Combinatorial Aspects of [Colored] Point Sets in the Plane

Doctoral Thesis
at
Graz University of Technology

submitted by

Birgit Vogtenhuber

November 11, 2011

Institute for Software Technology
Faculty of Computer Science
Graz University of Technology
A-8010 Graz, Austria

Advisor: Assoc.Prof. Dipl.-Ing. Dr.techn. Oswin Aichholzer
Reader: Catedrático de Universidad Dr. Ferran Hurtado

Abstract

Combinatorial geometry and graph theory are both areas in the field of discrete mathematics. The study of combinatorial aspects of point sets in the plane and graphs based on them is situated in the intersection of these areas and also includes aspects of computational geometry and graph drawing.

We study variations of the classical Erdős-Szekeres problems on convex k-gons and k-holes. Relaxing the convexity condition, we show structural results and bounds on the numbers of k-gons and k-holes. Most noteworthy, for constant k and sufficiently large n, we give a quadratic lower bound for the number of k-holes and show that this number is maximized by sets in convex position. For bichromatic point sets, we show that every sufficiently large such set contains a monochromatic 4-hole.

Continuing with investigations on bichromatic point sets, a famous problem is Zarankiewicz's conjecture on the crossing number of the complete bipartite graph. Studying rectilinear versions of this conjecture, we present several interesting observations. The main question remains unsettled even for these variations though. We also consider the existence of different kinds of compatible colored matchings for given bichromatic graphs of certain classes and present bounds on their cardinalities.

The study of Delaunay triangulations is a central topic in geometric graph theory. We investigate the question of finding a set of additional points W for a given set B with n points such that the Delaunay triangulation of the joint set $B \cup W$ does not contain any edge spanned by two points of B. Improving the previously best known bounds, we show that $|W| \leq 3n/2$ points are sufficient in general and $|W| \leq 5n/4$ points suffice if B is a convex set. We also present a lower bound of $|W| \geq n - 1$.

Investigating pointed pseudo-triangulations, we study the existence of compatible pointed pseudo-triangulations whose union is a maximal plane graph. We present a construction for pairs of such pseudo-triangulations for a given point set. A given pointed pseudo-triangulation in general does not admit such a compatible pseudo-triangulation. Finally, we generalize the pointedness property of pointed pseudo-triangulations to other types of graphs. We consider (1) redrawing the edges of a given plane straight-line graph such that all vertices become pointed; and (2) embedding a planar graph such that every vertex has all its incident edges within a small angle. We show that both tasks can be realized using Bézier-curves, biarcs or polygonal chains of length two as edges.

Kurzfassung

Die Graphentheorie und die kombinatorische Geometrie sind Teilbereiche der diskreten Mathematik. Die Untersuchung kombinatorischer Aspekte von Punktmengen und Graphen darauf ist ein wesentlicher, gemeinsamer Teil dieser beiden Forschungsgebiete und beinhaltet auch Aspekte der rechnerischen Geometrie sowie des Graphzeichnens.

Wir untersuchen Varianten des klassischen Erdős-Szekeres-Problems über (leere) konvexe k-Ecke (Vielecke mit k Punkten) in Punktmengen, bei denen die k-Ecke nicht notwendigerweise konvex sind. Es werden strukturelle Eigenschaften sowie Schranken für die Anzahl von (leeren) k-Ecken gezeigt. So wird bewiesen, daß für konstantes k und hinreichend großes n die Anzahl leerer k-Ecke zumindest quadratisch in n ist und von konvexen Punktmengen minimiert wird. Weiters wird gezeigt daß jede genügend große zweigefärbte Punktmenge monochromate leere Vierecke enthält.

Eine weitere klassische Fragestellung auf zweigefärbten Punktmengen ist die Zarankiewicz-Vermutung zur Kreuzungszahl des vollständigen bipartiten Graphen. Bei der Betrachtung geradliniger Versionen dieser Vermutung haben sich einige interessante Beobachtungen ergeben. Die Hauptfragestellung blieb allerdings auch für diese Varianten ungelöst. Des weiteren wird die Existenz verschiedener Arten von kompatiblen, gefärbten Paarungen für gegebene bichromate Graphen bestimmter Klassen untersucht, wobei Schranken für die Kardinalität solcher Paarungen präsentiert werden.

Delaunay-Triangulierungen sind ein wesentliches Thema in der Graphentheorie. Wir behandeln eine Fragestellung, bei der für eine gegebene Menge B mit n Punkten eine zuätzliche Punktmenge W gesucht wird, sodaß die Delaunay-Triangulierung der Vereinigungsmenge $B \cup W$ keine Kante in B enthält. Es werden die folgenden, verbesserten Schranken für die notwendige Anzahl von Punkten in W gezeigt: $|W| \leq 3n/2$ im allgemeinen, $|W| \leq 5n/4$ für konvexe Mengen B, und $|W| \geq n - 1$.

Weiters wird die Existenz von kompatiblen minimalen Pseudotriangulierungen untersucht, deren Vereinigung eine Triangulierung ergibt. Es wird ein Algorithmus zur Erzeugung von Paaren solcher Pseudotriangulierungen für eine gegebene Punktmenge präsentiert. Das letzte betrachtete Thema ist eine Verallgemeinerung des Konzepts der spitzen Winkel in minimalen Pseudotriangulierungen auf planare Graphen, mit folgenden Aufgabestellungen: (1) Neuzeichnen der Kanten eines ebenen Graphen, sodaß alle Knoten zu einem spitzen Winkel inzident sind; und (2) Zeichnen eines planaren Graphen, sodaß die inzidenten Kanten jedes Knotens innerhalb eines sehr kleinen Winkels liegen. Es wird gezeigt, daß beide Aufgabenstellungen gelöst werden können, indem man Bézierkurven, Doppelkreisbögen oder Polygonzüge der Länge zwei als Kanten verwendet.

Deutsche Fassung:
Beschluss der Curricula-Kommission für Bachelor-, Master- und Diplomstudien vom 10.11.2008
Genehmigung des Senates am 1.12.2008

EIDESSTATTLICHE ERKLÄRUNG

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst, andere als die angegebenen Quellen/Hilfsmittel nicht benutzt, und die den benutzten Quellen wörtlich und inhaltlich entnommene Stellen als solche kenntlich gemacht habe.

Graz, am ……………………………. ………………………………………………..
(Unterschrift)

Englische Fassung:

STATUTORY DECLARATION

I declare that I have authored this thesis independently, that I have not used other than the declared sources / resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

……………………………. ………………………………………………..
date (signature)

Acknowledgements

First and foremost, I wish to express my deepest gratitude to my parents, who have supported me in all possible ways throughout my whole life. Without your love and patience I would not be at this point now. Best thanks also go to my sisters, my grandma and all my family.

I am indebted to my supervisor and friend, Oswin Aichholzer, for introducing me to the fascinating world of research and for his faith in me, for employing me in his project and offering me the possibility to participate in many exciting research weeks with many different researchers. I highly value his motivating enthusiasm, his enormous support of my work, and the opportunity to learn from his knowledge and experience.

Particular thanks go to Ferran Hurtado. I am not only indebted to him for many inspiring discussions and for conveying, reviewing, and proofreading this thesis, but also for introducing me to the Spanish way of life, in particular tapas and Spanish wine.

I would like to thank all my co-authors, fellow researchers, and collegues. I really appreciated our meetings at conferences and research stays, where we had a lot of fun working in many different fields, at universities as well as in restaurants, cafés, or outdoors. I sincerely enjoyed the good working atmosphere and the many fruitful and inspiring discussions where many new ideas evolved. We obtained nice results, also celebrated them accordingly, some of them condensed into papers, some are still ongoing work. I learned a lot from all for you and hope that cooperations will continue in future.

Special thanks go to Pedro Ramos and his family for their great hospitality during several stays in Alcalá and during his sabatical in Graz. I wish to thank Jorge Urrutia and his family for sheltering me during my stay in México and for the long evenings with research, wine, and philosophical discussions. I am grateful to Gelasio Salazar for hosting me during a research visit in Waterloo, and for supporting me with lots of background information on crossing numbers. I also want to mention Ruy Fabila-Monroy and Clemens Huemer as we have so many topics in common and spent so much time with research and fun together.

I wish to thank my colleagues at the Institute of Software Technology at Graz University of Technology, with whom I enjoyed many joint lunches, chats, and laughes during the last years, plus countless coffees with and without cake. I thank Petra, who additionally helped me on my way through the jungle of administrive tasks. Especially I thank Alex, Thomas, and Wolfgang, with whom I share my office, for their friendship and the positive atmosphere. You were always available for questions, and I really appreciated the many research discussions and private conversations – thanks a lot for everything.

Last but not least, I want to thank all the other people who supported me, including the crews of Pastis and Down Under, who provided me with food and drinks, as well as occasional chats and laughs on the long evenings and weekends of thesis writing this autumn.

Contents

Chapter 1

Introduction

The area of *Combinatorial Geometry* is a field of discrete mathematics which studies questions on combinatorial properties and relations of discrete geometric objects. While the roots of combinatorial geometry are rather old, dating back at least to the time of classical mathematicians like Euler and Kepler, many advances have been made in young history, initiated for example by Minkowski, Fejes Toth, Rogers, and Erdős. Nowadays the field has become a recognized successful research discipline with close relations to computational geometry, and with many applications. A central structure in combinatorial geometry are finite sets of points in the plane. Even for this simple class there are many interesting and challenging research problems.

One example is the famous problem of Paul Erdős on empty convex k-gons [76]: "Is it true that for any k there is a smallest integer $h(k)$ such that any set of at least $h(k)$ points contains at least one empty convex k-gon?" The according problem for convex k-gons (which need not be empty) was raised by Esther Klein in the 1930's. As the positive solution of this problem by George Szekeres and Paul Erdős is said to have advanced the marriage between Szekeres and Klein, it became famous as the "Happy End Problem" [98]. While the existence question for convex k-gons is solved since then, the exact number $g(k)$ of points that are needed to guarantee the existence of a convex k-gon is still unknown.

Erdős' empty convex k-gon question turned out to be less happy. While Klein and Harborth answered it in the affirmative for $k \leq 5$, showing that $h(4) = 5$ and $h(5) = 10$, respectively, some years later Horton proved that there exist arbitrarily large point sets not containing any empty convex 7-gon. It took almost a quarter of a century after Horton's construction to solve the existence question for empty convex 6-gons: In 2007/08, Nicolás [118] and independently Gerken [89] finally proved that $h(6)$ is finite.

As an example of empty k-gons in point sets, Figure 1.1 illustrates an empty convex 5-gon and an empty convex 6-gon in this set of 22 points.

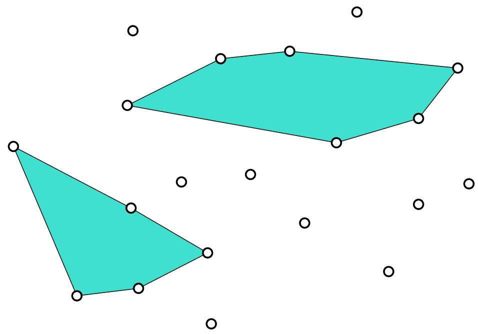


Figure 1.1: A pentahole (empty convex pentagon) and a hexahole in a point set.

Since the initial problem proposal by Klein, many variations have been arising which are generally known as Erdős-Szekeres type questions. The maybe first one was determining the number of k-gons and has already been considered in the 1970's by Erdős and Guy [78]. Other examples that came up later are generalizations to higher dimensions [41, 109] or to not necessarily convex k-gons [16, 31, 15], and the search for monochromatic empty k-gons in colored point sets [66, 124, 27].

An area that is strongly related to discrete and combinatorial geometry is *Graph Theory*, which studies the properties of graphs and their relations. The origins of this field date back to the 18^{th} century and started with a problem known as the "Seven Bridges of Königsberg". Euler's negative solution, showing that it is not possible to make a closed walk through Königsberg such that each of the seven bridges is used exactly once, is commonly seen as the foundation of graph theory. Solutions of graph-theoretic problems are often based on algorithms, and sometimes also on the extensive use of computers. On the other hand, graphs turned out to be a great means for modeling algorithmic problems, yielding the fact that by now graph theory plays an important role in computer science, especially in complexity theory. For an abstract graph, neither the position of the vertices, nor the shape of the edges are relevant. Determining these two properties, we obtain a drawing of the graph.

A classical problem in graph theory is how to draw a graph such that the number of crossings between its edges is minimized. This question originated from Paul Turán's "Brick Factory Problem", where he asked for the crossing number of the complete bipartite graph $K_{n,m}$. The according question for the complete graph K_n was posed later and independently by Anthony Hill. He published a construction where the vertices are arranged on two concentric circles and connected as illustrated in Figure 1.2. Hill's conjecture that these drawings reach the crossing number of K_n is by now known to be true for small values of n, but still open in general.

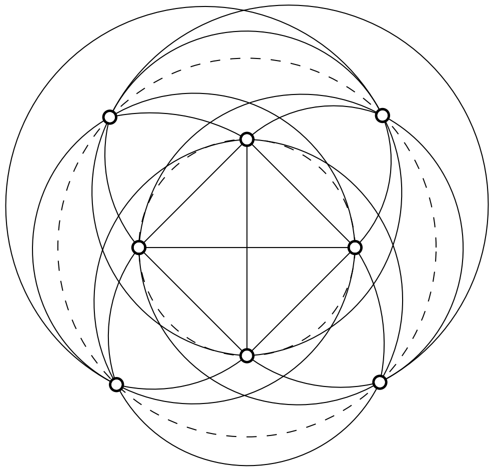


Figure 1.2: Hill's construction for drawings of K_n.

The according conjecture for $K_{n,m}$ by Zarankiewicz is older than the one for K_n and widely known as *Zarankiewicz's conjecture*. Like Hill's conjecture, it is still open (by now since more than 60 years), but has been confirmed for small values of n and m. Figure 1.3 illustrates the principle of Zarankiewicz's drawings of $K_{n,m}$ which attain the conjectured minimum number of crossings (due to the small number of points, the drawing shown in the figure is in fact known to be optimal).

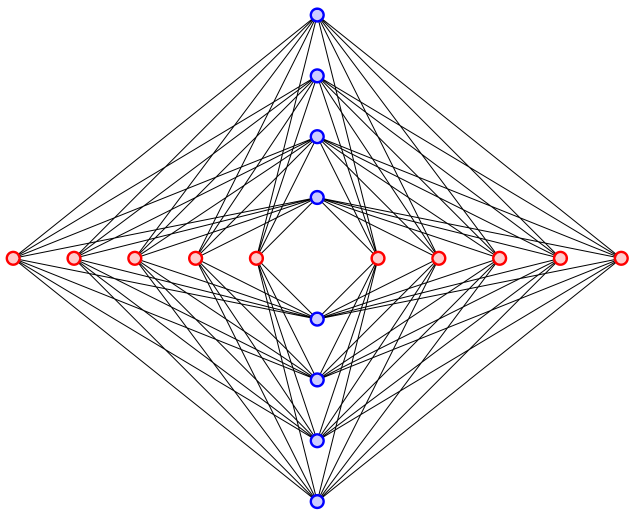


Figure 1.3: Zarankiewicz's construction for drawings of $K_{n,m}$.

There exist many variants of Turán's problem which consider other classes of graphs, drawings on different surfaces, or special kinds of drawings; see for example [46, 125]. Let us mention the rather popular research on the rectilinear crossing number (of some type of graph). There, the edges are straight-line segments and thus the number of crossings is determined solely by the placement of the points. This setting is also strongly related to the before-mentioned Erdős-Szekeres type questions.

Note that while Zarankiewicz's conjecture being true would imply that optimal drawings of $K_{n,m}$ can be rectilinear, minimizing the number of crossings of K_n is known to be in general not possible with straight-line drawings.

The questions considered in this thesis are situated in the intersection of the areas of combinatorial geometry and graph theory. The above-mentioned Erdős-Szekeres type questions and crossing number problems are among the "royal members" of questions in this area, and form the central topic of this work.

Meanwhile there exist a numerous amount of textbooks in the fields of combinatorial and computational geometry and graph theory. As some references for graph theory, let us mention the books by Biggs [49], Diestel [67], and Gross and Yellen [91]. Literature on computational and combinatorial geomerty are for example the text books by Goodman and O'Rourke [90], Pach and Agarwal [123], and de Berg et al. [58]. Classical references in this field are the books by Edelsbrunner [72] and Preparata and Shamos [129], and a very recent one is by Devadoss and O'Rourke [65]. We also mention the book on research problems by Brass et al. [53].

1.1 Outline of the thesis

In this work, we consider several questions within the area of combinatorial geometry and graph theory which we have investigated during the last years. Here we give a short overview of the considered questions and results, and of how they are organized in the following chapters.

In Chapter 2, we consider variations of the classical Erdős-Szekeres problems on the existence and number of convex k-gons and k-holes (empty k-gons) in a set of n points in the plane in general position. Allowing also non-convex k-gons and k-holes, we show bounds and structural results on maximizing and minimizing their numbers. Most noteworthy, for constant k and sufficiently large n, we give a quadratic lower bound for the number of k-holes, and show that this number is maximized by sets in convex position. We also provide improved lower bounds for the numbers of convex 5-holes and 6-holes.

In Chapter 3, we continue with Erdős-Szekeres type questions on colored point sets. In Section 3.1 we answer a question first posed by Hurtado [102] and Pach [122] to the positive, by showing that every sufficiently large bichromatic point set contains a monochromatic 4-hole.

Another classical question on bichromatic point sets is the beforementioned Zarankiewicz Conjecture about the crossing number of the complete bipartite graph, a question that is open since at least 1954. Section 3.2 deals with weaker versions of this conjecture where we (1) restrict the setting to straight-line drawings and (2) require linear separability of the red and blue point set. Although we present several interesting observations, the main question remains unsettled even in these restricted cases.

Switching from complete graphs to the other end of the range, we consider compatible crossing-free geometric graphs on bichromatic point sets. Two plane graphs on top of the same point set S are called compatible, if their union is again a plane graph. In Section 3.3, we investigate the question of finding colored compatible matchings (of several types) for given bichromatic graphs. For the classes of spanning trees, spanning cycles, spanning paths, and perfect matchings we show bounds on the numbers of edges that can always be obtained by such compatible matchings.

Chapter 4 continues with graphs on top of bichromatic point sets, but in a different setting which was introduced by Aronov et al. [38]. Given a set of black points B, $|B| = n$, our goal in Section 4.1 is to choose a set of white points W, such that the Delaunay triangulation of the joint set $B \cup W$ does not contain any black-black edge. We show the following improvements. $|W| \leq 3n/2$ points are always sufficient to block all edges of B. Moreover, if

B is convex, then $|W| \leq 5n/4$ white points always suffice. We also present a lower bound of $|W| \geq n - 1$.

In Section 4.2, we switch from triangulations to pointed pseudo-triangulations, and to compatibility in the classical sense. Given a pointed pseudo-triangulation PT, we want to find another pointed pseudo-triangulation PT' that is compatible to PT and at the same time as different from PT as possible, meaning that $PT \cup PT'$ form a maximal plane graph. We show that this is in general not possible by providing a construction for arbitrarily large point sets S and pointed pseudo-triangulations $PT(S)$ such that any pointed pseudo-triangulation $PT'(S) \neq PT(S)$ is incompatible with $PT(S)$. On the other hand, for any given set S, we construct pairs of pointed pseudo-triangulations $PT_1(S)$, $PT_2(S)$ whose union is a triangulation of S.

In Chapter 5 we generalize the pointedness property of pointed pseudo-triangulations to other types of graphs. Finding straight-line graphs of certain types for a given point set S where every point has an incident angle of at least some value α (independent of S) has been considered in [23]. Relaxing the straight-line condition for the edges and instead allowing simple curves as edges, we consider (1) redrawing a given plane straight-line graph such that the vertex positions remain the same and all vertices become pointed; and (2) embedding a given plane straight-line graph such that for every vertex v, all edges emanate from v in the (nearly) same direction. We show among others that both questions can be answered to the positive when using Bézier-curves, biarcs or polygonal chains of length two as edges.

Parts of this thesis have already been presented at conferences [16, 31, 15, 26, 18, 30, 34] and are or will be published in journals [27, 35, 19, 17, 32]. These publications are marked bold in the publication list of the curriculum vitae (page 191)

Research meetings and financial support

Research on the topic of Chapter 2 was initiated during the *Third Workshop on Discrete Geometry and its Applications* in Morelia (Michoacán, Mexico), August 2010.

Research for Section 3.1 was mainly carried out during the visit of C. Huemer and F. Hurtado in Graz (Austria) for the defense of C. Huemer, March 2008.

Research on the topic of Section 3.2 was partly carried out during a research week in Graz (Austria) in October 2007 and two research weeks in Alcalá de Hernares (Spain) in July 2008 and June 2009.

Research on Section 3.3 was initiated by F. Hurtado during the *i-Math Winter School: DocCourse on Discrete and Computational Geometry* in Bellaterra (Barcelona, Spain).

Research on the topic of Section 4.1 commenced during a visit of R. Fabila in Graz (Austria), March 2010.

Research for Section 4.2 was started during the *Fifth European Pseudo-Triangulation Research Week* in Ratsch (Austria), October 2008.

Research on the topic of Chapter 5 was initiated during the *Fourth European Pseudo-Triangulation Week* in Eindhoven (the Netherlands), 2007.

In addition, there have been further research weeks and research stays (both incoming and outgoing) which contributed to this work. I would like to thank all the organizers for their hospitality, and all the participants for the good atmosphere and fruitful discussions.

Research for this thesis has been supported by the Austrian Science Fund (FWF): S9205-N12, National Research Network 'Industrial Geometry'.

1.2 Definitions and notations

The following paragraphs contain basic definitions which are used throughout this thesis. We mostly follow the notation of the textbooks [65] and [129].

Point sets, k-gons, and k-holes

Let S be a set of n (labeled) points in the Euclidean plane. We say that S is in *general position* if no three points of S lie on a common line.

The *convex hull* of a point set S is the intersection of all convex subsets of the Euclidean plane which contain S. We denote the convex hull of S with $\mathrm{CH}(S)$, with h the number of extreme points of S, that is, the points of S that are on the boundary of $\mathrm{CH}(S)$, and with $i = n - h$ the number of non-extreme (interior) points of S. We say that a point of S is *on the convex hull of* S if it is on the boundary of $\mathrm{CH}(S)$.

If all points of S lie on the convex hull of S then we say that S is a *convex set* (or, equivalently, that the points of S are in *convex position*).

A *polygon* is a closed and connected subset of the Euclidean plane, whose boundary consists of a finite set of straight-line segments forming a closed curve; see Figure 1.4. The end points of these segments are called the *vertices* of the polygon, and the segments themselves are called *edges*. The edges are assumed to be maximal, in the sense that no two consecutive edges along the boundary of the polygon span an angle of π. Polygons can be seen as one of the most important non-trivial class of elements for computational geometry in the plane; see for example Chapter 1 in [65].

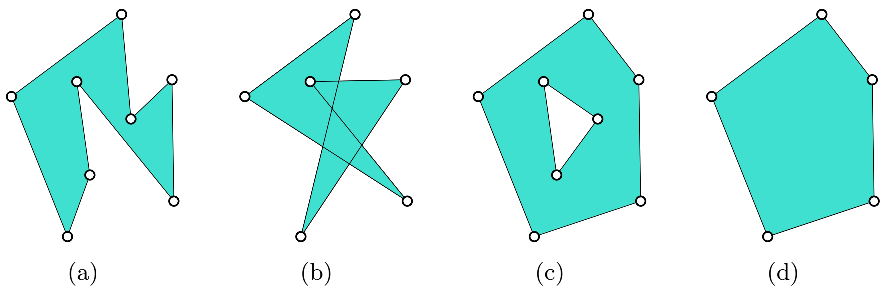


Figure 1.4: (a) a simple polygon; (b) a non-simple polygon; (c) not a polygon; (d) a convex polygon.

A polygon is called *simple*, if its boundary is non–self-intersecting. The boundary of a simple polygon partitions the Euclidean plane into the (finite) *interior* and the (infinite) *exterior* of the polygon. A simple polygon with k vertices is also called *k-gon*.

A vertex of a k-gon is called *convex*, if the angle between its two incident edges that lies in the interior of the k-gon is smaller than π. If this angle is larger than π, the vertex is called *reflex*. By definition, an interior angle of exactly π does not occur. A k-gon is called convex if all its vertices are convex (and *non-convex* otherwise). Last but not least, a k-gon which has exactly three convex vertices (and arbitrary many reflex vertices) is called a *pseudo-triangle*; see Figure 1.5.

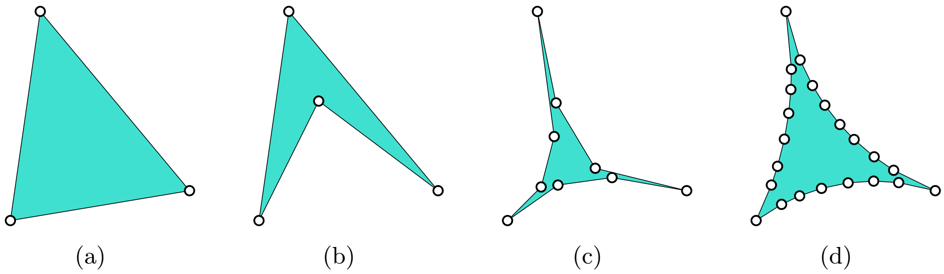


Figure 1.5: Several different pseudo-triangles.

When a subset S' of a point set S is the vertex set of a polygon P, we say that S *contains* P or that P is *spanned* by S'. A k-gon that is spanned by points of S and does not contain any points of S in its interior, is called a *k-hole* (of S); see Figure 1.6. Questions about the existence and number of k-gons and k-holes in point sets go back to Erdős [76, 78, 80] and have been considered in many variations; see for example the survey [9].

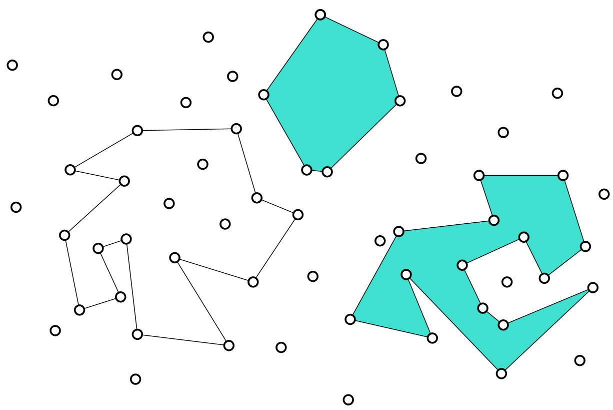


Figure 1.6: A non-convex 15-gon, a non-convex 15-hole, and a convex 6-hole in a point set.

Note that the convex hull CH(S) can also be seen as the (area-wise) largest simple polygon spanned by points of S. Thus, the points of S on

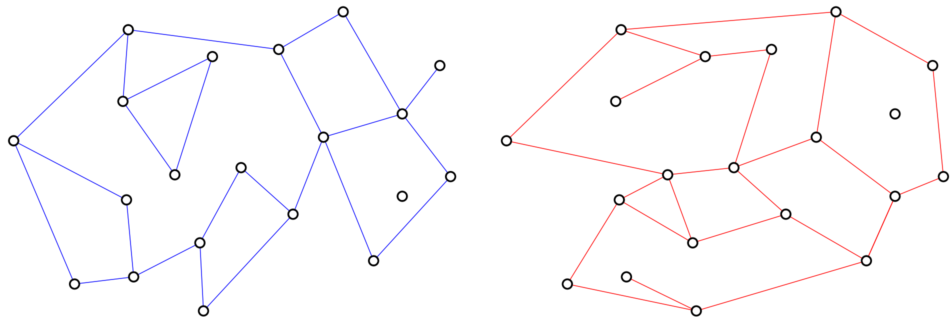
CH(S) are sometimes also referred to as *vertices* of CH(S), and the segments between two consecutive vertices are also called *edges* of CH(S).

Straight-line graphs, compatibility, and crossing-numbers

A *geometric graph* or *straight-line graph* $G(S)$ *on top of* a point set S is an undirected graph whose vertex set is S and whose edges are straight-line segments spanned by pairs of points of S. Note that, as S is in general position, no point of S can lie in the interior of an edge of $G(S)$. A geometric graph $G(S)$ is called *plane* or *crossing-free* if all its edges are non-crossing, meaning that they do not share a point in the interior of an edge (but might share a common endpoint).

Note that, with the exception of Chapter 5, all the graphs we will consider in this thesis are straight-line graphs. Thus, for the sake of brevity, we will sometimes omit the attribute 'straight-line'.

As already mentioned, two plane straight-line graphs $G(S)$ and $G'(S)$ on top of (the same) point set S are *compatible*, if their union $G(S) \cup G'(S)$ is again a plane graph; see for example [130, 131]. Accordingly, we call an edge $e \in (S \times S)$ compatible (to $G(S)$) if $G(S) \cup e$ is again crossing-free. For recent work on compatible graphs see e.g. [20, 105, 1] and references therein. An overview of results with different types of graph compatibility can be found in [100].



(a) (b)

Figure 1.7: Two compatible graphs with the same vertex set.

Let us come back to graphs with crossings. The *rectilinear crossing number* of a straight-line graph $G(S)$ is the number of crossings of $G(S)$, i.e., the number of pairs of edges in $G(S)$ that cross. The rectilinear crossing number of the point set S itself is denoted with $\overline{\text{cr}}(S)$ and is defined to be the number of crossings in the complete graph with vertex set S. The minimum of $\overline{\text{cr}}(S)$ over all point sets S with n points (in the Euclidean plane, in general position), also called the rectilinear crossing number of the complete graph K_n, is denoted with $\overline{\text{cr}}(n)$. Similarly, the minimum over all point sets S with n points (in the Euclidean plane, in general position) of $\overline{\text{cr}}(G(S))$ can be denoted with $\overline{\text{cr}}(G)$. Figure 1.8 shows two point sets which have the same cardinality but different rectilinear crossing numbers. The left set reaches the minimum $\overline{\text{cr}}(6) = 3$, while the right set has the maximum of 15 crossings. Recent results on the rectilinear crossing number can be found for example in [2, 3, 4, 22, 54]; see also the survey [5] and the webpage [8].

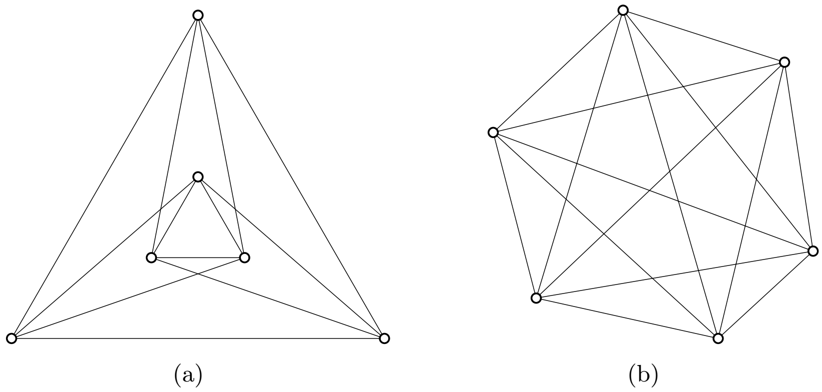


Figure 1.8: Point sets with the same number of points but different rectilinear crossing numbers.

Several according definitions exist for graphs whose edges need not be straight-line segments. In that case it has to be made clear whether a pair of edges is allowed to cross more than once, and, if yes, if such multiple crossings are all counted separately or just as one; see for example [125] for results on the various kinds of crossing numbers, and [46] for a recent survey on the roots of this topic.

Classes of plane straight-line graphs

A *triangulation* $T(S)$ is a maximal plane straight-line graph with vertex set S (maximal with respect to the number of edges). This is equivalent to all interior faces of $T(S)$ being triangles. Figure 1.9 shows an example triangulation. Triangulations are one of the oldest and most important data structures in theoretical and applied computational geometry as well as in related fields, where they are also called meshes; see the textbooks [73, 115].

The *face degree* $d(f)$ of a face f in a graph $G(S)$ is the number of edges in $G(S)$ which are incident to f (i.e., on the boundary of f). Similarly, the *vertex degree* $d(v)$ of a vertex $v \in S$ in a graph $G(S)$ is the number of edges

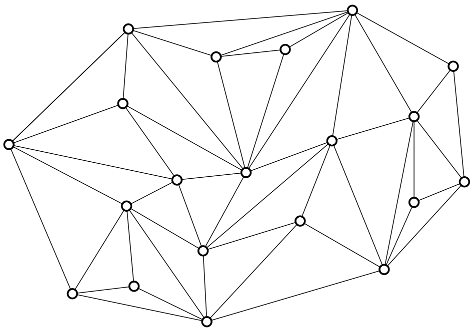


Figure 1.9: A triangulation.

in $G(S)$ which are incident to v. A vertex without any incident edges is called *isolated*

A plane straight-line graph $G(S)$ induces a cyclic order of incident edges around each vertex $v \in S$. We say that the angle α between two edges that are incident to v and consecutive in this sorting is *incident* to v, and vice versa. If v has vertex degree at most one, we define v to be incident to one angle (with value 2π). By this, the sum of all incident angles of a vertex equals 2π for every vertex. A vertex in $G(S)$ is called *pointed* if it is incident to an angle greater than π (a reflex angle). We say that a vertex is pointed to a face of the graph $G(S)$ if its large angle lies in this face. If all vertices in $G(S)$ are pointed, then $G(S)$ is called pointed as well. A vertex without an incident angle larger than π is called *non-pointed*.

A *pointed pseudo-triangulation* $PT(S)$ is a maximal pointed plane graph, meaning that $PT(S)$ is a plane graph in which every vertex is pointed, and that adding any edge to $PT(S)$ either makes a vertex non-pointed or produces a crossing (or both). All interior faces of a pointed pseudo-triangulation are pseudo-triangles; see again Figure 1.5 for pseudo-triangles and Figure 1.10 for a pointed pseudo-triangulation.

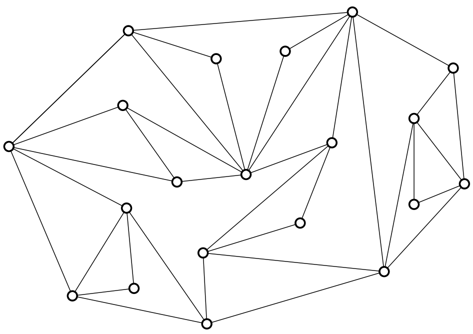


Figure 1.10: A pointed pseudo-triangulation.

A *(general) pseudo-triangulation* is a plane graph in which all interior faces are pseudo-triangles, but not all vertices need to be pointed; see Figure 1.11. Note that, as triangles are part of the class of pseudo-triangles, triangulations are also part of the class of pseudo-triangulations. With respect to their number of edges, triangulations are maximal pseudo-triangulations, while pointed pseudo-triangulations are minimal pseudo-triangulations. Although being a relatively young structure, pseudo-triangulations already have become an important geometric data structure with rich applications; see for example [127, 128, 138], and the survey [132].

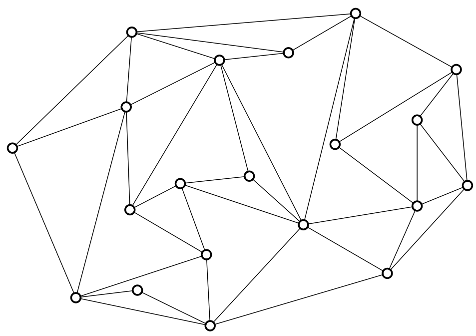


Figure 1.11: A (non-pointed) pseudo-triangulation with maximum face degree four.

A (simple) *path* in a graph $G(S)$ is a sequence of pairwise different vertices $v_1 \dots v_k$ such that $G(S)$ contains all edges $v_i v_{i+1}$, for $1 \le i < k$. The number of edges in a path is denoted as the *lenght* of the path. If $G(S)$ contains a path between u and v for any two vertices $u, v \in S$ then $G(S)$ is called *connected*. A minimal connected graph $T(S)$ is called *spanning tree*; see Figure 1.12.

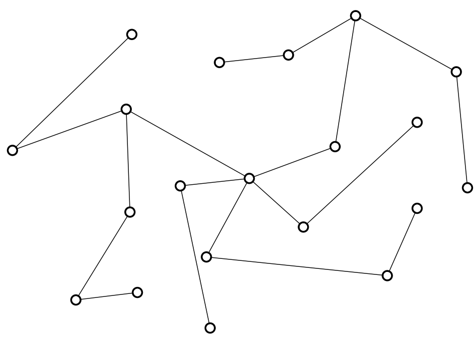


Figure 1.12: A spanning tree $T(S)$.

In a spanning tree $T(S)$, the vertices with vertex degree 1 are called the *leaves* (of $T(S)$). A spanning tree with $|S| = n$ vertices contains exactly $n-1$ edges and has at least two leaves. If a spanning tree has exactly two leaves, then all other vertices are incident to exactly two edges, and the tree is a *spanning path*. Figure 1.13 shows such a spanning path $P(S)$.

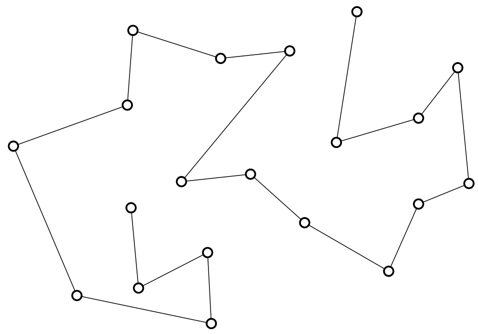


Figure 1.13: A spanning path $P(S)$.

A (simple) *cycle* in a graph $G(S)$ is a sequence of pairwise different vertices $v_1 \dots v_k$ such that $G(S)$ contains all edges $v_i v_{i+1}$, for $1 \leq i < k$, as well as the edge $v_k v_1$. If $G(S)$ does not contain any cycle then it is called *cycle-free*. In a cycle-free graph, there exists at most one path between any two vertices $u, v \in S$. Spanning trees can also be seen as maximal cycle-free graphs.

A connected graph $C(S)$ which consists only of one cycle containing all vertices of S os called a *spanning cycle* or a *polygonization* of S; see Figure 1.14. In a spanning cycle $C(S)$, every vertex has vertex degree 2.

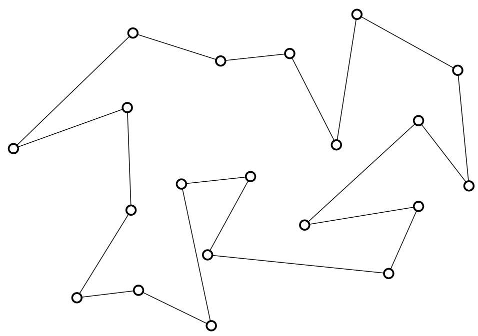


Figure 1.14: A spanning cylce $C(S)$.

A graph $M(S)$ in which every vertex $v \in S$ has vertex degree at most one is called a *matching*. If every vertex hat exactly degree one, then the matching is called *perfect* Note that a perfect matching $PM(S)$ for a point set S only exists if the number of points in S is even. Figure 1.15 shows a perfect matching. The *cardinality* of a matching $M(S)$ is the number of edges in $M(S)$. Perfect matchings are matchings of maximal cardinality.

For sets with an odd number of points, a matching of maximal cardinality leaves exactly one point of S unmatched.

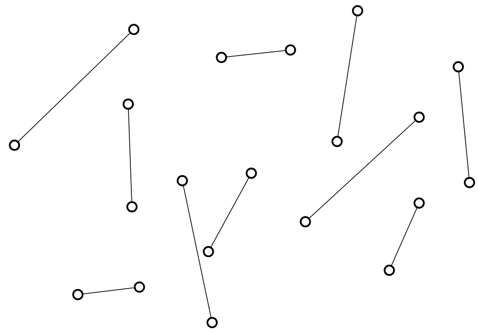


Figure 1.15: A perfect matching $PM(S)$.

Note that in general, spanning trees, spanning paths, spanning cycles, and (perfect) matchings need not be plane or straight-line. As throughout this thesis all considered such graphs are in fact plane straight-line graphs, we will mostly omit the terms plane and straight-line for these classes of graphs.

Chapter 2

On k-gons and k-holes

The first specific topic we consider in this thesis is a class of variations of the classical Erdős-Szekeres problems on the existence and number of k-gons and k-holes in sets of n points in the plane in general position.

Recall that a k-gon in a point set S is a simple polygon spanned by k points of S, and that a k-hole is an empty k-gon; that is, a k-gon which does not contain any points of S in its interior.

The classical problems deal exclusively with convex k-gons and k-holes. There, the main question is to find the minimum number of k-gons / k-holes any n-point set must contain. More precisely, for a given point set S, let $g_k(S)$ and $h_k(S)$ be the number of convex k-gons and k-holes in S, respectively. Then the minimum number of convex k-gons in any n-point set is

$$g_k(n) = \min_{\{S:|S|=n\}}(g_k(S)),$$

and the according minimum number of convex k-holes is

$$h_k(n) = \min_{\{S:|S|=n\}}(h_k(S)).$$

The other direction, namely the maximum number of convex k-gons or convex k-holes an n-point set might contain, trivially is $\binom{n}{k}$, obtained by any n-point set in convex position.

In the work presented in this chapter, we generalize the classical questions by also allowing non-convex k-gons and k-holes. In this context, we distinguish between convex, non-convex, and general (convex or non-convex) k-gons and k-holes.

For k-gons, there is a direct relation between the rectilinear crossing number and all types of 4-gons. We show that such a relation also exists for non-convex 5-gons and present an approach to obtain bounds on the numbers of the different types of k-gons for fixed $k \geq 5$. Also, we show that for fixed k and sufficiently large n, the number of general k-gons is $\Theta(n^k)$ and the number of non-convex k-gons is $O(n^k)$.

Switching topic-wise from k-gons to k-holes, and here first to the classical type of k-holes, convex k-holes, we provide improved lower bounds for the minimum numbers of convex 5-holes and 6-holes. As already mentioned before, the question of maximizing is trivially solved by convex sets

For non-convex k-holes, we give asymptotic upper and lower bounds on their maximum number. The minimum number of non-convex k-holes trivially is zero, attained again by point sets in convex position.

Last but not least, for general k-holes, we show that for any fixed k and sufficiently large n, their maximum number is as well achieved by sets in convex position (where the bound on the minimum necessary set size is tight for $k = 4$ and still quite good for $k = 5$). This fact is rather surprising. It implies that for this setting, the additional possibilities obtained by the relaxation from convex to general does not improve the result of the maximizing question. To the other extreme, we show that for sufficiently large k (in dependence of n) there exist point sets having far more general k-holes than convex ones. Also, we give a quadratic lower bound for the number of general k-holes for any $c < 1$ and $k \leq c \cdot n$, and present a class of point sets that do not admit more than $O(n^2(\sqrt{n}\log n)^{k-3})$ general k-holes (for constant k).

Most results of this chapter have been presented before [16, 31, 15], and are invited for journal publication in special issues [17, 32].

2.1 Introduction

In 1931 Esther Klein raised the following question which was (partially) answered in the classical paper by Erdős and Szekeres [80] in 1935: “Is it true that for any k there is a smallest integer $g(k)$ such that any set of $g(k)$ points contains at least one convex k-gon?” In mathematical history, this question is also known as “Happy End Problem”; see, e.g., [53, 98]. As observed by Klein, $g(4) = 5$, and Kalbfleisch et al. [107] solved the more involved case of $g(5) = 9$. The case $k = 6$ was only solved as recently as 2006 by Szekeres and Peters [139], who showed that $g(6) = 17$ by an exhaustive computer search. The well known Erdős–Szekeres Theorem [80] states that $g(k)$ is finite for any k. The current best bounds are $2^{k-2} + 1 \leq g(k) \leq \binom{2k-5}{k-2} + 1$ for $k \geq 5$, where the lower bound goes back to Erdős and Szekeres [81] and is conjectured to be tight. There have been many improvements on the upper bound, where the currently best bound has been obtained in 2005 by Tóth and Valtr [140].

Some years later Erdős and Guy [78] proposed a generalization of Klein's question. “What is the least number of convex k-gons determined by any set of n points in the plane?” The trivial solution for the case $k = 3$ is $\binom{n}{3}$. But already for 4-gons this question is highly non-trivial, as it is related to the search for the rectilinear crossing number [3]. In Section 2.3.1 we consider this relation in more detail.

In 1978 Erdős [76] raised the following question for convex k-holes: “What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?” As had been observed by Esther Klein, every set of 5 points determines a convex 4-hole, and 10 points always contain a convex 5-hole, a fact proved by Harborth [96]. However, in 1983 Horton showed that there exist arbitrarily large sets of points containing no convex 7-hole [99]. It took almost a quarter of a century after Horton's construction to answer the existence question for 6-holes. In 2007/08 Nicolás [118] and independently Gerken [89] proved that every sufficiently large point set contains a convex 6-hole Valtr [145] gives a simpler version of Gerken's proof, but it requires more points. As for a lower bound, it is known that at least 30 points are needed; that is, there exists a set of 29 points without empty convex hexagons [121].

Erdős also proposed the following variation of the problem [77]. “What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?” We know by Horton's construction that $h_k(n) = 0$ for $k \geq 7$. Table 2.1 shows the current best lower and upper bounds for $k = 3 \ldots 6$. All upper bounds in the table are due to Bárány and Valtr [44].

Concerning the lower bounds, Dehnhardt [63] showed in his thesis that

$$
\begin{aligned}
n^2 - 5n - 1 + \left\lfloor \tfrac{n-4}{6} \right\rfloor \;\le\; & h_3(n) \;\le\; 1.6196n^2 + o(n^2) \\
\tbinom{n-3}{2} + 6 \;\le\; & h_4(n) \;\le\; 1.9397n^2 + o(n^2) \\
\tfrac{3n}{4} - o(n) \;\le\; & h_5(n) \;\le\; 1.0207n^2 + o(n^2) \\
\left\lfloor \tfrac{n-1}{858} \right\rfloor - 2 \;\le\; & h_6(n) \;\le\; 0.2006n^2 + o(n^2)
\end{aligned}
$$

Table 2.1: Bounds on the number $h_k(n)$ of convex k-holes.

for $n \geq 13$, $h_3(n) \geq n^2 - 5n + 10$, $h_4(n) \geq \binom{n-3}{2} + 6$, and $h_5(n) \geq 3\left\lfloor \frac{n}{12} \right\rfloor$. As this thesis was published in German and is not easy to access, later on several weaker bounds have been published. A result of independent interest is by Pinchasi et al. [126], who showed $h_4(n) \geq h_3(n) - \frac{n^2}{2} - O(n)$. By this, an improvement for the lower bound of $h_3(n)$ would also imply a better lower bound for $h_4(n)$. Recently, García [86] showed a relation between 3-holes and convex 5-holes, where for $n \geq 70$ he also provided the currently best known lower bound $h_3(n) \geq n^2 - 5n - 1 + \left\lfloor \frac{n-4}{6} \right\rfloor$. For $h_4(n)$, the above mentioned lower bound from Dehnhardt [63] is still the best known one. Lower bounds of $h_5(n)$ and $h_6(n)$ can be found in Sections 2.5.1 and 2.6.4. See also [9] for a survey on the history of questions and results about k-gons and k-holes.

In the following sections, we generalize the above questions by allowing k-gons and k-holes to be non-convex. Thus whenever we refer to a (general) k-gon or k-hole, it might be convex or non-convex, and we will explicitly state it when we restrict investigations to one of these two classes. We remark that in some related literature, k-holes are assumed to be convex.

Note that for any $k \geq 3$, a set of k points in convex position obviously spans precisely one convex k-hole. In contrast, already a set of four points might span up to three non-convex 4-holes; see Figure 2.1. In general, a point set might admit exponentially many different polygonizations (spanning cycles) [70, 88, 136]. This implies that the number of k-gons and k-holes can be larger than $\binom{n}{k}$, a fact which makes the considered questions more challenging (and interesting) than they might appear on a first glance.

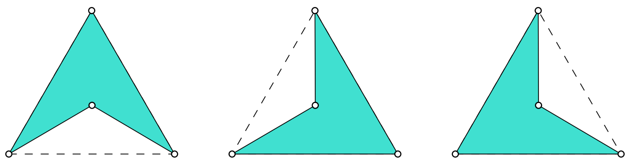


Figure 2.1: Three different (non-convex) 4-gons spanned by a set of four points with three extremal points (fixed order type).

Tables 2.2 and 2.3 summarize the currently best known known bounds on the numbers of different types of k-gons and k-holes, including the results obtained during the work for this thesis. Every cell in the table contains lower and upper bounds, also in explicit form if available, thus indicating for which values there are still gaps to close.

numbers of k-gons

	convex	non-convex	general	
	min	max	min	max
$k=4$	$\bar{cr}(n)$ $\Theta(n^4)$	$3\binom{n}{4}-3\bar{cr}(n)$ $\Theta(n^4)$	$\binom{n}{4}$ $\Theta(n^4)$	$3\binom{n}{4}-2\bar{cr}(n)$ $\Theta(n^4)$
$k=5$	$\Theta(n^5)$ [53]	$10\binom{n}{5}-2(n-4)\bar{cr}(n)$ $\Theta(n^5)$ [Sec. 2.3.1]	$\binom{n}{5}$ $\Theta(n^5)$ [Sec. 2.3.1]	$\Theta(n^5)$ [Sec. 2.3.1]
$k\geq 6$	$\Theta(n^k)$ [53]	$\Theta(n^k)$ [Sec. 2.3.2]	$\binom{n}{k}$ $\Theta(n^k)$ [Sec. 2.3.2]	$\Theta(n^k)$ [Sec. 2.3.2]

Table 2.2: Asymptotic bounds on the minimum and maximum numbers of convex, non-convex and general k-gons for sets of n points and constant k.

numbers of k-holes

	convex	non-convex	general	
	min	max	min	max
$k=4$	$\geq \frac{n^2}{2}-O(n)$ $\leq 1.9397n^2+o(n^2)$ $\Theta(n^2)$ [44, 63]	$\leq \frac{n^3}{2}-O(n^2)$ $\geq \frac{n^3}{2}-O(n^2\log n)$ $\Theta(n^3)$ [Sec. 2.4.2]	$\geq \frac{5}{2}n^2-O(n)$ $\leq \frac{n^3}{2}+O(n^2)$ $\Omega(n^2)$ [Sec. 2.4.3] $O(n^3)$ [Sec. 2.6.3]	$\binom{n}{4}$ $\Theta(n^4)$ [Sec. 2.4.1]
$k=5$	$\geq \frac{3}{4}n - o(n)$ $\leq 1.0207n^2+o(n^2)$ $\Omega(n)$ [Sec. 2.5.1] $O(n^2)$ [44]	$\leq n!/(n-4)!$ $\Theta(n^4)$ [Sec. 2.6.2]	$\geq 17n^2-O(n)$ $\leq O(n^{\frac{7}{2}})$ $\Omega(n^2)$ [Sec. 2.5.2] $O(n^{\frac{7}{2}})$ [Sec. 2.6.3]	$\binom{n}{5}$ $\Theta(n^5)$ [Sec. 2.5.3]
$k\geq 6$	$k=6: \geq \lfloor\frac{n-1}{858}\rfloor - 2$ $O(n^2)$ [44] $\Omega(n)$ [Sec. 2.6.4] $k\geq 7: \emptyset$ [99]	$\leq n!/(n-k+1)!$ $\Theta(n^{k-1})$ [Sec. 2.6.2]	$\geq n^2-O(n)$ $\leq O(n^{\frac{k+2}{2}})$ $\Omega(n^2)$ [Sec. 2.6.3] $O(n^{\frac{k+2}{2}})$ [Sec. 2.6.3]	$\binom{n}{k}$ $\Theta(n^k)$ [Sec. 2.6.1]

Table 2.3: Asymptotic bounds on the minimum and maximum numbers of convex, non-convex and general k-holes for sets of n points and constant k.

In the following we first investigate sets of small cardinality (Section 2.2). Then we present results on the numbers of the different types of k-gons (Section 2.3), including relations between the rectilinear crossing number and k-gons for (small) constant k. The remaining sections of this chapter are dedicated to k-holes. Matching with the section sub-numbering, Section 2.4 deals with the smallest case of possibly non-convex k-holes, 4-holes. Accordingly, in Section 2.5 we present results for 5-holes, and finally in Section 2.6 we present several results for the general case of $k \geq 6$.

2.2 Small sets

Even to determine the number of small holes is surprisingly intriguing. To get a first intuition about the numbers of k-gons and k-holes, we consider gons and holes of small size in point sets of small size. We start with the smallest possibly non-convex case, namely $k = 4$.

2.2.1 4-gons and 4-holes

Table 2.4, which is partly taken from [9], shows the numbers of 4-gons and 4-holes, respectively, for $n = 4, \ldots, 11$. The numbers were obtained by checking all point sets of the according cardinalities from the order type data base [33]. Given are the minimum number of convex 4-gons and 4-holes, the maximum number of non-convex 4-gons and 4-holes, the minimum and maximum number of (general) 4-gons and 4-holes, as well as the number of 4-tuples.

	numbers of 4-gons				numbers of 4-holes				
n	convex	non-convex	general		convex	non-convex	general		$\binom{n}{5}$
	min	max	min	max	min	max	min	max	
4	0	3	1	3	0	3	1	3	1
5	1	12	5	13	1	8	5	9	5
6	3	36	15	39	3	18	15	22	15
7	9	78	35	87	6	36	35	43	35
8	19	153	70	172	10	64	66	77	70
9	36	270	126	306	15	100	102	126	126
10	62	444	210	506	23	150	147	210	210
11	102	684	330	786	32	216	203	330	330

Table 2.4: Numbers of 4-gons and 4-holes for $n = 4, \ldots, 11$ points.

For counting convex 4-gons and 4-holes it is easy to see that their number is maximized by sets in convex position and gives $\binom{n}{4}$. Of course these sets do not contain any non-convex 4-gon or 4-hole. Also, also the minimum number of general 4-gons is $\binom{n}{4}$, obtained by sets in convex position. The reason for this is that a convex 4-tuple has exactly one polygonization, while a non-convex 4-tuple has three.

As already mentioned, the minimum number of convex 4-gons is identical to the rectilinear crossing number [3], a fact which immediately implies bounds on the numbers of general and non-convex 4-gons. The relation between the different kinds of 4-gons and the rectilinear crossing number is also described in Section 2.3.1.

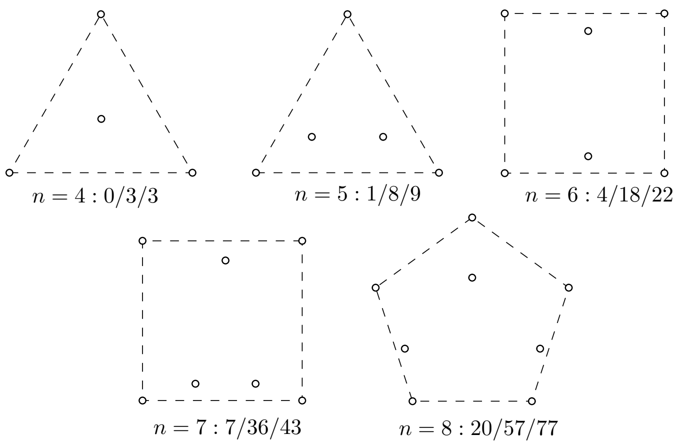


Figure 2.2: Point sets maximizing the number of 4-holes for $n = 4, \ldots, 8$. Shown are the number of convex, non-convex, and general 4-holes. With the exception of $n = 7$ all sets have a unique order type.

For minimizing the number of convex 4-holes, the currently best known bounds are $\frac{n^2}{2} - O(n) \leq h_4(n) \leq 1.9397n^2 + o(n^2)$; see [44, 63]. For $n = 4, \ldots, 7$ it can be seen from Table 2.4 that the minimum number of 4-holes is $\binom{n}{4}$. In contrast, $\binom{n}{4}$ is the maximum number of 4-holes for $n = 9, \ldots, 11$, so the structure of extremal sets seems to switch.

Figure 2.2 shows point sets maximizing the number of 4-holes for $n = 4, \ldots, 8$. The results for $n > 8$ suggest that sets in convex position might maximize the number of 4-holes for $n \geq 9$. Indeed, this will be the first result we prove for general 4-holes (Section 2.4.1).

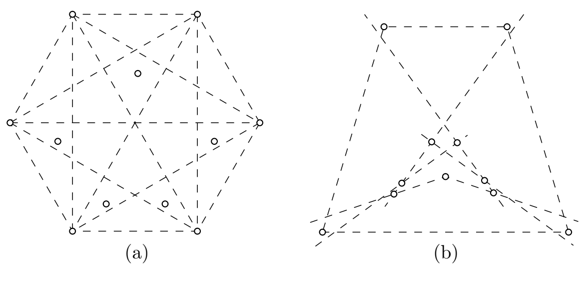


Figure 2.3: Two unique extremal sets for $n = 11$ points: (a) maximizes the number of non-convex 4-holes, and (b) minimizes the number of 4-holes.

Figure 2.3 shows two extremal sets for $n = 11$ points. Each set represents the unique order type which reaches the extreme value. The left set maximizes the number of non-convex 4-holes, namely 216, and consists of a convex 5-hole inside a convex 6-gon. The total number of 4-holes in this set is 267; i.e., it contains in addition 51 convex 4-holes. The set on the right side minimizes the number of general 4-holes. It contains 51 convex and 152 non-convex 4-holes, thus in total the minimum of 203 4-holes.

2.2.2 5-gons and 5-holes

To get some more impressions, we continue with $k = 5$. For $n \leq 11$, Table 2.5 shows the minimum and maximum numbers of convex, non-convex, and general 5-gons and 5-holes, and, for comparison, the number of 5-tuples. Again, the results were obtained using the order type data base [33].

	numbers of 5-gons				numbers of 5-holes				
n	convex min	non-convex max	general min	general max	convex min	non-convex max	general min	general max	$\binom{n}{5}$
5	0	8	1	8	0	8	1	8	1
6	0	48	6	48	0	31	6	31	6
7	0	156	21	157	0	76	21	77	21
8	0	408	56	410	0	157	56	160	56
9	1	900	126	909	0	288	126	292	126
10	2	1776	252	1790	1	492	252	501	252
11	7	3192	462	3228	2	779	462	802	462

Table 2.5: Numbers of 5-gons and 5-holes for $n = 5, \ldots, 11$ points.

As already mentioned in the last section, both the maximum numbers of convex 5-gons and 5-holes are $\binom{n}{5}$, obtained by sets in convex position. Likewise, the minimum numbers of non-convex 5-gons and 5-holes are zero.

From Table 2.5 we also see that the minimum numbers of general both 5-gons and 5-holes is $\binom{n}{5}$ for $5 \leq n \leq 11$. For 5-gons this is true for any number of points: Recall that a convex 5-tuple has exactly one polygonization, while a non-convex 5-tuple has at least four; see Figures 2.4 and 2.5. For 5-holes this is not the case. In fact, we will show that for sufficiently large n, the convex set maximizes the number of 5-holes; see Theorem 2.18 in Section 2.5.3.

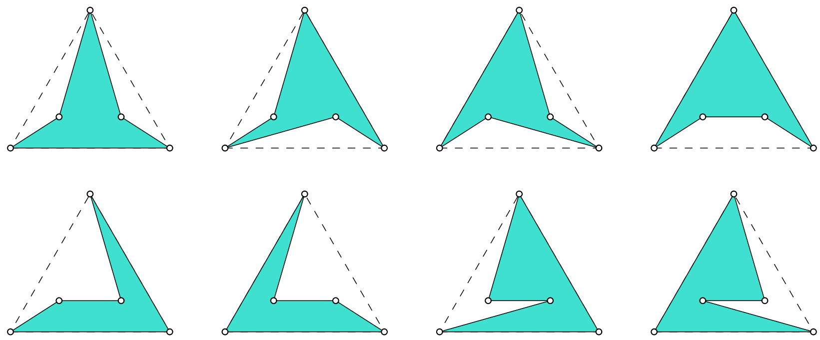


Figure 2.4: The eight different (non-convex) 5-gons spanned by a set of five points with three extremal points (fixed order type).

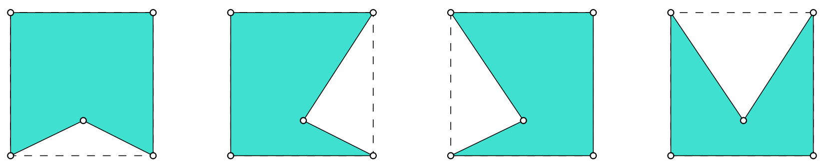


Figure 2.5: The four different (non-convex) 5-gons spanned by a set of five points with four extremal points (fixed order type).

2.3 k-gons

2.3.1 k-gons and the rectilinear crossing number

The rectilinear crossing number $\overline{\text{cr}}(S)$ of a set S of n points in the plane is the number of proper intersections in the drawing of the complete straight-line graph on S; see for example [125]. It is easy to see that the number of convex 4-gons is equal to $\overline{\text{cr}}(S)$ and is thus minimized by sets minimizing the rectilinear crossing number, a well known, difficult problem in discrete geometry; see [53] and [78] for details. Tight values for the minimum number of convex 4-gons are known for $n \leq 27$ points; see for example [8]. Asymptotically we have at least $c_4\binom{n}{4} = \Theta(n^4)$ convex 4-gons, where c_4 is a constant in the range $0.379972 \leq c_4 \leq 0.380488$. The currently best bounds on c_4 are by Ábrego, Fernández-Merchant, and Salazar [3]. As any 4 points in non-convex position span three non-convex 4-gons, altogether we get

- $\overline{\text{cr}}(S)$ convex 4-gons,
- $3\binom{n}{4} - 3\,\overline{\text{cr}}(S)$ non-convex 4-gons, and
- $3\binom{n}{4} - 2\,\overline{\text{cr}}(S)$ general 4-gons for a set S.

Thus, sets which minimize the rectilinear crossing number also minimize the number of convex 4-gons, and maximize both the number of non-convex 4-gons and the number of general 4-gons, and thus we obtain the following tight bounds:

- $\overline{\text{cr}}(n) \approx 0.38\binom{n}{4}$ for the minimum number of convex 4-gons,
- $3\binom{n}{4}$–$3\,\overline{\text{cr}}(n) \approx 1.86\binom{n}{4}$ for the maximum number of non-convex 4-gons, and
- $3\binom{n}{4}$–$2\,\overline{\text{cr}}(n) \approx 2.24\binom{n}{4}$ for the maximum number of general 4-gons for a set S.

Surprisingly, a similar relation can be obtained for the number of non-convex 5-gons. To see this, consider the three combinatorial different possibilities (order types) of arranging 5 points in the plane, as depicted in Figure 2.6.

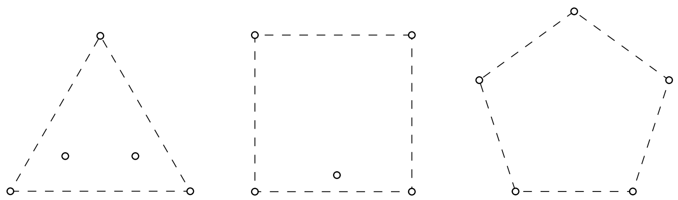


Figure 2.6: The three order types for $n = 5$. For each set its number of different 5-gons and the number of crossings for the complete graph is shown.

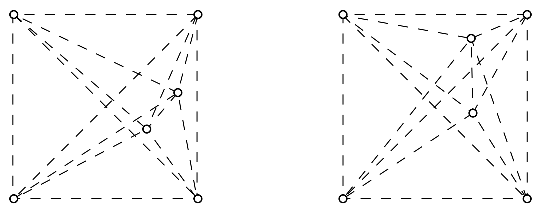
Theorem 2.1. *Let S be a set of $n \geq 5$ points in the plane in general position. Then S contains $10\binom{n}{5} - 2(n-4)\,\overline{\text{cr}}(S)$ non-convex 5-gons.*

Proof. We denote with $o_3(S)$, $o_4(S)$, and $o_5(S)$ the number of 5-tuples of points with 3, 4, and 5, respectively, points on their convex hull. Summing over all such sets we get $o_3(S) + o_4(S) + o_5(S) = \binom{n}{5}$.

Note that every four points spanning a crossing pair of edges in S show up in $(n-4)$ 5-tuples of points in S. Using the number of crossings for each order type from Figure 2.6 we get

$$\overline{\text{cr}}(S) = \frac{o_3(S) + 3o_4(S) + 5o_5(S)}{n-4}.$$

Considering the numbers of different 5-gons given in Figure 2.6, we see that the total number of non-convex 5-gons in S is $8o_3(S) + 4o_4(S)$. Using



$S_1 : 1/8$ $S_2 : 0/8$

Figure 2.7: Two point sets for $n = 6$, both with crossing number eight. One contains a convex 5-gon, the other one does not.

these three equations, it is straight forward to obtain by standard manipulation the following relation for the number of non-convex 5-gons in S:

$$8o_3(S) + 4o_4(S) = 10\binom{n}{5} - 2(n-4)\,\overline{\mathrm{cr}}(S).$$

□

Taking the constant c_4 for the rectilinear crossing number into account, we see that asymptotically we can have up to

$$10\binom{n}{5} - 2(n-4)c_4\binom{n}{4} = 10(1-c_4)\binom{n}{5} \approx 6.20\binom{n}{5}$$

non-convex 5-gons. This number is obtained for point sets minimizing the rectilinear crossing number and it is by a factor of approximately 6.20 larger than the maximum number of convex 5-gons.

For the number of convex 5-gons, no direct relation to the rectilinear crossing number is possible: There exist two different sets (order types) S_1 and S_2, both of cardinality 6 with 4 extremal points, with $\overline{\mathrm{cr}}(S_1) = \overline{\mathrm{cr}}(S_2) = 8$, where S_1 contains one convex 5-gon, while S_2 does not contain any convex 5-gon; see Figure 2.7.

Similarly, for $k \geq 6$, none of the three types of k-gons (convex, non-convex, and general) in a point set S can be expressed as a function of $\overline{\mathrm{cr}}(S)$. Still, we can use the rectilinear crossing number for obtaining bounds on these numbers. In the following let $g_k^t(S)$ be the number of k-gons of *type* t (convex, non-convex, or general) in a point set S.

Proposition 2.2. *Let $k \geq 4$, and c_1, c_2, and x arbitrary constants, and assume that Inequation (2.1) holds for all sets S with cardinality $|S| = k$.*

$$c_1 \leq g_k^t(S) + x \cdot \overline{\mathrm{cr}}(S) \leq c_2 \tag{2.1}$$

Then for every point set S with $|S| \geq k$, the following bounds hold for the number $g_k^t(S)$ of k-gons of type t in S.

$$g_k^t(S) \geq c_1 \cdot \binom{n}{k} - x \cdot \binom{n-4}{k-4} \cdot \overline{\mathrm{cr}}(S) \tag{2.2}$$

$$g_k^t(S) \leq c_2 \cdot \binom{n}{k} - x \cdot \binom{n-4}{k-4} \cdot \overline{\mathrm{cr}}(S) \tag{2.3}$$

Proof. Given some point set S with n points, consider all its $\binom{n}{k}$ subsets of size k $\{S_i \subseteq S : |S_i| = k\}$. Then we have the following equations.

$$\sum_i \overline{\mathrm{cr}}(S_i) = \binom{n-4}{k-4} \cdot \overline{\mathrm{cr}}(S) \tag{2.4}$$

$$\sum_i g_k^t(S_i) = g_k^t(S) \tag{2.5}$$

Using the first inequation in (2.1), Equation (2.5) can be transformed to the lower bound

$$\begin{aligned} g_k^t(S) &= \sum_i g_k^t(S_i) \\ &\geq \sum_i (c_1 - x \cdot \overline{\mathrm{cr}}(S_i)) \\ &= c_1 \cdot \binom{n}{k} - x \cdot \sum_i \overline{\mathrm{cr}}(S_i) \end{aligned}$$

which by applying (2.4) in turn gives the desired bound (2.2) Analogously, we obtain (2.3) if we combine the second inequation in (2.1) with Equations (2.4) and (2.5). $\square$

If in Inequation (2.2) from the above proposition x is negative, then the rectilinear crossing number of $\overline{\mathrm{cr}}(S)$ adds to the lower bound. Thus we can replace it by the minimum over all point sets of size n, $\overline{\mathrm{cr}}(n)$, by this obtaining a lower bound that is independent of S. Accordingly, if x is positive, we can generalize the Inequation (2.3) to a general upper bound.

Corollary 2.3. *Assume that for some constants $x \leq 0$ and c_1 arbitrary, the following inequation is fulfilled for all point sets S with cardinality $|S| = k$.*

$$c_1 \leq g_k^t(S) + x_1 \cdot \overline{\mathrm{cr}}(S)$$

Then for every point set S with $|S| \geq k$ the following lower bound holds for the number $g_k^t(S)$ of k-gons of type t in S.

$$g_k^t(S) \geq c_1 \cdot \binom{n}{k} - x \cdot \binom{n-4}{k-4} \cdot c_4 \binom{n}{4} = \left(c_1 - x \cdot c_4 \cdot \binom{k}{4}\right) \binom{n}{k} \tag{2.6}$$

Accordingly, assume that for some constants $x \geq 0$ and c_2 arbitrary, the following inequation is fulfilled for all point sets S with cardinality $|S| = k$.

$$c_2 \geq g_k^t(S) + x_1 \cdot \overline{\mathrm{cr}}(S)$$

Then for every point set S with $|S| \geq k$, the following bound applies to the number $g_k^t(S)$ of k-gons of type t in S.

$$g_k^t(S) \leq \left(c_2 - x \cdot c_4 \cdot \binom{k}{4}\right)\binom{n}{k} \tag{2.7}$$

In each of the bounds resulting from Proposition 2.2, either c_1 or c_2 is not needed. So of course, for independent optimization of the two bounds, it might be helpful to consider the pairs (c_1, x), and (c_2, x) independently, with possibly different optimal values for x. On the other hand, optimizing all three values c_1, c_2 and c_3 simultaneously results in bounds that are more easy to compare.

In the following we optimize in two different ways. On the one hand we try to minimize the difference between c_1 and c_2 to obtain most possibly small ranges for the number of (some type of) k-gons in sets with a certain crossing number. On the other hand we independently optimize (c_1, x) and (c_2, x) in order to obtain general bounds (meaning bounds that are independent from the crossing number of a given set) by applying Corollary 2.3.

Note that concerning the general bounds, lower bounds only make sense for the classical question about convex k-gons, as the numbers of general and non-convex k-gons are minimized by sets in convex position. Similarly, general upper bounds for non-convex or (general) k-gons are of interest while the maximum number of convex k-gons is $\binom{n}{k}$, again obtained by convex sets.

Recall that the number of k-gons (of whatever type) in a point set S only depends on the combinatorial properties and thus on the order type $OT(S)$ of the underlying point set S. Thus, we calculate pairs $(g_k^t(OT), \overline{\mathrm{cr}}(OT))$ for all possible order types OT of k points, by this obtaining all possible pairs $(g_k^t(S), \overline{\mathrm{cr}}(S))$ that can occur for any point set S with $|S| = k$. For the calculation, we use the order type database [33], that contains a complete list of the order types of up to 11 points. Having this, we can optimize the values c_1, c_2, and x that fulfill (2.1), obtaining the according relations (2.2) and (2.3).

As a first application of this, consider again $k = 4$. The two different order types together with their numbers of crossings and 4-gons are shown in Table 2.6. Following the lines of Proposition 2.2, we obtain the beforementioned tight relations for convex 4-gons with $c_1 = c_2 = 0$ and $x = -1$. For non-convex 4-gons we get $c_1 = c_2 = x = 3$, and for general 4-gons

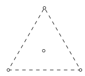
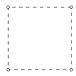
order type	$\overline{\text{cr}}$	convex 4-gons	non-convex 4-gons	general 4-gons
	0	0	3	3
	1	1	0	1

Table 2.6: Numbers of 4-gons and crossings for $n = 4$.

$c_1 = c_2 = 3$ and $x = 2$.

For $k = 5$, Table 2.7 shows the three different order types together with their numbers of crossings and 5-gons. As already mentioned before, we obtain an exact relation for the number of non-convex 5-gons with $x = 2$ and $c_1 = c_2 = 10$.

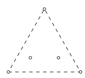
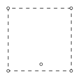
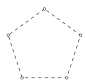
order type	$\overline{\text{cr}}$	convex 5-gons	non-convex 5-gons	general 5-gons
	1	0	8	8
	3	0	4	4
	5	1	0	1

Table 2.7: Numbers of 5-gons and crossings for $n = 5$.

For general 5-gons, the optimal difference of c_2 and c_1 is reached at $x = 1.75$, resulting in

$$\begin{aligned} g_5^{gen}(S) &\geq 9.25 \cdot \binom{n}{5} - 1.75 \cdot (n-4) \cdot \overline{\text{cr}}(S) \\ g_5^{gen}(S) &\leq 9.75 \cdot \binom{n}{5} - 1.75 \cdot (n-4) \cdot \overline{\text{cr}}(S). \end{aligned}$$

Independently optimizing just the second part of Inequation (2.1) gives the same results. Thus, applying Corollary 2.3, we obtain a general upper bound of $(9.75 - 8.75 \cdot c_4)\binom{n}{5} \approx 6.43\binom{n}{5}$ for the (maximum) number of general 5-gons in any n-point set. Note that by just assuming assuming the theoretic possible maximum of eight 5-gons for each 5-tuple, we would only obtain an upper bound of $8\binom{n}{5}$.

For convex 5-gons, the optimal difference of c_2 and c_1 is reached at $x = -0.25$, resulting in

$$\begin{aligned} g_5^{conv}(S) &\geq -0.75 \cdot \binom{n}{5} + 0.25 \cdot (n-4) \cdot \overline{\mathrm{cr}}(S) \\ g_5^{conv}(S) &\leq -0.25 \cdot \binom{n}{5} + 0.25 \cdot (n-4) \cdot \overline{\mathrm{cr}}(S). \end{aligned}$$

Optimization in order to obtain a general lower bound fails, only producing a the trivial lower bound of zero.

Similarly, for convex 6-gons, only obtain results for optimizing the difference between c_2 and c_1, obtaining

$$\begin{aligned} g_6^{conv}(S) &\geq -\binom{n}{6} + 0.041\dot{6} \cdot (n-4)(n-5) \cdot \overline{\mathrm{cr}}(S) \\ g_6^{conv}(S) &\leq -0.25 \cdot \binom{n}{6} + 0.041\dot{6} \cdot (n-4)(n-5) \cdot \overline{\mathrm{cr}}(S). \end{aligned}$$

For general 6-gons, the minimal difference between c_2 and c_1 is reached at $x = 2\frac{1}{3}$, resulting in

$$\begin{aligned} g_6^{gen}(S) &\geq 28\frac{1}{3} \cdot \binom{n}{6} - 1\frac{1}{6} \cdot (n-4)(n-5) \cdot \overline{\mathrm{cr}}(S) \\ g_6^{gen}(S) &\leq 36 \cdot \binom{n}{6} - 1\frac{1}{6} \cdot (n-4)(n-5) \cdot \overline{\mathrm{cr}}(S). \end{aligned}$$

Again, independently optimizing only parameters c_2 and x yields the same values, and applying Corollary 2.3, we can transform the second inequation to an upper bound of $(36 - 35 \cdot c_4)\binom{n}{6} \approx 22.7\binom{n}{6}$ for the (maximum) number of general 6-gons in any n-point set. A 6-tuple might span up to 29 general 6-gons, a value that is obtained by one of the sets with minimum crossing number $\overline{\mathrm{cr}}(6) = 3$. Thus, the obtained bound again improves over the trivial bound of $29\binom{n}{6}$ induced by this extreme case.

The optimal difference between c_1 and c_2 for non-convex 6-gons is reached at $x = 2\frac{4}{9}$, resulting in

$$\begin{aligned} g_6^{non-conv}(S) &\geq 29\frac{4}{9} \cdot \binom{n}{6} - 1\frac{2}{9} \cdot (n-4)(n-5) \cdot \overline{\mathrm{cr}}(S) \\ g_6^{non-conv}(S) &\leq 36\frac{6}{9} \cdot \binom{n}{6} - 1\frac{2}{9} \cdot (n-4)(n-5) \cdot \overline{\mathrm{cr}}(S). \end{aligned}$$

In this case, independent optimization for an upper bound produces different results, namely the same bound as for general 6-gons.

As one last example we consider the case $k = 7$. For convex 7-gons, optimizing the difference between c_1 and c_2 gives

$$\begin{aligned} g_7^{conv}(S) &\geq -1.1\overline{923076} \cdot \binom{n}{7} + 0.00\overline{641025} \cdot (n-4)(n-5)(n-6) \cdot \overline{\text{cr}}(S) \\ g_7^{conv}(S) &\leq -0.3\overline{461538} \cdot \binom{n}{7} + 0.00\overline{641025} \cdot (n-4)(n-5)(n-6) \cdot \overline{\text{cr}}(S). \end{aligned}$$

Concerning general 7-gons, for both, the difference between c_1 and c_2, and the upper general bound for $g_7^{gen}(S)$, the optimum is reached at $x = 3.5$, resulting in

$$\begin{aligned} g_7^{gen}(S) &\geq 85.5 \cdot \binom{n}{7} - \frac{7}{12} \cdot (n-4)(n-5)(n-6) \cdot \overline{\text{cr}}(S) \\ g_7^{gen}(S) &\leq 123.5 \cdot \binom{n}{7} - \frac{7}{12} \cdot (n-4)(n-5)(n-6) \cdot \overline{\text{cr}}(S). \end{aligned}$$

and the general upper bound of $(123.5 - 3.5 \cdot 35 \cdot c_4)\binom{n}{7} \approx 76.95\binom{n}{7}$. The maximum number of 7-gons a 7-tuple can have is 92, again obtained by a point set with minimum crossing number.

Considering non-convex 7-gons, the optimization yields again better bounds than for general 7-gons. With $x = 3.\overline{538461}$, $c_1 = 86.\overline{230769}$, and $c_2 = 123.\overline{846153}$ we obtain a smaller difference between c_1 and c_2, and also a better general upper bound of $(123.\overline{846153} - x = 3.\overline{538461} \cdot 35 \cdot c_4)\binom{n}{7} \approx 75.64\binom{n}{7}$.

From the above calculations it can be seen that for all sets of size $k \leq 7$, the point sets reaching the maximum number of general or non-convex k-gons are at the same time minimizing the number of crossings. The same is true for $n = 8$. But continuing the calculations until $k = 9$, it turned out that this is not true in general. The (combinatorially unique) point set containing the maximum number of 1282 general 9-gons has 38 crossings and thus does not reach the minimum crossing number $\overline{\text{cr}}(9) = 36$.

2.3.2 k-gons, polygonizations, and the double chain

Related to the topic of non-convex k-gons in a set S of n points is the number of planar polygonizations, also called spanning cycles. Spanning cycles can be considered as k-gons of maximal size (i.e., $k = n$). García et al. [88] construct a point set with $\Omega(4.64^n)$ spanning cycles, the so-called double chain $DC(n)$, which is currently the best known example; see Figure 2.8. The upper bound on the number of spanning cycles of any n-point set was improved several times during the last years, most recently to $O(70.21^n)$ [136]

and $O(68.664^n)$ [70], neglecting polynomial factors in the asymptotic expressions. The minimum is obtained by point sets in convex position, which have exactly one spanning cycle.

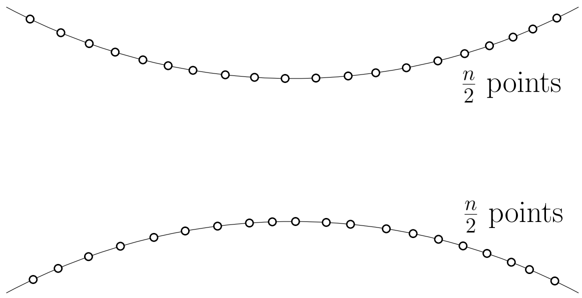


Figure 2.8: The so-called double chain $DC(n)$.

For the number of general k-gons this implies a lower bound of $\binom{n}{k}$, as well as an upper bound of $O\left(68.664^k\binom{n}{k}\right)$. For constant k, we obtain $\Theta(n^k)$. On the other hand, the double chain provides $\Omega(n^k)$ non-convex k-gons, where $k \geq 4$ is again a constant. To see this, choose one vertex from the upper chain of $DC(n)$ and $k-1 \geq 3$ vertices from the lower chain of $DC(n)$, and connect them to a simple, non-convex polygon. This gives at least $\frac{n}{2}\binom{n/2}{k-1} = \Omega(n^k)$ non-convex k-gons. As the lower bound on the maximal number of non-convex k-gons asymptotically matches the upper bound on the maximal number of general k-gons, we obtain the following result.

Proposition 2.4. *Let S be a set of n points in the plane in general position and $k \geq 3$ a constant. Then the maximum number of non-convex k-gons in S is $\Theta(n^k)$ and the maximum number of general k-gons in S is also $\Theta(n^k)$.*

2.4 4-holes

We switch from k-holes to k-gons. We start with results on 4-holes, the smallest variant of possibly non-convex k-holes, and with the question of maximizing the number of general 4-holes.

2.4.1 Maximizing the number of (general) 4-holes

Recall that for small n, the convex set minimizes the number of general 4-holes. For example, four points in convex position span exactly one 4-hole, while four points which are not in convex position span three non-convex 4-holes. Accordingly, five points span at least five 4-holes. On the other hand, the enumerative results for $n > 8$ from Section 2.2.1 suggest that for large enough n, sets in convex position might maximize the number of 4-holes, which in fact is true. The following lemma is one of the main ingredients for proving this statement.

Lemma 2.5. *Let Δ be a non-empty triangle in S. There are at most three non-convex 4-holes spanned by the three vertices of Δ plus a point of S in the interior of Δ.*

Proof. Let p_1, p_2, and p_3 be the vertices of Δ. Observe that any non-convex 4-hole has to use two edges of Δ. Thus there are three choices for the unused edge of Δ, and for each choice there is at most one way to complete the two used edges of Δ to a 4-hole. Assume to the contrary that two different 4-holes avoid the edge p_2p_3 and use points q_1 and q_2, respectively, in the interior. Then q_2 lies outside the two triangles formed by $p_1q_1p_2$ and $p_1q_1p_3$. Thus q_2 lies in the triangle formed by $p_2q_1p_3$; see Figure 2.9. But then q_1 must lie in one of the triangles spanned by $p_1q_2p_2$ and $p_1q_2p_3$, a contradiction. $\square$

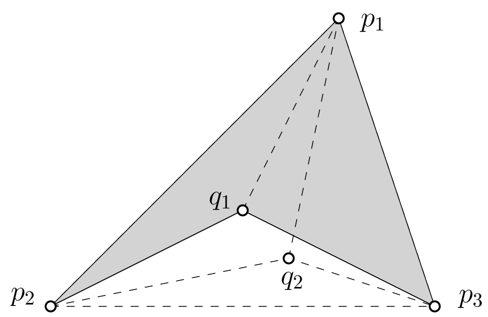


Figure 2.9: q_2 lies outside the 4-hole spanned by p_1,p_2,q_1, and p_3.

Note that for $k > 4$ the number of k-holes determined by the vertices of its convex hull is in general not constant. For example, a triangle Δ with i interior points can define $O(i)$ many 5-holes. For example, the point set shown in Figure 2.10 contains $(3i+2)$ such 5-holes; see also Figure 2.11.

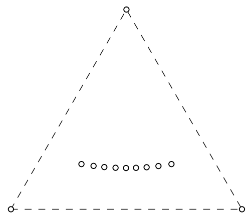


Figure 2.10: A point set where the extreme triangle spans $3i+2$ (non-convex) 5-holes.

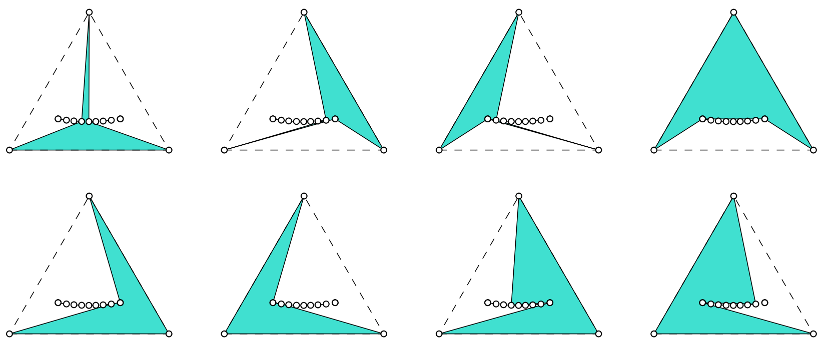


Figure 2.11: The eight different types of (non-convex) 5-holes spanned by the extreme triangle.

Theorem 2.6. *For $n \geq 9$ the number of 4-holes is maximized by a set of n points in convex position.*

Proof. In the following we assign every non-convex 4-tuple of points to the three vertices of its convex hull and call this the *representing* triangle of the potential non-convex 4-holes. By Lemma 2.5, any non-empty triangle represents at most three 4-holes, and any convex 4-tuple gives at most one 4-hole.

Let T be the number of non-empty triangles. As any non-empty triangle induces at least one 4-tuple in non-convex position, we get

$$\binom{n}{4} + 2T \tag{2.8}$$

as a first upper bound on the number of 4-holes of a point set.

Note that a triangle Δ with $k \geq 1$ interior points is counted $k+2$ times in (2.8), namely k times in the $\binom{n}{4}$ 4-tuples plus the extra 2 as Δ is non-empty. Thus for $k > 1$ we have over-counted the number of non-convex 4-holes assigned to Δ; cf. Lemma 2.5. Moreover, many of the convex 4-gons might not be empty and thus no 4-holes. Therefore we now analyze how many of the counted potential 4-holes can be reduced from (2.8). We will do this by assigning (possibly multiple) markers for over-counted 4-holes to convex 4-tuples and non-empty triangles.

As above, let Δ be a triangle with $k \geq 1$ interior points, and consider all 4-tuples consisting of the three vertices of Δ plus an extra point p. We distinguish two cases.

First let p be one of the n–k–3 points outside Δ. If the resulting 4-tuple is convex, we mark this 4-tuple, as it is not empty and thus no 4-hole. If the 4-tuple is non-convex, we mark the triangle which represents the potential non-convex 4-hole, as at least one of the three possible 4-holes of this 4-tuple is non-empty.

In the second case we consider the k points inside Δ. As argued above, Δ was counted $k+2$ times. But by Lemma 2.5, there are at most three 4-holes using one interior point of Δ and thus represented by Δ. Therefore we assign $k-1$ markers to Δ.

Altogether we have distributed $n-4$ markers while considering Δ. Repeating this for all non-empty triangles, we obtain a total of $(n-4) \cdot T$ markers.

A non-empty convex 4-tuple might have received up to 4 markers in this way, one from each of its sub-triangles. That is, we have at most 4 times as many markers as convex 4-tuples which we can reduce from the upper bound (2.8).

A non-empty triangle Δ with $k \geq 1$ interior points might have got $4 \cdot (k–1)$ markers: For its interior points, Δ received $k-1$ markers from the second case, and for each non-empty triangle formed by two vertices of Δ and one point inside Δ, we received one marker from the first case. As at least three of the considered inner triangles are empty (the ones spanned by an edge e of Δ and the interior point closest to e), the first case gives at most $3 \cdot (k-1)$ additional markers, resulting in a total of at most $4 \cdot (k–1)$ markers for Δ. As Δ was counted k+2 times, but represents at most three 4-holes (Lemma 2.5), we have at most $4 \cdot (k–1)$ markers for at least $(k+2)-3 = k-1$ over-counted objects. Thus, in both cases we over-counted reducible terms at most by a factor of 4. We therefore can reduce the number of potential 4-holes by one quarter of the distributed markers, namely by $\frac{n-4}{4} \cdot T$. This leads to the improved upper bound

$$\binom{n}{4} + 2T - \frac{n-4}{4} \cdot T$$

for the number of 4-holes. For $n \geq 12$ this is at most $\binom{n}{4}$, the number of 4-holes for a set of points in convex position. Together with the results from Table 2.4 for $n = 9, \ldots, 11$, this proves the theorem. □

2.4.2 Maximizing the number of non-convex 4-holes

In the previous section we have shown that the number of general 4-holes is maximized for sets in convex position. This obviously also holds for the number of convex 4-holes.

We now consider maximizing the number of non-convex 4-holes. From Lemma 2.5 we obtain the following.

Lemma 2.7. *The number of non-convex 4-holes of any set of n points is at most $\frac{n(n-1)(n-2)}{2} = \frac{n^3}{2} - O(n^2)$.*

Proof. By Lemma 2.5, any non-empty triangle generates at most three non-convex 4-holes, and there are at most $\binom{n}{3}$ such triangles in a set of n points. □

Theorem 2.8. *For every $n' > 0$ there exist sets of n points for some $n \in \{n', \ldots, 2n'\}$, with at least $\frac{n^3}{2} - O(n^2 \log n)$ non-convex 4-holes.*

Proof. We consider the special point sets $\mathcal{X}_m$, $m \geq 1$, with $|\mathcal{X}_m| = n = 2^{m+1} - 2$ points, introduced in [108]. The point sets are defined recursively in layers, starting with two points $\mathcal{X}_1 := \mathcal{R}_1$ in the first layer. An additional layer $\mathcal{R}_i$ is added to $\mathcal{X}_{i-1} := \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_{i-1}$ by placing two new points close to any point in $\mathcal{R}_{i-1}$ outside the convex hull of $\mathcal{X}_{i-1}$, such that the following conditions hold:

1. $\mathcal{X}_j = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_j$ is in general position,
2. $\mathcal{R}_j$ are the extremal points of $\mathcal{X}_j$, and
3. any triangle determined by $\mathcal{R}_j$ contains precisely one point of $\mathcal{X}_j$ in its interior.

See Figure 2.12 for an example and [108] for a detailed description of the construction. Furthermore, in [108] it is shown that every triangle spanned by $\mathcal{X}_m$ contains at most one interior point of $\mathcal{X}_m$; i.e., every non-empty triangle of $\mathcal{X}_m$ contains exactly one point. Using Lemma 2.5, the number of non-convex 4-holes of $\mathcal{X}_m$ is three times the number of non-empty triangles.

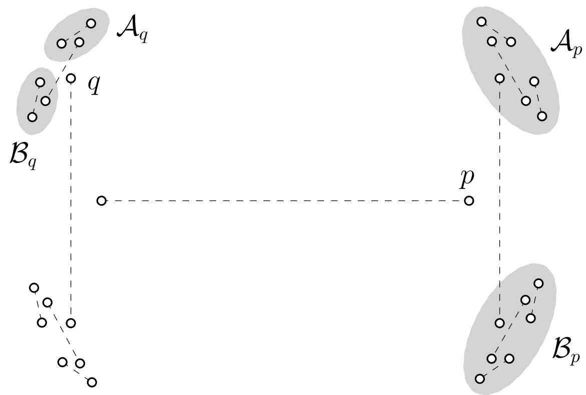


Figure 2.12: Example for $m = 4$ of the special point set defined in [108].

For each point of the set $\mathcal{X}_m$, we count the number of triangles that contain it. First, fix a point in the first layer $\mathcal{R}_1$, say p in Figure 2.12. Any triangle containing p is formed by one point of $\mathcal{A}_p$, one point of $\mathcal{B}_p$, and one point of the remaining set $\mathcal{C}_p = \mathcal{X}_m \backslash \{\mathcal{A}_p \cup \mathcal{B}_p \cup \{p\}\}$. We say that $\mathcal{A}_p$ and $\mathcal{B}_p$ are the *induced subsets* of p, and that $\mathcal{C}_p$ is the *remainder* (of $\mathcal{X}_m$) for p. As $a_1 := |\mathcal{A}_p| = |\mathcal{B}_p| = \frac{n-2}{4}$ and $c_1 := |\mathcal{C}_p| = n - 2 \cdot a_1 - 1 = \frac{n}{2}$, this gives $a_1^2 \cdot c_1$ triangles containing p, and thus the number of triangles containing a point of $\mathcal{R}_1$ is $2 \cdot a_1^2 \cdot c_1 = 2 \cdot (\frac{n-2}{4})^2 \cdot \frac{n}{2}$.

Now consider a point q in the second layer $\mathcal{R}_2$. Its induced subsets $\mathcal{A}_q$ and $\mathcal{B}_q$ have size $a_2 = \frac{n-6}{8}$, and the remainder $\mathcal{C}_q$ has $c_2 = n - 2 \cdot a_2 - 1 = \frac{3n+2}{4}$ points. In combination with $r_2 := |\mathcal{R}_2| = 4$ this gives a total of $4 \cdot (\frac{n-6}{8})^2 \cdot \frac{3n+2}{4}$ triangles containing a point of $\mathcal{R}_2$.

In general, $|\mathcal{R}_i| = r_i = 2^i$, and the size of the two induced subsets of any point p_i in $\mathcal{R}_i$ is

$$a_i = \frac{1}{r_{i+1}}(n - |\mathcal{X}_i|) = \frac{n - (2^{i+1} - 2)}{2^{i+1}}.$$

Thus, with the size of the corresponding remainder $\mathcal{C}_{p_i}$ of

$$c_i = n - 2 \cdot a_i - 1 = \frac{(2^i - 1)n + 2^i - 2}{2^i},$$

we get $r_i \cdot a_i^2 \cdot c_i$ triangles containing one point of $\mathcal{R}_i$.

Using that every non-empty triangle of $\mathcal{X}_m$ gives three non-convex 4-holes, and summing up over all layers $\mathcal{R}_i$, we obtain

$$
\begin{aligned}
3 \cdot \sum_{i=1}^{m} r_i \cdot a_i^2 \cdot c_i \quad &= \\
3 \cdot \sum_{i=1}^{m} 2^i \left(\frac{n - (2^{i+1} - 2)}{2^{i+1}} \right)^2 \frac{(2^i - 1)n + 2^i - 2}{2^i} \quad &= \\
\frac{1}{2} n^3 + \left(\frac{39}{4} - 3 \log_2(n+2) \right) n^2 - O(n \log n)
\end{aligned}
$$

for the total number of non-convex 4-holes of $\mathcal{X}_m$. □

2.4.3 Minimizing the number of (general) 4-holes

As already mentioned, we have $\frac{n^2}{2} - O(n) \leq h_4(n) \leq 1.9397n^2 + o(n^2)$ for the minimal number $h_4(n)$ of convex 4-holes. For non-convex 4-holes, the corresponding lower bound trivially is zero.

By checking all point sets of cardinality eight from the order type data base [33], we obtained the following observation for general 4-holes.

Observation 2.9. *Let S be a set of $n = 8$ points in the plane in general position, and $p_1, p_2 \in S$ two arbitrary points of S. Then S contains at least five 4-holes having p_1 and p_2 among their vertices.*

In fact, this bound is best possible in general. On the one hand, consider any set S with $n \geq 8$ points, and any two points $p_1, p_2 \in S$. Then p_1 and p_2 together with the six points of $S \backslash \{p_1, p_2\}$ that are closest to the segment $p_1 p_2$ form a set S' of eight points. S' does interfere with any point from $S' \backslash S$. Thus, as by Observation 2.9, S' contains at least five 4-holes having p_1 and p_2 among its vertices, S does as well.

On the other hand, there exist arbitrarily large point sets S such that there exist points $p_1, p_2 \in S$ which are only contained in at most five 4-holes. For example, consider the point set shown in Figure 2.13, and consider 4-holes having both p_1 and p_2 as vertices. Note that such a 4-hole cannot contain any of the points $p_7 \ldots p_n$. The reason is that every triangle $p_1 p_2 p_k$, $7 \leq k \leq n$, contains p_3, p_4, and at least one of p_5 and p_6, and thus cannot be completed to a 4-hole. But the set $\{p_1, \ldots, p_6\}$ contains only five 4-holes with both p_1 and p_2 in their vertex set. Thus, we cannot hope for anything better than the result stated in Observation 2.9.

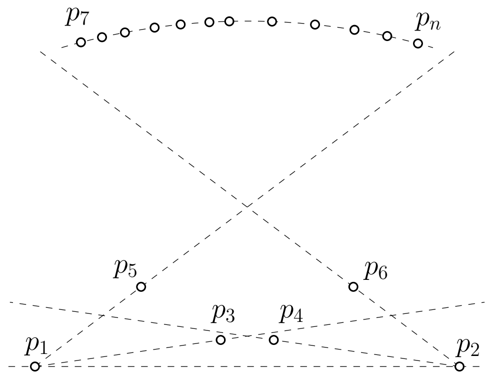


Figure 2.13: A point set containing only five 4-holes that have both p_1 and p_2 as vertices.

Based on Observation 2.9, we can derive a lower bound for the number of general 4-holes. Note that there exist sets which contain fewer than $1.94n^2$ convex 4-holes, while by the below result any set contains at least $2.5n^2 - O(n)$ general 4-holes.

Theorem 2.10. *Let S be a set of $n \geq 8$ points in the plane in general position. Then S contains at least $\frac{5}{2}n^2 - O(n)$ 4-holes.*

Proof. We consider the point set S in x-sorted order, $S = \{p_1, \ldots, p_n\}$, and sets $S_{i,j} = \{p_i, \ldots, p_j\} \subseteq S$. The number of sets $S_{i,j}$ having at least 8 points is

$$\sum_{i=1}^{n-7} \sum_{j=i+7}^{n} 1 = \sum_{i=1}^{n-7} n - i - 6 = \frac{n^2}{2} - \frac{13}{2}n + 21.$$

By Observation 2.9, each set $S_{i,j}$ contains at least five 4-holes having p_i and p_j among their vertices. Moreover, as p_i and p_j are the left- and rightmost points of $S_{i,j}$, they are also the left- and rightmost points for each of these 4-holes. This implies that any 4-hole of S counts for at most one set $S_{i,j}$, which gives a lower bound of $\frac{5}{2}n^2 - O(n)$ for the number of 4-holes in S. $\square$

2.5 5-holes

For the case of 5-holes, we start with a slight improvement to one of the original questions from Erdős, namely the number of convex 5-holes any point set must at least have.

2.5.1 Lower bounds for the number of convex 5-holes

Recall that $h_5(n) = \min_{|S|=n} h_5(S)$ is the number of convex 5-holes any point set of cardinality n has to have. The best upper bound $h_5(n) \leq 1.0207n^2 + o(n^2)$ can be found in [44]. Although $h_5(n)$ is conjectured to be quadratic in the size of S [53], to this date not even a super-linear lower bound exists. For quite some time, the best lower bound was $h_5(n) \geq \lfloor \frac{n-4}{6} \rfloor$, obtained by Bárány and Károlyi [42]. During the work for this thesis, we subsequently improved this bound several times. For the sake of completeness, and as the approaches to the problem use different techniques, in the following we present the proofs of our last three bounds, namely $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$ [31], $h_5(n) \geq \lceil \frac{3}{7}(n-11) \rceil$ [32], and finally $h_5(n) \geq \frac{3}{4}n + o(n)$.

The last bound was partly developed during the birthday conference for Ferran Hurtado (the XIV Spanish Meeting on Computational Geometry) in order to convert his present to science. It is based on joint work with Clemens Huemer et al. and has not been published yet. Note that all these lower bounds for $h_5(n)$ still remain linear in the set size $|S| = n$.

Theorem 2.11. *Let S be a set of $n \geq 12$ points in the plane in general position. Then $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$*

Proof. Dehnhardt [63] showed that every set of 12 points contains at least three convex 5-holes. To make use out of this observation, we sort the points of S from left to right, and split them into groups of 8 points (we assume that no two points of S have the same x-coordinate, as we slightly rotate S otherwise). To each of these groups we also associate the next (w.r.t. the sorting) 4 points on its right. In this way we obtain $\lfloor \frac{n-4}{8} \rfloor$ groups of 12 points each. Any two of these groups share at most 4 points, and thus any convex 5-hole can belong to at most one such group. By the result of Dehnhardt we thus have $h_5(n) \geq 3\lfloor \frac{n-4}{8} \rfloor$. □

Theorem 2.12. *Any set of n points in the plane in general position contains at least $h_5(n) \geq \lceil \frac{3}{7}(n-11) \rceil$ convex 5-holes.*

Proof. Consider an arbitrary set S of n points. Assume that there is an extreme point $p \in S$ which is incident to (at least) one convex 5-hole spanned by S. We count these convex 5-holes (solely) for p, remove p from S, and continue with $S_1 = S\backslash\{p\}$. Assume further that we can repeat this $i \geq 0$ times. This way we count (at least) i different convex 5-holes, and obtain a point set S_i of cardinality $|S_i| = n - i$, for which all extreme points of S_i are not incident to any convex 5-hole.

Now take any extreme point $p \in S_i$. Sort all other points of S_i radially around p (such that its neighbors on the convex hull $\mathrm{CH}(S_i)$ are the first point p_1 and the last point p_{n-i-1} in the sorting, respectively). Split the sorted set $S_i\backslash\{p\}$ into consecutive groups G_l, for $1 \leq l \leq \left\lfloor \frac{n-i-5}{7} \right\rfloor$, of seven points each such that the remainder R contains at least four points; see Figure 2.14. Then every group G_l together with the sorting anchor p and the first four points of G_{l+1} (or R, respectively) gives a set $G'_l \subset S_i$ of 12 points.

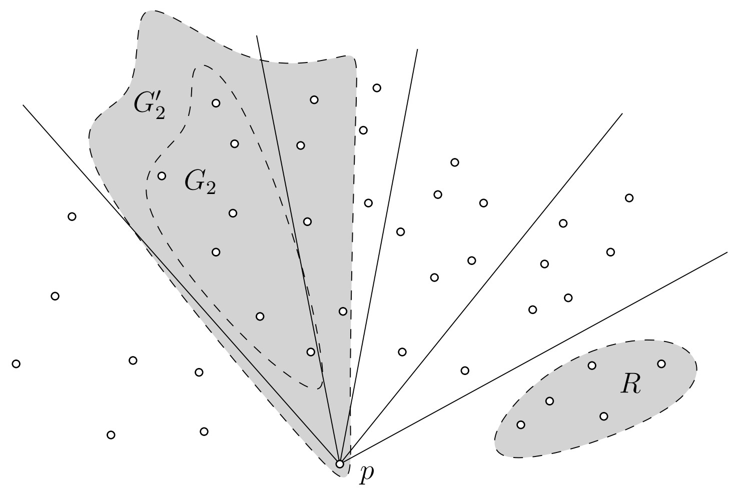


Figure 2.14: Splitting S_i into groups G_l of seven points each, plus a remainder R of at least four points.

We know by Dehnhardt [63] that every set of 12 points, and thus also every set G'_l, contains at least 3 convex 5-holes. As p is not incident to any convex 5-hole, all convex 5-holes in any set G'_l must be incident to at least one point of its underlying set G_l and can thus be counted (solely) for G_l. As R must have at least four points, there are exactly $\left\lfloor \frac{n-i-1-4}{7} \right\rfloor$ groups G_l, and at least three times that many convex 5-holes in S_i. Adding the convex 5-holes we counted for points of $S\backslash S_i$, of which there are at least i, we obtain a lower bound of

$$\begin{aligned} i + 3\left\lfloor \frac{n-i-5}{7} \right\rfloor &\geq i + 3\frac{n-i-5-6}{7} \\ &= \frac{3n+4i-33}{7} \end{aligned}$$

for the total number of convex 5-holes in S. This term is minimized for $i = 0$, which leads to a lower bound of $\lceil \frac{3}{7}(n-11) \rceil$ for the minimum number $h_5(n)$ of convex 5-holes in any set of n points. $\square$

Theorem 2.13. *For any integer $m \geq 1$, every set of $n = 2^m \cdot 12$ points in the plane in general position contains at least $h_5(n) \geq \frac{3n}{4} - o(n)$ convex 5-holes.*

Proof. Consider an arbitrary point set S of cardinality n and a halving line l of S which splits S into S_1 and S_2 and does not contain any point of S; see Figure 2.15.

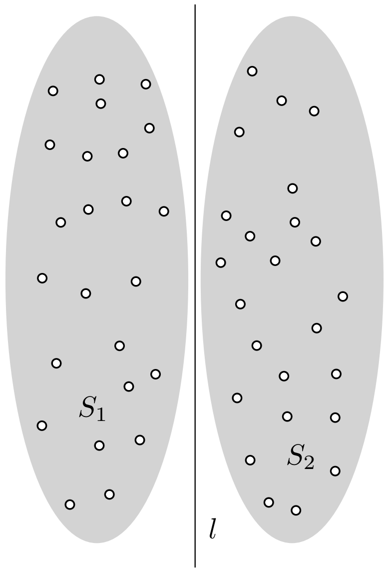


Figure 2.15: A point set S split by a halving line l into to equal sized point sets $S_1, S_2 \subset S$.

Let c be the number of convex 5-holes that are crossed by l. Then the number of convex 5-holes of S is c plus the numbers of convex 5-holes in S_1 and S_2, respectively. As $|S_1| = |S_2| = \frac{n}{2}$, we obtain (2.9) as a lower bound for $h_5(S)$.

$$h_5(S) \geq c + 2 \cdot h_5(\frac{n}{2}) \tag{2.9}$$

Next we use the result of Dehnhardt [63], that every set of 12 points contains at least three convex 5-holes, to construct a different lower bound for $h_5(S)$. To this end we consider a line l' that intersects l and cuts off a set $S' \subseteq S$, consisting of eight points from S_1 and four points from S_2; see Figure 2.16. Further let l'' be a line which is parallel to l' and splits $S' \cap S_1$

into two groups of four points, and let $S'' \subset S'$ be the set which is cut off by l''. Note that neither l' nor l'' contain any points of S.

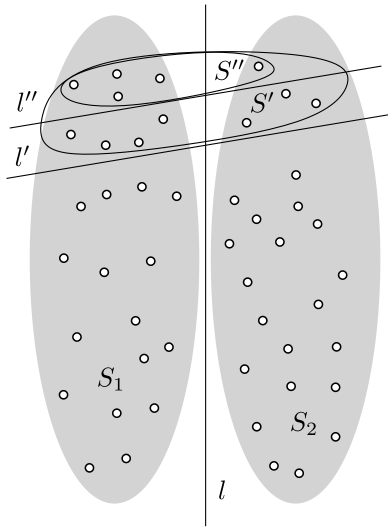


Figure 2.16: Cutting of eight points from S_1 and four points from S_2.

By Dehnhardt [63] we know that S' contains at least three convex 5-holes. We distinguish two cases.

Case 1. Assume that S' contains at least three convex 5-holes which are not intersected by l. Then each of these 5-holes contains only points from S_1 and thus at least one point above l''; see Figure 2.17 left. We count the three 5-holes for the set S_1 and continue on $S \backslash S''$.

Case 2. S' contains at most two convex 5-holes which are not intersected by l. Then we count all convex 5-holes in S' that are intersected by l for the halving line l and continue on $S \backslash S'$; see Figure 2.17 right.

Note that in both cases we cut off at least four points from S_1 and at most four points from S_2. Thus we can repeat this process until we have processed all $\frac{n}{2}$ points of S_1. Assume that we counted c_1 convex 5-holes for l. Then we have processed Case 2 at most c_1 times, and thus Case 1 has appeared at least $\frac{1}{4} \cdot \left(\frac{n}{2} - 8c_1\right)$ times. Thus the number of convex 5-holes in S_1 (i.e., not intersecting l) we counted is at least

$$h_5(S_1) \geq \frac{3}{4}\left(\frac{n}{2} - 8c_1\right). \tag{2.10}$$

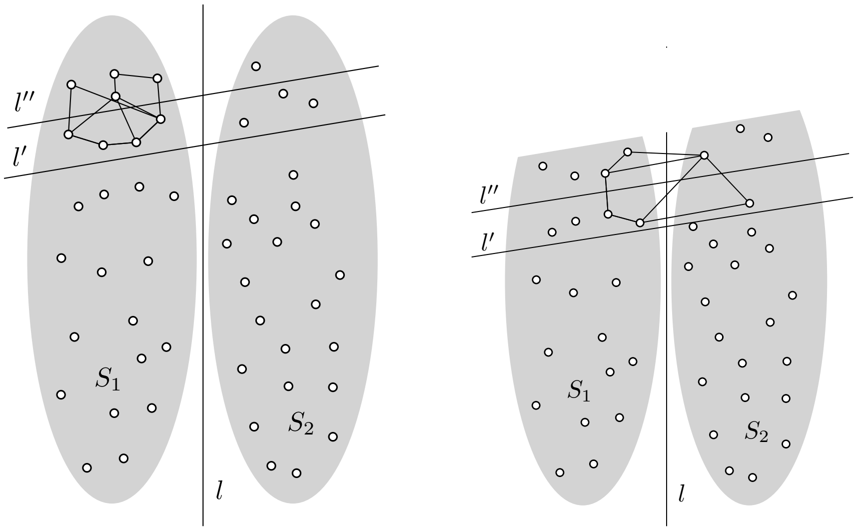


Figure 2.17: Two cases for counting convex 5-holes. Case 1 (left): at least three 5-holes in $S_1 \cap S'$; Case 2 (right): at least one 5-hole crossing the halving line l.

Repeating the same procedure for S_2 (considering lines l' which cut off eight points from S_2 and four points from S_1), we also obtain

$$h_5(S_2) \geq \frac{3}{4}\left(\frac{n}{2} - 8c_2\right), \tag{2.11}$$

where c_2 is the number of convex 5-holes which we counted as intersected by l. Note that any convex 5-hole intersected by l which we counted during processing S_1 might have occurred again when processing S_2. The total number c of convex 5-holes intersected by l is at least

$$c \geq \max\{c_1, c_2\} \geq \frac{c_1 + c_2}{2}.$$

Combining this with Inequations (2.10) and (2.11), we obtain

$$h_5(S) \geq \frac{3n}{4} - 6(c_1 + c_2) + \frac{c_1 + c_2}{2} = \frac{3n}{4} - 11 \cdot \frac{c_1 + c_2}{2}$$

as a second lower bound for the number of convex 5-holes in S. Combining this with (2.9), we obtain

$$h_5(S) \geq \max\left\{\left(\frac{c_1 + c_2}{2} + 2 \cdot h_5(\frac{n}{2})\right), \left(\frac{3n}{4} - 11 \cdot \frac{c_1 + c_2}{2}\right)\right\}. \tag{2.12}$$

Note that the first term in Inequation (2.12) is strictly monotonic incresing in $\frac{c_1+c_2}{2}$, while the second term is strictly monotonic decreasing in $\frac{c_1+c_2}{2}$.

Thus, the minimum of the lower bound in (2.12) is reached if both bounds are equal.

$$\begin{aligned}
\frac{c_1+c_2}{2} + 2\cdot h_5(\frac{n}{2}) &= \frac{3n}{4} - 11\cdot\frac{c_1+c_2}{2} \\
12\cdot\frac{c_1+c_2}{2} &= \frac{3n}{4} - 2\cdot h_5(\frac{n}{2}) \\
\frac{c_1+c_2}{2} &= \frac{n}{16} - \frac{1}{6}\cdot h_5(\frac{n}{2})
\end{aligned}$$

Plugging this value for $\frac{c_1+c_2}{2}$ into the first first term in Inequation (2.12), we obtain (2.13) as a lower bound for $h_5(S)$ for any S, and thus also as a lower bound for $h_5(n)$.

$$h_5(n) \geq \frac{n}{16} - \frac{1}{6}\cdot h_5(\frac{n}{2}) + 2\cdot h_5(\frac{n}{2}) = \frac{n}{16} + \frac{11}{6}\cdot h_5(\frac{n}{2}) \tag{2.13}$$

We resolve this recursion by repeatedly re-applying it until we reach the base case $h_5(12)$.

$$\begin{aligned}
h_5(n) &\geq \frac{n}{16} + \frac{11}{6}\cdot h_5(\frac{n}{2}) \\
&\geq \frac{n}{16} + \frac{11}{6}\left(\frac{n}{2\cdot 16} + \frac{11}{6}\cdot h_5(\frac{n}{2\cdot 2})\right) \\
&\geq \frac{n}{16} + \frac{n}{16}\cdot\frac{11}{12} + \left(\frac{11}{6}\right)^2\cdot h_5(\frac{n}{2^2}) \\
&\geq \frac{n}{16} + \frac{n}{16}\cdot\frac{11}{12} + \frac{n}{16}\cdot\left(\frac{11}{12}\right)^2 + \left(\frac{11}{6}\right)^3\cdot h_5(\frac{n}{2^3}) \\
&\geq \dots \\
&\geq \frac{n}{16}\cdot\sum_{i=0}^{m-1}\left(\frac{11}{12}\right)^i + \left(\frac{11}{6}\right)^m\cdot h_5\left(\frac{n}{2^m}\right) \\
&= \frac{n}{16}\cdot\frac{1-\left(\frac{11}{12}\right)^m}{1-\frac{11}{12}} + \left(\frac{11}{6}\right)^m\cdot h_5(12) \\
&= \frac{n}{16}\cdot 12\cdot\left(1-\left(\frac{11}{12}\right)^m\right) + \left(\frac{11}{6}\right)^m\cdot h_5(12) \\
&= \frac{3n}{4}\cdot\left(1-\left(\frac{11}{12}\right)^m\right) + \left(\frac{11}{6}\right)^m\cdot h_5(12).
\end{aligned}$$

As $h_5(12) \geq 3$, and as as $n = 2^m\cdot 12$ which implies $m = \log_2\left(\frac{n}{12}\right)$, continuing the calculation we get

$$\begin{aligned}
h_5(n) &\geq \frac{3n}{4} \cdot \left(1 - \left(\frac{11}{12}\right)^m\right) + \left(\frac{11}{6}\right)^m \cdot 3 \\
&= \frac{3n}{4} - \frac{3n}{4} \cdot \left(\frac{1}{2}\right)^m \cdot \left(\frac{11}{6}\right)^m + \left(\frac{11}{6}\right)^m \cdot 3 \\
&= \frac{3n}{4} - \frac{3n}{4} \cdot \left(\frac{12}{n}\right) \cdot \left(\frac{11}{6}\right)^m + 3 \cdot \left(\frac{11}{6}\right)^m \\
&= \frac{3n}{4} - 6 \cdot \left(\frac{11}{6}\right)^m = \frac{3n}{4} - 6 \cdot \left(\frac{11}{6}\right)^{\log_2\left(\frac{n}{12}\right)} \\
&= \frac{3n}{4} - 6 \cdot \left(\frac{n}{12}\right)^{\log_2\left(\frac{11}{6}\right)} = \frac{3n}{4} - \left(\frac{6}{12^{\log_2\left(\frac{11}{6}\right)}}\right) \cdot n^{\log_2\left(\frac{11}{6}\right)}
\end{aligned}$$

Considering the negative part of the last term, we have $\frac{11}{6} < 2$ and thus $\log_2\left(\frac{11}{6}\right) \lessapprox 0.874469117916 < 1$, which yields the claimed result:

$$h_5(n) \gtrapprox \frac{3n}{4} - 0.683031256499 \cdot n^{0.874469117916} = \frac{3n}{4} - o(n).$$

□

In the above proof we used a result by Dehnhardt [63], stating that every set of 12 points contains at least three convex 5-holes. In fact, Dehnhardt only showed $h_5(12) \geq 3$, and it was unclear whether $h_5(12) = 3$ or $h_5(12) = 4$. Using the order type database, Aichholzer found point sets of 12 points that contain only three convex 5-holes, settling this question. A point set attaining the lower bound $h_5(12) = 3$ is shown in Figure 2.18. The point set in Figure 2.19 is geometrically quite different but combinatorially similar, again with three convex 5-holes.

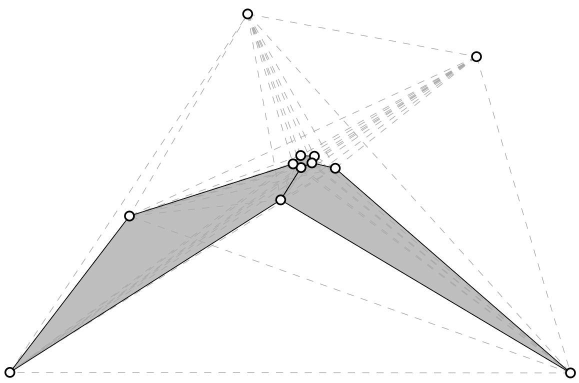


Figure 2.18: A set of 12 points containing only three convex 5-holes.

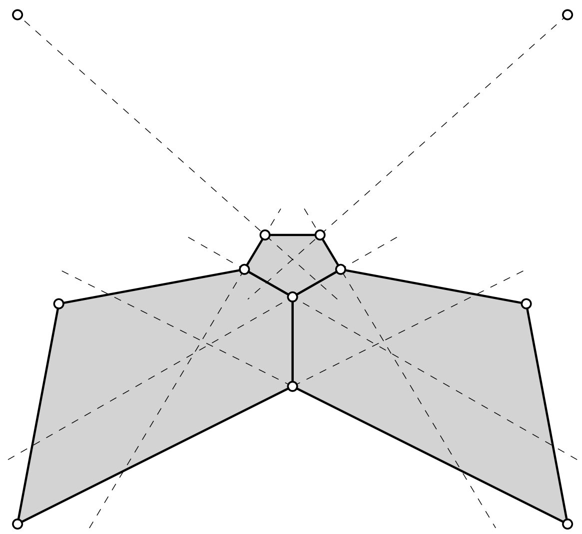


Figure 2.19: A symmetric set of 12 points containing only three convex 5-holes.

Figures 2.20 and 2.21 show point sets of size 13 and 14, respectively, which also contain few convex 5-holes. But here we do not know whether or not these are minimizing configurations. Combining the values from Table 2.5 in Section 2.2.2 with the result for $n = 12$, the first set size guaranteeing at least four convex 5-holes is 18. Still, we believe that the four convex 5-holes for $n = 13$ as well as the six convex 5-holes for $n = 14$ might in fact be minimizing.

For $n = 15$, the best we have been able to find by now are sets containing nine convex 5-holes. Here we think that there could as well be sets with less convex 5-holes. For most of the sets with nine convex 5-holes we found, a good drawing (in the sense of avoiding almost collinear point triples) is not possible. Figure 2.22 shows one of the best drawings we found with respect to this criterion.

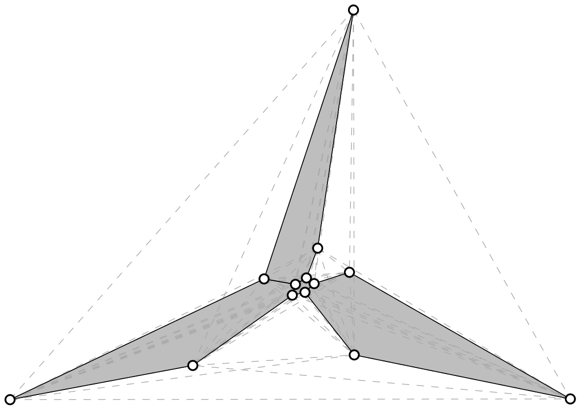


Figure 2.20: A set of 13 points containing only four convex 5-holes.

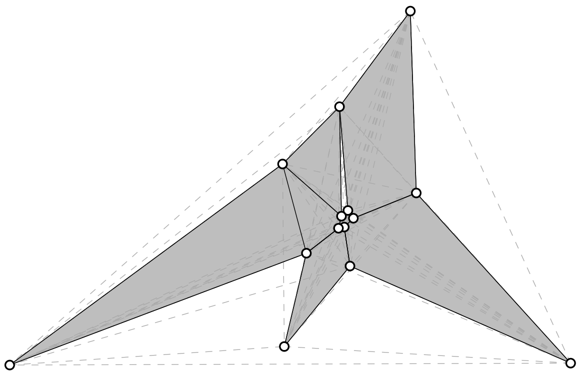


Figure 2.21: A set of 14 points containing only six convex 5-holes.

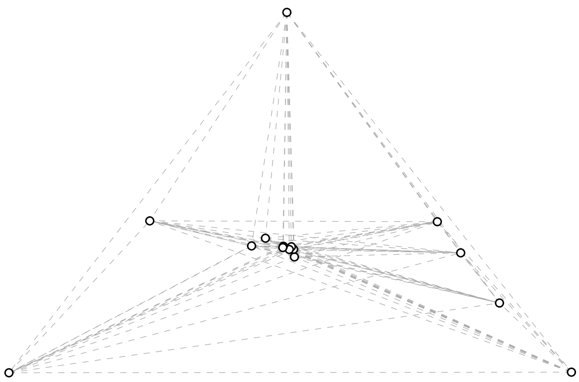


Figure 2.22: A set of 15 points containing only nine convex 5-holes.

2.5.2 A lower bound for the number of (general) 5-holes

We obtained the following observation for general 5-holes by checking all 14 309 547 according point sets from the order type data base [33].

Observation 2.14. *Let S be a set of $n = 10$ points in the plane in general position, and $p_1, p_2 \in S$ two arbitrary points of S. Then S contains at least 34 5-holes having p_1 and p_2 among their vertices.*

Based on this simple observation we derive the following lower bound for the number of 5-holes.

Theorem 2.15. *Let S be a set of $n \geq 10$ points in the plane in general position. Then S contains at least $17n^2 - O(n)$ 5-holes.*

Proof. We follow the lines of the proof of Theorem 2.10 in Section 2.4.3. We consider the point set S in x-sorted order, $S = \{p_1, \ldots, p_n\}$, and sets $S_{i,j} = \{p_i, \ldots, p_j\} \subseteq S$. The number of sets $S_{i,j}$ having at least 10 points is

$$\sum_{i=1}^{n-9} \sum_{j=i+9}^{n} 1 = \frac{n^2}{2} - O(n)$$

For each $S_{i,j}$ consider the eight points of $S_{i,j} \backslash \{p_i, p_j\}$ which are closest to the segment $p_i p_j$ to obtain a set of 10 points, including p_i and p_j. By Observation 2.14, each such set contains at least 34 5-holes which have p_i and p_j among their vertices. Moreover, as p_i and p_j are the left- and rightmost point of $S_{i,j}$, they are also the left- and rightmost point for each of these 5-holes. This implies that any 5-hole of S can count for at most one set $S_{i,j}$, which gives a lower bound of $17n^2 - O(n)$ for the number of 5-holes in S. □

2.5.3 Maximizing the number of (general) 5-holes

The results for small sets shown in Table 2.5 suggest that the number of (general) 5-holes is minimized by sets in convex position. In this section we will not only show that this is in fact not the case, but rather prove the contrary: Similar to the result in Section 2.4.1, sets in convex position maximize the number of 5-holes for sufficiently large n.

Lemma 2.16. *A point set S with triangular convex hull and i interior points contains at most $(4i+5)$ 5-holes which have the three extreme points among their vertices.*

Proof. Let Δ be the convex hull of S, a, b, and c the three extreme points of S (in counterclockwise order), and $I = S\backslash\{a, b, c\}$ the set of inner points of S, $|I| = i$. As all 5-holes we consider have a, b, and c among their vertices, they contain either one or two edges of Δ.

First, we derive an upper bound for the number of 5-holes that contain only one edge of Δ. If two points $p, q \in I$ form a 5-hole that contains only the edge ab of Δ, they have to be neighbored in a circular order of I around c; see Figure 2.23(a).

Let p be before q in the counterclockwise order around c. We say that p *starts a 5-hole (with ab)*. Note that q is uniquely defined by p and ab, and that the triangular area bounded by the lines cp, ap, and the edge ab must not contain any points of I.

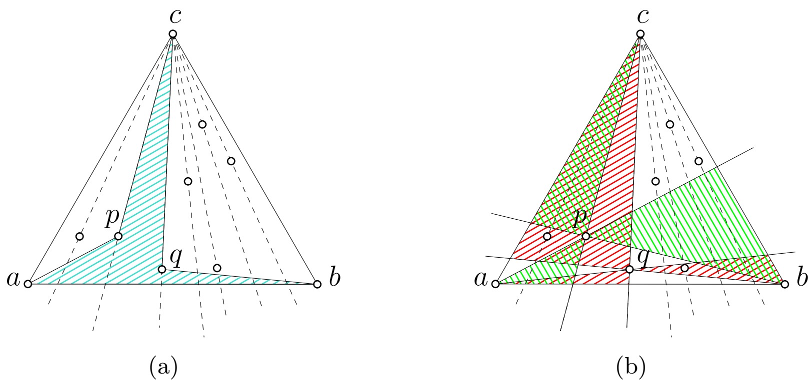


Figure 2.23: (a) A 5-hole containing only the edge ab of Δ. (b) Shaded areas have to be empty if p (or q, respectively) starts a 5-hole with each edge of Δ.

Assume that p starts a 5-hole with each edge of Δ, implying that the according areas for all three edges of Δ have to be empty; see Figure 2.23(b). Then any other point $q \in I$ can start 5-holes with at most two edges of Δ, as p lies in one of the three areas that would have to be empty for q; see again Figure 2.23(b). Using this fact, we conclude that at most one point of I might start three such 5-holes and all other inner points start at most two such 5-holes. This gives a total of at most $(2i+1)$ 5-holes that contain only one edge of Δ.

Second, we consider 5-holes that contain two edges of Δ where one of the two vertices of I is reflex and the other is convex. Assume that there exists such a 5-hole without the edge ab, and with p_{ab} as reflex vertex; see Figure 2.24(a).

Then the non-convex quadrilateral $ap_{ab}bc$ must not contain any points of I, which implies that all other such 5-holes without the edge ab have p_{ab}

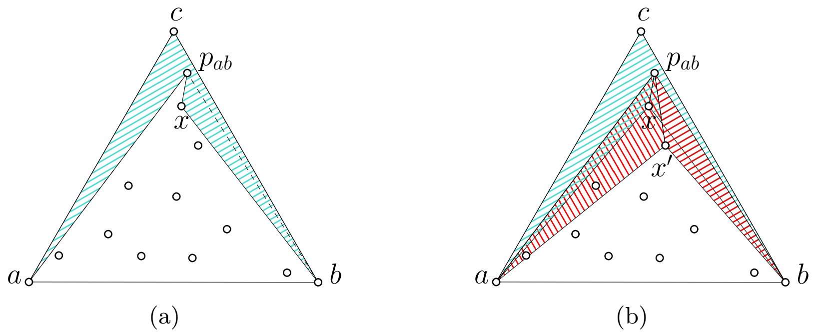


Figure 2.24: (a) A 5-hole $ap_{ab}xbc$ containing two edges of Δ. (b) Only one point of I can span two 5-holes for ab.

as reflex vertex as well. Let x be the convex vertex in a 5-hole without ab. We say that x *spans the 5-hole (for ab)*.

Note that a point x might span two 5-holes for ab, namely $axp_{ab}bc$ and $ap_{ab}xbc$. But this situation can happen for at most one point x, as all other points have to lie inside the triangle axb and thus for each of them, x lies inside exactly one of the two according possible 5-holes; see Figure 2.24(b).

Now assume that for every edge e of Δ, there exist 5-holes skipping (solely) e. Then for every edge e there is one unique point $p_e \in I$ that is the single reflex vertex in all 5-holes for e, and each non-convex quadrilateral spanned by $\Delta \backslash \{e\}$ and p_e is empty; see Figure 2.25.

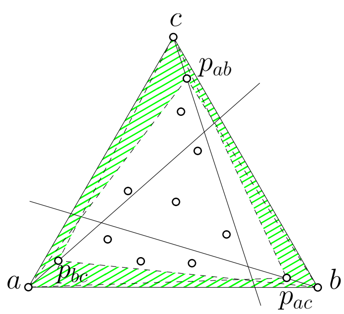


Figure 2.25: If for each combination of two sides of Δ there is a 5-hole where one vertex of I is convex, then the shaded area must be empty.

Assume further that there is a point y, that spans a 5-hole for each edge e of Δ. Note that if a point x lies below the supporting line of ap_{bc}, then the 5-gon $axp_{ab}bc$ contains p_{bc}. Accordingly, if x lies below the supporting line of bp_{ac}, then p_{ac} lies inside $ap_{ab}xbc$. Thus, no point inside the triangle formed by the supporting lines of ap_{bc}, bp_{ac}, and ab can span a 5-hole for ab because any such 5-gon contains either p_{bc} or p_{ac}. As similar statements hold for the other edges of Δ as well, y has to lie outside all these triangles, and thus inside the triangle formed by the supporting lines of ap_{bc}, bp_{ac}, and cp_{ab}.

Note that y can span only one 5-hole for each side, as for each reflex point there is a line l supporting one of the segments cp_{ab}, ap_{bc}, or bp_{ac} such that y and the reflex point lie on opposite sides of l.

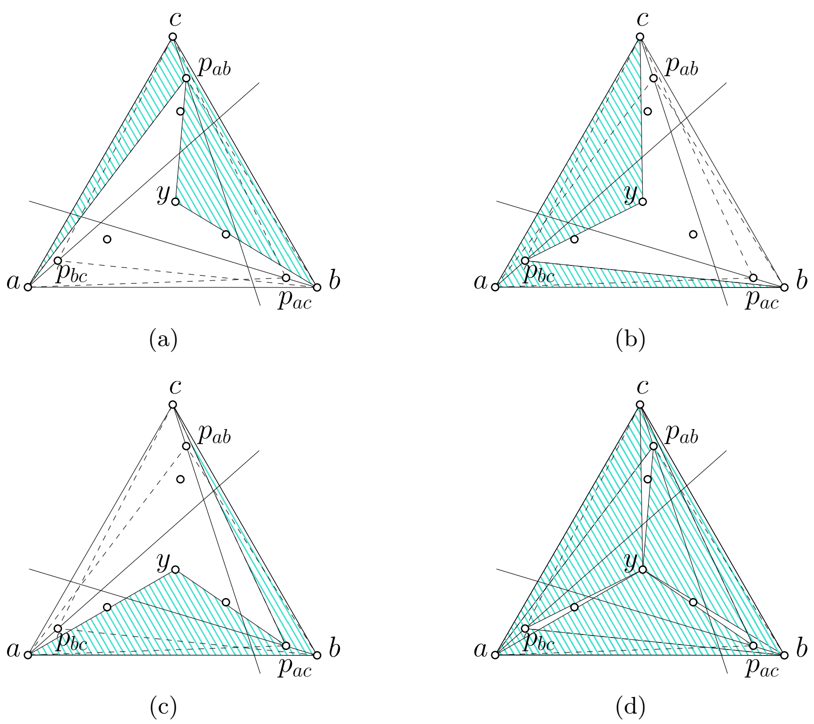


Figure 2.26: Three 5-holes spanned by y, each one leaving out a different side of Δ.

Figure 2.26 shows the three possible 5-holes spanned by y both separately and altogether. As by assumption, the whole shaded area in Figure 2.26(d) does not contain any points of I, all other points must be located in the non-shaded wedges.

Now, if a point lies in the wedge from y towards p_{ab}, then it cannot span a 5-hole for ac, as y lies inside one candidate and p_{ab} lies inside the other. Accordingly, a point in the wedge from y to p_{bc} cannot span a 5-hole for ab, and a point in the wedge from y to p_{ac} cannot span a 5-hole for bc. Thus, at most one point might span a 5-hole for e for each edge e of Δ, and we obtain an upper bound of $(2i+4)$ for the number of such 5-holes: at most two per point of I, plus one for the special point spanning a 5-hole for each edge of Δ, plus one additional per edge of Δ.

Finally, consider 5-holes that contain two edges of Δ, where the two additional vertices are both reflex, like the one shown in Figure 2.27.

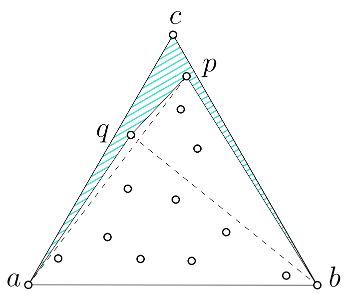


Figure 2.27: Remaining points of I have to be located in the white areas.

There is at most one such 5-hole per non-used side of Δ. Moreover, the existence of such a 5-hole for an edge e of Δ implies that there is no 5-hole for e where one of the vertices of I is convex. Thus, the upper bound for all 5-holes using two edges of Δ (with and without a point of I being convex) is still $(2i+4)$, and we obtain an upper bound for the total number of 5-holes of $(4i+5)$. □

Lemma 2.17. *Let Γ be a non-empty convex quadrilateral in S. There are at most four (non-convex) 5-holes spanned by the four vertices of Γ plus a point of S in the interior of Γ.*

Proof. Let $p_1, \ldots, p_4$ be the vertices of Γ. Observe that any non-convex 5-hole has to use three edges of Γ. Thus there are four choices for the unused edge of Γ, and for each choice there is at most one way to complete the three used edges of Γ to a 5-hole. Assume to the contrary that two different 5-holes avoid the edge p_1p_2 and use points q_1 and q_2, respectively, in the interior. Then q_2 lies in the triangle formed by $p_1p_2q_1$. But then q_1 must lie inside the polygon $p_1q_2p_2p_3p_4$, a contradiction. □

Taking into account the number of points on the convex hull of each 5-tuple, these two lemmas lead to the following theorem.

Theorem 2.18. *For $n \geq 86$ the number of 5-holes is maximized by a set of n points in convex position.*

Proof. In the following we assign every non-convex 5-tuple to the (three or four) vertices of its convex hull, and call this convex hull the *representing* triangle (or quadrilateral) of the potential non-convex 5-holes.

From Lemma 2.16 we know that a non-empty triangle Δ with $i > 0$ interior points represents at most $4i+5$ non-convex 5-holes. In addition, each of the $o = n-3-i$ points outside Δ might form a convex quadrilateral Γ with Δ. According to Lemma 2.17, each such Γ represents at most 4 non-convex 5-holes. Thus, altogether we obtain (2.14) as an upper bound for the number of non-convex 5-holes which have the vertices of Δ on their convex hull.

$$4o + 4i + 5 = 4n - 7 \tag{2.14}$$

Note that if a (convex) quadrilateral is non-empty, then its vertices form at least one triangle which is non-empty as well. Thus, summing up (2.14) over all non-empty triangles, we obtain an upper bound on the number of non-convex 5-holes.

Considering convex 5-holes, observe that every 5-tuple gives at most one convex 5-hole. Denote with N the number of 5-tuples that do *not* form a convex 5-hole, and with T the number of non-empty triangles. Then we get (2.15) as a first upper bound on the number of (general) 5-holes of a point set.

$$\binom{n}{5} - N + (4n-7) \cdot T \tag{2.15}$$

To obtain an improved upper bound from (2.15), we need to derive a good lower bound for N. For this, consider again a non-empty triangle Δ. As Δ is not empty, each of the $\binom{n-3}{2}$ 5-tuples that contain all three vertices of Δ is either not convex or not empty. On the other hand, for such a 5-tuple, all of its $\binom{5}{3}$ contained triangles might be non-empty. Thus, we obtain $T\binom{n-3}{2}/\binom{5}{3}$ as a lower bound for N, and thus (2.16) as an upper bound for the number of 5-holes.

$$\binom{n}{5} + \left(4n - 7 - \frac{\binom{n-3}{2}}{\binom{5}{3}}\right) \cdot T \tag{2.16}$$

For $n \geq 86$ this is at most $\binom{n}{5}$, the number of 5-holes for a set of points in convex position, which proves the theorem. $\square$

Note that for the lower bound of the set size n, the truth lies somewhere between 17 and 84: A least for $n \leq 16$, the number of general 5-holes is not maximized by convex sets.

2.6 k-holes

2.6.1 Maximizing the number of (general) k-holes

In the last two sections we have shown that the numbers of 4-holes and 5-holes are maximized for point sets in convex position if n is sufficiently large. After we had been able to show the result for 4-holes, we conjectured that this might true for any $k \geq 4$ [16]. The following theorem settles this conjecture in the affirmative.

Theorem 2.19. *For every $k \geq 4$ and $n \geq 2(k-1)!\binom{k}{4}+k-1$, the number of k-holes is maximized by a set of n points in convex position.*

Proof. Every non-convex k-hole has as its vertex set a non-convex k-tuple, and every non-convex k-tuple has at least one triangle formed by three extreme points (i.e., points on the convex hull of the k-tuple) that contains points of the k-tuple in its interior. So consider such a non-empty triangle Δ. We count the number of non-convex k-holes having the three vertices of Δ as extreme points. Note that any such k-hole can be reduced to a (not necessarily simple) non-empty $(k-1)$-gon by removing a reflex vertex from its boundary.

Denote by $\mathcal{K}$ the set of (not necessarily simple) non-empty $(k-1)$-gons having the vertices of Δ on their convex hull. First, $|\mathcal{K}|$ can be bounded from above by the number of (not necessarily simple) possibly empty $(k-1)$-gons having the three vertices of Δ on their boundary, which is

$$\frac{(k-2)!}{2}\binom{n-3}{k-4}.$$

Further, every $(k-1)$-gon in $\mathcal{K}$ can be completed to a (simple) non-convex k-hole in at most $k-1$ ways by adding a reflex vertex. Thus the number of non-convex k-holes having all vertices of Δ on their convex hull is bounded from above by

$$(k-1)\frac{(k-2)!}{2}\binom{n-3}{k-4} = \frac{(k-1)!}{2}\binom{n-3}{k-4}.$$

Considering convex k-holes, observe that every k-tuple gives at most one convex k-hole. Denote by N the number of k-tuples that do *not* form a convex k-hole, and by T the number of non-empty triangles. Then we get (2.17) as a first upper bound on the number of (general) k-holes of a point set.

$$\binom{n}{k} - N + \left(\frac{(k-1)!}{2}\binom{n-3}{k-4}\right) \cdot T \tag{2.17}$$

To obtain an improved upper bound from (2.17), we need to derive a good lower bound for N. To this end, consider again a non-empty triangle Δ. As Δ is not empty, none of the $\binom{n-3}{k-3}$ k-tuples that contain all three vertices of Δ forms a convex k-hole. On the other hand, for such a k-tuple, all of its $\binom{k}{3}$ contained triangles might be non-empty. We obtain $T \cdot \binom{n-3}{k-3} / \binom{k}{3}$ as a lower bound for N, and thus (2.18) as an upper bound for the number of k-holes.

$$\binom{n}{k} + \left(\frac{(k-1)!}{2} \binom{n-3}{k-4} - \frac{\binom{n-3}{k-3}}{\binom{k}{3}} \right) \cdot T \tag{2.18}$$

For $n \geq 2(k-1)!\binom{k}{4} + k - 1$ this is at most $\binom{n}{k}$, the number of k-holes of a set of n points in convex position, which proves the theorem. $\square$

The above theorem states that convexity maximizes the number of k-holes for $k = O(\frac{\log n}{\log\log n})$ and sufficiently large n. Moreover, the proof implies that any non-empty triangle in fact reduces the number of empty k-holes. Thus it follows that, for $k = O(\frac{\log n}{\log\log n})$ and n sufficiently large, the maximum number of convex k-holes is strictly larger than the maximum number of non-convex k-holes; see also the next section.

At the other extreme, for $k \approx n$ the statement does not hold: As already mentioned in the introduction, a set of k points spans at most one convex k-gon, but might admit exponentially many different non-convex k-gons. This leads to the question, for which k the situation changes. The following theorem provides a linear upper bound.

Theorem 2.20. *The number of k-holes in the double chain $DC(n)$ on n points is at least*

$$\binom{\frac{n-4}{2}}{\frac{n-k}{2}} \cdot \frac{n-k+2}{2} \cdot \Omega(4.64^k).$$

Proof. As already mentioned in Section 2.3.1, García et al. [88] showed that the double chain of n points ($n/2$ points on each chain) admits $\Omega(4.64^n)$ polygonizations. To estimate the number of k-holes of the double chain on n points, we first use this result for a double chain on k points ($k/2$ points on each chain), obtaining $\Omega(4.64^k)$ different k-polygonizations. Then we distribute the remaining $n-k$ points among all possible positions, meaning that for each k-polygonization, we obtain the double chain on n points with a k-hole drawn, as shown in Figure 2.29.

In their proof, García et al. count paths that start at the first vertex of the upper chain and end at the last vertex of the lower chain. Before the first vertex on the lower chain, they add an additional point q to complete these paths to polygonizations. We slightly extend this principle, by also adding an additional point p on the upper chain after the last vertex; see Figure 2.28.

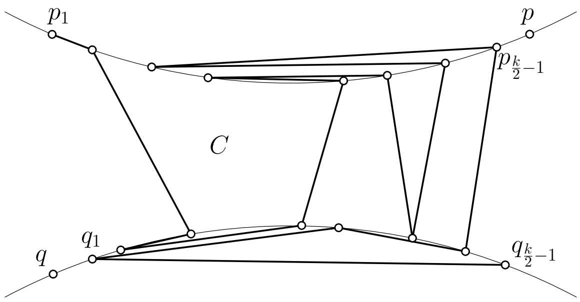


Figure 2.28: A path C in the double chain, using all but the vertices p and q.

Then we complete each path C to a polygonization in one of the following ways: Either we add p to C directly next to $p_{\frac{k}{2}-1}$ and then complete C via q, obtaining P_q, or we add q to C directly next to q_1, and close the polygonization via p, obtaining P_p; see again Figure 2.29.

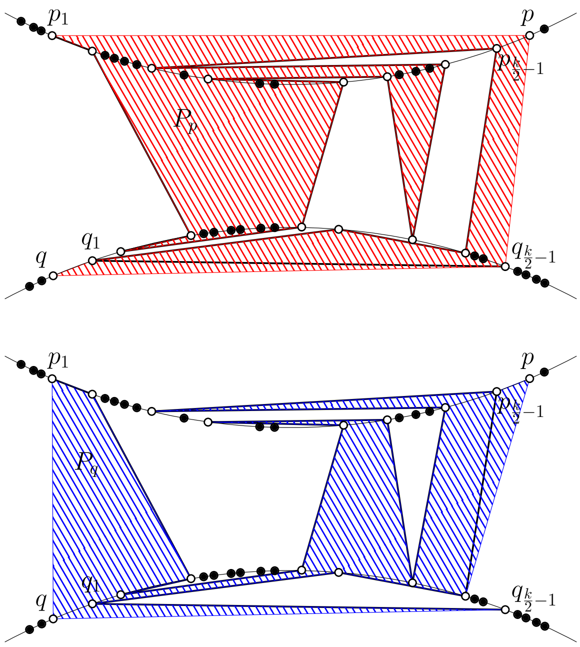


Figure 2.29: Two ways to complete a path to a polygonization.

Note that this changes the number of polygonizations only by a constant factor and thus does not influence the asymptotic bound. However, the interior of P_q is the exterior of its “complemented” polygonization P_p, meaning that if we place a point somewhere on the double chain and it lies inside P_q, then it lies outside P_p, and vice versa. It follows that, in one of the two polygonizations, at least half of the $k+2$ positions to insert points are outside the polygonization. Hence we can distribute the $\frac{n-k}{2}$ points on each chain to at least $\frac{k}{2}+1$ possible positions in total. Now, on one of the two chains we have at least $\frac{k}{4}+1$ positions; see again Figure 2.29. More precisely, there are $\frac{k}{4}+j+1$ positions on this chain (where $0 \leq j < \frac{k}{4}$) and (at least) $\max\{2, \frac{k}{4}-j\}$ positions on the other chain. The lower bound stems from the fact that the positions before the first and after last vertex of a chain are always possible. Placing a points on the b positions of one chain can be seen as placing a balls into b boxes. The number of ways to do so is $\binom{a+b-1}{a}$. Using this, we obtain

$$\binom{\frac{n-k}{2}+\frac{k}{4}+j}{\frac{n-k}{2}} \cdot \max\left\{\binom{\frac{n-k}{2}+1}{\frac{n-k}{2}}, \binom{\frac{n-k}{2}+\frac{k}{4}-j-1}{\frac{n-k}{2}}\right\}$$

possibilities to place the remaining points on the two chains. This factor is minimized for $j=\frac{k}{4}-2$, which yields the claimed lower bound of

$$\binom{\frac{n-4}{2}}{\frac{n-k}{2}} \cdot \frac{n-k+2}{2} \cdot \Omega(4.64^k)$$

for the number of k-holes of $DC(n)$. □

2.6.2 An upper bound for the number of non-convex k-holes

The following theorem shows that for sufficiently small k with respect to n, the maximum number of non-convex k-holes is smaller than the maximum number of convex k-holes.

Theorem 2.21. *For any constant $k \geq 3$, the number of non-convex k-holes in a set of n points is bounded by $O(n^{k-1})$ and there exist sets with $\Theta(n^{k-1})$ non-convex k-holes.*

Proof. We first show that there are at most $O(n^{k-1})$ non-convex k-holes by giving an algorithmic approach to generate all non-convex k-holes. We represent a non-convex k-hole by the counter-clockwise sequence of its vertices, where we require that the last vertex is reflex. Note that any non-convex k-hole has $r \geq 1$ such representations, where r is the number of its reflex vertices. Thus the number of different representations is an upper bound on the number of non-convex k-holes.

We have n possibilities to choose the first vertex v_1, $n-1$ for the second vertex v_2, and so on. Several of the sequences obtained might lead to non-simple polygons, but we are only interested in an upper bound. For the second-last vertex v_{k-1} we have $n-k+2$ possibilities, but the last vertex v_k is uniquely defined. As v_k is required to be reflex and the polygon has to be empty, we have to use the inner geodesic connecting v_{k-1} back to v_1. Only if this geodesic contains exactly one point, namely v_k, we do obtain one non-convex k-hole (again ignoring possible non-simplicity). Thus we obtain at most $n(n-1)(n-2)\dots(n-k+2) = n!/(n-k+1)! = O(n^{k-1})$ non-convex k-holes.

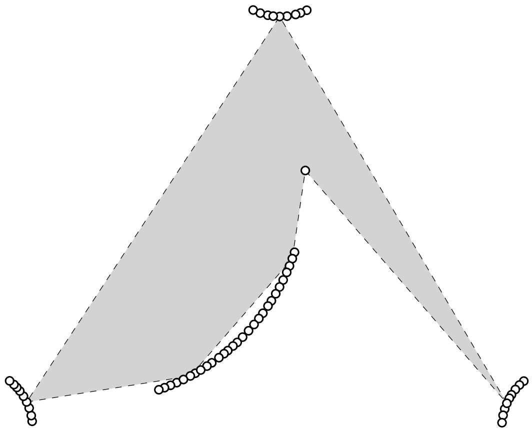


Figure 2.30: A set with $\Theta(n^{k-1})$ non-convex k-holes.

For an example which achieves this bound see Figure 2.30. Each of the four indicated groups of points contains a linear fraction of the point set; for example $\frac{n}{4}$ points. To show that in this example we have $\Omega(n^{k-1})$ non-convex k-holes it is sufficient to only consider the k-holes with triangular convex hull of the type indicated in the figure. For each of the three vertices of the convex hull of the k-hole we have a linear number of possible choices, and the $k-4$ non-reflex inner vertices can also be chosen from a linear number of vertices. Thus we obtain

$$\Omega\left(n^3 \cdot \binom{n}{k-4}\right) = \Omega(n^{k-1})$$

non-convex k-holes. □

2.6.3 On the minimum number of (general) k-holes

Every set of k points admits at least one polygonization. Using this obvious fact, we obtain the following result, which is a more general formulation of the according statements for 4-holes and 5-holes (but with worse constants).

Theorem 2.22. *Let S be a set of n points in the plane in general position. For every $c < 1$ and every $k \leq c \cdot n$, S contains $\Omega(n^2)$ k-holes.*

Proof. We follow the lines of the proof of Theorem 2.10 in Section 2.4.3. Consider the point set S in x-sorted order, $S = \{p_1, \ldots, p_n\}$, and sets $S_{i,j} = \{p_i, \ldots, p_j\} \subseteq S$. The number of sets $S_{i,j}$ of cardinality at least k is

$$\sum_{i=1}^{n-k+1} \sum_{j=i+k-1}^{n} 1 \;=\; \frac{(n-k+1)(n-k+2)}{2} \;=\; O(n^2).$$

For each $S_{i,j}$ use the $k-2$ points of $S_{i,j} \backslash \{p_i, p_j\}$ which are closest to the segment p_ip_j to obtain a subset of k points including p_i and p_j. Each such set contains at least one k-hole which has p_i and p_j among its vertices. Moreover, as p_i and p_j are the left and rightmost points of $S_{i,j}$, they are also the left and rightmost points of this k-hole. This implies that any k-hole of S can count for at most one set $S_{i,j}$, which gives a lower bound of $\Omega(n^2)$ for the number of k-holes in S. □

At the other extreme, we know that the minimum (over all n-point sets of the) number of (general) k-holes cannot be more than the minimum number of convex k-holes plus the maximum number of non-convex k-holes. The minimum number of convex k-holes is $O(n^2)$ for $k \leq 6$ (and zero for $k \geq 7$), and the maximum number of non-convex k-holes is $O(n^{k-1})$. As the latter dominates the former, this gives an upper bound of $O(n^{k-1})$ for the minimum number of general k-holes. But this bound is by far not tight, as the following theorem shows.

Lemma 2.23. *In an integer grid G of size $\sqrt{n} \times \sqrt{n}$, every edge is incident to at most $O(\sqrt{n}\log n)$ interior-empty triangles (meaning that they do not contain any point of G in their interior).*

Proof. We denote a shortest edge of G (an edge that does not have any points of G in its interior) as a slot. Further, we denote the slope of a line l spanned by points of G as the difference (d_x, d_y) of the coordinates of the endpoints of a slot on l. Note that a line with slope $(0,1)$ or $(1,0)$ contains exactly $\sqrt{n}$ points of G. A line with slope (d_x, d_y), $d_x, d_y \neq 0$, contains at most $\min\left\{\left\lceil \frac{\sqrt{n}}{|d_x|} \right\rceil, \left\lceil \frac{\sqrt{n}}{|d_y|} \right\rceil\right\}$ points of G.

Consider an arbitrary edge pq of G (which possibly contains some points of G in its interior), and its supporting line l. Let l' and l'' be the two lines parallel to l and spanned by points of G for which no point of G lies between l and l', and between l and l'', respectively; see Figure 2.31.

Both, l' and l'', contain at most $\sqrt{n}$ points of G, each of which spans an interior-empty triangle with pq. Further, each of the points on l spans a degenerate interior-empty triangle with pq. Any other triangle Δ incident to pq has its third point r strictly outside the strip bounded by l' and l''.

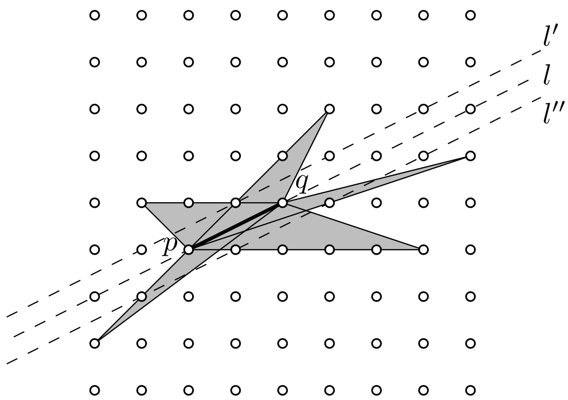


Figure 2.31: A slot pq in a 9×9 integer grid with the according lines l, l', and l''. Gray triangles are interior-empty and have the third point outside the strip $l'l''$.

A necessary condition for such a triangle Δ to be interior-empty is that both, pr and qr, pass through the same slot s on l' (or l''). Moreover, at least one of the supporting lines of pr and qr contains an endpoint of this slot s. To see this latter property, consider lines l_p and l_q from p and q to the according two endpoints of one slot s on l'. Because s is a slot, there are no points in the interior of the (triangular of half-strip) region bounded by l_p, l_q and l'. Thus, any point seen from both p and q through s must lie on the boundary of this region, more exactly on l_p or l_q; see again Figure 2.31.

Using this latter property, we can derive an upper bound on the number of points r that are visible from p and q via the same slot by counting the number of points on such supporting lines.

Consider first the case that pq is a horizontal edge, i.e., $q - p = (d_x, 0)$. Then the according lines through p (or q) and a point of a slot on l' (or l'') have slopes in $\{(0,1), (\pm 1, 1), (\pm 2, 1), \ldots, (\pm\sqrt{n}, 1)\}$. Assuming that all these lines really exist for pq in G, we obtain the following upper bound

$$\begin{aligned} 3\sqrt{n} + 2 \cdot \sum_{i=-\sqrt{n}}^{\sqrt{n}} \left\lceil \frac{\sqrt{n}}{|i|} \right\rceil &= 3\sqrt{n} + 2\sqrt{n} + 4 \cdot \sum_{i=1}^{\sqrt{n}} \left\lceil \frac{\sqrt{n}}{i} \right\rceil \\ &\leq 6\sqrt{n} + 4\sqrt{n}\log_e(n) \\ &= O(\sqrt{n}\log(n)) \end{aligned} \tag{2.19}$$

for the total number of interior-empty triangles incident to pq (the first $3\sqrt{n}$ have the third point on one of l, l', and l'').

For the general case of pq being an edge with $q-p = (d_x, d_y)$, its supporting line l has slope $(d'_x, d'_y) = (\frac{d_x}{\gcd(d_x,d_y)}, \frac{d_y}{\gcd(d_x,d_y)})$. The according slopes of the lines through p (or q) and a point of a slot on l' (or l'') differ from each other by a multiple of (d'_x, d'_y). Note that d'_x and d'_y are integers with $\max\{d'_x, d'_y\} \geq 1$. Thus, the according number of interior-empty triangles for a general edge cannot exceed the bound in 2.19 for a horizontal edge, which completes the proof. $\square$

Theorem 2.24. *For every constant $k \geq 4$ and every $n = m^2 \geq k$, there exist sets with n points in general position that admit at most $O(n^{\frac{k+1}{2}}(\log n)^{k-3})$ k-holes.*

Proof. The point set S we consider is the squared Horton set of size $\sqrt{n} \times \sqrt{n}$; see [144]. Roughly speaking, S is a grid which is perturbed such that every set of originally collinear points forms a Horton set. For any two points $p, q \in S$, the number of empty triangles that contain the edge pq is bound from above by the number of (possibly degenerate) interior-empty triangles incident to the according edge in the regular grid. By Lemma 2.23, this latter number is at most $O(\sqrt{n} \log n)$.

To estimate the number of k-holes in S, we will use triangulations and their dual: In the dual graph of a triangulation, every triangle is represented as a node, and two nodes are connected iff the corresponding triangles share an edge. For the triangulation of a k-hole, this gives a binary tree which can be rooted at any triangle that has an edge on the boundary of the k-hole; see [116]. It is well known that there are $C_{k-2} = O(4^k \cdot k^{-\frac{3}{2}})$ such rooted binary trees [116]. Although exponential in k, this bound is constant with respect to the size n of S.

Now pick an empty triangle Δ in S and an arbitrary rooted binary tree B. Consider all k-holes which contain Δ and admit a triangulation that is represented by B rooted at Δ. As the number of empty triangles incident to an edge in S is $O(\sqrt{n} \log n)$, each of the $n-3$ edges in B yields $O(\sqrt{n} \log n)$ possibilities to continue a triangulated k-hole, and we obtain an upper bound of $O((\sqrt{n} \log n)^{k-3})$ for the number of triangulations of k-holes for Δ that represent B.

Multiplying this by the (constant) number of rooted binary trees of size $k-2$ does not change the asymptotic and thus yields an upper bound of $O((\sqrt{n} \log n)^{k-3})$ for the number of all triangulations of all k-holes containing Δ. As any k-hole can be triangulated, this is also an upper bound for the number of k-holes containing Δ.

Finally, there are $O(n^2)$ empty triangles in S (see [44]), and thus we obtain $O(n^2(\sqrt{n} \log n)^{k-3}) = O(n^{\frac{k+1}{2}}(\log n)^{k-3})$ as an upper bound for the number of k-holes in S. $\square$

A notion similar to k-holes is that of islands. An island I is a subset of S, not containing points of $S \setminus I$ in its convex hull. A k-island is an island of k elements. For example, any two points form a 2-island, and any 3 points spanning an empty triangle are a 3-island of S. In particular, convex k-holes are also islands while non-convex k-holes need not be islands. In general, any k-tuple spans at most one island, while it might span many k-holes. In [82] it was shown that the number of k-islands of S is always $\Omega(n^2)$ and that the Horton set of n points contains only $O(n^2)$ k-islands (for sufficiently large n). Compare this with Theorems 2.22 and 2.24.

Note that the Horton set has $\Omega(n^3)$ 4-holes. A general super-quadratic lower bound for the number of 4-holes would solve a conjecture of Bárány in the affirmative, showing that every point set contains an edge that spans a super-constant number of 3-holes; see e.g. [53], Chapter 8.4, Problem 4. This would also imply a quadratic lower bound for the number of convex 5-holes. So far, not even a super-linear bound is known for the latter problem; see also Section 2.5.1.

2.6.4 An improved lower bound for the number of convex 6-holes

Gerken [89] showed that each set of at least 1717 points in general position contains a convex 6-hole. This immediately implies that each set of n points contains a linear number of convex 6-holes, namely at least $\lfloor \frac{n}{1717} \rfloor$. In the following we slightly improve on this bound. We start by showing a result for monochromatic convex 6-holes in two-colored point sets.

Lemma 2.25. *Each set of r red points and b blue points in general position in the plane with $r \geq 1716 \left\lceil \frac{b}{2} \right\rceil + 1717$ contains a convex red 6-hole.*

Proof. Consider a non-crossing perfect matching of the blue points; if b is odd, then allow one isolated point p. We extend the segments (in both directions) one by one, until each segment either hits another segment, the line of a previously extended segment or goes to infinity. If b is odd, we take an arbitrary segment through p and extend it as well. Altogether, this results in a decomposition of the plane into $\left\lceil \frac{b}{2} \right\rceil + 1$ convex regions. As the red points lie inside these regions, it follows by the pigeon-hole principle that at least one of these regions contains 1717 red points, and thus a red convex 6-hole by [89]. $\square$

Theorem 2.26. *Each set S of n points in general position in the plane contains at least $\lfloor \frac{n-1}{858} \rfloor - 2$ convex 6-holes.*

Proof. We prove the statement by contradiction. Assume that the point set S contains strictly less than $\lfloor \frac{n-1}{858} \rfloor - 2$ convex 6-holes, and color the points of S red. Now we eliminate all red convex 6-holes by placing an additional blue point inside each of them (one for each), such that the resulting two-colored point set is in general position. By this, at most

$$b \leq \left\lfloor \frac{n-1}{858} \right\rfloor - 3 \tag{2.20}$$

blue points are added, resulting in a bichromatic set with b blue and n red points. We transform the upper bound (2.20) for the number of blue points to a lower bound for the number n of red points, obtaining

$$n \geq 858(b+3) + 1 \geq 1716 \left\lceil \frac{b}{2} \right\rceil + 1717.$$

By Lemma 2.25, any such two-colored point set contains a convex red 6-hole, a contradiction. □

2.7 Discussion

We have shown various lower and upper bounds on the numbers of convex, non-convex, and general k-holes and k-gons in point sets. For some of the bounds we have classes of point sets reaching these bounds. For example the maximum number of general k-holes is obtained by point sets in convex position.

Several questions remain unsettled. For example, some of the presented bounds are not tight, as can be seen in Tables 2.2 and 2.3. Maybe the most intriguing open questions in this context are the following two. Is there a super-linear lower bound for the number of convex 5-holes (cf. Section 2.5.1)? And can we show a super-quadratic lower bound for the number of general k-holes (cf. Theorems 2.10, 2.15, and 2.22) for some constant $k \geq 4$?

Chapter 3

Bichromatic point sets

Continuing with Erdős-Szekeres type questions, we now consider more colorful versions. A point set S is called *l-chromatic* (or *l-colored*), if S is partitioned into l subsets, usually described as *color classes*. A (simple) polygon spanned by points of an l-chromatic set S is called *monochromatic* if all its vertices have the same color, i.e., belong to the same color class. Accordingly, an edge in a graph with vertex set S is called monochromatic if both its endpoints have the same color.

Devillers et al. [66] introduced the following questions: Does every sufficiently large l-chromatic point set contain convex monochromatic k-holes? And if yes, how many? Note that the analogous question for k-gons does not make sense, because the bounds from the uncolored case can be applied directly to each color class of S independently. In contrast, the colors are a strong additional constraint for finding monochromatic holes in S. For example, every set of at least five points contains a convex 4-hole, while the existence of convex monochromatic 4-holes in bichromatic point sets is still an open question, even for arbitrarily large bichromatic point sets.

In the first section of this chapter we consider a relaxed version of the latter question: We investigate monochromatic 4-holes in bichromatic point sets, where, analogously to Chapter 2, we allow the holes to be non-convex. This variation was posed before by Ferran Hurtado [102] and Janos Pach [122]. We have been able to solve it in the affirmative, showing that every bichromatic point set with at least 2079 points contains at least one monochromatic 4-hole (and thus linearly many).

Another classical question on bichromatic point sets that is also strongly related to Erdős-Szekeres type questions is the problem of computing the crossing number of the complete bipartite graph, which is also known as Zarankiewicz's conjecture [46, 93, 149]. As already mentioned, crossings of edges play an important role in the Erdős-Szekeres questions, as for a 4-gon this reflects exactly the difference between a convex and a non-convex configuration; see also Section 2.3.1. Zarankiewicz's conjecture is the two-colored version of trying to minimize the number of crossings: Denote with $K_{n,m}$ the complete bipartite graph on n red and m blue vertices. Zarankiewicz published a proof that the minimal number of crossings $\mathrm{cr}(n, m)$ in a drawing of $K_{n,m}$ is $\mathrm{cr}(n, m) = \mathrm{z}(n, m)$, with

$$\mathrm{z}(n, m) := \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor.$$

To show $\mathrm{cr}(n, m) \leq \mathrm{z}(n, m)$, Zarankiewicz provided a straight-line drawing of $K(n, m)$ obtaining $\mathrm{z}(n, m)$ crossings. Unfortunately, the proof of $\mathrm{cr}(n, m) \geq \mathrm{z}(n, m)$ contained an error and thus the statement changed from Zarankiewicz's theorem to Zarankiewicz's conjecture.

In the second part of this chapter, we work on restricted versions of the Zarankiewicz Conjecture, where (1) we confine the setting to straight-line drawings and (2) require linear separability of the red and blue point set in the drawing. But although we obtained several interesting relations, the main question remains unsettled even in these weaker cases.

A trivial but immanent fact concerning the research of the crossing number is that, as soon as the number of edges relative to the number of points (of each color class) is large enough, crossings in the complete (bipartite) graph are unavoidable. In the third part of this chapter, we change focus to the other end of the range and investigate compatible crossing-free graphs on bichromatic point sets. Abellanas et al. [20] show bounds on how many edges a compatible matching for an (uncolored) graph of a certain class can admit at least.

In Section 3.3, we consider the existence of (different kinds of) matchings for given bichromatic graphs. For several classes of graphs we provide upper and lower bounds for the numbers of edges that can always be obtained by such matchings. For example, we show that every bichromatic perfect matching with n edges admits a bichromatic disjoint compatible matching with at least $\lceil \frac{n-1}{2} \rceil$ edges and that there exist bichromatic perfect matchings for which any such compatible matching has at most $\frac{3n}{4}$ edges.

The results of Section 3.1 have been presented before [26, 24] and are published in [27]. The results of Section 3.2 have been obtained during several research weeks [21, 12]. Preliminary results can be found in [28, 120]. Research for Section 3.3 was started during the *i-Math Winter School: DocCourse on Discrete and Computational Geometry* [103].

3.1 Large bichromatic point sets admit monochromatic 4-holes

In the last chapter we considered problems on the existence and number of k-gons and k-holes in point sets. In this section we will focus on a variation of this topic where the point sets are colored and the k-holes are monochromatic.

Recall the following questions raised by Erdős (see Section 2.1 for background and results): "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole? And what is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?"

Devillers et al. [66] introduced several variations on these questions for colored point sets. In particular, he asked for the existence and number of convex monochromatic k-holes in l-chromatic point sets.

Concerning the case $l = 2$, in [66] the authors proved that any bichromatic set of n points in the plane determines at least $\lceil \frac{n}{4} \rceil - 2$ monochromatic empty triangles (3-holes) with pairwise disjoint interiors, which is tight. Later it was shown in [13] that any bichromatic set of n points contains at least $\Omega(n^{5/4})$ monochromatic 3-holes (no disjointness is required), which then has been improved to $\Omega(n^{4/3})$ [124]. It is conjectured [13, 14] that any bichromatic set of n points in $\mathbb{R}^2$ in general position spans a quadratic number of monochromatic 3-holes. For values of k larger than 3, Devillers et al. show that for $k \geq 5$ and any n there are bichromatic sets of n points where no convex monochromatic k-hole exists (Theorem 3.4 in [66]).

It is natural to wonder whether similar results are possible when there are more than two colors. In [66] (Theorem 3.3) this question has been settled by showing that already for three colors there are sets not even spanning any monochromatic 3-hole.

Hence, the interesting remaining case is the existence of (convex) monochromatic 4-holes in bichromatic point sets. Figure 3.1 shows a set with 18 points which does not contain a convex monochromatic 4-hole, and larger examples with 20 [52], 30 [85], 32 [92], and most recently 36 [101], points have been found. However, all these examples do contain non-convex monochromatic 4-holes, while the one in Figure 3.1 does not.

Note that every point set that admits a convex heptahole contains a convex monochromatic 4-hole for any bicoloration, because at least four of the vertices of the heptahole have the same color. Moreover, it is known that for $n \geq 64$, any bichromatic Horton set contains convex monochromatic 4-holes.

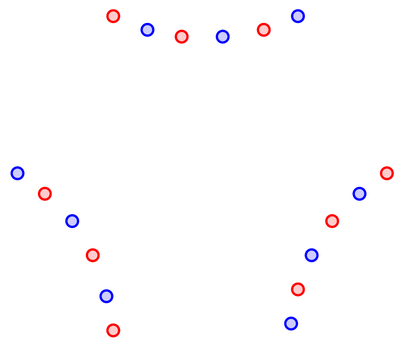


Figure 3.1: Example without monochromatic 4-holes.

These facts led to Conjecture 3.1 in [66], which states that for sufficiently large n any bichromatic set contains at least one convex monochromatic 4-hole.

For quite some time, this conjecture was even open for 4-holes which are not required to be convex, a weaker version that arose later [102, 122], as no progress for the original question had been obtained.[1] As an important step towards solving the initial problem we show that this relaxed version of the conjecture is true: if the cardinality of the bichromatic point set S is sufficiently large, there is always a (possibly non-convex) monochromatic 4-hole spanned by S. To this aim, we first prove several sufficient conditions and then show that for large point sets at least one of them must hold.

Throughout this section, if $S = R \cup B$ is a two-color partition of a point set, then we also write $S = (R, B)$, where R is the set of *red* points and B the set of *blue* points, respectively, with $r = |R|$, $b = |B|$, $n = r + b$, and $r, b \geq 0$.

Recall that we denote with $\mathrm{CH}(S)$ the convex hull of S, with h the number of points of S on the boundary of $\mathrm{CH}(S)$, and with $i = n - h$ the number of interior points of S. Similarly we define $h_r = |\partial\,\mathrm{CH}(R) \cap R|$ and $i_r = r - h_r$ for the red set, as well as $h_b = |\partial\,\mathrm{CH}(B) \cap B|$ and $i_b = b - h_b$ for the blue set.

3.1.1 Preliminaries on uncolored point sets

Let us start with a result on triangulations for (uncolored) point sets which is of interest on its own.

[1] This weaker form of the 4-hole conjecture was probably first proposed and popularized for many years by J. Pach.

Lemma 3.1. *Let S be a set of n points in general position in the plane, and let π be a fixed parity (even or odd). Then there exists a triangulation $T(S)$ of S such that the parity of the degrees in $T(S)$ of at least $2\lfloor \frac{n-1}{4} \rfloor$ points from S is π.*

Proof. Let us assume that all the points in S have different abscissa, which is always possible for a suitable choice of the coordinate system. Consider the points of S being sorted in x-order and group them into sets of five consecutive points such that two neighboring groups have one point in common. Each of the $\lfloor \frac{n-1}{4} \rfloor$ groups admits a convex 4-hole. Let Q be this set of 4-holes and note that two 4-holes in Q are interior disjoint and share at most one point of S but no edge. Each of these 4-holes can either be isolated from the others, not sharing a point with any other 4-hole of Q, or connected to a chain of 4-holes from Q. Moreover, from the x-sorting of the groups it follows that chains of 4-holes in Q cannot close a cycle.

Draw the 4-holes of Q and complete them arbitrarily to an initial triangulation $T(S)$ of S by adding edges. For the remainder of the proof the parity of a point $p \in S$ always refers to the parity of the degree of p in the current triangulation $T(S)$, which we are updating whenever necessary. Our goal is to show that we can assign two points of S with parity π to each 4-hole in Q. First note that flipping the diagonal inside a convex 4-hole, i.e., exchanging it with the second diagonal, changes the parity of all four involved points; hence, if we have t, $0 \leq t \leq 4$, points of parity π before the flip, then we get $4 - t$ afterwards.

For isolated 4-holes it is straightforward to obtain at least two points of parity π. Thus consider a chain of 4-holes in Q. By processing the 4-holes in the chain from left to right we assign to each 4-hole q two points of parity π not using the rightmost point of q. We will call this rightmost point the "connecting" point of the 4-hole. If, after a possible diagonal flip in q, the number of points with parity π in q is at least three, we choose two non-connecting points. Otherwise we can always flip the diagonal of q in such a way that the two points with parity π do not include the connecting point. As the connecting point is the only element that two 4-holes in the chain might share, the given order allows us to consider the 4-holes independently, meaning without having to take care of restrictions imposed by previous assignments. Therefore we have assigned two different points of parity π to each 4-hole in Q, and we obtain a total of at least $2\lfloor \frac{n-1}{4} \rfloor$ points with parity π. $\square$

For odd parity the preceding result can be slightly strengthened to a lower bound of $\frac{n-1}{2}$ by considering unused interior points and a case analysis of small sets. As the improvement is marginal, and we believe that a much better result is possible, we skip the details and instead formulate the following problem.

Open problem 3.2. *What is the maximum value of a constant c such that for any set S of n points in general position there exists a triangulation $T(S)$ in which at least $cn - o(n)$ points of S have odd (or even) degree?*

Remark. Since the preparation of the final version of the paper [27] which contains the results of this section, the bound of Lemma 3.1 has been improved to guarantee $10\left\lfloor \frac{n}{13} \right\rfloor - 2$ points with odd degree; see [25]. This improvement does not change our principle approach. In the following sections we will use this new bound in order to provide an up-to-date version of the results.

We now consider a fixed triangulation $T(S)$ of S and give lower bounds of how many triangles of $T(S)$ have to be "pierced" (meaning that they contain an obstacle in their interior) so that $T(S)$ does not contain any unpierced (i.e., empty) 4-gons. We will see that the number $\#_{odd}$ of points of S with odd degree in $T(S)$ will play a central role.

Lemma 3.3. *If for a triangulation $T(S)$ the number of pierced triangles is less than $n + \frac{\#_{odd} - 4h - 6}{6}$, then there exists an unpierced 4-gon in $T(S)$.*

Proof. Any two adjacent triangles form a 4-gon and at least one of these triangles has to be pierced to prevent unpierced 4-gons. Thus if we consider for a point $p \in S$ all triangles of $T(S)$ incident to p in cyclic order, then every other of these triangles has to be pierced. So if p is an interior point of S, then we need to pierce at least $\lceil \frac{\delta(p)}{2} \rceil$ incident triangles of p, and if p is an extremal point of S, then at least $\lceil \frac{\delta(p)}{2} \rceil - 1$, where $\delta(p)$ is the degree of p in $T(S)$. Note that here points with odd parity contribute a bigger share, as, for example, for inner points, two adjacent triangles have to get pierced.

Using this observation and summing up over all points of S, we over count each piercing at most three times, once for each corner of a triangle. So let S_E be the set of points with even edge degree in $T(S)$, and let S_O be the set of points with odd edge degree in $T(S)$, respectively. Assuming that there is no unpierced 4-gon in $T(S)$, we get the following as a lower bound for the number of piercings:

$$\frac{\sum\limits_{p \in S} \left\lceil \frac{\delta(p)}{2} \right\rceil - h}{3} = \frac{\sum\limits_{p \in S_E} \frac{\delta(p)}{2} + \sum\limits_{p \in S_O} \frac{\delta(p)+1}{2} - h}{3} = \frac{\sum\limits_{p \in S} \frac{\delta(p)}{2} + \frac{\#_{odd}}{2} - h}{3}$$
$$= n + \frac{\#_{odd} - 4h - 6}{6}\,.$$

The last equality stems from the fact that $\sum_{p \in S} \frac{\delta(p)}{2}$ is the number of edges in $T(S)$, and thus $3n - h - 3$ by Euler's formula. $\square$

3.1.2 Bichromatic sets with small convex hulls

Now let $S = (R, B)$ be a bichromatic set. We will triangulate R and use the results of the previous section to get bounds for piercing the resulting red quadrilaterals with blue points from B. From Lemma 3.3 we immediately get the following.

Lemma 3.4. *Let $S = (R, B)$ be a bichromatic point set, and let $T(R)$ be a triangulation of R. With $\#_{odd(R)}$ we denote the number of points of R with odd edge degree in $T(R)$. If $b < r + \frac{\#_{odd(R)} - 4h_r - 6}{6}$, then there exists at least one red 4-hole consisting of two adjacent triangles in $T(R)$.*

A consequence of Lemma 3.4 is a relation between the number of points in R of odd degree in $T(R)$ and the size of the convex hull of R. Namely, if $\#_{odd(R)} > 4h_r + 6 - 6(r - b)$, then there exists at least one red 4-hole in the triangulation $T(R)$. We can now combine this fact with the choice of an appropriate triangulation $T(R)$, with $\#_{odd(R)} \geq 10 \left\lfloor \frac{r}{13} \right\rfloor - 2$, whose existence is proven in Proposition 2 in [25], and we get the following.

Proposition 3.5. *Let $S = (R, B)$ be a bichromatic point set. If*

$$h_r < \frac{5 \left\lfloor \frac{r}{13} \right\rfloor - 4}{2} + \frac{3}{2}(r - b),$$

then S contains at least one red 4-hole.

Note that for this result the roles of R and B can of course be switched. Proposition 3.5 also shows that the worst case occurs if R and B have the same cardinality. In this case, or more generally for $r \geq b$, we can simplify the bound to $h_r < \frac{5r-8}{26} - 4$. In particular, this proves that if the convex hull of the larger subset has sub-linear size, we immediately get a monochromatic 4-hole.

3.1.3 Bichromatic sets with large discrepancy

In this section we consider the case that the cardinalities of the red set and the blue set differ significantly. As a first step we generalize a result of Sakai and Urrutia [133] on convex monochromatic 4-holes to simple but not necessarily convex 4-holes.

Lemma 3.6. *If in a bichromatic point set $r \geq \frac{3}{2}b + 4$, then there exists at least one red 4-hole in R.*

Proof. The proof is based on induction over b and is similar to the proof given in [133].

Induction base. The case $b = 0$ with $r \geq 4$ is trivially true. For $b = 1$ we have $r \geq 6$ as $r \in \mathbb{N}$. Fix one extremal point $p \in R$ and sort $R \setminus \{p\}$ around p. Connecting the points of $R \setminus \{p\}$ in their order around p and to p results in at least four red triangles. As only one of these triangles can be pierced by the only blue point, there exist at least two neighboring unpierced triangles and thus at least one red 4-hole in R.

Induction step $b \to b + 1$. Let $b \geq 2$. Consider the supporting line l through an edge of CH(B). Exactly two points of B lie on l. The remaining $b' = b - 2$ points of B lie on one side of l, say to the right. If more than three points of R lie to the left of l, then they span at least one red 4-hole in R. Otherwise we can apply induction on the $b' + r'$ points to the right of l because $r' \geq r - 3 \geq \frac{3}{2}b + 4 - 3 = \frac{3}{2}(b' + 2) + 4 - 3 = \frac{3}{2}b' + 4$. □

Let us recall that for a point set with an even number of extreme points, a quadrangulation is a maximal planar bipartite graph. If the size of the convex hull is odd, we allow one triangle. The number of 4-gons in a quadrangulation of a point set is given in the next observation in terms of n and h, a fact that will be used in Lemma 3.8. For more details and a proof see, e.g., [10].

Observation 3.7. *A quadrangulation $Q(S)$ on a point set S with n points and h extreme points contains $n - \lceil \frac{h}{2} \rceil - 1$ (empty) 4-gons.*

Note that Lemma 3.6 can be rephrased in the form that $\frac{b}{r-4} > \frac{2}{3}$ is a necessary condition for S to not contain any monochromatic 4-holes. In combination with Observation 3.7 this leads to an interesting iterative relation between the size of a set $S = (R, B)$ not containing a monochromatic 4-hole and the maximum discrepancy between R and B.

We are now ready to show that for sets with sufficiently large cardinality the factor of discrepancy has to be arbitrarily small in order to avoid monochromatic 4-holes.

Lemma 3.8. *For $k \in \mathbb{N}$, $k \geq 4$, let $f(k) = \frac{4}{3}k(2k - 1)$ and $g(k) = \frac{k}{k+2}$. Every bichromatic set $S = (R, B)$ with $|R| = r \geq f(k)$, $|B| = b \leq (r-4)g(k)$ contains at least one monochromatic 4-hole.*

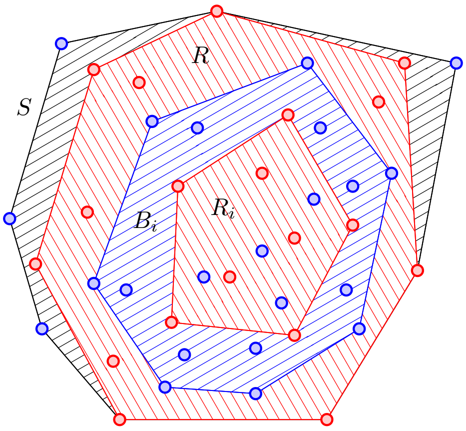


Figure 3.2: Red and blue layers in the proof of Lemma 3.8.

Proof. The lemma can be rephrased in the form that for $S = (R, B)$ with $|R| = r \geq f(k)$ and $|B| = b$ we have

$$\frac{b}{r-4} > g(k) \tag{3.1}$$

as a necessary condition for S to not contain a monochromatic 4-hole. We will prove this by induction over k.

Induction base. For the case $k = 4$ we have $f(k) = \frac{112}{3}$ and $g(k) = \frac{2}{3}$. This is equivalent to Lemma 3.6 with the additional restriction of $r \geq 38$.

Induction step $k \to k+1$. For $k \geq 4$ consider a set $S = (R, B)$ with $r \geq f(k+1)$, and assume that S does not contain a monochromatic 4-hole. Let b_i be the cardinality of the set $B_i \subseteq B$ of blue points in the interior of $\mathrm{CH}(R)$, and let r_i be the cardinality of the set $R_i \subseteq R$ of red points in the interior of $\mathrm{CH}(B_i)$; see Figure 3.2.

As $r \geq f(k+1) > f(k)$, we can apply the induction hypothesis (3.1) to the set (R, B_i). Note that blue points of $B \setminus B_i$ would not interfere with 4-holes in (R, B_i), as they are outside $\mathrm{CH}(R)$. Thus we get the bound

$$\begin{aligned} b_i > \frac{k}{k+2}(r-4) \geq & \frac{k}{k+2}(f(k+1)-4) \\ &= \frac{k}{k+2}\left(\frac{4}{3}(k+1)(2k+1)-4\right) \\ &= \cdots = \frac{4}{3}k(2k-1) = f(k). \end{aligned}$$

Repeating the argument, we can thus apply the induction hypothesis (3.1) to the set (B_i, R_i) and get

$$r_i > \frac{k}{k+2}(b_i - 4)\,. \tag{3.2}$$

Let $\alpha = \frac{b_i}{r-4}$ and thus $b_i = \alpha(r-4)$. By inserting this relation for b_i into (3.2), the inequation is rewritten as

$$r_i > \frac{k}{k+2}(\alpha(r-4)-4) = \alpha\frac{k(r-4)}{k+2} - \frac{4k}{k+2}\,. \tag{3.3}$$

Putting a quadrangulation on R, we use Observation 3.7 to obtain a necessary condition on $\alpha(r-4)$ for S to not contain a red 4-hole:

$$\alpha(r-4) = b_i \geq r - \left\lceil \frac{r-i_r}{2} \right\rceil - 1 \geq \frac{r}{2} + \frac{i_r}{2} - 2 \geq \frac{r}{2} + \frac{r_i}{2} - 2. \tag{3.4}$$

Inserting the lower bound (3.3) for r_i into (3.4), we get

$$\alpha(r-4) > \frac{r}{2} + \frac{1}{2}\left(\alpha\frac{k(r-4)}{(k+2)} - \frac{4k}{(k+2)}\right) - 2.$$

Using that $k \geq 4$ and thus $r \geq f(k+1) \geq 60$, standard manipulation shows that this is equivalent to

$$\alpha > \frac{k+2}{k+4} - \frac{4k}{(k+4)(r-4)}$$

as a necessary condition such that S does not contain a red 4-hole. As $r \geq f(k+1) \geq 2(k+1)(k+2)$, this implies that

$$\alpha > \frac{k+1}{k+3} \tag{3.5}$$

has to hold. Relation (3.5) holds for any fixed set S with $r \geq f(k+1)$, and as $B \supseteq B_i$ we have $b \geq b_i$, which consequently completes the induction step:

$$\frac{b}{r-4} \geq \frac{b_i}{r-4} = \alpha > \frac{k+1}{k+3} = g(k+1). \qquad \square$$

3.1.4 Putting things together

As a consequence of Lemma 3.8 we derive a lower bound on the number of extreme points of the red set, which guarantees the existence of 4-holes.

Lemma 3.9. *For $k \in \mathbb{N}$, $k \geq 4$, let $S = (R, B)$ be a set with at least $r \geq \frac{4}{3}(k+1)(2k+1)$ red points. Then $h_r \geq r\frac{2(2k+3)}{(k+2)(k+3)} + \frac{8k}{k+3}$ implies that S contains a monochromatic 4-hole.*

Proof. Consider a set $S = (R, B)$ with $r \geq \frac{4}{3}(k+1)(2k+1)$. As in the proof of Lemma 3.8, let b_i be the cardinality of the set $B_i \subseteq B$ of blue points in the interior of $\mathrm{CH}(R)$, and let r_i be the cardinality of the set $R_i \subseteq R$ of red points in the interior of $\mathrm{CH}(B_i)$; cf. Figure 3.2.

We apply Lemma 3.8 to (R, B_i) and $k+1$. If $b_i \leq \frac{k+1}{k+3}(r-4)$, we have a monochromatic 4-hole and we are done. So assume $b_i > \frac{k+1}{k+3}(r-4)$ which, together with the lower bound on r, implies that $b_i \geq \frac{4}{3}k(2k-1)$. We can thus now apply Lemma 3.8 to (B_i, R_i) (switching colors) and k. This implies that $r_i > \frac{k}{k+2}(b_i - 4)$, as otherwise we again have a monochromatic 4-hole and are done.

Combining these two lower bounds for b_i and r_i, respectively, we obtain

$$r_i > \frac{k}{k+2}\left(\frac{k+1}{k+3}(r-4) - 4\right) = r\frac{k(k+1)}{(k+2)(k+3)} - 8\frac{k}{k+3}\,. \tag{3.6}$$

As $h_r = r - i_r \leq r - r_i$ we can plug in relation (3.6) and get

$$h_r < r\frac{2(2k+3)}{(k+2)(k+3)} + \frac{8k}{k+3}$$

as a necessary condition such that S does not contain a monochromatic 4-hole, which proves the statement. □

By combining the results of Proposition 3.5 and Lemma 3.9 we finally obtain our main result.

Theorem 3.10. *Every bichromatic set $S = (R, B)$ with $n \geq 2079$ points contains a monochromatic 4-hole.*

Proof. Without loss of generality, assume that $r \geq b$. Moreover, from Proposition 3.5 we know that $h_r < \frac{5\lfloor\frac{r}{13}\rfloor - 4}{2} + \frac{3}{2}(r-b)$ is sufficient to obtain a monochromatic 4-hole. On the other hand, Lemma 3.9 provides $h_r \geq r\frac{2(2k+3)}{(k+2)(k+3)} + \frac{8k}{k+3}$ for $k \geq 4$ and $r \geq \frac{4}{3}(k+1)(2k+1)$ as a second sufficient condition. So if

$$\frac{5\left\lfloor\frac{r}{13}\right\rfloor - 4}{2} \geq r\frac{2(2k+3)}{(k+2)(k+3)} + \frac{8k}{k+3} \tag{3.7}$$

holds, then Theorem 3.10 follows. Using the inequation $r \geq \frac{4}{3}(k+1)(2k+1)$ it follows that inequation (3.7) is fulfilled for $k \geq 19$, and thus for any set with $r \geq 1040$. In other words, for any set with $n \geq 2079$ points. □

As already mentioned in Section 3.1.1, any set of n points admits a triangulation in which at least $10\left\lfloor\frac{n}{13}\right\rfloor - 2$ points have odd vertex degree [25], which is one ingredient for proving the lower bound of 2079 for the required number of points in Theorem 3.10. Note that even if we were able to find a triangulation which guarantees all points to have odd degree, the lower bound on the cardinality of the point set in Theorem 3.10 still would be 1159.

3.1.5 Discussion

We have shown that every sufficiently large bichromatic point set contains monochromatic 4-holes. In combination with the result in [25] this yields an upper bound of 2079 points needed to guarantee the existence of monochromatic 4-holes. At the other extreme, we know that for $n \leq 18$ points there exist bichromatic point sets without any monochromatic 4-holes. It would be nice to close this rather large gap between 18 and 2079.

Of course, the most challenging open question is the initial conjecture for convex monochromatic 4-holes. It seems that the techniques used in our approach cannot be generalized to the convex case, as convexity invalidates several of our lemmas and intermediate results.

It would also be interesting to establish a 3D version of these results for hexahedra consisting of two tetrahedra sharing a face. Let us recall in this respect that Urrutia [143] proved that in any four-colored point set in $\mathbb{R}^3$ in general position there is at least one empty monochromatic tetrahedron (in fact, a linear number of them).

Let us finally mention again Open Problem 3.2, which asks what is the maximum constant c such that for any point set there always exists a triangulation where $cn - o(n)$ points have odd (or even) degree. More generally, the question might be stated for any predefined parity assignment; see [25] for more details.

3.2 Zarankiewicz's conjecture

Continuing with bichromatic point sets and 4-gons, we now investigate non-monochromatic 4-tuples of points which have two points of each color. The question of this section goes back to the Hungarian mathematician Paul Turán, who had to work in a brick factory during World War II.

Problem 3.11 (The Brick Factory Problem [141])**.** *Assume there are n kilns and m storage yards in a brick factory. Further, assume that every kiln is connected to every storage yard by rail. What is the minimum number of crossings of the rail tracks?*

In the following we assume that in a (general) drawing of a graph G, the vertices are represented as points in the the Euclidean plane, and the edges are represented as non self-intersecting continuous curves that do not have any vertices in their interior. Further we assume that any two edges of G share at most a finite number of points and that they cross at each such point.

Recall that the rectilinear crossing number $\overline{\text{cr}}(G)$ of a graph G is the minimum of the number of crossings in any straight-line drawing of G. Accordingly, the crossing number $\text{cr}(G)$ of G is the minimum of the number of crossings in any (general) drawing of G (more exactly, the sum over all pairs e, e' of edges of crossings between e and e'). Trivially, $\text{cr}(G) \leq \overline{\text{cr}}(G)$.

A graph G is called *bipartite* if its vertex set can be partitioned into two *independent* subsets V_1 and V_2, meaning that neither V_1 nor V_2 spans any edge of G The *complete bipartite graph* $K_{n,m}$ contains an edge pq for all pairs of vertices p, q with $p \in V_1$ and $q \in V_2$.

With these definitions, Turán's problem can be reformulated as searching for the crossing number of the complete bipartite graph $K_{n,m}$, which we denote by $\text{cr}(n, m)$. Similarly, we denote the rectilinear crossing number of $K_{n,m}$ by $\overline{\text{cr}}(n, m)$.

In the 1950's, K. Zarankiewicz [148, 149] and K. Urbanik [142] independently proposed the same solution, both claiming the searched minimum $\text{cr}(n, m)$ to be equal to $\text{z}(n, m)$, with

$$\text{z}(n, m) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor .$$

For a while, this was known as Zarankiewicz's theorem. But some years later, Kainen and Ringel found an error in the proof for the lower bound $\text{z}(n, m) \leq \text{cr}(n, m)$ (see Guy [93]), and thus the statement was changed to Zarankiewicz's conjecture.

Conjecture 3.12 (Zarankiewicz's conjecture)**.** *The crossing number of the complete bipartite graph* $K_{n,m}$ *is* $\mathrm{cr}(n,m) = \mathrm{z}(n,m)$*.*

Although many people have been working on this conjecture, still surprisingly little is known. Kleitman [110] showed that it is true for $\min\{n,m\} \leq 6$. Woodall [147] proved it for $n \leq 8$ and $m \leq 10$, and from $\overline{\mathrm{cr}}(n,m) = \mathrm{z}(n,m)$ for some odd n it follows that also $\overline{\mathrm{cr}}(n+1,m) = \mathrm{z}(n+1,m)$.

Concerning lower bounds of $\mathrm{cr}(n,m)$, Kleitman's result for $\mathrm{cr}(5,m)$ together with some simple arguments implies $\mathrm{cr}(n,m) \geq 0.8 \cdot \mathrm{z}(n,m)$; see for example [61]. The latest asymptotic lower bound is by de Klerk et al. [62] who showed that

$$\lim_{n,m\to\infty} \frac{\mathrm{cr}(n,m)}{\mathrm{z}(n,m)} \geq 0.8594,$$

by this improving the previously best known bound of 0.83 for the same value [61].

Finally, the upper bound $\mathrm{cr}(n,m) \leq \mathrm{z}(n,m)$ is from the original proof of Zarankiewicz, by a construction containing $\mathrm{z}(n,m)$ crossings. The construction is surprisingly simple and yields straight-line drawings like the one illustrated in Figure 3.3. In this *Zarankiewicz cross* $C(n,m)$, the n independent vertices are placed on the x-axis (half of them on the positive part and the rest on the negative part), and the m independent vertices are placed on the y-axis (again half of them on the positive part and the other half on the negative part).

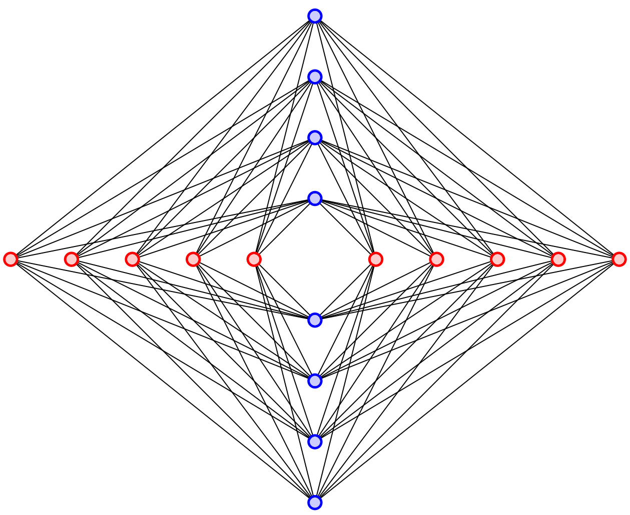


Figure 3.3: The Zarankiewicz cross $C(10,8)$.

We know by Fáry [83] that any planar graph G also admits a plane straight-line drawing in the Euclidean plane. In other words, $\mathrm{cr}(G) = 0$

implies $\overline{\text{cr}}(G) = 0$. Bienstock and Dean have shown that the same is true for any graph with $\text{cr}(G) \leq 3$ [48]. To the contrary, in the same paper they prove that there are classes of graphs with $\text{cr}(G) = 4$ but arbitrarily large rectilinear crossing number, implying that for any $\text{cr}(G) \geq 4$, the ratio $\overline{\text{cr}}(G)/\,\text{cr}(G)$ might be unbounded. For the complete graph K_n, Blažek and Koman [50] showed that $\text{cr}(n) \leq 0.375\binom{n}{4}$, while Ábrego et al. [3] proved $\overline{\text{cr}}(n) \geq 0.379972\binom{n}{4}$. Together this implies that for sufficiently large n, $\overline{\text{cr}}(n)$ is significantly larger than $\text{cr}(n)$.

Interestingly, Zarankiewicz's conjecture – if being true – would imply that for the complete bipartite graph the rectilinear crossing number would be equal to the (general) crossing number, obtained by the configuration from Figure 3.3.

As the original conjecture from Zarankiewicz is by now open since more than fifty years, it seems natural to restrict considerations to the (hopefully) simpler case of straight-line drawings.

3.2.1 Combinatorial types of 4-tuples and directed edges

In the following we draw $K_{n,m}$ with a bichromatic vertex set $S = R \cup B$ such that the red set R and the blue set B are the two independent sets of $K_{n,m}$, with $|R| = n$ and $|B| = m$, respectively. Instead of $R \cup B$ we also write (R, B). For the sake of brevity, we sometimes denote the number of crossings in a straight-line representation of $K_{n,m}$ with vertex set (R, B) as number of crossings of (R, B). For distinguishing between crossing and non-crossing edges in a straight-line representation of $K_{n,m}$, we assume that no point of S lies on the supporting line of any edge, and thus of any bichromatic segment spanned by S. Note that it is not necessary that the points of S are in general position. Especially, we need not care about collinearities among only red points or only blue points (cf. Figure 3.3).

A pair of possibly crossing edges of $K_{n,m}$ is spanned by four points of S, two red and two blue ones. From a combinatorial point of view, there are exactly four different combinatorial types of such 4-tuples: The points can form a convex 4-gon with two possibilities of color arrangements, or they can form a triangle with a red or blue point in the center. Figure 3.4 shows these 4-tuples.

Note that the first configuration in Figure 3.4 contains a crossing, while the other three configurations are plane. For simplicity, we use the following pictograms of these configurations for the numbers of the according 4-tuples in a point set S, as well as for the according types of configurations.

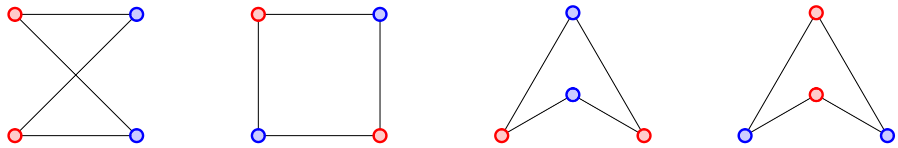


Figure 3.4: The four combinatorially different types of 4-tuples for two blue and two red points

If it is not clear which point set we are considering, we might also make it explicit by writing (R, B). Note that $(R, B) = \overline{\mathrm{cr}}(R, B)$.

In order to count 4-tuples, we investigate several types of segments spanned by points of S. A *red edge* is a directed segment defined by two red points of S. Similarly, a *blue edge* is a directed segment with two blue points of S as endpoints; see Figure 3.5. Note that, if S is not in general position, red edges might contain other red points in their interior. Likewise, blue edges might contain additional blue points. But by assumption, no blue point can lie on a line spanned by two red points, and vice versa.

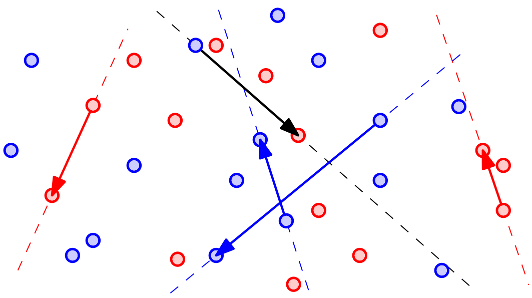


Figure 3.5: Monochromatic and bichromatic (black :-) edges in a point set $S = (R, B)$ with $|R| = 15$ and $|B| = 16$.

If a red edge $\vec{pq}$ has exactly j blue points on the right side of (the directed supporting line of) $\vec{pq}$ then it is called a *red j-edge*. For example, Figure 3.5 contains a red 0-edge and a red 2-edge. As no blue point can lie on the supporting line of $\vec{pq}$, $(m-j)$ blue points lie on the other side of $\vec{pq}$. Thus, inverting the direction of a red j-edge results in a red $(m-j)$-edge. We denote the number of red j-edges in a point set S by e_r^j. If for a red j-edge, j equals $\lfloor \frac{m}{2} \rfloor$ or $\lceil \frac{m}{2} \rceil$, then the supporting line of $\vec{pq}$ partitions B into two subsets of (nearly) the same size. In this case we say that $\vec{pq}$ *halves* B or that $\vec{pq}$ *spans a halving line* of B.

Accordingly, we call a blue edge $\vec{pq}$ with exactly k red points to the right of (the directed supporting line of) $\vec{pq}$ a *blue k-edge*, and denote their number by e_b^k. A blue k-edge is said to halve the red set if $k \in \left\{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\right\}$. Figure 3.5 contains a blue 8-edge which halves the red set of cardinality 15.

Last but not least, a *bichromatic edge* is a directed segment $\vec{pq}$ with one blue and one red endpoint (we do not care whether the segment is directed from the red to the blue point, or vice versa). If such an edge has exactly k red and j blue points on the right side, it is called a bichromatic (k, j)-edge. As no additional point can lie on the supporting line of $\vec{pq}$, exactly $(n-k-1)$ red and $(m-j-1)$ blue points lie on the other side. We denote the number of bichromatic (k, j)-edges by $g_{k,j}$.

But why do these types of edges help us count configurations of 4-tuples? Consider for example a red j-edge $\vec{pq}$. Any choice of two blue points from one side of $\vec{pq}$ together with $\vec{pq}$ gives either a convex 4-tuple of type , or a non-convex 4-tuple of type . As every such 4-tuple can uniquely be identified with $\vec{pq}$, summing up over all j-edges for $0 \le j \le m$ gives exactly all 4-tuples of types and .

$$+ \quad = \sum_{j=0}^{m} e_r^j \cdot \binom{j}{2}$$

Alternatively, again considering a red j-edge, any choice of one blue point from the right and one blue point from the left side gives either a convex 4-tuple of type or a non-convex 4-tuple of type . As each such 4-tuple occurs for exactly one pair of red points, and thus exactly two red edges, we obtain

$$2 \cdot \quad + 2 \cdot \quad = \sum_{j=0}^{m} e_r^j \cdot j \cdot (m-j).$$

Using the same arguments, summing up over blue k-edges yields bounds for + and $2 \cdot$ + $2 \cdot$, respectively.

Considering a bichromatic (k, j)-edge $\vec{pq}$, we can choose a blue and a red point from the right side of $\vec{pq}$. This might yield any of the four possible configurations of 4-tuples, but when summing up, the situation changes. The reason is that a tuple of type is encountered by each of the four bichromatic edges on its convex hull, while all other tuples have only two such edges and are thus counted only twice. This results in

$$2 \cdot \quad + 4 \cdot \quad + 2 \cdot \quad + 2 \cdot \quad = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} g_{k,j} \cdot k \cdot j.$$

The second possibility for counting based on bichromatic edges is choosing a red point from the right side and a blue point from the left side of a bichromatic edge. This might result in any configuration but . and any possible configuration is again counted twice (once for each interior bichromatic edge of the configuration where the other red point lies on the right side).

$$2\cdot$$  $$+2\cdot$$  $$+2\cdot$$  $$= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} g_{k,j}\cdot k\cdot(m-1-j) = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} g_{k,j}\cdot k\cdot(n-1-k).$$

Table 3.1 summarizes all obtained relations. It provides an easy to read overview with which multiplicity the different types of configurations are counted in each relation. Additionally, it contains the sum over all four types of configurations, which simply gives all possibilities to choose two red and two blue points from S.

					
(I)	1	1	1	1	$\binom{n}{2}\cdot\binom{m}{2}$
(II)	1	0	0	1	$\sum_{j=0}^{m} e_r^j \cdot \binom{j}{2}$
(III)	0	1	0	1	$\sum_{k=0}^{n} e_b^k \cdot \binom{k}{2}$
(IV)	0	2	2	0	$\sum_{j=0}^{m} e_r^j \cdot j \cdot (m-j)$
(V)	2	0	2	0	$\sum_{k=0}^{n} e_b^k \cdot k \cdot (n-k)$
(VI)	2	2	4	2	$\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} g_{k,j}\cdot k\cdot j$
(VII)	2	2	0	2	$\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} g_{k,j}\cdot k\cdot (m-1-j) = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} g_{k,j}\cdot (n-1-k)\cdot j$

Table 3.1: Counting configurations of 4-tuples with two red and two blue points in a bichromatic point set $S = (R, B)$.

Using Relation (I) from Table 3.1, we can rewrite the number of crossings $\overline{\mathrm{cr}}(R, B) = (R, B)$ as

$$(R, B) = \binom{n}{2} \cdot \binom{m}{2} - (R, B) - (R, B) - (R, B).$$

Thus, minimizing the rectilinear crossing number is equivalent to maximizing the sum

$$(R, B) + (R, B) + (R, B). \tag{3.8}$$

From Relation (IV) it follows that $(R, B) + (R, B)$ is maximized if all red edges halve the blue set. Similarly, Relation (V) implies that $(R, B) + (R, B)$ is maximized if all blue edges halve the red set. Thus,

$$\frac{1}{2} \cdot (\mathrm{IV}) + \frac{1}{2} \cdot (\mathrm{V}) = (R, B) + (R, B) + 2 \cdot (R, B) \tag{3.9}$$

is maximized by sets (R, B) where every monochromatic edge halves the set of the other color. Note that this is for example the case in the Zarankiewicz cross, and that the sum in (3.9) differs from the according sum for the Zarankiewicz Conjecture in (3.8) only by the factor 2 for (R, B).

Combining different relations from Table 3.1, it also becomes possible to calculate the rectilinear crossing number $\overline{\mathrm{cr}}(R, B) = (R, B)$ in terms of the numbers of monochromatic and bichromatic edges. For example, using Relations (II) and (III) for monochromatic edges together with Relation (VII) for monochromatic edges, we obtain $(R, B) = \frac{1}{2}(\mathrm{VII}) - (\mathrm{II}) - (\mathrm{III})$, and thus

$$(R, B) = \frac{1}{2} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} g_{k,j} \cdot (n - 1 - k) \cdot j - \sum_{j=0}^{m} e_r^j \cdot \binom{j}{2} - \sum_{k=0}^{n} e_b^k \cdot \binom{k}{2}.$$

Similarly, we can choose Relations (VI), (IV), and (V), obtaining

$$2 \cdot (R, B) = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} g_{k,j} \cdot k \cdot j - \sum_{j=0}^{m} e_r^j \cdot j \cdot (m - j) - \sum_{k=0}^{n} e_b^k \cdot k \cdot (n - k).$$

Note that the Zarankiewicz cross maximizes the two negative terms in this latter equation. If it would at the same time minimize the positive term, this would proof the rectilinear version of Zarankiewicz's conjecture. Unfortunately this is not the case.

The following section contains one of the possible reasons why Zarankiewicz's conjecture is still open, and why minimizing the rectilinear crossing number of the complete bipartite graph turns out to be not trivial either.

3.2.2 Point configurations with $\overline{\mathrm{cr}}(R, B) = \mathrm{z}(n, m)$

The Zarankiewicz cross $C(n, m)$ has several special properties. It has the same structure for both colors, a nice regular symmetric pattern, and strong geometric and combinatorial properties. For example, as already mentioned before, all monochromatic edges span halving lines for the point set of the other color. While at a first glance it seems as if these properties are somehow needed to obtain a small crossing number, in fact this is not at all the case.

For example, bending the horizontal arms of the Zarankiewicz cross (i.e., slightly rotating them around the origin) as illustrated in Figures 3.6 and 3.7 destroys the halving property of many red edges. Depending on how exactly the bend is done, e_r^j might be nonzero for all values $0 \leq j \leq m$. The number of -configurations decreases while the number of -configurations is increased. However, the number of -configurations (and thus the number of crossings) stays the same.

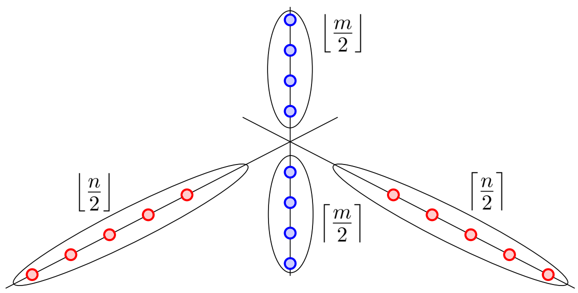


Figure 3.6: Point configuration of the stretched and bent Zarankiewicz cross: several red edges do not anymore halve the blue set.

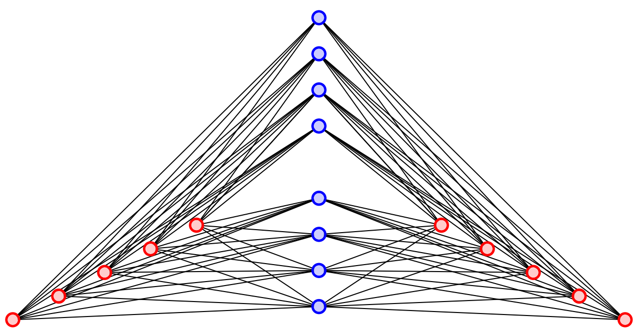


Figure 3.7: A stretched and bent Zarankiewicz cross with 10 red and 8 blue vertices.

While the example from Figure 3.7 is still rather close to the original Zarankiewicz cross, the situation gets more involved in the next example. Here, we additionally rotated the lower part of the blue set. Figures 3.8 and 3.9 show the principle of the point configuration and the resulting complete bipartite graph. Again, the number of crossings remains unchanged.

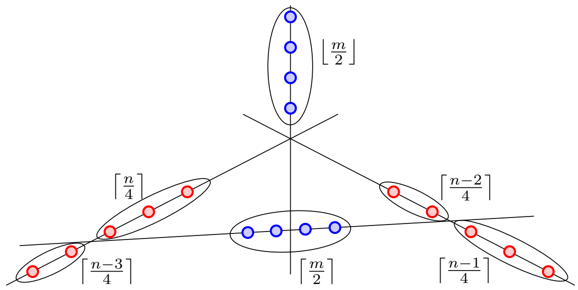


Figure 3.8: Point configuration for the bent cross with the lower blue part rotated

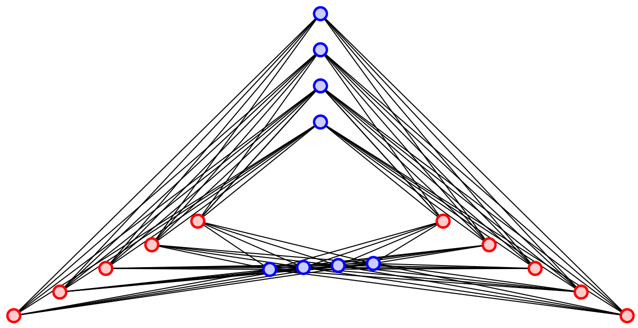


Figure 3.9: Complete bipartite graph for the bent cross with the lower blue part rotated

It turns out that the property of all blue edges halving the red set is not necessary either. In the configuration shown in Figure 3.10, it can be seen that the blue points in the lower part might be arranged such that no blue edge from the above part to the below part remains halving. Moreover, both red arms are bent once more. Figure 3.11 illustrates the according complete bipartite graph. Despite all these changes the number of crossings is still equal to $\mathrm{z}(n, m)$.

As a last example of this kind, Figure 3.12 illustrates that the red sets need not be partitioned equally, and the blue points might as well be all moved completely to one side. Depending on how the red points are partitioned, there might even be blue 0-edges. Altogether this yields a rather asymmetric configuration which is shown for $K_{10,8}$ in Figure 3.13. So let us count crossings. Two blue points from the upper part form a crossing with any two red points from the right side as well as any two red points from the left side. Similarly, two blue points from the lower part form a crossing with any two red points above the lower blue line as well as any two red points below this line. As the combination of one blue point from above and one blue point from below does not induce any crossing, the total number of crossings is still $\overline{\mathrm{cr}}(R, B) = \mathrm{z}(n, m)$.

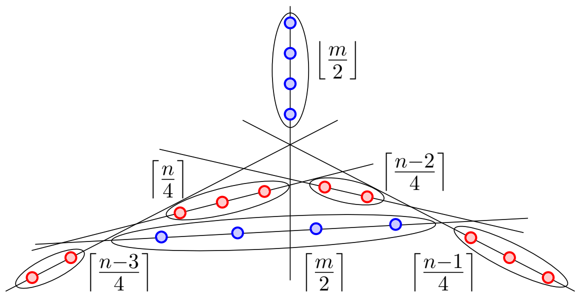


Figure 3.10: Point configuration where the red subsets are bent once more and the lower blue subset is stretched.

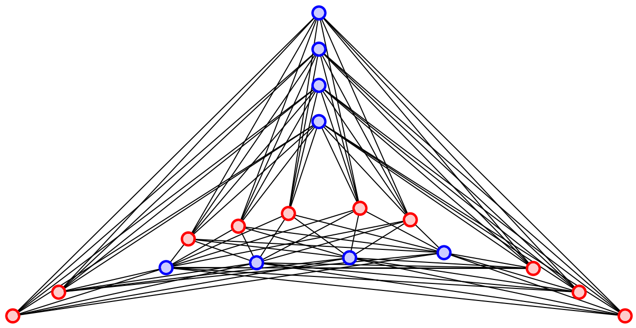


Figure 3.11: Complete bipartite graph for the configuration from Figure 3.10.

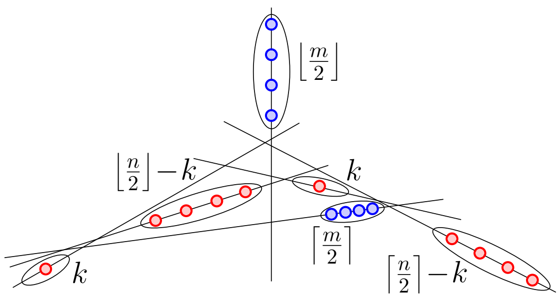


Figure 3.12: Point configuration where red subsets are changed to be not half at each side, and the lower blue set is moved to the side.

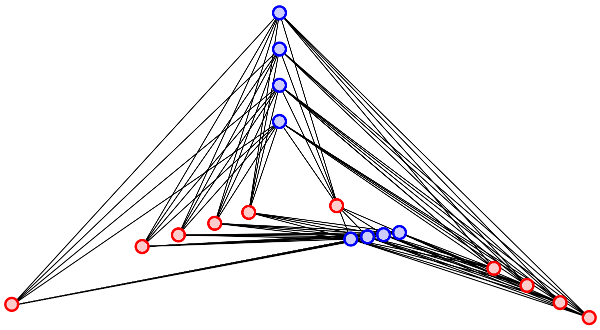


Figure 3.13: straight-line $K_{10,8}$ for the configuration from Figure 3.12.

Let us come back to the first example of this section, the bent Zarankiewicz cross. Stretching it and further bending the red arms, we still obtain a configuration having $\overline{\mathrm{cr}}(R, B) = \mathrm{z}(n, m)$; see Figures 3.14 and 3.15. But this configuration now has a new special property: The sets R and B are linearly separated from each other, meaning that there exists a straight line l such that R lies completely in one open half plane defined by l and B lies completely in the other.

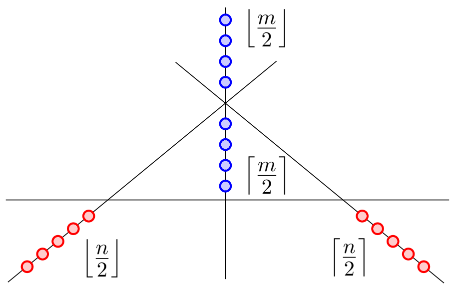


Figure 3.14: Point configuration for a variation of the Zarankiewicz cross where R and B are linearly separated.

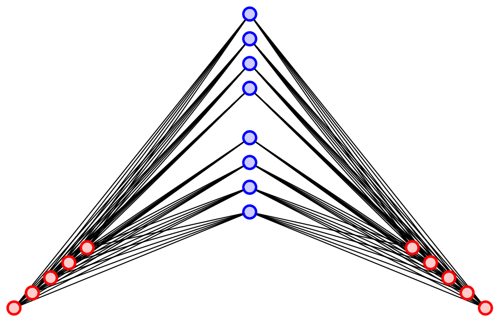


Figure 3.15: A linearly separated variation of the Zarankiewicz cross.

The bent Zarankiewicz cross is by far not the only linearly separated configuration obtaining crossing number $\mathrm{z}(n,m)$. Figure 3.16 shows a more general configuration with the same property, for which Figure 3.17 illustrates a drawing of $K_{10,8}$.

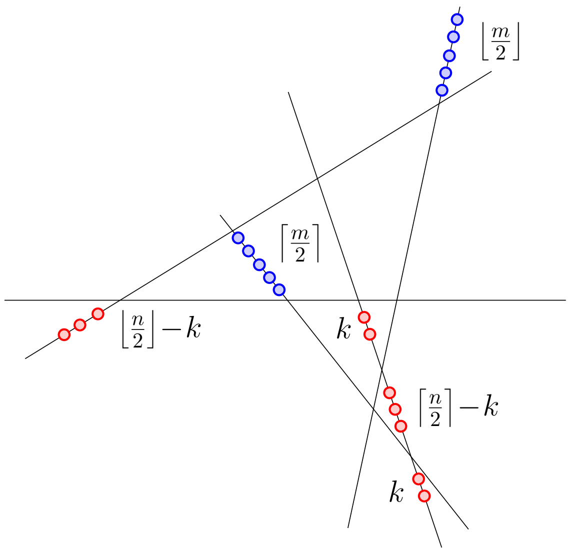


Figure 3.16: A more general linearly separated point configuration with $\overline{\mathrm{cr}}(R,B) = \mathrm{z}(n,m)$.

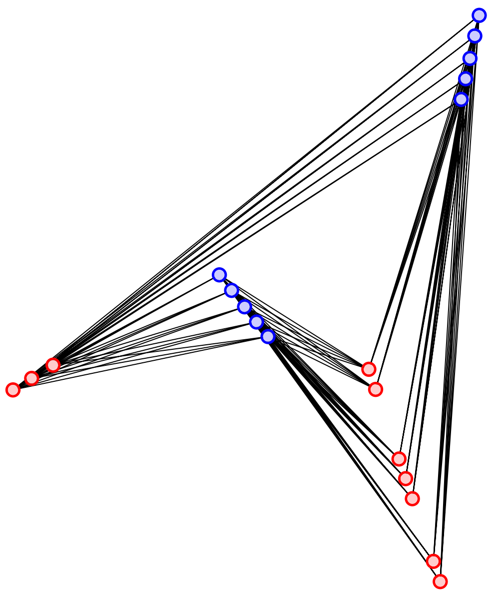


Figure 3.17: A more general linearly separated point configuration with $\overline{\mathrm{cr}}(R,B) = \mathrm{z}(n,m)$.

Note that the property of linear separability of the two sets R and B seems to be counter intuitive for crossing-minimizing rectilinear drawings of $K_{n,m}$. In the next section we restrict the considered configurations to such linearly separated point sets.

3.2.3 Linearly separable point sets

In this section, all considered point sets $S = (R, B)$ have linearly separated subsets R and B. Whenever we say linearly separated, we mean a separation of R and B. Despite the fact that there exist such sets for which $\overline{\mathrm{cr}}(R, B) = \mathrm{z}(n, m)$, they still have a property which is of special interest for us. Namely, they cannot contain any 4-tuple of type . This drastically changes the system of relations between the numbers of monochromatic and bichromatic edges on the one hand, and the numbers of different 4-tuples on the other hand. Table 3.2 contains the relations from Table 3.1 for linearly separated sets.

				
(I)	1	1	1	$\binom{n}{2} \cdot \binom{m}{2}$
(II)	1	0	1	$\sum_{j=0}^{m} e_r^j \cdot \binom{j}{2}$
(III)	0	1	1	$\sum_{k=0}^{n} e_b^k \cdot \binom{k}{2}$
(IV)	0	2	0	$\sum_{j=0}^{m} e_r^j \cdot j \cdot (m-j)$
(V)	2	0	0	$\sum_{k=0}^{n} e_b^k \cdot k \cdot (n-k)$
(VI)	2	2	2	$\sum_{j=0}^{m-1} \sum_{k=0}^{n-1} g_{k,j} \cdot k \cdot j$
(VII)	2	2	2	$\sum_{j=0}^{m-1} \sum_{k=0}^{n-1} g_{k,j} \cdot k \cdot (m-1-j)$

Table 3.2: Counting configurations of 4-tuples with two red and two blue points in a linearly separated bichromatic point set $S = (R, B)$.

While in the general case we needed the numbers of bichromatic edges to express the number of crossings , this number can now be expressed purely in terms of red j-edges and blue k-edges in several different ways.

$$\begin{aligned}
\text{(II)} + \text{(III)} - \text{(I)} &= 1 \cdot \quad = \sum_{j=0}^{m} e_r^j \cdot \binom{j}{2} + \sum_{k=0}^{n} e_b^k \cdot \binom{k}{2} - \binom{n}{2} \cdot \binom{m}{2} \\
2 \cdot \text{(I)} - \text{(IV)} - \text{(V)} &= 2 \cdot \quad = \binom{n}{2} \cdot \binom{m}{2} - \sum_{j=0}^{m} e_r^j \cdot j \cdot (m-j) \\
&\qquad - \sum_{k=0}^{n} e_b^k \cdot k \cdot (n-k) \\
2 \cdot \text{(II)} - \text{(V)} &= 2 \cdot \quad = \sum_{j=0}^{m} e_r^j \cdot \binom{j}{2} - \sum_{k=0}^{n} e_b^k \cdot k \cdot (n-k) \\
2 \cdot \text{(III)} - \text{(IV)} &= 2 \cdot \quad = \sum_{k=0}^{n} e_b^k \cdot \binom{k}{2} - \sum_{j=0}^{m} e_r^j \cdot j \cdot (m-j)
\end{aligned}$$

The last two of these equations might be interpreted as to show the asymmetry of the separated Zarankiewicz cross.

The number of bichromatic (k, j)-edges turn out to not only be unnecessary for the calculation, but also trivial, as the considered sums simply give all types of 4-tuples with the same multiplicity. The reason for this identity is quite interesting.

Proposition 3.13. *For every linearly separated set $S = (R, B)$, and every (k, j) with $0 \leq j \leq m-1$ and $0 \leq k \leq n-1$, there are exactly two (k, j)-edges.*

Proof. For the beginning we assume that the points of R are in general position. Consider an arbitrary value k with $0 \leq k \leq n-1$, and a directed line l which goes through exactly one red point p and has k red points to its right. We start rotating l clockwise around p. Whenever l hits a second point of R, we change the anchor of the rotation to the newly hit point; see Figure 3.18 for an illustration of this process. By this, whenever there is only one point of R on l, l has exactly k points of R to the right, and after one full rotation of 2π, we have encountered all possibilities for such lines. This process is called *k-rotation around R* [79].

Note that the same principle can be applied if R has collinear points. The only difference is that when a set of collinear points happens to lie on l, then the new rotation anchor has to be chosen properly among these points such that again k points lie to the right of l as soon as the rotation continues.

Now consider what happens with the blue set B during such a rotation of l around R. Assume that, as illustrated in Figure 3.18, we start with a

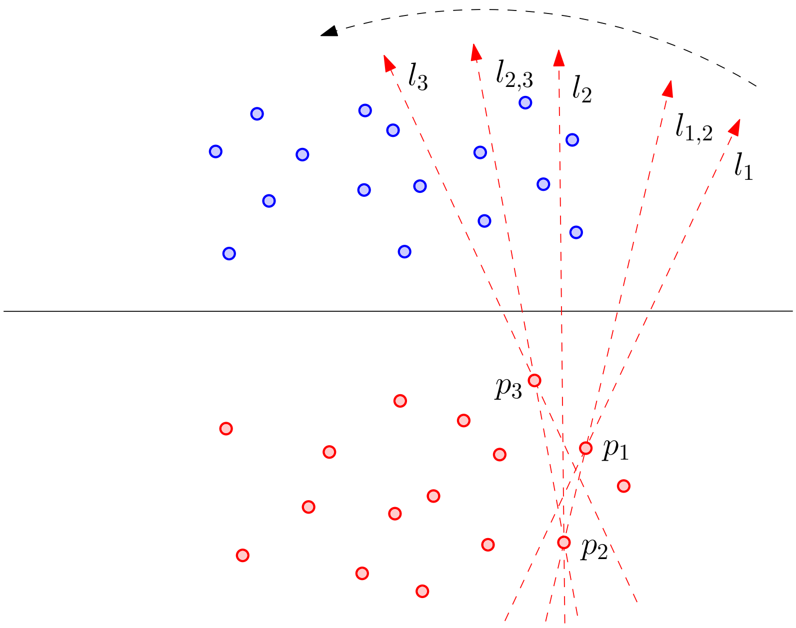


Figure 3.18: 2-rotation around R. For better readability, the lines as well as the rotation anchors are labeled in order of their appearance during the rotation.

line l that has all points of B to its left. While l is rotating around R, one after another of the points of B switches from the left side of l to its right side. After a rotation of π, all points of B are to the right of l. Thus, we have encountered exactly one bichromatic (k, j)-edge for each $0 \leq j \leq m - 1$, all of them directed from a red to a blue point. When continuing the rotation until we have completed a full rotation, l passes each point of B once more, by this encountering a bichromatic edge which is directed from blue to red. In total we have collected exactly two (k, j)-edges for every combination of j and k, one starting at a red point and one ending at a red point. $\square$

Let us come back to the number of crossings $\overline{\text{cr}}(R, B) = (R, B)$. As in the linearly separated case there are no convex 4-tuples of type , the number of crossings is now equivalent to the number of convex 4-tuples (with two red and two blue points). Thus, and by Relation (I), minimizing the rectilinear crossing number is now equivalent to maximizing the sum $(R, B) + (R, B)$ of all non-convex such 4-tuples.

Also, the sum of Relations (IV) and (V) now directly gives $(R, B) +$ (R, B), implying that a linearly separated set where all red edges halve the blue set and all blue edges halve the red set would minimize $\overline{\text{cr}}(n, m)$.

Note that the separated Zarankiewicz cross from Figure 3.15 does not fulfill the requirement of all monochromatic edges halving the set of the other color. While all of its blue edges halve the red set, all the $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ red edges between the left and the right red subsets are 0-edges.

But what about other examples? In the following paragraphs we show that, at least for n and m even, this approach is doomed to failure, as it is impossible to find a linearly separated set where the number of monochromatic edges that halve the set of the other color is larger than in the separated Zarankiewicz cross.

An upper bound on the number of monochromatic edges which halve the set of the other color

Let $H^R = H^R(R)$ be the straight-line graph with vertex set R which contains an edge pq for every pair of points $p, q \in R$ that spans a halving line for the blue set B. We denote by $h^r \leq \binom{|R|}{2}$ the number of edges in H^R. Similarly, let $H^B = H^B(B)$ be the straight-line graph with vertex set B that has an edge pq for every pair of points $p, q \in B$ spanning a halving line for R, and $h^b \leq \binom{|B|}{2}$ the number of edges in H^B. Further, denote the total number edges in $H^R \cup H^B$ by $h = h^r + h^b$.

Lemma 3.14. *If $|B|$ is even and C is a connected component of H^R, then all edges of C halve the blue set B "in the same way", meaning that the induced partition of B is identical among all edges of C.*

Proof. Consider three arbitrary red points $p, q, r \in C$, and the partition of B into B_l (left), B_c (center), and B_r (right) which is caused by the supporting lines of pq and pr (see Figure 3.19). Both pq and pr are halving B if and only if $B_c = \emptyset$ and $|B_l| = |B_r|$. Starting with an arbitrary edge of C and repeatedly applying this argument, all edges of C have to split B into the same two sets B_l and B_r. □

Lemma 3.15. *Let $|B|$ be even, and let C be a connected component of H^R whose edges split B into B_l and B_r. Then for every blue edge $\vec{pq}$ with $p \in B_l$ and $q \in B_r$, C lies completely on one side of (the supporting line of) $\vec{pq}$.*

Proof. Assume that there exists a blue edge $\vec{pq}$ with $p \in B_l$ and $q \in B_r$, for which the supporting line l of $\vec{pq}$ cuts through C. As C is connected, l also cuts through an edge $xy \in C$. Thus, p and q lie on the same side of xy, which is a contradiction to the claim that all edges of C split B into B_l and B_r. □

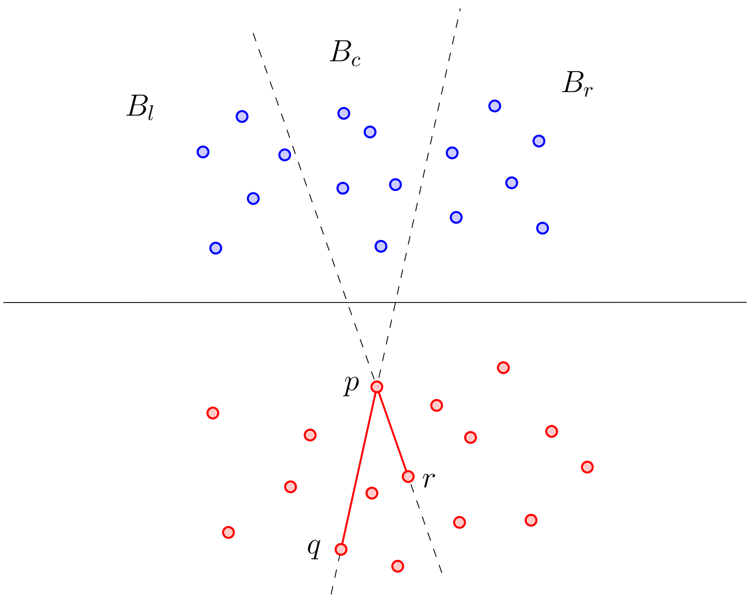


Figure 3.19: Two red halving lines pq and pr with their induced partitions of the blue set B.

Theorem 3.16. *If $|R| = |B| = n$ is even, then the number of (directed) monochromatic edges halving the set of the other color is at most*

$$e_r^{\frac{n}{2}} + e_b^{\frac{n}{2}} \leq 2 \cdot \binom{n}{2} + 4 \cdot \binom{\frac{n}{2}}{2}.$$

Proof. Let $\mathcal{C} = \{C_i = C_i(R_i) : i = 1 \ldots k\}$ be the set of connected components of H^R, with vertex sets R_i, $\bigcup_{i=1}^{k} R_i = R$, and denote with h_i^r the number of edges in C_i. Further, let $C_{\max} = C_{\max}(R_{\max}) \in \mathcal{C}$ be a component of H^R with $|R_{\max}| \geq |R_i|\ \forall\ i = 1 \ldots k$. We distinguish between two cases with respect to the cardinality of $R_{\max}$.

Case 1. $|R_{\max}| \geq \frac{n}{2}$, meaning that $C_{\max}$ contains more than half of all the red points. Consider the partition (B_l, B_r) of B that is induced by $C_{\max}$. Then for any blue edge $\vec{pq}$ with $p \in B_l$ and $q \in B_r$, all points of $R_{\max}$ lie on the same side of $\vec{pq}$, implying that $\vec{pq}$ does not halve R. Thus we obtain an upper bound on the number of edges in the blue graph H^B of

$$h^b \leq \binom{|B_l|}{2} + \binom{|B_r|}{2}$$

As $h^r \leq \binom{|R|}{2}$, we obtain the following upper bound for the total number of edges in $H^R \cup H^B$.

$$h = h^r + h^b \leq \binom{|R|}{2} + \binom{|B_l|}{2} + \binom{|B_r|}{2} = \binom{n}{2} + 2 \cdot \binom{\frac{n}{2}}{2}.$$

Case 2. $|R_{\max}| < \frac{n}{2}$, implying that $|R_i| < \frac{n}{2}$ for all components $C_i(R_i)$ of H^R. This yields an upper bound on the number edges in H^R of

$$h^r = \sum_{i=1\ldots k} h_i^r \leq \sum_{i=1\ldots k} \binom{|R_i|}{2} < 2 \cdot \binom{\frac{|R|}{2}}{2} = 2 \cdot \binom{\frac{n}{2}}{2}.$$

Thus, even under the assumption that the number of edges in H^B is $\binom{|B|}{2}$, the total number of edges in $H^R \cup H^B$ is still at most

$$h = h^r + h^b < 2 \cdot \binom{\frac{n}{2}}{2} + \binom{n}{2}.$$

As any edge pq in $H^R \cup H^B$ induces exactly two directed monochromatic edges which halve the set of the other color ($\vec{pq}$ and $\vec{qp}$), and as for every such directed edge its undirected version is contained in $H^R \cup H^B$, this gives the desired result. $\square$

Before stating a corollary of Theorem 3.16 and Lemma 3.15, let us introduce some more notation. Given a set $S = R \cup B$, we call a (directed) blue edge that is an i-edge for some $i \leq k$ a *blue* $(\leq k)$*-edge* of S. Similarly, a (directed) blue edge which is an i-edge for some $i \geq k$ is called *blue* $(\geq k)$*-edge* of S. We denote by $e_b^{\leq k} = \sum_{i=0}^{k} e_b^k$ the number of blue $(\leq k)$-edges of S, and by $e_b^{\geq k} = \sum_{i=k}^{|B|} e_b^k$ be the number of blue $(\geq k)$-edges of S. Of course, according definitions can be made for red edges.

Corollary 3.17. *Let $|R| = |B| = n$ even, and consider again H^R, its biggest connected component $C_{\max}$ (w.r.t. the number of vertices), and the partition (B_l, B_r) of B that is induced by $C_{\max}$. If $C_{\max}$ contains $n-k$ points, then every pair of points $p \in B_l$ and $q \in B_r$ spans a blue $(\leq k)$-edge $(\vec{q,p})$ and a blue $(\geq n-k)$-edge $(\vec{p,q})$. Thus, we obtain*

$$e_b^{\leq k} \;=\; e_b^{\geq n-k} \;\geq\; \frac{n^2}{4}.$$

Note that in the linearly separated Zarankiewicz cross shown in Figure 3.15, the number of monochromatic edges which halve the set of the other color is exactly the upper bound from Theorem 3.16. Moreover, all its non-halving edges are zero-edges, which also immediately follows from Corollary 3.17 (as H^B is equal to the complete graph K_n and thus connected).

Observation 3.18. *The proof of Lemma 3.14 is strongly based on the fact that B has an even number of points. If $|B|$ is odd, then two red halving edges that share an end point might as well split B into three sets B_l, B_c*

and B_r, with $|B_l| = |B_r|$ and $|B_c| = 1$ (see again Figure 3.19). Moreover, for a connected component C of H^R, it is possible that at least half of the edges of C split B differently. Figure 3.20 illustrates an example for this situation.

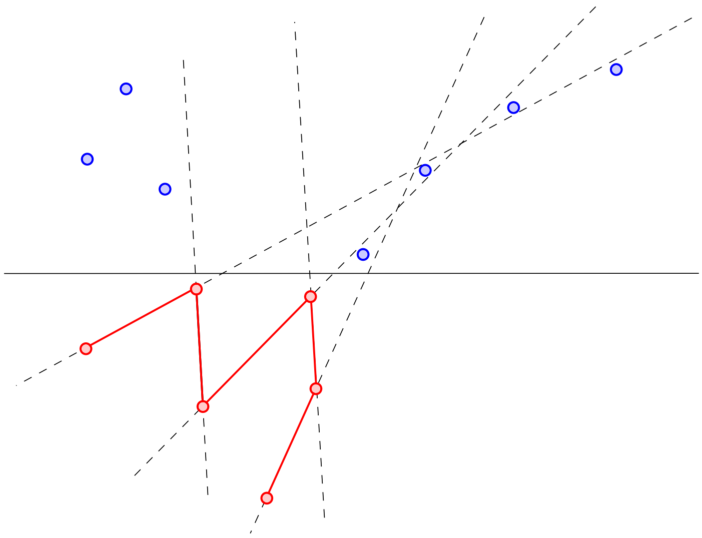


Figure 3.20: Different induced partitions of a blue set with odd cardinality by edges of one connected component of H^R.

In the next section we switch back to non-separated drawings. We investigate the rectilinear crossing number of $K_{n,m}$ for sets optimizing the rectilinear crossing number of the complete graph. As we consider point sets with cardinality $|S| = n + m$, we denote the complete graph by K_{n+m}, and its crossing number by $\overline{\text{cr}}(n + m)$

3.2.4 Point sets which optimize $\overline{\text{cr}}(n + m)$

While to our knowledge there are no publications dedicated to the rectilinear crossing number $\overline{\text{cr}}(n, m)$ of $K_{n,m}$ [134], quite a lot of results are known for the rectilinear crossing number $\overline{\text{cr}}(n + m)$ of the complete graph K_{n+m}; see the recent survey [5]. For example, for small values of $(n+m)$, point configurations are known which reach the minimum $\overline{\text{cr}}(n+m)$ [4, 54]. These sets differ strongly from the ones which are conjectured to optimize $\overline{\text{cr}}(n, m)$.

For these sets and fixed n and m, we calculated bicolorings $R \cup B$ that result in minimized crossing numbers $\overline{\text{cr}}(R, B)$. Surprisingly, for some of the sets we found bicolorings for which the number of crossings is not more than in the according Zarankiewicz cross, i.e., $\overline{\text{cr}}(R, B) = \text{z}(n, m)$.

Figures 3.21 and 3.22 illustrate the resulting complete bipartite graphs for optimal colorings of two such sets.

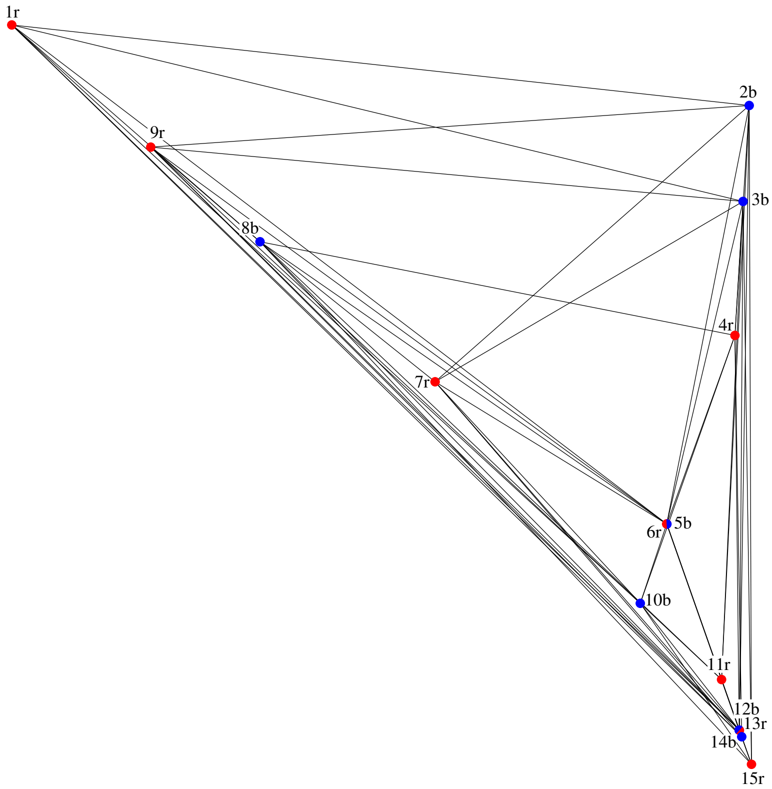


Figure 3.21: A drawing of $K_{8,7}$ with $\overline{\text{cr}}(R, B) = \text{z}(8, 7)$ on a set S with $\overline{\text{cr}}(S) = \overline{\text{cr}}(15)$.

Table 3.3 shows an overview of the obtained results. Note that for the listed results, at least one of the sets optimizing $\overline{\text{cr}}(n+m)$ admits a coloring with that many crossings (and often by far not all of them). For larger cardinality, there seem to be no more sets optimizing both $\overline{\text{cr}}(n+m)$ and $\overline{\text{cr}}(n, m)$. This seems reasonable, as $K_{n,m}$ only contains bichromatic edges, while K_{n+m} also contains all the monochromatic edges. Sets that have only few crossings for K_{n+m} also somehow have to minimize the number of crossings in which monochromatic edges are involved, while in $K_{n,m}$ these do not appear anyhow.

Open problem 3.19. *What are possible structures of "good" point sets, and what are their essential properties? And what can be good heuristics to find such sets?*

In the following section we discuss a possible approach towards a solution of this problem.

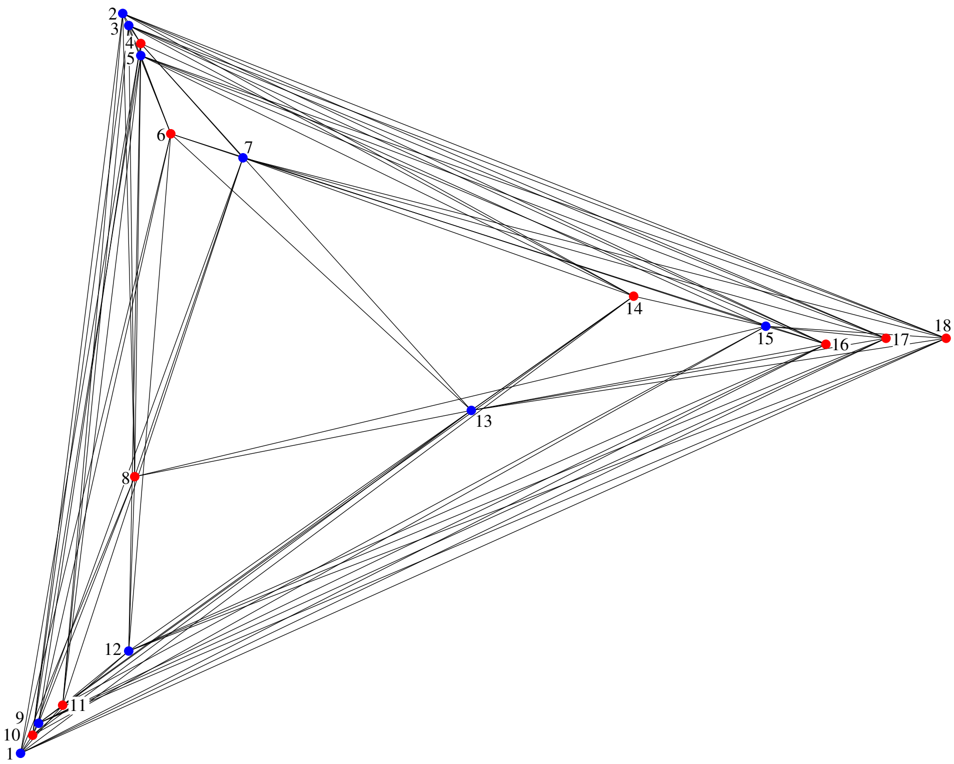


Figure 3.22: A drawing of $K_{9,9}$ with $\overline{\mathrm{cr}}(R, B) = \mathrm{z}(9, 9)$ on a set S with $\overline{\mathrm{cr}}(S) = \overline{\mathrm{cr}}(18)$. Exchanging the colors of vertices 15 and 16 gives the second coloring with this property for this set.

$n+m$	(n,m)	$\mathrm{z}(n,m)$	$\overline{\mathrm{cr}}(R,B)$	diff
5	(3,2)	0	0	0
6	(3,3)	1	1	0
7	(4,3)	2	2	0
8	(4,4)	4	4	0
9	(5,4)	8	8	0
10	(5,5)	16	16	0
11	(6,5)	24	24	0
12	(6,6)	36	38	+2
13	(7,6)	54	54	0
14	(7,7)	81	81	0
15	(8,7)	108	108	0
16	(8,8)	144	149	+5
17	(9,8)	192	192	0

$n+m$	(n,m)	$\mathrm{z}(n,m)$	$\overline{\mathrm{cr}}(R,B)$	diff
18	(9,9)	256	256	0
19	(10,9)	320	322	+2
20	(10,10)	400	411	+11
21	(11,10)	500	507	+7
22	(11,11)	625	629	+4
23	(12,11)	750	760	+10
24	(12,12)	900	922	+22
25	(13,12)	1080	1099	+19
26	(13,13)	1296	1306	+10
27	(14,13)	1512	1532	+20
28	(14,14)	1764	1798	+34
29	(15,14)	2058	2090	+32
30	(15,15)	2401	2423	+21

Table 3.3: Minimal numbers of crossings $\overline{\mathrm{cr}}(R, B)$ for sets minimizing the rectilinear crossing number and thus having $\overline{\mathrm{cr}}(S) = \overline{\mathrm{cr}}(n + m)$. The last column contains the difference between $\mathrm{z}(n, m)$ and $\overline{\mathrm{cr}}(R, B)$.

3.2.5 Crossing degrees

We continue with a different way to count crossings, namely per point. We denote the number of crossings in which a point $p \in S$ is involved, or, more exactly, the number of crossings in which the edges incident to p are involved as *crossing degree* $\mathrm{x}_p(R, B)$ *of* p.

Note that by summing up over all crossing degrees of the points of one color class, every crossing is counted exactly twice:

$$\sum_{r \in R} \mathrm{x}_r = \sum_{b \in B} \mathrm{x}_b = 2 \cdot \overline{\mathrm{cr}}(R, B)$$

This gives an average crossing degree of $\frac{2}{n}\,\overline{\mathrm{cr}}(R, B)$ for the red and $\frac{2}{m}\,\overline{\mathrm{cr}}(R, B)$ for the blue points. If we delete a point $p \in R$, then the number of crossings in the set is reduced by exactly $\mathrm{x}_p(R, B)$.

$$\overline{\mathrm{cr}}(R \backslash \{p\}, B) = \overline{\mathrm{cr}}(R, B) - \mathrm{x}_p(R, B).$$

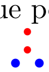
Of course, an according relation holds for deleting a point $p \in B$. The next proposition considers the contrary direction, namely adding a point to S.

Proposition 3.20 (Point Duplication)**.** *Consider a point $p \in R$ with crossing degree $\mathrm{x}_p(R, B)$, a red point q very close*[2] *to p such that $\overline{pq}$ is a halving line of B. Then the number of crossings of the complete bipartite graph with vertex set $S' = (R \cup \{p\}, B)$ is*

$$\overline{\mathrm{cr}}(R \cup \{p\}, B) = \overline{\mathrm{cr}}(R, B) + \mathrm{x}_p(R, B) + \binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lceil \frac{m}{2} \rceil}{2}$$

Proof. All $\overline{\mathrm{cr}}(R, B)$ crossings induced by S are also induced by S'. Further, the new point q is involved in all new crossings (the ones induced by S' but not by S). We divide these new crossings in two different groups and count them separately.

The first group are crossings induced by S' where p is not involved. As q is very close to p, their number is identical to the number of crossings induced by S where p is involved and thus $\mathrm{x}_p(R, B)$.

The second group are crossing that are spanned by p and q (and two blue points). Any choice of two blue points that are on different sides of pq forms a non-convex 4-gon of type  with p and q and thus no crossing.

[2]By "very close" we mean that the order type of $S \backslash \{p\} \cup \{q\}$ is identical to the order type of S

As p and q are very close together, any choice of two blue points that are on one side of pq forms a convex 4-gon of type  with p and q and thus a crossing. As pq is a halving line for the blue set, this gives $\binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lceil \frac{m}{2} \rceil}{2}$ new crossings. $\square$

In order to obtain good sets, one way is to take good sets with $(n-1)$ or $(n+1)$ red points and either duplicate a point with minimum crossing degree or delete a point with maximum crossing degree.

Observation 3.21 (Duplicating a red point with minimum crossing degree)**.** *Taking into account that the minimum crossing degree of a red point is at most the average crossing degree* $\frac{2}{n} \cdot \overline{\mathrm{cr}}(R, B)$*, and duplicating the point* $p \in R$ *with minimum crossing degree among all points in* R*, the above equation gives*

$$\overline{\mathrm{cr}}(R', B) \leq \frac{n+2}{n} \cdot \overline{\mathrm{cr}}(R, B) + \binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lceil \frac{m}{2} \rceil}{2}.$$

Now assume that n *is even and* $\overline{\mathrm{cr}}(R, B) = \mathrm{z}(n, m)$*. Then the right-hand side of this inequation gives* $\mathrm{z}(n+1, m)$*, and thus*

$$\overline{\mathrm{cr}}(R', B) \leq \mathrm{z}(n+1, m)$$

with equality iff the crossing degrees of all red points are identical.

Adding a point to a Zarankiewicz cross such that the result is again a Zarankiewicz cross is exactly duplicating a point with minimum crossing degree. The crossing degrees of the Zarankiewicz cross $C(n, m)$ can be explicitly calculated: Let R^-, R^+, B^- and B^+ be the red and blue sets of the four arms of the cross, with $|R^-| = \lfloor \frac{n}{2} \rfloor$, $|R^+| = \lceil \frac{n}{2} \rceil$, $|B^-| = \lfloor \frac{m}{2} \rfloor$ and $|B^+| = \lceil \frac{m}{2} \rceil$. Then the crossing degree of a vertex p only depends on the subset to which it belongs. We define $\mathrm{x}^C_{R^-}(n, m) := \mathrm{x}_p(C)$ for $p \in R^-$. Analogously, we define $\mathrm{x}^C_{R^+}(n, m)$, $\mathrm{x}^C_{B^-}(n, m)$, and $\mathrm{x}^C_{B^+}(n, m)$. These crossing degrees have the following values.

$$\begin{aligned}
\mathrm{x}^C_{R^-}(n, m) &= \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(\binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lceil \frac{m}{2} \rceil}{2}\right) \\
\mathrm{x}^C_{R^+}(n, m) &= \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\left(\binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lceil \frac{m}{2} \rceil}{2}\right) \\
\mathrm{x}^C_{B^-}(n, m) &= \left(\left\lfloor \frac{m}{2} \right\rfloor - 1\right)\left(\binom{\lfloor \frac{n}{2} \rfloor}{2} + \binom{\lceil \frac{n}{2} \rceil}{2}\right) \\
\mathrm{x}^C_{B^+}(n, m) &= \left(\left\lceil \frac{m}{2} \right\rceil - 1\right)\left(\binom{\lfloor \frac{n}{2} \rfloor}{2} + \binom{\lceil \frac{n}{2} \rceil}{2}\right)
\end{aligned}$$

Note that if n is even, then in the Zarankiewicz cross all red points have the same crossing degree $\mathrm{x}^C_{R^\pm}(n, m) := \mathrm{x}^C_{R^-}(n, m) = \mathrm{x}^C_{R^+}(n, m)$. Likewise, if m is even, the crossing degree of all blue points is $\mathrm{x}^C_{B^\pm}(n, m)$.

The crossing degrees of $C(n,m)$ together with point duplication and Observation 3.21 lead to some more observations on crossing properties of optimal sets.

Observation 3.22 (Questioning Zarankiewicz's conjecture)**.** *If there exists a point set $S=(R,B)$ and a point $p \in R$ with $\mathrm{x}_p(R,B)$ that fulfills one of the following strict inequalities, then the Zarankiewicz Conjecture is wrong.*

$$\begin{array}{rrcl} (1) & \mathrm{x}_p(R,B) & < & \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(\binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lceil \frac{m}{2} \rceil}{2}\right) \\ (2) & \mathrm{x}_p(R,B) & > & \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\left(\binom{\lfloor \frac{m}{2} \rfloor}{2} + \binom{\lceil \frac{m}{2} \rceil}{2}\right) \end{array}$$

Observation 3.23 (Questioning Zarankiewicz's conjecture for n even)**.**

If there exists a set $S=(R,B)$ with n even and $\overline{\mathrm{cr}}(R,B)=\mathrm{z}(n,m)$ and at least two different crossing degrees among the red points, then the Zarankiewicz Conjecture is false in general, and already for $(n-1,m)$ by deleting a red point p with $\mathrm{x}_p(R,B) > \frac{2}{n}\cdot\overline{\mathrm{cr}}(R,B)$.

If there exists a set $S=(R,B)$ with m even and $\overline{\mathrm{cr}}(R,B)=\mathrm{z}(n,m)$ and at least two different crossing degrees among the blue points, then the Zarankiewicz Conjecture is false in general, and already for $(n,m-1)$ by deleting a blue point p with $\mathrm{x}_p(R,B) > \frac{2}{m}\cdot\overline{\mathrm{cr}}(R,B)$.

If the Zarankiewicz Conjecture is true, then in every optimal set the following two conditions must hold:

(1) *If n is even then the crossing degree has to be identical for all points of R:*
$$\mathrm{x}_p(R,B) = \frac{2}{n}\cdot\overline{\mathrm{cr}}(R,B) = \frac{2}{n}\cdot\mathrm{z}(n,m) \quad \forall p \in R.$$

(2) *If m is even then the crossing degree has to be identical for all points of B:*
$$\mathrm{x}_p(R,B) = \frac{2}{m}\cdot\overline{\mathrm{cr}}(R,B) = \frac{2}{m}\cdot\mathrm{z}(n,m) \quad \forall p \in B.$$

Corollary 3.24. *The first two statements in Observation 3.23 show that if there exists a counter example to the rectilinear version of Zarankiewicz's conjecture for a combination (n,m) with n even (or m even), then there also exists a counter example for $(n-1,m)$ (or $(n,m-1)$).*

Seen the other way round, if we already know that the rectilinear version of Zarankiewicz's conjecture is true for (n,m) with n odd (or m odd), then it also has to be true for $(n+1,m)$ (or $(n,m+1)$).

Reconsider the point set $S_{8,7}$ shown in Figure 3.21. All red points have a crossing degree of $27 = \mathrm{x}^C_{R\pm}(8,7)$. As a consequence, point duplication can be applied to any of the red points, in each case obtaining a (9,7)-set with $\mathrm{z}(9,7) = 144$ crossings. None of the blue points has crossing degree $24 = \mathrm{x}^C_{B-}(8,7)$, so duplicating a blue point can only lead to (8,8)-sets with more than $\mathrm{z}(8,8) = 144$ crossings.

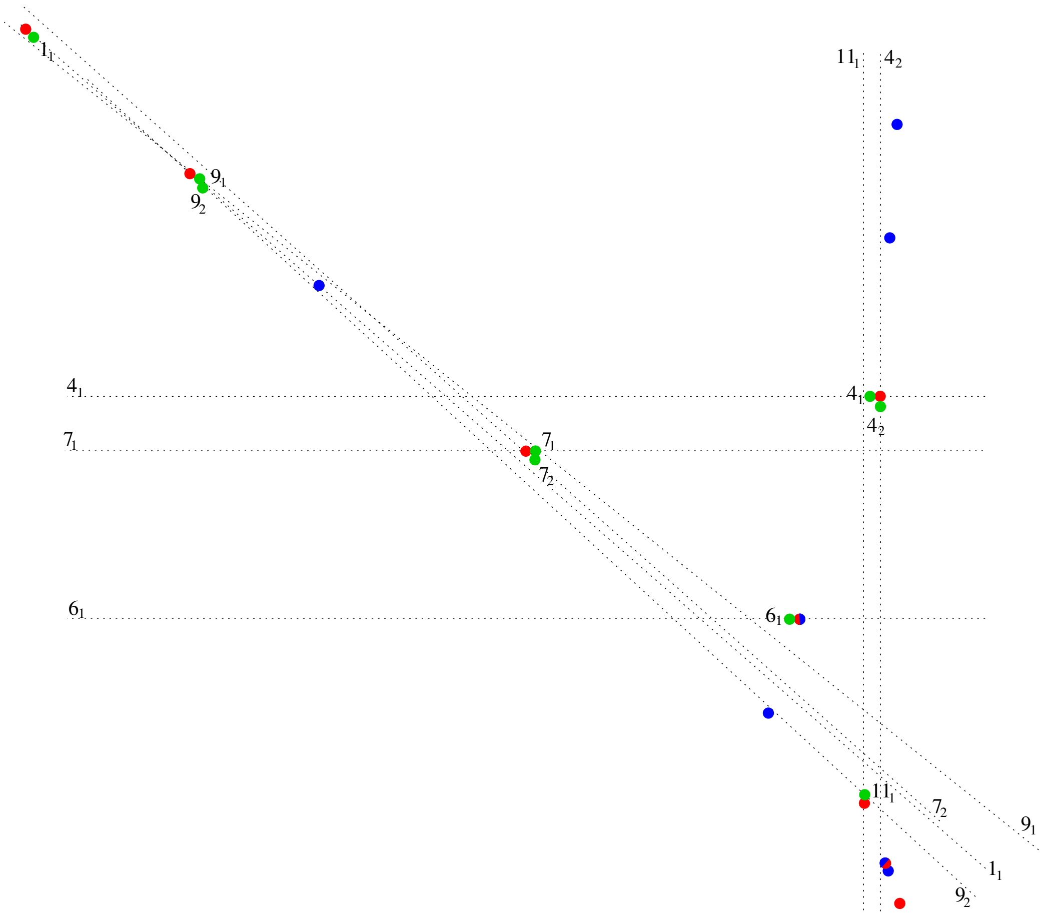


Figure 3.23: The locations of additional red points. For better visualization the additional points are drawn in green and in the correct relative direction, but not at their exact position. The dotted lines indicate the according halving lines.

Figure 3.23 roughly shows possible locations for additional (duplicated) red points. Note that in every extension both the duplicated and the additional point have (and must have) crossing degree $36 = \mathrm{x}^C_{R+}(9,7)$. The change of the crossing degrees of the other points strongly varies for different duplication locations. If a resulting (9,7)-set has a red point p with $\mathrm{x}_p = 27 = \mathrm{x}^C_{R-}(9,7)$ or a blue point q with $\mathrm{x}_q = 32 = \mathrm{x}^C_{B-}(9,7)$, then by duplicating the according point one obtains a (10,7)-set with $z(10, 7)$ crossings, or a (9,8)-set with $z(9, 8)$ crossings, respectively. In the examples we tried, the former occurred while the latter didn't.

While in $S_{8,7}$ from Figure 3.21 many points can be duplicated, and some even simultaneously without obtaining a crossing degree larger than the according Zarankiewicz number, this is not the case for the example in Figure 3.22. Here the occurring minimum crossing degree is 52, obtained by vertex 8. In the alternative coloring (obtained by exchanging the colors of vertices 15 and 16), the according minimum is higher (54, for vertex 6). In both configurations, none of the points reaches the value of $x^C_{B\pm}(9, 9) = 48$ which would be necessary to obtain a set $S_{9,10}$ with $\overline{\mathrm{cr}}(R, B) = z(9, 10)$.

One could pose the question whether or not duplicating a point with minimum crossing degree is always an optimal way of extending a point set. In general the answer is no: Consider once again the (9,9)-point set from Figure 3.22. With the before-mentioned alternative coloring, it has maximal crossing degree 62, obtained by vertex 1. Thus, by deleting this vertex we obtain a non-optimal (9,8)-set with $z(9, 8) + 2$ crossings. As the minimum crossing degree for a blue point in this set is 48, the best we can reach by point duplication is a (9,9)-set with $z(9, 9) + 2$ crossings. So in this case it is better to re-add vertex 1 than to duplicate a point.

Open problem 3.25 (Optimal positions for additional points)**.** *Is duplicating a point with minimum crossing degree always an optimal way to add a point to an "optimal" set (if the set has - at most - $z(n, m)$ crossings)? If no, what are properties of good positions for additional points? And what might be good heuristics for finding such positions?*

Maybe from good positions for points we can get ideas about better ways of getting a point set that falsifies Zarankiewicz's conjecture, or get more insight about why it is true ...

3.3 Compatible matchings for bichromatic plane geometric graphs

For the last section of this chapter, we consider again straight-line graphs on top of bichromatic point sets, but switch from graphs with many crossings to the other end of the range, namely crossing-free graphs.

We consider bichromatic point sets $S = R \cup B$ where the red set R and the blue set B have the same cardinality $|R| = |B| = n$. Similar to the setting in Section 3.2, an edge spanned by two points of S is called bichromatic if it has one red and one blue endpoint. A graph $G(S)$ is called bichromatic, if all its edges are bichromatic. Accordingly, an edge where both endpoints have the same color is called monochromatic, and a graph with only monochromatic edges is called monochromatic as well. Depending on the color of the endpoints, a monochromatic edge is called red or blue, respectively. If a graph $G(S)$ has only red (blue) edges, we say that $G(S)$ is red (blue). In either of these cases, we also call $G(S)$ *purely monochromatic*.

Recall that two plane graphs with the same vertex set S are compatible if their union is again a plane graph and that they are disjoint if their intersection does not contain any edge.

In the following we will consider exclusively plane straight-line graphs. Thus, the terms plane and straight-line are omitted for the sake of brevity.

As also mentioned in Section 4.2.3, there exist several results on compatible graphs, for bichromatic as well as for uncolored point sets. For example, Aichholzer et al. [11] showed that any (uncolored) geometric matching with n edges admits a compatible matching with approximately $\frac{4n}{5}$ edges. They also conjectured that for n even there always exists a compatible perfect matching, a conjecture which Ishaque et al. [105] recently proved do be true. In a similar direction (and as an inspiration for the results of this section), Abellanas et al. [20] showed upper and lower bounds for how many edges a compatible matching for a graph of a certain class can admit.

In a different work, Abellanas et al. [1] showed how many edges are needed at least to augment an (uncolored) connected graph to a plane 2-vertex or 2-edge connected graph. According results on bichromatic graphs have been obtained by Hurtado et al. [104], who also considered the question of augmenting a (disconnected) bichromatic graph to be connected. Among others, they provided an efficient algorithm for connecting a bichromatic perfect matching to a (valid bichromatic) spanning tree. Later, Hoffmann and Tóth [97] extended this work, showing how to augment a perfect matching to a (valid bichromatic) spanning tree with maximum vertex degree three.

In this section we investigate the following question. Given a bichromatic graph $G(S)$, we want to find a matching $M(S)$ (of some type) that is compatible with $G(S)$. Following the lines of [20], we call such a matching $G(S)$-compatible. Similarly, if $M(S)$ is disjoint from $G(S)$, we also say that it is $G(S)$-disjoint. We consider four classes of $G(S)$-compatible matchings:

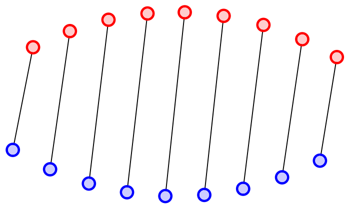
1. Bichromatic perfect matchings with a maximum number of edges not in $G(S)$
2. Bichromatic $G(S)$-disjoint matchings with a maximum number of edges
3. Monochromatic matchings with a maximum number of edges
4. Purely monochromatic matchings with a maximum number of edges

For a $G(S)$-compatible bichromatic matching $M(S)$, we denote the number of edges in $M(S)$ that are not in $G(S)$ by $d(G(S), M(S))$, and the total number edges in $M(S)$ by $e(G(S), M(S))$. Further, we denote the number of edges in a $G(S)$-compatible monochromatic matching $M(S)$ by $m(G(S), M(S)) = e(G(S), M(S))$, the number of red edges in $M(S)$ by $r(G(S), M(S))$, and the number of blue edges in $M(S)$ by $b(G(S), M(S))$. Similar to the work in [20], we focus on bounds for these numbers for the worst case examples for several different classes of bichromatic graphs. The classes of graphs we consider are spanning trees (*tree*), spanning paths (*path*), spanning cycles (*cycle*), and perfect matchings (*match*).

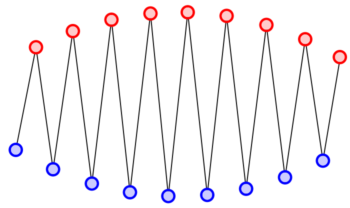
3.3.1 Bichromatic matchings

We start with bichromatic (perfect) matchings which are compatible to a given bichromatic graph. To simplify reading, we mostly omit the attribute bichromatic in this part.

It is well known that every set S with $|R| = |B|$ admits a bichromatic perfect matching [113]. On the other hand, there exists a large class of point sets with $|R| = |B|$, for which there is exactly one bichromatic perfect matching. Figure 3.24(a) illustrates such point set together with its unique perfect matching $M(S)$. Note that for any plane graph $G(S)$ that is obtained by adding edges to $M(S)$, we get $d(G(S), M(S)) = 0$. Especially, any perfect matching (and thus also $M(S)$) can be augmented to a spanning tree. Moreover, in this case, $M(S)$ can be extended to a spanning path as well; see Figure 3.24(b). Due to these observations, and to avoid trivial bounds, we restrict the considerations in this section to point sets admitting strictly more than one bichromatic perfect matching.



(a)



(b)

Figure 3.24: A point set S for which there exists exactly one perfect matching. (a) The unique perfect matching $M(S)$ for S. (b) A spanning path obtained by augmenting $M(S)$.

Let $\mathcal{S}$ be the class of point sets admitting at least two different bichromatic perfect matchings, and let $\mathcal{S}_n \subset \mathcal{S}$ be the sets with $|R| = |B| = n$. For a given class $\mathcal{C}$ of graphs, we denote by

$$e_{\mathcal{C}}(n) = \min_{S \in \mathcal{S}_n} \quad \min_{G(S) \in \mathcal{C}} \quad \max_{M(S)} \quad e(G(S), M(S))$$

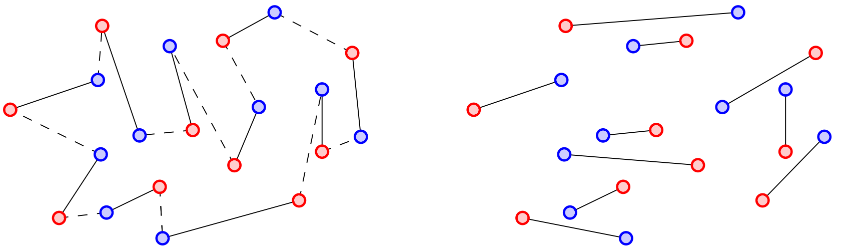
the maximum number such that for every point set $S \in \mathcal{S}_n$ and every graph $G(S) \in \mathcal{C}$, we can find a bichromatic disjoint compatible matching of (at least) this cardinality. Accordingly, we denote by

$$d_{\mathcal{C}}(n) = \min_{S \in \mathcal{S}_n} \quad \min_{G(S) \in \mathcal{C}} \quad \max_{M(S)} \quad d(G(S), M(S))$$

the maximum number such that for every point set $S \in \mathcal{S}_n$ and every graph $G(S) \in \mathcal{C}$, there exists a bichromatic compatible perfect matching $M(S)$ with (at least) this many edges disjoint from $G(S)$.

Note that any compatible perfect matching $M(S)$ for a given graph $G(S)$ contains a matching $M'(S)$ that is disjoint from $G(S)$ and has cardinality $e(G(S), M'(S)) = d(G(S), M(S))$. Thus $e_{\mathcal{C}}(n) \geq d_{\mathcal{C}}(n)$. The other direction need not be true, because a $G(S)$-disjoint matching can not necessarily be completed to a compatible perfect matching for $G(S)$.

As a first class $\mathcal{C}$ of graphs we consider spanning cycles. Obviously, any spanning cycle $C(S)$ contains a perfect matching as a sub graph, as it can be interpreted as the union of two disjoint compatible perfect matchings. Figure 3.25(a) illustrates the two perfect matchings in a spanning cycle (one is drawn with solid, the other one with dashed edges). Of course, these matchings do not have any edge disjoint from $C(S)$. To the contrary, the perfect matching in Figure 3.25(b) has 6 edges that are not in $C(S)$, inducing a disjoint compatible matching of cardinality 6. Note that the existence of a spanning cycle already implies that S admits at least two different perfect matchings and thus is a point set from $\mathcal{S}$.



(a) (b)

Figure 3.25: $C(S)$-compatible perfect matchings for a spanning cycle $C(S)$: (a) The perfect matchings induced by $C(S)$. (b) A $C(S)$-compatible perfect matching $M(S)$ with $d(C(S), M(S)) = 6$.

So what can we say about general bounds for $d_{cycle}(n)$ and $e_{cycle}(n)$? It turns out that there exist point sets S and spanning cycles $C(S)$ such that every bichromatic edge in S that is compatible with $C(S)$ is actually part of $C(S)$; see for example the spanning cycle shown in Figure 3.26. This implies that $e_{cycle}(n) = 0$ and thus also $d_{cycle}(n) = 0$.

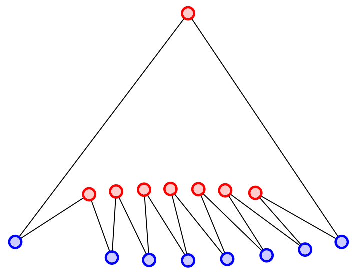
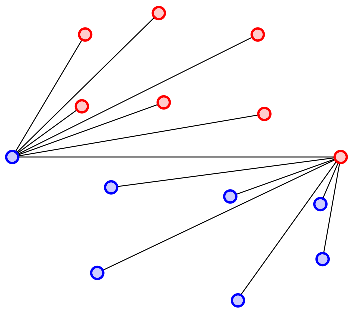


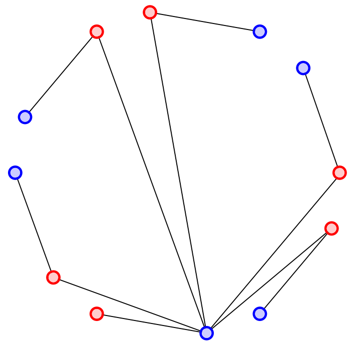
Figure 3.26: A spanning cycle $C(S)$ for which no bichromatic disjoint compatible edge exists.

Removing an edge from a spanning cycle, we obtain a spanning path. This immediately implies that there exist spanning paths that do not admit more than one bichromatic disjoint compatible edge. Thus, we obtain upper bounds of $d_{path}(n) \leq e_{path}(n) \leq 1$ for the class of spanning paths.

As spanning paths are a special case of spanning trees, the upper bounds for paths also directly apply for trees. Moreover, there exist arbitrarily large spanning trees (also with $|R| = |B|$) for which any compatible matching has only constant size. An example of such a spanning tree is illustrated in Figure 3.27(a). Furthermore, even if a spanning tree $T(S)$ admits a compatible perfect matching $M(S)$, it might still not be possible for $M(S)$ to contain a single edge that is disjoint from $T(S)$. Figure 3.27(b) illustrates a spanning tree for which this is the case. Altogether we obtain $d_{tree}(n) = 0$ and $e_{tree}(n) \leq 1$. Note that both point sets in Figure 3.27 admit many different perfect matchings.



(a)



(b)

Figure 3.27: (a) A spanning tree where any bichromatic compatible matching has at most 2 edges. (b) A spanning tree $T(S)$ where any bichromatic compatible perfect matching $M(S)$ has no edge that is disjoint from $T(S)$.

As a last class $\mathcal{C}$ of graphs we consider perfect matchings. As already mentioned, Hoffman and Tóth [97] showed that every bichromatic perfect matching $PM(S)$ can be augmented to a (valid bichromatic) spanning tree $T(S)$ with maximal vertex degree three. Figure 3.28 shows such a tree for the perfect matching from Figure 3.25(b), where the additional tree edges are drawn dashed.

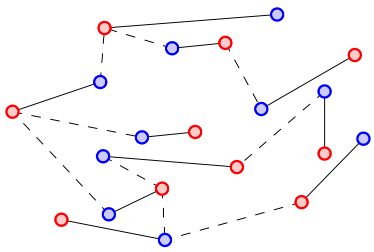


Figure 3.28: A perfect matching $PM(S)$ augmented to a spanning tree $T(S)$ with maximum vertex degree three.

Now consider the graph $A(S) = T(S) \backslash PM(S)$ of the augmenting edges. $A(S)$ contains exactly $n-1$ edges (as $T(S)$ has $2n-1$ edges of which n are in $M(S)$). Further, as every vertex is incident to exactly one edge of $M(S)$, the maximum vertex degree in $A(S)$ is at most two. In other words, $A(S)$ is a collection of paths $\mathcal{P}$ and isolated vertices. We know from before that every path with an even number of vertices contains a perfect matching (of its vertices). More general, a path P with $k_P + 1$ vertices has k_P edges, of which $\lceil \frac{k_P}{2} \rceil$ form a matching. Thus, $A(S)$ contains a matching with $\sum_{P \in \mathcal{P}} \lceil \frac{k_P}{2} \rceil \geq \lceil \frac{n-1}{2} \rceil$ edges, yielding a lower bound of $e_{match}(n) \geq \lceil \frac{n-1}{2} \rceil$ for the number of edges in a maximum $M(S)$-disjoint compatible matching for any perfect matching $M(S)$. Note that by the above arguments this bound is best possible, as it might happen that all paths in $A(S)$ have an even number of edges.

For an upper bound on $e_{match}(n)$ consider the perfect matching $PM(S)$ shown in Figure 3.29. Every red vertex that is incident to one of the small edges inside a triangle does not “see” any blue vertex except for the one it is matched to. Thus, in any disjoint compatible matching $M(S)$ of $PM(S)$ all these blue vertices must stay unmatched, implying an upper bound of $e_{match}(n) \leq e(PM(S), M(S)) \leq \frac{3n}{4}$.

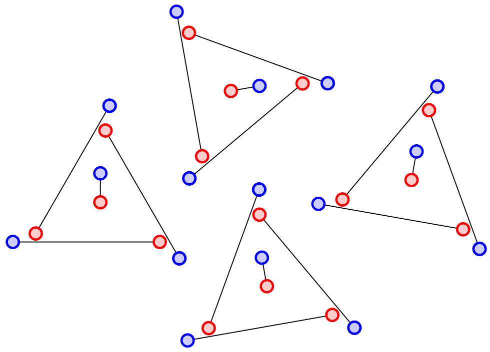


Figure 3.29: A perfect matching $PM(S)$ where any disjoint compatible matching $M(S)$ has at most $e(PM(S), M(S)) \leq \frac{3n}{4}$ edges.

The upper bound for disjoint matchings (induced by Figure 3.29) directly implies an according upper bound of $d_{match}(n) \leq \frac{3n}{4}$ for perfect matchings that are compatible to a given perfect matching. But for this case we can say even more. Consider the perfect matching $PM(S)$ that is illustrated in Figure 3.30. Every vertex that is incident to one of the short edges can only be compatibly matched in two ways; either to the other vertex of the edge it is incident to, or to the accordingly colored vertex of the long edge next to it. It is not possible to simultaneously match both vertices of a short edge of $PM(S)$ to the according vertices of the close-by long edge of $PM(S)$, because these two new edges would cross each other. Thus, any perfect matching $M(S)$ that is compatible with $PM(S)$ must contain all short edges of $PM(S)$, inducing an upper bound of $d_{match}(n) \leq d(PM(S), M(S)) \leq \frac{n}{2}$.

Note that this bound does not carry over to disjoint non-perfect matchings, because there we are not forced to match all vertices. Thus, for the graph in Figure 3.30, we can match one vertex of each short edge to the according vertex of the close-by long edge. Just leaving the second one unmatched yields a total number of $e(PM(S), M(S)) = \frac{3n}{4}$ edges in the constructed matching $M(S)$. Similarly, the lower bound for $e_{match}(n)$ cannot be reused to obtain a lower bound on $d_{match}(n)$ for the perfect matching case. For this case no non-trivial lower bound is known.

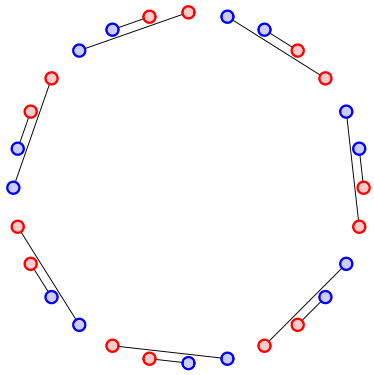


Figure 3.30: A perfect matching $PM(S)$ where any compatible perfect matching $M(S)$ has at most $d(PM(S), M(S)) \leq \frac{n}{2}$ edges that are disjoint from $PM(S)$.

Open problem 3.26. *Let $S = R \cup B$ be a bichromatic point set that admits at least two different bichromatic perfet matchings. Given a bichromatic perfect matching $PM(S)$, can we always find a compatible perfect matching $M(S)$ that contains edges which are not in $PM(S)$? How many disjoint edges can we guarantee?*

Table 3.4 summarizes the obtained results for the numbers of edges $e_{\mathcal{C}}(n)$ and $d_{\mathcal{C}}(n)$ that can be reached by a bichromatic $G(S)$-compatible ($G(S)$-disjoint) (perfect) matching, for any bichromatic graph $G(S) \in \mathcal{C}$.

$$
\begin{array}{rcccc}
d_{cycle}(n) & = & e_{cycle}(n) & = & 0 \\
d_{path}(n) & \leq & e_{path}(n) & \leq & 1 \\
0 & \leq & e_{tree}(n) & \leq & 1 \\
 & & d_{tree}(n) & = & 0 \\
\lceil \frac{n-1}{2} \rceil & \leq & e_{match}(n) & \leq & \frac{3n}{4} \\
0 & \leq & d_{match}(n) & \leq & \frac{n}{2}
\end{array}
$$

Table 3.4: Obtained results for $d_{\mathcal{C}}(n)$ and $e_{\mathcal{C}}(n)$.

3.3.2 Monochromatic matchings

We continue with (purely) monochromatic compatible matchings for bichromatic graphs. Dumitrescu and Steiger [71] showed that not every bichromatic point set admits a monochromatic perfect matching. More precisely, there exist point sets such that a linear number of the points must stay unmatched. This result implies that we cannot hope for a monochromatic perfect matching that is compatible to a given bichromatic graph.

In the following, we denote by

$$m_{\mathcal{C}}(n) = \min_{|S|=2n} \quad \min_{G(S)\in\mathcal{C}} \quad \max_{M(S)} \quad m(G(S), M(S))$$

the maximum number such that for a given class $\mathcal{C}$ and every graph $G(S) \in \mathcal{C}$, we can find a monochromatic compatible matching of (at least) this cardinality. Accordingly, we define

$$p_{\mathcal{C}}(n) = \min_{|S|=2n} \quad \min_{G(S)\in\mathcal{C}} \quad \max_{M(S)} \quad \{r(G(S), M(S)), b(G(S), M(S))\}$$

to be the maximum number such that for every graph $G(S) \in \mathcal{C}$ there exists a purely monochromatic compatible matching $M(S)$ with (at least) this many edges.

Let us start with some upper bound examples. Figure 3.31 shows a bichromatic spanning tree $T(S)$. Any monochromatic edge that is compatible with $T(S)$ must be incident to one of the two vertices of high degree (indicated by the dashed lines in the figure). Thus, any monochromatic $T(S)$-compatible matching $M(S)$ can contain at most one red and one blue edge, implying that $m_{tree}(n) \leq 2$ and $p_{tree}(n) \leq 1$.

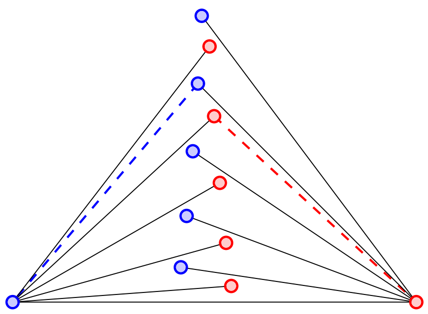


Figure 3.31: A bichromatic spanning tree $T(S)$ with $m(T(S), M(S)) = 2$ and $r(T(S), M(S)) = b(T(S), M(S)) = 1$ for every maximum monochromatic compatible matching $M(S)$.

While in the bichromatic setting, spanning trees and spanning paths had essentially the same bounds, this is not true for the monochromatic case. One important ingredient for the example in Figure 3.31 is that it has two vertices of high degree, while all other vertices are leaves. The "worst" examples that we could find for spanning paths and spanning cycles are illustrated in Figures 3.32 and 3.33 (for spanning paths just ignore one of the very short edges).

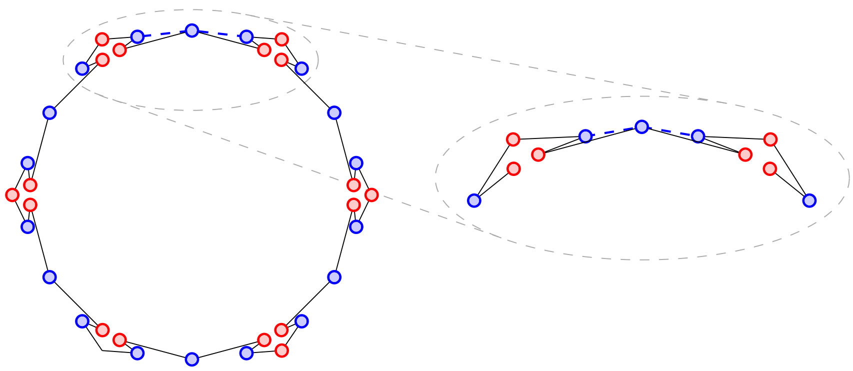


Figure 3.32: A bichromatic spanning cycle $C(S)$ with $m(C(S), M(S)) = \frac{5n}{6}$ for every maximum monochromatic compatible matching $M(S)$.

Consider first Figure 3.32. The blue vertices at the small spikes can only be matched with the blue vertices between the spikes. The dashed blue lines indicate possible matching edges. Thus, one third of the blue vertices have to stay unmatched, implying $b(C(S), M(S)) \leq \frac{n}{3}$, for any monochromatic $T(S)$-compatible matching $M(S)$. As all red vertices can be matched, we have $r(C(S), M(S)) \leq \frac{n}{2}$ and thus $m(C(S), M(S)) \leq \frac{5n}{6}$. The spanning cycle illustrated in Figure 3.33 is slightly worse for the total number of edges in a monochromatic compatible matching, but forces $\frac{n-7}{6}$ unmatched points of each color. Thus, it admits a maximum of $r(C(S), M(S)) = b(C(S), M(S)) \leq \frac{5n+7}{12}$ edges in any purely monochromatic matching. Altogether we obtain upper bounds of $m_{path}(n), m_{cycle}(n) \leq \frac{5n}{6}$ for monochromatic, and $p_{path}(n), p_{cycle}(n) \leq \frac{5n+7}{12}$ for purely monochromatic matchings.

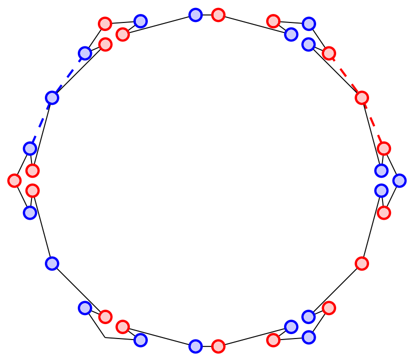


Figure 3.33: A bichromatic spanning cycle $C(S)$ with $r(C(S), M(S)) = b(C(S), M(S)) = \frac{5n+7}{12}$ for every maximum purely monochromatic compatible matching $M(S)$.

For the case of monochromatic compatible matchings for a bichromatic perfect matching we can partly reuse the examples from the bichromatic case. Inverting the vertex colors of the short edges in the example shown in Figure 3.30 yields the perfect matching $PM(S)$ shown in Figure 3.34. As at least one vertex per short edge stays unmatched in any monochromatic $PM(S)$-compatible matching, this yields $m_{match}(n) \leq \frac{3n}{4}$.

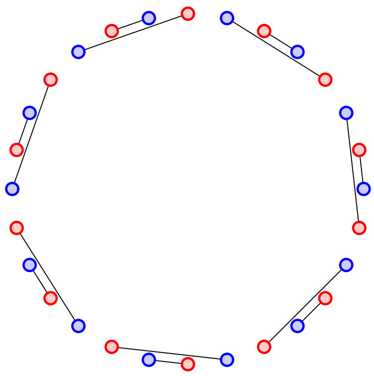


Figure 3.34: A bichromatic perfect matching $PM(S)$ where any monochromatic compatible matching $M(S)$ has at most $m(PM(S), M(S)) \leq \frac{3n}{4}$ edges.

Note that the perfect matching from Figure 3.34 admits a complete matching of each color. For an upper bound on the number of edges in a purely monochromatic matching, we can recycle the idea from Figure 3.29: If we invert the colors of every second triangle construction and combine them to a closed cycle, we obtain a perfect matching $PM(S)$ where every sixth point of each color must remain unmatched in any monochromatic $PM(S)$-compatible matching $M(S)$; see Figure 3.35. This gives an upper bound of $p_{match}(n) \leq \frac{5n}{12}$.

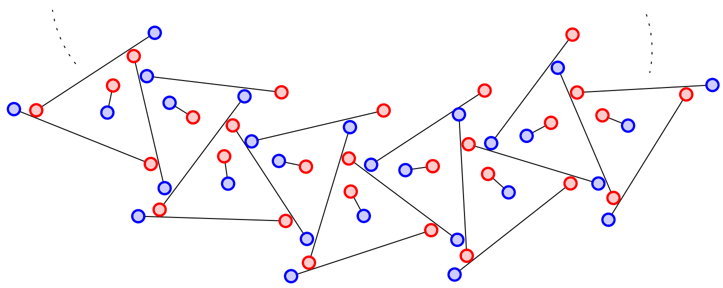


Figure 3.35: Scheme for a bichromatic perfect matching $PM(S)$ where any purely monochromatic compatible matching $M(S)$ has at most $p(PM(S), M(S)) \leq \frac{5n}{12}$ edges.

Continuing with the class of perfect matchings we come to lower bounds. Consider a perfect matching $PM(S)$. Assume w.l.o.g. that $PM(S)$ does not contain any vertical edge, and that for at least half of the edges, the left vertex is red. We augment the matching to a weakly simple polygon in the following way; see Figure 3.36. First, we add a bounding box around the $PM(S)$. Next, we extend all edges with a left red vertex to the right until it hits the bounding box or (an extension of) an edge. Then we extend all other edges to the right as well, but with a slight turn. The result is a so-called weakly simple polygon, which can be transformed to a simple polygon by slightly "inflating" the edges; see Figure 3.37. In the resulting polygon, all left endpoints of edges and all red right endpoints of edges appear as reflex vertices. All blue vertices which are right endpoints of matching edges are "hidden" in the interior of a polygon edge.

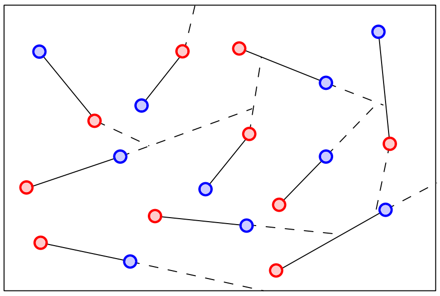
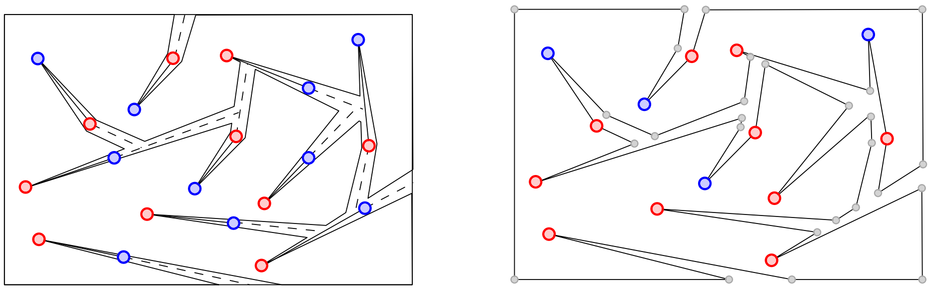


Figure 3.36: Transforming a perfect matching to a weakly simple polygon.



(a) (b)

Figure 3.37: (a) "Inflating" the edges. (b) The resulting simple polygon (convex vertices are colored gray).

Abellanas et al. [1] showed that for every simple polygon $P(V)$ with vertex set V and every subset $V' \subseteq V$ containing all reflex vertices of $P(V)$, there exists a perfect matching of the vertices V' where no edge is outside the boundary of $P(V)$. Applying this result to the set of reflex vertices of the constructed polygon, we obtain a matching $M(S)$ with at most $\frac{n}{2}$ bichromatic edges (one for every blue reflex vertex). Thus $M(S)$ contains at least $\frac{n}{4}$ red edges, implying $m_{match}(n) \geq p_{match}(n) \geq \frac{n}{4}$. Figure 3.38 shows a possible resulting matching.

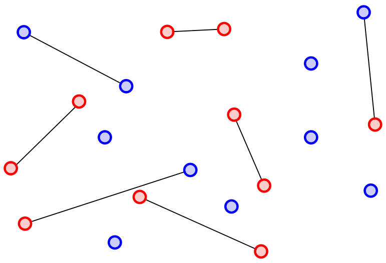
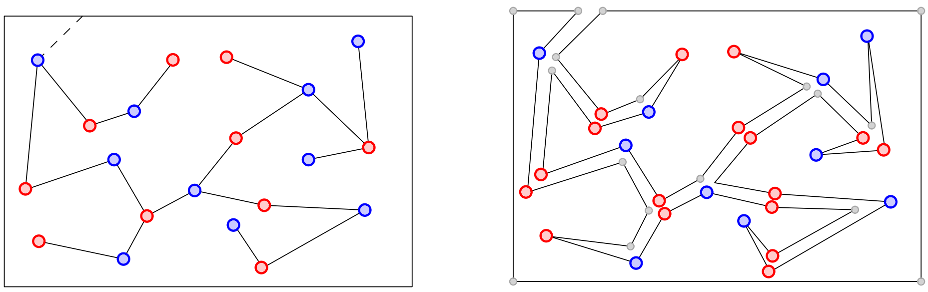


Figure 3.38: A resulting compatible matching.

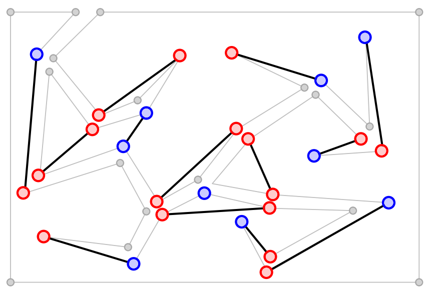
We reuse the principle of the above proof to provide a lower bound for the size of monochromatic compatible matchings for trees in dependence of the number of interior vertices (of one color) of the tree. The basic idea for this bound was developed during a research week which was also the starting point for the work [20]. Consider a bichromatic tree $T(S)$, let i_r and i_b be the numbers of interior red and blue vertices of $T(S)$, respectively, and assume w.l.o.g. that $i_r \geq i_b$. We generate a simple polygon for $T(S)$ by adding a bounding box, connecting $T(S)$ to the box and then inflating the whole construction. Every vertex v of $T(S)$ with vertex degree $d(v)$ corresponds to $d(v)$ vertices in the polygon P, at most one of them being reflex. We choose one of these $d(v)$ vertices for each vertex v of $T(S)$, (if v corresponds to a reflex vertex, we choose that one). Additionally, we choose a second vertex for each of the i_r red interior vertices; see Figure 3.39.



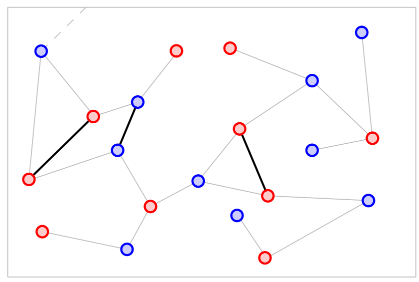
(a) (b)

Figure 3.39: (a) A spanning tree connected to its bounding box. (b) The resulting simple polygon (the non-selected vertices and the vertices on the bounding box are drawn gray).

Applying again the result of [1], we obtain a (nearly) perfect matching with $\lfloor \frac{2n+i_r}{2} \rfloor$ edges, at least $\lfloor \frac{i_r}{2} \rfloor$ of them with two red endpoints. But now it might happen that both red vertices corresponding to one red vertex in S are incident to such a red edge. Moreover, there even might occur cycles of such red edges, as can be seen in Figure 3.40(a).



(a)



(b)

Figure 3.40: (a) A (nearly) perfect matching of the selected vertex set. (b) A resulting monochromatic matching of $T(S)$.

If red edges translated to the original point set form a path of length k, $\lceil \frac{k}{2} \rceil$ of them can simultaneously occur in a monochromatic $T(S)$-compatible matching. In the case of a cycle of length k, $\lfloor \frac{k}{2} \rfloor$ of these edges can be used. A monochromatic matching for $T(S)$ is illustrated in Figure 3.40(b). In the worst case, (nearly) all red edges form 3-cycles, and we obtain only $\lceil \frac{i_r - 1}{6} \rceil$ red edges (and maybe no blue ones) for a monochromatic $T(S)$-compatible matching. At least, for $i_r \geq 14$, this lower bound is above the general upper bound for trees (which is just 2).

The above result for trees can immediately be used for a lower bound of $m_{path} \geq p_{path} \geq \lceil \frac{n-2}{6} \rceil$ for the number of edges in a maximum (purely) monochromatic matching that is compatible to a bichromatic spanning path. Note that a path does not have any vertex v of degree $d(v) > 2$. Thus, a cycle of odd length among the red edges (translated back to S) can only occur if the path ends inside this cycle. Unfortunately, as there might be a sequence of cycles $C_1, \ldots, C_l$ such that each C_{i+1} contains C_i in its interior, this observation does not help to improve the bound. Figure 3.41 illustrates such a sequence of cycles for a path.

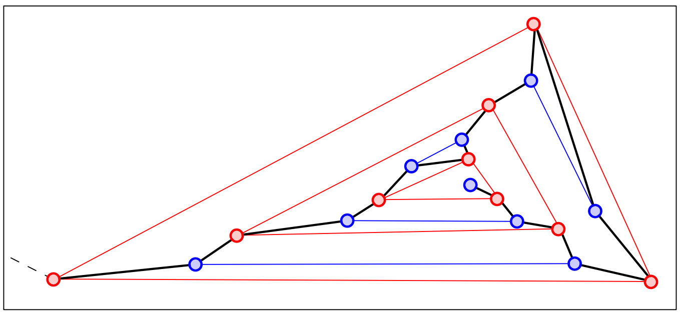
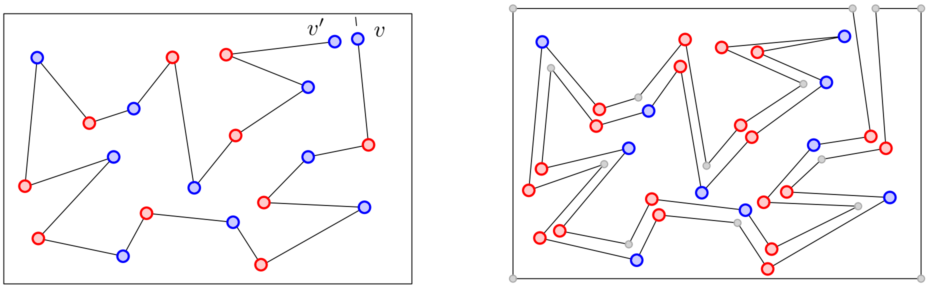


Figure 3.41: Multiple red 3-cycles for a path resulting from a matching of the according inflated polygon.

Finally, let us apply the above result to bichromatic spanning cycles. For creating a simple polygon P for a given spanning cycle $C(S)$, we duplicate an extreme vertex v (w.l.o.g. a blue one), and cut the cycle between v and its duplicate v'; see Figure 3.42(a).



(a) (b)

Figure 3.42: (a) A spanning cycle cut and connected to its bounding box. (b) The resulting simple polygon (the non-selected vertices are drawn grey).

Then we extend one of the incident edge of v until it hits the bounding box, and again inflate the resulting construction, by this hiding v. As illustrated in Figure 3.42(b), we select the according reflex vertex of P for each blue vertex of $C(S)$ and both corresponding vertices of P for each red vertex of $C(S)$. In the same way as above we obtain a total of $3n$ selected vertices, and thus at least $\lfloor \frac{n}{2} \rfloor$ red matching edges. But now the red edges, when translated back to S, cannot form any odd cycles because the "end" vertex v' is extreme and thus cannot lie inside such a cycle. For paths of red edges as well as for cycles of even length we can use every second edge for a matching of $C(S)$, implying that $m_{cycle}(n) \geq p_{cycle}(n) \geq \lceil \frac{n-1}{4} \rceil$.

Table 3.5 summarizes the obtained bounds on the numbers of edges $m_{\mathcal{C}}(n)$ and $p_{\mathcal{C}}(n)$ that can be reached by a (purely) monochromatic $G(S)$-compatible matching, with $G(S) \in \mathcal{C}$ and $\mathcal{C} \in \{tree, path, cycle, match\}$.

		$m_{tree}(n)$	$=$	2
		$p_{tree}(n)$	$=$	1
$\lceil \frac{n-2}{6} \rceil$	$\leq$	$m_{path}(n)$	$\leq$	$\frac{5n}{6}$
$\lceil \frac{n-2}{6} \rceil$	$\leq$	$p_{path}(n)$	$\leq$	$\frac{5n+7}{12}$
$\lceil \frac{n-1}{4} \rceil$	$\leq$	$m_{cycle}(n)$	$\leq$	$\frac{5n}{6}$
$\lceil \frac{n-1}{4} \rceil$	$\leq$	$p_{cycle}(n)$	$\leq$	$\frac{5n+7}{12}$
$\frac{n}{4}$	$\leq$	$m_{match}(n)$	$\leq$	$\frac{3n}{4}$
$\frac{n}{4}$	$\leq$	$p_{match}(n)$	$\leq$	$\frac{5n}{12}$

Table 3.5: Obtained results for $m_{\mathcal{C}}(n)$ and $p_{\mathcal{C}}(n)$.

Note that all of the lower bounds for monochromatic matchings stem from observations about purely monochromatic matchings. So let us conclude this section with the following question.

Open problem 3.27. *Given a class of graphs $\mathcal{C} \in \{tree, path, cycle, match\}$. Is it true that $m_{\mathcal{C}}(n) = p_{\mathcal{C}}(n)$, or are these values different for (some of) the classes?*

Chapter 4

Triangulations, Pointed Pseudo-Triangulations and Compatibility

In the first section of this chapter we continue with plane graphs on top of bichromatic point sets, but in a slightly different setting. While in the last section of the previous chapter, the task was to choose appropriate edges in order to obtain compatible graphs, now edges are already predefined by the Delaunay property.

The *Delaunay graph* $DG(S)$ [64] of a point set S in the plane is the set of all edges for which there exists a circle passing through both endpoints of the edge that does not contain any other point of S. We say that a point set S is in *strong general position*, if, in addition to no three points of S lying on a common line, also no four points of S lie on a common circle. It can be shown that if S is in strong general position, then $DG(S)$ is in fact a triangulation, the *Delaunay triangulation* $DT(S)$; see for example the textbook [58]. For every triangle pqr of $DT(S)$, its circumcircle does not contain any point of $S \backslash \{p, q, r\}$. The existence of an empty circle for an edge or a triangle is often referred to as *Delaunay property*.

A variation of this concept is the so-called *constrained Delaunay triangulation* (also known as *generalized Delaunay triangulation*; see [114] and Chapter 4.4.4. in [40]). Given a plane (uncolored) graph $G(S)$ (on top of a point set S), the constrained Delaunay triangulation is the compatible completion of $G(S)$ to a triangulation $T(S)$ such that $T(S)$ is as Delaunay as possible. More exactly, for every non-predefined edge e in the constrained Delaunay triangulation, there exists a circle C through the endpoints of e such that no point inside C is visible from e (the edges of $G(S)$ block visibility).

In the first part of this chapter we consider a different variation (related to witness proximity graphs; see below), which was introduced before by Aronov et al. [38]. Given a set B of black points, $|B| = n$, our goal is to appropriately choose a set of white points W, such that the Delaunay triangulation of the joint set $B \cup W$ does not contain any black-black edge (i.e., no edge between two points of B). If this is the case, we say that the set W *blocks* the set B.

Aronov et al. [38] proved that it is always possible to block B with $(2n–2)$ white points and that $4n/3$ white points suffice if B is in convex position. Improving over these bounds, in Section 4.1 we show how to place white points such that $|W| \leq 3n/2$ points are always sufficient to block B. We also prove that $|W| \leq 5n/4$ white points suffice to block B if B is in convex position. As a lower bound, we show that at least $n - 1$ white points are always necessary to block B.

In the second part of this chapter, Section 4.2, we switch from triangulations to pointed pseudo-triangulations, and to compatibility in the classical sense (see also Section 3.3).

Given some pointed pseudo-triangulation $PT(S)$ on top of a point set S in general position (no three points on a common line is sufficient here), we want to find a second pointed pseudo-triangulation $PT'(S)$ that is compatible to $PT(S)$. At the same time, we want $PT'(S)$ to be as different from $PT(S)$ as possible.

Recall that two graphs, and thus also two pointed pseudo-triangulations $PT(S)$ and $PT'(S)$ on top of the same point set S are compatible if their union $PT(S) \cup PT'(S)$ is again a plane graph. If, moreover, $PT(S) \cup PT'(S)$ is a triangulation (i.e., a maximal plane graph), then we say that $PT(S)$ and $PT'(S)$ are *maximally disjoint compatible*.

We show that for any set S there exist two maximally disjoint compatible pointed pseudo-triangulations. In contrast, we prove that there are arbitrary large point sets S and pointed pseudo-triangulations $PT(S)$ such that there exists no pointed pseudo-triangulation that is compatible to and different from $PT(S)$.

The results of this chapter have been presented before [18, 30], and the results from the Section 4.1 are accepted for journal publication in a special issue [19].

4.1 Blocking Delaunay triangulations

Proximity graphs were originally defined to capture different notions of proximity in a set of points [106]. A particular proximity graph in a set of points S is defined by assigning to every pair of points in S a region (or family of regions). Then, the edge pq is part of the graph if and only if at least one of the regions corresponding to the pair is empty of points of S. Examples of such graphs are the Gabriel graph, the relative neighborhood graph, and the Delaunay triangulation.

Recently, Dulieu [69] and Aronov et al. [38, 39] extended the notion of proximity graphs with the concept of *witness proximity graphs*. In this generalized setting, we have a second point set W. The points of W are the *witnesses* which account for the existence of an edge between two points of S. The authors consider two different possibilities. In the first one, an edge between two points $p, q \in S$ exists if some of the regions corresponding to the pair p, q do not contain any witness point. In the second version, an edge between two points of S exists if there exists a region containing a witness point. In this section we deal with the first version of the witness Delaunay graph: Given a set B of *black* points and a set W of *white* points (the witnesses), we follow the notation in [38] and consider the graph $DG^-(B, W)$. In this graph, two points $p, q \in B$ are adjacent if there exists a circle passing through p and q and not containing any point of W.

In the same paper, the following combinatorial problem is proposed: If B has size n, find the smallest c such that we can always guarantee the existence of a set W, with size cn, and such that $DG^-(B, W)$ does not contain any edges. This problem can also be formulated as a very natural stabbing problem: Given a set B of n points, consider the family $\mathcal{C}$ of circles defined by two points in B. Give a bound for the size of a set of points stabbing all circles in $\mathcal{C}$.

Let $\mathcal{D}$ be the set of Delaunay circles of B, i.e., empty circles passing through at least two points of B. In [38] it is shown that in order to pierce all circles in $\mathcal{C}$ it is sufficient to pierce all circles in $\mathcal{D}$. For the combinatorial problem mentioned previously, they show that in order to pierce the set of Delaunay circles generated by a set of n points, $2n - 2$ points are always sufficient, and n points are sometimes necessary. If points of B are in convex position, they improve the upper bound to $4n/3$. Similar problems have been studied from an algorithmic point of view in [7].

We focus on this combinatorial problem, with a slightly different language that we find more intuitive: For $DG^-(B, W)$ having no edges it is necessary and sufficient that points in W pierce all circles in $\mathcal{D}$. This in turn is equivalent to the fact that there is no edge connecting two black points in the Delaunay graph of $B \cup W$. If this is the case, we say that the set W *blocks* the set B.

In the following, we assume that the set $B \cup W$ is in strong general position. Further, we denote the Delaunay triangulation of a point set S with $DT(S)$, the Voronoi diagram of S with $V(S)$, and the Voronoi region of point $p \in S$ in $V(S)$ with $V_p(S)$.

4.1.1 An upper bound for general point sets

We start with a constructive approach for blocking general point sets that utilizes the duality between Delaunay triangulations and Voronoi diagrams.

Theorem 4.1. *Let B be a set of n black points in general position. There always exists a set W of at most $3n/2$ white points that blocks B.*

Proof. Let I be the biggest independent set in $DT(B)$, and $C = B \setminus I$ its complement. Because every triangulation is 4-colorable, we know that $C \leq 3n/4$. We are going to show that B can be blocked by adding two white points in a close neighborhood of each point in C.

First, for each $p \in C$ we choose a point $\eta(p) \in C$ among the neighbors of p in $DT(B)$. This is always possible, because if pqr is a triangle in $DT(B)$, then it cannot happen that q and r are both in I.

Then, for each $p \in C$ we choose a point x_p (not in B) in the interior of the Voronoi cell $V_p(B)$, and with the following conditions: (i) The ray $x_p p$ intersects the boundary of $V_p(B)$ in the Voronoi edge of $V(B)$ which separates $V_p(B)$ from $V_{\eta(p)}(B)$. Let y_p be this point of intersection. (ii) In the case in which $q = \eta(p)$ and $p = \eta(q)$, x_p and x_q have to be chosen in such a way that $y_p \neq y_q$ (see Figure 4.1).

Now we assign a segment e_p to each point $p \in C$ such that e_p is centered at y_p and contained in the edge of $V(B)$ separating the Voronoi regions of p and $\eta(p)$. If $q = \eta(p)$ and $p = \eta(q)$, we choose the intervals e_p and e_q small enough to be disjoint.

Next, we add two white points in $V_p(B)$ in the following way. Consider the circle that is centered at x_p and passes through p, and place the white points at the intersections of this circle with the line segments defined by x_p and the endpoints of e_p. Note that x_p does not belong to our set of white points.

Once we have done this for every point $p \in C$, we claim that in the Voronoi diagram of the resulting set no pair of black points have adjacent

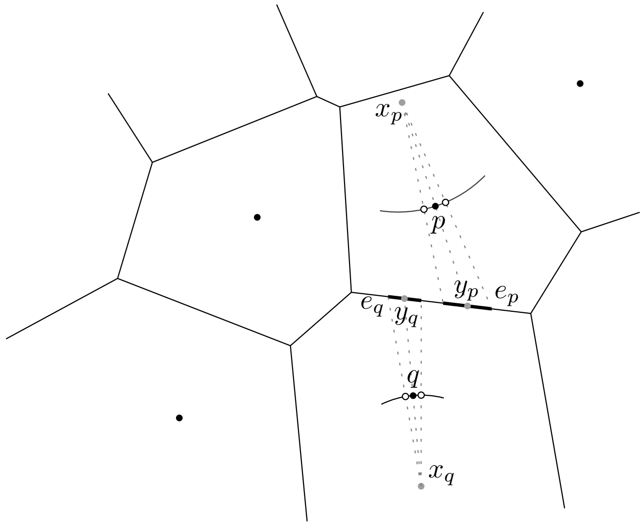


Figure 4.1: Blocking a black point by placing two white points in its Voronoi cell.

regions. The only area where p could be closer to some black point than (one of) the two shielding white points (constructed for it) is inside the wedge defined by the bisectors of p and these two white points. The apex of the wedge is $x_p \in V_p(B)$, and only point $q = \eta(p)$ has the possibility to be a Voronoi neighbor for p. But by construction, the intervals e_p and e_q are disjoint, so this does not happen. □

4.1.2 An upper bound for convex sets

For the special case of point sets in convex position we improve the $4n/3$ bound in [38] by translating the problem into a combinatorial setting.

We call a triangle of a triangulation an *ear* if it contains a vertex (the *tip*) which is not incident to inner edges. We call a triangle an *inner triangle* if it consists solely of inner edges.

Considering the properties of the Delaunay triangulation, we propose the following two simple operations to block Delaunay edges. *Blocking a single edge* is done by placing a white point arbitrarily close to the center of the edge. For inner edges this can be done on any of its two sides, and for edges of the convex hull the white point has to be placed slightly outside the convex hull. *Blocking a vertex p* is done by placing two white points outside the convex hull, one at each incident convex hull edge, and arbitrarily close to p (such that the two white points are Delaunay neighbors). In this way all Delaunay edges incident to p are blocked.

Reconsidering the presented blocking operations we transform the whole setting into a combinatorial framework. We call blocking a single edge *coloring an edge* with cost 1, and blocking a vertex *coloring a vertex* with cost 2, where the latter also colors all incident edges. Thus, our task can be rephrased as coloring all edges of a given triangulation with minimal cost. Let $C(n)$ denote the maximum minimal cost over all sets of n points in convex position. Clearly, an upper bound on the occurring cost is an upper bound on the number of white points needed in the geometric setting, while the inverse statement is not true in general. In fact, we can apply our combinatorial setting to any triangulation, not only to the Delaunay triangulation.

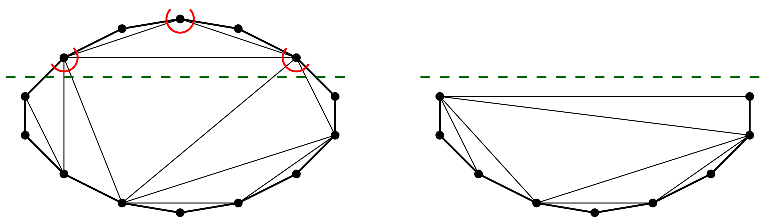


Figure 4.2: An $(14, 5, 6)$-cut and the re-triangulated subset.

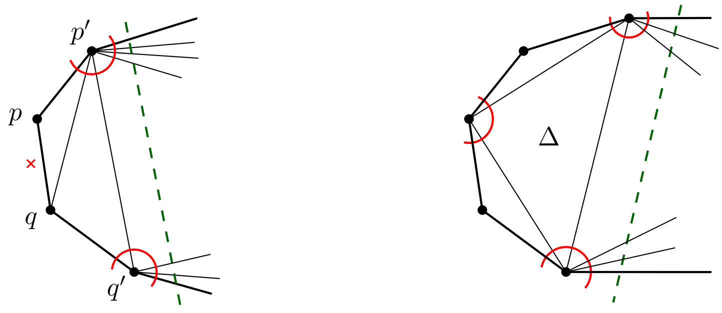
An (n, a, k)-*cut* of a triangulation T of a set of n points is a separation of the n points into two disjoint groups A and B with $|A| = a$ and $|B| = n - a$, plus a coloring of A with cost k such that any edge of T incident to a point in A is colored, see Figure 4.2.

Lemma 4.2. *If for a triangulation T of a convex n-gon, there exists an (n, a, k)-cut, then the cost of coloring T is at most $C(n - a) + k$.*

Proof. Let A and B be the two sets as defined for the (n, a, k)-cut. We use the coloring of A given by the cut and remove all colored vertices and edges. We complete the remaining graph of B to a full triangulation of the convex set B by (arbitrarily) re-triangulating the holes induced by removing A (cf. Figure 4.2, right), and color this triangulation of B with cost at most $C(n - a)$. Combining the two colorings of A and B (where we can ignore colored edges of B which are not part of T), we obtain a coloring of T with cost at most $C(n - a) + k$. □

Theorem 4.3. $C(n) \leq \frac{5n}{4}$.

Proof. We prove the statement by induction on the number n of vertices. For the induction base it is straightforward that for $n \leq 3$ we have $C(n) \leq n$.



(a) (b)

Figure 4.3: The two cases for a convex set: removing an ear (left), and removing an inner triangle with two incident ears (right).

So assume the statement is true for any set of size $n' < n$, and consider a triangulation T of n points. We distinguish two cases.

Case 1. Assume that there exists an ear of T with tip p such that a neighbor q of p (neighborhood is with respect to the convex hull) has precisely one incident inner edge, see Figure 4.3(left). We color the two other neighbors p' and q' of p and q, respectively, as well as the edge pq. With $A = \{p, q, p', q'\}$ this induces an $(n, 4, 5)$-cut of T. By Lemma 4.2 we have $C(n) \leq 5 + C(n-4) \leq 5 + \frac{5(n-4)}{4} = \frac{5n}{4}$, where the last inequality follows from the induction hypothesis.

Case 2. Otherwise, all neighbors of the tip of an ear are incident to at least two inner edges. This is equivalent to the fact that all ears are adjacent to inner triangles. As in any triangulation (of a convex set) the number of ears is by two larger than the number of inner triangles (this follows from considering the dual tree), there exists at least one inner triangle Δ which is adjacent to two ears. We color the three vertices of Δ, see Figure 4.3(b). The tips of the two ears incident to Δ together with the three vertices of Δ form our set A. As A has cardinality 5, this induces an $(n, 5, 6)$-cut of T, and similar as before we have $C(n) \leq 6 + C(n-5) < \frac{5n}{4}$. □

Corollary 4.4. *Let B be a set of n black points in convex position. There always exists a set W of at most $5n/4$ white points that blocks B.*

4.1.3 A lower bound for general point sets

In this section we provide a general lower bound on the number of points needed to block any given set, again using independent vertices.

Lemma 4.5. *The size of an independent set in the Delaunay triangulation of a set of n points is at most $\lfloor \frac{n+1}{2} \rfloor$.*

Proof. It is known that every Delaunay triangulation contains a perfect matching of its vertices [68]. Consider such a perfect matching M, and an independent set I. Then for every edge in M, at most one of its endpoints can be in I. If n is odd, then the non-matched point can be in I as well. □

Note that Lemma 4.5 describes a special property of the Delaunay triangulation, as there exist sets of n points, which can be triangulated in a way that the triangulation has an independent set of size as much as $\frac{2n-2}{3}$. For example, take a set of k white points and triangulate it arbitrarily. Place one black point in the interior of each white triangle. Further, place one black point outside but close to each convex hull edge. Complete the full set of $n = 3k-2$ points to a triangulation with k white and $2k-2$ independent black points.

Theorem 4.6. *For any set B of n black points, at least $n-1$ white points are necessary to block it*

Proof. Assume that the white point set W, $|W| = m$, blocks B. Then the joint Delaunay triangulation $DT(B \cup R)$ does not contain any edge between two black vertices, which implies that B is an independent set in $DT(B \cup R)$. Thus, by Lemma 4.5, we have $n \leq \lfloor \frac{n+m+1}{2} \rfloor$, and consequently $m \geq n-1$. □

4.1.4 Discussion

We have shown that for blocking a set B of n black points, $3n/2$ white points are sufficient for general sets B, and $5n/4$ white points are sufficient if the points of B are in convex position. Note that both proofs for the upper bounds are constructive, directly providing an algorithm. Moreover, we know that we need $n-1$ white points for blocking any set with n black points. As already mentioned, in [39] the authors show that there exist point sets B of n black points which need n white points to be blocked – Figure 4.4 shows the intuition behind their proof.

So far we have not been able to construct a set that needs more than n white points to be blocked, and to the best of our knowledge, no example is known that can in fact be blocked with only $n-1$ points. Thus we state the following conjecture.

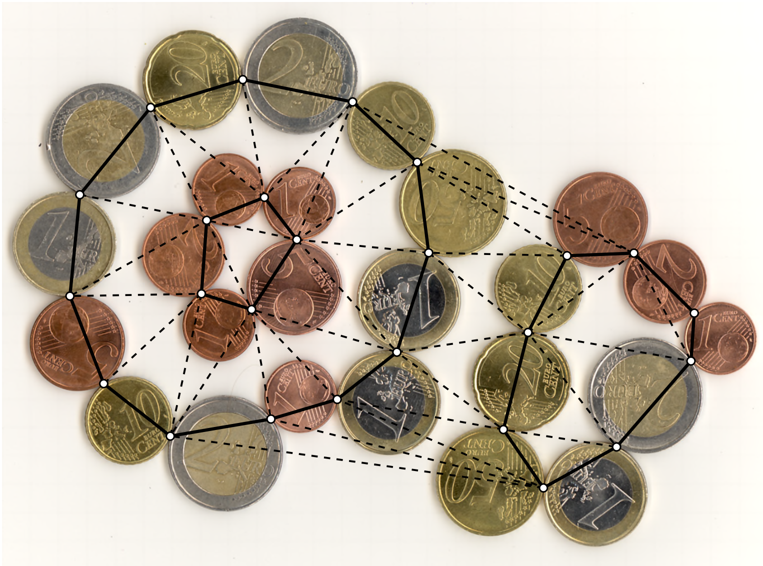


Figure 4.4: Euros proving a lower bound of n: the cycles induced by the coins are drawn solid. The dashed edges complete the Delaunay triangulation.

Conjecture 4.7. *For any set B of n black points in convex position, n white points are necessary and sufficient to block B.*

In fact, from what is currently known, the conjecture might be true even for general point sets.

Independently, the algorithmic issue of finding a minimal set of blocking white points arises as a natural question for future work. A related problem has been studied recently by de Berg et al. [59]: Given a set of black and white sites, it is NP-hard to compute the minimum number of white points that have to be removed so that the union of the Voronoi cells of the black points is a connected region.

4.2 Compatible pointed pseudo-triangulations

Recall that a pseudo-triangle is a simple polygon which has exactly three *corners* (vertices with interior angle smaller than π). A path along the boundary of a pseudo-triangle that has two of the corners as end points and does not contain the third one is called a *side-chain* of this pseudo-triangle. The non-incident corner is called *opposite* (to this side-chain). Further recall that a pseudo-triangulation is a plane straight-line graph whose edges partition the convex hull $\mathrm{CH}(S)$ into pseudo-triangles. It is called pointed if all its vertices are pointed. Note that for any pseudo-triangulation, all extreme vertices of S are pointed towards the outer face.

Pseudo-triangulations have been first introduced by Pocciola and Vegter [128] in a more general framework and by Streinu [137] in the context of geometric graphs. They are a rather young structure, with interesting properties and applications. See the recent survey [132] and references therein.

Streinu [138] showed that pointed pseudo-triangulations are minimal pseudo-triangulations, and at the same time they are maximal pointed plane straight-line graphs, where minimal and maximal is with respect to the number of edges. Further, any pointed pseudo-triangulation on top of a point set S with n points has $2n-3$ edges, independent of the number of interior points of S.

In this section we consider the simultaneous existence of two (different) pointed pseudo-triangulations on the same point set. Before we define the precise questions that we want to investigate, let us state two preliminary results.

Proposition 4.8. *If two pointed pseudo-triangulations $PT_1(S) = (S, E_1)$ and $PT_2(S) = (S, E_2)$ on top of the same point set S with i interior points are compatible, then they differ by at most i edges: $|E_1 \backslash E_2| = |E_2 \backslash E_1| \leq i$.*

Proof. The number of edges in any pointed pseudo-triangulation of a point set S with n points is $2n-3$. As S has i interior and $h = n-i$ extreme points, the number of edges in any maximal plane graph (triangulation) of S is $3n-h-3 = 2n+i-3$. Thus, considering an arbitrary pointed pseudo-triangulation $PT_1(S)$, the number of edges that can be added to obtain a maximal plane graph is $2n+i-3-2n+3 = i$. This implies that *any* plane straight-line graph compatible with $PT_1(S)$, and thus also any such pointed pseudo-triangulation, can have at most i edges that are not in $PT_1(S)$. $\square$

Corollary 4.9. *Two pointed pseudo-triangulations on top of the same point set S (with i interior points) are maximally disjoint compatible if and only if they differ by exactly i edges.*

4.2.1 Two compatible pointed pseudo-triangulations

We start with the following question. Given a point set S with n points, can we find two compatible pointed pseudo-triangulations which are maximally disjoint? As we have just seen, maximally disjoint means that the two pointed pseudo-triangulations have exactly $2n-3-i$ edges in common, where i is the number of interior points of S.

Theorem 4.10. *For every point set S with n points, i of them interior, there exist two pointed pseudo-triangulations $PT_1(S) = (S, E_1)$ and $PT_2(S) = (S, E_2)$ such that $PT_1(S)$ and $PT_2(S)$ are maximally disjoint compatible, that is, $|E_1 \backslash E_2| = |E_2 \backslash E_1| = i$.*

Proof. We prove the statement by construction of the two pointed pseudo-triangulations $PT_1(S)$ and $PT_2(S)$ by iteratively adding edges. We color edges of $PT_1(S)$ blue, those of $PT_2(S)$ red, and edges that are in both, $PT_1(S)$ and $PT_2(S)$, black; see Figures 4.6(a)-4.8. For the sake of brevity we sometimes refer to edges belonging to $PT_1(S)$, $PT_2(S)$, or both, by their color.

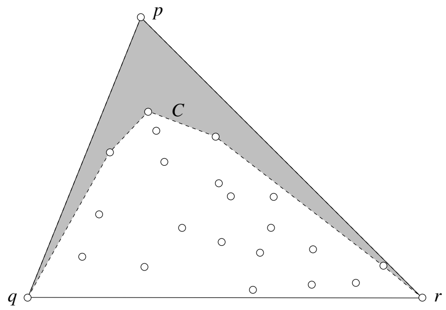
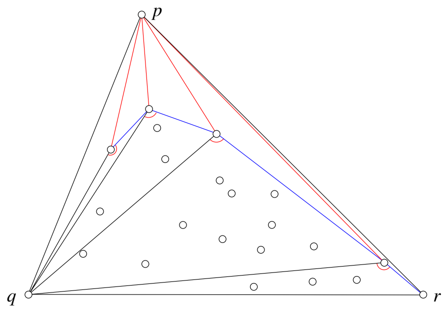


Figure 4.5: Empty pseudo-triangle (shaded) formed by p and the chain C (dashed).

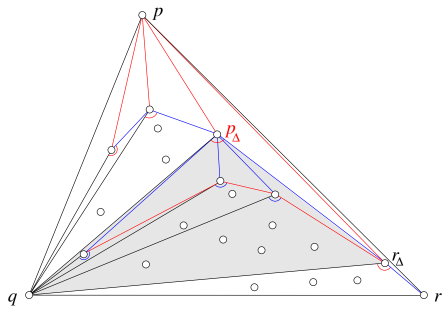
Assume that S has a triangular convex hull pqr, and $i > 0$ interior points[1]. We start by adding the boundary of the convex hull to both, $PT_1(S)$ and $PT_2(S)$. Now choose an extreme point $p \in S$, and consider the boundary of the convex hull of $S \backslash \{p\}$, consisting of the edge qr and a concave chain C connecting q and r in the interior of CH(S). The chain C together with p forms a pseudo-triangle with corners pqr that does not contain any point of S in its interior; see Figure 4.5.

[1]If S has more than three extreme points, we first triangulate these extreme points arbitrarily (adding the edges to both, $PT_1(S)$ and $PT_2(S)$), and then process each resulting non-empty triangle independently.

For our construction, we add black edges from q to all points of C except r (to both $PT_1(S)$ and $PT_2(S)$), red edges from p to all points of C except q and r (to $PT_2(S)$), and all edges on the chain C (except the one incident to q) with color blue (to $PT_1(S)$). See Figure 4.6(a) for the set of edges added in this step.



(a) First step of adding edges.



(b) Iterative construction step.

Figure 4.6: Constructing two compatible pointed pseudo-triangulations.

The union of $PT_1(S)$ and $PT_2(S)$ splits CH(S) into a set of triangles. Each triangle that contains further points of S in its interior has q as one corner, two of the triangle edges are black, and the third is blue. In each such triangle, one of the corners adjacent to the blue edge becomes pointed in the whole graph when removing this blue edge. We mark this corner red (indicated by a small arc in the figures).

We consider these triangles iteratively, in a similar way as the starting triangle. Throughout the process we keep the following invariants for each interior triangle Δ: q is a corner of Δ and both triangle edges incident to q are black. One of the other two corners is marked with color $c'_\Delta \in \{\text{red, blue}\}$ (red after the first step). Let this colored corner be p_Δ and the remaining corner r_Δ. The triangle edge $p_\Delta r_\Delta$ has the color $c_\Delta \in \{\text{red, blue}\} \setminus \{c'_\Delta\}$ (blue after the first step).

Let $S_\Delta \subset S$ be the set of points inside Δ plus the corners of Δ. Like in the first step, we consider the chain C_Δ on the convex hull of $S_\Delta \setminus \{p_\Delta\}$. We add black edges from q to all points of C_Δ except r_Δ, edges with color c_Δ from p_Δ to all points of C_Δ except q and r_Δ, and all edges on the chain C_Δ (except the one incident to q) with color c'_Δ. Inside every nonempty triangle, we mark the corner that becomes pointed in $PT_1(S) \cup PT_2(S)$, when removing the c'_Δ-colored triangle edge, with color c_Δ. See Figure 4.6(b) for one iterative step (inside the shaded triangle) and Figure 4.7 for the completed construction.

The construction results in a red-blue-black colored triangulation of S, and thus $PT_1(S)$ and $PT_2(S)$ are compatible.

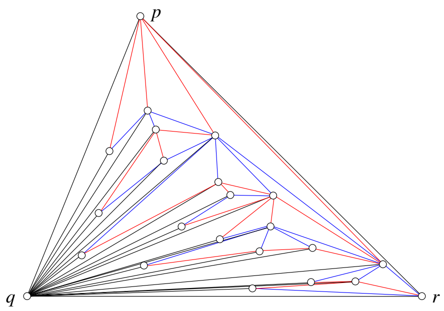


Figure 4.7: Completed construction.

Concerning the pointedness of the interior points of S, note that in each construction step, all points on the chain C_Δ (except q and r_Δ) become pointed towards p_Δ with respect to color c'_Δ. This pointedness cannot be destroyed in further recursive steps because there is nothing left to be processed between C_Δ and p_Δ (see again Figure 4.6(b)). With respect to color c_Δ, each point on C_Δ (except q and r_Δ) is pointed towards one of its adjacent c'_Δ-colored edges on C_Δ. If the according triangle is not empty then the point is marked with c_Δ for the next iteration and thus cannot get any additional incident c_Δ-colored edges in the relevant area. Thus all points of S are pointed in both $PT_1(S)$ and $PT_2(S)$.

Finally, every interior point of S appears on exactly one chain C_Δ as non-endpoint. For every non-endpoint of a chain C_Δ, exactly one red, one blue, and one black edge is added, which, including the initial three black edges on the convex hull of S, adds up to $|E_1| = |E_2| = 2n-3$. Together with pointedness and planarity, this proves that $PT_1(S)$ and $PT_2(S)$ are pointed pseudo-triangulations. Figure 4.8 shows the two resulting maximally disjoint compatible pointed pseudo-triangulations. $\square$

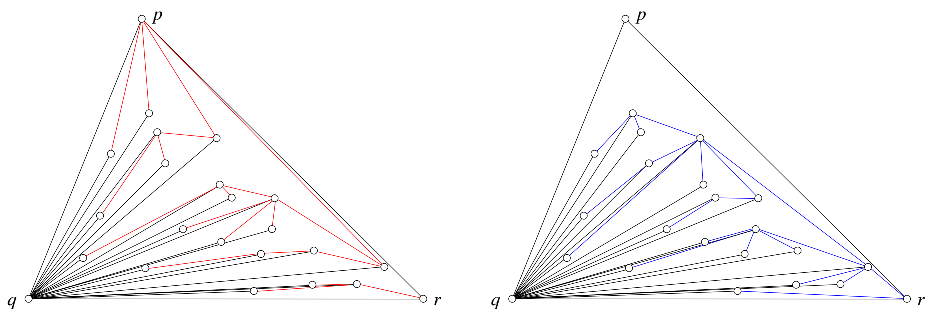


Figure 4.8: The resulting two maximally disjoint compatible pointed pseudo-triangulations.

4.2.2 Compatible pointed pseudo-triangulations for a given pointed pseudo-triangulation

We now consider a more restrictive setting. Given a point set S and a pointed pseudo-triangulation $PT(S) = (S, E)$, can we find a pointed pseudo-triangulation $PT'(S) = (S, E')$, such that $PT(S)$ and $PT'(S)$ are compatible and differ by at least k edges for some $k \geq 1$?

For point sets in convex position, this question is trivial. The answer is negative and positive at the same time, depending on the point of view. Every pointed pseudo-triangulation $PT(S)$ on top of a convex set is a triangulation and thus a maximal plane graph. It is not compatible to any pointed pseudo-triangulation $PT'(S) \neq PT(S)$. But we know from Proposition 4.8 that the maximum number of differing edges we can hope for is the number of interior points, which is zero in this case. Thus, by just taking $PT(S)$ itself, we have a pointed pseudo-triangulation that is maximally disjoint compatible with $PT(S)$. In the following we will only consider the non-trivial case, i.e., point sets S with $i > 0$ interior points.

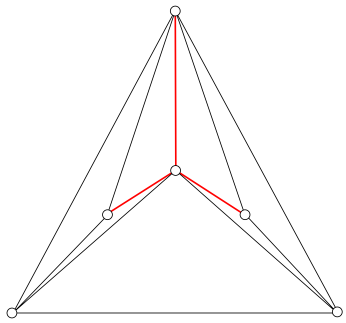


Figure 4.9: A point set with a pointed pseudo-triangulation for which there is no maximally disjoint compatible pointed pseudo-triangulation.

Actually, there are point sets with interior points and pointed pseudo-triangulations such that there doesn't exist any pointed pseudo-triangulation which is maximally disjoint compatible with the given one. In the example in Figure 4.9, the underlying point set of the black pointed pseudo-triangulation has three interior points and three compatible edges (drawn in red). As all red edges together would make their common point non-pointed, at most two of them can be part of a pointed pseudo-triangulation. So we're left with the question of how many differing edges we can guarantee, for example depending on the number of interior points of the given set.

Before giving a general answer to this question, let us introduce the concept of flips [51]. A *flip* in a pointed pseudo-triangulation $PT(S) = (S, E)$ is the exchange of an edge $e \in E$ by an edge $e' \in (S \times S) \backslash E$ such that the resulting graph $PT'(S) = (S, E\backslash\{e\} \cup \{e'\})$ is again a pointed pseudo-triangulation. The flip is called *compatible flip* if e' is compatible with $PT(S)$.

In a pointed pseudo-triangulation, every edge e that is not a convex hull edge is flippable, and there is a unique edge e' to which e can be flipped [132] (see Figure 4.10). For finding e', consider the two pseudo-triangles Δ_1 and Δ_2 that are incident to e, and the corners c_1 of Δ_1 and c_2 of Δ_2 that are opposite to (the side-chains of) e. The *geodesic* from c_1 to c_2 is the shortest path from c_1 to c_2 which lies inside and possibly on the boundary of $\Delta_1 \cup \Delta_2\backslash\{e\}$, and does not cross any edge of $PT(S)$ except possibly e. The only edge of this geodesic which is not in E is the flip-destination of e. If e' does not cross e, then the flip is compatible.

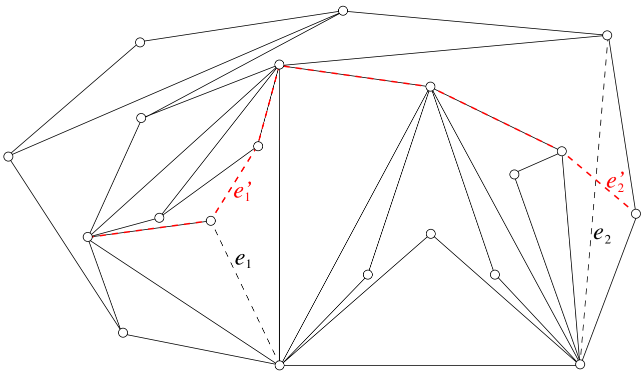
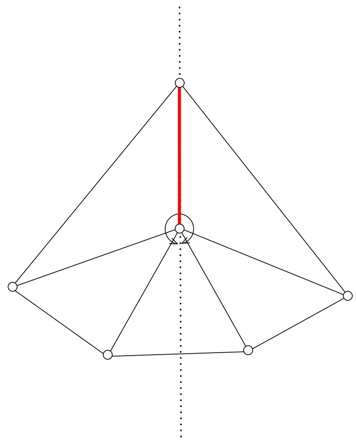


Figure 4.10: Flipping edges in a pointed pseudo-triangulation: Replacing the dashed black edges by the additional edges on the dashed red geodesics.

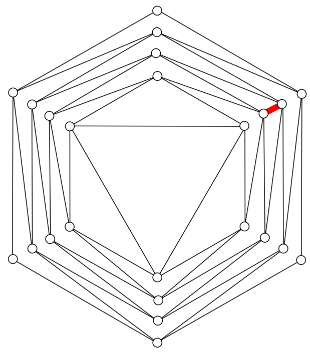
Obviously, if there are k "independent" compatible flips in a pointed pseudo-triangulation $PT(S)$ (compatible flips that can be processed simultaneously such that the result is a pointed pseudo-triangulation again), then $PT(S)$ has a compatible pointed pseudo-triangulation with at least k differing edges (see again Figure 4.10).

Two questions naturally arise. Given a point set S and a pointed pseudo-triangulation on top of S, can we always perform a compatible flip? And if this is true, is the flip graph of pointed pseudo-triangulations (w.r.t. compatible flips) connected, that is, can we flip any pointed pseudo-triangulation to any other by only using compatible flips? Note that connectivity of the flip graph is known in the unrestricted case [132].

Unfortunately, for compatible flips the answer to both questions is negative. There exist point sets (with interior points) and pointed pseudo-triangulations for them that do not admit any compatible flip. Thus the according flip graph is not connected (as it has isolated vertices). Figure 4.11(a) shows a small point set and a pointed pseudo-triangulation (black) which does not admit any compatible flip. The only compatible edge in this example is drawn in red. As the interior point has two incident edges on each side of the dotted line, adding the red edge and removing one black edge always leaves this point non-pointed and thus the resulting graph is not a pointed pseudo-triangulation.



(a)



(b)

Figure 4.11: Point sets and pointed pseudo-triangulations that do not admit any compatible flip.

Figure 4.11(b) contains a more complex example, showing that the basic concept of the small example can be used to build point sets with many interior points and according pointed pseudo-triangulations such that not a single edge can be compatibly flipped. Again, the black edges form a pointed pseudo-triangulation. The red edge, as well as all other compatible edges, cannot be part of any compatible flip.

Actually, the example in Figure 4.11(b) can be modified to contain an arbitrary number of (at least six) points. In the interior, it contains the graph from Figure 4.11(a) as a sub graph (the central triangle plus two of the adjacent triangles plus the pseudo-triangle adjacent to these triangles). Starting from this sub graph, we can iteratively make the example larger by adding first the last missing triangle, and then the desired number of pseudo-triangles (with the only restriction that one layer of pseudo-triangles has to be completed before the next one is started).

Corollary 4.11. *For every $n \geq 6$ and $i \leq \max\{1, n-6\}$, there exists a point set S with n points, i of them interior, such that there exists a pointed pseudo-triangulation $PT(S)$ that does not admit any compatible flip.*

Proof. Consider a point set S' and a pointed pseudo-triangulation $PT'(S')$ with i interior and $\min\{6, n-i\}$ extreme points according to the construction indicated in Figure 4.11(b). The additional extreme points for S can be added close to (and outside of) some convex hull edges of S'. Triangulating these parts outside $CH(S')$ arbitrarily results in a pointed pseudo-triangulation $PT(S)$ with the desired property. $\square$

In the graphs from Figure 4.11, every interior vertex p has degree four. Further, for each p and each compatible edge $e = pq$, the supporting line of e partitions the edges incident to p into two groups such that each group contains two edges. This provides the argument for why none of the compatible edges can be involved in a compatible flip.

Using the contrary argumentation, we can derive sufficient conditions for compatibly flippable edges.

Proposition 4.12. *Given a pointed pseudo-triangulation $PT(S) = (S, E)$ and a compatible edge $e = pq \in (S \times S) \backslash E$ that destroys the pointedness of p but not of q, consider the two non-empty subsets into which the set of incident edges of p is split by the supporting line of e. If one of these subsets contains only one edge e', then e' can be flipped to e in a compatible way.*

Proof. Consider the two pseudo-triangles Δ_1 and Δ_2 that are incident to e' and the corners c_1 of Δ_1 and c_2 of Δ_2 that are opposite to the side-chains on which e' lies. As the edge e does not destroy the pointedness of q, and as e' is the only edge incident to p on one side of the supporting line of e, e lies on the geodesic from c_1 to c_2 in $\Delta_1 \cup \Delta_2 \backslash e'$. Thus e' is flippable to e in a compatible way. $\square$

Corollary 4.13. *A necessary condition for a pointed pseudo-triangulation $PT(S) = (S, E)$ to not admit a compatible flip is that every interior point p of S has vertex degree $d(p) \geq 4$ in $PT(S)$. Every point with vertex degree $d(p) \leq 3$ has at least one incident edge that can be compatibly flipped.*

Proof. We have seen in the arguments for Corollary 4.11 that a vertex with degree four might not admit any compatible flip. In the other direction, if $PT(S)$ contains vertices with degree less than four, consider such a vertex p and the pseudo-triangle Δ in which p is pointed. The geodesic from p to its opposite corner in Δ spans a compatible edge $e = pq$ that destroys the pointedness of p and does not destroy the pointedness of q. The line through this edge e has at least one side with only one edge e' incident to p, and thus e' can be compatibly flipped to e. $\square$

Let us come back to the question of compatible pointed pseudo-triangulations. Consider again the example in Figure 4.11(a). Any pointed pseudo-triangulation on top of this point set that contains the red edge must not contain two of the black edges in order to keep the interior vertex pointed. Thus it has to also contain another non-black edge. But any non-black edge on top of S, except for the red one, is incompatible to the black pointed pseudo-triangulation. Accordingly, in the example in Figure 4.11(b), none of the compatible edges can be part of a pointed pseudo-triangulation compatible to the black one.

Corollary 4.14. *For every $n \geq 6$ and $i \leq \max\{1, n-6\}$, there exists a point set S with n points, i of them interior, such that there exists a pointed pseudo-triangulation $PT(S) = (S, E)$ that is incompatible to any pointed pseudo-triangulation $PT'(S) = (S, E')$ with $E' \neq E$.*

4.2.3 Discussion

We have shown that for every point set there exist pairs of maximally disjoint pointed pseudo-triangulations. On the other hand, there exist point sets (also with many interior points) and pseudo-triangulations without any non-identical compatible pointed pseudo-triangulation. We also introduced the concept of compatible flips in pointed pseudo-triangulations. We have shown that the flip graph for this constrained version of flips in pseudo-triangulations can be disconnected by proving that it even might contain singletons (also for point sets with many interior points).

Concerning compatibility, similar questions have been considered for example for spanning trees [87] and geometric matchings [11, 105]

Another related question are flip graphs of pointed pseudo-triangulations with other constraints than compatibility. For example, the flip graph of (pseudo-)triangulations of convex sets with bounded vertex degree is connected if and only if the degree bound is at least six; see [29] and Chapter 3 of the thesis [95]. The according question for general point sets is still unknown. Similarly, the question of flip graph connectivity for pointed pseudo-triangulations with bounded face degree (for the interior faces) is still an open problem; see again [95], Section 3.5.

Chapter 5

Pointed Drawings of Planar Graphs

Triangulations are maximal plane graphs. Consequently, the question of the previous section would not make sense for triangulations because two triangulations on top of a given point set have to be identical in order to be compatible. In contrast to triangulations, the pointedness property of pointed pseudo-triangulations makes them minimal among all pseudo-triangulations. In this chapter we consider a generalized version of pointedness for not necessarily straight-line drawings of graphs.

Questions concerning (certain types of) geometric graphs with constraints on the occurring incident angles have been considered for triangulations [47, 57, 74, 75], spanning cycles [6, 37, 84], spanning paths [43, 84] and other types of geometric graphs. The question of finding specific straight-line graphs that have incident angles of at least a certain size at every vertex has been investigated in [23].

Generalizing this approach further, we relax the straight-line condition by allowing several different types of simple curves as edges. We consider the questions of (1) redrawing a given plane straight-line graph such that the vertex positions remain the same and all vertices become pointed; and (2) embedding a given plane straight-line graph such that for every vertex v, all edges emanate from v in the (nearly) same direction. We show that both questions can be answered to the positive when using Bézier-curves, biarcs, or polygonal chains of length two as edges. When restricting the edges to circular arcs this is in general not possible. For the latter setting, we provide a method for constructing a pointed embedding that is based on the classical grid embedding algorithm of de Fraysseix, Pach and Pollack [60].

Due to this different setting (more or less abstract graphs on the one hand, and drawings with not necessarily straight-line edges on the other hand), the notation we use here is slightly different from the previous chapters.

The results of this chapter have been presented before [34] and will be published in [35].

5.1 Introduction

Throughout this chapter, let $G = (V, E)$ be a simple planar graph without loops, with finite vertex set V and thus a finite set of edges E. We use the natural understanding of a drawing of a graph. Vertices are represented as points in the plane and edges as continuous and (at least piecewise) differentiable curves connecting the points of adjacent vertices. A drawing is called *non-crossing* or *plane*, if the drawn edges do not intersect in their interior. If we consider only topological properties, that is, the order of the edges and consequently of the faces, we refer to this as *combinatorial embedding*. Given a (combinatorial) embedding of a graph G, the faces of G are defined as usual. By an *(abstract) plane graph*, we mean an equivalence class of plane drawings under homotopic deformations of the plane. For connected graphs, this amounts to a combinatorial embedding together with a designation of the outer face.

For a drawing $\mathcal{F}(G)$ of G we denote the placement of a vertex $v \in V$ by $\mathcal{F}(v)$, and the drawing of an edge $e \in E$ by $\mathcal{F}(e)$. Note that we consider embedded edges to be open, i.e., they do not contain their endpoints. For simplicity, and as there is no risk of confusion, in the figures we will denote embedded vertices just by v instead of $\mathcal{F}(v)$.

The *tangent* of an edge $\mathcal{F}(e)$ at a vertex $\mathcal{F}(v)$ is the limit of the tangents to $\mathcal{F}(e)$ when approaching $\mathcal{F}(v)$ along $\mathcal{F}(e)$. The *tangent ray* of $\mathcal{F}(e)$ at $\mathcal{F}(v)$ is the open ray along the tangent to $\mathcal{F}(e)$ at $\mathcal{F}(v)$ from $\mathcal{F}(v)$ towards $\mathcal{F}(e)$. A drawing gives us a cyclic order of incident edges around each vertex. The angle between two consecutive edges incident to a vertex $\mathcal{F}(v)$ is defined as the angle between the corresponding tangent rays at $\mathcal{F}(v)$ that does not contain the tangent ray of any other incident edge. We say that this angle is incident to $\mathcal{F}(v)$ (and vice versa). In the case of a degree two vertex there are two such angles between the two incident edges. If a vertex has degree at most one, we say that it is incident to one angle (having value 2π).

Similar to Section 1.2, we call a vertex in a drawing $\mathcal{F}(G)$ *pointed* if it is incident to an angle greater than π (see Figure 5.1). We say that a vertex is pointed to a face if its large angle lies in this face. If all vertices in a drawing are pointed we call the drawing pointed.

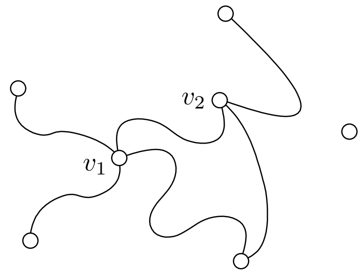


Figure 5.1: Drawing with a non-pointed vertex v_1 and a pointed vertex v_2.

For the special case of straight-line drawings this definition is identical to the classic definition of pointedness, a term which stems from the field of pseudo-triangulations. See again Section 1.2 for definition and background of pseudo-triangulations.

The graphs which can be drawn as pseudo-triangulations are well-characterized: A graph is called *generically rigid*, if its straight-line realization on a generic point set induces a rigid framework (edges represent fixed length rods and vertices represent joints). In two dimensions, there exists an easy combinatorial characterization of generically rigid graphs that become non-rigid after removing an arbitrary edge [112]. These graphs are called *Laman graphs*. Due to Streinu [138], a graph of a pointed pseudo-triangulation is a Laman graph. Conversely, as observed by Haas et al. [94], every plane Laman graph can be realized as a pointed pseudo-triangulation. As a consequence, subsets of plane Laman graphs are exactly the graphs that admit a pointed non-crossing straight-line drawing. An operation that preserves the Laman property is the so-called *Henneberg operation* of type 1: Adding a new vertex of degree 2 to an existing Laman graph will create another Laman graph. A simple example of a planar graph that has no pointed straight-line drawing without crossings is the complete graph with four vertices.

We consider various incarnations of the problem how to draw a plane graph pointed, using different kinds of edge shapes. With arbitrary smooth curves or polygonal chains, the task of constructing a pointed drawing of a given plane graph is trivial. As natural, but still quite simple edge shapes, we study circular arcs, tangent continuous biarcs, and quadratic Bézier curves. Let us briefly review the definition and basic properties of these curves. A *tangent continuous biarc* consists of two circular arcs that are joined in a way that they form a C^1 continuous curve. A *quadratic Bézier curve* b spanned by three points p_1, p_m and p_2 is defined by the equation

$$b(t) = (1-t)^2 p_1 + 2t(1-t)p_m + t^2 p_2, \quad t \in [0,1].$$

It lies completely inside the triangle $p_1p_mp_2$ (which is also called *control polygon* of b), has p_1 and p_2 as endpoints, and is tangent to p_1p_m at p_1 and to p_2p_m at p_2. The class of quadratic Bézier curves is the same as the class of parabolic arcs.

We also consider the "extent of pointedness". For example, can we guarantee a free angular space around each vertex bigger than a given fixed angle larger than π? For this stronger pointedness criterion we define the term ε-pointedness.

Let $\varepsilon > 0$ be a real number. A vertex in the drawing $\mathcal{F}(G)$ is called *ε-pointed* if it is incident to an angle greater than $2\pi - \varepsilon$. We call a drawing ε-pointed if every vertex is ε-pointed. In other words, all edges incident to an ε-pointed vertex emanate in a sector of angle ε.

Further, we propose a stronger version of the pointed drawing problem: Given a plane straight-line drawing $\mathcal{F}_s(G)$, can we redraw it as a plane pointed drawing with a certain family of edge shapes without moving the vertices? We call a drawing with this property a *pointed redrawing.* The motivation of a pointed redrawing is clear: we can benefit from the given drawing and keep its advantages (e.g., all vertices are placed on an integer grid or fulfill other optimality criteria).

A more general redrawing problem would start from a plane embedding with not necessarily straight edges. We have not considered this question.

Results

In Section 5.2, we consider the problem of *pointed redrawings.* We show that every plane straight-line drawing $\mathcal{F}_s(G)$ can be redrawn pointed and plane with Bézier curves as well as with tangent continuous biarcs. We show that this is not always possible with circular arcs as edges.

Section 5.3 then deals with *pointed drawings* of (abstract) plane graphs. We prove that every plane graph can be drawn ε-pointed with Bézier curves, for arbitrary small $\varepsilon > 0$. We show that by using biarcs as edges, every plane graph can be drawn such that for all vertices v, all incident edges share a common tangent ray at v. This is maybe one of the most beautiful results in this chapter from an aesthetic point of view. We further prove that every plane graph can be drawn pointed and plane with circular arcs as edges. For pointed drawings with biarcs, Bézier curves, or polygonal chains of length two, we give a tight bound on the number of edges that have to be drawn as non-straight curves (Theorem 5.14).

We summarize the results presented in this chapter in Table 5.1. All obtained drawings can be constructed algorithmically, with the exception of the method described in the proof of Theorem 5.6, which needs a disk packing of the plane graph in a preprocessing step.

edge shape	problem instance	obtained result
circular arcs	pointed drawing	possible, Theorem 5.11
	pointed redrawing	not possible, Theorem 5.4
tangent continuous biarcs	ε-pointed drawing	possible, Theorem 5.6
	pointed redrawing	possible, Theorem 5.3
quadratic Bézier curves	ε-pointed drawing	possible, Theorem 5.5
	pointed redrawing	possible, Theorem 5.1

Table 5.1: Overview of obtained results.

Related work and applications

Traditionally, graph drawing is mainly concerned with using the simplest class of curves for the edges: straight-line segments. According to Fáry's theorem [83], every (simple) plane graph has a plane straight-line drawing in the Euclidean plane. There is a vast literature dealing with the question of efficiently finding plane straight-line drawings that fulfill certain (optimality) criteria (see [45, 119] for an overview). Improving work of de Fraysseix, Pach and Pollack [60], Schnyder [135] proved that every plane graph with n vertices has a plane straight-line drawing where the vertices lie on a grid of size $(n-2) \times (n-2)$. The famous Koebe-Andreev-Thurston circle packing theorem [36, 111] states that every plane graph can be embedded with straight-line edges in a way such that its vertices correspond to interior disjoint disks, which touch if and only if the corresponding vertices are connected with an edge, see also [123, 55]. We will use both the procedure of de Fraysseix, Pach and Pollack [60] and circle packings as building blocks in our algorithm.

If we relax the condition that the given plane graph has to be simple, Fáry's theorem does not hold, for the simple reason that straight-line drawings are not well defined for loops, and multiple edges between two vertices are excluded. However, with more complex edge shapes, one can ask for crossing-free drawing of planar multigraphs with loops. The most natural approach is to allow circular arcs. Drawing multiple edges as circular arcs is no problem, as an edge in a straight-line drawing can be perturbed to any number of close-by circular arcs. Loops, however, require more space. The only circular arc between a vertex and itself is a full circle through this vertex. Thus, the vertex has to be incident to an angle of at least π, which then is sufficient for any number of nested loops at this vertex. This means that the simple graph containing only the non-loop edges must be drawn as a pointed graph. In Section 5.3.2, we show that such a drawing exists (Theorem 5.11) and as a consequence, we obtain a plane drawing with circular arcs for every planar multigraph (Corollary 5.12). Moreover, an $O(n) \times O(n^2)$ grid is sufficient to embed the vertices of these drawings.

Another potential application for constructing pointed drawings of graphs comes from drawing vertex labels. If the edges incident to a vertex point in all directions, it might be hard to place a label close to its vertex. Thus it is good to have some angular space without incident edges.

These results were presented at the 2007 Canadian Conference on Computational Geometry in Ottawa [34]. One of the results which we announced in this proceedings version cannot be maintained in full generality, see Theorem 5.14 below.

5.2 Pointed redrawings

We start with the redrawing problem. Throughout this section we consider a plane straight-line drawing as input of our problem instance. Let this drawing be $\mathcal{F}_s(G)$.

Theorem 5.1. *For every plane straight-line drawing $\mathcal{F}_s(G)$ of a simple planar graph G there exists a pointed plane redrawing $\mathcal{F}_q(G)$ with quadratic Bézier curves as edges: $\mathcal{F}_q(v) = \mathcal{F}_s(v)$ for all $v \in V$, and for every vertex $v \in V$, the cyclic order of the edges incident to v in $\mathcal{F}_s(G)$ is the same as in $\mathcal{F}_q(G)$.*

Proof. Without loss of generality assume that in $\mathcal{F}_s(G)$ no two vertices have identical x-coordinates or y-coordinates. Assume further, that the vertices are sorted by y-coordinates in increasing order.

We construct $\mathcal{F}_q(G)$ by iteratively replacing the straight-line edges of $\mathcal{F}_s(G)$ with quadratic Bézier curves. We first replace the edges incident to the bottom-most vertex v_1, then the edges incident to v_2, and so on. During the construction we maintain the following two invariants:

(1) For every vertex v_i, the tangent rays of all already redrawn edges lie in the open halfplane H_i^- below the horizontal line through $\mathcal{F}_q(v_i)$.

(2) The intermediate drawing is plane.

When replacing the edges incident to a vertex v_i, all edges incident to a vertex v_j, with $j < i$, have already been redrawn by our algorithm (as all vertices below v_i have already been processed). Let $E_i = e_1, \ldots, e_k$ be the edges which have not been replaced yet, sorted by absolute slope, such that e_1 has the smallest absolute slope. We redraw these edges in increasing order.

Let $e = v_i v_j,\ j > i$ be the current edge we want to process (see Figure 5.2). Due to invariant (1) and the processing order of the edges incident to v_i we can choose a point p_m in H_i^- such that the triangle $t = \mathcal{F}_q(v_i) p_m \mathcal{F}_q(v_j)$ does not contain any vertex or part of an edge of the current drawing in its interior. By convention we place p_m to the right of $\mathcal{F}_q(v_i)$ if e has positive slope, otherwise to the left. We use the triangle t as a control polygon for a quadratic Bézier curve b with endpoints $\mathcal{F}_q(v_i)$ and $\mathcal{F}_q(v_j)$, which we take as replacement for the straight-line edge $\mathcal{F}_s(e)$. Note that b is tangent to $\mathcal{F}_q(v_i) p_m$ at $\mathcal{F}_q(v_i)$, and thus invariant (1) still holds for v_i. As b lies completely inside t, and $t \backslash \{\mathcal{F}_q(v_j)\}$ lies completely inside H_j^-, invariant (1) for v_j and invariant (2) remain fulfilled as well.

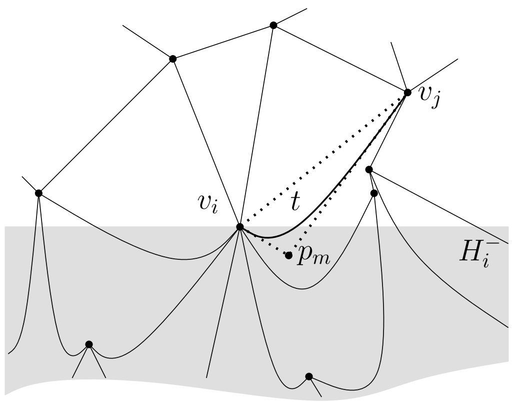


Figure 5.2: Constructing a plane pointed drawing where the edges are quadratic Bézier curves (intermediate step).

Having redrawn all edges in this way, we obtain a drawing whose pointedness follows directly from invariant (1), and that is plane follows from invariant (2).

The easiest way to establish that the cyclic order of edges is unchanged is to augment the drawing $\mathcal{F}_s(G)$ to a triangulation by adding edges and deleting the corresponding arcs in the final drawing. For a triangulation a with fixed outer face the order of the edges around a vertex is unique up to a global reflection [146]. Hence, this order has to be preserved in $\mathcal{F}_q(G)$. $\square$

The technique used in the proof of Theorem 5.1 can be modified to show a similar statement for (tangent continuous) biarcs due to the following observation.

Lemma 5.2. *Consider a triangle spanned by three points p_1, p_m and p_2. There exists a tangent continuous biarc connecting p_1 with p_2 that lies inside the triangle. Furthermore, the biarc is tangent to p_1p_m at one end and tangent to p_2p_m at the other end.*

Proof. Assume that the segment p_1p_m is shorter than p_2p_m. The first arc starts in p_1 with tangent direction p_1p_m and touches the line p_2p_m in some point $\tilde{p}$. This point on the segment p_2p_m has the property that the length of $p_m\tilde{p}$ is equal to the length of p_1p_m (see Figure 5.3). The center of the arc is found by intersecting the line l_1 perpendicular to p_1p_m through p_1 with the line l_2 perpendicular to p_2p_m through $\tilde{p}$. The second part of the biarc is given by the straight line segment $\tilde{p}p_2$ (a degenerate circular arc). $\square$

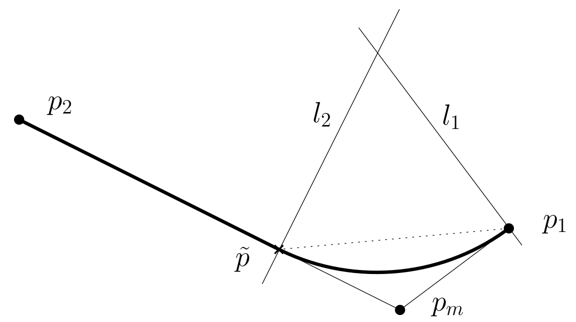


Figure 5.3: Drawing an edge as a tangent continuous biarc in a triangle.

Theorem 5.3. *For every plane straight-line drawing $\mathcal{F}_s(G)$ of a plane graph G there exists a pointed plane redrawing $\mathcal{F}_b(G)$ with tangent continuous biarcs as edges: $\mathcal{F}_b(v) = \mathcal{F}_s(v)$ for all $v \in V$, and for every vertex $v \in V$, the cyclic order of the edges incident to v in $\mathcal{F}_s(G)$ is the same as in $\mathcal{F}_b(G)$.*

Proof. We re-use the construction from the proof of Theorem 5.1. Whenever we have chosen an appropriate empty triangle for an edge replacement, we place a tangent continuous biarc in it (as described in Lemma 5.2). $\square$

We conclude this section with a negative result on pointed redrawings.

Theorem 5.4. *There is a planar graph $G = (V, E)$ with a plane straight-line drawing $\mathcal{F}_s(G)$, for which there are no pointed plane drawings $\mathcal{F}_c(G)$ with circular arcs as edges such that $\mathcal{F}_c(v) = \mathcal{F}_s(v)$ for all $v \in V$.*

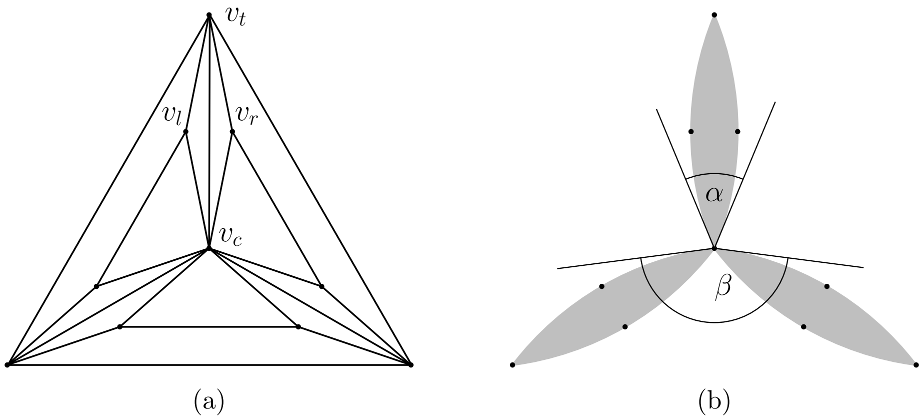


Figure 5.4: Example of a straight-line drawing that can not be redrawn pointed with circular arcs.

Proof. Consider the graph G shown in Figure 5.4(a). Vertex v_c is placed at the origin, vertex v_t at $(0, 2)$, vertex v_l at $(-0.2, 1)$, and vertex v_r at $(0.2, 1)$. The positions of the remaining vertices are obtained by rotating these vertices by ± 120 degrees. Since G is 3-connected and planar, its combinatorial embedding is fixed for any non-crossing drawing [146]. This implies that in any such drawing the edge between v_c and v_t has to pass through the narrow passage between v_l and v_r. Since we are restricted to circular arcs, the arc connecting v_t and v_c has to lie in the shaded region depicted in Figure 5.4(b). This region is the intersection of the disk touching v_t, v_l, v_c with the disk touching v_t, v_r, v_c. The region lies inside a wedge of angle $\alpha = 45.3$ degrees. Thus, the tangents of two arcs from v_c to the convex hull are separated by an angle of at most $\beta = 165.3$ degrees. But in order to make the vertex v_c pointed, one of these angles would have to be larger than π. $\square$

Larger examples can be constructed easily. As long as a straight-line drawing similar to Figure 5.4(a) is contained inside another drawing, a pointed redrawing with circular arcs is impossible. Moreover, with a construction similar to the one shown in Figure 5.4(a), but with many "spokes" (instead of just three), one can force the largest possible angle free of incident edges at the central vertex to be arbitrary small.

5.3 Pointed drawings

5.3.1 Pointed drawings with Bézier curves and biarcs

In the previous section the placement of the points was determined by a given plane straight-line drawing. If the location of the vertices can be chosen arbitrarily, we get the following easy consequence of Theorem 5.1.

Theorem 5.5. *For any $\varepsilon > 0$ and any plane graph G, there exists a plane drawing $\mathcal{F}_q(G)$ with quadratic Bézier curves where all vertices are ε-pointed.*

Proof. Consider an arbitrary straight-line drawing $\mathcal{F}_s(G)$. In the proof of Theorem 5.1 we showed a construction for a pointed drawing $\mathcal{F}'_q(G)$, in which for every vertex v and for every edge e incident to v, the tangent ray of $\mathcal{F}'_q(e)$ at $\mathcal{F}'_q(v)$ lies below the horizontal line through $\mathcal{F}'_q(v)$. By compressing the x-axis (i.e., scaling by a factor less than 1), the large angle at every vertex in the resulting drawing increases towards 2π. This modification produces no crossings. Moreover, every quadratic Bézier curve is transformed to a quadratic Bézier curve (with respect to the compressed control polygon). Thus, sufficiently compressing $\mathcal{F}'_q(G)$ results in the desired ε-pointed drawing $\mathcal{F}_q(G)$. □

By similar arguments, it is possible to obtain an ε-pointed drawing $\mathcal{F}_b(G)$ with biarcs. In this case the argumentation is more involved, because compressing a biarc in one direction does not result in another biarc. However, we can modify the proof of Theorem 5.1 in the following way: Recall that we used as invariant (1) that for every vertex v_i, the tangent rays of all already redrawn edges lie in the open halfplane H_i^- below the horizontal line through $\mathcal{F}_s(v_i)$. To obtain a stronger result, we consider vertical double-wedges centered at the embedded vertices with wedge angle ε, and redefine the region H_i^- to be the wedge below the horizontal line through the embedded vertex. We compress the x-axis until all edges of the compressed straight-line drawing lie strictly within the double-wedges of their endpoints, and apply the previous approach with the changed invariant to this compressed drawing.

A disadvantage of this construction is that the biarcs tend to consist of a circular arc with small radius and a circular arc with infinite radius. Thus, these drawings are not aesthetically pleasing. For this reason, we present a completely different approach, which also fulfills an even stronger criterion of pointedness. This criterion, namely that all arcs incident to a vertex share a common tangent at this vertex, implies ε-pointedness for any $\varepsilon > 0$.

Theorem 5.6. *Every plane graph $G = (V, E)$ has a plane pointed drawing $\mathcal{F}_b(G)$ with tangent-continuous biarcs as edges such that $\mathcal{F}_b(G)$ is pointed.*

Moreover, for every vertex v, all edges incident to v share a common tangent at $\mathcal{F}_b(v)$. The directions of these tangents can be independently specified for each vertex.

We emphasize that the locations of the vertices cannot be specified in this theorem.

Proof. According to the Koebe-Andreev-Thurston circle packing theorem [36, 111], every plane graph admits a disk packing, where each disk belongs to a vertex (which is the center of the disk), and two disks touch if and only if the corresponding vertices share an edge.

We start with such a disk packing of the graph G (see [55, 117, 56] for algorithmic aspects of such packings). To get our drawing $\mathcal{F}_b(V)$ of the vertices, we place every vertex v_i arbitrarily on the boundary of its disk D_i, avoiding touching points of the disks. The edges incident to v_i will emanate from $\mathcal{F}_b(v_i)$ perpendicular to D_i into the interior of D_i. Thus, we can obtain desired tangent direction for the edges by placing v_i on D_i appropriately. We can avoid the coincidence of v_i with a touching point by rotating the whole disk packing. (There are only finitely many rotations that have to be avoided.)

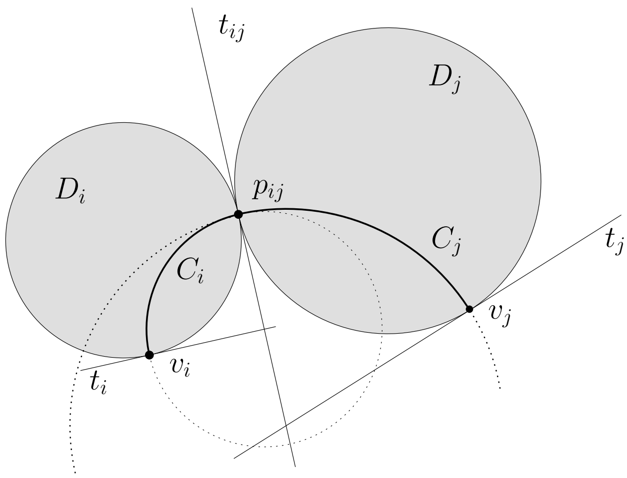


Figure 5.5: Construction of a tangent-continuous biarc from two touching disks D_i, D_j.

Now consider an edge $v_iv_j \in E$. For the embedded vertex $\mathcal{F}_b(v_i)$ let t_i be the tangent through $\mathcal{F}_b(v_i)$ to its disk D_i. Furthermore, let p_{ij} be the touching point of the two adjacent disks D_i and D_j and let t_{ij} be the tangent to D_i and D_j through p_{ij} (see Figure 5.5). We draw a circular arc C_i from $\mathcal{F}_b(v_i)$ to p_{ij} inside D_i, the center of C_i being the crossing of t_i and

t_{ij}. Similarly, we draw an arc C_j from $\mathcal{F}_b(v_j)$ to p_{ij} inside D_j, with center $t_j \cap t_{ij}$. Both arcs meet in p_{ij} with the same tangent (orthogonal to t_{ij}). Therefore, the concatenation of C_i and C_j gives a tangent-continuous biarc. We use C_iC_j as drawing for v_iv_j and apply this construction for all edges in E.

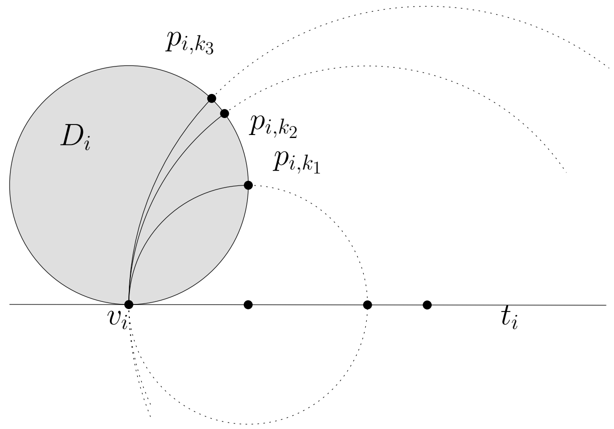


Figure 5.6: The situation at a vertex v_i that shows that the biarcs do not intersect.

It is left to show that the constructed drawing is non-crossing. Two biarcs could cross only within a disk of the disk packing. Consider all circular arcs incident to the embedded vertex $\mathcal{F}_b(v_i)$ as depicted in Figure 5.6. All corresponding circles have their centers on t_i and are passing through $\mathcal{F}_b(v_i)$, which lies on t_i as well. Thus, any two of these circles intersect only in $\mathcal{F}_b(v_i)$, and the constructed drawing is plane.

All biarcs incident to an embedded vertex $\mathcal{F}_b(v_i)$ have a common tangent orthogonal to t_i. We can determine this tangent by placing the vertex v_i on C_i appropriately, avoiding the finitely many touching points of D_i. $\square$

The above proof leaves some freedom to place the vertices on the boundaries of the corresponding disks. If in the drawing $\mathcal{F}_s(G)$ no two disk centers have the same x-coordinate, we can place each vertex on the bottommost point of the boundary of its disk. By this, all biarcs have positive curvature and we have no "S-shaped" biarcs (see Figure 5.7).

Another possibility is to place each vertex $v_i \in V$ farthest away from any touching point of its disk D_i. In this way we can guarantee the radius of any circular arc inside D_i to be at least $R_i \cdot \tan \frac{\pi}{2k_i}$, where R_i is the radius of D_i, and $k_i \geq 2$ is the degree of v_i. Unfortunately, in general, we have no control over the radii R_i in the disk packing.

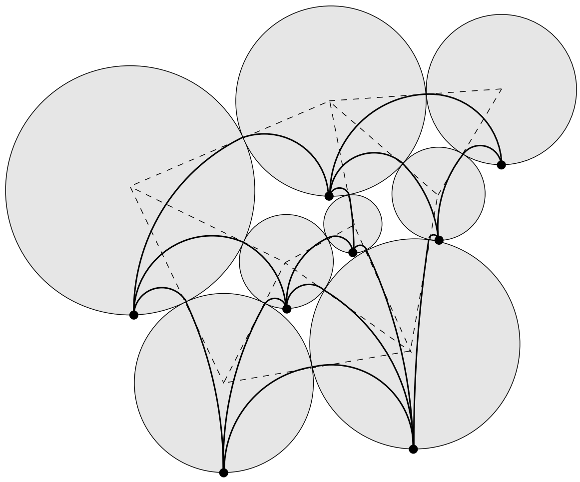


Figure 5.7: A pointed drawing with biarcs as edges, constructed from a disk packing.

5.3.2 Pointed drawings with circular arcs

We assume in this section that no two vertices will get the same y-coordinate in the drawing. The drawing we describe next uses the following special type of circular arcs.

Let p_1 and p_2 be two points, where p_2 has the larger y-coordinate. We call a circular arc between p_1 and p_2 *upper horizontally tangent* (short UHT-arc), if it has a horizontal tangent at p_2. We call a drawing of a triangle *upper horizontally tangent* (short UHT-triangle) if all of its edges are drawn as UHT-arcs (see Figure 5.9).

For any two points, the UHT-arc is uniquely defined. Hence, for every point triple the UHT-triangle is unique. The following lemmata show that under certain assumptions the UHT-triangles behave nicely.

Lemma 5.7. *Consider the UHT-arc μ between p_1 and p_2. Let h_1 be the horizontal line through p_1. Then the angle at p_1 between h_1 and μ is twice as large as the angle at p_1 between p_1p_2 and h_1.*

Proof. The situation stated in the lemma is depicted in Figure 5.8. Let α be the angle at p_1 between h_1 and p_1p_2, and let h_2 be the horizontal line through p_2. Then the angle α is the alternate angle to the angle at p_2 between h_2 and p_2p_1. Further, let p_t be the intersection of the tangents of μ at p_1 and p_2. The triangle $p_1p_2p_t$ is isosceles and hence the angle between p_1p_2 and p_1p_t is α as well. Thus, the angle between μ and h_1 equals 2α. □

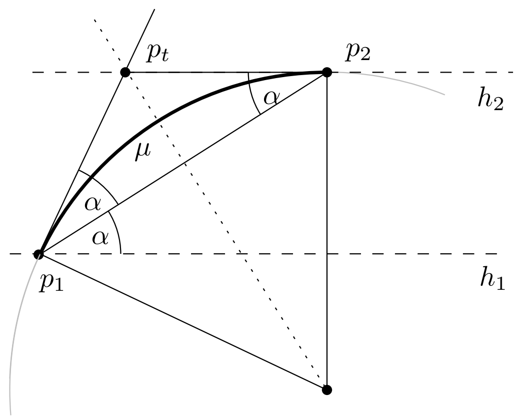


Figure 5.8: Construction used in the proof of Lemma 5.7.

In the following lemma, we restrict the straight-line edges to have an absolute slope less then or equal to 1. This implies that the angle between the tangent of an UHT-arc at the lower point and the horizontal line through this lower point is at most $\pi/2$. As a consequence, the UHT-arc is x-monotone and is contained in the axis-parallel bounding rectangle spanned by its endpoints.

Lemma 5.8. *Consider three points p_1, p_2, p_3, sorted by their increasing x-coordinates. If*

(i) *the absolute slopes of the line segments p_1p_2, p_2p_3 and p_1p_3 are smaller than 1, and*

(ii) *p_2 lies below the line through p_1 and p_3, or p_2 has the highest y-coordinate,*

then p_1, p_2, and p_3 span a non-crossing UHT-triangle that is oriented in the same way as the straight-line triangle $p_1p_2p_3$. That is, the clockwise order of the points around the faces is the same.

Proof. Let y_i be the y-coordinate of p_i, let h_i denote the horizontal line passing through p_i, and let a_{ij} denote the UHT-arc between p_i and p_j.

We prove the lemma by case distinction. Without loss of generality we assume that $y_1 < y_3$. Depending on the relative location of y_2 we obtain three cases (see Figure 5.9).

Case 1 ($y_2 < y_1$). a_{13} and a_{23} cannot intersect since they have a common tangent at p_3 and do not lie on the same circle. The other pairs of arcs have bounding rectangles with disjoint interior, and hence cannot intersect.

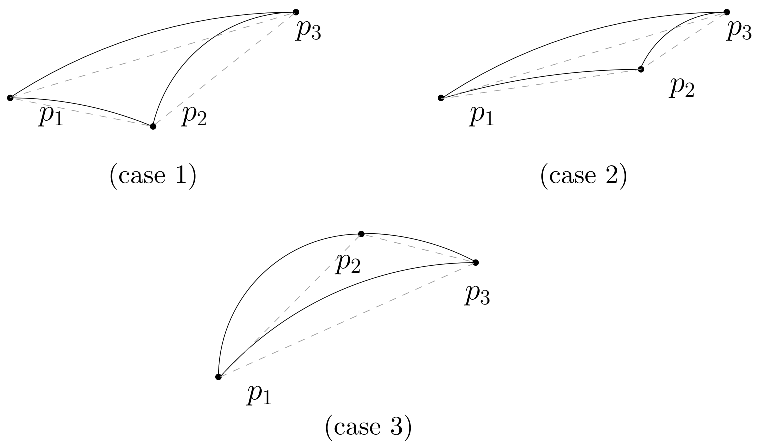


Figure 5.9: The three cases discussed in the proof of Lemma 5.8.

Case 2 ($y_1 < y_2 < y_3$). Again, a_{13} and a_{23} do not intersect since they have a common tangent at p_3 and do not lie on the same circle. The arcs a_{12} and a_{23} have bounding rectangles with disjoint interior, and therefore do not intersect either. Since p_2 lies below the line segment p_1p_3 (condition (ii)), p_2 lies below the arc a_{13} and p_1p_3 has larger slope than p_1p_2. Thus, and due to Lemma 5.7, the angle between the tangent of a_{13} and h_1 is larger than the angle between the tangent of a_{12} and h_1, meaning that a_{12} is incident to p_1 "below" a_{13}. As the second endpoint of p_2 of a_{12} lies below a_{13} as well, an intersection of a_{12} and a_{13} (to the right of p_1) would imply a second such intersection. This is impossible, because the two circles induced by a_{12} and a_{13} would intersect three times.

Case 3 ($y_3 < y_2$). The pairs a_{23}/a_{12}, and a_{23}/a_{13} have bounding rectangles with disjoint interior and therefore do not intersect. For the remaining pair of arcs a_{12} and a_{13} we apply again Lemma 5.7 and observe that a_{12} is incident to p_1 "above" a_{13}. As the second endpoint of p_2 of a_{12} lies above a_{13} as well, it follows that an intersection of a_{12} and a_{13} (to the right of p_1) would again imply that the two circles induced by a_{12} and a_{13} intersect three times, which is impossible.

Since in all three cases, the above-below order of the (x-monotone) incident edges at each vertex is preserved, the orientation of the UHT-triangle is identical to the orientation of the straight-line triangle. □

We continue by constructing a straight-line drawing that allows us to substitute its triangles by UHT-triangles. The basic idea goes back to de Fraysseix, Pach and Pollack [60].

Theorem 5.9 ([60]). *A plane triangulated graph has a plane straight-line drawing on a* $(2n-4) \times (n-2)$ *grid.*

Let us briefly review the incremental construction used in [60], see Figure 5.10. The vertices are inserted in a special (so-called canonical) order, such that the next vertex p_{k+1} that is inserted can be drawn on the outer face of the graph G_k induced by the first k vertices. Thereby as invariant it is maintained that the outer boundary of the graph G_k (drawn so far) forms a chain of pieces of slope ± 1, resting on a horizontal basis (Figure 5.10(a)). The next vertex p_{k+1} to be drawn is adjacent to a continuous subsequence of vertices on the outer boundary. To make space for the new edges incident to p_{k+1}, the boundary of G_k is split into three pieces, which are separated from each other by shifting them one unit apart (Figure 5.10(b)). The middle piece contains all neighbors of p_{k+1} except the first and the last one. In [60] it is shown that one can split the upper boundary of G_k at an arbitrary point and shift the pieces apart horizontally, by an arbitrary amount. If an appropriate part of G_k inside the shaded area is shifted along, no crossings are created. Any number of these shifting operation can be carried out in succession. Furthermore, during such a shifting operation, the endpoints of an edge can only be moved farther apart horizontally.

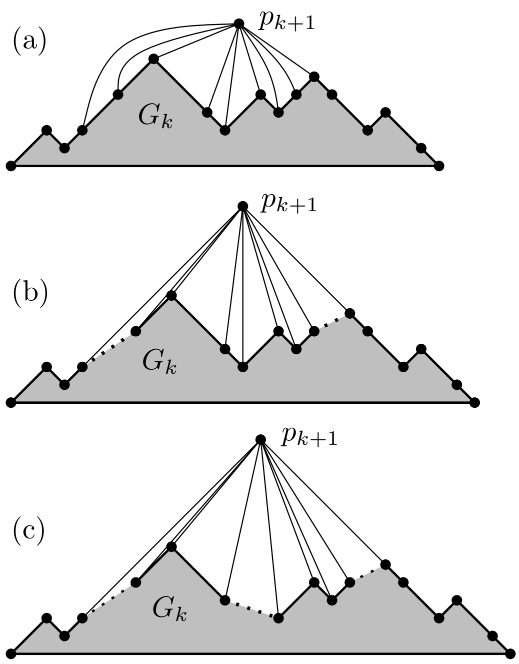


Figure 5.10: (a–b) The incremental step in the straight-line drawing algorithm of de Fraysseix, Pach and Pollack [60], and (c) the modification that prevents vertical edges.

We slightly modify this inductive procedure to prove the following theorem.

Theorem 5.10. *A plane triangulated graph has a plane straight-line drawing on a grid of size* $(4n-9) \times (2n-4)$*, with the following additional properties:*

(a) *No edge is vertical.*

(b) *No edge is horizontal.*

(c) *In each triangular face, the vertex with the middle* x*-coordinate is either the vertex with the highest* y*-coordinate, or it lies below the opposite edge.*

Proof. The newly created triangles in the incremental construction described above always fulfill property (c), which can be checked directly, and no horizontal edges are created (property (b)). The only horizontal edge is the bottom base edge. This horizontal edge can easily be avoided by starting the construction with a non-horizontal base triangle in the first step.

To prevent vertical edges, one can split the middle part into two pieces by vertical line through p_{k+1} and set them apart by two more units (Figure 5.10(c)). A boundary vertex on the vertical line can be assigned to either part. (Two units are necessary to ensure that the left and right part are separated in total by an even offset; this guarantees that the position of p_{k+1}, which is defined by the requirement that its leftmost and rightmost incident edges have slope $+1$ and -1 respectively, gets integer coordinates.)

Adding a vertex preserves the old y-coordinates and the order of the x-coordinates between adjacent vertices, as well as the cyclic order of the edges. As a consequence, properties (b) and (c) can be guaranteed to hold for previously added vertices after shifting. Property (a) is preserved because shifting decreases the absolute slope of an already inserted edge, and by the same reasoning, no edge becomes vertical. The dimensions of the grid increase by 4×2 units for each new vertex. The initial drawing of the graph G_3 with the first three vertices needs a 3×2 grid. □

We continue with the main result of this section.

Theorem 5.11. *Every plane graph* G *has a plane pointed drawing with circular arcs as edges.*

Proof. We assume that the graph G is a triangulation. (Otherwise we add edges such that G becomes a triangulation and delete these edges in the end.)

Given an n-vertex plane triangulated graph, the algorithm of Theorem 5.10 constructs drawings in which for every edge its absolute slope is less than $2n$. By scaling the x-axis by a factor of $2n$, we obtain a drawing in which all edges have slopes in the range between -1 and $+1$. In this scaled drawing, all triangles fulfill the conditions of Lemma 5.8. We substitute every straight-line edge by its corresponding UHT-arc. By this substitution, the order of the edges around a vertex is preserved, and every straight-line triangle is replaced by its corresponding UHT-triangle. Thus, and due to Lemma 5.8, the obtained circular drawing is crossing-free (Edges on the upper hull are non-crossing as they have bounding-rectangles with disjoint interior).

Around every vertex there is a number of edges that emanate in the horizontal direction, plus a number of additional edges that point upward. The latter type of edges have distinct tangent directions. Thus one can slightly bend every edge upward and achieve a pointed drawing with circular arcs. □

Due to Theorem 5.10, pointed drawings constructed as above lie on an $O(n) \times O(n^2)$ grid. An example of such a drawing is shown in Figure 5.11.

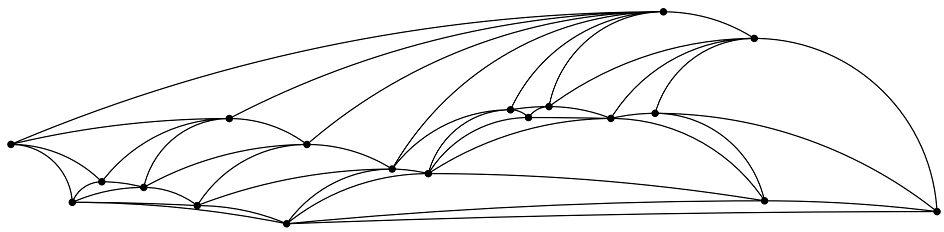


Figure 5.11: An example of a pointed drawing with circular arcs. The horizontal stretch factor was chosen just sufficiently to ensure that all straight edges have absolute slope less than 1, instead of $2n$.

As a consequence, we obtain the following result about multigraphs with loops, as mentioned in the introduction:

Corollary 5.12. *Every planar multigraph, possibly with loops, admits a plane drawing with circular arcs, whose vertices lie on an* $O(n) \times O(n^2)$ *grid.* □

Note that this is no longer true if we insist on a particular combinatorial embedding. For example, we cannot have three non-nested loops incident to a vertex.

5.3.3 Pointed drawings obtained via combinatorial pseudo-triangulations

A different way to find a pointed drawing uses the framework established by Haas et al. [94]. Let us recall some terminology first. A *combinatorial pseudo-triangulation* is a planar combinatorial embedding of an (abstract) connected planar graph G with an assignment of the tags *big/small* to the angles of G such that the following conditions are fulfilled.

(1) Every interior face has exactly three *small* angles.

(2) The outer face has only angles labeled *big*.

(3) Every vertex is incident to at most one angle labeled *big*. If it is incident to a big angle, it is called pointed (in the face where is has its big angle).

(4) A vertex of degree at most 2 is incident to one angle labeled *big*.

An angle assignment that fulfills these conditions is called *cpt-assignment*.

By [94, Theorem 6], every combinatorial pseudo-triangulation whose underlying graph is a Laman graph can be embedded as a pseudo-triangulation such that every angle with tag *big* is larger than π in the drawing, and every angle with tag *small* is smaller than π in the drawing. Furthermore, the shape of every face can be specified up to affine transformations [94, Theorem 11].

Lemma 5.13. *Every triangulation with n vertices can be turned into a combinatorial pointed pseudo-triangulation by subdividing $n-3$ edges, each with exactly one additional vertex. Furthermore, the underlying graph is a Laman graph.*

Proof. We construct a cpt-assignment by an iterative procedure that is guided by the canonical order of the plane graph (see Theorem 5.9). The assignment ensures that all graphs G_k have a valid cpt-assignment. This can be easily made true for G_3, which is a single triangle. Here, the three angles at the boundary get the tag *big*, and the three interior angles get the tag *small*. Assume now that we add the vertex v_{k+1} to G_k to obtain G_{k+1}. We therefore connect v_{k+1} with its neighbors on the boundary of G_k (see Figure 5.10(a)). All new edges that are not on the boundary of G_{k+1} are subdivided by adding a new vertex on each such edge. The cpt-assignment is extended such that every exterior angle of G_{k+1} gets the tag *big*, and all newly added faces have exactly three small angles. Every vertex that vanishes from the boundary will now realize its big angle in one of the newly created faces. The simple scheme how to assign the angles is depicted in Figure 5.12.

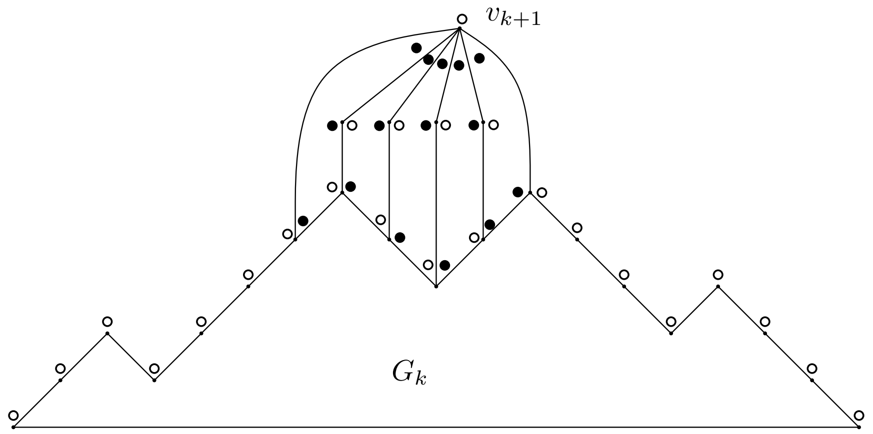


Figure 5.12: Extending the cpt-assignment during the construction of G by its canonical order.

From a different perspective we add the vertex v_{k+1} by linking it to G_k by two new edges. This operation is a Henneberg-1 step and it preserves the Laman property of the graph. The vertices that are introduced by subdividing edges are a result of additional Henneberg-1 steps. Since G_3 is a Laman graph, also $G_n = G$ is a Laman graph.

In every step we add two edges that are not subdivided, and there are $n-4$ steps necessary to go from G_3 to G_n. Thus, in total, we add $2n-3$ edges that are not subdivided. Since the triangulation G has $3n-6$ edges, $n-3$ of them are subdivided. □

Theorem 5.14. *Every plane graph G with n vertices has a plane pointed drawing with either quadratic Bézier curves, tangent continuous biarcs, or 2-chains (polygonal chains consisting of two line segments), which uses at most $n-3$ non-straight edges.*

Proof. We assume that the graph G is a triangulation. (Otherwise we add edges such that G becomes a triangulation and delete these edges in the end.) As a first step, we turn G into a combinatorial pointed pseudo-triangulation as done in Lemma 5.13. By this we creates four types of bounded faces:

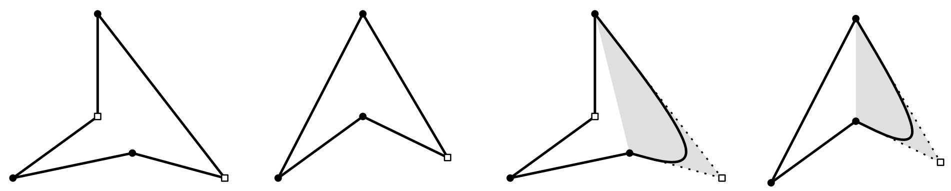
(i) triangles,

(ii) quadrilaterals with a degree-2 vertex with big angle,

(iii) quadrilaterals with a degree-2 vertex with small angle and the big angle is realized next to it,

(iv) pentagons with two non-adjacent degree-2 vertices, one of them with a big, one of them with a small angle.

We apply the algorithm of [94] to realize the combinatorial pseudo-triangulation. In all faces of type (iii) the interior of the triangle spanned by the degree-2 vertex and its two neighbors is empty, see Figure 5.13(b). The same is not necessarily true for the faces of type (iv). However, the algorithm of [94] allows us to specify the face shapes up to affine transformations. By giving all faces of type (iv) the shape shown in Figure 5.13(a) one assures that the interior of shaded triangle in Figure 5.13(c) is empty. This property is preserved under affine transformations.



(a) (b) (c) (d)

Figure 5.13: Affine shapes of faces used for the drawing (a–b) and control triangles for curve replacement inside these faces (c–d). The degree-2 vertices that came from edge subdivisions are marked as boxes.

What we have obtained so far is a pointed drawing, where at most $n-3$ edges are drawn as 2-chains, which proves the theorem for the case of polygonal chains.

For the case of Bézier curves or biarcs, we consider for each 2-chain p_1, p_m, p_2 (with p_m being the vertex of degree two) the triangle $\Delta = p_1 p_m p_2$. Δ lies in a pseudo-triangle in which p_m has a small angle. Due to the affine shape of the faces, Δ has an empty interior. We use these triangles as control polygons as shown in Figure 5.13 and replace the 2-chains by Bézier curves or biarcs (similar to Lemma 5.2). □

In general it is not possible to draw a planar graph pointed using a larger number of (non-crossing) straight-line edges, since every maximal pointed straight-line graph has at most $2n$–3 edges [138], and due to Euler's formula a triangulation has $3n - 6$ edges. In this sense, Theorem 5.14 is optimal.

We demonstrate the construction used in the proof of Theorem 5.14 by an example. Let G be the graph depicted in Figure 5.14(a). The big angles of the cpt-assignment and the subdivisions computed by our method are shown in Figure 5.14(b). This leads to the pointed pseudo-triangulation in Figure 5.14(c) and finally to a pointed drawing with only three Bézier curves as shown in Figure 5.14(d).

In a first version of this work, we made a stronger claim [34, Theorem 7]: for each inner vertex, the face in which it is pointed can be chosen arbitrarily.

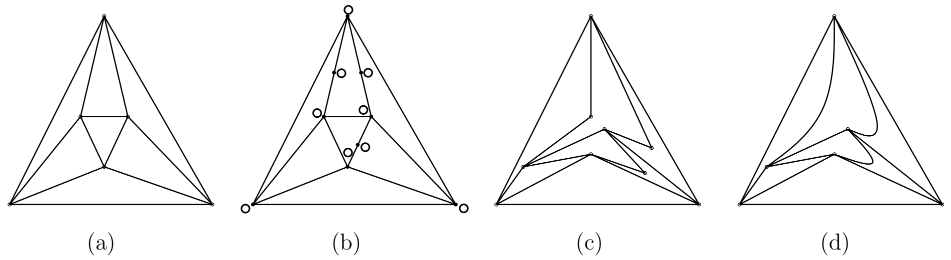


Figure 5.14: Construction of a pointed drawing with Bézier curves with help of a combinatorial pseudo-triangulation as example.

Figure 5.15(a) shows a counterexample where this is not true. It is not possible to make the three marked angles big with at most n–3 non-straight edges which are either quadratic Bézier curves or 2-chains. The reason is that a single quadratic Bézier curve bends by less than π. Therefore, in a triangle with three vertices that are pointed in the interior face, all three edges must be curved, see Figure 5.15(b): if we proceed clockwise along the boundary, the tangent direction turns right by less than π along each edge. At each vertex, it makes a left turn, by pointedness. With less than three curved edges, the tangent direction cannot complete a full right turn of 2π. The same argument works for 2-chains. By a similar argument, a triangle with two vertices that are pointed in the interior face needs at least two curved edges.

Applying these facts to our example, we see that all three edges in the triangle ABC must be curved. With a total of $n = 6$ vertices, we have thus exhausted our reservoir of at most n–$3 = 3$ non-straight edges. But then the two straight edges $A'B$ and $A'C$ together with the curved edge BC cannot make pointed interior angles at B and C in the triangle $A'BC$.

This example does not rule out the possibility that pointedness in the chosen faces can be achieved with more than $n-3$ curved edges, or with biarcs.

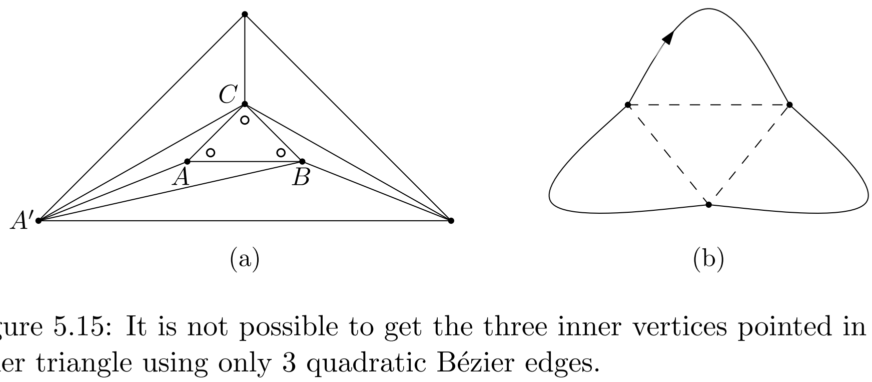


Figure 5.15: It is not possible to get the three inner vertices pointed in the inner triangle using only 3 quadratic Bézier edges.

Chapter 6

Conclusion

We considered questions in the intersection of combinatorial geometry and graph theory. The central theme are combinatorial aspects of point sets in the plane and graphs based on them, where in some questions the point sets were bichromatic.

During the work on this thesis, we obtained several interesting results. Some problems have been completely solved, while others are left partially open. Moreover, new problems as well as new interpretations of existing problems evolved. Let us below mention some of the questions that seem to be interesting and challenging for future research.

A central topic are Erdős-Szekeres type problems on k-gons and k-holes. In Chapter 2 we presented results on the numbers of not necessarily convex k-gons and k-holes. For example, we gave a quadratic lower bound and an upper bound of $O(n^{\frac{k+1}{2}}(\log n)^{k-3})$ for the minimum number of general k-holes, as well as an improved linear lower bound for the number of convex 5-holes. This immediately rises two questions, one of them classical, the other one new. Is there a super-linear lower bound for the number of convex 5-holes? And what about a super-quadratic lower bound for the number of general k-holes?

It turned out that the former question is strongly related to the latter one in the following way: A super-linear lower bound for the number of general 4-holes would solve a conjecture of Bárány in the affirmative, showing that every point set contains an edge that spans a super-constant number of 3-holes; see e.g. [53], Chapter 8.4, Problem 4. Bárány's conjecture in turn is equivalent to a conjecture stating that the number of convex 5-holes in a point set is always at least quadratic.

Adding colors to the points, we showed in Section 3.1 that every sufficiently large bichromatic point set contains a monochromatic (not necessarily convex) 4-hole. The maybe most challenging question for this topic is whether there exists a similar result for convex 4-holes.

Another classical question for bichromatic point sets is Zarankiewicz's conjecture on the crossing number of the complete bipartite graph $K_{n,m}$. Section 3.2 addresses restricted versions of this conjecture, where the edges are straight lines, and, in the underlying point set, the red set is linearly separable from the blue set. Despite several interesting observations, the following question remains open: Is it true that the rectilinear crossing number of the complete bipartite graph $K_{n,m}$ is $\mathrm{z}(n,m)$ if the underlying point set has to be linearly separable?

Compatible matchings for bichromatic graphs are the topic of Section 3.3. We presented bounds on the number of edges in a compatible matching that are always attainable for a graph of a given class. All our lower bounds concerning monochromatic matchings are obtained by constructing purely monochromatic matchings, i.e., matchings that only use one of the two color classes. This rises the question whether the attainable number of edges $m_{\mathcal{C}}(n)$ for monochromatic matchings is larger than the according number $p_{\mathcal{C}}(n)$ for purely monochromatic matchings for some class $\mathcal{C} \in \{tree, path, cycle, match\}$. Concerning bichromatic matchings, it is well known that every bichromatic set S with $|R| = |B|$ admits a bichromatic perfect matching [113], but also that there exist point sets admitting only one such matching. Thus, assume that S is a point set admitting at least two different perfect matchings. The most challenging open question here is the following. Given a bichromatic perfect matching $PM(S)$, does there always exists a compatible bichromatic perfect matching $PM'(S)$ that contains edges which are not in $PM(S)$?

Finally, let us state the major open problem from Section 4.1. There, we showed that for every set B of n points there exists a set W of at most $3n/2$ points ($5n/4$ if B is convex) such that the Delaunay triangulation of $B \cup W$ does not contain any edge with both end points in B. We also provided a lower bound of $|W| \geq n-1$ for the number of points that is always necessary. In fact, for all point sets we considered during the work on this topic, we found a solution with $|W| = n$. Is true in general that every point set B needs exactly $|W| = n$ points to be blocked? Or are there sets for which $|W| = n - 1$ suffices or $|W| \geq n + 1$ is necessary?

List of Figures

List of Tables

Bibliography

[1] M. Abellanas, A. García, F. Hurtado, J. Tejel, and J. Urrutia. Augmenting the connectivity of geometric graphs. *Computational Geometry*, 40:220–230, 2008.

[2] B. M. Ábrego, M. Cetina, S. Fernández-Merchant, J. Leaños, and G. Salazar. 3-symmetric and 3-decomposable geometric drawings of K_n. *Discrete Applied Mathematics*, 158(12):1240–1258, 2010.

[3] B. M. Ábrego, S. Fernández-Merchant, J. Leaños, and G. Salazar. A central approach to bound the number of crossings in a generalized configuration. *Electronic Notes in Discrete Mathematics*, 30:273–278, 2008.

[4] B. M. Ábrego, S. Fernández-Merchant, J. Leaños, and G. Salazar. The maximum number of halving lines and the rectilinear crossing number of K_n for $n <= 27$. *Electronic Notes in Discrete Mathematics*, 30:261–266, 2008.

[5] B. M. Ábrego, S. Fernández-Merchant, and G. Salazar. The rectilinear crossing number of K_n: Closing in (or are we?). *submitted to journal*, 2011.

[6] A. Aggarwal, D. Coppersmith, S. Khanna, R. Motwani, and B. Schieber. The angular-metric traveling salesman problem. *SIAM Journal on Computing*, 29(3):697–711, 1999.

[7] S. I. Ahmed, M. Hasan, and A. Sopan. Vindictive Voronoi games and stabbing Delaunay circles. In M. A. Mostafavi, editor, *Seventh International Symposium on Voronoi Diagrams in Science and Engineering, ISVD 2010, Quebec, Canada, June 28-30, 2010*, pages 124–131. IEEE Computer Society, 2010.

[8] O. Aichholzer. On the rectilinear crossing number. *www.ist.tugraz.at/aichholzer/research/rp/triangulations/crossing/*.

[9] O. Aichholzer. [Empty] [colored] k-gons - Recent results on some Erdős-Szekeres type problems. In *Proc. XIII Encuentros de Geometría Computacional ECG2009*, pages 43–52, Zaragoza, Spain, 2009.

[10] O. Aichholzer, F. Aurenhammer, P. Gonzalez-Nava, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, S. Ray, and B. Vogtenhuber. Matching edges and faces in polygonal partitions. *Computational Geometry: Theory and Applications*, 39(2):134–141, 2008.

[11] O. Aichholzer, S. Bereg, A. Dumitrescu, A. García, C. Huemer, F. Hurtado, M. Kano, A. Márquez, D. Rappaport, S. Smorodinsky, D. Souvaine, J. Urrutia, and D. Wood. Compatible geometric matchings. *Computational Geometry: Theory and Applications*, 42(6-7):617–626, 2009.

[12] O. Aichholzer, M. Cetina, T. Hackl, C. Huemer, D. Orden, P. Ramos, G. Salazar, B. Vogtenhuber, and P. Zubieta. Research week in Alcalá de Hernares, Spain, June 22–26 2009.

[13] O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, and J. Urrutia. Empty monochromatic triangles. In *Proc. 20th Annual Canadian Conference on Computational Geometry CCCG 2008*, pages 75–78, Montreal, Quebec, Canada, 2008.

[14] O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, and J. Urrutia. Empty monochromatic triangles. *Computational Geometry: Theory and Applications*, 42(9):934–938, 2009.

[15] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, P. Valtr, and B. Vogtenhuber. On k-gons and k-holes in point sets. In *Proc. 23th Canadian Conference on Computational Geometry CCCG'11*, pages 21–26, Toronto, Canada, 2011.

[16] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. In *Proc. 27th European Workshop on Computational Geometry EuroCG'11*, pages 115–118, Morschach, Switzerland, 2011.

[17] O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. *submitted to journal*, 2011.

[18] O. Aichholzer, R. Fabila-Monroy, T. Hackl, M. van Kreveld, A. Pilz, P. Ramos, and B. Vogtenhuber. Blocking delaunay triangulations. In *Proc. 22nd Annual Canadian Conference on Computational Geometry CCCG 2010*, pages 21–24, Winnipeg, Manitoba, Canada, 2010.

[19] O. Aichholzer, R. Fabila-Monroy, T. Hackl, M. van Kreveld, A. Pilz, P. Ramos, and B. Vogtenhuber. Blocking delaunay triangulations. *Computational Geometry, Theory and Applications (special issue for*

the 22^{nd} *Canadian Conference on Computational Geometry)*, 2011. to appear.

[20] O. Aichholzer, A. García, F. Hurtado, and J. Tejel. Compatible matchings in geometric graphs. In *Proc. XIV Encuentros de Geometría Computacional ECG2011*, pages 145–148, Alcalá de Hernares, Spain, 2011.

[21] O. Aichholzer, J. Garcia, H. Krasser, D. Orden, P. Ramos, and B. Vogtenhuber. Research week in Alcalá de Hernares, Spain, July 7–11 2008.

[22] O. Aichholzer, J. García, D. Orden, and P. Ramos. New lower bounds for the number of ($\leq k$)-edges and the rectilinear crossing number of K_n. *Discrete and Computational Geometry*, 38:1–14, 2007.

[23] O. Aichholzer, T. Hackl, M. Hoffmann, C. Huemer, F. Santos, B. Speckmann, and B. Vogtenhuber. Maximizing maximal angles for plane straight line graphs. *Computational Geometry: Theory and Applications*, 2011. to appear.

[24] O. Aichholzer, T. Hackl, M. Hoffmann, A. Pilz, G. Rote, B. Speckmann, and B. Vogtenhuber. Plane graphs with parity constraints. In *Lecture Notes in Computer Science, Proc.* 11^{th} *International Workshop on Algorithms and Data Structures (WADS)*, volume 5664, pages 13–24, Banff, Alberta, Canada, 2009.

[25] O. Aichholzer, T. Hackl, M. Hoffmann, A. Pilz, G. Rote, B. Speckmann, and B. Vogtenhuber. Plane graphs with parity constraints. *submitted to journal*, 2009.

[26] O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, and B. Vogtenhuber. Large bichromatic point sets admit empty monochromatic 4-gons. In *Proc.* 25^{th} *European Workshop on Computational Geometry EuroCG '09*, pages 133–136, Brussels, Belgium, 2009.

[27] O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, and B. Vogtenhuber. Large bichromatic point sets admit empty monochromatic 4-gons. *SIAM Journal on Discrete Mathematics (SIDMA)*, 23(4):2147–2155, 2010.

[28] O. Aichholzer, T. Hackl, H. Krasser, B. Vogtenhuber, and U. Wagner. Notes on bichromatic point sets, 2007. Mansucript.

[29] O. Aichholzer, T. Hackl, D. Orden, P. Ramos, G. Rote, A. Schulz, and B. Speckmann. Flip graphs of bounded-degree triangulations. In *Electronic Notes in Discrete Mathematics: Proc. European Conference*

on Combinatorics, Graph Theory and Applications EuroComb 2009, volume 34, pages 509–513, Bordeaux, France, 2009.

[30] O. Aichholzer, T. Hackl, and B. Vogtenhuber. Compatible pointed pseudo-triangulations. In *Proc.* 22nd *Annual Canadian Conference on Computational Geometry CCCG 2010*, pages 91–94, Winnipeg, Manitoba, Canada, 2010.

[31] O. Aichholzer, T. Hackl, and B. Vogtenhuber. On 5-holes and 5-gons. In *Proc. XIV Encuentros de Geometría Computacional ECG2011*, pages 7–10, Alcalá de Hernares, Spain, 2011.

[32] O. Aichholzer, T. Hackl, and B. Vogtenhuber. On 5-holes and 5-gons. *submitted to journal*, 2011.

[33] O. Aichholzer and H. Krasser. The point set order type data base: A collection of applications and results. In *Proc. 13th Canadian Conference on Computational Geometry CCCG'01*, volume 13, pages 17–20, 2001.

[34] O. Aichholzer, G. Rote, A. Schulz, and B. Vogtenhuber. Pointed drawings of planar graphs. In *Proc.* 19*th Annual Canadian Conference on Computational Geometry CCCG 2007*, pages 237–240, Ottawa, Ontario, Canada, 2007.

[35] O. Aichholzer, G. Rote, A. Schulz, and B. Vogtenhuber. Pointed drawings of planar graphs. *Computational Geometry, Theory and Applications (special issue for the* 19th *Canadian Conference on Computational Geometry)*, 2011. to appear.

[36] E. M. Andreev. On convex polyhedra in Lobačevskiĭ space. *Mathematics of the USSR-Sbornik*, 10(3):413–440, 1970.

[37] E. M. Arkin, S. P. Fekete, F. Hurtado, J. S. B. Mitchell, M. Noy, V. Sacristan, and S. Sethia. On the reflexivity of point sets. *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, 25:139–156, 2003.

[38] B. Aronov, M. Dulieu, and F. Hurtado. Witness (Delaunay) graphs. *Computational Geometry: Theory and Applications*, 44(6-7):329 – 344, 2011.

[39] B. Aronov, M. Dulieu, and F. Hurtado. Witness Gabriel graphs. *Computational Geometry*, 2011. to appear.

[40] F. Aurenhammer and R. Klein. Voronoi diagrams. In J. Sack and G. Urrutia, editors, *Handbook of Computational Geometry, Chapter V*, pages 201–290. Elsevier Science Publishing, 2000.

[41] I. Bárány and Z. Füredi. Empty simplices in Euclidean space. *Canadian Mathematical Bulletin*, 30:436–445, 1987.

[42] I. Bárány and G. Károlyi. Problems and results around the Erdős–Szekeres convex polygon theorem. *Discrete and Combinatorial Geometry*, LNCS 2098:91–105, 2001.

[43] I. Bárány, A. Pór, and P. Valtr. Paths with no small angles. *SIAM Journal on Discrete Mathematics (SIDMA)*, 23(4):1655–1666, 2009.

[44] I. Bárány and P. Valtr. Planar point sets with a small number of empty convex polygons. *Studia Scientiarum Mathematicarum Hungarica*, 41(2):243–266, 2004.

[45] G. D. Battista, P. Eades, R. R. Tamassia, and I. Tollis. *Graph Drawing: Algorithms for Geometric Representations of Graphs.* Prentice Hall, Englewood Cliffs, 1998.

[46] L. Beineke and R. Wilson. The early history of the Brick Factory Problem. *The Mathematical Intelligencer*, 32(2):41–48, 2010.

[47] M. W. Bern, D. Eppstein, and J. R. Gilbert. Provably good mesh generation. *Journal of Computer and System Sciences*, 48(3):384–409, 1994.

[48] D. Bienstock and N. Dean. Bounds for rectilinear crossing numbers. *Journal of Graph Theory*, 17(3):333–348, 1993.

[49] N. L. Biggs. *Algebraic Graph Theory*, volume 67 of *Cambridge Tracts in Mathematics.* Cambridge University Press, 1974 (2nd Edition 1993).

[50] J. Blažek and M. Koman. A minimal problem concerning complete plane graphs. *Theory of graphs and its applications. Proceedings of the Sympos. Prague*, pages 113–117, 1963.

[51] P. Bose and F. Hurtado. Flips in planar graphs. *Computational Geometry: Theory and Applications*, 42(1):60–80, 2009.

[52] P. Brass. Empty monochromatic fourgons in two-colored point sets. In *Geombinatorics XIV*, pages 5–7, 2004.

[53] P. Brass, W. Moser, and J. Pach. *Research Problems in Discrete Geometry.* Springer, 2005.

[54] M. Cetina, C. Hernández-Vélez, J. Leaños, and C. Villalobos. Point sets that minimize ($\leq k$)-edges, 3-decomposable drawings, and the rectilinear crossing number of K_{30}. *Discrete Mathematics*, 311(16):1646 – 1657, 2011.

[55] Y. Colin de Verdière. Un principe variationnel pour les empilements de cercles. *Inventiones Mathematicae*, 104(3):655–669, 1991.

[56] C. R. Collins and K. Stephenson. A circle packing algorithm. *Computational Geometry: Theory and Applications*, 25(3):233–256, 2003.

[57] Y. Dai, N. Katoh, and S.-W. Cheng. LMT-skeleton heuristics for several new classes of optimal triangulations. *Computational Geometry: Theory and Applications*, 17(1-2):51–68, 2000.

[58] M. de Berg, O. Cheong, M. van Kreveld, , and M. Overmars. *Computational Geometry: Algorithms and Applications*. Springer-Verlag Heidelberg, 2008. Third edition.

[59] M. de Berg, D. Gerrits, A. Khosravi, I. Rutter, C. Tsirogiannis, and A. Wolff. How Alexander the Great brought the Greeks together while inflicting minimal damage to the Barbarians. In *Proc. 26th European Workshop on Computational Geometry EuroCG'10*, pages 73–76, Dortmund, Germany, 2010.

[60] H. de Fraysseix, J. Pach, and R. Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10(1):41–51, 1990.

[61] E. de Klerk, J. Maharry, D. V. Pasechnik, R. B. Richter, and G. Salazar. Improved bounds for the crossing numbers of $K_{m,n}$ and K_n. *SIAM Journal on Discrete Mathematics (SIDMA)*, 20(1):189–202, 2006.

[62] E. de Klerk, D. V. Pasechnik, and A. Schrijver. Reduction of symmetric semidefinite programs using the regular $*$-representation. *Mathematical Programming*, 109(2):613–624, 2007.

[63] K. Dehnhardt. *Leere konvexe Vielecke in ebenen Punktmengen*. PhD thesis, TU Braunschweig, Germany, 1987.

[64] B. Delaunay. Sur la sphère vide. *Bulletin of Academy of Sciences of the USSR*, 7(6):793–800, 1934.

[65] S. Devadoss and J. O'Rourke. *Discrete and Computational Geometry*. Princeton University Press, 2011.

[66] O. Devillers, F. Hurtado, G. Károlyi, and C. Seara. Chromatic variants of the Erdős-Szekeres theorem on points in convex position. *Computational Geometry: Theory and Applications*, 26(3):193–208, 2003.

[67] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Heidelberg, 1997 (4th Edition 2010).

[68] M. Dillencourt. Toughness and delaunay triangulations. *Discrete and Computational Geometry*, 5(1):575–601, 1990.

[69] M. Dulieu. *Witness Proximity Graphs*. PhD thesis, Department of Computer Science and Engineering, Polytechnic Institute of New York University, USA, 2012. In preparation.

[70] A. Dumitrescu, A. Schulz, A. Sheffer, and C. D. Toth. Bounds on the maximum multiplicity of some common geometric graphs. In *Proc. 28th International Symposium on Theoretical Aspects of Computer Science (STACS 2011)*, Leibniz International Proceedings in Informatics (LIPIcs), pages 637–648, Dagstuhl, Germany, 2011.

[71] A. Dumitrescu and W. L. Steiger. On a matching problem in the plane. *Discrete Mathematics*, 211:183–195, 2000.

[72] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*, volume 10 of *EATCS Monographs in Computer Science*. Springer-Verlag, Heidelberg, 1987.

[73] H. Edelsbrunner. *Geometry and Topology for Mesh Generation*, volume 7 of *Cambridge Monographs on Applied and Computational Geometry*. Cambridge University Press, New York, USA, 2001.

[74] H. Edelsbrunner, T. S. Tan, and R. Waupotitsch. An $o(n^2 \log n)$ time algorithm for the minmax angle triangulation. *SIAM Journal on Scientific and Statistical Computing*, 13(4):994–1008, 1992.

[75] D. Eppstein. The farthest point Delaunay triangulation minimizes angles. *Computational Geometry: Theory and Applications*, 1(3):143–148, 1991.

[76] P. Erdős. Some more problems on elementary geometry. *Australian Mathematical Society Gazette*, 5:52–54, 1978.

[77] P. Erdős. Some old and new problems in combinatorial geometry. In M. Rosenfeld and J. Zaks, editors, *Annals of Discrete Mathematics, Convexity and Graph Theory*, volume 20, pages 129–136, 1984.

[78] P. Erdős and R. Guy. Crossing number problems. *The American Mathematical Monthly*, 88:52–58, 1973.

[79] P. Erdős, L. Lovász, A. Simmons, and E. Strauss. Dissection graphs on planar point sets. In *A Survey of Combinatorial Theory*, pages 139–149. American Elsevier, North Holland, Amsterdam, 1973.

[80] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[81] P. Erdős and G. Szekeres. On some extremum problems in elementary geometry. *Ann. Univ. Sci. Budapest. Eötvös, Sect. Math.*, 3/4:53–62, 1960.

[82] R. Fabila-Monroy and C. Huemer. Covering islands in plane point sets. In *Proc. XIV Encuentros de Geometría Computacional ECG2011*, pages 213–216, Alcalá de Hernares, Spain, 2011.

[83] I. Fáry. On straight line representations of planar graphs. *Acta Univ. Szeged. Sect. Sci. Math.*, 11:229–233, 1948.

[84] S. P. Fekete and G. J. Woeginger. Angle-restricted tours in the plane. *Computational Geometry: Theory and Applications*, 8(4):195–218, 1997.

[85] E. Friedmann. 30 two-colored points with no empty monochromatic convex fourgons. In *Geombinatorics XIV*, pages 53–54, 2004.

[86] A. García. A note on the number of empty triangles. In *Proc. XIV Encuentros de Geometría Computacional ECG2011*, pages 101–104, Alcalá de Hernares, Spain, 2011.

[87] A. García, C. Huemer, F. Hurtado, and J. Tejel. Árboles geométricos compatibles. In *Proc. XII Encuentros de Geometría Computacional*, pages 161–167, Valladolid, Spain, 2007.

[88] A. García, M. Noy, and J. Tejel. Lower bounds on the number of crossing-free subgraphs of K_n. *Computational Geometry: Theory and Applications*, 16:211–221, 2000.

[89] T. Gerken. Empty convex hexagons in planar point sets. *Discrete and Computational Geometry*, 39(1–3):239–272, 2008.

[90] J. E. Goodman and J. O'Rourke, editors. *Handbook of Discrete and Computational Geometry.* CRC Press LLC, 2004. Second Edition.

[91] J. Gross and J. Yellen. *Graph Theory and Its Applications.* Discrete Mathematics and Its Applications. CRC Press LLC, 1998 (2nd Edition 2005).

[92] R. V. Gulik. 32 two-colored points with no empty monochromatic convex fourgons. In *Geombinatorics XV*, pages 32–33, 2005.

[93] R. Guy. The decline and fall of Zarankiewicz's theorem. In *Proof Techniques in Graph Theory, Proceedings of the Second Ann Arbor Graph Theory Conference*, pages 63–69, Ann Arbor, Michigan, 1968. Academic Press, New York.

[94] R. Haas, D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, D. Souvaine, I. Streinu, and W. Whiteley. Planar minimally rigid graphs and pseudo-triangulations. *Computational Geometry, Theory and Applications*, 31(1–2):31–61, 2005.

[95] T. Hackl. *Relaxing and lifting triangulations*. PhD thesis, IST & IGI-TU Graz, Austria, 2010.

[96] H. Harborth. Konvexe Fünfecke in ebenen Punktmengen. *Elemente der Mathematik*, 33:116–118, 1978.

[97] M. Hoffmann and C. Tóth. Pointed and colored binary encompassing trees. In *Proc. 21st ACM Symposion on Computational Geometry*, pages 81–90, Pisa, Italy, 2005. ACM Press.

[98] P. Hoffmann. *The Man Who Loved Only Numbers*. Hyperion, New York, 1998.

[99] J. Horton. Sets with no empty convex 7-gons. *Canadian Mathematical Bulletin*, 26(4):482–484, 1983.

[100] C. Huemer. *Compatible Geometric Graphs: Problems on Trees and Matchings*. PhD thesis, Graz University of Technlolgy, Graz, Austria, 2008.

[101] C. Huemer and C. Seara. 36 two-colored points with no empty monochromatic convex fourgons. In *Geombinatorics XIX*, pages 5–6, 2009.

[102] F. Hurtado. Open problem session. 23^{st} European Workshop on Computational Geometry EWCG '05, Eindhoven, The Netherlands, 2005.

[103] F. Hurtado. Personal communication, February 2009.

[104] F. Hurtado, M. Kano, D. Rappaport, and C. D. Tóth. Encompassing colored planar straight line graphs. *Computational Geometry: Theory and Applications*, 39:14–23, 2008.

[105] M. Ishaque, D. L. Souvaine, and C. D. Toth. Disjoint compatible geometric matchings. In *Proceedings of the 27th annual ACM symposium on Computational geometry*, SoCG '11, pages 125–134, New York, NY, USA, 2011. ACM.

[106] J. W. Jaromczyk and G. T. T. Abstract. Relative neighborhood graphs and their relatives. In *Proceedings of IEEE*, volume 80, pages 1502–1517, 1992.

[107] J. Kalbfleisch, J. Kalbfleisch, and R. Stanton. A combinatorial problem on convex n-gons. In *Proc. Louisiana Conference on Combinatorics, Graph Theory and Computing*, pages 180–188, Louisiana State University, 1970.

[108] G. Károlyi, J. Pach, and G. Tóth. A modular version of the Erdős-Szekeres theorem. *Studia Scientiarum Mathematicarum Hungarica*, 38(1–4):245–260, 2001.

[109] M. Katchalski and A. Meir. On empty triangles determined by points in the plane. *Acta Mathematica Hungarica*, 51(3-4):323–328, 1988.

[110] D. J. Kleitman. The crossing number of $K_{5,n}$. *Journal of Combinatorial Theory*, 9(4):315 – 323, 1970.

[111] P. Koebe. Kontaktprobleme der konformen Abbildung. *Ber. Verh. Sächs. Akademie der Wissenschaften Leipzig, Math.-Phys. Klasse*, 88:141–164, 1936.

[112] G. Laman. On graphs and rigidity of plane skeletal structures. *Journal of Engineering Mathematics*, 4(4):331–340, 1970.

[113] L. Larson. *Problem-Solving Through Problems.* Springer, New York, 1983.

[114] D. T. Lee and A. Lin. Generalized Delaunay triangulation for planar graphs. *Discrete and Computational Geometry*, 1:201–217, 1986.

[115] J. A. D. Loera, J. Rambau, and F. Santos. *Triangulations: Structures for Algorithms and Applications*, volume 25 of *Algorithms and Computation in Mathematics.* Springer-Verlag, 2010.

[116] J. A. D. Loera, J. Rambau, and F. Santos. *Triangulations: Structures for Algorithms and Applications*, volume 25 of *Algorithms and Computation in Mathematics.* Springer-Verlag, 2010.

[117] B. Mohar. A polynomial time circle packing algorithm. *Discrete Mathematics*, 117(1–3):257–263, 1993.

[118] C. Nicolás. The empty hexagon theorem. *Discrete and Computational Geometry*, 38(2):389–397, 2007.

[119] T. Nishizeki and M. S. Rahman. *Planar Graph Drawing.* World Scientific, 2004.

[120] D. Orden and P. Ramos. Notes on Zarankiewicz's conjecture, 2007. Mansucript.

[121] M. H. Overmars. Finding sets of points without empty convex 6-gons. *Discrete and Computational Geometry*, 29:153 – 158, 2003.

[122] J. Pach. On simplices embracing a point. invited talk, Topological & Geometric Graph Theory (TGGT), Paris, France, 2008.

[123] J. Pach and P. K. Agarwal. *Combinatorial Geometry*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995.

[124] J. Pach and G. Tót. Monochromatic empty triangles in two-colored point sets. *Geometry, Games, Graphs and Education: The Joe Malkevitch Festschrift*, pages 195–198, 2008.

[125] J. Pach and G. Tóth. Which crossing number is it anyway? *Journal of Combinatorial Theory, Series B*, 80(2):225–246, 2000.

[126] R. Pinchasi, R. Radoičić, and M. Sharir. On empty convex polygons in a planar point set. *Journal of Combinatorial Theory, Series A*, 113(3):385–419, 2006.

[127] M. Pocchiola and G. Vegter. Pseudo-triangulations: theory and applications. In *SCG '96: Proceedings of the 12th Annual Symposium on Computational Geometry*, pages 291–300, New York, NY, USA, 1996. ACM.

[128] M. Pocchiola and G. Vegter. The visibility complex. *International Journal of Computational Geometry & Applications*, 6(3):279–308, 1996.

[129] F. P. Preparata and M. I. Shamos. *Computational Geometry - An Introduction*. Springer, 1985.

[130] D. Rappaport. Computing simple circuits from a set of line segments is NP-complete. *SIAM Journal on Computing*, 18:1128–1139, 1989.

[131] D. Rappaport, H. Imai, and G. T. Toussaint. Computing simple circuits from a set of line segments. *Discrete and Computational Geometry*, 5:289–304, 1990.

[132] G. Rote, F. Santos, and I. Streinu. Pseudo-triangulations — a survey. In J. E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry—Twenty Years Later*, volume 453 of *Contemporary Mathematics*, pages 343–410. American Mathematical Society, 2008.

[133] T. Sakai and J. Urrutia. Covering the convex quadrilaterals of point sets. *Graphs and Combinatorics*, 23(1):343–357, 2007.

[134] G. Salazar. Personal communication, September 2011.

[135] W. Schnyder. Embedding planar graphs on the grid. In *SODA '90: Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms*, pages 138–148, Philadelphia, PA, USA, 1990. Society for Industrial and Applied Mathematics.

[136] M. Sharir and A. Sheffer. Counting triangulations of planar point sets. *The Electronic Journal of Combinatorics*, 18(1):P70 1–74, 2011.

[137] I. Streinu. A combinatorial approach to planar non-colliding robot arm motion planning. In *41st Annual Symposium on Foundations of Computer Science*, pages 443–453, Los Alamitos, CA, 2000. IEEE Comput. Soc. Press.

[138] I. Streinu. Pseudo-triangulations, rigidity and motion planning. *Discrete and Computational Geometry*, 34(4):587–635, 2005.

[139] G. Szekeres and L. Peters. Computer solution to the 17-point Erdős–Szekeres problem. *The ANZIAM Journal*, 48(2):151–164, 2006.

[140] G. Tóth and P. Valtr. The Erdős–Szekeres theorem: upper bounds and related results. *Combinatorial and Computational Geometry, J.E. Goodman, J. Pach, and E. Welzl (Eds.),*, 52:557–568, 2005.

[141] P. Turán. A note of welcome. *Journal of Graph Theory*, 1(1):7–9, 1977.

[142] K. Urbaník. Solution du problème posé par P. Turán. *Colloquium Mathematicum*, 3:200–201, 1955.

[143] J. Urrutia. Coloraciones, tetraedralizaciones, y tetraedros vacios en coloraciones de conjuntos de puntos en R^3. In *Proc. X Encuentros de Geometría Computacional*, pages 95–100, Sevilla, Spain, 2003.

[144] P. Valtr. Convex independent sets and 7-holes in restricted planar point sets. *Discrete and Computational Geometry*, 7:135–152, 1992.

[145] P. Valtr. On empty hexagons. In J. E. Goodman, J. Pach, and R. Pollack, editors, *Surveys on Discrete and Computational Geometry—Twenty Years Later*, volume 453 of *Contemporary Mathematics*, pages 433–441. American Mathematical Society, 2008.

[146] H. Whitney. A set of topological invariants for graphs. *American Journal of Mathematics*, 55:235–321, 1933.

[147] D. R. Woodall. Cyclic-order graphs and Zarankiewicz's crossing-number conjecture. *Journal of Graph Theory*, 17(6):657–671, 1993.

[148] K. Zarankiewicz. The solution of a certain problem on graphs of P. Turán. *Bulletin de l'Académie Polonaise des Sciences*, 1:167–168, 1953.

[149] K. Zarankiewicz. On a problem of P. Turán concerning graphs. *Fundamenta Mathematicae*, 41:137–145, 1954.

Index

Curriculum Vitae of Birgit Vogtenhuber

Affiliation and address

Dipl.-Ing. Birgit Vogtenhuber
Research Assistant
Institute for Software Technology
Graz University of Technology
Inffeldgasse 16b, II
A-8010 Graz, Austria
Phone: `++43 316 873 5703`
Email: `bvogt@ist.tugraz.at`

Personal Information

Born February 18, 1979 in Vöcklabruck (Austria)
Austrian nationality

Education

Centre de Recerca Mathemàtica, Bellaterra, Spain
January - March 2009
Participation (with awarded grant) at 'I-Math Winter School – DocCourse Combinatorics and Geometry 2009: Discrete and Computational Geometry'

Graz University of Technology, Austria
2007 - present
PhD student, Advisor: Prof. O. Aichholzer
1999 - 2007
Dipl.-Ing. (MSc), Technical Mathematics, Advisor: Prof. O. Aichholzer
Study of Technical Mathematics (Information and Dataprocessing), Diploma Thesis at the Institute for Software Technology (with distinction)

Vienna University of Technology, Austria
1997 - 1999
Studies of Technical Mathematics (Computer Science)

High School BRG, Vöcklabruck, Austria
1989 - 1997
Matura (A-levels) (with distinction)

Professional Experience

AXIS Flight Training Systems GmbH, Lebring, Austria
2010 - present[1]
Part time software engineer

Graz University of Technology, Austria
2007 - present[1]
Part time research assistant at the Institute for Software Technology

[1]Leave of absence from January to March 2009

supported by the Austrian Science Fund (FWF): S9205-N12, National Research Network 'Industrial Geometry'.

APUS Software GmbH in Graz, Austria
2001 - 2010
Part time software engineer

Graz University of Technology, Austria
2002 - 2006
Teaching assistant at the Institute for Software Technology
2001
Teaching assistant at the Institute for Mathematics
2000 - 2002
Teaching assistant at the Institute for Theoretical Computer Science

Career Related Activities

Organization	Co-organizer of the annual European Pseudo-triangulation research week in 2008 and 2009. Member of the organizing committee of the 23rd European Workshop on Computational Geometry 2007, Graz, Austria.
Co-supervisor	Of 4 programming projects on algorithms, geometry, and optimization.
Sub-Referee	For Computational Geometry: Theory and Applications (Elsevier), and Journal of Discrete Algorithms (Elsevier).
Research stays	Freie Universität Berlin (Germany), TU Eindhoven (Netherlands), Universitat Politécnica de Catalunya (Barcelona, Spain), INRIA - Sophia Antipolis (France), Universidad de Alcalá (Alcalá de Hernares, Spain), Universidad Nacional Autónoma de México (Mexico City, Mexico), Universidad Michoacana San Nicolás de Hidalgo (Morelia, Mexico), University of Waterloo (Canada).

Research Interests

Data structures and algorithms in general, combinatorial geometry and geometric graph theory in particular, especially Erdős-Szekeres type questions and combinatorial properties of geometric graphs.

Publications overview

3+3+4 Articles in refereed journals (published + to appear + submitted).
17 Articles in refereed proceedings.
Diploma Thesis.

The publications marked bold in the following list arose from parts of this doctoral thesis. The list is in reverse chronological order (sorted alphabetically by authors within each year) and categorized in journal appearances, conference papers, and theses.

Publications

3+3+4 articles in refereed journals:

submitted

O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. *submitted to special issue for the* 27^{th} *European Workshop on Computational Geometry*, 2011.

O. Aichholzer, T. Hackl, M. Hoffmann, A. Pilz, G. Rote, B. Speckmann, and B. Vogtenhuber. Plane graphs with parity constraints. *submitted to journal*, 2011.

O. Aichholzer, T. Hackl, and B. Vogtenhuber. On 5-gons and 5-holes. *submitted to special issue for the XIV Encuentros de Geometría Computacional ECG2011*, 2011.

to appear

O. Aichholzer, R. Fabila-Monroy, T. Hackl, M. van Kreveld, A. Pilz, P. Ramos, and B. Vogtenhuber. Blocking Delaunay triangulations. *Computational Geometry, Theory and Applications (special issue for the* 22^{nd} *Canadian Conference on Computational Geometry)*, 2011.

O. Aichholzer, T. Hackl, M. Hoffmann, C. Huemer, F. Santos, B. Speckmann, and B. Vogtenhuber. Maximizing maximal angles for plane straight line graphs. *Computational Geometry: Theory and Applications*, 2011.

O. Aichholzer, G. Rote, A. Schulz, and B. Vogtenhuber. Pointed drawings of planar graphs. *Computational Geometry: Theory and Applications (special issue for the* 19^{th} *Canadian Conference on Computational Geometry)*, 2011.

2010

O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, and B. Vogtenhuber. Large bichromatic point sets admit empty monochromatic 4-gons. *SIAM Journal on Discrete Mathematics (SIDMA)*, 23(4): 2147–2155, 2010.

2008

O. Aichholzer, F. Aurenhammer, P. Gonzalez-Nava, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, S. Ray, and B. Vogtenhuber. Matching edges and faces in polygonal partitions. *Computational Geometry: Theory and Applications*, 39(2):134–141, 2008.

2007

O. Aichholzer, F. Aurenhammer, C. Huemer, and B. Vogtenhuber. Gray code enumeration of plane straight-line graphs. *Graphs and Combinatorics (Springer)*, 23(5):467–479, 2007.

O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, and B. Vogtenhuber. On the number of plane geometric graphs. *Graphs and Combinatorics (Springer)*, 23(1):67–84, 2007.

17 articles in refereed proceedings:

2011

O. Aichholzer, R. Fabila-Monroy, H. González-Aguilar, T. Hackl, M. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber. 4-holes in point sets. In *Proc.* 27^{th} *European Workshop on Computational Geometry EuroCG '11*, pages 115–118, Morschach, Switzerland, 2011.

O. Aichholzer, R. Fabila-Monroy, H. Gonzalez-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, P. Valtr, and B. Vogtenhuber. On k-gons and k-holes in point sets. In *Proc.* 23^{rd} *Annual Canadian Conference on Computational Geometry CCCG 2011*, pages 21–26, Toronto, Canada, 2011.

O. Aichholzer, T. Hackl, and B. Vogtenhuber. On 5-gons and 5-holes. In *Proc. XIV Encuentros de Geometría Computacional*, pages 7–10, Alcalá, Spain, 2011.

2010

O. Aichholzer, R. Fabila-Monroy, T. Hackl, M. van Kreveld, A. Pilz, P. Ramos, and B. Vogtenhuber. Blocking Delaunay triangulations. In *Proc.* 22^{nd} *Annual Canadian Conference on Computational Geometry CCCG 2010*, pages 21–24, Winnipeg, Manitoba, Canada, 2010.

O. Aichholzer, T. Hackl, and B. Vogtenhuber. Compatible pointed pseudo-triangulations. In *Proc.* 22^{nd} *Annual Canadian Conference on Computational Geometry CCCG 2010*, pages 91–94, Winnipeg, Manitoba, Canada, 2010.

O. Aichholzer, F. Aurenhammer, T. Hackl, C. Huemer, A. Pilz, and B. Vogtenhuber. 3-colorability of pseudo-triangulations. In *Proc.* 26^{th} *European Workshop on Computational Geometry EuroCG '10*, pages 21–24, Dortmund, Germany, 2010.

2009

O. Aichholzer, F. Aurenhammer, O. Devillers, T. Hackl, M. Teillaud, and B. Vogtenhuber. Lower and upper bounds on the number of empty cylinders and ellipsoids. In *Proc.* 25^{th} *European Workshop on Computational Geometry EuroCG '09*, pages 139–142, Brussels, Belgium, 2009.

O. Aichholzer, R. Fabila-Monroy, D. Flores-Peñaloza, T. Hackl, C. Huemer, J. Urrutia, and B. Vogtenhuber. Modem illumination of monotone polygons. In *Proc.* 25^{th} *European Workshop on Computational Geometry EuroCG '09*, pages 167–170, Brussels, Belgium, 2009.

O. Aichholzer, T. Hackl, M. Hoffmann, A. Pilz, G. Rote, B. Speckmann, and B. Vogtenhuber. Plane graphs with parity constraints. In *Lecture Notes in Computer Science, Proc.* 11^{th} *International Workshop on Algorithms and Data Structures (WADS)*, volume 5664, pages 13–24, Banff, Alberta, Canada, 2009.

O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, and B. Vogtenhuber. Large bichromatic point sets admit empty monochromatic 4-gons. In *Proc.* 25^{th} *European Workshop on Computational Geometry EuroCG '09*, pages 133–136, Brussels, Belgium, 2009.

2007

O. Aichholzer, T. Hackl, M. Hoffmann, C. Huemer, F. Santos, B. Speckmann, and B. Vogtenhuber. Maximizing maximal angles for plane straight line graphs. In *Proc.* 23^{rd} *European Workshop on Computational Geometry EuroCG '07*, pages 98–101, Graz, Austria, 2007.

O. Aichholzer, T. Hackl, M. Hoffmann, C. Huemer, A. Por, F. Santos, B. Speckmann, and B. Vogtenhuber. Maximizing maximal angles for plane straight line graphs. In *Lecture Notes in Computer Science, Proc.* 10^{th} *International Workshop on Algorithms and Data Structures (WADS)*, volume 4619, pages 458–469, Halifax, Nova Scotia, Canada, 2007.

O. Aichholzer, G. Rote, A. Schulz, and B. Vogtenhuber. Pointed drawings of planar graphs. In *Proc.* 19*th* *Annual Canadian Conference on Computational Geometry CCCG 2007*, pages 237–240, Ottawa, Ontario, Canada, 2007.

2006

O. Aichholzer, F. Aurenhammer, C. Huemer, and B. Vogtenhuber. Gray code enumeration of plane straight-line graphs. In *Proc.* 22^{nd} *European Workshop on Computational Geometry EuroCG '06*, pages 71–74, Delphi, Greece, 2006.

O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, and B. Vogtenhuber. On the number of plane graphs. In *Proc.* 17^{th} *Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 504–513, Miami, Florida, USA, 2006.

2005

O. Aichholzer, F. Aurenhammer, P. Gonzalez-Nava, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, S. Ray, and B. Vogtenhuber. Matching edges and faces in polygonal partitions. In *Proc.* 17^{th} *Annual Canadian Conference on Computational Geometry CCCG 2005*, pages 123–126, Windsor, Ontario, Canada, 2005.

O. Aichholzer, T. Hackl, C. Huemer, F. Hurtado, H. Krasser, and B. Vogtenhuber. Bounding the number of plane graphs. In *Proc.* 15^{th} *Annual Fall Workshop on Computational Geometry and Visualization*, pages 31–32, Philadelphia, Pennsylvania, USA, 2005.

Thesis:

B. Vogtenhuber. On plane straight-line graphs. Master's thesis, IST, TU Graz, Austria, January 2007. Supervisor: Oswin Aichholzer.