad \text{and} \quad V(s)^{-1} = S^-(s) - S^+(s).$$

To classify the introduced operators we introduce the following space, see [31].

Definition 4.1. *Let X and Y be Hilbert spaces and let $F(s) : \mathbb{C}^+_\sigma \to \mathcal{L}(X,Y)$ be an analytic function in s. $F(s)$ is an element of $\mathcal{A}(\mu,X,Y)$ if*

$$|F(s)| \leq C(\sigma)\,|s|^\mu \quad \text{for all } s \in \mathbb{C}_+$$

where $C : (0,\infty) \to (0,\infty)$ is an non-decreasing function such that

$$C(\sigma) \leq \frac{c}{\sigma^m} \quad \text{for all } \sigma \in (0,1].$$

An overview on the mapping properties and bounds of all discussed operators is given in Table 4.1.

Additionally we introduce the operator

$$H^-(s) = \begin{pmatrix} V(s) & -\left(\frac{1}{2}I + K(s)\right) \\ \left(\frac{1}{2}I + \widetilde{K}(s)^*\right) & D(s) \end{pmatrix}.$$

F	X	Y	μ
SL(s)	$[H^{-1/2}(\Gamma)]^4$	$[H^1(\mathbb{R}^3)]^4$	2
DL(s)	$[H^{1/2}(\Gamma)]^4$	$[H^1(\mathbb{R}^3\setminus\Gamma)]^4$	5/2
$V(s)$	$[H^{-1/2}(\Gamma)]^4$	$[H^{1/2}(\Gamma)]^4$	2
$D(s)$	$[H^{1/2}(\Gamma)]^4$	$[H^{-1/2}(\Gamma)]^4$	2
$K(s)$	$[H^{1/2}(\Gamma)]^4$	$[H^{1/2}(\Gamma)]^4$	5/2
$\widetilde{K}(s)$	$[H^{-1/2}(\Gamma)]^4$	$[H^{-1/2}(\Gamma)]^4$	3/2
$V(s)^{-1}$	$[H^{1/2}(\Gamma)]^4$	$[H^{-1/2}(\Gamma)]^4$	2
$D(s)^{-1}$	$[H^{-1/2}(\Gamma)]^4$	$[H^{1/2}(\Gamma)]^4$	2
$\mathrm{S}^{\pm}(s)$	$[H^{1/2}(\Gamma)]^4$	$[H^{-1/2}(\Gamma)]^4$	2
$\mathrm{T}^{\pm}(s)$	$[H^{-1/2}(\Gamma)]^4$	$[H^{1/2}(\Gamma)]^4$	2

Table 4.1: The operator $F(s)$ is an element of the space $\mathcal{A}(\mu,X,Y)$.

Theorem 4.14. *The operator* $H^-(s):[H^{-1/2}(\Gamma)]^4\times[H^{1/2}(\Gamma)]^4\to[H^{1/2}(\Gamma)]^4\times[H^{-1/2}(\Gamma)]^4$ *is* $[H^{-1/2}(\Gamma)]^4\times[H^{1/2}(\Gamma)]^4$*-elliptic, i. e.*

$$\begin{aligned}\operatorname{Re}\left[\langle\psi,\Theta_s V(s)\psi\rangle_\Gamma\right]+\operatorname{Re}\left[\left\langle\psi,-\Theta_s\left(\frac{1}{2}I+K(s)\right)\phi\right\rangle_\Gamma\right]\\ +\operatorname{Re}\left[\left\langle\left(\frac{1}{2}I+\widetilde{K}(s)^*\right)\psi,\Theta_s\phi\right\rangle_\Gamma\right]+\operatorname{Re}\left[\langle D(s)\phi,\Theta_s\phi\rangle_\Gamma\right]\\ \geq\frac{\underline{\sigma}^8\sigma}{|s|^2}\left(\|\psi\|^2_{-1/2,\Gamma}+\|\phi\|^2_{1/2,\Gamma}\right)\end{aligned}\tag{4.26}$$

for all $\psi\in[H^{-1/2}(\Gamma)]^4$ *and* $\phi\in[H^{1/2}(\Gamma)]^4$*. The operator* $H^-(s)$ *is therefore invertible with*

$$\left\|H^-(s)^{-1}\right\|_{[H^{1/2}(\Gamma)]^4\times[H^{-1/2}(\Gamma)]^4\to[H^{-1/2}(\Gamma)]^4\times[H^{1/2}(\Gamma)]^4}\leq\frac{|s|^2}{\underline{\sigma}^7\sigma}.\tag{4.27}$$

Moreover we have the bound

$$\left\|\left[\mathrm{SL}(s)\quad-\mathrm{DL}(s)\right]H^-(s)\right\|_{[H^{1/2}(\Gamma)]^4\times[H^{-1/2}(\Gamma)]^4\to[H^1(\Omega)]^4}\leq\frac{|s|^2}{\underline{\sigma}^7\sigma}.\tag{4.28}$$

Proof. $u=\mathrm{SL}(s)\,\psi-\mathrm{DL}(s)\,\phi$ fulfills $\mathcal{P}u=0$ in $\mathbb{R}^3\setminus\Gamma$. The application of Green's first formula (3.30) results in

$$\begin{aligned}\underline{\sigma}^4\sigma\,|||u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma}&\lesssim\operatorname{Re}\left[a_{\mathbb{R}^3\setminus\Gamma}(u,\Theta_s u)\right]\\ &=\operatorname{Re}\left[\left\langle[\gamma_1 u]_\Gamma,\Theta_s\gamma_0^+u\right\rangle_\Gamma\right]+\operatorname{Re}\left[\left\langle\gamma_1^-u,\Theta_s[\gamma_0 u]_\Gamma\right\rangle_\Gamma\right].\end{aligned}\tag{4.29}$$

Due to the jump conditions we have $[\gamma_1 u]_\Gamma = \psi$ and $[\gamma_0 u]_\Gamma = \phi$ and furthermore

$$\gamma_0^+ u = V(s)\psi - \left(\frac{1}{2}I + K(s)\right)\phi$$

and

$$\gamma_1^- u = D(s)\phi + \left(\frac{1}{2}I + \widetilde{K}(s)^*\right)\psi$$

resulting in

$$\begin{aligned}\underline{\sigma}^4\sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim & \operatorname{Re}\left[\langle \psi, \Theta_s V(s)\psi\rangle_\Gamma\right] \\ & - \operatorname{Re}\left[\left\langle \psi, \Theta_s \left(\frac{1}{2}I + K(s)\right)\phi \right\rangle_\Gamma\right] \\ & + \operatorname{Re}\left[\left\langle \left(\frac{1}{2}I + \widetilde{K}(s)^*\right)\psi, \Theta_s\phi \right\rangle_\Gamma\right] \\ & + \operatorname{Re}\left[\langle D(s)\psi, \phi\rangle_\Gamma\right].\end{aligned}$$

With the help of the trace theorem (Theorem 3.4), Lemma 3.13 and the norm estimate (3.11) we can estimate the traces by

$$\|[\gamma_1 u]_\Gamma\|^2_{-1/2,\Gamma} + \|[\gamma_0 u]_\Gamma\|^2_{1/2,\Gamma} \lesssim \frac{|s|^2}{\underline{\sigma}^4}\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \tag{4.30}$$

resulting in the ellipticity estimate (4.26).

Next we consider the operator equation

$$H^-(s)\begin{bmatrix}\psi\\ \phi\end{bmatrix} = \begin{bmatrix}g_1\\ g_2\end{bmatrix}.$$

Starting from estimate (4.29)

$$\underline{\sigma}^4\sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} = \operatorname{Re}\left[\left\langle [\gamma_1 u]_\Gamma, \Theta_s \gamma_0^+ u\right\rangle_\Gamma\right] + \operatorname{Re}\left[\left\langle \gamma_1^- u, \Theta_s [\gamma_0 u]_\Gamma\right\rangle_\Gamma\right],$$

we use the property $\gamma_0^+ u = g_1$ and $\gamma_1^- u = g_2$ and use Corollary 3.14 and estimate (3.11) resulting in

$$\underline{\sigma}^4\sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|}{\underline{\sigma}}\, |||\, u\, |||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma}\left(\|g_1\|_{1/2,\Gamma} + \|g_2\|_{-1/2,\Gamma}\right) \tag{4.31}$$

or

$$|||u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|}{\underline{\sigma}^5\sigma}\left(\|g_1\|_{1/2,\Gamma} + \|g_2\|_{-1/2,\Gamma}\right). \tag{4.32}$$

Estimate (3.11) results in (4.28), whereas estimate (4.30) results in (4.27). □

Proposition 4.15. *Let $s \in C_\sigma^+$, then the property*

$$H^-(s) \in \mathcal{A}(5/2, [H^{-1/2}(\Gamma)]^4 \times [H^{1/2}(\Gamma)]^4, [H^{-1/2}(\Gamma)]^4 \times [H^{1/2}(\Gamma)]^4)$$

holds.

Proof. For the operator itself the bound can be easily calculated since

$$\left\|H^-(s)\right\| \leq 2\max_{i,j}\left\|H_{ij}\right\|,$$

where the operator norms are induced by the natural spaces. □

Remark 4.7. *The inverse of the operator $H^-(s)$ can be stated explicitly by*

$$H^-(s)^{-1} = \begin{pmatrix} -S^+(s) & I \\ -I & T^-(s) \end{pmatrix}.$$

Proof. Using the non-symmetric representation of $\mathrm{T}^-(s)$ and $\mathrm{S}^+(s)$ results in

$$\begin{aligned}
&H^-(s)(H^-(s))^{-1} \\
&\quad= \begin{pmatrix} V(s) & -\frac{1}{2}I + K(s) \\ \frac{1}{2}I + \widetilde{K}(s)^* & D(s) \end{pmatrix} \begin{pmatrix} V(s)^{-1}\left(\frac{1}{2}I - K(s)\right) & I \\ -I & D(s)^{-1}\left(\frac{1}{2}I - \widetilde{K}(s)^*\right) \end{pmatrix} \\
&\quad= \begin{pmatrix} I & \mathbf{A} \\ \mathbf{B} & I \end{pmatrix}
\end{aligned}$$

and since

$$\begin{aligned}
\mathbf{A} &= -\left(\frac{1}{2}I + K(s)\right) D(s)^{-1}\left(\frac{1}{2}I - \widetilde{K}(s)^*\right) + V(s) \\
&= V(s) + -\left(\frac{1}{2}I - K(s)\right) D(s)^{-1}\left(\frac{1}{2}I - \widetilde{K}(s)^*\right) - D(s)^{-1}\left(\frac{1}{2}I - \widetilde{K}(s)^*\right) \\
&= \mathrm{T}^-(s) - \mathrm{T}^-(s) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{B} &= -D(s) + \left(\frac{1}{2}I + \widetilde{K}(s)^*\right) V(s)^{-1}\left(\frac{1}{2}I - K(s)\right) \\
&= -\left[D(s) + \left(\frac{1}{2}I - \widetilde{K}(s)^*\right) V(s)^{-1}\left(\frac{1}{2}I - K(s)\right)\right] - D(s)^{-1}\left(\frac{1}{2}I - \widetilde{K}(s)^*\right) \\
&= \mathrm{S}^-(s) - \mathrm{S}^-(s) = 0
\end{aligned}$$

we end up with the identity. □

Similarly we introduce the operator

$$H^{+}(s) = \begin{pmatrix} V & \left(\frac{1}{2}I - K\right) \\ \left(-\frac{1}{2}I + \widetilde{K}^{*}\right) & D \end{pmatrix} \tag{4.33}$$

which shares the same properties as $H^{-}(s)$.

Corollary 4.16. *For $s \in \mathbb{C}_{\sigma}^{+}$ the operator $H^{+}(s)$ is $[H^{-1/2}(\Gamma)]^4 \times [H^{1/2}(\Gamma)]^4$-elliptic, i. e. ,*

$$\operatorname{Re}\left[\left\langle H^{+}(s)\begin{bmatrix}\psi \\ \phi\end{bmatrix}, \begin{bmatrix}\Theta_{\bar{s}}\psi \\ \Theta_{s}\phi\end{bmatrix}\right\rangle\right] \gtrsim \frac{\sigma^{8}\sigma}{|s|^{2}}\left(\|\psi\|_{-1/2,\Gamma}^{2} + \|\phi\|_{1/2,\Gamma}^{2}\right)$$

for all $\psi \in [H^{-1/2}(\Gamma)$ and $\phi \in [H^{1/2}(\Gamma)]^4$. Furthermore the properties

$$\begin{aligned} H^{+}(s) \quad &\in \mathcal{A}(5/2, [H^{-1/2}(\Gamma)]^4 \times [H^{1/2}(\Gamma)]^4, [H^{1/2}(\Gamma)]^4 \times [H^{-1/2}(\Gamma)]^4), \\ H^{+}(s)^{-1} &\in \mathcal{A}(2, [H^{1/2}(\Gamma)]^4 \times [H^{-1/2}(\Gamma)]^4, [H^{-1/2}(\Gamma)]^4 \times [H^{1/2}(\Gamma)]^4) \end{aligned}$$

and

$$\begin{bmatrix}\mathrm{SL(s)} & -\mathrm{DL(s)}\end{bmatrix} H^{+}(s)^{-1} \in \mathcal{A}(2, [H^{1/2}(\Gamma)]^4 \times [H^{-1/2}(\Gamma)]^4, [H^{1}(\mathbb{R}^3 \setminus \Gamma)]^4).$$

hold.

Proof. Repeating the arguments as in the proof of Theorem 4.14 results in these properties. □

5 BOUNDARY INTEGRAL EQUATIONS

In this chapter we will discuss the application of boundary integral equations to the solution of boundary value problems in the Laplace domain. Starting from a representation formula we will derive boundary integral equations of the direct approach. Boundary integral equations resulting from indirect approaches are discussed briefly. In preparation for the return to time domain, the dependency of the boundary integral equations and it's solutions on the Laplace parameter s will be presented.

5.1 Representation formula

The fundamental solution given in (4.1) is a solution of the partial differential equation

$$\mathcal{P}_y G_s(x,y) = I\delta(y-x)$$

with the Dirac distribution δ and the identity matrix $I \in \mathbb{R}^{4\times 4}$. Insertion into Green's second formula (3.29) yields the representation formula

$$u = \mathrm{SL(s)}[\gamma_1 u]_\Gamma - \mathrm{DL(s)}[\gamma_0 u]_\Gamma \quad \text{in } \mathbb{R}^3 \setminus \Gamma \tag{5.1}$$

for all $u \in [H^1(\mathbb{R}^3 \setminus \Gamma)]^4$ satisfying $\mathcal{P}u = 0$. Setting $u \equiv 0$ in Ω^+ and taking the interior traces results in the well known integral equations related to interior boundary value problems,

$$\gamma_0^- u = V(s)\gamma_1^- u + \left(\frac{1}{2}I - K(s)\right)\gamma_0^- u, \tag{5.2}$$

$$\gamma_1^- u = \left(\frac{1}{2}I + \widetilde{K}(s)^*\right)\gamma_1^- u + D(s)\gamma_0^- u. \tag{5.3}$$

Reciprocal setting $u \equiv 0$ in Ω^- results in two integral equations for the exterior boundary value problems,

$$\gamma_0^+ u = \left(\frac{1}{2}I + K(s)\right)\gamma_0^+ u - V(s)\gamma_1^+ u, \tag{5.4}$$

$$\gamma_1^+ u = \left(\frac{1}{2}I - \widetilde{K}(s)^*\right)\gamma_1^+ u - D(s)\gamma_0^+ u. \tag{5.5}$$

5.2 Mixed boundary value problem

The interior boundary value problem with mixed boundary conditions is given as

$$\begin{aligned} \mathcal{P}u &= 0 && \text{in } \Omega^-, \\ \gamma_0^- u &= g_D && \text{on } \Gamma_D, \\ \gamma_1^- u &= g_N && \text{on } \Gamma_N. \end{aligned} \tag{5.6}$$

With the help of the representation formula (5.1) we can calculate the solution of the boundary value problem, if the complete Neumann and Dirichlet data are known. Thus we need to find the unknown Dirichlet datum $\gamma_0^- u$ on Γ_N and the unknown Neumann datum $\gamma_1^- u$ on Γ_D. The approach itself is based on the symmetric formulation [49]. For deriving bounds for the solution of the boundary integral equations techniques from [31] are used.

First we choose appropriate extensions $\tilde{g}_D \in [H^{1/2}(\Gamma)]^4$ and $\tilde{g}_N \in [H^{-1/2}(\Gamma)]^4$ of the given Dirichlet datum $g_D \in [H^{1/2}(\Gamma_D)]^4$ and the given Neumann datum $g_N \in [H^{-1/2}(\Gamma_N)]^4$ such that

$$g_D = \tilde{g}_D \quad \text{on } \Gamma_D, \qquad g_N = \tilde{g}_N \quad \text{on } \Gamma_N.$$

The boundary integral equations for the interior problem yield

$$\begin{aligned} \gamma_0^- u &= V(s)\gamma_1^- u + \left(\frac{1}{2}I - K(s)\right)\gamma_1^- u, \\ 0 &= \left(-\frac{1}{2}I + \widetilde{K}(s)^*\right)\gamma_1^- u + D(s)\gamma_0^- . \end{aligned}$$

We define the unknowns

$$\psi_1 = \gamma_0^- u - \tilde{g}_D \in [\widetilde{H}^{1/2}(\Gamma_N)]^4$$

and

$$\phi_1 = \gamma_1^- u - \tilde{g}_N \in [\widetilde{H}^{-1/2}(\Gamma_D)]^4.$$

Insertion leads to the boundary integral equations

$$\begin{aligned} V(s)\phi_1 - K(s)\psi_1 &= \left(\frac{1}{2}I + K(s)\right)\tilde{g}_D - V(s)\tilde{g}_N && \text{on } \Gamma_D, \\ \widetilde{K}(s)^*\phi_1 + D(s)\psi_1 &= \left(\frac{1}{2}I - \widetilde{K}(s)^*\right)\tilde{g}_N - D(s)\tilde{g}_D && \text{on } \Gamma_N. \end{aligned} \tag{5.7}$$

The operator

$$H^+(s) = \begin{pmatrix} V(s) & -K(s) \\ \widetilde{K}(s)^* & D(s) \end{pmatrix}$$

is $[H^{-1/2}(\Gamma)]^4 \times [H^{1/2}(\Gamma)]^4$ elliptic, see Theorem 4.14. The system of boundary integral equations (5.7) is therefore uniquely solvable. The operator $H^+(s)^{-1}$ is bounded by

$$\left\|H^+(s)^{-1}\right\|_{[H^{-1/2}(\Gamma)]^4\times[H^{1/2}(\Gamma)]^4\to[H^{1/2}(\Gamma)]^4\times[H^{-1/2}(\Gamma)]^4} \lesssim c(\underline{\sigma},\sigma)\,|s|^2,$$

see Proposition 4.15. The right hand side of (5.7) is bounded by

$$\left\|\left(\frac{1}{2}I+K(s)\right)\tilde{g}_D - V(s)\tilde{g}_N\right\|_{1/2,\Gamma} \lesssim |s|^2 c(\sigma)\left(\|g_D\|_{1/2,\Gamma_D}+\|g_N\|_{-1/2,\Gamma_N}\right)$$

and

$$\left\|\left(\frac{1}{2}I-\widetilde{K}(s)^*\right)\tilde{g}_N - D(s)\tilde{g}_D\right\|_{1/2,\Gamma} \lesssim |s|^{5/2} c(\sigma)\left(\|g_D\|_{1/2,\Gamma_D}+\|g_N\|_{-1/2,\Gamma_N}\right).$$

Combining these estimates yields an estimate for the solution of the boundary integral equations (5.7)

$$\|\psi_1\|_{-1/2,\Gamma}+\|\phi_1\|_{1/2,\Gamma} \lesssim c(\underline{\sigma},\sigma)\frac{|s|^{9/2}}{\sigma^2}\left(\|g_D\|_{1/2,\Gamma_D}+\|g_N\|_{-1/2,\Gamma_N}\right). \tag{5.8}$$

This bound can be further improved by an approach developed in [31]. The boundary integral equations (5.7) can be equivalently rewritten as

$$\begin{aligned}
\tilde{g}_D &= V(s)\phi_1 + \left(\frac{1}{2}I-K(s)\right)\psi_1 + V(s)\tilde{g}_N + \left(\frac{1}{2}I-K(s)\right)\tilde{g}_D && \text{on } \Gamma_D,\\
0 &= \left(-\frac{1}{2}I+\widetilde{K}(s)^*\right)\phi_1 + D(s)\psi_1 + \left(-\frac{1}{2}I+\widetilde{K}(s)^*\right)\tilde{g}_N + D(s)\tilde{g}_D && \text{on } \Gamma_N.
\end{aligned}$$

We replace the given Cauchy datum by the functions $\phi_2 = \widetilde{g}_N \in [H^{-1/2}(\Gamma)]^4$ and $\psi_2 = \widetilde{g}_D \in [H^{1/2}(\Gamma)]^4$ and can therefore rewrite the boundary integral equations (5.7) as

$$H^+_{mix}(s)\begin{bmatrix}\phi_1\\ \psi_1\\ \phi_2\\ \psi_2\end{bmatrix} = \begin{bmatrix}H^+(s) & H^+(s)\\ & I\end{bmatrix}\begin{bmatrix}\phi_1\\ \psi_1\\ \phi_2\\ \psi_2\end{bmatrix} = \begin{bmatrix}g_1\\ g_2\\ g_3\\ g_4\end{bmatrix} \tag{5.9}$$

with $g_1 = \widetilde{g}_D$, $g_2 = 0$, $g_3 = \widetilde{g}_N$ and $g_4 = \widetilde{g}_D$.

Theorem 5.1. *Let* $\mathbf{H} = [H^{-1/2}(\Gamma)]^4 \times [H^{1/2}(\Gamma)]^4 \times [H^{-1/2}(\Gamma)]^4 \times [H^{1/2}(\Gamma)]^4$ *and* $s \in \mathbb{C}^+_\sigma$, *then the property*

$$H^+_{mix}(s) \in \mathcal{A}(5/2,\mathbf{H},\mathbf{H}^*)$$

holds. Moreover, the operator is invertible with

$$H^+_{mix}(s)^{-1} \in \mathcal{A}(5/2,\mathbf{H}^*,\mathbf{H}).$$

Additionally, the property

$$\left[\mathrm{SL(s)} \quad -\mathrm{DL(s)}\right] H_{mix}^{+}(s)^{-1} \in \mathcal{A}(5/2, \mathbf{H}^*, [H^1(\Omega)]^4).$$

holds.

Proof. The boundedness property $H_{mix}^{+}(s) \in \mathcal{A}(5/2, \mathbf{H}, \mathbf{H}^*)$ is obtained straight forward by

$$\left\|H_{mix}^{+}(s)\right\|_{\mathbf{H}\to\mathbf{H}^*} \lesssim \max\Big\{ \|V(s)\|_{[H^{-1/2}(\Gamma)]^4\to[H^{1/2}(\Gamma)]^4}, \|D(s)\|_{[H^{1/2}(\Gamma)]^4\to[H^{-1/2}(\Gamma)]^4}, \\ \|K(s)\|_{[H^{1/2}(\Gamma)]^4\to[H^{1/2}(\Gamma)]^4}, \left\|\widetilde{K}(s)^*\right\|_{[H^{-1/2}(\Gamma)]^4\to[H^{-1/2}(\Gamma)]^4} \Big\}.$$

The proof of invertibility of the operator $H_{mix}^{+}(s)$ and the bound of the inverse is given in [31] for the wave equation. We follow this proof closely. We define

$$u = \mathrm{SL(s)}(\phi_1+\phi_2) - \mathrm{DL(s)}(\psi_1+\psi_2).$$

The operator equation (5.9) is equivalent to the following boundary value problem

$$\begin{aligned} \mathcal{P}u &= 0 \quad \text{in } \mathbb{R}^3\setminus\Gamma,\\ \gamma_0^- u &= g_1 \quad \text{on } \Gamma_D,\\ \gamma_1^+ u &= g_2 \quad \text{on } \Gamma_N, \end{aligned} \tag{5.10}$$

with the transmission conditions

$$[\gamma_1 u]|_\Gamma - g_3 \in [\widetilde{H}^{-1/2}(\Gamma_D)]^4, \qquad [\gamma_0 u]|_\Gamma - g_4 \in [\widetilde{H}^{1/2}(\Gamma_N)]^4.$$

Given a function u as a solution of (5.10), a solution for equation (5.9) is obtained by

$$(\phi_1, \psi_1, \phi_2, \psi) = ([\gamma_1 u]|_\Gamma - g_3, [\gamma_0 u]_\Gamma - g_4, g_3, g_4).$$

The boundary value problem (5.10) is equivalent to the following variational formulation:

Find $u \in [H^1(\mathbb{R}^3\setminus\Gamma)]^4$ with $\left(\gamma_0^- u, [\gamma_0 u]_\Gamma\right) = (g_1, g_4)$ on Γ_D such that

$$a_{\mathbb{R}^3\setminus\Gamma}(u,v) = \left\langle g_3, \gamma_0^- v\right\rangle_\Gamma + \langle g_2, [\gamma_0 v]_\Gamma\rangle_\Gamma$$

for all $v \in \mathbf{H}_0 = \{u \in \mathbf{H} | (\gamma_0^- u, [\gamma_0 u]|_\Gamma) = 0 \text{ on } \Gamma_D\}$.

This variational formulation can be analyzed by repeating the arguments of Theorem 3.12, resulting in the estimate

$$|||u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \le c(\underline{\sigma})\,|s|\left(\|(g_2,g_3)\|_{-1/2,\Gamma} + |s|^{1/2}\,\|(g_1,g_4)\|_{1/2,\Gamma}\right),$$

and so finally by estimating the traces (Theorem 3.4 and Corollary 3.14) we end up with

$$\|(\phi_1,\phi_2)\|_{-1/2,\Gamma} + \|(\psi_1,\psi_2)\|_{1/2,\Gamma} \le c(\underline{\sigma})\,|s|^{5/2}\left(\|(g_2,g_3)\|_{-1/2,\Gamma} + \|(g_1,g_4)\|_{1/2,\Gamma}\right).$$

□

5.3 Dirichlet boundary value problem

The Dirichlet problem

$$\begin{aligned} \mathcal{P}u &= 0 \qquad \text{in } \Omega^-, \\ \gamma_0^- u &= g_D \qquad \text{on } \Gamma \end{aligned} \tag{5.11}$$

can be solved by using the first boundary integral equation of (5.2) to find the unknown Neumann datum $t \in [H^{-1/2}(\Gamma)]^4$ satisfying

$$V(s)t = \left(\frac{1}{2}I + K(s)\right) g_D \quad \text{on } \Gamma. \tag{5.12}$$

Unique solvability follows from the ellipticity of the single layer boundary integral operator (Theorem 4.8). A bound for the unknown Neumann datum t can be obtained by composing the bound for the inverse single layer boundary integral operator and the double layer boundary integral operator resulting in

$$\|t\|_{-1/2,\Gamma} \lesssim c(\underline{\sigma}, \sigma)\, |s|^{9/2}\, \|g_D\|_{1/2,\Gamma}\,.$$

By using the estimate for the mixed problem (Theorem 5.1) the estimate can be improved to

$$\|t\|_{-1/2,\Gamma} \lesssim c(\underline{\sigma}, \sigma)\, |s|^{5/2}\, \|g_D\|_{1/2,\Gamma}\,.$$

The operator

$$\mathrm{S}^-(s) = V(s)^{-1}\left(\frac{1}{2}I + K(s)\right)$$

is the interior Steklov-Poincaré operator, which was already discussed in Section 4.4. The bound for the Steklov-Poincaré operator results in an improved bound for the solution t of the boundary integral equation (5.12)

$$\|t\|_{-1/2,\Gamma} \lesssim c(\underline{\sigma}, \sigma)\, |s|^{5/2}\, \|g_D\|_{1/2,\Gamma}\,,$$

see Proposition 4.13. This bound is obviously the best one. The solution

$$u = \mathrm{SL}(\mathrm{s})\,\phi - \mathrm{DL}(\mathrm{s})\, g_D$$

itself can be estimated as

$$\|u\|_{1,\Omega^-} \lesssim |s|^2\, \|g_D\|_{1/2,\Gamma}\,,$$

see Theorem 5.1.

Another popular approach to solve the interior Dirichlet boundary value problem is an indirect single layer approach. Using the ansatz $u = \mathrm{SL}(\mathrm{s})\,\phi$ results in the boundary integral equation

$$V(s)\phi = g_D \quad \text{on } \Gamma.$$

Unique solvability as well as the estimate

$$\|\phi\|_{-1/2,\Gamma} \lesssim c(\underline{\sigma},\sigma)\,|s|^2\,\|g_D\|_{1/2,\Gamma}$$

is obtained by the ellipticity estimate (Theorem 4.8). An estimate for the solution u is given by estimate (4.14), resulting in

$$\|u\|_{1,\Omega^-} \lesssim c(\underline{\sigma},\sigma)\,|s|^{3/2}\,\|g_D\|_{1/2,\Gamma}\,.$$

5.4 Neumann boundary value problem

The interior Neumann boundary value problem is given by

$$\begin{aligned} \mathcal{P}u &= 0 && \text{in } \Omega^-, \\ \gamma_1^- u &= g_N && \text{on } \Gamma. \end{aligned} \tag{5.13}$$

The equation for the unknown Dirichlet datum $\tilde{u} \in [H^{1/2}(\Gamma)]^4$ is given by

$$D(s)\tilde{u} = \left(\frac{1}{2}I - \widetilde{K}(s)^*\right) g_N \quad \text{on } \Gamma. \tag{5.14}$$

The ellipticity of the hyper-singular operator guarantees unique solvability. An estimate for the solution $\tilde{u} \in [H^{1/2}(\Gamma)]^4$ of the boundary integral equation is given by Proposition 4.13

$$\|\tilde{u}\|_{1/2,\Gamma} \lesssim c(\sigma,\underline{\sigma})\,|s|^2\,\|g_N\|_{-1/2,\Gamma}\,.$$

The solution $u = \mathrm{SL}(\mathrm{s})\,g_N - \mathrm{DL}(\mathrm{s})\,\tilde{u}$ can be estimated by

$$\|u\|_{1,\Omega^-} \lesssim c(\sigma,\underline{\sigma})\,|s|^2\,\|g_N\|_{-1/2,\Gamma}\,,$$

see Theorem 5.1.

An indirect double layer approach $u = -\,\mathrm{DL}(\mathrm{s})\,\psi$ would result in the boundary integral equation

$$D(s)\psi = g_N \quad \text{on } \Gamma.$$

Again, ellipticity of the hyper-singular operator guarantees unique solvability and $\psi \in [H^{1/2}(\Gamma)]^4$ can be estimated by

$$\|\psi\|_{1/2,\Gamma} \lesssim c(\sigma,\underline{\sigma})\,|s|^2\,\|g_N\|_{-1/2,\Gamma}\,,$$

see Theorem 4.10. The solution u can be estimated by

$$\|u\|_{1,\Omega^-} \lesssim c(\underline{\sigma},\sigma)\,|s|^2\,\|g_N\|_{-1/2,\Gamma}\,.$$

see estimate (4.22).

6 GALERKIN DISCRETIZATION OF BOUNDARY INTEGRAL EQUATIONS

In this chapter we will discuss the discretization of boundary integral equations introduced in Chapter 5. For the Galerkin discretizations unique solvability will be proven. The boundary integral equations will be discussed in Laplace domain only. In preparation for the return to time domain, the dependency of the boundary integral equations and it's solutions onto the Laplace parameter s will be presented. More precisely, estimates as in Remark 7.1 will be shown for the solutions of the discretized boundary integral equations. Similar estimates are given for the approximate solutions inside the domain, both in the corresponding norm and for a pointwise evaluation. Finally, similar estimates are given for the error estimates in the energy norm of the approximate solutions of all boundary integral equations.

6.1 Galerkin discretization

To discretize boundary integral equations, first a variational formulation has to be set up, which is furthermore discretized by restricting test and ansatz functions to finite dimensional subspaces. The theoretical background for the Galerkin discretization is well established, for more information we refer to [45, 52].

Let X be a Hilbert space, for an operator $A : X \to X^*$ and a given right hand side $f \in X^*$ we consider the following variational formulation.

Find $u \in X$ such that

$$\langle Au, v\rangle = \langle f, v\rangle \tag{6.1}$$

for all $v \in X$.

By introducing a finite dimensional subspace $X_h \subset X$ a Galerkin approximation $u_h \in X_h$ of the solution u is defined by:

Find $u_h \in X_h$ such that

$$\langle Au_h, v_h\rangle = \langle f, v_h\rangle \tag{6.2}$$

for all $v_h \in X_h$.

Restricting the test space to $X_h \subset X$ in the variational formulation (6.1) and subtracting the equation from equation (6.2) leads to the Galerkin orthogonality

$$\langle A(u_h - u), v_h \rangle = 0 \quad \text{for all } v_h \in X_h. \tag{6.3}$$

Stability of the Galerkin scheme (6.2) is unique solvability together with a uniform bound

$$\|u_h\|_X \leq C \|u\|_X \qquad \text{for all } u \in X$$

with a positive constant $C > 0$ independent of h and u.

Stability together with the approximation property

$$\inf_{v_h \in X_h} \|u - v_h\|_X \to 0 \quad \text{for } h \to 0$$

leads to convergence for arbitrary right hand sides.

We introduce an operator notation for the Galerkin discretization by utilizing Lemma 3.1. We define the operator $A_h : X_h \to X_h^*$ by

$$\langle A_h u_h, v_h \rangle := \langle A u_h, v_h \rangle \quad \text{for all } u_h \in X_h, v_h \in X_h.$$

Lemma 6.1 (Cea's Lemma)**.** *Let the discrete operator $A_h : X_h \to X_h^*$ be invertible, u the solution of the variational formulation* (6.1)*, u_h the solution of the variational formulation* (6.2)*. Then the following estimate holds*

$$\|u - u_h\|_X \leq (1 + \left\|A_h^{-1}\right\|_{X_h^* \to X_h} \|A\|_{X \to X^*}) \inf_{v_h \in X_h} \|u - v_h\|_X .$$

Proof. See [45, 52]. □

If the right hand side $f \in X^*$ is given as $f = Bg$ with $g \in Y$ and a bounded linear operator $B : Y \to X^*$, we introduce a finite dimensional subspace $Y_h \subset Y$ and approximate $g \in Y$ by a function $g_h \in Y_h$ resulting in the disturbed variational formulation:

Find $\tilde{u}_h \in X_h$ such that

$$\langle A\tilde{u}_h, v_h \rangle = \langle B g_h, v_h \rangle \tag{6.4}$$

for all $v_h \in X_h$.

The approximation property of the disturbed variational formulation (6.4) is given by the following lemma.

Lemma 6.2 (Strang-Lemma)**.** *Let the operator $A_h : X_h \to X_h^*$ be invertible and let $B_h : Y_h \to X_h^*$ be a bounded linear operator. Furthermore, let u be the solution of the variational formulation* (6.1)*, u_h the solution of the variational formulation* (6.2) *and $\tilde{u}_h$ the solution of the variational formulation* (6.4)*. The following error estimate holds:*

$$\|u - \tilde{u}_h\|_X \leq \left(1 + \left\|A_h^{-1}\right\|_{X_h^* \to X_h} \|A\|_{X \to X^*}\right) \inf_{v_h \in X_h} \|u - v_h\|_X + \left\|A_h^{-1}\right\|_{X_h^* \to X_h} \|B\|_{Y_h \to X_h^*} \|g - g_h\|_Y .$$

Proof. Subtraction of the variational formulations (6.2) and (6.4) leads to

$$\langle A(u_h - \tilde{u}_h), v_h \rangle = \langle B(g - g_h), v_h \rangle \quad \text{for all } v_h \in X_h.$$

Since A_h is invertible we immediately get the estimate

$$\|u - \tilde{u}_h\|_X \leq \left\|A_h^{-1}\right\|_{X_h^* \to X_h} \|B\|_{Y_h \to X_h^*} \|g - g_h\|_Y .$$

Combining this estimate with Lemma 6.1 in addition to the triangle inequality concludes the proof. □

The operator A usually denotes some boundary integral operator and thus appropriate discretization spaces on the boundary have to be introduced.

First we introduce a sequence of boundary discretizations $\Gamma_N = \cup_{\ell=1}^{N} \overline{\tau}_\ell$ with N disjoint plane triangles, which are assumed to be regular in the sense of Ciarlet [15]. The local mesh size is defined by

$$h_\ell := \left(\int_{\tau_\ell} \mathrm{d}s_x \right)^{1/2}$$

and the global mesh size is defined by $h = \max_{\ell=1,\ldots,N} h_\ell$. Let M be the number of nodes on the boundary with M_D and M_N denoting the nodes on the Dirichlet boundary Γ_D and the Neumann boundary Γ_N respectively. Likewise N_D denotes the number of boundary elements on the Dirichlet boundary Γ_D and N_N the number of boundary elements on the Neumann boundary Γ_N.

Define the discrete subspaces

$$S_h^{-1,0}(\Gamma_D) = \mathrm{span}\{\psi_i^{-1,0}\}_{i=1}^{4N_D} \subset [H^{-1/2}(\Gamma_D)]^4,$$
$$S_h^{0,1}(\Gamma_N) = \mathrm{span}\{\psi_i^{0,1}\}_{i=1}^{4M_N} \subset [\widetilde{H}^{1/2}(\Gamma_N)]^4$$

with piecewise constant basis functions $\psi_i^{-1,0}$ and piecewise linear continuous basis functions $\psi_i^{0,1}$. For convenience we introduce the space

$$\widetilde{S}_h^{0,1}(\Gamma_N) = S_h^{0,1}(\Gamma_N) \cap [\widetilde{H}^{1/2}(\Gamma_N)]^4.$$

This function space is used to discretize the solid displacement u^s and the pore pressure p, whereas the piecewise constant functions are used to discretize the unknown Neumann traces. Additionally, the given Neumann and Dirichlet data will be approximated by calculating L_2 projections into discrete spaces. The L_2 projection $P_h : X \to X_h$ is defined by

$$\langle P_h u, v_h \rangle_\Gamma = \langle u, v_h \rangle_\Gamma \quad \text{for all } v_h \in X_h.$$

We denote the L_2 projection into the space $S_h^{0,1}(\Gamma)$ by

$$P_h^{0,1} : [H^{-1/2}(\Gamma)]^4 \to S_h^{-1,0}(\Gamma). \tag{6.5}$$

For the approximation of the unknown Neumann datum we additionally introduce the space

$$S_h^{-1,1}(\Gamma_D) = \text{span}\{\psi_i^{-1,1}\}_{i=1}^{12N_D} \subset [\widetilde{H}^{-1/2}(\Gamma_D)]^4$$

with piecewise linear but discontinuous basis functions $\psi_i^{-1,1}$, and the appropriate projection operator

$$P_h^{-1,1} : [H^{-1/2}(\Gamma)]^4 \to S_h^{-1,1}(\Gamma). \tag{6.6}$$

The projection operator $P_h^{0,1}$ is used to approximate the unknown Dirichlet datum and the projection operator $P_h^{-1,1}$ is used to approximate the unknown Neumann datum. The choice of these projection operators results in an optimal convergence order of all involved unknowns, in particular for the point evaluation of the solution in the interior.

For the discrete spaces $S_h^{-1,0}(\Gamma)$ and $S_h^{0,1}(\Gamma)$ the following approximation properties hold.

Lemma 6.3. *The following approximation properties hold:*

$$\inf_{t_h \in S_h^{-1,0}} \|t - t_h\|_{-\alpha,\Gamma} \le c h^{\beta+\alpha} \|u\|_{\beta,\Gamma} \quad \textit{with } \alpha \in [0,1] \textit{ and } \beta \in [0,1],$$
$$\inf_{t_h \in S_h^{-1,1}} \|t - t_h\|_{-\alpha,\Gamma} \le c h^{\beta+\alpha} \|u\|_{\beta,\Gamma} \quad \textit{with } \alpha \in [0,2] \textit{ and } \beta \in [0,2],$$
$$\inf_{u_h \in S_h^{0,1}} \|u - u_h\|_{-\alpha,\Gamma} \le c h^{\beta+\alpha} \|u\|_{\beta,\Gamma} \quad \textit{with } \alpha \in [0,2] \textit{ and } \beta \in [0,2]$$

when assuming $u \in H^\beta_{pw}(\Gamma)$ *and* $t \in H^\beta_{pw}(\Gamma)$.

Proof. See [45, 52]. □

6.2 Bounds for discrete operators

In Section 4.2 the explicit behavior of the boundedness constants of different integral operators on the Laplace parameter s has been discussed. Bounds for the discrete operators are needed as well. For an exact Galerkin discretization of an operator the bound obviously remains the same. We introduce discrete subspaces $X_h \subset [H^{-1/2}(\Gamma)]^4$ and $Y_h \subset [H^{1/2}(\Gamma)]^4$.

Corollary 6.4. *Let A be an element of $\mathcal{A}(\mu, X, Y^*)$, then the Galerkin discretization A_h is an element of $\mathcal{A}(\mu, X_h, Y_h^*)$.*

The Corollary 6.4 gives us a bound for the Galerkin discretization of the boundary integral operators $V(s)$, $K(s)$, $\widetilde{K}(s)^*$ and $D(s)$. We denote the Galerkin discretizations of these operators by $V_h(s)$, $K_h(s)$, $\widetilde{K}_h(s)^*$ and $D_h(s)$ respectively. An overview of the bounds is given in Table 6.1.

However, most of the time we do not have an exact discretization of an operator. For example, the inverse of the discrete single layer integral operator $V_h(s)^{-1}$ is not the Galerkin discretization of the inverse single layer integral operator $(V(s)^{-1})_h$. Therefore, estimates for inverse operators cannot be transferred directly. However, the bound of the inverse discrete single layer integral operator and the inverse discrete hyper-singular operator are direct results of the ellipticity estimates as given in Theorem 4.8 and Theorem 4.10. The ellipticity estimates also hold for the discrete operators and allow us to formulate the following Corollary.

Corollary 6.5. *The inverse of the Galerkin discretization of the hyper-singular operator $D_h(s)^{-1}$ and the inverse of the Galerkin discretization of the single layer integral operator $V_h(s)^{-1}$ fulfill the following estimates:*

$$\left\|V_h(s)^{-1}\right\|_{X_h^* \to X_h} \lesssim \frac{|s|^2}{\sigma \underline{\sigma}^9},$$

$$\left\|D_h(s)^{-1}\right\|_{Y_h^* \to Y_h} \lesssim \frac{|s|^2}{\sigma \underline{\sigma}^8}.$$

Proof. The bound for $V_h(s)^{-1}$ is a direct consequence of the ellipticity estimate as given in Theorem 4.8:

$$\operatorname{Re}\left[\langle \psi, \Theta_s V(s) \psi \rangle_\Gamma\right] \geq c_1^V \frac{\sigma \underline{\sigma}^8}{|s|} \|\psi\|_{-1/2,\Gamma}^2 \quad \text{for all } \psi \in [H^{-1/2}(\Gamma)]^4.$$

For $\psi_h \in X_h \subset [H^{-1/2}(\Gamma)]^4$ we have

$$\operatorname{Re}\left[\langle \psi_h, \Theta_s V(s) \psi_h \rangle_\Gamma\right] \geq c_1^V \frac{\sigma \underline{\sigma}^8}{|s|} \|\psi_h\|_{-1/2,\Gamma}^2 \quad \text{for all } \psi_h \in X_h.$$

Introducing $\psi_h = V_h(s)^{-1} \phi_h$ results in

$$\begin{aligned}
\left\|V_h(s)^{-1}\phi_h\right\|_{-1/2,\Gamma}^2 &\lesssim \frac{|s|}{\sigma \underline{\sigma}^8} \operatorname{Re}\left[\langle \psi_h, \Theta_s V_h(s) \psi_h \rangle_\Gamma\right] \\
&\lesssim \frac{|s|}{\sigma \underline{\sigma}^8} \|\Theta_s \phi_h\|_{1/2,\Gamma} \left\|V_h(s)^{-1}\phi_h\right\|_{-1/2,\Gamma} && \text{(Duality estimate)} \\
&\lesssim \frac{|s|^2}{\sigma \underline{\sigma}^9} \|\phi_h\|_{1/2,\Gamma} \left\|V_h(s)^{-1}\phi_h\right\|_{-1/2,\Gamma} && \text{(Estimate (3.31))}
\end{aligned}$$

concluding the estimate for $V_h(s)^{-1}$.

As in the proof of Theorem 4.10 we define $u = -\mathrm{DL}(\mathrm{s})\,\phi_h$ for which we have the estimate

$$\underline{\sigma}^5\sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim |\langle D_h(s)\phi_h, \Theta_s\phi_h\rangle_\Gamma|$$

which can be further estimated by

$$\begin{aligned}
\underline{\sigma}^5\sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} &\lesssim \|D_h(s)\phi_h\|_{-1/2,\Gamma}\,\|\Theta_s\phi_h\|_{1/2,\Gamma} && \text{(Duality estimate)}\\
&\lesssim \|D_h(s)\phi_h\|_{-1/2,\Gamma}\,\|\Theta_s[u]|_\Gamma\|_{1/2,\Gamma} && \text{(Jump conditions (4.6))}\\
&\lesssim \frac{|s|}{\underline{\sigma}}\,\|D_h(s)\phi_h\|_{-1/2,\Gamma}\,\|u\|_{1,\Omega^-\cup\Omega^+} && \text{(Thm. 3.4, estimate (3.31))}\\
&\lesssim \frac{|s|}{\underline{\sigma}}\,\|D_h(s)\phi_h\|_{-1/2,\Gamma}\,|||\, u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} && \text{(Norm equivalence (3.11)).}
\end{aligned}$$

The norm equivalence (3.11) and the trace theorem (Theorem 3.4) conclude the proof:

$$\|\phi_h\|_{1/2,\Gamma} \lesssim \|[u]|_\Gamma\|_{1/2,\Gamma} \lesssim \frac{|s|}{\underline{\sigma}^2}\,|||\, u|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^2}{\underline{\sigma}^8\sigma}\,\|D_h(s)\phi_h\|_{-1/2,\Gamma}\,.$$

Replacing $\phi_h = D_h(s)^{-1}\psi_h$ results in the desired estimate. □

Additionally, the estimates (4.22) and (4.14) can be transferred to the Galerkin discretization of the single layer integral operator and the hyper-singular operator.

Corollary 6.6. *The following estimates hold for all $\phi_h \in X_h^*$ and $\psi_h \in Y_h^*$:*

$$\left\|\mathrm{SL}(\mathrm{s})\,V_h(s)^{-1}\phi_h\right\|_{1,\Omega^-\cup\Omega^+} \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{15/2}\sigma}\,\|\phi_h\|_{1/2,\Gamma}\,, \tag{6.7}$$

$$\left\|\mathrm{DL}(\mathrm{s})\,D_h(s)^{-1}\psi_h\right\|_{1,\Omega^-\cup\Omega^+} \lesssim \frac{|s|^2}{\underline{\sigma}^6\sigma}\,\|\psi_h\|_{-1/2,\Gamma}\,. \tag{6.8}$$

Proof. Setting $u = \mathrm{SL}(\mathrm{s})\,\psi_h$ with $\psi_h \in X_h \subset [H^{-1/2}(\Gamma)]^4$ results in

$$\begin{aligned}
\underline{\sigma}^5\sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} &\lesssim |\langle \psi_h, \Theta_s V(s)\psi_h\rangle_\Gamma| && \text{(Green's first formula (3.30))}\\
&\lesssim |\langle \psi_h, \Theta_s V_h(s)\psi_h\rangle_\Gamma| && \text{(Galerkin discretization)}\\
&\lesssim \frac{|s|}{\underline{\sigma}}\,\|\psi_h\|_{-1/2,\Gamma}\,\|V_h(s)\psi_h\|_{1/2,\Gamma}\,. && \text{(Duality estimate)}
\end{aligned}$$

Corollary 3.14 and introducing $\phi_h = V_h(s)\psi_h$ yield

$$\underline{\sigma}^5\sigma\, |||\, u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{3/2}}\,|||\, u\,|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma}\,\|\phi_h\|_{1/2}$$

and thus estimate (6.7).

To derive the estimate (6.8) we set $v = \mathrm{DL}(s)\,\phi_h$ with $\phi_h \in Y_h \subset [H^{1/2}(\Gamma)]^4$ and obtain

$$
\begin{aligned}
\underline{\sigma}^5\sigma\,|||\,u|||^2_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} &\lesssim |\langle D(s)\phi_h, \Theta_s\phi_h\rangle_\Gamma| && \text{(Green's first formula (3.30))}\\
&= |\langle D_h(s)\phi_h, \Theta_s\phi_h\rangle_\Gamma| && \text{(Galerkin discretization)}\\
&\lesssim \frac{|s|^{3/2}}{\underline{\sigma}^{5/2}}\,\|\phi_h\|_{1/2,\Gamma}\,\|D_h(s)\phi_h\|_{-1/2,\Gamma}. && \text{(Duality estimate)}
\end{aligned}
$$

Introducing $\psi_h = D_h(s)\phi_h$ and using the estimate (3.12) result in the desired estimate. □

6.3 A discrete Steklov–Poincaré operator

Bounds for the discrete versions of the Steklov–Poincaré operator and the Poincaré–Steklov operator are still missing. These operators consist of a combination of different boundary integral operators. Two different representations by boundary integral operators were introduced in Section 4.4. On the continuous level the different representations are equivalent to each other. In general, the equivalence is lost after discretisation.

In this section we will discuss the symmetric approximation of the Steklov–Poincaré operator. A bound for the non-symmetric approximation is a byproduct of the analysis of the mixed problem in Section 6.4.

The Dirichlet datum of the interior Neumann boundary value problem

$$
\begin{aligned}
\mathcal{P}u &= 0 \quad \text{in } \Omega^-\\
\gamma_1 u &= g \quad \text{on } \Gamma
\end{aligned}
$$

can be obtained by solving

$$\mathrm{S}^-(s)u = g \quad \text{on } \Gamma. \tag{6.9}$$

The Steklov–Poincaré operator can be expressed as

$$\mathrm{S}^-(s) = D(s) + \left(\frac{1}{2}I + \widetilde{K}(s)^*\right)V(s)^{-1}\left(\frac{1}{2}I + K(s)\right).$$

By introducing $t \in [H^{-1/2}(\Gamma)]^4$ as a solution of the operator equation

$$V(s)t = \left(\frac{1}{2}I + K(s)\right)u \quad \text{on } \Gamma,$$

the boundary integral equation (6.9) can be rewritten as

$$H^-(s)\begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} V(s) & -\left(\frac{1}{2}I + K(s)\right) \\ \left(\frac{1}{2}I + \widetilde{K}(s)^*\right) & D(s) \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Properties of the operator $H^-(s)$ were discussed in Theorem 4.14.

Introducing the discrete subspaces $X_h \subset [H^{-1/2}(\Gamma)]^4$ and $Y_h \subset [H^{1/2}(\Gamma)]^4$ the Galerkin discretization of the operator $H^-(s)$ is given by

$$\left\langle H_h^-(s)\begin{bmatrix} x_h \\ y_h \end{bmatrix}, \begin{bmatrix} v_h \\ w_h \end{bmatrix} \right\rangle_\Gamma = \left\langle H^-(s)\begin{bmatrix} x_h \\ y_h \end{bmatrix}, \begin{bmatrix} v_h \\ w_h \end{bmatrix} \right\rangle_\Gamma \quad \text{for all } x_h, v_h \in X_h, y_h, w_h \in Y_h$$

resulting in the discrete equations

$$H_h^-(s)\begin{pmatrix} t_h \\ u_h \end{pmatrix} = \begin{pmatrix} V_h(s) & -\left(\frac{1}{2}M_h + K_h(s)\right) \\ \left(\frac{1}{2}M_h + \widetilde{K}_h(s)^*\right) & D_h(s) \end{pmatrix} \begin{pmatrix} t_h \\ u_h \end{pmatrix} = \begin{pmatrix} 0 \\ g_h \end{pmatrix}.$$

Proposition 6.7. *The operator $H_h^-(s)$ fulfills the following property*

$$H_h^-(s) \in \mathcal{A}(5/2, X_h \times Y_h, X_h^* \times Y_h^*).$$

Additionally, the operator $H_h^-(s)$ is invertible with

$$H_h^-(s)^{-1} \in \mathcal{A}(2, X_h^* \times Y_h^*, X_h \times Y_h).$$

Finally, we have the property

$$\begin{bmatrix} \mathrm{SL(s)} & -\mathrm{DL(s)} \end{bmatrix} H_h^-(s)^{-1} \in \mathcal{A}(2, X_h^* \times Y_h^*, [H^1(\mathbb{R}^3 \setminus \Gamma)]^4).$$

Proof. Repeating the arguments of Theorem 4.14 results in these properties. □

Hence we have unique solvability of the following operator equation

$$H_h^-(s)\begin{bmatrix} t_h \\ u_h \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{6.10}$$

with $g_1 \in X_h^*$ and $g_2 \in Y_h^*$. For $g_1 = 0$ we obtain the Galerkin discretisation of equation (6.9). By eliminating t_h we can define a discrete approximation of the Steklov–Poincaré operator by

$$\mathrm{S}_h^-(s) = D_h(s) + \left(\frac{1}{2}M_h + \widetilde{K}_h(s)^*\right) V_h(s)^{-1} \left(\frac{1}{2}M_h + K_h(s)\right).$$

On the other hand, for $g_2 = 0$ we can eliminate u_h resulting in a symmetric approximation of the Poincaré–Steklov operator

$$\mathrm{T}_h(s) = V_h(s) + \left(\frac{1}{2}M_h + K_h(s)\right) D_h(s)^{-1} \left(\frac{1}{2}M_h + \widetilde{K}_h(s)^*\right)$$

as introduced in Section 4.4.

Corollary 6.8. *The following properties hold:*

$$\begin{aligned} \left[S_h^-(s)\right]^{-1} &\in \mathcal{A}(2, X_h^*, X_h), \\ [T_h(s)]^{-1} &\in \mathcal{A}(2, Y_h^*, Y_h). \end{aligned}$$

Starting from the operator $H^+(s)$ as introduced in (4.33) we can define a symmetric approximation of the exterior Steklov–Poincaré operator $\mathrm{S}_h^+(s)$ and of the interior Poincaré–Steklov operator $\mathrm{S}_h^-(s)$ in a similar way.

$F(s)$	X	Y	μ
$V_h(s)$	X_h	X_h^*	2
$D_h(s)$	Y_h	Y_h^*	2
$K_h(s)$	X_h	Y_h^*	5/2
$\widetilde{K}_h(s)^*$	Y_h	X_h^*	3/2
$V_h(s)^{-1}$	X_h^*	X_h	2
$D_h(s)^{-1}$	Y_h^*	Y_h	2
$[\mathrm{T}^\pm(s)]^{-1}$	X_h^*	X_h	2
$[\mathrm{S}^\pm(s)]^{-1}$	Y_h^*	Y_h	2

Table 6.1: The operator $F(s)$ is an element of the space $\mathcal{A}(\mu, X, Y)$.

The estimates for these operators are given in Table 6.1 in addition to bounds for already discussed operators.

6.4 Mixed boundary value problem

The interior boundary value problem with mixed boundary conditions (5.6) is given as

$$\begin{aligned} \mathcal{P}u &= 0 && \text{in } \Omega^-, \\ \gamma_0^- u &= g_D && \text{on } \Gamma_D, \\ \gamma_1^- u &= g_N && \text{on } \Gamma_N. \end{aligned} \tag{6.11}$$

The boundary integral equations which are related to the mixed boundary value problem (6.11) are deduced in (5.7) and are given as

$$\begin{aligned} V(s)\phi - K(s)\psi &= \left(\frac{1}{2}I + K(s)\right)\tilde{g}_D - V(s)\tilde{g}_N \quad \text{on } \Gamma_D,\\ \widetilde{K}(s)^*\phi + D(s)\psi &= \left(\frac{1}{2}I - \widetilde{K}(s)^*\right)\tilde{g}_N - D(s)\tilde{g}_D \quad \text{on } \Gamma_N, \end{aligned} \tag{6.12}$$

or equivalently as

$$H^+(s)\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{bmatrix}g_D\\ 0\end{bmatrix} - H^+(s)\begin{bmatrix}g_D\\ g_N\end{bmatrix}.$$

We test the boundary integral equations (6.12) with functions $(\eta,\xi) \in [\widetilde{H}^{-1/2}(\Gamma_N)]^4 \times [\widetilde{H}^{1/2}(\Gamma_D)]^4$. This results in the variational formulation:

Find $(\phi,\psi) \in [\widetilde{H}^{-1/2}(\Gamma_N)]^4 \times [\widetilde{H}^{1/2}(\Gamma_D)]^4$ such that

$$\left\langle H^+(s)\begin{bmatrix}\phi\\ \psi\end{bmatrix}, \begin{bmatrix}\eta\\ \xi\end{bmatrix}\right\rangle = \left\langle \begin{bmatrix}g_D\\ 0\end{bmatrix} - H^+(s)\begin{bmatrix}g_D\\ g_N\end{bmatrix}, \begin{bmatrix}\eta\\ \xi\end{bmatrix}\right\rangle$$

for all $(\eta,\xi) \in [\widetilde{H}^{-1/2}(\Gamma_N)]^4 \times [\widetilde{H}^{1/2}(\Gamma_D)]^4$.

The given Dirichlet datum $\widetilde{g}_D$ is projected in the discrete space $S_h^{0,1}(\Gamma)$ by using the L_2 projection $P_h^{0,1}$, see (6.5). The given Neumann datum $\widetilde{g}_N$ is approximated by using the L_2 projection $P_h^{-1,1}$, see (6.6). The corresponding Galerkin variational formulation has the form:

Find $(\phi_h,\psi_h) \in S_h^{-1,0}(\Gamma_N) \times \widetilde{S}_h^{0,1}(\Gamma)$ such that

$$\left\langle H^+(s)\begin{bmatrix}\phi_h\\ \psi_h\end{bmatrix}, \begin{bmatrix}\eta_h\\ \xi_h\end{bmatrix}\right\rangle_\Gamma = \left\langle \begin{bmatrix}P_h^{0,1}\widetilde{g}_D\\ 0\end{bmatrix}, \begin{bmatrix}\eta_h\\ \xi_h\end{bmatrix}\right\rangle_\Gamma - \left\langle H^+(s)\begin{bmatrix}P_h^{0,1}\widetilde{g}_D\\ P_h^{-1,1}\widetilde{g}_N\end{bmatrix}, \begin{bmatrix}\eta_h\\ \xi_h\end{bmatrix}\right\rangle_\Gamma \tag{6.13}$$

for all $(\eta_h,\xi_h) \in S_h^{-1,0}(\Gamma_D) \times \widetilde{S}_h^{0,1}(\Gamma_N)$.

Due to the ellipticity of the operator $H^+(s)$, see Corollary 4.16, the variational formulation (6.13) is uniquely solvable. Strang's lemma (Lemma 6.2) and the approximation property (Lemma 6.3) gives us the following corollary.

Corollary 6.9. *Let $\phi \in [H^1_{pw}(\Gamma)]^4$, $\psi \in [H^2_{pw}(\Gamma)]^4$, $g_N \in [H^1_{pw}(\Gamma_N)]^4$ and $g_D \in [H^2_{pw}(\Gamma_D)]^4$, then the following error estimate holds*

$$\|\phi-\phi_h\|_{-1/2,\Gamma} + \|\psi-\psi_h\|_{1/2,\Gamma} \le c(\underline{\sigma})\,|s|^{9/2}\,h^{3/2}\left(\|\phi\|_{1,\Gamma} + \|\psi\|_{2,\Gamma} + \|g_N\|_{1,\Gamma_N} + \|g_D\|_{2,\Gamma_D}\right).$$

With the help of the inverse inequality an error estimate in the L_2-norm for the error of ϕ can be deduced,

$$\|\phi-\phi_h\|_{0,\Gamma} \leq c(\underline{\sigma})\,|s|^{9/2}\,h\left(\|\phi\|_{1,\Gamma}+\|\psi\|_{2,\Gamma}+\|g_N\|_{1,\Gamma_N}+\|g_D\|_{2,\Gamma_D}\right),$$

see [52]. The dependancy on the parameter s is the same as for the natural norm. The Aubin–Nitsche trick gives us an error estimate in the L_2-norm of the error of ψ

$$\|\psi-\psi_h\|_{0,\Gamma} \leq c(s)h^2\left(\|\phi\|_{1,\Gamma}+\|\psi\|_{2,\Gamma}+\|g_N\|_{1,\Gamma_N}+\|g_D\|_{2,\Gamma_D}\right)$$

see [52]. The constant $c(s)$ depends on estimates of the operator $D(s):[H^1(\Gamma)]^4 \to [L_2(\Gamma)]^4$ and it's inverse and estimates for the operator $\widetilde{K}(s)^*:[H^{-1}(\Gamma)]^4 \to [H^{-1}(\Gamma)]^4$. The explicit behaviour of these estimates onto the parameter s has not been investigated yet and therefore the explicit behaviour of the error estimate onto the parameter s is not known.

The solution u of the interior mixed boundary value problem (5.6) is approximated by evaluating the representation formula (5.1)

$$u_h = \mathrm{SL}(\mathrm{s})\left(\phi_h + P_h^{-1,1}\widetilde{g}_N\right) - \mathrm{DL}(\mathrm{s})\left(\psi_h + P_h^{0,1}\widetilde{g}_N\right).$$

The error in a point $x\in\Omega$ can be estimated by the following lemma.

Lemma 6.10. *Let $\phi\in[H^1_{pw}(\Gamma)]^4$, $\psi\in[H^2_{pw}(\Gamma)]^4$, $g_N\in[H^1_{pw}(\Gamma_N)]^4$ and $g_D\in[H^2_{pw}(\Gamma_D)]^4$, then for $x\in\Omega$ the following error estimate holds*

$$|u(x)-u_h(x)| \leq c(\underline{\sigma},|s|)h^3\left(\|\phi\|_{1,\Gamma}+\|\psi\|_{2,\Gamma}+\|g_N\|_{1,\Gamma_N}+\|g_D\|_{2,\Gamma_D}\right).$$

Proof. In [52] the proof is given for an L_2 approximation of the Neumann datum by using piecewise constant discontinuous basis functions. Therefore the order of convergence is restricted to two. By reiterating the proof using the piecewise linear discontinuous basis functions as approximation the order can be increased to three without further assumptions onto the given Neumann datum g_N. □

The solution of the variational formulation (6.13) is obviously bounded. An improved bound, corresponding to s is obtained by reiterating the arguments from [31].

Theorem 6.11. *For $s\in\mathbb{C}^+_\sigma$ the solution of the variational formulation* (6.13) *is bounded by*

$$\|\phi_h\|_{-1/2,\Gamma}+\|\psi_h\|_{1/2,\Gamma} \leq c(\underline{\sigma})\,|s|^{5/2}\left(\|g_N\|_{-1/2,\Gamma_N}+\|g_D\|_{1/2,\Gamma_D}\right).$$

Moreover, $u_h = \mathrm{SL}(\mathrm{s})(\phi_h + P_h^{-1,1}\widetilde{g}_N) - \mathrm{DL}(\mathrm{s})(\psi_h + P_h^{0,1}\widetilde{g}_D)$ is bounded by

$$\|u_h\|_{1,\Omega} \leq c(\underline{\sigma})\,|s|^{5/2}\left(\|g_N\|_{-1/2,\Gamma_N}+\|g_D\|_{1/2,\Gamma_D}\right).$$

Proof. For the wave equation the corresponding bound is derived in [31]. We follow this proof closely. The proof is essentially an extension of the proof of Theorem 5.1 to discrete operators.

We start by definining the function

$$u_h = \mathrm{SL}(\mathrm{s})(\phi_h + P_h^{-1,1}\widetilde{g}_N) - \mathrm{DL}(\mathrm{s})(\psi_h + P_h^{0,1}\widetilde{g}_D).$$

Then the variational formulation (6.13) is equivalent to the following boundary value problem

$$\begin{aligned} \mathcal{P}\, u_h &= 0 && \text{in } \mathbb{R}^3 \setminus \Gamma, \\ \left\langle \gamma_0^- u_h, \eta \right\rangle_\Gamma &= \left\langle P_h^{0,1}\widetilde{g}_D, \eta \right\rangle_\Gamma && \text{for all } \eta \in S_h^{-1,0}(\Gamma_D) \\ \left\langle \gamma_1^+ u_h, v \right\rangle_\Gamma &= 0 && \text{for all } v \in S_h^{0,1}(\Gamma_N) \end{aligned} \tag{6.14}$$

with the transmission conditions

$$[\gamma_1 u_h]|_\Gamma - P_h^{-1,1}\widetilde{g}_N \in S_h^{-1,0}(\Gamma_D), \qquad [\gamma_0 u_h]|_\Gamma - P_h^{0,1}\widetilde{g}_D \in \widetilde{S}_h^{0,1}(\Gamma_N).$$

We denote the annihilator of a function space X by X°. The boundary value problem (6.14) is on the other hand equivalent to the following variational formulation:

Find $u_h \in [H^1(\mathbb{R}^3 \setminus \Gamma)]$ with $\left\langle \gamma_0^- u_h, v_h \right\rangle_\Gamma = \left\langle P_h^{0,1}\widetilde{g}_D, v_h \right\rangle_\Gamma$ for all $v_h \in S_h^{-1,0}(\Gamma_D)$ and $\langle [\gamma_0 u_h]_\Gamma, w_h \rangle_\Gamma = \left\langle P_h^{0,1}\widetilde{g}_D, w_h \right\rangle_\Gamma$ for all $w_h \in \left[\widetilde{S}_h^{0,1}(\Gamma_N)\right]^\circ$ such that

$$a_{\mathbb{R}^3\setminus\Gamma}(u_h, v) = \left\langle P_h^{-1,1}\widetilde{g}_N, \gamma_0^- v \right\rangle_\Gamma \tag{6.15}$$

for all $v \in [H^1(\mathbb{R}^3 \setminus \Gamma)]^4$ with $\gamma^- v \in \left[S_h^{-1,0}(\Gamma_D)\right]^\circ$ and $[\gamma_0 v]|_\Gamma \in \widetilde{S}_h^{0,1}(\Gamma_N)$.

The solution of this variational formulation can be estimated by repeating the arguments in the proof of Theorem 3.12. Finally the boundedness of the projection operators results in

$$|||u_h|||_{|\widetilde{s}|,\mathbb{R}^3\setminus\Gamma} \le c(\underline{\sigma})\, |s| \left(\|g_N\|_{-1/2,\Gamma_N} + |s|^{1/2} \|g_D\|_{1/2,\Gamma_D} \right).$$

Using estimates for traces, see Theorem 3.4 and Corollary 3.14, we obtain

$$\|\phi\|_{-1/2,\Gamma} + \|\psi_1\|_{1/2,\Gamma} \le c(\underline{\sigma})\, |s|^{5/2} \left(\|g_N\|_{-1/2,\Gamma_N} + \|g_D\|_{1/2,\Gamma_D} \right).$$

□

Pointwise evaluation

To obtain a convergence estimate for the pointwise error in the time domain, we need to have a bound for the point evaluation in the Laplace domain. These estimates are obtained by estimating the fundamental solution.

For this, an additional assumption on the material data is needed. We assume

$$\left|\alpha_1^2 - \alpha_2^2\right| \geq \frac{c(\sigma)}{|s|^2} \tag{6.16}$$

where α_1 and α_2 are given in (4.2) and correspond to the fast and slow compression wave. Assumption 6.16 is fulfilled by all materials considered within this work.

Lemma 6.12. *The point evaluation of the single layer potential at $\tilde{x} \in \Omega^-$ is a bounded linear functional with the absolute value bounded as*

$$|\mathrm{SL(s)}\,\phi(\tilde{x})| \leq c(\underline{\sigma}, \mathrm{dist}(\tilde{x}, \Gamma))\, |\mathrm{s}|^2 .$$

The point evaluation of the double layer potential at $\tilde{x} \in \Omega^-$ is a bounded linear functional with the absolute value bounded as

$$|\mathrm{DL(s)}\,\phi(\tilde{x})| \leq c(\underline{\sigma}, \mathrm{dist}(\tilde{x}, \Gamma))\, |\mathrm{s}|^2 .$$

Proof. The single layer potential can be estimated by

$$|\mathrm{SL(s)}\,\phi(\tilde{x})| = \left| \int_\Gamma G_s(\tilde{x}, y)\phi(y)\, \mathrm{d}s_y \right| \leq \|G_s(\tilde{x}, \cdot)\|_{\frac{1}{2},\Gamma} \|\phi\|_{-\frac{1}{2},\Gamma} .$$

As long as $\tilde{x} \in \Omega^-$, the fundamental solution $G_s(\tilde{x}, \cdot)$ is an element of $C^\infty(\Omega^+)$ and of $[H^1(\Omega^-)]^4$. The trace theorem (Theorem 3.4) can therefore be applied and yields

$$|\mathrm{SL(s)}\,\phi(\tilde{x})| \leq \|G_s(\tilde{x}, \cdot)\|_{1,\Omega^+} \|\phi\|_{-\frac{1}{2},\Gamma} .$$

With assumption (6.16) the different parts of the fundamental solution can be estimated by

$$\begin{aligned} \left|U_{ij}^E\right| &\leq c(\underline{\sigma}) e^{-\alpha|\tilde{x}-y|}, & \left|P_j\right| &\leq c(\underline{\sigma}) e^{-\alpha|\tilde{x}-y|}, \\ |U_i| &\leq c(\underline{\sigma})\, |s|\, e^{-\alpha|\tilde{x}-y|}, & |P^p| &\leq c(\underline{\sigma})\, |s|\, e^{-\alpha|\tilde{x}-y|}, \end{aligned}$$

with $\alpha = \max(\mathrm{Re}[\alpha_1], \mathrm{Re}[\alpha_2], \mathrm{Re}[\alpha_3])$. Due to assumption (4.3) $\alpha \geq c(\sigma) > 0$ which results in

$$\|G_s(\tilde{x}, \cdot)\|_{0,\Omega^+} \leq c(\sigma, \Omega^+)\, |s| .$$

Since $\alpha_{\{1,2,3\}} < c(\sigma)\,|s|$ all derivatives and thus the $[H^1(\Omega^+)]^4$-norm of the fundamental solution can be estimated by

$$\|G_s(\tilde{x},\cdot)\|_{1,\Omega^+} \leq c(\sigma,\Omega^+)\,|s|^2\,.$$

For the double layer potential we have

$$|\mathrm{DL}(s)\,\psi(x)| = \left| \int_\Gamma [\widetilde{\gamma}_1 G_s^*(\tilde{x},y)]^*\,\psi(y)\;\mathrm{d}s_y \right| \leq \|\widetilde{\gamma}_1 G_s^*(\tilde{x},y)\|_{-1/2,\Gamma}\,\|\psi\|_{\frac{1}{2},\Gamma}\,.$$

The adjoint of the fundamental solution is the fundamental solution of the adjoint problem. Therefore the fundamental solution fulfills the property $\widetilde{\mathcal{P}}G_s^*(\tilde{x},\cdot) = 0$ in Ω^+ for $x \in \Omega^-$. Lemma 3.15 results in

$$\|\widetilde{\gamma}_1 G_s^*(\tilde{x},y)\|_{-1/2,\Gamma} \lesssim |||G_s^*(\tilde{x},y)\,|||_{|s|,\Omega^+}\,.$$

By following the same estimates as above, we can show the desired estimate. □

6.5 Dirichlet boundary value problem

The Dirichlet problem (5.11)

$$\begin{aligned} \mathcal{P}u &= 0 && \text{in } \Omega^-, \\ \gamma_0^- u &= g_D && \text{on } \Gamma \end{aligned} \tag{6.17}$$

can be by solved by starting from the boundary integral equation (5.12), which results in in the following variational formulation:

Find $t_h \in S_h^{-1,0}(\Gamma)$ such that

$$\langle V(s)t_h,\tau_h\rangle_\Gamma = \left\langle \left(\frac{1}{2}I + K(s)\right)P_h^{0,1}g_D,\tau_h \right\rangle_\Gamma \tag{6.18}$$

for all $\tau_h \in S_h^{-1,0}(\Gamma)$.

Equation (6.18) can be rewritten in operator notation as

$$V_h(s)t_h = \left(\frac{1}{2}M_h + K_h(s)\right)P_h^{0,1}g_D. \tag{6.19}$$

Unique solvability of equation (6.19) follows from the ellipticity of the single layer potential (Theorem 4.8). A bound for this nonsymmetric realization of the Dirichlet to Neumann map can be obtained by refining the result for the mixed boundary value problem in Theorem 6.11.

Lemma 6.13. *The following property holds:*

$$V_h(s)^{-1}\left(\frac{1}{2}M_h + K_h(s)\right) \in \mathcal{A}\left(2, S_h^{0,1}(\Gamma), S_h^{-1,0}(\Gamma)\right).$$

Proof. Following the proof of Theorem 5.1 we first introduce

$$u_h = \mathrm{SL(s)}\,\phi_h - \mathrm{DL(s)}\,P_h^{0,1} g_D$$

and by estimating the variational formulation (6.15) we obtain

$$|||u_h|||_{\widetilde{|s|},\mathbb{R}^3\setminus\Gamma} \leq c(\underline{\sigma})\,|s|^{3/2}\,\|g_D\|_{1/2,\Gamma}\,. \tag{6.20}$$

Estimating the jump of the conormal derivative, Corollary 3.14, results in

$$\|\phi_h\|_{-1/2,\Gamma} \leq c(\underline{\sigma})\,|s|^2\,\|g_D\|_{1/2,\Gamma}\,,$$

and thus the desired estimate follow. □

For $\phi \in [H^1_{pw}(\Gamma)]^4$ and $g_D \in [H^2_{pw}(\Gamma)]^4$ Strang's Lemma 6.2 results in the error estimate

$$\|\phi_h - \phi\|_{-1/2,\Gamma} \leq c(\sigma) h^{3/2}\left(|s|^4\,\|\phi\|_{1,\Gamma} + |s|^{9/2}\,\|g_D\|_{2,\Gamma}\right). \tag{6.21}$$

An estimate for the L_2-norm can be obtained by the use of the inverse inequality,

$$\|\phi_h - \phi\|_{0,\Gamma} \leq c(\sigma) h\left(|s|^4\,\|\phi\|_{1,\Gamma} + |s|^{9/2}\,\|g_D\|_{2,\Gamma}\right), \tag{6.22}$$

see [52].

The solution inside the domain is given by the representation formula

$$u_h = \mathrm{SL(s)}\,\phi_h - \mathrm{DL(s)}\,P_h^{0,1} g_D$$

When assuming $u \in [H^{5/2}(\Omega)]^4$ and $\mathrm{dist}(\tilde{\mathrm{x}},\Gamma) > 0$ we have the error estimate

$$|u(\tilde{x}) - u_h(\tilde{x})| \leq c(\underline{\sigma},|s|)h^3\left(\|\phi\|_{1,\Gamma} + \|g_D\|_{2,\Gamma}\right) \tag{6.23}$$

for any $\tilde{x} \in \Omega$, see [52].

Indirect single layer approach

Another popular approach is an indirect single layer approach. Using the ansatz $\tilde{u} = \mathrm{SL}(\mathrm{s})\,\phi_h$ results in the variational formulation:

Find $\phi_h \in S_h^{-1,0}(\Gamma)$ such that

$$\langle V_h(s)\phi_h, \eta_h\rangle_\Gamma = \langle g_D, \eta_h\rangle_\Gamma$$

for all $\eta_h \in S_h^{-1,0}(\Gamma)$.

Unique solvability as well as the estimate

$$\|\phi_h\|_{-1/2,\Gamma} \leq c(\underline{\sigma})\,|s|^2\,\|g_D\|_{1/2,\Gamma}$$

is obtained by using the ellipticity estimate (Theorem 4.8). An estimate for the solution inside the domain is given by Corollary 6.6 which results in

$$\|\tilde{u}\|_{1,\Omega} \leq c(\underline{\sigma})\,|s|^2\,\|g_D\|_{1/2,\Gamma}\,.$$

An estimate for the pointwise evaluation inside the domain is given by combining Theorem 4.8 and Lemma 6.12, resulting in the estimate

$$|\tilde{u}(\tilde{x})| \lesssim c(\underline{\sigma})\,|s|^4\,\|g_D\|_{-1/2,\Gamma}$$

for $\tilde{x} \in \Omega$.

6.6 Neumann boundary value problem

According to Section 5.4 the Neumann problem (5.13)

$$\begin{aligned} \mathcal{P}u &= 0 && \text{in } \Omega^-, \\ \gamma_1^- u &= g_N && \text{on } \Gamma \end{aligned} \tag{6.24}$$

results in the following variational formulation:

Find $\psi_h \in S_h^{0,1}(\Gamma)$ such that

$$\langle D(s)\psi_h, v_h\rangle_\Gamma = \left\langle \left(\frac{1}{2}I - \widetilde{K}(s)^*\right) S_h^{-1,1} g_N, v_h \right\rangle_\Gamma$$

for all $v_h \in S_h^{0,1}(\Gamma)$,

or equivalently

$$D_h(s)\psi_h = \left(\frac{1}{2}M_h^T - \widetilde{K}_h(s)^*\right) P_h^{-1,1} g_N. \tag{6.25}$$

The ellipticity of the hyper-singular operator guarantees unique solvability. A bound for this nonsymmetric realization of the Neumann to Dirichlet operator is given in the following lemma.

Lemma 6.14. *The following property holds:*

$$D_h(s)^{-1}\left(\frac{1}{2}M_h^T - \widetilde{K}_h(s)^*\right) \in A(2, X_h, Y_h).$$

Proof. Following the proof of Theorem 5.1 we first introduce

$$u_h = \mathrm{SL}(\mathrm{s})\, P_h^{-1,1} g_N - \mathrm{DL}(\mathrm{s})\, \psi_h$$

and by estimating the equivalent variational formulation we obtain

$$|||u_h|||_{\widetilde{|s|}, \mathbb{R}^3 \setminus \Gamma} \leq c(\underline{\sigma})\, |s|\, \|g_N\|_{1/2,\Gamma}. \tag{6.26}$$

Notice that the estimate is better, corresponding to s, than the corresponding estimate for the Dirichlet problem (6.20). Combining the estimate (6.26) with the trace theorem, Theorem 3.4, and estimate (3.11) results in the estimate

$$\|\psi_h\|_{1/2,\Gamma} \leq c(\underline{\sigma})\, |s|^2\, \|g_N\|_{-1/2,\Gamma}.$$

Due to the slightly worse estimate for the jump of the Dirichlet trace we end up with a similar estimate as for the pure Dirichlet boundary value problem. □

When assuming $\psi \in [H^2_{pw}(\Gamma)]^4$ and $g_N \in [H^1_{pw}(\Gamma)]^4$ Strang's lemma (Lemma 6.2) results in the error estimate

$$\|\psi_h - \psi\|_{1/2,\Gamma} \leq c(\sigma) h^{3/2} \left(|s|^4 \|\psi\|_{2,\Gamma} + |s|^{7/2} \|g_N\|_{1,\Gamma}\right). \tag{6.27}$$

With the help of the Aubin-Nitsche trick an error estimate for the L_2-norm is given as

$$\|\psi_h - \psi\|_{0,\Gamma} \leq c(\sigma, s) h^2 \left(\|\psi\|_{2,\Gamma} + \|g_N\|_{1,\Gamma}\right). \tag{6.28}$$

see [52].

The solution inside the domain is approximated by

$$u_h = \mathrm{SL}(\mathrm{s})\, P_h^{-1,1} g_N - \mathrm{DL}(\mathrm{s})\, \psi.$$

When assuming $u \in [H^{5/2}(\Omega)]^4$ and $\mathrm{dist}(\tilde{\mathrm{x}}, \Gamma) > 0$ we have the error estimate

$$|u(\tilde{x}) - u_h(\tilde{x})| \leq c(\underline{\sigma}, |s|) h^3 \left(\|g_N\|_{1,\Gamma} + \|\psi\|_{2,\Gamma} \right) \tag{6.29}$$

for any $\tilde{x} \in \Omega$.

In [52] the estimate is done for a piecewise constant discontinuous approximation of the known Neumann datum, resulting in a lower convergence rate of two. By repeating the arguments when using an approximation of piecewise linear discontinuous basis functions of the known Neumann datum g_N the stated error estimate can be shown.

Indirect double layer approach

An indirect double layer approach $\tilde{u} = -\mathrm{DL}(\mathrm{s})\, \psi_h$ with $\psi_h \in S_h^{0,1}(\Gamma)$ results in the variational formulation:

Find $\psi_h \in S_h^{0,1}(\Gamma)$ such that

$$\langle D_h(s)\psi_h, v_h\rangle_\Gamma = \langle g_N, v_h\rangle_\Gamma$$

for all $v_h \in S_h^{0,1}(\Gamma)$.

Again ellipticity of the hyper-singular operator guarantees unique solvability. Theorem 4.10 and Corollary 6.6 yield the following estimates

$$\|\psi_h\|_{1/2,\Gamma} \lesssim c(\underline{\sigma})\, |s|^2\, \|g_N\|_{-1/2,\Gamma} \tag{6.30}$$

and

$$\|\tilde{u}\|_{1,\Omega} \lesssim c(\underline{\sigma})\, |s|^2\, \|g_n\|_{-1/2,\Gamma}\,.$$

An estimate for the pointwise evaluation inside the domain is given by combining estimate (6.30) and Lemma 6.12, resulting in the estimate

$$|\tilde{u}(\tilde{x})| \lesssim c(\underline{\sigma})\, |s|^4\, \|g_N\|_{-1/2,\Gamma}$$

for $\tilde{x} \in \Omega$.

7 CONVOLUTION QUADRATURE

Convolution quadrature is an approximation method for convolution integrals. It was developed by Christian Lubich in [32, 33] and applied to the wave equation in [34]. In the following chapter the method will be derived and important results will be stated.

7.1 The Convolution Quadrature Method (CQM)

Let $F(s)$ be an analytic function in the half-plane $\operatorname{Re}[s] > \rho_0$ such that the Laplace inversion formula

$$f(t) = \frac{1}{2\pi i} \int\limits_{\rho + i\mathbb{R}} e^{st} F(s) ds$$

exists for all $\rho > \rho_0$. $f(t)$ is a continuous and exponentially bounded function which vanishes for $t < 0$. To emphasize the dependency on the function $F(s)$ we denote the convolution as

$$F(\partial_t) g(t) := \int\limits_0^t f(t-\tau) g(\tau)\, \mathrm{d}\tau \tag{7.1}$$

The notation (7.1) emphasized the dependancy of the convolution onto the analytic function $F(s)$ in the Laplace domain. A justification for the notation comes from the fact that for $F(s) = s$ we have $\partial_t g = g'$ and from the composition rule

$$F(\partial_t) G(\partial_t) g = (F \cdot G)(\partial_t) g.$$

Parseval's formula gives us the following result:

Remark 7.1. *Assume that $F(s)$ is bounded by*

$$|F(s)| \leq C |s|^{\mu}$$

for all $\operatorname{Re}[s] \geq \sigma > 0$. The operator extends by density to a bounded linear operator

$$F(\partial_t) : H_0^{r+\mu}(0,T) \to H_0^r(0,T) \tag{7.2}$$

for all $r \in \mathbb{R}$. Insertion of the Laplace inversion formula into the convolution integral and applying Fubini's theorem results in

$$F(\partial_t)g(t) = \frac{1}{2\pi i}\int_0^t \int_{\rho+i\mathbb{R}} F(s)e^{s\tau}\,\mathrm{d}s\, g(t-\tau)\,\mathrm{d}\tau$$
$$= \frac{1}{2\pi i}\int_{\rho+i\mathbb{R}} F(s) \underbrace{\int_0^t e^{s\tau} g(t-\tau)\,\mathrm{d}\tau}_{=:y(t,s)}\,\mathrm{d}s.$$

The function $y(t,s)$ is the solution of the ordinary differential equation

$$y'(t) = sy(t) + g(t), \qquad y(0) = 0.$$

This ordinary differential equation can be discretized using a multistep method. We consider a constant time step grid with $t_n = n\Delta t$. A general linear multistep method is given by

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = \Delta t \sum_{j=0}^{k} \beta_j (s y_{n-j} + g((n-j)\Delta t)).$$

We multiply the sums with ξ^n and sum over n. We manipulate the resulting sum in the following way

$$\sum_{n=0}^{\infty}\sum_{j=0}^{k} \alpha_j y_{n-j}\xi^n = \sum_{j=0}^{k} \alpha_j \xi^j \sum_{n=0}^{\infty} y_{n-j}\xi^{n-j}.$$

The right hand side can be rewritten accordingly. Setting $y_n = 0$ and $g_n = 0$ for $n < 0$ and introducing $y(\xi) = \sum_{n\geq 0} y_n \xi^n$ and $g(\xi) = \sum_{n\geq 0} g(n\Delta t)$ results in

$$y(\xi)\sum_{j=0}^{k} \alpha_j \xi^j = h(sy(\xi) + g(\xi))\sum_{j=0}^{k} \beta_j \xi^j.$$

Introducing the quotient of the generating polynomials

$$\delta(\xi) = \frac{\sum_{j=0}^{k} \alpha_j \xi^j}{\sum_{j=0}^{k} \beta_j \xi^j}$$

we obtain

$$y(\xi) = \frac{g(\xi)}{\left(\frac{\delta(\xi)}{\Delta t} - s\right)}.$$

By utilizing Cauchy's integral formula we get an approximation of $F(\partial_t)$ as the n-th coefficient of a series expansion of

$$\frac{1}{2\pi i}\int\limits_{\rho+i\mathbb{R}} \frac{F(s)}{\frac{\delta(\xi)}{\Delta t}-s} g(\xi)\,\mathrm{d}s = F\left(\frac{\delta(\xi)}{\Delta t}\right) g(\xi).$$

Using the series expansion

$$F\left(\frac{\delta(\xi)}{\Delta t}\right) = \sum_{n=0}^{\infty} \omega_n \xi^n, \quad |\xi| < 1 \tag{7.3}$$

an approximation is defined by

$$\left(F\left(\partial_t^{\Delta t}\right) g^{\Delta t}\right)(t_n) := \sum_{j=0}^{n} \omega_{n-j} g(t_j). \tag{7.4}$$

The convergence order of the underlying multistep method is transferred to the convolution quadrature under the following assumptions, see [34]. The linear multistep method has to be A-stable, i. e. , $\mathrm{Re}\,[\delta(\xi)] > 0$ for $|\xi| < 1$ and $\delta(\xi)$ is not allowed to have poles on the unit circle. Due to A-stability we are restricted to multistep methods of order 2. We will use the backwards difference formula of order one (BDF1) and of order 2 (BDF2) in this work. Both fulfill the stated assumptions. The generating polynomials are given as

$$\delta_{BDF2}(\xi) = \frac{3}{2} - 2\xi + \frac{1}{2}\xi^2 \quad \text{and} \quad \delta_{BDF1}(\xi) = 1 - \xi.$$

Theorem 7.1. *Let $F(s) \in A(\mu, X, Y)$. The generating polynomial of the multistep method $\delta(\xi)$ has no poles along the unit circle and $\mathrm{Re}\,[\delta(\xi)] > 0$ for $|\xi| < 1$. For $g \in H_0^r(0,T)$ with $r > \frac{1}{2} + \max(\mu, 0)$, and $\beta = \min\left((r-\mu)\frac{p}{p+1}, r, p\right)$ we have*

$$\left\|F(\partial_t^{\Delta t})g(t) - F(\partial_t)g(t)\right\|_Y \leq C\Delta t^{\beta} \log(\Delta t)\, \|g\|_{H^r(0,T)} \quad \text{for } 0 \leq t \leq T$$

and

$$\left(h\sum_{i=0}^{N} \left\|F(\partial_t^{\Delta t})g(i\Delta t) - F(\partial_t)g(i\Delta t)\right\|_Y^2\right)^{1/2} \leq C\Delta t^{\beta}\, \|g\|_{H^r(0,T)}.$$

If the first two terms in the definition of β are strictly greater than p, the $\log(\Delta t)$ term in the first error estimate can be omitted.

Proof. See [34]. □

7.2 A decoupled system

Several methods have been presented to speed up the evaluation of the approximation (7.4), defined by a convolution quadrature approach. Different approaches are given in, e. g. , [22, 23, 25, 30]. The approach presented in this chapter was developed in [7]. The method was further extended in [5].

The weights in the series expansion (7.3) can be calculated by the Cauchy integral

$$\omega_k = \frac{1}{2\pi i} \oint_C F\left(\frac{\delta(\xi)}{\Delta t}\right) \xi^{-k-1}\, \mathrm{d}s$$

as proposed in [32]. Choosing the contour as a circle around the origin with $\xi = \lambda e^{i2\pi\alpha}$ and approximating the resulting integral with the trapezoidal rule in the points $\alpha_k = k/(N+1)$ results in the approximate weights

$$\omega_k^\lambda = \frac{\lambda^{-i}}{N+1} \sum_{j=0}^{N} F\left(\frac{e^{-i\frac{2\pi j}{N+1}}}{\Delta t}\right) e^{i\frac{2\pi jk}{N+1}}. \tag{7.5}$$

We will introduce additional approximations that allow us to decouple the set of equations. Starting from the definition of the convolution quadrature approximation

$$\left(F\left(\partial_t^{\Delta t}\right) g\right)(t_n) := \sum_{j=0}^{n} \omega_{n-j} g(t_j)$$

and using the approximate weights (7.5) we extend the sum to N by setting $\omega_j^\lambda = 0$ for $j < 0$. We end up with a new approximation

$$\left(F\left(\partial_{t,\lambda}^{\Delta t}\right) g\right)(t_n) := \sum_{j=0}^{N} \omega_{n-j}^\lambda g(t_j). \tag{7.6}$$

Introducing $\xi_{N+1} = e^{i\frac{2\pi}{N+1}}$ and $s_j = \dfrac{\delta(\xi_{N+1}^{-j})}{\Delta t}$ the new operator can be written as

$$\left(F\left(\partial_{t,\lambda}^{\Delta t}\right) g\right)(t_n) = \sum_{j=0}^{N} \frac{\lambda^{-n+j}}{N+1} \sum_{k=0}^{N} F(s_k)\xi^{k(n-j)} g_j = \sum_{k=0}^{N} \frac{\lambda^{-n}}{N+1} F(s_k)\xi^{kn} \sum_{j=0}^{N} \lambda^j \xi^{-kj} g_j$$

with $g_j = g(t_j)$.

The weighted discrete Laplace transform is given by

$$\mathcal{L}_{\Delta t}(g)_k := \sum_{j=0}^{N} \lambda^j \xi^{-kj} g_j.$$

Starting from the equation

$$\left(F\left(\partial_{t,\lambda}^{\Delta t}\right)g\right)(t_n) = h_n$$

and applying the weighted inverse Laplace transformation we end up with the set of decoupled equations

$$F(s_k)\mathcal{L}_{\Delta t}(g)_k = \mathcal{L}_{\Delta t}(h)_k \quad \text{for } k = 0,\ldots,N.$$

The error of the additional approximation can be bounded by the following lemma.

Lemma 7.2. *Let the multistep method be either the BDF2 or the BDF1. Let N denote the number of time steps,* $0 < \lambda < 1$ *and* $F \in \mathcal{A}(\mu,X,Y)$*, then*

$$\left\| F(\partial_{t,\lambda}^{\Delta t})g - F(\partial_t^{\Delta t})g \right\|_Y \leq C\frac{\lambda^{N+1}}{1-\lambda^{N+1}}\Delta t^{-1}\|g\|_{H^\mu(0,T)}$$

with C *depending on* T*.*

Proof. In [7] the proof was done for the inverse single layer potential and the backward difference formula of order 2. The extension to the general case is straight forward. □

Remark 7.2. *The discrete operator fulfills the composition rule, thus we have*

$$F(\partial_{t,\lambda}^{\Delta t})G(\partial_{t,\lambda}^{\Delta t}) = (FG)(\partial_{t,\lambda}^{\Delta t}).$$

Proof. See [7]. □

7.3 Galerkin discretization in space and convolution quadrature in time

In the previous chapters the convolution quadrature method was discussed. This method can be used to discretize convolution integrals arising from boundary integral equations. The necessary properties for all boundary integral operators have been established in Chapter 4, the properties have been transferred to their Galerkin discretizations in Chapter 6 and finally all necessary properties for several boundary integral formulations and their Galerkin discretizations have been established in Chapter 5 and Chapter 6. All boundary integral equations will be discretized in time by the convolution quadrature and in space by the Galerkin method. The necessary theory is established in this chapter. We will first discuss an abstract setting and finally apply this theory to different boundary integral formulations.

Let X be a Hilbert space and the operator

$$\hat{A} : X \to X^* \ \in \mathcal{A}(\mu,X,X^*). \tag{7.7}$$

We start with the operator equation

$$\hat{A}\hat{u} = \hat{f}$$

for $\hat{u} = \mathcal{L}u \in X$ and $\hat{f} = \mathcal{L}f \in X^*$ in Laplace domain and the corresponding Galerkin approximation

$$\hat{A}_h\hat{u}_h = \hat{f}_h,$$

as defined in (6.2).

Let the operator A_h be invertible with the property

$$\hat{A}_h^{-1} \in \mathcal{A}(\nu, X_h^*, X_h). \tag{7.8}$$

The Laplace inversion formula yields

$$A_h(\partial_t)u_h = f_h$$

with $u_h = \mathcal{L}^{-1}\hat{u}_h$ and $f_h = \mathcal{L}^{-1}\hat{f}_h$. Applying a time discretization as defined in (7.6) results in the fully discretized system

$$A_h(\partial_{t,\lambda}^{\Delta t})u_h^\lambda = f_h. \tag{7.9}$$

With the help of the composition rule, see Remark 7.2, this equation can be rewritten as

$$u_h^\lambda = A_h^{-1}(\partial_{t,\lambda}^{\Delta t}) f_h.$$

We have the mapping property $A_h^{-1} : H^{r+\nu}(0,T;X_h^*) \to H^r(0,T;X_h)$ and thus for $f_h \in H^{r+\nu}(0,T;X_h^*)$ we end up with $u_h \in H^r(0,T;X_h)$. The error of the Galerkin approximation is bounded, see Lemma 6.1,

$$\|u - u_h\|_{H^r(0,T;X)} \leq c \inf_{v_h \in X_h} \|u - v_h\|_{H^{r+\mu+\nu}(0,T;X)} . \tag{7.10}$$

where u is the solution of the equation

$$A(\partial_t)u = f. \tag{7.11}$$

The error in space and time is estimated in the following lemma.

Lemma 7.3. *Let the multistep method be either BDF1 or BDF2 and p its order. Let u be the solution of equation* (7.11) *and let u_h^λ be the solution of equation* (7.9). *We assume conditions* (7.7) *and* (7.8) *fulfilled, $f \in H^{\nu+p+1}(0,T,X)$ and $0 < \lambda < 1$. Introducing $\alpha = 1/2 + \mu + \nu + \varepsilon$ and $\beta = \nu + p + 1 + \varepsilon$ results in the following error estimate*

$$\begin{aligned} \left\| u_h^\lambda(t_i) - u(t_i) \right\|_X \leq c \Big[&\inf_{v_h \in X_h} \|u - v_h\|_{H^\alpha(0,T;X)} \\ &+ \Delta t^p \|f\|_{H^\beta(0,T;X^*)} \\ &+ \frac{\lambda^{N+1}}{1 - \lambda^{N+1}} \Delta t^{-1} \|f\|_{H^\nu(0,T;X^*)} \Big] \end{aligned}$$

for all $i = 1, \ldots, N$, with c depending on T.

Proof. The error can be split up as

$$\begin{aligned}\left\|u(t_n)-u_{h,n}^{\lambda}\right\|_X &= \left\|A^{-1}(\partial_t)f - A_h^{-1}(\partial_{t,\lambda}^{\Delta t})f\right\|_X \\ &\leq \left\|A^{-1}(\partial_t)f - A_h^{-1}(\partial_t)f\right\|_X \\ &+ \left\|A_h^{-1}(\partial_t)f - A_h^{-1}(\partial_t^{\Delta t})f\right\|_X \\ &+ \left\|A_h^{-1}(\partial_t^{\Delta t})f - A_h^{-1}(\partial_{t,\lambda}^{\Delta t})f\right\|_X\end{aligned}$$

and thus combining estimate (7.10), Theorem 7.1 and Lemma 7.2 yields the result. □

Remark 7.3. *Let the assumptions of Lemma 7.3 be valid and furthermore assume* $\lambda^N \sim \Delta t^{p+1}$*. For* $\alpha = 1/2+\mu+\nu+\varepsilon$ *and* $\beta = \nu+p+1+\varepsilon$ *the following error estimate holds*

$$\left\|u_h^{\lambda}(t_i)-u(t_i)\right\|_X \leq c\left(\inf_{v_h\in X_h}\|u-v_h\|_{H^{\alpha}(0,T;X)} + \Delta t^p\,\|f\|_{H^{\beta}(0,T;X^*)}\right).$$

8 TIME DOMAIN

In this chapter different boundary value problems will be discussed in time domain. Parseval's formula allows us to transfer the results from Laplace domain to time domain. We will prove unique solvability for all discussed problems, formulate the boundary integral equations, prove unique solvability for the continuous integral equations and for their Galerkin discretizations. Finally, by application of Lemma 7.3 and Remark 7.3, error estimates for the error of the space and time discretization can be given.

8.1 The mixed boundary value problem

The mixed problem in time domain is given as

$$
\begin{aligned}
\hat{\mathcal{P}}\hat{u}(x,t) &= 0 && \text{for } x\in\Omega^-, \quad t\in(0,T),\\
\gamma_0^-\hat{u}(x,t) &= \hat{g}_D(x,t) && \text{for } x\in\Gamma_D, \quad t\in(0,T),\\
\hat{\gamma}_1^-\hat{u}(x,t) &= \hat{g}_N(x,t) && \text{for } x\in\Gamma_N, \quad t\in(0,T),\\
\hat{u}(x,0) &= 0 && \text{for } x\in\Omega,\\
\hat{u}'(x,0) &= 0 && \text{for } x\in\Omega.
\end{aligned}
\tag{8.1}
$$

For $\hat{u}(x,t)\in H_0^r(0,T;[H^1(\Omega)]^4)$ unique solvability is a direct consequence of unique solvability in Laplace domain, see Corollary 3.12. Remark 7.1 gives us the following bound for the solution:

$$\|\hat{u}\|_{r,1,\Omega} \le c\left(\|\hat{g}_D\|_{r+\frac{3}{2},-\frac{1}{2},\Gamma_D} + \|\hat{g}_N\|_{r+1,-\frac{1}{2},\Gamma_N}\right).$$

The notation of the norm was introduced in (3.13).

The system of integral equations in time domain is given by

$$
\begin{aligned}
V(\partial_t)\hat{\phi} - K(\partial_t)\hat{\psi} &= \left(\frac{1}{2}I(\partial_t)+K(\partial_t)\right)\tilde{g}_D - V(\partial_t)\tilde{g}_N \quad && \text{on } \Gamma_D\times(0,T),\\
\widetilde{K}^*(\partial_t)\phi + D(\partial_t)\psi &= \left(\frac{1}{2}I(\partial_t)-\widetilde{K}^*(\partial_t)\right)\tilde{g}_N - D(\partial_t)\tilde{g}_D \quad && \text{on } \Gamma_N\times(0,T)
\end{aligned}
$$

with the unknown Neumann datum $\hat{\phi}\in H_0^r(0,T,[\widetilde{H}^{-1/2}(\Gamma_D)]^4)$, the unknown Dirichlet datum $\hat{\psi}\in H_0^r(0,T,[\widetilde{H}^{1/2}(\Gamma_N)]^4)$, the extension of the given Dirichlet datum $\hat{g}_D\in$

$H_0^{r+3/2}(0,T,[H^{1/2}(\Gamma_D)]^4)$ to $\widetilde{g}_D \in H_0^{r+3/2}(0,T,[H^{1/2}(\Gamma)]^4)$ and the extension of the given Neumann datum $\hat{g}_N \in H_0^{r+1}(0,T,[H^{-1/2}(\Gamma_N)]^4)$ to $\widetilde{g}_N \in H_0^{r+1}(0,T,[H^{-1/2}(\Gamma)]^4)$ for any $t \in (0,T)$ and $r \in \mathbb{R}$.

The system of boundary integral equations is again uniquely solvable due to Theorem 4.14 and the solution is bounded by

$$\left\|\hat{\phi}\right\|_{r,-\frac{1}{2},\Gamma_N} + \|\hat{\psi}\|_{r,\frac{1}{2},\Gamma_D} \leq C\left(\frac{|s|}{\sigma}\|\hat{g}_N\|_{r+1,-\frac{1}{2},\Gamma_N}\frac{|s|^{3/2}}{\sigma}\|\hat{g}_D\|_{r+\frac{3}{2},\frac{1}{2},\Gamma_D}\right)$$

for all $r \in \mathbb{R}$. The discrete decoupled system at time steps $t_n = n\Delta t$, $n = 1,\dots,N+1$ is given by

$$\begin{aligned} V_h(\partial_{t,\lambda}^{\Delta t})\hat{\phi}_h^{\Delta t} - K_h(\partial_{t,\lambda}^{\Delta t})\hat{\psi}_h^{\Delta t} &= \left(\frac{1}{2}M_h(\partial_{t,\lambda}^{\Delta t}) + K_h(\partial_{t,\lambda}^{\Delta t})\right)P_h^{0,1}\tilde{g}_D - V_h(\partial_{t,\lambda}^{\Delta t})P_h^{-1,1}\tilde{g}_N \quad \text{on } \Gamma_D,\\ \widetilde{K}_h^*(\partial_{t,\lambda}^{\Delta t})\phi_h^{\Delta t} + D_h(\partial_{t,\lambda}^{\Delta t})\psi_h^{\Delta t} &= \left(\frac{1}{2}M_h^\top(\partial_{t,\lambda}^{\Delta t}) - \widetilde{K}_h^*(\partial_{t,\lambda}^{\Delta t})\right)P_h^{-1,1}\tilde{g}_N - D_h(\partial_{t,\lambda}^{\Delta t})P_h^{0,1}\tilde{g}_D \quad \text{on } \Gamma_N. \end{aligned} \tag{8.2}$$

Unique solvability can be proven due to Theorem 4.14 and the composition rule, Remark 7.2. The Galerkin spaces will be chosen as stated in Section 6.4. When assuming $u(t,\cdot) \in [H^{5/2}(\Omega)]^4$, Remark 7.3 combined with Corollary 6.9, Theorem 5.1 and Theorem 6.11 results in the following error estimate for the discrete solution at the time step $t_n = n\Delta t$.

$$\begin{aligned} &\left\|\psi(t_n) - \psi_h^{\Delta t}(t_n)\right\|_{-\frac{1}{2},\Gamma_D} + \left\|\phi(t_n) - \phi_h^{\Delta t}(t_n)\right\|_{\frac{1}{2},\Gamma_N}\\ &\leq ch^{3/2}\left[\|\psi\|_{11/2+\varepsilon,1,\Gamma} + \|\phi\|_{11/2+\varepsilon,2,\Gamma}\right] + c\Delta t^p\left[\|\hat{g}_N\|_{\frac{7}{2}+p+\varepsilon,-\frac{1}{2},\Gamma_N}\|\hat{g}_D\|_{\frac{7}{2}+p+\varepsilon,\frac{1}{2},\Gamma_D}\right] \end{aligned} \tag{8.3}$$

For a reduced order in space or time a reduced order of convergence can be deduced.

8.2 Dirichlet boundary value problem

The Dirichlet problem in time domain is given by

$$\begin{aligned} \hat{\mathcal{P}}\hat{u}(x,t) &= 0 && \text{for } x \in \Omega^-, \quad t \in (0,T),\\ \gamma_0^-\hat{u}(x,t) &= \hat{g}_D(x,t) && \text{for } x \in \Gamma, \quad t \in (0,T),\\ \hat{u}(x,0) &= 0 && \text{for } x \in \Omega,\\ \hat{u}'(x,0) &= 0 && \text{for } x \in \Omega. \end{aligned} \tag{8.4}$$

For $u \in H_0^r(0,T;[H^1(\Omega)]^4)$ unique solvability is a direct consequence of unique solvability in Laplace domain, see Corollary 3.12. The corresponding boundary integral equation is given by

$$V(\partial_t)\hat{\psi} = \left(\frac{1}{2}I(\partial_t) + K(\partial_t)\right)\hat{g}_D$$

with the unknown Neumann datum $\psi \in H_0^r(0,T,[H^{-1/2}(\Gamma)]^4)$. Unique solvability is guaranteed by Theorem 4.8 with the estimate

$$\|\hat{\psi}\|_{r,-\frac{1}{2},\Gamma} \leq c\,\|\hat{g}_D\|_{r+2,\frac{1}{2},\Gamma}$$

which is obtained with the help of Lemma 6.13. Choosing discrete subspaces as in Section 6.5 we end up with the discrete equation at the timesteps $t_n = n\Delta t$

$$V_h(\partial_{t,\lambda}^{\Delta t})\hat{\psi}_h = \left(\frac{1}{2}M_h(\partial_{t,\lambda}^{\Delta t}) + K_h(\partial_{t,\lambda}^{\Delta t})\right)P_h^{0,1}\hat{g}_D,$$

which is unique solvable with the estimate

$$\|\hat{\psi}_h\|_{r,-\frac{1}{2},\Gamma} \leq c\,\|\hat{g}_D\|_{r+2,\frac{1}{2},\Gamma}.$$

When assuming $\hat{\psi} \in H_0^{5+\varepsilon}(0,T,[H^1(\Gamma)]^4)$ and $\hat{g}_D \in H_0^{9/2+\varepsilon}(0,T,[H_{pw}^2(\Gamma)]^4)$ we can apply estimate (6.21) and Lemma 6.13, thus obtaining

$$\left\|\hat{\psi}(t_n) - \hat{\psi}_h^{\Delta t}(t_n)\right\|_{-\frac{1}{2},\Gamma} \leq c\left[h^{3/2}\left(\|\hat{\psi}\|_{9/2+\varepsilon,1,\Gamma} + \|\hat{g}_D\|_{5+\varepsilon,2,\Gamma}\right) + \Delta t^p\,\|\hat{g}_D\|_{3+p+\varepsilon,\frac{1}{2},\Gamma}\right]. \tag{8.5}$$

8.3 Neumann problem

The Neumann problem in time domain is given by

$$\begin{aligned}
\hat{\mathcal{P}}\hat{u}(x,t) &= 0 && \text{for } x \in \Omega^-, \quad t \in (0,T), \\
\gamma_0^-\hat{u}(x,t) &= \hat{g}_N(x,t) && \text{for } x \in \Gamma, \quad t \in (0,T), \\
\hat{u}(x,0) &= 0 && \text{for } x \in \Omega, \\
\hat{u}'(x,0) &= 0 && \text{for } x \in \Omega.
\end{aligned} \tag{8.6}$$

The corresponding boundary integral equation in time domain is given by

$$D(\partial_t)\hat{\phi} = \left(\frac{1}{2}I(\partial_t) - \widetilde{K}^*(\partial_t)\right)P_h^{-1,1}\hat{g}_N$$

with the unknown Dirichlet datum $\hat{\phi} \in H^r(0,T,[H^{1/2}(\Gamma)]^4)$. Theorem 4.10 guarantees unique solvability with the estimate

$$\|\hat{\phi}\|_{r,\frac{1}{2},\Gamma} \leq c\,\|\hat{g}_N\|_{r+2,-\frac{1}{2},\Gamma},$$

which is a direct consequence of Lemma 6.14. We choose the same discrete subspaces as in Section 6.6 and end up with a full discretized system

$$D_h(\partial_{t,\lambda}^{\Delta t})\hat{\phi}_h = \left(\frac{1}{2}M_h^T(\partial_{t,\lambda}^{\Delta t}) - \widetilde{K}_h^*(\partial_{t,\lambda}^{\Delta t})\right)P_h^{-1,1}\hat{g}_N$$

for the timesteps $t_n = n\Delta t$. The discrete system is uniquely solvable with the following estimate for the solution.

$$\left\|\hat{\phi}_h\right\|_{r,\frac{1}{2},\Gamma} \leq c\left\|\hat{g}_N\right\|_{r+2,-\frac{1}{2},\Gamma}$$

Assuming $\hat{\phi} \in H_0^{9/2+\varepsilon}(0,T,[H_{pw}^1(\Omega)]^4)$ and $\hat{g}_N \in H_0^{5/2+\max(p,3/2)+\varepsilon}(0,T,[H_{pw}^2(\Gamma)]^4)$ and using estimate (6.27) and Lemma 6.14, we end up with the error estimate

$$\left\|\hat{\phi}(t_n) - \hat{\phi}_h^{\Delta t}(t_n)\right\|_{\frac{1}{2},\Gamma} \leq ch^{3/2}\left(\left\|\hat{\phi}\right\|_{\frac{9}{2}+\varepsilon,1,\Gamma} + \left\|\hat{g}_N\right\|_{4,1,\Gamma}\right) + c\Delta t^p \left\|\hat{g}_N\right\|_{\frac{5}{2}+p+\varepsilon,\frac{1}{2},\Gamma}. \qquad (8.7)$$

9 NUMERICAL EXAMPLES

In this chapter the convergence results from the Chapter 8 are confirmed with the help of numerical examples. In addition to the presented approach we study a collocation approach, which is derived in Section 9.2. The error of the space discretization is discussed first, followed by a discussion of the error in time.

9.1 On the implementation

The discussed algorithms were implemented in the software library HyENA [27]. The integral operators were realized by the Duffy transformation, see [20]. The double layer potential, adjoint double layer potential and the hyper-singular operator where regularized through partial integration, see [39, 40].

To reduce computational and storage complexity fast methods have been utilized. The first fast methods, which were developed, are the Fast Multipole Method, see [42] and references therein, and the Panel-Clustering method [24]. In the HyENA library the $\mathcal{H}$-Matrices [11, 21, 44] are utilized, to be more precise the Adaptive Cross Approximation (ACA) [10] as implemented in the AHMED library [9] is used.

The different parameters differ greatly in the order of magnitude, see e. g. , Table 9.1 for Berea sand stone. A direct discretization leads to system matrices with condition numbers higher than 10^{20}. Direct solvers still succeed, at least most of the time, whereas iterative solvers, as they are used in our code, simply fail. A variable transformation from [28] is applied, which results in reasonable conditioned matrices. Additionally preconditioners are applied. The single layer potential is preconditioner by an artificial multilevel preconditioner [50] and the hyper-singular operator is preconditioned by an operator of inverse order [53].

9.2 A collocation approach

The collocation approach is still very popular especially in engineering applications. Starting from an operator equation

$$Au = f$$

with $A : X \to X^*$ and $f \in X^*$ we can restrict u_h to X_h. Instead of using test functions we require the equation to be fulfilled in collocation points x_j, $j \in I$ resulting in the discrete system

$$A^C u_h(x_j) = f(x_j) \quad \text{for all } j \in J.$$

For a basis $\phi_i, i \in I$, of the space X_h the matrix has the entries

$$A^C_{ij} = (A\phi_i)(x_j) \quad i, j \in I$$

On general Lipschitz domains the solvability of the equation and thus the stability of the numerical scheme is still an open question. Stability is only known for special cases, for a more detailed discussion see, e. .g. , [2, 18, 43].

For the mixed boundary value problem

$$\begin{aligned} \mathcal{P}u &= 0 && \text{in } \Omega^-, \\ \gamma_0^- u &= g_D && \text{on } \Gamma_D, \\ \gamma_1^- u &= g_N && \text{on } \Gamma_N. \end{aligned} \tag{9.1}$$

we start with the first integral equation, see [38],

$$0 = V(s)\gamma_1 u - (\sigma I + K(s))\, u \tag{9.2}$$

where

$$\sigma(x) = \lim_{\varepsilon \to 0} \int_{y \in \Omega : |y-x| = \varepsilon} [\widetilde{\gamma}_1 G_s^*(x,y)]^* \, ds_y.$$

The term σ degenerates to $1/2$ on C^2 surfaces, see [38], however on corners and edges this simple relation is not true. Thus the jump term σ is equal to $1/2$ almost everywhere.

We choose appropriate extensions $\widetilde{g}_D \in [H^{1/2}(\Gamma)]^4$ and $\widetilde{g}_N \in [H^{-1/2}(\Gamma)]^4$ for the Dirichlet and Neumann data $g_D \in [H^{1/2}(\Gamma_D)]^4$ and $g_N \in [H^{-1/2}(\Gamma_N)]^4$ such that $\widetilde{g}_D = g_D$ on Γ_D and $\widetilde{g}_N = g_N$ on Γ_N. We define the unknowns

$$\tilde{t} = \gamma_1 u - \widetilde{g}_N \text{ and } \tilde{u} = u - \widetilde{g}_D.$$

Insertion into the first integral equation (9.2) results in

$$V(s)\tilde{t} - (\sigma I + K(s))\tilde{u} = (\sigma I + K(s))\widetilde{g}_D - V(s)\widetilde{g}_N. \tag{9.3}$$

As for the Galerkin approach we choose lowest order ansatz functions $\widetilde{t}_h \in S_h^{-1,0}(\Gamma_D)$ and $\widetilde{u}_h \in S_h^{0,1}(\Gamma_N)$ to approximate the unknown functions. Equation (9.3) is discretized using the collocation approach. To end up with a quadratic system the collocation points are chosen in the following way. On Γ_D we choose the centre of the triangles x_i, $i = 1, \ldots, N_D$ and for Γ_N we choose the points of the mesh itself y_j, $i = 1, \ldots, M_N$. We denote the number of elements on the Dirichlet boundary with N_D and the number of elements on the Neumann boundary with N_N respectively. The number of nodes on the Dirichlet and Neumann boundary are denoted by M_D and M_N. In the same way as for the Galerkin approach the Dirichlet data $\widetilde{g}_D$ is approximated by $P_h^{0,1}\widetilde{g}_D$ and the Neumann data $\widetilde{g}_N$ is approximated by $P_h^{-1,1}\widetilde{g}_N$.

We end up with the linear system of equations

$$\begin{bmatrix} V_{DD}^C(s) & K_{ND}^C(s) \\ V_{DN}^C(s) & \sigma_{NN}+K_{NN}^C(s) \end{bmatrix} \begin{bmatrix} \underline{\widetilde{t}} \\ \underline{\widetilde{u}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}I_{DD}^C + K_{DD}^C(s) & -V_{ND}^C(s) \\ K_{DN}^C(s) & -V_{NN}^C(s) \end{bmatrix} \begin{bmatrix} P_h^{0,1}\widetilde{g}_D \\ P_h^{-1,1}\widetilde{g}_N \end{bmatrix}$$

with the system matrices

$$\begin{aligned}
[V_{DD}^C(s)]_{ii} &= (V(s)\psi_i^{-1,0})(x_i) && \text{for } i = 1,\dots,N_D, \\
[V_{DN}^C(s)]_{ik} &= (V(s)\psi_i^{-1,0})(x_k) && \text{for } k = 1,\dots,M_N \text{ and } i = 1,\dots,N_D, \\
[K_{ND}^C(s)]_{ki} &= (K(s)\psi_k^{0,1})(x_i) && \text{for } i = 1,\dots,N_D \text{ and } k = 1,\dots,M_N, \\
[K_{NN}^C(s)]_{kk} &= (K(s)\psi_k^{0,1})(x_k) && \text{for } k = 1,\dots,M_N, \\
[\sigma_{NN}]_{kk} &= (\sigma\psi_k^{0,1})(x_k) && \text{for } k = 1,\dots,M_N
\end{aligned}$$

and the matrices for the right hand side

$$\begin{aligned}
[V_{NN}^C(s)]_{\ell k} &= (V(s)\psi_\ell^{-1,1})(x_k) && \text{for } k = 1,\dots,M_N \text{ and } \ell = 1,\dots,N_N, \\
[V_{ND}^C(s)]_{\ell i} &= (V(s)\psi_\ell^{-1,1})(x_i) && \text{for } i = 1,\dots,N_D \text{ and } \ell = 1,\dots,N_N, \\
[K_{DD}^C(s)]_{ji} &= (K(s)\psi_j^{0,1})(x_i) && \text{for } i = 1,\dots,N_D \text{ and } j = 1,\dots,M_D, \\
[I_{DD}]_{ij} &= (\psi_j^{0,1})(x_i) && \text{for } i = 1,\dots,N_D \text{ and } j = 1,\dots,M_D, \\
[K_{DN}^C(s)]_{jk} &= (K(s)\psi_j^{0,1})(x_k) && \text{for } k = 1,\dots,M_N \text{ and } j = 1,\dots,M_D.
\end{aligned}$$

A formula for the evaluation of the jump term $\sigma(x)$ is given in [37]. On the right hand side the jump term is evaluated in the centre of the triangles, thus the jump term is simply $1/2$.

The final system of linear equations is given in Laplace domain. A convolution quadrature approach, see Chapter 7, is used obtain a solution in time domain.

9.3 Laplace domain

We start by examining different problems for a fixed Laplace parameter s in Laplace domain. The resulting error represents the error in space. For a fixed frequency s an analytical solution is well known, namely the fundamental solution $G(\tilde{x},\cdot)$ fixed in a point $\tilde{x}\in\mathbb{R}^3\setminus\overline{\Omega}$. Actually each column of the fundamental solution is a solution of the problem. We chose the last column. The unit cube $\Omega=(-0.5,0.5)^3$ is chosen as the domain Ω. As material we chose Berea sand stone as given in Table 9.1, see [29].

	λ	μ	ρ	ϕ	α	ρ_f	R	κ
rock	$4\cdot 10^9$	$6\cdot 10^9$	2458	0.19	0.778	1000	$4.885\cdot 10^8$	$1.9\cdot 10^{-10}$

Table 9.1: Material properties of Berea sand stone.

Starting from an initial mesh consisting of 12 triangles the surface of the cube is uniformly refined. We chose the point $\tilde{x}$ equal to $(0.3,0.13,1.5)$. The solution is evaluated on 413 nodes residing on a second cube $(-0.375,0.375)^3$ inside the domain Ω. The error is observed for the unknowns of the solid and the fluid separately. The solid displacement u and the pore pressure p is examined on the boundary Γ and pointwise in the domain Ω. Furthermore we study the Neumann trace split up into the traction $t=\gamma_1^\mu u$ and the flux $q=-\gamma_1^p p$, see (2.17) and (2.18) respectively.

9.3.1 Dirichlet boundary value problem

In this chapter we discuss the Dirichlet problem (5.11) The Laplace parameter is fixed at $s=2+1i$. We compare the results of the Galerkin approach given in (6.19)

$$V_h(s)\phi_h=\left(\frac{1}{2}M_h+K_h\right)P_h^{0,1}g_D$$

with the collocation approach discussed in Section 9.2

$$V_h^C(s)\phi_h^C=\left(\frac{1}{2}I_h^C+K_h^C\right)P_h^{0,1}g_D.$$

In case of the Dirichlet problem the original boundary integral equations are the same, only the space discretization is different. We compare the relative $L_2(\Gamma)$-error on the boundary of the unknown Neumann data ϕ_h and ϕ_h^C. For the Galerkin approach the theory implies a convergence rate of one, see (6.22). The Neumann trace ϕ is split up into the traction $t=(\phi_1,\phi_2,\phi_3)$ and the flux $q=-\phi_4$. We split up the discrete Neumann traces accordingly. We denote the relative L_2-error by

$$e_t=\frac{\|t-t_h\|_{0,\Gamma}}{\|t\|_{0,\Gamma}},\qquad e_q=\frac{\|q-q_h\|_{0,\Gamma}}{\|q\|_{0,\Gamma}}.$$

The results are stated in Table 9.2.

	Collocation				Galerkin			
#DOF	e_t	eoc	e_q	eoc	e_t	eoc	e_q	eoc
48	4.75e-01		6.73e-01		4.78e-01		6.67e-01	
192	2.50e-01	0.92	3.41e-01	0.98	2.55e-01	0.91	3.55e-01	0.91
768	1.18e-01	1.08	1.46e-01	1.22	1.21e-01	1.08	1.55e-01	1.20
3072	5.73e-02	1.05	6.39e-02	1.20	5.80e-02	1.06	6.66e-02	1.21
12288	2.81e-02	1.03	2.89e-02	1.14	2.83e-02	1.04	2.97e-02	1.16

Table 9.2: Comparison of the relative $L_2(\Gamma)$-errors for the Neumann trace.

The collocation approach produces slightly lower errors than the Galerkin approach, the difference however is almost negligible. The convergence orders match quite well with the theoretical bounds. Furthermore we compare the errors for a point evaluation of the unknown function

$$U = \mathrm{SL}(\mathrm{s})\,\phi_h - \mathrm{DL}(\mathrm{s})\,P_h^{0,1} g_D$$

inside the domain. We split the error into the solid displacement $u = (U_1, U_2, U_3)^\top$ and the pore pressure $p = U_4$.

The pointwise error is defined as

$$pe_u = \frac{\sqrt{\sum_i |u(\tilde{x}_i) - u_h(\tilde{x}_i)|^2}}{\sqrt{\sum_i |u(\tilde{x}_i)|^2}}, \qquad pe_t = \frac{\sqrt{\sum_i |t(\tilde{x}_i) - t_h(\tilde{x}_i)|^2}}{\sqrt{\sum_i |t(\tilde{x}_i)|^2}}, \tag{9.4}$$

where $i = 1, \ldots, 413$ are the different evaluation nodes. Equation (6.23) implies an optimal convergence order of three for the Galerkin approach. The results are stated in Table 9.3.

	Collocation				Galerkin			
#DOF	pe_u	eoc	pe_p	eoc	pe_u	eoc	pe_p	eoc
48	6.79e-02		1.14e-01		6.44e-02		1.13e-01	
192	1.65e-02	2.05	2.03e-02	2.49	1.36e-02	2.25	1.90e-02	2.58
768	2.49e-03	2.73	1.86e-03	3.45	1.04e-03	3.70	8.67e-04	4.45
3072	5.11e-04	2.28	4.23e-04	2.13	8.00e-05	3.70	6.82e-05	3.67
12288	1.19e-04	2.10	1.02e-04	2.05	8.10e-06	3.30	7.34e-06	3.22

Table 9.3: Comparison of the relative ℓ_2-errors for the point evaluation.

The convergence rate for the Galerkin approach starts rather high but seems to retreat to three. The convergence rate of the collocation approach on the other hand tends towards two. The errors for the Galerkin approach are therefore significantly better. To achieve the desired convergence rate for the Galerkin approach it is important to project the incoming Dirichlet data g_D into the discrete subspace $S_h^{0,1}$. Replacing this L_2-projection by an interpolation into the space $S_h^{0,1}$ results convergence orders as stated in Tables 9.5 and 9.4.

	Collocation				Galerkin			
#DOF	e_t	eoc	e_q	eoc	e_t	eoc	e_q	eoc
48	5.61e-01		7.48e-01		5.64e-01		7.45e-01	
192	2.76e-01	1.02	4.05e-01	0.89	2.77e-01	1.03	4.06e-01	0.88
768	1.28e-01	1.12	1.78e-01	1.18	1.27e-01	1.12	1.78e-01	1.19
3072	5.99e-02	1.09	7.51e-02	1.24	5.99e-02	1.09	7.50e-02	1.24
12288	2.88e-02	1.05	3.24e-02	1.21	2.88e-02	1.06	3.24e-02	1.21

Table 9.4: $L_2(\Gamma)$-errors of the Galerkin approach with interpolation of the given Dirichlet data.

The $L_2(\Gamma)$-error as presented in Table 9.4 increases slightly, the impact is however not as significant.

	Collocation				Galerkin			
#DOF	pe_u	eoc	pe_p	eoc	pe_u	eoc	pe_p	eoc
48	1.96e-01		1.81e-01		1.81e-01		1.71e-01	
192	5.03e-02	1.96	4.36e-02	2.05	4.55e-02	2.00	4.01e-02	2.09
768	1.26e-02	1.99	1.05e-02	2.05	1.10e-02	2.05	9.15e-03	2.13
3072	3.15e-03	2.00	2.63e-03	2.00	2.71e-03	2.02	2.26e-03	2.02
12288	7.85e-04	2.00	6.58e-04	2.00	6.73e-04	2.01	5.61e-04	2.01

Table 9.5: ℓ_2-errors of the Galerkin approach with interpolation of the given Dirichlet data.

From the errors in table 9.5 we deduce that the pointwise error increases for both approaches. The impact on the Galerkin approach however is more significant, since the convergence rate for the Galerkin approach is reduced to two. This effect was already studied in [52].

9.3.2 Neumann boundary value problem

In this section the Neumann problem in Laplace domain (5.13) is discussed. The Laplace parameter is fixed at $100+200i$. The Galerkin approach for the Neumann problem (6.28)

$$D_h \psi_h = \left(\frac{1}{2} M_h^T - \widetilde{K}_h^*\right) P_h^{-1,1} g_N$$

is compared with the collocation approach from Section 9.2

$$\left(\frac{1}{2}\sigma_h + K_h^C\right)\psi_h^C = VP_h^{-1,1} g_N.$$

In addition to different spatial discretizations, different boundary integral equations are compared. The unknown function is split up into the solid displacement $u = (\psi_1, \psi_2, \psi_3)$ and the pore pressure $p = \psi_4$. Estimate (6.28) implies a convergence rate of two of the Galerkin approach. The error is given as

$$e_u = \frac{\|u - u_h\|_{0,\Gamma}}{\|u\|_{0,\Gamma}}, \qquad e_p = \frac{\|p - p_h\|_{0,\Gamma}}{\|p\|_{0,\Gamma}}.$$

The errors are presented in Table 9.6.

	Collocation				Galerkin			
#DOF	e_u	eoc	e_p	eoc	e_u	eoc	e_p	eoc
32	4.32e+01		1.01e+01		2.55e-01		3.90e+00	
104	1.34e+01	1.68	2.35e+00	2.10	9.21e-02	1.47	1.72e+00	1.18
392	3.40e+00	1.98	7.51e-01	1.65	1.99e-02	2.21	4.04e-01	2.09
1544	9.01e-01	1.92	2.18e-01	1.78	4.26e-03	2.22	6.35e-02	2.67
6152	2.41e-01	1.90	5.74e-02	1.93	9.73e-04	2.13	1.06e-02	2.58

Table 9.6: Comparison of the relative L_2-errors for the Dirichlet trace.

The convergence rate of the Galerkin approach for the solid displacement is in good agreement with the theory. The convergence rate of the pore pressure is higher than expected. The collocation approach results in significant larger errors, the convergence rate however seems to tend towards two as well. Additionally the errors for the point evaluation inside the domain are given in Table 9.7.

	Collocation				Galerkin			
#DOF	pe_u	eoc	pe_p	eoc	pe_u	eoc	pe_p	eoc
32	3.54e+01		3.61e+01		8.74e-02		7.51e+00	
104	1.12e+01	1.65	4.85e+00	2.90	2.89e-02	1.60	2.91e+00	1.37
392	2.91e+00	1.95	9.75e-01	2.31	2.73e-03	3.40	3.15e-01	3.21
1544	7.78e-01	1.91	2.33e-01	2.06	2.17e-04	3.65	2.36e-02	3.74
6152	2.08e-01	1.91	5.79e-02	2.01	2.30e-05	3.24	1.40e-03	4.07

Table 9.7: Comparison of the relative ℓ_2-errors for the point evaluation.

Estimate (6.29) predicts an optimal convergence rate of three for the Galerkin approach. The convergence rate is indeed achieved. The error for the collocation approach is again much higher and the convergence rate seems to be restricted to two.

Remark 9.1. *To achieve the presented convergence rates for the Galerkin approach in Table 9.7 the accuracy for the evaluation of the matrix entries for the hyper-singular operator had to be increased quite significantly.*

To obtain a convergence rate of three for the Galerkin approach, the given Neumann data has to be projected into the discrete subspace $S_h^{-1,1}$ of linear discontinuous basis functions. Projecting the given Neumann data into the discrete subspace $S_h^{-1,0}$ of constant basis functions results in a lower convergence rate in the interior, see [52]. The resulting errors for the collocation and the Galerkin approach are presented in Table 9.9 and Table 9.8.

	Collocation				Galerkin			
#DOF	e_u	eoc	e_p	eoc	e_u	eoc	e_p	eoc
32	1.60e+02		4.11e+01		1.35e+02		2.36e+01	
104	3.88e+01	2.04	8.86e+00	2.21	2.82e+01	2.26	9.62e+00	1.30
392	9.01e+00	2.11	2.86e+00	1.63	7.19e+00	1.97	3.23e+00	1.58
1544	2.15e+00	2.06	8.20e-01	1.80	1.81e+00	1.99	9.52e-01	1.76
6152	5.31e-01	2.02	2.11e-01	1.96	4.55e-01	2.00	2.56e-01	1.89

Table 9.8: The $L_2(\Gamma)$-errors for the Galerkin discretization with the given right hand side $P_h^{-1,0} g_N$.

Comparing the errors presented in Table 9.8 with the errors given in Table 9.6 shows that both approaches suffer severely by this change. Especially the error of the Galerkin approach increases significantly. Again the error for the pore pressure is slightly better in the collocation approach, whereas the solid displacement results in a slightly smaller error when calculated with the Galerkin approach.

The point evaluation for both approaches is now restricted to a convergence rate of two, see Table 9.9. The error of the pore pressure is even slightly better for the collocation approach, however the error of the solid displacement is still slightly higher. The errors for the collocation approach did increase slightly, the impact was however small in comparison to the Galerkin approach.

9.3.3 Mixed boundary value problem

In this section the numerical results for the mixed problem in Laplace domain are studied. The Laplace parameter is fixed at $s = 20 + 15i$. We compare the Galerkin approach discussed in Section 6.4 and the collocation approach discussed in Section 9.2.

	Collocation				Galerkin			
#DOF	pe_u	eoc	pe_p	eoc	pe_u	eoc	pe_p	eoc
32	1.31e+02		3.98e+02		1.11e+02		3.27e+02	
104	3.18e+01	2.04	1.10e+01	5.18	2.31e+01	2.26	1.42e+01	4.53
392	7.38e+00	2.11	3.48e+00	1.66	5.89e+00	1.97	4.03e+00	1.82
1544	1.77e+00	2.06	8.80e-01	1.98	1.48e+00	1.99	9.40e-01	2.10
6152	4.37e-01	2.02	2.09e-01	2.07	3.72e-01	2.00	2.28e-01	2.04

Table 9.9: The ℓ_2-errors for the Galerkin discretization with the given right hand side $P_h^{-1,0} g_N$.

#DOF	e_u	eoc	e_p	eoc	e_t	eoc	e_q	eoc
24	2.26e-01		4.25e-01		5.02e-01		4.59e-01	
100	8.04e-02	1.49	1.70e-01	1.32	2.82e-01	0.83	2.48e-01	0.89
420	1.56e-02	2.37	3.93e-02	2.11	1.47e-01	0.94	1.34e-01	0.88
1732	3.30e-03	2.24	8.65e-03	2.18	7.40e-02	0.99	6.84e-02	0.98
7044	7.55e-04	2.13	1.96e-03	2.14	3.71e-02	1.00	3.43e-02	0.99

Table 9.10: L_2-error of the Cauchy data on the boundary for the Galerkin approach.

The results for the Galerkin approach are stated in Table 9.10. The convergence rates are in good agreement with the theory. The errors for the collocation approach, as given in Table 9.11, behave in a similar way. In general the errors of the Galerkin approach are slightly smaller or equal to the errors of the collocation approach.

#DOF	e_u	eoc	e_p	eoc	e_t	eoc	e_q	eoc
24	5.66e-01		5.26e-01		5.03e-01		4.64e-01	
100	1.49e-01	1.92	1.95e-01	1.43	2.83e-01	0.83	2.47e-01	0.91
420	4.21e-02	1.83	5.28e-02	1.89	1.47e-01	0.95	1.34e-01	0.88
1732	1.19e-02	1.83	1.35e-02	1.97	7.40e-02	0.99	6.84e-02	0.98
7044	3.27e-03	1.86	3.40e-03	1.99	3.71e-02	1.00	3.43e-02	0.99

Table 9.11: L_2-error of the Cauchy data on the boundary for the collocation approach.

Additionally the error of the point evaluation is given in Table 9.12. The error behaves in a similar way as for the Dirichlet and the Neumann problem. We have a convergence rate of two for the collocation approach and a convergence rate of three for the Galerkin approach. This results in a smaller error for the Galerkin approach.

	Collocation				Galerkin			
#DOF	pe_u	eoc	pe_p	eoc	pe_u	eoc	pe_p	eoc
24	4.16e-01		3.04e-01		1.03e-01		1.90e-01	
100	9.89e-02	2.07	6.82e-02	2.16	4.09e-02	1.33	4.58e-02	2.05
420	2.91e-02	1.77	1.34e-02	2.35	3.47e-03	3.56	3.68e-03	3.64
1732	8.47e-03	1.78	3.43e-03	1.97	4.28e-04	3.02	2.67e-04	3.79
7044	2.36e-03	1.85	8.57e-04	2.00	6.09e-05	2.81	2.80e-05	3.25

Table 9.12: l_2-error of the point evaluation for the mixed problem.

9.4 Time domain

In this chapter we discuss numerical results for the convergence of solutions in time domain. To the best of our knowledge no pure analytical solution in time domain is known. We therefore start with a fixed discretization in space and refine only in time. The finest level is taken as a reference solution. The error and the corresponding convergence rates reflect the error in time. As domain Ω we chose the cube $(-0.5, 0.5)^3$. The surface of the cube is refined with 12 elements. No further refinements in space are necessary.

9.4.1 Dirichlet boundary value problem

The Dirichlet datum is given as

$$u(t,x) = 10^{-6} e^{-5(1308.73t - \langle a,x\rangle - 3)^2} \text{ and } p(t,x) = 0$$

with $a = (1,2,1)^\top$. The solid displacement represents an incoming wave. At time zero the solid displacement is not equal to zero, the maximal value at time zero is however smaller than $2e-9$. The wave travels at a speed of 1308.73 m/s. The length of the time interval is chosen as $T = 4e-3$. Starting with 64 time steps we calculate up to 2048 time steps. The finest level is chosen as a reference solution. In Table 9.13 first the number of time steps, denoted by N, is given, followed by the pointwise errors and finally the error of the traction and the flux on the boundary. The relative pointwise error is defined as

$$pet_u = \frac{\sqrt{\sum_{i,j} \left|u(\tilde{x}_i,t_j) - u_h(\tilde{x}_i,t_j)\right|^2}}{\sqrt{\sum_{i,j} \left|u(\tilde{x}_i,t_j)\right|^2}}, \qquad pet_t = \frac{\sqrt{\sum_{i,j} \left|t(\tilde{x}_i,t_j) - t_h(\tilde{x}_i,t_j)\right|^2}}{\sqrt{\sum_{i,j} \left|t(\tilde{x}_i,t_j)\right|^2}} \tag{9.5}$$

where $i = 1,\dots,413$ is an index for the different evaluation nodes and $j = 1,\dots,N$ an index for the different time steps. On the boundary the error for the flux is given as

$$e_q^{-1/2} = \frac{\sqrt{\sum_j \left\langle V((q-q_{ref})(t_j)),(q-q_{ref})(t_j)\right\rangle_\Gamma^2}}{\sqrt{\sum_j \left\langle Vq_{ref}(t_j),q_{ref}(t_j)\right\rangle_\Gamma^2}}, \tag{9.6}$$

with the reference solution q_{ref}. The error for the traction is defined in the same way and denoted by $e_t^{-1/2}$. The single layer potential V is evaluated for a fixed $s = 1$ and thus the error is equivalent to the $[H^{-1/2}(\Gamma)]^4$-norm. Both approaches are discretized in time using a BDF 2 scheme, which results in an optimal convergence order of two.

N	pet_u	eoc	pet_p	eoc	$e_t^{-1/2}$	eoc	$e_q^{-1/2}$	eoc
128	2.36e-02	1.65	1.39e-01	1.75	9.90e-02	1.74	1.57e-01	1.71
256	6.13e-03	1.95	3.64e-02	1.93	2.60e-02	1.93	4.13e-02	1.92
512	1.52e-03	2.01	9.06e-03	2.00	6.46e-03	2.01	1.03e-02	2.01
1024	3.62e-04	2.07	2.16e-03	2.07	1.54e-03	2.07	2.45e-03	2.07
2048	7.24e-05	2.32	4.32e-04	2.32	3.08e-04	2.32	4.90e-04	2.32

Table 9.13: Dirichlet problem - Collocation approach - BDF 2.

Both the collocation approach, see errors presented in Table 9.13, as well as the Galerkin approach, see errors presented in Table 9.14, converge with a convergence rate of two.

N	pet_u	eoc	pet_p	eoc	$e_t^{-1/2}$	eoc	$e_q^{-1/2}$	eoc
128	2.70e-02	1.65	1.25e-01	1.68	1.04e-01	1.63	1.37e-01	1.74
256	7.10e-03	1.93	3.24e-02	1.95	2.68e-02	1.95	3.56e-02	1.95
512	1.77e-03	2.00	8.04e-03	2.01	6.66e-03	2.01	8.86e-03	2.01
1024	4.24e-04	2.07	1.92e-03	2.07	1.59e-03	2.07	2.11e-03	2.07
2048	8.48e-05	2.32	3.83e-04	2.32	3.17e-04	2.32	4.23e-04	2.32

Table 9.14: Dirichlet problem - Galerkin approach - BDF 2.

For the Galerkin approach the numerical results confirm the theoretical convergence rates. Theoretically the scheme is stable for any time step. However this was not observed in our numerical examples. Both approaches tend to become unstable if the matrix entries are not computed with enough accuracy. By increasing the Gauss points we are able to calculate rather a long time, especially for the Galerkin approach. The collocation approach tends to be more sensitive. For this Dirichlet problem 4096 time steps are not stable with the collocation approach, refining once in space however results in a stable scheme again.

Remark 9.2. *For an error estimate in the* $L_2(\Gamma)$ *an estimate*

$$\left\|V_h^{-1}\left(\frac{1}{2}M_h+K_h\right)\right\|_{[H^1(\Gamma)]^4\to[L_2(\Gamma)]^4} \leq c(\underline{\sigma})\,|s|^{\mu}$$

needs to be proven. The $L_2(\Gamma)$ *errors of the solution of the Galerkin and the collocation approach are given in Table 9.15, which indicate such a bound exists for both approaches.*

	Collocation				Galerkin			
N	e_t	eoc	e_q	eoc	e_t	eoc	e_q	eoc
128	1.24e-01	1.74	1.59e-01	1.71	1.16e-01	1.68	1.38e-01	1.74
256	3.21e-02	1.94	4.19e-02	1.93	3.00e-02	1.95	3.58e-02	1.94
512	7.97e-03	2.01	1.04e-02	2.01	7.45e-03	2.01	8.90e-03	2.01
1024	1.90e-03	2.07	2.49e-03	2.07	1.78e-03	2.07	2.12e-03	2.07
2048	3.80e-04	2.32	4.98e-04	2.32	3.56e-04	2.32	4.25e-04	2.32

Table 9.15: Dirichlet problem - $L_2(\Gamma)$ error - BDF 2.

9.4.2 Neumann boundary value problem

For the Neumann problem we prescribed the given Neumann data as

$$t(t,x) = 10^{-6}(10t-\langle a,x\rangle-3)\,\langle a,n\rangle\, e^{-5(1308.73t-\langle a,x\rangle-3)^2} \text{ and } q(t,x)=0$$

with $a=(1,2,1)^\top$ and n as the normal vector. The length of the interval is again taken as $T=4e-3$. We compare the pointwise errors in the interior defined by (9.4). The error on the boundary was measured by the norm

$$e_u^{1/2} = \frac{\sqrt{\sum_j \left\langle D((u-u_{ref})(t_j)),(u-u_{ref})(t_j)\right\rangle_\Gamma^2}}{\sqrt{\sum_j \left\langle Du_{ref}(t_j),u_{ref}(t_j)\right\rangle_\Gamma^2}}, \tag{9.7}$$

with the reference solution u_{ref}. The error of the pore pressure is measured in the same norm and denoted by $e_p^{1/2}$. The hyper-singular operator was evaluated for $s=1$ and thus the norm is equivalent to the $H^{1/2}(\Gamma)$-norm.

The errors presented in Tables 9.16 and 9.17 indicate that the collocation as well as the Galerkin approach yield the desired convergence rates for the point evaluation in the interior as well as the given norm on the boundary.

The error for the Galerkin approach is significantly smaller than the error for the collocation approach.

N	pet_u	eoc	pet_p	eoc	$e_t^{1/2}$	eoc	$e_q^{1/2}$	eoc
256	1.16e-01	1.66	1.59e-01	1.63	1.39e-01	1.67	7.14e-02	1.81
512	3.08e-02	1.91	4.24e-02	1.90	3.70e-02	1.91	1.85e-02	1.95
1024	7.74e-03	1.99	1.07e-02	1.99	9.31e-03	1.99	4.61e-03	2.00
2048	1.85e-03	2.07	2.56e-03	2.06	2.22e-03	2.07	1.10e-03	2.06
4096	3.70e-04	2.32	5.11e-04	2.32	4.45e-04	2.32	2.21e-04	2.32

Table 9.16: Neumann problem - Collocation approach - BDF 2.

N	pet_u	eoc	pet_p	eoc	$e_t^{1/2}$	eoc	$e_q^{1/2}$	eoc
256	6.72e-03	1.92	2.30e-02	1.93	1.82e-02	1.92	2.67e-02	1.93
512	1.70e-03	1.98	5.81e-03	1.99	4.61e-03	1.98	6.74e-03	1.99
1024	4.22e-04	2.01	1.44e-03	2.01	1.14e-03	2.01	1.67e-03	2.01
2028	1.01e-04	2.07	3.43e-04	2.07	2.73e-04	2.07	3.98e-04	2.07
4096	2.01e-05	2.32	7.07e-05	2.28	5.46e-05	2.32	7.97e-05	2.32

Table 9.17: Neumann problem - Galerkin approach - BDF 2.

9.4.3 Mixed boundary value problem

For the mixed problem the Dirichlet boundary Γ_D is chosen as the face $x_1 = -0.5$ and the Neumann boundary as $\Gamma_N = \Gamma \setminus \Gamma_D$. On the Dirichlet boundary the wave

$$u(t,x) = 10^{-6} e^{-5(1308.73t - \langle a,x \rangle - 3)^2} \text{ and } p(t,x) = 0$$

is prescribed. On the Neumann boundary the incoming wave

$$t(t,x) = 10^{-6}(10t - \langle a,x \rangle - 3) \langle a,n \rangle \, e^{-5(1308.73t - \langle a,x \rangle - 3)^2} \text{ and } q(t,x) = 0$$

is prescribed. Again $a = (1,2,1)^\top$ and n is the normal vector.

The errors for the collocation approach, see Chapter 9.2, are stated in Table 9.18. The errors are given in a $H^{1/2}(\Gamma_N)$ norm, see (9.7), for the solid displacement u and the pore pressure p and a $H^{-1/2}(\Gamma_D)$ norm, see (9.6), for the traction t and the flux q.

The errors for the Galerkin approach (8.2) are presented in Table 9.19. The errors converge with the expected convergence rate of two.

Comparing the errors of the Galerkin and the collocation approach, the errors for the Galerkin approach are in general significantly better by almost one order of magnitude.

Moreover the error of the point evaluation inside the domain Ω is presented in Table 9.20. The relative error is defined in (9.5).

N	$e_u^{1/2}$	eoc	$e_p^{1/2}$	eoc	$e_t^{-1/2}$	eoc	$e_q^{-1/2}$	eoc
256	3.51e-02	1.85	4.35e-02	1.87	3.01e-02	1.85	4.91e-02	1.81
512	9.01e-03	1.96	1.11e-02	1.97	7.71e-03	1.96	1.27e-02	1.95
1024	2.24e-03	2.01	2.76e-03	2.01	1.92e-03	2.00	3.18e-03	2.00
2028	5.36e-04	2.07	6.58e-04	2.07	4.61e-04	2.06	7.61e-04	2.06
4096	1.07e-04	2.32	1.32e-04	2.32	9.68e-05	2.25	1.52e-04	2.32

Table 9.18: Mixed Problem - Collocation approach - BDF 2.

N	$e_u^{1/2}$	eoc	$e_p^{1/2}$	eoc	$e_t^{-1/2}$	eoc	$e_q^{-1/2}$	eoc
256	6.87e-03	1.96	8.18e-03	1.96	8.25e-03	1.94	1.15e-02	1.96
512	1.73e-03	1.99	2.05e-03	1.99	2.08e-03	1.99	2.89e-03	1.99
1024	4.28e-04	2.01	5.09e-04	2.01	5.16e-04	2.01	7.16e-04	2.01
2028	1.02e-04	2.07	1.21e-04	2.07	1.23e-04	2.07	1.71e-04	2.07
4096	2.04e-05	2.32	2.43e-05	2.32	2.48e-05	2.31	3.42e-05	2.32

Table 9.19: Mixed Problem - Galerkin approach - BDF 2.

Again the errors for the Galerkin approach are significantly better.

In comparison the errors for the BDF 1 multistep methods utilizing the Galerkin approach are presented in Table 9.21. The reference solution which was used to calculated the error was calculated with 8192 time steps with the BDF 2 multistep method. Thus the reference solution should be far more exact. Indeed no increase in the convergence number in the last level is observed.

As expected the convergence number tends towards one and the errors are significantly worse than the errors presented in Table 9.19 for the BDF 2 multistep method.

N	pet_u	eoc	pet_p	eoc	pet_t	eoc	pet_q	eoc
	Collocation				Galerkin			
256	1.64e-02	1.84	4.38e-02	1.83	4.38e-03	1.96	9.17e-03	1.94
512	4.20e-03	1.96	1.13e-02	1.95	1.10e-03	1.99	2.31e-03	1.99
1024	1.05e-03	2.00	2.82e-03	2.00	2.72e-04	2.01	5.73e-04	2.01
2028	2.50e-04	2.07	6.75e-04	2.07	6.49e-05	2.07	1.37e-04	2.07
4096	5.01e-05	2.32	1.35e-04	2.32	1.30e-05	2.32	2.75e-05	2.31

Table 9.20: Mixed Problem - Point evaluation - BDF 2.

N	$e_u^{1/2}$	eoc	$e_p^{1/2}$	eoc	$e_t^{-1/2}$	eoc	$e_q^{-1/2}$	eoc
256	1.10e-01	0.87	1.19e-01	0.85	1.33e-01	0.83	1.32e-01	0.81
512	5.78e-02	0.93	6.30e-02	0.92	7.12e-02	0.91	7.13e-02	0.89
1024	2.96e-02	0.97	3.24e-02	0.96	3.69e-02	0.95	3.71e-02	0.94
2028	1.50e-02	0.98	1.65e-02	0.98	1.88e-02	0.97	1.89e-02	0.97
4096	7.53e-03	0.99	8.29e-03	0.99	9.46e-03	0.99	9.56e-03	0.99

Table 9.21: Mixed Problem - Galerkin approach - BDF 1.

10 CONCLUSIONS AND OUTLOOK

We have derived a boundary element approach for poroelasticity. We started from a system of partial differential equations in Laplace domain, and derived the symmetric formulation. By applying an inverse Laplace transformation this set of boundary integral equations was transferred into time domain. Furthermore the set of boundary integral equations was discretized by a Galerkin approximation in space and a convolution quadrature approximation in time. Unique solvability, stability and convergence of the fully discretized system was shown. The convergence order coincide with the convergence orders obtained by numerical examples.

A few open points remain: Error estimates for the convolution quadrature methods were derived in the energy norms. An extension to a set of norms is desireable. Especially error estimates for the $L_2(\Gamma)$-norm are more practical. To show such an error estimate for the boundary integral equation related to the Dirichlet boundary value problem an estimate for the operator

$$V(s)^{-1}\left(\frac{1}{2}I+K(s)\right) : [H^1(\Gamma)]^4 \to [L_2(\Gamma)]^4 \tag{10.1}$$

is needed. It is well known that the operator (10.1) is bounded, however the explicit dependency onto the Laplace parameter s is not yet known. The bound can be calculated by techniques utilized in [38]. If such a bound also holds for the discrete operator is still an open question.

To obtain reasonable results with the Galerkin method for the mixed boundary value problem the accuracy of the computation of matrix entries is quite demanding. The partially integrated kernel of the hyper–singular operator is rather complex, combined with the Duffy transformation, see [20], which utilizes a Gauss product approach, the requirements on the computation time are so far quite high. For example optimized integration formulae, see [16, 54], could decrease the computation time.

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