Dipl.-Math. Andreas Weinmann

Analysis of nonlinear geometric subdivision schemes on polyhedral meshes

DISSERTATION

zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften

Doktoratsstudium der Naturwissenschaften im Rahmen der Doktoratsschule „Mathematik und Wissenschaftliches Rechnen”

Technische Universität Graz

Betreuer:
Univ.-Prof. Dr. Johannes Wallner

Institut für Geometrie

Graz, im September 2010

EIDESSTATTLICHE ERKLÄRUNG

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst, andere als die angegebenen Quellen/Hilfsmittel nicht benutzt, und die den benutzten Quellen wörtlich und inhaltlich entnommene Stellen als solche kenntlich gemacht habe.

Graz, am ....

(Unterschrift)

Englische Fassung:

STATUTORY DECLARATION

I declare that I have authored this thesis independently, that I have not used other than the declared sources / resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

....

date (signature)

Abstract

This thesis is concerned with nonlinear geometric subdivision schemes and interpolatory multiscale transformations derived from nonlinear subdivision schemes.

The first subdivision scheme goes back to de Rham, who refines a polygon by an iterative linear procedure. De Rham shows that his univariate scheme converges and that the produced limit is C^1. Beginning in the early 90s, a framework for the systematic analysis of linear schemes has been developed. In the bivariate setting, the analysis splits into two parts: One considers on the one hand regular meshes, and on the other hand so-called irregular faces or vertices. This is necessary since a closed surface with non-zero Euler characteristic cannot be modeled by a regular mesh.

Linear subdivision schemes are computationally quite cheap and are therefore widely used to process vector-valued data. For data which are not defined in a vector space, but in some nonlinear geometry, say the space of diffusion tensors or the space of rigid motions, one tries to find modified, necessarily nonlinear, subdivision rules so as to apply to these data.

For the univariate case, results on the smoothness of such geometric, nonlinear, subdivision schemes were obtained by J. Wallner and N. Dyn using the method of proximity inequalities. On regular grids, multivariate nonlinear schemes whose dilation matrices are multiples of the identity were treated by P. Grohs. In this thesis we show a smoothness result for nonlinear, geometric schemes based on general dilation matrices, essentially stating that a geometric scheme is as smooth as a linear scheme it is in proximity with.

Furthermore, we consider geometric schemes acting on general (not necessarily regular) meshes. Here we deal with the singularities in the meshes and obtain, as a central result of this thesis, C^1 smoothness.

In general, the convergence statements for geometric, nonlinear, subdivision schemes only hold for dense enough input data. However, for a certain class of curve subdivision schemes acting in Cartan-Hadamard manifolds we show convergence for arbitrary input data.

Finally, we analyze interpolatory multiscale transformations based on subdivision. Linear transforms for regular grids were investigated by Donoho; he also proposed nonlinear geometric versions which act on manifold valued functions defined on Euclidean space. These transforms were analyzed by Grohs and Wallner. In this thesis we propose a transformation which acts on functions between manifolds. We characterize the decay of detail coefficients in terms of the Hölder-Zygmund smoothness of the corresponding function.

Kurzfassung

Diese Dissertation befasst sich mit nichtlinearen Unterteilungsalgorithmen und Multi-Skalen-Transformationen, die auf solchen Unterteilungsalgorithmen basieren.

Der erste Unterteilungsalgorithmus geht auf de Rham zurück, der einen Polygonzug iterativ durch ein lineares Verfahren verfeinerte. De Rham zeigte die Konvergenz seines Verfahrens und wies nach, dass der erzeugte Limes C^1 ist. In den neunziger Jahren entstand eine systematische Theorie zur Analyse von linearen Unterteilungsalgorithmen, die auch den mehrdimensionalen Fall beinhaltet. Im Zweidimensionalen besteht die Analyse aus zwei Teilen: Einerseits betrachtet man reguläre Netze, andererseits sogenannte singuläre Knoten beziehungsweise singuläre Facetten. Dies ist notwendig, da eine geschlossene Fläche mit nichttrivialer Euler-Charakteristik nicht mithilfe eines regulären Netzes modelliert werden kann.

Lineare Unterteilungsalgorithmen sind aus Sicht des Rechenaufwands sehr billig und finden daher häufig zur Verarbeitung von vektorraumwertigen Daten Verwendung. Liegen die Daten in einer Mannigfaltigkeit, z.B. im Raum der Diffusionstensoren oder der starren Bewegungen, so versucht man, Unterteilungsalgorithmen so zu modifizieren, dass sie auch Daten in solchen nichtlinearen Geometrien verarbeiten.

Im Eindimensionalen erzielten J. Wallner und N. Dyn mithilfe von „Proximity Inequalities" Ergebnisse zur Glattheit solcher geometrischer, nichtlinearer Unterteilungsalgorithmen. Im Fall regulärer Netze wurden Unterteilungsalgorithmen, deren Dilatationsmatrix ein Vielfaches der Identität ist, von P. Grohs behandelt. In dieser Dissertation wird ein Glattheitsresultat für nichtlineare Algorithmen mit beliebiger Dilatationsmatrix bewiesen; es besagt im wesentlichen, dass ein geometrischer Unterteilungsalgorithmus Glattheitseigenschaften von einem in der Nähe liegenden linearen Unterteilungsalgorithmus erbt.

Ein weiterer Teil dieser Arbeit besteht in der Glattheitsanalyse von nichtlinearen Algorithmen für den allgemeinen Fall nicht notwendigerweise regulärer Netze. Wir behandeln den Fall singulärer Knoten beziehungsweise Facetten und zeigen C^1 Glattheit der durch nichtlineare Unterteilung entstehenden Limiten; dies ist das zentrale Ergebnis der Dissertation.

Für geometrische, nichtlineare Unterteilungsalgorithmen gibt es im Allgemeinen nur Konvergenzsätze für den Fall genügend dichter Eingangsdaten. Für spezielle Riemannsche Mannigfaltigkeiten, sogenannte Cartan-Hadamard-Mannigfaltigkeiten, und eine gewisse Klasse von Unterteilungsalgorithmen beweisen wir in dieser Arbeit für den eindimensionalen Fall ein Konvergenzresultat ohne Einschränkung an die Eingangsdaten.

Abschließend betrachten wir interpolierende Multi-Skalen-Transformationen, die auf Unterteilungsalgorithmen fußen. Für reguläre Gitter und lineare Algorithmen wurden solche Transformationen von Donoho untersucht; er schlug auch eine nichtlineare, geometrische Variante vor, die auf Funktionen operiert, die Werte in einer Mannigfaltigkeit annehmen und im Euklidschen Raum definiert sind. Diese geometrischen Transformationen wurden von Grohs und Wallner untersucht. In dieser Dissertation stellen wir eine Transformation vor, die auf Funktionen zwischen zwei Mannigfaltigkeiten agiert. Wir charakterisieren die Hölder-Zygmund-Glattheit einer Funktion durch das Abklingen der sogenannten Detail-Koeffizienten, die durch die Transformation gewonnen werden.

Acknowledgements

First of all, I want to thank Johannes Wallner. He provided me with this topic, was always willing to read my manuscripts and to discuss problems. Also, he offered me the opportunity to participate in scientific meetings.

Furthermore, I want to thank my colleagues Christian, Mirko, Oleg, Oliver and Philipp at TU Graz for the nice atmosphere.

Finally, I want to thank Eva for her support.

This work has been carried out under the support of the Austrian Science Fund (FWF, Grant P-19780).

Contents

Introduction

The first subdivision scheme goes back to de Rham [7]. This article takes the point of view that a subdivision scheme is a refinement procedure for polygons, and each step doubles the number of vertices. He shows that iterated application of his scheme produces a sequence of (finer) control polygons which converge to a C^1 smooth function in the limit. Further early work was done by Chaikin [5] which is with a view towards application in computer graphics. Both articles deal with linear schemes which means that the refinement procedure is linear. For general linear curve schemes a framework for their analysis has been developed in the early 90s, [4, 11].

The case of subdivision schemes which refine polyhedral meshes instead of polygons is much more involved. First of all, the reader should note that a closed surface with non-zero Euler characteristic cannot be modeled by a regular quadrilateral or regular triangular mesh. Therefore, such meshes need to have singularities (i.e., vertices or faces in whose neighborhood the combinatorics is not regular). Hence the analysis of surface subdivision schemes has to treat the situation near such a singularity.

By the locality of the considered schemes the analysis splits into two parts: The first part is to consider the regular mesh case, and the second one consists of the analysis near singularities. Even the regular mesh case is more extensive than the univariate case since one can also consider schemes based on dilation matrices different to multiples of the identity. For work on the regular mesh case we refer to [4, 49, 26, 21, 18]. The breakthrough in the analysis near singularities is the article [47] of U. Reif. Further references are [68, 67, 44] and the comprehensive book [45]. Reif's work contains the first complete analysis of the first surface subdivision schemes introduced by Catmull and Clark [3] and Doo and Sabin [10], and thus solves a problem which had been open for more than fifteen years.

Subdivision has a wide range of applications. An overview of its use in Computer Graphics and Geometric Modeling can be found in [69]. Note that subdivision is used to 'model everything that moves' in 3D animated movies [50]. But subdivision schemes also have other applications; for instance, linear subdivision schemes defined on regular meshes are applied to produce scaling functions in wavelet analysis [6]. Furthermore, subdivision schemes are also used in the numerical solution of PDEs [24, 23].

In Chapter 1 we give a brief introduction to linear subdivision schemes and sum up the results on linear subdivision we need in the subsequent chapters. Furthermore, we set up a framework we need in the analysis of nonlinear schemes.

The theory of linear subdivision schemes is very extensive. In contrast, only in recent years nonlinear subdivision has been subject to systematic analysis. To get an impression of the diversity of this field, we exemplarily refer to [22], [57], [63] and the references therein. We stick to geometric, nonlinear, subdivision in this work.

Geometric subdivision aims at handling data in nonlinear geometries such as Lie groups, symmetric spaces, or Riemannian manifolds. Examples are the Euclidean motion group, hyperbolic space, Grassmannians or the space of positive definite matrices. The latter space is especially interesting in diffusion tensor imaging, where data are modeled as positive matrices sitting on a spatial grid. Other instances of geometric data in connection with subdivision are given by Ur Rahman et al. [53].

In Chapter 2 we recall well known constructions which provide means of deriving geometric subdivision schemes, using characteristics of linear ones. We also introduce a new geometric analogue which is particularly suited to subdivision in Riemannian manifolds and which in a natural way retains the symmetries of the linear scheme it is derived from.

In the analysis of nonlinear subdivision schemes some peculiar behavior can be observed; for instance, the paper [65] treats a nonlinear (but not geometric) scheme where the smoothness of the produced limit functions depends on the input data. It turns out that in this respect geometric schemes behave quite tame, i.e., their smoothness does not depend on input data. However, the input should be dense enough which is, in general, needed to guarantee that the scheme converges. A framework for the analysis of geometric, nonlinear, subdivision schemes by means of so-called proximity inequalities has been introduced by Wallner and Dyn in [57]. They show convergence and C^1 smoothness of a large class of geometric curve subdivision schemes. Their technique splits into two parts: The first part is to show that the geometric scheme fulfills proximity conditions with a linear scheme. The second part is to show that an arbitrary (not necessarily geometric) scheme which fulfills proximity conditions inherits convergence (for dense enough input) and C^1 smoothness from the linear scheme it is in proximity with. This technique has been extended to higher smoothness in [56]. In the multivariate regular setting, such results were only known for subdivision schemes with dilation matrices which are scalar multiples of the identity [14] before the author's paper [60] .

In Chapter 3 we obtain results for schemes based on arbitrary dilation matrices which operate on regular meshes. In fact, we show that the limit function obtained by a nonlinear subdivision scheme which meets proximity conditions belongs to the Hölder-Zygmund class Lip_α where α is real number arbitrarily close to but smaller than the smoothness index of the linear scheme which the nonlinear scheme is derived from. This applies to the geometric analogues of linear schemes considered in this thesis. This chapter is contained in the paper [60].

In Chapter 4 we deal with an essential part of the theory which was missing: Convergence and C^1 smoothness of nonlinear subdivision rules for irregular meshes. We show that a certain class of such schemes converges and produces C^1 limit functions. This analysis is based on a local proximity inequality similar to the one in [57]: If a nonlinear scheme is in proximity with a linear scheme which converges, respectively produces C^1 limit functions, then the nonlinear scheme does the same for sufficiently dense input data. This result applies to the geometric schemes considered in this thesis. This chapter is based on the paper [62].

As already mentioned, convergence of geometric subdivision schemes in general is only guaranteed if input data is dense enough. For a certain class of Riemannian manifolds, so-called Cartan-Hadamard manifolds, and a certain class of geometric curve schemes we can show convergence for all input data. This result is presented in Chapter 5. It is contained in the paper [58].

The last chapter of this thesis is concerned with interpolatory multiscale transforms. In [9], Donoho analyzes linear *interpolatory wavelet transforms*. In particular he characterizes smoothness properties of a function by decay properties of the detail coefficients which are derived from the function via the transformation. Interpolatory transforms can also be defined in a reasonable manner in the setting of geometric subdivision [53]. In [17], Grohs and Wallner show an analogue of Donoho's result for the class of Hölder-Zygmund functions in the geometric setting; they consider manifold-valued functions defined on Euclidean space.

In Chapter 6, we treat manifold-valued functions defined on a two-dimensional manifold. We define a multiscale transform, where both the choice of sample points and the prediction operator are based on nonlinear geometric subdivision. We characterize the Hölder-Zygmund smoothness of a function in terms of the detail coefficient decay w.r.t. our transform, in particular near irregular points. This chapter is based on the paper [61].

1 Linear subdivision schemes

This thesis mainly treats geometric nonlinear subdivision schemes for arbitrary meshes. Most of the geometric schemes we treat are intrinsically defined and therefore only use intrinsic information without referring to some ambient vector space. Nevertheless, the way we follow for constructing such geometric schemes uses quantities derived from linear schemes. Furthermore, the analysis of a geometric scheme is based on comparison with a linear scheme 'nearby'. For this reason we start with a brief introduction to linear subdivision. One aim is to provide the information and terminology necessary to define geometric analogues of linear schemes in Chapter 2. The other aim is to build a basis for the analysis of nonlinear schemes in the Chapters 3, 4 and 5.

In this chapter we define linear subdivision schemes for (general) two-dimensional meshes. We explain how the analysis of convergence and smoothness of a scheme can be reduced to the particular case of k-regular meshes. Although a regular mesh is a k-regular mesh for certain k, this case is treated separately due to its different behavior. Another reason is that the regular mesh (or synonymously, regular grid) case immediately generalizes to higher dimensions. These two cases of meshes are treated in detail.

We introduce our notation concerning meshes, as it is used later on. The *combinatorics* (or the *connectivity*) of a mesh is an abstract triple $K = (V, E, F)$ consisting of a set of vertices V, edges E and faces F. A *mesh* (K, h) in some set M consists of the combinatorics K and a vertex based positioning function $h : V \to M$. Typically, $M = \mathbb{R}^3$, e.g. in Computer Graphics; in this thesis we are interested in the case when M is a smooth manifold. The set $h(V)$ is also called vertices, since $h(v)$ represents the geometric position of the vertex v. When we want to emphasize the difference between V and $h(V)$ we speak of abstract and realized vertices, respectively.

We always assume that the mesh under consideration has so-called 2-*manifold topology* which means that any edge has either one or two faces adjacent and that for each vertex the set of neighboring vertices is nonempty and connected. The *valence* of a vertex or a face is the number of edges it is adjacent to.

The n-ring of a face or a vertex are those vertices in the mesh which can be reached from the face or the vertex by passing at most n faces. We use the notation $\mathcal{N}_n(v)$ and $\mathcal{N}_n(F)$ for the n-ring of a vertex v, and of a face F, respectively.

A *subdivision scheme* S consists of a *topological refinement rule* and a *geometric refinement rule*. For a given connectivity (V_0, E_0, F_0), the topological rule generates a new connectivity (V_1, E_1, F_1). The geometric rule computes new vertex positions from old ones. In other words, it acts as an operator on the positioning functions producing $h_1 : V_1 \to M$ from input $h_0 : V_0 \to M$. The geometric refinement rule should not be confused with a geometric subdivision scheme which is just a subdivision scheme acting in a geometry, i.e. acting on positioning functions $h : V \to M$ where M is a smooth manifold. A subdivision scheme is linear, if M is a vector space and if the operator acting on positioning functions is linear. A subdivision scheme S is *interpolatory* if $V_i \subset V_{i+1}$ and old (realized) vertex positions are not changed during the subdivision process. In that case subdivision adds new vertices to the existing ones.

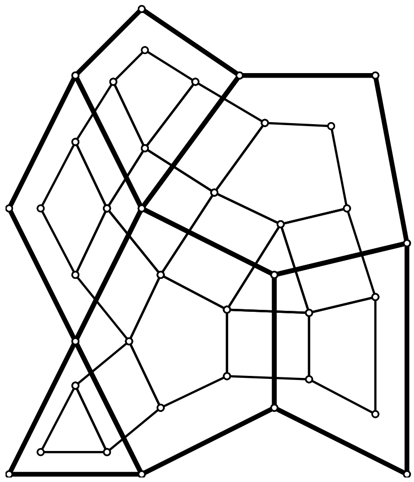


Figure 1: dual refinement

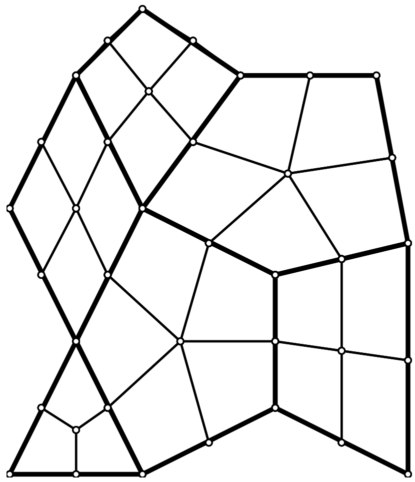


Figure 2: primal refinement

Quadrilateral-based primal and dual quadrisection schemes

The historically first categories of topological refinement rules were primal quadrilateral-based and dual quadrilateral-based topological rules based on quadrisection, of which the schemes of Catmull/Clark and Doo/Sabin are examples. For a visualization, see Figures 1 and 2. A *primal quadrilateral quadrisection scheme* produces a new connectivity by a so-called face split: This means old vertices are retained and a new vertex is inserted for each old edge and face. New edges are inserted between new vertices originating from edge and face if they were adjacent, and between new vertices originating from vertex and edge if they were incident. After one round of subdivision all new faces have valence four. Vertices originating from faces and edges also have valence four while 'vertex'-vertices inherit the valence of their predecessors.

A *dual quadrilateral quadrisection scheme* generates a new connectivity by a so-called vertex-split: A new vertex is created for each pair of face and adjacent vertex. Vertices arising from the same old vertex share a new edge if the faces they stem from shared an edge, and vertices arising from the same old face share a new edge if the vertices they stem from shared an edge. A new face originates either from an edge, or a face, or a vertex, whereby 'edge'-faces have valence four and the others inherit the valence of their predecessors. All new vertices have valence four and therefore in a second subdivision step, irregular faces can only originate from old faces.

In case of quad-based refinement, vertices and faces are called *regular* if they have valence four, otherwise they are called *irregular*.

The geometric rule computes new vertex positions from old ones. In the case of linear subdivision, we consider *affine invariant* rules, meaning that a new vertex position $h_1(w)$ is an affine combination of finitely many previous ones:

$$h_1(w) = \sum_{v \in V_0} \alpha_{v,w} h_0(v), \qquad \text{where} \sum_{v \in V_0} \alpha_{v,w} = 1. \tag{1.1}$$

We call the mapping $v \mapsto \alpha_{v,w}$ the *stencil* of w, and denote its *support* by $\operatorname{supp}_S(w)$. We further require that the rule only depends on the connectivity in a local neighborhood

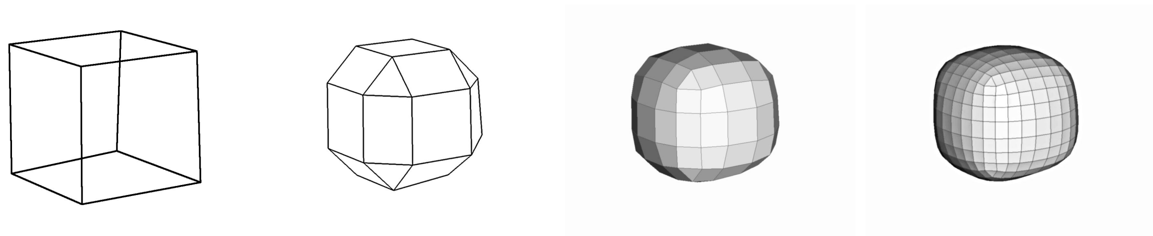


Figure 3: Dual Refinement: The neighborhood of the irregular face becomes regular.

of globally fixed size [68]. We explain what we mean by this: Since a new vertex can be identified with either a face, an edge or a vertex of the old mesh in the primal case, or a face-vertex pair of the old mesh in the dual case, there is a natural notion of a neighborhood of a new vertex in the old mesh. We require that there is an integer n, such that, for any new vertex w, the choice of the stencil $\alpha_{\cdot,w}$ in (1.1) only depends on the combinatorics of the n-neighborhood of w. Furthermore, we require that $\operatorname{supp}_S(w)$ is contained in this n-neighborhood of w. This is to avoid pathologies, and can be seen as an *uniform locality condition.*

After a few iterations, the greater part of the combinatorics becomes regular, i.e., faces and vertices have valence 4. However, there are remaining isolated singularities, *extraordinary vertices* in the primal case, and *extraordinary faces* in the dual case, which are surrounded by regular connectivity. For a visualization, see Figure 3. In this work we are interested in limit properties of the subdivision scheme, i.e., for an initial mesh (K_0, h_0), we consider the sequence of its i-times subdivided meshes (K_i, h_i), where we are interested in what happens if $i \to \infty$. Now the uniform locality condition guarantees that the information needed for the limit behavior of the subdivision process near an irregular vertex/face is contained in a certain n-neighborhood of the irregular vertex/face on each subdivision level i, where n does not depend on the subdivision level i. If we choose the level i high enough, this n-neighborhood has regular combinatorics except for one irregular vertex/face. This means that the analysis of subdivision on a general mesh reduces to the analysis of two special kinds of meshes: Firstly, a mesh with regular combinatorics, and secondly, a mesh with only one irregular vertex/face. Such a mesh is called k-regular, where k is the valence of the extraordinary face/vertex. These two kinds of meshes are discussed in Chapter 1.1 and Chapter 1.2, respectively. An extensive treatment of the above reduction process can be found in D. Zorin's thesis [66].

Schemes based on other types of topological rules

There are also various other types of topological refinement rules. In the community of people who apply subdivision to graphics, the systematic classification of different topological rules has gained quite a lot of interest and is presently still in progress; see e.g. [54, 25, 8] and the references therein. It is not our aim in this thesis to contribute to this classification.

Nevertheless, we consider schemes not based on quadrilateral quadrisection since they have gained considerable importance in applications. First of all, triangle-based quadrisection schemes like Loop's scheme or the Butterfly scheme are widely used. See Figure 4 for a visualization of the topological refinement. Secondly, the quadrilateral-based primal

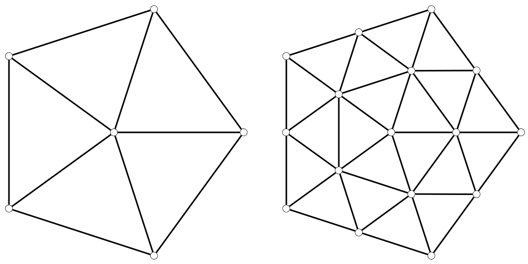


Figure 4: Quadrisection for triangle-based meshes.

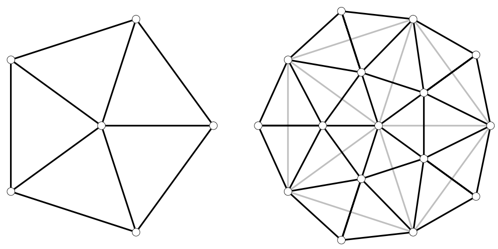


Figure 5: $\sqrt{3}$-refinement near an irregular vertex.

and dual schemes we considered before, approximately increase the number of vertices by a factor of 4 per subdivision step. Certain different topological rules, like e.g. $\sqrt{2}$-schemes or $\sqrt{3}$-schemes, show the attractive feature of slow refinement which can be desirable in certain applications. In contrast, there are also topological rules which provide a fast 'zoom in', like e.g. $\sqrt{7}$-schemes. For example, subdivision schemes with fast refinement are applied in the context of pyramid schemes for modeling the function of the retina [59, 34] and for representing terrain data in geophysical applications [48]. So considering more general topological rules provides more flexibility in the refinement speed which is important in some applications. Last, but not least, certain schemes based on more general topological rules provide anisotropic features, when applied in multiscale transforms [40].

We try to introduce the reader to these classes of schemes by providing some examples. The reader who is interested in a systematic treatment is referred to [54, 25, 8]. In Chapter 1.2, we build a framework for the analysis near extraordinary vertices/faces, which incorporates our examples and certain classes of the topological refinement rules in [54, 25, 8].

Primal $\sqrt{3}$*-subdivision* is triangle-based, i.e., after the first subdivision step the mesh consists of triangles. In case of triangle-based topological rules, a vertex is regular, if it has valence 6; a face is regular if it has valence 3. The topological rule works as follows (cf. Figure 5): A new vertex is inserted for each old vertex and each old face. Two vertices which are descendants of neighboring faces and two vertices which are descendants of a face and a neighboring old vertex are connected by an edge. Faces are given by three new vertices of the following form: They originate from two neighboring faces and the third vertex originates from an old vertex adjacent to both old faces.

$\sqrt{2}$*-subdivision* is quad-based. For a visualization, see Figure 6. Similar to $\sqrt{3}$-schemes, for *primal* $\sqrt{2}$-subdivision, a new vertex is given by each old vertex and each old face. New edges are obtained between two new vertices which originate from an old face and a neighboring old vertex. A new face is defined by four new vertices of the following form: Two vertices originate from two adjacent old faces and the other two originate from two old vertices which are both adjacent to the old edge shared by the two old faces.

The *dual* $\sqrt{2}$-topological refinement rule (cf. Figure 7) ist obtained by applying the primal rule to obtain combinatorics (V_1, E_1, F_1), and then to exchange faces and vertices to obtain the combinatorics (F_1, E_1, V_1). More graphically, a new vertex is obtained for each old edge. New edges are given between any two new vertices which originate from two old edges which share a common old vertex.

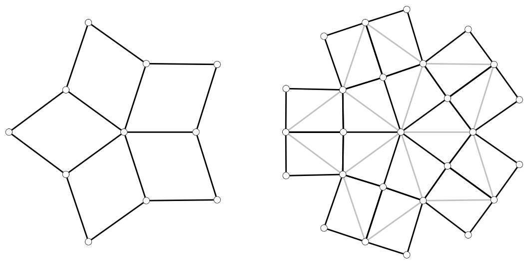


Figure 6: Primal $\sqrt{2}$-refinement.

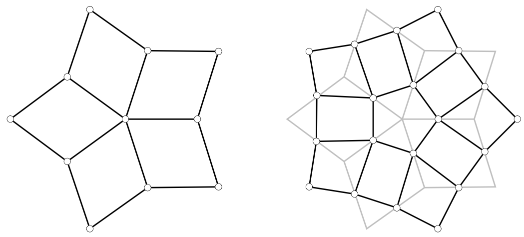


Figure 7: Dual $\sqrt{2}$-refinement.

Primal $\sqrt{7}$-subdivision is triangle-based. This topological refinement has a certain asymmetry which requires the orientability of the initial mesh. Each old vertex defines a new vertex. Furthermore, three new vertices are inserted per old face, called face-vertices. The choice of new edges and faces is best explained with the help of figure Figure 8. Since the initial mesh is orientable, we can a choose a consistent orientation on the set of old faces. This orientation allows us to define a one-to-one correspondence between face-vertices and triples of the form (old vertex, old edge, old face). Then edges are inserted between two new vertices in the following cases: Firstly, between a vertex originating from a vertex v and face-vertex whose associated triple has the form $(*, *, v)$. Secondly, between face-vertices whose associated triples have the same edge. Thirdly, between face-vertices whose associated triples have the same vertex and whose associated triples have faces which are neighbors in the old mesh. Note that the way of choosing the correspondence between face-vertices and the triples influences the choice of new edges and thus the combinatorics of the output mesh.

As in the case of primal and dual quadrisection, the geometric rule is required to be affine invariant; a new vertex position $h_1(w)$ is computed by

$$h_1(w) = \sum_{v \in V_0} \alpha_{v,w} h_0(v), \qquad \text{where} \sum_{v \in V_0} \alpha_{v,w} = 1.$$

We furthermore require the uniform locality condition already formulated for primal and dual quadrisection schemes. This allows us to apply the same arguments as for primal/dual quadrisection schemes in order to reduce the general mesh case to the cases of regular combinatorics and k-regular combinatorics.

Examples of $\sqrt{3}$-subdivision schemes are Kobbelt's $\sqrt{3}$-scheme [31], Labsik and Greiner's interpolatory $\sqrt{3}$-subdivision schemes and the schemes constructed in [41]. Examples of $\sqrt{2}$-schemes can be found in [33]. An example of a dual $\sqrt{2}$-scheme is the well-known simplest (or mid-edge) subdivision scheme of Peters and Reif [43]. Methods to produce $\sqrt{7}$-schemes have been proposed by Oswald [40].

1.1 Linear subdivision schemes on regular grids

We saw that local limit properties of a linear subdivision scheme can be inferred from studying regular and k-regular meshes. For the regular mesh part, it is enough to consider a mesh with (abstract) vertex set $\mathbb{Z}^2$. This is because of the locality of the subdivision rules; it is not necessary to consider regular meshes which are e.g. topologically isomorphic to the torus.

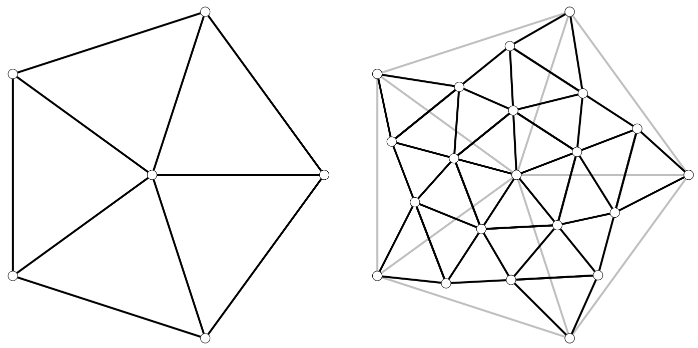


Figure 8: $\sqrt{7}$-refinement.

For such a regular input mesh with abstract vertex set $\mathbb{Z}^2$, a subdivision scheme produces an output mesh, which again has a combinatorics with abstract vertex set $\mathbb{Z}^2$; see Figure 9 for a visualization in case of quadrisection and $\sqrt{2}$-refinement. So a (linear) subdivision scheme can be seen as a (linear) operator on the space of sequences with index set $\mathbb{Z}^2$. A sequence corresponds to the vertex-based positioning function above.

With this is mind, we give the following definition of the subdivision operator. We explain afterwards how a subdivision scheme defines a subdivision operator.

Definition 1.1. *A linear subdivision operator $S = S_{a,M}$ is a linear operator on the vector space of sequences with index set $\mathbb{Z}^d$ given by*

$$S_{a,M}p(\alpha) = \sum\nolimits_{\beta\in\mathbb{Z}^d} a(\alpha - M\beta)p(\beta), \tag{1.2}$$

where the mask $a : \mathbb{Z}^d \to \mathbb{R}$ *has finite support, and M is a dilation matrix.*

A $d \times d$ integer matrix M is called *dilation matrix*, if $\|M^{-n}\| \to 0$ as $n \to \infty$. M is called *isotropic*, if it is $\mathbb{C}$-diagonalizable and all eigenvectors have equal modulus. We assume that

$$\sum\nolimits_{\alpha\in\mathbb{Z}^d} a(\alpha) = |\det M|. \tag{1.3}$$

This guarantees affine invariance, if $S_{a,M}$ converges.

In the above definition we let d be arbitrary since this causes no trouble and provides more generality. If we set $d = 2$, we are in the regular mesh case. The topological refinement rule is encoded in the dilation matrix. For example, $\sqrt{2}$-schemes can be realized by the following dilation matrices:

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The left-hand matrix yields a clockwise rotation of the grid, the right-hand matrix a counter-clockwise rotation of the grid. For $\sqrt{7}$ schemes the following dilation matrices are possible:

$$M = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}.$$

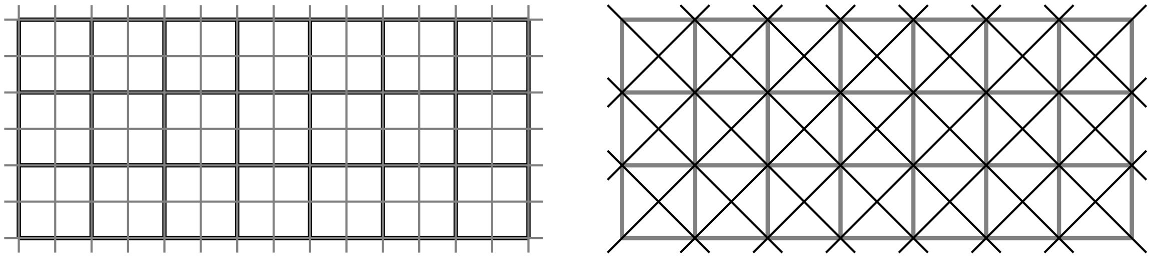


Figure 9: Primal quadrisection (left) and $\sqrt{2}$-refinement (right), regular case.

Note that all four matrices displayed here are isotropic.

The weights for affine averaging which were given by the stencils $\alpha_{v,w}$ in (1.1) are now stored in the mask a. The condition $\sum_v \alpha_{v,w} = 1$ in (1.1) guarantees that (1.3) is fulfilled.

We study the subdivision operator defined by (1.2). We formalize the notion of convergence of a subdivision scheme defined by a subdivision operator. Since the operator acts on sequences with index set $\mathbb{Z}^d$, we can also consider it on the space of bounded sequences $l^\infty(\mathbb{Z}^d)$, where, by the finiteness of the mask, $S_{a,M}$ is a bounded linear operator.

Definition 1.2. *We say that a linear subdivision scheme S converges if, for arbitrary bounded input p on $\mathbb{Z}^d$, there is a uniformly continuous function f on $\mathbb{R}^d$ such that*

$$\|f(M^{-k}\cdot) - S^k p\|_{l^\infty(\mathbb{Z}^d)} \to 0 \qquad \text{as } k \to \infty.$$

Here f is sampled on $M^{-k}\mathbb{Z}^d$ and a sequence on $\mathbb{Z}^d$ is generated by the assignment $\alpha \to M^k\alpha$.

If it exists, the limit function for the Dirac sequence δ_0 as input is denoted by ϕ. This compactly supported function fulfills the refinement equation

$$\phi = \sum a(\alpha)\phi(M \cdot - \alpha)$$

and thus ϕ is called the *refinable function*. Convergence of a linear scheme is equivalent to convergence for the special input δ_0. This is a consequence of the locality, continuity, shift-invariance and linearity of the subdivision operator. This is not the case for a nonlinear scheme as we will see later. For input p we can write the limit as

$$p * \phi = \textstyle\sum_{k\in\mathbb{Z}^d} p(k)\phi(\cdot - k),$$

where ϕ is the refinable function.

Furthermore, convergence of a linear scheme is equivalent to the convergence of the cascade algorithm:

$$\phi_{k+1} = \sum a(\alpha)\phi_k(M \cdot - \alpha), \tag{1.4}$$

where ϕ_0 belongs to a certain class of non-zero compactly supported continuous input functions [21]. It turns out that the limit of this iteration coincides with the limit function of subdivision for input δ_0; see [21].

Convergence of a subdivision scheme is also equivalent to a certain smoothness index $\nu_{a,M}$ of the scheme being greater than zero [21]. We define this index after giving some preliminaries.

We say that a subdivision scheme satisfies *sum rules* of order k if for every $\alpha \in \mathbb{Z}^d$ and any polynomial q of total degree lower than k we have the identity

$$\sum\nolimits_{\beta \in M\mathbb{Z}^d} a(\alpha+\beta)q(\alpha+\beta) = \sum\nolimits_{\beta \in M\mathbb{Z}^d} a(\beta)q(\beta).$$

Sum rules of order k imply *polynomial reproduction* of order k for the subdivision scheme S. This means that subdivision applied to a sample p of a polynomial of total degree less than k is again a polynomial of total degree less than k. In general, the converse is not true [26], i.e., there are schemes which have polynomial reproduction of order k and do not fulfill sum rules of order k. A characterization of the equivalence of both notions in terms of the mask can be found in [26]. As a result, if the scheme is *stable*, both notions coincide. A scheme is called stable if the translation invariant subspace generated by ϕ is isomorphic to $l^\infty(\mathbb{Z}^d)$ via the operator producing limit functions, i.e., there are $c, C > 0$ such that, for all $p \in l^\infty(\mathbb{Z}^d)$,

$$c\|p\|_\infty \le \|p * \phi\|_\infty \le C\|p\|_\infty.$$

Sum rules are important to us since the maximal sum rule order determines an a priori upper bound for the smoothness index of a scheme. So assume that the maximal sum-rule order of S is k. Then we consider the spectral quantity $\rho_k = \rho_k(a, M)$ defined by

$$\rho_k = \max\{\lim_{n\to\infty} \|\nabla^\mu S^n \delta_0\|_\infty^{1/n} : \mu \text{ is a multiindex with } |\mu| = k\}, \tag{1.5}$$

and define the *smoothness index of the scheme* by

$$\nu_{a,M} = -\log_{\lambda_{\max}} \rho_k, \tag{1.6}$$

where $\lambda_{\max}$ is the greatest modulus of the eigenvalues of M. We use the usual multiindex notation here: For a multiindex $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}_0^d$, and $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, we let $|\mu| = \sum_{i=1}^d \mu_i$, $\mu! = \prod_{i=1}^d \mu_i!$, and $\xi^\mu = \prod_{i=1}^d \xi_i^{\mu_i}$. Furthermore we use the difference ∇^μ defined by

$$\nabla^\mu p = \nabla_{e_1}^{\mu_1} \ldots \nabla_{e_d}^{\mu_d} p,$$

where $\nabla_{e_1} p = p - p(\cdot - e_i)$ is the backward difference in direction of the canonical basis vector e_i.

The smoothness index may seem artificial at first glance. However, it provides a lower bound for the smoothness of the refinable function. Furthermore, if the scheme S is stable and the corresponding dilation matrix is isotropic, the smoothness index determines the smoothness of the scheme. We have

$$\text{always}: \qquad \nu_{a,M} \le \nu(\phi), \tag{1.7}$$

$$\text{for stable } S \text{ and isotropic } M: \qquad \nu_{a,M} = \nu(\phi). \tag{1.8}$$

Here $\nu(\phi)$ is the *Hölder-index* of the refinable function ϕ given by

$$\nu(\phi) := \max\{\lambda : \phi \in \mathrm{Lip}_\lambda\}.$$

This result can be found in [18]. Here Lip_λ is the space of Hölder-Zygmund functions of order λ. One possible definition of these spaces is to choose some integer k strictly greater than λ and to consider those bounded continuous functions which fulfill:

$$\text{There is } C > 0 \text{, such that} \qquad \|\nabla_y^k f\|_\infty < C|y|_{\mathbb{R}^d}^\lambda \tag{1.9}$$

for all $y \in \mathbb{R}^d$, where $\nabla_y f = f - f(\cdot - y)$. Then the Lip-seminorm $|\cdot|_{\mathrm{Lip}_\lambda,k}$ and the Lip-norm $\|\cdot\|_{\mathrm{Lip}_\lambda,k}$ are given by

$$|f|_{\mathrm{Lip}_\lambda,k} = \inf\{C : C \text{ fulfills (1.9)}\} \quad \text{and} \quad \|f\|_{\mathrm{Lip}_\lambda,k} = \|f\|_\infty + |f|_{\mathrm{Lip}_\lambda,k}.$$

With this norm Lip_λ becomes a Banach space. It turns out that for different choices of k, we obtain the same linear space of functions and that different norms are equivalent. For this and more information on Hölder-Zygmund spaces we refer to the book [51].

Let us put this in the form of a theorem:

Theorem 1.3. *Let S be a subdivision scheme based on a general dilation matrix M. S converges if and only if $\nu_{a,M} > 0$. Then the smoothness index $\nu_{a,M}$ provides a lower bound for the Hölder-index of any limit function produced by S. If S is stable and M is isotropic, then $\nu_{a,M}$ is the Hölder-index of any nontrivial limit function produced by S.*

We are in the middle of the discussion of the smoothness index $\nu_{a,M}$ of a scheme S. Besides from agreeing with the Hölder index of the refinable function for isotropic dilation and stable S, there is an important connection to the cascade algorithm which does not rely on stability: If the dilation matrix is isotropic, $\nu_{a,M} > k$ if and only if the cascade algorithm (1.4) converges in C^k for all input functions belonging to a certain natural class of C^k functions [19, Theorem 4.3].

We provide some information we need in the analysis of nonlinear schemes later on. For $l < k$, we define the quantity ρ_l analogous to (1.5) by replacing k by l. The connection between ρ_k and ρ_l for $k \neq l$ is given by B. Han, [18, Theorem 3.1]:

Theorem 1.4. *Let $S_{a,M}$ be a linear subdivision operator satisfying sum rules of order k. Let $\lambda_{\min}$ be the smallest modulus of the eigenvalues of M. Then*

$$\rho_l = \max(\rho_k, \lambda_{\min}^{-l}) \quad \text{for any nonnegative integer } l < k. \tag{1.10}$$

We later use this connection between different ρ_l's in the proof of our smoothness theorem for nonlinear schemes on regular meshes.

We briefly point out some ways to compute, or at least estimate, the smoothness exponent (1.6). A very efficient way of computing the L_2-analogue of the smoothness index is given in [18], where also a detailed exposition of this topic and references can be found. The problem is reduced to finding the eigenvalues of a certain matrix which is computationally not that expensive. Estimates from below for our L_∞ setting, i.e., for $\nu_{a,M}$, are obtained via embedding theorems. For example, for $d = 2$, $\nu_{a,M}$ is bounded from below by the smoothness index of the scheme in the L_2-setting minus one. Fortunately, there is also a quite fast algorithm for exactly determining $\nu_{a,M}$ in case that the Fourier transform of the mask is a positive function; see also [18].

Another approach is to use the joint spectral radius criterion [21] and to brute-force compute upper and lower bounds of the corresponding spectral radii. These bounds converge very slowly, which makes such computations very expensive. However, this method also works exactly in our L_∞-setting, at least in theory.

There is another, slightly different, approach to the smoothness analysis of subdivision schemes going back to Dyn [11] which is based on iteration of the following theorem formulated for the univariate case $d = 1$. Here the scale of C^k spaces is considered for measuring smoothness.

Theorem 1.5. *Let S be a univariate affinely invariant subdivision scheme with dilation factor 2. If the derived scheme $S^{[1]}$ converges, then the scheme S produces C^1 limit functions.*

Here the derived scheme is given by

$$S^{[1]}\nabla = 2\nabla S.$$

Applying the derived scheme iteratively to the differences of the input data yields an approximation of the derivative of the limit function if it exists.

The derived scheme exists if and only if S is affinely invariant, which is equivalent to the fact that first order sum rules are fulfilled. The existence of the k-th derived scheme is equivalent to k-th order sum rules being fulfilled. By iterated application of Theorem 1.5, one gets the following statement.

Corollary 1.6. *Let S be a univariate convergent subdivision scheme with dilation factor 2. If the derived scheme $S^{[k]}$ exists and converges, then the scheme S produces C^k limit functions.*

A criterion for the convergence of a scheme S is the contractivity of the commutator scheme $\frac{1}{2}S^{[1]}$, i.e.,

$$\text{There is } n \in \mathbb{N} \text{ such that} \qquad \|(\tfrac{1}{2}S^{[1]})^n\|_\infty < 1. \quad \Rightarrow \quad S \text{ converges.} \tag{1.11}$$

Extending this approach to the multivariate situation is not so straightforward. The case of dilation matrix $M = nI$ still works in a certain sense: The statement has to be modified but the technique of using the derived scheme to approximate derivatives still works. In the case of more general isotropic dilation certain problems arise at that point. We consider the case of dilation matrix $M = nI$ first. Multivariate differences for an $\mathbb{R}^s$-valued sequence $p \in l^\infty(\mathbb{Z}^d)^s$ are elements $\Delta p \in l^\infty(\mathbb{Z}^d)^{sd}$ given by

$$\Delta p(\alpha) = (p(\alpha + e_1) - p(\alpha), \ldots, p(\alpha + e_s) - p(\alpha))^T$$

So Δp is an $\mathbb{R}^{sd}$-valued sequence on $\mathbb{Z}^d$. With this preparation, derived schemes can (at least formally) be recursively defined by

$$S^{[l]}\Delta = N\Delta S^{[l-1]}, \qquad S^{[0]} = S.$$

Note that the coefficients in the masks of such derived schemes are no longer scalars, but d^l- matrices. This is the reason why such schemes are called *vector subdivision schemes.*

Derived schemes are no longer uniquely defined in the multivariate setting. It turns out that a k-th derived scheme exists if and only if k-th order sum rules are fulfilled [37].

A multivariate version of Corollary 1.6 and (1.11) reads:

Theorem 1.7. *Let S be a multivariate scheme with dilation matrix NI, with sum rule order $k+1$ (or, equivalently, such that the $k+1$-st derived scheme exists). If for all l with $0 < l \leq k+1$, the spectral radius $\rho(S^{[l]}|_{\Delta^l})$ of the l-th derived scheme restricted to l times differenced input data fulfills*

$$\rho(\tfrac{1}{N} S^{[l]}|_{\Delta^l}) < 1$$

then S produces C^k limit functions.

This theorem can be found in the thesis [13] and generalizes results for dilation factor 2. References to previous work can be found in [13]. The core of the argument is that the k-th derived scheme applied to k times differenced input approximates the k-th order derivatives of the limit function. This is no longer the case, if M is a general isotropic dilation matrix; see [49]. However, we can use the result on the smoothness-exponent (1.7) to obtain a result similar to Theorem 1.7. We decide to do this, although it is not the scope of the thesis, since the auxiliary lemmas we need are used elsewhere later on. For a scheme S on $\mathbb{Z}^d$ with isotropic dilation matrix M, we let $m = \sqrt[d]{\det(M)}$ and we define (again formally) derived schemes by

$$S^{[l]}\Delta = m\Delta S^{[l-1]}, \qquad S^{[0]} = S.$$

This implies that, for all $n \in \mathbb{N}$,

$$(S^{[l+1]})^n \Delta^{l+1} = m^{n(l+1)} \Delta^{l+1} S^n. \tag{1.12}$$

Also in this case it turns out that a k-th derived scheme exists if and only if k-th order sum rules are fulfilled [37].

Theorem 1.8. *Let S be a multivariate scheme on $\mathbb{Z}^d$ with isotropic dilation matrix M which has sum rule order $k+1$ (which is equivalent to the existence of a $k+1$-st derived scheme). Let $m = \sqrt[d]{\det(M)}$. If the spectral radius $\rho(S[k+1]|_{\Delta^{k+1}})$ of the $k+1$-st derived scheme restricted to the space of $k+1$ times differenced input data $\Delta^{k+1} l^\infty(\mathbb{Z}^d)$ fulfills*

$$\rho(\tfrac{1}{m} S^{[k+1]}|_{\Delta^{k+1}}) < 1 \tag{1.13}$$

then S produces C^k limit functions.

Conversely, if S is stable and produces C^k limits, then (1.13) is true.

For the proof we need the following very interesting fact: A stable scheme with isotropic dilation matrix which produces C^k limits has smoothness index $\nu_{a,M} > k$. This is a consequence of [20, Corollary 4.2]. So if the smoothness index equals an integer k, the limit functions cannot be C^k. Or, if the limit functions are C^k, the smoothness index is strictly greater than k.

Proof of Theorem 1.8. We start with the second part, i.e., we assume that S is stable and produces C^k limit functions. By [20, Corollary 4.2] $\nu_{a,M} > k$ and a fulfills at least $k+1$-st order sum rules. Assume the maximal sum rule order of a equals i. Since we are in the case of isotropic dilation, (1.10) reads

$$\rho_{k+1} = \max(\rho_i, m^{-(k+1)}). \tag{1.14}$$

We have two cases: If $\rho_{k+1} = \rho_i$, then $m^{-\nu_{a,M}} = \rho_{k+1}$ by the definition of $\nu_{a,M}$. The second case is that $m^{-(k+1)} \geq \rho_i$. Then $\rho_{k+1} = m^{-(k+1)}$. Putting both cases together, $\rho_{k+1} = m^{-(k+\varepsilon)}$, where $\varepsilon = \nu_{a,M} - k > 0$ in the first case, and $\varepsilon = 1$ in the second case, respectively. By (1.15) (shown independently in Lemma 1.10), for any $s > 1$,

$$\|\Delta^{k+1} S^n p\|_\infty \leq C(\rho_{k+1} s)^n \|\Delta^{k+1} p\|_\infty,$$

where $C > 0$ is independent of $p \in l^\infty(\mathbb{Z}^d)$ and n. Note that the differences ∇ and Δ carry the same information. Then, by (1.12),

$$\begin{aligned}\|(S^{[k+1]})^n \Delta^{k+1} p\|_\infty &= \|m^{n(k+1)} \Delta^{k+1} S^n p\|_\infty \leq C m^{n(k+1)} (\rho_{k+1} s)^n \|\Delta^{k+1} p\|_\infty \\ &= C m^{n(k+1)} (m^{-(k+\varepsilon)} s)^n \|\Delta^{k+1} p\|_\infty = C(m^{(1-\varepsilon)} s)^n \|\Delta^{k+1} p\|_\infty.\end{aligned}$$

Choosing $s > 1$ small enough yields

$$\|(\tfrac{1}{m} S^{[k+1]})^n \Delta^{k+1} p\|_\infty \leq C(m^{-\varepsilon} s)^n \|\Delta^{k+1} p\|_\infty = C\gamma^n \|\Delta^{k+1} p\|_\infty,$$

where $\gamma = m^{-\varepsilon} s < 1$ and C are independent of n and p. This yields (1.13).

We consider the first part of Theorem 1.8, i.e., we assume that $\rho := \rho(\frac{1}{m} S^{[k+1]}|_{\Delta^{k+1}}) < 1$. Letting $p = \delta_0$, and applying the definition of the spectral radius, there is for any $\epsilon > 0$ a constant $C > 0$, independent of n, such that for any multiindex μ of order $k+1$,

$$\|\nabla^\mu S^n \delta_0\|_\infty \leq C m^{-kn} (\rho + \varepsilon)^n \|\nabla^\mu \delta_0\|_\infty$$

This means at least that $\rho_{k+1} < m^{-k}(\rho + \epsilon)$, and so $\rho_{k+1} < m^{-k}$. By (1.14), $\rho_i < m^{-k}$, and so $\nu_{a,M} > -\log_m(m^{-k}) = k$. Then the scheme produces C^k limit functions. □

For later use we also formulate the following lemma.

Lemma 1.9. *Let S be a linear convergent scheme with dilation matrix NI. If a derived scheme $S^{[1]}$ converges, then there is a constant $C \geq 1$ such that*

$$\|\Delta S^k p\|_\infty \leq C(1/2)^k \|\Delta p\|_\infty, \qquad \textit{for all bounded } p : \mathbb{Z}^d \to \mathbb{R}.$$

Proof. For any $f \in l^\infty(\mathbb{Z}^d)$, $\|S^{[1]k} \Delta f\|_\infty \leq D$, with D independent of k since $S^{[1]}$ converges. Restrict f to $B = \{-n, \dots, n\}^2$, where n is big enough such that B controls the limit on the unit square. We apply the Banach-Steinhaus-Theorem to the operators $(S^{[1]})^k$ restricted to the finite dimensional space of sequences vanishing outside $3B$ and on 0. This yields that $\|2^k \Delta S^k f'\|_\infty = \|S^{[1]k} \Delta f'\|_\infty \leq C' \|\Delta f'\|_\infty$, for all such sequences f'. Here C' is independent of f'. For general f, we find f', such that on B, $\nabla f = \nabla f'$. □

Besides from Theorem 1.8 the next lemma is needed in Chapter 3. Lemma 1.10 starts from (1.5) and establishes the inequality (1.15). The main point here is that differences are incorporated in the right-hand side of (1.15) and that general input data are considered; this is going to be important for the analysis of nonlinear schemes. We did not find this statement in the literature, even it is possibly already known.

Lemma 1.10. *Assume that $S = S_{a,M}$ is a linear convergent subdivision rule which satisfies sum rules of order k. Then for every $s > 1$ there is $C \geq 1$ such that, for all $p \in l^\infty(\mathbb{Z}^d)$ and all $n \in \mathbb{N}_0$,*

$$\sup_{|\mu|=k} \|\nabla^\mu S^n p\|_\infty \leq C(\rho_k s)^n \sup_{|\mu|=k} \|\nabla^\mu p\|_\infty. \tag{1.15}$$

Proof. By the definition of ρ_k in Equation (1.5) there is a constant $C > 0$ such that, for $s > 1$,

$$\|\nabla^\mu S^n \delta_0\|_\infty \leq C(\rho_k s)^n \quad \text{for all multiindices } \mu \text{ with } |\mu| = k. \tag{1.16}$$

The constant C depends on the choice of s but not on the exponent n. We use the notation $l(\mathbb{Z}^d)$ for the space of sequences on $\mathbb{Z}^d$. We consider the mapping

$$p \mapsto \{\nabla^\mu S^n p\}_{|\mu|=k} \tag{1.17}$$

from $l(\mathbb{Z}^d)$ to $l(\mathbb{Z}^d)^r$, where $r = \binom{k+d-1}{k}$ is the number of different multiindices with $|\mu| = k$. This mapping is linear. We show that this mapping only depends on the k-th order differences of the input p, i.e., it only depends on $\{\nabla^\mu p\}_{|\mu|=k}$: Since S satisfies sum rules of order k, S leaves the set of samples of polynomials of degree lower than k invariant (see [26], Theorem 5.2). A sample of a polynomial p with $\deg(p) < k$ is characterized by the vanishing of all differences of order k, i.e., $\nabla^\mu p = 0$ for all μ with $|\mu| = k$. These two observations guarantee that the property $\nabla^\mu p = 0$ for all multiindices μ with order k implies $\nabla^\mu S p = 0$ whenever $|\mu| = k$. This implies that the mapping (1.17) only depends on the k-th order differences of p.

With these observations at hand we use the locality of the subdivision scheme S and construct a scenario which allows us to apply the principle of uniform boundedness which then yields (1.15). To that end, we consider the 'discrete simplex' $T = \{\alpha \in \mathbb{N}_0^d : |\alpha| < k\}$, and choose $N > 2k$ so large that the limit function of subdivision on $[-1,1]^d$ for input p only depends on the values of p on $Q = \{-N, \ldots, N\}^d$ (It is well known that for finitely supported masks such an N exists). We start with (possibly unbounded) data $p \in l(\mathbb{Z}^d)$ and find $p' \in l(\mathbb{Z}^d)$ with

$$\nabla^\mu p = \nabla^\mu p' \ (\mu \text{ with } |\mu| = k) \quad \text{and} \quad p'|_T = 0. \tag{1.18}$$

This is done by finding a polynomial with degree lower than k which agrees with p on T and subtracting it from p. We use the notation $l(A)$ for the space of sequences on $\mathbb{Z}^d$ vanishing outside $A \subset \mathbb{Z}^d$. We consider the projection operator $P : l(\mathbb{Z}^d \setminus T) \to l(Q \setminus T)$, which sets values outside Q to 0. We get a constant C which is independent of p such that

$$\sup_{|\mu|=k} \|\nabla^\mu P p\|_\infty \leq C \sup_{|\mu|=k} \|\nabla^\mu p\|_\infty.$$

We consider the family of operators

$$(\rho_k s)^{-n}\nabla^\mu S^n : l(Q\setminus T)\to l^\infty(\mathbb{Z}^d),$$

indexed by the multiindex μ and the exponent n. This family is bounded on any sequence q. The principle of uniform boundedness yields a constant C, independent of q, n, and μ, such that

$$\sup_{|\mu|=k}\|\nabla^\mu S^n q\|_\infty \le C(\rho_k s)^n \sup_{|\mu|=k}\|\nabla^\mu q\|_\infty$$

for $q\in l(Q\setminus T)$.

We consider general $p\in l^\infty(\mathbb{Z}^d)$ and choose a sequence p' according to (1.18) and define $q\in l(Q\setminus T)$ by $q = Pp'$. Then we use the above estimates to get

$$\sup_{|\mu|=k}\|\nabla^\mu S^n q\|_\infty \le C(\rho_k s)^n \sup_{|\mu|=k}\|\nabla^\mu p\|_\infty.$$

Furthermore, for any multiindex μ of order k, we have that $\nabla^\mu S^n q = \nabla^\mu S^n p$ on $\{-k,\dots,k\}^d$. In view of the translation invariance of S, this implies (1.15). □

The next proposition is also needed in Chapter 3. Its purpose is to estimate Lip-seminorms of the limit functions by differences of data.

Proposition 1.11. *Assume that $S_{a,M}$ is a linear convergent subdivision operator which has maximal sum rule order k. Then for every γ which is smaller that the smoothness index $\nu_{a,M}$, the mapping $p\mapsto p*\phi(M^m\cdot)$ of data on level m to limit functions is a bounded linear operator from $l^\infty(\mathbb{Z}^d)$ to Lip_γ for every input level m. The growth of the bounds of the Lip_γ-seminorms in m can be estimated by differences of input data as follows: For all $s>1$ there is $C\ge 1$ such that*

$$|p*\phi(M^m\cdot)|_{\mathrm{Lip}_\gamma,k} \le C|\lambda_{\max}|^{m\gamma}s^m \sup_{|\mu|=k}\|\nabla^\mu p\|_\infty, \tag{1.19}$$

where C is independent of m, and $\lambda_{\max}$ is an eigenvalue of M of greatest modulus.

Proof. Since the refinable function ϕ is a Lip_γ function, we have, for every $s>1$, a constant $C>0$ such that, for every nonnegative integer m,

$$|\phi(M^m\cdot)|_{\mathrm{Lip}_\gamma,k} \le C|\lambda_{\max}|^{m\gamma}s^m.$$

As a consequence, the Lip_γ-seminorm for arbitrary bounded input data p can be estimated by

$$|p*\phi(M^m\cdot)|_{\mathrm{Lip}_\gamma,k} \le C|\lambda_{\max}|^{m\gamma}s^m\|p\|_\infty.$$

This is due to the compact support of ϕ. Since S satisfies k-th order sum rules, $q*\phi$ is a polynomial with $\deg(q*\phi)<k$ for any sample q of a polynomial of degree lower than k (see e.g. the discussion around Theorem 2.1 in [26]). Therefore, the directional difference $\nabla_y^k p*\phi$ of the limit function for input p only depends on the k-th order differences $\{\nabla^\mu p\}_{|\mu|=k}$.

We use the notation of the proof of Lemma 1.10, and define, for $p \in l^\infty(\mathbb{Z}^d)$, the sequence $q \in l(Q \setminus T)$ by $q = Pp'$, where p' is chosen according to (1.18). Then in the cube $[-1,1]^d$, the limit functions $p * \phi$ and $q * \phi$ are equal. If we consider the smaller cube $[-1/2,1/2]^d$, we find a step size $h > 0$, such that the difference $\nabla_y^k p * \phi$ and $\nabla_y^k q * \phi$ agree for all vectors $y \in \mathbb{R}^d$ with $\|y\| \leq h$.

We consider the family of operators $l(Q \setminus T) \to \mathrm{Lip}_\gamma$,

$$q \mapsto |\lambda_{\max}|^{-m\gamma} s^{-m} q * \phi(M^m \cdot),$$

which is indexed by the exponent m. This family is bounded on every sequence $q \in l(Q \setminus T)$. Therefore, the principle of uniform boundedness yields a constant $C > 0$, which is independent of q and m such that

$$|q * \phi(M^m \cdot)|_{\mathrm{Lip}_\gamma, k} \leq C |\lambda_{\max}|^{m\gamma} s^m \sup_{|\mu|=k} \|\nabla^\mu p\|_\infty.$$

This yields (1.19). □

1.2 Linear subdivision schemes near extraordinary points

In the beginning of this chapter we saw that limit properties of a linear subdivision scheme can be inferred from studying regular and k-regular meshes. After treating the regular mesh case in Chapter 1.1 we are now going to analyze k-regular meshes in order to see what happens near singularities. We introduce a setup in the spirit of Reif's framework near extraordinary points [47] (but we have to incorporate some discrete component since, in the nonlinear case, we do not have a finite set of a priori known surface patches).

We start with primal and dual quadrisection schemes restricting ourselves to a certain class of schemes which we call standard schemes. Our notion of standard schemes differs slightly from that in [45]. Afterwards we consider the more general case of shift-invariant schemes where we also allow for other topological refinement rules. One difference between standard schemes and the more general schemes discussed afterwards is that the latter may have arbitrary isotropic dilation and may be triangle based. Furthermore, for the latter class of schemes we impose weaker conditions on the eigenstructure of the subdivision matrix.

Standard schemes

We first consider primal and dual quadrisection schemes. In this case the setup simplifies. In the next part we consider the more general situation and extend our setup.

Our final objective is to derive convergence and smoothness results for nonlinear schemes acting on meshes with irregular combinatorics. To that end we first define a parametric notion of convergence near the singularity in a k-regular mesh.

Subdivision operators $S_{a,M}$ based on quadrisection have dilation matrix $M = 2I$. We consider Definition 1.2. There are two notions in this parametric definition of convergence which are not a priori determined by our definition of a subdivision scheme S for general meshes: The grid $\mathbb{Z}^2$ with its refinements $\frac{1}{2^n}\mathbb{Z}^2$, as well as the domain of the limit function $\mathbb{R}^2$. We define substitutes for these two objects for k-regular meshes.

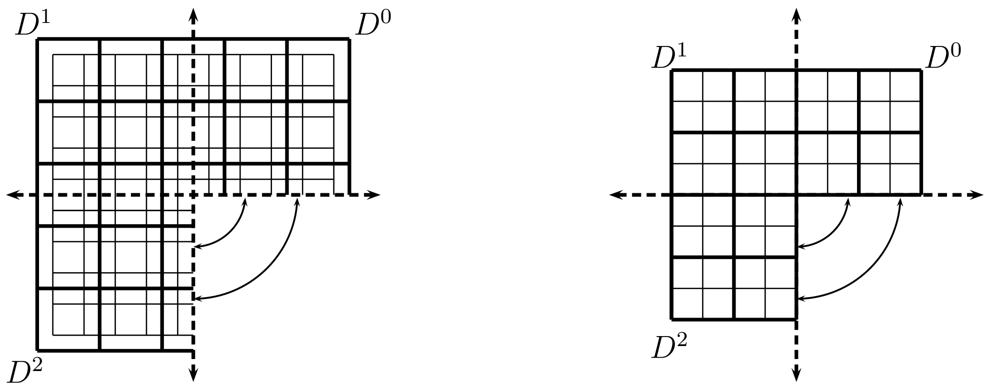


Figure 10: 3-regular mesh combinatorics. *left:* Dual subdivision rules. *right:* Primal subdivision rules. We show the sectors D_0, D_1, D_2, the identification of common boundaries of D_2 and D_0 being indicated by arrows. Combinatorics of 3-regular meshes at level 0 (thick lines, vertex set V_0) and level 1 (thin lines, vertex set V_1) are shown.

We start with the domain D of the limit function: We glue k copies $D^0, \dots, D^{k-1}$ of the positive quadrant $\Omega = [0, \infty[\, \times\, [0, \infty[$ together by identifying the y-axis of D^j with the x-axis of D^{j+1} (indices modulo k).

$$D = \Omega \times \mathbb{Z}_k,$$

where $\mathbb{Z}_k$ are the integers modulo k. We refer to D^j as its j-th sector (see Figure 10 for a visualization).

D becomes a metric space by defining the distance of points by the length of the shortest path which connects them, with the metric in the single sectors being that of $\mathbb{R}^2$. We introduce the map R^j which keeps the j-th quadrant and rotates the $(j+1)$-st by 90 degrees. Thus, it bijectively maps two successive sectors to the upper half plane. Next, we define domains $V_0, V_1, \dots$ which serve as a substitute for the grid $\mathbb{Z}^2$ and its refinements $\frac{1}{2^n}\mathbb{Z}^2$ (see Figure 10). In each sector we consider the set

$$\tilde{V}_n = 2^{-n}(\mathbb{N}_0 \times \mathbb{N}_0) \qquad \text{(primal case)}, \quad \text{or}$$
$$\tilde{V}_n = (2^{-n-1}, 2^{-n-1}) + 2^{-n}(\mathbb{N}_0 \times \mathbb{N}_0) \qquad \text{(dual case)}.$$

Then we obtain $V_0, V_1, \dots$ by

$$V_n = \tilde{V}_n \times \mathbb{Z}_k,$$

with the appropriate identifications at boundaries. So V_{n+1} arises from V_n by dilation with factor 2 (see Figure 10). R^j maps parts of V_n to vertices of the regular grid $2^{-n}\mathbb{Z}^2$ (or to a translated regular grid in the dual case). We choose edges and faces such that R^j maps the combinatorics to a part of the regular grid. So we obtain combinatorics with one single vertex of valence k (primal case) or one single face of valence k (dual case).

The action of a subdivision scheme S on such k-regular input meshes is interpreted in the following way: It transforms vertex data $h : V_n \to \mathbb{R}^d$ at level n to new vertex data

$$S_n h : V_{n+1} \to \mathbb{R}^d.$$

We explicitly distinguish the operations on different levels since we find it more convenient for the analysis of nonlinear schemes. The dilation operator σ defined by $\sigma f(x) = f(2x)$ obviously obeys $S_{n+1} \circ \sigma = \sigma \circ S_n$. We now can define convergence near a singularity:

Definition 1.12. *A subdivision rule S converges on the bounded k-regular mesh $p : V_0 \to \mathbb{R}^d$, if there is a uniformly continuous function $f_p : D \to \mathbb{R}^d$ such that*

$$\|f_p|_{V_i} - S_{i-1,0}p\|_\infty \text{ converges to } 0, \text{ as } i \to \infty.$$

Here $S_{i,l}$ is short for

$$S_{i,l} = S_i \circ \ldots \circ S_l \quad \text{for} \quad i \geq l,$$

and $S_{i,l}$ is the identity if $i < l$. $S_{i-1,0}$ maps data on subdivision level 0 to data on level i, by performing i steps of subdivision. For the limit we use the notation

$$S_{\infty,0}p := f_p.$$

The first step in the convergence and smoothness analysis is to split the neighborhood of the singularity into so-called rings D_i (see Figure 11). We let

$$D_i = D_i(r) = \{(x, y, j) \in D : \ 2^{-i-1}r \leq \max(x, y) \leq 2^{-i}r\},$$

where the 'radius' r denotes some scaling factor which is explained later on. The segments $D_i^j(r)$ are given as the intersection of the i-th ring with the j-th sector:

$$D_i^j = D_i^j(r) = D_i(r) \cap D^j.$$

Furthermore, we define the i-th inner area D_i' by

$$D_i' = \bigcup_{l \geq i} D_l \cup \{0\}. \tag{1.20}$$

Finally, we let

$$D_{-1} = D \setminus D_0' \quad \text{and} \quad D' = D_0'. \tag{1.21}$$

With this preparation at hand, we formulate our notion of *standard algorithms.*

To formulate the assumptions on the schemes we consider, we need the notion of the *control set* $\mathrm{ctrl}^i(U)$ of a set $U \subset D$ which is defined by D. Zorin in [68], and which is a set of vertices in the i-th level mesh which determine the limit function on U. This means that the limit function on U only depends on data on $\mathrm{ctrl}^i(U)$.

Our Setup. We impose the following conditions on linear subdivision schemes. The major restriction in contrast to Reif's setup for standard algorithms [45] comes from the fact that we do not take the point of view of iteratively generating control points of surface patches. In the nonlinear case, this view is not possible since such a finite dimensional space of patches is not available in general. This also explains that our notion of a subdivision matrix, given below, differs from [45]. For us, a standard subdivision scheme S is a linear, primal or dual quadrilateral scheme, which is based on quadrisection, with the following properties:

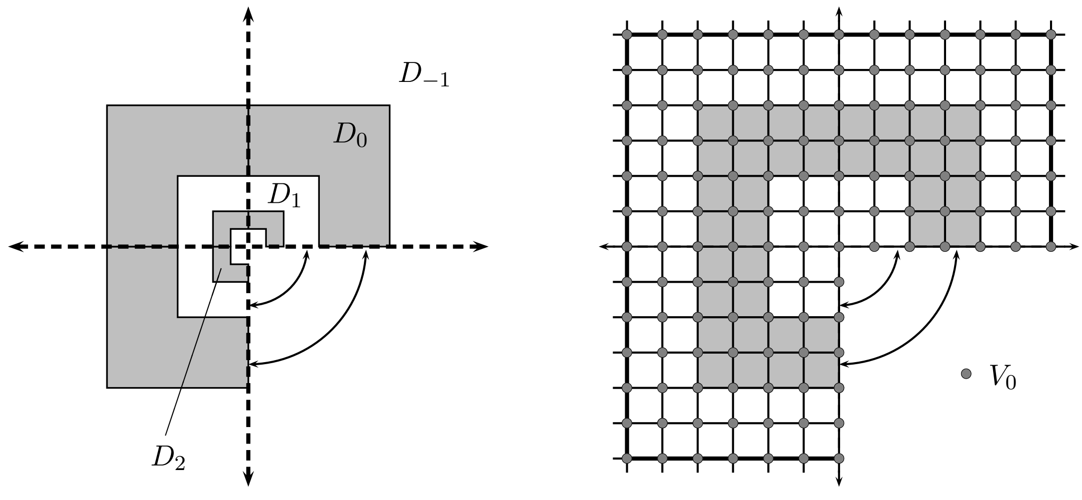


Figure 11: Parametrization near an extraordinary vertex of valence 3 (primal and dual quadrisection). *Left:* The domain D is obtained by gluing three quadrants together. The first three rings D_0, D_1 and D_2 are visualized. *Right:* The thick line circumscribes the control set of D_0 on level 0. The particular choice of the size parameter $r = 4$ is valid for Kobbelt's interpolatory quad scheme.

(1) For regular connectivity, the derived scheme $S^{[1]}$ converges.

(2) There is a 'radius' $r > 0$, such that the control sets $\mathrm{ctrl}^i(D_i^j(r))$ are vertices of a regular connectivity. The subdivision matrix A maps data on $\mathrm{ctrl}^i(D_i'(r))$, controlling the i-th inner area $D_i'(r)$, to data on $\mathrm{ctrl}^{i+1}(D_{i+1}'(r))$.

(3) The subdivision matrix A has eigenstructure

$$1 > \lambda = \lambda > |\mu_3| \geq \ldots$$

with the geometric multiplicity of λ being 2. The characteristic map χ, defined below, fulfills:

$$\chi|_{D' \setminus \{0\}} \text{ is regular and injective.}$$

We simply write D_i^j instead of $D_i^j(r)$. Examples of schemes which meet these requirements are the *generalized Lane-Riesenfeld schemes* [70], which the classical *Doo-Sabin* [10] and *Catmull-Clark scheme* [3] are particular examples of. Those two schemes are generalized and analyzed in [44]. An example of an interpolatory scheme is *Kobbelt's interpolatory quad scheme* [30], which was analyzed by Zorin in [67].

The notion of a *characteristic map* has been introduced by Reif in [47]. Our definition is slightly different and follows Prautzsch [46]. The limit function of subdivision on D', which is the union of all rings D_i and 0, is determined by data on $\mathrm{ctrl}^0(D')$. We choose two linearly independent eigenvectors to the subdominant eigenvalue of A. (all such choices of eigenvectors essentially lead to the same characteristic map.) Each one determines input data on $\mathrm{ctrl}^0(D')$, say $h_0', h_0'' : \mathrm{ctrl}^0(D') \to \mathbb{R}$. We use h_0',h_0'' to define 2D input data on

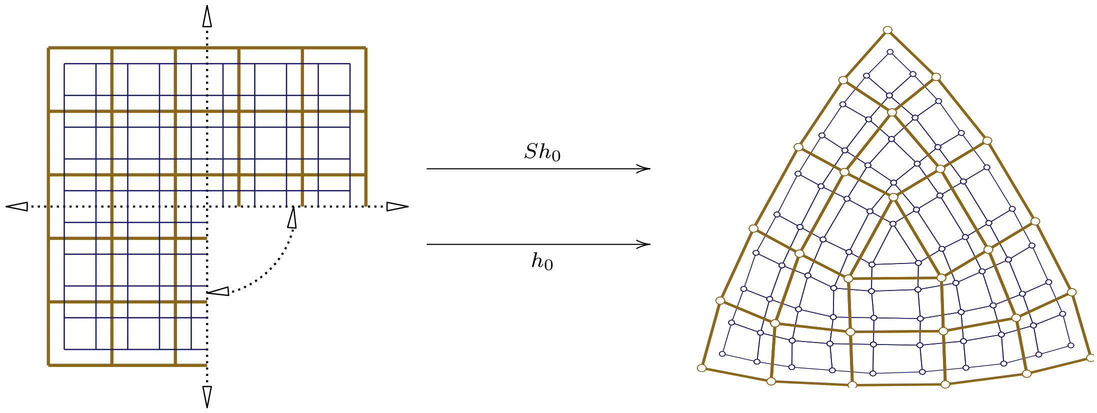


Figure 12: Characteristic input h_0 and one round of subdivision for the Doo-Sabin scheme. The limit function for h_0 defines the characteristic map.

$\mathrm{ctrl}^0(D'_0)$ by $h_0 = (h'_0, h''_0)$. The limit function $\chi : \ P \to \mathbb{R}^2$ of these data h_0 is called the characteristic map, i.e.,

$$\chi = S_{\infty,0} h_0;$$

see Figure 12. We have the following important scaling property of the characteristic map

$$\chi(2\cdot) = \lambda\chi(\cdot). \tag{1.22}$$

The following theorem is due to Reif [47]. A proof which immediately generalizes to $\mathbb{R}^d$, $d \geq 2$, has been given by Prautzsch [46].

Theorem 1.13. *For a standard scheme S and input data $p_0 : \mathrm{ctrl}^0(D') \to \mathbb{R}^d$, let $S_{\infty,0}p_0$ be the limit function of subdivision. Then the map*

$$S_{\infty,0}p_0 \circ \chi^{-1} : \ \chi(D') \to \mathbb{R}^d \quad \textit{is} \quad C^1.$$

For almost all input data p_0, the image $S_{\infty,0}p_0(D')$ is a two-dimensional submanifold of $\mathbb{R}^d$ locally around the (extraordinary) limit point $S_{\infty,0}p_0(0)$.

Shift-invariant schemes for more general topological refinement rules

We generalize the notions introduced for standard schemes to a setting which incorporates more general topological refinement rules. Furthermore, we relax the conditions on the eigenvalues of the subdivision matrix. Because of the greater generality the setup becomes somewhat more abstract, but we are concerned with the same issues as for standard schemes. As in the case of standard schemes we try to find a parametric notion of convergence for a scheme near a singularity. We consider primal quadrilateral and primal triangular based schemes in detail, and point out a setup for dual schemes using $\sqrt{2}$-schemes as an example.

We assume that for regular meshes the scheme can be represented by a subdivision operator with *isotropic* dilation matrix M which is associated with a rotation of the regular quadrilateral lattice in the plane or the regular triangular lattice in the plane, respectively. For $\sqrt{2}$-schemes and $\sqrt{3}$-schemes the corresponding angle is $\pm 45°$ (see Figure 9), in case of $\sqrt{7}$-schemes it is $\pm\arctan(\sqrt{3}/5)$ which is not a rational multiple of π [40].

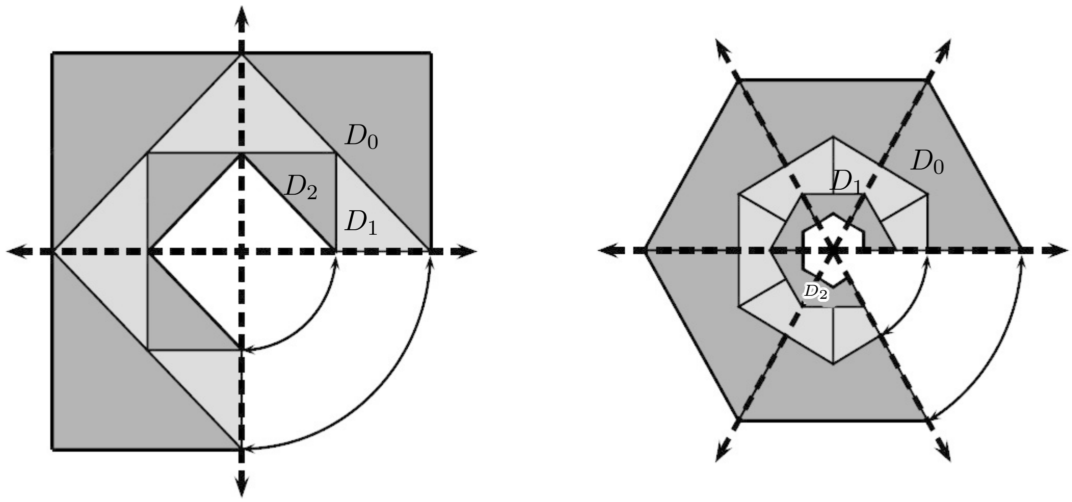


Figure 13: Domains for limit functions near extraordinary points for $\sqrt{2}$ schemes (left) and $\sqrt{3}$ schemes (right). The rings D_0 and D_2 are shown in dark gray, the ring D_1 in light gray. The union of all rings and the origin is the domain D'.

We define a domain for the limit function of subdivision for a k-regular input mesh. We refer to Figure 13 for a visualization. In the quadrilateral case, as for standard schemes, we use k copies of the positive quadrant $\Omega = [0, \infty[\times[0, \infty[$ in $\mathbb{R}^2$ to define the domain D. In the triangular case, we have to define a substitute for the positive quadrant: We let Ω be the sector with opening angle $\pi/3$, and cyclically glue k copies of this sector Ω to obtain the domain D for the limit function. To sum up, in both cases,

$$D = \Omega \times \mathbb{Z}_k$$

with identification of points according to the gluing which is done as follows:

In each sector we have polar coordinates (x, ϕ) where $0 \leq \phi \leq 90°$ ($60°$, resp.). The points $(x, 90°)$ of the first sector and the points $(x, 0°)$ of the second sector are identified, and so on, where the points $(x, 90°)$ in the k-th sector and $(x, 0°)$ in the first sector are also identified. In the triangular case, $(x, 90°)$ is replaced by $(x, 60°)$. In this way we obtain polar coordinates on D where angles vary between $0°$ and $(k\ 90)°$ (or $(k\ 60)°$ in the triangular case). For example, a point in D with polar coordinates $(x, 110°)$ comes from the second sector and has angle $20°$ in that sector. The i-th copy of Ω in D is referred to as i-th sector.

Next we define the domain V_0 for the initial k-regular mesh which serves as a substitute for the grid $\mathbb{Z}^2$ we used in the regular mesh case. In the quadrilateral case, we let Σ be the unit square $[0, 1] \times [0, 1] \subset \Omega$ (the equilateral triangle of length 1 in the triangular case). In the quadrilateral case, there is a quadrangulation of the sector Ω such that each quadrilateral is congruent to Σ. We glue these sector-wise quadrangulations together to obtain a quadrangulation of D. The vertices of this quadrangulation define the set V_0 which serves as domain for the initial k-regular mesh; see Figure 14 for a visualization. In the triangular case, we have a regular triangulation of Ω where each triangle is congruent to Σ. We glue these sector-wise triangulations together to obtain a triangulation of D. The vertices of this triangulation define the set V_0 which serves as a domain for the initial k-regular mesh.

We define the domains $V_1, V_2, \ldots$ for the subdivided k-regular meshes (which serve as a substitute for the refined grids $M^{-n}\mathbb{Z}^2$). To that end we introduce the notions of *dilation* and *rotation* on D : In polar coordinates, dilation by a factor $\lambda > 0$ is given by $(x, \phi) \to (\lambda x, \phi)$. Rotation about an angle ψ is given by $(x, \phi) \to (x, \phi + \psi)$. The dilation matrix M now induces a 'similarity transform' $G = G_{m^{-1},\psi}$ with dilation $m^{-1} = |\det M|^{-1}$ and rotation angle ψ which is the same as the rotation angle in the regular case. We define

$$V_i = G^i V_0$$

(In the standard scheme case, we used the definition $G = G_{\lambda,\psi}$ with rotation angle $\psi = 0$ and the dilation factor $\lambda = 1/2$).

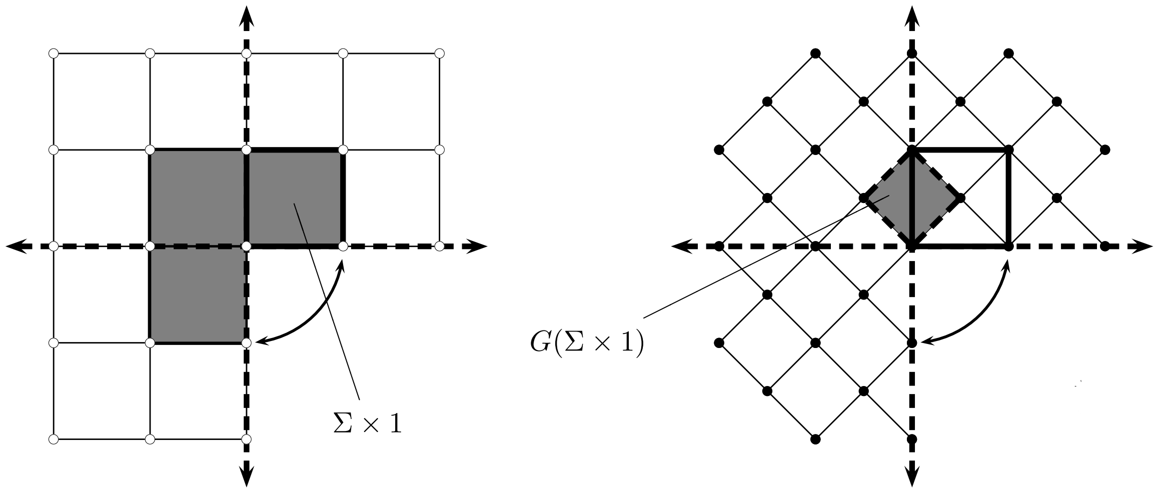


Figure 14: Primal $\sqrt{2}$-subdivision for valence $k = 3$ near a central irregular vertex. Vertex sets V_0 at level 0 (left) and $V_1 = GV_0$ at level 1 (right).

Convergence of a scheme can now be defined by Definition 1.12. As in the case of standard schemes, the action of a subdivision scheme S on input on level n is denoted by S_n. We let $S_{n,m} = S_n \cdots S_m$, and $S_{\infty,n}$ is the operator mapping data on level n to the corresponding limit function.

We define subsets of D which are needed for analysis purposes. As in the case of standard schemes, we use the symbol r to denote some scaling factor which is explained later on and which should not be confused with the radial component of some polar coordinate. We start with the neighborhood $D'(r)$ of the singular point 0 of the domain D given by

$$D'(r) := r\Sigma \times \mathbb{Z}_k,$$

which means that $D'(r)$ is obtained as scaling of the union of all copies of the unit square in all sectors (or equilateral triangles in the triangular case).

Using the similarity transform G from above we obtain rings $D_i(r)$ (see Figure 13) as follows:

$$D_0(r) = D'(r) \setminus GD'(r), \quad D_1(r) = GD'(r) \setminus G^2 D'(r), \quad \ldots.$$

The segments $D_i^j(r)$ and the i-th inner area $D_i'(r)$ are given by

$$D_i^j(r) = G^i(\Omega \times j) \cap D_i(r), \qquad D_i'(r) = G^i D'(r). \tag{1.23}$$

The set D_{-1} is given in analogy to (1.21), respectively.

We now formulate the setup for our class of more general schemes. Note that that assumptions on the eigenvalues of the subdivision matrix are also weakened.

Our Setup. We consider subdivision schemes S with the following properties near a singularity of valence k:

(1) For regular connectivity, the smoothness index $\nu_{a,M}$ is greater than 1.

(2) There is a 'radius' $r > 0$, such that the control sets $\mathrm{ctrl}^i(D_i^j(r))$ are vertices of a regular connectivity. The subdivision matrix A maps data on $\mathrm{ctrl}^i(D_i'(r))$, controlling the i-th inner area $D_i'(r)$, to data on $\mathrm{ctrl}^{i+1}(D_{i+1}'(r))$.

(3) The largest eigenvalue of A is 1, and A has a unique pair of complex conjugate subdominant Jordan blocks (if the corresponding eigenvalue is real, we assume that there are exactly two Jordan blocks of highest multiplicity). The characteristic map $\chi : D \setminus \{0\} \to \mathbb{R}^2$, which is defined below, is regular and injective on the punctured set $U \setminus \{0\}$, where U is some neighborhood of 0.

We simply write D_i instead of $D_i(r)$, D_i^j instead of $D_i^j(r)$, ... in the following.

In order to define the *characteristic map* we consider a (complex) Jordan vector v of highest multiplicity to a subdominant (complex) eigenvalue λ. If λ is real, we choose v such that $\mathrm{Re}\, v$ and $\mathrm{Im}\, v$ are linearly independent. The characteristic map $\chi : D \to \mathbb{R}^2$, is obtained as limit of subdivision with input data $[\mathrm{Re}\, v, \mathrm{Im}\, v]$.

Concerning the subdivision matrix we want to point out that the ordering of the columns and rows of the matrix must be in correspondence to the similarity $G_{\lambda,\psi}$.

To mention some sources of examples, $\sqrt{3}$ and $\sqrt{7}$ schemes can be found in [41] and [40]. Kobbelt's $\sqrt{3}$ scheme serves as an example in [41].

The conditions on the eigenvalues in (iii) are, for example, fulfilled for so-called shift-invariant algorithms [45]. An algorithm is shift-invariant if it is invariant w.r.t. shifting the sector-index. We explain this: Assume that we have data p_0 on V_0. Then each $v \in V_0$ is of the form (x, j) with $x \in \Omega$ and $j \in \mathbb{Z}_k$. We can apply a shift by $l \in \mathbb{Z}_k$ meaning that $(x, j) \to (x, j + l)$. This induces a shift on the data p_0; we call the result q_0. Then the shift by k applied to the limit of a shift invariant algorithm for input p_0 equals the limit for input q_0. This assumption is absolutely natural for a subdivision scheme in our sense, since for a general mesh no a priori ordering near a singularity is available (the book [45] has a somewhat different view towards subdivision which explains why shift-invariance is a property there).

There is another interesting scheme we would like to incorporate into our framework: Peters' and Reif's simplest subdivision scheme (mid-edge subdivision)[43]. This scheme is a dual $\sqrt{2}$-scheme. We have only treated primal schemes above. Except for the choice of the discrete domain V_0 the framework presented above can remain unchanged. We

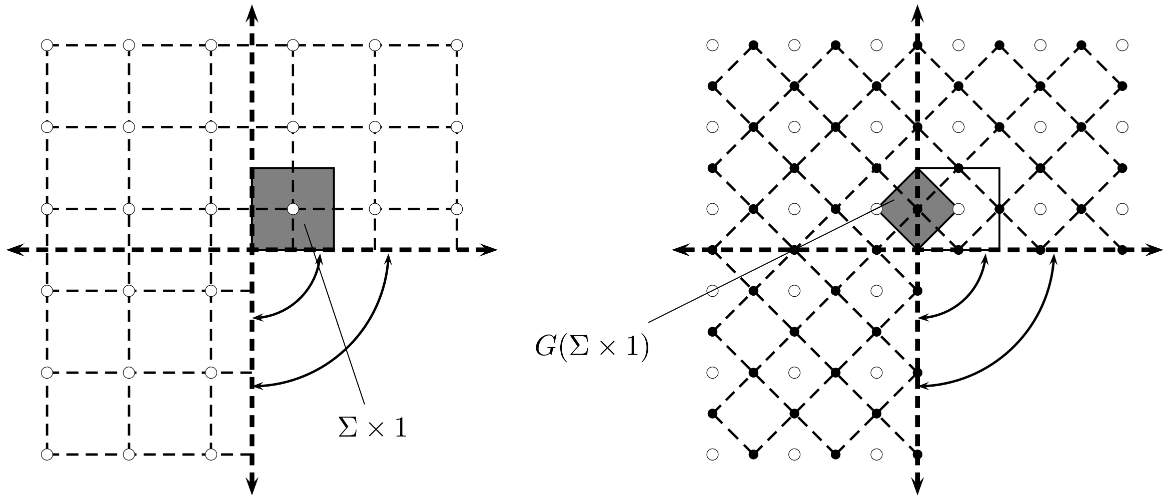


Figure 15: Choice of vertex sets V_0 (left) and $V_1 = GV_0$ (right) for dual $\sqrt{2}$-subdivision near an irregular vertex of valence $k = 3$.

explain how to choose V_0 and the refinements V_i such that the class of dual $\sqrt{2}$-schemes also fits into our framework; see Figure 15. We let

$$V_0 = ((\tfrac{1}{2}, \tfrac{1}{2}) + \mathbb{N}_0 \times \mathbb{N}_0) \times \mathbb{Z}_k, \quad \text{and} \quad V_{i+1} = G_{1/\sqrt{2}, \pm\pi/4} V_i.$$

This means that V_0 are the midpoints of the copies of Σ, and that the refinements V_i are obtained by i times application of the similarity transform with dilation factor $1/\sqrt{2}$ and rotation angle $\pm\pi/4$ to V_0. Proceeding similarly, one can also include other dual schemes; in particular dual quadrilateral schemes based on quadrisection.

The following theorem of U. Reif is analogous to Theorem 1.13 and is also valid in the case of our more general dilation matrices. This has also been observed in the papers [41, 40].

Theorem 1.14. *Let S be a subdivision scheme fulfilling the assumptions above. For input data $p_0 : \mathrm{ctrl}^0(D') \to \mathbb{R}^d$, let $S_{\infty,0}p_0$ be the limit function of subdivision for input p_0. Then the map*

$$S_{\infty,0}p_0 \circ \chi^{-1} : \ \chi(D') \to \mathbb{R}^d \quad \text{is} \quad C^1.$$

For almost all input data p_0, the image $S_{\infty,0}p_0(D')$ is a two-dimensional submanifold of $\mathbb{R}^d$ locally around the (extraordinary) limit point $S_{\infty,0}p_0(0)$.

2 Nonlinear geometric subdivision schemes

In this chapter we introduce geometric subdivision schemes which handle data in nonlinear geometries such as Riemannian manifolds or Lie groups. These schemes are necessarily nonlinear. The constructions we present here use a linear rule as a template and modify it such that the resulting scheme is able to process the geometric data. Various such constructions have been proposed in the literature; see [53, 57, 55] and the references cited there. We present some of them later on. The following construction is new.

Intrinsic mean subdivision:

Intrinsic mean subdivision is particularly suited to subdivision in Riemannian manifolds. The idea of an intrinsic mean in a Riemannian manifold M, also called Riemannian center of mass, goes back to Cartan. For details, we refer to [27]. In the context of 'meshless geometric subdivision', intrinsic midpoints of surfaces were used in [36].

We consider a linear scheme S. The geometric rule of S is given by the stencils $\alpha_{v,w}$ which determine new vertex positions by

$$h_1(w) = \sum_{v \in V_0} \alpha_{v,w} h_0(v), \qquad \text{where} \qquad \sum_{v \in V_0} \alpha_{v,w} = 1.$$

For the construction of the intrinsic mean analogue T of S, we retain the topological rule. For defining the geometric rule, observe that in Euclidean space the weighted center of mass $h_1(w)$ is the minimizer of a quadratic function:

$$h_1(w) = \operatorname{argmin}_q \sum\nolimits_v \alpha_{v,w} \|h_0(v) - q\|_2^2.$$

Replacing the Euclidean distance by the Riemannian distance, we obtain the rule

$$h_1(w) = \operatorname{argmin}_q \sum\nolimits_v \alpha_{v,w} \operatorname{dist}(h_0(v), q)^2. \tag{2.1}$$

This minimizer is called (weighted) *Riemannian center of mass* or *intrinsic mean*. Using the rule (2.1) naturally preserves the symmetries present in the coefficients $\alpha_{v,w}$.

Existence and uniqueness of $h_1(w)$ is guaranteed if the contributing old vertex positions $h_0(v)$ lie in a small enough ball. For estimates on the sizes of these Riemannian balls we refer to Kendall [28, 27]. It is a general issue that a geometric scheme, in general, is only defined for dense enough input data; a fact we also encounter for all the schemes presented later on. By dense enough we mean that the values $h_0(v)$ which contribute to the calculation of $h_1(w)$ are nearby. However, in certain Riemannian manifolds and for certain schemes the above construction is globally defined. We discuss this topic in Chapter 5.

We have the following nice property:

$$\sum\nolimits_v \alpha_{v,w} \; \exp^{-1}_{h_1(w)}(h_0(v)) = 0. \tag{2.2}$$

Here *exp* is the Riemannian exponential function. If the old vertex positions $p_0(v)$ sit in a small enough Riemannian ball, the balance condition (2.2) even characterizes the center of mass (2.1). This property could also serve as a definition if no distance is available, like in a Lie group.

Obviously, (2.2) is equivalent to

$$h_1(w) = \exp_{h_1(w)} \big(\sum\nolimits_v \alpha_{v,w} \ \exp^{-1}_{h_1(w)}(h_0(v)) \big). \tag{2.3}$$

The following gradient descent converges to the intrinsic mean $h_1(w)$:

$$y_{j+1} = \exp_{y_j} \big(\sum\nolimits_v \alpha_{v,w} \ \exp^{-1}_{y_j}(h_0(v)) \big), \tag{2.4}$$

whenever the initial point y_0 is chosen in a small enough ball. This provides a way of computing the mean.

In case of regular mesh subdivision the intrinsic mean analogue of $S = S_{a,M}$ given by

$$Sp(\alpha) = \sum\nolimits_{\beta \in \mathbb{Z}^d} a(\alpha - M\beta) p(\beta),$$

reads

$$Tp(\alpha) = \operatorname{argmin}_q \sum\nolimits_{\beta \in \mathbb{Z}^d} a(\alpha - M\beta) \ \operatorname{dist}(p(\alpha), q)^2. \tag{2.5}$$

The equations (2.3) and (2.4) allow us to establish a connection between the intrinsic mean analogue and the $\log - \exp$ analogue we explain next.

Log-exp subdivision:

For log-exp subdivision the data is supposed to take values in a Lie group, a Riemannian manifold or a symmetric space, see [58]. It was proposed in [53]. In order to define the log-exp analogue T of a linear scheme S, we again retain the topological rule. In order to define the geometric rule, we use the affine invariance of S, i.e. $\sum_{v \in V_0} \alpha_{v,w} = 1$ and rewrite the geometric rule of S :

$$h_1(w) = \sum\nolimits_v \alpha_{v,w} h_0(v) = x(w) + \sum\nolimits_v \alpha_{v,w}(h_0(v) - x(w)), \tag{2.6}$$

where $x(w)$ is an arbitrary point in Euclidean space.

The operation 'point + vector' and the operation 'point − point' in (2.6) are replaced by *exp* and its inverse, respectively, which are available in a Lie group, a Riemannian manifold or a symmetric space. We obtain

$$h_1(w) = \exp_{x(w)}(\sum\nolimits_v \alpha_{v,w} \ \exp^{-1}_{x(w)}(h_0(v))), \tag{2.7}$$

with base points $x(w)$ in the manifold. For Lie groups, $\exp_x(v) = x \exp(x^{-1} v)$. The choice of base points should match with the connectivity of the mesh: a vertex of a refined mesh is combinatorically associated with a vertex, edge or face of the original mesh. It makes sense to let w's ancestor determine $x(w)$, e.g. as intrinsic edge midpoint or face midpoint. One possible face midpoint is the midpoint of diagonals.

For obtaining C^1 smoothness on general meshes it turns out that the choice of base points is rather arbitrary: $x(w)$ should just be chosen to lie in a neighborhood (of globally fixed size) of w. For obtaining smoothness higher than C^2 for regular meshes this is not the case. Here a special choice of base points is necessary.

In the regular mesh case, the corresponding subdivision operator reads:

$$Tp(\alpha) = \exp_q\Big(\sum\nolimits_{\beta\in\mathbb{Z}^d} a(\alpha - M\beta)\log_q p(\beta)\Big), \tag{2.8}$$

where q is some base point dependent on α.

Ur Rahman et al. [53] choose $q = p(\gamma)$ as base point, where $\alpha = M\gamma + r$. Numerical experiments in [64] suggest that no high order smoothness can be expected for that choice of base-points for non-interpolatory schemes. In that paper, Xie and Yu propose a different strategy. They choose the base point as the result of an auxiliary interpolatory scheme Q, which fulfills sufficiently high order sum rules. Then (2.8) reads

$$Tp(\alpha) = \exp_{Qp(\alpha)}\Big(\sum\nolimits_{\beta\in\mathbb{Z}^d} a(\alpha - M\beta)\log_{Qp(\alpha)} p(\beta)\Big). \tag{2.9}$$

The correct choice of base-points is also the topic of the paper [16].

As intrinsic mean subdivision, log-exp subdivision is only well defined for dense enough input. This is due to the fact that the log-function and the exp-function are in general only locally defined.

We owe the connection between log-exp subdivision and intrinsic mean subdivision: By (2.4) the action of the log-exp analogue at a point in a Riemannian manifold can be interpreted as first step in the iteration to the intrinsic mean on the one hand. On the other hand, by (2.3), the intrinsic mean analogue can be interpreted as log-exp analogue with a very special choice of base points, namely the means itself. So for analysis purposes, intrinsic mean subdivision can be seen as an instance of log-exp subdivision.

Subdivision using the geodesic analogue:

This construction works for Riemannian manifolds and was proposed in [57]. The idea is to write the linear rule

$$h_1(w) = \sum_v \alpha_{v,w} h_0(v), \qquad \text{where} \qquad \sum_v \alpha_{v,w} = 1, \tag{2.10}$$

as an iterated process of affine averaging. To explain this, we use as an example the Doo-Sabin scheme on regular meshes (producing the tensor product of the quadratic B-spline) which has the form

$$h_1(w) = \tfrac{9}{16}h_0(v_0) + \tfrac{3}{16}h_0(v_1) + \tfrac{3}{16}h_0(v_2) + \tfrac{1}{16}h_0(v_3). \tag{2.11}$$

With the affine averaging operator av_λ given by

$$\mathrm{av}_\lambda(x, y) = (1-\lambda)x + \lambda y, \qquad \lambda \in \mathbb{R},$$

we can rewrite (2.11) as

$$h_1(w) = \mathrm{av}_{1/4}(\mathrm{av}_{1/4}(h_0(v_0), h_0(v_1)), \mathrm{av}_{1/4}(h_0(v_2), h_0(v_3)))$$

This representation is by far not unique. It is shown in [57] that as a consequence of the affine invariance every rule of the form (2.10) can be written as an iterated process of affine averaging:

$$h_1(w) = \mathrm{av}_{\lambda_0}(\mathrm{av}_{\lambda_1}(\ldots), \mathrm{av}_{\lambda_2}(\ldots, \mathrm{av}_{\lambda_k}(\ldots))). \tag{2.12}$$

The replacement for the affine averaging operator av_λ in linear case is given by the geodesic averaging operator g-av_λ . For two nearby points x and y in a Riemannian manifold we consider the geodesic c joining x and y which is parametrized such that $c(0) = x$ and $c(1) = y$. Then the geodesic averaging operator g-av_λ is defined by

$$\text{g-av}_\lambda(x, y) = c(\lambda).$$

Replacing each occurrence of av by g-av in (2.12) we obtain a geodesic analogue T of the scheme S :

$$h_1(w) = \text{g-av}_{\lambda_0}(\text{g-av}_{\lambda_1}(\ldots), \text{g-av}_{\lambda_2}(\ldots, \text{g-av}_{\lambda_k}(\ldots))). \tag{2.13}$$

This analogue is again well-defined for dense enough input data. This restriction is due to the fact that, in general, shortest geodesics need not exist and even if they do exist, they do not have to be unique.

Subdivision using the projection analogue:

This analogue works for surfaces M embedded in some $\mathbb{R}^n$ or nice closed sets M in some $\mathbb{R}^n$ like compact sets with smooth boundary. In applications, the complements of these second sets might be considered as obstacles the subdivision surface is not allowed to intersect. The general idea is that there is some projection mapping available, such that one can use a linear subdivision scheme S in $\mathbb{R}^n$ for data in M and afterwards project the output data back to M. This way of subdivision in a surface is pointed out in [57], and a detailed treatment can be found in [15]. The case of subdivision in the presence of obstacles can be found in [55].

We discuss the notion of a *projection* (or retraction) P: If we have dense enough input data it is reasonable to assume that the output of S applied to data in M does not lie too far from M. So it is sufficient that P is defined in an (open) ϵ-neighborhood U of M. The notion of a projection is formalized by requiring $P \circ P = P$. Since it shall be a projection to M, we require $P(U) \subset M$. Then the projection analogue T of S is defined by

$$T = P \circ S. \tag{2.14}$$

The projection is also required to be at least continuous. The smoothness of the projection limits the smoothness of the resulting geometric scheme [15].

We give some examples of projection mappings. In the case that M is a surface one can use a closest point projection, or if a surface in $\mathbb{R}^3$ is given as a level set $f(x, y, z) = 0$ of a smooth function f then one case use gradient flow for projecting. The projection in the obstacle case can be defined as follows: If $x \in M$, then $Px = x$, and otherwise P is a projection to the boundary of M. This mapping is only continuous which limits the smoothness we can expect to C^1.

General bundle framework:

We briefly recall a general framework set up in [17] which we use in Chapter 6. The framework applies to the log-exp analogues above and thus to the intrinsic mean analogue via its interpretation as log-exp analogue with a special choice of base points.

It is assumed that the manifold M is the base space of a smooth vector bundle $\pi : E \to M$ with a smooth bundle norm (e.g. in a Lie group the trivial bundle with the Lie algebra as fiber and some canonically extended norm on the Lie algebra, or the tangent bundle of a Riemannian manifold with the norm induced by the Riemannian scalar product). The substitutes of addition and subtraction are given by an operation $\oplus : E \to M$, which is defined in a neighborhood of the zero section of the bundle, and an operation $\ominus : M \times M \to E$, which is defined near the diagonal (e.g., the Lie group exponential or the Riemannian exponential and their inverses). Furthermore, $y \ominus x \in \pi^{-1}(\{x\})$ and $x \oplus (y \ominus x) = y$ have to be fulfilled. Then the geometric analogue of (2.6) (w.r.t. this bundle) is given by

$$p_1(w) = x(w) \oplus \sum\nolimits_v \alpha_{v,w}(p_0(v) \ominus x(w)). \tag{2.15}$$

3 Analysis of nonlinear schemes with general dilation on regular grids

In this chapter we obtain convergence and smoothness results on regular grids for schemes which are ‘not to far apart’ from convergent and smooth linear schemes. The nearness is formalized by so-called proximity conditions. We consider schemes based on general dilation matrices. The results are applied to the geometric schemes explained in Chapter 2. Such results have been obtained by [57, 56, 64] in the univariate case, and by [14] in the multivariate case for standard dilation matrices, i.e., multiples of the identity matrix. Our method of proof is not via derived schemes as in the above mentioned references; that derived schemes exhibit problems in the case of general dilation has already been observed by Sauer [49].

The material of this chapter is contained in [60].

3.1 Statement of the results

The fact that nonlinear subdivision is well-defined only for dense enough data entails considerable technicalities in the proofs. The exact formulation of the *proximity* between a nonlinear scheme and the linear scheme it is derived from is similarly technical. We introduce the following notions: For a subset N of Euclidean space and a positive real number σ, we consider the class $P_{N,\sigma}$ of σ-dense data which lie in N :

$$P_{N,\sigma} = \Big\{ p \in l^\infty(\mathbb{Z}^d, N) \ : \ \|\nabla_{e_i} p\|_\infty \le \sigma \text{ for all canonical basis vectors } e_i \Big\}.$$

Typically N is a surface in Euclidean space or some open set in Euclidean space obtained as image of a chart. Further, we consider the quantity

$$\Omega_j(p) = \sum_{\gamma \in \Gamma_j} \prod_{i=1}^{j} \sup_{|\mu|=i} (\|\nabla^\mu p\|_\infty)^{\gamma_i}, \quad \text{where} \ \ \Gamma_j = \{\gamma \in \mathbb{N}_0^j \mid \gamma_1 + 2\gamma_2 + \cdots + j\gamma_j = j+1\}. \tag{3.1}$$

For illustration, consider the cases $j = 1$ and $j = 2$:

$$\Omega_1(p) = \sup_{|\mu|=1} \|\nabla^\mu p\|^2, \quad \Omega_2(p) = \sup_{|\mu|=1} \|\nabla^\mu p\|^3 + \sup_{|\mu|=1} \|\nabla^\mu p\| \sup_{|\mu|=2} \|\nabla^\mu p\|.$$

Using this notation, we define proximity between subdivision rules S, T which operate on data living in a Euclidean vector space.

Definition 3.1. *Subdivision rules S and T obey proximity inequalities of order k in the domain $P_{N,\sigma}$ if there is a constant $C > 0$ such that, for all $p \in P_{N,\sigma}$,*

$$\sup_{|\mu|=j-1} \|\nabla^\mu (Sp - Tp)\|_\infty \le C\Omega_j(p) \quad \textit{for } j = 1, \ldots, k. \tag{3.2}$$

This definition can locally be applied to a geometric scheme acting in an abstract manifold N by going to Euclidean space using charts (if $N \subset \mathbb{R}^n$ it can be applied directly). The linear scheme is applied w.r.t. to the chart representation of the data and the output of the geometric scheme is transfered to Euclidean space by the chart.

The above conditions have been successfully applied to the analysis of geometric curve subdivision schemes and multivariate schemes based on a dilation matrix of the form NI. [57, 56, 64, 14]. It turns out that also in our setting, allowing dilation matrices to be arbitrary, we can use them to obtain convergence and smoothness of T. However, we need to follow a somewhat different path in our argumentation later on.

We use the following definition of convergence of a nonlinear scheme where we postulate that iterated subdivision is well-defined.

Definition 3.2. *A subdivision scheme T is called convergent for input data p if $T^n p$ is well-defined for all n, and if there is a uniformly continuous function f_p such that*

$$\|f_p(M^{-k}\cdot) - T^k p\|_{l^\infty(\mathbb{Z}^d)} \to 0 \qquad \text{as } k \to \infty.$$

Here f_p is sampled on $M^{-k}\mathbb{Z}^d$ and a sequence on $\mathbb{Z}^d$ is generated from this sample by the change of coordinates $\alpha \to M^k\alpha$.

The statement of our first result, the following convergence theorem, is rather technical. This is mainly due to the fact, that in the nonlinear case, where the scheme is in general not globally defined, it must be guaranteed that the subdivision process is well-defined on each intermediate level of subdivision. Note that we also obtain a nice sequence of uniformly continuous functions converging to the limit of nonlinear subdivision.

Theorem 3.3. *Consider a convergent linear subdivision rule $S_{a,M}$ acting on a regular grid. We assume that S is in first order proximity with the subdivision rule T w.r.t. the class of data $P_{N,\sigma}$. We assume that, for all $p \in P_{N,\sigma}$, the subdivided data Tp takes its values in some set N' with $N \subset N' \subset \mathbb{R}^n$. Assume further that there is a subset $N'' \subset N$ and $\sigma' > 0$ such that the σ'-neighborhood $U_{\sigma'}(N'')$ obeys*

$$U_{\sigma'}(N'') \cap N' \subset N.$$

Then there is a denseness bound $\sigma'' > 0$ such that the subdivision rule T converges for bounded data $p \in P_{N'',\sigma''}$. Furthermore, using the notation of Definition 3.2

$$T^n p * \phi(M^n\cdot) \to f_p \qquad \text{as uniformly continuous functions.}$$

Here ϕ is the refinable function generated by S.

Since we consider quite general sets N in this theorem we have to assume the existence of the set N'' with the above properties. However, if, for example, N is a ball of radius r, then N'' can be chosen as the ball with the same center and radius $r - \sigma'$. Then, for this particular choice of N, the theorem says that, if S and T fulfill proximity conditions w.r.t. $P_{N,\sigma}$, then T converges for dense enough input in the smaller ball N''. The next statement is the main result of the present chapter. It concerns smoothness.

Theorem 3.4. *Assume that the linear subdivision rule $S_{a,M}$ (acting on a regular grid) has maximal sum rule order k and that it is in k-th order proximity with a subdivision rule T w.r.t. to some domain $P_{N,\sigma}$ of σ-dense data. If T converges for input data p, then the limit f_p is a Lip_γ function for all $\gamma < \nu_{a,M}$.*

The proofs of these theorems are given in Chapter 3.2. The main reason for deriving these theorems is that they apply to the geometric subdivision rules introduced above. When transferring the data we operate in into $\mathbb{R}^n$ by means of some coordinate representation, we get nonlinear subdivision rules which work in $\mathbb{R}^n$. Knowing that this produces rules which are in proximity to linear rules, we conclude:

Theorem 3.5. *The previous theorems regarding convergence and smoothness apply to geometric subdivision rules which are the log-exp analogue or the intrinsic mean analogue of the linear rule $S_{a,M}$. In the log-exp case, the choice of base points must follow [16] (of which* (2.9) *is a special case). Furthermore, they apply to the projection analogue, where for the smoothness result it is required that the projection mapping is C^{k+1}.*

Corollary 3.6. *If the linear subdivision scheme $S_{a,M}$ is stable and M is isotropic, the analogues mentioned in Theorem 3.5 produce limits f_p whose smoothness index $\nu(f_p)$ is at least as high as the smoothness index $\nu(\phi)$ of the refinable function ϕ of the linear scheme. In particular, if the linear scheme produces C^k limits, then the geometric analogues also produce C^k limits.*

3.2 Convergence and smoothness analysis

Here we prove Theorem 3.3, Theorem 3.4 and its corollaries.

The first lemma shows the two important inequalities (3.3) and (3.4). The estimate (3.3) establishes a certain contractivity of the nonlinear scheme T. A similar estimate is also an important intermediate step in all previous smoothness proofs for geometric nonlinear schemes in the literature. In addition, we obtain the estimate (3.4) which is central in the proof of Theorem 3.4.

Lemma 3.7. *Assume that $S_{a,M}$ is a linear convergent subdivision scheme with maximal sum rule order k. Assume furthermore that $S_{a,M}$ and the (nonlinear) scheme T fulfill k-order proximity conditions w.r.t. some class $P_{N,\sigma}$ of σ-dense input.*

Then for any $s > 1$, we can find $C > 0$ and $\sigma'' > 0$ such that the following is true: For input $p \in P_{N,\sigma''}$, for which we assume that $T^n p$ is defined for all n and that $T^n p \in P_{N,\sigma}$ for all n, and for any $j \in \{1, \dots, k\}$ we have the inequality

$$\sup_{|\mu|=j} \|\nabla^\mu T^n p\|_\infty \leq C \max(\rho_k, |\lambda_{\min}|^{-j})^n s^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty, \tag{3.3}$$

where C is independent of p. Here $\lambda_{\min}$ is an eigenvalue of the dilation matrix M of minimal modulus. In particular there is a constant $L > 0$ with

$$\Omega_j(T^n p) \leq L(\rho_j \rho_1 s)^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2. \tag{3.4}$$

Proof. If the statement holds for some $s > 1$, it obviously holds for any $s' > s$. So we can fix $s > 1$ such that $\rho_j s < 1$ for all $j = 1, \dots, k$. For every $j \in \{1, \dots, k\}$ there is, by Lemma 1.10, a constant C'_j (dependent on s) such that

$$\sup_{|\mu|=j} \|\nabla^\mu S^n p\|_\infty \leq C'_j (\rho_j s)^n \sup_{|\mu|=j} \|\nabla^\mu p\|_\infty.$$

We let $C' = \max_{1\le j\le k} C'_j$. Furthermore, we denote the proximity constants from (3.2) by C_P.

For the next estimate, we consider $j \in \{1,\dots,k\}$ and a multiindex μ of order j. We apply Lemma 1.10 and (3.2) in order to obtain, for every $n \in \mathbb{N}$, the estimate

$$\begin{aligned}\|\nabla^\mu T^n p\|_\infty &\le \sum_{l=0}^{n-1} \|\nabla^\mu S^l (T-S) T^{n-l-1} p\|_\infty + \|\nabla^\mu S^n p\|_\infty \\ &\le 2C' \sum_{l=0}^{n-1} \rho_j^l s^l \sup_{|\eta|=j-1} \|\nabla^\eta (T-S) T^{n-l-1} p\|_\infty + C' \rho_j^n s^n \sup_{|\mu|=j} \|\nabla^\mu p\|_\infty \\ &\le 2C' C_P \sum_{l=0}^{n-1} \rho_j^l s^l \Omega_j(T^{n-l-1}p) + C' \rho_j^n s^n \sup_{|\mu|=j} \|\nabla^\mu p\|_\infty. \end{aligned} \tag{3.5}$$

Recall that by Theorem 1.4, $\rho_m = \max(\rho_k, |\lambda_{\min}|^{-m})$ for $m < k$. We use induction on 'the order of differences' j to show (3.3) and start with the case $j = 1$. We show (3.3) for the case $j = 1$ for the constants

$$C = C_1 := 2C' \quad \text{and} \quad \sigma'' = \sigma''_1 := \min(\sigma, \frac{\rho_1 s(1-\rho_1 s)}{8C'^2 C_P}, 1). \tag{3.6}$$

To that end, we perform induction on the subdivision level n. The case $n = 0$ is clear, since $C' \ge 1$. As to general n assume that (3.3) holds for all smaller values than n (still, $j = 1$). Observing that we set $C = 2C'$ in (3.6), we have

$$\Omega_1(T^{n-l-1}p) = \sup_{|\mu|=1} \|\nabla^\mu T^{n-l-1} p\|_\infty^2 \le 4C'^2 (\rho_1 s)^{2n-2l-2} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \tag{3.7}$$

by the induction hypothesis. This implies, using (3.5),

$$\sup_{|\mu|=1} \|\nabla^\mu T^n p\|_\infty \le C' \rho_j^n s^n \Big(8C'^2 C_P \big(\sum_{l=0}^{n-1} (\rho_1 s)^{n-l-2} \big) \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty + 1 \Big) \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty. \tag{3.8}$$

Applying the geometric series, we get

$$\sum\nolimits_{l=0}^{n-1} (\rho_1 s)^{n-l-2} \le (\rho_1 s)^{-1} (1-\rho_1 s)^{-1}. \tag{3.9}$$

Our choice of σ''_1 in (3.6) implies that

$$\sup\nolimits_{|\mu|=1} \|\nabla^\mu p\|_\infty \le \sigma''_1 \le 1/8\ C'^{-2} C_P^{-1} \rho_1 s (1-\rho_1 s). \tag{3.10}$$

Plugging (3.9) and (3.10) into (3.8), we obtain (3.3) for the case $j = 1$.

We perform the induction step. As an induction hypothesis we assume that (3.3) is valid for $i = 1,\dots,j-1$ instead of j with constants $C = 2^i C'$ and σ''_{j-1}. This means that we consider input $p \in P_{N,\sigma''_{j-1}}$, for which iterated subdivision using T is defined and for which $T^n p \in P_{N,\sigma}$ for all $n \in \mathbb{N}$. We assume that, for such input and $i = 1,\dots,j-1$,

$$\sup_{|\mu|=i} \|\nabla^\mu T^n p\|_\infty \le 2^i C' \max(\rho_k, |\lambda_{\min}|^{-i})^n s'^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty, \tag{3.11}$$

for any $s' > 1$. We choose s' as

$$s' = s^{1/(j+1)}. \tag{3.12}$$

Recall that s is chosen in a way such that $\rho_1 s < 1$, as well as $\rho_j s < 1$. We perform induction on n to show (3.11) for $i = j$ for the constants

$$C = 2^j C' \quad \text{and} \quad \sigma'' = \sigma''_j = \min(\sigma''_{j-1}, \frac{2^{j-2}(1-\rho_1 s)\rho_j s}{DC_P}, 1),$$

where we define the constant D by

$$D = 2^{j+1} C'^{j+1} |\Gamma_j|.$$

The choice of D will become clear later on. The case $n = 0$ is obvious. For the induction step we assume that (3.11) is valid for $i = j$ and n replaced by smaller values than n. There is only one $\gamma \in \Gamma_j$ with $\gamma_j \neq 0$, namely $\gamma = (1, 0, \ldots, 0, 1)$. Using the induction hypothesis its contribution to (3.1) can be estimated as follows:

$$\sup_{|\mu|=j} \|\nabla^\mu T^{n-l-1} p\|_\infty \sup_{|\mu|=1} \|\nabla^\mu T^{n-l-1} p\|_\infty \leq 2^{j+1} C'^2 (\rho_j \rho_1 s^2)^{n-l-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2.$$

For the other summands $\gamma \in \Gamma_j$ (with $\gamma_j = 0$) we obtain, using the induction hypothesis,

$$\begin{aligned}
\prod_{i=1}^{j} \sup_{|\mu|=i} \|\nabla^\mu T^{n-l-1} p\|_\infty^{\gamma_i} &\leq \prod_{i=1}^{j} 2^{i\gamma_i} C'^{\gamma_i} (\rho_i s')^{\gamma_i (n-l-1)} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^{\gamma_i} \\
&\leq 2^{j+1} C'^{j+1} s'^{(j+1)(n-l-1)} \prod_{i=1}^{j} \rho_i^{\gamma_i (n-l-1)} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^{\gamma_i} \\
&\leq 2^{j+1} C'^{j+1} s^{(n-l-1)} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \cdot \prod_{i=1}^{j} \rho_i^{\gamma_i (n-l-1)}.
\end{aligned} \tag{3.13}$$

The last inequality is a consequence of $\sup_{|\mu|=1} \|\nabla^\mu p\|_\infty \leq 1$ which is due to our choice of σ''. Next, we show the estimate

$$\prod\nolimits_{i=1}^{j} \rho_i^{\gamma_i} \leq \rho_j \rho_1. \tag{3.14}$$

We recall that Theorem 1.4 states that $\rho_i = \max(|\lambda_{\min}|^{-i}, \rho_k)$; this implies in particular that $\rho_k \leq \rho_{k-1} \leq \ldots \leq \rho_1 < 1$. We distinguish different cases: If $j \leq -\log_{|\lambda_{\min}|} \rho_k$, which means that $\rho_k \leq |\lambda_{\min}|^{-j}$, we apply Theorem 1.4 and obtain that $\rho_i = |\lambda_{\min}|^{-i}$ for all $1 \leq i \leq j$. As a consequence,

$$\prod\nolimits_{i=1}^{j} \rho_i^{\gamma_i} = |\lambda_{\min}|^{-j-1} = \rho_j \rho_1.$$

This shows (3.14) in case that $j \leq -\log_{|\lambda_{\min}|} \rho_k$. So we can assume that $j > -\log_{|\lambda_{\min}|} \rho_k$, i.e., $\rho_k > |\lambda_{\min}|^{-j}$. Then Theorem 1.4 implies $\rho_k = \rho_j$. If there is some non-zero factor γ_{i_0} such that $i_0 \geq -\log_{|\lambda_{\min}|} \rho_k$, then

$$\prod\nolimits_{i=1}^{j} \rho_i^{\gamma_i} \leq \rho_k \cdot \prod\nolimits_{i=1, i \neq i_0}^{j} \rho_i^{\gamma_i} \leq \rho_k \rho_1 = \rho_j \rho_1.$$

This is true since $\rho_{i_0} = \rho_k = \rho_j$ and $\rho_i \le \rho_1 < 1$. If $\gamma_i \ne 0$ only for i smaller than $-\log_{|\lambda_{\min}|} \rho_k$, then

$$\prod_{i=1}^{j} \rho_i^{\gamma_i} = |\lambda_{\min}|^{-j-1} = |\lambda_{\min}|^{-j} |\lambda_{\min}|^{-1} < \rho_k \rho_1 = \rho_j \rho_1.$$

This shows (3.14). Using the estimate (3.14) in (3.13), we obtain

$$\begin{aligned} \Omega_j(T^{n-l-1}p) &\le (2^{j+1}C'^2 + (|\Gamma_j| - 1)2^{j+1}C'^{j+1})(\rho_j\rho_1 s^2)^{n-l-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \\ &\le D(\rho_j \rho_1 s^2)^{n-l-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2. \end{aligned} \tag{3.15}$$

We use (3.5) and (3.15) to obtain

$$\begin{aligned} \sup_{|\mu|=j} \|\nabla^\mu T^n p\|_\infty &\le 2C'C_P \sum_{l=0}^{n-1} \rho_j^l s^l \Omega_j(T^{n-l-1}p) + C'\rho_j^n s^n \sup_{|\mu|=j} \|\nabla^\mu p\|_\infty \\ &\le 2C'C_P D \sum_{l=0}^{n-1} (\rho_j s)^{n-1} (\rho_1 s)^{n-l-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 + 2^{j-1} C' \rho_j^n s^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty \\ &\le C'\rho_j^n s^n \Big(2C_P D (1-\rho_1 s)^{-1} (\rho_j s)^{-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty + 2^{j-1} \Big) \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty \\ &\le 2^j C' \rho_j^n s^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty. \end{aligned}$$

The last inequality is valid since, by the choice of σ_j'', the term in brackets is smaller than 2^j. So the induction w.r.t. both n and j is complete. Finally, the statement (3.4) is shown by (3.7) and (3.15). □

With these preparations at hand we can prove Theorem 3.3.

Proof of Theorem 3.3. We choose $s > 1$ such that $s\rho_1 < 1$. We let ϕ_0 be the piecewise linear B-Spline. Since both ϕ_0 and the refinable function ϕ associated with S reproduce constant functions and have compact support, the inequality

$$\|p * \phi_0 - p * \phi\|_\infty \le C_1 \sup\nolimits_{|\mu|=1} \|\nabla^\mu p\|_\infty$$

holds for all bounded input data p with C_1 not depending on p. Furthermore,

$$\|p * \phi_0\|_\infty \le C_2 \|p\|_\infty \quad \text{and} \quad \|p * \phi\|_\infty \le C_3 \|p\|_\infty,$$

where the constants are the corresponding operator norms. Let C_4 be the constant from the first order proximity condition, and C_5 be the constant from (3.3). We use the symbol σ_1'' for the constant from (3.6). Then we let

$$\sigma'' = \min\Big(\sigma_1'', \frac{\sigma'}{4C_1C_5}, \Big(\frac{\sigma'(1-\rho_1^2 s^2)}{2C_3C_4C_5^2} \Big)^{1/2}, \frac{\sigma}{C_5} \Big). \tag{3.16}$$

We show that, for input data $p \in P_{N'',\sigma''}$, $T^n p$ is defined for all $n \in \mathbb{N}_0$, and that $T^n p \in P_{N,\sigma}$. Then the assumptions of Lemma 3.7 are met and we can use this lemma to deduce convergence. We use induction on n. The case $n = 0$ is clear. As induction hypothesis we assume that for all $k = 0, \dots, n-1$, $T^k p$ is well-defined and that it belongs to $P_{N,\sigma}$. Furthermore, we assume that $T^k p$ takes values in $U_{\sigma'}(N'')$. Then $T^n p$ is defined, and for $k = 0, \dots, n-1$,

$$\begin{aligned}\|T^{k+1}p * \phi(M^{k+1}\cdot) - T^k p * \phi(M^k\cdot)\|_\infty &= \|(T-S)T^k p * \phi(M^k\cdot)\|_\infty \\ &\leq C_3 C_4 \sup_{|\mu|=1} \|\nabla^\mu T^k p\|_\infty^2.\end{aligned}$$

Using Lemma 3.7 and the above estimate we obtain, for $m < n$,

$$\begin{aligned}&\|T^n p * \phi_0(M^n\cdot) - T^m p * \phi_0(M^m\cdot)\|_\infty \\ &\leq \|T^n p * (\phi_0(M^n\cdot) - \phi(M^n\cdot))\|_\infty + \sum_{k=m}^{n-1} \|T^{k+1}p * \phi(M^{k+1}\cdot) - T^k p * \phi(M^k\cdot)\|_\infty \\ &\quad + \|T^m p * (\phi_0(M^m\cdot) - \phi(M^m\cdot))\|_\infty \\ &\leq C_1 \sup_{|\mu|=1} \|\nabla^\mu T^n p\|_\infty + C_3 C_4 \sum_{k=m}^{n-1} \sup_{|\mu|=1} \|\nabla^\mu T^k p\|_\infty^2 + C_1 \sup_{|\mu|=1} \|\nabla^\mu T^m p\|_\infty \\ &\leq 2C_1 C_5 (\rho_1 s)^m \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty + C_3 C_4 C_5^2 \sum_{k=m}^{n-1} (\rho_1 s)^{2k} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \\ &\leq \tfrac{\sigma'}{2}(\rho_1 s)^m + C_3 C_4 C_5^2 (\rho_1 s)^m (1 - \rho_1^2 s^2)^{-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2 \leq \sigma'(\rho_1 s)^m. \end{aligned} \tag{3.17}$$

The last inequality is true because of the choice of σ'' in (3.16) and because, by assumption, $\|\nabla^\mu p\|_\infty \leq \sigma''$. If we let $m = 0$ in (3.17), we obtain that $T^n p$ takes values in $U_{\sigma'}(N'')$. Furthermore, $\sup_{|\mu|=1} \|\nabla^\mu T^n p\|_\infty \leq C_5(\sigma/C_5) = \sigma$. This completes the induction.

A straightforward consequence of (3.17) is that $T^n p * \phi_0(M^n\cdot)$ is a Cauchy sequence, which implies the convergence of T for input data p which belong to $P_{N'',\sigma''}$. Furthermore,

$$\|T^n p * (\phi_0(M^n\cdot) - \phi(M^n\cdot))\|_\infty \leq C_1 \sup_{|\mu|=1} \|\nabla^\mu T^n p\|_\infty,$$

and the right hand side approaches 0 as $n \to \infty$. This implies that the sequence of uniformly continuous functions $T^n p * \phi(M^n\cdot)$ converges to the limit of T for input p as $n \to \infty$. Hence, we also have that the limit f_p of the nonlinear scheme is uniformly continuous. This completes the proof. □

Our next objective is the proof of our main result on smoothness of nonlinear subdivision schemes in the regular grid case.

Proof of Theorem 3.4. Since, by assumption, T converges for data p, subdivided data eventually gets dense. So we can w.l.o.g. assume that p itself is already dense enough.

We show that $T^n p * \phi(M^n\cdot)$ is a Cauchy sequence in Lip_γ. Then Theorem 3.3 implies that the limit function of T belongs to Lip_γ. We choose $s > 1$ such that $s^2\rho_1 < 1$. We let C_1 be the constant of Proposition 1.11 and C_2 be the proximity constant of (3.2), and we denote the constant of (3.4) by L. We use Proposition 1.11 to estimate

$$\begin{aligned}
&|T^{n+1}p * \phi(M^{n+1}\cdot) - T^n p * \phi(M^n\cdot)|_{\mathrm{Lip}_\gamma,k} \\
&\qquad = |T^{n+1}p * \phi(M^{n+1}\cdot) - ST^n p * \phi(M^{n+1}\cdot)|_{\mathrm{Lip}_\gamma,k} \\
&\qquad \le C_1|\lambda_{\max}|^{\gamma n} s^n \sup_{|\mu|=k} \|\nabla^\mu (S-T)T^n p\|_\infty \\
&\qquad \le 2C_1C_2|\lambda_{\max}|^{\gamma n} s^n \Omega_k(T^n p).
\end{aligned} \tag{3.18}$$

By (3.4),

$$\Omega_k(T^n p) \le L(\rho_k \rho_1 s)^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2. \tag{3.19}$$

By the definition of the smoothness index $\nu_{a,M}$, we have $\rho_k = |\lambda_{\max}|^{-\nu_{a,M}}$. Therefore, $\rho_k|\lambda_{\max}|^\gamma < 1$. Using this fact and plugging (3.19) into (3.18) we get

$$|T^{n+1}p * \phi(M^{n+1}\cdot) - T^n p * \phi(M^n\cdot)|_{\mathrm{Lip}_\gamma,k} \le 2C_1C_2L\ r^n \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2,$$

where $r = \rho_k|\lambda_{\max}|^\gamma s^2 |\lambda_{\min}|^{-1} < 1$. We apply this estimate to obtain

$$\begin{aligned}
&\|T^{n+l}p * \phi(M^{n+l}\cdot) - T^n p * \phi(M^n\cdot)\|_{\mathrm{Lip}_\gamma,k} \\
&= |T^{n+l}p * \phi(M^{n+l}\cdot) - T^n p * \phi(M^n\cdot)|_{\mathrm{Lip}_\gamma,k} + \|T^{n+l}p * \phi(M^{n+l}\cdot) - T^n p * \phi(M^n\cdot)\|_\infty \\
&\le C_1C_2L\ r^n(1-r)^{-1} \sup_{|\mu|=1} \|\nabla^\mu p\|_\infty^2. + \|T^{n+l}p * \phi(M^{n+l}\cdot) - T^n p * \phi(M^n\cdot)\|_\infty,
\end{aligned}$$

where the second term tends to 0 by Theorem 3.3. Therefore, $T^n p * \phi(M^n\cdot)$ is a Cauchy sequence in Lip_γ. This completes the proof. □

Proof of Theorem 3.5. It remains to verify the proximity inequalities. The geometric analogues considered in this corollary are instances of the so-called g-f-analogues introduced in [64]. Therefore the proximity inequalities for the intrinsic mean analogue (2.5), the log-exp analogue (2.9), and the projection analogue (2.14) follow directly from Theorems 5.8 and 5.9 of [16]. □

Proof of Corollary 3.6. Theorem 3.5 ensures that the mentioned analogues produce limits f_p whose smoothness index $\nu(f_p)$ is at least as high as the smoothness index $\nu_{a,M}$ of the linear scheme. Then the smoothness index of the refinable function $\nu(\phi)$ equals the smoothness index $\nu_{a,M}$ [18]. The second statement of the corollary follows from the fact that if $S_{a,M}$ produces C^k limits, then the corresponding smoothness index $\nu_{a,M}$ is strictly greater than k [20]. □

4 Analysis of nonlinear schemes on irregular combinatorics

In this chapter we consider meshes with irregular combinatorics. We show that a nonlinear scheme which is ‘not too far apart’ from a linear one converges (for dense enough input data) and produces C^1 limits, if the linear scheme meets certain assumptions. The geometric schemes considered in Chapter 2 are schemes which are ‘not too far apart’ from the linear ones they are derived from. This allows us to conclude that, also on irregular meshes, they converge (for dense enough input data) and produce C^1 limits. We start by stating our main result. The remainder of the chapter is concerned with its proof where we first consider (nonlinear) perturbations of standard schemes and then proceed to the more general class of shift-invariant schemes, both defined in Chapter 1.2.

The results of this chapter are contained in the papers [62, 60].

4.1 Statement of the results

The main result of this chapter is the following:

Theorem 4.1. *The geometric analogues of Chapter 2, i.e., the intrinsic mean analogue, the log-exp analogue, the geodesic analogue and the projection analogue, of the linear schemes defined in Chapter 1.2 (both standard schemes and shift-invariant schemes) converge provided input data are dense enough and the mesh under consideration has bounded face and vertex valences. These limits are even* C^1*, if considered w.r.t. the characteristic parametrization.*

There are two ways to treat convergence (for dense enough input) and smoothness issues for geometric schemes: The first is to embed the manifold into $\mathbb{R}^d$, the second is to go to a chart neighborhood. In both cases the geometric scheme T, which then acts in Euclidean space, is shown to meet a proximity condition with a linear scheme S. For meshes of arbitrary combinatorics with a fixed, but arbitrary, bound L on the valence of vertices and faces in the mesh, we define the *class of σ-dense meshes $P_{N,\sigma}$* with values in N as all those meshes (K_i, h_i) whose positioning function h_i is bounded and has values in N, and where the distance of neighboring vertices in N is smaller than σ. We use the following (local) proximity condition.

Definition 4.2. *We consider a subset $N \subset \mathbb{R}^d$, a denseness bound $\sigma > 0$, and two subdivision schemes S, T with the same topological rule.*

Then S and T satisfy a (local) proximity condition w.r.t. $P_{N,\sigma}$, if there is a constant C, such that for all input meshes (K_0, h_0) which belong to $P_{N,\sigma}$, and all vertices $w \in V_1$, $h_1^T(w)$ only depends on $h_0|_{\operatorname{supp}(\alpha_{\cdot,w})}$ and

$$\|h_1^S(w) - h_1^T(w)\| \leq C \sup_{v_1,v_2 \in \operatorname{supp}(\alpha_{\cdot,w})} \|h_0(v_1) - h_0(v_2)\|^2, \tag{4.1}$$

Here h_1^S and h_1^T are the resulting positioning functions after refinement of (K_0, h_0) using S and T, respectively.

Here $\mathrm{supp}(\alpha_{\cdot,w})$ is the support of the stencil $\alpha_{v,w}$ which are those vertices v which contribute to the calculation of $h_1^S(w)$. The proximity conditions used in [57] and [14] are slightly weaker than the one in Definition 4.2; the difference is that we have to use locality in order to show smoothness for the general mesh case.

Like in the analysis of linear schemes we can restrict the analysis of a nonlinear scheme T (which is in local proximity with a linear scheme S) to the neighborhood of an extraordinary face or vertex, respectively, when the regular mesh case has already been treated (which is done in Chapter 3). This restriction is possible for the following reasons: By the locality of the proximity condition (4.1), T is also a local scheme, and a new vertex generated by T depends only on the old ones in the support of the according stencil of S. In contrast to the linear case, this is not enough, because some additional technicality arises in connection with the 'dense enough' assumption for input data, which can be overcome as follows: If we have an input mesh with an upper bound on the valence of faces and vertices, we can postulate the input data even denser, such that after the first subdivision steps, the mesh is still dense enough near the singularity, but the connectivity around the latter is that of a k-regular mesh locally near the singularity. Therefore, a convergence or smoothness statement for k-regular meshes implies a corresponding statement for the general case. This means that in order to prove Theorem 4.1, it is enough to prove Theorem 4.4, Theorem 4.5, and Corollary 4.6. The first one is a convergence theorem. The formulation is rather technical, which is mainly due to the fact that nonlinear schemes are in general not globally defined and therefore we have to guarantee the well-definedness of the data during the subdivision process. For our terminology concerning k-regular meshes we refer to Chapter 1.2.

Definition 4.3. *A subdivision scheme T is called convergent for k-regular input data p_0, defined on V_0, if iterated subdivision using T is well-defined, and if there is a uniformly continuous function f, defined on D, such that*

$$\|f|_{V_k} - T_{k-1,0}p_0\|_\infty \to 0 \qquad \text{as } k \to \infty.$$

Theorem 4.4. *Let S be a linear subdivision scheme as introduced in Chapter 1.2, and let S and T fulfill a local proximity condition w.r.t. some $P_{N,\sigma}$. Assume that $T_n p_n$ takes its values in a set N' for all k-regular data $p_n \in P_{N,\sigma}$ where N' is some set with $N \subset N' \subset \mathbb{R}^n$. Assume further that there is a subset $N'' \subset N$ and $\sigma' > 0$ such that the σ'-neighborhood $U_{\sigma'}(N'')$ obeys*

$$U_{\sigma'}(N'') \cap N' \subset N.$$

Then there is a denseness bound $\sigma'' > 0$ such that T converges for data $p_0 \in P_{N'',\sigma''}$ given on V_0, and

$$S_{\infty,i+1}T_{i,0}p_0 \to T_{\infty,0}p_0 \quad \text{as} \quad i \to \infty, \tag{4.2}$$

where convergence is understood in the sense of uniform convergence.

Since we consider quite general sets N in this theorem we have to assume the existence of the set N'' with the above properties. However, if, for example, N is a ball of radius r, then N'' can be chosen as the ball with the same center and radius $r - \sigma'$. Then, for this particular choice of N, the theorem says that, if S and T fulfill proximity conditions w.r.t. $P_{N,\sigma}$, then T converges for dense enough input in the smaller ball N''.

Theorem 4.5. *We consider a linear subdivision scheme S as introduced in Chapter 1.2 on k-regular input data. Assume that S and the scheme T fulfill a local proximity inequality w.r.t. some class $P_{N,\sigma}$ of σ-dense input. Then the limit of subdivision using T is smooth. More precisely, the limit function $T_{\infty,0}p_0 \circ \chi^{-1}$ for data p_0 on V_0 is well-defined and C^1 in a neighborhood of the (extraordinary) point $\chi(0)$.*

Corollary 4.6. *The geometric subdivision rules defined in Chapter 2 converge for dense enough input data on k-regular combinatorics and produce C^1 limits w.r.t. the characteristic parametrization.*

The purpose of Chapter 4.2 is to prove these statements for standard schemes, whereas Chapter 4.4 proves the above statements for the more general schemes of Chapter 1.2.

4.2 Analysis of standard schemes

In this part we prove Theorem 4.4 and Theorem 4.5 for standard schemes. We start with some preparations and formulate some auxiliary lemmas we need for both the convergence and the smoothness result. Then we prove the main results. These are contained in [62].

As in Chapter 1.2 we interpret k-regular meshes as functions $p_n : V_n \to \mathbb{R}^d$. We introduce the following (nonlinear) difference operator: For some $n \in \mathbb{N}_0$, we consider bounded input $p_n \in l^\infty(V_n, \mathbb{R}^d)$ and a subset $B \subset V_n$. We define

$$\Delta_B p_n(v) = \sup\{\|p_n(v) - p_n(w)\|_{\mathbb{R}^d} : w \text{ is a face neighbor of } v \text{ in } B\},$$

and furthermore

$$\mathcal{D}_B(p_n) := \sup\{\Delta_B p_n(v) : \; v \in B\}.$$

We drop the index B, if $B = V_n$. The quantity $\mathcal{D}_B$ obviously satisfies the triangle inequality. Then the class $P_{N,\delta}$ of σ-dense input can be written as

$$P_{N,\sigma} = \cup_{n\in\mathbb{N}_0}\{p_n : V_n \to N \mid p_n \text{ is bounded, } \mathcal{D}(p_n) \leq \sigma\}.$$

Here N denotes some subset of $\mathbb{R}^d$. With view towards application to geometric schemes, one can think of N as a chart neighborhood or a surface embedded in Euclidean space. Then the local proximity condition in Definition 4.2 reads: There is $C > 0$ such that for all $n \in \mathbb{N}_0$, and all n-th level data $p_n \in P_{N,\sigma}$,

$$\|S_n p_n(v) - T_n p_n(v)\| \leq C \sup_{v_1,v_2 \in \text{supp}_S(v)} \|p_n(v_1) - p_n(v_2)\|^2.$$

Note that any n-th level control set of $U \subset D$ w.r.t. S also controls the limit of subdivision using T on this set U due to the locality of the proximity condition. We consider the sequence of sets $V_n' = V_n$, $V_n' = \text{ctrl}^n(D_i^j)$ or $V_n' = \text{ctrl}^n(D_n)$, where $n = 0, 1, 2, \ldots$. For those sets, a local proximity condition implies that there is a constant F such that for every level n and for data $p_n \in P_{N,\sigma}$,

$$\|S_n p_n(v) - T_n p_n(v)|_{V_{n+1}'}\|_\infty \leq F(\mathcal{D}_{V_n'}(p_n))^2, \tag{4.3}$$

This follows immediately from the locality of S, using the triangle inequality and the fact that $(a+b)^2 \leq 2\,(a^2+b^2)$ for $a, b \in \mathbb{R}$.

We state a technical lemma which is a key ingredient in the proof of both the convergence and smoothness result. The sequence g_n in the lemma should be thought of as the data the nonlinear scheme produces.

Lemma 4.7. *Let S be a standard scheme. Let $V_n' \subset V_n$ $(n = 0, 1, 2, \dots)$ be a sequence of subsets, such that subdivision of data p_n on V_n' determines $S_n p_n$ on V_{n+1}'. We assume that there are constants $C \geq 1$ and $\gamma \in (0,1)$ such that the following is true: For all levels n, all n-th level data $p_n \in l^\infty(V_n', \mathbb{R}^d)$, and all $k \geq n$,*

$$\mathcal{D}_{V_k'}(S_{k-1,n} p_n) \leq C\gamma^{k-n} \mathcal{D}_{V_n'}(p_n). \tag{4.4}$$

Let $m \in \mathbb{N}_0$ and suppose there is a constant $C' > 0$ such that for a sequence $\{g_n\}_{n=0}^{m+1}$ with $g_n \in l^\infty(V_n', \mathbb{R}^d)$ we have the inequalities:

$$\|g_{n+1} - S_n g_n\|_\infty \leq C'\gamma(\mathcal{D}_{V_n'}(g_n))^2 \tag{4.5}$$

for all $0 \leq n \leq m$, and

$$\mathcal{D}_{V_0'}(g_0) \leq \frac{1-\gamma}{8C'C^2}. \tag{4.6}$$

Then, for all $1 \leq k \leq m+1$,

$$\mathcal{D}_{V_k'}(g_k) \leq 2C\gamma^k \mathcal{D}_{V_0'}(g_0).$$

Proof. We use induction on k. For $k = 1$, Equations (4.5), (4.4) and (4.6) consecutively yield

$$\begin{aligned}\mathcal{D}_{V_1'}(g_1) \leq& \mathcal{D}_{V_1'}(g_1 - S_0 g_0) + \mathcal{D}_{V_1'}(S_0 g_0) \leq 2C'\gamma(\mathcal{D}_{V_0'}(g_0))^2 + C\gamma\mathcal{D}_{V_0'}(g_0) \\ \leq& C(2C'\mathcal{D}_{V_0'}(g_0) + 1)\gamma\mathcal{D}_{V_0'}(g_0) \leq 2C\gamma\mathcal{D}_{V_0'}(g_0).\end{aligned}$$

We proceed with the induction step. Using (4.4) and (4.5) we have

$$\begin{aligned}\mathcal{D}_{V_k'}(g_k) &\leq \sum_{l=1}^{k} \mathcal{D}_{V_k'}(S_{k-1,l} g_l - S_{k-1,l-1} g_{l-1}) + \mathcal{D}_{V_k'}(S_{k-1,0} g_0) \\ &\leq \sum_{l=1}^{k} C\gamma^{k-l} \mathcal{D}_{V_l'}(g_l - S_{l-1,l-1} g_{l-1}) + \mathcal{D}_{V_k'}(S_{k-1,0} g_0) \\ &\leq \sum_{l=1}^{k} 2C\gamma^{k-l} \cdot \gamma C'(\mathcal{D}_{V_{l-1}'}(g_{l-1}))^2 + C\gamma^k \mathcal{D}_{V_0'}(g_0)\end{aligned}$$

We use the induction hypothesis and (4.6) to obtain

$$\begin{aligned}\mathcal{D}_{V_k'}(g_k) \leq& \sum_{l=1}^{k} 8CC'\gamma^{k-l+1} C^2 \gamma^{2(l-1)} (\mathcal{D}_{V_0'}(g_0))^2 + C\gamma^k \mathcal{D}_{V_0'}(g_0) \\ \leq& C\mathcal{D}_{V_0'}(g_0) \left[C^2 \sum_{l=1}^{k} 8C'\gamma^{k+l-1} \mathcal{D}_{V_0'}(g_0) + \gamma^k\right] \\ \leq& C\gamma^k \mathcal{D}_{V_0'}(g_0) \left[\frac{8C'C^2}{1-\gamma} \mathcal{D}_{V_0'}(g_0) + 1\right] \leq 2C\gamma^k \mathcal{D}_{V_0'}(g_0).\end{aligned}$$

This completes the proof. □

The next lemma uses differences to express the condition on the subdominant eigenvalues of the subdivision matrix. It is possibly not new, but we did not find it in the literature.

Lemma 4.8. *Let $A: \mathbb{R}^m \to \mathbb{R}^m$ be a matrix with single eigenvalue 1 for the eigenvector $v_1 = (1, \dots, 1)^T$, and assume that all other eigenvalues have smaller modulus. We set $\Delta'(b) := \sup_{1 \le k,j \le m} |b_k - b_j|$ for $b \in \mathbb{R}^m$. Then for every $\varepsilon > 0$ there is $C > 1$ such that, for all $l \in \mathbb{N}$, and all $b \in \mathbb{R}^m$,*

$$\Delta'(A^l b) \le C(|\lambda_2| + \varepsilon)^l \Delta'(b), \tag{4.7}$$

where λ_2 is a subdominant eigenvalue of A. If all eigenvalues μ with $|\mu| = |\lambda_2|$ have equal algebraic and geometric multiplicity, then we can choose $\varepsilon = 0$.

Proof. With the Jordan normal form J of A we have $AV = VJ$, where the generalized eigenvectors of A are stored in $V = (v_1, \dots, v_m)$. We assume that J is ordered by modulus and denote the dual basis of V by $\{v_i^*\}_{i=1}^m$. Then we can write

$$A^l b = A^l V V^{-1} b = \sum_{i=1}^m A^l v_i v_i^*(b).$$

Since $A^l v_1 = (1, \dots, 1)^T$,

$$\begin{aligned} |(A^l b)_j - (A^l b)_k| &= \left| \sum\nolimits_{i=2}^m [(A^l v_i)_j - (A^l v_i)_k] v_i^*(b) \right| \\ &\le \sum\nolimits_{i=2}^m |(A^l v_i)_j - (A^l v_i)_k| \sup\nolimits_{2 \le i \le m} |v_i^*(b)|. \end{aligned} \tag{4.8}$$

For estimating the first factor, consider a Jordan block D of A of size α with eigenvalue μ, eigenvector w_0, and ordered generalized eigenvectors $w_1, \dots, w_{\alpha-1}$. Then for integers $\beta > \alpha > \gamma \ge 0$ and $1 \le j, k \le m$,

$$|(A^\beta w_\gamma)_j - (A^\beta w_\gamma)_k| \le \left| \mu^{\beta-\gamma} \sum\nolimits_{\delta=0}^{\gamma} \tbinom{\beta}{\gamma-\delta} \mu^\delta \right| \ \sup\nolimits_{0 \le \delta \le \gamma} |(w_\delta)_j - (w_\delta)_k|.$$

For $\varepsilon > 0$ there is a constant $C_\mu > 0$ such that for all $\beta > \alpha > \gamma \ge 0$,

$$\left| \mu^{\beta-\gamma} \sum\nolimits_{\delta=0}^{\gamma} \tbinom{\beta}{\gamma-\delta} \mu^\delta \right| \le C_\mu (|\mu| + \varepsilon)^\beta,$$

since the sum on the left-hand side is a polynomial in β. Thus, for $\varepsilon > 0$ there is $C_0 > 0$, such that

$$\sum\nolimits_{i=2}^m |(A^l v_i)_j - (A^l v_i)_k| \le C_0 (|\lambda_2| + \varepsilon)^l \sup\nolimits_{1 \le i \le m} |(v_i)_j - (v_i)_k|.$$

For the second factor on the right hand side of (4.8) we have, for $2 \le i \le m$,

$$\begin{aligned} |v_i^*(b)| &= |v_i^*(b - b_1 v_1)| \le \|v_i^*\| \|b - b_1 v_1\|_\infty \\ &\le (\sup\nolimits_{2 \le i \le m} \|v_i^*\|) \cdot \sup\nolimits_{2 \le i \le m} |b_i - b_1|, \end{aligned}$$

where $\|\cdot\|$ is the norm of the linear functionals on $l^\infty(\{1, \dots, m\}, \mathbb{C})$. From this, the lemma follows. □

Convergence analysis

In this part we prove Theorem 4.4 for standard schemes. The proof of this theorem consists of two parts: The first one is to show that the contractivity of the differences of data generated by S implies the convergence of data generated by T, if the input is dense enough. The second part is to show this contractivity for a standard scheme S.

In order to show the first part, we consider the map

$$E : D \to \mathbb{R}^2.$$

E bijectively maps the entire domain D to the plane by first squeezing the j-th quadrant into a sector of opening angle $2\pi/k$ with a shear transformation and then rotating it by an angle of $2\pi j/k$. We connect the points $E(V_n)$, by straight lines according to the k-regular connectivity and obtain a set of faces $\mathcal{F}_n$.

Note that the notion of convergence of a scheme for input p_0 is invariant under reparametrization with the help of E. So let us consider the whole subdivision process w.r.t. $E(V_n) \subset \mathbb{R}^2$ instead of $V_n \subset D$.

The next statement is clear by the definition of the vertex sets V_n, the face sets $\mathcal{F}_n$, and by the locality of S.

Lemma 4.9. *Let S be a standard scheme acting as operators $S_n : l^\infty(E(V_n), \mathbb{R}^d) \to l^\infty(E(V_{n+1}), \mathbb{R}^d)$. Then $\|S_n\|$ is uniformly bounded, and each face of $\mathcal{F}_n$ is convex. Furthermore there are constants $C_1, C_2, R > 0$ such that for all $n \in \mathbb{N}_0$,*

(i) the infimum d' of distances of neighboring vertices in $E(V_n)$ satisfies $C_1 2^{-n} \leq d' \leq \max_{F\in\mathcal{F}_n} \operatorname{diam} F \leq C_2 2^{-n}$;

(ii) the value $S_n p_n(v)$ is an affine combination of the local values $\{p_n(w) : w \in B(v, 2^{-n}R) \cap E(V_n)\}$, where $B(x, r)$ is the open ball with radius r around x.

Next, we need interpolation operators to extend the discrete data to continuous functions. For every $n \in \mathbb{N}_0$, we define the *interpolation operator*

$$I_n : \; l^\infty(E(V_n), \mathbb{R}^d) \to C_u(\mathbb{R}^2, \mathbb{R}^d),$$

where C_u denotes the space of uniformly continuous functions, as follows: We split each face $F \in \mathcal{F}_n$ into triangles, each of them determined by F's barycenter and an edge. We get data for the barycenter by the barycenter of the data on the neighboring vertices. Then we use linear interpolation on the triangles. For x, y in a face, we obviously have

$$\sup_{x,y\in F_n} \|I_n p_n(x) - I_n p_n(y)\|_{\mathbb{R}^d} \leq \mathcal{D}(p_n).$$

Furthermore, these operators have the following properties:

Proposition 4.10. *Let S be a standard scheme. Suppose there is $\gamma \in (0, 1)$ and $C \geq 1$ such that for any $l \in \mathbb{N}$, $p_l \in l^\infty(E(V_l), \mathbb{R}^d)$, and $n \geq l$,*

$$\mathcal{D}(S_{n-1,l}\, p_l) \leq C\gamma^{n-l}\mathcal{D}(p_l). \tag{4.9}$$

Then, the sequence $\{I_n S_{n-1,l} p_l\}_{n\in\mathbb{N}_0}$ *converges to* $S_{\infty,l}p_l$ *in* $C_u(\mathbb{R}^2,\mathbb{R}^d)$*. In addition, there are constants* $C_B, C_I > 0$*, independent of* $l \in \mathbb{N}_0$ *and* p_l*, such that*

$$\|I_{l+1}S_l p_l - I_l p_l\| \le C_B \mathcal{D}(p_l),$$
$$\|S_{\infty,l}p_l|_{E(V_l)} - p_l\| \le \|S_{\infty,l}p_l - I_l p_l\| \le C_I \mathcal{D}(p_l).$$

Proof. We start by estimating $\|I_{m+1}S_m g_m - I_m g_m\|$ for general bounded g_m, defined on $E(V_m)$. Let $x \in \mathbb{R}^2$, and choose faces F_m of $\mathcal{F}_m$ and F_{m+1}, of $\mathcal{F}_{m+1}$, resp., which contain x. In addition denote by v_m, resp., v_{m+1}, a vertex of F_m, resp., F_{m+1}, nearest to x. Then,

$$\begin{aligned}\|I_{m+1}S_m g_m(x) - I_m g_m(x)\| &\le \|I_{m+1}S_m g_m(x) - I_{m+1}S_m g_m(v_{m+1})\| + \\ &+ \|S_m g_m(v_{m+1}) - g_m(v_m)\| + \|I_m g_m(v_m) - I_m g_m(x)\| \\ &\le \mathcal{D}(S_m g_m) + \mathcal{D}(g_m) + \|S_m g_m(v_{m+1}) - g_m(v_m)\|.\end{aligned}$$

In order to estimate the last summand on the right hand side, note that the value $S_m g_m(v_{m+1})$ is uniquely determined by $g_m|_{E(V_m)\cap B(v_{m+1},2^{-m}R)}$, where R is the constant from Lemma 4.9. With the constant C_2 of the same lemma it follows that $d(v_m, v_{m+1}) \le \frac{3}{2}C_2 2^{-m}$. Consequently, $\max\{d(v_m,y) :\ y \in E(V_m)\cap B(v_{m+1}, 2^{-m}R)\} \le \frac{3}{2}C_2 2^{-m} + 2^{-m}R$. Lemma 4.9*(i)* now implies that the number of faces in $\mathcal{F}_m$ not disjoint to $B(v_m, (\frac{3}{2}C_2 + R)2^{-m})$ is bounded by $D \in \mathbb{N}$, where D is independent of m or v_m. With $B^* := B(v_{m+1}, 2^{-m}R)$, we can write $S_m g_m(v_{m+1}) = \sum_{q\in E(V_m)\cap B^*} \alpha_q g_m(q)$ with $\sum_{q\in E(V_m)\cap B^*} \alpha_q = 1$ and $\sum_{q\in E(V_m)\cap B^*} |\alpha_q| \le \|S_m\|$. We obtain

$$\begin{aligned}\|S_m g_m(v_{m+1}) - g_m(v_m)\| &= \left\|\sum\nolimits_{q\in E(V_m)\cap B^*} \alpha_q(g_m(q) - g_m(v_m))\right\| \\ &\le \sum\nolimits_{q\in E(V_m)\cap B^*} |\alpha_q| \cdot \max\nolimits_{q\in E(V_m)\cap B^*} \|g_m(q) - g_m(v_m)\| \le \|S_m\| D \mathcal{D}(g_m).\end{aligned}$$

Altogether, it follows that

$$\|I_{m+1}S_m g_m - I_m g_m\| \le \mathcal{D}(S_m g_m) + (\|S_m\|D + 1)\mathcal{D}(g_m).$$

Equipped with this inequality, we estimate, for $n \ge l$,

$$\begin{aligned}\|I_{n+1}S_{n,l}p_l - I_n S_{n-1,l}p_l\|_\infty &\le \mathcal{D}(S_{n,l}p_l) + (\|S_n\|D + 1)\mathcal{D}(S_{n-1,l}p_l) \\ &\le C\gamma^{n-l}(\|S_n\|D + 2)\mathcal{D}(p_l).\end{aligned}$$

For $n'' \ge n' \ge n \ge l$ we make use of the geometric series and get

$$\|I_{n''+1}S_{n'',l}p_l - I_{n'}S_{n'-1,l}p_l\|_\infty \le C(\sup_{n\in\mathbb{N}_0} \|S_n\|D + 2)\gamma^{n-l}\frac{1}{1-\gamma}\mathcal{D}(p_l). \tag{4.10}$$

We immediately see the first inequality of the proposition if we let $n'' = n' = l$. Furthermore, (4.10) implies that $\{I_n S_{n-1,l}p_l\}_{n>l}$ is Cauchy in the space of bounded continuous functions. Since these functions are uniformly continuous, so is the limit, called f for the moment. Now, $\|f|_{E(V_n)} - S_{n-1,l}p_l\|_\infty \le \|f - I_n S_{n-1,l}p_l\| \to 0$ for $n \to \infty$. Thus f equals $S_{\infty,l}p_l$. Letting $n' = l$ in (4.10) yields the estimate

$$\|f - I_l p_l\| = \lim_{n''\to\infty} \|I_{n''+1}S_{n'',l}p_l - I_l p_l\| \le \frac{1}{1-\gamma}(\sup_{n\in\mathbb{N}_0} \|S_n\|D + 2)\mathcal{D}(p_l).$$

This proves the last statement of the proposition. □

Proposition 4.11. *Let S be a standard scheme acting on data defined on $E(V_n)$, and suppose* (4.9) *holds true, i.e.,*

$$\mathcal{D}(S_{l-1,n}p_n) \leq C\gamma^{l-n}\mathcal{D}(p_n),$$

for some $0 < \gamma < 1$, $C \geq 1$, and any data p_n. Let furthermore T and N'' be as in Theorem 4.4, with V_n replaced by its image under E. Assume also that S and T fulfill a local proximity condition w.r.t. some $P_{N,\sigma}$. Then there is $\sigma'' > 0$ such that for any input $p_0 \in P_{N'',\sigma''}$ on level 0, $T_{l-1,0}p_0$ $(l \in \mathbb{N})$ is defined and

$$\mathcal{D}(T_{l-1,0}p_0) \leq 2C\gamma^l\mathcal{D}(p_0), \tag{4.11}$$

with the same C and γ as in (4.9)*. For such p_0, $\{T_{l-1,0}p_0\}_{l\in\mathbb{N}}$ converges and the sequence $\{I_lT_{l-1,0}p_0\}_{l\in\mathbb{N}}$ converges to the same limit in $C_u(\mathbb{R}^2, \mathbb{R}^d)$.*

Proof. We denote the constant of (4.3) by F, and use N' and σ' from Theorem 4.4. C_B is the constant from Proposition 4.10. We define the denseness bound σ'' by

$$\sigma'' = \min\left\{\frac{(1-\gamma)\gamma}{8FC^2}, \frac{\sigma}{2C}, \frac{1-\gamma}{4C_BC}\sigma', \left(\frac{1-\gamma^2}{8FC^2}\sigma'\right)^{\frac{1}{2}}\right\}.$$

The reason for this choice of σ'' will become clear during the proof.

We intend to use Lemma 4.7 and induction on l. The condition (4.4) of Lemma 4.7 is fulfilled by our assumption on S. We start with $l = 1$, and let $g_0 = p_0$. Since $p_0 \in l^\infty(E(V_0), N'')$, $N'' \subset N$, and $\mathcal{D}(p_0) < \sigma'' < \sigma$, data p_0 lie in the domain of T_0. We let $g_1 = T_0p_0$. Now the proximity condition (4.3) ensures the condition (4.5) of Lemma 4.7 with constant $C' = \frac{F}{\gamma}$. Since $\mathcal{D}(p_0) \leq \frac{(1-\gamma)\gamma}{8FC^2}$, the condition (4.6) of Lemma 4.7 is fulfilled. We apply Lemma 4.7 and obtain

$$\mathcal{D}(T_0p_0) \leq 2C\gamma\mathcal{D}(p_0) < \sigma. \tag{4.12}$$

The use of Proposition 4.10, the proximity condition (4.3) and our choice of σ'' consecutively yield

$$\begin{aligned}\|I_1T_0p_0 - I_0p_0\| &\leq \|I_1T_0p_0 - I_1S_0p_0\| + \|I_1S_0p_0 - I_0p_0\| \\ &\leq \|T_0p_0 - S_0p_0\| + C_B\mathcal{D}(p_0) \leq F\mathcal{D}(p_0)^2 + C_B\mathcal{D}(p_0) \leq \sigma'.\end{aligned}$$

It follows that T_0p_0 takes its values in N, and together with (4.12) that T_0p_0 is in the domain of T_1.

We now perform the induction step. We assume that $g_m = T_{m-1,0}p_0$ is defined, that $T_{m-1,0}p_0$ takes its values in N, and that $T_{m-1,0}p_0$ is in the domain of T_m, for $0 \leq m \leq l$.

We let $g_{l+1} = T_{l,0}p_0$. Then the proximity condition (4.3) implies condition (4.5) of Lemma 4.7, again with $C' = \frac{F}{\gamma}$. So Lemma 4.7 yields

$$\mathcal{D}(T_{l,0}p_0) \leq 2C\gamma^{l+1}\mathcal{D}(p_0) < \sigma. \tag{4.13}$$

We use the proximity condition (4.3), Proposition 4.10, the induction hypothesis, and our choice of σ'' to obtain

$$\begin{aligned}
&\|I_{l+1}T_{l,0}p_0 - I_0p_0\| \\
&\le \sum\nolimits_{m=0}^{l} \|I_{m+1}T_{m,0}p_0 - I_{m+1}S_mT_{m-1,0}p_0\| + \|I_{m+1}S_mT_{m-1,0}p_0 - I_mT_{m-1,0}p_0\| \\
&\le F\sum\nolimits_{m=0}^{l} \mathcal{D}(T_{m-1,0}p_0)^2 + C_B\sum\nolimits_{m=0}^{l} \mathcal{D}(T_{m-1,0}p_0) \\
&\le 4FC^2\left(\sum\nolimits_{m=0}^{\infty}\gamma^{2m}\right)\mathcal{D}(p_0)^2 + 2C_BC\sum\nolimits_{m=0}^{\infty}\gamma^m\mathcal{D}(p_0) \\
&\le \frac{4FC^2}{1-\gamma^2}\mathcal{D}(p_0)^2 + \frac{2C_BC}{1-\gamma}\mathcal{D}(p_0) < \sigma'. \qquad (4.14)
\end{aligned}$$

Thus $T_{l,0}p_0$ takes its values in N, and together with (4.13) $T_{l,0}p_0$ is in the domain of T_{l+1}. This completes the induction step.

For the convergence statement, we assume that $l'' \ge l' \ge l$. As in (4.14), we get

$$\begin{aligned}
\|I_{l''+1}T_{l'',0}p_0 - I_{l'+1}T_{l',0}p_0\| &\le \frac{4FC^2}{1-\gamma^2}\mathcal{D}(T_{l-1,0}p_0)^2 + \frac{2C_BC}{1-\gamma}\mathcal{D}(T_{l-1,0}p_0) \\
&\le \frac{16FC^4}{1-\gamma^2}\gamma^{2l}\mathcal{D}(p_0)^2 + \frac{4C_BC^2}{1-\gamma}\gamma^l\mathcal{D}(p_0).
\end{aligned}$$

Since the right hand side approaches 0 as $l \to \infty$, the sequence $\{I_lT_{l-1,0}p_0\}_{l\in\mathbb{N}}$ is Cauchy in $C_u(\mathbb{R}^2, \mathbb{R}^d)$ and therefore convergent. □

Lemma 4.12. *With the notation and assumptions of Proposition 4.11, the sequence* $S_{\infty,l}T_{l-1,0}p_0$ *converges to* $T_{\infty,0}p_0$ *in* $C_u(\mathbb{R}^2, \mathbb{R}^d)$ *as* $l \to \infty$.

Proof. For $\varepsilon > 0$, choose $L \in \mathbb{N}$ such that for all $l \ge L$, $\|T_{\infty,0}p_0 - I_lT_{l-1,0}p_0\| < \frac{\varepsilon}{2}$. By Proposition 4.10 there is $C_I > 0$ such that

$$\|S_{\infty,l}T_{l-1,0}p_0 - I_lT_{l-1,0}p_0\| \le C_I\mathcal{D}(T_{l-1,0}p_0) \le 2C_IC\gamma^l\mathcal{D}(p_0).$$

Now choose $L_0 > L$ such that $2C_IC\gamma^{L_0} < \frac{\varepsilon}{2}$. Then for all $l \ge L_0$, $\|T_{\infty,0}p_0 - S_{\infty,l}T_{l-1,0}p_0\| < \varepsilon$. □

We have collected sufficient results to show Theorem 4.4.

Proof of Theorem 4.4 for standard schemes. It remains to show (4.9) for the operators $\{S_n\}_{n\in\mathbb{N}_0}$, i.e.,

$$\mathcal{D}(S_{l-1,n}p_n) \le C\gamma^{l-n}\mathcal{D}(p_n), \qquad (4.15)$$

for some $0 < \gamma < 1$, $C \ge 1$, and any data p_n. Then Theorem 4.4 immediately follows from Proposition 4.11.

We consider some l-th level data $S_{l-1,n}p_n$ and split the domain D into the l-th inner area D'_l, the rings $D_{l-1}, \dots, D_n$, and the 'outer area' $D_- = D \setminus D'_n$. Then we have

$$\mathcal{D}(S_{l-1,n}p_n) = \qquad (4.16)$$
$$\max\left(\mathcal{D}_{\mathrm{ctrl}^l(D'_l)}(S_{l-1,n}p_n), \max_{m=n,\dots,l-1}\mathcal{D}_{\mathrm{ctrl}^l(D_m)}(S_{l-1,n}p_n), \mathcal{D}_{\mathrm{ctrl}^l(D_-)}(S_{l-1,n}p_n)\right).$$

Before we consider $S_{l-1,n}p_n$ separately on the items of the splitting, we need some preparation: Lemma 4.8 yields constants $C_2 \geq 1$ and $\gamma'' \in (0,1)$ such that for all i, j with $i \geq j$,

$$\sup_{v,w \in \mathrm{ctrl}^i(D'_i)} \|S_{i-1,j}p_j(v) - S_{i-1,j}p_j(w)\|_{\mathbb{R}^d} \leq C_2(\gamma'')^{i-j} \sup_{v,w \in \mathrm{ctrl}^j(D'_j)} \|p_j(v) - p_j(w)\|_{\mathbb{R}^d}.$$

Since the sets $\mathrm{ctrl}^i(D'_i)$ are finite, the triangle inequality yields a constant $C_3 > 0$ such that

$$\mathcal{D}_{\mathrm{ctrl}^i(D'_i)}(S_{i-1,j}p_j) \leq C_3(\gamma'')^{i-j}\mathcal{D}_{\mathrm{ctrl}^j(D'_j)}(p_j). \tag{4.17}$$

We consider $S_{l-1,n}p_n$ separately on the items of the splitting and begin with D'_l. We obtain using (4.17) that

$$\mathcal{D}_{\mathrm{ctrl}^l(D'_l)}(S_{l-1,n}p_n) \leq C_3(\gamma'')^{l-n}\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(p_n) \leq C_3(\gamma'')^{l-n}\mathcal{D}(p_n). \tag{4.18}$$

We continue with the rings $D_{l-1}, \ldots, D_n$. For the ring D_m, where $m = n, \ldots, l-1$, we consider its segments D^s_m, $s \in \mathbb{Z}_k$. Since S converges on regular parts of the mesh, and since $S_{l-1,n}p_n|_{\mathrm{ctrl}^l(D^s_m)}$ is obtained by $l-m$ steps of regular mesh subdivision from $S_{m-1,n}p_n|_{\mathrm{ctrl}^m(D^s_m)}$ by our assumptions on S, we obtain

$$\mathcal{D}_{\mathrm{ctrl}^l(D^s_m)}(S_{l-1,n}p_n) \leq C_1(\gamma')^{l-m}\mathcal{D}_{\mathrm{ctrl}^m(D^s_m)}(S_{m-1,n}p_n). \tag{4.19}$$

Here the constants $C_1 > 0$ and $0 < \gamma' < 1$ are independent of the levels l, n, m and data p_n. Since $\mathrm{ctrl}^l(D^s_m) \subset \mathrm{ctrl}^l(D'_m)$, we use (4.17) and get

$$\mathcal{D}_{\mathrm{ctrl}^l(D^s_m)}(S_{l-1,n}p_n) \leq C_1(\gamma')^{l-m}\mathcal{D}_{\mathrm{ctrl}^l(D'_m)}(S_{m-1,n}p_n) \leq C_1C_3(\gamma')^{l-m}(\gamma'')^{m-n}\mathcal{D}(p_n). \tag{4.20}$$

Since the n-th level control set of the outer area D_- has regular connectivity we can proceed as in (4.19) to get

$$\mathcal{D}_{\mathrm{ctrl}^l(D_-)}(S_{l-1,n}p_n) \leq C_1(\gamma')^{l-n}\mathcal{D}(p_n). \tag{4.21}$$

If we summarize (4.18), (4.20), (4.21) and define the constants $C = C_1C_3$ and $\gamma = \max(\gamma', \gamma'')$, we get using (4.16) that (4.15) is true. This completes the proof. □

Remark 4.13. Proposition 4.10 and Proposition 4.11 are actually valid in a more general setting: If the requirements of Lemma 4.9, where we can replace the 2 by $m > 1$, are fulfilled for a sequence of arbitrary operators S_n, point sets, and face sets, then Proposition 4.10 is still valid, and works as a convergence proof. Subsequently, Proposition 4.11 carries over to this more general setting with the same proof.

Smoothness analysis

So far we have shown convergence for a nonlinear scheme T, which is in proximity to a standard scheme S. In this part we analyze C^1 smoothness.

More precisely, we reconsider the sequence $S_{\infty,n}T_{n-1,0}p_0$ which converges to $T_{\infty,0}p_0$ in $C(D, \mathbb{R}^d)$ by Lemma 4.12. We reparametrize each sequence member with the inverse of Reif's characteristic parametrization χ over the relevant set $D' \subset D$ and show that then convergence is true even in the space $C^1(\chi(D'), \mathbb{R}^d)$. The main statement is the following which was stated as Theorem 4.5 above:

Theorem. *Let S be a standard subdivision scheme, and assume that S and T fulfill a local proximity condition w.r.t. $P_{N,\sigma}$. Then for $p_0 \in P_{N'',\sigma''}$ (which ensures convergence of T for p_0 by Proposition 4.11), the function $T_{\infty,0}p_0 \circ \chi^{-1}$ is continuously differentiable, where $T_{\infty,0}p_0 : D' \to N$ is the limit function of T, and $\chi : D' \to \mathbb{R}^2$ is the characteristic map.*

Notice that if T converges, data eventually get dense enough. So for showing smoothness, a 'dense enough' assumption is no restriction. In order to show Theorem 4.5 we first prove a series of lemmas.

Lemma 4.14. *Let T be in proximity to a standard scheme S w.r.t. $P_{N,\sigma}$, and let $p_0 \in P_{N'',\sigma''}$ (which ensures convergence of T for p_0 by Proposition 4.11). Then there is a constant $C_1 \geq 1$ such that, for all $i \geq l$, and all $j \in \mathbb{Z}_k$,*

$$\mathcal{D}_{\mathrm{ctrl}^i(D_l^j)}(T_{i-1,0}p_0) \leq C_1 2^{-i+l} \mathcal{D}_{\mathrm{ctrl}^l(D_l^j)}(T_{l-1,0}p_0). \tag{4.22}$$

Furthermore there is a constant $C_2 \geq 1$ such that, for any $l \in \mathbb{N}$,

$$\mathcal{D}_{\mathrm{ctrl}^l(D_l')}(T_{l-1,0}p_0) \leq C_2 \lambda^l \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0). \tag{4.23}$$

Here λ is the subdominant eigenvalue of the subdivision matrix A.

Proof. We begin with the first statement. Note that $T_{l,0}p_0$ is defined for any $l \in \mathbb{N}$. For any $i \geq l$, we have that $S_{i-1,l}T_{l-1,0}p_0|_{\mathrm{ctrl}^i(D_l^j)}$ is determined by $T_{l-1,0}p_0|_{\mathrm{ctrl}^l(D_l^j)}$ by means of subdivision w.r.t. regular connectivity. Lemma 1.9 and the triangle inequality yield a constant $C' > 0$ such that

$$\mathcal{D}_{\mathrm{ctrl}^i(D_l^j)}(S_{i-1,l}T_{l-1,0}p_0) \leq C' 2^{-i+l} \mathcal{D}_{\mathrm{ctrl}^l(D_l^j)}(T_{l-1,0}p_0).$$

This constant C' is independent of the level i, the segment index j, the ring index l and data p_0. We apply Lemma 4.7 with $\gamma = 1/2$ to the sets $\{\mathrm{ctrl}^i(D_l^j)\}_{i \geq l}$. In Lemma 4.7, we start on subdivision level l instead of level 0. The condition (4.4) of Lemma 4.7 is fulfilled by Lemma 1.9. The locality of the proximity condition and the fact that $p_0 \in P_{N'',\sigma''}$ guarantee that the remaining conditions of Lemma 4.7 are met. We conclude that (4.22) holds true.

We show the second statement. From Lemma 4.8 we get a constant $C' > 0$ such that

$$\mathcal{D}_{\mathrm{ctrl}^l(D_l')}(S_{l-1,0}p_0) \leq C' \lambda^l \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0).$$

Then we apply Lemma 4.7 for the sequence $\{\mathrm{ctrl}^l(D_l')\}_{l \in \mathbb{N}_0}$; the assumptions of Lemma 4.7 are fulfilled, which can be seen by a similar argument as above, and (4.23) follows. □

Proposition 4.15. *Let a standard scheme S and a (nonlinear) scheme T fulfill a local proximity condition w.r.t. $P_{N,\sigma}$. Let furthermore $\chi : D' \to \mathbb{R}^2$ be the characteristic map, and $p_0 \in P_{N'',\sigma''}$ (which ensures convergence of T for p_0 by Proposition 4.11). Then $S_{\infty,i}T_{i-1,0}p_0 \circ \chi^{-1} \in C^1(\chi(D'), N)$. In addition, there is a constant $C \geq 1$ such that, for all $i \geq n$, and all $j \in \mathbb{Z}_k$,*

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(D_n^j)}\|_{C^1(\chi(D_n^j),\mathbb{R}^d)} \leq C\gamma^i \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2. \tag{4.24}$$

Here we let $\gamma = \max(2^{-1}, \lambda)$, where λ denotes the subdominant eigenvalue of the subdivision matrix A.

Proof. Theorem 1.13 implies that $S_{\infty,0}p_0 \circ \chi^{-1} \in C^1(\chi(D'), N)$. By the scaling property of the characteristic map, i.e., $\chi(\cdot/2^m) = \lambda^m \chi$, and since S produces C^1-limits on regular connectivities, $S_{\infty,i}T_{i-1,0}p_0 \circ \chi^{-1}$ is C^1.

In order to prove (4.24) we first show that there is a constant $C_3 > 0$, which is independent of i, j, and n, such that

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0)|_{D_n^j}\|_{C^1(D_n^j,\mathbb{R}^d)} \leq C_3 2^i \|(T_i - S_i)T_{i-1,0}p_0\|_\infty. \tag{4.25}$$

On a regular connectivity, the scheme S commutes with translation. Furthermore, its corresponding mask has finite support. Hence $S_{\infty,i}$ is a bounded linear operator from $l^\infty(\mathrm{ctrl}^i(D_n^j), \mathbb{R}^d)$ to $C^1(D_n^j, \mathbb{R}^d)$. Scaling a grid by two at most doubles the C^1-norm, so for any bounded data $f_i \in l^\infty(\mathrm{ctrl}^i(D_n^j), \mathbb{R}^d)$, we get

$$\|S_{\infty,i}f_i\|_{C^1(D_n^j,\mathbb{R}^d)} \leq 2^i \|S_{\infty,0}\|_{l^\infty \to C^1} \|f_i\|_\infty.$$

This implies (4.25).

Since χ is a diffeomorphism in a neighborhood of D_0^j, all $h \in C^1(D_0^j, \mathbb{R}^d)$ obey the inequality $\|h \circ \chi^{-1}|_{\chi(D_0^j)}\|_{C^1} \leq D\|h\|_{C^1}$ for some $D > 0$, which is independent of h. Using the scaling relation $\chi(\cdot/2^n) = \lambda^n \chi$ again, yields a constant $C_4 > 0$ which is independent of i, j, and n such that, using (4.25),

$$\begin{aligned}\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(D_n^j)}\|_{C^1(\chi(D_n^j),\mathbb{R}^d)} \\ \leq 2^{i-n}\lambda^{-n}C_4 \|(T_i - S_i)T_{i-1,0}p_0|_{\mathrm{ctrl}^{i+1}(D_n^j)}\|_\infty.\end{aligned}$$

We now use the proximity condition (4.3), and obtain that there is a constant $C_5 > 0$ such that

$$\begin{aligned}\|(T_i - S_i)T_{i-1,0}p_0|_{\mathrm{ctrl}^{i+1}(D_n^j)}\|_\infty &\leq C_5 \left(\mathcal{D}_{\mathrm{ctrl}^i(D_n^j)}(T_{i-1,0}p_0)\right)^2 \\ &\leq C_5 C_2^2 C_1^2 \lambda^{2n} 2^{-2i+2n} \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2.\end{aligned}$$

For the last inequality we used Lemma 4.14 and C_1 and C_2 denote the constants from this lemma. Altogether,

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(D_n^j)}\|_{C^1(\chi(D_n^j),\mathbb{R}^d)} \leq C_1^2 C_2^2 C_4 C_5 \lambda^n 2^{-i+n} \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2.$$

This completes the proof. □

Proposition 4.16. *Let T be in proximity to a standard scheme S w.r.t. $P_{N,\sigma}$, and let $p_0 \in P_{N'',\sigma''}$ (which ensures convergence of T for p_0 by Proposition 4.11). Then $\{S_{\infty,i+1}T_{i,0}p_0\}_{i\in\mathbb{N}_0}$ is a Cauchy sequence in $C^1(\chi(D'), \mathbb{R}^d)$.*

Proof. The linear operators

$$L_i : \; l^\infty(\mathrm{ctrl}^i(D_i'), \mathbb{R}^d) \to C^1(\chi(D_i'), \mathbb{R}^d),$$

assigning the limit function of subdivision by S w.r.t. the characteristic parametrization to data on $\mathrm{ctrl}^i(D_i')$, are bounded, since they operate on finite dimensional space. We consider, for $i, k \in \mathbb{N}_0$, the isometric isomorphism

$$\begin{aligned}&V_{i,k} : \; l^\infty(\mathrm{ctrl}^i(D_i'), \mathbb{R}^d) \to l^\infty(\mathrm{ctrl}^k(D_k'), \mathbb{R}^d), \\ &V_{i,k}p_i(x) = p_i(2^{-i+k}x).\end{aligned}$$

We have $V_{i,k} \circ L_i = L_k \circ V_{i,k}$. Now the scaling property of the rings of the characteristic map yields, for any $i \in \mathbb{N}_0$ and any $p_i \in l^\infty(\mathrm{ctrl}^i(D_i'), \mathbb{R}^d)$, the estimate $\|L_i p_i\|_{C^1} \le \|L_0\|\lambda^{-i}\|p_i\|_\infty$, where λ again denotes the subdominant eigenvalue of the subdivision matrix. Using the constant $C_3 > 0$ from the proximity condition (4.3), we obtain

$$\begin{aligned}\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(D_i')}\|_{C^1} & \\ = \|L_{i+1}(T_i - S_i)T_{i-0,0}p_0\|_{C^1} &\le \lambda^{-i}\|L_0\|C_3\mathcal{D}_{\mathrm{ctrl}^i(D_i')}(T_{i-1,0}p_0)^2 \\ \le C'\lambda^{-i}\lambda^{2i}\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2 &\le C'\lambda^i\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2, \qquad (4.26)\end{aligned}$$

where $C' = C_2^2 C_3\|L_0\|$ with the constant C_2 of Lemma 4.14. We know from Proposition 4.15 that for any $i \in \mathbb{N}_0$, the limit function $S_{\infty,i+1}T_{i,0}p_0 \circ \chi^{-1} \in C^1(\chi(D'), \mathbb{R}^d)$. We use both (4.24) and (4.26) and see that

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i,0}p_0) \circ \chi^{-1}|_{\chi(D')}\|_{C^1} \le C\gamma^i\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2$$

for some $C > 0$ and $\gamma := \max(2^{-1}, \lambda)$. This implies, for $k, l \in \mathbb{N}_0$,

$$\|(S_{\infty,k}T_{k-1,0}p_0 - S_{\infty,l}T_{l-1,0}p_0) \circ \chi^{-1}|_{\chi(D')}\|_{C^1} \le C\gamma^{\min(k,l)}\frac{1}{1-\gamma}\mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2, \qquad (4.27)$$

which completes the proof. □

Finally we are able to show Theorem 4.5.

Proof of Theorem 4.5 for standard schemes. By Lemma 4.12, $S_{\infty,n}T_{n-1,0}p_0$ converges to $T_{\infty,0}p_0$ on D' in the sup norm. Since this sequence is Cauchy on $\chi(D')$ with respect to the C^1 norm by Proposition 4.16, its limit $T_{\infty,0}p_0 \circ \chi^{-1}$ must be continuously differentiable. □

We can add a condition which guarantees that $T_{\infty,0}p_0(D')$ locally is a submanifold around the extraordinary point $T_{\infty,0}p_0(0)$. Note that the statement below is not as strong as the respective statement in the linear case.

Corollary 4.17. *Let a standard scheme S and a (nonlinear) scheme T be in proximity w.r.t. $P_{N,\sigma}$, and let $p_0 \in P_{N'',\sigma''}$ (which ensures convergence of T for p_0 by Proposition 4.11). Assume that the Jacobian $J_0(S_{\infty,0}p_0 \circ \chi^{-1})$ in the extraordinary point 0 of the limit function of linear subdivision using S fulfills $\|J_0(S_{\infty,0}p_0 \circ \chi^{-1})(x)\|_\infty \ge \xi\|x\|_\infty$ for some $\xi > 0$. Assume further that*

$$\mathcal{D}_{\mathrm{ctrl}^0(D')}(p_0) < (\xi(1-\gamma)/C)^{\frac{1}{2}},$$

where C is the constant from (4.27), $\gamma = \max(2^{-1}, \lambda)$, and λ is the subdominant eigenvalue of the subdivision matrix A. Then also the nonlinear scheme T produces a 2-dimensional manifold locally around the extraordinary point.

Proof. From (4.27) it follows that

$$\|(S_{\infty,0}p_0 - T_{\infty,0}p_0) \circ \chi^{-1}|_{\chi(D')}\|_{C^1} \le C(1-\gamma)^{-1}\mathcal{D}_{\mathrm{ctrl}^0(D')}(p_0)^2.$$

Thus, for any $x \in \mathbb{R}^2$ with $\|x\| = 1$,

$$\begin{aligned}\|J_0(T_{\infty,0}p_0 \circ \chi^{-1})(x)\| &\ge \|J_0(S_{\infty,0}p_0 \circ \chi^{-1})(x)\| - \|(S_{\infty,0}p_0 - T_{\infty,0}p_0) \circ \chi^{-1}\|_{C^1} \\ &\ge \xi - C(1-\gamma)^{-1}\mathcal{D}_{\mathrm{ctrl}^0(D')}(p_0)^2 > 0.\end{aligned}$$

This shows that the Jacobian is regular in 0, which completes the proof. □

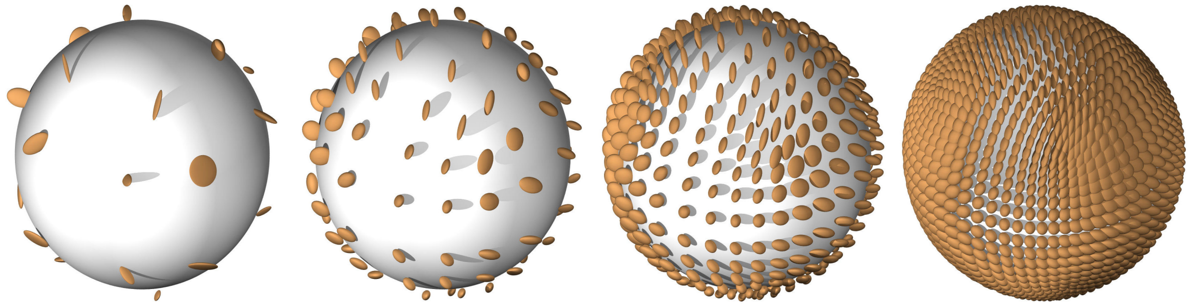


Figure 16: Initial rounds of subdivision for a subdivided cube connectivity.

We still owe the proof of Corollary 4.6.

Proof of Corollary 4.6 for standard schemes. By Theorem 4.4 and Theorem 4.5, convergence and smoothness are ensured, if a local proximity condition holds. Although Wallner and Dyn's and Grohs' proximity inequality in [57] and [15], respectively, is slightly weaker, they actually prove our local proximity condition for the geodesic analogue in [57], Lemma 5, and the projection analogue in [15], Theorem 4, respectively. A proof which works for the log-exp analogue is the proof of [14], Proposition 7.2. Then the local proximity condition (4.1) for the intrinsic mean analogue is a consequence of its interpretation as log-exp analogue with special base points. □

4.3 An application

As an application we show how subdivision in the geometric setting can be used to generate manifold-valued smooth functions on smooth two-dimensional manifolds. To that end, we consider two meshes with the same connectivity. Let us assume the first mesh has its values in the smooth manifold N. Let the second mesh 'cover' a smooth 2-manifold M, and assume that its positioning function is one-to-one. Then we have a map from the positions in M to that in N. Now, let S be a standard scheme, T be an analogue acting in M, and T' an analogue acting in N. Iterated application of both T and T' simultaneously yields a sequence of mappings, defined in discrete subsets of M, with values in N. The first steps of this process are visualized in Figure 16 and Figure 17. Here we used the projection analogue on spheres on the one hand, and intrinsic mean subdivision in the Riemannian manifold of positive matrices on the other hand. The theoretical basis is given by the following corollary, formulated near extraordinary points of valence k.

Corollary 4.18. *Let T and T' be analogues of the same standard scheme S, and let input data $p_0 : V_0 \to M$ and $p_0' : V_0 \to N$ be dense enough. If $T_{\infty,0}p_0 \circ \chi^{-1} : \chi(D') \to M$ is injective and regular, then*

$$T'_{\infty,0}p'_0 \circ (T_{\infty,0}p_0)^{-1} : T_{\infty,0}p_0(D') \to T'_{\infty,0}p'_0(D') \tag{4.28}$$

is a C^1 mapping.

Note that Corollary 4.17 gives a sufficient condition for regularity near the extraordinary point. Then, at least in a small neighborhood, we also have injectivity.

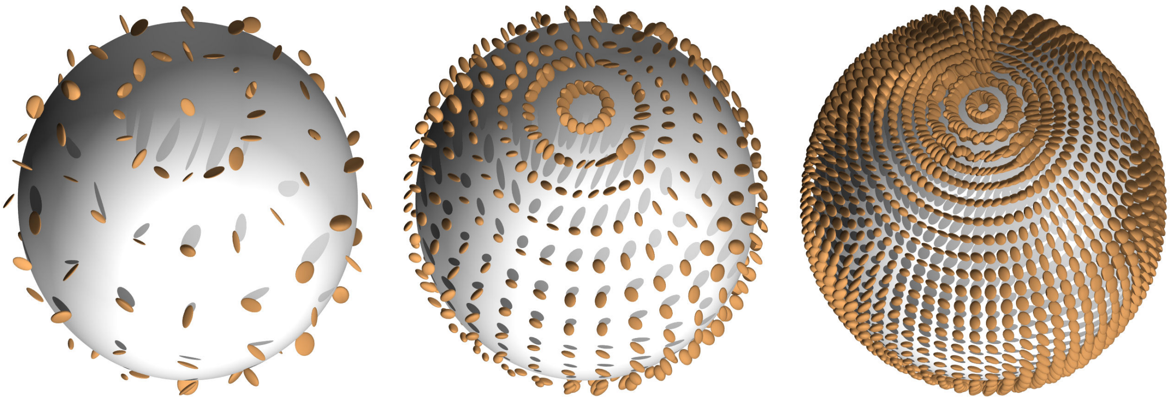
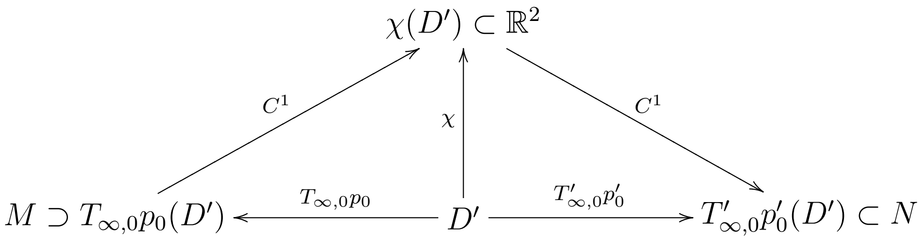


Figure 17: Two rounds of subdivision analogous to the Doo-Sabin scheme using projection and intrinsic means. One can clearly observe the oscillation near the valence 16 extraordinary face, especially in the data position. Also note that 8 valence 3 faces are nearby.

Proof. Consider the following commutative diagram:



This means that $T'_{\infty,0}p'_0 \circ (T_{\infty,0}p_0)^{-1} = T'_{\infty,0}p'_0 \circ \chi^{-1} \circ (T_{\infty,0}p_0 \circ \chi^{-1})^{-1}$, where χ is the characteristic map. Now the statement follows from Theorem 4.5. □

4.4 Analysis of shift-invariant schemes

The purpose of this part is to prove Theorem 4.4 and Theorem 4.5 for nonlinear schemes meeting local proximity inequalities with a linear shift-invariant scheme introduced in Chapter 1.2. It is actually not difficult to generalize the result concerning convergence of standard schemes. This is done first. Then we show C^1 smoothness w.r.t. the characteristic parametrization, which involves some work since in contrast to standard schemes the conditions on the eigenvalues of the subdivision matrix are relaxed. This part is contained in [60].

Our first task is to establish contractivity of a nonlinear scheme T which is in proximity to a linear shift-invariant scheme S (from Chapter 1.2) near the singularity. The next lemma uses Lemma 4.8 and the proximity condition (4.1) to get a grip on the differences of data near the extraordinary point obtained during the subdivision process.

Lemma 4.19. *Assume that a linear shift-invariant scheme S from Chapter 1.2 and the scheme T fulfill the local proximity condition (4.1) w.r.t σ-dense input $P_{N,\sigma}$. Then for $s > 1$ there is a constant $C > 0$ and $\sigma'' > 0$ such that the following is true: If the input* *data p_0 belongs to $P_{N,\sigma''}$, if iterated subdivision for input p_0 is defined, and if $T_{l-1,0}p_0$ stays within $P_{N,\sigma}$ for all $l < n$, then*

$$\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(T_{n-1,0}p_0) \leq C(\lambda s)^n \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0), \tag{4.29}$$

where λ is the modulus of a subdominant eigenvalue of the subdivision matrix of S.

Proof. We start by rephrasing (4.7). For any $s > 1$ there is a constant $C_L \geq 1$ such that for all levels n the following is true: The linear scheme is contractive for data on the control sets of the inner areas D'_n (defined by (1.23)) in the following sense

$$\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(S_{n-1,0}p_0) \leq C_L(\lambda s)^n \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0). \tag{4.30}$$

To see this, we consider the definition of the subdivision matrix A of the scheme S in Chapter 1.2. The subdivision matrix A maps data on $\mathrm{ctrl}^0(D'_0)$ to subdivided data on $\mathrm{ctrl}^1(D'_1)$. Therefore, A^n maps data on $\mathrm{ctrl}^0(D'_0)$ to n-times subdivided data on $\mathrm{ctrl}^n(D'_n)$. In this interpretation, (4.7) estimates differences of $\dim(A)$ many subdivided data items by $\dim(A)$ many input data items. Therefore, application of the triangle inequality and enlarging the constant C of (4.7) yields (4.30).

We also rewrite the local proximity condition (4.1) in the following way: There is a constant C such that for σ-dense input $p_n \in P_{N,\sigma}$ on some data level n,

$$\|S_n p_n(w) - T_n p_n(w)\|_\infty \leq C \sup_{v_1,v_2 \in \mathrm{supp}(\alpha_{\cdot,w})} \|p_n(v_1) - p_n(v_2)\|^2, \tag{4.31}$$

where $\mathrm{supp}(\alpha_{\cdot,w})$ denotes the set of vertices on level n which contribute to the calculation of $S_n p(w)$. The locality of the proximity condition guarantees that any n-th level control set of $U \subset D$ w.r.t. S also controls the limit of subdivision using T on this set U. This fact and estimating differences on $\mathrm{supp}(\alpha_{\cdot,w})$ by differences of neighboring vertices yields a constant C_P such that, for σ-dense input $p_n \in P_{N,\sigma}$,

$$\|(S_n p_n - T_n p_n)|_{\mathrm{ctrl}^{n+1}(D'_{n+1})}\|_\infty \leq C_P(\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(p_n))^2. \tag{4.32}$$

Here we also used that $(a+b)^2 \leq 2\,(a^2+b^2)$ for $a, b \in \mathbb{R}$.

With these preparations we define the 'denseness'-bound σ'' by

$$\sigma'' = \frac{(1-\lambda s)\lambda s}{8 C_P C_L^2}. \tag{4.33}$$

For data p_0 meeting the requirements of the lemma we show that

$$\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(T_{n-1,0}p_0) \leq 2C_L(\lambda s)^n \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0), \tag{4.34}$$

using induction on n; this implies (4.29) with $C = 2C_L$. We start with $n = 1$ and estimate

$$\begin{aligned}
\mathcal{D}_{\mathrm{ctrl}^1(D'_1)}(T_0p_0) &\leq \mathcal{D}_{\mathrm{ctrl}^1(D'_1)}(T_0p_0 - S_0p_0) + \mathcal{D}_{\mathrm{ctrl}^1(D'_1)}(S_0p_0) \\
&\leq 2\|T_0p_0 - S_0p_0|_{\mathrm{ctrl}^1(D'_1)}\|_\infty + \mathcal{D}_{\mathrm{ctrl}^1(D'_1)}(S_0p_0) \\
&\leq 2C_P(\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0))^2 + C_L(\lambda s)\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0) \\
&\leq C_L(2C_P\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0) + \lambda s)\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0) \\
&\leq 2C_L(\lambda s)\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0).
\end{aligned}$$

The second inequality estimates differences by twice the sup-norm of data. For the third inequality we used proximity in the form of (4.32) and the contractivity of the linear scheme near the singularity in the form of (4.30). For the fourth inequality notice that $C_L \geq 1$. The last inequality is a consequence of our choice of σ'' in (4.33). As induction hypothesis we assume that (4.34) is true for all $l < n$. We now show (4.34) by estimating

$$\begin{aligned}
&\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(T_{n-1,0}p_0) \\
&\quad \leq \sum\nolimits_{l=1}^{n} \mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(S_{n-1,l}T_{l-1,0}p_0 - S_{n-1,l-1}T_{l-2,0}p_0) + \mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(S_{n-1,0}p_0) \\
&\quad \leq \sum\nolimits_{l=1}^{n} C_L(\lambda s)^{n-l}\mathcal{D}_{\mathrm{ctrl}^l(D'_l)}(T_{l-1,0}p_0 - S_{l-1}T_{l-2,0}p_0) + \mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(S_{n-1,0}p_0) \\
&\quad \leq \sum\nolimits_{l=1}^{n} 2C_L(\lambda s)^{n-l}C_P(\mathcal{D}_{\mathrm{ctrl}^{l-1}(D'_{l-1})}(T_{l-2,0}p_0))^2 + C_L(\lambda s)^n\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0).
\end{aligned}$$

For the second inequality we used the contractivity of S near the singularity in the sense of (4.30). For the third inequality we estimated differences by twice the sup-norm and then applied the proximity inequality (4.32). We use the induction hypothesis and obtain

$$\begin{aligned}
&\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(T_{n-1,0}p_0) \\
&\quad \leq \sum\nolimits_{l=1}^{n} 8C_LC_P(\lambda s)^{n-l}C_L^2(\lambda s)^{2(l-1)}(\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0))^2 + C_L(\lambda s)^n\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0) \\
&\quad \leq C_L\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0)\left[C_L^2\sum\nolimits_{l=1}^{k} 8C_P(\lambda s)^{n+l-2}\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0) + (\lambda s)^n\right] \\
&\quad \leq C_L(\lambda s)^n\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0)\left[\frac{8C_PC_L^2}{(1-\lambda s)\lambda s}\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0) + 1\right] \\
&\quad \leq 2C_L(\lambda s)^n\mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0).
\end{aligned}$$

For the first inequality we use the contractivity of T which is the induction hypothesis. The last inequality is true by our choice of σ''. This completes the induction. □

We have collected all information to show Theorem 4.4 (for shift-invariant schemes), which we recall here.

Theorem. *Let S be a linear subdivision scheme as introduced in Chapter 1.2, and let S and T fulfill a local proximity condition w.r.t. some $P_{N,\sigma}$. Assume that T_np_n takes its values in a set N' for all data $p_n \in P_{N,\sigma}$ where N' is some set with $N \subset N' \subset \mathbb{R}^n$. Assume further that there is a subset $N'' \subset N$ and $\sigma' > 0$ such that the σ'-neighborhood $U_{\sigma'}(N'')$ obeys*

$$U_{\sigma'}(N'') \cap N' \subset N.$$

Then there is a denseness bound $\sigma'' > 0$ such that T converges for data $p_0 \in P_{N'',\sigma''}$ given on V_0, and

$$S_{\infty,i+1}T_{i,0}p_0 \to T_{\infty,0}p_0 \quad \text{as} \quad i \to \infty, \tag{4.35}$$

where convergence is understood in the sense of uniform convergence.

Proof. We split the proof of this statement into several parts. In part (1) we obtain the contractivity of the nonlinear scheme T, where we assume that $T_{n,0}p_0$ is defined for all n and certain input data p_0. In part (2) we define interpolation operators which extend the discrete data on different levels to continuous functions and derive some properties. In part (3) we define the constant σ'' and explain our choice of σ''. In part (4) we apply the interpolation operators from part (2) to show that that iterated subdivision by T is well defined for σ''-dense data p_0 in $P_{N'',\sigma''}$, thus justifying the assumption of (1). Furthermore, we use the proximity of S and T and the contractivity of T to derive the convergence of T for data in $P_{N'',\sigma''}$. In part (5), we use part (4) and the interpolation operator from part (2) to show (4.35).

(1) In this part we obtain contractivity of T. We denote a subdominant eigenvalue of the subdivision matrix of the linear scheme S by the symbol λ, and we let M be the dilation matrix corresponding to S. We choose $s > 1$ such that

$$\gamma := s \max(|\lambda|, 1/\sqrt{\det M}) < 1.$$

We show that there is σ_1'' and $C_1 \geq 1$ such that the following is true: If input data p_0 on level 0 belongs to $P_{N,\sigma''}$, if iterated subdivision for input p_0 is defined, and if $p_l = T_{l-1,0}p_0$ stays within $P_{N,\sigma}$ for all $l < n$, then

$$\mathcal{D}(T_{n-1,l}p_l) \leq C_1 \gamma^{n-l} \mathcal{D}(p_l). \tag{4.36}$$

This is a consequence of the corresponding statement near the singularity which is formulated in Lemma 4.19 and the corresponding statement for the regular mesh case which is Lemma 3.7. The constant C_1 is the product of the corresponding constants of Lemma 3.7 and Lemma 4.19, and σ_1'' is obtained as follows: We apply Lemma 4.19 for the denseness bound σ used in the statement of the theorem. We obtain a constant $\sigma''_{\text{Lemma 4.19}}$. Then we apply Lemma 3.7 for this constant, i.e., we replace the σ in Lemma 3.7 by $\sigma''_{\text{Lemma 4.19}}$. The resulting denseness bound is denoted by σ_1''.

In order to conclude (4.36), one has to show that 'no interaction takes place between the neighborhood of the singularity and the regular part': To that end we split the domain D into the inner area D'_n (defined by (1.23)), the rings D_i, $i = 0, \dots, n-1$ and the 'outer' ring

$$D_{-1} = D \setminus D'.$$

The union of the corresponding n-th level control sets equals V_n, and control sets of neighboring items of the splitting overlap (recall that control sets were defined w.r.t. the linear scheme S and that any n-th level control set of $U \subset D$ w.r.t. S also controls the limit of subdivision using T on this set U).

We consider (4.36) separately on the items of the splitting: The control set of the outer ring D_{-1} intersected with each sector has regular combinatorics on all data levels. Therefore the validity of (4.36) on $\mathrm{ctrl}^n(D_{-1})$ is a consequence of Lemma 3.7. On D'_n, (4.36) is a direct consequence of Lemma 4.19 applied to $\mathrm{ctrl}^n(D'_n)$. We consider the rings D_i : For each segment D_i^j of the i-th ring we consider its n-th level control set and get

$$\begin{aligned}\mathcal{D}_{\mathrm{ctrl}^n(D_i^j)}(T_{n-1,0}p_0) &\leq C_{\text{Lemma 3.7}}(s \det M^{-1})^{(n-i)/2} \mathcal{D}_{\mathrm{ctrl}^i(D_i^j)}(T_{i-1,0}p_0) \\ &\leq C_{\text{Lemma 3.7}} C_{\text{Lemma 4.19}} (s \det M^{-1})^{(n-i)/2} (s\lambda)^i \mathcal{D}_{\mathrm{ctrl}^0(D_0)}(p_0) \leq C_1 \gamma^n \mathcal{D}(p_0).\end{aligned}$$

Altogether, this shows (4.36) and completes part (1).

(2) The convergence of subdivision with T is quite intricate, mostly due to the fact that the well-definedness of iterated application of T has to be guaranteed. That is why we need interpolation operators I_i which map data on level i to a uniformly continuous function on the domain D. The domain D is perfectly suited to smoothness analysis across sector boundaries (not near the central point). However, in this part we are only concerned with convergence and we use a homeomorphism $E : D \to \mathbb{R}^2$ to reparametrize data on each level, and to reparametrize limit functions. E maps the entire domain D to the plane by first squeezing the j-th sector into a sector of opening angle $2\pi/k$ by means of a shear transformation and then rotating it by an angle of $2\pi j/k$. It is straightforward to see that there are constants c_1, c_2 such that for $x, y \in D$,

$$c_1 \operatorname{dist}(x, y) \le \operatorname{dist}(E(x), E(y)) \le c_2 \operatorname{dist}(x, y).$$

This implies that convergence of a scheme is invariant under reparametrization by means of E.

The points $E(V_i)$ are still associated with a k-regular combinatorics. By connecting points in $E(V_i)$ with straight lines according to the combinatorics we get a realization of its edges and faces in $\mathbb{R}^2$. For defining the interpolation operator $\bar{I}_i$ which maps data on $E(V_i)$ to a function on $\mathbb{R}^2$ we split each face into triangles, each of them determined by the face's barycenter and an edge. We get data for the barycenter by the barycenter of the data on the neighboring vertices. Then we use linear interpolation on the triangles. For x, y in a face and data p_n defined on $E(V_n)$, we obviously have

$$\sup_{x,y \text{ belong to the same face}} \|\bar{I}_n p_n(x) - \bar{I}_n p_n(y)\|_{\mathbb{R}^d} \le \mathcal{D}(p_n). \tag{4.37}$$

Furthermore the infimum d' of distances of neighboring vertices in $E(V_i)$ satisfies

$$c_3 (\det M)^{-i/2} \le d' \le \{\operatorname{diam} F : F \text{ is a face on level } i\} \le c_4 (\det M)^{-i/2}, \tag{4.38}$$

where the constants c_3, c_4 are independent of the the level i. In addition, there is a constant R for all levels i such that the value

$$S_i p_i(v) \text{ is an affine average of } \{p_i(w) : \ w \in B(v, (\det M)^{-i/2} R)\}. \tag{4.39}$$

Here the considered points w are elements of $E(V_i)$, and $B(x, r)$ is the open ball with radius r around x.

Interpolation operators I_i mapping data on V_i to functions on D are obtained from the operators $\bar{I}_i$ by reversing the reparametrization E.

The interpolation operators I_i have the following properties: There are constants $C_B, C_I > 0$, which depend neither on i nor on bounded data p_i on level i, such that

$$\|I_{i+1} S_i p_i - I_i p_i\| \le C_B \mathcal{D}(p_i), \tag{4.40}$$

$$\|S_{\infty,i} p_i|_{V_i} - p_l\| \le \|S_{\infty,i} p_i - I_i p_i\| \le C_I \mathcal{D}(p_i). \tag{4.41}$$

When showing (4.40) and (4.41) we may replace I_i by $\bar{I}_i$, and we may reparametrize both data and limit functions using the map E. This is justified, since a reparametrization does not effect the statements. We begin with (4.40). For arbitrary $x \in \mathbb{R}^2$ we choose faces F_i

and F_{i+1} containing x on levels i and $i+1$, respectively. We consider vertices v_i of F_i and v_{i+1} of F_{i+1} and estimate, using (4.37),

$$\begin{aligned}
&\|\bar{I}_{i+1}S_ip_i(x) - \bar{I}_ip_i(x)\| \\
&\leq \|\bar{I}_{i+1}S_ip_i(x) - \bar{I}_{i+1}S_ip_i(v_{i+1})\| + \|S_ip_i(v_{i+1}) - p_i(v_i)\| + \|\bar{I}_ip_i(v_i) - \bar{I}_ip_i(x)\| \\
&\leq \mathcal{D}(S_ip_i) + \mathcal{D}(p_i) + \|S_ip_i(v_{i+1}) - p_i(v_i)\|.
\end{aligned}$$

In order to estimate the last summand on the right hand side, note that by (4.39) the value $S_ip_i(v_{i+1})$ is uniquely determined by $p_i|_{E(V_i)\cap B(v_{i+1},(\det M)^{-i/2}R)}$. With the constant c_4 of (4.38) it follows that $\operatorname{dist}(v_i, v_{i+1}) \leq 2c_4(\det M)^{-i/2}$. Consequently, $\max\{\operatorname{dist}(v_i, y) : y \in E(V_i) \cap B(v_{i+1}, 2^{-i}R)\} \leq 2c_4(\det M)^{-i/2} + (\det M)^{-i/2}R$. The left hand inequality in (4.38) now implies that the number of faces on level i which are not disjoint to the ball $B(v_i, (2c_4 + R)(\det M)^{-i/2})$ is bounded by some integer D which is independent of the level i and v_i. With $B^* := B(v_{i+1}, (\det M)^{-i/2}R)$, we can rewrite (4.39) as $S_ip_i(v_{i+1}) = \sum_{q\in E(V_i)\cap B^*} \alpha_q p_i(q)$, where $\sum_{q\in E(V_i)\cap B^*} \alpha_q = 1$ and $\sum_{q\in E(V_i)\cap B^*} |\alpha_q| \leq \|S_i\|$. We obtain

$$\begin{aligned}
\|S_ip_i(v_{i+1}) - p_i(v_i)\| &= \left\|\sum\nolimits_{q\in E(V_i)\cap B^*} \alpha_q(p_i(q) - p_i(v_i))\right\| \\
&\leq \sum\nolimits_{q\in E(V_i)\cap B^*} |\alpha_q| \cdot \max\nolimits_{q\in E(V_i)\cap B^*} \|p_i(q) - p_i(v_i)\| \leq \|S_i\| D\mathcal{D}(p_i).
\end{aligned}$$

Altogether, it follows that

$$\|\bar{I}_{i+1}S_ip_i - \bar{I}_ip_i\| \leq \mathcal{D}(S_ip_i) + (\|S_i\|D + 1)\mathcal{D}(p_i). \tag{4.42}$$

This implies (4.40), since $\|S_i\|$ is uniformly bounded in i.

We show (4.41) for the interpolation operators $\bar{I}_i$. Equipped with (4.42), we estimate, for $n \geq i$,

$$\begin{aligned}
\|\bar{I}_{n+1}S_{n,i}p_i - \bar{I}_nS_{n-1,i}p_i\|_\infty &\leq \mathcal{D}(S_{n,i}p_i) + (\|S_n\|D + 1)\mathcal{D}(S_{n-1,i}p_i) \\
&\leq C_1\gamma^{n-i}(\|S_n\|D + 2)\mathcal{D}(p_i),
\end{aligned}$$

where we used the contractivity of S which follows, for example, from part (1), since S can be seen as a scheme in proximity to S. For $n'' \geq n' \geq n \geq i$ we make use of the geometric series and get

$$\|\bar{I}_{n''+1}S_{n'',i}p_i - \bar{I}_{n'}S_{n'-1,i}p_i\|_\infty \leq C(\sup_{n\in\mathbb{N}_0} \|S_n\|D + 2)\gamma^{n-i}\frac{1}{1-\gamma}\mathcal{D}(p_i). \tag{4.43}$$

Thus $\{\bar{I}_nS_{n-1,i}p_i\}_{n>i}$ is a Cauchy sequence in the space of bounded continuous functions. Since these functions are uniformly continuous, so is the limit, called f for the moment. Now, $\|f|_{E(V_n)} - S_{n-1,i}p_i\|_\infty \leq \|f - \bar{I}_nS_{n-1,i}p_i\| \to 0$ for $n \to \infty$. Thus f equals $S_{\infty,i}p_i$. Letting $n' = i$ in (4.43) yields the estimate

$$\|f - \bar{I}_ip_i\| = \lim_{n''\to\infty} \|\bar{I}_{n''+1}S_{n'',i}p_i - \bar{I}_ip_i\| \leq \frac{1}{1-\gamma}(\sup_{n\in\mathbb{N}_0} \|S_n\|D + 2)\mathcal{D}(p_i).$$

This implies (4.41).

(3) We define the constant σ'' which guarantees convergence by

$$\sigma'' = \min\Big(\sigma_1'', \frac{\sigma}{C_1}, \frac{1-\gamma}{2C_BC_1}\sigma', \Big(\frac{1-\gamma^2}{2C_PC_1^2}\sigma'\Big)^{\frac{1}{2}}\Big). \tag{4.44}$$

Here σ' and σ are the constants from the statement of the present theorem. The constant C_B is given by (4.40), and the symbol C_P denotes the proximity constant as used in (4.32). We take C_1, σ_1'' and the contractivity factor γ from part (1), see (4.36). For σ_1''-dense input data p_0, contractivity of T in the sense of (4.36) is guaranteed whenever iterated subdivision for input p_0 is defined, and $T_{l-1,0}p_0$ stays within $P_{N,\sigma}$. The choice of the other items in (4.44) guarantees these two properties as shown in part (4). The second item is important in the estimates (4.45) and (4.47). The last two items are important in the estimates (4.46) and (4.48).

(4) We apply the interpolation operators from part (2) to show that iterated subdivision using T is well defined for σ''-dense data p_0 in $P_{N'',\sigma''}$ and that $T_{i,0}p_0$ stays within $P_{N,\sigma}$ for all i. We use induction on the subdivision level i. We consider input data $p_0 \in P_{N'',\sigma''}$. Since $\mathcal{D}(p_0) < \sigma'' < \sigma$, subdivision by T for input p_0 is defined. From (4.36) we get that

$$\mathcal{D}(T_0p_0) \le C_1\gamma\mathcal{D}(p_0) \le C_1\sigma'' \le \sigma. \tag{4.45}$$

The last inequality is a consequence of the choice of σ''.

Now we use the interpolation operators from part (2) and get

$$\begin{aligned}\|I_1T_0p_0 - I_0p_0\| &\le \|I_1T_0p_0 - I_1S_0p_0\| + \|I_1S_0p_0 - I_0p_0\| \\ &\le \|T_0p_0 - S_0p_0\| + C_B\mathcal{D}(p_0) \\ &\le C_P\mathcal{D}(p_0)^2 + C_B\mathcal{D}(p_0) \le \tfrac{\sigma'}{2} + \tfrac{\sigma'}{2}.\end{aligned} \tag{4.46}$$

Here we used (4.40) for the second inequality and the proximity condition (4.32) for the third inequality. The last inequality is a consequence of our choice of σ''. From the assumptions of the theorem it follows that T_0p_0 takes its values in N. Combining this fact with (4.45), we get that $T_0p_0 \in P_{N,\sigma}$ and thus T_0p_0 is in the domain of T_1. This serves as the induction base (i=0).

We use as an induction hypothesis that $T_{n-1,0}p_0$ is well-defined, that $T_{n-1,0}p_0$ takes its values in N, and that $T_{n-1,0}p_0$ is in the domain of T_n, for $n = 1, \dots, i$.

From (4.36) we get

$$\mathcal{D}(T_{i,0}p_0) \le C_1\gamma^{i+1}\mathcal{D}(p_0) \le C_1\sigma'' \le \sigma. \tag{4.47}$$

The last inequality is a consequence of the choice of σ''.

Now we use the interpolation operators from part (2) and get

$$\begin{aligned}&\|I_{i+1}T_{i,0}p_0 - I_0p_0\| \\ &\le \sum\nolimits_{n=0}^{i} \|I_{n+1}T_{n,0}p_0 - I_{n+1}S_nT_{n-1,0}p_0\| + \|I_{n+1}S_nT_{n-1,0}p_0 - I_nT_{n-1,0}p_0\| \\ &\le C_P\sum\nolimits_{n=0}^{i} \mathcal{D}(T_{n-1,0}p_0)^2 + C_B\sum\nolimits_{n=0}^{i}\mathcal{D}(T_{n-1,0}p_0) \\ &\le C_PC_1^2\Big(\sum\nolimits_{n=0}^{\infty}\gamma^{2n}\Big)\mathcal{D}(p_0)^2 + C_BC_1\sum\nolimits_{n=0}^{\infty}\gamma^n\mathcal{D}(p_0) \\ &\le \frac{C_PC_1^2}{1-\gamma^2}\mathcal{D}(p_0)^2 + \frac{C_BC_1}{1-\gamma}\mathcal{D}(p_0) \le \tfrac{\sigma'}{2} + \tfrac{\sigma'}{2}.\end{aligned} \tag{4.48}$$

Here we used (4.40) and the proximity condition (4.32) for the second inequality. The last inequality is a consequence of our choice of σ''. From the assumptions of the theorem it follows that $T_{i,0}p_0$ takes its values in N. Combining this fact with (4.47) we get that $T_{i,0}p_0 \in P_{N,\sigma}$ and thus $T_{i,0}p_0$ is in the domain of T_{i+1} which means that $T_{i+1,0}p_0$ is well-defined. This completes the induction.

As a consequence, for σ''-dense input in $P_{N'',\sigma''}$, $T_{i,0}p_0$ exists for all i and T is contractive for such input in the sense of (4.36). Toward convergence, we choose $i'' \geq i' \geq i$ and estimate, similar to (4.48),

$$\begin{aligned}\|I_{i''+1}T_{i'',0}p_0 - I_{i'+1}T_{i',0}p_0\| &\leq \frac{C_P C_1^2}{1-\gamma^2}\mathcal{D}(T_{i',0}p_0)^2 + \frac{C_B C_1}{1-\gamma}\mathcal{D}(T_{i',0}p_0) \\ &\leq \frac{C_P C_1^4}{1-\gamma^2}\gamma^{2i}\mathcal{D}(p_0)^2 + \frac{C_B C_1^2}{1-\gamma}\gamma^i\mathcal{D}(p_0).\end{aligned}$$

Since the right hand side approaches 0 as $i \to \infty$, the sequence $\{I_i T_{i-1,0}p_0\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $C(D, \mathbb{R}^d)$ and therefore convergent. Each sequence member is uniformly continuous, which implies the same for the limit. Thus T converges for input in $P_{N'',\sigma''}$.

(5) It remains to show (4.35). We consider $\varepsilon > 0$, and choose the index L large enough such that for all indices $i \geq L$, $\|T_{\infty,0}p_0 - I_i T_{i-1,0}p_0\| < \frac{\varepsilon}{2}$. With (4.41) we estimate, for $i \geq L$,

$$\|S_{\infty,i}T_{i-1,0}p_0 - I_i T_{i-1,0}p_0\| \leq C_I \mathcal{D}(T_{i-1,0}p_0) \leq C_I C_1 \gamma^i \mathcal{D}(p_0).$$

Now we choose $L_0 > L$ such that $C_I C_1 \gamma^{L_0}\sigma'' < \frac{\varepsilon}{2}$. Then for all $i \geq L_0$, $\|T_{\infty,0}p_0 - S_{\infty,i}T_{i-1,0}p_0\| < \varepsilon$. This proves (4.35). □

Our next task is to prove Theorem 4.5 which is a smoothness statement w.r.t. the characteristic parametrization. To that end, we need the following two lemmas concerning the characteristic parametrization of limit functions.

Concerning the constants in the proofs of the remainder of this chapter we employ the following conventions: We use generic constants c, C which can change from line to line.

Lemma 4.20. *Let λ be a subdominant eigenvalue of the subdivision matrix A of a linear subdivision scheme as defined in Chapter 1.2 (which has the single dominant eigenvalue 1). If we choose the ring index n_0 sufficiently large, we get a constant $C > 0$ such that, for all $n \geq n_0$ and each C^1 function $f : D_n \to \mathbb{R}^d$,*

$$\|f \circ \chi^{-1}\|_{C^1(\chi(D_n),\mathbb{R}^d)} \leq C|\lambda|^{-n}(\det M)^{-n/2}\|f\|_{C^1(D_n,\mathbb{R}^d)} \tag{4.49}$$

(M is the dilation matrix, D_n is the n-th ring). The constant C does not depend on the ring index $n \geq n_0$.

Proof. By our assumptions on the linear scheme S, its characteristic map χ is one-to-one in a neighborhood of the point 0. So we find an index n_0, such that χ is one-to-one on D'_{n_0}. In the following we assume that n_0 is chosen such that this requirement is fulfilled. We write χ_n for the restriction of χ to the ring D_n.

Our argument is based on the following fact which we verify only at the end of the proof: There is a ring index n_0 and a constant $C > 0$ such that the differential of the characteristic map χ obeys

$$\|d_x\chi_n(v)\| \geq C|\lambda|^n(\det M)^{n/2}\|v\|, \tag{4.50}$$

where C is independent of the ring index $n \geq n_0$ and the point $x \in D_n$. We use the Euclidean norm for the tangent vectors v; $\|d_x\chi_n\|$ is the induced operator norm. In other words, (4.50) states that differentials are lower bounded, uniformly for all $x \in D_n$, with constant C independent of the ring. If (4.50) is proved, we can apply the inverse function theorem to obtain a constant $C > 0$ such that

$$\sup_{y\in\chi(D_n)} \|d_y\chi_n^{-1}\| \leq C|\lambda|^{-n}(\det M)^{-n/2}, \tag{4.51}$$

where C is independent of the ring index $n \geq n_0$. Using the submultiplicativity of operator norms we get, for $y \in \chi(D_n)$,

$$\|d_y(f \circ \chi_n^{-1})\| \leq \|d_{\chi_n^{-1}(y)}f\| \cdot \|d_y\chi_n^{-1}\| \leq C|\lambda|^{-n}(\det M)^{-n/2}\|f\|_{C^1(D_n,\mathbb{R}^d)}.$$

This implies (4.49), since sup-norms of functions do not change under reparametrization.

To show (4.50) we need some preparations. We consider a Jordan block of the subdivision matrix A corresponding to a subdominant eigenvalue λ. We denote its multiplicity by m and order the Jordan vectors w_i, such that w_0 is the eigenvector. For the Jordan vector with the highest multiplicity, we have the expression

$$A^n w_{m-1} = \sum_{i=0}^{m-1} \binom{n}{i} \lambda^{n-i} w_{m-i-1}. \tag{4.52}$$

Since $\binom{n}{i}$ grows as n^i as $n \to \infty$, the dominating term in this expression is given by $\binom{n}{m-1}\lambda^{n-m+1}w_0$. We define vectors v_i in the following way: If the subdominant eigenvalues of A are complex conjugate numbers, we use the vectors w_i to define new vectors v_i where each component consists of the tuple of real numbers consisting of the real and the imaginary part of the corresponding component of w_i. If the subdominant eigenvalues of A are real and equal, we use vectors w_i as above and a second set of vectors $\bar{w}_i$ corresponding to the second subdominant Jordan block with the same ordering as above. We define new vectors v_i where each component consists of the tuple of real numbers consisting of the corresponding components of w_i and $\bar{w}_i$, respectively.

Then the characteristic map χ is the limit of subdivision for the input data stored in the vector v_{m-1}. We define $\xi_n : D_0 \to \mathbb{R}^2$ by

$$\chi_n = \xi_n \circ (G^n)^{-1}. \tag{4.53}$$

Then ξ_n is the limit function on D_0 of linear (regular mesh) subdivision for 0-th level input data obtained from $A^n v_{m-1}$.

We let $\psi : D_0 \to \mathbb{R}^2$ be the limit function for input data on level 0 obtained from v_0, and let $f_i : D_0 \to \mathbb{R}^2$ be the limit functions for the other v_i. All these limits are C^1 on

D_0, since they were obtained by regular mesh subdivision. Furthermore, the finiteness of the control set $\mathrm{ctrl}^0(D_0)$ yields

$$\|S_{\infty,0}p_0\|_{C^1(D_0)} \leq C\|p_0\|_\infty,$$

for arbitrary input data p_0 on $\mathrm{ctrl}^0(D_0)$. Knowing this and the fact that $A^n v_{m-1}$ is dominated by $\binom{n}{m-1}\lambda^{n-m+1}v_0$ for $n \to \infty$, which is a consequence of (4.52), we see that the sequence of mappings

$$\binom{n}{m-1}^{-1}\lambda^{m-n-1}\xi_n(\cdot) \to \psi(\cdot) \quad \text{in } \; C^1(D_0),$$

as n tends to ∞. This implies that ψ is regular, since we assumed that ξ_n (which is a reparametrization and restriction of the characteristic map) is regular for sufficiently large n. This fact allows us to estimate the Jacobian of ξ_n from below as follows: We start out by using the inverse triangle inequality to estimate

$$\begin{aligned}\|d_x\xi_n(v)\| &= \|\tbinom{n}{m-1}\lambda^{n-m+1}d_x\psi(v) + \sum_{i=0}^{m-2}\tbinom{n}{i}\lambda^{n-i}d_x f_{m-i-1}(v)\| \\ &\geq \tbinom{n}{m-1}|\lambda|^{n-m+1}\|d_x\psi(v)\| - \sum_{i=0}^{m-2}\tbinom{n}{i}|\lambda|^{n-i}\|d_x f_{m-i-1}\|\|v\|. \end{aligned} \tag{4.54}$$

We use that $\binom{n}{i}$ grows as n^i as $n \to \infty$ to estimate the binomial coefficients. Due to the compactness of D_0 we find a constant $C > 0$ such that for all points $x \in D_0$ and all functions f_i the differentials obey $\|d_x f_i\| \leq C$. Since ψ is regular we get a lower constant $c > 0$ such that, for all $x \in D_0$, $\|d_x\psi(v)\| \geq c\|v\|$. Making the constant c smaller (which comes form estimating the binomial coefficients and multiplying with λ^{m-1}) these estimates help us to get

$$\|d_x\xi_n(v)\| \geq c\, n^{m-1}|\lambda|^n\|v\| - C\, \sum_{i=0}^{m-2} n^i|\lambda|^n\|v\| \geq n^{m-1}|\lambda|^n\|v\|(c - Cn^{-1}). \tag{4.55}$$

If we now choose n_0 large enough, there is a constant $c > 0$ which does not depend on the index $n > n_0$ such that

$$\|d_x\xi_n(v)\| \geq cn^{m-1}|\lambda|^n\|v\|. \tag{4.56}$$

With (4.53) we get, for $x \in D_n$,

$$\|d_x\chi_n(v)\| \geq c|\lambda|^n\|d_x G^{-n}(v)\| \geq c|\lambda|^n \det M^{n/2}\|v\|.$$

This proves (4.50). □

Lemma 4.21. *Let p_n be input data on the control set $\mathrm{ctrl}^n(D'_n)$ of the inner area D'_n for data level n. Then for large enough n_0, and $s > 1$, there is a constant $C > 0$, which does not depend on the level $n \geq n_0$ and data p_n, such that*

$$\|S_{\infty,n}p_n \circ \chi^{-1}\|_{C^1(\chi(D'_n),\mathbb{R}^d)} \leq C|\lambda|^{-n}s^n\|p_n|_{\mathrm{ctrl}^n(D'_n)}\|_\infty. \tag{4.57}$$

Here λ is a subdominant eigenvalue of the subdivision matrix.

Proof. We use the notation of the proof of Lemma 4.20 and choose the integer n_0 so large that χ is regular and injective on $D'_{n_0} \setminus \{0\}$ and such that Lemma 4.20 works. Over the characteristic parametrization, the subdivision scheme S produces C^1 limit functions. As in Lemma 4.20, from the finiteness of the control set $\mathrm{ctrl}^{n_0}(D'_{n_0})$ we conclude that the differential of limit functions w.r.t. the characteristic parametrization can be estimated by

$$\sup_{x \in \chi(D'_{n_0})} \|d_x(S_{\infty,n_0} p_{n_0} \circ \chi^{-1})\| \leq C \|p_{n_0}|_{\mathrm{ctrl}^{n_0}(D'_{n_0})}\|_\infty, \tag{4.58}$$

where the constant C is independent of the n_0-th level input data p_{n_0} given on $\mathrm{ctrl}^{n_0}(D'_{n_0})$. In order to derive (4.57) from (4.58) we consider input p_n on level $n > n_0$, given on the control sets $\mathrm{ctrl}^n(D'_n)$ of the inner area D'_n. Reparametrizing this discrete data with the help of the similarity transform G, i.e., applying G^{n_0-n}, yields data $\bar{p}_{n_0}$ on level n_0. The limit function $S_{\infty,n} p_n$(over D'_n) equals $S_{\infty,n_0} \bar{p}_{n_0} \circ G^{n-n_0}$. Our objective is to get the estimate

$$\sup_{x \in \chi(D'_n)} \|d_x(S_{\infty,n} p_n \circ \chi^{-1})\| \leq C|\lambda|^{-n} s^n \sup_{x \in \chi(D'_{n_0})} \|d_x(S_{\infty,n_0} \bar{p}_{n_0} \circ \chi^{-1})\| \tag{4.59}$$

with the constant C not depending on the level $n > n_0$. If this estimate is established, then (4.57) is a direct consequence of (4.58) if we keep in mind that a reparametrization of any function does not change its sup-norm. To show (4.59), we split D'_n and D'_{n_0} into rings and show (4.59) on the rings. More precisely, we show, letting $r = n - n_0$, that

$$\sup_{x \in \chi(D_{l+r})} \|d_x(S_{\infty,n} p_n \circ \chi^{-1})\| \leq C|\lambda|^{-r} s^r \sup_{x \in \chi(D_l)} \|d_x(S_{\infty,n_0} \bar{p}_{n_0} \circ \chi^{-1})\| \tag{4.60}$$

with the constant C not depending on the $l > n_0$ and $r > 0$. Although the exponents of $|\lambda|$ and s in (4.59) and (4.60) differ by n_0 this does not affect the estimate since the resulting constant $|\lambda|^{n_0} s^{n_0}$ is independent of l and r, or n, respectively. Although (4.60) does not consider the central point 0, it nevertheless implies (4.59), since we know that both the function $S_{\infty,n} p_n \circ \chi^{-1}$ and the function $S_{\infty,n_0} \bar{p}_{n_0} \circ \chi^{-1}$ are continuously differentiable in 0.

In order to show (4.60) we consider the maps ξ_{l+r} and ξ_l introduced in the proof of Lemma 4.20. Those maps are reparametrizations of the characteristic map on the rings D_{l+r} and D_l, respectively, such that both maps are defined on D_0. We use the mapping

$$Z_{l,r} := \xi_l \circ \xi_{l+r}^{-1} : \chi(D_{l+r}) \to \chi(D_l)$$

to reparametrize limit functions defined on $\chi(D_{l+r})$ and to obtain functions defined on $\chi(D_l) \subset \chi(D'_{n_0})$ where we have the estimate (4.58). In order to analyze the mappings $Z_{l,r}$ we need some preparations. First, the estimate (4.56) together with the inverse function theorem shows that there is a constant $C > 0$, independent of the indices $l > n_0$ and $r > 0$, such that

$$\sup_{y \in \chi(D_{l+r})} \|d_y \xi_{l+r}^{-1}\| \leq C(l+r)^{1-m} |\lambda|^{-l-r}. \tag{4.61}$$

Secondly, we proceed similar to (4.54) and (4.55) in Lemma 4.20, but estimate from above, instead of from below, to get a constant C which does not depend on l and $x \in D_0$ such that

$$\|d_x \xi_l\| \leq C|\lambda|^l l^{m-1} \|d_x \psi\|. \tag{4.62}$$

Using the chain rule and both (4.61) and (4.62), we obtain

$$\sup_{y\in\chi(D_{l+r})} \|d_y Z_{l,r}\| \le C\big((l+r)^{1-m}|\lambda|^{-l-r}\big)\cdot\big(|\lambda|^l l^{m-1}\big) \le C\,|\lambda|^{-r}s^r,$$

where C is independent of $l > n_0$ and $r > 0$. Since $S_{\infty,n}p_n \circ \chi^{-1} = S_{\infty,n_0}\bar{p}_{n_0} \circ \chi^{-1} \circ Z_{l,r}$ on the ring $\chi(D_{l+r})$, we can apply the chain rule to estimate

$$\begin{aligned}\sup_{x\in\chi(D_{l+r})} \|d_x(S_{\infty,n}p_n \circ \chi^{-1})\| &\le \sup_{x\in\chi(D_{l+r})} \|d_x Z_{l,r}\| \sup_{x\in\chi(D_l)} \|d_x(S_{\infty,n_0}\bar{p}_{n_0} \circ \chi^{-1})\| \\ &\le C|\lambda|^{-r}s^r \sup_{x\in\chi(D_l)} \|d_x(S_{\infty,n_0}\bar{p}_{n_0} \circ \chi^{-1})\|,\end{aligned}$$

where the constant C does not depend on $l > n_0$ and $r > 0$. This proves (4.60), which completes the proof. □

We proof the main result of this part.

Proof of Theorem 4.5. We use the ring index n_0 of Lemma 4.20 which guarantees that the estimates of Lemma 4.20 and Lemma 4.21 are valid.

We show that the functions $S_{\infty,i}T_{i-1,0}p_0 \circ \chi^{-1}$ form a Cauchy sequence in the Banach space $C^1(\chi(D'_{n_0}), \mathbb{R}^d)$. Since this sequence (with each member reparametrized by χ) converges to the limit of subdivision in the space $C(D, \mathbb{R}^d)$ according to Theorem 4.4, it also converges to the reparametrized limit of subdivision in the space $C(\chi(D'_{n_0}), \mathbb{R}^d)$. So if the sequence is Cauchy in C^1 its limit agrees with the reparametrized limit of subdivision, which must then be a C^1 function.

In order to show that the sequence $S_{\infty,i}T_{i-1,0}p_0 \circ \chi$ is Cauchy we show that there is a constant C, which does not depend on the level $i \ge n_0$, such that

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}\|_{C^1(\chi(D'_{n_0}),\mathbb{R}^d)} \le C\gamma^i \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0)^2, \tag{4.63}$$

for $\gamma = s^2 \max((\det M)^{-1/2}, |\lambda|)$, and $s > 1$ chosen such that $\gamma < 1$. If (4.63) is shown, the geometric series yields the desired statement.

We consider $(i+1)$-st level data q_{i+1} given by

$$q_{i+1} := (T_i - S_i)T_{i-1,0}p_0.$$

According to (4.63), we have to estimate the C^1 norm of the limit function $S_{\infty,i+1}q_{i+1}$ of linear subdivision using S for input data q_{i+1} w.r.t. the characteristic parametrization. In order to get fine enough estimates, we split the n_0-th inner area D'_{n_0} into the rings D_n ($n_0 \le n \le i$) and the $(i+1)$-st inner area D'_{i+1}. We estimate $S_{\infty,i+1}q_{i+1} \circ \chi^{-1}$ on the domains $\chi(D_n)$ and $\chi(D'_{i+1})$ separately.

We begin with the rings D_n. We fix n with $n_0 \le n \le i$. From Lemma 4.19 we get a constant $C > 0$ which does not depend on the ring index n such that

$$\mathcal{D}_{\mathrm{ctrl}^n(D'_n)}(T_{n-1,0}p_0) \le C|\lambda|^n s^n \mathcal{D}_{\mathrm{ctrl}^0(D'_0)}(p_0). \tag{4.64}$$

In Chapter 1.2 we assumed that the control sets $\mathrm{ctrl}^n(D_n^j)$ of the segments D_n^j have regular combinatorics. Therefore, the limit function w.r.t. linear subdivision using S on the domain D_n is obtained from n-th level data on $\mathrm{ctrl}^n(D_n)$ by means of subdivision on a regular part of the mesh. By the locality of the proximity inequality, the same is true for using T instead of S. Then Lemma 3.7 implies that

$$\begin{aligned}\mathcal{D}_{\mathrm{ctrl}^i(D_n)}(T_{i-1,0}p_0) &\leq C \ \det M^{(n-i)/2} s^{i-n} \mathcal{D}_{\mathrm{ctrl}^n(D_n)}(T_{n-1,0}p_0)\\ &\leq C \ \det M^{(n-i)/2} |\lambda|^n s^i \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0).\end{aligned}$$

For the second inequality we used (4.64). The constants C do not depend on i. The proximity inequality and the above estimate yield

$$\begin{aligned}\mathcal{D}_{\mathrm{ctrl}^{i+1}(D_n)}(q_{i+1}) &\leq C \ \mathcal{D}_{\mathrm{ctrl}^i(D_n)}(T_{i-1,0}p_0)^2\\ &\leq C \ \det M^{n-i} |\lambda|^{2n} s^{2i} \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2, \qquad (4.65)\end{aligned}$$

where the occurring constants do not depend on the index i. We turn to estimating C^1 norms. From the scaling relation and the translation invariance of the scheme S in regular parts of a mesh we get a constant C which is again independent of i and the level n, where $n_0 \leq n \leq i$, such that

$$\|S_{\infty,i+1}q_{i+1}\|_{C^1(D_n,\mathbb{R}^d)} \leq C \ \det M^{i/2} \|q_{i+1}|_{\mathrm{ctrl}^{i+1}(D_n)}\|_\infty. \qquad (4.66)$$

These facts together with Lemma 4.20 imply

$$\begin{aligned}\|S_{\infty,i+1}q_{i+1} \circ \chi^{-1}\|_{C^1(\chi(D_n),\mathbb{R}^d)} &\leq C \ |\lambda|^{-n} (\det M)^{-n/2} \|S_{\infty,i+1}q_{i+1}\|_{C^1(D_n,\mathbb{R}^d)}\\ &\leq C \ |\lambda|^{-n} (\det M)^{(i-n)/2} \|q_{i+1}|_{\mathrm{ctrl}^{i+1}(D_n)}\|_\infty.\\ &\leq C \ |\lambda|^n s^{2i} (\det M)^{(n-i)/2} \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2.\end{aligned}$$

The constants C do not depend on the indices n and i. For the first inequality we used the estimate (4.49) of Lemma 4.20. The second and the third inequality are a consequence of (4.66) and (4.65), respectively. This proves (4.63) on the rings $\chi(D_n)$ with ring index $n_0 \leq n \leq i$.

It remains to consider the $(i+1)$-st inner area D'_{i+1}. We obtain

$$\begin{aligned}\|S_{\infty,i+1}q_{i+1} \circ \chi^{-1}\|_{C^1(\chi(D'_{i+1}),\mathbb{R}^d)} &\leq C \ |\lambda|^{-i} s^i \|q_{i+1}|_{\mathrm{ctrl}^{i+1}(D'_{i+1})}\|_\infty\\ &\leq C \ |\lambda|^{-i} s^i \mathcal{D}_{\mathrm{ctrl}^i(D_i')}(T_{i-1,0}p_0)^2\\ &\leq C \ |\lambda|^i s^{2i} \mathcal{D}_{\mathrm{ctrl}^0(D_0')}(p_0)^2,\end{aligned}$$

where the constants C are independent of i. We use Lemma 4.21 for the first estimate. The second inequality is obtained by applying the local proximity inequality, and Lemma 4.19 gives the last inequality. This estimate proves (4.63) on $\chi(D'_{i+1})$, which completes the proof.

Note that if we have pure eigenvalues, χ is already invertible on D'_0 and Lemma 4.21 is true for any $n \in \mathbb{N}_0$. So we can choose $n_0 = 0$ in that case. □

Finally, we show Corollary 4.6.

Proof of Corollary 4.6. It remains to verify the local proximity condition (4.1). This follows directly from [57, Lemma 5] for the geodesic analogue, [15, Theorem 4] for the projection analogue, from [14, Proposition 7.2] for the log-exp analogue, and for the intrinsic mean analogue by its interpretation as log-exp analogue with special choice of base points. □

5 Convergence of schemes in Cartan-Hadamard manifolds

This part of the thesis is concerned with convergence of geometric subdivision schemes for all input data. The results of this chapter are contained in [58]. This circle of problems is much more involved than the convergence results for dense enough input data. This is due to the fact that convergence for dense enough input data is a local problem and thus only local properties of manifolds (which are locally homeomorphic to $\mathbb{R}^n$) enter the scene. In contrast, when we are interested in convergence for all input data, the global structure of the manifold is important. Furthermore, the geometric analogues we considered in Chapter 2 are, in general, even only well defined for dense enough input data. So if we want to speak about convergence for all input data, we need manifolds with additional properties which guarantee that the geometric scheme under consideration is well defined for all input data. A class of manifolds meeting this requirement for the intrinsic mean analogue is the class of so-called Cartan-Hadamard manifolds provided the mask of the scheme is positive. Cartan-Hadamard (CH-)manifolds are complete simply connected Riemannian manifolds with nonpositive sectional curvatures. Examples are the spaces of positive $n \times n$-matrices which are e.g. treated in Chapter XII of [32]. CH-manifolds have the nice feature that the Riemannian exp-map is a diffeomorphism. Furthermore, the weighted intrinsic mean of finitely many points is globally well defined [29, Ch. 8, Thm. 9.1] .

In this chapter we thus consider intrinsic mean subdivision schemes with positive weights in CH-manifolds. The main result of this chapter is the following theorem for the curve case:

Theorem 5.1. *Assume that T is an 'intrinsic mean' curve subdivision scheme analogous to an affinely invariant linear scheme S with positive mask. Let T act on data in a CH-manifold. If S is contractive in the strong sense, i.e., for bounded input p,*

$$\sup_i \operatorname{dist}(Sp_i, Sp_{i-1}) \leq \gamma \sup_i \operatorname{dist}(p_i, p_{i-1}), \qquad \gamma < 1, \tag{5.1}$$

then the same inequalities true with S replaced by T (with the same constant γ), and T converges for all input data.

Note that the derived scheme $S^{[1]}$ of any linear scheme S yields a constant $\gamma = S^{[1]}/N$ (where N is the dilation factor of S) which can be used for Equation (5.1) provided $\gamma < 1$. This yields the following corollary.

Corollary 5.2. *If T operates in a CH-manifold and is the intrinsic mean analogue of an affinely invariant linear curve scheme S with positive mask and the derived scheme fulfills $\|S^{[1]}\| < N$, then T converges for all input data.*

As an example we consider the Lane-Riesenfeld schemes which produce B-Splines as limit functions. For these schemes the dilation factor is 2. It is well known that $\|S^{[1]}\| = 1 < 2$. This together with the smoothness result of Theorem 3.5 implies the following corollary:

Corollary 5.3. *The intrinsic mean subdivision schemes which inherit their weights from the Lane-Riesenfeld curve schemes converge for all input data in a CH-manifold and the limits are as smooth as the corresponding B-splines.*

For the proof of Theorem 5.1 we need the following lemma from differential geometry:

Lemma 5.4. *Let M be a CH-manifold and let c_i be nonnegative weights with $\sum_{i=1}^n c_i = 1$. We consider two sets of points p_i and q_i and regard the corresponding intrinsic means $m_{(p_i,c_i)}$ and $m_{(q_i,c_i)}$ of these two sets of points with the same weights c_i. Then*

$$\operatorname{dist}(m_{(p_i,c_i)}, m_{(q_i,c_i)}) \leq \sum_{i=1}^{n} c_i \operatorname{dist}(p_i, q_i). \tag{5.2}$$

Remark 5.5. A similar statement with constants on the right hand side is true locally without requiring the manifold to be a CH-manifold. These constants depend on the sectional curvature of the manifold in question. We refer to [27] for details.

Proof. We follow the proof of Corollary 1.6. in [27]. We use the notation $m_p = m_{(p_i,c_i)}$ and $m_q = m_{(q_i,c_i)}$. Consider the real-valued mapping P_q defined on the CH-manifold given by

$$P_q(x) = \tfrac{1}{2} \sum_i c_i \operatorname{dist}(x, q_i)^2.$$

By [27, Theorem 1.2], its gradient reads

$$\operatorname{grad} P_q(x) = -\sum_i c_i \exp_x^{-1} q_i. \tag{5.3}$$

It is a consequence of [27, (1.5.1)] and the last sentence in the proof of [27, Theorem 1.2] that for all x in the manifold

$$\operatorname{dist}(x, m_q) \leq |\operatorname{grad} P_q(x)|. \tag{5.4}$$

Here the nonpositive sectional curvature of the CH-manifold is used. Combining (5.3) and (5.4), and letting $x = m_p$, yields

$$\operatorname{dist}(m_p, m_q) \leq \Big|\sum_i c_i \exp_{m_p}^{-1} q_i\Big|. \tag{5.5}$$

CH-manifolds have the additional important property that the exp-mapping is not decreasing distances [32], i.e., for points x, y, z in the manifold,

$$|\exp_z^{-1} x - \exp_z^{-1} y| \leq \operatorname{dist}(x, y). \tag{5.6}$$

In our current notation the balance condition (2.2) for the intrinsic mean m_p reads

$$\sum_i c_i \exp_{m_p}^{-1}(p_i) = 0. \tag{5.7}$$

Starting from (5.5) we use (5.6) and (5.7) to estimate

$$\begin{aligned}\operatorname{dist}(m_p, m_q) &\leq \Big|\sum_i c_i \exp_{m_p}^{-1} q_i\Big| \\ &= \Big|\sum_i c_i \exp_{m_p}^{-1} q_i - \sum_i c_i \exp_{m_p}^{-1} p_i\Big| \\ &\leq \sum_i c_i \left|\exp_{m_p}^{-1} q_i - \exp_{m_p}^{-1} p_i\right| \\ &\leq \sum_i c_i \operatorname{dist}(q_i, p_i).\end{aligned}$$

□

We are going to prove Theorem 5.1.

Proof of Theorem 5.1. We show that for data p,

$$\sup_i \operatorname{dist}(Tp_i, Tp_{i-1}) \leq \gamma \sup_i \operatorname{dist}(p_i, p_{i-1}). \tag{5.8}$$

Locality of T implies that w.l.o.g. we can assume that data are bounded. From (5.8) we immediately conclude that data eventually gets dense enough after sufficiently many subdivision steps. This means that if we have shown (5.8) the statement of the theorem is a consequence of Theorem 3.3.

To that end we consider the scheme S with mask $\{a_i\}_{i\in\mathbb{Z}}$. Depending on the dilation factor N, we obtain N essentially different sets of nonnegative averaging coefficients $b^{(k)} = \{a_{k-lN}\}_l$, where $k \in \mathbb{Z}$. These coefficients are used to define the weights for the intrinsic means. In order to estimate

$$\operatorname{dist}(Tp_i, Tp_{i-1}) = \operatorname{dist}(m_{(p_l, b_l^{(i)})}, m_{(p_l, b_l^{(i-1)})}) \tag{5.9}$$

we are going to apply Lemma 5.4. If we compare the right-hand side of (5.9) and the left-hand side of (5.2) we see that on the one hand we have one sequence of points and two sets of weights, on the other hand we have two sequences of points and one set of weights. To overcome this problem we define a new sequence of weights w_j using the weights $b_l^{(i)}$ and $b_l^{(i-1)}$, and we derive new sequences of points x_j and y_j from the sequence p_l. Below, we are going to show that the following means coincide:

$$m_{(p_l, b_l^{(i)})} = m_{(x_j, w_j)} \quad \text{and} \quad m_{(p_l, b_l^{(i-1)})} = m_{(y_j, w_j)}. \tag{5.10}$$

We define the weights w_j, and x_j, y_j as follows: For $l \in \mathbb{Z}$, we denote the partial sums of the weights $b^{(i)}$ and $b^{(i-1)}$ by

$$C_l = \sum_{r\leq l} b_r^{(i)} \quad \text{and} \quad D_l = \sum_{r\leq l} b_r^{(i-1)}.$$

We merge the two monotonously increasing sequences C_l and D_l and order them according to their values. We get a monotonously increasing sequence W_j and two monotonously increasing mappings $j_C, j_D : \mathbb{Z} \to \mathbb{Z}$ of indices such that

$$W(j_C(l)) = C(l) \quad \text{and} \quad W(j_D(l)) = D(l).$$

We define the weights w_j by

$$w_j = W_j - W_{j-1}.$$

Surely, w_j defines a nonnegative weight sequence. In order to define the point x_j we consider the smallest index $k_0 \geq j$ in the sequence W which 'comes from the sequence C', i.e., $k_0 = \min\{k \geq j : k \in j_C(\mathbb{Z})\}$. The preimage $j_C^{-1}(k_0)$ defines the index in the sequence of points p we use to define x_j. For y_j we proceed in an analogous way. Thus we let

$$x_j = p_{\min\{m : j_C(m) \geq j\}} \quad \text{and} \quad y_j = p_{\min\{m : j_D(m) \geq j\}}.$$

We show (5.10): We consider the sequence members p_i as formal objects, meaning that $p_r = p_s$ if and only if $r = s$. We start with the left-hand side of (5.10). We consider a point p_l, examine the sequence x to find the indices j with $x_j = p_l$, and show that the sum of corresponding weights $\sum w_j = b_l^{(i)}$.

For fixed l, $x_j = p_l$ if and only if $j_C(l-1) + 1 \leq j \leq j_C(l)$. This is a consequence of j_C being monotonously increasing. So x is of the form $(\ldots, p_{l-1}, p_l, p_l, \ldots, p_l, p_l, p_{p_{l+1}} \ldots)$, and

$$\sum_{j=j_C(l-1)+1}^{j_C(l)} w_j = W_{j_C(l)} - W_{j_C(l-1)} = C_l - C_{l-1} = b_l^{(i)}.$$

Therefore, for all x,

$$\sum_l b_l^{(i)} \operatorname{dist}(x, p_l)^2 = \sum_l \sum_{j=j_C(l-1)+1}^{j_C(l)} w_j \operatorname{dist}(x, x_j)^2 = \sum_j w_j \operatorname{dist}(x, x_j)^2.$$

This implies that $m_{(p_l, b_l^{(i)})} = m_{(x_j, w_j)}$. The second part of (5.10) follows by the same argument.

Application of (5.9), (5.10) and (5.2) yields

$$\operatorname{dist}(Tp_i, Tp_{i-1}) \leq \sum_j w_j \operatorname{dist}(x_j, y_j). \tag{5.11}$$

We investigate the right-hand expression of (5.11): We are going to show that

$$\sum_j w_j \operatorname{dist}(x_j, y_j) \leq \sum_l |C_l - D_l| \cdot \sup_i \operatorname{dist}(p_i, p_{i-1}). \tag{5.12}$$

We fix j and consider two neighboring sequence members W_j and W_{j-1} ($W_j > W_{j-1}$) and examine subsequences of the form $\{W_{j_C(l)}, \ldots, W_{j_D(l)}\}$, the values being between C_l and D_l. We want to find the indices l with

$$\{W_{j-1}, W_j\} \subset \{W_{j_C(l)}, \ldots, W_{j_D(l)}\}. \tag{5.13}$$

Clearly, the index l fulfills (5.13) if and only if $j_C(l) < j \leq j_D(l)$. This is the case if and only if $\min\{m : j_D(m) \geq j\} \leq l < \min\{m : j_C(m) \geq j\}$. With the roles of C_l and D_l

exchanged we obtain $\{W_{j-1}, W_j\} \subset \{W_{j_D(l)}, \ldots, W_{j_C(l)}\}$ if and only if $\min\{m : j_C(m) \geq j\} \leq l < \min\{m : j_D(m) \geq j\}$. This means that the number of indices

$$\begin{aligned}
&\#\{l : \{W_{j-1}, W_j\} \subset \{W_{\min(j_C(l), j_D(l))}, \ldots, W_{\max(j_C(l), j_D(l))}\}\ \} \\
&\quad = |\min\{m : j_C(m) \geq j\} - \min\{m : j_D(m) \geq j\}|.
\end{aligned} \tag{5.14}$$

Furthermore, by definition,

$$x_j = p_{\min\{m : j_C(m) \geq j\}} \quad \text{and} \quad y_j = p_{\min\{m : j_D(m) \geq j\}}.$$

Note that the right-hand side of (5.14) is the absolute value of the difference of the indices of the sequence p appearing in x_j and y_j. We show (5.12):

$$\begin{aligned}
\sum_j w_j \operatorname{dist}(x_j, y_j) &= \sum_j (W_j - W_{j-1}) \operatorname{dist}(p_{\min\{m : j_C(m) \geq j\}}, p_{\min\{m : j_D(m) \geq j\}}) \\
&\leq \sum_j (W_j - W_{j-1}) \cdot |\min\{m : j_C(m) \geq j\} - \min\{m : j_D(m) \geq j\}| \cdot \sup_i \operatorname{dist}(p_i, p_{i-1}) \\
&= \sum_l \sum_{j=\min(j_C(l), j_D(l))+1}^{\max(j_C(l), j_D(l))} (W_j - W_{j-1}) \cdot \sup_i \operatorname{dist}(p_i, p_{i-1}) \\
&= \sum_l |C_l - D_l| \cdot \sup_i \operatorname{dist}(p_i, p_{i-1}).
\end{aligned}$$

Here we have used the triangle inequality. The last but one equality is a consequence of (5.14). This completes the proof of the estimate (5.12). We conclude, using (5.11), that

$$\operatorname{dist}(Tp_i, Tp_{i-1}) \leq \sum_l |C_l - D_l| \cdot \sup_n \operatorname{dist}(p_n, p_{n-1}). \tag{5.15}$$

We are going to show that

$$\sum_l |C_l - D_l| \leq \gamma. \tag{5.16}$$

Then plugging (5.16) into (5.15) yields (5.8) which completes the proof. To show (5.16) we define the 'sign sequence' σ_l by

$$\sigma_l = \operatorname{sign}(C_l - D_l). \tag{5.17}$$

Then we have

$$\begin{aligned}
\sum_l |C_l - D_l| &= \sum_l \sigma_l (C_l - D_l) \\
&= \sum_l \sigma_l \big(\sum_{r \leq l} b_r^{(i)} - \sum_{r \leq l} b_r^{(i-1)} \big) \\
&= \sum_r \big(\sum_{l \geq r} \sigma_l \big) b_r^{(i)} - \sum_r \big(\sum_{l \geq r} \sigma_l \big) b_r^{(i-1)}.
\end{aligned} \tag{5.18}$$

We define a test data sequence $q : \mathbb{Z} \to \mathbb{R}$ by $q_r = \sum_{l \geq r} \sigma_l$. Obviously, this sequence is bounded and $|q_r - q_{r-1}| \leq 1$ for all r. By (5.18),

$$\begin{aligned} \sum_l |C_l - D_l| &= \sum_r q_r b_r^{(i)} - \sum_r q_r b_r^{(i-1)} = \big| \sum_r q_r b_r^{(i)} - \sum_r q_r b_r^{(i-1)} \big| \\ &= |Sq_i - Sq_{i-1}| \leq \gamma \sup_r |q_r - q_{r-1}| = \gamma. \end{aligned} \tag{5.19}$$

This shows (5.16) which completes the proof. □

6 Interpolatory multiscale transforms for functions between manifolds

This last chapter uses geometric subdivision schemes to define interpolatory multiscale transforms for functions between manifolds. We consider the case when the domain manifold is a 2-manifold. The purpose of this chapter is to characterize the smoothness of a function by the decay of its detail coefficients which are derived by the transform. This chapter is contained in [61].

Such results have quite a history. In [9], D. Donoho analyzes linear *interpolatory wavelet transforms*. He characterizes smoothness properties of a function by decay properties of the detail coefficients which are derived from the function via the transformation. Interpolatory transforms can also be defined in a reasonable manner in the setting of geometric subdivision which has been observed by Donoho et al.[53]. In [17], Grohs and Wallner show an analogue of Donoho's result concerning the decay of detail coefficients for the class of Hölder-Zygmund functions in the geometric setting. More precisely, they consider a continuous function

$$f : \mathbb{R}^n \to M,$$

where M in a manifold. This function is sampled on the grid $2^{-i}\mathbb{Z}^n$ to obtain a grid function

$$f_i : 2^{-i}\mathbb{Z}^n \to M$$

A geometric subdivision scheme T is applied to f_i and the (generalized) difference $\ominus$ (defined at the end of Chapter 2) between this prediction Tf_i and the finer sample f_{i+1} on the grid $2^{-i-1}\mathbb{Z}^n$ gives the i-th level detail coefficients:

$$d_i = f_i \ominus Tf_{i-1}.$$

Then the transform reads

$$f \to (f_0, d_0, d_1, \ldots). \tag{6.1}$$

The corresponding linear transform is obtained by replacing T by a linear scheme S, and $\ominus$ by $-$.

Smoothness of a function f is related to decay of the coefficients d_i by the following theorem which is part of the results of [9] and the result of [17]. In this context smoothness of a function is measured by its membership in the Hölder-Zygmund classes Lip_α.

Theorem 6.1. *Let S be a linear interpolatory subdivision scheme on the regular mesh which produces Lip_α limits, and assume that f is a continuous function on $\mathbb{R}^d$ with image contained in a compact subset. Let $\gamma < \alpha$.*

Then $f \in \mathrm{Lip}_\gamma$ if and only if the coefficients d_i w.r.t. the linear scheme decay as $O(2^{-\gamma i})$, i.e., there is $C > 0$ such that $2^{\gamma i}\|d_i\|_\infty \leq C$ for all i.

Assume furthermore that T is a geometric analogue of S, and that f_0 is dense enough such that the geometric version of the transform is defined. Then the detail coefficients w.r.t. T also decay as $O(2^{-\gamma i})$ if and only if $f \in \mathrm{Lip}_\gamma$.

In contrast, we treat domains which are not necessarily Euclidean, i.e., we deal with manifold-valued functions, defined on a two-dimensional manifold. We consider a multiscale transform, where both the choice of sample points and the prediction operator are based on nonlinear geometric subdivision. Since 2-manifolds with non-zero Euler characteristics cannot be covered with regular quad meshes or triangular meshes, we must be able to process irregular combinatorics.

The chapter is organized as follows. In Chapter 6.1 we give the definition of the transform. The remainder of the chapter is devoted to the characterization of Hölder-Zygmund functions in terms of the detail coefficient decay w.r.t. our transform, in particular near irregular points. The results are stated in Chapter 6.2, where we also give some examples. The proofs are collected in Chapter 6.3.

6.1 Definition of a multiscale transformation for geometric data

In the following let N be a two-dimensional smooth *domain* manifold, and let M be a smooth *target* manifold of arbitrary dimension. We explain a way of sampling continuous functions from N to M: Consider a mesh (K_0, p_0) which covers N. We use an interpolatory subdivision scheme T', which processes data in N and which is a geometric analogue of a linear scheme S. Application of T' yields a sequence of subdivided meshes $(K_1, p_1), (K_2, p_2), \ldots$. The (realized) vertex sets $X_i = p_i(V_i)$ in N are nested.

We assume that two (realized) vertices $p_i(v)$ and $p_i(w)$ never coincide, i.e., we assume that p_i is injective. Sufficient conditions for injectivity are given in Chapter 6.2.

We propose the following discrete interpolatory multiscale transform: We point-sample a continuous function $f : N \to M$ on X_i and let

$$f_i = f|_{X_i}.$$

So f_i is an M-valued function defined in the discrete subset $X_i \subset N$.

To define a prediction operator T we use a second interpolatory analogue T'' of S which processes data in the target manifold M. T'' is applied to the mesh $(K_i, f \circ p_i)$ whose realized vertex set is $f(X_i)$. The result is a mesh (K_{i+1}, g_{i+1}) where g_{i+1} has values

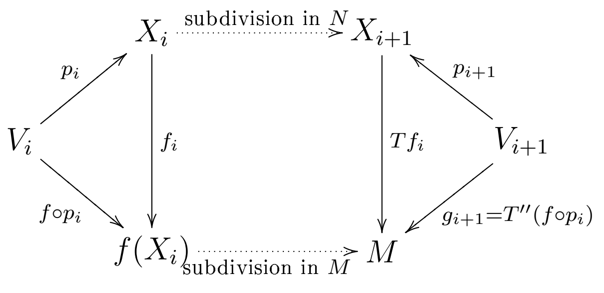


Figure 18: Definition of the prediction operator for a multiscale transform based on interpolatory geometric subdivision.

in M. By our assumption on the injectivity of p_{i+1}, the function $g_{i+1} \circ p_{i+1}^{-1} : X_{i+1} \to M$ is well defined. We define the prediction operator T by

$$Tf_i = g_{i+1} \circ p_{i+1}^{-1}.$$

Using the geometric operation $\ominus$ (defined at the end of Chapter 2) pointwise, detail coefficients are defined by

$$d_i = f_{i+1} \ominus Tf_i.$$

Our multiscale transform is now defined by

$$R : f \to (f_0, d_0, d_1 \ldots). \tag{6.2}$$

Note that the well-definedness of the transform depends on the well-definedness of the subdivision operators T' and T'', which in general can only be guaranteed for dense enough input data. This translates to the fact that we cannot arbitrarily choose the coarsest level for sampling (as in the linear case), but there is a bound on the maximal 'zoom out'. It turns out, however, that the guaranteed theoretical bounds are very pessimistic in contrast to what can be observed in practice.

In applications, we have the following finite version of the transform. It reads

$$R_n : f_n \to (f_0, d_0, \ldots, d_{n-1}). \tag{6.3}$$

A special case occurs if M is a vector space and T'' is a linear scheme. Then the multiscale transform is linear.

On the other hand, if $N = \mathbb{R}^2$ and the initial covering of N is given by the $\mathbb{Z}^2$ lattice, choosing T' as an interpolatory linear scheme which reproduces linear functions yields the multiscale transform (6.1) which was defined in [53].

6.2 Statement of the results and examples

In order not to introduce additional technical problems, we formulate our results for the case when N is compact. However, considering compact sets N and using a local definition of Hölder-Zygmund functions seems a straightforward way to generalize the results to non-compact N.

Our main theorem is Theorem 6.4. Its formulation needs the following notions: the smoothness index of a linear subdivision scheme, Hölder-Zygmund functions between manifolds, a certain non-degeneracy property referring to a mesh covering a manifold, and the quantities $\|d_i\|_{i,\gamma}$ ($i \in \mathbb{N}_0$) which encode the decay of the coefficients under the transformation (6.2). We define these objects first and then state the theorem.

Non-degeneracy Property of a Covering Mesh

Consider the initial mesh covering the manifold N in Chapter 6.1. In order to formulate the non-degeneracy property, we need a certain differentiable manifold Q. Q is obtained by imposing a smooth structure on the mesh by considering it as a topological space in the canonical way and using the characteristic maps χ_v as charts (which are defined in each 1-ring neighborhood N_v of a vertex v):

$$\chi_v : N_v \subset Q \to \mathbb{R}^2.$$

For analysis purposes, we consider the mapping κ from the manifold Q to N, which is given as the limit of subdivision. We request the following *non-degeneracy property*:

$$\kappa : Q \to N \text{ is regular and injective.} \tag{6.4}$$

Obviously, this property guarantees that no vertices of the initial mesh or its subdivided meshes coincide in N as required in Chapter 6.1. Furthermore, it guarantees that κ is onto, and thus invertible. This follows e.g. from degree theory [35][1].

If N has non-zero Euler characteristic, we can weaken (6.4) by dropping the injectivity assumption which then is fulfilled automatically. Again, this a consequence of degree theory [35][2].

Corollary 4.17 yields a way to infer the regularity of κ from properties of initial data p_0 using the regularity of the according limit of S (if p_0 does not satisfy this condition, there is still the chance that $p_1, p_2, \ldots$ do). So (6.4) can be effectively verified for given initial data p_0 (or the following $p_1, p_2, \ldots$).

Definition of the decay measure $\|d_i\|_{i,\gamma}$

Our decay conditions near extraordinary vertices are only slightly more involved than the very simple decay conditions in Theorem 6.1. To formulate these conditions we again need the notion of the *control set* $\mathrm{ctrl}^i(U)$ of a set $U \subset D$ which is defined by D. Zorin in [68], and which is a set of vertices in the i-th level mesh which determine the limit function on U. This means that the limit function on U only depends on data on $\mathrm{ctrl}^i(U)$.

First we consider k-regular meshes and use the notation of Chapter 1.2. For fixed i, we split the domain D into the rings D_j $(0 \le j < i)$ and the inner area $D \backslash (D_0 \cup \ldots \cup D_{i-1})$. For their i-th level control sets we use the notation

$$\begin{aligned} V_i^j &= \mathrm{ctrl}^i(D_j), && j < i, \\ V_i^i &= \mathrm{ctrl}^i(D \backslash (D_0 \cup \ldots \cup D_{i-1})). && \end{aligned} \tag{6.5}$$

The corresponding subsets of X_i (defined at the beginning of Chapter 6.1) are denoted by $X_i^j = p_i(V_i^j)$. We take the difference $d_{i-1} = f_i \ominus T f_{i-1}$ and measure each component with its bundle norm. Then we define

$$\|d_i\|_{i,\gamma} = \max_j (\lambda^{-j} 2^{i-j})^{\gamma} \|s_i|_{X_{i+1}^j}\|_\infty, \qquad \text{where } s_i(x) = \|d_i(x)\|. \tag{6.6}$$

Here λ is the subdominant eigenvalue of the subdivision matrix A (of our considered standard scheme). It turns out that this is the appropriate quantity to measure the detail coefficient decay near extraordinary vertices with.

Note that our definition is essentially a weighted sup-norm, where the weights depend on the 'distance' to an extraordinary vertex.

[1] For the reader's convenience we give the following short direct argument: Consider a curve $\gamma : [0,1] \to N$ connecting a point $x = \gamma(0)$ in the image $\kappa(Q)$ and an arbitrary point $y = \gamma(1)$ in N. Consider the maximal parameter t_0 such that for all smaller parameters $t < t_0$ the curve $\gamma([0,t])$ stays in $\kappa(Q)$. The compactness of N implies that $\gamma([0,t_0]) \subset \kappa(Q)$. So there is $p \in Q$ with $\kappa(p) = \gamma(t_0)$ and κ is a local diffeomorphism. Now, if t_0 were not 1, the inverse function theorem and the continuity of γ would guarantee that there is a neighborhood U of $\kappa(p) \subset \kappa(N)$ and $\varepsilon > 0$ such that $\gamma([t_0 - \varepsilon, t_0 + \varepsilon]) \subset U$. This is a contradiction and therefore κ is onto.

[2] As above, we give a short argument for the reader's convenience: By the regularity of κ and the compactness of Q, it follows that κ is a smooth finite covering. Then the Euler characteristic of the covering space Q must be a multiple of the Euler characteristic of N. But this is a contradiction to the fact that the manifolds N and Q are homeomorphic.

The definition of $\|\cdot\|_{i,\gamma}$ naturally extends to an arbitrary mesh and the corresponding subdivided meshes: Near extraordinary vertices, we locally use the above definition and obtain a global definition by 'gluing'. Therefore, we do not introduce complicated notation for that situation.

Smoothness Index of a Linear Subdivision Scheme

We assume that S is a standard scheme or a triangular quadrisection scheme fulfilling all the requirements of a a standard scheme except for being quadrilateral based. Let ν be the smoothness index of S on regular meshes, i.e., the greatest number such that S produces Lip_γ limits for all $\gamma < \nu$. Now we consider the subdivision matrix A for a valence k vertex. We order the eigenvalues according to their modulus by $1, \lambda, \lambda, \mu_3, \mu_4, \ldots$. Then we let $\nu' = \min(\log_\lambda |\mu_3|, 2)$ (subdivision schemes with $\log_\lambda |\mu_3| > 2$ are not desirable anyway [45]). We call

$$\omega = \min(\nu, \nu') \tag{6.7}$$

the *smoothness index* of S near an extraordinary vertex of valence k. For a general mesh, take the minimum of the smoothness indices of all extraordinary vertices.

On the Definition of Hölder-Zygmund Classes for Functions between Manifolds

Here we first follow Triebel [52] to define Hölder-Zygmund functions from the compact manifold N to $\mathbb{R}$. We equip N with an auxiliary Riemannian structure. We consider finitely many exponential charts $\exp_{p_i}^{-1}$ (whose images are balls of the same radius r) covering N and a subordinate C^∞ partition of unity $\{\varphi_i\}$. We say a continuous function $f : N \to \mathbb{R}$ belongs to the *Hölder-Zygmund class* $\mathrm{Lip}_\alpha(N, \mathbb{R})$ if $(f\varphi_i) \circ \exp_{p_i}$ is a Lip_α-function on $\mathbb{R}^2$, if we consider it extended by 0 outside the ball of radius r; Lip_α was defined in Chapter 1.1 with the help of (1.9).

Note that this definition does not depend on the chosen Riemannian structure. It also does not depend on the chosen centers of the balls, nor on the radius r, nor the partition of unity [52]. So the imposed Riemannian structure is only a tool for defining the Hölder-Zygmund Classes, and does not prejudice the subdivision scheme we are going to employ: If N is, for example, a Lie group we can still use a Lie group scheme.

We are going to define the class $\mathrm{Lip}_\alpha(N, M)$ where both N and M are smooth manifolds and N is compact. We equip both N and M with an auxiliary Riemannian structure.

Definition 6.2. *Let $f : N \to M$ be a continuous function. Choose finitely many open geodesic balls $B(x_i, r)$ which cover N, and finitely many balls $B(y_j, R)$ which cover* $\operatorname{im} f$, *such that each $f(B(x_i, r))$ is contained in one of the balls $B(y_j, R)$. We choose a partition of unity $\{\varphi_i\}$ subordinate to the balls $B(x_i, r)$. The continuous function $f : N \to M$ belongs to the class of Hölder-Zygmund functions between N and M,*

$$f \in \mathrm{Lip}_\gamma(N, M) \quad \Longleftrightarrow \quad f_i \in \mathrm{Lip}_\gamma(\mathbb{R}^m, \mathbb{R}^n), \text{ for all } i.$$

Here f_i is obtained from $(g_i\varphi_i) \circ \exp_{x_i} : B(0, r) \to \mathbb{R}^n$ by extending with 0 outside the ball, and $g_i = \exp_{y_j}^{-1} \circ f|_{B(x_i, r)}$.

Note that in the above definition, the main purpose of introducing the Riemannian structure is to obtain nice charts. Concerning well-definedness we have the following statement, whose proof is given later on.

Proposition 6.3. *The definition of* $\mathrm{Lip}_\gamma(N, M)$ *does not depend on the imposed Riemannian structure, the particular choice of balls, or the partition of unity.*

We formulate our main result:

Theorem 6.4. *Let S be an interpolatory linear standard scheme or a triangular scheme based on quadrisection which fulfills all the assumptions imposed on a standard scheme except for being triangular based instead of quad based. Assume furthermore that the two interpolatory schemes T' and T'' (acting in N and M, resp.,) both fulfill the local proximity conditions* (4.1) *w.r.t. S. Assume that an initial mesh covering N has the non-degeneracy property* (6.4)*, and that the smoothness index of S fulfills $\omega > 1$ on its mesh combinatorics. Then the multiscale transform R defined by T', T'' and the initial mesh covering N has the following property: The smoothness of a continuous function $f : N \to M$ is related to the decay of detail coefficients d_i w.r.t. R by*

$$f \in \mathrm{Lip}_\gamma(N, M) \quad \textit{if and only if} \quad \sup_{i\in\mathbb{N}_0} \|d_i\|_{i,\gamma} \leq C \tag{6.8}$$

for $0 < \gamma < \omega$. Here $\|\cdot\|_{i,\gamma}$ is defined by (6.6)*.*

In Chapter 6.1 we already encountered the fact that nonlinear subdivision schemes are in general only defined for dense enough input. By choosing a high enough index i_0, the samples of the continuous function f on all levels X_i with $i \geq i_0$ are dense enough such that the multiscale transform is well defined if we start on level i_0 instead of level 0. Then the statement of the theorem holds if we choose the i_0-th level mesh as initial mesh. As the statement is an asymptotic one in i, the initial level i_0 does not matter anyway.

Remark 6.5. We want to point out that by considering N as a smooth (meaning C^∞) manifold, Theorem 6.4 does not apply to the case when N itself is a subdivision surface in $\mathbb{R}^3$. The central technical reason is our use of geodesic balls in the definition of the Hölder-Zygmund classes. This is done to obtain 'nice' chart neighborhoods. However, a subdivision surface already brings nice chart neighborhoods. Although we omit this case in this paper to avoid further technical complications, we strongly conjecture that the above theorem is also true when N is a subdivision surface.

Remark 6.6. Modifications of our proofs would also work for C^1 schemes with $\omega = 1$. However, this would produce an additional case in most situations which we want to omit. Furthermore, we want to point out that we do not know how to prove the above theorem if the scheme is not C^1, or $\omega < 1$.

For the geometric situation we have the following result:

Corollary 6.7. *If T' and T'' are geometric (bundle) analogues of a linear scheme S which operate in N and M, respectively, then* (6.8) *is valid in this geometric setting.*

Linear schemes which meet our requirements are the modified butterfly scheme and Kobbelt's interpolatory quad scheme [30]. The butterfly scheme was proposed by Dyn et al.[12]. It was modified by Zorin [72] to produce smooth limits near extraordinary vertices. An analysis of both schemes can be found in [67].

As a consequence of Corollary 6.7, the Riemannian analogues (2.1) and (2.7) of the modified butterfly scheme and of Kobbelt's interpolatory quad scheme fulfill (6.8). Other analogues meeting the requirements of the corollary are the *projection analogue* and the *geodesic analogue* analyzed in [57].

The exact value of the smoothness index ω defined by (6.7) depends on the valences of the vertices in the combinatorics K. For its numerical evaluation in case of Kobbelt's scheme we refer to [67].

The modified butterfly scheme has some properties which are very nice for our purposes:

Corollary 6.8. *Let T' and T'' be geometric (bundle) analogues of the modified butterfly scheme in N and M, respectively, and assume that the initial mesh which covers N fulfills* (6.4)*. Then for continuous $f : N \to M$ and and any positive γ, which is smaller than the smoothness index of the butterfly scheme on regular meshes,*

$$f \in \mathrm{Lip}_\gamma(N, M) \quad \textit{if and only if} \quad \|d_i\|_\infty \leq C 2^{-i\gamma}.$$

Here d_i are the coefficients of the multiscale transform (6.2)*.*

The above corollary involves the smoothness index of the butterfly scheme on regular meshes which is known to lie in the interval $[1.44, 2]$. The lower bound is given in [18], and the upper bound is clear since the 4-point scheme does not produce C^2 limits. Note that the statement of Corollary 6.8 does not depend on the valences of the vertices in the combinatorics K, and that the decay conditions are as in the regular mesh case. This corollary is proved at the very end of Chapter 6.3.

6.3 Proofs

The main part of this section is devoted to the proof of Theorem 6.4. We begin by providing some information on the invariance properties of Hölder-Zygmund functions.

For an open subset $U \subset \mathbb{R}^n$ and $0 < \alpha \leq 1$ we define *the Hölder classes* $C^{1,\alpha}(U, \mathbb{R}^d)$ as the space of C^1 functions $f : U \to \mathbb{R}^d$ such that, for the differential of f, $\|d_x f - d_y f\| \leq C\|x - y\|^\alpha$, for all $x, y \in U$.

We need the following properties of Hölder-Zygmund and Hölder classes which mainly concern invariance under composition and multiplication.

Proposition 6.9. *Assume that $0 < \gamma < 2$ and that $0 < \alpha \leq 1$ such that $\alpha \geq \gamma - 1$. Consider $f \in \mathrm{Lip}_\gamma(\mathbb{R}^n, \mathbb{R}^d)$. Let U, V be open sets in $\mathbb{R}^n$, and let $g : U \to V$ be a C^1 diffeomorphism with $g \in C^{1,\alpha}(U, \mathbb{R}^n)$. Furthermore, assume that U', V' are open sets in $\mathbb{R}^d$, and that $h : U' \to V'$ is a C^1 diffeomorphism with $h \in C^{1,\alpha}(U', \mathbb{R}^d)$. Let $K \subset W \subset \mathbb{R}^n$ be a compact set contained in the open set W, and $f' : W \to \mathbb{R}^d$ be a continuous bounded function which fulfills $\|\Delta_t^2 f'(x)\| < C\|t\|^\gamma$ for all $x \in K$ and $\|t\| < t_0$, where $B(y, 2t_0) \subset W$ for all $y \in K$. Under the assumption that all sets are connected and contain* 0*, we have the following statements.*

- *(i) If $u \in \mathrm{Lip}_\gamma(\mathbb{R}^n)$ with $\operatorname{supp} u \subset \operatorname{int} K$, then the product $uf' : \mathbb{R}^n \to \mathbb{R}^d$ (extended by* 0 *outside K) belongs to $\mathrm{Lip}_\gamma(\mathbb{R}^n, \mathbb{R}^d)$.*

(ii) *If $L \subset U$ is compact, then there is an open neighborhood N of $g(L)$, such that $g^{-1} \in C^{1,\alpha}(N, \mathbb{R}^n)$.*

(iii) *If f is compactly supported in V, then $f \circ g \in \mathrm{Lip}_\gamma(\mathbb{R}^n, \mathbb{R}^d)$. Furthermore, $\|f \circ g\|_{\mathrm{Lip}_\gamma} \leq C\|g\|_{C^{1,\alpha}(\operatorname{supp} f)}\|f\|_{\mathrm{Lip}_\gamma}$.*

(iv) *If f has compact support and $\operatorname{im} f \subset U'$, then $h \circ f \in \mathrm{Lip}_\gamma(\mathbb{R}^n, \mathbb{R}^d)$.*

Proof. Note that for $0 < \alpha < 1$ the Hölder spaces $C^{1,\alpha}(\mathbb{R}^n)$ and the Hölder-Zygmund spaces $\mathrm{Lip}_{1+\alpha}(\mathbb{R}^n)$ coincide (which is, in general, no longer true, if we replace $\mathbb{R}^n$ by an open set U).

In order to avoid pathologies (arising from the choice of domains), the Hölder functions and the Hölder-Zygmund functions in the statements are compactly supported or defined in a neighborhood of the open set of interest− not only on the open set itself. This allows us to use certain results for the $\mathbb{R}^n$ case rather than having to deal with problems at the boundaries of the domain. In particular, certain proofs given for the $\mathbb{R}^n$ case which are based on differences and moduli of continuity (which are quantities of a local nature) carry over to our setting.

In case $\gamma \neq 1$, (i) is a straightforward computation. For $\gamma = 1$, we can use the representation [2, Equ. (2.4)] and proceed in a way analogous to the proof of Proposition 3 in [2]. This is justified, since our setup allows to apply [2, Equ. (2.2)].

We come to (ii). The corresponding statement for the $\mathbb{R}^n$ case is stated as Theorem 2.1 in [1] and is there attributed to Norton [38]. The argumentation in [1] is a local one, and choosing N as a set with compact closure in $g(U)$ yields (ii).

For $\gamma \neq 1$, statements (iii) and (iv) in the $\mathbb{R}^n$ case are Lemma 2.2 and Lemma 2.3 of [1]. Again, by the locality of the arguments in the proof of these lemmas, and by the compactness of $\operatorname{supp} f$, (iii) and (iv) hold true as stated.

The $\mathbb{R}^n$ statement analogous to (iii) for $\gamma = 1$ is the *composition theorem* of [39]. Its proof which is based on certain moduli of continuity also applies to the situation in (iii).

A statement similar to (iv) in the $\mathbb{R}^n$ case for $\gamma = 1$ is Theorem 2 of [2]. The difference is that only the case $d = 1$ is stated. However, the moduli η and ν employed in [2] can be generalized to arbitrary dimension d in the obvious way. Then the generalization to arbitrary d of Proposition 4 and Theorem 6 in [2] remains valid. An analysis of the proofs of Proposition 4 and Theorem 6 of [2] shows that they also apply to the situation in (iv) (every $C^{1,\alpha}$ function fulfills the condition [2, Equ. (1.1)]). □

We decided to give a not too detailed proof because following the lines in the references and checking that they are of local nature, and thus apply to our situation, is easily possible while explicitly writing down the argumentation would mostly copy the lines in the references and take up lots of space.

With the help of Proposition 6.9 we are able to show Proposition 6.3.

Proof of Proposition 6.3. It is sufficient to show the result for connected N. We assume that the conditions of Definition 6.2 are fulfilled for a function f and geodesic balls $B(x_i, r)$ and $B(y_j, R)$, respectively. We consider another such set of balls $B'(z_k, r')$ and $B'(v_l, R')$ with respect to different Riemannian metrics on N and M, respectively. Consider the partition of unity $\{\varphi_i\}$ and the functions f_i as in Definition 6.2, and an analogous partition of unity $\{\varphi'_k\}$ and the corresponding functions f'_k corresponding to the different choice of balls. We have to show that, for all k, $f'_k \in \mathrm{Lip}_\gamma(\mathbb{R}^m, \mathbb{R}^n)$.

To that end, we choose some small enough R'' and finitely many balls $B'(q_t, R'')$ which cover $f(N)$ such that, for each t, there is j and l with $B'(q_t, R'') \subset B(y_j, R)$ and $B'(q_t, R'') \subset B'(v_l, R')$. Then we choose some small enough r'' and finitely many balls $B'(p_s, r'')$ which cover N such that, for each s, there is i and k with $B'(p_s, r'') \subset B(x_i, r)$ and $B'(p_s, r'') \subset B'(z_k, r')$, and such that there is t with $f(B'(p_s, r'')) \subset B'(q_t, R'')$. We let $\{\varphi''_s\}$ be a partition of unity subordinate to the balls $B'(p_s, r'')$.

We construct the functions f''_s following Definition 6.2, using the balls $B'(p_s, r'')$, $B'(q_t, R'')$ and the partition of unity $\{\varphi''_s\}$. The statements (i),(iii), and (iv) of Proposition 6.9 together yield $f''_s \in \mathrm{Lip}_\gamma(\mathbb{R}^m, \mathbb{R}^n)$ for all s.

Consider now f'_k. Modulo a change of exponential charts, we can write $f'_k = \sum_s \psi_s f''_s$ with smooth functions ψ_s with compact support. By Proposition 6.9 (iii) and (iv), this change of exponential charts leaves the Lip_γ property invariant. By Proposition 6.9(i), multiplication with ψ_s leaves the Lip_γ property invariant. Thus $f'_k \in \mathrm{Lip}_\gamma(\mathbb{R}^m, \mathbb{R}^n)$. □

We have shown that our definition of Hölder-Zygmund functions between manifolds is consistent.

Recall that, for a function p_n on V_n for some k-regular mesh and a subset B of V_n, we use the notation

$$\mathcal{D}_B(p_n) = \sup\{\|p_n(v) - p_n(w)\| : v \text{ and } w \text{ are neighbors in } B\}.$$

We drop the index B, if $B = V_n$. $\mathcal{D}_B$ gives an upper bound on the coarseness of the corresponding mesh on B. Theorem 4.5 is only concerned with C^1 smoothness. We need the following generalization concerning Hölder functions.

Theorem 6.10. *Let S be a linear subdivision scheme which meets the requirements of Theorem 6.4, and let T be in proximity with S. Let $\omega > 1$ be the smoothness index of S for a k-regular mesh. If T converges for k-regular input p_0 (which is guaranteed if p_0 is dense enough in the sense that $\mathcal{D}(p_0)$ is small) then its limit is in $C^{1,\alpha-1}$ w.r.t. the characteristic parametrization, whenever $1 < \alpha < \omega$.*

Proof. We first consider linear subdivision and then use the results to obtain the corresponding statement for the nonlinear case.

We consider the limit function $h = S_{\infty,0}p_0$ for input p_0 and its restriction $h_m = h|_{D_m}$ to the ring D_m. As before, λ denotes the subdominant eigenvalue of the subdivision matrix A and μ denotes the modulus of the sub-subdominant eigenvalue(s). We are ordering the eigenvalues of A by their modulus, $1 > \lambda = \lambda > |\mu_3| \geq \ldots \geq |\mu_r| \geq \ldots$. Then h_0 can be represented as $h_0 = \sum_r \sum_{j=0}^{l_r} \beta_r^j e_r^j$ with $\{e_r^j\}$ being the eigen-rings of the subdivision scheme [45] and β_r^j being coefficients. Here the index r corresponds to the eigenvalues and the index j corresponds to the Jordan block of the corresponding eigenvalue. The limit function on the m-th ring has the nice representation

$$\begin{aligned} h_m &= \beta_0 + \beta_1 \lambda^m e_1(2^m \cdot) + \beta_2 \lambda^m e_2(2^m \cdot) \\ &+ \sum_r \sum_{l=0}^{l_r} \binom{m}{l} \mu_r^{m-l} \sum_{i=l}^{l_r} \beta_r^i e_r^{i-l}(2^i \cdot) =: h'_m + h''_m. \end{aligned} \tag{6.9}$$

See Chapter 4.6 of [45] for details.

Consider now the function $h_m \circ \chi^{-1}$, i.e., we look at the characteristic parametrization of the limit. By [45], the differential of h''_m as defined by (6.9) fulfills $d(h''_m \circ \chi^{-1}) = O(\lambda^{-m}(\mu s)^m)$ uniformly on D_m as $m \to \infty$ for every $s > 1$.

Assume that α is a real number with $1 < \alpha < \omega$. Since limits on regular meshes are $C^{1,\alpha-1}$, for all points x, y in, say, three consecutive rings $\chi(D_{m-1}) \cup \chi(D_m) \cup \chi(D_{m+1})$ the Hölder condition

$$\|d_x(h \circ \chi^{-1}) - d_y(h \circ \chi^{-1})\| \leq C\|x - y\|^{\alpha-1} \tag{6.10}$$

is fulfilled for some constant $C > 0$ which is independent of the particular m.

We consider the situation near the central point 0. We write h' for the function defined on each D_m by h'_m ($m \in \mathbb{N}$) and by β_0 in 0 (h'_m is defined in (6.9)). Analogously, we define h'', the only difference being that $h''(0) = 0$. Then $h' \circ \chi^{-1}$ is an affine-linear function and therefore $d_x(h' \circ \chi^{-1}) - d_y(h' \circ \chi^{-1}) = 0$. Hence

$$\|d_x(h \circ \chi^{-1}) - d_0(h \circ \chi^{-1})\|/\|x\|^{\alpha-1} = \|d_x(h'' \circ \chi^{-1})\|/\|x\|^{\alpha-1}.$$

Now, consider $x \in \chi(D_m)$. Two consecutive rings are λ-homothetic. So there are $k, K > 0$ which are independent of x and m such that $k\lambda^m \leq \|x\| \leq K\lambda^m$. Therefore, there are $C_1, C_2 > 0$ such that

$$\begin{aligned}\|d_x(h''_m \circ \chi^{-1})\|/\|x\|^{\alpha-1} &\leq C_1\|d_x(h''_m \circ \chi^{-1})\|/\lambda^{m(\alpha-1)} \\ &\leq C_2\lambda^{-m}(\mu s)^m/\lambda^{m(\alpha-1)} = (s(\mu/\lambda^\alpha))^m.\end{aligned}$$

We choose $s > 1$ such that $\rho = s\lambda^{\nu-\alpha} < 1$. Then $s(\mu/\lambda^\alpha) = (\mu/\lambda^\nu)(s\lambda^{\nu-\alpha}) = \rho$. This is because the first factor equals 1 by definition of ν. Then, $\|d_x(h_m \circ \chi^{-1})\|/\|x\|^{\alpha-1} \leq C_2\rho^m \leq C_2$. This implies that the Hölder condition (6.10) holds also in 0.

For points x and y, which lie in two rings, say $\chi(D_r)$ and $\chi(D_s)$, with $|r - s| > 2$, we estimate differentials by

$$\|d_x(h \circ \chi^{-1}) - d_y(h \circ \chi^{-1})\| \leq \|d_x(h \circ \chi^{-1}) - d_0(h \circ \chi^{-1})\| + \|d_y(h \circ \chi^{-1}) - d_0(h \circ \chi^{-1})\|.$$

By the contraction of the rings, $\|x - y\|^\alpha \geq c\max(\|x\|^\alpha, \|y\|^\alpha)$ for some $c > 0$ which is independent of x and y as long as $|r - s| > 2$. This yields a (larger) constant C' such that (6.10) still holds with C replaced by C'. Altogether, this implies that the limit of linear subdivision is a $C^{1,\alpha-1}$ function.

Since we now know that S produces $C^{1,\alpha-1}$ limits for $\alpha < \omega$, we can base the proof for the nonlinear case upon the perturbation arguments used in the proof of Theorem 4.5. We assume $\alpha < \omega$. We point out where modifications are necessary. First of all, note that for a function u on $\mathbb{R}^n$ and some $h > 0$, we have $c, C > 0$ such that the dilated function $u(h\cdot)$ can be estimated by $ch^\alpha\|u\|_{C^{1,\alpha-1}} \leq \|u(h\cdot)\|_{C^{1,\alpha-1}} \leq Ch^\alpha\|u\|_{C^{1,\alpha-1}}$. ($C$ is a generic constant, which can change from line to line from now on.) With this in mind, we can use the argumentation of Proposition 4.15 to obtain that

$$\begin{aligned}&\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(D_n)}\|_{C^{1,\alpha-1}} \\ &\qquad \leq C(2^{i-n}\lambda^{-n})^\alpha\|(T_i - S_i)T_{i-1,0}p_0|_{\mathrm{ctrl}^{i+1}(D_n)}\|_\infty.\end{aligned}$$

Invoking this estimate yields a statement analogous to (4.24) for the rings near the extraordinary vertex:

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(D_n)}\|_{C^{1,\alpha-1}} \leq C\gamma^{(2-\alpha)i}\mathcal{D}_{\mathrm{ctrl}^0(D')}(p_0)^2, \tag{6.11}$$

where $\gamma := \max(2^{-1}, \lambda)$. The $C^{1,\alpha-1}$ version of (4.26) reads

$$\|(S_{\infty,i+1}T_{i,0}p_0 - S_{\infty,i}T_{i-1,0}p_0) \circ \chi^{-1}|_{\chi(D'_i)}\|_{C^{1,\alpha-1}} \leq C\lambda^{(2-\alpha)i}\mathcal{D}_{\mathrm{ctrl}^0(D')}(p_0)^2. \tag{6.12}$$

The estimates (6.11) and (6.12) now imply that the limit using T is $C^{1,\alpha-1}$. This follows with minor modifications from the proofs of Proposition 4.16 and Theorem 4.5. □

The next proposition treats Euclidean space data defined over a 2-manifold. It is a special case of our main result.

Proposition 6.11. *Let the interpolatory scheme T' act on the smooth compact 2-manifold N and assume that it is in proximity to a linear interpolatory scheme S. Assume that the initial mesh (K_0, p_0) in N fulfills the non-degeneracy property (6.4). Let $\omega > 1$ be the smoothness index of S for that mesh. We apply the linear version of the transform (6.2) to a continuous function $f : N \to \mathbb{R}^d$. Then for any γ with $0 < \gamma < \omega$ we have the characterization*

$$f \in \mathrm{Lip}_\gamma(N, \mathbb{R}^d) \quad \text{if and only if} \quad \sup_{i\in\mathbb{N}_0} \|d_i\|_{i,\gamma} \leq C. \tag{6.13}$$

Furthermore, $\|f_0\|_\infty + \sup_{i\in\mathbb{N}_0} \|d_i\|_{i,\gamma}$ provides an equivalent norm on $\mathrm{Lip}_\gamma(N, \mathbb{R}^d)$.

Proof. The proof of this statement takes some time. We split it into several parts. Part (1) reduces the statement to a statement involving only one extraordinary vertex. In parts (2)–(5) we show the reduced statement: Part (2) is the 'only if'-part in case $\gamma \neq 1$. The 'if'-part of the statement is treated in part (3). In part (4) we explain why $\|f_0\|_\infty + \sup_{i\in\mathbb{N}_0} \|d_i\|_{i,\gamma}$ defines an equivalent norm on $\mathrm{Lip}_\gamma(N, \mathbb{R}^d)$ in case $\gamma \neq 1$. In Part (5) we show the 'only if'-part and treat the norm equivalence for $\gamma = 1$.

We need the sets V_i and X_i which were defined in Chapter 1.2 and at the beginning of Chapter 6.1, respectively. The subsets V_i^j and X_i^j are given by (6.5) and the lines following (6.5), respectively. We let C be a generic constant which can change from line to line.

(1) We reduce the statement to a more accessible situation near extraordinary vertices. To that purpose, consider the neighborhood of an extraordinary vertex $x \in X_0 \subset N$ and the corresponding point $0 \in V_0$ in the glued domain D. Denote by $\bar{X}_i = \chi(V_i)$ the image of V_i under characteristic parametrization. With the diffeomorphism κ of (6.4), $\chi \circ \kappa^{-1}$ is a local diffeomorphism mapping x to $0 \in \mathbb{R}^2$. Thus $\chi \circ \kappa^{-1}$ sends neighbors of $x \in X_i \subset X_0$ to neighbors of $0 \in \bar{X}_i$. For a visualization see Figure 19.

Now choose finitely many small geodesic balls $B(y_j, r)$ which cover N, such that each $\kappa^{-1}(B(y_j, r))$ is completely contained in some characteristic chart neighborhood. Let $\{\psi_j\}$ be C^∞ functions such that each ψ_j is supported in $B(y_j, r)$ and equal to 1 on $B(y_j, r - \varepsilon)$, where $\varepsilon > 0$ is so small such that the balls $B(y_j, r - \varepsilon)$ still cover N. If $f \in \mathrm{Lip}_\gamma(N, \mathbb{R}^d)$ then $f\psi_j$ is compactly supported in $B(y_i, r)$ and the extension of its chart representation with 0 outside the ball is in $\mathrm{Lip}_\gamma(\mathbb{R}^2, \mathbb{R}^d)$. Let us denote this extension also by $f\psi_j$.

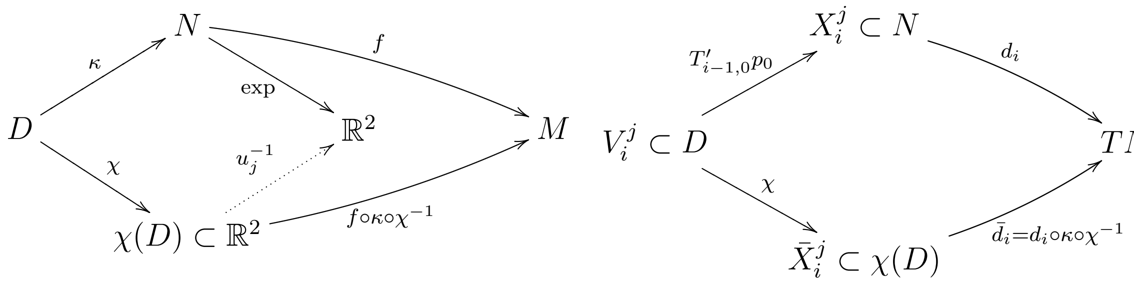


Figure 19: Setup for the proofs of Proposition 6.11 and Theorem 6.4.

The mapping $u_j = \chi \circ \kappa^{-1} \circ \exp_{y_j}^{-1}$ is a diffeomorphism from $B(0, r)$ into $\mathbb{R}^2$. Its image contains the compact set $u_j(\operatorname{supp} f\psi_j)$. Since κ is a limit of subdivision, Theorem 6.10 implies that the inverse u_j^{-1} is $C^{1,\alpha-1}$ for all $\alpha < \omega$. Therefore, for α with $\gamma < \alpha < \omega$, Proposition 6.9 implies that $(f\psi_j) \circ u_j^{-1} \in \mathrm{Lip}_\gamma(\mathbb{R}^2, \mathbb{R}^d)$ (with the usual 0-extension). This means that a Hölder-Zygmund function on N transforms to a Hölder-Zygmund function near 0 in the image of a characteristic chart.

Conversely, if we have a Hölder-Zygmund function g in the image of a characteristic chart which is compactly supported in $u_j(B(0, r))$, we use Proposition 6.9 to obtain that $g \circ u_j$ is Hölder-Zygmund on N (with extension by 0). For a Hölder-Zygmund function g defined on $\chi(D)$ which is not necessarily supported in $u_j(B(0, r))$ we can multiply g with $\psi_j \circ u_j^{-1}$ to obtain a function that has support in $u_j(B(0, r))$ and apply the above to obtain a Hölder-Zygmund function on N.

We define the details $\bar{d}_i$ and the control sets $\bar{X}_i^j$ analogous to the details d_i and the control sets X_i^j, only by replacing $X_i \subset N$ by $X'_i \subset \mathbb{R}^2$. Then, locally near an extraordinary vertex, the details d_i of f given on N and the details $\bar{d}_i$ of $f \circ \kappa \circ \chi^{-1}$ are equal.

If a ball $B(y_j, r)$ in N does not contain an extraordinary vertex, then we are in the regular mesh case. But this is a special instance of a 4-regular mesh in case of quad meshes, and a 6-regular mesh in case of triangular meshes which is treated by the general k-regular case.

Summing up, it is enough to show the following reduced statement for the k-regular mesh for a continuous function f with compact support in a neighborhood of $\chi(D')$:

$$f \in \mathrm{Lip}_\gamma(\chi(D), \mathbb{R}^d) \quad \text{if and only if} \quad \sup_{i\in\mathbb{N}} \|f_i - S_{i-1}f_{i-1}\|_{i,\gamma} \leq C. \tag{6.14}$$

We also show that $\|f_0\|_\infty + \sup_{i\in\mathbb{N}_0} \|f_i - S_{i-1}f_{i-1}\|_{i,\gamma}$ provides an equivalent norm on

$$\mathrm{Lip}_\gamma^K(\chi(D), \mathbb{R}^d) = \{f \in \mathrm{Lip}_\gamma(\chi(D), \mathbb{R}^d) : \operatorname{supp} f \subset K\} \tag{6.15}$$

for some fixed but arbitrary neighborhood K of 0. Then the corresponding statement in the proposition follows from Proposition 6.9(iii).

For the further proof we let $d = 1$, since the right hand expression in (6.14) is equivalent (lower and upper constants) to the maximum of the corresponding component-wise expressions.

(2) We show the 'only if'-part of (6.14) for $\gamma \neq 1$. So our assumption is that $f \in \mathrm{Lip}_\gamma(\chi(D), \mathbb{R})$. f_i denotes the restriction of f to $\bar{X}_i$. The subdivision scheme S acts on functions on V_i as a linear operator S_i and thus also on functions on $\bar{X}_i$. We denote this

operator on functions on $\bar{X}_i$ by S_i, too. We abuse notation and also use $S_{\infty,i}$ to denote the operator which maps input $\bar{X}_i \to \mathbb{R}$ to its limit $\chi(D) \to \mathbb{R}$.

Consider the restriction of f_i to the sets $\bar{X}_i^j$ (the index i corresponds to level i and the index j to the ring j near an irregular vertex). In the course of the proof we have to estimate the norm of $(f_i - S_{i-1}f_{i-1})|_{\bar{X}_i^j}$. We have to distinguish two cases depending on whether $l := i - j$, (i.e., the difference between level and ring index) is small or not.

If we choose l sufficiently large, say $l \geq l_0$, we get that

$$\|(f_i - S_{i-1}f_{i-1})|_{\bar{X}_i^j}\|_\infty \leq \|(f - S_{\infty,i-1}f_{i-1})|_{\chi(D_j'')}\|_\infty, \tag{6.16}$$

where we let $D_j'' = D_{j-1} \cup D_j \cup D_{j+1}$. This is a consequence of S being interpolatory and the fact that the control sets $\bar{X}_i^j$ on level i of $\chi(D_j)$ are contained in D_j''.

For $l = i - j < l_0$, we find $r \in \mathbb{N}$ such that $\bar{X}_i^j \subset \chi(D'_{j-r})$, where $D'_l := D' \backslash (D_0 \cup \ldots \cup D_{l-1})$. ($D'$ was defined as the union of all the rings D_i, $i \in \mathbb{N}$, and 0 in Chapter 1.2.) Then

$$\|(f_i - S_{i-1}f_{i-1})|_{\bar{X}_i^j}\|_\infty \leq \|(f - S_{\infty,i-1}f_{i-1})|_{\chi(D'_{j-r})}\|_\infty. \tag{6.17}$$

Observe that showing

$$\|(f - S_{\infty,i-1}f_{i-1})|_{\chi(D_j)}\|_\infty \leq C\lambda^{j\gamma}2^{(j-i)\gamma} \quad \text{and} \tag{6.18}$$

$$\|(f - S_{\infty,i-1}f_{i-1})|_{\chi(D_i')}\|_\infty \leq C\lambda^{i\gamma} \tag{6.19}$$

is enough to complete this part of the proof. This is because (6.18) and (6.19) together imply that (6.18) is valid with D_j replaced by D_j'' or by D_i', respectively, if we enlarge the constant C. Then (6.16) and (6.17) imply $\|(f_i - S_{i-1}f_{i-1})|_{\bar{X}_i^j}\|_\infty \leq C\lambda^{j\gamma}2^{(j-i)\gamma}$, where C is independent of i and j. This is the right-hand side of (6.14).

We show the approximation estimates (6.18) and (6.19). We consider the two cases $\gamma > 1$ and $\gamma < 1$, and use the fact that in both cases the spaces of Hölder-Zygmund functions and Hölder functions on Euclidean space coincide with equivalent norms (which is not true for $\gamma = 1$). If $\gamma > 1$, we write $f = f(v) + d_v f(\cdot - v) + g(\cdot)$ with $g(x) = O(\|x - v\|^\gamma)$ for $x \to v$ by our assumption. The linear bounded operator which first samples f and then maps the result to the limit of subdivision reproduces constants. Furthermore, it reproduces linear functions $f : \chi(D) \to \mathbb{R}$. So, for a vertex $v \in \bar{X}_{i-1}$, we have $S_{\infty,i-1}f_{i-1} = f(v) + d_v f(\cdot - v) + h(\cdot)$ for some h with $h(x) = O(\|x - v\|^\gamma)$ by Theorem 6.10. Then, if v is a point in $\bar{X}_{i-1}$ nearest to x, we obtain

$$f(x) - S_{\infty,i-1}f_{i-1}(x) = g(x) - h(x) = O(\|x - v\|^\gamma) \text{ for } x \to v. \tag{6.20}$$

If $\gamma < 1$, the estimate (6.20) is shown in the same way, without using differentials.

In order to estimate $\|x - v\|$ in (6.20) we introduce the notation $\sigma(A, B) = \sup_{x \in A} \inf_{v \in B} \|x - v\|$. By the definition of V_k, $\sigma(D_k', V_k^k) = O(2^{-k})$ and $\sigma(D_r, V_k^r) = O(2^{-k})$ as $k \to \infty$, uniformly in r for $r < k$. Because the characteristic map is a diffeomorphism on each ring D_k, fulfilling the scaling relation $\chi(2^{-1}\cdot) = \lambda\chi(\cdot)$, we have that $\sigma(\chi(D_k'), \bar{X}_k^k) = O(\lambda^k)$ and that $\sigma(\chi(D_r), \bar{X}_k^r) = O(\lambda^r 2^{r-k})$ as $k \to \infty$ uniformly in r for $r < k$. So for $x \in \chi(D'_{i-1})$, we get $\inf_{v \in \bar{X}_{i-1}} \|x - v\| = O(\lambda^i)$. Also, for $j \leq i - 1$, and $x \in \chi(D_j)$, we obtain that $\inf_{v \in \bar{X}_{i-1}} \|x - v\| = O(\lambda^j 2^{j-i})$.

Then plugging $\|x-v\| \le C\lambda^j 2^{(j-i)}$ into (6.20) and enlarging the constant C yields both (6.18) and (6.19). This completes part (2) of the proof.

(3) We show the 'if'-part of (6.14). The continuous functions $g_i = S_{\infty,i} f_i$ uniformly converge to f on $\chi(D)$ for the following reason: Since S is interpolatory, for a vertex $v_i \in \bar{X}_i$ nearest to x we get

$$\begin{aligned}\|g_i(x)-f(x)\| &\le \|S_{\infty,i}f_i(x)-S_{\infty,i}f_i(v_i)\| + \|f(x)-f(v_i)\| \\ &\le C\sup\{\|f(v)-f(w))\| : v,w \text{ neighboring vertices}\} + \|f(x)-f(v_i)\|,\end{aligned}$$

and the right hand side tends to 0 as $i \to \infty$.

The right-hand side of (6.14) implies that, for $i > j$,

$$\|g_i - g_{i-1}|_{\chi(D_j)}\|_\infty \le \|S_{\infty,0}\|\|f_i - Sf_{i-1}|_{\bar{X}_i^j}\|_\infty \le C'\|S_{\infty,0}\|2^{(j-i)\gamma}\lambda^{j\gamma}. \tag{6.21}$$

Here C' is the constant in the decay condition which depends on f. In this part, we continue to use the symbol C as a generic constant which can change from term to term, but we only employ it if it does not depend on f. We use (6.21) to quantify the distance between f and the approximants g_i on the ring $\chi(D_j)$:

$$\begin{aligned}\|f - g_i|_{\chi(D_j)}\|_\infty &\le \sum\nolimits_{k=i+1}^{\infty} \|g_k - g_{k-1}|_{\chi(D_j)}\|_\infty \\ &\le C'\|S_{\infty,0}\| \sum\nolimits_{k=i+1}^{\infty} 2^{(j-k)\gamma}\lambda^{j\gamma} \le C'C2^{(j-i)\gamma}\lambda^{j\gamma}.\end{aligned} \tag{6.22}$$

Now we consider the inner domains $\chi(D_j')$. Using the right-hand side of (6.14), an estimate analogous to (6.21) yields

$$\|g_i - g_{i-1}|_{\chi(D_j')}\|_\infty \le C'C\lambda^{i\gamma}, \tag{6.23}$$

whenever $i \le j$. Then,

$$\begin{aligned}\|f - g_i|_{\chi(D_j')}\|_\infty &\le \sum\nolimits_{k=i+1}^{\infty} \|g_k - g_{k-1}|_{\chi(D_j')}\|_\infty \\ &\le C\|S_{\infty,0}\|(\sum\nolimits_{k=i+1}^{j} \lambda^{k\gamma} + \sum\nolimits_{k=j+1}^{\infty} 2^{(j-k)\gamma}\lambda^{j\gamma}) \\ &\le C'C\lambda^{i\gamma}\sum\nolimits_{k=1}^{\infty} \max(2^{-1},\lambda)^{k\gamma} \le C'C\lambda^{i\gamma}.\end{aligned} \tag{6.24}$$

We proceed to estimate second differences, beginning on the rings $\chi(D_j)$. By enlarging the constant C in (6.22), the statement of (6.22) remains valid for sufficiently small ϵ-neighborhoods U_j of $\chi(D_j)$. We choose the neighborhoods U_j in such a way that each U_j is a scaled copy of the neighborhood U_0 where the scaling factor equals λ^j. Then there is $h_0 > 0$ such that, for any j, all $x \in \chi(D_j)$, and all t with $\|t\| < \lambda^j h_0$, the second difference $\Delta_t^2 f(x)$ only depends on $f|U_j$.

We let α be a real number with $\gamma < \alpha < \omega$. Consider the modulus of continuity $\omega_2^j(h,f) := \sup_{\|t\|<h} \|(\Delta_t^2 f)|_{\chi(D_j)}\|$, for $h < \lambda^j h_0$. We have the estimate

$$\begin{aligned}\omega_2^j(h,f) &\le \omega_2^j(h, f-g_n) + \sum\nolimits_{i=0}^{n-1} \omega_2^j(h, g_{i+1}-g_i) + \omega_2^j(h,g_0) \\ &\le 4\|(f-g_n)|_{\chi(U_j)}\|_\infty + \sum\nolimits_{i=0}^{n-1} h^\alpha \|g_{i+1}-g_i\|_{\alpha,j} + \omega_2^j(h,g_0),\end{aligned} \tag{6.25}$$

where $\|\cdot\|_{\alpha,j} := \sup_h h^{-\alpha}\omega_2^j(h,\cdot) + \|\cdot|_{\chi(U_j)}\|_\infty$.

With the help of (6.22) and (6.24) we can estimate the first summand on the right-hand side of (6.25) by

$$\|(f-g_n)|_{\chi(U_j)}\|_\infty \le CC'\lambda^{\min(n,j)\gamma}2^{-\max(n-j,0)\gamma}. \tag{6.26}$$

We note that the last summand in (6.25) can be estimated by $\omega_2^j(h,g_0) \le CC'h^\gamma$ by Theorem 6.10. We consider the sum in (6.25). By the locality of the subdivision scheme S, the limit function locally is a linear combination of finitely many generating functions. Furthermore, on a regular mesh, an integer shift of those generating functions is a generating system for the shifted functions. Near 0 in a k-regular mesh, changing to a finer resolution only dilates the generating systems, we get $\|g_{i+1}-g_i\|_{\alpha,j} \le C2^{(i-j)\alpha}\lambda^{-j\alpha}\|g_{i+1}-g_i|_{\chi(U_j)}\|_\infty$ in case that $i > j$. If $i \le j$, we obtain $\|g_{i+1}-g_i\|_{\alpha,j} \le C\lambda^{-i\alpha}\|g_{i+1}-g_i|_{\chi(U_j\cup D_j')}\|_\infty$. By combining these estimates, we get

$$\begin{aligned}\sum_{i=0}^{n-1} h^\alpha\|g_{i+1}-g_i\|_{\alpha,j} &\le C\sum_{i=0}^{n-1} h^\alpha\lambda^{-\min(i,j)\alpha}2^{\max(i-j,0)\alpha}\|g_{i+1}-g_i|_{\chi(A_j)}\|_\infty \\ &\le CC'\sum_{i=0}^{n-1} h^\alpha\lambda^{-\min(i,j)(\alpha-\gamma)}2^{\max(i-j,0)(\alpha-\gamma)}.\end{aligned} \tag{6.27}$$

Here $A_j = U_j$ for $i > j$, and $A_j = U_j \cup D_j'$ for $i \le j$. For the second inequality we used the estimates (6.21) and (6.23). We further discuss this upper bound in (6.27). We consider n with $n > j$ and set $h = 2^{j-n}\lambda^j$. Then,

$$\begin{aligned}h^{-\gamma}\sum_{i=0}^{n-1} h^\alpha\lambda^{-\min(i,j)(\alpha-\gamma)}2^{\max(i-j,0)(\alpha-\gamma)} &= \sum_{i=0}^{n-1}\lambda^{(j-\min(i,j))(\alpha-\gamma)}2^{(\max(i-j,0)+j-n)(\alpha-\gamma)} \\ &\le \sum_{i=0}^{j-1}\lambda^{(j-i)(\alpha-\gamma)} + \sum_{i=j}^{n-1}2^{(i-n)(\alpha-\gamma)} \le C,\end{aligned} \tag{6.28}$$

where C is independent of n and j. We plug (6.28) into (6.27) and the result into (6.25). For $h = 2^{j-n}\lambda^j$ and $j < n$ we obtain, using also (6.26),

$$h^{-\gamma}\omega_2^j(h,f) \le 4C'C + CC' + CC', \tag{6.29}$$

where the constants do not depend on j and n. Since the sequence $h_n = 2^{j-n}\lambda^j$ nicely tends to 0, it follows that there is h_0' with $0 < h_0' < h_0$ such that, for all j and h with $0 < h < h_0'\lambda^j$,

$$h^{-\gamma}\omega_2^j(h,f) \le C'C. \tag{6.30}$$

Having estimated second differences on the rings $\chi(D_j)$ we now consider the neighborhood of the central point. Instead of D_j we consider the central domain D_j', and employ the second modulus of continuity $\tilde\omega_2^j(h,f) := \sup_{\|t\|<h}\|(\Delta_t^2 f)|_{\chi(D_j')}\|$, for $h < \lambda^j h_0$. Analogous to (6.25) we estimate

$$\begin{aligned}\tilde\omega_2^j(h,f) &\le \tilde\omega_2^j(h,f-g_j) + \sum_{i=0}^{j-1}\tilde\omega_2^j(h,g_{i+1}-g_i) + \tilde\omega_2^j(h,g_0) \\ &\le 4\|(f-g_j)|_{\chi(U_j')}\|_\infty + \sum_{i=0}^{j-1} h^\alpha\|g_{i+1}-g_i\|_{\alpha,j} + \omega_2^j(h,g_0),\end{aligned} \tag{6.31}$$

where the above definition of $\|\cdot\|_{\alpha,j}$ is modified by replacing U_j by U'_j. By (6.24), the first summand on the right-hand side of (6.31) is bounded from above by $4\|(f-g_j)|_{\chi(U_j)}\|_\infty \leq CC'\lambda^{j\gamma}$. Similar to (6.27) and (6.28), letting $h=(c\lambda)^j$, for some c with $0<c<1$, which is small enough to guarantee that $\tilde{\omega}_2^j$ is defined, we obtain, using (6.23),

$$\begin{aligned} h^{-\gamma}\sum_{i=0}^{j-1} h^\alpha\|g_{i+1}-g_i\|_{\alpha,j} &\leq Ch^{-\gamma}\sum_{i=0}^{j-1} h^\alpha\lambda^{-i\alpha}\|g_{i+1}-g_i|_{\chi(U'_j)}\|_\infty \\ &\leq CC'h^{-\gamma}\sum_{i=0}^{j-1} h^\alpha\lambda^{-i(\alpha-\gamma)}. \\ &= CC'\sum_{i=0}^{j-1}\lambda^{(j-i)(\alpha-\gamma)} \leq CC'. \end{aligned} \tag{6.32}$$

Here the constants C, C' are independent of j. Combining these two estimates and plugging them into (6.31), we get, on the inner domain $\chi(D'_j)$,

$$h^{-\gamma}\tilde{\omega}_2^j((c\lambda)^j, f) \leq C'C \tag{6.33}$$

uniformly in j. Firstly, this yields the decay condition $\|\Delta_t^2 f(0)\| \leq C'C\|t\|^\gamma$ in the central point. Furthermore, if we consider some x in the j-th ring $\chi(D_j)$, and some y in the i-th ring with $j-i\geq 2$ then (6.33) ensures that $\|f(x)-2f(\frac{x+y}{2})+f(y)\| \leq CC'\|x-y\|^\gamma$. If the distance is smaller, then (6.30) applies. In summary, this shows that $f\in \mathrm{Lip}_\gamma(\chi(D),\mathbb{R}^d)$.

(4) We explain why in the case $\gamma\neq 1$ the expression $\|f\|'_\gamma = \|f_0\|_\infty + \sup_{i\in\mathbb{N}_0}\|f_i - S_{i-1}f_{i-1}\|_{i,\gamma}$ is an equivalent norm on $\mathrm{Lip}_\gamma^K(\chi(D),\mathbb{R}^d)$ (which is defined by (6.15)). By (2) and (3) the subspace of continuous functions where $\|\cdot\|'_\gamma < \infty$ coincides with Lip_γ^K. It is a straightforward computation that $\|\cdot\|'_\gamma$ defines a norm. The constants C occurring in (3) do not depend on f (for constants depending on f, we used the symbol C'). This implies existence of $C>0$, independent of f, such that

$$\|f\|_{\mathrm{Lip}_\gamma^K} \leq C\|f\|'_\gamma. \tag{6.34}$$

Since part (3) includes the case $\gamma=1$, (6.34) is also valid for $\gamma=1$.

For the converse part, we have to analyze the proof of part (2). In the beginning of part (2), we reduce the statement of part (2) to (6.18) and (6.19). Examining this reduction we see that the occurring constants 'C' do not depend on f. It remains to analyze the constants occurring in the proof of (6.18) and (6.19): By careful examination, it turns out that f only influences constants via the $O()$-term in (6.20). This means that we have to look at the Hölder constants of the functions g and h occurring in part (2). By definition, those Hölder constants are bounded by some multiple of the Hölder norm of f. Summing up, there is $C>0$, independent of f such that

$$\|f\|'_\gamma \leq C\|f\|_{\mathrm{Lip}_\gamma^K}, \tag{6.35}$$

in case $\gamma\neq 1$. Thus those norms are equivalent for $\gamma\neq 1$ (The inequality (6.35) for the case $\gamma=1$ is shown at the end of part (5)).

(5) It remains to show the 'only if'-part of (6.14) for $\gamma=1$. To that purpose, we use interpolation theory. We refer to [42] for a thorough treatment in connection with Hölder-Zygmund classes. It is well known that Lip_1 is the interpolation space $[\mathrm{Lip}_{1-\epsilon},\mathrm{Lip}_{1+\epsilon}]_{1/2}$.

This notation means the following: For two Banach spaces X and Y with $Y \subset X$, the symbol $[X,Y]_\theta$ denotes the space of all $f \in X$ such that Peetre's K-functional $K(f,t) \le Ct^\theta$, for $0 < t \le 1$, where

$$K(f,t) = \inf_{g \in Y} \|f-g\|_X + t\|g\|_Y.$$

The interpolation space becomes a Banach space with norm $\|\cdot\| = \sup_t t^{-\theta} K(\cdot,t)$.

We proceed in the following way: We assume that $f \in \mathrm{Lip}_1 \subset \mathrm{Lip}_{1-\varepsilon}$. Then for every t with $0 < t \le 1$ there is $g_t \in \mathrm{Lip}_{1+\varepsilon}$ such that $t^{-1/2}\|f-g_t\|_{\mathrm{Lip}(1-\varepsilon)} + t^{1/2}\|g_t\|_{\mathrm{Lip}(1+\varepsilon)} < C$, where C does not depend on t.

We let $h_t = f - g_t$. We consider the coefficients under the multiscale transform of f, h_t and g_t on $\bar{X}_i^j$ for arbitrary but fixed i. We denote these coefficients on $\bar{X}_i^j$ by $d(f), d(g_t), \ldots$. By (4) we have $\|d(h_t)\|_\infty < C' 2^{(j-i)(1-\varepsilon)} \lambda^{j(1-\varepsilon)} \|h_t\|_{\mathrm{Lip}(1-\varepsilon)}$, and $\|d(g_t)\|_\infty < C'' 2^{(j-i)(1+\varepsilon)} \lambda^{j(1+\varepsilon)} \|g_t\|_{\mathrm{Lip}(1+\varepsilon)}$. By applying the triangle inequality and letting $t^{1/2} = 2^{(j-i)\varepsilon}\lambda^{j\varepsilon}$ we get

$$\begin{aligned} 2^{i-j}\lambda^{-j}\|d(f)\|_\infty &\le 2^{i-j}\lambda^{-j}\|d(h_t)\|_\infty + 2^{i-j}\lambda^{-j}\|d(g_t)\|_\infty \\ &\le C' 2^{-(j-i)\varepsilon}\lambda^{-j\varepsilon}\|h_t\|_{\mathrm{Lip}(1-\varepsilon)} + C'' 2^{(j-i)\varepsilon}\lambda^{j\varepsilon}\|g_t\|_{\mathrm{Lip}(1+\varepsilon)} \\ &\le \max(C',C'')(t^{-1/2}\|f-g_t\|_{\mathrm{Lip}(1-\varepsilon)} + t^{1/2}\|g_t\|_{\mathrm{Lip}(1+\varepsilon)}) \\ &\le C\|f\|_{\mathrm{Lip}_1}. \end{aligned} \tag{6.36}$$

For the last inequality we have used the equivalence of the norm induced by the K-functional and the norm induced by second differences. (6.36) means that we have the desired decay of the multiscale coefficients if $f \in \mathrm{Lip}_1$.

Furthermore, the coefficient based norm $\|\cdot\|_1'$ from part (4) obeys

$$\|f\|_1' \le C\|f\|_{\mathrm{Lip}_1}.$$

The other direction, i.e., $\|f\|_{\mathrm{Lip}_1} \le C\|f\|_1'$, was already established in (6.34). Hence $\|f\|_1'$ is an equivalent norm on Lip_1. □

Having collected all this information we are now able to prove Theorem 6.4.

Proof of Theorem 6.4. This proof is quite long which is the reason why we split it into several parts. In part (1) we reduce the statement to a statement only involving one extraordinary vertex. We proceed similar to the proof of Proposition 6.11 which is the reason for keeping this part short. Part (2) is the 'only if'-part of the reduced statement, and part (3) is its 'if'-part (which is actually the hard estimate).

For a visualization of the setup we refer to Figure 19. We use the notation of the proof of Proposition 6.11. Furthermore, we use the symbol C for a generic constant which can change from line to line.

(1) Similar to the proof of Proposition 6.11 we reduce the statement to the situation near an extraordinary vertex. We show that a certain way of 'applying charts' does neither affect the Hölder-Zygmund classes nor the decay of detail coefficients.

We cover $f(N)$ with balls $B(z_k, R)$, and N with balls $B(y_j, r)$ such that each $f(B(y_j, r))$ is completely contained in one of the $B(z_k, R)$'s and such that the image of each $B(y_j, r)$ under κ^{-1} is completely contained in some characteristic chart neighborhood.

We let ψ_j be C^∞ functions supported in $B(y_j, r)$ and equal to 1 in $B(y_j, r-\varepsilon)$, where $\varepsilon > 0$ is so small that the balls $B(y_j, r-\varepsilon)$ still cover N. Then the extension by 0 of $g_j = \exp_{z_k}^{-1} \circ (f\psi_j) \circ \exp_{y_j}^{-1}$ is in $\mathrm{Lip}_\gamma(\mathbb{R}^2, \mathbb{R}^d)$. Except for applying charts, g_j agrees with f on $B(y_j, r-\varepsilon)$.

With the mapping $u_j = \chi \circ \kappa^{-1} \circ \exp_{y_j}^{-1}$ already defined in part (1) of the proof of Proposition 6.11 we obtain that the 0-extension of $g_j \circ u_j^{-1}$ is in $\mathrm{Lip}_\gamma(\mathbb{R}^2, \mathbb{R}^d)$, by Proposition 6.9.

Conversely, assume that we have Hölder-Zygmund functions g_j (of order γ) such that each g_j is supported in a neighborhood of $\chi(D')$ and maps to $\mathbb{R}^d$. We also write f, ψ_j and $B(y_j, r)$ for their corresponding reparametrizations by charts. We assume that each g_j agrees with $f \circ u_j^{-1}$ on $u_j(\operatorname{supp} \psi_j)$. Then we restrict g_j to $u_j(B(y_j, r))$ and multiply the result with $\psi_j \circ u_j^{-1}$ to obtain a Hölder-Zygmund function g_j' with support in $u_j(\operatorname{supp} \psi_j)$. Then $g_j' \circ u_j$ (extension by 0) is Hölder-Zygmund and agrees with f on $B(y_j, r-\varepsilon)$. Furthermore, the coefficients of the multiscale transform for g_j around 0 and the transform of f near the corresponding extraordinary vertex agree.

After going to charts for M, the following statement implies the theorem. For a k-regular mesh and for a continuous function f with compact support in a neighborhood of $\chi(D')$ we have

$$f \in \mathrm{Lip}_\gamma(\chi(D), \mathbb{R}^d) \quad \text{if and only if} \quad \sup_{i\in\mathbb{N}} \|f_i - T_{i-1} f_{i-1}\|_{i,\gamma} \le C. \tag{6.37}$$

There is one more thing to explain here: We let the scheme T act in a chart which allows us to write an ordinary minus sign in (6.37). The right-hand side expression in (6.37), $\|f_i - T_{i-1} f_{i-1}\|_{i,\gamma}$, which is based on the Euclidean norm, is bounded both from above and below by constants times $\|f_i \ominus T_{i-1} f_{i-1}\|_{i,\gamma}$, which is based on the smooth bundle norm. This is true locally (because in finite dimensional spaces every two norms are equivalent and the bundle norm is smooth) and also globally because the image of f is compact.

(2) We show the 'only if'-part of (6.37), assuming $f \in \mathrm{Lip}_\gamma(\chi(D), \mathbb{R}^d)$. Equation (6.14) yields that $\sup_{i\in\mathbb{N}_0} \|f_i - S_{i-1} f_{i-1}\|_{i,\gamma} \le C'$. We consider the sets $\bar{X}_i^j$ and observe $\|f_i - S_{i-1} f_{i-1}|_{\bar{X}_i^j}\|_\infty \le C' 2^{(j-i)\gamma} \lambda^{\gamma j}$. Since S and T fulfill the proximity condition (4.1),

$$\begin{aligned} \|f_i - T_{i-1} f_{i-1}|_{\bar{X}_i^j}\|_\infty &\le \|f_i - S_{i-1} f_{i-1}|_{\bar{X}_i^j}\|_\infty + \|(S_{i-1} f_{i-1} - T_{i-1} f_{i-1})|_{\bar{X}_i^j}\|_\infty \\ &\le C' 2^{(j-i)\gamma} \lambda^{j\gamma} + C \mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2. \end{aligned} \tag{6.38}$$

Here we let $\tilde{X}_i^j = \bar{X}_i^j$, if $i \ge j$, and $\tilde{X}_i^j = \bar{X}_i^i$, if $i < j$. Then $\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})$ is the difference of function values of f on neighboring vertices in $\tilde{X}_{i-1}^j$. Neighboring vertices in $\tilde{X}_{i-1}^j$ have distance of order $2^{-\min(i-j-1,0)} \lambda^{\min(j,i-1)}$; this was shown at the very end of part (2) of the proof of Proposition 6.11.

If $f \in \mathrm{Lip}_\gamma$, then $f \in \mathrm{Lip}_{\gamma/2+\varepsilon}$, when we choose $\varepsilon > 0$ such that $\varepsilon < \max(1-\gamma/2, \gamma/2)$. This choice of ε guarantees that $\gamma/2 + \varepsilon < 1$. Then the Lipschitz norm based on first differences is an equivalent norm on $\mathrm{Lip}_{\gamma/2+\varepsilon}$. Hence, since $f \in \mathrm{Lip}_{\gamma/2+\varepsilon}$, and all f_i's are samples of f, we get, with the above order of distances of neighboring vertices, that

$$\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1}) \le C 2^{-\min(i-j-1,0)(\gamma/2+\varepsilon)} \lambda^{\min(j,i-1)(\gamma/2+\varepsilon)}. \tag{6.39}$$

Plugging (6.39) into (6.38) yields the decay of the detail coefficients w.r.t T which is required by (6.37).

(3) We now consider the 'if'-part of (6.37), i.e., we assume that a continuous function f has coefficient decay as stated by (6.37). We again look at the control sets $\tilde{X}_i^j$. By assumption, the decay conditions read:

$$\|(f_i - T_{i-1}f_{i-1})|_{\tilde{X}_i^j}\| \leq C_f 2^{-(i-j)\gamma}\lambda^{j\gamma}, \qquad \text{for } i > j \tag{6.40}$$

$$\|(f_i - T_{i-1}f_{i-1})|_{\tilde{X}_i^i}\| \leq C_f \lambda^{i\gamma}. \tag{6.41}$$

Here C_f is a constant which depends on the continuous function f, but is neither dependent on the 'ring-index' j nor on the detail level i. Our aim is to show that (6.40) and (6.41) imply that for $i > j$,

$$\|(f_i - S_{i-1}f_{i-1})|_{\tilde{X}_i^j}\| \leq C' 2^{-(i-j)\gamma}\lambda^{j\gamma}, \tag{6.42}$$

and the same for $i = j$, but with the right hand side replaced by $C'\lambda^{i\gamma}$. Here the constant C' should not depend on i or j. Once (6.42) is proved we apply (6.14), and obtain that $f \in \mathrm{Lip}_\gamma$ as desired.

It remains to show (6.42) which will take some time. We start by invoking the proximity and decay conditions to obtain the following estimate for $i + 1 > j$:

$$\begin{aligned} \|(f_{i+1} - S_i f_i)|_{\tilde{X}_{i+1}^j}\| &\leq \|(f_{i+1} - T_i f_i)|_{\tilde{X}_{i+1}^j}\| + \|(T_i f_i - S_i f_i)|_{\tilde{X}_{i+1}^j}\| \\ &\leq C_f 2^{-(i+1-j)\gamma}\lambda^{j\gamma} + C_{pr}\mathcal{D}_{\tilde{X}_i^j}(f_i)^2. \end{aligned} \tag{6.43}$$

Here C_{pr} is the proximity constant. This estimate is valid for dense enough input, which we can always achieve by going to a finer sampling level since f is continuous. Analogously, if $i + 1 \leq j$,

$$\|(f_{i+1} - S_i f_i)|_{\tilde{X}_{i+1}^j}\| \leq 2C_f\lambda^{i\gamma} + C_{pr}\mathcal{D}_{\tilde{X}_i^j}(f_i)^2. \tag{6.44}$$

From (6.43) and (6.44) we can conclude (6.42) if we know the estimates

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq C 2^{-(i-j)\gamma/2}\lambda^{j\gamma/2} \qquad (i > j), \tag{6.45}$$

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq C\lambda^{i\gamma/2} \qquad (i \leq j), \tag{6.46}$$

for some constant $C > 0$. We are thus left with proving (6.45) and (6.46). We write, for $i > j > i_0$,

$$\begin{aligned} f_i &= (f_i - S_{i-1}f_{i-1}) + \ldots + (S_{i-1,j+1}f_{j+1} - S_{i-1,j}f_j) + (S_{i-1,j}f_j - S_{i-1,j-1}f_{j-1}) \\ &\quad + \ldots + (S_{i-1,i_0+1}f_{i_0+1} - S_{i-1,i_0}f_0) + S_{i-1,i_0}f_{i_0} \\ &= \sum_{k=i_0+1}^{i} S_{i-1,k}(f_k - S_{k-1}f_{k-1}) + S_{i-1,i_0}f_{i_0}. \end{aligned} \tag{6.47}$$

Here i_0 is a nonnegative integer which will be specified later on. By Lemma 4.14, there is a constant C_S such that for any subdivision level k and input p_k on level k,

$$\begin{aligned}
&\mathcal{D}_{\tilde{X}_i^j}(S_{i-1,k}p_k) \le C_S 2^{-(i-k)} \mathcal{D}_{\tilde{X}_k^j}(p_k) && (i \ge k > j),\\
&\mathcal{D}_{\tilde{X}_i^j}(S_{i-1,k}p_k) \le C_S 2^{-(i-j)} \lambda^{j-k} \mathcal{D}_{\tilde{X}_k^j}(p_k) && (i > j > k),\\
&\mathcal{D}_{\tilde{X}_i^j}(S_{i-1,k}p_k) \le C_S \lambda^{i-k} \mathcal{D}_{\tilde{X}_k^j}(p_k) && (j \ge i \ge k).
\end{aligned} \tag{6.48}$$

Furthermore,

$$\begin{aligned}
\mathcal{D}_{\tilde{X}_k^j}(f_k - S_{k-1}f_{k-1}) &\le \mathcal{D}_{\tilde{X}_k^j}(f_k - T_{k-1}f_{k-1}) + \mathcal{D}_{\tilde{X}_k^j}(T_{k-1}f_{k-1} - S_{k-1}f_{k-1})\\
&\le 2\|(f_k - T_{k-1}f_{k-1})|_{\tilde{X}_k^j}\| + 2\|(T_{k-1}f_{k-1} - S_{k-1}f_{k-1})|_{\tilde{X}_k^j}\|.
\end{aligned}$$

We use the telescoping sum (6.47) to estimate $\mathcal{D}_{\tilde{X}_i^j}(f_i)$ and apply both (6.48) and the previous inequality to the single terms: If $i > j > i_0$ we get

$$\begin{aligned}
\mathcal{D}_{\tilde{X}_i^j}(f_i) &\le 2\|(f_i - T_{i-1}f_{i-1})|_{\tilde{X}_i^j}\| + 2\|(T_{i-1}f_{i-1} - S_{i-1}f_{i-1})|_{\tilde{X}_i^j}\|\\
&+ 2\sum\nolimits_{k=j+1}^{i-1} C_S 2^{-(i-k)} (\|(f_k - T_{k-1}f_{k-1})|_{\tilde{X}_k^j}\| + \|(T_{k-1}f_{k-1} - S_{k-1}f_{k-1})|_{\tilde{X}_k^j}\|)\\
&+ 2\sum\nolimits_{k=i_0+1}^{j} C_S 2^{-(i-j)} \lambda^{j-k} (\|(f_k - T_{k-1}f_{k-1})|_{\tilde{X}_k^j}\| + \|(T_{k-1}f_{k-1} - S_{k-1}f_{k-1})|_{\tilde{X}_k^j}\|)\\
&+ C_S 2^{-(i-j)} \lambda^{j-i_0} \mathcal{D}_{\tilde{X}_{i_0}^j}(f_{i_0}).
\end{aligned}$$

Using (6.40), proximity and again (6.48), we further obtain

$$\begin{aligned}
\mathcal{D}_{\tilde{X}_i^j}(f_i) &\le 2C_f 2^{-(i-j)\gamma} \lambda^{j\gamma} + 2C_S C_f \Big(\sum_{k=j+1}^{i-1} 2^{-(i-k)} 2^{-(k-j)\gamma} \lambda^{j\gamma} + \sum_{k=i_0+1}^{j} 2^{-(i-j)} \lambda^{j-k} \lambda^{k\gamma} \Big)\\
&+ 2C_{pr} \mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2 + 2C_S C_{pr} \Big(\sum_{k=j+1}^{i-1} 2^{-(i-k)} \mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2\\
&\qquad + \sum_{k=i_0+1}^{j} 2^{-(i-j)} \lambda^{j-k} \mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2 \Big)\\
&+ C_S 2^{-(i-j)} \lambda^{j-i_0} \mathcal{D}_{\tilde{X}_{i_0}^j}(f_{i_0}) =: A + B + C.
\end{aligned} \tag{6.49}$$

Here the symbols A, B, C refer to the first line, second plus third lines, and fourth line, resp., in (6.49). Analogously, we obtain, for $j \le i_0$,

$$\begin{aligned}
\mathcal{D}_{\tilde{X}_i^j}(f_i) &\le 2C_f 2^{-(i-j)\gamma} \lambda^{j\gamma} + 2C_S C_f \sum\nolimits_{k=i_0+1}^{i-1} 2^{-(i-k)} 2^{-(k-j)\gamma} \lambda^{j\gamma}\\
&+ 2C_{pr} \mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2 + 2C_S C_{pr} \sum\nolimits_{k=i_0+1}^{i-1} 2^{-(i-k)} \mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2\\
&+ C_S 2^{-(i-i_0)} \mathcal{D}_{\tilde{X}_{i_0}^j}(f_{i_0}) =: A + B + C.
\end{aligned} \tag{6.50}$$

Furthermore, for $j \geq i$,

$$\begin{aligned} \mathcal{D}_{\tilde{X}_i^j}(f_i) \leq 2C_f \lambda^{i\gamma} + 2C_S C_f \sum\nolimits_{k=i_0+1}^{i} \lambda^{i-k}\lambda^{k\gamma} \\ + 2C_{pr}\mathcal{D}_{\tilde{X}_{i-1}^j}(f_{i-1})^2 + 2C_S C_{pr} \sum\nolimits_{k=i_0+1}^{i} \lambda^{i-k}\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2 \\ + C_S \lambda^{i-i_0}\mathcal{D}_{\tilde{X}_{i_0}^j}(f_{i_0}) =: A + B + C. \end{aligned} \tag{6.51}$$

We estimate the terms called 'A' in the formulas (6.49), (6.50), and (6.51): Since $\gamma/2 < 1$, we can estimate A in (6.49) by $A \leq 2C_S C_f \cdot 2^{-(i-j)\gamma/2}\lambda^{j\gamma/2} \cdot Z$, where

$$Z = \sum\nolimits_{k=j+1}^{i} 2^{-(i-k)(1-\gamma/2)} 2^{-(k-j)\gamma/2}\lambda^{j\gamma/2} + \sum\nolimits_{k=i_0+1}^{j} 2^{-(i-j)(1-\gamma/2)}\lambda^{(j-k)(1-\gamma/2)}\lambda^{k\gamma/2}.$$

In order to estimate Z we get rid of the dependence on the index j by introducing $q = \max(2^{-1}, \lambda) < 1$: We obtain

$$Z \leq \sum_{k=i_0+1}^{i} q^{(i-k)(1-\gamma/2)} q^{k\gamma/2} \leq \sum_{k=i_0+1}^{i} q^{k\gamma/2} \leq (1-q^{\gamma/2})^{-1}.$$

This yields an upper bound on Z independent of i, j, and i_0. Proceeding in an analogous way for (6.50) and (6.51) yields a constant $D \geq 1$, independent of i, j, and i_0 such that

$$A \leq D 2^{-(i-j)\gamma/2}\lambda^{j\gamma/2} \qquad \text{in case of (6.49) and (6.50),} \tag{6.52}$$

$$A \leq D\lambda^{i\gamma/2} \qquad \text{in case of (6.51).} \tag{6.53}$$

We are ready to estimate $\mathcal{D}_{\tilde{X}_i^j}(f_i)$. We choose i' such that

$$(C_S C_{pr}) 18 D^2 \cdot \max(\lambda, 2^{-1})^{i'\frac{\gamma}{2}} (1 - \max(\lambda, 2^{-1})^{\frac{\gamma}{2}})^{-1} < \frac{1}{9}. \tag{6.54}$$

This reason for this choice becomes clear later on.

Note that f is continuous, thus uniformly continuous because of its compact support. Therefore we can choose the initial level i_0 for our estimates such that

$$\mathcal{D}(f_i) \leq \min(1, (18D^2 C_S C_{pr})^{-1}(i'+1)^{-1}\min(\lambda, 2^{-1})^{i'}) =: D' \quad \text{for all } i \geq i_0. \tag{6.55}$$

We show that, for all $i \geq i_0$,

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq \min(3D 2^{-(i-j)\gamma/2}\lambda^{(j-i_0)\gamma/2}, D') \qquad (i > j > i_0), \tag{6.56}$$

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq \min(3D 2^{-(i-i_0)\gamma/2}, D') \qquad (j \leq i_0), \tag{6.57}$$

$$\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq \min(3D\lambda^{(i-i_0)\gamma/2}, D') \qquad (j \geq i). \tag{6.58}$$

Once (6.56)–(6.58) are proved, we obtain (6.45) and (6.46) which in turn completes the proof.

It remains to show (6.56)–(6.58) for which we use induction on i. The case $i = i_0$ is clear. We assume that (6.56)–(6.58) hold for the values $i_0, \dots, i-1$. We get, for $i > j > i_0$, using the decomposition (6.49), (6.52), and (6.55),

$$\begin{aligned}\mathcal{D}_{\tilde{X}_i^j}(f_i) &\leq A + B + C \\ &\leq D2^{-(i-j)\gamma/2}\lambda^{j\gamma/2} + B + C_S 2^{-(i-j)}\lambda^{(j-i_0)} \cdot \frac{\min(\lambda, 2^{-1})^{i'}}{18D^2 C_S C_{pr}} \\ &\leq D2^{-(i-j)\gamma/2}\lambda^{(j-i_0)\gamma/2} + B + D2^{-(i-j)\gamma/2}\lambda^{(j-i_0)\gamma/2}. \end{aligned} \tag{6.59}$$

Analogously, using (6.50), (6.51) and (6.52), (6.53), we obtain, for $i \geq i_0 \geq j$, that $\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq 2D2^{-(i-i_0)\gamma/2} + B$, and for $j \geq i$ that $\mathcal{D}_{\tilde{X}_i^j}(f_i) \leq 2D\lambda^{(i-i_0)\gamma/2} + B$.

We only consider the case $i > j > i_0$, since the other cases are analogous. In this case, it remains to show that $B \leq D2^{-(i-j)\gamma/2}\lambda^{(j-i_0)\gamma/2}$. We use the induction hypothesis (6.56)–(6.58) and see that for $i_0 + i' < j$, the definition of B in (6.49) implies

$$\begin{aligned} B &\leq 2C_S C_{pr}\Big(\sum\nolimits_{k=j+1}^{i} 2^{-(i-k)}\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2 + \sum\nolimits_{k=i_0+i'+1}^{j} 2^{-(i-j)}\lambda^{j-k}\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2 \\ &\qquad + \sum\nolimits_{k=i_0+1}^{i_0+i'} 2^{-(i-j)}\lambda^{j-k}\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2\Big) \\ &\leq 18D^2 C_S C_{pr}\Big(\sum\nolimits_{k=j+1}^{i} 2^{-(i-k)}2^{-(k-1-j)\gamma}\lambda^{(j-i_0)\gamma} + \sum\nolimits_{k=i_0+i'+1}^{j} 2^{-(i-j)}\lambda^{j-k}\lambda^{(k-1-i_0)\gamma}\Big) \\ &\qquad + 2C_S C_{pr}(i'+1)2^{-(i-j)}\lambda^{j-(i_0+i')}D'. \end{aligned}$$

For the third sum, we used that, by the induction hypothesis, $\mathcal{D}_{\tilde{X}_{k-1}^j}(f_{k-1})^2 \leq D'^2 \leq D'$. We now expand the definition of D' in (6.55) in the last term and get

$$\begin{aligned} B &\leq (C_S C_{pr})18D^2(2^{-(i-j)\gamma/2}\lambda^{(j-i_0)\gamma/2}) \cdot \\ &\quad \cdot \Big(\sum_{k=j+1}^{i} 2^{-(i-k)(1-\frac{\gamma}{2})}2^{-(k-1-j)\frac{\gamma}{2}}\lambda^{(j-i_0)\frac{\gamma}{2}} + \sum_{k=i_0+i'+1}^{j} 2^{-(i-j)(1-\frac{\gamma}{2})}\lambda^{(j-k)(1-\frac{\gamma}{2})}\lambda^{(k-1-i_0)\frac{\gamma}{2}}\Big) \\ &\quad + 2C_S C_{pr}(18D^2 C_S C_{pr})^{-1}(i'+1)(i'+1)^{-1}2^{-(i-j)}\lambda^{j-(i_0+i')}\min(\lambda, 2^{-1})^{i'}. \end{aligned} \tag{6.60}$$

We estimate the sums in brackets:

$$\begin{aligned} &\sum_{k=j+1}^{i} 2^{-(i-k)(1-\frac{\gamma}{2})}2^{-(k-1-j)\frac{\gamma}{2}}\lambda^{(j-i_0)\frac{\gamma}{2}} + \sum_{k=i_0+i'+1}^{j} 2^{-(i-j)(1-\frac{\gamma}{2})}\lambda^{(j-k)(1-\frac{\gamma}{2})}\lambda^{(k-1-i_0)\frac{\gamma}{2}} \\ &\qquad \leq \sum_{k=j+1}^{i} 2^{-(k-1-j)\frac{\gamma}{2}}\lambda^{(j-i_0)\frac{\gamma}{2}} + \sum_{k=i_0+i'+1}^{j} \lambda^{(k-1-i_0)\frac{\gamma}{2}} \\ &\qquad \leq \sum_{k=i_0+i'+1}^{i} \max(\lambda, 2^{-1})^{(k-1-i_0)\frac{\gamma}{2}} = \max(\lambda, 2^{-1})^{i'\frac{\gamma}{2}}(1-\max(\lambda, 2^{-1})^{\frac{\gamma}{2}})^{-1}. \end{aligned}$$

Plugging this into (6.60) and simplifying the last term yields

$$\begin{aligned} B &\leq (C_S C_{pr})18D^2(2^{-(i-j)\gamma/2}\lambda^{(j-i_0)\gamma/2}) \cdot \\ &\qquad \cdot \max(\lambda, 2^{-1})^{i'\frac{\gamma}{2}}(1-\max(\lambda, 2^{-1})^{\frac{\gamma}{2}})^{-1} + \tfrac{1}{9}2^{-(i-j)}\lambda^{j-i_0}. \end{aligned}$$

We further apply (6.54) to obtain

$$B \leq \tfrac{1}{9} 2^{-(i-j)\gamma/2} \lambda^{(j-i_0)\gamma/2} + \tfrac{1}{9} 2^{-(i-j)} \lambda^{j-i_0} \leq \tfrac{2}{9} 2^{-(i-j)\gamma/2} \lambda^{(j-i_0)\gamma/2}. \tag{6.61}$$

For $i_0 + i' \geq j$ as well as the other two cases (6.57) and (6.58) one proceeds in an analogous way. This completes the induction step and shows (6.56)–(6.58). The proof of Theorem 6.4 is done. □

We conclude with the proofs of Corollary 6.7 and Corollary 6.8.

Proof of Corollary 6.7. By Theorem 6.4 a local proximity condition (4.1) must be shown for a geometric (bundle) analogue of S given by (2.15). This is done in [17]. □

Proof of Corollary 6.8. The smoothness index $\omega = \min(\nu, \nu')$ was defined by (6.7). By [71], $\nu' = 2$, and since $\nu < 2$, $\omega = \nu$. Further, the subdominant eigenvalue λ of the subdivision matrix equals 1/2. So letting $\lambda = 1/2$ in (6.6) completes the proof. □

References

[1] Bojarski, B., Hajlasz, P., and Strzelecki, P. Sard's theorem for mappings in Hölder and Sobolev spaces. *Manuscripta Mathematica 118* (2005), 383–397.

[2] Bourdaud, G., and de Cristoforis, M. L. Functional calculus in Hölder-Zygmund spaces. *Trans. Amer. Math. Soc. 354* (2002), 4109–4129.

[3] Catmull, E., and Clark, J. Recursively generated B-spline surfaces on arbitrary topological meshes. *Computer Aided Design 10* (1978), 350–355.

[4] Cavaretta, A. S., Dahmen, W., and Micchelli, C. A. *Stationary Subdivision.* Mem. Amer. Math. Soc., No. 453, 1991.

[5] Chaikin, G. An algorithm for high speed curve generation. *Computer Graphics and Image Processing 2* (1974), 346–349.

[6] Daubechies, I. *Ten Lectures on Wavelets.* SIAM, 1992.

[7] de Rham, G. Sur quelques fonctions différentiables dont toutes les valeurs sont des valeurs critiques. *Celebrazioni Archimedee del Secolo XX, Siracusa II* (1961), 11–16.

[8] Destelle, F., Gérot, C., and Montanvert, A. A topological lattice refinement descriptor for subdivision surfaces. In *International Conference on Mathematical Methods for Curves and Surfaces* (Tonsberg, Norway, 2008).

[9] Donoho, D. L. Interpolating wavelet transforms. Preprint, Department of Statistics, Stanford University, 1992. `http://citeseer.comp.nus.edu.sg/39237.html`.

[10] Doo, D., and Sabin, M. A. Behaviour of recursive subdivision surfaces near extraordinary points. *Computer Aided Design 10* (1978), 356–360.

[11] Dyn, N. Subdivision schemes in CAGD. *Advances in Numerical Analysis 2* (1992), 36–104.

[12] Dyn, N., Levin, D., and Gregory, J. A. A butterfly subdivision scheme for surface interpolation with tension control. *ACM Trans. on Graphics 9* (1990), 160–169.

[13] Grohs, P. *Smoothness Analysis of Nonlinear Subdivision Schemes on Regular Grids.* PhD thesis, Graz Technical University, 2007.

[14] Grohs, P. Smoothness analysis of subdivision schemes on regular grids by proximity. *SIAM J. Numer. Anal. 46* (2008), 2169–2182.

[15] Grohs, P. Smoothness equivalence properties of univariate subdivision schemes and their projection analogues. *Numerische Mathematik 113/2* (2009), 163–180.

[16] Grohs, P. Proximity analysis of nonlinear subdivision schemes and applications to manifold-valued subdivision via the Riemannian center of mass. *SIAM J. Math. Analysis 42* (2010), 729–750.

[17] Grohs, P., and Wallner, J. Interpolatory wavelets for manifold-valued data. *Appl. Comput. Harmon. Anal. 27* (2009), 325–333.

[18] Han, B. Computing the smoothness exponent of a symmetric multivariate refinable function. *SIAM J. Matrix Anal. Appl. 24* (2002), 693–714.

[19] Han, B. Vector cascade algorithms and refinable function vectors in Sobolev spaces. *J. Approx. Theory 124* (2003), 44–88.

[20] Han, B. Solutions in Sobolev spaces of vector refinement equations with a general dilation matrix. *Adv. Comput. Math. 24* (2006), 375–403.

[21] Han, B., and Jia, R.-Q. Multivariate refinement equations and convergence of subdivision schemes. *SIAM J. Math. Anal. 29* (1998), 1177–1199.

[22] Harizanov, S., and Oswald, P. Stability of nonlinear subdivision and multiscale transforms. *Constr. Approx. (to appear)* (2010).

[23] Harten, A., and Osher, S. Uniformly high-order accurate nonoscillatory schemes I. *SIAM J. Numer. Anal. 24* (1987), 279–309.

[24] Holmstrom, M. Solving hyperbolic PDEs using interpolating wavelets. *SIAM J. Sci. Comput. 21* (2000), 405–420.

[25] Ivrissimtzis, I. P., Dodgson, N. A., and Sabin, M. A. A generative classification of mesh refinement rules with lattice transformations. *Comput. Aided Geom. Des. 21* (2004), 99–109.

[26] Jia, R.-Q. Approximation properties of multivariate wavelets. *Math. Comput. 67* (1998), 647–665.

[27] Karcher, H. Riemannian center of mass and mollifier smoothing. *Comm. Pure Appl. Math. 30* (1977), 509–541.

[28] Kendall, W. S. Probability, convexity, and harmonic maps with small image. i: Uniqueness and fine existence. *Proc. Lond. Math. Soc., III. Ser. 61* (1990), 371–406.

[29] Kobayashi, S., and Nomizu, K. *Foundations of differential geometry, Vol II.* Wiley, 1969.

[30] Kobbelt, L. Interpolatory subdivision on open quadrilateral nets with arbitrary topology. *Computer Graphics Forum 15* (1996), 409–420.

[31] Kobbelt, L. $\sqrt{3}$-subdivision. *Computer Graphics Proceedings (SIGGRAPH)* (2000), 103–112.

[32] Lang, S. *Fundamentals of Differential Geometry.* Springer, 1999.

[33] Li, G., and Ma, W. Composite $\sqrt{2}$ subdivision surfaces. *Comput. Aided Geom. Des. 24* (2007), 339–369.

[34] Lundmark, A., Wadströmer, N., and Li, H. Hierarchical subsampling giving fractal regions. *IEEE Transactions on Image Processing 10* (2001), 167–173.

[35] Milnor, J. *Topology from the differentiable viewpoint.* Princeton University Press, 1997.

[36] Moenning, C., Memoli, F., Sapiro, G., Dyn, N., and Dodgson, N. Meshless geometric subdivision. *Graphical Models 69* (2007), 160–179.

[37] Möller, H. M., and Sauer, T. Multivariate refinable functions of high approximation order via quotient ideals of Laurent polynomials. *Adv. Comput. Math. 20* (2004), 205–228.

[38] Norton, A. A critical set with nonnull image has large Hausdorff dimension. *Trans. Amer. Math. Soc. 296* (1986), 367–376.

[39] Norton, A. The Zygmund Morse-Sard theorem. *Journal of Geometric Analysis 4* (1994), 403–424.

[40] Oswald, P. Designing composite triangular subdivision schemes. *Comput. Aided Geom. Design 22* (2005), 659–679.

[41] Oswald, P., and Schröder, P. Composite primal/dual -subdivision schemes. *Comput. Aided Geom. Design 20* (2003), 135–164.

[42] Peetre, J. *New Thoughts on Besov Spaces.* Duke University mathematics series, 1976.

[43] Peters, J., and Reif, U. The simplest subdivision scheme for smoothing polyhedra. *ACM Trans. Graph. 16* (1997), 420–431.

[44] Peters, J., and Reif, U. Analysis of algorithms generalizing B-spline subdivision. *SIAM J. Numer. Anal. 35* (1998), 728–748.

[45] Peters, J., and Reif, U. *Subdivision Surfaces.* Springer, 2008.

[46] Prautzsch, H. Smoothness of subdivision surfaces at extraordinary points. *Adv. Comput. Math. 9* (1998), 377–389.

[47] Reif, U. A unified approach to subdivision algorithms near extraordinary vertices. *Comput. Aided Geom. Design 12* (1995), 153–174.

[48] Sahr, K., and White, D. Discrete global grid system. In *Proc. 30th Symposium on the Interface Computing Science and Statistics, vol. 30* (1998), pp. 269–278.

[49] Sauer, T. Differentiability of multivariate refinable functions and factorization. *Adv. Comput. Math. 25* (2006), 211–235.

[50] Schröder, P., and Zorin, D. Subdivision surfaces in "Geri's Game". In *Subdivision for Modeling and Animation*, ACM SIGGRAPH 1999 Courses. 1999. http://www.multires.caltech.edu/teaching/courses/subdivision.

[51] Triebel, H. *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland, 1978.

[52] Triebel, H. Spaces of Besov-Hardy-Sobolev type on complete Riemannian manifolds. *Arkiv för Matematik 24* (1985), 299–337.

[53] Ur Rahman, I., Drori, I., Stodden, V. C., Donoho, D. L., and Schröder, P. Multiscale representations for manifold-valued data. *Multiscale Mod. Sim. 4* (2005), 1201–1232.

[54] Velho, L. Stellar subdivision grammars. In *SGP '03: Proceedings of the 2003 Eurographics/ACM SIGGRAPH symposium on Geometry processing* (Aire-la-Ville, Switzerland, Switzerland, 2003), Eurographics Association, pp. 188–199.

[55] Waller, J., and H.Pottmann. Intrinsic subdivision with smooth limits for graphics and animation. *ACM Transactions on Graphics 25* (2006), 356–374.

[56] Wallner, J. Smoothness analysis of subdivision schemes by proximity. *Constr. Approx. 24* (2006), 289–318.

[57] Wallner, J., and Dyn, N. Convergence and C^1 analysis of subdivision schemes on manifolds by proximity. *Comput. Aided Geom. Design 22* (2005), 593–622.

[58] Wallner, J., Nava Yazdani, E., and Weinmann, A. Convergence and smoothness analysis of subdivision rules in Riemannian and symmetric spaces. *Adv. Comp. Math. (to appear)* (2010).

[59] Watson, A., and Ahumada Jr., A. A hexagonal orthogonal-oriented pyramid as a model of image presentation in visual cortex. *IEEE Trans. Biomed. Eng. 36* (1989), 97–106.

[60] Weinmann, A. Smoothness of nonlinear subdivision schemes for arbitrary dilation matrices. Geometry Preprint 2009/01, TU Graz, April 2009. http://www.geometrie.tugraz.at/weinmann/ArbDil.pdf.

[61] Weinmann, A. Interpolatory multiscale representation for functions between manifolds. Geometry Preprint 2010/04, TU Graz, January 2010. http://www.geometrie.tugraz.at/weinmann/Decay.pdf.

[62] Weinmann, A. Nonlinear subdivision schemes on irregular meshes. *Constr. Approx. 31* (2010), 395–415.

[63] Xie, G., and Yu, T. Smoothness analysis of nonlinear subdivision schemes of homogeneous and affine invariant type. *Constr. Approx. 22* (2005), 219–254.

[64] Xie, G., and Yu, T. Smoothness equivalence properties of general manifold-valued data subdivision schemes. *Multiscale Mod. Sim. 7* (2008), 1073–1100.

[65] Yu, T. How data dependent is a nonlinear subdivision scheme? *SIAM Journal on Numerical Analysis 44* (2006), 936–948.

[66] Zorin, D. *Stationary subdivision and multiresolution surface representations.* PhD thesis, CALTECH, 1997.

[67] Zorin, D. A method for analysis of C^1-continuity of subdivision surfaces. *SIAM J. Numer. Anal. 37* (2000), 1677–1708.

[68] Zorin, D. Smoothness of stationary subdivision on irregular meshes. *Constr. Approx. 16* (2000), 359–397.

[69] Zorin, D. Session: Interactive shape editing. In *Modeling with multiresolution subdivision surfaces*, ACM SIGGRAPH 2006 Courses. 2006, pp. 30–50. http://doi.acm.org/10.1145/1185657.1185673.

[70] Zorin, D., and Schröder, P. A unified framework for primal/dual quadrilateral subdivision schemes. *Comput. Aided Geom. Design 18* (2001), 429–454.

[71] Zorin, D., Schröder, P., and Sweldens, W. Interpolating subdivision for meshes of arbitrary topology. Technical Report CS-TR-96-06, Department of Computer Science, Caltech, 1996. `http://resolver.caltech.edu/CaltechCSTR:1996.cs-tr-96-06`.

[72] Zorin, D., Schröder, P., and Sweldens, W. Interpolating subdivision for meshes with arbitrary topology. *Computer Graphics Proceedings (SIGGRAPH 96)* (1996), 189–192.