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Symmetry breaking in graphs and groups

DISSERTATION

zur Erlangung des akademischen Grades eines Doktors der technischen Wissenschaften

Doktoratsstudium der Technischen Wissenschaften im Rahmen der Doktoratsschule “Mathematik und Wissenschaftliches Rechnen”

Graz University of Technology

Technische Universität Graz

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Graz, im Mai 2014

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Abstract

This thesis is concerned with open conjectures in graph theory which revolve around colourings of graphs, colour preserving automorphisms, and the concepts of "distinguishing number" and "motion" which naturally arise in this context. Interestingly part of our results could be achieved using probabilistic methods: in many cases we could show the existence of objects having a certain property by proving the much stronger statement that a randomly chosen objects has that property almost surely.

A colouring of a graph G is called distinguishing if it is not preserved by any non-trivial automorphism of G. The distinguishing number is the least number of colours used by a distinguishing colouring. The motion of G is the least number of vertices moved by a non-trivial automorphism of G. For finite graphs Russel and Sundaram showed that the two concepts are related. More precisely they proved that if G is a finite graph with motion m and $|\mathrm{Aut}\, G| \leq d^{\frac{m}{2}}$ for some $d \in \mathbb{N}$, then the distinguishing number of G is at most d.

If G is locally finite and has infinite motion then the inequality holds for every $d \geq 2$. Tucker conjectured that the conclusion also remains true in this case, that is, every locally finite graph with infinite motion has distinguishing number at most 2.

We show that Tucker's conjecture is true for graphs with growth $\mathcal{O}(2^{(1-\varepsilon)\frac{\sqrt{n}}{2}})$. Furthermore, we investigate random 2-colourings of locally finite graphs. We prove that random colourings are good candidates for being distinguishing, since they are almost surely only preserved by a sparse subgroup of $\mathrm{Aut}\, G$. This holds even in the more general setting of a subdegree finite, closed permutation group of a countable set. It also turns out that random colourings are almost surely distinguishing for many classes of locally finite graphs. Finally, we show that local finiteness is indeed necessary for the validity of Tucker's conjecture by giving non-locally finite counterexamples.

Acknowledgements

There are many people without whom writing this dissertation would not have been possible—far too many to mention them all. Nevertheless, I would like to take this opportunity and express my gratitude to a few people who contributed to this thesis in one way or another.

I would like to begin by thanking my supervisor Johannes Wallner who has always been supportive and helpful. I especially cherished the freedom he gave me in choosing my thesis topic. Not every graduate student gets the opportunity to work with such independence, and I feel that I have greatly benefited from it and that it helped me to grow as a scientist.

Besides my supervisor, I would like to express my gratitude to Wilfried Imrich for introducing me to the topic of symmetry breaking and distinguishing numbers which turned out to be a fruitful thesis topic. His comments and suggestions have been a great help in writing the research papers on which this dissertation is based.

I am grateful to Rögnvaldur Möller for making me feel at home during my stay in Iceland, and to Simon Smith for doing the same during my stay in Australia. I thoroughly enjoyed the insightful discussions during these stays whose outcomes now are part of my dissertation.

Gratitude is also due to Wolfgang Woess who got me interested in the topic of infinite graphs in the first place, and to Agelos Georgakopoulos who has always been a mentor to me. Had I not met them, I probably still would not know a thing about infinite graphs.

Finally I would like to thank all of my colleagues from the Institute of Geometry and everyone in the DK Discrete Mathematics for providing such a great, friendly atmosphere, and for the opportunities to present my ideas to fellow academics and learn more about diverse areas of mathematics.

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1 Introduction

A colouring of the vertices of a graph G is called distinguishing if it is not preserved by any non-trivial automorphism of G. The notion has been introduced by Albertson and Collins [1], but problems involving distinguishing colourings have been around for much longer. A classic example is Rubin's key problem [22] which can be summed up as follows.

Problem 1.1. A blind professor wants to distinguish the keys on his key ring by using different handle shapes. How many different shapes does he need to uniquely determine each key?

Obviously, the solution of the problem amounts to finding a distinguishing colouring of the cycle C_n where n is the number of keys and colours correspond to the different shapes. The solutions for some small values of n are shown in Figure 1. It may be surprising that, if the number of keys is at most 5, then 3 different colours are needed while for 6 or more keys 2 colours always suffice.

A distinguishing colouring clearly exists for every graph (simply colour every vertex with a different colour). Finding a distinguishing colouring with the minimum number of colours can however be challenging.

In this thesis we focus on infinite, locally finite graphs with infinite motion, that is, every non-trivial automorphism moves infinitely many vertices. Specifically we investigate the following conjecture of Tucker [26].

Conjecture 1.2. *Let G be an infinite, connected, locally finite graph with infinite motion. Then there is a distinguishing 2-colouring of G.*

This conjecture generalises a result on finite graphs due to Russel and Sundaram [24].

Lemma 1.3. *Let G be a finite graph and assume that every non-trivial automorphism moves at least m vertices. If $|\operatorname{Aut} G| \leq d^{\frac{m}{2}}$, then G has a distinguishing colouring with d colours.*

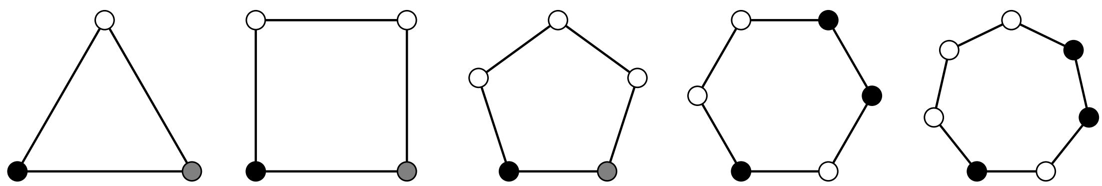

Figure 1: Distinguishing colourings of C_3, C_4, C_5, C_6, and C_7. It is easy to check that there are no distinguishing colourings with fewer colours. It is also easy to extend the idea of the colouring of C_6 and C_7 to larger cycles.

A proof of this result can be found in Section 3.3. The connection to Tucker's conjecture is also outlined there.

The conjecture is known to be true for many classes of infinite graphs including trees [27], tree-like graphs [13], and graphs with countable automorphism group [14]. In [25] it is shown that graphs satisfying the so-called distinct spheres condition have infinite motion as well as distinguishing number two. Examples of such graphs include leafless trees, graphs with infinite diameter and primitive automorphism group, vertex-transitive graphs of connectivity 1, and Cartesian products of graphs where at least two factors have infinite diameter.

The proof of Lemma 1.3 does not depend on the actual graph structure but only on the action of the automorphism group. A graph and its complement, for example, always have the same automorphism group while their graph structure usually differs. Hence it is reasonable to generalise Conjecture 1.2 to a group theoretical setting. The following conjecture appeared in [14].

Conjecture 1.4. *Let Γ be a group acting faithfully on a countable set S. If Γ has infinite motion, is closed in the permutation topology and subdegree finite, then there is a 2-colouring which is not preserved by the action of any non-trivial element of Γ.*

The notions of closedness and subdegree finiteness will be explained later. For now we only remark that the automorphism group of a locally finite graph $G = (V, E)$ always has those properties. Hence, by setting $S = V$ and $\Gamma = \operatorname{Aut} G$, we recover Conjecture 1.2 from Conjecture 1.4 which therefore is indeed a generalisation.

The aim of this thesis is to make further progress towards Conjectures 1.2 and 1.4. In Chapter 4 we investigate graphs with bounded growth. We show that if a graph does not grow faster than $\mathcal{O}(2^{(1-\varepsilon)\frac{\sqrt{n}}{2}})$, then it cannot be a counterexample to Tucker's conjecture. This is achieved by inductively constructing a distinguishing colouring, using the result for finite graphs as a tool. The results in this chapter can be found in [5, 19].

In Chapter 5 we pursue a different approach. Rather than using the result for finite graphs as a tool, we use its (probabilistic) proof as a motivation to study random colourings of locally finite graphs. It turns out that such colourings are almost surely distinguishing for many graph classes, and even if they are not, their stabiliser is almost surely a very sparse subgroup of the automorphism group. This suggests the following conjecture.

Conjecture 1.5. *Let G be an infinite, connected, locally finite graph with infinite motion, then a random 2-colouring of G is almost surely distinguishing.*

Many results in Chapter 5 are formulated in the more general setting of a subdegree finite closed group Γ acting on a countable set S, thus also providing progress towards Conjecture 1.4. The results of Chapter 5 have been published in [18].

In Chapter 6 we investigate colourings of graphs with infinite vertex degrees. We show that in all of the above conjectures the requirement of local finiteness or subdegree finiteness is necessary by giving appropriate counterexamples. Most of this is unpublished joint work with Möller.

Furthermore we consider uncountable graphs or rather groups acting on potentially uncountable sets and show a statement similar to Lemma 1.3 in this setting. This result has appeared in [5] and its proof is essentially due to Imrich.

Finally, in Chapter 7 we pose several interesting open problems related to the infinite motion conjecture.

2 Notions and notations

Throughout this thesis, $\mathbb{N}$ denotes the set of positive integers, while $\mathbb{N}_0$ stands for the set of non-negative integers, that is, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbol log denotes the base 2 logarithm. Greek letters are used predominantly for group related variables while the Latin alphabet is used for graphs or more generally for sets on which the groups act.

2.1 Graph theoretical notions

This section contains some basic graph theoretical concepts. The exposition follows the textbook [6] whose terminiology will also be used for notions that are not explicitly defined.

Throughout this thesis, $G = (V, E)$ denotes a graph with (usually countably infinite) vertex set V and edge set $E \subseteq \binom{V}{2}$, where $\binom{V}{2}$ is the set of all 2-element subsets of V. For the sake of simplicity we write uv instead of $\{u, v\}$ for an edge connecting vertices u and v. Two vertices u and v are called neighbours if $uv \in E$. The *neighbourhood* $N(v)$ of a vertex v is the set of neighbours of v.

From the above definition of the edge set it is clear that all graphs in consideration are *simple*, that is, they contain no loops or multiple edges. Furthermore, unless explicitly stated otherwise, all graphs are *locally finite*, meaning that every vertex has only finitely many neighbours.

A *walk* in a graph is a sequence $v_1, e_1, v_2, e_2, v_3, \dots, e_{n-1}, v_n$ where $v_i \in V$ and $e_i = v_i v_{i+1} \in E$ for $1 \leq i \leq n$. We say that such a walk connects v_1 to v_n. If all v_i are distinct, then the walk is called a *path*. The *length* of a walk is the number of edges contained in it. We say that a graph is connected, if for any two vertices there is a path connecting them. All graphs considered in this thesis are assumed to be connected.

It is possible to equip the vertex set with a natural metric. The distance $d(u, v)$ is defined as the minimal length of a walk connecting u and v. The closed ball with centre v and radius r with respect to this metric is denoted by $B_v(r)$. Since $B_v(r) = B_v(\lfloor r \rfloor)$, we can restrict ourselves to $r \in \mathbb{N}_0$. The *sphere* $S_v(r)$ with centre v and radius r consists of all vertices whose distance from v is exactly r. If $r \in \mathbb{N}_0$ then $S_v(r) = B_v(r) \setminus B_v(r-1)$, otherwise the sphere is empty. Both $B_v(r)$ and $S_v(r)$ depend on the graph G. However, since G is usually clear from the context we omit this dependency in the notation for the sake of readability.

A concept central to this thesis is the notion of the automorphism group of a graph. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. A function $\varphi\colon V_G \to V_H$ such that $uv \in E_G$ implies $\varphi(u)\varphi(v) \in E_H$ is called a *graph homomorphism* from G to H. A *graph endomorphism* is a graph homomorphism from G to itself. A *graph automorphism* is a bijective graph endomorphism whose inverse is a homomorphism as well. Clearly the automorphisms of a graph form a group. This group is denoted by $\operatorname{Aut} G$.

2.2 Group actions and permutation groups

In this section we briefly introduce some notions related to group actions. For a more extensive introduction see for example [2].

Let Γ be a group with group operation $\circ$ and neutral element id and let S be a set. A *left action of* Γ *on* S is a mapping

$$\begin{aligned} \Gamma \times S &\to S \\ (\gamma, s) &\mapsto \gamma s \end{aligned}$$

such that

$$\begin{gathered} \forall s \in S \colon \mathrm{id} s = s, \\ \forall \gamma_1, \gamma_2 \in \Gamma \colon \forall s \in S \colon (\gamma_1 \circ \gamma_2) s = \gamma_1(\gamma_2 s). \end{gathered}$$

Analogously we can define a *right action*, simply replacing left multiplication with right multiplication.

Clearly every group acts on itself from the left and from the right by left and right multiplication, respectively. The two actions coincide if and only if Γ is abelian. Another example of a group action which plays a central role in this thesis is the action of $\operatorname{Aut} G$ on V, where $G = (V, E)$ is a graph. By convention, automorphisms act from the left.

For a "generic" example of a group action consider the following. Take a countable set S and let Sym_S be the symmetric group on S, that is, Sym_S consists of all bijective mappings from S onto itself with composition as the group operation. Clearly Sym_S acts on S by bijective mappings and so does every subgroup of Sym_S. The elements of Sym_S are called *permutations*, and subgroups of Sym_S are called *permutation groups*.

To see that this is indeed a generic example, observe that every group action gives rise to a group homomorphism from Γ to Sym_S. The action of Γ is *faithful* if different group elements act by different permutations on S, that is, if the homomorphism mentioned above is injective. In this case we do not distinguish between $\gamma \in \Gamma$ and the corresponding permutation of S and consider Γ a permutation group. We

An important notion throughout this thesis is the notion of stabilisers.

Definition 2.1. Let Γ be a group acting on a set S and let $s \in S$. The *stabiliser of* s *in* Γ is defined as

$$\Gamma_s = \{\gamma \in \Gamma \mid \gamma s = s\}.$$

The following result on stabilisers is well known.

Proposition 2.2. *Let* Γ *be a group acting on a set* S *and let* $s \in S$*. Then the stabiliser* Γ_s *is a subgroup of* Γ*.*

Proof. Clearly $\Gamma_s \neq \emptyset$ since $\mathrm{id} \in \Gamma_s$. Now let $\gamma, \delta \in \Gamma_s$. Then

$$(\gamma \circ \delta)s = \gamma(\delta s) = \gamma s = s,$$

hence $\gamma \circ \delta \in \Gamma_s$.

Finally assume that $\gamma^{-1} \notin \Gamma_s$. Then

$$(\gamma^{-1} \circ \gamma)s = \gamma^{-1}(\gamma s) = \gamma^{-1}s \neq s.$$

But this contradicts the fact that $\mathrm{id}s = s$. $\square$

If $S' \subseteq S$, then we denote by $\Gamma_{S'}$ the *setwise stabiliser of S' in* Γ, that is,

$$\Gamma_{S'} = \{\gamma \in \Gamma \mid \forall s \in S' \colon \gamma s \in S'\}.$$

It is not hard to see that $\Gamma_{S'}$ is the stabiliser of S' with respect to the action of Γ on the power set of S defined by

$$\gamma T = \{\gamma s \mid s \in T\}$$

for $\gamma \in \Gamma$ and $T \subseteq S$. Hence in particular setwise stabilisers are subgroups of Γ by Proposition 2.2.

The *pointwise stabiliser of S' in* Γ is the set $\Gamma_{(S')} = \bigcap_{s \in S'} \Gamma_s$. Pointwise stabilisers are intersections of stabiliser subgroups and hence also subgroups of Γ.

The *kernel* of an action of a group Γ on a set S is defined as $\Gamma_{(S)}$. By the above remark, this is a subgroup of Γ. The following proposition shows that this subgroup is normal. Moreover it states that faithful group actions—or equivalently actions of permutation groups—cover all possible group actions on a set.

Proposition 2.3. *Let Γ be a group acting on a set S. Then the kernel $\Gamma_{(S)}$ is a normal subgroup of Γ and the group $\Gamma/\Gamma_{(S)}$ acts faithfully on S in a natural way.*

Proof. By Proposition 2.2 we know that $\Gamma_{(S)}$ is a subgroup of Γ. Now let $\gamma \in \Gamma_{(S)}$ and let $\delta \in \Gamma$. Then

$$(\delta \circ \gamma \circ \delta^{-1})s = \delta(\gamma(\delta^{-1}s)) = \delta(\delta^{-1}s) = (\delta \circ \delta^{-1})s = \mathrm{id}s = s$$

for every $s \in S$. Hence $\delta \circ \gamma \circ \delta^{-1} \in \Gamma_{(S)}$ and thus $\Gamma_{(S)}$ is a normal subgroup of Γ.

Define an action of $\Gamma/\Gamma_{(S)}$ on S by $(\gamma \circ \Gamma_{(S)})s = \gamma s$ for every $\gamma \in \Gamma$ and $s \in S$. This is well defined because $\Gamma_{(S)}$ fixes every $s \in S$, and it is a group action because Γ acts on S. $\square$

Definition 2.4. Let Γ be a group acting on a set S and let Δ be a subset of Γ. We denote by $\Delta s = \{\gamma s \mid \gamma \in \Delta\}$ the *orbit of s under* Δ.

Note that in the above definition we do not require Δ to be a subgroup of Γ. If it is a subgroup, then it is well known that

$$s \sim t \iff s \in \Delta t$$

is an equivalence relation on S whose equivalence classes are the orbits.

Definition 2.5. Let Γ be a group acting on a set S. A *suborbit* is a set of the form $\Gamma_s t$, where $s, t \in S$, that is, it is an orbit under a point stabiliser. We say that (the action of) Γ is *subdegree finite*, if all suborbits are finite.

The property of being subdegree finite is a property of the action of Γ rather than the group itself. However, we are mostly interested in the case where $\Gamma \leq \mathrm{Sym}_S$. In this setting it does make sense to speak of a subdegree finite group because the action on S is known.

Many results on distinguishing numbers of graphs remain true if we take a subdegree finite permutation group acting on a set instead of $\mathrm{Aut}\, G$ acting on the vertex set. There are several examples of this in Chapter 5, where results on locally finite graphs follow from results for subdegree finite permutation groups.

The automorphism group of a locally finite graph $G = (V, E)$ (acting on its vertex set) is easily seen to be subdegree finite. Simply observe that every automorphism is an isometry. Since in a locally finite graph there are only finitely many vertices at a given distance from v, it follows that $\Gamma_v w$ is finite for every pair $v, w \in V$.

2.3 The permutation topology

In this section we describe a family of metrics on a group Γ of permutations of a countable set S and discuss some of the properties of the induced topology. The way the metrics are constructed may seem familiar to many readers. In fact, the construction is similar to the construction of the p-adic norm, and a similar approach can also be used to equip the end space of a locally finite graph with a metric. It turns out that every metric in this family induces the same topology on Γ, the so called permutation topology. This topology was first studied in the 1950s by Karass and Solitar [15] and Maurer [20] and is a rather natural topology for groups of permutations. Another way of introducing the same topology is to equip the set S with the discrete topology and consider the topology of pointwise convergence on Γ. The paper [21] by Möller gives a good overview on the permutation topology on closed, subdegree finite permutation groups.

For the construction of the metric, let S be a countable set and let Γ be a group of permutations of S. Let $(S_i)_{i\in\mathbb{N}}$ be a sequence of finite subsets of S such that $S_i \subset S_{i+1}$ and $\lim_{i\to\infty} S_i = \bigcup_{i\in\mathbb{N}} S_i = S$. For two permutations $\gamma_1, \gamma_2 \in \Gamma$ define the confluent of γ_1 and γ_2 as

$$\mathrm{conf}(\gamma_1, \gamma_2) = \min\{i \in \mathbb{N} \mid \exists s \in S_i \colon \gamma_1\gamma_2^{-1}s \neq s\} - 1,$$

that is, the confluent is the maximum i such that γ_1 and γ_2 coincide on S_i and it is zero if they differ on S_1. Note that the value of $\mathrm{conf}(\gamma_1, \gamma_2)$ clearly depends on the choice of the sequence S_i.

Now define the distance between γ_1 and γ_2 as

$$\delta(\gamma_1, \gamma_2) = \begin{cases} 0 & \text{if } \gamma_1 = \gamma_2, \\ 2^{-\mathrm{conf}(\gamma_1,\gamma_2)} & \text{otherwise.} \end{cases}$$

The following proposition shows that the term distance is justified. In fact, δ even satisfies the ultrametric triangle inequality $\delta(\gamma_1, \gamma_3) \leq \max\{\delta(\gamma_1, \gamma_2), \delta(\gamma_2, \gamma_3)\}$. As we mentioned earlier, the topology induced by δ does not depend on the choice of the sequence S_i.

Proposition 2.6. *The function δ as defined above is an ultrametric on Γ. All such metrics induce the same topology on Γ, which makes Γ a topological group.*

Proof. It is readily verified that $\delta(\gamma_1, \gamma_2)$ is symmetric, non-negative, and zero if and only if $\gamma_1 = \gamma_2$. Furthermore, if $r = \min\{\mathrm{conf}(\gamma_1, \gamma_2), \mathrm{conf}(\gamma_2, \gamma_3)\}$ then both $\gamma_1\gamma_2^{-1}$ and $\gamma_2\gamma_3^{-1}$ fix S_r pointwise and hence so does $\gamma_1\gamma_2^{-1}\gamma_2\gamma_3^{-1} = \gamma_1\gamma_3^{-1}$. Thus

$$\delta(\gamma_1, \gamma_3) \leq 2^{-r} = \max\{\delta(\gamma_1, \gamma_2), \delta(\gamma_2, \gamma_3)\},$$

so δ is an ultrametric.

Clearly, every sequence S_i induces a different metric on Γ but we claim that all of them induce the same topology.

Indeed, let Δ be an open neighbourhood of a permutation $\gamma \in \Gamma$ in the topology which comes from the distance δ defined using the sequence $(S_i)_{i\in\mathbb{N}}$. Then there is a natural number n such that Δ contains a δ-ball with centre γ and radius 2^{-n}. This implies that Δ contains all automorphisms γ' such that $\gamma\gamma'^{-1}$ fixes S_n pointwise.

Now consider a different sequence $(S'_i)_{i\in\mathbb{N}}$ of finite subsets of S whose union is S and use this sequence to define another metric δ'. Then there is an index m such that $S_n \subset S'_m$. So if a permutation γ' fulfils $\delta'(\gamma, \gamma') \leq 2^{-m}$ then it certainly holds that $\delta(\gamma, \gamma') \leq 2^{-n}$. In other words, Δ contains a δ'-ball with centre γ and radius 2^{-m}.

So we have proved that an open set with respect to the metric δ is also open with respect to the metric δ'. Since the converse can be shown in a completely analogous way we conclude that the respective topologies must coincide.

Finally, it is easy to see that this topology makes Γ a topological group. Simply note that both left and right multiplication as well as taking inverses are isometries. □

Definition 2.7. Let $\Gamma \leq \mathrm{Sym}_S$ be a group of permutations of a countable set S. We say that Γ is *closed*, if it is closed as a subset of Sym_S with respect to the permutation topology.

It is a well known fact that in an ultrametric space any two balls are either contained in one another or disjoint. In particular, distinct balls with the same radius must be disjoint. From this it follows that for any ball Δ with radius ϱ, the subballs of Δ with radius $\varrho' < \varrho$ form a partition of Δ. The following lemma states that this partition is countable if we partition the whole space, and finite if Γ is subdegree finite and Δ is a strict subset of Γ.

Lemma 2.8. *There are only countably many distinct balls of radius $\varrho < 1$ in Γ. If Γ is subdegree finite, then each ball of radius $\varrho < 1$ only has finitely many distinct subballs of radius $\varrho' < \varrho$.*

Proof. By the definition of δ, balls of radius ϱ are exactly the cosets with respect to the pointwise stabiliser of S_i where i is the unique integer such that $2^{-i+1} > \varrho \geq 2^{-i}$. Since S_i is finite, there are only countably many possibilities to choose the image of S_i. So the set of cosets—and hence also the set of balls with radius ϱ—is at most countable.

Now let $\Delta \subseteq \Gamma$ be a ball of radius $\varrho < 1$. Since multiplication by a group element is an isometry, we may without loss of generality assume that the centre of Δ is id. This implies that Δ is the pointwise stabiliser of S_i where $2^{-i+1} > \varrho \geq 2^{-i}$.

A subball of Δ with radius ϱ' is a coset of Δ with respect to the stabiliser of S_j where j is the unique natural number such that $2^{-j+1} > \varrho' \geq 2^{-j}$. Hence it suffices to show that there is only a finite number of such cosets.

To see that this is the case note that every automorphism in Δ fixes S_1. Furthermore note that Γ is subdegree finite, hence the orbit of each $s \in S$ under Δ is finite. Since S_j is finite there are only finitely many possibilities to choose an image of S_j. □

We can use the previous lemma to show that small balls in a closed, subdegree finite permutation group Γ are compact. From this result we can derive a multitude of topological properties of Γ.

Lemma 2.9. *If Γ is closed and subdegree finite, then Γ is locally compact. More specifically, balls of radius $\varrho < 1$ are compact.*

Proof. Since in a metric space compactness and sequential compactness are equivalent, it suffices to show that every sequence has a convergent subsequence. So assume we have a sequence $(\gamma_i)_{i\in\mathbb{N}}$ of pairwise different permutations all of which lie inside a ball Δ of radius $\varrho < 1$.

Let $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \varrho$. Then, by Lemma 2.8, Δ has only finitely many subballs of radius 2^{-k_0} and hence we can find an infinite subsequence of γ_i which is completely contained in one of the subballs Δ_0, say.

The ball Δ_0 again has only finitely many subballs of radius 2^{-k_0-1} so we can find an infinite sub-subsequence which lies completely in a subball Δ_1 of Δ_0. Proceeding inductively we obtain a sequence of nested balls $(\Delta_k)_{k\in\mathbb{N}}$ in Γ, each ball containing infinitel many γ_i, where the radius of Δ_k is 2^{-k_0-k}.

Now we define a permutation γ as follows: to determine γs for $s \in S_{k_0+k}$ look at the coset Δ_k. All permutations in this coset map s to the same vertex t. Choose $\gamma s = t$. Since the sets Δ_k are nested, γ is well defined.

It follows easily from subdegree finiteness that γ is bijective and hence a permutation. Simply observe that if γ_i and γ_j are in Δ_k then $\gamma_i s = \gamma_j s$ and hence $\gamma_i^{-1}\gamma_j s = s$ for every $s \in S_0$. By subdegree finiteness there are only finitely many possible values for $\gamma_i^{-1}\gamma_j t$ for every $t \in S$ and hence there are only finitely many values for $\gamma_i^{-1}u$ (recall that γ_j is bijective) for every $u \in S$. Now choose k such that all of the possible values are contained in S_{k_0+k}. Then all permutations in Δ_k map the same vertex to u and hence u has a preimage under γ.

If we can find a subsequence of γ_i which converges to γ in the set Sym_S of all permutations of S, then it follows that $\gamma \in \Gamma$ since Γ is closed in Sym_S. Furthermore in this case we found a convergent subsequence of γ_i, which completes the proof of the lemma.

To construct such a subsequence choose i_k such that $i_k > i_{k-1}$ and $\gamma_{i_k} \in \Delta_k$. Since γ coincides with γ_{i_k} on S_{k+i_0} it follows that $\delta(\gamma_{i_k}, \gamma) \to 0$ as $k \to \infty$, so γ_{i_k} converges to γ. □

Various nice topological properties of Γ follow from the above results by well known theorems of topology which can for example be found in [11]. In the sequel we only use separability, local compactness and σ-compactness. However we present a more extensive list of nice topological properties to emphasise how well behaved Γ is as a topological space.

Corollary 2.10. *Let Γ be a closed, subdegree finite group of permutations of a set S. Then Γ equipped with the permutation topology has the following properties:*

- *locally compact, that is, every point has a compact neighbourhood,*
- *σ-compact, that is, it can be covered by countably many compact sets,*
- *separable, that is, there is a countable dense subset,*
- *Lindelöf, that is, every cover of the space with open sets has a countable subcover,*
- *second countable, that is, there is a countable basis of the topology,*
- *totally disconnected, that is, for any two points there are disjoint open neighbourhoods whose union covers all of Γ,*
- *complete (with respect to the metric δ), that is, every Cauchy sequence converges.*

Proof. The group is locally compact because small balls are compact. It is σ-compact because there are only countably many distinct balls of radius $r < 1$. The Lindelöf property follows from the fact that every σ-compact space is Lindelöf. Separablity and second countability are equivalent to Lindelöf for metric spaces. Total disconnectedness follows from the fact that in an ultrametric space balls are both open and closed. The metric is complete because every Cauchy sequence eventually stays within a small ball. Since this ball is compact, it must contain an accumulation point of the sequence which must be the limit of the sequence because it is Cauchy. □

3 Motion and distinguishing numbers

3.1 Distinguishing numbers of graphs and groups

As mentioned earlier, we are investigating the problem of finding a colouring of a graph which is not preserved by any non-trivial automorphism. By a *colouring of a graph* G we simply mean a map c from its vertex set to a set C of colours. Usually C will be finite. We speak of a C-colouring or a $|C|$-colouring, since C_1-colourings and C_2-colourings are the same up to relabelling the colours if C_1 and C_2 have the same size. The set of all C-colourings of G is denoted by $\mathcal{C}(G, C)$. The case of 2-colourings, that is, $|C| = 2$ is of particular interest to us.

Definition 3.1. Let $G = (V, E)$ be a graph, let $c\colon V \to C$ be a C-colouring of G and let $\gamma \in \operatorname{Aut} G$. We say that γ *preserves* c if $c(\gamma v) = c(v)$ for every $v \in V$. Otherwise we say that c *breaks* γ.

The colouring c breaks $\Delta \subseteq \operatorname{Aut} G$, if it breaks every non-trivial element of Δ.

Note that in the above definition Δ need not be a subgroup of Γ. The reason for this is, that sometimes it is more convenient to be able to split up the group into arbitrary parts instead of just subgroups.

Definition 3.2. Let $G = (V, E)$ be a graph and let $c\colon V \to C$ be a colouring of G. Then c is called *distinguishing*, if the only automorphism φ of G that preserves c is the identity.

The *distinguishing number of* G is the minimal number of colours needed for a distinguishing colouring. It is denoted by $D(G)$. If $D(G) \leq k$ for some $k \in \mathbb{N}$ then we say that *G is k-distinguishable.*

The above definitions implicitly use a natural action of $\operatorname{Aut} G$ on the set of C-colourings of G. If the automorphism group acts on V from the left then we can define a right action of $\operatorname{Aut} G$ on $\mathcal{C}(G, C)$ as follows. For $c \in \mathcal{C}(G, C)$ and $\gamma \in \operatorname{Aut} G$ define the action of γ on c by $(c, \gamma) \mapsto c\gamma$ where $c\gamma(v) = c(\gamma v)$. This action gives an alternative definition of a distinguishing colouring.

Proposition 3.3. *A colouring c is distinguishing if and only if its stabiliser with respect to the above action is trivial, that is,* $(\operatorname{Aut} G)_c = \{\mathrm{id}\}$.

Proof. An automorphism γ is contained in the stabiliser if and only if $c(v) = c\gamma(v) = c(\gamma v)$ for every $v \in V$. $\square$

Note that $D(G) = 1$ for all asymmetric graphs. This means that almost all finite graphs have distinguishing number one, because almost all graphs are asymmetric, see Erdős and Rényi [7]. Clearly $D(G) \geq 2$ for all other graphs. Again, it is natural to conjecture that almost all of them have distinguishing number two. This is supported by the observations of Conder and Tucker [4].

However, for the complete graph K_n, and the complete bipartite graph $K_{n,n}$ we have $D(K_n) = n$, and $D(K_{n,n}) = n + 1$. Furthermore, as we have already seen in Chapter 1, the distinguishing number of the 5-cycle is 3, but cycles C_n of length $n \geq 6$ have distinguishing number 2.

This compares with more general results of Klavžar, Wong and Zhu [16] and of Collins and Trenk [3], which assert that $D(G) \leq \Delta(G)+1$, where Δ denotes the maximum degree of G. Equality holds if and only if G is a K_n, $K_{n,n}$ or C_5.

For $V' \subseteq V$ a *partial C-colouring* of $G = (V, E)$ with *domain* V' is a map $c' \colon V' \to C$. We denote by $\mathcal{C}(V', C)$ the set of all partial C-colourings with domain V'. There is an action of $\operatorname{Aut} G$ on the set of all partial colourings defined similarly to the action on the colourings above, that is, $c'\gamma(v) = c'(\gamma v)$ for $v \in V'$. Clearly, if c' is a partial C-colouring with domain V' then $c'\gamma$ is a partial C-colouring with domain $\{\gamma v \mid v \in V'\}$

This implies that $\operatorname{Aut} G$ does not act on $\mathcal{C}(V', C)$ because unless an automorphism stabilises V' setwise it does not map colourings with domain V' to colourings with the same domain. Furthermore the stabiliser of $c' \in \mathcal{C}(V', C)$ with respect to the above action is always contained in the setwise stabiliser $(\operatorname{Aut} G)_{V'}$. Now assume that we have a partial colouring c' with domain V' and let γ be an automorphism that moves V' to a disjoint set. Then it is possible that we can extend c' to a colouring c of the whole vertex set which is preserved by γ although γ is not contained in the stabiliser of c'. To prevent such things from happening, we use a different notion of stabilisers for partial colourings.

Definition 3.4. Let $G = (V, E)$ be a graph, let $V' \subseteq V$, and let $c' \colon V' \to C$ be a partial C-colouring of G. Let $\gamma \in \operatorname{Aut} G$. We say that γ *preserves* c' if there are colourings c_1 and c_2 of V such that $c_1(v) = c_2(v) = c'(v)$ for every $v \in V'$ and $c_1\gamma = c_2$. Otherwise we say that c' *breaks* γ.

This definition deals with the problem mentioned before. If we can extend a partial colouring c' to a colouring c which is preserved by $\gamma \in \operatorname{Aut} G$, then setting $c_1 = c_2 = c$ shows that γ preserves c'. Conversely, if c' breaks γ then we cannot find such a colouring c. Note however that the colourings c_1 and c_2 in the above definition do not necessarily coincide.

We now define the stabiliser of a partial colouring completely analogously to the definition of the stabiliser of a colouring.

Definition 3.5. The *stabiliser* $(\operatorname{Aut} G)_{c'}$ *of a partial colouring* c' consists of all automorphisms which preserve c'.

Although the definitions look very similar, stabilisers of partial colourings behave differently to stabilisers of colourings. They do not come from any group action. In

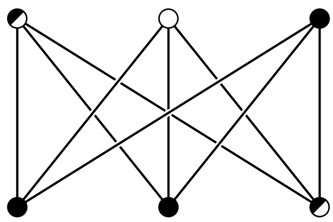

Figure 2: A partial colouring of $K_{3,3}$. Observe that for every uncoloured vertex (drawn half black, half white) there is an automorphism $\gamma_1 \in (\operatorname{Aut} G)_{c'}$ which maps it to a black vertex and an automorphism $\gamma_2 \in (\operatorname{Aut} G)_{c'}$ which maps it to a white vertex. It follows immediately that the stabiliser cannot be a subgroup because $\gamma_1 \circ \gamma_2^{-1}$ maps a white vertex to a black vertex and hence does not preserve the partial colouring.

particular observe that—unlike the stabiliser of a colouring—the stabiliser of a partial colouring need not be a subgroup of $\operatorname{Aut} G$ (see Figure 2).

So far in this section we were concerned with automorphism groups of graphs, but the attentive reader will have noticed that the graph structure did not play a role. Indeed one can formulate all of the above in the more general setting of a group Γ of permutations of a countable set S.

A *C-colouring of the set S* in this context is a function $c\colon S \to C$. A *partial C-colouring of S* is a map $c'\colon S' \to C$ where $S' \subseteq S$. The set S' is called the *domain* of the partial C-colouring. The set of all C-colourings of S and the set of all partial C-colourings of S with domain S' are denoted by $\mathcal{C}(S, C)$ and $\mathcal{C}(S', C)$ respectively.

In analogy to Definitions 3.1 and 3.2 we define distinguishing colourings of a permutation group.

Definition 3.6. Let $\Gamma \leq \operatorname{Sym}_S$. An element $\gamma \in \Gamma$ *preserves a colouring c of S*, if $c(\gamma s) = c(s)$ for every $s \in S$. Otherwise c *breaks* γ. We say that c breaks $\Delta \subseteq \Gamma$ if it breaks every non-trivial element of Δ.

A colouring of S is called Γ-*distinguishing* if it is only preserved by the identity element of Γ. We omit the Γ in "Γ-distinguishing" if the group is clear from the context.

The *distinguishing number of* Γ is the mininal number of colours needed for a Γ-distinguishing colouring of S. It is denoted by $D(\Gamma)$. If $D(\Gamma) \leq k$ for some $k \in \mathbb{N}$ then we say that Γ *is k-distinguishable.*

Observe that again there is a group action of Γ on the set of C-colourings of S hiding in this definition: for $\gamma \in \Gamma$ and for $c \in \mathcal{C}(S, C)$ define the colouring $c\gamma$ by $(c\gamma)(s) = c(\gamma s)$ for all $s \in S$.

It is easy to check that this is a right action. An analogous statement to Proposition 3.3 holds for permutation groups as well, that is, a colouring c of S is Γ-distinguishing if and only if its stabiliser in Γ is trivial.

Finally, we have the following definition of the stabiliser of a partial colouring of S following the spirit of Definitions 3.4 and 3.5.

Definition 3.7. Let Γ be a group of permutations of a set S and let $c'\colon S' \to C$ be a partial C-colouring of S with domain S'. An element $\gamma \in \Gamma$ *preserves* c' if there are C-colourings c_1 and c_2 of S such that $c_1\gamma = c_2$ and for every $s \in S'$ it holds that $c_1(s) = c_2(s) = c'(s)$. Otherwise we say that c' *breaks* γ.

The *stabiliser* $\Gamma_{c'}$ of a partial C-colouring is the set of all $\gamma \in \Gamma$ which preserve c'.

3.2 Stabilisers of colourings

In this section we outline some basic properties of stabilisers of colourings, partial colourings, and subsets of S. We start with a well known result about the stabiliser of a single element s of S which can for example be found in [28].

Lemma 3.8. *Let Γ be a closed, subdegree finite group of permutations of a countable set S. Then for every $s \in S$ the stabiliser Γ_s is a compact subgroup of Γ.*

Proof. It is clear that the stabiliser must be a subgroup of Γ so we only need to show that it is compact. In the construction of the metric δ in Section 2.3 choose $S_1 = \{s\}$. Then Γ_s is the ball centred at id with radius $\varrho = \frac{1}{2}$. Hence it is compact by Lemma 2.9. □

A similar result can also be obtained for the setwise stabiliser of a finite subset $S' \subseteq S$. In fact, the following lemma exactly tells us when a closed and subdegree finite group of permutations of a countable set is compact.

Lemma 3.9. *Let Γ be a closed, subdegree finite group of permutations of a countable set S. Then the following are equivalent:*

1. *Γ is compact.*
2. *Γ setwise stabilises some finite subset S' of S.*
3. *The orbit of some element $s \in S$ is finite.*
4. *All orbits under the action of Γ are finite.*

Proof. Clearly $4 \Rightarrow 3$. The implication $3 \Rightarrow 2$ follows from the fact that Γ stabilises every orbit setwise. The converse implication follows from the fact that the orbit of $s \in S'$ must be contained in S' if the set is setwise stabilised. So we only need to show the implications $3 \Rightarrow 1 \Rightarrow 4$ in order to prove the equivalence of the statements.

First assume that there is some $s \in S$ such that the orbit Γs is finite. Clearly Γ is the union of the (finitely many) cosets with respect to the stabiliser Γ_s. All of the cosets are compact, because the stabiliser is compact by Lemma 3.8. Hence we have decomposed Γ into finitely many compact sets and Γ itself must be compact.

To see that $1 \Rightarrow 4$, let Γ be compact and assume that there is some $s \in S$ whose orbit is infinite. Then we can find an infinite sequence $(\gamma_i)_{i\in\mathbb{N}}$ of permutations in Γ such that no two permutations map s to the same point. Since Γ is compact, this sequence must have a convergent subsequence. This is impossible because no two permutations coincide on s, which gives a lower bound on their distance. □

Next we would like to turn to stabilisers of colourings of S. In general such a stabiliser is not compact, but we can show that it is always a closed subgroup of Γ.

Lemma 3.10. *Let Γ be a group of permutations of a countable set S. Then the stabiliser Γ_c of a colouring c of S is a closed subgroup of Γ.*

Proof. Again it is clear that the stabiliser of c is a subgroup of γ since $c\gamma = c \circ \gamma$ defines a right action of Γ on the set $\mathcal{C}(S)$ of colourings of S. Hence we only need to show that it is closed.

Consider a permutation $\gamma \notin \Gamma_c$. There must be some $s \in S$ such that $c(s) \neq c(\gamma s)$. This point is contained in some set S_i, where $(S_i)_{i\in\mathbb{N}}$ is the non-decreasing sequence of finite subsets of S, which was used to construct the metric in Section 2.3. Every permutation γ' with $\delta(\gamma', \gamma) < 2^{-i}$ coincides with γ on S_i. This implies that no permutation in the ball $B_\gamma(2^{-i})$ is contained in Γ_c. So γ has an open neighbourhood which is disjoint to Γ_c and hence the complement of Γ_c is open. □

What happens if we consider partial colourings instead of colourings? It is readily verified that the stabiliser of a partial colouring c' is in general not a subgroup of Γ, so we cannot hope for a verbatim extension of Lemma 3.10 to partial colourings. But it turns out that apart from the group property everything generalises nicely. If the domain of the partial colouring is finite, we even get a better result: in this case the stabiliser is a set that is both closed and open in the permutation topology.

Lemma 3.11. *Let Γ be a group of permutations of a countable set S and let c' be a partial colouring of S. Then the stabiliser of c' is closed. If the domain of c' is finite then the stabiliser is also open.*

Proof. Denote by S' the domain of c'. Clearly, a permutation $\gamma \in \Gamma$ preserves c' if and only if there is a colouring c'' of the set

$$T = S' \cup \gamma^{-1} S'$$

such that for every $s \in S'$ it holds that $c''(\gamma^{-1}s) = c''(s) = c'(s)$.

If S' is finite then so is T and hence T is contained in S_i for some $i \in \mathbb{N}$. Consider a permutation γ' such that $\delta(\gamma, \gamma') < 2^{-i}$. It follows from the definition of δ that $\gamma' s = \gamma s$ for every $s \in T$. Hence a colouring of T with the above property exists for γ if and only if it exists for γ'. It follows that if $\gamma \in \Gamma_{c'}$ then the ball with centre γ and radius 2^{-i} is completely contained in the stabiliser of c', showing that the stabiliser is open. Conversely, if $\gamma \notin \Gamma_{c'}$ then this ball is completely contained in the complement of the stabiliser, proving that the complement is open as well.

Now let us turn to the case where S' is infinite. In this case choose a sequence S'_i of finite subsets of S' such that $S'_i \subseteq S'_{i+1}$ and $\lim_{i\to\infty} S'_i = S'$. Let c'_i be the colouring with domain S'_i which coincides with c' on S'_i. We know that $\Gamma_{c'_i}$ is closed because of the first part of the proof. If we can show that $\Gamma_{c'} = \bigcap_{i\in\mathbb{N}} \Gamma_{c'_i}$ then it is closed because it is the intersection of closed sets.

But this is easy: if a permutation is contained in $\Gamma_{c'}$ then it is clearly contained in every $\Gamma_{c'_i}$ (simply use the same colourings to extend c' and c'_i). If a permutation γ is not contained in $\Gamma_{c'}$ then this means that there is no partial colouring with domain T such that $c''(\gamma s) = c''(s)$ for each $s \in S'$. since we can colour every $s \in T \setminus S'$ arbitrarily this implies that there are two elements $s, t \in S'$ with different colours such that $\gamma s = t$. now choose i large enough that $s, t \in S'_i$. Clearly $\gamma \notin \Gamma_{c'_i}$ and hence γ is not contained in the intersection. □

3.3 The motion lemma and the infinite motion conjecture

In this section we introduce the notion of motion. Its connection to distinguishing numbers is the central topic of this thesis. We introduce motion in terms of permutation groups. Analogous definitions for a graph $G = (V, E)$ and its automorphism group are obtained by setting $S = V$ and $\Gamma = \operatorname{Aut} G$.

Definition 3.12. Let γ be a permutation of a set S. The *motion* $m(\gamma)$ is the cardinality of the set $\{s \in S \mid \gamma s \neq s\}$.

The motion of a set $\Delta \subseteq \mathrm{Sym}_S$ is the least motion of a non-trivial element contained in Δ.

Technically the motion could be any cardinal number. However, we are mostly concerned with permutations of countable sets. In this case the motion is contained in $\mathbb{N} \cup \{\aleph_0\}$, where $\aleph_0$ denotes countable infinity. If there is no possible confusion with other infinite cardinals we write ∞ instead of $\aleph_0$.

If Δ is the automorphism group of a graph then instead of the motion of Δ, we simply speak about the motion of the graph G.

Definition 3.13. Let $G = (V, E)$ be a graph. then the *motion of* G is the motion of $\operatorname{Aut} G$ acting on V.

Consider the case where $S' \subseteq S$ and $\Delta \subseteq \Gamma_{S'}$. Then the elements of Δ can also be seen as permutations of S', possibly with the same permutation occurring more than once. This viewpoint is useful because it allows us to break all permutations in Δ that act non-trivially on S' with a partial colouring with domain S'.

Definition 3.14. If γ fixes $S' \subset S$ as a set, we define the *restriction* $\gamma|_{S'}$ of γ to S' to be the permutation which γ induces on S'.

For a set $\Delta \subseteq \Gamma_{S'}$ of permutations we define the *restriction* $\Delta|_{S'}$ to be the set of all distinct permutations $\gamma|_{S'}$ where $\gamma \in \Delta$. Note that $\Delta|_{S'}$ may contain fewer elements than Δ, because there may be multiple elements of Δ inducing the same permutation on S'. This permutation is only present once in $\Delta|_{S'}$.

The *restricted motion* $m(\Delta)|_{S'}$ is the motion of $\Delta|_{S'}$ seen as a subset of $\mathrm{Sym}_{S'}$.

3.3.1 The motion lemma

In Chapter 1 we already mentioned the a by Russell and Sundaram [24] connecting the motion of a finite graph to its distinguishing number. Let us recall the statement of this result.

Lemma 1.3. *Let G be a finite graph and assume that every non-trivial automorphism moves at least m vertices. If $|\mathrm{Aut}\, G| \leq d^{\frac{m}{2}}$, then G has a distinguishing colouring with d colours.*

This lemma can be seen as the finite analogue of Tucker's conjecture. We outline this connection in the next section. Before that, however, we have a look at its proof as well as several generalisations which can be obtained in a very similar way.

In order to prove Lemma 1.3, Russell and Sundaram [24] first defined the cycle norm of an automorphism γ. For the definition of the cycle norm recall that every permutation can be written as a product of disjoint cycles.

Definition 3.15. Let $G = (V, E)$ be a graph and let γ be an automorphism of G. Assume that

$$\gamma = (v_{11}v_{12} \dots v_{1l_1})(v_{21} \dots v_{2l_2}) \dots (v_{k1} \dots v_{kl_k}),$$

is the decomposition of γ (seen as a permutation of V) into disjoint cycles. The *cycle norm* $\mathrm{cn}(\gamma)$ of γ is defined as

$$\mathrm{cn}(\gamma) = \sum_{i=1}^{k} (l_i - 1).$$

The *cycle norm* $\mathrm{cn}(G)$ *of the graph* G is defined as

$$\mathrm{cn}(G) = \min_{\gamma \in \mathrm{Aut}\,(G) \setminus \{\mathrm{id}\}} \mathrm{cn}(\gamma).$$

Note that in the above definition the graph structure did not play a role. This means that we can define the cycle norm of a permutation of a set S and the cycle norm of a set of such permutations in a completely analogous way.

There is a close relation between the cycle norm of a graph and its motion. Assume that an automorphism γ has cycle norm k. From the definition of $\mathrm{cn}(\gamma)$ it should be obvious that the motion of γ is obtained from the cycle norm by adding the number of non-trivial cycles. This immediately gives

$$\mathrm{cn}(\gamma) + 1 \leq m(\gamma) \leq 2\,\mathrm{cn}(\gamma).$$

Minimising over all $\gamma \in \mathrm{Aut}\, G$ we get

$$\mathrm{cn}(G) + 1 \leq m(G) \leq 2\,\mathrm{cn}(G).$$

Next let us elaborate on the connection between the cycle norm and distinguishing colourings. Let G be a graph, let $\gamma \in \mathrm{Aut}\, G$ and let c be a colouring of G. It is an easy observation that γ preserves c if and only if every cycle of γ is monochromatic. Choose the colouring c randomly by assigning a colour to each vertex uniformly at random, such that the colours of different vertices are independent. Then the probability that each cycle is monochromatic is $d^{-\operatorname{cn}(\gamma)}$.

We now reprove Theorem 2 of [24] with $\geq$ instead of $>$. In fact, the only difference from the original proof is the insertion of the middle term in Equation (3.2) below.

Theorem 3.16. *Let G be a finite graph, and $d^{\operatorname{cn}(G)} \geq |\operatorname{Aut}(G)|$. Then G is d-distinguishable, that is, $D(G) \leq d$.*

Proof. Let c be a random d-colouring of G, the probability distribution being given by selecting the colour of each vertex independently and uniformly in the set $\{1, \dots, d\}$. For a fixed automorphism $\gamma \in \operatorname{Aut}(G) \setminus \{\mathrm{id}\}$ consider the probability that the random colouring c is preserved by γ:

$$\Pr[c\gamma = c] = d^{-\operatorname{cn}(\gamma)} \leq d^{-\operatorname{cn}(G)}. \tag{3.1}$$

Collecting these events yields the inequality

$$\Pr[\exists\gamma \in \operatorname{Aut} G \setminus \{\mathrm{id}\} \colon c\gamma = c] \leq (|\operatorname{Aut}(G)| - 1)d^{-\operatorname{cn}(G)} < |\operatorname{Aut}(G)|d^{-\operatorname{cn}(G)}. \tag{3.2}$$

By hypothesis the last term is at most 1. This implies that the probability that a random colouring is not distinguishing is strictly less than 1, and there exists a distinguishing colouring c. $\square$

Since $m(G) \leq 2\operatorname{cn}(G)$ it is clear that Theorem 3.16 implies Lemma 1.3. We now state some generalisations of Lemma 1.3. In the rest of the thesis we are only concerned with 2-colourings, hence we state them only for $d = 2$. However, the proofs are completely analogous to the above proof, so all of the generalisations also hold for d-colourings.

The first generalisation we would like to mention is obtained by observing that the graph structure did not play a role anywhere in the proof. Hence with the exact same proof we can show an analogous statement for permutation groups.

Lemma 3.17. *Let S be a finite set and let Γ be a group of permutations of S with motion m. Assume that $2^{\frac{m}{2}} \geq |\Gamma|$. Then there is a Γ-distinguishing 2-colouring of S.*

To further generalise the above result, observe that the proof did not depend on the group structure either, that is, we can show the same result for arbitrary sets of permutation using the exact same arguments. We have to be careful about the inequality though, because the identity element need not be contained in the set of permutations. Finally, to get to the most general version of Lemma 1.3, observe that instead of colourings of the whole set it suffices to consider partial colourings.

Lemma 3.18. *Let S be a (possibly infinite) set and let Δ be a set of permutations of S. Let $S' \subset S$ be a finite set that is fixed by every $\gamma \in \Delta$. If*

$$2^{\frac{m(\Delta)|_{S'}}{2}} > \left|\Delta|_{S'}\right|,$$

then there is a partial 2-colouring of S with support S' which breaks Δ.

3.3.2 Tucker's conjecture

In the previous section we showed that there is a connection between the motion of a finite graph and its distinguishing number. An analogous connection for infinite graphs has been conjectured to be true by Tucker [26], as already mentioned in the introduction.

Conjecture 1.2. *Let G be an infinite, connected, locally finite graph with infinite motion. Then there is a distinguishing 2-colouring of G.*

To see the analogy to Lemma 1.3, recall that the only assertion in the condition of the lemma (for $d = 2$) is that

$$|\mathrm{Aut}\, G| \leq 2^{\frac{m(G)}{2}}.$$

Let us take a closer look at the above inequality for locally finite connected graphs with infinite motion. Let $G = (V, E)$ be such a graph. Then the vertex set of G must be countable, and hence

$$|\mathrm{Aut}\, G| \leq |V|^{|V|} = \aleph_0^{\aleph_0}.$$

On the other hand the motion is infinite. Since V is countable, we have $m(G) = \aleph_0$ and the right hand side of the above equation evaluates to

$$2^{\frac{m(G)}{2}} = 2^{\frac{\aleph_0}{2}} = 2^{\aleph_0}.$$

It is a well known fact that $2^{\aleph_0} = \aleph_0^{\aleph_0}$. Thus the inequality in the condition of Lemma 1.3 holds for every countable graph with infinite motion. Hence Conjecture 1.2 can really be seen as an infinite analogue to Lemma 1.3. By completely analogous arguments Conjecture 1.4 can be seen as an infinite analogue to Lemma 3.17.

If the inequality in the condition of the conjecture is strict, then the following theorem of Halin [10], which is independent of the continuum hypothesis, tells us that the automorphism group must be countable.

Theorem 3.19. *Let G be a locally finite graph. Then $|\mathrm{Aut}\, G| < 2^{\aleph_0}$ if and only if there is a finite subset of V whose pointwise stabiliser is trivial.* □

Clearly, if there is such a set then an automorphism is uniquely determined by the image of this set. Since there are only countably many possibilities to map a finite set to a countable set the automorphism group is at most countable.

A similar result holds for closed permutation groups by the following result of Evans [8]. Again this is independent of the continuum hypothesis.

Theorem 3.20. *If Γ and Δ are closed permutation groups on a countable set S and $\Delta \subseteq \Gamma$, then either $|\Gamma : \Delta| = 2^{\aleph_0}$ or Δ contains the pointwise stabiliser of some finite set in Γ.* □

Taking $\Delta = \{\mathrm{id}\}$ in the above theorem, we obtain that a closed permutation group Γ either has cardinality $2^{\aleph_0}$, or there is some finite subset of S whose pointwise stabiliser is trivial. In particular, Theorem 3.19 and all of its implications remain true in the more general setting of closed permutation groups.

The above results also imply that Conjectures 1.2 and 1.4 are true if the inequality is strict. The following theorem has been known for a while. The proof we give here has first appeared in [14].

Theorem 3.21. *Let Γ be a group of permutations of a set S and assume that Γ has infinite motion. If Γ is countable, then there is a Γ-distinguishing 2-colouring of S*

Proof. We inductively construct a distinguishing 2-colouring. For the construction let $(\gamma_i)_{i\in\mathbb{N}}$ be an enumeration of all non-trivial elements of Γ.

Inductively select $s_i \in S$ such that for every $j \leq i$ it holds that $\gamma_j s_j \neq s_i$ and $\gamma_i s_i \neq s_j$. Note that this in particular implies that γ_i is not contained in the stabiliser of s_i. Such an s_i always exist because there are infinitely many elements s such that $\gamma_i s \neq s$ and only finitely many which we are not allowed to choose due to the above restrictions.

Now define a 2-colouring which assigns one colour to all of the s_i and the other colour to the rest of S. This colouring breaks all permutations in Γ because clearly s_i and $\gamma_i s_i$ are assigned different colours for every $i \in \mathbb{N}$. □

Chapter 5 contains an alternative proof of this theorem using probabilistic methods. It is worth noting that the proof does not depend on the group structure. In particular, a similar proof can be given if we replace the group Γ by an arbitrary countable set of permutations.

4 Growth, motion, and distinguishability

Although our graphs are usually infinite, as long as they are locally finite, all balls and spheres of finite radius are finite. Hence the following definition makes sense.

Definition 4.1. Let $G = (V, E)$ be a locally finite graph and let $v_0 \in V$. The *growth function* of G with respect to the base point v_0 is defined as

$$\mathrm{growth}_{v_0}(n) = |B_{v_0}(n)|.$$

While this function is defined for all real values of n it is constant between two consecutive integers. Hence we might as well consider it a function from $\mathbb{N}$ to $\mathbb{N}$. The number of vertices in $B_{v_0}(n)$ is a strictly increasing function of n, because

$$|B_{v_0}(n)| = \sum_{i=0}^{n} |S_{v_0}(i)|$$

and

$$|S_{v_0}(i)| \geq 1.$$

Note that the growth function does not only depend on the graph G, but also on the base point v_0. However, we are not interested in the exact values of the growth function. Instead we consider the growth rate which describes its asymptotics. We use the usual Landau notation to describe the asymptotic behaviour. A sequence $g(n)$ has growth

- $\mathcal{O}(f(n))$ if $g(n)$ is bounded from above by $cf(n)$ for some constant c,
- $\Omega(f(n))$ if $g(n)$ is bounded from below by $cf(n)$ for some constant c,
- $o(f(n))$ if $\frac{g(n)}{f(n)}$ converges to zero, and
- $\omega(f(n))$ if $\frac{f(n)}{g(n)}$ converges to zero.

Definition 4.2. Let $G = (V, E)$ be a graph and let $v_0 \in V$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a monotonically increasing function. We say that the *growth of* G is $\mathcal{O}(f(n))$ if

$$\mathrm{growth}_{v_0}(n) = \mathcal{O}(f(n)).$$

Similarly we can define graphs of growth $o(f(n))$, $\Omega(f(n))$ and $\omega(f(n))$.

Under mild restrictions on the function f it is relatively easy to see that the asymptotics of the growth of G do not depend on the base point.

Proposition 4.3. *Let $G = (V, E)$ be a graph and let $f\colon \mathbb{N} \to \mathbb{N}$ be a monotonically increasing function. If there is a function $g\colon \mathbb{N} \to \mathbb{N}$ such that*

$$\forall n, k\colon f(n+k) \leq f(n)g(k)$$

then G has growth $\mathcal{O}(f(n))$ for one base point v_0 if and only if it has growth $\mathcal{O}(f(n))$ for every base point.

Proof. Let $v_0, v_1 \in V$ and assume that $\text{growth}_{v_0}(n) = \mathcal{O}(f(n))$. Let $k = d(v_0, v_1)$. Then clearly

$$B_{v_1}(n) \subseteq B_{v_0}(n+k).$$

From this we infer that

$$\text{growth}_{v_1}(n) \leq \text{growth}_{v_0}(n+k) \leq cf(n+k) \leq cg(k)f(n) = \mathcal{O}(f(n))$$

because $g(k)$ is just an additional multiplicative constant. □

Analogous proofs can be given for growth $o(f(n))$, $\Omega(f(n))$ and $\omega(f(n))$. The following example shows that the existence of the function g is vital for the validity of the above proposition.

Let G be a graph with vertex set $V = V_1 \uplus V_2 \uplus V_3 \uplus \ldots$ such that $|V_i| = (i+1)! - i!$ and all possible edges between V_i and V_{i+1} for every $i \in \mathbb{N}$. Clearly V_1 consists of a single vertex v. It is easy to check that it is not possible to find a function g as in the proposition. Furthermore

$$\text{growth}_v(n) = n!$$

but for $d(v, w) = k$ and $n > k$ it holds that

$$\text{growth}_w(n) = (n+k)! \approx n^k n!.$$

So for this particular graph growth does depend on the chosen base point, even if we are only interested in its asymptotics.

However, for all growth functions f considered in this thesis there is a function g as asserted in Proposition 4.3. In particular, the asymptotic behaviour of the growth function does not depend on the base point, and it makes sense to speak about the growth of the graph rather than its growth with respect to a base point.

Definition 4.4. Let G be a graph. We say that G has *polynomial growth* if the growth of G is $\mathcal{O}(n^c)$ for some constant c. If $c = 1$ then G has *linear growth*, if $c = 2$ it has *quadratic growth*.

The graph G has *exponential growth* if its growth is $\Omega(c^n)$ for some constant c.

Finally, by a graph with *intermediate growth* we mean a graph whose growth is superpolynomial, but still not exponential, that is, the growth is $\omega(n^c)$ and $o(c^n)$ for every constant $c > 1$.

4.1 Linear growth

In this section we discuss a result due to Imrich et al. [14] which shows that Conjecture 1.2 is true for graphs with linear growth. We include a proof not only for the sake of completeness but also because some of the core ideas turn out to be useful in the proofs of the more sophisticated results in the following sections.

Theorem 4.5. *Let $G = (V, E)$ be a locally finite, connected graph with linear growth and infinite motion. Then G is 2-distinguishable.*

The following lemma is the key to the proof of the above theorem and probably the most important observation in this section.

Lemma 4.6. *Let G be a graph with infinite motion, $\gamma \in \operatorname{Aut} G$. Denote by $V_{\mathit{fix}} \subseteq V$ the set of fixed points of γ. Then the graph $G - V_{\mathit{fix}}$, which is obtained from G by removing V_{fix} and all incident edges, has only infinite components.*

Proof. If there were a finite component C then we could define an automorphism γ' which coincides with γ on this component and fixes every vertex $v \notin C$. This automorphism is easily seen to have finite motion, which contradicts G having infinite motion. □

Let us have a look at some implications of this result.

Corollary 4.7. *Let G be a graph with infinite motion, let $V' \subseteq V$ be a finite set of vertices, and denote by $\partial V'$ the set of vertices in $V \setminus V'$ which have a neighbour in V'. If an automorphism γ fixes $\partial V'$ pointwise, then it must also fix V' pointwise.*

Proof. If this was not the case then V' would be a finite component of $G - V_{\text{fix}}$. □

The special case where $V' = B_{v_0}(n)$ and hence $\partial V' = S_{v_0}(n+1)$ is of particular interest. If additionally v_0 is a fixed point of γ, we get the following result.

Corollary 4.8. *Let $G = (V, E)$ be an infinite, locally finite, connected graph with infinite motion. Let $\gamma, \gamma' \in \operatorname{Aut} G$ and assume that there is $v_0 \in V$ such that $\gamma(v_0) = v_0$. Then*

1. *for every $i \in \mathbb{N}$ it holds that γ fixes $S_{v_0}(i)$ as a set,*
2. *$m(\gamma)|_{S_{v_0}(i)} > 0$ implies that $\forall j > i \colon m(\gamma)|_{S_{v_0}(j)} > 0$,*
3. *$\gamma|_{S_{v_0}(j)} = \gamma'|_{S_{v_0}(j)}$ if and only if $\gamma|_{B_{v_0}(j)} = \gamma'|_{B_{v_0}(j)}$.* □

Proof. The first property follows from the fact that every automorphism of a graph preserves all distances between vertices.

The second property immediately follows from Corollary 4.7 with $V' = B_{v_0}(j-1)$ and $\partial V' = S_{v_0}(j)$.

For the third property note that $S_{v_0}(j) \subseteq B_{v_0}(j)$. Thus it is clear that if γ and γ' coincide on the ball then they must also coincide on the sphere. Conversely assume that there were two automorphisms γ and γ' which coincide on the sphere but not on the ball. Then $\gamma^{-1} \circ \gamma'$ acts trivially on $S_{v_0}(j)$ but non-trivially on $B_{v_0}(j)$. Hence there must be some $i < j$ such that $\gamma^{-1} \circ \gamma'$ acts non-trivially on $S_{v_0}(i)$. This contradicts the second statement of the corollary. □

We are now ready to prove the main result of this section.

Proof of Theorem 4.5. We claim that the stabiliser of every vertex $v \in V$ is finite. Since the vertex set is countable, the stabiliser can only have countably many conjugacy classes and hence the automorphism group itself must be countable. So if the point stabilisers are finite, we can conclude from Theorem 3.21 that G is 2-distinguishable.

In order to prove the claim recall that linear growth means that the size of $B_v(n)$ is bounded from above by cn for some constant c. Since $|B_v(n)| = \sum_{i=0}^{n} |S_v(i)|$ there must be infinitely many n such that $|S_v(n)| \leq c$.

Assume that there are more than $c!$ different automorphisms in the stabiliser of v and select a set of $c! + 1$ such automorphisms.All of them fix every $S_v(n)$ setwise and by the pigeonhole principle two of them must induce the same permutation on $S_v(n)$ whenever $|S_v(n)| \leq c$. This implies that there must be two automorphisms in the set whose actions on $S_v(n)$ coincide for infinitely many values of n. By Corollary 4.8 these two automorphisms must coincide on every $B_v(n)$ and hence they cannot be distinct. □

Note that in Lemma 4.6 we do not require that the graph is connected or locally finite, in fact it may even be uncountable. If G is locally finite and connected then we can say even more about $G - V_{\text{fix}}$.

A *ray* is a one sided infinite path, that is, an infinite sequence $(v_n)_{n\in\mathbb{N}}$ of vertices where v_n is connected to v_{n+1} by an edge. It is a well known fact that any connected, locally finite, infinite graph contains a ray. Hence it is clear that every component of $G - V_{\text{fix}}$ must contain a ray. The following result says that if $V_{\text{fix}} \neq \emptyset$ then we can even find a ray which is mapped to a disjoint ray in every component of $G - V_{\text{fix}}$.

Lemma 4.9. *Let G be a connected locally finite graph with infinite motion, let $\gamma \in \operatorname{Aut} G$ and assume that there is a vertex $v \in V$ such that $\gamma v = v$. Then every component of $G - V_{\mathit{fix}}$ contains a ray R which is mapped to a disjoint ray R'.*

Proof. Let C be a component of $G - V_{\text{fix}}$. First note that there must be a ray in C since G is locally finite and C is infinite by Lemma 4.6.

Any two vertices in C are connected by a path which does not use any vertex in V_{fix}. Clearly the image of such a path is again a path which does not contain any vertex in V_{fix}. Hence if some vertex in C has an image outside of C then so do all vertices of C. So in this case each ray in C is mapped to a disjoint ray.

Now assume that C is fixed by γ. Choose a fixed point v_0 of γ which is adjacent to some vertex in C. Note that such a vertex v_0 must exist because there is a path connecting C to v. Consider the graph G' which is obtained from C by adding v_0 and all edges between v_0 and C.

Using breadth-first-search, construct a spanning tree T of G' with root v_0. Note that γ acts on G' as an automorphism. Since every automorphism is an isometry, for every $w \in C$ the vertices w and γw have the same distance from v_0 in G'. Thus they also have the same distance from v_0 in T.

Choose a ray R in T which starts at a neighbour of v_0 but does not use v_0. Then all vertices in R have different distances from v_0. Since no $w \in R$ is mapped to itself it is clear that R must be mapped to a disjoint ray. □

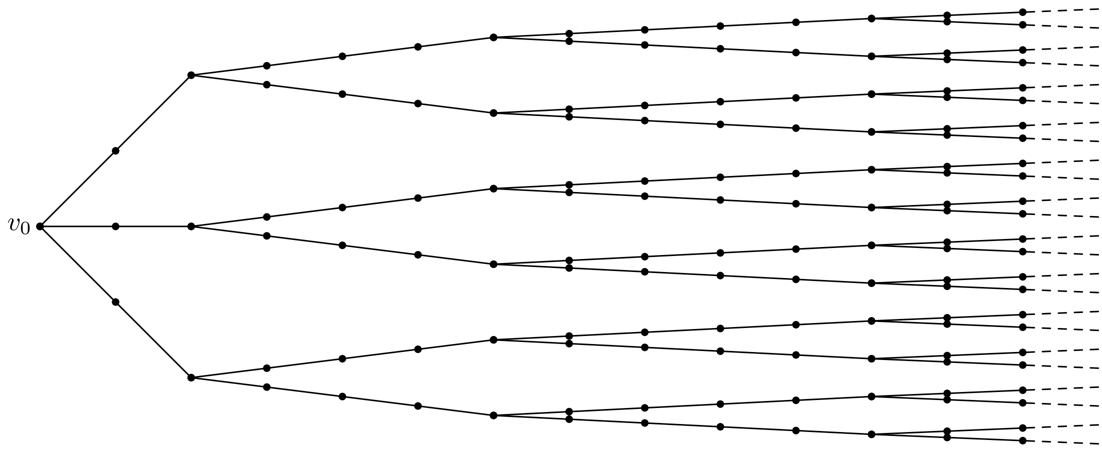

Figure 3: Replacing the egdes of T_3 by paths. The lengths of the paths are determined by the desired growth function f and the distance from the root v_0.

In the last part of this chapter we discuss why the proof of Theorem 4.5 does not work once the growth becomes non-linear. Some of the key ideas (most prominently Lemma 4.6 and its corollaries) still work for graphs of larger growth. However, the following example from [5] shows that even if the growth is only slightly non-linear, the vertex stabilisers can become uncountable, causing the proof to break down.

Theorem 4.10. *Let $f\colon \mathbb{N} \to \mathbb{N}$ be a strictly increasing function of growth $\omega(n)$. Then there exists an infinite, locally finite, connected graph G with uncountable automorphism group and infinite motion whose growth is $\mathcal{O}(f)$.*

Remark 4.11. Since $\operatorname{Aut} G$ is uncountable, Theorem 4.5 implies that G cannot have linear growth.

Proof. We construct G from the 3-regular tree T, that is, the tree in which every vertex has degree 3. First, choose an arbitrary vertex v_0 of T. Our strategy is to replace the edges of T by paths such that $B_{v_0}(n)$ contains at most $6f(n)$ vertices. To get an idea of the construction see Figure 3.

Since $f(n) = \omega(n)$, it is clear that for every $i \in \mathbb{N}$ there is an integer n_i such that

$$\forall n \geq n_i \colon f(n) \geq 2^i n.$$

Furthermore, $n_0 = 0$ because f is strictly increasing.

We obtain G by replacing every edge of T by a path. The length of the path is determined by the distance from the edge to v_0: if x is the endpoint of the edge lying closer to v_0 and $d(v_0, x) = i$, then we replace the edge by a path of length $n_{i+1} - n_i$.

Now let $n \in \mathbb{N}$. There is some i such that $n_i \leq n < n_{i+1}$. By our construction it is clear that for every $k \leq n$ the sphere $S_{v_0}(k)$ in G contains at most $3 \cdot 2^i$ vertices. We conclude that

$$|B_{v_0}(n)| = \sum_{k=0}^{n} |S_{v_0}(k)| \leq 3 \cdot 2^i \cdot (n+1) \leq 6 \cdot 2^i \cdot n \leq 6f(n).$$

Every automorphism of T that fixes v_0 induces an automorphism of G. This correspondence is bijective unless all the path lengths in the construction were equal. But in this case $\operatorname{Aut} G = \operatorname{Aut} T$. Thus, $\operatorname{Aut} G$ is uncountable. Furthermore, G inherits infinite motion from T. □

4.2 Non-linear growth

Although we cannot assume that the automorphism groups of our graphs are countable, we prove that infinite, locally finite, connected graphs with infinite motion and non-linear, but moderate, growth are still 2-distinguishable, that is, they have distinguishing number either 1 or 2. All results in this section have appeared in [5], the main result being the following extension of Theorem 4.5 to graphs of almost quadratic growth.

Theorem 4.12. *Let G be a graph with growth $o(\frac{n^2}{\log n})$. Then G is 2-distinguishable.*

The proof of this theorem consists of two stages. First, in Lemma 4.13 we show how to colour a part of the vertices in order to break all automorphisms that move a distinguished vertex v_0.

In the second step we need to break the remaining automorphisms by colouring the rest of the vertices. Lemma 4.17 shows how to colour some of the remaining vertices in order to break more automorphisms. Iteration of this procedure yields a distinguishing colouring.

So let us start by constructing a partial colouring whose stabiliser is contained in the stabiliser of a vertex v_0.

Lemma 4.13. *Let $G = (V, E)$ be an infinite, locally finite, connected graph with infinite motion and $v_0 \in V$. Then, for every $k \in \mathbb{N}$, one can 2-colour all vertices in $B_{v_0}(k+3)$ and $S_{v_0}(\lambda k + 4)$, $\lambda \in \mathbb{N}$, such that this partial colouring breaks all automorphisms which move v_0.*

Proof. If $k = 1$, then we colour v_0 black and all $v \in V \setminus \{v_0\}$ white, whence all automorphisms that move v_0 are broken. So, let $k \geq 2$. First, we colour all vertices in $S_{v_0}(0)$, $S_{v_0}(1)$, and $S_{v_0}(k+2)$ black and the remaining vertices in $B_{v_0}(k+3)$ white. Moreover, we colour all vertices in $S_{v_0}(\lambda k + 4)$, $\lambda \in \mathbb{N}$, black and claim that, no matter how we colour the remaining vertices, v_0 is the only black vertex that has only black neighbours and only white vertices at distance $r \in \{2, 3, \ldots, k+1\}$, see Figure 4.

It clearly follows from this claim that this colouring breaks every automorphism that moves v_0. It only remains to verify the claim.

Consider a vertex $v \in V \setminus \{v_0\}$. If v is not in $S_{v_0}(1)$, then it is easy to see that v cannot have the aforementioned properties. So, let v be in $S_{v_0}(1)$ and assume it has only black neighbours and only white vertices at distance 2. Then it cannot be neighbour to any vertex in $S_{v_0}(2)$, but must be neighbour to all vertices in $B_{v_0}(1)$ except itself. Therefore, the transposition of the vertices v and v_0 is a non-trivial automorphism of G with finite support. Since G has infinite motion, this is not possible. □

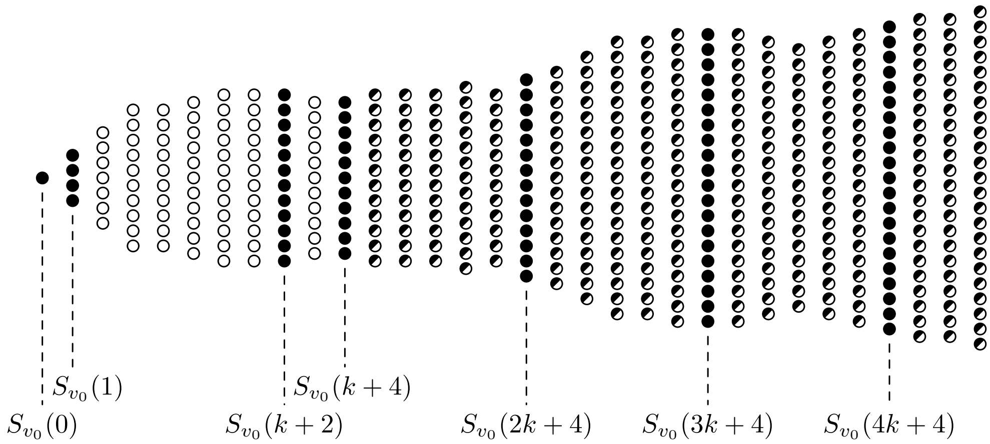

Figure 4: Breaking all automorphisms that move v_0. Note that there are still many vertices left uncoloured (drawn half black, half white). These vertices are later used to break the automorphisms that fix v_0.

Before we proceed to the second step we need some auxiliary results on colourings. The following implication of Lagrange's theorem is well known.

Lemma 4.14. *Let Γ be a finite group acting on a set S. If a colouring of S breaks some element of Γ, then it breaks at least half of the elements of Γ.*

Proof. The elements of Γ that preserve a given colouring form a subgroup. If some element of Γ is broken, then this subgroup is proper and thus, by Lagrange's theorem, cannot contain more than half of the elements of Γ. □

If the action is non-trivial, then we can always find a colouring that breaks at least one element. Hence, we have the following result.

Lemma 4.15. *If Γ is a finite group acting non-trivially on a set S, then there exists a 2-colouring of S that breaks at least half of the elements of Γ.*

The proof of Lemma 4.15 is based on the fact that Γ is a group. But a very similar result holds for any finite family of non-trivial automorphisms, as the following lemma shows. Note that we do not only drop the group structure but also allow elements to appear more than once in the family.

Lemma 4.16. *Let $G = (V, E)$ be a finite graph. If Δ is a finite set equipped with a mapping $\phi : \Delta \to \mathrm{Aut}\,(G) \setminus \{\mathrm{id}\}$, then there exists a 2-colouring of G that breaks $\phi(\delta)$ for at least half of the elements $\delta \in \Delta$.*

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$. For every $k \in \{1, 2, \ldots, n\}$, let Δ_k be the set of all $\delta \in \Delta$ with $\mathrm{supp}(\phi(\delta)) \subseteq \{v_1, v_2, \ldots, v_k\}$. We show by induction that the assertion holds for all Δ_k and, in particular, for Δ.

Because Δ_1 is the empty set, the assertion is true for Δ_1. Suppose it is true for Δ_{k-1}. Then we can choose a 2-colouring of G that breaks $\phi(\delta)$ for at least half of the elements of Δ_{k-1}. This remains true, even when we change the colour of v_k.

Note that, for every $\delta \in \Delta_k \setminus \Delta_{k-1}$, $\phi(\delta)$ either maps v_k into a white vertex in $\{v_1, v_2, \ldots, v_{k-1}\}$ or into a black vertex in $\{v_1, v_2, \ldots, v_{k-1}\}$. We colour v_k with the colour which appears less frequently as colour of $\phi(\delta)v_k$.

By construction this 2-colouring also breaks $\phi(\delta)$ for at least half of the elements of $\Delta_k \setminus \Delta_{k-1}$ and, hence, for at least half of the elements of Δ_k. □

We can use Lemma 4.16 to break some of the automorphisms that preserve the partial colouring of Lemma 4.13 in the following way.

Lemma 4.17. *Let $G = (V, E)$ be an infinite, locally finite, connected graph with infinite motion and $v_0 \in V$. Moreover, let $\varepsilon > 0$. Then there exists $k \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$ and for every $n \in \mathbb{N}$ that is sufficiently large and fulfils*

$$|S_{v_0}(n)| \leq \frac{n}{(1+\varepsilon)\log n},$$

one can 2-colour all vertices in $S_{v_0}(m+1), S_{v_0}(m+2), \ldots, S_{v_0}(n)$, but not those in $S_{v_0}(\lambda k + 4)$, $\lambda \in \mathbb{N}$, such that all automorphisms are broken that fix v_0 and act non-trivially on $B_{v_0}(m)$.

Figure 5 illustrates which vertices are actually used for the colouring.

Proof. First, choose an integer $k > 1 + \frac{1}{\varepsilon}$. Then

$$\frac{k-1}{k} > \frac{1}{1+\varepsilon}.$$

Let $m \in \mathbb{N}$. Then there is an $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0 \colon (n-m) \cdot \frac{k-1}{k} \geq n \cdot \frac{1}{1+\varepsilon} + 1.$$

Choose $n \geq n_0$ such that n fulfils the inequality in the condition of the lemma. Then, the number of spheres $S_{v_0}(m+1), S_{v_0}(m+2), \ldots, S_{v_0}(n)$ that are not of the type $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, is at least

$$\left\lfloor (n-m) \cdot \frac{k-1}{k} \right\rfloor \geq \left\lfloor n \cdot \frac{1}{1+\varepsilon} + 1 \right\rfloor > \frac{n}{1+\varepsilon}.$$

Our goal is to 2-colour the vertices in these spheres in order to break all automorphisms that fix v_0 and act non-trivially on $B_{v_0}(m)$.

Every automorphism in the stabiliser of v_0 fixes $B_{v_0}(n)$ setwise. Let Δ be the group of permutations of $B_{v_0}(n)$ obtained by restricting the stabiliser of v_0 to this set. Since

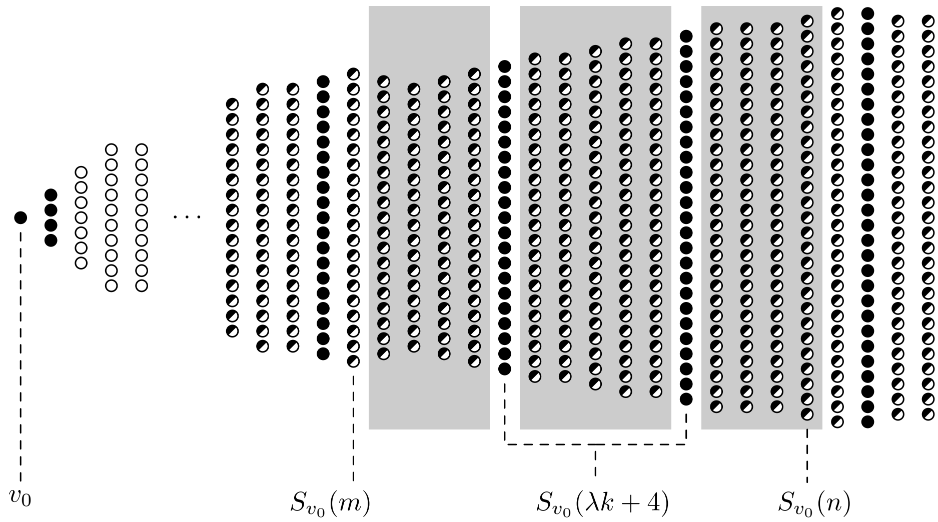

Figure 5: Breaking automorphisms that fix v_0. The grey boxes indicate, which vertices are coloured in order to break all automorphisms that fix v_0 and act non-trivially on $B_{v_0}(m)$.

by Corollary 4.8 two elements of Δ are equal if and only if their action on $S_{v_0}(m)$ is the same, we get the following bound on the size of Δ:

$$\begin{aligned}|S_{v_0}(n)|! &\leq |S_{v_0}(n)|^{|S_{v_0}(n)|-1}\\ &\leq \left(\frac{n}{(1+\varepsilon)\log n}\right)^{\frac{n}{(1+\varepsilon)\log n}-1}\\ &\leq n^{\frac{n}{(1+\varepsilon)\log n}-1}\\ &= 2^{\left(\frac{n}{(1+\varepsilon)\log n}-1\right)\log n}\\ &\leq 2^{\frac{n}{1+\varepsilon}-1}.\end{aligned}$$

It is clear that, if an element $\sigma \in \Delta$ acting non-trivially on $B_{v_0}(m)$ is broken by a suitable 2-colouring of some spheres in $B_{v_0}(n)$, then all $\gamma \in \operatorname{Aut} G$ with $\gamma|_{B_{v_0}(n)} = \sigma$ are broken at once. Thus breaking all $\sigma \in \Delta$ that act non-trivially on $B_{v_0}(m)$ by a suitable 2-colouring of some spheres in $B_{v_0}(n)$ will break all $\gamma \in \operatorname{Aut} G$ that fix v_0 and act non-trivially on $B_{v_0}(m)$.

Note that by Corollary 4.8 any element $\sigma \in \Delta$ that acts non-trivially on the ball $B_{v_0}(m)$, also acts non-trivially on every sphere $S_{v_0}(m+1), \ldots, S_{v_0}(n)$. This implies that we can break σ by breaking the action of σ on any one of the spheres $S_{v_0}(m+1), \ldots, S_{v_0}(n)$.

Consider the subset $\Sigma \subseteq \Delta$ of all elements that act non-trivially on $B_{v_0}(m)$. As already remarked, every $\sigma \in \Sigma$ acts non-trivially on each sphere $S_{v_0}(m+1), \ldots, S_{v_0}(n)$. Hence, we can apply Lemma 4.16 to break at least half of the elements of Σ by a suitable colouring of $S_{v_0}(m+1)$. What remains unbroken is a subset $\Sigma' \subseteq \Sigma$ of cardinality at most $\frac{|\Sigma|}{2}$. Now, we proceed to the next sphere. We can break at least half of the elements of Σ' by a suitable colouring of $S_{v_0}(m+2)$. What still remains unbroken, is a subset $\Sigma'' \subseteq \Sigma$ of cardinality at most $\frac{|\Sigma|}{4}$.

Iterating the procedure, but avoiding spheres of the type $S_{v_0}(\lambda k+4)$, we end up with the empty subset $\emptyset \subseteq \Sigma$ after at most $\log|\Sigma|+1 \leq \log|\Delta|+1 \leq \frac{n}{1+\varepsilon}$ steps. This is less than the number of spheres not of the type $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, between $S_{v_0}(m+1)$ and $S_{v_0}(n)$. Thus, we remain within the ball $B_{v_0}(n)$. Hence, we have broken all $\sigma \in \Sigma$ and, therefore, all $\gamma \in <(\operatorname{Aut} G)_{v_0}$ that act non-trivially on $B_{v_0}(m)$. □

We now apply Lemma 4.17 iteratively to break all automorphisms that fix v_0, and hence also all automorphisms that preserve the partial colouring given by Lemma 4.13.

Theorem 4.18. *Let $G=(V,E)$ be an infinite, locally finite, connected graph with infinite motion and $v_0 \in V$. Moreover, let $\varepsilon > 0$. If there exist infinitely many $n \in \mathbb{N}$ such that*

$$|S_{v_0}(n)| \leq \frac{n}{(1+\varepsilon)\log n},$$

then G is 2-distinguishable.

Proof. Consider the integer k provided by Lemma 4.17. First, we use Lemma 4.13 to 2-colour all vertices in $B_{v_0}(k+3)$ and in $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, such that this partial colouring breaks all automorphisms that do not fix v_0.

Let $m_1 = k+3$. Among all $n \in \mathbb{N}$ that satisfy the inequality in the condition of the theorem we choose a number $n_1 \in \mathbb{N}$ that is larger than m_1 and sufficiently large to apply Lemma 4.17. Hence, we can 2-colour all vertices in $S_{v_0}(m_1+1), S_{v_0}(m_1+2), \ldots, S_{v_0}(n_1)$, except those in $S_{v_0}(\lambda k+4)$, $\lambda \in \mathbb{N}$, in order to break all automorphisms that fix v_0 and act non-trivially on $B_{v_0}(m_1)$. Next, let $m_2 = n_1$ and choose an $n_2 \in \mathbb{N}$ to apply Lemma 4.17 again. Iteration of this procedure yields a 2-colouring of G.

If an automorphism $\gamma \neq \mathrm{id}$ moves v_0, then our colouring breaks γ by Lemma 4.13. If it fixes v_0, consider a vertex v with $\gamma v \neq v$. Since G is connected and $m_1 < m_2 < m_3 < \ldots$, there is an $i \in \mathbb{N}$ such that v is contained in $B_{v_0}(m_i)$. Hence, γ acts non-trivially on $B_{v_0}(m_i)$ and is again broken by our colouring. □

Finally, we have the following result which is clearly a strengthening of Theorem 4.12 (and hence also implies the theorem) because under the conditions asserted in Theorem 4.12 the inequality in the corollary below is true infinitely often for every choice of v_0.

Corollary 4.19. *Let $G=(V,E)$ be an infinite, locally finite, connected graph with infinite motion and $v_0 \in V$. Moreover, let $\varepsilon > 0$. If there exist infinitely many $n \in \mathbb{N}$ such that*

$$|B_{v_0}(n)| \leq \frac{n^2}{(2+\varepsilon)\log_2 n},$$

then the G is 2-distinguishable.

Proof. Let $n_1 < n_2 < n_3 < \ldots$ be an infinite sequence of integers that fulfil the inequality. Note that, for every $k \in \mathbb{N}$,

$$\sum_{i=1}^{n_k} \frac{i}{(1+\frac{\varepsilon}{2})\log i} > \frac{{n_k}^2}{(2+\varepsilon)\log n_k} \geq |B_{v_0}(n_k)| > \sum_{i=1}^{n_k} |S_{v_0}(i)| \,.$$

Since

$$\lim_{k\to\infty}\left(\left(\sum_{i=1}^{n_k} \frac{i}{(1+\frac{\varepsilon}{2})\log i}\right) - \frac{{n_k}^2}{(2+\varepsilon)\log n_k}\right) = \infty,$$

we infer that

$$\lim_{k\to\infty}\sum_{i=1}^{n_k}\left(\frac{i}{(1+\frac{\varepsilon}{2})\log_2 i} - |S_{v_0}(i)|\right) = \infty,$$

and that, for infinitely many $i \in \mathbb{N}$,

$$|S_{v_0}(i)| < \frac{i}{(1+\frac{\varepsilon}{2})\log i} \,.$$

Hence, we can apply Theorem 4.18 to show that G is 2-distinguishable. □

4.3 Intermediate growth

In this section we improve Theorem 4.12 even further. This yields a result which is currently the strongest known growth condition for the validity of Conjecture 1.2.

Theorem 4.20. *Let G be a connected, locally finite graph with infinite motion and growth $\mathcal{O}\left(2^{(1-\varepsilon)\frac{\sqrt{n}}{2}}\right)$. Then G is 2-distinguishable.*

Before proving Theorem 4.20 we would like to provide a sketch of the proof to explain the main ideas some of which may seem familiar from the previous section.

By Lemma 4.13 we can assume that there is a vertex v_0 which is fixed by every automorphism that we still need to break. By Corollary 4.8 every such automorphism fixes every sphere $S_{v_0}(i)$ as a set, so it makes sense to speak of restricted motion.

Now assume that we would like to break the set Δ of all automorphisms that act non-trivially on $S_{v_0}(m)$. We know by Corollary 4.8 that every $\gamma \in \Delta$ also acts non-trivially on every higher sphere. We choose k "large enough" (we will specify later, how large it must be) and split up the set of spheres $S_{v_0}(m+1), \ldots, S_{v_0}(m+k)$ in some small sets P_i and a remainder set P_r. Following a suggestion of Imrich we partition Δ into several sets Δ_i of automorphisms whose motion on one of the spheres $S_{v_0}(m+1), \ldots, S_{v_0}(m+k)$ is small and a remainder set Δ_r in which every automorphism has large restricted motion on each of those spheres.

Since the cardinality of the sets Δ_i is small, we can apply Lemma 3.18 to break all of Δ_i by a colouring of P_i although the motion of the elements of Δ_i may be small. Similarly we can break all automorphisms in Δ_r by a colouring of P_r since the motion is large.

Having broken all automorphisms in Δ we proceed inductively, breaking all automorphisms which act non-trivially on $S_{v_0}(m+k)$. In the limit we obtain a colouring which breaks every non-trivial automorphism because every such automorphism has to act non-trivially on some sphere.

We now turn to a detailed proof of Theorem 4.20. We will be using the following slightly weaker version of Lemma 4.13.

Lemma 4.21. *Let $G = (V, E)$ be an infinite, locally finite, connected graph with infinite motion, $v_0 \in V$. For every $\delta > 0$ there is a partial colouring c of the vertices of G with the following properties:*

1. *c is Δ-distinguishing for $\Delta = \{\gamma \in \operatorname{Aut} G \mid \gamma(v_0) \neq v_0\}$.*
2. *There is k_0 such that less than δk of the spheres $S_{v_0}(m+1), \dots, S_{v_0}(m+k)$ are coloured for every $k > k_0$ and every $m \in \mathbb{N}$.*

Proof of Theorem 4.20. First of all apply Lemma 4.21 with $\delta = \frac{\varepsilon}{2}$ and an arbitrarily chosen vertex v_0. This gives a colouring of a small fraction of the spheres which breaks all automorphisms that do not fix v_0. Recall from the statement of the theorem that $\delta > 0$ is arbitrary, hence we can assume $0 < \varepsilon < 1$. As mentioned before, every unbroken automorphism must fix every sphere $S_{v_0}(i)$ as a set.

Now assume that all spheres up to $S_{v_0}(m)$ have already been coloured while $S_{v_0}(m+1)$ is still uncoloured. We know that there is a constant c such that for large n

$$|B_{v_0}(n)| \leq c\, 2^{(1-\varepsilon)\frac{\sqrt{n}}{2}}.$$

By increasing the constant c we can guarantee that this inequality holds for every n. Next note that

$$\sqrt{m+k} \leq \sqrt{m} + \sqrt{k}$$

and hence

$$\begin{aligned} |B_{v_0}(m+k)| &\leq c\, 2^{(1-\varepsilon)\frac{\sqrt{m+k}}{2}} \\ &\leq c\, 2^{(1-\varepsilon)\frac{\sqrt{m}}{2}}\, 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}} \\ &= \tilde{c}\, 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}}, \end{aligned}$$

where $\tilde{c}$ depends on c and m. Note that this implies

$$|S_{v_0}(i)| < \tilde{c}\, 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}} \tag{4.1}$$

for every $i \leq m+k$.

Now choose k larger than the value k_0 given by Lemma 4.21 and large enough that each of the following inequalities holds:

$$\log \tilde{c} < \frac{\varepsilon\sqrt{k}}{8}, \tag{4.2}$$

$$\log k < \frac{\varepsilon\sqrt{k}}{8}, \tag{4.3}$$

$$4\sqrt{k} < \frac{1}{2}\,\varepsilon\left(1-\frac{\varepsilon}{2}\right)k, \tag{4.4}$$

$$\tilde{c}\,\frac{\sqrt{k}}{2} < \frac{\varepsilon}{4}\,k. \tag{4.5}$$

These inequalities are by no means independent. For example it is easy to see that if $\tilde{c}$ is large (which usually is the case) then (4.5) implies (4.2) and (4.4). However, we need all four inequalities in the proof so we might as well explicitly require them.

Next consider the spheres $S_{v_0}(m+1), \ldots, S_{v_0}(m+k)$. We know that at least $(1-\frac{\varepsilon}{2})k$ of these spheres are still uncoloured, denote those spheres by $S_1, \ldots, S_l$ ordered in a way that S_i lies closer to v_0 than S_{i+1}.

Define

$$\kappa = \left\lceil 2\sqrt{k}\left(1-\frac{\varepsilon}{2}\right)\right\rceil,$$

$$r = \left\lceil (1-\varepsilon)\frac{\sqrt{k}}{2}\right\rceil + 1.$$

We now show that it is possible to split up the spheres $S_1, \ldots S_l$ into r sets such that the first $r-1$ sets each contain κ spheres and the last set still contains $\mathcal{O}(k)$ spheres.

For $1 \leq i \leq r-1$ let P_i be the set of vertices contained in $S_{(i-1)\kappa+1}, \ldots, S_{i\kappa}$. The vertices contained in $S_{(r-1)\kappa}, \ldots, S_l$ are collected in the set P_r. Obviously P_i contains κ spheres for $i < r$. Let us check how many spheres there are in P_r:

$$\begin{aligned}
l - \sum_{i=1}^{r-1} \kappa &\geq \left(1-\frac{\varepsilon}{2}\right)k - \kappa(r-1) \\
&\geq \left(1-\frac{\varepsilon}{2}\right)k - \left(2\sqrt{k}\left(1-\frac{\varepsilon}{2}\right)+1\right)\left((1-\varepsilon)\frac{\sqrt{k}}{2}+1\right) \\
&= \left(1-\frac{\varepsilon}{2}\right)k - \left(\left(1-\frac{\varepsilon}{2}\right)(1-\varepsilon)k + 2\sqrt{k}\left(1-\frac{\varepsilon}{2}\right) + (1-\varepsilon)\frac{\sqrt{k}}{2}+1\right) \\
&\geq \varepsilon\left(1-\frac{\varepsilon}{2}\right)k - \left(2+\frac{1}{2}+1\right)\sqrt{k} \\
&\geq \varepsilon\left(1-\frac{\varepsilon}{2}\right)k - 4\sqrt{k} \\
&> \frac{\varepsilon}{2}\left(1-\frac{\varepsilon}{2}\right)k.
\end{aligned}$$

The last inequality follows from (4.4). So altogether we have partitioned the spheres $S_1, \dots, S_l$ into $r-1$ sets of κ spheres and a set of more than $\frac{\varepsilon}{2}\left(1-\frac{\varepsilon}{2}\right)k$ spheres.

Next we would like to partition the set Δ of automorphisms that act non-trivially on $S_{v_0}(m)$ into sets Δ_i such that

$$m(\Delta_i)|_{P_i} > 2\log|(\Delta_i|_{P_i})|\,.$$

This enables us to apply Lemma 3.18 to break all permutations in Δ_i by a colouring of the set P_i. If we colour every P_i according to this colouring, we obtain a partial colouring of G which breaks every automorphism that acts non-trivially on $S_{v_0}(m)$.

In order to define the sets Δ_i, let

$$\Delta_i' = \left\{\gamma \in \Delta \mid \exists S_j \subseteq P_{i'}, r \geq i' > i \text{ such that } m(\gamma)|_{S_j} \leq 2^i\right\}.$$

In words, Δ_i' contains all automorphisms γ which move at most 2^i vertices in some sphere that lies above P_i. Define $\Delta_i = \Delta_i' \setminus \Delta_{i-1}'$ for $1 \leq i < r$ and $\Delta_r = \Delta \setminus \bigcup_{i<r}\Delta_i$ where $\Delta_0' = \emptyset$. Note that for $S_j \subseteq P_i$ and $\gamma \in \Delta_i$ it holds that

$$m(\gamma)|_{S_j} > 2^{i-1}$$

because otherwise γ would be contained in Δ_{i-1}'.

Now that we have partitioned both the uncoloured spheres and the automorphisms that we wish to break, let us check if we can apply Lemma 3.18 to the sets Δ_i and P_i. We establish an upper bound for $|(\Delta_i|_{P_i})|$ and a lower bound for the restricted motion of an automorphism $\gamma \in \Delta_i$ on P_i.

Clearly $|(\Delta_i|_{P_i})| \leq |(\Delta_i'|_{P_i})|$. First we consider the case $i < r$. The case $i = r$ is treated later.

To estimate the cardinality of $\Delta_i'|_{P_i}$, observe that by Corollary 4.8 every permutation in a sphere S_j in $P_{i+1}, \dots, P_r$ induces a unique permutation on P_i. Hence we only need to count the permutations which move at most 2^i elements in one of these spheres. There are at most k such spheres and the cardinality of each of them is bounded from above by $\left\lfloor \tilde{c}2^{(1-\varepsilon)\frac{\sqrt{k}}{2}} \right\rfloor$ according to (4.1). Thus, we get the following estimate:

$$\begin{aligned}
|(\Delta_i|_{P_i})| &\leq k\binom{\left\lfloor \tilde{c}2^{(1-\varepsilon)\frac{\sqrt{k}}{2}} \right\rfloor}{2^i}\left(2^i\right)! \\
&\leq k\frac{\left(\tilde{c}2^{(1-\varepsilon)\frac{\sqrt{k}}{2}}\right)^{2^i}}{(2^i)!}\left(2^i\right)! \\
&= 2^{\log k + \left(\log \tilde{c} + (1-\varepsilon)\frac{\sqrt{k}}{2}\right)2^i} \\
&\leq 2^{\left(\log k + \log \tilde{c} + (1-\varepsilon)\frac{\sqrt{k}}{2}\right)2^i} \\
&< 2^{\left(1-\frac{\varepsilon}{2}\right)\frac{\sqrt{k}}{2}2^i}.
\end{aligned}$$

The last inequality follows from (4.2) and (4.3).

In order to estimate the restricted motion of $\gamma \in \Delta_i$ on P_i, recall that there are κ spheres in P_i and γ moves at least 2^{i-1} elements in each of the spheres. Hence we get the following inequality:

$$m(\gamma)|_{P_i} \geq \kappa 2^{i-1} \geq 2\sqrt{k}\left(1 - \frac{\varepsilon}{2}\right) 2^{i-1}.$$

If we combine the two estimates, we obtain

$$\frac{m(\Delta_i)|_{P_i}}{2} \geq \sqrt{k}\left(1 - \frac{\varepsilon}{2}\right) 2^{i-1} > \log |(\Delta_i|_{P_i})|.$$

This is exactly the inequality in the condition of Lemma 3.18. So for $1 \leq i < r$ we can apply the lemma in order to break all elements of Δ_i by a suitable colouring of P_i.

Finally we need to verify that the inequality also holds for Δ_r and P_r. By Corollary 4.8 the number of permutations in $\Delta_r|_{P_r}$ is bounded by the number of permutations of S_{m+k}, that is

$$|(\Delta_r|_{P_r})| \leq |S_{m+k}|! \leq \left\lfloor \tilde{c} 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}} \right\rfloor ! \leq 2^{\left(\log \tilde{c} + (1-\varepsilon)\frac{\sqrt{k}}{2}\right) \tilde{c} 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}}} < 2^{\tilde{c}(1-\frac{\varepsilon}{2})\frac{\sqrt{k}}{2} 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}}}.$$

The last inequality easily follows from (4.2).

In order to estimate the motion note that every $\gamma \in \Delta_r$ moves at least 2^{r-1} vertices in each sphere in P_r. Since there are more than $\frac{\varepsilon}{2}\left(1 - \frac{\varepsilon}{2}\right) k$ spheres in P_r we get

$$m(\gamma)|_{P_r} > \frac{\varepsilon}{2}\left(1 - \frac{\varepsilon}{2}\right) k 2^{r-1} \geq \frac{\varepsilon}{2}\left(1 - \frac{\varepsilon}{2}\right) k 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}}.$$

Putting these estimates together we obtain the inequality in the condition of Lemma 3.18:

$$\frac{m(\Delta_r)|_{P_r}}{2} > \frac{\varepsilon}{4}\left(1 - \frac{\varepsilon}{2}\right) k 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}} > \tilde{c}\left(1 - \frac{\varepsilon}{2}\right) \frac{\sqrt{k}}{2} 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}} > \log |(\Delta_r|_{P_r})|$$

where the middle inequality is a direct consequence of (4.5). This proves that we can apply Lemma 3.18 to find a 2-colouring of P_r which breaks every automorphism in Δ_r.

So we have shown that we can break all of Δ by a 2-colouring of the part of the spheres $S_{v_0}(m+1), \dots, S_{v_0}(m+k)$ that has not been coloured when we applied Lemma 4.13.

Iteratively proceed by breaking all automorphisms that fix v_0 and act non-trivially on $S_{v_0}(m+k)$. Clearly in the limit this yields a colouring that breaks every non-trivial automorphism. Simply note that every non-trivial automorphism that fixes v_0 has to act non-trivially on some sphere $S_{v_0}(n)$ and thus also on every higher sphere. $\square$

The reader may have noticed that in the proof we have only used that the size of the spheres is bounded by $2^{(1-\varepsilon)\frac{\sqrt{n}}{2}}$. Since the ball $B_{v_0}(n)$ is the union of all spheres of radius at most n one might wonder if the same proof gives a better bound on the growth of the graph. This is however not the case, because

$$\sum_{k=1}^{n} 2^{(1-\varepsilon)\frac{\sqrt{k}}{2}} \leq c \int_0^n 2^{(1-\varepsilon)\frac{\sqrt{x}}{2}} \, \mathrm{d}x \leq c' \sqrt{n} 2^{(1-\varepsilon)\frac{\sqrt{n}}{2}} \leq 2^{(1-\frac{\varepsilon}{2})\frac{\sqrt{n}}{2}}$$

for large values of n and suitable constants c, c'. So if the sphere of radius n has size $\mathcal{O}\left(2^{(1-\varepsilon)\frac{\sqrt{n}}{2}}\right)$, then the same holds true for the ball of radius n with a slightly different ε.

4.4 Growth of ends

In this section we would like to outline a possible generalisation of Theorem 4.20 to graphs with countably many ends. For readers not familiar with the notion of ends we refer to [6] for an accessible introduction to the topic. Before stating this extension, however, we need to introduce the notion of the growth of an end.

If we consider an end ω of a graph and a base vertex v_0, we can define the set

$$S_{v_0}^{\omega}(n) = \{v \in S_{v_0}(n) \mid v \text{ lies in the same component of } G \setminus B_{v_0}(n-1) \text{ as } \omega\}.$$

An end ω has *growth* $\mathcal{O}(f(n))$ if the cardinality of $S_{v_0}^{\omega}(n)$ is $\mathcal{O}(f(n))$. In general one has to be careful about this definition because it may depend on the base point v_0. However, as long as f is non-decreasing and for every $k \in \mathbb{N}$ there is a constant c_k such that $f(n+k) \leq c_k f(n)$ there is no such dependency. Compare this to Proposition 4.3, the only diference is that we need to explicitly require monotonicity. Clearly, the growth function $f(n) = 2^{(1-\varepsilon)\frac{\sqrt{n}}{2}}$ has these properties.

Theorem 4.22. *Let G be a connected graph with countably many ends each of which has growth $\mathcal{O}(2^{(1-\varepsilon)\frac{\sqrt{n}}{2}})$ for the same fixed ε. If G has infinite motion, then G is 2-distinguishable.*

Proof. First of all—just as in the proof of Theorem 4.20—find a partial colouring c which fixes a vertex v_0. The only difference is that we choose $\delta = \frac{\varepsilon}{4}$ rather than $\delta = \frac{\varepsilon}{2}$.

The rest of the proof consists of two steps. First we extend c to a partial colouring that breaks every automorphism of G which does not fix the set of ends of G pointwise, still leaving a large fraction of the vertices uncoloured. Then we use the same argument as in the proof of Theorem 4.20 in order to colour the rest of the vertices such that the remaining automorphisms are broken.

For the first step choose an increasing sequence n_i such that the spheres $S_{v_0}(n_i)$ are still uncoloured and $n_i - n_{i-1} > \frac{4}{\varepsilon}$. Consider the set of spheres $S_{v_0}(n_i)$. We wish to colour those spheres such that every automorphism that fixes v_0 and preserves the colouring also fixes every end of G. Note that after colouring these spheres the fraction of uncoloured spheres is still at least $1 - \frac{\varepsilon}{2}$.

It is not hard to see that the sets $S_{v_0}^{\omega}(n_i)$ carry a rooted tree structure. Consider v_0, the root, which is connected by an edge to every $S_{v_0}^{\omega}(n_1)$. Draw an edge from $S_{v_0}^{\omega}(n_{i-1})$ to $S_{v_0}^{\omega}(n_i)$. To see that this is indeed a tree just note that if $S_{v_0}^{\omega_1}(n) = S_{v_0}^{\omega_2}(n)$, then $S_{v_0}^{\omega_1}(m) = S_{v_0}^{\omega_2}(m)$ for every $m < n$, so there cannot be any circles.

Next, note that every automorphism $\gamma \in \operatorname{Aut}(G)$ that fixes v_0 but does not fix all ends also acts as a non-trivial automorphism on this rooted tree. By [27] the distinguishing number of infinite leafless trees is at most 2, therefore it is possible to 2-colour the sets $S_{v_0}^{\omega}(n_i)$ such that every such automorphism is broken. It is also worth noting that so far we did not use the countability of the end space of G, nor did we use the growth condition on the ends.

For the second step of the proof let us check which automorphisms of G have not yet been broken. Denote the set of such automorphisms by Δ. We already know that every

$\gamma \in \Delta$ must fix v_0 as well as every end of G. Lemma 4.9 implies that every automorphism of G moves some ray of G into a disjoint ray. Hence every automorphism in Δ permutes some rays which belong to the same end ω.

For an end ω of G let Δ^ω be the set of permutations in Δ which move some rays in ω. Note that these sets are not necessarily disjoint but their union is all of Δ. Also note that every automorphism $\gamma \in \Delta^\omega$ acts non-trivially on every $S_{v_0}^\omega(n)$ from some index n_0 on.

Furthermore, let $(\omega_i)_{i\in\mathbb{N}}$ be an enumeration of the ends of G. Choose a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f^{-1}(i)$ is infinite for every natural number i. Assume that all spheres up to $S_{v_0}(m)$ have been coloured in the first $i-1$ steps. In the i-th step we would like to colour some more spheres in order to break all automorphisms in $\Delta^{\omega_{f(i)}}$ that act non-trivially on $S_{v_0}^{\omega_{f(i)}}(n)$ for every $n > m$. Since we only coloured an $\frac{\varepsilon}{2}$-fraction of all spheres so far, this can be achieved by exactly the same arguments as in the proof of Theorem 4.20.

As we already mentioned, every automorphism that was not broken in the first step acts non-trivially on the rays of some end. Since, in the procedure described above, every end is considered infinitely often, it is clear that every such automorphism is broken eventually. This completes the proof. $\square$

The same proof still works if we can partition the (possibly uncountably many) ends into countably many classes such that the combined growth of all ends contained in each class $\mathcal{C}$ is $\mathcal{O}(2^{(1-\varepsilon)\frac{\sqrt{n}}{2}})$.

Theorem 4.23. *Let G be a locally finite connected graph. Assume that there is a decomposition $(\mathcal{C}_i)_{i\in\mathbb{N}}$ of the set of ends of G and an $\varepsilon > 0$ such that for every i it holds that $\left|S_{v_0}^{\mathcal{C}_i}(n)\right| = \mathcal{O}(2^{(1-\varepsilon)\frac{\sqrt{n}}{2}})$, where*

$$S_{v_0}^{\mathcal{C}_i}(n) = \bigcup_{\omega\in\mathcal{C}_i} S_{v_0}^\omega(n).$$

Then G is 2-distinguishable.

This can be seen as a generalisation of both Theorem 4.20 (all ends in the same class) and Theorem 4.22 (every end has its own class).

5 Random colourings

All the results mentioned so far were achieved by deterministically colouring vertices in order to break certain automorphisms. In this chapter we pursue a different approach. We investigate how random colourings behave with respect to automorphism breaking. The idea suggests itself, especially since the proof of Lemma 1.3 uses probabilistic methods. As it turns out, in all of the examples mentioned so far a random colouring is almost surely distinguishing. The same is true in many other graph classes where the validity of Conjecture 1.2 has been proved.

Besides investigating properties of random colourings for various graph classes we also show that Conjecture 1.5 is "almost true" in the following sense.

Recall that a colouring is distinguishing if and only if its stabiliser is trivial. While the stabiliser of a random colouring may be non-trivial we can show that at least it is almost surely very sparse in two ways:

- it is almost surely nowhere dense in the permutation topology, and
- it is almost surely a null set with respect to the corresponding Haar measure.

Before proving these results let us define what we mean by a random colouring. In the finite case there are only finitely many colourings, hence we can choose one uniformly at random (which is what we did in the proof of Lemma 1.3). If the graph is infinite then this is not possible. There is, however, a probability measure on the set of colourings with similar properties.

The product measure of countably many uniform 0–1 random variables can be seen as a probability measure on the set of 2-colourings of a countable set. This measure has the property that every vertex receives its colour uniformly at random and that the colours on disjoint vertex sets are independent. From now on a random colourings are always chosen according to this measure $\mathbb{P}$. Furthermore, instead of just considering graphs and their automorphism groups we work in the more general setting of a subdegree finite, closed group Γ of permutations of a countable set S. The graph case can always be recovered by setting $S = V$ and $\Gamma = \operatorname{Aut} G$.

From Theorem 3.21 we know that a countable group of permutations with infinite motion of a countable set admits a distinguishing 2-colouring. The following theorem shows that almost every 2-colouring has this property. Its proof closely follows the lines of the proof of Lemma 1.3.

Theorem 5.1. *Let Γ be a countable group of permutations with infinite motion of a countable set S and let c be a random colouring of S. Then c is almost surely Γ-distinguishing.*

Proof. For any given permutation $\gamma \in \Gamma$ it follows from infinite motion that there are infinitely many disjoint pairs $(s_i, \gamma s_i) \in S \times S$ such that $s_i \neq \gamma s_i$. For γ to preserve the colouring it is necessary that all of those pairs are monochromatic. However, for each pair this only happens with probability $\frac{1}{2}$. So there is almost surely a non-monochromatic pair $(s_i, \gamma s_i)$ and hence γ is almost surely broken by c.

In order to see that c is almost surely distinguishing we use σ-subadditivity of the probability measure $\mathbb{P}$:

$$\mathbb{P}[\exists \gamma \colon c\gamma = c] \leq \sum_{\gamma \in \Gamma} \mathbb{P}[c\gamma = c] = 0$$

because every summand is 0. □

Just like for Lemma 1.3 the proof of the above result is easily seen to be independent of the group structure of Γ.

5.1 Sparsity of the stabilisers of random colourings

The proof of Theorem 5.1 breaks down if Γ is uncountable, because summation is no longer possible. However, we know from Section 2.3 that a closed, subdegree finite group of permutations is always separable with respect to the permutation topology. Applying our argument to a dense countable subset yields the following.

Theorem 5.2. *Let Γ be a separable group of permutations of a countable set S with infinite motion and let c be a random colouring of S. Then Γ_c is almost surely nowhere dense in Γ.*

Proof. Choose a dense countable subset of Γ. By the same arguments as before, the random colouring c almost surely breaks every automorphism in this subset. By Lemma 3.10 the stabiliser of c is a closed subgroup, hence its complement is almost surely an open dense set. This implies that the stabiliser must be almost surely nowhere dense in Γ. □

In [14] it is shown that every closed permutation group has a dense subgroup which admits a distinguishing 2-colouring. Observing that the subgroup generated by a countable set is again countable, we get the same result for every separable permutation group.

So far we have shown that, if Γ is closed and subdegree finite, then the stabiliser subgroup of a random colouring is almost surely topologically sparse, which was more or less a direct consequence of separability. But it turns out that under suitable conditions the set of unbroken permutations is small in at least one more way: it is almost surely a null set with respect to the Haar measure, a natural measure fore locally compact topological groups. It was introduced by Haar [9] who proved the following.

Theorem 5.3. *Let Γ be a locally compact topological group. Then there is a non-trivial measure $\mathbb{H}$ on the Borel σ-Algebra such that*

1. *compact sets have finite measure, and*

2. *the measure is left translation invariant, that is* $\mathbb{H}(\gamma\Delta) = \mathbb{H}(\Delta)$ *for every measurable* $\Delta \subseteq \Gamma$.

The measure $\mathbb{H}$ *is unique up to multiplication by a constant.*

Definition 5.4. A non-trivial measure $\mathbb{H}$ satisfying properties 1 and 2 from the above theorem is called a *left Haar measure*, where "left" refers to the invariance under left multiplication.

A right Haar measure can be defined in an analogous way. It is worth noting that left and right Haar measures need not coincide, that is, there may be left Haar measures which are not invariant under right multiplication and vice versa. However, our result is true for both left and right Haar measures. In fact, we do not even need the translation invariance, so the proof still works for a measure which only satisfies property 1 of Theorem 5.3.

Theorem 5.5. *Let* Γ *be a closed, subdegree finite group of permutations of a countable set* S*, and assume that the motion of* Γ *is infinite. Then a random colouring* c *almost surely breaks almost every (with respect to the Haar measure) element of* Γ.

The basic ideas of the proof again come from the proof of Theorem 5.1, the main difference being that we replace the sum by an integral with respect to the Haar measure. In the proof we need the following version of Fubini's theorem which is sometimes also known as Tonelli's theorem.

Theorem 5.6. *Let* $(X, \mathcal{X}, \mu)$ *and* $(Y, \mathcal{Y}, \nu)$ *be* σ*-finite measure spaces and let* $f\colon X \times Y \to \mathbb{R}$ *be a non-negative,* $(\mathcal{X} \times \mathcal{Y})$*-measurable function. Then*

$$\int_X \left(\int_Y f(x,y) \,\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x,y) \,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y). \qquad \square$$

For a proof see [23].

Proof of Theorem 5.5. First of all recall that Γ is locally compact by arguments in Section 2.3. So we can define a Haar measure $\mathbb{H}$ on Γ.

We now claim that for a random colouring c the expected value of $\mathbb{H}(\Gamma_c)$ is 0. Since $\mathbb{H}(\Gamma_c)$ is a non-negative random variable, this implies that $\mathbb{H}(\Gamma_c) = 0$ almost surely, thus proving the lemma.

To see that the expected value is indeed 0 we calculate

$$\begin{aligned} \mathbb{E}(\mathbb{H}(\Gamma_c)) &= \int_{\mathcal{C}(S)} \mathbb{H}(\Gamma_c) \,\mathrm{d}\mathbb{P}(c) \\ &= \int_{\mathcal{C}(S)} \int_\Gamma I_{[c\gamma = c]} \,\mathrm{d}\mathbb{H}(\gamma) \,\mathrm{d}\mathbb{P}(c). \end{aligned}$$

We know that Γ is the union of countably many compact balls by Lemma 2.8 and Lemma 2.9, hence Γ is a σ-compact group. Compact sets have finite Haar measure, so the Haar measure on Γ is σ-finite.

In order to be able to apply Theorem 5.6, we still need to show that the function which we would like to integrate is measurable. Since it is the indicator function of the set

$$U = \{(c, \gamma) \in \mathcal{C}(S) \times \Gamma \mid c\gamma = c\}$$

it suffices to show that U is measurable. For this purpose let $(S_i)_{i\in\mathbb{N}}$ be a sequence of finite subsets of S such that $\lim_{i\to\infty} S_i = S$. For each partial colouring c' with domain S_i define

$$U_i(c') = \{(c, \gamma) \in \mathcal{C}(S) \times \Gamma_{c'} \mid \forall s \in S_i \colon c(s) = c'(s)\}.$$

Observe that $C = \{c \in \mathcal{C}(S) \mid \forall s \in S_i \colon c(s) = c'(s)\}$ is a cylinder set and thus both open and closed. The set $\Gamma_{c'}$ is both open and closed by Lemma 3.11. Since $U_i(c') = C \times \Gamma_{c'}$ it is clearly contained in the product σ-algebra. Now let

$$U_i = \bigcup_{c'\in\mathcal{C}(S_i)} U_i(c').$$

This set is measurable because it is the finite union of measurable sets. We claim that

$$U = \bigcap_{i\in\mathbb{N}} U_i.$$

To see that this is indeed the case consider $(c, \gamma) \in U$. Clearly c coincides with some partial colouring c' on S_i and γ preserves this partial colouring because it preserves c. Hence (c, γ) is contained in every U_i and thus also in the intersection.

Conversely, let $(c, \gamma) \in \bigcap_{i\in\mathbb{N}} U_i$. Assume that γ does not preserve c. Then there is $s \in S$ such that $c(s) \neq c(\gamma s)$. Take i large enough that s and γs are contained in S_i. Clearly, γ does not preserve the partial colouring c' which coincides with c on S_i. Hence $(c, \gamma) \notin U_i$, a contradiction to $(c, \gamma) \in \bigcap_{i\in\mathbb{N}} U_i$.

Altogether we have shown that U can be written as a countable intersection of measurable sets. So it is measurable itself and hence the indicator function $I_U = I_{[c\gamma=c]}$ is measurable as well.

This implies that we can apply Fubini's theorem to the iterated integral above and obtain

$$\begin{aligned}\mathbb{E}(\mathbb{H}(\Gamma_c)) &= \int_\Gamma \int_{\mathcal{C}(S)} I_{[c\gamma=c]} \,\mathrm{d}\mathbb{P}(c) \,\mathrm{d}\mathbb{H}(\gamma) \\ &= \int_\Gamma \mathbb{P}[c\gamma = c] \,\mathrm{d}\mathbb{H}(\gamma).\end{aligned}$$

We already observed earlier that the probability that a given permutation preserves a random colouring is 0 unless $\gamma = \mathrm{id}$. Hence we integrate over the characteristic function of the singleton set $\{\mathrm{id}\}$ and this integral is easily seen to be 0. □

5.2 The distinct spheres condition and a useful equivalence relation

The distinct spheres condition was introduced in [25] as a sufficient condition for 2-distinguishability.

Definition 5.7. Let $G = (V, E)$ be a graph. We say that G satisfies the *distinct spheres condition* if there is a vertex $v_0 \in V$ such that for any two distinct vertices $x, y \in V$ we have

$$d(v_0, x) = d(v_0, y) \implies S_x(n) \neq S_y(n) \text{ for infinitely many } n.$$

The following result from [25] states that the distinct spheres condition is sufficient for 2-distinguishability not only for locally finite graphs but for general countable graphs.

Theorem 5.8. *Let G be a connected, countable graph satisfying the distinct spheres condition. Then G is 2-distinguishable.* □

We show later (Theorem 5.18) that for locally finite graphs satisfying this condition a random 2-colouring is almost surely distinguishing. The following equivalence relation is one of our main tools.

Let S be a countable set and let Γ be a subdegree finite group of permutations of S with infinite motion. Define an equivalence relation $\sim_\Gamma$ on the set S as follows: two points $s, t \in S$ are called Γ-equivalent, if the following holds:

- there is a permutation $\varphi \in \Gamma$ such that $\varphi s = t$ and
- for all but finitely many $x \in S$ the orbits $\Gamma_s x$ and $\varphi \Gamma_s x$ coincide.

Note that the latter requirement is true for φ if and only if it is true for every γ such that $\gamma s = t$ because in this case $\gamma = \varphi \gamma_s$ for a suitable $\gamma_s \in \Gamma_s$. Hence the second condition does not depend on the choice of φ.

Proposition 5.9. *The relation $\sim_\Gamma$ is indeed an equivalence relation.*

Proof. To show reflexivity simply choose $\varphi = \mathrm{id}$.

For symmetry assume that $s \sim_\Gamma t$ and let $\varphi \in \Gamma$ such that $\varphi s = t$. Note that $\Gamma_s = \varphi^{-1} \Gamma_t \varphi$, so for $\Gamma_s x = \varphi \Gamma_s x$ we have

$$\Gamma_t \varphi x = \varphi \Gamma_s x = \Gamma_s x = \varphi^{-1} \Gamma_t \varphi x.$$

This implies that $\Gamma_t y = \varphi^{-1} \Gamma_t y$ for all but finitely many values of $y = \varphi x$, that is $t \sim_\Gamma s$.

Finally, we need to show transitivity. Assume that $s \sim_\Gamma t$ and that $t \sim_\Gamma u$ and let φ and ψ be the corresponding permutations. By definition this implies that for all but finitely many $x \in S$ it holds that $\varphi \Gamma_s x = \Gamma_s x$ and $\psi \Gamma_t \varphi x = \Gamma_t \varphi x$. Using the fact that $\varphi \Gamma_s = \Gamma_t \varphi$ we obtain

$$\psi \varphi \Gamma_s x = \psi \Gamma_t \varphi x = \Gamma_t \varphi x = \varphi \Gamma_s x = \Gamma_s x$$

for all but finitely many $x \in S$. □

We denote the equivalence class of $s \in S$ with respect to $\sim_\Gamma$ by $[s]_\Gamma$. With the above notation we have the following lemma.

Lemma 5.10. *Let S, Γ, and $\sim_\Gamma$ be defined as above. Assume that Γ has infinite motion and let c be a random 2-colouring of S. Then c almost surely fixes every equivalence class with respect to $\sim_\Gamma$, that is*

$$\Gamma_c \subseteq \bigcap_{s \in S} \Gamma_{[s]_\Gamma}$$

where $\Gamma_{[s]_\Gamma}$ denotes the setwise stabiliser of $[s]_\Gamma$.

Proof. For $t \nsim_\Gamma s$ and $u \in S$ consider the event

$$A_{stu} = [\exists \gamma \in \Gamma_c \colon \gamma s = t, \gamma t = u].$$

If we can show that the probability of A_{stu} is 0 we are done, because in this case

$$\mathbb{P}[\exists s \nsim_\Gamma t, \gamma \in \Gamma \colon \gamma s = t] = \mathbb{P}\Bigg(\bigcup_{s \in S} \bigcup_{\substack{t \in S \\ s \nsim_\Gamma t}} \bigcup_{u \in S} A_{stu}\Bigg) \leq \sum_{s \in S} \sum_{\substack{t \in S \\ t \nsim_\Gamma s}} \sum_{u \in S} \mathbb{P}(A_{stu}) = 0.$$

So let us take a closer look at $\mathbb{P}(A_{stu})$. If there is no permutation in Γ which maps s to t and t to u, then this probability clearly is 0. So assume that there is such a permutation γ. Let $v \in S$. Since s is mapped to t the set $\Gamma_s v$ must be mapped to the set $\gamma\Gamma_s v$. Note that the set $\gamma\Gamma_s v$ does not depend on the particular choice of γ, that is, it is the same for every $\gamma \in \Gamma$ with $\gamma s = t$. In particular this implies that if the set $\Gamma_s v \setminus \gamma\Gamma_s v$ is non-empty, then it must be mapped to the disjoint set $\gamma\Gamma_s v \setminus \gamma^2\Gamma_s v$ by every automorphism which maps s to t. The set $\gamma^2\Gamma_s v$ depends only on u, that is, the image of s under γ^2 and not on the particular choice of γ.

There are infinitely many points v for which these difference sets are non-empty because $s \nsim_\Gamma t$ and each of the sets is finite because of subdegree finiteness. Hence we can choose infinite sequences of non-empty, disjoint sets $P_i := \Gamma_s v_i \setminus \gamma\Gamma_s v_i$ and $Q_i = \gamma\Gamma_s v_i \setminus \gamma^2\Gamma_s v_i$ such that all of the P_i and Q_j are also pairwise disjoint for all $i, j \in \mathbb{N}$.

Now assume that there is a colour preserving permutation which maps s to t. This can only happen if the sets P_i and Q_i contain the same number of vertices of each colour for every i. Let $n_i := |P_i| = |Q_i|$ and denote by p_i and q_i the number of elements of P_i and Q_i with colour 0 respectively. Then the probability that the colour distributions on P_i and Q_i coincide, can be expressed as

$$\begin{aligned}
\mathbb{P}[p_i = q_i] &= \sum_{j=0}^{n_i} \mathbb{P}[p_i = j \mid q_i = j]\, \mathbb{P}[q_i = j] \\
&= \sum_{j=0}^{n_i} \mathbb{P}[p_i = j]\, \mathbb{P}[q_i = j] \\
&= \sum_{j=0}^{n_i} \binom{n_i}{j} 2^{-n_i} \binom{n_i}{j} 2^{-n_i},
\end{aligned}$$

where the second equality follows from the fact that P_i and Q_i are disjoint and hence their colourings are independent. To get an estimate for the last sum observe that

$$\binom{n_i}{j} 2^{-n_i} \leq \frac{1}{2}$$

and

$$\sum_{j=0}^{n_i} \binom{n_i}{j} 2^{-n_i} = 1.$$

Hence

$$\mathbb{P}[p_i = q_i] \leq \frac{1}{2} \sum_{j=0}^{n_i} \binom{n_i}{j} 2^{-n_i} \leq \frac{1}{2}.$$

Recall that in order to have an automorphism which maps s to t and t to u we need $p_i = q_i$ for every $i \in \mathbb{N}$. These events are independent because all of the sets are disjoint. Hence we have

$$\mathbb{P}[\exists \gamma \in \Gamma_c \mid \gamma s = t, \gamma t = u] \leq \prod_{i \in \mathbb{N}} \mathbb{P}[p_i = q_i] = 0.$$

This completes the proof. □

Remark 5.11. Note that Lemma 5.10 can be iterated as follows. Let $\Gamma^0 = \Gamma$ and denote by $\sim_0$ the relation $\sim_{\Gamma^0}$. Inductively, for $i \geq 0$ define

$$\Gamma^{i+1} = \bigcap_{s \in S} \Gamma^i_{[s]_i},$$

where $[s]_i$ is the equivalence class of s with respect to $\sim_i$ and $\Gamma^i_{[s]_i}$ is its setwise stabiliser. Define $\sim_{i+1} = \sim_{\Gamma^{i+1}}$.

Let c be a random colouring of S. Inductively applying Lemma 5.10 we obtain that almost surely $\Gamma_c \subseteq \Gamma^i$ for each $i \in \mathbb{N}_0$. Define

$$\Gamma^\infty = \lim_{i \to \infty} \Gamma^i = \bigcap_{i \in \mathbb{N}_0} \Gamma^i.$$

Then almost surely

$$\Gamma_c \subseteq \Gamma^\infty.$$

Remark 5.12. The set of permutations in Γ, that fix all equivalence classes with respect to $\sim_\Gamma$ setwise, is a group Δ. If there is a finite equivalence class then Lemma 3.9 implies that Δ is compact and hence the stabiliser of a random colouring is almost surely compact.

But even if Δ is not compact, Δ is the limit of a sequence of compact subgroups. To see this, note that for a fixed $s \in S$ every permutation $\gamma \in \Delta$ must fix all but finitely many suborbits $\Delta_s x$ setwise. Let $\Delta_s x_i$ be an enumeration of all suborbits and define

$$\Delta_i = \{\gamma \in \Delta \mid \forall j > i \colon \gamma \Delta_s x_j = \Delta_s x_j\}.$$

Then Δ_i is compact by Lemma 3.9 because the Δ_i-orbit of x_j is contained in the finite suborbit $\Delta_s x_j$ for $j > i$. Clearly the sequence Δ_i is non-decreasing and every $\gamma \in \Delta$ is contained in some Δ_i. Thus

$$\Delta = \lim_{i\to\infty} \Delta_i = \bigcup_{i\in\mathbb{N}_0} \Delta_i.$$

The above remark tells us that in order to prove Conjecture 1.5 it suffices to consider compact groups. More precisely, we have the following.

Corollary 5.13. *Assume that for every compact, subdegree finite permutation group with infinite motion a random colouring is almost surely distinguishing. Then the same is true for every subdegree finite permutation group with infinite motion.*

Proof. With the above notation every non-trivial permutation is contained in some Δ_i. Since Δ_i is compact the stabiliser in Δ_i of a random colouring c is almost surely trivial. By σ-subadditivity of the probability measure we get

$$\mathbb{P}[\Gamma_c \text{ is not trivial}] \leq \sum_{i=1}^{\infty} \mathbb{P}[(\Delta_i)_c \text{ is not trivial}] = 0.$$

□

5.3 Random colourings of graphs

The last section of this chapter is devoted to random colourings of graphs. First of all recall that the automorphism group of a locally finite graph is always a closed, subdegree finite group of permutations of the vertex set. Hence all results from Section 5.1 apply to automorphism groups of locally finite graphs.

Theorem 5.14. *Let G be a locally finite graph with infinite motion and let c be a random colouring of G. Then $(\operatorname{Aut} G)_c$ is almost surely a nowhere dense, closed subgroup with Haar measure 0.* □

Instead of colouring all vertices randomly, we could first colour part of the vertices deterministically. This will make the stabiliser subgroup of the resulting colouring compact, as the following theorem shows.

Theorem 5.15. *Let G be a locally finite graph with infinite motion. Then there is a colouring of G which is only stabilised by a nowhere dense, compact subgroup of $\operatorname{Aut} G$ with measure 0.*

Proof. First apply Lemma 4.13 in order to break all automorphisms which move a given vertex v_0. This gives a partial colouring c' of the graph which by Lemma 3.11 is only preserved by a closed subset of $\operatorname{Aut} G$. The set $\Gamma_{c'}$ is completely contained in $(\operatorname{Aut} G)_{v_0}$, which is compact by Lemma 3.8.

It is easy to see that $\Gamma_{c'}$ has infinite motion on the set of yet uncoloured vertices. Now let c be the colouring obtained by randomly colouring all vertices that have not been coloured yet. We apply Theorem 5.5 to show that c almost surely breaks almost every remaining automorphism of G.

The stabiliser $\Gamma_c \subseteq \Gamma_{c'}$ forms a closed and hence compact subgroup of $(\operatorname{Aut} G)_{v_0}$. Since it has measure 0 in $(\operatorname{Aut} G)_{v_0}$, it must also have measure 0 in $\operatorname{Aut} G$. The property of being nowhere dense also carries over from $(\operatorname{Aut} G)_{v_0}$ to $\operatorname{Aut} G$. □

The following definition is a weaker version of the equivalence relation of Lemma 5.10. It is easy to verify that it is indeed an equivalence relation. Also note the similarity of the definitions of sphere equivalence and the distinct spheres condition.

Definition 5.16. Call two vertices u and v *sphere equivalent* ($u \sim_S v$), if there is an automorphism of G which maps u to v and an integer $n_0 \in \mathbb{N}$ such that $S_u(n) = S_v(n)$ for every $n \geq n_0$.

Considering Lemma 5.10, it is not surprising that the stabiliser of a random colouring is almost surely contained in the setwise stabiliser of each equivalence class with respect to the above relation.

Lemma 5.17. *Let G be a locally finite graph with infinite motion. A random colouring almost surely setwise fixes all equivalence classes with respect to $\sim_S$.*

Proof. Recall that the automorphism group of a locally finite graph is always subdegree finite. For $\Gamma = \operatorname{Aut} G$ the relation $\sim_\Gamma$ defined in Section 5.2 is finer than $\sim_S$. Since by Lemma 5.10 a random colouring almost surely fixes every equivalence class with respect to $\sim_\Gamma$, it also almost surely fixes every equivalence class with respect to $\sim_S$. □

In the remainder of this section we focus on examples of graphs, where a random colouring is almost surely distinguishing. The above lemma is one of our main tools.

5.3.1 The distinct spheres condition

As mentioned earlier, graphs satisfying the distinct spheres condition have infinite motion and are 2-distinguishable and hence they support Conjecture 1.2. The similarity between the distinct spheres condition and sphere equivalence suggests that for such graphs a random 2-colouring is also likely to be distinguishing. And indeed we can show that if a locally finite graph satisfies the distinct spheres condition, then it supports Conjecture 1.5, that is, a random colouring is almost surely distinguishing.

Theorem 5.18. *If a locally finite graph $G = (V, E)$ satisfies the distinct spheres condition, then a random 2-colouring c is almost surely distinguishing.*

Proof. Let γ be an automorphism of G which is contained in the setwise stabiliser of each equivalence class with respect to $\sim_S$. By Lemma 5.17 it suffices to show that $\gamma = \mathrm{id}$.

Assume that γ is non-trivial. If $\gamma v_0 \nsim_S v_0$ then γ is not contained in the setwise stabiliser of all equivalence classes with respect to $\sim_S$.

So assume $\gamma v_0 \sim_S v_0$. If $\gamma v_0 = v_0$ then γ stabilises all spheres with centre v_0 setwise. As $\gamma \neq \mathrm{id}$ there must be some $n \in \mathbb{N}$ such that γ acts non-trivially on $S_{v_0}(n)$.

If $\gamma v_0 \neq v_0$ but $\gamma v_0 \sim_S v_0$ then there is some $n_0 \in \mathbb{N}$ such that γ stabilises $S_{v_0}(n)$ for $n > n_0$. Because γ acts non-trivially on $B_{v_0}(n)$, it must also act non-trivially on the boundary $S_{v_0}(n)$.

Since $x \nsim_S y$ for any two vertices $x, y \in S_{v_0}(n)$ we can conclude that γ is not contained in the setwise stabiliser of all equivalence classes with respect to $\sim_S$. $\square$

Corollary 5.19. *Let G be an infinite, locally finite graph. Then each of the following properties implies that a random 2-colouring is almost surely distinguishing:*

- *G is a leafless tree,*
- *G can be written as a product of two infinite factors,*
- *the automorphism group of G acts primitively on the vertex set,*
- *G is vertex-transitive and has connectivity 1.*

Proof. All of these graphs satisfy the distinct spheres condition by [25]. $\square$

5.3.2 Graphs with a global tree structure

Trees probably are the most elementary example of a family of graphs which is known to satisfy Conjecture 1.2. As we have seen, leafless trees also satisfy Conjecture 1.5. The following corollary to Theorem 5.18 shows that the same holds true for arbitrary trees with infinite motion.

Corollary 5.20. *A random colouring of a locally finite tree with infinite motion is almost surely distinguishing.*

Proof. Since we assume infinite motion we can ignore finite subtrees and consider the subgraph induced by those vertices whose removal results in at least 2 infinite components. On this set the relation $\sim_S$ is easily seen to be trivial. Alternatively one could note that the resulting graph is a leafless tree and hence satisfies the distinct spheres condition. $\square$

Tree-like graphs are graphs with the following property: there is a vertex $v_0 \in V$ such that every vertex $v \in V$ has a neighbour w such that v lies on every shortest w-v_0-path. It is readily verified that this class of graphs again staisfies the distinct spheres condition.

Corollary 5.21. *A random colouring of a locally finite, tree-like graph is almost surely distinguishing.* $\square$

It is a well known fact that every graph has an end faithful spanning tree [6], that is, the ends of a graph can be seen as the ends of a spanning tree of the same graph. We now show that this large-scale tree structure is also almost surely preserved by every automorphism that preserves a random colouring. First of all we show that if G has more than one end, then Γ_c is almost surely compact and hence by Lemma 3.9 stabilises a finite set which plays the role of a root. Thus translations can only happen on a small scale. Then we show that such an automorphism almost surely fixes every end. Both of these results are again consequences of Lemma 5.17.

Lemma 5.22. *Let c be a random colouring of a locally finite graph with at least two ends. Then $(\operatorname{Aut} G)_c$ is almost surely compact.*

Proof. By Lemma 3.9 it suffices to show that there is a finite orbit which is the case if the equivalence class of some vertex v with respect to $\sim_S$ is finite.

So let $v \in V$. There is a ball $B_v(n_0)$ such that $G - B_v(n_0)$ has at least two infinite components. Assume that there is a vertex $w \sim_S v$ such that $d(v, w) \geq 2n_0 + 1$ and assume that $S_v(n) = S_w(n)$ for every $n > N$.

Now note that if u and w lie in a different components of $G - B_v(n)$, then every path from w to u has to pass through $B_v(n_0)$. This implies that $d(v, u) < d(w, u)$ since a shortest path from v to u takes n_0 steps before exiting $B_v(n_0)$ while a shortest w-u-path takes $n_0 + 1$ steps to reach $B_v(n_0)$.

So all vertices equivalent to v must lie within the ball $B_v(2n_0)$ which is finite. □

Lemma 5.23. *Let c be a random colouring of a locally finite graph. Then $(\operatorname{Aut} G)_c$ almost surely only contains automorphisms which fix the set of ends of G pointwise.*

Proof. For one-ended graphs there is nothing to show, so we may assume that G has at least 2 ends. A random colouring is almost surely only preserved by automorphisms which setwise stabilise the equivalence classes with respect to $\sim_S$. Hence it suffices to show that every such automorphism also fixes the set Ω of ends of G pointwise.

Assume that γ is contained in the setwise stabiliser of each equivalence class and that $\gamma\omega \neq \omega$ for some end ω of G. Let $(v_i)_{i\in\mathbb{N}}$ be a sequence of vertices converging to ω. The sequence $(\gamma v_i)_{i\in\mathbb{N}}$ converges to $\gamma\omega$ and hence v_i and γv_i lie in different infinite components of $G \setminus B_{v_0}(n)$ for large n and i. By similar arguments as in the proof of the previous theorem this implies that $v_i \not\sim_S \gamma v_i$ for large values of i.

So γ does not stabilise the equivalence classes with respect to $\sim_S$ setwise, a contradiction. □

5.3.3 Cartesian products

Another class of graphs where 2-distinguishability results are known are Cartesian products. The *Cartesian product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$ where $V = V_1 \times V_2$ and two vertices (v_1, v_2) and (w_1, w_2) are adjacent either if $v_1w_1 \in E_1$ and $v_2 = w_2$, or if $v_1 = w_1$ and $v_2w_2 \in E_2$. In this case we write $G = G_1 \square G_2$.

The Cartesian product of finitely many factors is obtained by iterating this construction. One has to be a bit more careful when considering products with infinitely many factors. However, it can be shown that a locally finite, connected graph cannot be such a product. In particular, since all graphs in this section are locally finite and connected, we do not need to deal with the difficulties that arise when dealing with products graphs with infinitely many factors.

It is easy to see that the Cartesian product is associative and commutative, that is, the graphs obtained by changing the order in which Cartesian products are taken are isomorphic. We use this fact throughout this section without explicitly mentioning it.

A G_1-*layer* of $G = G_1 \,\Box\, G_2$ is the subgraph of G induced by the set $\{(v, v_2) \mid v \in V_1\}$ where $v_2 \in V_2$ is fixed. Analogously define a G_2-layer.

Throughout this section we state state some well known facts about Cartesian products of graphs without proving them. All of the results and their proofs can be found in [12].

The first fact that we need is that the distance between two vertices in a Cartesian product is the sum of the distances of the projections to the factors. Hence a composition of shortest paths in the factors is a shortest path in the Cartesian product. This can be used to show the following result.

Lemma 5.24. *Let G be a locally finite, connected graph with infinite motion which is not prime with respect to the Cartesian product. Choose a decomposition $G = G_1 \,\Box\, G_2$ such that G_1 is infinite. Let c be a random colouring of G. Then c almost surely fixes every G_1-layer setwise.*

Proof. Once again we would like to use Lemma 5.17. So assume that there are two sphere equivalent vertices $v \sim_S w$ of G which lie in different G_1-layers.

Let $R = (v = v_0 v_1 v_2 v_3 \ldots)$ be a geodesic ray (that is, $d(v_0, v_i) = i$) starting in v which remains inside the same G_1-layer forever. Denote by $R' = (v'_0 v'_1 v'_2 v'_3 \ldots)$ the ray in the layer of w which is obtained from R by only changing the G_2-coordinates.

Then $d(w, v_i) < d(w, v'_i)$ while $d(v, v_i) > d(v, v'_i)$ for every $i \in \mathbb{N}$. The spheres $S_v(r)$ and $S_w(r)$ are supposed to be equal for $r \geq r_0$ which implies that $d(v, v_r) = d(w, v_r)$ and $d(v, v'_r) = d(w, v'_r)$ for large enough values of r. But then we have

$$d(v, v'_r) > d(v, v_r) = d(w, v_r) > d(w, v'_r) = d(v, v'_r),$$

a contradiction. $\square$

It is known that each graph G has a unique decomposition into prime graphs with respect to the Cartesian product. It is easy to see that, if G is locally finite, then it only has finitely many factors. Hence an infinite, locally finite graph must have at least one infinite prime factor. If there is more than one infinite prime factor, then G can be decomposed into two infinite factors and in this case G is 2-distinguishable by Corollary 5.19. However, this fact can also be seen as a corollary to Lemma 5.24.

Corollary 5.25. *Let G be a locally finite, connected graph with more than one infinite prime factor. Then a random colouring of G is almost surely distinguishing.*

Proof. If G has two infinite prime factors then it can be written as $G = G_1 \,\Box\, G_2$ where both G_1 and G_2 are infinite. Now every vertex is uniquely defined by its G_1-layer and its G_2-layer. Both of these layers are almost surely fixed by a random colouring. Hence for every vertex v the probability that the stabiliser of a random colouring is contained in the stabiliser of v is 1.

Since there are only countably many vertices this implies that the stabiliser of a random colouring is almost surely trivial. $\square$

As a direct consequence we get the following result about powers of locally finite graphs.

Corollary 5.26. *Let G be a locally finite Cartesian power of an infinite, locally finite, connected graph. Then a random colouring of G is almost surely distinguishing.* $\square$

Finally, the following result states that in order to prove Conjecture 1.5, it suffices to consider prime graphs.

Corollary 5.27. *If a random colouring is almost surely distinguishing for every locally finite, connected, prime graph with infinite motion, then it is almost surely distinguishing for every locally finite, connected graph with infinite motion.*

Proof. By Corollary 5.25 it suffices to consider graphs with only one infinite prime factor. Let $G = G_1 \Box G_2$ be a factorisation of such a graph where G_1 is the unique infinite prime factor and let c be a random colouring of G.

By Lemma 5.24 all G_1-layers are almost surely setwise fixed by every automorphism in $(\operatorname{Aut} G)_c$. By assumption c is almost surely distinguishing for G_1 because G_1 is an infinite prime graph. Hence c almost surely fixes every G_1-layer pointwise. $\square$

5.3.4 Growth bounds

In the last part of this chapter we are concerned with growth bounds. We show the following probabilistic version of Theorem 4.20.

Theorem 5.28. *Let G be a graph with infinite motion and growth $\mathcal{O}\left(2^{(\frac{1}{2}-\varepsilon)\sqrt{n}}\right)$. Then a random colouring of G is almost surely distinguishing.*

In order to prove this result we need the following probabilistic version of Lemma 1.3. The proof given for Lemma 1.3 in Section 3.3 also works for this result.

Lemma 5.29. *Let S be a finite set and let Δ be a set of non-trivial permutations of S with motion $\geq m$. Let c be a random colouring of S. Then*

$$\mathbb{P}[\exists \gamma \in \Delta \colon c\gamma = c] \leq |\Delta|\, 2^{-\frac{m}{2}}. \qquad \square$$

Proof of Theorem 5.28. Let c be a random colouring of G, and choose a vertex $v_0 \in V$. For every $v \in V$ let $\Delta_{v_0}^v$ be the set of automorphisms which map v_0 to v. Clearly, $(\Delta_{v_0}^v)_{v \in V}$ is a countable decomposition of $\operatorname{Aut} G$. Hence we only need to show that $\Delta_{v_0}^v$ almost surely contains no automorphism γ such that $\gamma c = c$.

For $v \nsim_S v_0$ this follows from Lemma 5.17. If $v \sim_S v_0$, then it follows from the following claim:

($*$) Let Δ_k be the set of automorphisms that fix $S_{v_0}(i)$ setwise but not pointwise, for every $i \geq k$. Then a random colouring almost surely breaks every automorphism in Δ_k.

Assume that ($*$) is true and let $\gamma \in \Delta_{v_0}^v$ for some $v \sim_S V_0$. For every $v \sim_S v_0$ there is some index i such that $S_{v_0}(i) = S_v(i)$. Hence γ setwise fixes $S_{v_0}(i)$ if i is large enough. Furthermore, since G has infinite motion, γ acts non-trivially on infinitely many of the spheres. If it acts non-trivially on some sphere $S_{v_0}(k)$ then it also acts non-trivially on $S_{v_0}(i)$ for each $i > k$. Hence γ is contained in some set Δ_k.

By $(*)$ a random colouring c almost surely breaks all of Δ_k and there are only countably many possible values for k. Hence c almost surely breaks every automorphism in the union of the Δ_k. This implies that a random colouring almost surely breaks all of $\Delta^v_{v_0}$. This completes the proof of the theorem.

So we only need to show that $(*)$ holds for every k. Let $n > k$. Because of the growth condition on the graph we know that there is some constant c such that

$$|B_{v_0}(n^2)| \leq c\, 2^{(\frac{1}{2}-\varepsilon)n}.$$

This in particular implies that the same upper bound holds for the size of each sphere $S_{v_0}(i)$ for $i < n^2$. For $1 \leq j \leq n-1$, define

$$\begin{aligned}
R_j &= B_{v_0}((j+1)n) \setminus B_{v_0}(jn),\\
\Lambda'_j &= \{\gamma \in \Delta_k \mid \gamma \text{ moves at most } 2^j \text{vertices in some } S_{v_0}(i) \text{ for } i > (j+1)n\},\\
\Lambda_j &= \Lambda'_j \setminus \Lambda'_{j-1}.
\end{aligned}$$

Let $\Pi_j = \Lambda_j|_{R_j}$, that is, Π_j is the set of different permutations induced by Λ_j on R_j.

The next step is to estimate the probability that a random colouring of R_j breaks all automorphisms in Λ_j or, equivalently, all permutations in Π_j. Since we would like to use Lemma 5.29 we need to establish estimates for the cardinality of Π_j and for the restricted motion of Π_j on R_j.

To estimate the number of different permutations, observe that any two automorphisms that coincide on S_i for some $i > (j+1)n$ must also coincide on R_j. Hence it suffices to estimate the number of permutations on S_i which move less than 2^j vertices and add up those estimates. Since the size of S_i is bounded by $2^{(\frac{1}{2}-\varepsilon)n}$, the number of such permutations is bounded by

$$\binom{c\, 2^{(\frac{1}{2}-\varepsilon)n}}{2^j}(2^j)! \leq \frac{2^{2^j(\frac{1}{2}-\varepsilon)n+2^j\log c}}{(2^j)!}(2^j)! = 2^{2^j(\frac{1}{2}-\varepsilon)n+2^j\log c}.$$

Adding up those estimates for $(j+1)n \leq i \leq n^2$, we obtain

$$|\Pi_j| \leq n^2 2^{2^j(\frac{1}{2}-\varepsilon)n+2^j\log c}.$$

In order to estimate the motion m of Π_j on R_j observe that an element of Λ_j moves at least 2^{j-1} vertices in every sphere S_i for $jn < i < (j+1)n$. Otherwise it would be contained in Λ'_{j-1}. Adding up those estimates, we get

$$m \geq n2^{j-1}.$$

Let X_j denote the event that there is a permutation $\pi \in \Pi_j$ that preserves a random colouring c of R_j. Plugging the estimates from above into Lemma 5.29, we obtain

$$\begin{aligned}
\log \mathbb{P}[X_j] &\leq \log|\Pi_j| - \frac{m}{2}\\
&\leq 2\log n + 2^j(\frac{1}{2}-\varepsilon)n + 2^j\log c - 2^{j-1}n\\
&= -\varepsilon 2^j n + 2^j \log c + 2\log n.
\end{aligned}$$

If we choose n large enough, this implies that

$$\log \mathbb{P}[X_j] \leq -\varepsilon 2^{j-1} n \leq -\varepsilon n.$$

The probability that for every j a random colouring of R_j breaks Π_j is now given by

$$\prod_{j=1}^{n-1} (1 - \mathbb{P}[X_j]) \geq (1 - 2^{-\varepsilon n})^n.$$

This probability tends to 1 as n goes to infinity. Finally observe that if n is large enough, then

$$\Delta_k = \bigcup_{j=1}^{n-1} \Lambda_j,$$

because the motion on $B_{v_0}(n^2)$ is bounded by the number of vertices in $B_{v_0}(n^2)$. The set Λ'_j contains all automorphisms whose motion is at most 2^j, hence for n large enough and $j \geq \frac{n}{2}$ it is true that $\Delta_k = \Lambda'_j$. □

6 Graphs with infinite degrees

6.1 Non-locally finite counterexamples

In this section we give some examples to show that we cannot drop the local finiteness condition in Conjectures 1.2 and 1.5. In the case of Conjecture 1.5 this is even the case for trees, as the following example shows.

Theorem 6.1. *Denote by T_∞ the regular tree with countably infinite degree. If c is a random colouring of T_∞ with finitely many colours, then there is almost surely an automorphism of T_∞ which preserves c.*

Proof. First of all note that in a random colouring every vertex almost surely has infinitely many neighbours of each colour. Hence it suffices to find a non-trivial automorphism preserving a colouring with this property.

Let c be such a colouring and choose a vertex v_0 of T_∞. Define $\gamma v_0 = v_0$. Next choose an arbitrary colour-preserving permutation π of the neighbours of v_0 and define $\gamma v = \pi v$ for every neighbour v of v_0.

Assume that γ has already been defined for all vertices v with $d(v, v_0) \leq n$. For a vertex v with $d(v, v_0) = n$ let $(w_i^{(v,j)})_{i\in\mathbb{N}}$ be an enumeration of the neighbours of v with colour j which lie further away from v_0 then v. Recall that there are always countably many such neighbours, hence the sequence is infinite.

Define $\gamma w_i^{(v,j)} = w_i^{(\gamma v,j)}$. Clearly this assignment is bijective if the assignment on $S_{v_0}(n)$ is bijective, which is the case since we started with a permutation for $n = 1$. It is also straightforward to check that it preserves adjacency and colours.

Proceeding inductively we obtain the desired automorphism. $\square$

Another example of a graph where a random colouring with finitely many colours is almost surely not distinguishing is the Rado graph, also known as the infinite random graph.

It can be obtained by the following random process. Take a countable set V of vertices and independent 0-1-random variables x_{uv} for each pair $\{u, v\} \in \binom{V}{2}$. The edge set is $\{uv \mid x_{uv} = 1\}$. It is possible to show that the result of this random process almost surely has the following property.

(R) For any two disjoint finite sets $U, U' \subseteq V$ there is a vertex $v \in V \setminus (U \cup U')$ such that $uv \in E$ for every $u \in U$ and $u'v \notin E$ for every $u' \in U'$.

It can be shown that any two countable graphs with property (R) are isomorphic, hence this property characterises the Rado graph.

Theorem 6.2. *A random colouring of the Rado graph with finitely many colours is almost surely not distinguishing.*

Proof. Note that all the colour classes are almost surely infinite. Secondly, since the colours are chosen independently from the edges, the induced graph of every colour class is isomorphic to the Rado graph. Hence it is not surprising that the randomly coloured Rado graph has the following property.

($*$) For any two disjoint finite sets $U, U' \subseteq V$ and for every colour c there is a vertex $v \in V \setminus (U \cup U')$ such that $uv \in E$ for every $u \in U$, $u'v \notin E$ for every $u' \in U'$, and v has colour c.

For the proof of property ($*$) observe that the probability that a vertex has the right neighbours in $U \cup U'$ and the correct colour is $p = \frac{1}{k} 2^{-|U|-|U'|}$, where k is the number of colours. Let $v_1, \dots v_N$ be vertices not contained in $U \cup U'$. Then the probability that none of these vertices has the right neighbours and the correct colour is $(1-p)^N$, which tends to 0 as N goes to infinity, because p is strictly larger than 0.

Now we use property ($*$) to show that there is a non-trivial automorphism which preserves the colouring. In fact we show even more, namely that the randomly coloured Rado graph is homogeneous. That is, whenever we have a colour preserving automorphism of two finite induced subgraphs, then this automorphism can be extended to a colour preserving automorphism of the whole graph. Clearly this implies that there is a non-trivial colour preserving automorphism since the induced subgraphs of two singletons are always isomorphic.

So assume that we have a colour preserving automorphism γ of two finite induced subgraphs. Let $(v_i)_{i \in \mathbb{N}}$ be an enumeration of the vertices which have no image or no preimage under γ. We proceed inductively.

In step i, denote by V_i the set of vertices whose image under γ has already been defined. If $v_i \notin V_i$ then we find a vertex v such that

- for $w \in V_i$ there is an edge connecting γw to v if and only if wv_i is an edge, and
- v and v_i have the same colour.

Note that such a vertex exists by ($*$). We set $\gamma v_i = v$.

If $v_i \notin \gamma V_i$, that is, v_i has no preimage under γ yet, then we can use an analogous argument to find a vertex v such that

- for $w \in \gamma V_i$ there is an edge connecting $\gamma^{-1} w$ to v if and only if v_i is an edge, and
- v and v_i have the same colour.

Clearly we can choose $\gamma v = v_i$.

Hence after step i the vertex v_i has both an image and a preimage under γ. If we let i go to infinity we end up with a function $\gamma \colon V \to V$. By construction γ is bijective. There is an edge from γu to γv if and only if uv is an edge, because both u and v are eventually contained in V_i. Thus γ is an automorphism. Furthermore it preserves colours by construction. □

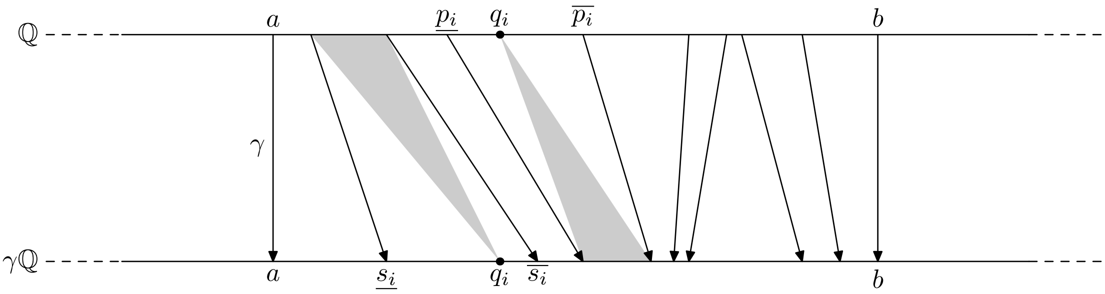

Figure 6: Finding an image and preimage for q_i. Arrows indicate, which values of γ have already been chosen inside the interval $I = [a, b]$. Note that the arrows cannot cross because the function is order preserving by construction. The grey triangles show the possible choices for γq_i and $\gamma^{-1} q_i$.

A similar construction shows that (for any finite set C of colours) the randomly C-coloured Rado graph is universal for the set of C-coloured graphs, that is, any countable coloured graph is an induced subgraph of the randomly C-coloured Rado graph.

We now proceed to show that local finiteness is also necessary in Conjecture 1.2. It can easily be seen that any countable tree with infinite motion admits a distinguishing 2-colouring. Hence trees are not sufficient to show that we cannot drop the assumption in Conjecture 1.2. The Rado graph is also no counterexample as it was shown to be 2-distinguishable in [13].

The construction that we use relies on the following result from [17] which also shows that there are permutation groups of countable sets whose distinguishing number is infinite. In particular Conjecture 1.4 becomes false if we drop the assumption of subdegree finiteness.

Theorem 6.3. *Let Γ be the group of all bijective, order preserving functions $\gamma\colon \mathbb{Q} \to \mathbb{Q}$. Then Γ has infinite motion but its distinguishing number is infinite.*

Proof. Clearly Γ has infinite motion because if some element $q \in \mathbb{Q}$ is moved by $\gamma \in \Gamma$, then all elements that lie between q and γq must be moved as well.

It remains to show that there is no distinguishing colouring with a finite number of colours. Let $c\colon \mathbb{Q} \to C$ be a colouring of the rationals with a finite number of colours. It is easy to see that we can find an open interval I such that the preimage of each colour is either dense in I or it does not intersect I at all. We can choose I such that the interval boundaries are rational.

We now use the interval I to define a colour preserving function $\gamma \in \Gamma$. For $q \in \mathbb{Q} \setminus I$ let $\gamma q = q$. In order to define γ on I let $(q_i)_{i\in\mathbb{N}}$ be an enumeration of all elements of $\mathbb{Q} \cap I$.

We now inductively define γ on I. Assume that we have already chosen γq_j and $\gamma^{-1} q_j$ for each $j < i$. The process of finding an image and a preimage of q_i is sketched in Figure 6. Denote by P_i the set of all $q \in \mathbb{Q}$ such that γq has already been defined. Note that

$$\overline{p_i} = \min\{q \in P_i \mid q \geq q_i\} \qquad \text{and} \qquad \underline{p_i} = \max\{q \in P_i \mid q \leq q_i\}$$

exist. This is because there are only finitely many candidates for the minimum and maximum, namely the boundary points of the interval and the finite set of points inside I whose image has already been chosen. If $\overline{p_i} = \underline{p_i}$, then they must both coincide with q_i and hence we have already chosen the image of q_i in an earlier step. Otherwise we can choose $\gamma q_i \in (\gamma\underline{p_i}, \gamma\overline{p_i})$ such that $c(q_i) = c(\gamma q_i)$ because the preimage of $c(q_i)$ is dense in I and hence also in $(\underline{p_i}, \overline{p_i}) \subseteq I$. Furthermore we can choose $\gamma q_i \neq q_i$.

Next we define $\gamma^{-1}q_i$. For this purpose let S_i be the set of all $q \in \mathbb{Q}$ whose preimage has already been defined. The elements

$$\overline{s_i} = \min\{q \in P_i \mid q \geq q_i\} \qquad \text{and} \qquad \underline{s_i} = \max\{q \in P_i \mid q \leq q_i\}$$

exist for the analogous reasons as $\overline{p_i}$ and $\underline{p_i}$. If $\overline{s_i} = \underline{s_i}$ then they must both coincide with q_i and hence we have already chosen the preimage of q_i in an earlier step. Otherwise we can choose $\gamma^{-1}q_i \in (\gamma^{-1}\underline{s_i}, \gamma^{-1}\overline{s_i})$ such that $c(q_i) = c(\gamma^{-1}q_i)$.

If we repeat this construction then we end up with a function $\gamma\colon \mathbb{Q} \to \mathbb{Q}$. This function is bijective because every $q \in \mathbb{Q}$ has a unique image and a unique preimage. It preserves the colouring and the order by construction and it is not the identity because in the first step we can choose $\gamma q_1 \neq q_1$.

We have showed that for any colouring c of $\mathbb{Q}$ with a finite number of colours there is a non-trivial permutation $\gamma \in \Gamma$ which preserves c. Hence the distinguishing number of this group must be infinite. □

Clearly the group Γ of the above theorem is the full automorphism of a directed graph. Simply draw an arrow from q to r if $q \leq r$. The underlying undirected graph is the complete countable graph which also has infinite distinguishing number but only finite motion.

One standard way to turn a directed graph into a graph is replacing every vertex v by an isomorphic copy of an asymmetric graph H. In this graph fix two vertices x^{in} and x^{out}. Denote by x^{in}_v and x^{out}_v the vertices corresponding to x^{in} in the copy of H that replaced the vertex v. Finally connect x^{out}_v to x^{in}_w whenever there is a directed edge from v to w.

If we apply this construction to the directed graph obtained from the order of the rationals, then we end up with a undirected graph G with the same automorphism group. Automorphisms of this graph simply permute the copies of H in the same way as the corresponding order automorphism of $\mathbb{Q}$ permuted the rationals. Clearly we can choose H to be finite. If H has n vertices then there are only k^n different colourings of H with k colours. In particular every colouring of G with k colours corresponds to a colouring of $\mathbb{Q}$ with n^k colours and hence it cannot be distinguishing.

However, there is an even more elegant way to use Theorem 6.3 to show that Conjecture 1.2 is not true for arbitrary countable graphs.

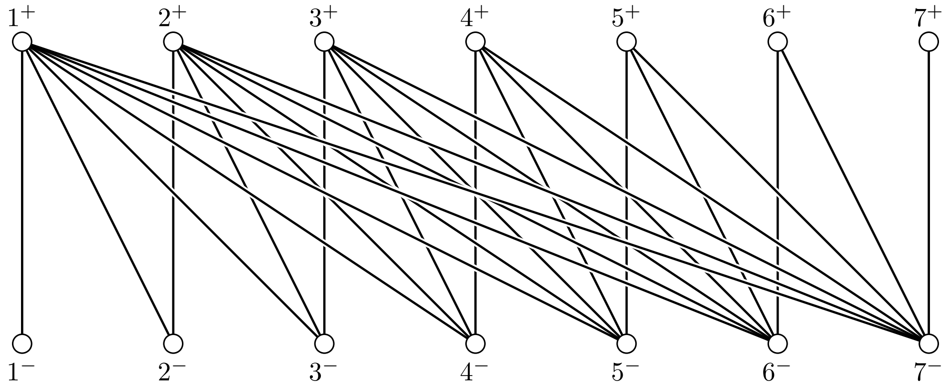

Figure 7: An induced subgraph of the graph in Theorem 6.4. Note that edges only go from top left to bottom right. In fact, by the definition of the graph all such edges are present and every edge is of this type.

Theorem 6.4. *There is a countable, connected, transitive graph with infinite motion which has no distinguishing colouring with a finite number of colours.*

Proof. Let $\mathbb{Q}^+$ and $\mathbb{Q}^-$ be two disjoint copies of $\mathbb{Q}$. Denote the element corresponding to $q \in \mathbb{Q}$ in these copies by q^+ and q^- respectively. Consider the graph $G = (V, E)$ where $V = \mathbb{Q}^+ \cup \mathbb{Q}^-$ and $q^+r^- \in E$ whenever $q \leq r$. Figure 7 shows an small subgraph of this graph to give an idea of what it looks like.

Clearly the graph is countable and connected. To see that it is transitive note that if γ is an order automorphism of $\mathbb{Q}$, then the maps $\gamma_\uparrow$ and $\gamma_\downarrow$ where $\gamma_\uparrow(q^+) = (\gamma(q))^+$, $\gamma_\uparrow(q^-) = (\gamma(q))^-$, $\gamma_\downarrow(q^+) = (-\gamma(q))^-$, and $\gamma_\downarrow(q^-) = (-\gamma(q))^+$ are automorphisms of G.

We claim that there are no further automorphims of G. To prove this claim, note that G is bipartite with bipartition $\mathbb{Q}^+ \cup \mathbb{Q}^-$. Hence every automorphism of G either fixes $\mathbb{Q}^+$ and $\mathbb{Q}^-$ setwise, or swaps the two sets. Furthermore every edge q^+q^- must be mapped to an edge r^+r^- because q^+ is the unique vertex with the property $N(q^+) = \bigcap_{v \sim q^-} N(v)$ and vice versa. So the action on $\mathbb{Q}^+$ uniquely determines an automorphism of G.

It is not hard to see that $q \leq r$ if and only if $N(q^+) \subseteq N(r^+)$. This implies that $N(g(q^+)) \subseteq N(g(r^+))$ for every automorphism φ of G. If φ fixes $\mathbb{Q}^+$ setwise we conclude that φ preserves the order on $\mathbb{Q}^+$, hence it is $\gamma_\uparrow$ for a suitable order automorphism γ. An analogous argument shows that if φ swaps $\mathbb{Q}^+$ and $\mathbb{Q}^-$, then $\varphi = \gamma_\downarrow$ for an order automorphism γ of $\mathbb{Q}$.

Every map of the type $\gamma_\uparrow$ and $\gamma_\downarrow$ moves infinitely many vertices, hence G has infinite motion.

Finally assume that there is a distinguishing colouring c of G with $n \in \mathbb{N}$ colours. In particular this colouring would break every automorphism of the form $\gamma_\uparrow$. Hence the map $q \mapsto (c(q^+), c(q^-))$ would be a distinguishing colouring of $\mathbb{Q}$ with $n^2 < \infty$ colours, a contradiction to Theorem 6.3. $\square$

6.2 Sets with higher cardinality

Recall that by Theorem 5.1 any countable set Δ of permutations of a countable set S is 2-distinguishable. In this section we prove an uncountable analogue of this result. The proof is essentially due to Imrich and has been published in [5]. We stress that in contrast to all results in this thesis so far, in the following theorem the set S does not have to be countable. In fact, if it is countable we recover the special case of Theorem 5.1.

Theorem 6.5. *Let Γ be a group acting on a set S. Then $\aleph_0 \leq |\Gamma| \leq m(\Gamma)$ implies $D(\Gamma) = 2$.*

Proof. Set $\mathfrak{n} = |\Gamma|$, and let ζ be the smallest ordinal number of cardinality $\mathfrak{n}$. Furthermore, choose a well ordering $\prec$ of $\Delta = \Gamma \setminus \{\mathrm{id}\}$ of order type ζ. Then for every $\alpha \in \Delta$ the cardinality of the set $\Delta_\alpha = \{\beta \in \Delta \mid \beta \prec \alpha\}$ is strictly smaller than $\mathfrak{n} \leq m(\Gamma)$.

We now define a colouring of S by transfinite recursion. In each step of the recursion we find the minimal $\alpha \in \Delta$ such that α preserves the partial colouring c defined so far. We then find an $s \in S$ such that $\alpha s \neq s$ and neither s nor αs have been coloured so far. Colouring s and αs with different colours clearly breaks α.

The recursion ends if either all elements of Δ are broken or if we cannot find an s as stated. Thus it remains to show that such an s always exists.

Denote by S' the support of the partial colouring c. If α is the minimal element of Δ with respect to $\prec$ such that α preserves c then there was at most one step for each $\beta \in \Delta_\alpha$. In each of those steps only two elements of S were assigned a colour. Hence $|S'| \leq 2|\Delta_\alpha|$. Since $m(\Gamma)$ is infinite and $|\Delta_\alpha| < m(\Gamma)$ this implies that $|S'| < m(\Gamma)$.

In particular, if S'' is the set of elements moved by α then

$$|S'' \setminus (S' \cup \alpha^{-1} S')| \geq m(\Gamma) - 2|S'| = m(\Gamma).$$

Thus in every step there is an element $s \in S$ which is moved by α such that neither s nor αs have been coloured so far. $\square$

Note that just like in Theorem 5.1, the group structure did not play any role in the proof. Hence we can get an analogous result by taking an arbitrary set of permutations instead of the group Γ.

Furthermore, if the generalised continuum hypothesis holds, then we have the following result. Note the similarity to Lemma 1.3.

Corollary 6.6. *Let Γ be a group acting on a set S. If the generalised continuum hypothesis holds, and if $|\Gamma| < 2^{m(G)}$, then $D(G) = 2$.*

Proof. For finite values of $m(\Gamma)$ this follows from Lemma 1.3.

If $m(\Gamma)$ is infinite then under the assumption of the general continuum hypothesis $2^{m(\Gamma)}$ is the successor of $m(\Gamma)$. Hence $|\Gamma| \leq m(\Gamma)$, and the assertion of the corollary follows from Theorem 6.5. $\square$

In particular, if S is countable we get the following result.

Corollary 6.7. *Let* Γ *be a group acting on a countably infinite set* S *with infinite motion. If the continuum hypothesis holds, and if* $|\mathrm{Aut}\,(G)| < 2^{m(G)}$*, then* $D(G) = 2$*.* $\square$

7 Outlook and open questions

In the last three chapters we have seen how both deterministic and probabilistic methods can be used to make progress towards Conjecture 1.2. In its full generality, however, the conjecture is still wide open.

The main goal of this final chapter is to highlight some interesting questions, whose answers could help to settle the infinite motion conjecture and give more insight in the structure of infinite permutation groups.

In light of Corollary 5.13 it is clear that knowing more about compact subdegree finite groups brings us closer to a solution of Conjectures 1.2, 1.4, and 1.5. Lemma 3.9 tells us that for such groups every orbit must be finite. Clearly, if the group has infinite motion, then the action on those orbits cannot be independent. However, very little is known about their interplay. We formulate this as a (admittedly very vague) question.

Question 7.1. What can we say (in terms of structure) about the action of compact subdegree finite permutation groups with infnite motion? What about point stabilisers in such groups?

The second question we pose is related to the distinct spheres condition. Recall that a graph $G = (V, E)$ satisfies the distinct spheres condition if there is a vertex v_0 such that for every pair of distinct vertices $u, v \in V$ the condition $d(v_0, u) = d(v_0, v)$ implies that $S_u(n) \neq S_v(n)$ for infinitely many values of n. If an automorphism γ maps u to v, then it maps the set $S_u(n) \setminus S_v(n)$ to a subset of $S_v(n)$. Clearly the two sets are disjoint. If G satisfies the distinct spheres condition, then it is easy to see that they are non-empty. As a consequence, we can enumerate infinitely many vertices which must be moved whenever u is mapped to v. This infinite sequence can be used to define a distinguishing colouring.

For many graphs with infinite motion which do not satisfy the distinct spheres condition it is still possible to find such an infinite sequence of vertices. There are however examples of locally finite graphs with infinite motion where this fails. Consider for example the following graph.

Let V_0, V_1, V_2 be sets of vertices with $|V_1| = 2|V_0|$ and $|V_2| = 2^{|V_0|}$. Assign to each $v \in V_0$ two vertices v^{l} and v^{r} in V_1 such that the sets $\{u^{\mathrm{l}}, u^{\mathrm{r}}\}$ and $\{v^{\mathrm{l}}, v^{\mathrm{r}}\}$ are disjoint for $u \neq v$. Connect every $v \in V_0$ to v^{l} and v^{r}. To each vertex $z \in V_2$ assign a subset $V_z \subseteq V_0$ and connect z to v^{l} if $v \in V_z$ and to v^{r} otherwise. Figure 8 shows this construction for $|V_0| = 3$.

Iterate this construction with V_2 playing the role of V_0. This gives an infinite, locally finite graph with infinite motion which does not satisfy the distinct spheres condition. Furthermore, for every permutation π of V_0 and every $z \in V_2$ there is an automorphism of G which acts like π on V_0 and fixes z. In particular we cannot give an infinite sequence

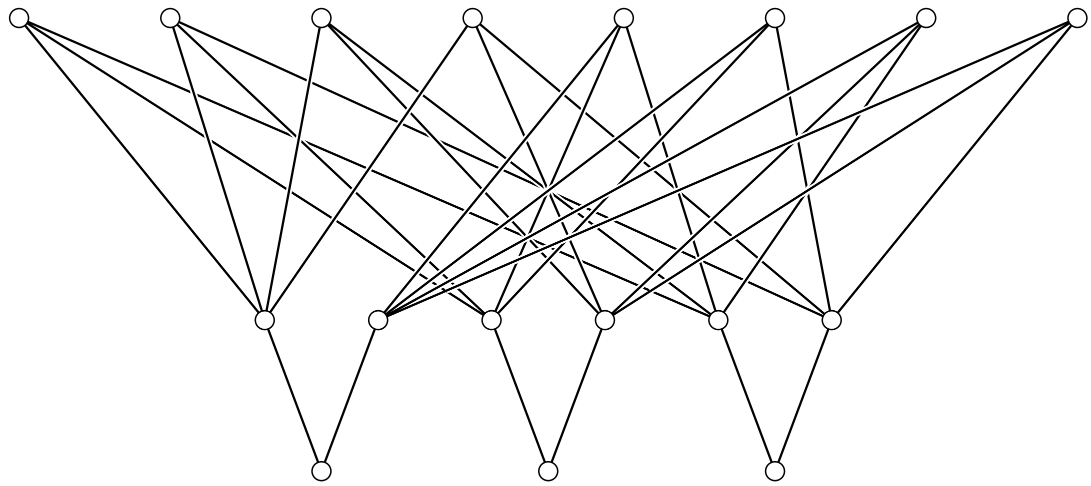

Figure 8: Avoiding distinct spheres. By iterating the construction above we get a graph with infinite motion which does not satisfy the distinct spheres condition.

of vertices which must be moved, if some particular vertex u is mapped to another vertex v.

However, one can show that this graph satisfies the following generalisation of the distinct spheres condition which was a joint discovery with Simon Smith.

($*$) For every pair u, v of vertices there is an infinite sequence $(U_i)_{i \in \mathbb{N}}$ of finite subsets of V such that for every automorphism γ with $\gamma u = v$ it holds that $\gamma U_i \neq U_i$ for every $i \in \mathbb{N}$.

It is not hard to see that such a graph must have infinite motion, and that there is a distinguishing 2-colouring. The proof is very similar to the proof of Theorem 3.21. However, it seems hard to come up with an example of a locally finite graph with infinite motion which does not satisfy ($*$). This brings us to the following question.

Question 7.2. Is there an infinite, locally finite graph with infinite motion which does not satisfy ($*$)?

Clearly, if such a graph does not exist, then this would immediately imply that Tucker's conjecture is true. But even if we can come up with an example of such a graph it may be another step towards proving (or disproving) the conjecture, because it may give more insight on the structure of locally finite graphs with infinite motion.

The next question is related to the random colourings studied in Chapter 5. The reader my have noticed that for all graphs studied there and in Section 6.1 a random 2-colouring of the vertices was either almost surely distinguishing or almost surely not distinguishing. Hence it is only natural to ask if such a 0–1-law holds more generally.

Question 7.3. Let $G = (V, E)$ be a countable graph with infinite motion and let c be a random colouring of G. Is there a 0–1-law for the probability that c is distinguishing, that is, is it always true that

$$\mathbb{P}[c \text{ is distinguishing}] \in \{0, 1\}?$$

The final question we would like to ask has already been posed by Imrich et al. [14] and is related to Conjecture 1.4. As we have seen, this conjecture becomes false when we drop the requirement of subdegree finiteness. The question whether closedness is really necessary, however, is still open.

Question 7.4. Is there a subdegree finite non-closed permutation group with infinite motion and distinguishing number > 2?

Similarly to Question 7.2, if the answer to this question is no, then Conjecture 1.4 and hence also Conjecture 1.2 is certainly true. A positive answer to this question has no immediate consequence on the status of the conjectures. Nevertheless it is possible that such an example can be used to construct a counterexample.

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