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ON A CLASS OF PSEUDO-ANALYTIC FUNCTIONS: Representations, generalizations and applications

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Dedication

To my family

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Abstract

In this thesis we study and apply the methods of representing pseudo-analytic functions by differential operators in complex variables and bicomplex variables. We consider the Bers-Vekua equation

$$w_{\overline{z}} = C(z,\overline{z})\overline{w}. \tag{0.1}$$

For the equation (0.1) I.N. Vekua developed a complete theory where the solutions are represented by means of certain integral operators. However the explicit determination of the required resolvents may be difficult. Many mathematicians used the results proved by I.N. Vekua to get the representations of solutions of this equation by differential operators. These representations not only permit a detailed investigation of the function theoretic properties of the solutions but also enable us to solve some boundary value problems explicitly.

Chapter 1 is aimed to investigate the representation of solutions of a class of equations of type (0.1) with the coefficients C satisfying

$$m^2(\log C)_{z\overline{z}} - C\overline{C} = 0, \quad m \in \mathbb{N}.$$

By changing variables we can reduce these equations to the following form

$$w_{\overline{z}} = \frac{m}{1 - z\overline{z}}\overline{w}, \qquad m \in \mathbb{N}. \tag{0.2}$$

We will study the Bers-Vekua equation (0.2). Applying the method of P. Berglez [11] or the method of K.W. Bauer on the determination of Vekua resolvents [6] we can derive a representation of solutions of this equation by differential operators of Bauer-type.

Then we use the representation of solutions of the equation (0.2) to solve a Dirichlet boundary value problem and a class of generalized Riemann-Hilbert boundary value problems for the equation (0.2) in Chapter 2.

In Chapter 3 we consider some consequences and applications of the representation of solutions of the equation (0.2) by differential operators of Bauer-type.

Chapter 4 is devoted to study a class of bicomplex pseudo-analytic functions which are solutions of a system in bicomplex variables of the form

$$\begin{cases} \partial_{z^*} V(z) = \mathcal{C}(z,z^*)V^*(z), \\ \partial_{z^\star} V(z) = \partial_{z^\dagger} V(z) = 0, \end{cases} \tag{0.3}$$

where z is a bicomplex variable and $z^*, z^\star, z^\dagger$ are bicomplex conjugations of z.

We obtain a class of coefficients $\mathcal{C}$ for which all solutions of the system (0.3) can be represented by differential operators. Some applications of this representation of solutions of the system (0.3) such as solving the Dirac equation on a pseudo-sphere and using the generalization of the Weierstrass formulae to generate surfaces via solutions of linear equations are given also.

Zusammenfassung

In dieser Arbeit werden Methoden zur Darstellung pseudoanalytischer Funktionen im komplexen und bikomplexen Fall untersucht und angewendet, die sich gewisser Differentialoperatoren bedienen. Wir betrachten die Bers-Vekua Gleichung

$$w_{\overline{z}} = C(z,\overline{z})\overline{w}. \tag{0.1}$$

Für die Gleichung (0.1) entwickelte I.N. Vekua eine vollständige Theorie zur Lösungsdarstellung unter Verwendung gewisser Integraloperatoren. Allerdings ist die explizite Bestimmung der dazu notwendigen Resolventen oft sehr schwierig. In vielen Arbeiten wurden die Ergebnisse von I.N. Vekua dazu verwendet um Lösungsdarstellung unter Verwendung von Differentialoperatoren zu erlangen. Diese Darstellungen erlauben nicht nur eine detaillierte Untersuchung der funktionentheoretischen Eigenschaften der Lösungen sondern auch die explizite Lösung von Randwertproblemen für diese Gleichung.

Im 1. Kapitel werden Darstellungen für Lösungen einer Klasse von Gleichungen vom Typ (0.1) untersucht, wobei die Koeffizienten C der Bedingung

$$m^2(\log C)_{z\overline{z}} - C\overline{C} = 0, \quad m \in \mathbb{N}.$$

genügen. Mit Hilfe einer geeigneten Variablentransformation kann diese Gleichung in die Form

$$w_{\overline{z}} = \frac{m}{1-z\overline{z}}\overline{w}, \quad m \in \mathbb{N}. \tag{0.2}$$

übergeführt werden. Unter Verwendung der Methode von P. Berglez [11] oder der Methode von K.W. Bauer zur Bestimmung der Vekua-Resolventen [6] können wir für diese Gleichung eine Lösungsdarstellung unter Verwendung von Differentialoperatoren vom Bauer-Typ herleiten.

In Kapitel 2 verwenden wir diese Darstellung der Lösungen von (0.2) um ein Dirichlet'sches Randwertproblem und eine Klasse von Riemann-Hilbert'schen Randwertproblemen für die Gleichung (0.2) zu lösen.

Im 3. Kapitel betrachten wir einige Folgerungen und Anwendungen dieser Lösungsdarstellungen für die Gleichung (0.2) unter Verwendung von Differentialoperatoren vom Bauer'schen Typ.

Das 4. Kapitel ist der Untersuchung einer Klasse von bikomplexen pseudoanalytischen Funktionen gewidmet, die Lösungen eines Systems von Differentialgleichungen von der Gestalt

$$\begin{cases} \partial_{z^*} V(z) = \mathcal{C}(z,z^*)V^*(z), \\ \partial_{z^\star} V(z) = \partial_{z^\dagger} V(z) = 0, \end{cases} \tag{0.3}$$

sind, wobei z eine bikomplexe Variable ist und $z^*, z^\star, z^\dagger$, die bikomplexen Konjugierten von z bezeichnen.

Wir erhalten eine Klasse von Koeffizienten $\mathcal{C}$ für die alle Lösungen des Systems (0.3) unter Verwendung von Differentialoperatoren angegeben werden können. Abschließend werden einige Anwendungen dieser Lösungsdarstellungen für das System (0.3) angegeben. So zum Beispiel Lösungen der Dirac Gleichung auf einer Pseudosphäre oder die Verallgemeinerung der Weierstrass'schen Formeln zur Darstellung von Flächen durch Lösungen linearer Gleichungen.

CONTENTS

Introduction **1**

1 Representation of the solutions of a class of pseudo-analytic functions **5**
1.1 Representation of solutions after P. Berglez 5
1.2 Representation of the solutions by integral operators 14
1.3 Determination of the Vekua resolvents 17
1.4 Representation of the solutions by differential operators of Bauer-type . . 20

2 A class of boundary value problems **27**
2.1 The Dirichlet boundary value problem 27
2.2 A class of the generalized Riemann-Hilbert boundary value problems . . 30

3 Consequences and applications **35**
3.1 Connection between the generating functions and the solutions 35
3.2 Representation of the solutions in the neighbourhood of an isolated singularity 44
3.3 A generating pair of the equation (M) in the sense of L.Bers 46
3.4 Ising field theory on a pseudo-sphere 47
3.5 Connection between solutions of the equation (M) with different parameters 49
3.6 Representation for solutions of the inhomogeneous equation 53

4 Representation of bicomplex pseudo-analytic functions **57**
4.1 An introduction to bicomplex algebra 57
4.1.1 Bicomplex numbers 58
4.1.2 The idempotent representation 59
4.1.3 Power series in the space of bicomplex numbers 60
4.1.4 Integrals and holomorphic functions in bicomplex variables . . . 62
4.2 Representation of bicomplex pseudo-analytic functions by integral operators 65
4.3 Representation of bicomplex pseudo-analytic functions by differential operators 74
4.3.1 Representation theorems for solutions of second order equations after P. Berglez 74
4.3.2 Representation theorem for bicomplex pseudo-analytic functions . 75
4.4 Applications 79
4.4.1 Representation of solutions of the Dirac equation on a pseudo-sphere 79
4.4.2 Generalized Weierstrass representation for surfaces 87

5 Conclusions **91**

References **93**

Introduction

The pseudo-analytic function theory was independently developed by two prominent mathematicians, L. Bers (see [1], [18], [19]) and I.N. Vekua (see [43]).
After L. Bers every complex function W defined in a subdomain of a simply connected domain $\mathcal{D} \subset \mathbb{R}^2$ admits the unique representation $W = \phi F + \psi G$, where ϕ and ψ are real-valued functions and a pair of complex functions F and G is a so-called *generating pair*.
The (F,G)-derivative of a function W exists if and only if $\phi_{\overline{z}} F + \psi_{\overline{z}} G = 0$. This condition can be rewritten in the following form

$$W_{\overline{z}} = a_{(F,G)} W + b_{(F,G)} \overline{W} \tag{0.4}$$

where $a_{(F,G)}, b_{(F,G)}$ are the characteristic coefficients of the pair (F,G)

$$a_{(F,G)} = -\frac{\overline{F} G_{\overline{z}} - F_{\overline{z}} \overline{G}}{F\overline{G} - \overline{F} G}, \qquad b_{(F,G)} = \frac{F G_{\overline{z}} - F_{\overline{z}} G}{F\overline{G} - \overline{F} G}.$$

Solutions of the equation (0.4) are called (F,G)-pseudo-analytic functions (or, simply, *pseudo-analytic functions*).
On the other hand after I.N. Vekua a *generalized analytic function* is a function

$$W(z) = u(x,y) + iv(x,y)$$

satisfying a system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + au + bv = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + cu + dv = 0$$

where a,b,c,d are real valued functions of the real variables x and y. This system can be rewritten in the complex form which is called the Bers-Vekua equation

$$W_{\overline{z}} = \alpha W + \beta \overline{W} \tag{0.5}$$

where $\alpha = \frac{1}{4}[a+b+i(c-b)]$, $\beta = \frac{1}{4}[a-d+i(c+b)]$.

Thus, the class of pseudo-analytic functions in the sense of Bers corresponding to the pair (F,G) coincides with the class of generalized analytic functions in the sense of Vekua. In the special case $\alpha = \beta = 0$, the solutions of the equation (0.5) are called *analytic functions* or *holomorphic functions*.
By transformation $W = we^A$, with $\alpha = A_{\overline{z}}$, we obtain from (0.5) the equation

$$w_{\overline{z}} \ = \ C(z,\overline{z})\overline{w}, \qquad (C = \beta e^{\overline{A}-A}). \tag{0.6}$$

For the equation (0.6) I.N. Vekua developed a complete theory [44] where the solutions are represented by means of certain integral operators. In special cases these representations of solutions may be converted to a form free of integrals by integration by parts. K.W. Bauer pointed out that if the coefficient C in the equation (0.6) is analytic and satisfies certain conditions then it is possible to derive general representation theorems for the solutions of the equation (0.6) defined in a simply connected domains $\mathcal{D}$ by differential operators [9]. Moreover, by using another method not depending on the Vekua resolvents, P. Berglez presented a necessary and sufficient condition on the coefficients C for the existence of the representation of solutions of the equation (0.6) by such operators [11].

The thesis is organized as follows. Chapter 1 is aimed to investigate the representation of solutions of a class of type (0.6). Using the result of P. Berglez, we can construct a Liouville system. After solving the Liouville system we obtain coefficients C for which all solutions of the equation (0.6) can be represented by differential operators. A special solution of this system leads to the fact that there exists a class of coefficients C satisfying the Liouville equation

$$m^2(\log C)_{z\overline{z}} - C\overline{C} = 0, \quad m \in \mathbb{N} \tag{0.7}$$

such that for these coefficients all solutions of (0.6) can be represented by differential operators.

This condition was investigated by K.W. Bauer [6]. He considered the equation (0.6) with the coefficients C satisfying the condition (0.7). From this condition we get the general representation of C and then using a suitable transformation we can reduce the equation (0.6) to the equation

$$w_{\overline{z}} = \frac{m}{1 - z\overline{z}}\overline{w}, \qquad m \in \mathbb{N}. \tag{0.8}$$

Therefore instead of (0.6) we consider the differential equation (0.8). Applying the method of P. Berglez [11] or the method of K.W. Bauer on the determination of the Vekua resolvents [6] we can derive a representation of all solutions of this equation by differential operators of Bauer-type.

Then we use this representation to solve a Dirichlet boundary value problem (BVP) and a class of generalized Riemann-Hilbert BVPs for the equation (0.8) in a disk in Chapter 2.

In Chapter 3 using some properties of the representation of the solutions we also derive a generalized representation theorem for solutions of the equation (0.8) in a neighbourhood of an isolated singularity. Some problems related to the equation (0.8) are also investigated: finding a generating pair in the sense of Bers; finding a special class of the chiral components in the Ising field theory; finding transformations between the solutions of the equation (0.8) with different parameters; finding inhomogeneous equations corresponding to the equation (0.8) such that all solutions of these equations can be represented by differential operators.

Chapter 4 is devoted to study a class of *bicomplex pseudo-analytic functions* which are solutions of a system in bicomplex variables of the form

$$\begin{cases} \partial_{z^*}V(z) = \mathcal{C}(z,z^*)V^*(z), \\ \partial_{\bar{z}_1}V(z) = \partial_{\bar{z}_2}V(z) = 0, \end{cases} \tag{0.9}$$

where z is a bicomplex variable and $z_1, z_2 \in \mathbb{C}$ are components of z.

First we introduce some concepts in bicomplex algebra (see, e.g., [37], [38]). We define the resolvents of Vekua type in bicomplex variables and hence we can derive the representation theorem for a class of bicomplex pseudo-analytic functions using integral operators. Then applying the representation theorems for solutions of a second order partial differential equations [11] we also obtain a class of coefficients $\mathcal{C}$ for which all solutions of system (0.9) can be represented by differential operators.

Using a so-called *idempotent representation* in a space of bicomplex functions we obtain an interesting result, that is, a Dirac equation on the pseudo-sphere is equivalent to a system of type (0.9). This implies that using the representation of the solutions of system (0.9) by differential operators we can solve the Dirac equation on a pseudo-sphere. Another application of this representation is using the generalization of the Weierstrass formulae to generate surfaces via solutions of linear equations.

1 REPRESENTATION OF THE SOLUTIONS OF A CLASS OF PSEUDO-ANALYTIC FUNCTIONS

In this chapter we deal with the Bers-Vekua equation $Dw := w_{\overline{z}} - C(z,\overline{z})\overline{w} = 0$ defined in a domain $\mathcal{D} \subset \mathbb{C}$. For a certain class of coefficients C and domains $\mathcal{D}$ we show how to get the explicit representation of solutions of this problem. Using a necessary and sufficient condition on the coefficients C, see [11], we can obtain certain differential operators for which every solution of $Dw = 0$ defined in $\mathcal{D}$ can be generated from a so-called *generating function* g holomorphic in $\mathcal{D}$. On the other hand after I.N.Vekua all solutions of the above Bers-Vekua equation can be represented using integral operators [44]. Applying the method of K.W. Bauer we can determine the Vekua resolvents for a certain class of the Bers-Vekua equations and hence every solution of these equations can be represented as the image of the generating function g under differential operators of Bauer-type [9].

1.1 Representation of solutions after P. Berglez

In this thesis we use the following notations. We denote a complex variable by

$$z = x + iy$$

where x and y are real variables, i is the imaginary unit. Complex conjugates are denoted by

$$\overline{z} = x - iy.$$

We use the formal differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and sometimes write $w_z, w_{\overline{z}}$ instead of $\frac{\partial w}{\partial z}, \frac{\partial w}{\partial \overline{z}}$, respectively.
Denote the space of all holomorphic functions in $\mathcal{D}$ by $H(\mathcal{D})$.
Consider the Bers-Vekua equation

$$w_{\overline{z}} \;=\; C(z,\overline{z})\overline{w}, \qquad z \in \mathcal{D} \tag{1.1}$$

where $\mathcal{D}$ is a simply connected domain in $\mathbb{C}$ and $C(z,\bar{z})$ is an analytic function of the real variables x and y.
Let

$$K_1^m = \sum_{k=0}^{m} \alpha_k(z,\bar{z})\frac{\partial^k}{\partial z^k}, \quad K_2^n = \sum_{l=0}^{n} \beta_l(z,\bar{z})\frac{\partial^l}{\partial \bar{z}^l}, \quad m,n \in \mathbb{N},$$

be given differential operators, where $\alpha_k,\ k = 0,1,\ldots,m$, and $\beta_l,\ l = 0,1,\ldots,n$, are continuously differentiable in $\mathcal{D}$. If $w = K_1^m g(z) + K_2^n \overline{g(z)}$ is a solution of the equation (1.1) in $\mathcal{D}$ for all functions $g(z) \in H(\mathcal{D})$, then $n = m-1$ (see [10]).
We call K_1^m and K_2^{m-1} the *differential operators of Bauer-type*.
P. Berglez gave the necessary and sufficient condition on the coefficients C such that the solutions of the equation (1.1) can be represented by differential operators [11] which is quoted as follows.

Theorem 1.1 (P.Berglez)**.**
Denote

$$A_m := \frac{1}{C}, \quad B_m := -C\overline{C},$$

where $C \neq 0$ is the coefficient in (1.1).
For the solutions of the Bers-Vekua equation (1.1) there exists a representation using differential operators of Bauer-type if and only if with

$$A_{k-1} = A_k B_k, \quad B_{k-1} = B_k + [\log(A_k B_k)]_{z\bar{z}}, \quad k = m, m-1, \ldots, 1$$

the condition

$$B_0 \equiv 0 \quad in \quad \mathcal{D}$$

is satisfied.
The solution w of (1.1) is then given by

$$w = K_m^1 g + C\overline{K_{m-1}^1 g}, \quad g \in H(\mathcal{D}),$$

with

$$K_m^1 = F_{m-1}^1 \ldots F_0^1, \quad F_k^1 = \frac{\partial}{\partial z} + (\log A_k)_z, \quad k = 0,1,\ldots,m-1.$$

Using this result we can construct a Liouville system.
Since

$$A_k = A_{k+1}B_{k+1}, \qquad B_k = B_{k+1} + [\log(A_{k+1}B_{k+1})]_{z\bar{z}}$$
$$= B_{k+1} + (\log A_k)_{z\bar{z}}, \quad \text{for } k = m-1,\ldots,1.$$

Therefore $\quad \log(A_k)_{z\bar{z}} = B_k - B_{k+1}, \quad \text{for } k = m-1,\ldots,1.$
Denote

$$C_1 := C, \qquad \lambda_1 := -1,$$
$$B_{m-(k-1)} := \lambda_k C_k^2, \qquad \lambda_k := \frac{(k-1)^2 - m^2}{m^2}, \quad k \geq 2.$$

• Step 1:

$$\begin{aligned} B_m &= -C\overline{C} =: -C_1\overline{C}_1, \qquad A_{m-1} = -\overline{C} =: -\overline{C}_1 \\ \Rightarrow &(\log A_{m-1})_{z\bar{z}} = B_{m-1} - B_m \\ \Rightarrow &[\log(-\overline{C}_1)]_{z\bar{z}} = \lambda_2 C_2^2 + C_1\overline{C}_1. \end{aligned}$$

This implies that

$$m^2(\log C_1)_{z\bar{z}} = m^2 C_1\overline{C}_1 + m^2\lambda_2\overline{C}_2^2 \tag{1.2}$$

with $m^2 + m^2\lambda_2 = 1$.

• Step 2:

$$\begin{aligned} B_{m-2} &= B_{m-1} + [\log(A_{m-1}B_{m-1})]_{z\bar{z}} \\ \Rightarrow \lambda_3 C_3^2 &= \lambda_2 C_2^2 + [\log(-\lambda_2\overline{C}_1 C_2^2)]_{z\bar{z}}. \end{aligned}$$

This implies that

$$m^2(\log C_2)_{z\bar{z}} = -\frac{m^2}{2}C_1\overline{C}_1 - m^2\lambda_2 C_2^2 + \frac{m^2}{2}\lambda_3 C_3^2 \tag{1.3}$$

with $-\frac{m^2}{2} - m^2\lambda_2 + \frac{m^2}{2}\lambda_3 = 1$.

• Step 3:
For $3 \le k \le m-1$

$$\begin{aligned} B_{m-k} &= B_{m-(k-1)} + [\log(A_{m-(k-1)}B_{m-(k-1)})]_{z\bar{z}} \\ B_{m-k} &= B_{m-(k-1)} + [\log A_{m-(k-1)}]_{z\bar{z}} + [\log B_{m-(k-1)}]_{z\bar{z}} \\ B_{m-k} &= 2B_{m-(k-1)} - B_{m-(k-2)} + [\log B_{m-(k-1)}]_{z\bar{z}} \\ \Rightarrow \lambda_{k+1}C_{k+1}^2 &= 2\lambda_k C_k^2 \; - \; \lambda_{k-1}C_{k-1}^2 \; + \; [\log(\lambda_k C_k^2)]_{z\bar{z}}. \end{aligned}$$

This implies that

$$\begin{aligned} m^2(\log C_k)_{z\bar{z}} &= \frac{m^2}{2}\lambda_{k-1}C_{k-1}^2 - m^2\lambda_k C_k^2 + \frac{m^2}{2}\lambda_{k+1}C_{k+1}^2 \\ \text{with } \; \frac{m^2}{2}\lambda_{k-1} - m^2\lambda_k &+ \frac{m^2}{2}\lambda_{k+1} = 1, \quad \text{for all } 3 \le k \le m-1. \end{aligned} \tag{1.4}$$

• Step 4:

$$\begin{aligned} B_0 &= B_1 + [\log(A_1B_1)]_{z\bar{z}} \\ B_0 &= 2B_1 - B_2 + [\log(\lambda_m C_m^2)]_{z\bar{z}}. \end{aligned}$$

This implies that

$$m^2(\log C_m)_{z\bar{z}} = \frac{m^2}{2}\lambda_{m-1}C_{m-1}^2 - m^2\lambda_n C_m^2 + B_0 \tag{1.5}$$

with $\frac{m^2}{2}\lambda_{m-1} - m^2\lambda_m = 1$.

Assume that for some $m \in \mathbb{N}$, the condition $B_0 \equiv 0$ satisfies, then from (1.2-1.5) we get the Liouville system

$$\begin{cases} m^2(\log C_1)_{z\bar{z}} &= m^2 C_1\overline{C}_1 + m^2\lambda_2\overline{C}_2^2, \\ m^2(\log C_2)_{z\bar{z}} &= -\frac{m^2}{2}C_1\overline{C}_1 - m^2\lambda_2 C_2^2 + \frac{m^2}{2}\lambda_3 C_3^2, \\ m^2(\log C_k)_{z\bar{z}} &= \frac{m^2}{2}\lambda_{k-1}C_{k-1}^2 - m^2\lambda_k C_k^2 + \frac{m^2}{2}\lambda_{k+1}C_{k+1}^2, \quad 3 \le k \le m-1, \\ m^2(\log C_m)_{z\bar{z}} &= \frac{m^2}{2}\lambda_{m-1}C_{m-1}^2 - m^2\lambda_m C_m^2, \end{cases} \tag{1.6}$$

with

$$\begin{cases} m^2 + m^2\lambda_2 = 1, \\ -\frac{m^2}{2} - m^2\lambda_2 + \frac{m^2}{2}\lambda_3 = 1, \\ \frac{m^2}{2}\lambda_{k-1} - m^2\lambda_k + \frac{m^2}{2}\lambda_{k+1} = 1, \quad 3 \le k \le m-1, \\ \frac{m^2}{2}\lambda_{m-1} - m^2\lambda_m = 1. \end{cases}$$

Some results on Liouville systems and the solutions can be found in, e.g., [22], [33], [34]. According to Theorem 1.1 we can say that if the system (1.6) has a solution $(C_1, C_2, \ldots, C_m)$, then with $C := C_1$ all solutions of the equation (1.1) can be represented by differential operators of Bauer-type.

From the above construction we have

$$A_k = A_{k+1}B_{k+1} = \cdots = A_m B_m B_{m-1} \ldots B_{k+1}.$$

Hence

$$(\log A_k)_z = [\log(A_m B_m)]_z + \sum_{j=k+1}^{m-1} (\log B_j)_z$$

$$\Leftrightarrow (\log A_k)_z = (\log \overline{C}_1)_z + 2\sum_{j=2}^{m-k} (\log C_j)_z.$$

Therefore

$$F_k^1 = \frac{\partial}{\partial z} + (\log A_k)_z = \frac{\partial}{\partial z} + (\log \overline{C}_1)_z + 2\sum_{j=2}^{m-k} (\log C_j)_z, \quad k = 0, 1, \ldots, m-1.$$

Summarising the above results we have the following theorem.

Theorem 1.2.
Denote

$$C_1 := C, \qquad \lambda_1 := -1,$$

$$B_{m-(k-1)} := \lambda_k C_k^2, \qquad \lambda_k := \frac{(k-1)^2 - m^2}{m^2}, \quad k \geq 2.$$

with $B_k,\ k = m-1, \ldots, 0$*, as in Theorem 1.1. The condition* $B_0 \equiv 0$ *is satisfied if and only if* $C_1, C_2, \ldots, C_m$ *satisfy the Liouville system (1.6).*
If this system has a solution $(C_1, C_2, \ldots, C_m)$*, then with* $C = C_1$ *all solutions of the equation (1.1) can be represented by differential operators of Bauer-type. The solution* w *of the equation (1.1) is given by*

$$w = K_m^1 g + C\overline{K_{m-1}^1 g}, \quad g \in H(\mathcal{D}),$$

with

$$K_m^1 = F_{m-1}^1 \ldots F_0^1, \quad F_k^1 = \frac{\partial}{\partial z} + (\log \overline{C}_1)_z + 2\sum_{j=2}^{m-k} (\log C_j)_z, \quad k = 0, 1, \ldots, m-1.$$

Consider a Liouville system of the type

$$m^2 (\log U_k)_{z\overline{z}} = \sum_{j=1}^{m} a_{kj} U_j^2, \quad \text{with} \sum_{j=1}^{m} a_{kj} = 1, \quad k = 1, \ldots, m. \tag{1.7}$$

It is easy to see that if U is a real-valued solution of $m^2(\log U)_{z\overline{z}} = U^2$ then the system (1.7) has a special solution

$$U_1 = U_2 = \cdots = U_m = U.$$

The Liouville system (1.6) maybe has many solutions. As long as we can find the solutions of this system, we obtain the pseudo-analytic functions which can be represented by differential operators of Bauer-type. In this work we only consider its special solution which is indicated in the following corollary.

Corollary 1.1.
If U *is a solution of the Liouville equation*

$$m^2 (\log U)_{z\overline{z}} = U^2$$

then the system (1.6) has a special solution

$$C_1 = U e^{iv},\ C_2 = C_3 = \cdots = C_m = U$$

where v is a real-valued solution of the Laplace equation.
Therefore with $C = Ue^{iv}$ all solutions of the equation (1.1) can be represented by differential operators of Bauer-type. The solution w of the equation (1.1) is given by

$$w = K_m^1 g + C\overline{K_{m-1}^1 g}, \quad g \in H(\mathcal{D}) \tag{1.8}$$

with $K_m^1 = F_{m-1}^1 \dots F_0^1, \quad F_k^1 = \dfrac{\partial}{\partial z} - iv_z + (2m-2k-1)(\log U)_z, \quad k = 0,1,\dots,m-1.$

In [6], K.W. Bauer considered the equation of type (1.1)

$$w_{\overline{z}} \;=\; C(z,\overline{z})\overline{w}, \qquad z \in \mathcal{D}$$

where C satisfies the differential equation

$$m^2(\log C)_{z\overline{z}} - C\overline{C} = 0, \quad m > 0. \tag{1.9}$$

The coefficient $C \neq 0$ can be represented in the form $C = Ue^{iv}$, with U and v are real-valued functions. Substituting this into (1.9) we have

$$m^2(\log U)_{z\overline{z}} + iv_{z\overline{z}} = U^2.$$

This implies that $v_{z\overline{z}} = 0$ or v is a harmonic function and U satisfies the Liouville equation

$$m^2(\log U)_{z\overline{z}} = U^2.$$

So the coefficients C satisfying the equation (1.9) coincide with the coefficients C given in Corollary 1.1. Therefore the solutions of the equation (1.1) with the condition (1.9) are given by (1.8).
We get the following representation of C, see [6],

$$C = \frac{m|f'|}{1 - f\overline{f}}\frac{g}{\overline{g}}, \quad f(z), g(z) \text{ holomorphic }, (1 - f\overline{f})f'g \neq 0.$$

For

$$W(\zeta,\overline{\zeta}) = \frac{w(z,\overline{z})}{g(z)\sqrt{f'(z)}}, \quad \text{and} \quad \zeta = f(z),$$

we obtain

$$W_{\overline{\zeta}} = \frac{m}{1 - \zeta\overline{\zeta}}\overline{W}. \tag{1.10}$$

Therefore the equation (1.1) with coefficients C satisfying the condition (1.9) can be reduced to the equation (1.10).
From now on we consider the following equation which is called the Bers-Vekua equation (M) or shortly the equation (M)

$$w_{\overline{z}} = \frac{m}{1 - z\overline{z}}\overline{w}, \quad z \in K_R,\ m \in \mathbb{N} \tag{M}$$

where $K_R = \{z \in \mathbb{C} \big| |z| < R < 1\}$.
Applying Corollary 1.1, with $v = 0$, to the equation (M), the solution w of the equation (M) is given by

$$w = K_m^1 g + \frac{m}{1 - z\overline{z}} \overline{K_{m-1}^1 g}, \qquad g \in H(K_R) \tag{1.11}$$

with

$$K_m^1 = F_{m-1}^1 \ldots F_0^1, \quad F_j^1 = \frac{\partial}{\partial z} + (2m - 2j - 1)(\log \frac{m}{1 - z\overline{z}})_z, \quad j = 0, 1, \ldots, m-1.$$

Denote the coefficients in F_j^1 by $c_j \in \mathbb{N}^* = \mathbb{N} \setminus \{0\},\ j = 0, 1, \ldots, m-1$

$$F_j^1 = \partial_z + (2m - 2j - 1)\frac{\overline{z}}{1 - z\overline{z}} =: \partial_z + c_j \frac{\overline{z}}{1 - z\overline{z}}, \quad j = 0, 1, \ldots, m-1.$$

Next we are going to calculate $K_m^1 g,\ K_{m-1}^1 g$. To do this we need the following lemma.

Lemma 1.1.
Assume that

$$F_j^1 = \partial_z + c_j \frac{\overline{z}}{1 - z\overline{z}}, \quad c_j \in \mathbb{N}^*,\ j = 0, 1, \ldots, k-1.$$

Then $K_k^1 g := F_{k-1}^1 \ldots F_1^1 F_0^1 g,\ k \geq 1,$ *has the form*

$$K_k^1 g(z) = g^{(k)}(z) + \sum_{j=0}^{k-1} a_j \left(\frac{\overline{z}}{1 - z\overline{z}}\right)^{k-j} g^{(j)}(z), \qquad a_j \in \mathbb{N}^*,\ j = 0, 1, \ldots, k-1. \tag{1.12}$$

Proof.
We shall prove by induction.
The statement of Lemma 1.1 is true for $k = 1$. We assume that $K_k^1 g,\ k > 1$, has the form

$$K_k^1 g(z) = F_{k-1}^1 \ldots F_1^1 F_0^1 g(z) = g^{(k)}(z) + \sum_{j=0}^{k-1} a_j \left(\frac{\overline{z}}{1 - z\overline{z}}\right)^{k-j} g^{(j)}(z), \quad a_j \in \mathbb{N}^*.$$

Then we have to prove that $K_{k+1}^1 g$ can be written as follows

$$K_{k+1}^1 g(z) = g^{(k+1)}(z) + \sum_{j=0}^{k} \tilde{a}_j \left(\frac{\overline{z}}{1 - z\overline{z}}\right)^{k+1-j} g^{(j)}(z), \quad \tilde{a}_j \in \mathbb{N}^*,\ j = 0, 1, \ldots, k. \tag{1.13}$$

Indeed

$$
\begin{aligned}
K^1_{k+1}g(z) &= \big(\partial_z + c_k\frac{\bar z}{1-z\bar z}\big)\left[g^{(k)}(z)+\sum_{j=0}^{k-1}a_j\left(\frac{\bar z}{1-z\bar z}\right)^{k-j}g^{(j)}(z)\right]\\
&= g^{(k+1)}(z)+\sum_{j=0}^{k-1}a_j\left[(k-j)\big(\frac{\bar z}{1-z\bar z}\big)^{k-j-1}\big(\frac{\bar z}{1-z\bar z}\big)^2g^{(j)}(z)+\big(\frac{\bar z}{1-z\bar z}\big)^{k-j}g^{(j)}(z)\right]\\
&= g^{(k+1)}(z)+(c_k+a_{k-1})\frac{\bar z}{1-z\bar z}g^{(k)}(z)+\\
&\sum_{j=1}^{k-1}[(k-j)a_j+a_{j-1}+a_jc_k]\big(\frac{\bar z}{1-z\bar z}\big)^{k-j-1}g^{(j)}(z)+[a_0c_k+ka_0]\big(\frac{\bar z}{1-z\bar z}\big)^{k+1}g(z).
\end{aligned}
$$

Denote

$$\tilde a_0 := a_0c_k+ka_0,\quad \tilde a_k := c_k+a_{k-1} \text{ and } \tilde a_j := (k-j)a_j+a_{j-1}+a_jc_k,\ j=1,\dots,k-1,$$

then the expression (1.13) of $K^1_{k+1}g$ holds. Therefore $K^1_k g$ has the form (1.12). The assertion follows. □

So if we denote $a_m = b_{m-1} = 1$ and then apply Lemma 1.1, $K^1_m g$, $K^1_{m-1}g$ can be written as follows

$$K^1_m g = F^1_{m-1}\dots F^1_0 = \sum_{j=0}^{m}a_j\left(\frac{\bar z}{1-z\bar z}\right)^{m-j}g^{(j)}, \tag{1.14}$$

$$K^1_{m-1} g = F^1_{m-2}\dots F^1_0 = \sum_{j=0}^{m-1}b_j\left(\frac{\bar z}{1-z\bar z}\right)^{m-1-j}g^{(j)} \tag{1.15}$$

where $a_j\in\mathbb{N}^*,\ j=0,\dots,m-1,\ b_j\in\mathbb{N}^*,\ j=0,\dots,m-2$, are unknown coefficients. Therefore inserting the expressions (1.14) and (1.15) into (1.11) all the solutions of the equation (M) can be written in the form

$$w=\sum_{j=0}^{m}a_j\left(\frac{\bar z}{1-z\bar z}\right)^{m-j}g^{(j)}(z)+\frac{m}{1-z\bar z}\sum_{j=0}^{m-1}b_j\left(\frac{z}{1-z\bar z}\right)^{m-j-1}\overline{g^{(j)}(z)}. \tag{1.16}$$

From the expression (1.16) we have

$$
\begin{aligned}
w_{\bar z} = &\sum_{j=0}^{m-1}(m-j)a_j\frac{\bar z^{m-j-1}}{(1-z\bar z)^{m-j+1}}g^{(j)}(z)+m^2b_0\frac{z^m}{(1-z\bar z)^{m+1}}\overline{g(z)}+\\
&\sum_{j=1}^{m-1}[m(m-j)b_j+mb_{j-1}]\frac{z^{m-j}}{(1-z\bar z)^{m-j+1}}\overline{g^{(j)}(z)}+mb_{m-1}\frac{1}{1-z\bar z}\overline{g^{(m)}(z)},
\end{aligned}
\tag{1.17}
$$

and

$$\frac{m}{1-z\overline{z}}\overline{w} = \sum_{j=0}^{m-1} m^2 b_j \frac{\overline{z}^{m-j-1}}{(1-z\overline{z})^{m-j+1}} g^{(j)}(z) + \sum_{j=0}^{m} m a_j \frac{z^{m-j}}{(1-z\overline{z})^{m-j+1}} \overline{g^{(j)}(z)}. \tag{1.18}$$

Substituting the expressions (1.17) and (1.18) into the equation (M) we obtain the system

$$\begin{cases} (m-j)a_j & = m^2 b_j, \quad j=0,\dots,m-1, \\ m^2 b_0 & = m a_0, \\ m b_{m-1} & = m a_m, \\ m(m-j)b_j + m b_{j-1} & = m a_j, \quad j=1,\dots,m-1. \end{cases}$$

From this system we get

$$b_{j-1} = \frac{j(2m-j)}{m-j} b_j, \quad j=1,\dots,m-1.$$

By hypothesis $b_{m-1} = 1$,

$$\begin{aligned} b_j &= \frac{(j+1)(j+2)\dots(m-1)(2m-j-1)(2m-j-2)\dots(m+1)}{(m-j-1)!} b_{m-1} \\ &= \frac{\dfrac{(m-1)!}{j!}\dfrac{(2m-j-1)!}{m!}}{(m-j-1)!}. \end{aligned}$$

Then

$$b_j = \frac{(2m-j-1)!}{j!(m-j-1)!m}, \quad j=0,\dots,m-2. \tag{1.19}$$

Therefore

$$a_j = \frac{m^2}{m-j} b_j = \frac{(2m-j-1)!m}{j!(m-j)!}, \quad j=0,\dots,m-1. \tag{1.20}$$

Substituting (1.19) and (1.20) into (1.16) we obtain a solution of the equation (M)

$$w = \sum_{j=0}^{m} m B_j^m \left(\frac{\overline{z}}{1-z\overline{z}}\right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1} (m-j) B_j^m \frac{z^{m-j-1}}{(1-z\overline{z})^{m-j}} \overline{g^{(j)}(z)} \tag{1.21}$$

where $B_j^m = \dfrac{(2m-j-1)!}{j!(m-j)!}$, and $g \in H(K_R)$.
Summarising the above results we have the following theorem.

Theorem 1.3.
Consider the Bers-Vekua equation (M)

$$w_{\overline{z}} = \frac{m}{1 - z\overline{z}}\overline{w}, \quad m \in \mathbb{N}.$$

Then
• *For every solution w of the equation (M) defined in* $K_R = \{z \big| |z| < R < 1\}$ *there exists a function* $g \in H(K_R)$ *such that for w, the representation (1.21) holds.*
• *On the other hand for every function* $g \in H(K_R)$ *the expression in (1.21) represents a solution of the equation (M) defined in* K_R.

The function g in Theorem 1.3 is called a *generating function* of the solution w.

In the sequel using the results of I.N. Vekua [44] and K.W. Bauer [9] we also derive an explicit representation of the solutions of the equation (M) by differential operators of Bauer-type. Moreover with an additional condition on the generating function g, for each solution w the existence of the generating function is unique. First we derive the representation of the solutions by integral operators. Then after computing the Vekua resolvents we convert this representation to a form free of integrals and hence we get the representation of solutions by differential operators of Bauer-type.

1.2 Representation of the solutions by integral operators

Consider the Bers-Vekua equation (1.1)

$$w_{\overline{z}} \; = \; C(z,\overline{z})\overline{w}, \qquad z \in \mathcal{D},$$

where $\mathcal{D}$ is a simply connected domain.
The details for the statements and their proofs in this subsection can be found in [44]. Now we shall introduce some notations.
Let $f(x_1,\ldots,x_n)$ be an analytic function of the real variables $x_1,\ldots,x_n$ in some domain Ω of the space of n dimensions. Then there exists a unique function $F(z_1,\ldots,z_n)$ of the complex variables $z_1 = x_1 + iy_1,\ldots,z_n = x_n + iy_n$, analytic in a domain Ω^* of the space of $2n$ dimensions, which coincides with $f(x_1,\ldots,x_n)$ when $y_1 = \cdots = y_n = 0$ (obviously $\Omega \subset \Omega^*$). The function $F(z_1,\ldots,z_n)$ is called the *analytic continuation* of the function $f(x_1,\ldots,x_n)$ from the domain of real values of the arguments $x_1,\ldots,x_n$ into the domain of complex values.
Let $F(z_1,\ldots,z_n)$ be an analytic function of the complex variables $z_1,\ldots,z_n$ in a domain Ω_{2n} of $2n$-dimensional space. Denote

$$\overline{\Omega}_{2n} = \{(\zeta_1,\ldots,\zeta_n)|(\overline{\zeta_1},\ldots,\overline{\zeta_n}) \in \Omega_{2n}\},$$

and define
$$F^*(\zeta_1,\dots,\zeta_n) := \overline{F(\overline{\zeta_1},\dots,\overline{\zeta_n})}, \quad (\zeta_1,\dots,\zeta_n)\in\overline{\Omega}_{2n}.$$
Obviously, $F^*(\zeta_1,\dots,\zeta_n)$ is an analytic function of $\zeta_1,\dots,\zeta_n$ in $\overline{\Omega}_{2n}$. We call $F^*(\zeta_1,\dots,\zeta_n)$ the *conjugate function* of $F(z_1,\dots,z_n)$. And $F(z_1,\dots,z_n)$ is also the conjugate to $F^*(\zeta_1,\dots,\zeta_n)$. We denote by $\overline{\mathcal{D}}$ the mirror image of $\mathcal{D}$ with respect to the real axis. If $\mathcal{D}$ is symmetrical with respect to this axis then $\mathcal{D}$ and $\overline{\mathcal{D}}$ are obviously the same.
By hypothesis the coefficient $C(z,\bar{z})$ of the equation (1.1) is an analytic function of the real variables x and y. If we continue analytically this function into a complex domain, we obtain an analytic function $C(z,\zeta)$ of the two complex variables $z\in\mathcal{D},\zeta\in\overline{\mathcal{D}}$
$$z = x+iy, \qquad \zeta = x-iy.$$
I.N. Vekua proved in [44] that *every solution of the equation (1.1) in $\mathcal{D}$ also can be continued analytically into the domain* $(\mathcal{D},\overline{\mathcal{D}})$, i.e., (1.1) is satisfied for $z\in\mathcal{D},\zeta\in\overline{\mathcal{D}}$ by some function $w(z,\zeta)$, analytic in z and ζ. In this case $\dfrac{\partial w}{\partial\bar{z}}$ is equal to the partial derivative $\dfrac{\partial w}{\partial\zeta}$ and (1.1) takes the form
$$\frac{\partial w(z,\zeta)}{\partial\zeta} \;=\; C(z,\zeta)w^*(\zeta,z), \quad (z,\zeta)\in\mathcal{D}\times\overline{\mathcal{D}}, \tag{1.22}$$
where $w^*(\zeta,z)$ is the conjugate function of $w(z,\zeta)$.
The equation (1.22) is called the *complex form* of the equation (1.1).
If $w(z,\zeta)$ is an analytic function of z and ζ, with $z\in\mathcal{D},\zeta\in\overline{\mathcal{D}}$, satisfying the differential equation (1.22), then $w(z,\bar{z})$ is an analytic function of the real variables x,y in $\mathcal{D}$, satisfying the differential equation (1.1). Therefore first we derive a formula which gives all the solutions of (1.22), analytic in z and ζ, with $z\in\mathcal{D},\zeta\in\overline{\mathcal{D}}$.
Assume that $w(z,\zeta)$ is such a solution of (1.22). We can now transform (1.22) as follows
$$\frac{\partial}{\partial\zeta}\Big[w(z,\zeta)-\int_{\zeta_0}^{\zeta}C(z,\tau)w^*(\tau,z)d\tau\Big] \;=\; 0.$$
This implies that
$$w(z,\zeta) \;=\; \varphi(z)+\int_{\zeta_0}^{\zeta}C(z,\tau)w^*(\tau,z)d\tau, \tag{1.23}$$
with $\varphi(z)$ is an analytic function of z in $\mathcal{D}$ and ζ_0 is a fixed point in $\overline{\mathcal{D}}$.
We now pass from (1.23) to the adjoint equation
$$w^*(\zeta,z) \;=\; \varphi^*(\zeta)+\int_{z_0}^{z}C^*(\zeta,t)w(t,\zeta)dt, \tag{1.24}$$
with $z_0=\overline{\zeta_0}$. If we substitute the expression (1.24) into the right-hand side of (1.23), we get
$$w(z,\zeta) \;=\; \varphi(z)+\int_{\zeta_0}^{\zeta}C(z,\tau)\varphi^*(\tau)d\tau+\int_{z_0}^{z}dt\int_{\zeta_0}^{\zeta}C(z,\tau)C^*(\tau,t)w(t,\tau)d\tau. \tag{1.25}$$

Denote

$$\Phi(z,\zeta) \;:=\; \varphi(z)+\int_{\zeta_0}^{\zeta} C(z,\tau)\varphi^*(\tau)d\tau, \tag{1.26}$$

then the equation (1.25) results in

$$w(z,\zeta)-\int_{z_0}^{z} dt\int_{\zeta_0}^{\zeta} C(z,\tau)C^*(\tau,t)w(t,\tau)d\tau = \Phi(z,\zeta). \tag{1.27}$$

As may be seen, every solution $w(z,\zeta)$ of the equation (1.22), analytic in z,ζ in the domain $(\mathcal{D},\overline{\mathcal{D}})$, also satisfies the Volterra integral equation (1.27). The right-hand side of this integral equation contains a function $\varphi(z)$ which is continuous, analytic in $\mathcal{D}$ and uniquely determined by $w(z,\zeta)$

$$\varphi(z) \;=\; w(z,\zeta_0). \tag{1.28}$$

An integral equation of the type (1.27) is well known and had been solved in [44]. Its solution has the form

$$w(z,\zeta) \;=\; \Phi(z,\zeta)+\int_{z_0}^{z} dt\int_{\zeta_0}^{\zeta} \Gamma(z,\zeta,t,\tau)\Phi(t,\tau)d\tau, \tag{1.29}$$

where $\Gamma(z,\zeta,t,\tau)$ is called the *main Vekua resolvent* of integral equation (1.27). The main resolvent satisfies the integral equation

$$\Gamma(z,\zeta,t,\tau) \;=\; C(z,\tau)C^*(\tau,t)+\int_{\tau}^{\zeta} d\eta\int_{t}^{z} C(\xi,\tau)C^*(\tau,t)\Gamma(z,\zeta,\xi,\eta)d\xi. \tag{1.30}$$

Note that $\Gamma(z,\zeta,t,\tau)$ is an analytic function of the four variables z,ζ,t,τ in the domain $z,t\in\mathcal{D}$, $\zeta,\tau\in\overline{\mathcal{D}}$.
Substituting (1.26) into (1.29) we obtain

$$\boxed{w(z,\zeta) \;=\; \varphi(z)+\int_{z_0}^{z}\Gamma_1(z,\zeta,t,\zeta_0)\varphi(t)dt+\int_{\zeta_0}^{\zeta}\Gamma_2(z,\zeta,z_0,\tau)\varphi^*(\tau)d\tau} \tag{1.31}$$

where

$$\begin{aligned}\Gamma_1(z,\zeta,t,\tau) \;&=\; \int_{\tau}^{\zeta}\Gamma(z,\zeta,t,\eta)d\eta,\\ \Gamma_2(z,\zeta,t,\tau) \;&=\; C(z,\tau)+\int_{t}^{z} C(\xi,\tau)\Gamma_1(z,\zeta,\xi,\tau)d\xi \;=\; \frac{\Gamma(z,\zeta,t,\tau)}{C^*(\tau,t)}.\end{aligned}$$

Γ_1,Γ_2 are called the *first* and *second Vekua resolvent*, respectively, and they have the following properties

$$\frac{\partial\Gamma_1(z,\zeta,t,\tau)}{\partial\zeta}-C(z,\zeta)\Gamma_2^*(\zeta,z,\tau,t) \;=\; 0, \tag{1.32}$$

$$\frac{\partial \Gamma_2(z,\zeta,t,\tau)}{\partial \zeta} - C(z,\zeta)\Gamma_1^*(\zeta,z,\tau,t) = 0, \tag{1.33}$$

$$\Gamma_2|_{\zeta=\tau} = \Gamma_2(z,\tau,t,\tau) = C(z,\tau), \tag{1.34}$$

$$\Gamma_2|_{z=t} = \Gamma_2(t,\zeta,t,\tau) = C(t,\tau). \tag{1.35}$$

I.N. Vekua proved that *the formula (1.31) gives all solutions of the equation (1.22), analytic in* z,ζ *in the domain* $(\mathcal{D},\overline{\mathcal{D}})$.
If we replace ζ by $\bar{z}$ in (1.31) we obtain the representation of solutions of the equation (1.1) by integral operators, analytic in the real variables x and y in $\mathcal{D}$. Applying this method to the equation (M) we also get the representation of solutions of the equation (M) by integral operators. However our aim is to derive an explicit representation of the solutions of the equation (M) by differential operators of Bauer-type. Then we need to determine the first and second resolvents Γ_1,Γ_2 by using the method of K.W. Bauer ([5], [6]). This will be done in the next section.

1.3 Determination of the Vekua resolvents

Lemma 1.2.
Γ_1,Γ_2 *are solutions of an equation*

$$W_{z\zeta} - \frac{C_z}{C}W_\zeta - CC^*W = 0. \tag{1.36}$$

Proof.
To prove this lemma we need the properties (1.32) and (1.33). Differentiating the two sides of the equation (1.33) with respect to z we get

$$\frac{\partial^2\Gamma_2(z,\zeta,t,\tau)}{\partial z\partial\zeta} - \frac{\partial C(z,\zeta)}{\partial z}\Gamma_1^*(\zeta,z,\tau,t) - C(z,\zeta)\frac{\partial\Gamma_1^*(\zeta,z,\tau,t)}{\partial z} = 0. \tag{1.37}$$

By definition

$$\frac{\partial\Gamma_1^*(\zeta,z,\tau,t)}{\partial z} = \frac{\partial\overline{\Gamma_1(\overline{\zeta},\bar{z},\bar{\tau},\bar{t})}}{\partial z} = \overline{\left[\frac{\partial\Gamma_1(\overline{\zeta},\bar{z},\bar{\tau},\bar{t})}{\partial\bar{z}}\right]},$$

and from the property (1.32) we have

$$\overline{\left[\frac{\partial\Gamma_1(\overline{\zeta},\bar{z},\bar{\tau},\bar{t})}{\partial\bar{z}}\right]} = \overline{C(\overline{\zeta},\bar{z})}.\overline{\Gamma_2^*(\bar{z},\overline{\zeta},\bar{t},\bar{\tau})} = C^*(\zeta,z)\Gamma_2(z,\zeta,t,\tau).$$

This implies that

$$\frac{\partial\Gamma_1^*(\zeta,z,\tau,t)}{\partial z} = C^*(\zeta,z)\Gamma_2(z,\zeta,t,\tau). \tag{1.38}$$

From the property (1.33) we have

$$\Gamma_1^*(\zeta,z,\tau,t) \;=\; \frac{1}{C(z,\zeta)}\frac{\partial\Gamma_2(z,\zeta,t,\tau)}{\partial\zeta}. \tag{1.39}$$

Substituting (1.38) and (1.39) into (1.37) we obtain

$$\frac{\partial^2\Gamma_2(z,\zeta,t,\tau)}{\partial z\partial\zeta} - C_z(z,\zeta)\frac{1}{C(z,\zeta)}\frac{\partial\Gamma_2(z,\zeta,t,\tau)}{\partial\zeta} - C(z,\zeta)C^*(\zeta,z)\Gamma_2(z,\zeta,t,\tau) \;=\; 0.$$

Therefore Γ_2 and Γ_1 (prove analogously) are solutions of the equation (1.36) and Lemma 1.2 is proved.

□

From the properties (1.34) and (1.35) together with the equation (1.36) we can determine Γ_2 and then Γ_1.
If we know one solution $W(z,\zeta,t,\tau)$ of (1.36) with the initial conditions

$$W|_{\zeta=\tau} \;=\; C(z,\tau), \qquad\qquad W|_{z=t} \;=\; C(t,\tau), \tag{1.40}$$

it follows that

$$\Gamma_2 \;=\; W, \qquad\qquad \Gamma_1^* \;=\frac{1}{C(z,\zeta)}W_\zeta.$$

In the case of the equation (M), the analytic continuation of the coefficient $C(z,\bar{z})$ has the form $C(z,\zeta) = \dfrac{m}{1-z\zeta}$. Then the equation (1.36) reads

$$\omega^2 W_{z\zeta} - \zeta\omega W_\zeta - m^2 W \;=\; 0 \quad \text{with } \omega := (1-z\zeta). \tag{1.41}$$

We are going to seek a solution W with the initial conditions (1.40) in the following form

$$W \;=\; \frac{m}{1-z\tau}H(\lambda)$$

with

$$\lambda = \lambda(z,\zeta,t,\tau),\; H|_{\zeta=\tau} = H|_{z=t} = 1.$$

We have

$$\begin{aligned} W_\zeta &= \frac{m}{1-z\tau}H'\lambda_\zeta, \\ W_{z\zeta} &= \frac{m\tau}{(1-z\tau)^2}H'\lambda_\zeta + \frac{m}{1-z\tau}(H''\lambda_z\lambda_\zeta + H'\lambda_{z\zeta}). \end{aligned}$$

Substituting these expressions into the equation (1.41) we obtain

$$\omega^2\lambda_z\lambda_\zeta H'' + \left[\omega^2\lambda_{z\zeta} + \frac{\omega(\tau-\zeta)}{1-z\tau}\lambda_\zeta\right]H' - m^2 H \;=\; 0. \tag{1.42}$$

Choose $\lambda = \dfrac{d(z-t)(\zeta-\tau)}{1-z\zeta}$, where d is a function not depending on z and ζ, then

$$\begin{aligned}\lambda_z &= d(\zeta-\tau)\frac{(1-t\zeta)}{(1-z\zeta)^2}, \quad \lambda_\zeta = d(z-t)\frac{(1-z\tau)}{(1-z\zeta)^2},\\ \lambda_{z\zeta} &= d\frac{(1-z\tau)}{(1-z\zeta)^2} - d(z-t)\frac{\tau}{(1-z\zeta)^2} + d(z-t)(1-z\tau)\frac{2\zeta}{(1-z\zeta)^3}.\end{aligned}$$

Therefore the equation (1.42) results in

$$[d(1-t\tau)\lambda+\lambda^2]H'' + [d(1-t\tau)+\lambda]H' - m^2H = 0.$$

Choose $d = \dfrac{-1}{1-t\tau}$, $(1-t\tau \neq 0)$ then we have the hypergeometric differential equation

$$\lambda(1-\lambda)H'' + [\gamma-(\alpha+\beta+1)\lambda]H' - \alpha\beta H = 0, \tag{1.43}$$

with $H|_{\lambda=0} = 1,\ \alpha = m,\ \beta = -m,\ \gamma = 1.$
Some properties of the hypergeometric differential equations and their solutions can be found in, e.g., [2], [24] or [42].
A solution $H(\lambda)$ of the hypergeometric equation (1.43) is given by

$$H(\lambda) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{\lambda^k}{k!}$$

where $(x)_k,\ k\in\mathbb{N}$, is the Pochhammer symbol defined by

$$(x)_k = \begin{cases} 1 & \text{if } k = 0,\\ x(x+1)\dots(x+k-1) & \text{if } k > 0.\end{cases}$$

For $\alpha = m,\ \beta = -m,\ \gamma = 1$ we have

$$H(\lambda) = 1 + \sum_{k=1}^{m}(-1)^k\frac{m(m+k-1)!}{(k!)^2(m-k)!}\lambda^k \tag{1.44}$$

with

$$\lambda = -\frac{(z-t)(\zeta-\tau)}{(1-z\zeta)(1-t\tau)}.$$

Therefore

$$H(\lambda(z,\zeta,t,\tau)) = 1 + \sum_{k=1}^{m}(-1)^k\frac{m(m+k-1)!}{(k!)^2(m-k)!}(-1)^k\frac{(z-t)^k(\zeta-\tau)^k}{(1-z\zeta)^k(1-t\tau)^k}.$$

So we have

$$\Gamma_2(z,\zeta,t,\tau) \;=\; W(z,\zeta,t,\tau) \;=\; \frac{m}{1-z\tau}H(\lambda)$$

$$= \frac{m}{1-z\tau}\left[1+\sum_{k=1}^{m}\frac{m(m+k-1)!}{(k!)^2(m-k)!}\frac{(z-t)^k(\zeta-\tau)^k}{(1-z\zeta)^k(1-t\tau)^k}\right].$$

This implies that

$$\frac{\partial\Gamma_2(z,\zeta,t,\tau)}{\partial\zeta} \;=\; \sum_{k=1}^{m}\frac{m^2(m+k-1)!k}{(k!)^2(m-k)!}\left(\frac{z-t}{1-t\tau}\right)^k\frac{(\zeta-\tau)^{k-1}}{(1-z\zeta)^{k+1}}. \tag{1.45}$$

By definition of the conjugate function and the relation (1.39), we get

$$\overline{\Gamma_1(\overline{\zeta},\overline{z},\overline{\tau},\overline{t})} \;=\; \frac{1}{C(z,\zeta)}\frac{\partial\Gamma_2(z,\zeta,t,\tau)}{\partial\zeta}. \tag{1.46}$$

Substituting (1.45) into (1.46), we obtain

$$\Gamma_1(\overline{\zeta},\overline{z},\overline{\tau},\overline{t}) \;=\; \sum_{k=1}^{m}\frac{m(m+k-1)!k}{(k!)^2(m-k)!}\left(\frac{\overline{z}-\overline{t}}{1-\overline{t}\overline{\tau}}\right)^k\frac{(\overline{\zeta}-\overline{\tau})^{k-1}}{(1-\overline{z}\overline{\zeta})^k},$$

$$\Rightarrow \Gamma_1(z,\zeta,t,\tau) \;=\; \sum_{k=1}^{m}\frac{m(m+k-1)!k}{(k!)^2(m-k)!}\left(\frac{\zeta-\tau}{1-t\tau}\right)^k\frac{(z-t)^{k-1}}{(1-z\zeta)^k}.$$

To sum up we have

$$\Gamma_1(z,\zeta,t,\tau) \;=\; \sum_{k=1}^{m}\frac{m(m+k-1)!k}{(k!)^2(m-k)!}\left(\frac{\zeta-\tau}{1-t\tau}\right)^k\frac{(z-t)^{k-1}}{(1-z\zeta)^k}, \tag{1.47}$$

$$\Gamma_2(z,\zeta,t,\tau) \;=\; \frac{m}{1-z\tau}\left[1+\sum_{k=1}^{m}\frac{m(m+k-1)!}{(k!)^2(m-k)!}\frac{(z-t)^k(\zeta-\tau)^k}{(1-z\zeta)^k(1-t\tau)^k}\right]. \tag{1.48}$$

After having the resolvents Γ_1 and Γ_2 we shall convert the representation (1.31) of the solutions to a form free of integrals. Hence we get the representation of the solutions of equation (M) by differential operators of Bauer-type.

1.4 Representation of the solutions by differential operators of Bauer-type

In (1.31) we can choose $z_0 = \zeta_0 = 0$, then the following formula

$$w(z,\zeta) \;=\; \varphi(z)+\int_0^z\Gamma_1(z,\zeta,t,0)\varphi(t)dt+\int_0^\zeta\Gamma_2(z,\zeta,0,\tau)\varphi^*(\tau)d\tau \tag{1.49}$$

gives all the analytic solutions of the equation

$$\frac{\partial w(z,\zeta)}{\partial \zeta} \;=\; \frac{m}{1-z\zeta} w^*(\zeta,z), \quad z,\zeta \in K_R. \tag{1.50}$$

Next we are going to calculate the two integrals in the formula (1.49).
The first integral in (1.49) is

$$\begin{aligned}\int_0^z \Gamma_1(z,\zeta,t,0)\varphi(t)dt \;&=\; \int_0^z \sum_{k=1}^{m} \frac{m(m+k-1)!k}{(k!)^2(m-k)!} \frac{\zeta^k (z-t)^{k-1}}{(1-z\zeta)^k} \varphi(t)dt \\ &=\; \sum_{k=1}^{m} \frac{m(m+k-1)!k}{(k!)^2(m-k)!} \frac{\zeta^k}{(1-z\zeta)^k} \int_0^z (z-t)^{k-1}\varphi(t)dt. \end{aligned} \tag{1.51}$$

Now we introduce the space $H_{K_R}(k,0)$, see [27], of all functions $g(z) \in H(K_R)$ satisfying

$$g(0) = g'(0) = \cdots = g^{(k-1)}(0) = 0.$$

In order to calculate the integrals on the right-hand side of (1.51), we need the following lemma.

Lemma 1.3.
For any function $\varphi(t) \in H(K_R)$ there exists a unique function $g(t) \in H_{K_R}(m,0)$ such that $\varphi(t) = g^{(m)}(t)$.

Proof.
The statement of the Lemma 1.3 can be obtained easily by considering the function

$$g(z) \;=\; \frac{1}{(m-1)!} \int_0^z (z-t)^{m-1}\varphi(t)dt.$$

□

Applying Lemma 1.3 for the function $\varphi(t) \in H(K_R)$, there exists a unique function $g(t) \in H_{K_R}(m,0)$ such that $\varphi(t) = g^{(m)}(t)$.
Denote

$$I_k \;=\; \int_0^z (z-t)^{k-1}\varphi(t)dt, \quad 1 \le k \le m,$$

then using the Lemma 1.3 we get

$$I_k = \int_0^z (z-t)^{k-1} g^{(m)}(t)dt = (k-1)\int_0^z (z-t)^{k-2} g^{(m-1)}(t)dt$$
$$= \cdots = (k-1)! g^{(m-k)}(t)\Big|_0^z = (k-1)! g^{(m-k)}(z).$$

Hence

$$I_k = (k-1)! g^{(m-k)}(z), \ 1 \le k \le m. \tag{1.52}$$

Inserting I_k into the expression (1.51) we obtain

$$\int_0^z \Gamma_1(z,\zeta,t,0)\varphi(t)dt = \sum_{k=1}^{m} \frac{m(m+k-1)!k}{(k!)^2(m-k)!}\frac{\zeta^k}{(1-z\zeta)^k}(k-1)! g^{(m-k)}(z).$$

Denote $j := m-k$ then

$$\int_0^z \Gamma_1(z,\zeta,t,0)\varphi(t)dt = \sum_{j=0}^{m-1} \frac{m(2m-j-1)!}{(m-j)!j!}\left(\frac{\zeta}{1-z\zeta}\right)^{m-j} g^{(j)}(z). \tag{1.53}$$

The second integral in (1.49) is

$$\int_0^\zeta \Gamma_2(z,\zeta,0,\tau)\varphi^*(\tau)d\tau, \tag{1.54}$$

where $\Gamma_2(z,\zeta,0,\tau)$ is given by (1.48).
In order to calculate the integral (1.54) we need the following lemma.

Lemma 1.4.
Denote

$$T = \sum_{k=1}^{m} (-1)^k \frac{m(m+k-1)!}{(k!)^2(m-k)!}, \tag{1.55}$$

then $T+1=0$.

From now on we denote the binomial coefficients C_k^m as

$$C_k^m =: \frac{m!}{k!(m-k)!}, \ k,m \in \mathbb{N}, \ k \le n.$$

Proof.
We can rewrite T in the form

$$T = \frac{1}{(m-1)!}\sum_{k=1}^{m}(-1)^k C_k^m \frac{(m+k-1)!}{k!}.$$

Now we consider the expansion

$$(1-x)^m = \sum_{k=0}^{m} C_k^m (-1)^k x^k,$$

$$\Rightarrow (1-x)^m x^{m-1} = \sum_{k=0}^{m}(-1)^k C_k^m x^{m+k-1}. \tag{1.56}$$

Differentiating the two sides of (1.56) of order $m-1$ with respect to x and then substituting $x=1$ we obtain the equality

$$\sum_{k=0}^{m}(-1)^k C_k^m \frac{(m+k-1)!}{k!} = 0$$

$$\Leftrightarrow \sum_{k=1}^{m}(-1)^k C_k^m \frac{(m+k-1)!}{k!} = -(m-1)!$$
$$\Leftrightarrow \frac{1}{(m-1)!}\sum_{k=1}^{m}(-1)^k C_k^m \frac{(m+k-1)!}{k!} = -1.$$

Thus Lemma 1.4 is proved. □

Remark 1.1.
There is another way to prove Lemma 1.4 *using the formula* (1.44) *for the solution of the hypergeometric equation*

$$H(\lambda) = 1 + \sum_{k=1}^{m}(-1)^k \frac{m(m+k-1)!}{(k!)^2(m-k)!}\lambda^k.$$

We obviously see that

$$H|_{\lambda=1} = H(1) = T+1.$$

Since the value of the hypergeometric series at $\lambda = 1$ *is equal to zero, we have* $T = -1$.

Now we use the above result to compute $\Gamma_2(z,\zeta,0,\tau)$ as follows

$$\begin{aligned}
\Gamma_2(z,\zeta,0,\tau) &= \frac{m}{1-z\tau}\left[1+\sum_{k=1}^{m}\frac{m(m+k-1)!}{(k!)^2(m-k)!}\frac{z^k(\zeta-\tau)^k}{(1-z\zeta)^k}\right] \\
&= \frac{m}{1-z\tau}\sum_{k=1}^{m}\frac{m(m+k-1)!}{(k!)^2(m-k)!}\left[(\frac{z\zeta-z\tau}{1-z\zeta})^k-(-1)^k\right] \\
&= \frac{m}{1-z\tau}\sum_{k=1}^{m}\frac{m(m+k-1)!}{(k!)^2(m-k)!}\left[\left(\frac{1-z\tau}{1-z\zeta}-1\right)^k-(-1)^k\right] \\
&= \frac{m}{1-z\tau}\sum_{k=1}^{m}\frac{m(m+k-1)!}{(k!)^2(m-k)!}\frac{1-z\tau}{1-z\zeta}\sum_{p=0}^{k-1}\left(\frac{1-z\tau}{1-z\zeta}-1\right)^p(-1)^{k-1-p} \\
&= \sum_{k=1}^{m}\frac{m^2(m+k-1)!}{(k!)^2(m-k)!}\sum_{p=0}^{k-1}(-1)^{k-1-p}\frac{z^p(\zeta-\tau)^p}{(1-z\zeta)^{p+1}}.
\end{aligned}$$

Therefore

$$\int_0^{\zeta}\Gamma_2(z,\zeta,0,\tau)\varphi^*(\tau)d\tau = \sum_{k=1}^{m}\frac{m^2(m+k-1)!}{(k!)^2(m-k)!}\sum_{p=0}^{k-1}\frac{(-1)^{k-1-p}z^p}{(1-z\zeta)^{p+1}}\int_0^{\zeta}(\zeta-\tau)^p\varphi^*(\tau)d\tau. \tag{1.57}$$

Now we denote

$$J_p = \int_0^{\zeta}(\zeta-\tau)^p\varphi^*(\tau)d\tau, \qquad 0\le p\le k-1.$$

Here $\varphi^*(t)$ is the conjugate function of the function $\varphi(t)$. So using Lemma 1.3 and the definition of $\varphi^*(t)$ we can write

$$J_p = \int_0^{\zeta}(\zeta-\tau)^p\overline{g^m(\overline{\tau})}d\tau = \overline{\int_0^{\overline{\zeta}}(\overline{\zeta}-\overline{\tau})^p g^m(\overline{\tau})d\overline{\tau}},$$

where $g(t)\in H_{K_R}(m,0)$ is chosen as in Lemma 1.3.
By using the property of the complex conjugate and changing the variable in integral we

have

$$
\begin{aligned}
\overline{J_p} &= \int\limits_0^{\overline{\zeta}} (\overline{\zeta}-t)^p g^{(m)}(t)dt = (\overline{\zeta}-t)^p g^{(m-1)}(t)\Big|_0^{\overline{\zeta}} + p\int\limits_0^{\overline{\zeta}} (\overline{\zeta}-t)^{p-1} g^{(m-1)}(t)dt \\
&= p\int\limits_0^{\overline{\zeta}} (\overline{\zeta}-t)^{p-1} g^{(m-1)}(t)dt = \cdots = p! g^{(m-p-1)}(\overline{\zeta}).
\end{aligned}
$$

Hence $J_p = p!\overline{g^{(m-p-1)}(\overline{\zeta})},\ 0 \le p \le k-1.$ Substituting this into (1.57) we have the following form of the integral (1.54)

$$
\int\limits_0^{\zeta} \Gamma_2(z,\zeta,0,\tau)\varphi^*(\tau)d\tau = \sum_{j=0}^{m-1} A_j \frac{z^{m-1-j}}{(1-z\zeta)^{m-j}} \overline{g^{(j)}(\overline{\zeta})}.
$$

Therefore we obtain the form of $w(z,\zeta)$

$$
w = g^{(m)}(z) + \sum_{j=0}^{m-1} \frac{m(2m-j-1)!}{(m-j)!j!}\left(\frac{\zeta}{1-z\zeta}\right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1} A_j \frac{z^{m-1-j}}{(1-z\zeta)^{m-j}} \overline{g^{(j)}(\overline{\zeta})}. \tag{1.58}
$$

Inserting the expression (1.58) into the equation (1.50) we have the following representation

$$
\begin{aligned}
w(z,\zeta) &= g^{(m)}(z) + \sum_{j=0}^{m-1} \frac{m(2m-j-1)!}{(m-j)!j!}\left(\frac{\zeta}{1-z\zeta}\right)^{m-j} g^{(j)}(z) \\
&\quad + \sum_{j=0}^{m-1} \frac{(2m-j-1)!}{j!(m-j-1)!} \frac{z^{m-1-j}}{(1-z\zeta)^{m-j}} \overline{g^{(j)}(\overline{\zeta})},
\end{aligned} \tag{1.59}
$$

where $g \in H_{K_R}(m,0)$.
The expression (1.59) gives all solutions of the equation (1.50) analytic in z,ζ in the domain K_R. Replacing ζ by $\overline{z}$ in (1.59) we obtain the following theorem.

Theorem 1.4.
Consider the differential equation (M)

$$
w_{\overline{z}} = \frac{m}{1-z\overline{z}}\overline{w}, \qquad m \in \mathbb{N},\ z \in K_R.
$$

Denote the coefficients of $g^{(j)}(z)$ and $\overline{g^{(j)}(z)}$ in (1.59) by $a_j(z,\overline{z})$ and $b_j(z,\overline{z})$, respectively. Then for every solution w of the equation (M) in K_R, analytic in the variables x and y, there *exists a unique generating function* $g \in H_{K_R}(m,0)$ *such that* w *has the representation*

$$\begin{aligned} w &= \sum_{j=0}^{m} a_j(z,\bar{z}) g^{(j)}(z) + \sum_{j=0}^{m-1} b_j(z,\bar{z}) \overline{g^{(j)}(z)} := \\ &\sum_{j=0}^{m} m B_j^m \left(\frac{\bar{z}}{1 - z\bar{z}} \right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1} (m-j) B_j^m \frac{z^{m-j-1}}{(1-z\bar{z})^{m-j}} \overline{g^{(j)}(z)}, \end{aligned} \tag{1.60}$$

where $B_j^m = \dfrac{(2m-j-1)!}{j!(m-j)!}$.

Conversely for each function $g \in H_{K_R}(m,0)$ *(1.60) represents a solution of the equation (M) in* K_R.

2 A CLASS OF BOUNDARY VALUE PROBLEMS

In this chapter we consider some boundary value problems for pseudo-analytic functions which can be represented by differential operators of Bauer-type. We show that these problems are equivalent to certain ordinary differential equations for the generating functions defined on the boundary of the domain under consideration. For the Bers-Vekua equation (M) we shall solve these differential equations explicitly using Fourier expansions for the functions involved. Once the generating function is determined on the boundary we can express it in the whole domain. This method can be applied to the Dirichlet boundary value problem and a class of the generalized Riemann-Hilbert boundary value problems for the pseudo-analytic functions which are solutions of the equation (M). The boundary value problems for such pseudo-analytic functions and poly-pseudoanalytic functions are treated in [17]. Applying this method to the more general boundary value problems for other classes of the Bers-Vekua equations is an open question.

2.1 The Dirichlet boundary value problem

We consider the boundary value problem

$$w_{\overline{z}} = C\,\overline{w} \quad \text{in} \quad \mathcal{D}, \tag{2.1}$$
$$\operatorname{Re}(w) = \Psi \quad \text{on} \quad \partial\mathcal{D}, \tag{2.2}$$

where C is an arbitrary analytic function defined in $\mathcal{D}$ and Ψ is Hölder-continuous on $\partial\mathcal{D}$. I.N. Vekua [44] presented theorems concerning the existence of solutions of this problem. He proved that this boundary value problem is equivalent to a singular integral equation for a certain density function, the kernel of which depends on the coefficient C.

In the following we will show that for the certain problem with $C = \dfrac{m}{1-z\overline{z}}$ whose solutions have the representation using the Bauer-type operators in the form (1.60) this boundary value problem can be solved explicitly in a direct way.
According to Theorem 1.4, the generating function $g \in H_{K_R}(m,0)$ is determined uniquely by the solution w, then we can state that solving the boundary value problem (2.1)-(2.2) is equivalent to finding the suitable generating function g.
Now the boundary condition (2.2) in connection with the representation (1.60) for w leads

to the differential equation

$$\operatorname{Re}\left\{\sum_{j=0}^{m} a_j(\xi) g^{(j)}(\xi) + \sum_{j=0}^{m-1} b_j(\xi)\overline{g^{(j)}(\xi)}\right\} = \Psi(\xi), \tag{2.3}$$

where $\xi \in \partial K_R$ and $a_j(\xi) := a_j(z,\bar{z})|_{\partial K_R}$, $b_j(\xi) := b_j(z,\bar{z})|_{\partial K_R}$ are used.

With respect to the condition $g \in H_{K_R}(m,0)$ we use for g the expansion

$$g(z) = \sum_{k=m}^{\infty} \gamma_k z^k, \quad \gamma_k \in \mathbb{C}.$$

In particular we have

$$g(\xi) = \sum_{k=m}^{\infty} \gamma_k \xi^k \quad \text{on} \quad \partial K_R. \tag{2.4}$$

Now we are going to calculate the coefficients γ_k, for $k \geq m$. With $g(\xi)$ in the form (2.4) the boundary condition (2.3) can be written as

$$\operatorname{Re}\left\{\sum_{j=0}^{m} a_j(\xi) \sum_{k=m}^{\infty} \frac{k!}{(k-j)!}\gamma_k \xi^{k-j} + \sum_{j=0}^{m-1} b_j(\xi) \sum_{k=m}^{\infty} \frac{k!}{(k-j)!}\overline{\gamma}_k \overline{\xi}^{k-j}\right\} = \Psi(\xi). \tag{2.5}$$

Since the coefficients a_j and b_j in (2.3) are known explicitly, we can solve the differential equation (2.3) for the function g in the following way.
Inserting the coefficients a_j and b_j into the differential equation (2.5) we have

$$\operatorname{Re}\left\{\sum_{j=0}^{m} \frac{B_j^m}{(1-\xi\overline{\xi})^{m-j}} \times \left[m\overline{\xi}^{m-j} \sum_{k=m}^{\infty} \frac{k!\gamma_k \xi^{k-j}}{(k-j)!} + (m-j)\xi^{m-j-1} \sum_{k=m}^{\infty} \frac{k!\overline{\gamma}_k \overline{\xi}^{k-j}}{(k-j)!}\right]\right\} = \Psi(\xi).$$

Introducing the real parameter $t \in [0,2\pi]$ by $\xi = Re^{it} \in \partial K_R$ we obtain

$$\operatorname{Re}\left\{\sum_{k=m}^{\infty} c_k \gamma_k R^{k-m} e^{i(k-m)t} + \sum_{k=m}^{\infty} d_k \overline{\gamma}_k R^{k-m+1} e^{-i(k-m+1)t}\right\} = \Psi(t), \tag{2.6}$$

with

$$\begin{aligned} c_k &= \sum_{j=0}^{m} m B_j^m \frac{k!}{(k-j)!} \frac{R^{2(m-j)}}{(1-R^2)^{m-j}} > 0, \\ d_k &= \sum_{j=0}^{m-1} (m-j) B_j^m \frac{k!}{(k-j)!} \frac{R^{2(m-j-1)}}{(1-R^2)^{m-j}} > 0. \end{aligned} \tag{2.7}$$

Now we use $\gamma_k = \alpha_k + i\beta_k,\ \alpha_k, \beta_k \in \mathbb{R},\ k \geq m$, and $e^{it} = \cos t + i \sin t,\ t \in \mathbb{R}$, for which we get

$$\begin{aligned} \mathrm{Re}(w)|_{\partial K_R} \;=\; & c_m\alpha_m + \sum_{k=1}^{\infty}(c_{m+k}\alpha_{m+k} + d_{m+k-1}\alpha_{m+k-1})R^k\cos(kt) \\ & - \sum_{k=1}^{\infty}(c_{m+k}\beta_{m+k} + d_{m+k-1}\beta_{m+k-1})R^k\sin(kt). \end{aligned} \tag{2.8}$$

Now the boundary function Ψ is assumed to possess a uniformly convergent Fourier series of the form

$$\Psi(t) \;=\; \varphi_0 + \sum_{k=1}^{\infty}(\varphi_k\cos(kt) + \psi_k\sin(kt)). \tag{2.9}$$

Comparing the two expressions (2.8) and (2.9) we are led to the following linear system of the coefficients α_k and β_k

$$\begin{cases} c_m\alpha_m & = \quad \varphi_0, \\ (c_{m+k}\alpha_{m+k} + d_{m+k-1}\alpha_{m+k-1})R^k & = \quad \varphi_k, \quad k = 1,2,\ldots \\ -(c_{m+k}\beta_{m+k} + d_{m+k-1}\beta_{m+k-1})R^k & = \quad \psi_k, \quad k = 1,2,\ldots \end{cases}$$

Here $\beta_m \in \mathbb{R}$ can be chosen arbitrarily and then the remaining coefficients can be calculated recursively as follows

$$\begin{aligned} \alpha_m \quad &= \quad \frac{\varphi_0}{c_m}, \\ \alpha_{m+k} \quad &= \quad \frac{\varphi_k - d_{m+k-1}\alpha_{m+k-1}R^k}{c_{m+k}R^k}, \quad k = 1,2,\ldots \\ \beta_{m+k} \quad &= \quad -\frac{\psi_k + d_{m+k-1}\beta_{m+k-1}R^k}{c_{m+k}R^k}, \quad k = 1,2,\ldots \end{aligned} \tag{2.10}$$

To sum up we have the following theorem.

Theorem 2.1.
The boundary value problem

$$\begin{aligned} w_{\bar{z}} \;&=\; \frac{m}{1 - z\bar{z}}\overline{w} \quad in \quad K_R = \{z \big| |z| < R,\ 0 < R < 1\}, \\ Re(w) \;&=\; \Psi \quad on \quad \partial K_R = \{z \big| |z| = R\}, \end{aligned}$$

with Ψ in the form (2.9) has the solution

$$w \;=\; \sum_{j=0}^{m} mB_j^m \left(\frac{\bar{z}}{1 - z\bar{z}}\right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1}(m-j)B_j^m \frac{z^{m-j-1}}{(1 - z\bar{z})^{m-j}}\overline{g^{(j)}(z)},$$

where $B^m_j = \dfrac{(2m-j-1)!}{j!(m-j)!}$ *and the generating function g has the following form*

$$g(z) \;=\; \sum_{k=m}^{\infty} (\alpha_k + i\beta_k) z^k.$$

Here $\beta_m \in \mathbb{R}$ *can be chosen arbitrarily and the coefficients* $\alpha_k,\ k \geq m$, *and* $\beta_k,\ k \geq m+1$, *are given recursively in (2.10).*

2.2 A class of the generalized Riemann-Hilbert boundary value problems

Using the representation of solutions of the Bers-Vekua equation (M) we can solve explicitly a class of the generalized Riemann-Hilbert boundary value problem given as follows

$$w_{\bar{z}} \;=\; C\overline{w} \quad \text{in} \quad K_R, \tag{2.11}$$

$$\mathrm{Re}(\overline{\lambda(z)}w) \;=\; \Phi \quad \text{on} \quad \partial K_R, \tag{2.12}$$

with $C = \dfrac{m}{1-z\bar{z}},\ \lambda(z) = z^p,\ m \in \mathbb{N},\ p \in \mathbb{N}^*$.
After I.N. Vekua [43] this problem is called Problem **A**. If $\Phi \equiv 0$ we have the homogeneous Problem **Å**. In order to solve this problem we need the introduction of the so-called *index* of the problem which we shall define now.
Let $\Delta_\Gamma f(t)$ denote the increment of the function $f(t)$ as the point t describes once the curve Γ in the direction leaving the domain G on the left, where Γ denotes the boundary of the simply connected domain G.

Definition 2.1.
The number n defined by

$$n := \frac{1}{2\pi}\Delta_{\partial K_R} \arg\lambda(t)$$

is called the index of the function $\lambda(t)$ *with respect to the boundary* ∂K_R *of the domain* K_R *or the index of the boundary value Problem* ***A***.

The existence of the solutions of the Problem **A** is proved by I.N. Vekua in [43] and is quoted in the following.

Theorem 2.2 (I.N. Vekua)**.**
In the case of a simply-connected domain if the index $n \geq 0$ *then the inhomogeneous Problem* ***A*** *is always soluble and its general solution is given by the formula*

$$w(z) \;=\; w_0(z) + \sum_{j=1}^{2n+1} \mu_j w_j(z), \tag{2.13}$$

where $\mu_j,\ j=1,\dots,2n+1,$ *are constants and* $\{w_1,\dots,w_{2n+1}\}$ *is the complete system of solutions of the homogeneous Problem* $\mathring{\mathbf{A}}$ *and* w_0 *is a particular solution of the non-homogeneous Problem* $\mathbf{A}$.

Here *the complete system of solutions* of the homogeneous Problem $\mathring{\mathbf{A}}$ is a basis of the space of its solutions.
Now we consider the boundary condition (2.12) in connection with the representation of the solutions in the form (1.60)

$$\mathrm{Re}\left\{\overline{\xi}^p\left[\sum_{j=0}^{m} a_j(\xi)g^{(j)}(\xi)+\sum_{j=0}^{m-1} b_j(\xi)\overline{g^{(j)}(\xi)}\right]\right\} \;=\; \Phi(\xi), \tag{2.14}$$

where $a_j(\xi), b_j(\xi)$ are defined as in (2.3).
Since the generating function $g(z)$ belongs to the space $H_{K_R}(m,0)$, $g(z)$ can be expanded into the power series

$$g(z) \;=\; \sum_{k=m}^{\infty}\widetilde{\gamma}_k z^k, \quad \widetilde{\gamma}_k\in\mathbb{C}.$$

We shall find the generating function $g\in H_{K_R}(m,0)$ provided the functions g on the boundary has the form

$$g(\xi) \;=\; \sum_{k=m}^{\infty}\widetilde{\gamma}_k\xi^k, \quad \text{on} \quad \partial K_R, \tag{2.15}$$

with $\widetilde{\gamma}_k=\widetilde{\alpha}_k+i\widetilde{\beta}_k,\ \widetilde{\alpha}_k,\widetilde{\beta}_k\in\mathbb{R}$.
Next we are going to calculate the coefficients $\widetilde{\gamma}_k,$ for $k\geq m$. For the function g in the form (2.15) and the coefficients a_j and b_j given in (2.3), the equation (2.14) becomes

$$\mathrm{Re}\left\{\sum_{j=0}^{m}\frac{B_j^m}{(1-\xi\overline{\xi})^{m-j}}\times\right.$$
$$\left.\left[m\overline{\xi}^{p+m-j}\sum_{k=m}^{\infty}\frac{k!\widetilde{\gamma}_k\xi^{k-j}}{(k-j)!}+(m-j)\xi^{m-j-1}\sum_{k=m}^{\infty}\frac{k!\overline{\widetilde{\gamma}}_k\overline{\xi}^{p+k-j}}{(k-j)!}\right]\right\} \;=\; \Phi(\xi).$$

We introduce the real parameter $t\in[0,2\pi]$ by $\xi=Re^{it}\in\partial K_R$ and assume that the function Φ on the boundary has the following form

$$\Phi(t) \;=\; \widetilde{\varphi}_0+\sum_{k=1}^{\infty}(\widetilde{\varphi}_k\cos(kt)+\widetilde{\psi}_k\sin(kt)) \quad \text{on } \partial K_R. \tag{2.16}$$

Using the notations c_k, d_k as in (2.7) we obtain

$$\mathrm{Re}\left\{\sum_{k=m}^{\infty}c_k\widetilde{\gamma}_kR^{k-m+p}e^{i(k-m-p)t}+\sum_{k=m}^{\infty}d_k\overline{\widetilde{\gamma}}_kR^{k-m+p+1}e^{-i(k-m+p+1)t}\right\} \;=\; \Phi(\xi).$$

Since $e^{it} = \cos t + i\sin t,\ t \in \mathbb{R}$, we get

$$\begin{aligned}\operatorname{Re}(\bar{z}^p w)|_{\partial K_R} = &\sum_{k=m}^{\infty} c_k R^{k-m+p}\big[\widetilde{\alpha}_k \cos(k-m-p)t - \widetilde{\beta}_k \sin(k-m-p)t\big] \\ &+ \sum_{k=m}^{\infty} d_k R^{k-m+p+1}\big[\widetilde{\alpha}_k \cos(k-m+p+1)t - \widetilde{\beta}_k \sin(k-m+p+1)t\big].\end{aligned}$$

For convenience, we split the above sums as follows

$$\begin{aligned}\operatorname{Re}(\bar{z}^p w)|_{\partial K_R} &= c_{m+p}\widetilde{\alpha}_{m+p}R^{2p} \\ &+ \sum_{k=1}^{p}\big[c_{m+p-k}R^{2p-k}\widetilde{\alpha}_{m+p-k} + c_{m+p+k}R^{2p+k}\widetilde{\alpha}_{m+p+k}\big]\cos(kt) \\ &+ \sum_{k=1}^{p}\big[c_{m+p-k}R^{2p-k}\widetilde{\beta}_{m+p-k} - c_{m+p+k}R^{2p+k}\widetilde{\beta}_{m+p+k}\big]\sin(kt) \\ &+ \sum_{k=p+1}^{\infty}\big[c_{m+p+k}R^{2p}\widetilde{\alpha}_{m+p+k} + d_{m-p-1+k}\widetilde{\alpha}_{m-p-1+k}\big]R^k\cos(kt) \\ &- \sum_{k=p+1}^{\infty}\big[c_{m+p+k}R^{2p}\widetilde{\beta}_{m+p+k} + d_{m-p-1+k}\widetilde{\beta}_{m-p-1+k}\big]R^k\sin(kt).\end{aligned} \tag{2.17}$$

Substituting the two expressions (2.16) and (2.17) into (2.14), we obtain the following linear system for the coefficients $\widetilde{\alpha}_k$ and $\widetilde{\beta}_k$

$$\begin{cases} c_{m+p}\widetilde{\alpha}_{m+p}R^{2p} = \widetilde{\varphi}_0, \\ c_{m+p-k}R^{2p-k}\widetilde{\alpha}_{m+p-k} + c_{m+p+k}R^{2p+k}\widetilde{\alpha}_{m+p+k} = \widetilde{\varphi}_k,\ k=1,2,\dots,p \\ c_{m+p-k}R^{2p-k}\widetilde{\beta}_{m+p-k} - c_{m+p+k}R^{2p+k}\widetilde{\beta}_{m+p+k} = \widetilde{\psi}_k,\ k=1,2,\dots,p \\ \big[c_{m+p+k}R^{2p}\widetilde{\alpha}_{m+p+k} + d_{m-p-1+k}\widetilde{\alpha}_{m-p-1+k}\big]R^k = \widetilde{\varphi}_k,\ k=p+1,p+2,\dots \\ \big[c_{m+p+k}R^{2p}\widetilde{\beta}_{m+p+k} + d_{m-p-1+k}\widetilde{\beta}_{m-p-1+k}\big]R^k = \widetilde{\psi}_k,\ k=p+1,p+2,\dots \end{cases}$$

Here $\widetilde{\alpha}_k \in \mathbb{R}\ (m \le k \le m+p-1)$ and $\widetilde{\beta}_k \in \mathbb{R}\ (m \le k \le m+p)$ can be chosen arbitrarily and then the remaining coefficients can be calculated recursively in a unique way as follows

$$\begin{aligned}
\widetilde{\alpha}_{m+p} &= \frac{\widetilde{\varphi}_0}{c_{m+p}R^{2p}}, \\
\widetilde{\alpha}_{m+p+k} &= \begin{cases} \dfrac{\widetilde{\varphi}_k - c_{m+p-k}R^{2p-k}\widetilde{\alpha}_{m+p-k}}{c_{m+p+k}R^{2p+k}} & \text{for} \quad 1 \le k \le p, \\[2ex] \dfrac{\widetilde{\varphi}_k - d_{m-p-1+k}R^{k}\widetilde{\alpha}_{m-p-1+k}}{c_{m+p+k}R^{2p+k}} & \text{for} \quad k \ge p+1, \end{cases} \\
\widetilde{\beta}_{m+p+k} &= \begin{cases} -\dfrac{\widetilde{\psi}_k - c_{m+p-k}R^{2p-k}\widetilde{\beta}_{m+p-k}}{c_{m+p+k}R^{2p+k}} & \text{for} \quad 1 \le k \le p, \\[2ex] -\dfrac{\widetilde{\psi}_k + d_{m-p-1+k}R^{k}\widetilde{\beta}_{m-p-1+k}}{c_{m+p+k}R^{2p+k}} & \text{for} \quad k \ge p+1. \end{cases}
\end{aligned} \tag{2.18}$$

Therefore the boundary value problem (2.11)-(2.12) can be solved explicitly.

Theorem 2.3.
The boundary value problem

$$\begin{aligned}
w_{\bar{z}} &= \frac{m}{1-z\bar{z}}\overline{w} \quad \textit{in} \quad K_R = \{z \,\big|\, |z| < R, \ 0 < R < 1\}, \\
Re(\bar{z}^p w) &= \Phi \quad \textit{on} \quad \partial K_R = \{z \,\big|\, |z| = R\},
\end{aligned}$$

with Φ *in the form (2.16), has the solution*

$$w = \sum_{j=0}^{m} mB_j^m \left(\frac{\bar{z}}{1-z\bar{z}}\right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1}(m-j)B_j^m \frac{z^{m-j-1}}{(1-z\bar{z})^{m-j}}\overline{g^{(j)}(z)}$$

where $B_j^m = \dfrac{(2m-j-1)!}{j!(m-j)!}$ *and the generating function g has the following form*

$$g(z) = \sum_{k=m}^{\infty}(\widetilde{\alpha}_k + i\widetilde{\beta}_k)z^k.$$

Here $\widetilde{\alpha}_k \in \mathbb{R}$ $(m \le k \le m+p-1)$ *and* $\widetilde{\beta}_k \in \mathbb{R}$ $(m \le k \le m+p)$ *can be chosen arbitrarily and the coefficients* $\widetilde{\alpha}_k$ $(k \ge m+p)$ *and* $\widetilde{\beta}_k$ $(k \ge m+p+1)$ *are given by (2.18).*

We have used the representations of the pseudo-analytic functions to solve the Dirichlet boundary value problem and a class of Riemann-Hilbert boundary value problems. Thought only some special classes of the boundary value problems are applied, the solutions of these problems have been solved in an explicite forms. From Theorem 2.3

we can see that the number of the arbitrary coefficients in the formula of the solution of the Riemann-Hilbert boundary value problem is equal to the dimension of the space of solutions of the corresponding homogeneous problem in the Theorem 2.2 of Vekua. The Dirichlet boundary value problem considered in Section 2.1 is a special case of the Riemann-Hilbert boundary value problem and its index is $n = 0$. According to Theorem 2.2, this problem always has a solution. This agrees with the fact that the number of the arbitrary coefficients in the formula of the solution of the Dirichlet bounday value problem is 1.

3 CONSEQUENCES AND APPLICATIONS OF THE REPRESENTATION OF SOLUTIONS BY DIFFERENTIAL OPERATORS OF BAUER-TYPE

In Chapter 3 we study some problems related to the Bers-Vekua equation (M). First we construct a connection between the generating functions and a given solution of the equation (M). Using this connection we can derive a representation theorem for solutions of the equation (M) in the neighbourhood of an isolated singularity. The representation formulae for the solutions of other partial differential equations in the neighbourhood of isolated singularities can be found in, e.g., [3], [4], [7], [8], [12]. Using the representation of the solutions of the equation (M) we can find a generating pair of the equation (M) in the sense of L. Bers and a special class of the chiral components in the Ising field theory.
Then we consider further differential equations connected with the equation (M) such as the Bers-Vekua equation of type (M) with different parameters and an inhomogeneous equation corresponding to the equation (M). We shall construct connections between the solutions of the Bers-Vekua equation (M) with different parameters. This problem for other Bers-Vekua equations can be found in [9], [14].
For the inhomogeneous equation corresponding to the equation (M) of type

$$w_{\overline{z}} - \frac{m}{1 - z\overline{z}}\overline{w} = \Phi(z, \overline{z})$$

the question arises that for which functions $\Phi(z,\overline{z})$ there exists a representation of all solutions by differential operators. We shall give some classes of functions $\Phi(z,\overline{z})$ for which the above inhomogeneous equation can be solved explicitly.

3.1 Connection between the generating functions and the solutions

Theorem 3.1 (Connection between the generating functions and a given solution)**.**
For every given solution w of the equation (M) in the form (1.21), the derivative $g^{(2m)}(z)$ of the generating function g is uniquely determined by

$$g^{(2m)}(z) \;=\; \frac{1}{(1 - z\overline{z})^m} \frac{\partial^m}{\partial z^m} \left[(1 - z\overline{z})^m w \right]. \tag{3.1}$$

Proof.
Multiplying the two sides of the equality (1.21) by $(1-z\bar{z})^m$, we have

$$(1-z\bar{z})^m w = \sum_{j=0}^{m} mB_j^m\, \bar{z}^{m-j}(1-z\bar{z})^j g^{(j)}(z) + \sum_{j=0}^{m-1}(m-j)B_j^m z^{m-j-1}(1-z\bar{z})^j \overline{g^{(j)}(z)} \quad (3.2)$$

where $B_j^m = \dfrac{(2m-j-1)!}{j!(m-j)!}$.
We denote the first term and the second term on the right hand side of (3.2) by

$$A := \sum_{j=0}^{m} mB_j^m\, \bar{z}^{m-j}(1-z\bar{z})^j g^{(j)}(z),$$
$$B := \sum_{j=0}^{m-1}(m-j)B_j^m z^{m-j-1}(1-z\bar{z})^j \overline{g^{(j)}(z)}.$$

Then taking the derivative of the equality (3.2) of order m with respect to z we have

$$\frac{\partial^m}{\partial z^m}\left[(1-z\bar{z})^m w\right] = \frac{\partial^m}{\partial z^m}A + \frac{\partial^m}{\partial z^m}B. \quad (3.3)$$

First we consider the derivative of A of order m

$$\begin{aligned}
\frac{\partial^m}{\partial z^m}A &= \sum_{j=0}^{m} mB_j^m\, \bar{z}^{m-j}\frac{\partial^m}{\partial z^m}\left[(1-z\bar{z})^j g^{(j)}(z)\right] \\
&= \sum_{j=0}^{m} mB_j^m\, \bar{z}^{m-j}\left[\sum_{i=0}^{m} C_i^m[(1-z\bar{z})^j]^{(i)}[g^{(j)}(z)]^{(m-i)}\right] \\
&= \sum_{j=0}^{m} \frac{(2m-j-1)!m}{j!(m-j)!}\,\bar{z}^{m-j}\sum_{i=0}^{j} C_i^m \frac{j!}{(j-i)!}(1-z\bar{z})^{j-i}(-\bar{z})^i g^{(m+j-i)}(z) \\
&= \sum_{j=0}^{m}\sum_{i=0}^{j}(-1)^i\frac{(2m-j-1)!m}{(j-i)!(m-j)!}C_i^m\,\bar{z}^{m-(j-i)}(1-z\bar{z})^{j-i}g^{(m+j-i)}(z).
\end{aligned}$$

Let $j-i=q$ then $j=i+q\geq q$.
Hence the above equality reads

$$\frac{\partial^m}{\partial z^m}A = \sum_{q=0}^{m}\sum_{j=q}^{m}(-1)^{j-q}\frac{(2m-j-1)!m}{q!(m-j)!}C_{j-q}^m\,\bar{z}^{m-q}(1-z\bar{z})^q g^{(m+q)}(z). \quad (3.4)$$

Now we consider the derivative of B of order m

$$\begin{aligned}
\frac{\partial^m}{\partial z^m}B &= \sum_{j=0}^{m-1}(m-j)B_j^m\frac{\partial^m}{\partial z^m}\left[z^{m-j-1}(1-z\bar{z})^j\right]\overline{g^{(j)}(z)} \\
&= \sum_{j=0}^{m-1}(m-j)B_j^m\left[\sum_{i=0}^{m} C_i^m[(1-z\bar{z})^j]^{(i)}[z^{m-j-1}]^{(m-i)}\right]\overline{g^{(j)}(z)}.
\end{aligned}$$

Denote

$$T_i :=\ [(1-z\overline{z})^j]^{(i)}[z^{m-j-1}]^{(m-i)},$$

then we see that $T_i \neq 0,\ 0 \leq i \leq m$, if and only if

$$\begin{cases}[(1-z\overline{z})^j]^{(i)} \neq 0 \\ (z^{m-j-1})^{(m-i)} \neq 0\end{cases} \Leftrightarrow \begin{cases} i \leq j \\ m-i \leq m-j-1\end{cases} \Leftrightarrow \begin{cases} i \leq j \\ i \geq j+1\end{cases}$$

This is impossible!
Therefore $T_i \equiv 0$ for all $0 \leq i \leq m$.
This implies

$$\frac{\partial^m}{\partial z^m} B \equiv 0 \quad \text{for all} \quad m \in \mathbb{N}^*. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we have

$$\boxed{\frac{\partial^m}{\partial z^m}\left[(1-z\overline{z})^m w\right] \ = \ \sum_{q=0}^{m}\left[\sum_{j=q}^{m}(-1)^{j-q}\frac{(2m-j-1)!m}{q!(m-j)!}C^m_{j-q}\overline{z}^{m-q}(1-z\overline{z})^q\right]g^{(m+q)}(z).} \tag{3.6}$$

It is easy to see that the coefficient of $g^{(2m)}(z)$ in (3.6) is equal to $(1-z\overline{z})^m$. In order to prove the formula (3.1), that is,

$$g^{(2m)}(z) \ = \ \frac{1}{(1-z\overline{z})^m}\frac{\partial^m}{\partial z^m}\left[(1-z\overline{z})^m w\right],$$

we have to point out that all the coefficients of $g^{(m+q)}(z)$ for $q = 0, 1, \ldots, m-1$ (except the coefficient of $g^{(2m)}(z)$) are equal to zero.
That means we have to show

$$\sum_{j=q}^{m}(-1)^{j-q}\frac{(2m-j-1)!m}{q!(m-j)!}C^m_{j-q} \ = \ 0 \quad \text{for} \quad 0 \leq q \leq m-1$$

$$\Leftrightarrow \qquad \sum_{j=q}^{m}(-1)^j\frac{(2m-j-1)!}{(j-q)!(m-j)!(m-j+q)!} \ = \ 0 \quad \text{for} \quad 0 \leq q \leq m-1.$$

Set $q := m-(s+1)$, $0 \leq s \leq m-1$, then we need to prove

$$\sum_{j=m-(s+1)}^{m}(-1)^j\frac{(2m-j-1)!}{(j-m+s+1)!(m-j)!(2m-j-(s+1))!} \ = \ 0 \ \text{ for } \ 0 \leq s \leq m-1,$$

$$\Leftrightarrow \sum_{j=m-(s+1)}^{m}(-1)^j\frac{(2m-j-1)(2m-j-2)\ldots(2m-j-s)}{(j-m+s+1)!(m-j)!} \ = \ 0 \ \text{ for } \ 0 \leq s \leq m-1,$$

$$\Leftrightarrow \frac{(m+s)(m+s-1)\dots(m+1)}{0!(s+1)!} - \frac{(m+s-1)(m+s-2)\dots m}{1!s!} + \cdots$$
$$\cdots + (-1)^{s+1}\frac{(m-1)(m-2)\dots(m-s)}{(s+1)!0!} \;=\; 0. \tag{3.7}$$

Indeed, we consider the expansion for $x \in \mathbb{R}$

$$x^{m-1}(x-1)^{s+1} \;=\; x^{m-1}\left[C_0^{s+1}x^{s+1} - C_1^{s+1}x^s + \cdots + (-1)^{s+1}C_{s+1}^{s+1}\right], \quad m,s \in \mathbb{N}^*,$$

$$x^{m-1}(x-1)^{s+1} \;=\; \frac{(s+1)!}{0!(s+1)!}x^{m+s} - \frac{(s+1)!}{1!s!}x^{m+s-1} + \cdots + (-1)^{s+1}\frac{(s+1)!}{(s+1)!0!}x^{m-1}. \tag{3.8}$$

Taking the derivative of the two sides of the equality (3.8) of order s with respect to x and then substituting $x = 1$, we obtain the equality (3.7) immediately.
Hence the coefficients of $g^{(m+q)}(z)$, with $0 \le q \le m-1$, are equal to zero.
To sum up we have

$$g^{(2m)}(z) \;=\; \frac{1}{(1-z\overline{z})^m}\frac{\partial^m}{\partial z^m}\left[(1-z\overline{z})^m w\right].$$

Therefore Theorem 3.1 is proved. □

If we consider the zero-solution $w = 0$ of the equation (M) then from Theorem 3.1 we have $g^{(2m)}(z) = 0$. Therefore g is a polynomial of degree $2m-1$

$$g(z) \;=\; a_0 + a_1 z + a_2 z^2 + \cdots + a_{2m-1}z^{2m-1},\; a_j \in \mathbb{C}, j = 0,1,\dots,2m-1.$$

In the following theorem we describe exactly the generating function of the zero-solution of the equation (M).

Theorem 3.2 (The generating function of the zero-solution)**.**
A function $g \in H(K_R)$ is the generating function of the zero-solution of the equation (M) if and only if g has the form

$$g(z) \;=\; \sum_{j=0}^{2m-1} a_j z^j,\; a_j \in \mathbb{C}, \tag{3.9}$$

with $a_j = -\overline{a}_{2m-1-j}$ for $j = 0,1,\dots,m-1$.

Proof.
• *Necessary condition.* We show that if the solution w is identically equal to zero then g has the form (3.9).
By hypothesis, the solution and its derivatives of any order are equal to zero at $z = 0$. This implies

$$\frac{\partial^q w}{\partial z^q}(0) \;=\; 0,\; 0 \le q \le m-1.$$

From the representation formula (1.60), we have

$$
\begin{aligned}
w &= \sum_{j=0}^{m} mB_j^m\, \overline{z}^{m-j}(1-z\overline{z})^{j-m} g^{(j)}(z) + \sum_{j=0}^{m-1}(m-j)B_j^m z^{m-j-1}(1-z\overline{z})^{j-m}\overline{g^{(j)}(z)}
\end{aligned}
$$

where $\quad B_j^m = \dfrac{(2m-j-1)!}{j!(m-j)!}$,

$$
\begin{aligned}
\Rightarrow \frac{\partial^q w}{\partial z^q} &= \sum_{j=0}^{m} mB_j^m\, \overline{z}^{m-j}\frac{\partial^q}{\partial z^q}\left[(1-z\overline{z})^{j-m} g^{(j)}(z)\right] \\
&\quad + \sum_{j=0}^{m-1}(m-j)B_j^m \frac{\partial^q}{\partial z^q}\left[z^{m-j-1}(1-z\overline{z})^{j-m}\right]\overline{g^{(j)}(z)}.
\end{aligned} \tag{3.10}
$$

When $z=0$ the first sum on the right-hand side of (3.10) has only one non-zero term which corresponds to the case $j=m$

$$
\frac{\partial^q}{\partial z^q}\left[g^{(m)}(z)\right]\bigg|_{z=0} = g^{(m+q)}(0). \tag{3.11}
$$

Next we consider the derivatives in the second sum in the right-hand side of (3.10)

$$
\frac{\partial^q}{\partial z^q}\left[z^{m-j-1}(1-z\overline{z})^{j-m}\right] = \sum_{i=0}^{q}\frac{C_i^q(m-j-1)!(j-m)!z^{m-j-q+i-1}}{(m-j-q+i-1)!(j-m-i)!}(1-z\overline{z})^{j-m-i}(-\overline{z})^i \tag{3.12}
$$

for each $j=0,1,\ldots,m-1$.

When $z=0$, there is only one term different from zero on the right-hand side of (3.12), which corresponds to the case $i=0$ and $j=m-q-1$. Hence the second sum on the right-hand side of (3.10) has only one non-zero term

$$
\frac{(m+q)!}{(m-q-1)!(1-z\overline{z})^{q+1}}\overline{g^{(m-q-1)}(z)}\bigg|_{z=0} = \frac{(m+q)!}{(m-q-1)!}\overline{g^{m-q-1}(0)}. \tag{3.13}
$$

From (3.11) and (3.13) we have

$$
\begin{aligned}
\frac{\partial^q w}{\partial z^q}(0) &= g^{(m+q)}(0) + \frac{(m+q)!}{(m-q-1)!}\overline{g^{m-q-1}(0)} \\
0 &= (m+q)!\, a_{m+q} + \frac{(m+q)!}{(m-q-1)!}(m-q-1)!\, \overline{a}_{m-q-1} \\
0 &= (m+q)![a_{m+q} + \overline{a}_{m-q-1}].
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_{m+q} &= -\overline{a}_{m-q-1} \qquad \text{for } q=0,1,\ldots,m-1, \\
\Leftrightarrow a_j &= -\overline{a}_{2m-1-j} \qquad \text{for } j=0,1,\ldots,m-1.
\end{aligned}
$$

Thus the necessary condition follows.
• *Sufficient condition.* If g has the form (3.9) then g is a generating function of the zero-solution.
Since the expression in (1.60) is linear with respect to g and g has the form (3.9) (by the hypothesis), it is enough to prove the sufficient condition with the following form of g

$$g = a_q z^q - \overline{a}_q z^{2m-1-q}, \qquad 0 \le q \le m-1.$$

This means we have to prove the following equality

$$\sum_{j=0}^{m} mB_j^m \,\overline{z}^{m-j}(1-z\overline{z})^{j-m} g^{(j)} + \sum_{j=0}^{m-1}(m-j)B_j^m z^{m-j-1}(1-z\overline{z})^{j-m}\overline{g^{(j)}} \equiv 0 \tag{3.14}$$

with $g = a_q z^q - \overline{a}_q z^{2m-1-q}$, $0 \le q \le m-1$.
Substituting g into the equation (3.14), we have

$$\begin{aligned}\sum_{j=0}^{m} mB_j^m \,\overline{z}^{m-j}(1-z\overline{z})^{j-m}[a_q z^q - \overline{a}_q z^{2m-1-q}]^{(j)} + \\ + \sum_{j=0}^{m-1}(m-j)B_j^m z^{m-j-1}(1-z\overline{z})^{j-m}\overline{[a_q z^q - \overline{a}_q z^{2m-1-q}]^{(j)}} = 0.\end{aligned}$$

The above equation can be rewritten as

$$T_1 a_q + T_2\, \overline{a}_q = 0,$$

where T_1 and T_2 read as follows

$$T_1 = \sum_{j=0}^{m} mB_j^m \,\overline{z}^{m-j}(1-z\overline{z})^{j}(z^q)^{(j)} - \sum_{j=0}^{m-1}(m-j)B_j^m z^{m-j-1}(1-z\overline{z})^{j}\overline{(z^{2m-q-1})^{(j)}},$$

$$T_2 = \sum_{j=0}^{m-1}(m-j)B_j^m z^{m-j-1}(1-z\overline{z})^{j}\overline{(z^q)^{(j)}} - \sum_{j=0}^{m} mB_j^m \,\overline{z}^{m-j}(1-z\overline{z})^{j}(z^{2m-q-1})^{(j)}.$$

We have to prove $T_1 = 0$ and $T_2 = 0$.
First, we consider the equation

$$T_2 = 0.$$

This equation can be rewritten as follows

$$\begin{aligned}\sum_{j=0}^{q}(m-j)B_j^m z^{m-j-1}(1-z\overline{z})^{j}\frac{q!}{(q-j)!}\overline{z}^{q-j} \\ = \sum_{j=0}^{m} mB_j^m \,\overline{z}^{m-j}(1-z\overline{z})^{j}\frac{(2m-q-1)!}{(2m-q-1-j)!}z^{2m-q-1-j}\end{aligned} \tag{3.15}$$

which we have to prove. On the left-hand side of (3.15) j only runs from 0 to q because

$$(z^q)^{(j)} = 0 \quad \text{for} \quad j > q.$$

Dividing the two sides of the equation (3.15) by z^{m-q-1} we obtain

$$\sum_{j=0}^{q}(m-j)B_j^m \frac{q!}{(q-j)!} z^{q-j}\,\bar{z}^{q-j}(1-z\bar{z})^j = \sum_{j=0}^{m} mB_j^m \frac{(2m-q-1)!}{(2m-q-1-j)!}\bar{z}^{m-j}z^{m-j}(1-z\bar{z})^j.$$

With $\lambda = z\bar{z},\ \lambda \in \mathbb{R}_+$, this equality can be written in the form

$$\sum_{j=0}^{q}\frac{(2m-j-1)!}{(m-j-1)!}C_j^q\lambda^{q-j}(1-\lambda)^j = \frac{(2m-q-1)!}{(m-1)!}\sum_{j=0}^{m}\frac{(2m-j-1)!}{(2m-q-j-1)!}C_j^m\lambda^{m-j}(1-\lambda)^j. \tag{3.16}$$

Denote the left-hand side and the right-hand side of (3.16) by $\mathcal{L}$ and $\mathcal{R}$, respectively. Now we are going to prove the equality (3.16).
In order to do that, we first consider the expansion

$$a^{m-1}[a+(1-b)]^m = \sum_{j=0}^{m} C_j^m a^{2m-j-1}(1-b)^j, \quad a,b \in \mathbb{R}. \tag{3.17}$$

Taking the derivative of order q with respect to a of the two sides of the expansion (3.17) we get

$$\sum_{j=0}^{q} C_j^q (a^{m-1})^{(q-j)}([a+(1-b)]^m)^{(j)} = \sum_{j=0}^{m} C_j^m (a^{2m-j-1})^q (1-b)^j$$

$$\Leftrightarrow \sum_{j=0}^{q} C_j^q \frac{(m-1)!}{(m-1-q+j)!}a^{m-1-q+j}\frac{m!}{(m-j)!}[a+(1-b)]^{m-j} = \sum_{j=0}^{m} C_j^m \frac{(2m-j-1)!}{(2m-j-1-q)!}a^{2m-j-1-q}(1-b)^j.$$

In the case $a = b = \lambda$, we obtain

$$\sum_{j=0}^{q} C_j^q \frac{(m-1)!m!}{(m-j)!(m-q-1+j)!}\lambda^{m-q-1+j} = \sum_{j=0}^{m} C_j^m \frac{(2m-j-1)!}{(2m-j-1-q)!}\lambda^{2m-q-1-j}(1-\lambda)^j. \tag{3.18}$$

Dividing the two sides of the equation (3.18) by λ^{m-q-1} we have

$$\sum_{j=0}^{m}\frac{(2m-j-1)!}{(2m-q-1-j)!}C_j^m\lambda^{m-j}(1-\lambda)^j = \sum_{j=0}^{q}\frac{(m-1)!m!}{(m-j)!(m-q-1+j)!}C_j^q\lambda^j,$$

and then multiplying by $\dfrac{(2m-q-1)!}{(m-1)!}$ we have

$$\begin{aligned}\mathcal{R} &= \frac{(2m-q-1)!}{(m-1)!}\sum_{j=0}^{q}\frac{(m-1)!m!}{(m-j)!(m-q-1+j)!}C_j^q\lambda^j \\ &= \sum_{j=0}^{q}\frac{(2m-q-1)!}{(m-q-1+j)!}\frac{q!}{(q-j)!}C_j^m\lambda^j. \end{aligned} \tag{3.19}$$

Next we consider the following expansion

$$a^{2m-q-1}[a+(1-b)]^q = \sum_{j=0}^{q}C_j^q a^{2m-j-1}(1-b)^j, \quad a,b\in\mathbb{R}. \tag{3.20}$$

Taking the derivative of order m with respect to a of the two sides of the expansion (3.20) we obtain

$$\sum_{j=0}^{m}C_j^m(a^{2m-q-1})^{(m-j)}([a+(1-b)]^q)^{(j)} = \sum_{j=0}^{q}C_j^q(a^{2m-j-1})^{(m)}(1-b)^j,$$

$$\begin{aligned}\Leftrightarrow \sum_{j=0}^{q}C_j^m\frac{(2m-q-1)!}{(m-q-1+j)!}a^{m-q-1+j}\frac{q!}{(q-j)!}[a+(1-b)]^{q-j} \\ = \sum_{j=0}^{q}C_j^q\frac{(2m-j-1)!}{(m-j-1)!}a^{m-j-1}(1-b)^j. \end{aligned} \tag{3.21}$$

In the sum on the left-hand side of the equation (3.21), j runs from 0 to q only because

$$([a+(1-b)]^q)^{(j)} = 0 \quad \text{if} \quad j>q.$$

By choosing $a=b=\lambda$ the equation (3.21) becomes

$$\sum_{j=0}^{q}C_j^q\frac{(2m-j-1)!}{(m-j-1)!}\lambda^{m-j-1}(1-\lambda)^j = \sum_{j=0}^{q}C_j^m\frac{(2m-q-1)!}{(m-q-1+j)!}\frac{q!}{(q-j)!}\lambda^{m-q-1+j}$$

and then dividing the two sides by λ^{m-q-1}, we get

$$\begin{aligned}\mathcal{L} &= \sum_{j=0}^{q}\frac{(2m-j-1)!}{(m-j-1)!}C_j^q\lambda^{q-j}(1-\lambda)^j \\ &= \sum_{j=0}^{q}\frac{(2m-q-1)!}{(m-q-1+j)!}\frac{q!}{(q-j)!}C_j^m\lambda^j. \end{aligned} \tag{3.22}$$

From the two formulae (3.19) and (3.22), the equality (3.16) is proved. Therefore

$$T_2 \; = \; 0.$$

To prove the statement $T_1 = 0$ we have to show that

$$m\sum_{j=0}^{q}\frac{(2m-j-1)!}{(m-j)!}C_j^q\lambda^{q-j}(1-\lambda)^j$$
$$= \frac{(2m-q-1)!}{(m-1)!}\sum_{j=0}^{m-1}\frac{(2m-j-1)!}{(2m-q-j-1)!}C_j^{m-1}\lambda^{m-1-j}(1-\lambda)^j,$$

where $\lambda = z\bar{z}$.
We use the same method which we has been used in order to prove $T_2 = 0$. Instead of using the expansions (3.17) and (3.20) we use the suitable expansions as follows.
We first consider the expansion

$$a^m[a+(1-b)]^{m-1} \; = \; \sum_{j=0}^{m-1} C_j^{m-1}a^{2m-j-1}(1-b)^j, \quad a,b\in\mathbb{R},$$

and then take the derivative of the two sides of this expansion of order q with respect to a.
The second expansion is

$$a^{2m-q-1}[a+(1-b)]^q \; = \; \sum_{j=0}^{q} C_j^q a^{2m-j-1}(1-b)^j, \quad a,b\in\mathbb{R},$$

and then we take the derivative of the two sides of order $m-1$ with respect to a. Therefore we can prove that

$$T_1 = 0.$$

That means the equality (3.14) is proved and thus the sufficient condition follows. □

Corollary 3.1.
Suppose that $\hat{g}$ is a generating function of a given solution w of the equation (M). Then every generating function g of the solution w is given by

$$g(z) \; = \; \hat{g}(z) + \sum_{j=0}^{2m-1} a_j z^j, a_j \in \mathbb{C}, \tag{3.23}$$

with $a_j = -\bar{a}_{2m-1-j}, \quad$ *for* $\quad j = 0,1,\ldots,m-1.$

3.2 Representation of the solutions in the neighbourhood of an isolated singularity

In Chapter 1 we have proved that all solutions of the equation (M)

$$w_{\overline{z}} = \frac{m}{1-z\overline{z}}\overline{w}, \quad m \in \mathbb{N},$$

in K_R can be represented by differential operators of Bauer-type

$$\begin{aligned} w &=: H_m g + H_{m-1}^{*}\overline{g} \\ &= \sum_{j=0}^{m} mB_j^m \left(\frac{\overline{z}}{1-z\overline{z}}\right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1}(m-j)B_j^m \frac{z^{m-j-1}}{(1-z\overline{z})^{m-j}}\overline{g^{(j)}(z)}, \end{aligned} \tag{3.24}$$

where $B_j^m = \dfrac{(2m-j-1)!}{j!(m-j)!}$, and $g \in H(K_R)$.
We have also found the connection between the generating functions and the given solution and the form of the generating functions of the zero-solution.
Using the Theorems 3.1 and 3.2 we can get a general representation theorem for the solutions of the equation (M) in the neighbourhood of an isolated singularity $z_0 \in K_R$.
Let

$$\widetilde{U}(z_0) = \{z \in \mathbb{C} | 0 < |z-z_0| < \rho\} \subset K_R,$$

be a punctured neighbourhood of the point z_0 and let w be a solution of the equation (M) in $\widetilde{U}(z_0)$.
Then for the given solution w, a derivative $g^{(2m)}(z)$ of a generating function g of w can be expanded into Laurent series in $\widetilde{U}(z_0)$

$$g^{(2m)}(z) = \sum_{-\infty}^{+\infty} \tilde{a}_j (z-z_0)^j. \tag{3.25}$$

After integrating $2m$ times the equality (3.25) we obtain

$$g(z) = g_1(z) + p(z)\log(z-z_0), \tag{3.26}$$

where $p(z)$ is a polynomial in z of degree $2m-1$,

$$p(z) = \sum_{j=0}^{2m-1} b_j z^j, \quad b_j \in \mathbb{C},$$

and $g_1(z)$ is a holomorphic, single-valued function in $\widetilde{U}(z_0)$.
The function $\log(z-z_0)$ is a multi-valued function in $\widetilde{U}(z_0)$ and therefore the second term in the right-hand side of (3.26) is also multi-valued, unless the factor $p(z)$ satisfies certain

conditions.
Now we build a solution w of the equation (M) with the generating function g according to (3.26) and postulate that w is a single-valued function in $\widetilde{U}(z_0)$.
Inserting the expression (3.26) for g into the formula (3.24) we get

$$\begin{aligned} w &= H_m[g_1(z) + p(z)\log(z - z_0)] + H^*_{m-1}\overline{[g_1(z) + p(z)\log(z - z_0)]} \\ &= \Psi + H_m[p(z)]\log(z - z_0) + H^*_{m-1}[\overline{p(z)}]\log(\overline{z - z_0}), \end{aligned}$$

where Ψ denotes a function which is single-valued in $\widetilde{U}(z_0)$.
With $z = z_0 + re^{i\vartheta}$, $\vartheta = \vartheta_0 + 2n\pi$, $n \in \mathbb{Z}$, and thus $\log(z - z_0) = \ln r + i\vartheta$ we have

$$w = \Psi + \big(H_m[p(z)] + H^*_{m-1}[\overline{p(z)}]\big)\ln r + \big(H_m[p(z)] - H^*_{m-1}[\overline{p(z)}]\big)i\vartheta.$$

Since w has to be single-valued in $\widetilde{U}(z_0)$ we have to require

$$\upsilon := H_m[p(z)] - H^*_{m-1}[\overline{p(z)}] = 0.$$

Setting $p(z) = iq(z)$ we have

$$\upsilon = i\big(H_m[q(z)] + H^*_{m-1}[\overline{q(z)}]\big) = 0.$$

We see that $-i\upsilon$ is a solution of the equation (M) with the generating function $q(z)$. Since υ is the zero-solution, $q(z)$ is a generating function of the zero-solution of the equation (M). According to Theorem 3.2 we see that q has the form

$$q(z) = \sum_{j=0}^{2m-1} a_j z^j, \quad \text{with } a_j \in \mathbb{C},\ a_j = -\overline{a}_{2m-1-j},\ j = 0,1,\ldots,m-1.$$

This means that the polynomial $p(z) = iq(z)$ is of the form

$$p(z) = \sum_{j=0}^{2m-1} b_j z^j, \quad \text{with } b_j = \overline{b}_{2m-1-j},\ j = 0,1,\ldots,m-1.$$

To sum up we get the general representation theorem for solutions of the equation (M) in the neighbourhood of an isolated singularity.

Theorem 3.3.
Let w be a solution of the equation (M) in

$$\widetilde{U}(z_0) = \{z \in \mathbb{C} | 0 < |z - z_0| < \rho\} \subset K_R,$$

with an isolated singularity z_0. Then w can be represented in $\widetilde{U}(z_0)$ by

$$w = \sum_{j=0}^{m} mB^m_j \left(\frac{\overline{z}}{1 - z\overline{z}}\right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1} (m - j)B^m_j \frac{z^{m-j-1}}{(1 - z\overline{z})^{m-j}} \overline{g^{(j)}(z)},$$

where

$$B_j^m = \frac{(2m-j-1)!}{j!(m-j)!}$$

and the generating function g has the form

$$g(z) = g_1(z) + p(z)\log(z-z_0),$$

with $g_1(z)$ *is a holomorphic function in* $\widetilde{U}(z_0)$ *and* $p(z)$ *is a polynomial of the form*

$$p(z) = \sum_{j=0}^{2m-1} b_j z^j, \quad b_j \in \mathbb{C},\ b_j = \overline{b}_{2m-1-j},\ j = 0,1,\ldots,m-1.$$

3.3 A generating pair of the equation (M) in the sense of L.Bers

The concepts and notations of pseudo-analytic functions introduced in the following can be found in the books of Lipman Bers [18] and Vladislav V. Kravchenko [32].
The notion of a generating pair in the sense of Lipman Bers which is a couple of complex functions, is independent in the sense that at any point the value of any complex function defined there can be represented as a real linear combination of the generating functions. In pseudo-analytic function theory they play the same role as 1 and i in the theory of analytic functions.

Definition 3.1.
A pair of complex functions F *and* G *in* Ω*, possessing Hölder continuous partial derivatives with respect to the real variables* x *and* y*, is said to be a generating pair if it satisfies the inequality*

$$Im(\overline{F}G) > 0 \qquad in\ \ \Omega.$$

The following expressions are known as *characteristic coefficients* of the pair (F,G)

$$a_{(F,G)} = -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad b_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G},$$

$$A_{(F,G)} = -\frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \qquad B_{(F,G)} = -\frac{FG_z - F_zG}{F\overline{G} - \overline{F}G}.$$

The equation

$$w_{\overline{z}} = a_{(F,G)}w + b_{(F,G)}\overline{w} \tag{3.27}$$

is called a *Bers-Vekua equation* (sometimes, Carleman-Bers-Vekua equation). This equation represents a generalization of the Cauchy-Riemann system and is the main object of the study of pseudo-analytic function theory.

In the special case when F,G are two independent solutions of the equation (1.1), that is, F,G satisfy the equation

$$w_{\overline{z}} = C\overline{w}$$

and $F\overline{G} - \overline{F}G \neq 0$, then

$$a_{(F,G)} = -\frac{\overline{F}(C\overline{G}) - (C\overline{F})\overline{G}}{F\overline{G} - \overline{F}G} = 0,$$
$$b_{(F,G)} = \frac{F(C\overline{G}) - (C\overline{F})G}{F\overline{G} - \overline{F}G} = C.$$

In view of the equation (3.27), we can say (F,G) is the generating pair of the equation (1.1)

$$w_{\overline{z}} = C\overline{w}, \quad (a_{(F,G)} = 0; b_{(F,G)} = C).$$

In order to determine the generating pair of the equation (M) in the sense of L. Bers, we choose F,G as two independent solutions of the equation (M).
We have proved that all solutions of the equation (M) in K_R, can be represented as

$$\begin{aligned} w(z,\overline{z}) = g^{(m)}(z) &+ \sum_{j=0}^{m-1} \frac{(2m-j-1)!m}{j!(m-j)!} \left(\frac{\overline{z}}{1-z\overline{z}}\right)^{m-j} g^{(j)}(z) \\ &+ \sum_{j=0}^{m-1} \frac{(2m-j-1)!}{j!(m-j-1)!} \frac{z^{m-1-j}}{(1-z\overline{z})^{m-j}} \overline{g^{(j)}(z)}, \end{aligned} \tag{3.28}$$

where $g \in H(K_R)$.
Choose $g = 1$ we have

$$F = \frac{(2m-1)!}{(m-1)!} \frac{[\overline{z}^m + z^{m-1}]}{(1-z\overline{z})^m},$$

and for $g = i$,

$$G = \frac{i(2m-1)!}{(m-1)!} \frac{[\overline{z}^m - z^{m-1}]}{(1-z\overline{z})^m}.$$

Then (F,G) is the generating pair of the equation (M) in the sense of L.Bers.

3.4 Ising field theory on a pseudo-sphere

The Ising field theory on the pseudo-sphere which was considered in [23] can be written in terms of a free massive Majorana fermion $(\psi, \overline{\psi})$ as

$$\mathcal{A} = \frac{1}{2\pi} \int_{|z|<1} d^2x \left[\psi\overline{\partial}\psi + \overline{\psi}\partial\overline{\psi} + \frac{2ir}{1-z\overline{z}}\overline{\psi}\psi\right].$$

We introduce the parameter R related to the Gaussian curvature $\hat{R}$ by

$$\hat{R} = -\frac{1}{R^2},$$

and the notation r related to the mass parameter m and Gaussian curvature $\hat{R}$

$$r = mR.$$

Then the chiral components ψ and $\overline{\psi}$ obey the linear field equations

$$\partial_{\overline{z}}\psi(x) = \frac{ir}{1-z\overline{z}}\overline{\psi}(x), \qquad \partial_z\overline{\psi}(x) = \frac{-ir}{1-z\overline{z}}\psi(x), \tag{3.29}$$

where $(z,\overline{z})$ are complex coordinates on the unit disk $|z| < 1$.
We consider the first equation of the system (3.29)

$$\partial_{\overline{z}}\psi = \frac{ir}{1-z\overline{z}}\overline{\psi}. \tag{3.30}$$

Let $\psi = e^{i\theta}w,\ \theta \in \mathbb{R}$ then

$$e^{i\theta}\partial_{\overline{z}}w = \frac{ir}{1-z\overline{z}}e^{-i\theta}\overline{w} \qquad \Leftrightarrow \qquad \partial_{\overline{z}}w = \frac{ir}{1-z\overline{z}}e^{-2i\theta}\overline{w}.$$

Choose $\theta = \frac{\pi}{4}$ then $ie^{-2i\theta} = 1$ and if $r \in \mathbb{N}$ we obtain an equation which has the same type as the equation (M)

$$\partial_{\overline{z}}w = \frac{r}{1-z\overline{z}}\overline{w}. \tag{3.31}$$

Hence we can solve the solution w of the equation (3.31) explicitly. This implies that the solution ψ of the equation (3.30) is given by

$$\psi(z,\overline{z}) = e^{i\frac{\pi}{4}}\left[\sum_{j=0}^{r} rB_j^r\left(\frac{\overline{z}}{1-z\overline{z}}\right)^{r-j} g^{(j)}(z) + \sum_{j=0}^{r-1}(r-j)B_j^r\frac{z^{r-1-j}}{(1-z\overline{z})^{r-j}}\overline{g^{(j)}(z)}\right],$$

where $\quad B_j^r \; = \; \dfrac{(2r-j-1)!}{j!(r-j)!}.$
Therefore we obtain the following lemma.

Lemma 3.1. *Assume that the parameter r in (3.29) is a nonnegative integer then we can solve explicitly the chiral components, which obey (3.29),*

$$\psi = e^{i\frac{\pi}{4}}\left[\sum_{j=0}^{r} rB_j^r\left(\frac{\overline{z}}{1-z\overline{z}}\right)^{r-j} g^{(j)}(z) + \sum_{j=0}^{r-1}(r-j)B_j^r\frac{z^{r-1-j}}{(1-z\overline{z})^{r-j}}\overline{g^{(j)}(z)}\right],$$

$$\overline{\psi} = e^{-i\frac{\pi}{4}}\left[\sum_{j=0}^{r} rB_j^r\left(\frac{z}{1-z\overline{z}}\right)^{r-j}\overline{g^{(j)}(z)} + \sum_{j=0}^{r-1}(r-j)B_j^r\frac{\overline{z}^{r-1-j}}{(1-z\overline{z})^{r-j}}g^{(j)}(z)\right],$$

where $\quad B_j^r \; = \; \dfrac{(2r-j-1)!}{j!(r-j)!}.$

3.5 Connection between solutions of the equation (*M*) with different parameters

In this section we shall find differential operators of first order which map solutions of the equation (*M*)

$$w_{\overline{z}} = \frac{m}{1-z\overline{z}}\overline{w}, \quad z \in \mathcal{D},\ m \in \mathbb{N},$$

to solutions of the equation

$$\upsilon_{\overline{z}} = \frac{m+1}{1-z\overline{z}}\overline{\upsilon}, \quad z \in \mathcal{D},\ m \in \mathbb{N}, \tag{3.32}$$

and

$$\upsilon_{\overline{z}} = \frac{m-1}{1-z\overline{z}}\overline{\upsilon}, \quad z \in \mathcal{D},\ m \in \mathbb{N}, \tag{3.33}$$

respectively.
Assume that w is a solution of the equation (*M*). We shall seek a solution υ of the equation (3.32) of the form

$$\upsilon := \alpha w_z + \beta w + \gamma \overline{w},$$

where α, β and γ are unknown coefficients. Inserting this expression into the equation (3.32) and using the fact that w is a solution of the equation (*M*) we obtain that α, β, γ obey the following system

$$\begin{cases} \alpha_{\overline{z}} = 0, \\ \gamma = \dfrac{m+1}{1-z\overline{z}}\overline{\alpha}, \\ \dfrac{\alpha m^2}{(1-z\overline{z})^2} + \beta_{\overline{z}} = \dfrac{m+1}{1-z\overline{z}}\overline{\gamma}, \\ \alpha\dfrac{m\overline{z}}{(1-z\overline{z})^2} + \beta\dfrac{m}{1-z\overline{z}} + \gamma_{\overline{z}} = \dfrac{m+1}{1-z\overline{z}}\overline{\beta}. \end{cases} \tag{3.34}$$

From the first equation of the system (3.34), α is a holomorphic function in variable z. We can choose specially $\alpha = z$. Then from the second equation of the system (3.34) it follows

$$\gamma = \frac{m+1}{1-z\overline{z}}\overline{z}.$$

Inserting α and γ into the third equation of the system (3.34) we obtain

$$\beta_{\overline{z}} = \frac{2m+1}{(1-z\overline{z})^2}z,$$

from which

$$\beta = \frac{2m+1}{1-z\overline{z}} + \varphi(z),\ \varphi(z) \text{ is an arbitrary holomorphic function,}$$

follows.
Now the last equation of the system (3.34) is satisfied if we choose $\varphi = -m$.
Therefore

$$\upsilon = zw_z + \left(\frac{2m+1}{1-z\overline{z}} - m\right)w + \frac{(m+1)\overline{z}}{1-z\overline{z}}\overline{w}$$

is a solution of the equation (3.32) if w is a solution of the equation (M).
Analogously we can prove that

$$\upsilon = zw_z + \left(\frac{1-2m}{1-z\overline{z}} + m\right)w + \frac{(m-1)\overline{z}}{1-z\overline{z}}\overline{w}$$

is a solution of the equation (3.33) if w is a solution of the equation (M).

Denote the set of the solutions of the equation (M), defined in $\mathcal{D}$ by $G_m(\mathcal{D})$ and the set of the solutions of the equations (3.32), (3.33) by $G_{m+1}(\mathcal{D})$, $G_{m-1}(\mathcal{D})$, respectively. Summarising the above results we have the following theorem.

Theorem 3.4.
Let $w \in G_m(\mathcal{D})$, *then*
a) $zw_z + \left(\frac{2m+1}{1-z\overline{z}} - m\right)w + \frac{(m+1)\overline{z}}{1-z\overline{z}}\overline{w} \ \in G_{m+1}(\mathcal{D}), m \in \mathbb{N}$,
b) $zw_z + \left(\frac{1-2m}{1-z\overline{z}} + m\right)w + \frac{(m-1)\overline{z}}{1-z\overline{z}}\overline{w} \ \in G_{m-1}(\mathcal{D}), m \in \mathbb{N}^*$,
c) $i\left(zw_z - \overline{z}w_{\overline{z}} + \frac{1}{2}w\right) \ \in G_m(\mathcal{D}), m \in \mathbb{N}$.

In the sequel we shall give another method to derive the differential operators of first order which map solutions of the equation (M) to solutions of the equations (3.32) and (3.33), respectively.
We consider the transformation

$$z = \frac{u-i}{u+i},$$

where u is a new complex variable and $i^2 = -1$, then the equation (M) becomes

$$w_{\overline{u}} = \frac{m(u+i)}{(\overline{u}-i)(u-\overline{u})}\overline{w}.$$

Let $W = \dfrac{iw}{u+i}$ then we have

$$W_{\overline{u}} = \frac{-m\overline{W}}{u-\overline{u}}, \quad m \in \mathbb{N}. \tag{3.35}$$

If we set $\alpha = iu$ then the equation (3.35) is of the form

$$W_{\overline{u}} = \frac{m\overline{\alpha'}}{\alpha+\overline{\alpha}}\overline{W}, \quad m \in \mathbb{N}, \tag{3.36}$$

where α is a holomorphic function satisfying the condition $(\alpha+\overline{\alpha})\alpha' \neq 0$, and α' denotes the derivative of α.
For the equation (3.36) K.W.Bauer established the connection between solutions corresponding to different parameters [9]. We need the two following theorems.

Theorem 3.5 (K.W.Bauer)**.**
For every solution W of the differential equation (3.36) defined in $\mathcal{D}$, there exists a function $f(u) \in H(\mathcal{D})$, such that

$$W := Q_m^\star f = \sum_{k=0}^{m} \frac{(-1)^{m-k}(2m-1-k)!}{k!(m-k)!(\alpha+\overline{\alpha})^{m-k}}[mR^k f - (m-k)\overline{R^k f}], \qquad (3.37)$$

with $R = \dfrac{1}{\alpha'}\dfrac{\partial}{\partial u}$.
Conversely, for each function $f(u) \in H(\mathcal{D})$, (3.37) represents a solution of (3.36) in $\mathcal{D}$.

Theorem 3.6 (K.W.Bauer)**.**
If we denote the set of the solutions of (3.36) defined in $\mathcal{D}$ by $F_m(\mathcal{D})$ and if we use the differential operators

$$R = \frac{1}{\alpha'}\frac{\partial}{\partial u}, \qquad S = \frac{1}{\overline{\alpha'}}\frac{\partial}{\partial \overline{u}},$$

and let $W = Q_m^\star f \in F_m(\mathcal{D})$. Then,

a) $(R + \frac{m+1}{m}S - \frac{2m+1}{\alpha+\overline{\alpha}})W = Q_{m+1}^\star f \in F_{m+1}(\mathcal{D}),\ m \in \mathbb{N},$
$RW + \frac{m+1}{\alpha+\overline{\alpha}}\overline{W} - \frac{2m+1}{\alpha+\overline{\alpha}}W = Q_{m+1}^\star f \in F_{m+1}(\mathcal{D}),\ m \in \mathbb{N}^*,$

b) $(R + \frac{m-1}{m}S + \frac{2m-1}{\alpha+\overline{\alpha}})W = Q_{m-1}^\star (R^2 f) \in F_{m-1}(\mathcal{D}),\ m \in \mathbb{N}^*,$

c) $i(R-S)W = Q_m^\star(iRf) \in F_m(\mathcal{D}),\ m \in \mathbb{N}.$

Now using the two theorems of K.W.Bauer and the fact that under linear transformations all solutions of the equation (M) can be transformed to a set of all solutions of the equation (3.36) and vice versa, we can find the desired differential operators of first order which give relations between sets of solutions of the Bers-Vekua equation(M) with different parameters.
If $w = P_m^\star g \in G_m(\mathcal{D})$ then

$$\begin{cases} P_{m+1}^\star g = & \frac{u+i}{i}\left[(R + \frac{m+1}{m}S - \frac{2m+1}{\alpha+\overline{\alpha}})(\frac{iw}{u+i})\right] \in G_{m+1}(\mathcal{D}), \\ P_{m-1}^\star g = & \frac{u+i}{i}\left[(R + \frac{m-1}{m}S + \frac{2m-1}{\alpha+\overline{\alpha}})(\frac{iw}{u+i})\right] \in G_{m-1}(\mathcal{D}). \end{cases}$$

Changing the variable u to z we obtain the following theorem.

Theorem 3.7.
Denote

$$\tilde{R} = \frac{-1}{2}(1-z)^2\frac{\partial}{\partial z}; \qquad \tilde{S} = \frac{-1}{2}(1-\bar{z})^2\frac{\partial}{\partial \bar{z}}. \tag{3.38}$$

Let $w = P^{\star}_{m} g \in G_m(\mathcal{D})$. *Then*

a) $\left[\tilde{R} + \frac{m+1}{m}\tilde{S} + \big(1 + \frac{(2m+1)(1-\bar{z})}{1-z\bar{z}}\big)\frac{1-z}{2}\right]w = P^{\star}_{m+1} g \in G_{m+1}(\mathcal{D}), m \in \mathbb{N},$

b) $\left[\tilde{R} + \frac{m-1}{m}\tilde{S} + \big(1 - \frac{(2m-1)(1-\bar{z})}{1-z\bar{z}}\big)\frac{1-z}{2}\right]w = P^{\star}_{m-1} g \in G_{m-1}(\mathcal{D}), m \in \mathbb{N}^*,$

c) $i\left[\tilde{R} - \tilde{S} - \frac{1-z}{2}\right]w \in G_m(\mathcal{D}), m \in \mathbb{N}.$

Proof.
To prove the statement a) we show that

$$P^{\star}_{m+1} g = \left[\tilde{R} + \frac{m+1}{m}\tilde{S} + \big(1 + \frac{(2m+1)(1-\bar{z})}{1-z\bar{z}}\big)\frac{1-z}{2}\right]w$$

is a solution of the equation (3.32).
We have

$$\begin{aligned}
P^{\star}_{m+1} g &= \left[-\frac{(1-z)^2}{2}\frac{\partial}{\partial z} + \frac{m+1}{m}\frac{(1-\bar{z})^2}{2}\frac{\partial}{\partial \bar{z}} + \big(1 + \frac{(2m+1)(1-\bar{z})}{1-z\bar{z}}\big)\frac{1-z}{2}\right]w \\
&= -\frac{(1-z)^2}{2}\frac{\partial w}{\partial z} - \frac{m+1}{2}\frac{(1-\bar{z})^2}{1-z\bar{z}}\overline{w} + \frac{1-z}{2}\left[1 + \frac{(2m+1)(1-\bar{z})}{1-z\bar{z}}\right]w. \\
\Rightarrow \frac{\partial P^{*}_{m+1} g}{\partial \bar{z}} &= -\frac{(1-z)^2}{2}w_{z\bar{z}} - \frac{m+1}{2}\left[\frac{(1-\bar{z})(-2+z+z\bar{z})}{(1-z\bar{z})^2}\overline{w} + \frac{(1-\bar{z})^2}{1-z\bar{z}}\overline{(w_z)}\right] \\
&\quad - \frac{(2m+1)(1-z)^2}{2(1-z\bar{z})^2}w + \frac{1-z}{2}\left[1 + \frac{(2m+1)(1-\bar{z})}{1-z\bar{z}}\right]\frac{m}{1-z\bar{z}}\overline{w}.
\end{aligned}$$

Inserting these expressions into the left-hand side of the equation (3.32) and denote by

$$T := \frac{\partial P^{\star}_{m+1} g}{\partial \bar{z}} - \frac{m+1}{1-z\bar{z}}\overline{P^{\star}_{m+1} g},$$

then $T = 0$. Indeed,

$$\begin{aligned}
T = &-\frac{(1-z)^2}{2}w_{z\bar{z}} - \frac{m+1}{2}\left[\frac{(1-\bar{z})(-2+z+z\bar{z})}{(1-z\bar{z})^2}\overline{w} + \frac{(1-\bar{z})^2}{1-z\bar{z}}\overline{(w_z)}\right] \\
&- \frac{(2m+1)(1-z)^2}{2(1-z\bar{z})^2}w + \frac{1-z}{2}\left[1 + \frac{(2m+1)(1-\bar{z})}{1-z\bar{z}}\right]\frac{m}{1-z\bar{z}}\overline{w} \\
&+ \frac{m+1}{1-z\bar{z}}\left[\frac{(1-\bar{z})^2}{2}\overline{(w_z)} + \frac{m+1}{2}\frac{(1-z)^2}{(1-z\bar{z})}w - \frac{1-\bar{z}}{2}\big(1 + \frac{(2m+1)(1-z)}{1-z\bar{z}}\big)\overline{w}\right].
\end{aligned}$$

A coefficient of $\overline{(w_z)}$ in the expression of T is

$$T_{\overline{(w_z)}} = -\frac{m+1}{2}\frac{(1-\overline{z})^2}{1-z\overline{z}} + \frac{(m+1)(1-\overline{z})^2}{2(1-z\overline{z})} = 0.$$

Since w is a solution of the equation (M), we have

$$w_{z\overline{z}} = \frac{m\overline{z}}{(1-z\overline{z})^2}\overline{w} + \frac{m^2}{(1-z\overline{z})^2}w.$$

Hence

$$\begin{aligned} T = &-\frac{(1-z)^2}{2}\left[\frac{m\overline{z}}{(1-z\overline{z})^2}\overline{w} + \frac{m^2}{(1-z\overline{z})^2}w\right] - \frac{m+1}{2}\frac{(1-\overline{z})(-2+z+z\overline{z})}{(1-z\overline{z})^2}\overline{w} \\ &-\frac{(2m+1)(1-z)^2}{2(1-z\overline{z})^2}w + \frac{m}{2}\frac{1-z}{1-z\overline{z}}\left[1+\frac{(2m+1)(1-\overline{z})}{1-z\overline{z}}\right]\overline{w} \\ &+\frac{m+1}{1-z\overline{z}}\left[\frac{m+1}{2}\frac{(1-z)^2}{(1-z\overline{z})}w - \frac{1-\overline{z}}{2}\left(1+\frac{(2m+1)(1-z)}{1-z\overline{z}}\right)\overline{w}\right]. \end{aligned}$$

A coefficient of w in the expression of T is

$$T_w = -\frac{(1-z)^2}{2}\frac{m^2}{(1-z\overline{z})^2} - \frac{(2m+1)(1-z)^2}{2(1-z\overline{z})^2} + \frac{(m+1)^2(1-z)^2}{2(1-z\overline{z})^2} = 0.$$

And a coefficient of $\overline{w}$ is

$$\begin{aligned} T_{\overline{w}} = &-\frac{(1-z)^2}{2}\frac{m\overline{z}}{(1-z\overline{z})^2} - \frac{m+1}{2}\frac{(1-\overline{z})(-2+z+z\overline{z})}{(1-z\overline{z})^2} \\ &+\frac{m}{2}\frac{1-z}{1-z\overline{z}}\left[1+\frac{(2m+1)(1-\overline{z})}{1-z\overline{z}}\right] - \frac{m+1}{1-z\overline{z}}\frac{1-\overline{z}}{2}\left[1+\frac{(2m+1)(1-z)}{1-z\overline{z}}\right]. \end{aligned}$$

This coefficient is also equal to zero.
Hence $T = 0$ and this implies that $P^{\star}_{m+1}g$ is a solution of the equation (3.32)

$$\upsilon_{\overline{z}} = \frac{m+1}{1-z\overline{z}}\overline{\upsilon}, \quad z \in \mathcal{D},\ m \in \mathbb{N}.$$

Therefore the statement a) of the theorem has been proved. Analogously we can prove the statements b) and c) of the theorem.

□

3.6 Representation for solutions of the inhomogeneous equation

In this section we shall find functions Φ for which the inhomogeneous differential equation of the form

$$w_{\overline{z}} - \frac{m}{1-z\overline{z}}\overline{w} = \Phi(z,\overline{z}) \tag{3.39}$$

can be solved explicitly. In [44] I.N. Vekua gave the representation for solutions of the inhomogeneous differential equation

$$w_{\bar{z}} = Aw + B\overline{w} + \Phi$$

using the integral operators. But the determination of the integrals containing the function Φ is difficult in general. In [9] K.W. Bauer proved that it is possible to get representations (by differential operators) for the solutions of the inhomogeneous equation of type

$$\frac{(\varphi+\overline{\psi})^2}{\varphi'\overline{\psi'}} w_{z\bar{z}} - n(n+1)w = \Phi(z,\bar{z}),$$

where φ, ψ are holomorphic or meromorphic functions in $\mathcal{D}$ and satisfy $(\varphi+\overline{\psi})\varphi'\psi' \neq 0$ in $\mathcal{D}$, if the function $\Phi(z,\bar{z})$ satisfies certain conditions.
Using the method of K.W.Bauer, we are seeking functions $\Phi(z,\bar{z})$ such that all solutions of the inhomogeneous equation (3.39) can be represented by differential operators.
Case 1
Denote

$$D_m := w_{\bar{z}} - \frac{m}{1-z\bar{z}}\overline{w}, \quad m \in \mathbb{N}, \quad z = x+iy \in \mathcal{D}.$$

The homogeneous equation $D_m = 0$ has been solved in Chapter 1 and its solutions in $\mathcal{D}$ can be represented by

$$w = \sum_{j=0}^{m} mB_j^m \left(\frac{\bar{z}}{1-z\bar{z}}\right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1} (m-j)B_j^m \frac{z^{m-1-j}}{(1-z\bar{z})^{m-j}} \overline{g^{(j)}(z)}, \tag{3.40}$$

where $B_j^m = \dfrac{(2m-j-1)!}{j!(m-j)!}$ and $g \in H(\mathcal{D})$.
First we assume that

$$\Phi = \frac{\overline{\Phi_k}(z,\bar{z})}{1-z\bar{z}} \tag{3.41}$$

where $\Phi_k(z,\bar{z}), k \in \mathbb{N}\setminus\{m\}$, is a solution of the following homogeneous equation

$$D_k := (\Phi_k)_{\bar{z}} - \frac{k}{1-z\bar{z}}\overline{\Phi_k} = 0. \tag{3.42}$$

To get a general solution of the inhomogeneous equation (3.39) we need to find a particular solution. We shall find the particular solution w_0^k of (3.39) in the following form

$$w_0^k = \lambda\Phi_k(z,\bar{z}), \quad \lambda \in \mathbb{R}.$$

Substituting this expression into the equation (3.39) with the right-hand side given by (3.41), we obtain $\lambda = \dfrac{1}{k-m}$. Hence

$$w_0^k = \frac{1}{k-m}\Phi_k(z,\bar{z}).$$

Now we assume further that for $p \in \mathbb{N}$

$$\Phi = \sum_{\substack{k=1\\k\neq m}}^{p} \frac{\overline{\Phi_k}(z,\overline{z})}{1-z\overline{z}},$$

where $\Phi_k(z,\overline{z}), k = 1,\dots,p,\ k \neq m$, is the solution of the homogeneous differential equation $D_k = 0$ defined in $\mathcal{D}$. Then

$$w_0 = \sum_{\substack{k=1\\k\neq m}}^{p} \frac{1}{k-m}\Phi_k(z,\overline{z}) \tag{3.43}$$

represents a particular solution in $\mathcal{D}$ of the inhomogeneous differential equation

$$w_{\overline{z}} - \frac{m}{1-z\overline{z}}\overline{w} = \sum_{\substack{k=1\\k\neq m}}^{p} \frac{\overline{\Phi_k}(z,\overline{z})}{1-z\overline{z}}.$$

Combining this result with the representation formula of solutions of the equation (M) we have the following theorem.

Theorem 3.8.
Consider the inhomogeneous equation

$$w_{\overline{z}} - \frac{m}{1-z\overline{z}}\overline{w} = \sum_{\substack{k=1\\k\neq m}}^{p} \frac{\overline{\Phi_k}(z,\overline{z})}{1-z\overline{z}}, \tag{3.44}$$

where $\Phi_k(z,\overline{z}), k = 1,\dots,p,\ k \neq m$, is the solution of the homogeneous differential equation (3.42) defined in $\mathcal{D}$. Then all solutions $\widetilde{w}$ of the equation (3.44) can be represented in the form

$$\widetilde{w} = \sum_{\substack{k=1\\k\neq m}}^{p} \frac{1}{k-m}\Phi_k(z,\overline{z}) + w,$$

where w is a solution of the homogeneous equation (M) given by (3.40).

Case 2
We assume that

$$\Phi := x^k\widetilde{\Phi} = \Big(\frac{z+\overline{z}}{2}\Big)^k\widetilde{\Phi}, \quad k \geq 0,$$

where $D_m(\widetilde{\Phi}) = 0$. Then the equation (3.39) becomes

$$D_m w := w_{\overline{z}} - \frac{m}{1-z\overline{z}}\overline{w} = x^k\widetilde{\Phi}. \tag{3.45}$$

Denote

$$D_m^k w = D_m^{k-1}(Dw),$$

then we have

$$D_m^{k+2} w = D_m^{k+1}(D_m w) = D_m^{k+1}(x^k \widetilde{\Phi}) = 0.$$

This implies w is a solution of the iterated Bers-Vekua equation (cf. [13], [16])

$$D_m^{k+2} w = 0.$$

P. Berglez proved that the solution w of this iterated equation has the following form

$$w = \sum_{j=0}^{k+1} x^j \widetilde{\Phi}_j$$

where $D_m(\tilde{\Phi}_j) = 0, j = 0, 1, \cdots, k+1$.
Substituting this expression into the equation (3.45) we obtain

$$\begin{cases} \widetilde{\Phi}_j &= 0, \qquad j = 0, 1, \ldots, k, \\ \widetilde{\Phi}_{k+1} &= \dfrac{2\widetilde{\Phi}}{k+1}. \end{cases}$$

Hence a particular solution of the inhomogeneous equation (3.45) is

$$w = \frac{2x^{k+1}\tilde{\Phi}}{k+1}.$$

In the case the right-hand side of the inhomogeneous equation (3.39) is of the form

$$\Phi = \sum_{k=1}^{q} \Big(\frac{z+\bar{z}}{2}\Big)^k \widetilde{\Phi}_k, \quad q \in \mathbb{N},$$

then it has a particular solution

$$\widetilde{w}_0 = 2 \sum_{k=1}^{q} \frac{x^{k+1}\widetilde{\Phi}_k}{k+1}.$$

Summarizing the above result we have the following theorem.

Theorem 3.9.
Consider the inhomogeneous differential equation

$$w_{\bar{z}} - \frac{m}{1-z\bar{z}}\overline{w} = \sum_{k=1}^{q} \Big(\frac{z+\bar{z}}{2}\Big)^k \widetilde{\Phi}_k, \tag{3.46}$$

where $D_m(\Phi_k) = 0$. Then all solutions $\widetilde{w}$ of the equation (3.46) can be represented in the form

$$\tilde{w} = 2 \sum_{k=1}^{q} \frac{x^{k+1}\widetilde{\Phi}_k}{k+1} + w,$$

where w is a solution of the equation (M) given by (3.40).

4 REPRESENTATION OF BICOMPLEX PSEUDO-ANALYTIC FUNCTIONS

In this chapter we study a class of bicomplex pseudo-analytic functions which are solutions of a system in bicomplex space

$$\begin{cases} \partial_{z^*} V(z) = \mathcal{C}(z, z^*) V^*(z), \\ \partial_{\bar{z}_1} V(z) = \partial_{\bar{z}_2} V(z) = 0, \end{cases}$$

where z is a bicomplex variable and $z_1, z_2 \in \mathbb{C}$ are the components of z.
Since the two components of the so-called *idempotent representation* of each bicomplex number are complex numbers, many results in the theory of functions of a complex variable are still true in bicomplex algebra [37]. Using this fact together with the results of I.N. Vekua [44] we can construct the bicomplex form of this system and define the resolvents of Vekua type in bicomplex variables. Then we derive the representation of these bicomplex pseudo-analytic functions by integral operators.
On the other hand, using the representation theorems for solutions of second order partial differential equations of P. Berglez [11] we obtain a condition on coefficients $\mathcal{C}$ such that these bicomplex pseudo-analytic functions can be represented by differential operators. In [15] P. Berglez considered other classes of bicomplex pseudo-analytic functions which obey specific bicomplex Bers-Vekua equation and derived different representations for solutions of such a Bers-Vekua equation.
For a special class of bicomplex pseudo-analytic functions we give an explicit representation by differential operators. Some applications such as solving a Dirac equation on a pseudo-sphere and using the generalization of the Weierstrass formulae to generate surfaces are given.

4.1 An introduction to bicomplex algebra

In this section we introduce some basic definitions and notations in the space of bicomplex numbers (see, e.g., [37], [38], [40] or [41]).

4.1.1 Bicomplex numbers

Let us denote the imaginary unit in the space of complex numbers $\mathbb{C}$ by i_1 and thus denote

$$\mathbb{C}(i_1) := \mathbb{C} = \{x + i_1 y \mid x, y \in \mathbb{R},\ i_1^2 = -1\}.$$

Let, then, i_2 denote the second imaginary unit with the properties

$$i_2^2 = -1, \qquad i_1 i_2 = i_2 i_1, \qquad \alpha i_2 = i_2 \alpha, \quad \forall \alpha \in \mathbb{R}.$$

Denote the space of bicomplex numbers by $\mathbb{T}$. Then

$$\mathbb{T} := \{z | z = z_1 + i_2 z_2,\ z_1, z_2 \in \mathbb{C}(i_1)\}$$

becomes a commutative algebra with the multiplication given by

$$(z_1 + i_2 z_2)(z_3 + i_2 z_4) = (z_1 z_3 - z_2 z_4) + i_2(z_1 z_4 + z_2 z_3).$$

It is also convenient to write the set of bicomplex numbers as

$$\mathbb{T} := \{x_0 + i_1 x_1 + i_2 x_2 + j x_3 \big| x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where the imaginary units i_1, i_2 and j are governed by the rules

$$\begin{gathered} i_1^2 = i_2^2 = -1, \\ i_1 i_2 = i_2 i_1 = j, \quad j^2 = 1, \\ i_1 j = j i_1 = -i_2, \quad i_2 j = j i_2 = -i_1. \end{gathered}$$

The bicomplex numbers have several representations, we shall mostly represent them by usual complex pairs.

Definition 4.1.
Define the function $||\ \cdot || : \mathbb{T} \to \mathbb{R}_+$ *as follows:*
For every $z = z_1 + i_2 z_2 \in \mathbb{T}$ *with* $z_1 = x_1 + i_1 x_2,\ z_2 = x_3 + i_1 x_4$,

$$||z|| := (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2} = \left(|z_1|^2 + |z_2|^2\right)^{1/2}.$$

Theorem 4.1 ([37]).
The function $||\ \cdot ||$ *defined as above is a norm on the linear space* $\mathbb{T}$*. With this norm,* $\mathbb{T}$ *becomes a Banach space.*

Definition 4.2.
Let ζ_1 *and* ζ_2 *be elements in* $\mathbb{T}$*. If* $\zeta_1 \neq 0,\ \zeta_2 \neq 0$*, and* $\zeta_1 \zeta_2 = 0$ *then* ζ_1 *and* ζ_2 *are called divisors of zero.*

$\mathbb{T}$ is a commutative ring and has divisors of zero. A set of divisors of zero in the space of bicomplex numbers is called the *null-cone* and is denoted by

$$\begin{aligned}\mathcal{O}_2 &= \{z_1 + i_2 z_2 \in \mathbb{T} \,|\, z_1^2 + z_2^2 = 0\} \\ &= \{z_1(i_1 - i_2) \,|\, z_1 \in \mathbb{C}(i_1)\}.\end{aligned}$$

Now we introduce the conjugations in the space of bicomplex numbers. There are three conjugations in $\mathbb{T}$. Normally the complex conjugation is given by its action over the imaginary unit, thus one expects at least two conjugations on $\mathbb{T}$ but one more candidate could arise from composing them. Hence for $z = z_1 + i_2 z_2 \in \mathbb{T}$ there are three conjugations defined as follows

$$\begin{aligned}z^* &= z_1 - i_2 z_2, \\ z^\star &= \bar{z}_1 + i_2 \bar{z}_2, \\ z^\dagger &= \bar{z}_1 - i_2 \bar{z}_2.\end{aligned}$$

In the next subsections, we present some properties of bicomplex numbers and functions of a bicomplex variable. The proofs of these properties can be found in [37], [39].

4.1.2 The idempotent representation

Definition 4.3.
Let ζ be an element in $\mathbb{T}$. If $\zeta^2 = \zeta$ then ζ is called an idempotent element.

Theorem 4.2 ([37])**.**
We have four and only four idempotent elements in $\mathbb{T}$, and they are

$$0, \quad 1, \quad e_1 := \frac{1 + i_1 i_2}{2}, \quad e_2 := \frac{1 - i_1 i_2}{2}.$$

Corollary 4.1.
Let e_1, e_2 be the two idempotent elements given as above, then

$$e_1^2 = e_1, \qquad e_2^2 = e_2, \qquad e_1 e_2 = 0.$$

Theorem 4.3 ([37])**.**
Every element $z = z_1 + i_2 z_2$ in $\mathbb{T}$ has the following unique representation

$$z = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2. \tag{4.1}$$

Definition 4.4.
The expression (4.1) is called the idempotent representation of the element $z = z_1 + i_2 z_2$ in $\mathbb{T}$. The numbers $z_1 - i_1 z_2$ and $z_1 + i_1 z_2$ are the idempotent components of z.

This representation is very useful because the addition, multiplication and division can be done term-by-term.

Theorem 4.4 ([37]).
Let $z = z_1 + i_2 z_2$ and $u = u_1 + i_2 u_2$ be elements in $\mathbb{T}$. Assume the idempotent representations of z and u are

$$z = \zeta_1 e_1 + \zeta_2 e_2, \qquad u = \eta_1 e_1 + \eta_2 e_2.$$

Then
(a) $(\zeta_1 e_1 + \zeta_2 e_2) + (\eta_1 e_1 + \eta_2 e_2) = (\zeta_1 + \eta_1) e_1 + (\zeta_2 + \eta_2) e_2,$
(b) $(\zeta_1 e_1 + \zeta_2 e_2)(\eta_1 e_1 + \eta_2 e_2) = (\zeta_1 \eta_1) e_1 + (\zeta_2 \eta_2) e_2,$
(c) $(\zeta_1 e_1 + \zeta_2 e_2)^n = (\zeta_1)^n e_1 + (\zeta_2)^n e_2,$
(d) If $\eta_1 \neq 0$ *and* $\eta_2 \neq 0$, *then*

$$\frac{\zeta_1 e_1 + \zeta_2 e_2}{\eta_1 e_1 + \eta_2 e_2} = \frac{\zeta_1}{\eta_1} e_1 + \frac{\zeta_2}{\eta_2} e_2.$$

Corollary 4.2.
An element $z = z_1 + i_2 z_2$ is non-invertible if and only if $z_1 - i_1 z_2 = 0$ or $z_1 + i_1 z_2 = 0$.

4.1.3 Power series in the space of bicomplex numbers

First we give a definition of bicomplex power series for which it seems to be easier to introduce holomorphic functions of a bicomplex variable. The holomorphic functions of a bicomplex variable have many striking similarities to holomorphic functions of a complex variable, for example, holomorphic functions of both complex and bicomplex variables can be defined either as functions which are represented locally by power series or as functions which have a derivative.

Definition 4.5.
Let α_k, z and z_0 denote elements in $\mathbb{T}$. A power series in the bicomplex variable z is an infinite series of the form

$$\sum_{k=0}^{\infty} \alpha_k (z - z_0)^k. \tag{4.2}$$

If we assume $z_0 = 0$ and $\alpha_k = p_k + i_2 q_k,\ z = z_1 + i_2 z_2,\ p_k, q_k, z_1, z_2 \in \mathbb{C}(i_1)$ then the power series (4.2) is

$$\sum_{k=0}^{\infty} (p_k + i_2 q_k)(z_1 + i_2 z_2)^k.$$

Now using the idempotent representation of bicomplex numbers we have the following theorem.

Theorem 4.5 ([37]).
The idempotent component series of the bicomplex power series (4.2) are the complex power series

$$\sum_{k=0}^{\infty}(p_k - i_1 q_k)(z_1 - i_1 z_2)^k, \tag{4.3}$$

$$\sum_{k=0}^{\infty}(p_k + i_1 q_k)(z_1 + i_1 z_2)^k. \tag{4.4}$$

Theorem 4.6.
The bicomplex power series (4.2) converges at $z_1 + i_2 z_2$ if and only if the complex power series (4.3) and (4.4) converge at $z_1 - i_1 z_2$ and $z_1 + i_1 z_2$, respectively, and vice versa.

Since the idempotent components of a bicomplex power series are power series in complex variables, and since many known theorems give information about the convergence and divergence of complex power series, it is possible to determine the convergence and divergence of bicomplex power series. The region of convergence of the bicomplex power series is a special cartesian set in $\mathbb{T}$, a so called *discus* which will be defined in the following, rather than the ball.

Let $a = a_1 + i_2 a_2$ be an element in $\mathbb{T}$ which has the idempotent representation $a = (a_1 - i_1 a_2)e_1 + (a_1 + i_1 a_2)e_2$, and r_1, r_2 be positive real numbers.

Definition 4.6.

$$D(a; r_1, r_2) = \{z = \zeta_1 e_1 + \zeta_2 e_2 \in \mathbb{T} \big| |\zeta_1 - (a_1 - i_1 a_2)| < r_1;\ |\zeta_2 - (a_1 + i_1 a_2)| < r_2\}$$

is called the open discus with center $a = a_1 + i_2 a_2$ and radii r_1 and r_2.

$$\overline{D}(a; r_1, r_2) = \{z = \zeta_1 e_1 + \zeta_2 e_2 \in \mathbb{T} \big| |\zeta_1 - (a_1 - i_1 a_2)| \le r_1;\ |\zeta_2 - (a_1 + i_1 a_2)| \le r_2\}$$

is called the closed discus with center $a = a_1 + i_2 a_2$ and radii r_1 and r_2.

With the following theorem it is possible to determine the convergence and divergence of bicomplex power series.

Theorem 4.7 ([37]).
Let r_1 and r_2 be the radii of the circles of convergence of the two series (4.3) and (4.4), respectively. Then the series in (4.2) converges absolutely at every point in the discus $D(0; r_1, r_2)$, and it diverges at every point in the complement of $\overline{D}(0; r_1, r_2)$. The radii of convergence r_1 and r_2 may have the values 0 and ∞.

4.1.4 Integrals and holomorphic functions in bicomplex variables

Definition 4.7.
Let f be a bicomplex-valued function of a bicomplex variable $z_1 + i_2 z_2$. The function f defined on $X \subset \mathbb{T}$ is called a $\mathbb{T}$- holomorphic function if for each $a = a_1 + i_2 a_2 \in X$ there exists a discus $D(a; r_1, r_2) \subset X$, with $r_1, r_2 > 0$, and a power series such that

$$f(z_1 + i_2 z_2) = \sum_{k=0}^{\infty} (p_k + i_2 q_k)[(z_1 + i_2 z_2) - (a_1 + i_2 a_2)]^k$$

for all $z_1 + i_2 z_2$ in $D(a; r_1, r_2)$.
A set of $\mathbb{T}$-holomorphic functions on X is denoted by $H_X^{\mathbb{T}}$.

Theorem 4.8 ([37]).
A function f is $\mathbb{T}$-holomorphic in $D(a; r_1, r_2)$ if and only if there exist two complex holomorphic functions $f_1 : D(a_1 - i_1 a_2, r_1) \to \mathbb{C}$ and $f_2 : D(a_1 + i_1 a_2, r_2) \to \mathbb{C}$ such that

$$f(z_1 + i_2 z_2) = f_1(z_1 - i_1 z_2) e_1 + f_2(z_1 + i_1 z_2) e_2.$$

There is an equivalent definition of a $\mathbb{T}$-holomorphic function, that is, a $\mathbb{T}$-holomorphic function is a $\mathbb{T}-$*differentiable function*. The definition of the derivative at z_0 of a function $f : X \to \mathbb{T},\ X \subset \mathbb{T}$, of a bicomplex variable is formally the same as for a function of a complex variable, but many differences arise in the details because the null-cone $\mathcal{O}_2$ contains many points rather than a single point as in the complex case.

Definition 4.8.
A function $f : X \to \mathbb{T}$ with $X \subseteq \mathbb{T}$ open, is called $\mathbb{T}$-differentiable at $z_0 \in X$ with derivative equal to $f'(z_0) \in \mathbb{T}$ if the limit

$$\lim_{\substack{z \to z_0 \\ z - z_0 \notin \mathcal{O}_2}} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0)$$

exists.
We also say that the function f is $\mathbb{T}$-holomorphic in X if and only if f is $\mathbb{T}$-differentiable at each point of X.

The differential operators are defined as follows

$$\begin{aligned}
\partial_z &= \frac{1}{2}(\partial_{z_1} - i_2 \partial_{z_2}), \\
\partial_{z^*} &= \frac{1}{2}(\partial_{z_1} + i_2 \partial_{z_2}), \\
\partial_{z^\star} &= \frac{1}{2}(\partial_{\bar{z}_1} - i_2 \partial_{\bar{z}_2}), \\
\partial_{z^\dagger} &= \frac{1}{2}(\partial_{\bar{z}_1} + i_2 \partial_{\bar{z}_2}).
\end{aligned}$$

Theorem 4.9 ([39]).
f is $\mathbb{T}$-holomorphic if and only if f is continuously differentiable and satisfies the system

$$\begin{cases} \partial_{z^*} f(z) = 0, \\ \partial_{z^\star} f(z) = 0, \\ \partial_{z^\dagger} f(z) = 0, \end{cases}$$

or equivalent to

$$\begin{cases} \partial_{z^*} f(z) = 0, \\ \partial_{\overline{z}_1} f(z) = \partial_{\overline{z}_2} f(z) = 0. \end{cases}$$

Theorem 4.10 ([37]).
Let $f : X \to \mathbb{T}$ be a $\mathbb{T}$-holomorphic function then we have

$$\partial_{z_1+i_2z_2} f(z_1+i_2z_2) = \partial_{z_1-i_1z_2} f_1(z_1-i_1z_2)e_1 + \partial_{z_1+i_1z_2} f_2(z_1+i_1z_2)e_2.$$

Note that the derivative of a $\mathbb{T}$-holomorphic function is also a $\mathbb{T}$-holomorphic function. So we have the following definition of derivatives of higher orders of a $\mathbb{T}$-holomorphic function.

Definition 4.9. *Let f be a $\mathbb{T}$-holomorphic function in open set X, then we define*

$$f^{(k)}(z) = \partial_z[f^{(k-1)}(z)], \qquad z \in X,\ k \in \mathbb{N}^*.$$

Definition 4.10. *A function $f : U \to \mathbb{T}$, with $U \subset \mathbb{T}^n$, is called $\mathbb{T}$-holomorphic in n variables $(z_1, z_2, \ldots, z_n) \in U$ if f is $\mathbb{T}$-differentiable with respect to one variable with all other variables held constant.*

Integrals of functions with values in $\mathbb{T}$

The theory of integrals of functions with values in $\mathbb{T}$ is introduced in [37]. We quote here some definitions and main theorems.

Definition 4.11.
Let X be a domain in $\mathbb{T}$, $f : X \to \mathbb{T}$ be a continuous function, and let $\tau : [c,d] \to X$ be a curve γ with a continuous derivative $\tau' : [c,d] \to X$. Let P_n denote a subdivision

$$c = t_0 < t_1 < \cdots < t_{i-1} < t_i < \cdots < t_n = d$$

of $[c,d]$, and let t_i^* be a point such that $t_{i-1} \le t_i^* \le t_i$.
Form of the sum

$$S(f,P_n) = \sum_{i=1}^{n} f[\tau(t_i^*)][\tau(t_i) - \tau(t_{i-1})].$$

If $\lim_{n\to\infty} S(f,P_n)$ *exists and has the same value for every choice of the points* t_i^* *and for every sequence* $P_1, P_2, \dots$ *of subdivision of* $[c,d]$ *whose norms have the limit zero, then* f *has an integral on* γ*, denoted by* $\int_\gamma f(\tau)d\tau$*, and*

$$\int_\gamma f(\tau)d\tau = \lim_{n\to\infty} S(f,P_n).$$

Theorem 4.11 ([37]).
If $f : X \to \mathbb{T}$ *is continuous and the curve* γ*, defined by* $\tau : [c,d] \to X,\ t \mapsto \tau(t)$*, has a continuous derivative, then* f *has an integral on* γ *and* $\int_c^d f[\tau(t)]\tau'(t)dt$ *exists and*

$$\int_\gamma f(\tau)d\tau = \int_c^d f[\tau(t)]\tau'(t)dt.$$

Theorem 4.12 ([37]).
Let X *be a domain in* $\mathbb{T}$ *which is star-shaped with respect to a point* τ^**, and let* $f : X \to \mathbb{T}$ *be a* $\mathbb{T}$*- holomorphic function. If* γ *is a curve* $\tau : [c,d] \to X$ *which has a continuous derivative, then* $\int_\gamma f(\tau)d\tau$ *is independent of the path.*

In the case $X = X_1 \times X_2 := \{z = z_1 + i_2 z_2 \in \mathbb{T} \big| z_1 - i_1 z_2 \in X_1,\ z_1 + i_1 z_2 \in X_2\}$, where X_1 and X_2 are simply connected domains in the complex plane, we have the idempotent representation for the integral of $\mathbb{T}$-holomorphic functions.

Theorem 4.13 ([37]).
Let f *be a* $\mathbb{T}$*-holomorphic function in* $X = X_1 \times X_2$*. Assume that the idempotent representation of* f *is*

$$\begin{gathered} f(z) = f(z_1 + i_2 z_2) = f_1(z_1 - i_1 z_2)e_1 + f_2(z_1 + i_1 z_2)e_2, \\ z_1 - i_1 z_2 \in X_1,\ z_1 + i_1 z_2 \in X_2. \end{gathered} \tag{4.5}$$

Let γ *be the curve with trace in* X *which is defined as*

$$\gamma : z_1 + i_2 z_2 = [z_1(t) - i_1 z_2(t)]e_1 + [z_1(t) + i_1 z_2(t)]e_2, \quad c \le t \le d.$$

Then γ_1 *and* γ_2 *defined as*

$$\begin{cases} \gamma_1 : & z_1 - i_1 z_2 = z_1(t) - i_1 z_2(t), \quad c \le t \le d, \\ \gamma_2 : & z_1 + i_1 z_2 = z_1(t) + i_1 z_2(t), \quad c \le t \le d, \end{cases}$$

are two curves which have continuous derivatives and whose traces are in X_1 *and* X_2*, respectively.*
Then the integrals of f_1 *and* f_2 *on the curves* γ_1 *and* γ_2 *exist and*

$$\int_\gamma f(z)dz = \int_{\gamma_1} f_1(z_1 - i_1 z_2)d(z_1 - i_1 z_2)e_1 + \int_{\gamma_2} f_2(z_1 + i_1 z_2)d(z_1 + i_1 z_2)e_2. \tag{4.6}$$

4.2 Representation of bicomplex pseudo-analytic functions by integral operators

Definition 4.12.
A bicomplex pseudo-analytic function w in a domain $X \subset \mathbb{T}$ is a solution of a system of the type

$$\begin{cases} \partial_{z^*} w &= a_1 w + b_1 w^* + c_1 w^\star + d_1 w^\dagger, \\ \partial_{z^\star} w &= a_2 w + b_2 w^* + c_2 w^\star + d_2 w^\dagger, \\ \partial_{z^\dagger} w &= a_3 w + b_3 w^* + c_3 w^\star + d_3 w^\dagger. \end{cases}$$

In this section we consider a special class of bicomplex pseudo-analytic functions. They are solutions of a system

$$\begin{cases} \partial_{z^*} V(z) = \mathcal{C}(z,z^*) V^*(z), \qquad z \in D(0;r_1,r_2), \\ \partial_{\overline{z}_1} V = \partial_{\overline{z}_2} V = 0 \end{cases} \tag{E}$$

where $D(0;r_1,r_2) = \{z = z_1 + i_2 z_2 \in \mathbb{T} \big| |z_1 - i_1 z_2| < r_1, |z_1 + i_1 z_2| < r_2\}$ is an open discus with the center at the origin and radii r_1 and r_2; and $\mathcal{C}(z,z^*)$ is a $\mathbb{T}$-valued function analytic in two complex variables $z_1,\ z_2$.
We shall establish the representation formula of these pseudo-analytic functions using integral operators.
First we construct a function $\mathbb{T}$-holomorphic in two bicomplex variables from a given function which is bicomplex-valued and analytic in two complex variables.

Analytic continuation

Let $f(z) = f(z_1,z_2),\ z = z_1 + i_2 z_2 \in D(0;r_1,r_2)$, be a bicomplex-valued function which is analytic in two complex variables z_1, z_2.
Denote

$$\zeta_1 = z_1 - i_1 z_2, \qquad \zeta_2 = z_1 + i_1 z_2, \qquad |\zeta_1| < r_1, |\zeta_2| < r_2,$$

then the function $f(\zeta_1,\zeta_2)$ is also analytic in two complex variables ζ_1,ζ_2. Hence $f(\zeta_1,\zeta_2)$ can be expanded into the power series in variables ζ_1,ζ_2,

$$f(\zeta_1,\zeta_2) = \sum_{i,j \geq 0} \alpha_{ij} \zeta_1^i \zeta_2^j, \qquad |\zeta_1| < r_1, |\zeta_2| < r_2, \tag{4.7}$$

where $\alpha_{ij} = a_{ij} e_1 + b_{ij} e_2 \in \mathbb{T},\ a_{ij}, b_{ij} \in \mathbb{C}(i_1)$.
Let $Z_1 = z_1 + i_2\sigma_1, Z_2 = z_2 + i_2\sigma_2 \in \mathbb{T}$, where $\sigma_1,\sigma_2 \in \mathbb{C}(i_1)$. Then with α_{ij} in (4.7) we define a function $F(Z_1,Z_2)$ as follows

$$F(Z_1,Z_2) := \sum_{i,j \geq 0} \alpha_{ij} (Z_1 - i_1 Z_2)^i (Z_1 + i_1 Z_2)^j.$$

Denote $\mathcal{X} = \{(Z_1,Z_2) \in \mathbb{T}^2 \big| Z_1 + i_2 Z_2 \in D(0;r_1,r_2),\ Z_1 - i_2 Z_2 \in D(0;r_2,r_1)\}$.
We will prove that $F(Z_1,Z_2)$ is a $\mathbb{T}$-holomorphic function in the variables Z_1,Z_2 in $\mathcal{X}$.
Indeed, assume that the idempotent representations of Z_1,Z_2 are $Z_1 = \xi_1 e_1 + \xi_2 e_2$ and $Z_2 = \mu_1 e_1 + \mu_2 e_2$, then

$$F(Z_1,Z_2) = \sum_{i,j\geq 0} \alpha_{ij}[(\xi_1 e_1 + \xi_2 e_2) - i_1(\mu_1 e_1 + \mu_2 e_2)]^i[(\xi_1 e_1 + \xi_2 e_2) + i_1(\mu_1 e_1 + \mu_2 e_2)]^j$$

$$\Rightarrow F(Z_1,Z_2) = \sum_{i,j\geq 0} a_{ij}(\xi_1 - i_1\mu_1)^i(\xi_1 + i_1\mu_1)^j e_1 + \sum_{i,j\geq 0} b_{ij}(\xi_2 - i_1\mu_2)^i(\xi_2 + i_1\mu_2)^j e_2. \qquad (4.8)$$

Since

$$Z_1 + i_2 Z_2 = (\xi_1 - i_1\mu_1)e_1 + (\xi_2 + i_1\mu_2)e_2 \in D(0;r_1,r_2)$$

and

$$Z_1 - i_2 Z_2 = (\xi_1 + i_1\mu_1)e_1 + (\xi_2 - i_1\mu_2)e_2 \in D(0;r_2,r_1),$$

it implies that

$$\begin{aligned} |\xi_1 - i_1\mu_1| &< r_1, \qquad & |\xi_2 + i_1\mu_2| &< r_2, \\ |\xi_1 + i_1\mu_1| &< r_2, \qquad & |\xi_2 - i_1\mu_2| &< r_1. \end{aligned}$$

On the other hand, with the conditions $|\xi_k - i_1\mu_k| < r_1, |\xi_k + i_1\mu_k| < r_2,\ k = 1,2$, the two series on the right-hand side of (4.8) converge. This implies that the function $F(Z_1,Z_2)$ is $\mathbb{T}$-holomorphic in two variables Z_1,Z_2 in $\mathcal{X}$.
If $z = (z_1,z_2) \in D(0;r_1,r_2)$ then $(z_1,z_2) \in \mathcal{X}$ and the function $F(Z_1,Z_2)$ coincides with $f(z_1,z_2)$ when $\sigma_1 = \sigma_2 = 0$. We call the function $F(Z_1,Z_2)$ the *analytic continuation* of the function $f(z_1,z_2)$ from $D(0;r_1,r_2)$ into the domain $\mathcal{X}$ of two bicomplex variables.
Now we change the variables

$$Z_1 = \frac{1}{2}(z+u), \qquad Z_2 = \frac{1}{2i_2}(z-u), \qquad z \in D(0;r_1,r_2),\ u \in D(0;r_2,r_1).$$

Then we obtain a $\mathbb{T}$-holomorphic function $F(z,u)$ of the two bicomplex variables z,u and the power series of $F(z,u)$ is given by

$$F(z,u) = \sum_{i,j\geq 0} \alpha_{ij}(ze_1 + ue_2)^i(ue_1 + ze_2)^j = \left(\sum_{i,j\geq 0} a_{ij}\zeta_1^i\eta_1^j\right)e_1 + \left(\sum_{i,j\geq 0} b_{ij}\eta_2^i\zeta_2^j\right)e_2,$$

where ζ_1,ζ_2 and η_1,η_2 denote the idempotent components of z and u, respectively

$$z := \zeta_1 e_1 + \zeta_2 e_2, \qquad u := \eta_1 e_1 + \eta_2 e_2,$$

satisfying $|\zeta_1| < r_1, |\zeta_2| < r_2$ and $|\eta_1| < r_2, |\eta_2| < r_1$.
When $u = z^*$ we have $F(z,z^*) \equiv f(z)$.

Bicomplex conjugation

Let $F(z_1,\dots,z_n)$ be a $\mathbb{T}-$holomorphic function of the bicomplex variables $(z_1,\dots,z_n)$ in some domain $\Omega\subset\mathbb{T}^n$. We denote by Ω^* the following domain

$$\Omega^*=\{(u_1,\dots,u_n)\big|(u_1^*,\dots,u_n^*)\in\Omega\}$$

where u_i^* denotes the first bicomplex conjugation of $u_i,\ i=1,\dots,n$.
Define

$$F^*(u_1,\dots,u_n)=[F(u_1^*,\dots,u_n^*)]^*,\qquad (u_1,\dots,u_n)\in\Omega^*.$$

We call $F^*(u_1,\dots,u_n)$ the *conjugate function* of $F(z_1,\dots,z_n)$.

Bicomplex form of the system (E)

By hypothesis the coefficient $\mathcal{C}(z,z^*)$ of the system (E) is analytic in two complex variables z_1,z_2 in the discus $D(0;r_1,r_2)$. If we continue analytically this function into the bicomplex domain $\mathcal{X}$ we obtain a $\mathbb{T}$-holomorphic function $\mathcal{C}(z,u)$ of the two bicomplex variables z,u and $\mathcal{C}(z,u)$ can be expanded into a power series

$$\begin{aligned}\mathcal{C}(z,u)&=\sum_{i,j\geq 0}\beta_{ij}(ze_1+ue_2)^i(ue_1+ze_2)^j,\qquad \beta_{ij}=c_{ij}e_1+d_{ij}e_2,\quad c_{ij},d_{ij}\in\mathbb{C}(i_1)\\ &=\Bigg(\sum_{i,j\geq 0}c_{ij}\zeta_1^i\eta_1^j\Bigg)e_1+\Bigg(\sum_{i,j\geq 0}d_{ij}\eta_2^i\zeta_2^j\Bigg)e_2,\end{aligned}\tag{4.9}$$

where $|\zeta_1|<r_1,|\zeta_2|<r_2$ and $|\eta_1|<r_2,|\eta_2|<r_1$.
Assume that $V(z)=V(z_1,z_2)$ is a solution of the system (E). Then V is analytic in the complex variables z_1,z_2 and hence analytic in the complex variables ζ_1,ζ_2. $V(\zeta_1,\zeta_2)$ can be expanded into a power series

$$V(\zeta_1,\zeta_2)=\sum_{i,j\geq 0}\alpha_{ij}\zeta_1^i\zeta_2^j,\qquad |\zeta_1|<r_1,|\zeta_2|<r_2,\tag{4.10}$$

where $\alpha_{ij}=a_{ij}e_1+b_{ij}e_2\in\mathbb{T},\quad a_{ij},b_{ij}\in\mathbb{C}(i_1)$.
Denote

$$\mathcal{G}:=D(0;r_1,r_2)\times D(0;r_2,r_1).$$

Let $V(z,u)$ be the analytic continuation of $V(z_1,z_2)$ into the domain $\mathcal{G}$. So we have the idempotent representation of the power series of $V(z,u)$

$$V(z,u)=\Bigg(\sum_{i,j\geq 0}a_{ij}\zeta_1^i\eta_1^j\Bigg)e_1+\Bigg(\sum_{i,j\geq 0}b_{ij}\eta_2^i\zeta_2^j\Bigg)e_2,\tag{4.11}$$

where $|\zeta_1|<r_1,|\zeta_2|<r_2$ and $|\eta_1|<r_2,|\eta_2|<r_1$.
The function $V(z,u)$ is $\mathbb{T}$-holomorphic in $(z,u)\in\mathcal{G}$.

Lemma 4.1.
If $V(z)$ is a solution of the system (E) then the analytic continuation $V(z,u)$ of $V(z)$ satisfies the following bicomplex Bers-Vekua equation

$$\frac{\partial V(z,u)}{\partial u} = \mathcal{C}(z,u)V^*(u,z), \qquad (F)$$

where $V^*(u,z)$ is the conjugate of $V(z,u)$.

Proof.
By hypothesis $V(z)$ is a solution of (E) and then (4.10) holds. Now we have to prove that $V(z,u)$ given by (4.11) satisfies the equation (F).
By definition we have

$$\begin{aligned}\partial_{z^*} &= \frac{1}{2}(\partial_{z_1} + i_2\partial_{z_2}) = \frac{1}{2}[(\partial_{\zeta_1} + \partial_{\zeta_2}) + i_2(-i_1\partial_{\zeta_1} + i_1\partial_{\zeta_2})] \\ &= \frac{1}{2}[\partial_{\zeta_1}(1 - i_1 i_2) + \partial_{\zeta_2}(1 + i_1 i_2)].\end{aligned}$$

This implies

$$\partial_{z^*} = e_1\partial_{\zeta_2} + e_2\partial_{\zeta_1}.$$

Therefore

$$\begin{aligned}\partial_{z^*}V(z) &= (e_1\partial_{\zeta_2} + e_2\partial_{\zeta_1})V(\zeta_1,\zeta_2) \\ &= \partial_{\zeta_2}\left(\sum_{i,j\geq 0} a_{ij}\zeta_1^i\zeta_2^j\right)e_1 + \partial_{\zeta_1}\left(\sum_{i,j\geq 0} b_{ij}\zeta_1^i\zeta_2^j\right)e_2 \\ &= \left(\sum_{i,j\geq 0} j a_{ij}\zeta_1^i\zeta_2^{j-1}\right)e_1 + \left(\sum_{i,j\geq 0} i b_{ij}\zeta_1^{i-1}\zeta_2^j\right)e_2. \end{aligned} \qquad (4.12)$$

From (4.10) we have

$$V^*(z) = [V(z)]^* = \left(\sum_{i,j\geq 0} b_{ij}\zeta_1^i\zeta_2^j\right)e_1 + \left(\sum_{i,j\geq 0} a_{ij}\zeta_1^i\zeta_2^j\right)e_2.$$

Hence

$$\mathcal{C}(z,z^*)V^*(z) = \left(\sum_{i,j\geq 0} c_{ij}\zeta_1^i\zeta_2^j\right)\left(\sum_{i,j\geq 0} b_{ij}\zeta_1^i\zeta_2^j\right)e_1 + \left(\sum_{i,j\geq 0} d_{ij}\zeta_1^i\zeta_2^j\right)\left(\sum_{i,j\geq 0} a_{ij}\zeta_1^i\zeta_2^j\right)e_2. \qquad (4.13)$$

Using the hypothesis that $V(z)$ is a solution of the system (E) and the expressions (4.12) and (4.13) we obtain

$$\begin{cases}\sum_{i,j\geq 0} j a_{ij}\zeta_1^i\zeta_2^{j-1} &= \left(\sum_{i,j\geq 0} c_{ij}\zeta_1^i\zeta_2^j\right)\left(\sum_{i,j\geq 0} b_{ij}\zeta_1^i\zeta_2^j\right), \\ \sum_{i,j\geq 0} i b_{ij}\zeta_1^{i-1}\zeta_2^j &= \left(\sum_{i,j\geq 0} d_{ij}\zeta_1^i\zeta_2^j\right)\left(\sum_{i,j\geq 0} a_{ij}\zeta_1^i\zeta_2^j\right),\end{cases} \qquad (4.14)$$

for all $|\zeta_1| < r_1, |\zeta_2| < r_2$.
Now we consider the equation (F). From (4.11) we have

$$V^*(u,z) = [V(u^*,z^*)]^* = \left(\sum_{i,j\geq 0} b_{ij}\zeta_1^i\eta_1^j\right)e_1 + \left(\sum_{i,j\geq 0} a_{ij}\eta_2^i\zeta_2^j\right)e_2.$$

Combining this representation of $V^*(u,z)$ with the idempotent representation of $\mathcal{C}(z,u)$ we get the idempotent representation of the right-hand side of (F)

$$\mathcal{C}(z,u)V^*(u,z) = \left(\sum_{i,j\geq 0} c_{ij}\zeta_1^i\eta_1^j\right)\left(\sum_{i,j\geq 0} b_{ij}\zeta_1^i\eta_1^j\right)e_1 + \left(\sum_{i,j\geq 0} d_{ij}\eta_2^i\zeta_2^j\right)\left(\sum_{i,j\geq 0} a_{ij}\eta_2^i\zeta_2^j\right)e_2. \tag{4.15}$$

On the other hand, since $V(z,u)$ is $\mathbb{T}-$holomorphic in bicomplex variable u, the left-hand side of (F) has the idempotent representation

$$\begin{aligned}\frac{\partial V(z,u)}{\partial u} &= \frac{\partial}{\partial \eta_1}\left(\sum_{i,j\geq 0} a_{ij}\zeta_1^i\eta_1^j\right)e_1 + \frac{\partial}{\partial \eta_2}\left(\sum_{i,j\geq 0} b_{ij}\eta_2^i\zeta_2^j\right)e_2 \\ &= \left(\sum_{i,j\geq 0} ja_{ij}\zeta_1^i\eta_1^{j-1}\right)e_1 + \left(\sum_{i,j\geq 0} ib_{ij}\eta_2^{i-1}\zeta_2^j\right)e_2. \end{aligned}\tag{4.16}$$

Using (4.14) we have

$$\begin{aligned}\sum_{i,j\geq 0} ja_{ij}\zeta_1^i\eta_1^{j-1} &= \left(\sum_{i,j\geq 0} c_{ij}\zeta_1^i\eta_1^j\right)\left(\sum_{i,j\geq 0} b_{ij}\zeta_1^i\eta_1^j\right), \\ \sum_{i,j\geq 0} ib_{ij}\eta_2^{i-1}\zeta_2^j &= \left(\sum_{i,j\geq 0} d_{ij}\eta_2^i\zeta_2^j\right)\left(\sum_{i,j\geq 0} a_{ij}\eta_2^i\zeta_2^j\right), \end{aligned}\tag{4.17}$$

for all $|\zeta_1| < r_1, |\zeta_2| < r_2$ and $|\eta_1| < r_2, |\eta_2| < r_1$.
From (4.15), (4.16) and (4.17) we have $V(z,u)$ satisfies the equation (F)

$$\frac{\partial V(z,u)}{\partial u} = \mathcal{C}(z,u)V^*(u,z).$$

Thus Lemma 4.1 is proved. □

Integral representation formula

If $V(z,u)$ is a $\mathbb{T}-$holomorphic function of z, u for $(z,u) \in \mathcal{G}$, satisfying the differential equation (F), then $V(z,z^*)$ is an analytic function of the complex variables (z_1,z_2) in $D(0;r_1,r_2)$, satisfying the system (E).
Our problem now is to derive a formula giving all the solutions of the equation (F),

$\mathbb{T}-$holomorphic in z and u for $(z,u) \in \mathcal{G}$.
We can now transform (F) as follows

$$\frac{\partial}{\partial u}\left[V(z,u) - \int_{u_0}^{u} \mathcal{C}(z,\tau)V^*(\tau,z)d\tau\right] = 0 \tag{4.18}$$

with u_0 is a fixed point in $D(0;r_2,r_1)$.
Denote

$$G(z,u) := V(z,u) - \int_{u_0}^{u} \mathcal{C}(z,\tau)V^*(\tau,z)d\tau.$$

For each $\mathbb{T}$-holomorphic function $G(z,u)$, denote the first and second idempotent components of the power series of G by $G_1(\zeta_1,\eta_1)$ and $G_2(\zeta_2,\eta_2)$, respectively. Then

$$\frac{\partial G}{\partial u} = \frac{\partial G_1}{\partial \eta_1}e_1 + \frac{\partial G_2}{\partial \eta_2}e_2.$$

$$\frac{\partial G}{\partial u} = 0 \Leftrightarrow \begin{cases} \dfrac{\partial G_1}{\partial \eta_1} & = 0, \\ \dfrac{\partial G_2}{\partial \eta_2} & = 0. \end{cases}$$

This implies that G_1 and G_2 do not depend on η_1 and η_2, respectively. So if the derivative of the function $G(z,u)$ with respect to u is equal to zero then G is a $\mathbb{T}-$holomorphic function of one bicomplex variable z. Therefore from (4.18) we have

$$V(z,u) - \int_{u_0}^{u} \mathcal{C}(z,\tau)V^*(\tau,z)d\tau = \varphi(z), \tag{4.19}$$

where $\varphi(z)$ is a $\mathbb{T}$-holomorphic function of z in $D(0;r_1,r_2)$.
Since the uniqueness of the idempotent representation of bicomplex-valued functions, each equation in bicomplex variables is equivalent to two equations in complex variables which have the same type as the original equation. So all statements in the following can be proved by using the results in complex analysis of I.N. Vekua [44], which we used in Chapter I for the complex form (1.22) of the equation (1.1).
We now pass from (4.19) to the adjoint equation

$$V^*(u,z) = \varphi^*(u) + \int_{z_0}^{z} \mathcal{C}^*(u,t)V(t,u)dt, \quad (z_0 = u_0^*).$$

This implies that

$$V^*(\tau,z) = \varphi^*(\tau) + \int_{z_0}^{z} \mathcal{C}^*(\tau,t)V(t,\tau)dt. \tag{4.20}$$

Substituting (4.20) into (4.19) we obtain an integral equation

$$V(z,u) - \int_{z_0}^{z} dt \int_{u_0}^{u} \mathcal{C}(z,\tau)\mathcal{C}^*(\tau,t)V(t,\tau)d\tau = \Phi(z,u), \tag{4.21}$$

where

$$\Phi(z,u) = \varphi(z) + \int_{u_0}^{u} \mathcal{C}(z,\tau)\varphi^*(\tau)d\tau. \tag{4.22}$$

Assume that we have the following idempotent representations

$$\begin{aligned}
&z = \zeta_1 e_1 + \zeta_2 e_2, \qquad z_0 = \zeta_1^0 e_1 + \zeta_2^0 e_2, \qquad u = \eta_1 e_2 + \eta_2 e_2,\\
&u_0 = \eta_1^0 e_2 + \eta_2^0 e_2, \quad t = \xi_1 e_1 + \xi_2 e_2, \qquad \tau = \mu_1 e_1 + \mu_2 e_2,\\
&V(z,u) = V_1(\zeta_1,\eta_1)e_1 + V_2(\zeta_2,\eta_2)e_2, \qquad \mathcal{C}(z,\tau) = \mathcal{C}_1(\zeta_1,\mu_1)e_1 + \mathcal{C}_2(\zeta_2,\mu_2)e_2,\\
&\Phi(z,u) = \Phi_1(\zeta_1,\eta_1)e_1 + \Phi_2(\zeta_2,\eta_2)e_2, \qquad \varphi(z) = \varphi_1(\zeta_1)e_1 + \varphi_2(\zeta_2)e_2.
\end{aligned}$$

By definition of bicomplex conjugation we get

$$\mathcal{C}^*(\tau,t) = [\mathcal{C}(\tau^*,t^*)]^* = \mathcal{C}_2(\mu_1,\xi_1)e_1 + \mathcal{C}_1(\mu_2,\xi_2)e_2.$$

Then the integral equation (4.21) is equivalent to the two following equations

$$V_1(\zeta_1,\eta_1) - \int_{\zeta_1^0}^{\zeta_1} d\xi_1 \int_{\eta_1^0}^{\eta_1} \mathcal{C}_1(\zeta_1,\mu_1)\mathcal{C}_2(\mu_1,\xi_1)V_1(\xi_1,\mu_1)d\mu_1 = \Phi_1(\zeta_1,\eta_1), \tag{4.23}$$

where $\Phi_1(\zeta_1,\eta_1) = \varphi_1(\zeta_1) + \int\limits_{\eta_1^0}^{\eta_1} \mathcal{C}_1(\zeta_1,\mu_1)\varphi_2(\mu_1)d\mu_1,$

and

$$V_2(\zeta_2,\eta_2) - \int_{\zeta_2^0}^{\zeta_2} d\xi_2 \int_{\eta_2^0}^{\eta_2} \mathcal{C}_2(\zeta_2,\mu_2)\mathcal{C}_1(\mu_2,\xi_2)V_2(\xi_2,\mu_2)d\mu_2 = \Phi_2(\zeta_2,\eta_2), \tag{4.24}$$

where $\Phi_2(\zeta_2,\eta_2) = \varphi_2(\zeta_2) + \int\limits_{\eta_2^0}^{\eta_2} \mathcal{C}_2(\zeta_2,\mu_2)\varphi_1(\mu_2)d\mu_2.$

The equations (4.23), (4.24) of the Volterra type in the complex domain have solutions of the forms, see [44],

$$V_1(\zeta_1,\eta_1) = \Phi_1(\zeta_1,\eta_1) + \int_{\zeta_1^0}^{\zeta_1} d\xi_1 \int_{\eta_1^0}^{\eta_1} \Gamma^1(\zeta_1,\eta_1,\xi_1,\mu_1)\Phi_1(\xi_1,\mu_1)d\mu_1,$$

$$V_2(\zeta_2,\eta_2) = \Phi_2(\zeta_2,\eta_2) + \int_{\zeta_2^0}^{\zeta_2} d\xi_2 \int_{\eta_2^0}^{\eta_2} \Gamma^2(\zeta_2,\eta_2,\xi_2,\mu_2)\Phi_2(\xi_2,\mu_2)d\mu_2,$$

where $\Gamma^1(\zeta_1,\eta_1,\xi_1,\mu_1)$ and $\Gamma^2(\zeta_2,\eta_2,\xi_2,\mu_2)$ are called the *main Vekua resolvents* of the integral equations (4.23) and (4.24), respectively.
Denote $\Gamma(z,u,t,\tau) = \Gamma^1(\zeta_1,\eta_1,\xi_1,\mu_1)e_1 + \Gamma^2(\zeta_2,\eta_2,\xi_2,\mu_2)e_2$.
Then a solution $V(z,u)$ of the equation (4.21) has the form

$$V(z,u) = \Phi(z,u) + \int_{z_0}^{z} dt \int_{u_0}^{u} \Gamma(z,u,t,\tau)\Phi(t,\tau)d\tau. \tag{4.25}$$

We call $\Gamma(z,u,t,\tau)$ the *main bicomplex resolvent* of the integral equation (4.21). $\Gamma(z,u,t,\tau)$ is a $\mathbb{T}$-holomorphic function in four variables z,u,t,τ and it satisfies the integral equation

$$\Gamma(z,u,t,\tau) = \mathcal{C}(z,\tau)\mathcal{C}^*(\tau,t) + \int_\tau^u d\eta \int_t^z \mathcal{C}(\xi,\tau)\mathcal{C}^*(\tau,t)\Gamma(z,u,\xi,\eta)d\xi.$$

Substituting (4.22) into (4.25) we obtain

$$V(z,u) = \varphi(z) + \int_{z_0}^z \Gamma_1(z,u,t,u_0)\varphi(t)dt + \int_{u_0}^u \Gamma_2(z,u,z_0,\tau)\varphi^*(\tau)d\tau, \tag{4.26}$$

where

$$\Gamma_1(z,u,t,\tau) = \int_\tau^u \Gamma(z,\zeta,t,\eta)d\eta,$$

$$\Gamma_2(z,u,t,\tau) = \mathcal{C}(z,\tau) + \int_t^z \mathcal{C}(\xi,\tau)\Gamma_1(z,u,\xi,\tau)d\xi = \frac{\Gamma(z,u,t,\tau)}{\mathcal{C}^*(\tau,t)}.$$

We call $\Gamma_1(z,u,t,\tau)$ and $\Gamma_2(z,u,t,\tau)$ the *first and second bicomplex resolvents*.
We have shown that if $V(z,u)$ is a solution of the equation (F) then it can be represented by the formula (4.26).
Furthermore we shall prove that for any $\mathbb{T}$-holomorphic function $\varphi(z)$ the formula (4.26) satisfies the equation (F). For this purpose, we shall show that every solution of the integral equation (4.21) also satisfies the differential equation (F).
Differentiating the two sides of (4.21) with respect to u we get

$$\frac{\partial V(z,u)}{\partial u} - \mathcal{C}(z,u)W(z,u) = 0, \tag{4.27}$$

where

$$W(z,u) = \int_{z_0}^{z} \mathcal{C}^*(u,t)V(t,u)dt + \varphi^*(u).$$

Now, it has to be shown that $W(z,u) = V^*(u,z)$ or $W^*(u,z) = V(z,u)$.
First of all,

$$W^*(u,z) = \int_{u_0}^{u} \mathcal{C}(z,\tau)V^*(\tau,z)d\tau + \varphi(z).$$

It follows that

$$\frac{\partial W^*(u,z)}{\partial u} - \mathcal{C}(z,u)V^*(u,z) = 0. \tag{4.28}$$

Subtracting the equation (4.28) from (4.27), we get

$$\frac{\partial U(z,u)}{\partial u} + \mathcal{C}(z,u)U^*(u,z) = 0, \tag{4.29}$$

where

$$U(z,u) = V(z,u) - W^*(u,z).$$

Since $V(z,u_0) = W^*(u_0,z) = \varphi(z)$ we have $U(z,u_0) = 0$. Thus $U(z,u)$ is a $\mathbb{T}$-holomorphic solution of the homogeneous differential equation (4.29), which satisfies the condition $U(z,u_0) = 0$. Such the solution satisfies the homogeneous integral equation

$$U(z,u) - \int_{z_0}^{z} dt \int_{u_0}^{u} \mathcal{C}(z,\tau)\mathcal{C}^*(\tau,t)U(t,\tau)d\tau = 0.$$

This implies $U \equiv 0$, i.e., $V(z,u) = W^*(u,z)$.
Thus formula (4.26) gives all solutions of the differential equation (F), $\mathbb{T}$-holomorphic in $(z,u) \in \mathcal{G}$. When $u = z^*$ in (4.26) we obtain a solution $V(z)$ of the system (E).
Summarising the above results we have the following theorem.

Theorem 4.14.
Consider the system (E)

$$\begin{cases} \partial_{z^*}V(z) = \mathcal{C}(z,z^*)V^*(z), & m \in \mathbb{N},\ z \in D(0;r_1,r_2), \\ \partial_{\overline{z}_1}V = \partial_{\overline{z}_2}V = 0, \end{cases}$$

where $\mathcal{C}(z,z^*)$ is a $\mathbb{T}$-valued function analytic in two complex variables z_1,z_2.
If $V(z)$ is a solution of the system (E) in $D(0;r_1,r_2)$ then it can be represented by integral operators as follows

$$V(z) = \varphi(z) + \int_{z_0}^{z} \Gamma_1(z,z^*,t,z_0^*)\varphi(t)dt + \int_{z_0^*}^{z^*} \Gamma_2(z,z^*,z_0,\tau)\varphi^*(\tau)d\tau, \qquad (4.30)$$

where $\varphi(z)$ is an arbitrary $\mathbb{T}$-holomorphic function in $D(0;r_1,r_2)$, Γ_1 and Γ_2 are the first and second bicomplex resolvents.
Conversely formula (4.30) gives all solutions of the system (E) in the discus $D(0;r_1,r_2)$.

Remark 4.1.
For a certain class of coefficients of the system (E), $\mathcal{C}(z) = \dfrac{m}{1-zz^*}$, $z \in D(0;1,1)$, we can use the formula (4.30) and the same method in Chapter I to determine the first and second bicomplex resolvents. Then we convert this formula to a form free of integrals. Therefore we also obtain a representation for solutions of the system (E) by differential operators

$$V(z) \;=\; \sum_{j=0}^{m} mB_j^m \left(\frac{z^*}{1-zz^*}\right)^{m-j} g^{(j)}(z) + \sum_{j=0}^{m-1}(m-j)B_j^m \frac{z^{m-j-1}}{(1-zz^*)^{m-j}}\overline{g^{(j)}(z)},$$

where $B_j^m \;=\; \dfrac{(2m-j-1)!}{j!(m-j)!}$ and $g \in H^{\mathbb{T}}_{D(0;1,1)}(m,0)$.
Here we denote by $H^{\mathbb{T}}_{D(0;1,1)}(m,0)$ the space of all $\mathbb{T}$-holomorphic functions in $D(0;1,1)$ satisfying

$$g^{(m-1)}(0) = \cdots = g'(0) = g(0) = 0.$$

4.3 Representation of bicomplex pseudo-analytic functions by differential operators

To represent bicomplex pseudo-analytic functions which obey the system (E) by differential operators we need the results of P. Berglez on second order partial differential equations [11] which are quoted in the following.

4.3.1 Representation theorems for solutions of second order equations after P. Berglez

Using suitable transformations we can reduce a formally hyperbolic differential equation of type

$$U_{\zeta_1\zeta_2}+\tilde{a}_1(\zeta_1,\zeta_2)U_{\zeta_1}+\tilde{a}_2(\zeta_1,\zeta_2)U_{\zeta_2}+\tilde{a}_3(\zeta_1,\zeta_2)U=0$$

to one of the two following equations

$$Lw:=w_{\zeta_1\zeta_2}+(\log A_n)_{\zeta_1}w_{\zeta_2}+B_n w=0, \tag{4.31}$$

$$\tilde{L}\tilde{w}:=\tilde{w}_{\zeta_1\zeta_2}+(\log\tilde{A}_{n'})_{\zeta_2}\tilde{w}_{\zeta_1}+\tilde{B}_{n'}\tilde{w}=0, \tag{4.32}$$

with A_n, $\tilde{A}_{n'}$, B_n, $\tilde{B}_{n'}$ are analytic functions in $\mathcal{D}\times\overline{\mathcal{D}}$.

Remark 4.2. *Using the transformation $\tilde{w}=A_n w$ the equation (4.31) becomes the equation (4.32) with $\tilde{A}_{n'}=\dfrac{1}{A_n}$, $\quad \tilde{B}_{n'}=B_n-(\log A_n)_{\zeta_1\zeta_2}$.*

Definition 4.13.
Let $K_n,\tilde{K}_{n'}$ be two differential operators in $\mathcal{D}\times\overline{\mathcal{D}}$ given by

$$K_n:=\sum_{j=0}^{n}a_j(\zeta_1,\zeta_2)\frac{\partial^j}{\partial\zeta_1^j},\qquad \tilde{K}_{n'}:=\sum_{j=0}^{n'}b_j(\zeta_1,\zeta_2)\frac{\partial^j}{\partial\zeta_2^j},\quad n,n'\in\mathbb{N},$$

where a_j, $j=0,1,\dots,n$, and b_j, $j=0,1,\dots,n'$, are analytic functions in $\mathcal{D}\times\overline{\mathcal{D}}$ satisfying $a_j\neq 0, b_j\neq 0$ in $\mathcal{D}\times\overline{\mathcal{D}}$.
If $K_n g$, for $g(\zeta_1)\in H(\mathcal{D})$, is a solution of the equation (4.31) then we call K_n a $\mathcal{B}_I^n$-operator for the equation (4.31).
If $\tilde{K}_{n'}h$, for $h(\zeta_2)\in H(\overline{\mathcal{D}})$, is a solution of the equation (4.31) then we call $\tilde{K}_{n'}$ a $\mathcal{B}_{II}^{n'}$-operator for the equation (4.32).

Theorem 4.15 (P. Berglez)**.**
For the equation (4.31) there exists a $\mathcal{B}_I^n$-operator $K_n, n\in\mathbb{N}$, if and only if with

$$A_{j-1}=A_jB_j,\qquad B_{j-1}=B_j+(\log A_jB_j)_{\zeta_1\zeta_2},\quad j=n,n-1,\dots,1$$

the condition $B_0 \equiv 0$ in $\mathcal{D} \times \overline{\mathcal{D}}$ is satisfied.
The operator K_n is then given by

$$K_n = F_{n-1}F_{n-2}\dots F_0 \ \textit{ with } \ F_j = \frac{\partial}{\partial \zeta_1} + (\log A_j)_{\zeta_1}, \ j = 0,1,\dots,n-1.$$

Theorem 4.16 (P. Berglez)**.**
For the equation (4.32), there exists a $\mathcal{B}_{II}^{n'}$-operator $\tilde{K}_{n'}, n' \in \mathbb{N}$ if and only if with

$$\tilde{A}_{j-1} = \tilde{A}_j\tilde{B}_j, \qquad \tilde{B}_{j-1} = \tilde{B}_j + (\log \tilde{A}_j\tilde{B}_j)_{\zeta_1\zeta_2}, \quad j = n', n'-1, \dots, 1$$

the condition $\tilde{B}_0 \equiv 0$ in $\mathcal{D} \times \overline{\mathcal{D}}$ is satisfied.
The operator $\tilde{K}_{n'}$ is then given by

$$\tilde{K}_{n'} = \tilde{A}_{n'}\tilde{F}_{n'-1}\tilde{F}_{n'-2}\dots\tilde{F}_0 \ \textit{ with } \ \tilde{F}_j = \frac{\partial}{\partial \zeta_2} + (\log \tilde{A}_j)_{\zeta_2}, \ j = 0,1,\dots,n'-1.$$

Theorem 4.17 (P. Berglez)**.**
If there exist a $\mathcal{B}_I^n$-operator K_n and a $\mathcal{B}_{II}^{n'}$-operator $\tilde{K}_{n'}$ for the equation (4.31) then for all solutions w of (4.31) defined in $\mathcal{D} \times \overline{\mathcal{D}}$ there exist functions $g \in H(\mathcal{D})$ and $h \in H(\overline{\mathcal{D}})$ such that

$$w = K_n g + \tilde{K}_{n'} h.$$

4.3.2 Representation theorem for bicomplex pseudo-analytic functions

We consider the system (E)

$$\begin{cases} \partial_{z^*} V = \mathcal{C} V^*, \\ \partial_{\overline{z}_1} V = \partial_{\overline{z}_2} V = 0, \end{cases}$$

where $z \in D(0; r_1, r_2)$ and $\mathcal{C}$ is a bicomplex-valued function analytic in two variables z_1, z_2. Denote the idempotent representations of the functions $\mathcal{C}(z, z^*)$ and $V(z)$ by

$$\mathcal{C} = \mathcal{C}_1 e_1 + \mathcal{C}_2 e_2, \qquad V = V_1 e_1 + V_2 e_2.$$

Since $\partial_{z^*} = \partial_{\zeta_2} e_1 + \partial_{\zeta_1} e_2$, the first equation of the system (E) becomes

$$\begin{cases} \partial_{\zeta_2} V_1 = \mathcal{C}_1 V_2, \\ \partial_{\zeta_1} V_2 = \mathcal{C}_2 V_1. \end{cases} \tag{4.33}$$

Thus, finding solutions $V(z_1, z_2)$ of the system (E) is equivalent to finding analytic solutions (V_1, V_2) of the system (4.33).
From the system (4.33), V_1 is a solution of the following second order differential equation

$$\partial_{\zeta_1\zeta_2} V_1 - \frac{\partial_{\zeta_1}\mathcal{C}_1}{\mathcal{C}_1}\partial_{\zeta_2} V_1 - \mathcal{C}_1\mathcal{C}_2 V_1 = 0. \tag{4.34}$$

The following theorem gives a condition on the coefficients $\mathcal{C}$ such that all bicomplex pseudo-analytic functions satisfying the system (E) can be represented by differential operators.

Theorem 4.18.
If the coefficient $\mathcal{C}$ in the system (E) satisfies the condition

$$m^2(\log\mathcal{C})_{zz^*} = (1+2ki_1i_2)\mathcal{C}\mathcal{C}^*, \quad \text{with } k\in\mathbb{N} \text{ and } p := \sqrt{k^2+m^2}\in\mathbb{N}, \tag{4.35}$$

then the solutions of the system (E) can be represented by differential operators of Bauer-type.
An idempotent representation of a solution $V(z)$ of the system (E) is then given by

$$V(z) = \left[L_{p+k}f + (\tilde{L}_{p-k-1}f)^*\right]e_1 + \frac{1}{\mathcal{C}^*}\left[(L_{p+k}f)^* + \tilde{L}_{p-k-1}f\right]_z e_2, \tag{4.36}$$

where f is a $\mathbb{T}$-holomorphic function in $D(0;r_1,r_2)$ and

$$L_{p+k} = T_{p+k-1}T_{p+k-2}\dots T_0, \qquad \tilde{L}_{p-k-1} = \mathcal{C}^*\tilde{T}_{p-k-2}\tilde{T}_{p-k-3}\dots\tilde{T}_0$$

with

$$T_j = \partial_z + [\log(C^{p+k-j-1}(C^*)^{p+k-j})]_z, \quad j=0,1,\dots,p+k-1,$$
$$\tilde{T}_j = \partial_z + [\log(C^{p-k-j-1}(C^*)^{p-k-j})]_z, \quad j=0,1,\dots,p-k-2.$$

Proof.
Using the idempotent representation of $\mathcal{C}$ and the fact that $i_1i_2 = e_1 - e_2$, we can rewrite the condition (4.35) as follows

$$\begin{cases} m^2(\log\mathcal{C}_1)_{\zeta_1\zeta_2} = (1+2k)\mathcal{C}_1\mathcal{C}_2, \\ m^2(\log\mathcal{C}_2)_{\zeta_1\zeta_2} = (1-2k)\mathcal{C}_1\mathcal{C}_2. \end{cases} \tag{4.37}$$

It is easy to see that the bicomplex pseudo-analytic functions satisfying the system (E) can be represented by differential operators if and only if the solutions of the equation (4.34) can be represented by differential operators. So we shall show that with conditions (4.37), all solutions of the equation (4.34) can be represented by differential operators of Bauer-type.
Applying Theorem 4.15 and Theorem 4.16 we can point out that with (4.37) the conditions $B_0 \equiv 0$ and $\tilde{B}_0 \equiv 0$ are satisfied.
For the second order differential equation (4.34) we have

$$A_n = \frac{1}{\mathcal{C}_1}, \qquad B_n = -\mathcal{C}_1\mathcal{C}_2,$$
$$\tilde{A}_{n'} = \frac{1}{A_n} = \mathcal{C}_1, \qquad \tilde{B}_{n'} = B_n - (\log A_n)_{\zeta_1\zeta_2} = -\mathcal{C}_1\mathcal{C}_2 - (\log\frac{1}{\mathcal{C}_1})_{\zeta_1\zeta_2}.$$

From (4.37) we have

$$(\log\mathcal{C}_1)_{\zeta_1\zeta_2} = \frac{(1+2k)\mathcal{C}_1\mathcal{C}_2}{m^2}, \qquad (\log\mathcal{C}_2)_{\zeta_1\zeta_2} = \frac{(1-2k)\mathcal{C}_1\mathcal{C}_2}{m^2}.$$

Assume that for $1 \le j < n$ we have

$$A_{n-j} = M_{n-j}\mathcal{C}_1^{j-1}\mathcal{C}_2^{j}, \quad B_{n-j} = \left[-1 + \frac{\sum_{i=1}^{j} i(1-2k)}{m^2} + \frac{\sum_{i=1}^{j-1} i(1+2k)}{m^2}\right]\mathcal{C}_1\mathcal{C}_2.$$

We shall prove that

$$A_{n-(j+1)} = M_{n-(j+1)}\mathcal{C}_1^{j}\mathcal{C}_2^{j+1}, \quad B_{n-(j+1)} = \left[-1 + \frac{\sum_{i=1}^{j+1} i(1-2k)}{m^2} + \frac{\sum_{i=1}^{j} i(1+2k)}{m^2}\right]\mathcal{C}_1\mathcal{C}_2.$$

Indeed, we have

$$A_{n-(j+1)} = A_{n-j}B_{n-j} = M_{n-(j+1)}\mathcal{C}_1^{j}\mathcal{C}_2^{j+1},$$

with $\quad M_{n-(j+1)} = M_{n-j}\left[-1 + \frac{\sum_{i=1}^{j} i(1-2k)}{m^2} + \frac{\sum_{i=1}^{j-1} i(1+2k)}{m^2}\right].$

On the other hand we have

$$\begin{aligned} B_{n-(j+1)} &= B_{n-j} + \left[\log A_{n-(j+1)}\right]_{\zeta_1\zeta_2} = B_{n-j} + j(\log\mathcal{C}_1)_{\zeta_1\zeta_2} + (j+1)(\log\mathcal{C}_2)_{\zeta_1\zeta_2} \\ &= B_{n-j} + \frac{j(1+2k)\mathcal{C}_1\mathcal{C}_2}{m^2} + \frac{(j+1)(1-2k)\mathcal{C}_1\mathcal{C}_2}{m^2} \\ &= \left[-1 + \frac{\sum_{i=1}^{j+1} i(1-2k)}{m^2} + \frac{\sum_{i=1}^{j} i(1+2k)}{m^2}\right]\mathcal{C}_1\mathcal{C}_2. \end{aligned}$$

Thus we have proved that

$$A_j = M_j\mathcal{C}_1^{n-j-1}\mathcal{C}_2^{n-j}, \quad j = 0,1,\dots,n-1. \tag{4.38}$$

$$\begin{aligned} B_{n-j} &= \left[-1 + \frac{\sum_{i=1}^{j} i(1-2k)}{m^2} + \frac{\sum_{i=1}^{j-1} i(1+2k)}{m^2}\right]\mathcal{C}_1\mathcal{C}_2 \\ &= \frac{j^2 - 2kj - m^2}{m^2}\mathcal{C}_1\mathcal{C}_2, \qquad \text{for all} \quad 0 \le j \le n. \end{aligned}$$

The condition $B_0 \equiv 0$ is satisfied when

$$\begin{cases} j = n \\ j^2 - 2kj - m^2 = 0 \end{cases} \quad \Leftrightarrow \begin{cases} n^2 - 2kn - m^2 = 0 \\ n \in \mathbb{N} \end{cases} \quad \Leftrightarrow n = k + p,\ p = \sqrt{k^2 + m^2}.$$

This implies that the $\mathcal{B}_I^n$-operator K_n of the equation (4.34) exists and its order is $n=k+p$. Analogously we can prove that

$$\begin{aligned}\tilde{A}_j &= M'_j C_1^{n'-j+1} C_2^{n'-j},\\ \tilde{B}_{n'-j} &= \frac{j^2+2(k+1)j+2k+1-m^2}{m^2} C_1 C_2.\end{aligned} \tag{4.39}$$

Hence $\tilde{B}_0 \equiv 0$ if

$$\begin{cases} j = n' \in \mathbb{N}, \\ j^2+2(k+1)j+2k+1-m^2 = 0. \end{cases} \quad \Leftrightarrow n' = p-k-1,\ p = \sqrt{k^2+m^2}.$$

Therefore the $\mathcal{B}_{II}^{n'}$-operator $\tilde{K}_{n'}$ of the equation (4.34) exists and its order is $n' = p-k-1$. According to Theorem 4.17 a solution V_1 of the equation (4.34) is given by

$$V_1(\zeta_1,\zeta_2) = K_{p+k} g(\zeta_1) + \tilde{K}_{p-k-1} h(\zeta_2),$$

where $g \in H(\mathcal{D})$ and $h \in H(\overline{\mathcal{D}})$ and

$$\begin{aligned} K_{p+k} &= F_{p+k-1}F_{p+k-2}\dots F_0 \ \text{ with } \ F_j = \frac{\partial}{\partial\zeta_1} + (\log A_j)_{\zeta_1},\ j=0,1,\dots,p+k-1,\\ \tilde{K}_{p-k-1} &= \mathcal{C}_1 \tilde{F}_{p-k-2}\tilde{F}_{p-k-3}\dots\tilde{F}_0 \ \text{ with } \ \tilde{F}_j = \frac{\partial}{\partial\zeta_2} + (\log\tilde{A}_j)_{\zeta_2},\ j=0,1,\dots,p-k-2. \end{aligned}$$

Using the expressions (4.38) and (4.39) we have

$$\begin{aligned} F_j &= \frac{\partial}{\partial\zeta_1} + (\log \mathcal{C}_1^{p+k-j-1}\mathcal{C}_2^{p+k-j})_{\zeta_1},\\ \tilde{F}_j &= \frac{\partial}{\partial\zeta_2} + (\log \mathcal{C}_1^{p-k-j}\mathcal{C}_2^{p-k-j-1})_{\zeta_2}. \end{aligned}$$

Denote $f(z) = g(\zeta_1)e_1 + h(\zeta_2)e_2$ then f is a bicomplex-valued function and $\mathbb{T}$-holomorphic in $D(0;r_1,r_2)$, and denote

$$L_{p+k} = T_{p+k-1}T_{p+k-2}\dots T_0, \qquad \tilde{L}_{p-k-1} = \mathcal{C}^* \tilde{T}_{p-k-2}\tilde{T}_{p-k-3}\dots\tilde{T}_0,$$

with

$$\begin{aligned} T_j &= \partial_z + [\log(C^{p+k-j-1}(C^*)^{p+k-j})]_z, \quad j=0,1,\dots,p+k-1,\\ \tilde{T}_j &= \partial_z + [\log(C^{p-k-j-1}(C^*)^{p-k-j})]_z, \quad j=0,1,\dots,p-k-2. \end{aligned}$$

Then $Ve_1 := V_1 e_1$ can be rewritten as follows

$$Ve_1 = \left[L_{p+k} f + (\tilde{L}_{p-k-1} f)^*\right] e_1.$$

It is easy to see that if Ve_1 is given then Ve_2 can be determined by Ve_1. Indeed we have

$$\begin{aligned}
&\partial_{z^*}(Ve_1) = \mathcal{C}(V^*)e_1 = \mathcal{C}(Ve_2)^* \\
\Rightarrow\ &\mathcal{C}^*(Ve_2) = \left[\partial_{z^*}(Ve_1)\right]^* \\
\Rightarrow\ &Ve_2 = \frac{1}{\mathcal{C}^*}\partial_z(Ve_1)^*.
\end{aligned}$$

Therefore a solution $V(z)$ of the system (E) is given by

$$V = \left[L_{p+k}f + (\tilde{L}_{p-k-1}f)^*\right]e_1 + \frac{1}{\mathcal{C}^*}\left[(L_{p+k}f)^* + \tilde{L}_{p-k-1}f\right]_z e_2.$$

Thus Theorem 4.18 is proved. □

Corollary 4.3.
If the coefficient $\mathcal{C}$ of the system (E) satisfies the condition (4.35) with $k = 0$ then we have that $p = m$ and a solution $V(z)$ of the system (E) is given by

$$V(z) = L_m f + \frac{1}{\mathcal{C}^*}\left(L_m f\right)^*_z, \tag{4.40}$$

where f is a $\mathbb{T}$-holomorphic function in $D(0;r_1,r_2)$ and

$$L_m = T_{m-1}T_{m-2}\dots T_0,$$

with

$$T_j = \partial_z + (\log C^{m-j-1}(C^*)^{m-j})_z, \quad j = 0,1,\dots,m-1.$$

4.4 Applications

4.4.1 Representation of solutions of the Dirac equation on a pseudo-sphere

In [36] the Dirac operator on the Poincaré disk is given by

$$D_k = \begin{pmatrix} 0 & 2(1-\xi\overline{\xi})\partial_\xi - (2k-1)\overline{\xi} \\ 2(1-\xi\overline{\xi})\partial_{\overline{\xi}} + (2k+1)\xi & 0 \end{pmatrix}, \tag{4.41}$$

where $\xi = x + i_1 y \in \mathbb{C}(i_1) = \mathbb{C},\ k \in \mathbb{R},\ |\xi| < 1$.
Consider the Dirac equation

$$(m - D_k)w = 0,\ m \in \mathbb{N}, \tag{4.42}$$

where the Dirac operator D_k is given in (4.41) and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{C}^2$.
The equation (4.42) is equivalent to the following system

$$\begin{cases} 2(1-\xi\overline{\xi})\partial_\xi w_2 - (2k-1)\overline{\xi} w_2 = m w_1, \\ \\ 2(1-\xi\overline{\xi})\partial_{\overline{\xi}} w_1 + (2k+1)\xi w_1 = m w_2. \end{cases} \tag{4.43}$$

We consider the first equation of the system (4.43)

$$2(1-\xi\overline{\xi})\partial_\xi w_2 - (2k-1)\overline{\xi} w_2 = m w_1,$$

$$\Rightarrow \partial_\xi w_2 - \frac{2k-1}{2}\frac{\overline{\xi}}{1-\xi\overline{\xi}} w_2 = \frac{m}{2(1-\xi\overline{\xi})} w_1. \tag{4.44}$$

Let $\varphi(\xi) := -\frac{2k-1}{2}\frac{\overline{\xi}}{1-\xi\overline{\xi}}$ and then define

$$\Phi(\xi) := (1-\xi\overline{\xi})^{\frac{2k-1}{2}}.$$

This implies that the function $\Phi(\xi)$ has a property $\frac{\partial_\xi \Phi}{\Phi} = \varphi(\xi)$.
The equation (4.44) reads

$$\begin{aligned} \partial_\xi w_2 + \frac{\partial_\xi \Phi}{\Phi} w_2 &= \frac{m w_1}{2(1-\xi\overline{\xi})}, \\ \Rightarrow \Phi\partial_\xi w_2 + \partial_\xi \Phi w_2 &= \frac{\Phi m w_1}{2(1-\xi\overline{\xi})}, \\ \Rightarrow \partial_\xi(\Phi w_2) &= \frac{(1-\xi\overline{\xi})^{\frac{2k-1}{2}} m w_1}{2(1-\xi\overline{\xi})}. \end{aligned}$$

We can rewrite the last equation as follows

$$\partial_\xi\left[(1-\xi\overline{\xi})^{\frac{2k-1}{2}} w_2\right] = \frac{m w_1}{2}(1-\xi\overline{\xi})^{\frac{2k-3}{2}}. \tag{4.45}$$

Analogously, the second equation of the system (4.43) can be rewritten in the form

$$\partial_{\overline{\xi}}\left[(1-\xi\overline{\xi})^{-\left(\frac{2k+1}{2}\right)} w_1\right] = \frac{m w_2}{2}(1-\xi\overline{\xi})^{-\left(\frac{2k+3}{2}\right)}. \tag{4.46}$$

In the two equations (4.45) and (4.46), denote

$$\begin{cases} V_2 &= (1-\xi\overline{\xi})^{\frac{2k-1}{2}} w_2, \\ V_1 &= (1-\xi\overline{\xi})^{-\left(\frac{2k+1}{2}\right)} w_1. \end{cases} \tag{4.47}$$

The system (4.43) now becomes

$$\begin{cases} \partial_{\overline{\xi}} V_1 = \dfrac{m}{(1-\xi\overline{\xi})^{1+2k}} V_2, \\ \partial_{\xi} V_2 = \dfrac{m}{(1-\xi\overline{\xi})^{1-2k}} V_1. \end{cases} \tag{4.48}$$

Since the coefficients $\dfrac{m}{(1-\xi\overline{\xi})^{1+2k}}$ and $\dfrac{m}{(1-\xi\overline{\xi})^{1-2k}}$ are analytic in variables $x,\ y$, this system always has a solution (V_1,V_2) analytic in variables $x,\ y$. Continue this system analytically into the complex domain of the variables

$$\zeta_1 = x + i_1 y, \qquad \zeta_2 = x - i_1 y,$$

we have a system of the form

$$\begin{cases} \partial_{\zeta_2} V_1 = \dfrac{m}{(1-\zeta_1\zeta_2)^{1+2k}} V_2, \\ \\ \partial_{\zeta_1} V_2 = \dfrac{m}{(1-\zeta_1\zeta_2)^{1-2k}} V_1. \end{cases} \tag{4.49}$$

Denote

$$\begin{aligned} &z = z_1 e_1 + z_2 e_2 \quad \text{with} \quad z_1 = \zeta_1 - i_1\zeta_2,\ z_2 = \zeta_1 + i_1\zeta_2, \\ &V(z) = V(\zeta_1,\zeta_2) = V_1 e_1 + V_2 e_2, \end{aligned}$$

then V becomes a bicomplex-valued function which is a solution of the system

$$\begin{cases} \partial_{z^*} V = \left[\dfrac{m}{(1-zz^*)^{1+2k}} e_1 + \dfrac{m}{(1-zz^*)^{1-2k}} e_2\right] V^*, \\ \partial_{\overline{z}_1} V = \partial_{\overline{z}_2} V = 0, \end{cases} \tag{4.50}$$

in $D(0;1,1)$.
Therefore if we can solve the system (4.50) then we obtain all solutions of the Dirac equation (4.42).
We now consider the coefficient $\mathcal{C}$ in the case of the system (4.50)

$$\mathcal{C} = \mathcal{C}_1 e_1 + \mathcal{C}_2 e_2 = \frac{m}{(1-zz^*)^{1+2k}} e_1 + \frac{m}{(1-zz^*)^{1-2k}} e_2.$$

It is easy to check that this coefficient satisfies the condition (4.35). According to Theorem 4.18, all bicomplex pseudo-analytic functions which are solutions of the system (4.50) can be represented by differential operators of Bauer-type

$$V(z) = \left[L_{p+k} f + (\tilde{L}_{p-k-1} f)^*\right] e_1 + \frac{1}{\mathcal{C}^*}\left[(L_{p+k} f)^* + \tilde{L}_{p-k-1} f\right]_z e_2.$$

In this problem we can calculate the two operators L_{p+k} and $\tilde{L}_{p-k-1}$, and then get an explicitly representation for the solution $V(z)$ of the system (4.50).
Ve_1 is a solution of an equation

$$\partial_{zz^*}(Ve_1) - \frac{\mathcal{C}_z}{\mathcal{C}}\partial_{z^*}(Ve_1) - \mathcal{C}\mathcal{C}^*(Ve_1) = 0$$

where $Ve_1 = V_1$ is the first idempotent component of V.
Since $e_1e_2 = 0$, we only care about the first idempotent components of $\frac{\mathcal{C}_z}{\mathcal{C}}$ and $\mathcal{C}\mathcal{C}^*$ in the above equation. Therefore we conclude that Ve_1 is a solution of the following equation

$$\partial_{zz^*}(Ve_1) - (1+2k)\frac{z^*}{1-zz^*}\partial_{z^*}(Ve_1) - \frac{m^2 Ve_1}{(1-zz^*)^2} = 0. \tag{4.51}$$

On the other hand, from the formula (4.36) we have

$$Ve_1 = \left[L_{p+k}f + (\tilde{L}_{p-k-1}f)^*\right]e_1. \tag{4.52}$$

Now we determine the two operators L_{p+k} and $\tilde{L}_{p-k-1}$. We have

$$T_j = \partial_z + [\log(C^{p+k-j-1}C^{*p+k-j})]_z =: \partial_z + c_j\frac{z^*}{1-zz^*},\ c_j \in \mathbb{T},\ j = 0,1,\dots,p+k-1,$$
$$\tilde{T}_j = \partial_z + [\log(C^{p-k-j-1}C^{*p-k-j})]_z =: \partial_z + d_j\frac{z^*}{1-zz^*},\ d_j \in \mathbb{T},\ j = 0,1,\dots,p-k-2$$

where c_j, d_j are coefficients of $\frac{z^*}{1-zz^*}$ in $T_j, \tilde{T}_j$, respectively whose idempotent components are integer numbers.
Applying Lemma 1.1 for the operators L_{p+k} and $\tilde{L}_{p-k-1}$ we obtain the result that $L_{p+k}f$ and $\tilde{L}_{p-k-1}f$ have the following forms

$$L_{p+k}f = T_{p+k-1}T_{p+k-2}\dots T_0 = \sum_{j=0}^{p+k}\tilde{c}_j\left(\frac{z^*}{1-zz^*}\right)^{p+k-j} f^{(j)},$$
$$\tilde{L}_{p-k-1}f = \mathcal{C}^*\tilde{T}_{p-k-2}\tilde{T}_{p-k-3}\dots\tilde{T}_0 = \mathcal{C}^*\sum_{j=0}^{p-k-1}\tilde{d}_j\left(\frac{z^*}{1-zz^*}\right)^{p-k-1-j} f^{(j)},$$

where $\tilde{c}_{p+k} = 1, \tilde{d}_{p-k-1} = 1$ and $\tilde{c}_j, j = 0,1,\dots,p+k-1,\ \tilde{d}_j,\ j = 0,1,\dots,p-k-2$, are unknown bicomplex coefficients with idempotent components are integer numbers.
Substituting these expressions into (4.52) we have

$$Ve_1 = \sum_{j=0}^{p+k}\tilde{c}_j\left(\frac{z^*}{1-zz^*}\right)^{p+k-j} f^{(j)}(z)e_1 + \mathcal{C}\sum_{j=0}^{p-k-1}\tilde{d}_j^*\left(\frac{z}{1-zz^*}\right)^{p-k-1-j}[f^{(j)}(z)]^*e_1. \tag{4.53}$$

In the formula (4.53), we only care about the first idempotent components of $\tilde{c}_j, \tilde{d}_j^*$ and $\mathcal{C}$. So without loss of generality we can assume that $\tilde{c}_j, \tilde{d}_j^*$ are integer numbers and Ve_1 can be rewritten as

$$Ve_1 = \sum_{j=0}^{p+k} P_j \left(\frac{z^*}{1-zz^*}\right)^{p+k-j} f^{(j)}(z)e_1 + \sum_{j=0}^{p-k-1} Q_j \frac{z^{p-k-1-j}}{(1-zz^*)^{p+k-j}} [f^{(j)}(z)]^* e_1$$

where f is a $\mathbb{T}-$holomorphic function in $D(0;r_1,r_2)$, $P_{p+k}=1, Q_{p-k-1}=m$ and $P_j, j= 0,1,\dots,p+k-1, Q_j, j=0,1,\dots,p-k-2$, are unknown integer coefficients.
For the convenience we denote $r := p+k$ and $s := p-k$. Then

$$Ve_1 = \widehat{W} + \tilde{W}, \tag{4.54}$$

where

$$\widehat{W} = \sum_{j=0}^{r} P_j \left(\frac{z^*}{1-zz^*}\right)^{r-j} f^{(j)}(z)e_1, \qquad \widetilde{W} = \sum_{j=0}^{s-1} Q_j \frac{z^{s-1-j}}{(1-zz^*)^{r-j}} [f^{(j)}(z)]^* e_1.$$

Since Ve_1 is a solution of the equation (4.51), $\widehat{W}$ and $\widetilde{W}$ are solutions of the following equation

$$\partial_{zz^*} W - (r-s+1)\frac{z^*}{1-zz^*}\partial_{z^*} W - \frac{rs}{(1-zz^*)^2} W = 0. \tag{4.55}$$

$$\widehat{W} = P_0 \frac{z^{*r}}{(1-zz^*)^r} f(z)e_1 + \sum_{j=1}^{r-1} P_j \left(\frac{z^*}{1-zz^*}\right)^{r-j} f^{(j)}(z)e_1 + P_r f^{(r)}(z)e_1,$$

$$\partial_{z^*}\widehat{W} = P_0 r \frac{z^{*(r-1)}}{(1-zz^*)^{r+1}} f(z)e_1 + \sum_{j=1}^{r-1} P_j (r-j) \frac{z^{*(r-j-1)}}{(1-zz^*)^{r-j+1}} f^{(j)}(z)e_1,$$

$$\begin{aligned}\partial_{zz^*}\widehat{W} = {} & P_0 r(r+1) \frac{z^{*r}}{(1-zz^*)^{r+2}} f(z)e_1 \\ & + \sum_{j=1}^{r-1} (r-j+1)[(r-j)P_j + P_{j-1}] \frac{z^{*(r-j)}}{(1-zz^*)^{r-j+2}} f^{(j)}(z)e_1 \\ & + P_{r-1} \frac{1}{(1-zz^*)^2} f^{(r)}(z)e_1.\end{aligned}$$

Substituting the above expressions into the equation (4.55) we have an equality which holds for all $z \in D(0;1,1)$ and $f^{(j)}(z)e_1, j=0,1,\dots,r$

$$\begin{aligned} & \left[P_0 r(r+1) - (r-s+1)P_0 r - rsP_0\right] \frac{z^{*r}}{(1-zz^*)^{r+2}} f(z)e_1 \\ & + \sum_{j=1}^{r-1} \left\{(r-j+1)[(r-j)P_j + P_{j-1}] - (r-s+1)P_j(r-j) - rsP_j\right\} \frac{z^{*(r-j)}}{(1-zz^*)^{r-j+2}} f^{(j)}(z)e_1 \\ & + [P_{r-1} - rsP_r] \frac{1}{(1-zz^*)^2} f^{(r)}(z)e_1 = 0.\end{aligned}$$

Hence we get a system

$$\begin{cases} P_0 r(r+1) - (r-s+1)P_0 r - rsP_0 = 0, \\ (r-j+1)[(r-j)P_j + P_{j-1}] - (r-s+1)P_j(r-j) - rsP_j = 0,\ 1 \le j \le r-1, \\ P_{r-1} - rsP_r = 0. \end{cases} \tag{4.56}$$

Solving the system (4.56) we obtain

$$P_{j-1} = \frac{j(r+s-j)}{r-j+1}P_j,\ 1 \le j \le r,$$
$$\Rightarrow P_j = \frac{r!}{(s-1)!}\frac{(r+s-j-1)!}{j!(r-j)!}P_r,\ 0 \le j \le r-1.$$

Since $P_r = 1,\ r = p+k,\ s = p-k$, we get

$$P_j = \frac{(p+k)!}{(p-k-1)!}\frac{(2p-j-1)!}{j!(p+k-j)!},\ j = 0,1,\dots,p+k. \tag{4.57}$$

Analogously we have $\widetilde{W}^*$ is a solution of the equation

$$\partial_{zz^*}\widetilde{W}^* - (r-s+1)\frac{z}{1-zz^*}\partial_z\widetilde{W}^* - \frac{rs}{(1-zz^*)^2}\widetilde{W}^* = 0. \tag{4.58}$$

$$\widetilde{W}^* = Q_0\frac{z^{*(s-1)}}{(1-zz^*)^r}f(z)e_1 + \sum_{j=1}^{s-1}Q_j\frac{z^{*(s-j-1)}}{(1-zz^*)^{r-j}}f^{(j)}(z)e_1,$$

$$\partial_z\widetilde{W}^* = Q_0 r\frac{z^{*s}}{(1-zz^*)^{r+1}}f(z)e_1 + \sum_{j=1}^{s-1}[Q_j(r-j)+Q_{j-1}]\frac{z^{*(s-j)}}{(1-zz^*)^{r-j+1}}f^{(j)}(z)e_1$$
$$+ Q_{s-1}\frac{1}{(1-zz^*)^{r-s+1}}f^{(s)}(z)e_1,$$

$$\partial_{zz^*}\widetilde{W}^* = Q_0\left[\frac{rsz^{*(s-1)}(1-zz^*)}{(1-zz^*)^{r+2}} + \frac{r(r+1)zz^{*s}}{(1-zz^*)^{r+2}}\right]f(z)e_1$$
$$+ \sum_{j=1}^{s-1}[Q_j(r-j)+Q_{j-1}]\left[\frac{(s-j)z^{*(s-j-1)}(1-zz^*)}{(1-zz^*)^{r-j+2}} + \frac{(r-j+1)zz^{*(s-j)}}{(1-zz^*)^{r-j+2}}\right]f^{(j)}(z)e_1$$
$$+ Q_{s-1}\frac{(r-s+1)z}{(1-zz^*)^{r-s+2}}f^{(s)}(z)e_1.$$

Substituting these expressions into the equation (4.58) we obtain

$$Q_{j-1} = \frac{j(r+s-j)}{s-j}Q_j,\ 1 \le j \le s-1,$$
$$\Rightarrow Q_j = \frac{(s-1)!}{r!}\frac{(r+s-j-1)!}{j!(s-j-1)!}Q_{s-1},\ 0 \le j \le s-2.$$

Since $Q_{s-1} = m,\ r = p+k,\ s = p-k$, we get

$$Q_j = \frac{(p-k-1)!m}{(p+k)!}\frac{(2p-j-1)!}{j!(p-k-1-j)!},\ j = 0,1,\dots,p-k-1. \tag{4.59}$$

From (4.54), (4.59) and (4.57) we have

$$Ve_1 = \sum_{j=0}^{p+k} P_j\left(\frac{z^*}{1-zz^*}\right)^{p+k-j} f^{(j)}(z)e_1 + \sum_{j=0}^{p-k-1} Q_j\frac{z^{p-k-1-j}}{(1-zz^*)^{p+k-j}}[f^{(j)}(z)]^*e_1 \tag{4.60}$$

with $P_j, j = 0,1,\dots,p+k$, and $Q_j, j = 0,1,\dots,p-k-1$, are given in (4.57) and (4.59), respectively.
Therefore

$$Ve_2 = \frac{1}{C^*}\partial_z(Ve_1)^*$$

with

$$\begin{aligned}\partial_z(Ve_1)^* = &\sum_{j=0}^{p+k-1} P_j(p+k-j)\frac{z^{p+k-j-1}}{(1-zz^*)^{p+k-j+1}}[f^{(j)}(z)]^*e_2 \\ &+\sum_{j=0}^{p-k-1} Q_j z^{*(p-k-1-j)}\left[\frac{(p+k-j)z^*}{(1-zz^*)^{p+k-j+1}} f^{(j)}(z) + \frac{1}{(1-zz^*)^{p+k-j}} f^{(j+1)}(z)\right]e_2.\end{aligned} \tag{4.61}$$

Denote the second term on the right-hand side of (4.61) by

$$T := \sum_{j=0}^{p-k-1} Q_j z^{*(p-k-1-j)}\left[\frac{(p+k-j)z^*}{(1-zz^*)^{p+k-j+1}} f^{(j)}(z) + \frac{1}{(1-zz^*)^{p+k-j}} f^{(j+1)}(z)\right]e_2.$$

Then we can rewrite T as follows

$$\begin{aligned}T =&Q_0(p+k)\frac{z^{*(p-k)}}{(1-zz^*)^{p+k+1}} f(z)e_2 + \sum_{j=1}^{p-k-1}[Q_j(p+k-j)+Q_{j-1}]\frac{z^{*(p-k-j)}}{(1-zz^*)^{p+k-j+1}} f^{(j)}e_2 \\ &+Q_{p-k-1}\frac{1}{(1-zz^*)^{2k+1}} f^{(p-k)}(z)e_2.\end{aligned}$$

From the formula (4.59) we have

$$Q_j(p+k-j)+Q_{j-1} = \frac{(p-k)!m}{(p+k-1)!}\frac{(2p-j-1)!}{j!(p-k-j)!}, \quad \text{for} \quad 1 \le j \le p-k-1.$$

Hence

$$T = \sum_{j=0}^{p-k}\frac{(p-k)!m}{(p+k-1)!}\frac{(2p-j-1)!}{j!(p-k-j)!}\frac{z^{*(p-k-j)}}{(1-zz^*)^{p+k-j+1}} f^{(j)}e_2.$$

Therefore

$$Ve_2 = \sum_{j=0}^{p+k-1} R_j \frac{z^{p+k-1-j}}{(1-zz^*)^{p-k-j}} [f^{(j)}(z)]^* e_2 + \sum_{j=0}^{p-k} S_j \left(\frac{z^*}{1-zz^*}\right)^{p-k-j} f^{(j)}(z) e_2 \qquad (4.62)$$

with

$$\begin{aligned} R_j &= \frac{(p+k)!}{(p-k-1)!m} \frac{(2p-j-1)!}{j!(p+k-j-1)!}, \quad j = 0,1,\dots,p+k-1, \\ S_j &= \frac{(p-k)!}{(p-k-1)!} \frac{(2p-j-1)!}{j!(p-k-j)!}, \quad j = 0,1,\dots,p-k. \end{aligned} \qquad (4.63)$$

Theorem 4.19.
If V is a solution of the system (4.50) in $D(0;r_1,r_2)$ then V can be represented as follows

$$\begin{aligned} V(z) = & \left\{ \sum_{j=0}^{p+k} P_j \left(\frac{z^*}{1-zz^*}\right)^{p+k-j} f^{(j)}(z) + \sum_{j=0}^{p-k-1} Q_j \frac{z^{p-k-1-j}}{(1-zz^*)^{p+k-j}} [f^{(j)}(z)]^* \right\} e_1 \\ & + \left\{ \sum_{j=0}^{p+k-1} R_j \frac{z^{p+k-1-j}}{(1-zz^*)^{p-k-j}} [f^{(j)}(z)]^* + \sum_{j=0}^{p-k} S_j \left(\frac{z^*}{1-zz^*}\right)^{p-k-j} f^{(j)}(z) \right\} e_2 \end{aligned} \qquad (4.64)$$

with $z \in D(0;r_1,r_2)$, f is a $\mathbb{T}-$holomorphic function in $D(0;r_1,r_2)$, P_j, Q_j and R_j, S_j are given as follows

$$\begin{aligned} P_j &= \frac{(p+k)!}{(p-k-1)!} \frac{(2p-j-1)!}{j!(p+k-j)!}, \quad j = 0,1,\dots,p+k, \\ Q_j &= \frac{(p-k-1)!m}{(p+k)!} \frac{(2p-j-1)!}{j!(p-k-1-j)!}, \quad j = 0,1,\dots,p-k-1, \\ R_j &= \frac{(p+k)!}{(p-k-1)!m} \frac{(2p-j-1)!}{j!(p+k-j-1)!}, \quad j = 0,1,\dots,p+k-1, \\ S_j &= \frac{(p-k)!}{(p-k-1)!} \frac{(2p-j-1)!}{j!(p-k-j)!}, \quad j = 0,1,\dots,p-k. \end{aligned} \qquad (4.65)$$

Conversely for each $\mathbb{T}-$holomorphic function f in $D(0;r_1,r_2)$, formula (4.19) gives all solutions of the system (4.50) in $D(0;r_1,r_2)$.

If we denote the idempotent representation of the function f by $f(z) = f_1(\zeta_1)e_1 + f_2(\zeta_2)e_2$ then we obtain the following corollary.

Corollary 4.4.
Solutions w of the Dirac equation (4.42) are given by

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

with

$$w_1 = (1-\xi\overline{\xi})^{\frac{2k+1}{2}}\left[\sum_{j=0}^{p+k} P_j\left(\frac{\overline{\xi}}{1-\xi\overline{\xi}}\right)^{p+k-j} f_1^{(j)}(\xi) + \sum_{j=0}^{p-k-1} Q_j \frac{\xi^{p-k-1-j}}{(1-\xi\overline{\xi})^{p+k-j}}\overline{f_2^{(j)}(\xi)}\right]$$

and

$$w_2 = \frac{1}{(1-\xi\overline{\xi})^{\frac{2k-1}{2}}}\left[\sum_{j=0}^{p+k-1} R_j \frac{\overline{\xi}^{p+k-1-j}}{(1-\xi\overline{\xi})^{p-k-j}} f_1^{(j)}(\xi) + \sum_{j=0}^{p-k} S_j\left(\frac{\xi}{1-\xi\overline{\xi}}\right)^{p-k-j}\overline{f_2^{(j)}(\xi)}\right],$$

where P_j, Q_j and R_j, S_j are given in (4.65).

4.4.2 Generalized Weierstrass representation for surfaces

The generalization of the Weierstrass formulae to generic surfaces in $\mathbb{R}^3$ has been proposed by B.G. Konopelchenko (see, e.g., [28], [30]). It starts with the linear system (two-dimensional Dirac equation)

$$\partial_\xi \psi_1 = \mathcal{P}\psi_2, \qquad \partial_{\overline{\xi}}\psi_2 = -\mathcal{P}\psi_1, \tag{4.66}$$

where $\mathcal{P}(\xi,\overline{\xi})$ is a real-valued function, ψ_1, ψ_2 are, in general, complex functions of the complex variable $\xi = x + i_1 y$, and the bar denotes the complex conjugation.
Then one defines the three real-valued functions $X_1(\xi,\overline{\xi})$, $X_2(\xi,\overline{\xi})$ and $X_3(\xi,\overline{\xi})$ by the formulae

$$\begin{aligned} X_1 + i_1 X_2 &= 2i_1\int_{\xi_0}^{\xi}(\overline{\psi}_1^2 d\xi' - \overline{\psi}_2^2 d\overline{\xi}'), \\ X_1 - i_1 X_2 &= 2i_1\int_{\xi_0}^{\xi}(\psi_2^2 d\xi' - \psi_1^2 d\overline{\xi}'), \\ X_3 &= -2i_1\int_{\xi_0}^{\xi}(\psi_2\overline{\psi}_1 d\xi' + \psi_1\overline{\psi}_2 d\overline{\xi}'). \end{aligned} \tag{4.67}$$

By virtue of (4.66), the integrals (4.67) do not depend on the choice of the curve of integration in a simply connected domain. Then one treats $\xi, \overline{\xi}$ as local coordinates on a surface and (X_1, X_2, X_3) as coordinates of its immersion in $\mathbb{R}^3$. Formulae (4.67) induce a surface in $\mathbb{R}^3$ via the solutions of the system (4.66) with the Gaussian (K) and mean (H) curvatures

$$K = -\frac{\left[\log(|\psi_1|^2 + |\psi_2|^2)\right]_{\xi\overline{\xi}}}{(|\psi_1|^2 + |\psi_2|^2)^2}, \qquad H = \frac{\mathcal{P}(\xi,\overline{\xi})}{|\psi_1|^2 + |\psi_2|^2}.$$

The type of this system has recently appeared in many papers and surveys on the theory of representation for surfaces (see, e.g., [20], [25], [31], [35]). The study of surfaces and their dynamics is an important part of many interesting phenomena in mathematics and especially in physics such as surface waves, deformation of membranes, dynamics of vortex sheets, etc. Quantum field theory and statistical physics are also important applications of surfaces.

In the sequel we shall give a method to solve the system (4.66) with a special class of coefficients $\mathcal{P}$. We assume that $\mathcal{P}$ is analytic in the real variables x and y. If the real variables x and y are continued into a complex domain we obtain a function $\mathcal{P}(\eta_1,\eta_2)$ of the two complex variables

$$\eta_1 = x - i_1 y \quad \text{and} \quad \eta_2 = x + i_1 y.$$

Then the system (4.66) becomes

$$\begin{cases} \partial_{\eta_2} V_1 = \mathcal{P} V_2, \\ \partial_{\eta_1} V_2 = -\mathcal{P} V_1. \end{cases} \tag{4.68}$$

Denote $V = V_1 e_1 + V_2 e_2$. Since V_1, V_2 are holomorphic functions in variables η_1, η_2 then V is a solution of a system

$$\begin{cases} \partial_{z^*} V = i_1 i_2 \mathcal{P} V^*, \\ \partial_{\bar{z}_1} V = \partial_{\bar{z}_2} V = 0. \end{cases} \tag{4.69}$$

Assume further that $\mathcal{P}(\xi,\overline{\xi})$ given in (4.66) satisfies the condition

$$m^2 (\log \mathcal{P})_{\xi\overline{\xi}} = -\mathcal{P}^2,\ m \in \mathbb{N}^*. \tag{4.70}$$

Then the coefficient $\mathcal{P}(\eta_1,\eta_2)$ in (4.69) also satisfies

$$m^2 (\log \mathcal{P})_{zz^*} = -\mathcal{P}^2,\ m \in \mathbb{N}^*. \tag{4.71}$$

In this case $\mathcal{C} = i_1 i_2 \mathcal{P}$ satisfies the condition (4.35) with $k = 0$

$$m^2 (\log \mathcal{C})_{zz^*} = \mathcal{C}\mathcal{C}^*.$$

According to Corollary 4.3, all the solutions of the system (4.69) can be represented by differential operators of Bauer-type as follows

$$V = L_m f + \frac{1}{\mathcal{C}^*}(L_m f)_z^* = L_m f - \frac{1}{i_1 i_2 \mathcal{P}}(L_m f)_z^*,$$

where f is a $\mathbb{T}$-holomorphic function and

$$L_m = T_{m-1} T_{m-2} \dots T_0,$$

with

$$T_j = \partial_z + [\log(C^{m-j-1}(C^*)^{m-j})]_z, \quad j = 0,1,\dots,m-1.$$

Once V can be represented explicitly, V_1 and V_2 can be represented explicitly also.

Example 4.1.

Now we consider a special case of $\mathcal{P}$ which satisfies the condition (4.71) with $m = 1$

$$\mathcal{P}(\eta_1,\eta_2) = \frac{\sqrt{g_1'(\eta_1)g_2'(\eta_2)}}{1+g_1(\eta_1)g_2(\eta_2)},$$

where $\eta_1,\eta_2 \in \mathbb{C}$ are the two idempotent components of the bicomplex variable z and $g_1(\eta_1), g_2(\eta_2)$ are holomorphic functions satisfying

$$[1+g_1(\eta_1)g_2(\eta_2)]g_1'(\eta_1)g_2'(\eta_2) \neq 0.$$

A solution V of the system (4.69) corresponding to this coefficient $\mathcal{P}$ is then given by

$$V = L_1 f - \frac{1}{i_1 i_2 \mathcal{P}}(L_1 f)_z^* = f'(z) + (\log \mathcal{P})_z f + i_1 i_2 \mathcal{P} f^*.$$

We assume that the idempotent representation of the $\mathbb{T}$-holomorphic function f is

$$f = f_1(\eta_1)e_1 + f_2(\eta_2)e_2,$$

where $f_1(\eta_1)$ and $f_2(\eta_2)$ are the two holomorphic functions. Then a solution (V_1,V_2) of the system (4.68) is given by

$$\begin{aligned}
V_1 &= f_1'(\eta_1) + \left[\log \mathcal{P}(\eta_1,\eta_2)\right]_{\eta_1} f_1(\eta_1) + \mathcal{P}(\eta_1,\eta_2) f_2(\eta_2) \\
&= f_1'(\eta_1) + \left[\frac{1}{2}\frac{g_1''(\eta_1)}{g_1'(\eta_1)} - \frac{g_1'(\eta_1)g_2(\eta_2)}{1+g_1(\eta_1)g_2(\eta_2)}\right] f_1(\eta_1) + \frac{\sqrt{g_1'(\eta_1)g_2'(\eta_2)}}{1+g_1(\eta_1)g_2(\eta_2)} f_2(\eta_2), \\
V_2 &= f_2'(\eta_2) + \left[\log \mathcal{P}(\eta_1,\eta_2)\right]_{\eta_2} f_2(\eta_2) - \mathcal{P}(\eta_1,\eta_2) f_1(\eta_1) \\
&= f_2'(\eta_2) + \left[\frac{1}{2}\frac{g_2''(\eta_2)}{g_2'(\eta_2)} - \frac{g_1(\eta_1)g_2'(\eta_2)}{1+g_1(\eta_1)g_2(\eta_2)}\right] f_2(\eta_2) - \frac{\sqrt{g_1'(\eta_1)g_2'(\eta_2)}}{1+g_1(\eta_1)g_2(\eta_2)} f_1(\eta_1).
\end{aligned}$$

If we choose especially

$$f_1(\eta_1) = \frac{1}{\sqrt{g_1'(\eta_1)}}, \qquad f_2(\eta_2) = \frac{2g_2(\eta_2)}{\sqrt{g_2'(\eta_2)}}$$

then we obtain

$$V_1 = \frac{\sqrt{g_1'(\eta_1)}g_2(\eta_2)}{1+g_1(\eta_1)g_2(\eta_2)}, \qquad V_2 = \frac{\sqrt{g_2'(\eta_2)}}{1+g_1(\eta_1)g_2(\eta_2)}.$$

Moreover, if we choose

$$\begin{aligned}
&\eta_2 = \xi \in \mathbb{C},\ \eta_1 = \overline{\xi}, \quad \text{and} \\
&g_2 = \omega, \quad g_1 = \omega^*, \quad \text{where } \omega^*(\overline{\xi}) = \overline{\omega(\xi)},
\end{aligned}$$

then we have solutions V_1, V_2 of the system (see, e.g., [21], [26])

$$\begin{cases} \partial_\xi V_1 = \mathcal{P} V_2 \\ \partial_{\overline{\xi}} V_2 = -\mathcal{P} V_1 \end{cases} \qquad \text{where } \mathcal{P} = \frac{|\partial_\xi \omega|}{1+|\omega|^2} \tag{4.72}$$

in the form

$$V_1 = \varepsilon\omega \frac{(\partial_{\overline{\xi}} \overline{\omega})^{1/2}}{1+|\omega|^2}, \qquad V_2 = \varepsilon \frac{(\partial_\xi \omega)^{1/2}}{1+|\omega|^2}, \quad \varepsilon = \pm 1. \tag{4.73}$$

In this case the solutions of the system (4.72) have the property $\mathcal{P}(\xi, \overline{\xi}) = |V_1|^2 + |V_2|^2$, and the mean curvature of the corresponding surface is $H = 1$.

Example 4.2.

We consider an example given in [29] when $\mathcal{P} = \dfrac{1}{2\cosh x}$.
It is easy to check that $\mathcal{P}$ satisfies the condition (4.70). Then a solution (V_1, V_2) of the system (4.68) is given by

$$\begin{aligned} V_1 &= f_1'(\eta_1) + \left[\log \mathcal{P}(\eta_1, \eta_2)\right]_{\eta_1} f_1(\eta_1) + \mathcal{P}(\eta_1, \eta_2) f_2(\eta_2) \\ &= f_1'(\eta_1) - \frac{\sinh(\frac{\eta_1+\eta_2}{2})}{2\cosh(\frac{\eta_1+\eta_2}{2})} f_1(\eta_1) + \frac{1}{2\cosh(\frac{\eta_1+\eta_2}{2})} f_2(\eta_2), \\ V_2 &= f_2'(\eta_2) + \left[\log \mathcal{P}(\eta_1, \eta_2)\right]_{\eta_2} f_2(\eta_2) - \mathcal{P}(\eta_1, \eta_2) f_1(\eta_1) \\ &= f_2'(\eta_2) - \frac{\sinh(\frac{\eta_1+\eta_2}{2})}{2\cosh(\frac{\eta_1+\eta_2}{2})} f_2(\eta_2) - \frac{1}{2\cosh(\frac{\eta_1+\eta_2}{2})} f_1(\eta_1). \end{aligned}$$

If we choose $f_1(\eta_1) = \exp(-\frac{\eta_1}{2})$ and $f_2(\eta_2) = 2\exp(\frac{\eta_2}{2})$ and take into consideration that $\eta_1 = x - i_1 y,\ \eta_2 = x + i_1 y$, then we obtain

$$V_1 = \frac{1}{2\cosh x} \exp\left(\frac{i_1 y + x}{2}\right), \qquad V_2 = \frac{1}{2\cosh x} \exp\left(\frac{i_1 y - x}{2}\right).$$

This implies that the corresponding surface is given by

$$X_1 = -\frac{\sin y}{\cosh x}, \quad X_2 = -\frac{\cos y}{\cosh x}, \quad X_3 = -\tanh x$$

which is the unit sphere $X_1^2 + X_2^2 + X_3^2 = 1$.

5 CONCLUSIONS

In this thesis, some classes of the pseudo-analytic functions in complex and bicomplex variables which can be represented by differential operators have been studied. There are different ways to get the representation for the pseudo-analytic functions in complex variables which are solutions of a certain Bers-Vekua equation of type $w_{\overline{z}} - C\overline{w} = 0$, see, e.g., [9], [11], [44]. After P. Berglez, we have the necessary and sufficient condition on the coefficient C of the Bers-Vekua equation for which all solutions of this equation can be represented by differential operators of Bauer-type. Using this result we have obtained a Liouville system from which we can find the coefficients C such that all solutions of the corresponding Bers-Vekua equations can be represented by differential operators of Bauer-type. In the case of bicomplex variables, applying the theorems of P. Berglez concerning the existence of the operators of Bauer-type for the second order partial differential equations we also obtain a class of bicomplex pseudo-analytic functions which can be represented by differential operators.

In this work, we have derived the representations of some special classes of the pseudo-analytic functions in complex and bicomplex variables. One of the most interesting applications of such a representation in a complex variable is to solve boundary value problems [17]. The advantage of using the representation of solutions by differential operators is an explicitness of the solutions of the boundary value problems. In the case of a bicomplex variable, we have obtained the representation of a class of bicomplex pseudo-analytic functions which has some applications connecting to the physical problems.

Further work can be done on the studies of an efficient method to find a larger class of pseudo-analytic functions which can be represented by differential operators. In addition, it is interesting to answer the open question from this thesis, e.g., solving the more general boundary value problems for the pseudo-analytic functions which can be represented by differential operators of Bauer-type.

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