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Boundary Element Methods for Eddy Current Transmission Problems

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Abstract

In this thesis we discuss the derivation and analysis of boundary element methods for the simulation of eddy current problems. The eddy current problem leads to a transmission problem, for which we derive different formulations based on Maxwell's equations. The electric field and the magnetic field intensity are both governed by a second order partial differential equation: the electromagnetic wave equation. In this thesis we derive boundary integral equations which describe solutions of the electromagnetic wave equation. We observe that the standard boundary integral operators tend to be instable when considering small wave numbers. We deduce an alternative boundary integral equation and prove its stability, if the wave number tends to zero. For the eddy current transmission problem with piecewise constant material parameters, we derive two different boundary integral formulations, which are based on the principle of symmetric coupling. In the first formulation the unknowns are given by traces of the electric field, in the second formulation the unknowns are given by traces of the magnetic field intensity. Moreover we present a non-symmetric indirect formulation based on the magnetic field intensity. For the discretization of the boundary integral formulations we introduce suitable boundary element spaces for the test and ansatz functions. Based on the Galerkin method, we deduce the discrete versions of the derived boundary integral formulations. We illustrate them by some numerical examples.
As an application we consider Magnetic Induction Tomography. The corresponding forward problem leads to an eddy current problem. For this specific eddy current problem, we derive a reduced formulation and investigate the error between the full eddy current model and the reduced formulation. We further introduce a boundary element method for the reduced model and present some numerical examples. Finally, we deal with the inverse problem of Magnetic Induction Tomography. We formulate the inverse problem as a shape reconstruction problem. We define the shape functional for the reduced formulation and compute its shape derivative.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der Herleitung und Analysis von Randelementmethoden zur Simulation eines Wirbelstromproblems. Das Wirbelstromproblem wird als Transmissionsproblem modelliert, für welches wir verschiedene Formulierungen ausgehend von den Maxwellschen Gleichungen herleiten. Die elektrische und magnetische Feldstärke erfüllen hierbei eine partielle Differentialgleichung zweiter Ordnung: die elektromagnetische Wellengleichung. Lösungen der elektromagnetischen Wellengleichung können durch Randintegralgleichungen beschrieben werden. Hierbei beobachten wir, dass die Standard-Randintegraloperatoren für die elektromagnetische Wellengleichung instabil sind, wenn man kleine Wellenzahlen betrachtet. Wir geben eine alternative Randintegralgleichung an und beweisen, dass diese stabil ist, wenn man die Wellenzahl gegen Null gehen lässt.
Für das Wirbelstrom-Transmissionsproblem mit stückweise konstanten Materialparametern leiten wir zwei verschiedene Randintegralformulierungen her, die auf dem Prinzip der symmetrischen Kopplung beruhen. In der ersten Formulierung sind die Unbekannten durch Spuren der elektrischen Feldstärke am Rand gegeben, in der zweiten Formulierung sind die Unbekannten durch Spuren der magnetischen Feldstärke gegeben. Weiters wird eine nicht symmetrische, indirekte Formulierung basierend auf der magnetischen Feldstärke vorgestellt. Für die Diskretisierung der Randintegralformulierungen werden für Ansatz- und Testfunktionen entsprechende Randelementräume eingeführt. Basierend auf der Galerkin Methode werden für die eingeführten Randintegralformulierungen entsprechende diskrete Formulierungen angegeben und durch numerische Beispiele illustriert.
Als Anwendung eines Wirbelstromproblems untersuchen wir die Magnetische Induktions Tomographie. Das zugehörige Vorwärtsproblem führt auf ein Wirbelstromproblem. Wir leiten für dieses spezielle Problem eine reduzierte Formulierung her und untersuchen den Fehler zur Lösung der vollständigen Wirbelstromformulierung. Für die reduzierte Formulierung wird eine Randelementmethode hergeleitet und numerische Beispiele dazu gezeigt. Zum Schluss beschäftigen wir uns mit dem inversen Problem der Magnetischen Induktions Tomographie. Wir formulieren das Inverse Problem als Formoptimierungsproblem, geben das zugehörige Formfunktional für die reduzierte Formulierung an und berechnen dessen Formableitung.

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CONTENTS

1 INTRODUCTION

1.1 Motivation

A conducting object under the influence of a time varying magnetic field produces electric currents. This fact was observed by Francois Arago (1786-1853) in an experiment, where he discovered that a magnetized needle is driven by a moving conducting body. The first who found an explanation for this effect was Michael Faraday (1791-1867), who studied time varying currents and magnetic fields. A similar experiment was carried out by Jean Bernard Léon Foucault (1819-1868), where a copper plate was rotated between two magnetic poles. He observed that the magnetic poles slow down the movement of the plate, like an 'invisible break'. He deduced that due to the rotation currents are generated in the conducting plate and that the work which is lost due to the slowdown of the plate transforms to heat inside the conducting body. [1]

Nowadays the effects observed by Arago and Foucault are well understood and we are able to describe them by well elaborated mathematical models. The basis for these mathematical models are Maxwell's equations, which are a set of equations, which describe the origin and interaction of electric and magnetic fields. Numerical methods for Maxwell's equations, which have been developed in the past and present century, make it possible to simulate electrodynamic processes. This enables us to predict the behavior of electric and magnetic fields in a certain setting, this is important for many engineering applications.

For the numerical simulation we have various methods at hand, among the most popular are the finite element method, the finite difference method and the boundary element method. The goal of this thesis is to describe and analyze boundary element methods for the simulation of eddy current problems.

[1] Foucault, Léon. Notice sur les travaux, XVII. De la chaleur produite par l'action de l'aimant sur le corps en mouvement, 1863. The digitalized document is available at http://num-scd-ulp.u-strasbg.fr:8080/498/.

1.2 State of the Art

Boundary integral equation methods are well established as a tool for simulations in electromagnetic engineering. For problems in electrostatics or magnetostatics, where the governing equation reduces to a potential equation, results about the analysis of boundary integral equations and boundary element methods are well known.

A first important result concerning the representation of electric and magnetic fields by boundary potentials in the time harmonic case was derived by Stratton and Chu [67]. For scattering or eddy current problems important results concerning the analysis of boundary integral equations go back to J.C. Nédélec [52]. The analysis of boundary integral equations is based on the definition of appropriate trace spaces on the boundary. For domains with a smooth boundary results about the appropriate trace spaces are known for quite long time, for Lipschitz domains the analysis of the trace spaces has been done quite recently [12, 13, 15].

The simulation of the eddy current problem, i.e. we have given a conducting body which is exposed to a time harmonic magnetic field, leads to a transmission problem in the whole space $\mathbb{R}^3$. When using boundary integral equations one can reduce the transmission problem to a problem on the boundary of the conducting domain provided the conductivity and permeability is constant inside the conducting domain. In the engineering literature we can find numerous examples for the use of boundary integral equation methods to solve the eddy current problem (see e.g. [38, 39, 74]). A boundary integral formulation for the eddy current transmission problem based on the principle of symmetric coupling can be found in [32].

When dealing with practical applications it is important to be able to treat structures consisting of different materials. In this thesis boundary integral equation formulations are given, with which we are able to cover problems, where different materials are involved. A theoretic treatment of this case in a slightly different setting can also be found in [9].

The behaviour of mathematical methods for numerical simulations usually depends on the parameter range we are using, e.g. if we are dealing with low or high frequency problems. The parameters are also determined by the type of application we are dealing with, i.e. for the simulation of power transformers the parameter range is different as when we are considering the simulation of eddy currents in biological tissues. For small parameters, i.e. conductivity or frequency, the standard boundary element formulations tend to be instable. For example when the frequency goes to zero, the electric and the magnetic field decouple. This can cause problems in the numerical simulation using standard boundary integral equation approaches, which is a well known fact in the engineering literature [5,16,42,45,69,73]. We are going to present a formulation, which is stable when the wave number κ is small and which is also valid for the case $\kappa = 0$.

1.3 Outline of the Thesis

This thesis is organized as follows: In the second chapter we give an introduction to the modeling of electromagnetic processes using Maxwell's equations. We derive the eddy current model from Maxwell's equations. We will also look at the model application of Magnetic Induction Tomography (MIT). We derive and analyze two different mathematical models for the forward problem of Magnetic Induction Tomography, a reduced model and the full eddy current model.

In the third chapter of the thesis basic results from functional analysis will be given, which will be needed later on for the analysis of boundary integral equations. We will also introduce Sobolev spaces in the domain and on the boundary.

In the fourth chapter we turn to boundary integral equations. First we introduce the representation formula for the solution of a scalar equation of the type

$$-\Delta u(x) + \kappa^2 u(x) = 0,$$

and give the basic results on boundary integral operators and equations. These results are all standard and well known [59, 66], thus we will only give a brief summary. After this we will come to the main part of the third chapter, which will be devoted to boundary integral equations for the electromagnetic wave equation, which is an equation of the type

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) + \kappa^2 \mathbf{U}(x) = 0.$$

We will derive a representation formula for solutions of the electromagnetic wave equation. This representation formula states that the solution in a bounded or an unbounded domain can be represented by certain surface potentials of boundary traces of the function. This representation formula is also known as Stratton-Chu formula. Starting from the representation formula we derive boundary integral equations for the electromagnetic wave equation. We will refer to the results in [52]. We summarize the basic results about Steklov-Poincaré operators for the electromagnetic wave equation.

The fifth chapter is devoted to boundary element formulations for eddy current transmission problems. We consider the eddy current problem in the following setting:

We have a conducting domain, consisting of different materials and we have a primary magnetic field, e.g. generated by a coil outside of the conducting domain. We derive different boundary integral formulations, which are based on different physical quantities.

To be able to perform simulations, we need to discretize the boundary integral equations, this topic will be covered in the sixth chapter. We introduce boundary element spaces, which are used to represent the traces of the electric and magnetic field on a boundary element mesh. We derive the linear systems resulting from the discretization of the boundary integral formulations derived in the previous two chapters. We illustrate all boundary element formulations by numerical results. We give iteration numbers and convergence results. For the eddy current problem we verify the methods by using a benchmark problem.

The last section deals with the inverse problem of Magnetic Induction Tomography. We are going to consider Magnetic Induction Tomography using a shape reconstruction approach. To describe the deformation of the shape we are going to use the velocity method, in this setting we then compute the shape derivatives for a reduced model.

2 BOUNDARY VALUE AND TRANSMISSION PROBLEMS IN ELECTROMAGNETISM

Electromagnetic phenomena are described by Maxwell's equations in a very general way. To be able to carry out a numerical simulation of electromagnetic fields one usually derives a simplified model from the general set of Maxwell's equations, which is suited for a specific type of problem or for a particular application. In this section we derive such simplified models starting from the full set of Maxwell's equations.
We will consider the following setting: We have a given time-harmonic primary magnetic field and a conducting body. The primary magnetic field induces eddy currents inside the conducting body, the eddy currents themselves again produce a secondary magnetic field. The aim is to compute the eddy currents inside the conducting domain and the secondary magnetic field outside the domain.
There are several technical applications which make use of this phenomenon. Eddy current imaging for crack detection is commonly used. In this section we will present a biomedical application, which is Magnetic Induction Tomography (see [28, 60]).

2.1 Maxwell's Equations

Maxwell's equations describe the interaction and mutual dependence of the following physical quantities:

- $\mathbf{E}(t,x)\dots$ electric field (V/m)
- $\mathbf{D}(t,x)\dots$ displacement field (C/m^2)
- $\mathbf{H}(t,x)\dots$ magnetic field intensity (A/m)
- $\mathbf{B}(t,x)\dots$ magnetic field or magnetic induction (T)
- $\mathbf{j}(t,x)\dots$ electric current (A/m^2)
- $\rho(t,x)\dots$ electric charge (C/m^3)

The Gauss law states that the sources of electric (displacement) fields are charges, i.e.

$$\operatorname{div}\mathbf{D}(t,x) = \rho(t,x). \tag{2.1}$$

In nature, no isolated magnetic charges have been discovered, this fact is expressed mathematically as

$$\operatorname{div} \mathbf{B}(t,x) = 0, \tag{2.2}$$

which is also called the magnetic Gauss law. A consequence of this is that all magnetic field lines are closed.
Faraday discovered that a time varying magnetic field produces an electric current, this is known as Faraday's law of electromagnetic induction,

$$\mathbf{curl}\, \mathbf{E}(t,x) = -\frac{\partial}{\partial t}\mathbf{B}(t,x). \tag{2.3}$$

Finally, Ampere's law states that

$$\mathbf{curl}\, \mathbf{H}(t,x) = \mathbf{j}(t,x) + \frac{\partial}{\partial t}\mathbf{D}(t,x). \tag{2.4}$$

In this thesis we will deal with time-harmonic and stationary fields, this means that we assume that all the excitation fields are either time-harmonic or stationary. So we assume that the time-dependent part of a state variable can be expressed by sinusoidal functions

$$\mathbf{F}(t,x) = \Re(\mathbf{F}(x)e^{i\omega t}), \quad \mathbf{F} = \mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{j}, \rho.$$

By inserting this representation into the time dependent Maxwell's equations (2.1)-(2.4) we obtain the following set of equations:

$$\mathbf{curl}\, \mathbf{E}(x) = -i\omega\mathbf{B}(x), \tag{2.5}$$
$$\mathbf{curl}\, \mathbf{H}(x) = \mathbf{j}(x) + i\omega\mathbf{D}(x), \tag{2.6}$$
$$\operatorname{div} \mathbf{D}(x) = \rho(x), \tag{2.7}$$
$$\operatorname{div} \mathbf{B}(x) = 0. \tag{2.8}$$

In addition we have the continuity equation

$$\operatorname{div} \mathbf{j}(x) = -i\omega\rho(x),$$

which follows from the Maxwell's equations in the case $\omega \neq 0$. Furthermore, we assume the constitutive relations

$$\mathbf{D}(x) = \varepsilon(x)\mathbf{E}(x), \quad \mathbf{j}(x) = \sigma(x)\mathbf{E}(x), \quad \mathbf{B}(x) = \mu(x)\mathbf{H}(x). \tag{2.9}$$

In particular we presume that all materials are isotropic and linear. $\varepsilon(F/m)$ is called the permittivity, $\sigma(S/m)$ is the conductivity and $\mu(H/m)$ describes the permeability. When considering time-harmonic problems the conductivity of the considered material can dependen on the frequency ω. The permittivity of vacuum is

$$\varepsilon_0 = 8.85418\ldots \cdot 10^{-12}\frac{As}{Vm},$$

and the permeability of vacuum is given by

$$\mu_0 = 4\pi \cdot 10^{-7}\frac{H}{m}.$$

Moreover, we introduce the (given) 'impressed' current $\mathbf{j}_i$. In what follows we will assume that there are no charges. By using these material laws we can reformulate the set of Maxwell's equations as a system of partial differential equations with two unknowns, these are the electric field $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$:

$$\mathbf{curl}\,\mathbf{E}(x) = -i\omega\mu(x)\mathbf{H}(x), \tag{2.10}$$

$$\mathbf{curl}\,\mathbf{H}(x) = \mathbf{j}_i(x) + (\sigma(x) + i\omega\varepsilon(x))\mathbf{E}(x), \tag{2.11}$$

$$\operatorname{div}(\varepsilon(x)\mathbf{E}(x)) = \rho(x), \tag{2.12}$$

$$\operatorname{div}(\mu(x)\mathbf{B}(x)) = 0. \tag{2.13}$$

In (2.11) the linear combination of the conductivity and permittivity appears, we introduce

$$\kappa(x) := \sigma(x) + i\omega\varepsilon(x)$$

and call $\kappa(x)$ complex conductivity.
The system (2.10)-(2.13) is a system of first oder differential equations with the two unknowns $\mathbf{E}$ and $\mathbf{H}$. By inserting (2.10) into (2.11) we obtain a second order partial differential equation for the electric field $\mathbf{E}$,

$$\mathbf{curl}\,\left[\mu(x)^{-1}\mathbf{curl}\,\mathbf{E}(x)\right] + i\omega(\sigma(x) + i\omega\varepsilon(x))\mathbf{E}(x) = 0, \tag{2.14}$$

$$\operatorname{div}(\varepsilon(x)\mathbf{E}(x)) = 0. \tag{2.15}$$

Similar, as for the electric field, we can obtain a second order partial differential equation for the magnetic field intensity $\mathbf{H}$ by inserting (2.11) into (2.10), i.e.

$$\mathbf{curl}\,[\kappa(x)^{-1}\mathbf{curl}\,\mathbf{H}(x)] + i\omega\mu(x)\mathbf{H}(x) = \mathbf{curl}\,[\kappa(x)^{-1}\mathbf{j}(x)], \tag{2.16}$$

$$\operatorname{div}(\mu(x)\mathbf{H}(x)) = 0. \tag{2.17}$$

This shows that both formulations lead to a system of second order partial differential equations of the same type, however, we see that the roles of $\mu(x)$ and $\kappa(x)$ are interchanged and in the $\mathbf{H}$-field formulation we have a possibly nonzero right hand side. Thus, in a certain setting the two formulations can have different mathematical properties. In this thesis we are going to deal with both formulations and analyze their characteristics. Furthermore, we see that, if we solve a formulation for $\mathbf{E}$ or $\mathbf{H}$ we can get the other unknown by inserting into (2.10) or (2.6), provided ω is not zero. If ω is zero, which means that we are in the static case, we have

$$\mathbf{curl}\,\mathbf{E}(x) = 0, \tag{2.18}$$

$$\mathbf{curl}\,\mathbf{H}(x) = \mathbf{j}_i(x) + \sigma(x)\mathbf{E}(x), \tag{2.19}$$

$$\operatorname{div}(\varepsilon(x)\mathbf{E}(x)) = \rho(x), \tag{2.20}$$

$$\operatorname{div}(\mu(x)\mathbf{B}(x)) = 0. \tag{2.21}$$

We observe that in the static case the electric and the magnetic field decouple when the conductivity is zero.

2.2 The Eddy Current Model

We are now looking at a more specific setting: Let us decompose the full space $\mathbb{R}^3$ into a bounded conducting domain Ω, and a non-conducting unbounded air domain $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}$. For the air domain we have $\varepsilon(x) = \varepsilon_0, \sigma(x) = 0$ and $\mu(x) = \mu_0$ for $x \in \Omega^c$. Furthermore, we assume that the impressed current $\mathbf{j}_i(x)$ has only support in the non-conducting domain Ω^c, and we assume that there are no charges, i.e. $\rho(x) \equiv 0$. The eddy current model is obtained from (2.5)-(2.8) by neglecting the displacement currents $i\omega\mathbf{D}$ in the exterior domain and by setting $\operatorname{div}\mathbf{E}(x) = 0$ in Ω^c. Physically speaking, this means that the propagation of electromagnetic waves in the exterior domain, and thus the contribution to the energy of the electromagnetic field, is neglected. Inserting this information into the $\mathbf{E}$-field formulation (2.14)-(2.15), we get the following set of equations:

$$\mathbf{curl}\,\left[\mu(x)^{-1}\mathbf{curl}\,\mathbf{E}(x)\right] + i\omega(\sigma(x) + i\omega\varepsilon(x))\mathbf{E}(x) = 0, \quad x \in \Omega, \tag{2.22}$$

$$\mathbf{curl}\,\left[\mu_0^{-1}\mathbf{curl}\,\mathbf{E}(x)\right] = -i\omega\mathbf{j}_i(x), \quad x \in \Omega^c, \tag{2.23}$$

$$\operatorname{div}\mathbf{E}(x) = 0, \quad x \in \Omega^c. \tag{2.24}$$

Remark 2.1. *If we assume, that the electric current* $\mathbf{j}$ *is solenoidal, i.e.* $\operatorname{div}\mathbf{j} = 0$ *and if* ω *is not zero, then the Gauss law*

$$\operatorname{div}(\varepsilon(x)\mathbf{E}(x)) = 0, \quad x \in \Omega$$

becomes redundant in the conducting domain Ω*, since it follows from (*2.22*).*

For the $\mathbf{H}$-field formulation (2.16)-(2.17) the eddy current model leads to

$$\mathbf{curl}\,[\kappa(x)^{-1}\mathbf{curl}\,\mathbf{H}(x)] + i\omega\mu(x)\mathbf{H}(x) = \mathbf{curl}\,[\kappa(x)^{-1}\mathbf{j}(x)], \quad x \in \Omega, \tag{2.25}$$

$$\mathbf{curl}\,\mathbf{H}(x) = \mathbf{j}_i(x), \quad x \in \Omega^c, \tag{2.26}$$

$$\operatorname{div}(\mu_0\mathbf{H}(x)) = 0, \quad x \in \Omega^c. \tag{2.27}$$

Remark 2.2. *Similar as in the* $\mathbf{E}$*-field-formulation, the magnetic Gauss law becomes redundant in the conducting domain, since it follows from (*2.25*).*

Typical criteria, which are used to check wether the eddy current model is justified, are the conditions

$$L\omega\sqrt{\mu\varepsilon} << 1, \qquad \omega\frac{\varepsilon}{\sigma} << 1 \tag{2.28}$$

where L is the diameter of Ω. In the exterior domain, $\mathbf{E}$ and $\mathbf{H}$ propagate as undamped waves with the wave number $\omega\sqrt{\varepsilon_0\mu_0}$. Hence, the first condition requires that the size of the conductor is small compared to the wave length. An alternative condition is that the size of the conductor is large compared with the skin depth δ which is defined as

$$\delta = \frac{1}{\sqrt{\omega\sigma\mu}}.$$

A mathematical study of the modeling error for the eddy current model can be found in [4].

2.2.1 Transmission and Radiation Conditions

On the boundary $\Gamma = \partial\Omega$ of the conducting domain the continuity of the tangential traces of the fields $\mathbf{E}$ and $\mathbf{H}$ is required, i.e.

$$[n(x) \times (\mathbf{E}(x) \times n(x))] = 0, \quad [n(x) \times (\mathbf{H}(x) \times n(x))] = 0, \quad x \in \Gamma, \tag{2.29}$$

where $[.]$ denotes the jump of a function. Furthermore we have

$$[\mathbf{D}(x) \cdot n(x)] = 0, \quad [\mathbf{j}(x) \cdot n(x)] = 0, \quad [\mathbf{B}(x) \cdot n(x)] = 0, \quad x \in \Gamma. \tag{2.30}$$

Since Ω^c is an unbounded domain we have to impose radiation conditions

$$|\mathbf{E}(x)| = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty, \tag{2.31}$$

$$|\mathbf{H}(x)| = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty. \tag{2.32}$$

In [4] it has been shown that if $\mathbf{E}$ and $\mathbf{H}$ are solutions of the eddy current model and satisfy the radiation conditions (2.31) and (2.32) then it holds

$$|\mathbf{E}(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |\mathbf{H}(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty.$$

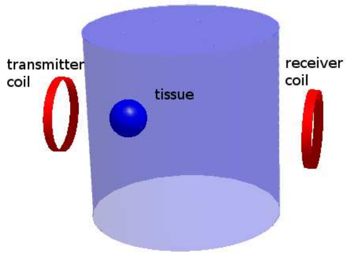


Figure 2.1: Magnetic Induction Tomography setting.

2.3 The Forward Problem of Magnetic Induction Tomography

Problems with low frequencies and small conductivities typically arise in medical applications. An exemplary application, which we are going to analyze is the Magnetic Induction Tomography (MIT), which is a noninvasive and contactless imaging method (see [34,47,48]). It is based on the fact that a time harmonic current induces eddy currents inside a conducting domain. The imaging method works as follows: excitation coils generate a time harmonic magnetic field. This field induces eddy currents which generate a magnetic field and which perturbs the primary magnetic field. Around the body an array of receiver coils is placed (see Figure 2.1), in which the perturbed magnetic field is measured. Out of this information one can gain knowledge about the conductivity distribution inside the body by solving an inverse problem. The solution of the inverse problem usually requires the evaluation of the forward map.
In this section we will discuss two different models for the forward problem of Magnetic Induction Tomography, the eddy current model and a reduced model. The reduced model is based on a quasi-static approximation and reduces the eddy current model to a potential equation. The reduced model is employed when dealing with low frequencies and conductivities as it is the case in Magnetic Induction Tomography. Moreover, we will provide error estimates for the error between the reduced and the eddy current model.

For the solution of the forward problem we split the magnetic and the electric fields into a 'primary' field and a 'secondary' field:

$$\mathbf{E}(x) = \mathbf{E}_s(x) + \mathbf{E}_p(x), \quad \mathbf{H}(x) = \mathbf{H}_s(x) + \mathbf{H}_p(x).$$

The primary electric and magnetic fields are the fields of a coil in free space without the presence of a conducting object. The primary electric field $\mathbf{E}_p$ and the primary magnetic field $\mathbf{H}_p$ can be retrieved by solving

$$-\Delta \mathbf{E}_p(x) = -i\omega\mu_0 \mathbf{j}_i(x), \quad x \in \mathbb{R}^3, \quad -\Delta \mathbf{H}_p(x) = \mathbf{curl}\, \mathbf{j}_i(x), \quad x \in \mathbb{R}^3.$$

Hence we can represent the primary fields by the Newton potentials

$$\mathbf{E}_p(x) = -\frac{i\omega\mu_0}{4\pi} \int\limits_{\mathbb{R}^3} \frac{\mathbf{j}_i(y)}{|x-y|} ds_y, \quad \mathbf{H}_p(x) = \frac{1}{4\pi} \mathbf{curl}_x \int\limits_{\mathbb{R}^3} \frac{\mathbf{j}_i(y)}{|x-y|} ds_y, \quad x \in \mathbb{R}^3. \quad (2.33)$$

For certain types of coil geometries, i.e. for the support of $\mathbf{j}_i$, those integrals can be evaluated analytically. If no analytical formula exists the primary field can be computed by solving a related boundary value problem.

2.3.1 The Eddy Current Model

In Magnetic Induction Tomography one usually deals with low frequencies and very low conductivities, for human tissue the conductivity is usually in the range of $0.1 \ldots 10 S/m$. So the conditions for the applicability of the eddy current model (2.28) are satisfied. We will now formulate the eddy current model for the MIT setting.

E-field formulation

Applying the eddy current model to the setting described in the previous section, this gives

$$\mathbf{curl} \left[\frac{1}{\mu(x)} \mathbf{curl}\, \mathbf{E}_s(x) \right] + i\omega\kappa(x) \mathbf{E}_s(x) = -i\omega\kappa(x) \mathbf{E}_p(x), \quad x \in \Omega, \quad (2.34)$$

$$\mathbf{curl} \left[\frac{1}{\mu_0} \mathbf{curl}\, \mathbf{E}_s(x) \right] = 0, \quad x \in \Omega^c, \quad (2.35)$$

$$\operatorname{div} \mathbf{E}_s(x) = 0, \quad x \in \Omega^c. \quad (2.36)$$

In addition we have to impose transmission conditions:

$$n(x) \times (\mathbf{E}_s^{\text{ext}}(x) \times n(x)) - n(x) \times (\mathbf{E}^{\text{int}}(x) \times n(x)) = -n(x) \times (\mathbf{E}_p(x) \times n(x)), \quad x \in \Gamma, \quad (2.37)$$

$$n(x) \times (\mathbf{H}_s^{\text{ext}}(x) \times n(x)) - n(x) \times (\mathbf{H}^{\text{int}}(x) \times n(x)) = -n(x) \times (\mathbf{H}_p(x) \times n(x)), \quad x \in \Gamma. \quad (2.38)$$

Since in the eddy current model we neglect the displacement currents in the exterior domain we have the boundary condition:

$$\kappa(x)\mathbf{E}(x) \cdot n(x) = 0, \quad x \in \Gamma. \tag{2.39}$$

We define a space in which we seek the solution of the above transmission problem. We denote by N_{cc} the number of connected components of the boundary $\Gamma = \partial\Omega$. $\tilde{\Gamma}_i, i = 1, \ldots, N_{cc}$ then name the connected components of Γ. We seek the solution in the space

$$\mathcal{V} = \left\{ \mathbf{U} \in \mathbf{H}(\mathbf{curl}; \mathbb{R}^3) : \operatorname{div} \mathbf{U}(x) = 0, \quad x \in \Omega^c, \quad \int\limits_{\tilde{\Gamma}_i} \mathbf{U} \cdot n ds_x = 0, \quad i = 1, \ldots N_{cc} \right\}.$$

Theorem 2.3. *For $\omega > 0$ and $\Re(\kappa(x)) > 0$ for all $x \in \Omega$ the variational problem to find*

$$\begin{aligned}\int\limits_{\mathbb{R}^3} \frac{1}{\mu(x)} \mathbf{curl}\, \mathbf{E}_s(x) \cdot \mathbf{curl}\, \mathbf{F}(x) dx + i\omega \int\limits_{\Omega} \kappa(x) \mathbf{E}_s(x) \cdot \mathbf{F}(x) dx \\ = -i\omega \int\limits_{\Omega} \kappa(x) \mathbf{E}_p(x) \cdot \mathbf{F}(x) dx\end{aligned}$$

for all $\mathbf{F} \in \mathcal{V}$ has a unique solution.

A proof can be found in [4, 9].

H-field formulation

We will now derive a formulation for the **H**-field in the eddy current setting. For this we assume that the conducting object Ω is simply connected. Neglecting the displacement currents in the exterior domain in (2.6) leads to

$$\mathbf{curl}\, \mathbf{H}_s(x) = 0, \quad x \in \Omega^c.$$

Hence we obtain that $\mathbf{H}_s(x) = -\nabla\phi(x), x \in \Omega^c$, which means that in the exterior domain we look for a gradient field, i.e. we have to solve the transmission problem

$$\mathbf{curl}\, [\kappa(x)^{-1} \mathbf{curl}\, \mathbf{H}_s(x)] + i\omega\mu(x)\mathbf{H}_s(x) = -i\omega\mu(x)\mathbf{H}_p(x), \quad x \in \Omega, \tag{2.40}$$

$$-\Delta\phi(x) = 0, \quad x \in \Omega^c, \tag{2.41}$$

with the transmission conditions

$$n(x)\times(\nabla\phi(x)\times n(x)) - n(x)\times(\mathbf{H}(x)\times n(x)) = -n(x)\times(\mathbf{H}_p(x)\times n(x)), \quad x\in\Gamma, \tag{2.42}$$

$$\frac{\partial}{\partial n(x)}\phi(x) - \mathbf{H}(x)\cdot n(x) = -\mathbf{H}_p(x)\cdot n(x), \quad x\in\Gamma. \tag{2.43}$$

For ϕ we impose the radiation conditions

$$\phi(x) = \mathcal{O}\left(\frac{1}{|x|}\right), \quad |\nabla\phi(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

A-ϕ formulation

In the so-called $\mathbf{A}$-ϕ-formulation (see [7]) the electric field $\mathbf{E}$ is decomposed into a gradient field and a vector potential field. In the case of low frequency applications the gradient field part dominates and as we let ω tend to zero the vector potential vanishes. In the next section we are going to derive a reduced model, which is based on the $\mathbf{A}$-ϕ-formulation.
Since $\mathbf{B}$ is divergence–free, we can represent the magnetic flux density $\mathbf{B}$ as the curl of a magnetic vector potential $\mathbf{A}$,

$$\mathbf{B}(x) = \mu_0\mathbf{H}(x) = \mathbf{curl}\,\mathbf{A}(x) \quad \text{for } x\in\mathbb{R}^3.$$

From

$$\mathbf{curl}\,\mathbf{E}(x) = -i\omega\mu_0\mathbf{H}(x) = -i\omega\mathbf{curl}\,\mathbf{A}(x)$$

we conclude the existence of a scalar potential ϕ satisfying

$$\mathbf{E}(x) + i\omega\mathbf{A}(x) = -\nabla\phi(x) \quad \text{for } x\in\mathbb{R}^3, \tag{2.44}$$

where ϕ is uniquely determined by the Coulomb gauge

$$\mathrm{div}\mathbf{A}(x) = 0 \quad \text{for } x\in\mathbb{R}^3. \tag{2.45}$$

By using the decomposition (2.44) we can write the primary field $\mathbf{E}_p$ as

$$\mathbf{E}_p(x) = -i\omega\mathbf{A}_p(x) \quad \text{for } x\in\mathbb{R}^3,$$

while for the secondary field $\mathbf{E}_s$ we obtain

$$\mathbf{E}_s(x) = -i\omega\mathbf{A}_s(x) - \nabla\phi(x) \quad \text{for } x\in\mathbb{R}^3.$$

Now we can rewrite the transmission problem (2.34)-(2.36) in terms of the $\mathbf{A}$–ϕ–formulation:

$$\mathbf{curl}\,\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}_s(x) + \kappa(x)[i\omega\mathbf{A}_s(x) + \nabla\phi(x)] = -i\omega\kappa(x)\mathbf{A}_p(x), \quad x \in \Omega, \tag{2.46}$$

$$\mathbf{curl}\,\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{A}_s(x) = 0, \quad x \in \Omega^c, \tag{2.47}$$

$$\nabla \cdot \mathbf{A}_s(x) = 0, \quad x \in \mathbb{R}^3. \tag{2.48}$$

When applying the divergence operator to equation (2.46), this gives

$$-\nabla \cdot [\kappa(x)(i\omega\mathbf{A}_s(x) + \nabla\phi(x))] = i\omega\nabla \cdot [\kappa(x)\mathbf{A}_p(x)] \quad \text{for } x \in \Omega. \tag{2.49}$$

In addition, we rewrite the transmission boundary condition (2.37) in terms of $\mathbf{A}$ and ϕ and obtain

$$\kappa(x)\left(i\omega\mathbf{A}_s(x) + \nabla\phi(x)\right) \cdot n(x) = -i\omega\kappa(x)\mathbf{A}_p(x) \cdot n(x) \quad \text{for } x \in \Gamma. \tag{2.50}$$

2.3.2 The Reduced Model

The solution of the forward problem using the eddy current model as described in the previous section is computationally rather expensive. Since in most solution algorithms for the inverse problem the forward problem has to be solved quite often, we are interested in a simplified model which also allows a more efficient solution of the forward problem, see also [25].

In the parameter range of Magnetic Induction Tomography numerical examples [22] indicate that $\mathbf{A}_s$ is very small compared to $\nabla\phi$. Therefore we neglect $\mathbf{A}_s$ in (2.49) and (2.50), i.e. we conclude the Neumann boundary value problem

$$-\nabla \cdot [\kappa(x)\nabla\widetilde{\phi}(x)] = i\omega\nabla \cdot [\kappa(x)\mathbf{A}_p(x)] \quad \text{for } x \in \Omega, \tag{2.51}$$

$$\kappa(x)\frac{\partial\widetilde{\phi}(x)}{\partial n(x)} = -i\omega\kappa(x)\mathbf{A}_p(x) \cdot n(x) \quad \text{for } x \in \Gamma, \tag{2.52}$$

where $\widetilde{\phi}$ now denotes the scalar potential in the reduced model. Since $\widetilde{\phi}$ is not uniquely determined by the Neumann boundary value problem (2.51) and (2.52), we introduce the scaling condition

$$\int\limits_{\Gamma} \widetilde{\phi}(x)ds_x = 0. \tag{2.53}$$

Moreover, by neglecting $\mathbf{A}_s$ in (2.46) we obtain

$$\mathbf{curl}\left[\frac{1}{\mu_0}\mathbf{curl}\,\widetilde{\mathbf{A}}_s(x)\right] = -\kappa(x)[i\omega\mathbf{A}_p(x) + \nabla\widetilde{\phi}(x)], \quad x \in \mathbb{R}^3,$$

$$\nabla \cdot \widetilde{\mathbf{A}}_s(x) = 0 \quad \text{for } x \in \mathbb{R}^3.$$

Using the vector identity **curl curl** $= -\Delta + \nabla \operatorname{div}$ we get

$$-\Delta\widetilde{\mathbf{A}}_s(x) = -\mu_0\kappa(x)[i\omega\mathbf{A}_p(x) + \nabla\widetilde{\phi}(x)] \quad \text{for } x \in \mathbb{R}^3.$$

Hence we conclude

$$\widetilde{\mathbf{A}}_s(x) = -\frac{\mu_0}{4\pi}\int\limits_\Omega \kappa(y)\frac{i\omega\mathbf{A}_p(y) + \nabla\widetilde{\phi}(y)}{|x-y|}dy \quad \text{for } x \in \mathbb{R}^3. \tag{2.54}$$

The electric field can finally be obtained by

$$\widetilde{\mathbf{E}}_s(x) = -i\omega\widetilde{\mathbf{A}}_s(x) - \nabla\widetilde{\phi}(x) \quad \text{for } x \in \mathbb{R}^3. \tag{2.55}$$

This means that the solution of the full eddy current model reduces to the solution of a Neumann boundary value problem for the Laplace equation, and the evaluation of a Newton potential. Both models are summarized in Table 2.1.

It remains to estimate the error when considering the reduced model instead of the eddy current model. In particular we have to consider the differences $\phi - \widetilde{\phi}$ and $\mathbf{A}_s - \widetilde{\mathbf{A}}_s$, respectively. For this, we first introduce the Newton potential operator

$$(N_0u)(x) = \frac{1}{4\pi}\int\limits_\Omega \frac{u(y)}{|x-y|}dy \quad \text{for } x \in \Omega.$$

In the case of a vector–valued function $\boldsymbol{u}$ we consider the Newton potential $N_0\boldsymbol{u}$ component–wise.

Lemma 2.4. *Assume* $\Omega \subset B_r(0)$*. The Newton potential operator* $N_0 : L_2(\Omega) \to L_2(\Omega)$ *is bounded satisfying*

$$\|N_0\| := \sup_{0\neq u \in L_2(\Omega)} \frac{\|N_0u\|_{L_2(\Omega)}}{\|u\|_{L_2(\Omega)}} \le \frac{r^2}{\sqrt{3}}.$$

Proof. By using the Hölder inequality we have

$$\|N_0u\|^2_{L_2(\Omega)} = \int\limits_\Omega \left|\frac{1}{4\pi}\int\limits_\Omega \frac{u(y)}{|x-y|}dy\right|^2 dx \le \frac{1}{(4\pi)^2}\|u\|^2_{L_2(\Omega)}\int\limits_\Omega\int\limits_\Omega \frac{1}{|x-y|^2}dydx.$$

The assertion then follows from Schmidt's inequality, i.e.

$$\int\limits_\Omega \frac{1}{|x-y|^2}dy \le \int\limits_{B_r(0)} \frac{1}{|x-y|^2}dy \le 4\pi r \quad \text{for } x \in \mathbb{R}^3.$$

Reduced model

$$
\begin{aligned}
\mathbf{A}_p(x) &= \frac{\mu_0}{4\pi}\int\limits_{\mathbb{R}^3}\frac{\mathbf{j}_i(y)}{|x-y|}ds_y \quad \text{for } x\in\mathbb{R}^3,\\
-\nabla\cdot[\kappa(x)\nabla\widetilde{\phi}(x)] &= i\omega\nabla\cdot[\kappa(x)\mathbf{A}_p(x)] \quad \text{for } x\in\Omega,\\
\kappa(x)\frac{\partial\widetilde{\phi}(x)}{\partial n(x)} &= -i\omega\kappa(x)\mathbf{A}_p(x)\cdot n(x) \quad \text{for } x\in\Gamma, \qquad \int\limits_\Gamma \widetilde{\phi}(x)ds_x = 0,\\
\widetilde{\mathbf{A}}_s(x) &= -\frac{\mu_0}{4\pi}\int\limits_\Omega \kappa(y)\frac{i\omega\mathbf{A}_p(y)+\nabla\widetilde{\phi}(y)}{|x-y|}dy \quad \text{for } x\in\mathbb{R}^3,\\
\widetilde{\mathbf{E}}_s(x) &= -i\omega\widetilde{\mathbf{A}}_s(x)-\nabla\widetilde{\phi}(x) \quad \text{for } x\in\mathbb{R}^3.
\end{aligned}
$$

Eddy current model

$$
\begin{aligned}
\mathbf{E}_p(x) &= -i\omega\frac{\mu_0}{4\pi}\int\limits_{\mathbb{R}^3}\frac{\mathbf{j}_i(y)}{|x-y|}dy \quad \text{for } x\in\mathbb{R}^3,\\
\mathbf{curl}\,\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{E}_s(x)+i\omega\kappa(x)\mathbf{E}_s(x) &= -i\omega\kappa(x)\mathbf{E}_p(x) \quad \text{for } x\in\Omega,\\
\mathbf{curl}\,\frac{1}{\mu_0}\mathbf{curl}\,\mathbf{E}_s(x) &= 0 \quad \text{for } x\in\Omega^c,\\
\nabla\cdot\mathbf{E}_s(x) &= 0 \quad \text{for } x\in\Omega^c
\end{aligned}
$$

Table 2.1: Comparison of the reduced model and the eddy current model.

In particular we have

$$
\int\limits_\Omega\int\limits_\Omega \frac{1}{|x-y|^2}dydx \le \int\limits_\Omega 4\pi r dx \le \int\limits_{B_r(0)} 4\pi r dx = (4\pi)^2\frac{r^4}{3},
$$

which concludes the proof. □

Let $\mathbf{A}_s$ be the solution of the eddy current model (2.46)–(2.48), in particular by using (2.48) we can rewrite (2.46) as

$$
-\Delta\mathbf{A}_s(x) = -\mu_0\kappa(x)[i\omega\mathbf{A}_s(x)+i\omega\mathbf{A}_p(x)+\nabla\phi(x)] \quad \text{for } x\in\mathbb{R}^3. \tag{2.56}
$$

Hence we can write $\mathbf{A}_s$ as Newton potential

$$\mathbf{A}_s(x) = -\mu_0 N_0(\kappa(i\omega\mathbf{A}_s + i\omega\mathbf{A}_p + \nabla\phi))(x). \tag{2.57}$$

Correspondingly, we have

$$\widetilde{\mathbf{A}}_s(x) = -\mu_0 N_0(\kappa(i\omega\mathbf{A}_p + \nabla\widetilde{\phi}))(x) \tag{2.58}$$

where $\nabla\widetilde{\phi}$ is chosen such that

$$\mathrm{div}\widetilde{\mathbf{A}}_s(x) = 0 \quad \text{for } x \in \mathbb{R}^3. \tag{2.59}$$

We therefore conclude

$$\mathbf{A}_s - \widetilde{\mathbf{A}}_s = -\mu_0 N_0(\kappa(i\omega\mathbf{A}_s + \nabla\phi^\delta)), \quad \phi^\delta := \phi - \widetilde{\phi}. \tag{2.60}$$

Theorem 2.5. *Let us define*

$$\kappa_{\min} := \sqrt{\inf_{x\in\Omega} \Re(\kappa(x))^2 + \inf_{x\in\Omega} \Im(\kappa(x))^2}, \tag{2.61}$$

$$\kappa_{\max} := \sup_{x\in\Omega} |\kappa(x)|, \tag{2.62}$$

$$q := \mu_0\omega\kappa_{\max}\left(1 + \frac{\kappa_{\max}}{\kappa_{\min}}\right)\frac{r^2}{\sqrt{3}}. \tag{2.63}$$

Let $\phi, \widetilde{\phi} \in H^1(\Omega)$ be the weak solutions of the Neumann type boundary value problems (2.49)–(2.50) *and* (2.51)–(2.52), *respectively. Then there holds the error estimate*

$$\|\nabla\phi^\delta\|_{L_2(\Omega)} \le \frac{\kappa_{\max}}{\kappa_{\min}}\omega\|\mathbf{A}_s\|_{L_2(\Omega)}. \tag{2.64}$$

If we assume $q < 1$, then there holds

$$\|\mathbf{A}_s\|_{L_2(\Omega)} \le \frac{q}{1-q}\|\mathbf{A}_p\|_{L_2(\Omega)}, \tag{2.65}$$

and

$$\|\mathbf{A}_s - \widetilde{\mathbf{A}}_s\|_{L_2(\Omega)} \le \frac{q^2}{1-q}\|\mathbf{A}_p\|_{L_2(\Omega)}. \tag{2.66}$$

Proof. From (2.49) and (2.51) we first conclude that $\phi^\delta := \phi - \widetilde{\phi}$ is a solution of the partial differential equation

$$-\nabla\cdot[\kappa(x)\nabla\phi^\delta(x)] = i\omega\nabla\cdot[\kappa(x)\mathbf{A}_s(x)] \quad \text{for } x \in \Omega$$

with the Neumann boundary condition

$$\kappa(x)\left[\frac{\partial\phi^\delta(x)}{\partial n(x)} + i\omega\mathbf{A}_s(x)\cdot n(x)\right] = 0 \quad \text{for } x \in \Gamma.$$

Hence, for $\psi \in H^1(\Omega)$ the weak formulation of the above Neumann boundary value problem reads

$$\int_\Omega \kappa(x)\nabla\phi^\delta(x)\cdot\nabla\psi(x)dx = i\omega\int_\Omega \nabla\cdot[\kappa(x)\mathbf{A}_s(x)]\psi(x)dx + \int_\Gamma \kappa(x)\frac{\partial\phi^\delta(x)}{\partial n(x)}\psi(x)ds_x$$
$$= \int_\Gamma \kappa(x)\left[i\omega\mathbf{A}_s(x)\cdot n(x) + \frac{\partial\phi^\delta(x)}{\partial n(x)}\right]\psi(x)ds_x - i\omega\int_\Omega \kappa(x)\mathbf{A}_s(x)\cdot\nabla\psi(x)dx$$
$$= -i\omega\int_\Omega \kappa(x)\mathbf{A}_s(x)\cdot\nabla\psi(x)dx.$$

For $\psi = \phi^\delta$ we therefore have

$$\int_\Omega \kappa(x)|\nabla\phi^\delta(x)|^2dx = -i\omega\int_\Omega \kappa(x)\mathbf{A}_s(x)\cdot\nabla\phi^\delta(x)dx,$$

from which (2.64) follows, i.e.

$$\|\nabla\phi^\delta\|_{L_2(\Omega)} \le \frac{\kappa_{\max}}{\kappa_{\min}}\omega\|\mathbf{A}_s\|_{L_2(\Omega)}.$$

Moreover, with (2.57) and by using Lemma 2.4 we further have

$$\begin{aligned}
\|\mathbf{A}_s\|_{L_2(\Omega)} &= \mu_0\|N_0(\kappa(i\omega\mathbf{A}_s + i\omega\mathbf{A}_p + \nabla\phi))\|_{L_2(\Omega)} \\
&\le \mu_0\kappa_{\max}\frac{r^2}{\sqrt{3}}\|i\omega\mathbf{A}_s + i\omega\mathbf{A}_p + \nabla\phi\|_{L_2(\Omega)} \\
&\le \mu_0\kappa_{\max}\frac{r^2}{\sqrt{3}}\left[\omega\left(\|\mathbf{A}_s\|_{L_2(\Omega)} + \|\mathbf{A}_p\|_{L_2(\Omega)}\right) + \|\nabla\phi\|_{L_2(\Omega)}\right]. \quad (2.67)
\end{aligned}$$

The variational formulation of the Robin type boundary value problem (2.49) and (2.50) reads, for $\psi \in H^1(\Omega)$,

$$\int_\Omega \kappa(x)\nabla\phi(x)\cdot\nabla\psi(x)dx$$
$$= i\omega\int_\Omega \nabla\cdot[\kappa(x)(\mathbf{A}_p(x) + \mathbf{A}_s(x))]\psi(x)dx + \int_\Gamma \kappa(x)\frac{\partial\phi(x)}{\partial n(x)}\psi(x)ds_x$$
$$= \int_\Gamma \kappa(x)\left[i\omega(\mathbf{A}_p(x) + \mathbf{A}_s(x))\cdot n(x) + \frac{\partial\phi(x)}{\partial n(x)}\right]\psi(x)ds_x$$
$$- i\omega\int_\Omega \kappa(x)(\mathbf{A}_p(x) + \mathbf{A}_s(x))\cdot\nabla\psi(x)dx$$
$$= -i\omega\int_\Omega \kappa(x)(\mathbf{A}_p(x) + \mathbf{A}_s(x))]\cdot\nabla\psi(x)dx.$$

For $\psi = \phi$ we therefore have

$$\int_\Omega \kappa(x)|\nabla\phi(x)|^2 dx = -i\omega \int_\Omega \kappa(x)(\mathbf{A}_p(x) + \mathbf{A}_s(x)) \cdot \nabla\phi(x) dx,$$

from which the estimate

$$\|\nabla\phi\|_{L_2(\Omega)} \le \frac{\kappa_{\max}}{\kappa_{\min}} \omega \|\mathbf{A}_p + \mathbf{A}_s\|_{L_2(\Omega)}$$

follows. From (2.67) we therefore conclude

$$\|\mathbf{A}_s\|_{L_2(\Omega)} \le \mu_0 \kappa_{\max} \frac{r^2}{\sqrt{3}} \omega \left(1 + \frac{\kappa_{\max}}{\kappa_{\min}}\right) \left(\|\mathbf{A}_s\|_{L_2(\Omega)} + \|\mathbf{A}_p\|_{L_2(\Omega)}\right),$$

which immediately results in the estimate (2.65) when we assume $q < 1$.

Finally, by using (2.60) and Lemma 2.4 we have

$$\begin{aligned} \|\mathbf{A}_s - \widetilde{\mathbf{A}}_s\|_{L_2(\Omega)} &= \mu_0 \|N_0(\kappa(i\omega\mathbf{A}_s + \nabla\phi^\delta))\|_{L_2(\Omega)} \\ &\le \mu_0 \frac{r^2}{\sqrt{3}} \|\kappa(i\omega\mathbf{A}_s + \nabla\phi^\delta)\|_{L_2(\Omega)} \\ &\le \mu_0 \kappa_{\max} \frac{r^2}{\sqrt{3}} \left(\omega\|\mathbf{A}_s\|_{L_2(\Omega)} + \|\nabla\phi^\delta\|_{L_2(\Omega)}\right) \\ &\le \mu_0 \kappa_{\max} \frac{r^2}{\sqrt{3}} \omega \left(1 + \frac{\kappa_{\max}}{\kappa_{\min}}\right) \|\mathbf{A}_s\|_{L_2(\Omega)} = q\|\mathbf{A}_s\|_{L_2(\Omega)} \end{aligned}$$

due to (2.64). Now, (2.66) follows from (2.65). □

Remark 2.1. *As an example we may consider a test problem with the following parameters:*

$$0.1 \le \kappa(x) \le 1 \quad \text{for } x \in \Omega, \quad \Omega \subset B_{0.1}(0), \quad \omega = 10^5.$$

In this case, we have

$$q = 7.98 \cdot 10^{-3}, \qquad \frac{q^2}{1-q} = 6.42 \cdot 10^{-5}.$$

Note that $\|\mathbf{A}_p\|_{L_2(\Omega)} = 3.609 \cdot 10^{-6}$*, which was obtained by using some finite element discretization.*

Corollary 2.6. *In addition we have an estimate for the error in an arbitrary point* $x \in \mathbb{R}^3$*, i.e.*

$$|\boldsymbol{A}_s(x) - \widetilde{\boldsymbol{A}}_s(x)| \le \frac{q^2}{1-q} \|\boldsymbol{A}_p\|_{L_2(B_r(0))}. \tag{2.68}$$

Proof. By using (2.60) we have, for $x \in \Omega$,

$$\begin{aligned}
|\mathbf{A}_s(x) - \widetilde{\mathbf{A}}_s(x)| &= \frac{\mu_0}{4\pi}\left|\int\limits_\Omega \kappa(y)\frac{i\omega\mathbf{A}_s(y) + \nabla\phi^\delta(y)}{|x-y|}dy\right| \\
&\leq \frac{\mu_0}{4\pi}\|\kappa(i\omega\mathbf{A}_s + \nabla\phi^\delta)\|_{L_2(\Omega)}\left(\int\limits_\Omega \frac{1}{|x-y|^2}dy\right)^{1/2} \\
&\leq \frac{\mu_0}{4\pi}\kappa_{\max}\sqrt{4\pi r}\left(\omega\|\mathbf{A}_s\|_{L_2(\Omega)} + \|\nabla\phi^\delta\|_{L_2(\Omega)}\right) \\
&\leq \frac{\mu_0}{4\pi}\kappa_{\max}\sqrt{4\pi r}\omega\left(1 + \frac{\kappa_{\max}}{\kappa_{\min}}\right)\|\mathbf{A}_s\|_{L_2(\Omega)} = \frac{\sqrt{12\pi r}}{4\pi r^2}q\|\mathbf{A}_s\|_{L_2(\Omega)}.
\end{aligned}$$

□

2.3.3 The Static Case

In the following chapters we want to look at the eddy current problem for low frequencies, therefore we need to understand what happens in the static case, i.e. in the case $\omega = 0$.
The primary electric field is proportional to the frequency ω, therefore as we let ω tend to zero, the primary electric field $\mathbf{E}_p$ tends to zero as well. Inserting this information in Faraday's and Gauss law this leads to

$$\begin{aligned}
\mathbf{curl}\,\mathbf{E}_s(x) &= 0, \quad x \in \mathbb{R}^3, \\
\operatorname{div}(\varepsilon(x)\mathbf{E}_s(x)) &= 0, \quad x \in \mathbb{R}^3.
\end{aligned}$$

From this we immediately see that $\mathbf{E}_s(x) \equiv 0$. In the magnetostatic case the magnetic field intensity is described by.

$$\begin{aligned}
\mathbf{curl}\,\mathbf{H}_s(x) &= 0, \quad x \in \mathbb{R}^3, \\
\operatorname{div}(\mu(x)\mathbf{H}(x)) &= 0, \quad x \in \mathbb{R}^3.
\end{aligned}$$

From the first equation we deduce $\mathbf{H}_s(x) = \nabla\phi(x)$. So we are left with solving the problem

$$-\operatorname{div}(\mu(x)\nabla\phi(x)) = \operatorname{div}(\mu(x)\mathbf{H}_p(x)).$$

Note that if $\mu(x)$ is constant in the whole space $\mathbb{R}^3$ we get $\mathbf{H}_s(x) \equiv 0$. We see that as long as ω is greater than zero, the condition $\operatorname{div}\mathbf{B} = 0$ is incorporated in the equation, but if $\omega = 0$ we have to pose this as an additional constraint.

3 MATHEMETICAL PRELIMINARIES

In the first section of this chapter we will briefly give fundamental results from functional analysis, which are essential tools for most of the proofs of the upcoming sections. One important result, which will be used very often is the famous Lax-Milgram lemma, the other result is Brezzi's theorem (see [10]). The second section deals with Sobolev spaces, which are the basis for the mathematical analysis of boundary value and transmission problems. First we introduce Sobolev spaces for scalar problems, after that we state the basic results about trace operators and the corresponding Sobolev spaces on the boundary for Maxwell's equations. An extensive and careful summary on Sobolev spaces for smooth domains as well as Lipschitz domains can be found in [2, 37, 46]. For the introduction of the Maxwell trace spaces on Lipschitz domains we take the paper [15] as a basis.

3.1 Functional Analytic Basics

Definition 3.1. *Let X be a Hilbert space. A bounded linear form $a(.,.) : X \times X \to \mathbb{C}$ is called X-elliptic if*

$$|a(u, u)| \geq c_1 \|u\|_X^2, \quad \forall\, u \in X \tag{3.1}$$

holds.

Theorem 3.2 (Lax-Milgram)**.** *Let X be a Hilbert space and X' denote its dual space. Let $a(.,.) : X \times X \to \mathbb{C}$ be a bounded bi-linear form. If $a(.,.)$ is X-elliptic, then the variational problem to find $u \in X$ such that*

$$a(u, v) = f(v) \quad \forall v \in X,$$

has a unique solution in X and we have

$$\|u\|_X \leq c\|f\|_{X'}.$$

Remark 3.3. *The Lax-Milgram lemma also holds if we use the ellipticity definition*

$$\Re(a(u, u)) \geq c\|u\|_X^2, \quad \forall u \in X.$$

This is a stronger requirement then using (3.1).

When considering saddle point problems, the following theorem provides a very useful tool to prove unique solvability. A proof for this theorem can be found in [10] for the case that $a(.,.)$ and $b(.,.)$ are real valued bi-linear forms. However, by using the Lax-Milgram lemma for complex-valued bi-linear forms the proof easily carries over to the complex-valued case, which is considered here.

Theorem 3.4 (Brezzi). *Let X and Q be Hilbert spaces and $a : X \times X \to \mathbb{C}$ and $b : X \times Q \to \mathbb{C}$ be bounded bilinear forms. Let*

$$X_0 = \{u \in X : b(u,q) = 0 \quad \forall q \in Q\}$$

denote the kernel of $b(.,.)$. If we assume that $a(.,.)$ is elliptic on X_0, i.e.

$$|a(u,u)| \geq c\|u\|_X^2, \quad \forall u \in X_0,$$

and $b(.,.)$ satisfies the LBB (Ladyshenskaya-Babuška-Brezzi) condition

$$\sup_{u \in X} \frac{|b(u,q)|}{\|u\|_X} \geq c\|q\|_Q, \quad \forall q \in Q,$$

then the variational problem to find $(u,p) \in X \times Q$

$$\begin{aligned} a(u,v) + b(v,p) &= f(v), \quad \forall v \in X, \\ b(u,q) &= g(q), \quad \forall q \in Q \end{aligned}$$

has a unique solution and we have the estimate

$$\|u\|_X + \|p\|_Q \leq c(\|f\|_{X'} + \|g\|_{Q'})$$

with a constant $c > 0$.

3.2 Sobolev Spaces for Scalar Problems

The mathematical analysis of boundary value problems requires the introduction of appropriate function spaces. In this section we introduce the Sobolev spaces we need when we deal with scalar problems of the type

$$-\Delta u + \kappa^2 u = 0.$$

There are mainly two methods for introducing Sobolev spaces, one is based on the Fourier transform while the other one is established by the concept of weak derivatives. For some cases both concepts result in Sobolev spaces which coincide. Here we will focus on introducing Sobolev spaces by means of the Fourier transform, however we will also give a few important results which establish the connection between the

two kinds of Sobolev spaces.
For the introduction of Sobolev spaces we have to specify the class of domains we will look at. In what follows we will assume that Ω is a Lipschitz domain, this means that Ω has to meet the following requirements ([46]):

Definition 3.5. *An open set $\Omega \subset \mathbb{R}^d$ is called Lipschitz hypograph if there exists a Lipschitz function $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$ such that*

$$\Omega = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \ x_d < \zeta(x') \ \text{ for all } x' = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}\}.$$

An open set $\Omega \subset \mathbb{R}^d$ is called Lipschitz domain if its boundary is compact and there exist finite families $\{W_j\}$ and $\{\Omega_j\}$ such that

1. *$\{W_j\}$ is a finite open cover of Γ.*
2. *Every Ω_j can be transformed to a Lipschitz hypograph by a rigid body motion (i.e. rotation and translation).*
3. *We have $W_j \cap \Omega = W_j \cap \Omega_j$ for every j.*

Remark 3.6. *A more restrictive definition is the Lipschitz polyhedron: We call a Lipschitz domain Ω Lipschitz polyhedron if it is simply connected and it is bounded by a finite number of polygons $\Gamma_i, i = 1, \dots, N_\Gamma$.*

We will now define the function space $H^s(\Omega)$ by using Fourier transforms and the Bessel potential operator, we will proceed as in [46].

Definition 3.7. *For any $s \in \mathbb{R}$ we define the space*

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}^*(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |\xi|)^s |\hat{u}(\xi)|^2 d\xi < \infty \right\},$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing functions, and $\mathcal{S}^*(\mathbb{R}^d)$ is the space of all linear functionals on $\mathcal{S}(\mathbb{R}^d)$. $\mathcal{S}^*(\mathbb{R}^d)$ is also called the Schwartz space of temperate distributions. Now we can define $H^s(\Omega)$ as

$$H^s(\Omega) := \{u \in [C^\infty_{comp}(\mathbb{R}^d)]^* : \ u = \tilde{u}|_\Omega, \ \tilde{u} \in H^s(\mathbb{R}^d)\},$$

with the norm

$$\|u\|_{H^s(\Omega)} := \inf_{\tilde{u} \in H^s(\mathbb{R}^d), \tilde{u}|_\Omega = u} \|\tilde{u}\|_{H^s(\mathbb{R}^d)}.$$

The dual spaces are denoted by

$$\widetilde{H}^s(\Omega) := [H^{-s}(\Omega)]', \quad \widetilde{H}^{-s}(\Omega) = [H^s(\Omega)]'$$

for $s > 0$.
As already stated there is another way for introducing Sobolev spaces, which is based on the concept of weak derivatives:

Definition 3.8. *A function* $g = \partial^\alpha u$ *is called weak derivative of* $u \in L_1^{loc}(\Gamma)$*, if*

$$(g, \phi)_\Omega = (-1)^{|\alpha|} (u, \partial^\alpha \phi)_\Omega$$

is satisfied for all $\phi \in C_0^\infty(\Omega)$ *with the multi-index* $\alpha = (\alpha_1, \dots, \alpha_d)$*. For* $k \in \mathbb{N}$ *the Sobolev space* $W_2^k(\Omega)$ *is defined as*

$$W_2^k(\Omega) = \{u \in L_2(\Omega) : \partial^\alpha u \in L_2(\Omega),\ |\alpha| \leq k\}.$$

Remark 3.9. *Let* Ω *be a Lipschitz domain, then due to [46] we have that*

$$H^s(\Omega) = W_2^s(\Omega).$$

holds for all $s \geq 0$*.*

We have now introduced Sobolev spaces in the domain Ω, as a next step we define Sobolev spaces on the boundary of a Lipschitz domain:
Let Ω be a Lipschitz hypograph, then we can define the space

$$H^s(\Gamma) = \{u \in L_2(\Gamma) :\ u_\zeta(x') = u(x', \zeta(x')) \in H^s(\mathbb{R}^{d-1})\} \tag{3.2}$$

for $s \in [0, 1]$. If Ω can be obtained by applying the rigid body motion κ to a Lipschitz hypograph $\tilde{\Omega}$, i.e. $\Omega = \kappa(\tilde{\Omega})$, we define

$$u_\zeta(x') = u(\kappa^{-1}(x', \zeta(x'))).$$

If Ω is a Lipschitz domain we choose a partition of unity $\phi_j \in \mathcal{C}^\infty_{\text{comp}}(W_j)$ with

$$\sum_j \phi_j(x) \equiv 1, \quad \forall\, x \in \Gamma,$$

and we define the $H^s(\Gamma)$ Norm by

$$\|u\|^2_{H^s(\Gamma)} = \sum_j \|\phi_j u\|^2_{H^s(\Gamma_j)}$$

with $\Gamma_j = \partial\Omega_j$. For $s \in [0, 1]$ we define

$$H^s(\Gamma) = \overline{\mathcal{C}(\Gamma)}^{\|\cdot\|_{H^s(\Gamma)}}. \tag{3.3}$$

For negative indices $-s < 0$ the space $H^{-s}(\Gamma)$ is defined as the dual space of $H^s(\Gamma)$, which is equipped with the norm

$$\|u\|_{H^{-s}(\Gamma)} := \sup_{v \in H^s(\Gamma)} \frac{\langle u, v\rangle_\Gamma}{\|v\|_{H^s(\Gamma)}}.$$

Important results, which are needed for the solution of boundary value problems for partial differential operators of the type $-\Delta + \kappa^2$, are given by the following Green's formulae:

Theorem 3.10 (Green). *For a given bounded Lipschitz domain Ω we have the following formulae for partial integration:*
1. Green Formula

$$-\langle \Delta u, v\rangle_\Omega = \langle \nabla u, \nabla v\rangle_\Omega - \langle \gamma_1^{\mathrm{int}} u, \gamma_0^{\mathrm{int}} v\rangle_\Gamma \tag{3.4}$$

2. Green Formula

$$-\langle \Delta u, v\rangle_\Omega + \langle u, \Delta v\rangle_\Omega = \langle \gamma_1^{\mathrm{int}} v, \gamma_0^{\mathrm{int}} u\rangle_\Gamma - \langle \gamma_1^{\mathrm{int}} u, \gamma_0^{\mathrm{int}} v\rangle_\Gamma \tag{3.5}$$

Green's first and second formula establish a connection between the Dirichlet trace operator

$$\gamma_0^{\mathrm{int}} u(x) = \lim_{\Omega \ni \tilde{x} \to x \in \Gamma} u(x), \quad \gamma_0^{\mathrm{ext}} u(x) = \lim_{\Omega^c \ni \tilde{x} \to x \in \Gamma} u(x)$$

and the Neumann trace operator

$$\gamma_1^{\mathrm{int}} u(x) = \lim_{\Omega \ni \tilde{x} \to x \in \Gamma} \nabla_{\tilde{x}} u(\tilde{x}) \cdot n(x), \quad \gamma_1^{\mathrm{ext}} u(x) = \lim_{\Omega^c \ni \tilde{x} \to x \in \Gamma} \nabla_{\tilde{x}} u(\tilde{x}) \cdot n(x).$$

3.3 Trace Operators and Function Spaces for Maxwell Problems

In this section we give the definition of Sobolev spaces for the domain and the boundary, which are needed for the analysis of boundary value and transmission problems related to Maxwell's equations and we define the important trace operators.

Notation 3.1. *In this chapter we deal with vector-valued functions, the Sobolev spaces which were introduced in the previous section can also be applied to vector valued functions. We denote Sobolev spaces for vector valued functions by bold letters, so let X be any of the Sobolev spaces, which were already defined, then we can define vector-valued equivalent as*

$$\boldsymbol{X} := [X]^3 = \{\mathbf{U} = (U_1, U_2, U_3) : U_i \in X, i = 1, 2, 3\}.$$

The energy space for the fields $\mathbf{E}$ and $\mathbf{H}$ in a bounded Lipschitz domain Ω is defined by

$$\mathbf{H}(\mathbf{curl}; \Omega) = \{\mathbf{U} \in \mathbf{L}^2(\Omega) : \ \mathbf{curl}\,\mathbf{U} \in \mathbf{L}^2(\Omega)\}.$$

Remark 3.11. *Functions in $\mathbf{H}(\mathbf{curl}; \Omega)$ have the following property: If we consider a $\mathbf{H}(\mathbf{curl}; \Omega)$-function in a domain Ω and we cut the domain Ω into two parts, the tangential component of this function is continuous along the cutting surface.*

The electric and magnetic Gauss' law motivate to choose the magnetic field **B** and the displacement field **D** as elements of the space

$$\mathbf{H}(\mathrm{div};\Omega) = \{\mathbf{U} \in \mathbf{L}^2(\Omega) : \ \mathrm{div}\,\mathbf{U} \in \mathbf{L}^2(\Omega)\}.$$

Remark 3.12. *Functions in $\mathbf{H}(\mathrm{div};\Omega)$ have the following property: If we consider a $\mathbf{H}(\mathrm{div};\Omega)$-function in a domain Ω and cut the domain Ω into two parts, the normal component of this function is continuous along the cutting surface.*

Remark 3.13. *The spaces $H^1(\Omega), \mathbf{H}(\mathbf{curl};\Omega), \mathbf{H}(\mathrm{div};\Omega)$ and $L_2(\Omega)$ are connected via the de-Rham sequence, which states that*

$$H^1(\Omega) \stackrel{\nabla}{\to} \mathbf{H}(\mathbf{curl};\Omega) \stackrel{\mathbf{curl}}{\to} \mathbf{H}(\mathrm{div};\Omega) \stackrel{\mathrm{div}}{\to} L_2(\Omega).$$

Moreover if Ω is a simple connected domain, then the de-Rham sequence is exact, i.e. the range of an operator in the sequence is the kernel of the next operator in the sequence (a proof can be found in [63]). A consequence of the exact sequence is that for
$\mathbf{U} \in \mathbf{H}(\mathbf{curl};\Omega)$

$$\mathbf{curl}\,\mathbf{U} = 0 \quad \Rightarrow \quad \mathbf{U} = \nabla\phi, \quad \textit{with} \quad \phi \in H^1(\Omega),$$

and for $\mathbf{U} \in \mathbf{H}(\mathrm{div};\Omega)$

$$\mathrm{div}\,\mathbf{U} = 0 \quad \Rightarrow \quad \mathbf{U} = \mathbf{curl}\,\mathbf{V}, \quad \textit{with} \quad \mathbf{V} \in \mathbf{H}(\mathbf{curl};\Omega),$$

provided Ω is a simple connected domain.

For unbounded domains we define the space

$$\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}\,;\Omega) = \{\mathbf{U} \in \mathbf{L}^2_{\mathrm{loc}}(\Omega) : \ \mathbf{curl}\,\mathbf{U} \in \mathbf{L}^2_{\mathrm{loc}}(\Omega)\}.$$

In what follows we will introduce the trace spaces for $\mathbf{H}(\mathbf{curl};\Omega)$ on the boundary, when Ω is a Lipschitz domain. For Lipschitz polyhedra those spaces have been introduced and analyzed in [12, 13], for Lipschitz domains this was done in [15], we take this paper as a basis for the introduction of those spaces.

Definition 3.14. *We define the tangential trace by*

$$\boldsymbol{\gamma}_t\mathbf{U}(x) = n(x) \times (\mathbf{U}(x)|_\Gamma \times n(x)), \quad x \in \Gamma \tag{3.6}$$

the twisted tangential trace by

$$\boldsymbol{\gamma}_\times\mathbf{U}(x) = \mathbf{U}(x)|_\Gamma \times n(x), \quad x \in \Gamma \tag{3.7}$$

and the Neumann trace by

$$\boldsymbol{\gamma}_N\mathbf{U}(x) = (\mathbf{curl}\,\mathbf{U}(x))|_\Gamma \times n(x), \quad x \in \Gamma. \tag{3.8}$$

The tangential trace $\boldsymbol{\gamma}_t$ and the twisted tangential trace are connected to each other by the operator $\mathbf{R}\mathbf{u} = \mathbf{u} \times n$,i.e.

$$\boldsymbol{\gamma}_t \mathbf{U} = -\mathbf{R}\boldsymbol{\gamma}_\times \mathbf{U}, \quad \boldsymbol{\gamma}_\times \mathbf{U} = \mathbf{R}\boldsymbol{\gamma}_t \mathbf{U}. \tag{3.9}$$

Let us first consider domains Ω with a smooth boundary. The trace operators $\boldsymbol{\gamma}_t, \boldsymbol{\gamma}_\times$ obviously map functions in the domain to tangential functions on the boundary, therefore we define the space

$$\mathbf{H}_t^{1/2}(\Gamma) = \left\{ \mathbf{u} \in \mathbf{H}^{1/2}(\Gamma) : \ \mathbf{U}(x) \cdot n(x) = 0, \quad x \in \Gamma \right\}$$

and its dual space

$$\mathbf{H}_t^{-1/2}(\Gamma) = \left[\mathbf{H}_t^{1/2}(\Gamma) \right]'$$

with the pivot space

$$\mathbf{L}_t^2(\Gamma) = \left\{ \mathbf{U} \in \mathbf{L}^2(\Gamma) : \ \mathbf{U} \cdot n = 0 \right\}.$$

For smooth boundaries Γ we get that the mappings

$$\gamma_t, \gamma_\times : \mathbf{H}^1(\Omega) \to \mathbf{H}_t^{1/2}(\Gamma)$$

are continuous. The partial integration formula

$$\langle \mathbf{curl}\, \mathbf{U}, \mathbf{V} \rangle_\Omega - \langle \mathbf{U}, \mathbf{curl}\, \mathbf{V} \rangle_\Omega = \langle \boldsymbol{\gamma}_t \mathbf{U}, \mathbf{V} \rangle$$

yields the extension to a continuous mapping on the space $\mathbf{H}(\mathbf{curl}; \Omega)$, i.e.

$$\boldsymbol{\gamma}_t, \boldsymbol{\gamma}_\times : \mathbf{H}(\mathbf{curl}; \Omega) \to \mathbf{H}_t^{-1/2}(\Gamma).$$

As a next step we define derivatives on the surface for domains with a smooth boundary:

Definition 3.15. *Let Γ be smooth and $u \in C^1(\Gamma)$, then there exists an extension U^* into the domain with*

$$u = U^*|_\Gamma.$$

By the aid of this extension we define the surface gradient as

$$\nabla_\Gamma u = \boldsymbol{\gamma}_t(\nabla U^*) \tag{3.10}$$

and the vectorial surface curl as

$$\mathbf{curl}_\Gamma u = \boldsymbol{\gamma}_\times(\nabla U^*). \tag{3.11}$$

In addition we define the surface gradient and the scalar surface rotation as the adjoint operators

$$\langle \mathrm{div}_\Gamma\, \mathbf{U}, u \rangle_\Gamma = -\langle \mathbf{U}, \nabla_\Gamma u \rangle_\Gamma, \tag{3.12}$$

$$\langle \mathrm{curl}_\Gamma \mathbf{U}, u \rangle_\Gamma = \langle \mathbf{U}, \mathbf{curl}_\Gamma u \rangle_\Gamma. \tag{3.13}$$

Remark 3.16. *For smooth boundaries* Γ *and functions* $\mathbf{u} \in \boldsymbol{C}^1(\Gamma)$ *the surface divergence* $\operatorname{div}_\Gamma$ *and the scalar surface curl* $\operatorname{curl}_\Gamma$ *can also be computed by*

$$\operatorname{div}_\Gamma \mathbf{u} = (\operatorname{div} \mathbf{U}^*)|_\Gamma,$$
$$\operatorname{curl}_\Gamma \mathbf{u} = (\mathbf{curl}\, \mathbf{U}^* \cdot n)|_\Gamma,$$

where $\mathbf{U}^*$ *denotes the extension of* $\mathbf{u}$ *into the domain* Ω.

The definitions of the surface derivatives pave the way to the definition of the Sobolev spaces for smooth boundaries Γ,

$$\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) = \{\mathbf{u} \in \mathbf{H}_t^{-1/2}(\Gamma) : \operatorname{div}_\Gamma \mathbf{u} \in H^{-1/2}(\Gamma)\}, \tag{3.14}$$
$$\mathbf{H}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma) = \{\mathbf{u} \in \mathbf{H}_t^{-1/2}(\Gamma) : \operatorname{curl}_\Gamma \mathbf{u} \in H^{-1/2}(\Gamma)\}. \tag{3.15}$$

These spaces have been introduced for smooth boundaries and studied in [3]. In the case of Lipschitz domains the problem arises that the normal vector is a discontinuous function. Hence the normal vector n on Lipschitz domains is only an element of $\mathbf{L}^\infty(\Gamma)$, so that for $\mathbf{U} \in \mathbf{H}^{-1/2}(\Gamma)$ the scalar product $\mathbf{U} \cdot n$ is not defined.
Furthermore we need to extend the definitions of $\nabla_\Gamma, \mathbf{curl}_\Gamma$ to Lipschitz domains. This can be done by using the formalism as introduced in the definition of the Lipschitz domain (we refer to Definition 3.1 in [15]). From now on we will assume that Γ is the boundary of a Lipschitz domain Ω.

Definition 3.17. *We set*

$$V_t = \boldsymbol{\gamma}_t(\boldsymbol{H}^{1/2}(\Gamma)), \quad V_\times = \boldsymbol{\gamma}_\times(\boldsymbol{H}^{1/2}(\Gamma)), \tag{3.16}$$

with the norms

$$\|\boldsymbol{\lambda}\|_{V_t} = \inf_{\mathbf{u} \in H^{1/2}(\Gamma)} \left\{ \|\mathbf{u}\|_{H^{1/2}(\Gamma)} : \boldsymbol{\gamma}_t \mathbf{u} = \boldsymbol{\lambda} \right\}, \tag{3.17}$$
$$\|\boldsymbol{\lambda}\|_{V_\times} = \inf_{\mathbf{u} \in H^{1/2}(\Gamma)} \left\{ \|\mathbf{u}\|_{H^{1/2}(\Gamma)} : \boldsymbol{\gamma}_\times \mathbf{u} = \boldsymbol{\lambda} \right\}. \tag{3.18}$$

Note that in the case that Γ is smooth we have $V_t = V_\times = \mathbf{H}_t^{1/2}(\Gamma)$, but in the case that Γ has an edge those spaces can be different, this effect is illustrated in Figure 3.1.

Remark 3.18. *For Lipschitz polyhedra the spaces* V_t *and* $V_\times$ *are also denoted by* $\boldsymbol{H}_{\|}^{1/2}(\Gamma)$ *and* $\boldsymbol{H}_{\perp}^{1/2}(\Gamma)$.

The mappings $\boldsymbol{\gamma}_t : \mathbf{H}^1(\Omega) \to V_t$ and $\boldsymbol{\gamma}_\times : \mathbf{H}^1(\Omega) \to V_\times$ are surjective and therefore we get that the mappings

$$\boldsymbol{\gamma}_t : \mathbf{H}(\mathbf{curl}; \Omega) \to V_t', \quad \boldsymbol{\gamma}_\times : \mathbf{H}(\mathbf{curl}; \Omega) \to V_\times'$$

are continuous. These preparatory definitions now pave the way to the definition of the spaces $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$ for boundaries of Lipschitz domains:

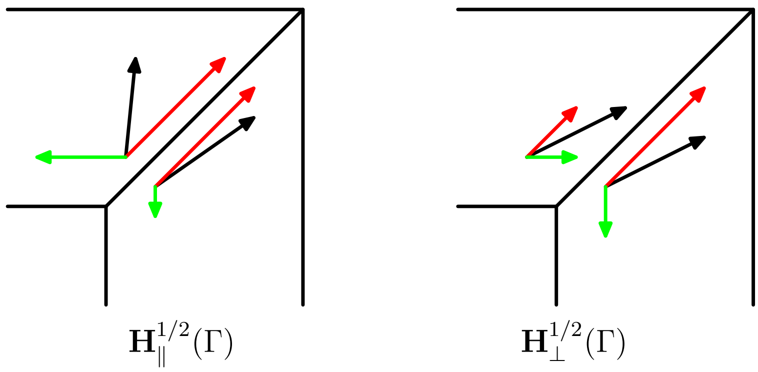


Figure 3.1: Continuity along edges for functions in the spaces $\mathbf{H}_{\|}^{1/2}(\Gamma)$ and $\mathbf{H}_{\perp}^{1/2}(\Gamma)$.

Definition 3.19. *For a Lipschitz domain Ω with boundary Γ we define the trace spaces*

$$\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \{\mathbf{u} \in V'_\times : \ \mathrm{div}_\Gamma\, \mathbf{u} \in H^{-1/2}(\Gamma)\}, \tag{3.19}$$

$$\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = \{\mathbf{u} \in V'_t : \ \mathrm{curl}_\Gamma \mathbf{u} \in H^{-1/2}(\Gamma)\}. \tag{3.20}$$

Remark 3.20. *In the case that Ω is a Lipschitz polyhedron we know that the surface Γ is bounded by a bounded number of Lipschitz polygons $\Gamma_k, k = 1, \ldots, N_\Gamma$. For this case there exists a more precise characterization of the spaces (3.19) and (3.20) (see [12, 13]), which is based on the functionals*

$$N_{lk}^{\|}(\mathbf{u}) = \int\limits_{\Gamma_l}\int\limits_{\Gamma_k} \frac{|\mathbf{u}(x)\cdot \boldsymbol{t}_{lk}(x) - \mathbf{u}(y)\cdot \boldsymbol{t}_{lk}(y)|}{|x-y|^3} ds_x ds_y,$$

$$N_{lk}^{\perp}(\mathbf{u}) = \int\limits_{\Gamma_l}\int\limits_{\Gamma_k} \frac{|\mathbf{u}(x)\cdot (\boldsymbol{t}_{lk}(x)\times n(x)) - \mathbf{u}(y)\cdot (\boldsymbol{t}_{lk}(y)\times n(y))|}{|x-y|^3} ds_x ds_y,$$

where $\boldsymbol{t}_{lk}$ is the unit vector along the edge $e_{lk} = \Gamma_l \cap \Gamma_k$. By the help of these functionals the following trace spaces are defined by

$$\boldsymbol{H}_{\|}^{1/2}(\Gamma) = \{\mathbf{u} \in \boldsymbol{H}_{t,pw}^{1/2}(\Gamma) : \ N_{lk}^{\|}(\mathbf{u}) < \infty, \textit{for all edges } e_{lk}\},$$

$$\boldsymbol{H}_{\perp}^{1/2}(\Gamma) = \{\mathbf{u} \in \boldsymbol{H}_{t,pw}^{1/2}(\Gamma) : \ N_{lk}^{\perp}(\mathbf{u}) < \infty, \textit{for all edges } e_{lk}\},$$

where $\boldsymbol{H}_{t,pw}^{1/2}(\Gamma)$ is the space of functions $\mathbf{u} \in \boldsymbol{L}_t^2(\Gamma)$ for which we have $\mathbf{u}|_{\Gamma_i} \in \boldsymbol{H}^{1/2}(\Gamma_i)$ for $i = 1, \ldots, N_\Gamma$.

Due to [15] we have the following theorem:

Theorem 3.21. *The mappings*

$$\boldsymbol{\gamma}_t : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$$

and

$$\boldsymbol{\gamma}_\times : \mathbf{H}(\mathbf{curl};\Omega) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$$

are linear and continuous.

Corollary 3.22. *For the Neumann trace $\boldsymbol{\gamma}_N$ it immediately follows that*

$$\boldsymbol{\gamma}_N : \mathbf{H}(\mathbf{curl}^2;\Omega) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$$

with the space

$$\mathbf{H}(\mathbf{curl}^2;\Omega) = \{\mathbf{U} \in \mathbf{H}(\mathbf{curl};\Omega) : \ \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} \in \mathbf{L}^2(\Omega)\}.$$

Furthermore we see that there exists an isometry, which maps between the two spaces $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ and $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$:

Lemma 3.23. *The mapping*

$$\mathbf{R} : \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$$

is bijective and isometric.

In addition the following duality property holds:

$$[\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)]' = \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma), \quad [\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)]' = \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma). \quad (3.21)$$

Definition 3.24. *The trace operator $\gamma_n : C^\infty_{comp}(\Omega) \to L_2(\Gamma)$ is defined by*

$$\gamma_n \mathbf{U}(x) = \mathbf{U}(x)|_\Gamma \cdot n(x), \quad x \in \Gamma.$$

The application of Gauss's theorem leads to the following result:

Proposition 3.25. *The mapping*

$$\gamma_n : \mathbf{H}(\mathrm{div};\Omega) \to H^{-1/2}(\Gamma)$$

is continuous.

An essential tool for the solution of boundary value problems related to the partial differential operator **curl curl** are given the following partial integration formulae:

Theorem 3.26 (Partial Integration). *Let Ω be a bounded Lipschitz domain, then there hold the partial integration formulae:*
1. Green Formula:

$$\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}, \mathbf{V}\rangle_\Omega = \langle \mathbf{curl}\,\mathbf{U}, \mathbf{curl}\,\mathbf{V}\rangle_\Omega - \langle \boldsymbol{\gamma}_N \mathbf{U}, \boldsymbol{\gamma}_t \mathbf{V}\rangle_\Gamma \tag{3.22}$$

2. Green Formula:

$$\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}, \mathbf{V}\rangle_\Omega - \langle \mathbf{U}, \mathbf{curl}\,\mathbf{curl}\,\mathbf{V}\rangle_\Omega = \langle \boldsymbol{\gamma}_N \mathbf{V}, \boldsymbol{\gamma}_t \mathbf{U}\rangle_\Gamma - \langle \boldsymbol{\gamma}_N \mathbf{U}, \boldsymbol{\gamma}_t \mathbf{V}\rangle_\Gamma \tag{3.23}$$

The spaces $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ can be characterized by the following famous decomposition:

Lemma 3.27 (Hodge Decomposition). *We have the decomposition*

$$\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = \nabla_\Gamma \mathcal{H}(\Gamma) \oplus \mathbf{curl}_\Gamma H^{1/2}(\Gamma) \tag{3.24}$$

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) = \mathbf{curl}_\Gamma \mathcal{H}(\Gamma) \oplus \nabla_\Gamma H^{1/2}(\Gamma) \tag{3.25}$$

with the space

$$\mathcal{H}(\Gamma) = \{v \in H^1(\Gamma)/\mathbb{C} : \Delta_\Gamma v \in H^{-1/2}(\Gamma)/\mathbb{C}\}, \quad \psi \in H^{1/2}(\Gamma) \setminus \mathbb{C}$$

For a proof we refer once more to [15] in the case of a Lipschitz domain and to [13] in the case of a Lipschitz polyhedra.

4 BOUNDARY INTEGRAL EQUATIONS FOR THE ELECTROMAGNETIC WAVE EQUATION

This chapter is devoted to the derivation of boundary integral formulations for Maxwell's equations. More precisely we deal with a partial differential equation of the type

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) + \kappa^2\mathbf{U}(x) = 0, \tag{4.1}$$

which we call electromagnetic wave equation. As we have seen in Chapter 2, in the time-harmonic case the electric field $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$ are governed by the electromagnetic wave equation (4.1).
Although the partial differential operator corresponding to (4.1) is not strongly elliptic, i.e. the principal symbol of $L = \mathbf{curl}\,\mathbf{curl} + \kappa^2$ is not invertible, we can find a fundamental solution for the operator L and derive a representation formula for a solution of (4.1). This famous formula is called Stratton-Chu representation formula (see [67]). A mathematical analysis of the Stratton-Chu formula and the resulting boundary integral operators has been done in [17, 52] for smooth domains and can be found in [33] for the case of Lipschitz domains.
At the beginning of this chapter we will give the basic results for boundary integral equations for scalar problems as some of the results will be needed later. This paves the way to the derivation of the Stratton-Chu representation formula. Based on the representation formula we take a closer look at the corresponding boundary integral operators. For special parameters considered here we can prove ellipticity results for some boundary integral operators. In the last section we will look at the behaviour of the boundary integral operators, when considering small κ. For this case we present a new formulation for solving a boundary value problem for (4.1). We prove that the formulation is stable as $\kappa \to 0$.

4.1 Boundary Integral Equations for Scalar Problems

When considering stationary problems the electric or magnetic field can be represented by a scalar potential. This means that the electric field $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$ can be written as the gradient of a function, which is governed by the potential equation. Also in the quasi-static approximation of the MIT-problem (see Section 2.3.2) a potential equation has to be solved. This motivates why we have to deal with boundary value problems with a partial differential equation of the type

$$-\Delta u(x) + \kappa^2 u(x) = 0, \quad x \in \Omega, \quad \kappa \in \mathbb{C}, \tag{4.2}$$

which includes the scalar potential equation for the case $\kappa = 0$. We will now introduce a representation formula for solutions of (4.2) and give a short recap of the important results related to boundary integral operators.

4.1.1 Boundary Value Problems

Let us consider the partial differential equation (4.2) in a bounded domain $\Omega \subset \mathbb{R}^3$, the corresponding variational form is defined by

$$a_\kappa(u, v) := \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)} dx + \kappa^2 \int_\Omega u(x)\overline{v(x)} dx. \tag{4.3}$$

Theorem 4.1. *For $\kappa \in \mathbb{C}$ with $\Re(\kappa) \neq 0$ the bi-linear form $a(u, v)$ is $H^1(\Omega)$-elliptic, i.e. there exists a constant $c_1 > 0$ such that*

$$|a_\kappa(u, u)| \geq c_1 \|u\|^2_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega) \tag{4.4}$$

holds.

Proof. We set $\kappa = \kappa_R + i\kappa_I$ with $\kappa_R, \kappa_I \in \mathbb{R}$, then we have

$$\begin{aligned} |a_\kappa(u, u)| &= \left| \|\nabla u\|^2_{\mathbf{L}^2(\Omega)} + \kappa^2 \|u\|^2_{L_2(\Omega)} \right| \\ &= \left[\left(\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} + (\kappa_R^2 - \kappa_I^2)\|u\|^2_{L_2(\Omega)} \right)^2 + \left(2\kappa_R\kappa_I \|u\|^2_{L_2(\Omega)} \right)^2 \right]^{1/2}. \end{aligned}$$

We prove the statement by distinction of different cases:

1. If $\kappa_I = 0$ the ellipticity follows immediately with $c_1 = \min(1, |\kappa|^2)$, since we have
$$|a_\kappa(u, u)| = \|\nabla u\|^2_{\mathbf{L}^2(\Omega)} + \kappa_R^2 \|u\|^2_{L_2(\Omega)} \geq \min(1, |\kappa|^2)\|u\|^2_{H^1(\Omega)}.$$
2. In the case $\kappa_R = \kappa_I$ we have $|\kappa|^2 = 2\kappa_R\kappa_I$, with which we obtain
$$\begin{aligned} |a_\kappa(u, u)| &= \left[\left(\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} \right)^2 + \left(|\kappa|^2 \|u\|^2_{L_2(\Omega)} \right)^2 \right]^{1/2} \\ &\geq \left[\frac{1}{2} \left(\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} + |\kappa|^2 \|u\|^2_{L_2(\Omega)} \right)^2 \right]^{1/2} \\ &\geq \frac{1}{\sqrt{2}} \min(1, |\kappa|^2)\|u\|^2_{H^1(\Omega)}, \end{aligned}$$
hence we have $c_1 = \frac{\min(1,|\kappa|^2)}{\sqrt{2}}$.

3. For the case $\kappa_R^2 > \kappa_I^2 > 0$ we get

$$
\begin{aligned}
|a_\kappa(u,u)| &= \left[\left(\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} + (\kappa_R^2 - \kappa_I^2)\|u\|^2_{L_2(\Omega)}\right)^2 + \left(2\kappa_R\kappa_I\|u\|^2_{L_2(\Omega)}\right)^2\right]^{1/2} \\
&\geq \left[\left(\|\nabla u\|^2_{\mathbf{L}^2(\Omega)}\right)^2 + \left(2\kappa_R\kappa_I\|u\|^2_{L_2(\Omega)}\right)^2\right]^{1/2} \\
&\geq \frac{1}{\sqrt{2}}\min(1, 2|\kappa_R\kappa_I|)\|u\|^2_{H^1(\Omega)},
\end{aligned}
$$

and hence the ellipticity constant is given by $c_1 = \frac{\min(1,2|\kappa_R\kappa_I|)}{\sqrt{2}}$.

4. In the case $0 < \kappa_R^2 < \kappa_I^2$ we first assume that $\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} \geq 2(\kappa_I^2 - \kappa_R^2)\|u\|^2_{L_2(\Omega)}$, this gives the ellipticity estimate

$$
\begin{aligned}
|a_\kappa(u,u)| &\geq \left[\left(\frac{1}{2}\|\nabla u\|^2_{\mathbf{L}^2(\Omega)}\right)^2 + \left(2\kappa_R\kappa_I\|u\|^2_{L_2(\Omega)}\right)^2\right]^{1/2} \\
&\geq \frac{1}{\sqrt{2}}\left(\frac{1}{2}\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} + 2|\kappa_R\kappa_I|\|u\|^2_{L_2(\Omega)}\right) \geq \frac{\min(1, 4|\kappa_R\kappa_I|)}{2\sqrt{2}}\|u\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Now we assume $\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} < 2(\kappa_I^2 - \kappa_R^2)\|u\|^2_{L_2(\Omega)}$, this leads us to the estimate

$$
\begin{aligned}
|a_\kappa(u,u)| &\geq 2|\kappa_R\kappa_I|\|u\|^2_{\mathbf{L}^2(\Omega)} > |\kappa_R\kappa_I|\|u\|^2_{L_2(\Omega)} + \frac{|\kappa_R\kappa_I|}{2(\kappa_I^2 - \kappa_R^2)}\|\nabla u\|^2_{\mathbf{L}^2(\Omega)} \\
&\geq \min\left(|\kappa_R\kappa_I|, \frac{|\kappa_R\kappa_I|}{2(\kappa_I^2 - \kappa_R^2)}\right)\|u\|^2_{H^1(\Omega)}.
\end{aligned}
$$

□

For the case $\kappa = 0$ we only have ellipticity in the semi-norm:

$$a_0(u,u) \geq |u|^2_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega).$$

Combining this result with Theorem 4.1 yields the unique solvability of the following Dirichlet boundary value problem:

Theorem 4.2. *For a bounded Lipschitz domain Ω and $\kappa \in \mathbb{C}$ with $\Re(\kappa) \neq 0$ or $\kappa = 0$ and given Dirichlet data $g \in H^{1/2}(\Gamma)$, the boundary value problem*

$$-\Delta u(x) + \kappa^2 u(x) = 0, \quad x \in \Omega, \quad \gamma_0^{\text{int}} u(x) = g(x), \quad x \in \Gamma, \tag{4.5}$$

has a unique solution $u \in H^1(\Omega)$.

For the case $\Re(\kappa) \neq 0$ the result follows immediately from the Lax-Milgram lemma, the proof for the case $\kappa = 0$ can be found for example in [66].

Remark 4.3. *In the case $\Re(\kappa) = 0$, which means $\kappa^2 < 0$ the equation (4.2) coincides with the Helmholtz equation. The boundary value problem (4.5) is then not uniquely solvable for a countable set of wave numbers κ, which correspond to the eigenvalues of the homogeneous Dirichlet eigenvalue problem for the Laplace equation. If the wave number does not correspond to one of those eigenvalues, we can prove the unique solvability of the Dirichlet boundary value problem (see [17, 24]).*

When dealing with transmission problems we also have to consider boundary value problems in unbounded domains, i.e. the exterior boundary value problem for the Laplace equation in the unbounded exterior domain Ω^c:

$$-\Delta u(x) = 0, \quad x \in \Omega^c, \quad \gamma_0^{\text{ext}} u(x) = g(x), \quad x \in \Gamma. \tag{4.6}$$

To ensure the unique solvability of the boundary value problem (4.6) we have to impose the radiation condition

$$|u(x)| = \mathcal{O}\left(\frac{1}{|x|}\right), \quad \text{as } |x| \to 0, \tag{4.7}$$

which also guarantees that we obtain a physical meaningful solution. The following proposition and its proof can be found in [46]:

Proposition 4.4. *The boundary value problem (4.6) together with the radiation condition (4.7) has a unqiue solution.*

4.1.2 Representation Formula

Definition 4.5. *For $u \in H^{-1/2}(\Gamma)$ the scalar single layer potential is defined by the surface integral*

$$\Psi^{\kappa}_{\text{SL}}(u)(x) = \int\limits_{\Gamma} U^*_{\kappa}(x,y)u(y)ds_y, \quad x \in \mathbb{R}^3 \setminus \Gamma, \tag{4.8}$$

and for $v \in H^{1/2}(\Gamma)$ the scalar double layer potential is defined by

$$\Psi^{\kappa}_{\text{DL}}(v)(x) = \int\limits_{\Gamma} \frac{\partial}{\partial n(y)} U^*_{\kappa}(x,y)v(y)ds_y, \quad x \in \mathbb{R}^3 \setminus \Gamma, \tag{4.9}$$

with the fundamental solution

$$U^*_{\kappa}(x,y) = \frac{1}{4\pi}\frac{e^{-\kappa|x-y|}}{|x-y|}, \quad \text{for } x \neq y, \tag{4.10}$$

and for $\kappa \in \mathbb{C}$.

By means of the above defined potentials we can find a representation formula for solutions of the partial differential equation (4.2) in the bounded domain Ω and in the unbounded domain Ω^c. The details of the derivation can be found in [59, 66].

Theorem 4.6 (Representation Formula)**.** *Let $u \in H^1(\Omega)$ be a solution of (4.2), then it has the representation*

$$u(x) = \Psi_{\mathrm{SL}}^{\kappa}(\gamma_1^{\mathrm{int}} u)(x) - \Psi_{\mathrm{DL}}^{\kappa}(\gamma_0^{\mathrm{int}} u)(x), \quad x \in \Omega. \tag{4.11}$$

If $u \in H^1_{loc}(\Omega^c)$ is a solution of (4.2), which satisfies the radiation condition (4.7) in the unbounded exterior domain Ω^c, then it can be represented by the formula

$$u(x) = -\Psi_{\mathrm{SL}}^{\kappa}(\gamma_1^{\mathrm{ext}} u)(x) + \Psi_{\mathrm{DL}}^{\kappa}(\gamma_0^{\mathrm{ext}} u)(x), \quad x \in \Omega^c. \tag{4.12}$$

4.1.3 Boundary Integral Equations

To obtain boundary integral equations the traces of the potential operators arising in the representation formula have to be studied carefully. For smooth and for Lipschitz domains this has been done in [19, 37, 46, 59, 66].

Theorem 4.7. *By applying the trace operators γ_0 and γ_1 to the single layer potential (4.8) and to the double layer potential (4.9) we obtain linear and continuous boundary integral operators, which have the following mapping properties:*

$$\gamma_0^{\mathrm{int}} \Psi_{\mathrm{SL}}^{\kappa} = V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \tag{4.13}$$

$$\gamma_1^{\mathrm{int}} \Psi_{\mathrm{SL}}^{\kappa} = \frac{1}{2} I + K'_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \tag{4.14}$$

$$\gamma_0^{\mathrm{int}} \Psi_{\mathrm{DL}}^{\kappa} = -\frac{1}{2} I + K_\kappa : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \tag{4.15}$$

$$\gamma_1^{\mathrm{int}} \Psi_{\mathrm{DL}}^{\kappa} = -D_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \tag{4.16}$$

Furthermore we have the following jump properties:

$$[\gamma_0 \Psi_{\mathrm{SL}}^{\kappa} w] = 0, \quad [\gamma_1 \Psi_{\mathrm{SL}}^{\kappa} w] = -w, \quad \textit{for } w \in H^{-1/2}(\Gamma) \tag{4.17}$$

and

$$[\gamma_0 \Psi_{\mathrm{DL}}^{\kappa} v] = v, \quad [\gamma_1 \Psi_{\mathrm{DL}}^{\kappa} v] = 0, \quad \textit{for } v \in H^{1/2}(\Gamma). \tag{4.18}$$

Applying the interior traces to the representation formula (4.11) leads to the following system of boundary integral equations for the bounded domain Ω

$$\begin{pmatrix} \gamma_0^{\mathrm{int}} u \\ \gamma_1^{\mathrm{int}} u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I - K & V \\ D & \frac{1}{2} I + K' \end{pmatrix} \begin{pmatrix} \gamma_0^{\mathrm{int}} u \\ \gamma_1^{\mathrm{int}} u \end{pmatrix}.$$

For the exterior domain Ω^c we get the system of boundary integral equations

$$\begin{pmatrix} \gamma_0^{\text{ext}} u \\ \gamma_1^{\text{ext}} u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K & -V \\ -D & \frac{1}{2}I - K' \end{pmatrix} \begin{pmatrix} \gamma_0^{\text{ext}} u \\ \gamma_1^{\text{ext}} u \end{pmatrix}$$

by applying the exterior trace operators to (4.12).

Theorem 4.8. *For $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ the operators $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $D_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are elliptic*

$$\langle V_\kappa w, w\rangle_\Gamma \geq c_1^V \|w\|^2_{H^{-1/2}(\Gamma)}, \quad \forall w \in H^{-1/2}(\Gamma),$$
$$\langle D_\kappa v, v\rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)}, \quad \forall v \in H^{1/2}(\Gamma).$$

For $\kappa = 0$ we have the ellipticity estimates

$$\langle V_0 w, w\rangle_\Gamma \geq c_1^V \|w\|^2_{H^{-1/2}(\Gamma)}, \quad \forall w \in H^{-1/2}(\Gamma),$$
$$\langle D_0 v, v\rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)}, \quad \forall v \in H_*^{1/2}(\Gamma),$$

with the space $H_*^{1/2}(\Gamma) = \{v \in H^{1/2}(\Gamma) : \langle v, 1\rangle_\Gamma = 0\}$.

The proof can be found in [46, 66].

Remark 4.9. *The hypersingular operator of the Laplace equation D_0 is not elliptic on the full space $H^{1/2}(\Gamma)$, therefore we introduce the stabilization*

$$\langle \widetilde{D}_0 u, v\rangle_\Gamma = \langle D_0 u, v\rangle_\Gamma + \alpha\langle u, 1\rangle_\Gamma \langle v, 1\rangle_\Gamma, \tag{4.19}$$

with some positive constant $\alpha \in \mathbb{R}_+$. The operator $\widetilde{D}_0$ is then elliptic on the whole space $H^{1/2}(\Gamma)$:

$$\langle \widetilde{D}_0 u, u\rangle_\Gamma \geq \tilde{c}_1^D \|u\|^2_{H^{1/2}(\Gamma)}, \quad \forall u \in H^{1/2}(\Gamma).$$

The operator, which maps the Dirichlet trace of a function, which is governed by (4.2) in the interior domain Ω, to its Neumann trace, is called interior Steklov-Poincaré operator:

$$\mathcal{S}_\kappa^{\text{int}} \gamma_0^{\text{int}} u = \gamma_1^{\text{int}} u, \quad \mathcal{S}_\kappa^{\text{int}} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

For the case $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ or $\kappa = 0$, the Dirichlet boundary value problem has a unique solution, thus the operator $\mathcal{S}_\kappa^{\text{int}}$ is well defined. Note that for $\kappa \in \mathbb{C}$ with $\Re(\kappa) = 0$ the operator $\mathcal{S}_\kappa^{\text{int}}$ may not be well defined, a detailed study of this case can be found in [72].
The properties of the Steklov-Poincaré operator are well known, see e.g. [19,37,46,66].

Out of the Calderon identities for the interior domain we can find a symmetric representation of the Steklov-Poincaré operator by using boundary integral operators:

$$\gamma_1^{\text{int}} u = \mathcal{S}_\kappa^{\text{int}} \gamma_0^{\text{int}} u = \left[D_\kappa + (\frac{1}{2}I + K_\kappa')V_\kappa^{-1}(\frac{1}{2}I + K_\kappa) \right] \gamma_0^{\text{int}} u. \tag{4.20}$$

In a similar way we can derive a Steklov-Poincaré operator for the exterior traces:

$$\gamma_1^{\text{ext}} u = -\mathcal{S}_\kappa^{\text{ext}} \gamma_0^{\text{ext}} u = -\left[D_\kappa + (\frac{1}{2}I - K_\kappa')V_\kappa^{-1}(\frac{1}{2}I - K_\kappa) \right] \gamma_0^{\text{ext}} u. \tag{4.21}$$

In the case $\kappa = 0$ we use the stabilized hypersingular operator $\widetilde{D}_0$ for the representation of the Steklov-Poincaré operators $\mathcal{S}_0^{\text{int}}$ and $\mathcal{S}_0^{\text{ext}}$:

$$\mathcal{S}_0^{\text{int}} = \widetilde{D}_0 + (\frac{1}{2}I + K_0')V_0^{-1}(\frac{1}{2}I + K_0), \quad \mathcal{S}_0^{\text{ext}} = \widetilde{D}_0 + (\frac{1}{2}I - K_0')V_0^{-1}(\frac{1}{2}I - K_0).$$

A simple consequence of Theorem 4.8 is the following result:

Corollary 4.10. *For $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ and $\kappa = 0$ the operators $\mathcal{S}_\kappa^{\text{int}}, \mathcal{S}_\kappa^{\text{ext}}$ are $H^{1/2}(\Gamma)$-elliptic with the ellipticity constant c_1^D:*

$$\langle \mathcal{S}_\kappa^{\text{int}} v, v \rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)}, \quad \forall v \in H^{1/2}(\Gamma),$$
$$\langle \mathcal{S}_\kappa^{\text{ext}} v, v \rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)}, \quad \forall v \in H^{1/2}(\Gamma).$$

4.2 Boundary Value Problems for the Electromagnetic Wave Equation

We will now consider the electromagnetic wave equation, i.e. a partial differential equation of the type (4.1) in a bounded domain Ω. For the electrogmagnetic wave equation we define the variational form

$$\mathbf{a}_\kappa(\mathbf{U}, \mathbf{V}) := \int_\Omega \mathbf{curl}\, \mathbf{U}(x) \cdot \overline{\mathbf{curl}\, \mathbf{V}(x)} dx + \kappa^2 \int_\Omega \mathbf{U}(x) \cdot \overline{\mathbf{V}(x)} dx. \tag{4.22}$$

Theorem 4.11. *For $\kappa \in \mathbb{C}$ with $\Re(\kappa) \neq 0$ the bi-linear form $\boldsymbol{a}_\kappa(.,.)$ is $\mathbf{H}(\mathbf{curl};\Omega)$-elliptic, i.e. there exists a constant $C_1 > 0$ such that*

$$|\boldsymbol{a}_\kappa(\mathbf{U}, \mathbf{U})| \geq C_1 \|\mathbf{U}\|^2_{\mathbf{H}(\mathbf{curl};\Omega)} \tag{4.23}$$

holds for all $\mathbf{U} \in \mathbf{H}(\mathbf{curl};\Omega)$.

The proof is analogous to the proof for Theorem 4.1. By using the Lax-Milgram lemma we get the following result:

Theorem 4.12. *For* $\kappa \in \mathbb{C}$ *with* $\Re(\kappa) \neq 0$ *and for* $\mathbf{F} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *the boundary value problem*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) + \kappa^2 \mathbf{U}(x) = 0, \quad x \in \Omega, \quad \boldsymbol{\gamma}_t^{\mathrm{int}} \mathbf{U}(x) = \mathbf{F}(x), \quad x \in \Gamma \tag{4.24}$$

has a unique solution in $\mathbf{H}(\mathbf{curl}; \Omega)$.

In contrast to the scalar equation (4.2) the boundary value problem (4.24) is not uniquely solvable for the case $\kappa = 0$. In fact, we easily see that any gradient function $\nabla\phi$ satisfies (4.1) for the case $\kappa = 0$. Hence we have to impose the additional condition $\operatorname{div} \mathbf{U} = 0$.

Theorem 4.13. *For* $\mathbf{F} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *the boundary value problem*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) = 0, \quad \operatorname{div} \mathbf{U}(x) = 0, \quad x \in \Omega, \quad \boldsymbol{\gamma}_t^{\mathrm{int}} \mathbf{U}(x) = \mathbf{F}(x), \quad x \in \Gamma \tag{4.25}$$

has a unique solution in $\mathbf{H}(\mathbf{curl}; \Omega)$.

When considering the exterior boundary value problem for $\kappa = 0$ we have to impose an additional radiation condition:

Theorem 4.14. *For* $\mathbf{F} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *the boundary value problem*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) = 0, \quad \operatorname{div} \mathbf{U}(x) = 0, \quad x \in \Omega^c, \quad \boldsymbol{\gamma}_t^{\mathrm{int}} \mathbf{U}(x) = \mathbf{F}(x), \quad x \in \Gamma \tag{4.26}$$

with the radiation condition

$$\mathbf{U}(x) = \mathcal{O}\left(\frac{1}{|x|}\right), \quad \text{as } |x| \to \infty \tag{4.27}$$

has a unique solution in $\mathbf{H}(\mathbf{curl}; \Omega^c)$.

The statement follows from Brezzi's Theorem (Theorem 3.4).

4.3 The Stratton-Chu Representation Formula

In this section we are going to derive a representation formula for solutions of the electromagnetic wave equation, i.e. an equation of the type (4.1). This representation formula was derived by Stratton and Chu [67], we can also find a derivation in [52]. For the derivation of the Stratton-Chu representation formula here we take

the derivation in [52] as a basis. We are going to derive an extended version of the Stratton-Chu representation formula, which includes an additional equation, which is usually omitted in the other derivations. This extended representation formula turns out to be also valid for the case $\kappa = 0$. The derivation is based on the decomposition of a solution of the electromagnetic wave equation into a gradient field and a vector valued remainder, which is fixed by some gauging condition. With this decomposition we can transform the electromagnetic wave equation into a system of partial differential equations. For this system we can find a fundamental solution, which is based on the fundamental solution (4.10) for the scalar equation (4.2).
As in the scalar case we define a single and a double layer potential, which are now applied to vector valued functions in contrast to the scalar case.

Definition 4.15. *For* $\mathbf{u} \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *the vectorial single layer potential is defined by the surface integral*

$$\mathbf{\Psi}_{\mathrm{SL}}^{\kappa}(\mathbf{u})(x) = \int_\Gamma U_\kappa^*(x,y)\mathbf{u}(y)ds_y, \quad x \in \mathbb{R}^3 \setminus \Gamma, \tag{4.28}$$

and for $\mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *the Maxwell double layer potential is defined by*

$$\mathbf{\Psi}_{\mathrm{DL}}^{\kappa}(\mathbf{v})(x) = \mathbf{curl}_x \int_\Gamma U_\kappa^*(x,y)\mathbf{R}\mathbf{v}(y)ds_y, \quad x \in \mathbb{R}^3 \setminus \Gamma \tag{4.29}$$

with the fundamental solution $U_\kappa^*(x,y)$ *as defined in (4.10).*

We are now going to prove two auxiliary results, which we will need for the derivation of the Stratton-Chu representation formula. The first result proves an alternative representation of the Maxwell double layer potential:

Lemma 4.16. *For any vector* $\boldsymbol{a} \in \mathbb{R}^3$ *and for a function* $\mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *it holds*

$$\int_\Gamma \gamma_{N,y}(U_\kappa^*(x,y)\boldsymbol{a}) \cdot \mathbf{v}(y)ds_y = -\boldsymbol{a} \cdot \mathbf{\Psi}_{\mathrm{DL}}^{\kappa}(\mathbf{v})(x), \quad x \notin \Gamma.$$

Proof. The result can easily be obtained by using some basic results from vector calculus:

$$
\begin{aligned}
\int_\Gamma \gamma_{N,y}(U^*_\kappa(x,y)\mathbf{a})\cdot\mathbf{v}(y)ds_y &= \int_\Gamma (n(y)\times(\mathbf{curl}_y(U^*_\kappa(x,y)\mathbf{a}))\cdot\mathbf{v}(y)ds_y \\
&= -\int_\Gamma (\mathbf{curl}_y(U^*_\kappa(x,y)\mathbf{a})\cdot(\mathbf{Rv}(y))ds_y \\
&= -\int_\Gamma (\nabla_y U^*_\kappa(x,y)\times\mathbf{a})\cdot(\mathbf{Rv}(y))ds_y \\
&= \int_\Gamma (\mathbf{a}\times\nabla_y U^*_\kappa(x,y))\cdot(\mathbf{Rv}(y))ds_y \\
&= -\mathbf{a}\cdot\int_\Gamma \nabla_x U^*_\kappa(x,y)\times(\mathbf{Rv}(y))ds_y \\
&= -\mathbf{a}\cdot\left(\mathbf{curl}_x\int_\Gamma U^*_\kappa(x,y)\mathbf{Rv}(y)ds_y\right).
\end{aligned}
$$

□

The second result contains an interesting fact, which establishes a connection between the vector valued Maxwell double layer potential (4.29) and the scalar double layer potential (4.9). It states that the Maxwell double layer potential applied to the surface gradient of a scalar function can be rewritten by using the vector valued single layer potential (4.28) plus the gradient of the scalar double layer potential:

Lemma 4.17. *For any $\phi\in H^{1/2}(\Gamma)$ we have the relation*

$$\mathbf{\Psi}^\kappa_{\mathrm{DL}}(\nabla_\Gamma\phi)(x) = -\kappa^2\mathbf{\Psi}^\kappa_{\mathrm{SL}}(n\phi)(x) - \nabla\Psi^\kappa_{\mathrm{DL}}(\phi)(x), \quad x\notin\Gamma. \tag{4.30}$$

Proof. Due to Chapter 3 we have the relation $\mathbf{curl}_\Gamma\phi = \mathbf{R}\nabla_\Gamma\phi$. Inserting this information into the definition of the Maxwell double layer potential (4.16) we obtain

$$\mathbf{\Psi}^\kappa_{\mathrm{DL}}(\nabla_\Gamma\phi)(x) = \mathbf{curl}_x\int_\Gamma U^*_\kappa(x,y)\mathbf{curl}_\Gamma\phi(y)ds_y.$$

Using the symmetry of the fundamental solution gives $\nabla_x U^*_\kappa(x,y) = -\nabla_y U^*_\kappa(x,y)$ and hence

$$\begin{aligned}\mathbf{e}_i \cdot (\mathbf{\Psi}^\kappa_{\mathrm{DL}}(\nabla_\Gamma \phi)(x)) &= \mathbf{e}_i \cdot \mathbf{curl}_x \int_\Gamma U^*_\kappa(x,y)\mathbf{curl}_\Gamma \phi(y) ds_y \\ &= -\mathbf{e}_i \cdot \int_\Gamma \nabla_y U^*_\kappa(x,y) \times \mathbf{curl}_\Gamma \phi(y) ds_y.\end{aligned}$$

The identity (4.30) then can easily be retrieved by using

$$\mathbf{curl}\,\mathbf{curl} = -\Delta + \nabla \operatorname{div}$$

and the fact that $U^*_\kappa(x,y)$ is a solution of $-\Delta u + \kappa^2 u = 0$ for $x \neq y$:

$$\begin{aligned}\mathbf{e}_i \cdot (\mathbf{\Psi}^\kappa_{\mathrm{DL}}(\nabla_\Gamma \phi)(x)) &= \int_\Gamma (\nabla_y U^*_\kappa(x,y) \times \mathbf{e}_i) \cdot \mathbf{curl}_\Gamma \phi(y) ds_y \\ &= \int_\Gamma (\mathbf{curl}_y (U^*_\kappa(x,y)\mathbf{e}_i)) \cdot \mathbf{curl}_\Gamma \phi(y) ds_y \\ &= \int_\Gamma \operatorname{curl}_\Gamma (\mathbf{curl}_y (U^*_\kappa(x,y)\mathbf{e}_i))\phi(y) ds_y \\ &= \int_\Gamma n(y) \cdot (\mathbf{curl}_y \mathbf{curl}_y (U^*_\kappa(x,y)\mathbf{e}_i))\phi(y) ds_y \\ &= \int_\Gamma n(y) \cdot (-\Delta(U^*_\kappa(x,y)\mathbf{e}_i) + \nabla_y \operatorname{div}_y (U^*_\kappa(x,y)\mathbf{e}_i))\phi(y) ds_y \\ &= \int_\Gamma n(y) \cdot \left[-\kappa^2 (U^*_\kappa(x,y)\mathbf{e}_i) + \nabla_y \frac{\partial}{\partial y_i} U^*_\kappa(x,y) \right] \phi(y) ds_y \\ &= -\kappa^2 \int_\Gamma U^*_\kappa(x,y)\phi(y) n_i(y) + n(y) \cdot \left[\nabla_y \frac{\partial}{\partial y_i} U^*_\kappa(x,y) \right] \phi(y) ds_y \\ &= -\kappa^2 \int_\Gamma U^*_\kappa(x,y)\phi(y) n_i(y) ds_y - \frac{\partial}{\partial x_i} \int_\Gamma \frac{\partial}{\partial n(y)} U^*_\kappa(x,y)\phi(y) ds_y.\end{aligned}$$

□

With these auxiliary results we are now ready to formulate and prove the Stratton-Chu representation formula. For the proof we pursue the same ansatz as in [52]. There the field is decomposed into a gradient field and a vector potential, which is defined by the Lorentz gauge.

Theorem 4.18 (Stratton-Chu representation formula)**.** *Let* $\mathbf{U} \in \mathbf{H}(\mathbf{curl};\Omega)$ *be a function which satisfies*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) + \kappa^2\mathbf{U}(x) = 0, \quad x \in \Omega \tag{4.31}$$

for some $\kappa \in \mathbb{C}$*. Then* $\mathbf{U}$ *can be represented by the formula*

$$\mathbf{U}(x) = \mathbf{\Psi}^{\kappa}_{\mathrm{SL}}(\boldsymbol{\gamma}^{\mathrm{int}}_N\mathbf{U})(x) + \mathbf{\Psi}^{\kappa}_{\mathrm{DL}}(\boldsymbol{\gamma}^{\mathrm{int}}_t\mathbf{U})(x) + \nabla\Psi^{\kappa}_{\mathrm{SL}}(\gamma^{\mathrm{int}}_n\mathbf{U})(x), \quad x \in \Omega. \tag{4.32}$$

In addition we have the relation

$$0 = \Psi^{\kappa}_{\mathrm{SL}}(\mathrm{div}_\Gamma\,\boldsymbol{\gamma}^{\mathrm{int}}_N\mathbf{U})(x) + \kappa^2\Psi^{\kappa}_{\mathrm{SL}}(\gamma^{\mathrm{int}}_n\mathbf{U})(x), \quad x \in \Omega, \tag{4.33}$$

which establishes a connection between the traces $\boldsymbol{\gamma}^{\mathrm{int}}_N\mathbf{U}$ *and* $\gamma^{\mathrm{int}}_n\mathbf{U}$*.*

Proof. We assume that the function $\mathbf{U} \in \mathbf{H}(\mathbf{curl};\Omega)$ satisfies (4.31) and introduce the decomposition

$$\mathbf{U}(x) = \kappa^2\mathbf{A}(x) + \nabla\phi(x), \quad x \in \Omega$$

in combination with the gauging condition

$$\mathrm{div}\,\mathbf{A}(x) + \phi(x) = 0, \quad x \in \Omega$$

with $\mathbf{A} \in \mathbf{H}(\mathbf{curl};\Omega)$ and $\phi \in H^1(\Omega)$. We insert this decomposition into (4.31), and together with the gauging condition we get the new system of equations

$$\mathcal{L}\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix} = \begin{pmatrix}\mathbf{curl}\,\mathbf{curl} + \kappa^2 & \nabla\\ \mathrm{div} & I\end{pmatrix}\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}. \tag{4.34}$$

Applying Green's second formula to the operator gives

$$\left\langle \mathcal{L}\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\ \psi\end{pmatrix}\right\rangle_\Omega = \left\langle \begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix}, \mathcal{L}^*\begin{pmatrix}\mathbf{V}\\ \psi\end{pmatrix}\right\rangle_\Omega - T\left(\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\ \psi\end{pmatrix}\right),$$

with the adjoint operator

$$\mathcal{L}^*\begin{pmatrix}\mathbf{V}\\ \psi\end{pmatrix} = \begin{pmatrix}\mathbf{curl}\,\mathbf{curl} + \kappa^2 & -\nabla\\ -\,\mathrm{div} & I\end{pmatrix}\begin{pmatrix}\mathbf{V}\\ \psi\end{pmatrix} \tag{4.35}$$

and the boundary trace operator

$$T(\begin{pmatrix}\mathbf{A}\\ \phi\end{pmatrix}, \begin{pmatrix}\mathbf{V}\\ \psi\end{pmatrix}) = -\langle\boldsymbol{\gamma}_t\mathbf{A}, \boldsymbol{\gamma}_N\mathbf{V}\rangle_\Gamma + \langle\boldsymbol{\gamma}_N\mathbf{A}, \boldsymbol{\gamma}_t\mathbf{V}\rangle_\Gamma - \langle\phi, \gamma_n V\rangle_\Gamma - \langle\gamma_n\mathbf{A}, \psi\rangle_\Gamma. \tag{4.36}$$

By making use of the fact that $U_\kappa^*(x,y)$ is a fundamental solution of (4.2) we get the fundamental solution corresponding to the operator (4.34), which given by the four by four matrix

$$\mathbf{U}_\kappa^*(x,y) = \begin{pmatrix} U_\kappa^*(x,y) I_3 & \nabla U_\kappa^*(x,y) \\ \nabla U_\kappa^*(x,y)^\top & \kappa^2 U_\kappa^*(x,y) \end{pmatrix}. \tag{4.37}$$

Note that this fundamental solution is also valid for the case $\kappa = 0$. By inserting the fundamental solution and by using Lemma 4.16 we get the representations

$$A_i(x) = \mathbf{e}_i \cdot (\mathbf{\Psi}_{\mathrm{DL}}^\kappa \boldsymbol{\gamma}_t \mathbf{A})(x) + \mathbf{e}_i \cdot \Psi_{\mathrm{SL}}^\kappa(\boldsymbol{\gamma}_N \mathbf{A})(x) \\ - \mathbf{\Psi}_{\mathrm{SL}}^\kappa(\phi n_i)(x) + \frac{\partial}{\partial x_i} \Psi_{\mathrm{SL}}^\kappa(\gamma_n \mathbf{A})(x), \quad x \in \Omega \tag{4.38}$$

and

$$\phi(x) = \int_\Gamma \nabla_\Gamma U_\kappa^*(x,y) \boldsymbol{\gamma}_N \mathbf{A}(y) ds_y - \Psi_{\mathrm{DL}}^\kappa(\phi)(x) - \kappa^2 \Psi_{\mathrm{SL}}^\kappa(\gamma_n \mathbf{A})(x). \tag{4.39}$$

As a next step we add the representation formula for $\mathbf{A}$ and ϕ to obtain a representation for $\mathbf{U}$ and use Lemma 4.17. This leads us to the Stratton-Chu representation formula

$$\mathbf{U}(x) = \kappa^2 \mathbf{A}(x) + \phi(x) = \mathbf{\Psi}_{\mathrm{SL}}^\kappa(\boldsymbol{\gamma}_N \mathbf{U})(x) + \mathbf{\Psi}_{\mathrm{DL}}^\kappa(\boldsymbol{\gamma}_t \mathbf{U})(x) + \nabla \Psi_{\mathrm{SL}}^\kappa(\gamma_n \mathbf{U})(x), \quad x \in \Omega.$$

Applying the divergence operator to (4.31) and inserting the gauging condition we conclude that ϕ satisfies

$$-\Delta \phi(x) + \kappa^2 \phi(x) = 0, \quad x \in \Omega$$

and due to Section 4.1 it has the representation

$$\phi(x) = \Psi_{\mathrm{SL}}^\kappa(\gamma_1 \phi)(x) - \Psi_{\mathrm{DL}}^\kappa(\gamma_0 \phi)(x), \quad x \in \Omega.$$

Inserting this into (4.39) gives the relation

$$\Psi_{\mathrm{SL}}^\kappa(\operatorname{div}_\Gamma \boldsymbol{\gamma}_N \mathbf{U})(x) + \kappa^2 \Psi_{\mathrm{SL}}^\kappa(\gamma_n \mathbf{U}(x)) = 0, \quad x \in \Omega,$$

which finishes the proof. □

Remark 4.19. *The relation*

$$0 = \Psi_{\mathrm{SL}}^\kappa(\operatorname{div}_\Gamma \boldsymbol{\gamma}_N^{\mathrm{int}} \mathbf{U})(x) + \kappa^2 \Psi_{\mathrm{SL}}^\kappa(\gamma_n^{\mathrm{int}} \mathbf{U})(x), \quad x \in \Omega, \tag{4.40}$$

is equivalent to $\operatorname{div} \mathbf{U}(x) = 0$ *for* $x \in \Omega$, *which is automatically satisfied for a solution of the electromagnetic wave equation (4.1).*

Remark 4.20. *For a function* $\mathbf{U} \in \mathbf{H}(\mathbf{curl};\Omega)$*, which satisfies the electromagnetic wave equation (4.31) we have the relation*

$$\operatorname{div}_\Gamma \boldsymbol{\gamma}_N^{\text{int}}\mathbf{U}(x) = -\kappa^2\gamma_n^{\text{int}}\mathbf{U}(x), \quad x \in \Gamma, \tag{4.41}$$

which is a consequence of the application of the Stokes formula for the surface (see [9]). If we assume that $\kappa \neq 0$ *we can replace the trace* $\gamma_n\mathbf{U}$ *in the representation formula by* $\boldsymbol{\gamma}_N\mathbf{U}$*, i.e.*

$$\mathbf{U}(x) = (\boldsymbol{\Psi}_{\text{SL}}^\kappa - \frac{1}{\kappa^2}\nabla_\Gamma \circ \Psi_{\text{SL}}^\kappa \circ \operatorname{div}_\Gamma)\boldsymbol{\gamma}_N^{\text{int}}\mathbf{U}(x) + \boldsymbol{\Psi}_{\text{DL}}^\kappa(\boldsymbol{\gamma}_t^{\text{int}}\mathbf{U})(x). \tag{4.42}$$

Since $\frac{1}{\kappa^2} \to \infty$ *as* $\kappa \to 0$ *we see that this formulation might cause problems, when* κ *is small.*

For the solution of the exterior boundary value problem with $\kappa = 0$ we have a similar result, we refer to [52] for a proof:

Theorem 4.21. *For a function* $\mathbf{U} \in \mathbf{H}(\mathbf{curl};\Omega^c)$*, which satisfies*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) = 0, \quad \operatorname{div}\mathbf{U}(x) = 0, \quad x \in \Omega^c \tag{4.43}$$

and the radiation condition

$$\mathbf{U}(x) = \mathcal{O}\left(\frac{1}{|x|}\right), \quad \text{as } |x| \to \infty,$$

we have the representation

$$\mathbf{U}(x) = -\boldsymbol{\Psi}_{\text{SL}}^0(\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{U})(x) - \boldsymbol{\Psi}_{\text{DL}}^0(\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{U})(x) - \nabla\Psi_{\text{SL}}^0(\gamma_n^{\text{ext}}\mathbf{U})(x), \quad x \in \Omega^c. \tag{4.44}$$

Remark 4.22. *For the exterior trace* $\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{U}$ *of a function* $\mathbf{U}$*, which satisfies (4.43) we get analogue to (4.41)*

$$\operatorname{div}_\Gamma \boldsymbol{\gamma}_N^{\text{ext}}\mathbf{U}(x) = 0, \quad x \in \Gamma. \tag{4.45}$$

4.4 Boundary Integral Operators

For the derivation of boundary integral equations for the electromagnetic wave equation the trace operators $\boldsymbol{\gamma}_t, \boldsymbol{\gamma}_N$ and γ_n are applied to the Stratton-Chu formula. The following results have been derived in [52] for domains with a smooth boundary Γ, for polyhedra we refer to [33] for the upcoming results.

Theorem 4.23. *The operators*

$$\boldsymbol{\gamma}_t^{\text{int}}\boldsymbol{\Psi}_{\text{SL}}^{\kappa} = A_\kappa : \mathbf{H}_{\parallel}^{-1/2}(\text{div}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\text{curl}_\Gamma, \Gamma),$$
$$\boldsymbol{\gamma}_N^{\text{int}}\boldsymbol{\Psi}_{\text{SL}}^{\kappa} = (\frac{1}{2}I + B_\kappa) : \mathbf{H}_{\parallel}^{-1/2}(\text{div}_\Gamma, \Gamma) \to \mathbf{H}_{\parallel}^{-1/2}(\text{div}_\Gamma, \Gamma),$$
$$\boldsymbol{\gamma}_t^{\text{int}}\boldsymbol{\Psi}_{\text{DL}}^{\kappa} = (\frac{1}{2}I + C_\kappa) : \mathbf{H}_{\perp}^{-1/2}(\text{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\text{curl}_\Gamma, \Gamma),$$
$$\boldsymbol{\gamma}_N^{\text{int}}\boldsymbol{\Psi}_{\text{DL}}^{\kappa} = N_\kappa : \mathbf{H}_{\perp}^{-1/2}(\text{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\parallel}^{-1/2}(\text{div}_\Gamma, \Gamma)$$

define continuous linear mappings. Furthermore we have the jump properties

$$\boldsymbol{\gamma}_t^{\text{int}}\boldsymbol{\Psi}_{\text{SL}}^{\kappa} - \boldsymbol{\gamma}_t^{\text{ext}}\boldsymbol{\Psi}_{\text{SL}}^{\kappa} = 0, \quad \boldsymbol{\gamma}_N^{\text{int}}\boldsymbol{\Psi}_{\text{SL}}^{\kappa} - \boldsymbol{\gamma}_N^{\text{int}}\boldsymbol{\Psi}_{\text{SL}}^{\kappa} = I,$$
$$\boldsymbol{\gamma}_t^{\text{int}}\boldsymbol{\Psi}_{\text{DL}}^{\kappa} - \boldsymbol{\gamma}_t^{\text{ext}}\boldsymbol{\Psi}_{\text{DL}}^{\kappa} = I, \quad \boldsymbol{\gamma}_N^{\text{int}}\boldsymbol{\Psi}_{\text{DL}}^{\kappa} - \boldsymbol{\gamma}_N^{\text{int}}\boldsymbol{\Psi}_{\text{DL}}^{\kappa} = 0,$$

and the relation

$$\langle C_\kappa \boldsymbol{u}, \boldsymbol{\mu}\rangle_\Gamma = -\langle \boldsymbol{u}, B_\kappa \boldsymbol{\mu}\rangle_\Gamma \tag{4.46}$$

for all $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\text{curl}_\Gamma, \Gamma)$ *and* $\boldsymbol{\mu} \in \mathbf{H}_{\parallel}^{-1/2}(\text{div}_\Gamma, \Gamma)$.

Definition 4.24. *If we apply the trace operator* $\boldsymbol{\gamma}_t$ *to the representation formula (4.42) we obtain the following operator for* $\kappa \neq 0$

$$S_\kappa := A_\kappa - \frac{1}{\kappa^2}\nabla_\Gamma \circ V_\kappa \circ \text{div}_\Gamma\,. \tag{4.47}$$

We call S_κ *Maxwell single layer potential.*

Corollary 4.25. *The operator* $S_\kappa : \mathbf{H}_{\parallel}^{-1/2}(\text{div}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\text{curl}_\Gamma, \Gamma)$ *defines a continuous linear mapping.*

In [52] an alternative representation for the Maxwell hypersingular operator N_κ was derived by using integration by parts. In fact we see that the operator N_κ exhibits a similar structure as S_κ:

Remark 4.26. *For* N_κ *with* $\kappa \in \mathbb{C}$ *we have the representation*

$$\langle N_\kappa \mathbf{u}, \mathbf{v}\rangle_\Gamma = \kappa^2 \langle A_\kappa \mathbf{R}\mathbf{u}, \mathbf{R}\mathbf{v}\rangle_\Gamma + \langle V_\kappa \text{curl}_\Gamma \mathbf{u}, \text{curl}_\Gamma \mathbf{v}\rangle_\Gamma, \quad \mathbf{u}, \mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\text{curl}_\Gamma, \Gamma). \tag{4.48}$$

We can also write

$$\langle N_\kappa \mathbf{u}, \mathbf{v}\rangle_\Gamma = \kappa^2 \langle S_\kappa \mathbf{R}\mathbf{u}, \mathbf{R}\mathbf{v}\rangle_\Gamma. \tag{4.49}$$

when $\kappa \neq 0$. *We observe that although the operator* N_κ *seems to have strong similarities to* S_κ, *the behaviour for* $\kappa \to 0$ *is very different. We also see that the operator* N_0 *is well defined for* $\kappa = 0$ *in contrast to* S_κ.

In the case of $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ we have the following ellipticity results for the Maxwell single layer potential S_κ and for the Maxwell hypersingular operator N_κ:

Theorem 4.27. *For $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ there exist positive constants $c_1^S > 0$ and $c_1^N > 0$ such that*

$$|\langle S_\kappa \boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\Gamma| \geq c_1^S \|\boldsymbol{\lambda}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}, \quad \forall \boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \tag{4.50}$$

$$|\langle N_\kappa \mathbf{u}, \mathbf{u}\rangle_\Gamma| \geq c_1^N \|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)}, \quad \forall \mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma). \tag{4.51}$$

For the special case $\kappa = \kappa_R + i\kappa_I$ with $\kappa_R = \kappa_I$ we have $c_1^S = c_1 \frac{\min(1,|\kappa|)}{2}$ and $c_1^N = c_2 \frac{\min(1,|\kappa|)}{2}$, where c_1 and c_2 do not depend on κ. This means for this case we see the explicit dependence on κ in the ellipticity constants c_1^S and c_1^N.

For the operator A_κ we do not have the ellipticity on the full space $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, though the operator A_κ is elliptic on a subspace of $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$:

Theorem 4.28. *For κ with $\Re(\kappa) > 0$ and for $\kappa = 0$ the operator A_κ is $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$-elliptic, i.e. there exists a constant c_1^A such that*

$$|\langle A_\kappa \boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\Gamma| \geq c_1^A \|\boldsymbol{\lambda}\|^2_{\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma,\Gamma)}, \quad \forall \boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma). \tag{4.52}$$

For the proof of Theorem 4.27 and 4.28 we refer to [9] for the case $\kappa = \kappa_R + \kappa_I$ with $\kappa_R = \kappa_I$. For the other cases the proof works in an analogous way due to Theorem 4.11. The proof for the ellipticity of S_κ can also be found in [14]. The ellipticity for N_κ immediately follows from the ellipticity of S_κ by using the relation (4.49).
A simple consequence of Lemma 4.17 is the following relation:

Lemma 4.29. *For $\phi \in H^{1/2}(\Gamma)$ we have the relation*

$$\boldsymbol{\gamma}_t^{\mathrm{int}} \boldsymbol{\Psi}_{\mathrm{DL}}^0 (\nabla_\Gamma \phi) = (\frac{1}{2} I + C_0) \nabla_\Gamma \phi = \nabla_\Gamma (\frac{1}{2} I - K_0)\phi.$$

Lemma 4.30. *For any $\boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ we have*

$$\Im(\langle S_\kappa \boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\Gamma) \geq 0, \quad \Im(\langle N_\kappa \mathbf{u}, \mathbf{u}\rangle_\Gamma) \leq 0,$$
$$\Im(\langle S_\kappa^{-1} \boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\Gamma) \leq 0, \quad \Im(\langle N_\kappa^{-1} \mathbf{u}, \mathbf{u}\rangle_\Gamma) \geq 0.$$

A proof of Lemma 4.30 can be found in [71].

4.5 Boundary Integral Equations

Having derived boundary integral operators we are now ready to formulate boundary integral equations for the electromagnetic wave equation. Applying the trace operators $\boldsymbol{\gamma}_t^{\text{int}}$ and $\boldsymbol{\gamma}_N^{\text{int}}$ to the Stratton-Chu representation formula (4.32) yield the following boundary integral equations for the case $\kappa \neq 0$:

$$\boldsymbol{\gamma}_t^{\text{int}}\mathbf{E}(x) = S_\kappa(\boldsymbol{\gamma}_N^{\text{int}}\mathbf{E})(x) + (\frac{1}{2}I + C_\kappa)\boldsymbol{\gamma}_t^{\text{int}}\mathbf{E}(x), \quad x \in \Gamma, \tag{4.53}$$

$$\boldsymbol{\gamma}_N^{\text{int}}\mathbf{E}(x) = (\frac{1}{2}I + B_\kappa)\boldsymbol{\gamma}_N^{\text{int}}(x) + N_\kappa(\boldsymbol{\gamma}_t^{\text{int}}\mathbf{E})(x), \quad x \in \Gamma. \tag{4.54}$$

For the exterior domain we obtain

$$\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E}(x) = -S_\kappa\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}(x) + (\frac{1}{2}I - C_\kappa)(\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E})(x), \quad x \in \Gamma, \tag{4.55}$$

$$\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}(x) = (\frac{1}{2}I - B_\kappa)(\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E})(x) - N_\kappa\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E}(x), \quad x \in \Gamma \tag{4.56}$$

by applying the traces $\boldsymbol{\gamma}_t^{\text{ext}}$ and $\boldsymbol{\gamma}_N^{\text{ext}}$ to the representation formula for the exterior domain.

4.5.1 Steklov-Poincaré Operator

For the solution of transmission problems we need to define an operator, which maps the tangential trace of a function to its Neumann trace, we call this operator analogue to the scalar case Steklov-Poincaré operator: For $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ we define for the interior traces

$$\mathcal{S}_\kappa^{\text{int}}\boldsymbol{\gamma}_t^{\text{int}}\mathbf{E}(x) = \boldsymbol{\gamma}_N^{\text{int}}\mathbf{E}(x), \quad \mathcal{S}_\kappa^{\text{int}} : \mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma, \Gamma). \tag{4.57}$$

For such $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ we know that the boundary value problem (4.24) has a unique solution and therefore the Steklov-Poincaré operator is well defined. By the aid of the boundary integral equations (4.53)-(4.54) the Dirichlet-to-Neumann map can be realized as

$$\mathcal{S}_\kappa^{\text{int}} = [S_\kappa]^{-1}(\frac{1}{2}I - C_\kappa), \tag{4.58}$$

or in the symmetric version as

$$\mathcal{S}_\kappa^{\text{int}} = N_\kappa + (\frac{1}{2}I + B_\kappa)[S_\kappa]^{-1}(\frac{1}{2}I - C_\kappa). \tag{4.59}$$

Proposition 4.31. *For $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ the Steklov-Poincaré operator is elliptic*

$$|\langle \mathcal{S}_\kappa^{\text{int}} \mathbf{u}, \mathbf{u} \rangle_\Gamma| \geq c \|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma, \Gamma)}, \quad \forall \mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma, \Gamma).$$

Proof. Since we have $\Re(\langle S_\kappa^{-1} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle_\Gamma) \geq 0$ and $\Im(\langle S_\kappa^{-1} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle_\Gamma) \geq 0$ it immediately follows for all $\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma, \Gamma)$ that

$$\begin{aligned} |\langle \mathcal{S}_\kappa^{\text{int}} \mathbf{u}, \mathbf{u} \rangle_\Gamma| &= |\langle N_\kappa \mathbf{u}, \mathbf{u} \rangle_\Gamma + \langle S_\kappa^{-1} (\frac{1}{2} I - C_\kappa) \mathbf{u}, (\frac{1}{2} I - C_\kappa) \mathbf{u} \rangle_\Gamma| \\ &\geq |\langle N_\kappa \mathbf{u}, \mathbf{u} \rangle_\Gamma| \geq c_1^N \|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma, \Gamma)}. \end{aligned}$$

□

Let us now consider the Steklov-Poincaré operator for the exterior domain for the case $\kappa = 0$. Due to Theorem 4.14 the Steklov-Poincaré operator

$$\mathcal{S}_0^{\text{ext}} \boldsymbol{\gamma}_t^{\text{ext}} \mathbf{U}(x) = \boldsymbol{\gamma}_N^{\text{ext}} \mathbf{U}(x),$$

which maps the tangential trace of a solution of the boundary value problem (4.26)-(4.27) to its Neumann trace, is well defined. Out of the relation (4.45) we get the mapping property

$$\mathcal{S}_0^{\text{ext}} \boldsymbol{\gamma}_t^{\text{ext}} \mathbf{E} = \boldsymbol{\gamma}_N^{\text{ext}} \mathbf{E}, \quad \mathcal{S}_0^{\text{ext}} : \mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma 0, \Gamma). \tag{4.60}$$

Let us now look at the boundary integral equations for the case $\kappa = 0$, out of the Stratton-Chu representation formula (4.44) we get the following system of boundary integral equations

$$A_0 \boldsymbol{\gamma}_N^{\text{ext}} \mathbf{E}(x) + (\frac{1}{2} I + C_0) \boldsymbol{\gamma}_t^{\text{ext}} \mathbf{E}(x) + \nabla_\Gamma V_0 (\gamma_n^{\text{ext}} \mathbf{E})(x) = 0, \quad x \in \Gamma, \tag{4.61}$$

$$(\frac{1}{2} I + B_0) \boldsymbol{\gamma}_N^{\text{ext}} \mathbf{E}(x) + N_0 (\boldsymbol{\gamma}_t \mathbf{E})(x) = 0, \quad x \in \Gamma. \tag{4.62}$$

If we test the third summand in the left hand side of the boundary integral equation (4.61) with a test function $\boldsymbol{\mu}$ from the space $\mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma 0, \Gamma)$, we observe we observe that it vanishes:

$$\langle \nabla_\Gamma V_0 \gamma_n^{\text{ext}} \mathbf{E}, \boldsymbol{\mu} \rangle_\Gamma = -\langle V_0 \gamma_n^{\text{ext}} \mathbf{E}, \text{div}_\Gamma \boldsymbol{\mu} \rangle_\Gamma = 0.$$

Furthermore we know that A_0 is elliptic on the space $\mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma 0, \Gamma)$, therefore the variational problem:
Find $\boldsymbol{\gamma}_N^{\text{ext}} \mathbf{E} \in \mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma 0, \Gamma)$ such that

$$\langle A_0 \boldsymbol{\gamma}_N^{\text{ext}} \mathbf{E}, \boldsymbol{\mu} \rangle_\Gamma = -\langle (\frac{1}{2} I + C_0) \boldsymbol{\gamma}_t^{\text{ext}} \mathbf{E}, \boldsymbol{\mu} \rangle_\Gamma, \quad \forall \boldsymbol{\mu} \in \mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma 0, \Gamma), \tag{4.63}$$

has a unique solution. Thus we can define the Steklov-Poincaré operator for the case $\kappa = 0$, which maps $\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E}$ to the solution $\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}$ of the variational problem (4.63). Note that the operator $\mathcal{S}_0^{\text{ext}}$ is not invertible, since for the solution $\mathbf{E}(x) = \nabla\phi$ we have

$$\mathcal{S}_0^{\text{ext}}\nabla_\Gamma\phi(x) = 0,$$

which is due to $\boldsymbol{\gamma}_N^{\text{ext}}(\nabla\phi) = 0$.

4.6 Stabilization for small κ

In this section we investigate the behaviour of boundary integral equations when considering small κ. First we analyze the so-called Dirichlet boundary value problem where we prescribe the tangential trace on the boundary. We will introduce a boundary integral formulation which is stable for small κ. The formulation, which we are going to present, is also considered in [70] for the scattering case. A similar approach was introduced in [68] for the scattering of composite objects. We will show that this approach can also be applied to the Neumann boundary value problem where we prescribe the Neumann trace.

4.6.1 Prescribing the tangential trace

In this section we will deal with the solution of the following boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}(x) + \kappa^2\mathbf{E}(x) = 0, \quad x \in \Omega, \quad \boldsymbol{\gamma}_t^{\text{int}}\mathbf{E}(x) = \mathbf{F}(x), \quad x \in \Gamma. \tag{4.64}$$

One possibility to solve this boundary value problem would be to determine the unknown boundary data $\boldsymbol{\gamma}_N^{\text{int}}\mathbf{E}$ by solving the boundary integral equation

$$S_\kappa(\boldsymbol{\gamma}_N^{\text{int}}\mathbf{E})(x) = (\frac{1}{2}I - C_\kappa)\mathbf{F}(x), \quad x \in \Gamma.$$

However, as already stated, the operator S_κ is not defined for $\kappa = 0$, therefore we want to derive a boundary integral formulation which is also valid for $\kappa = 0$. We will now keep the trace $\gamma_n^{\text{int}}\mathbf{E}$ and apply the tangential trace operator to the 'extended' representation formula (5.16). This results in the following system on Γ

$$\begin{aligned} A_\kappa(\boldsymbol{\gamma}_N^{\text{int}}\mathbf{E}) + \nabla_\Gamma V_\kappa(\gamma_n^{\text{int}}\mathbf{E}) &= (\frac{1}{2}I - C_\kappa)\mathbf{F}, \\ V_\kappa(\operatorname{div}_\Gamma \boldsymbol{\gamma}_N^{\text{int}}\mathbf{E}) + \kappa^2 V_\kappa(\gamma_n^{\text{int}}\mathbf{E}) &= 0, \end{aligned}$$

or in equivalent form

$$\begin{pmatrix} A_\kappa & \nabla_\Gamma V_\kappa \\ V_\kappa \operatorname{div}_\Gamma & \kappa^2 V_\kappa \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}_N \mathbf{E} \\ \gamma_n \mathbf{E} \end{pmatrix} = \begin{pmatrix} (\frac{1}{2}I - C_\kappa)\mathbf{F} \\ 0 \end{pmatrix}. \tag{4.65}$$

For the unique solvability of (4.65) we will make use of the following result.

Lemma 4.32. *For $\kappa = 0$ or $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ there exists a constant $c_S > 0$ such that*

$$\sup_{0 \neq \boldsymbol{\lambda} \in \mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} \frac{\langle V_\kappa \operatorname{div}_\Gamma \boldsymbol{\lambda}, \phi\rangle_\Gamma}{\|\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}} \geq c_S \|\phi\|_{H^{-1/2}(\Gamma)} \tag{4.66}$$

holds for all $\phi \in H_{**}^{-1/2}(\Gamma) = \{u \in H^{-1/2}(\Gamma) : \langle u, V_\kappa 1\rangle_\Gamma = 0\}$.

Proof. Due to [11] there exists a constant $c > 0$ such that

$$\|\nabla_\Gamma \phi\|_{\mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)} \geq c\|\phi\|_{H^{1/2}(\Gamma)}$$

holds for all $\phi \in H_{**}^{1/2}(\Gamma) = \{u \in H^{1/2}(\Gamma) : \langle u, 1\rangle_\Gamma = 0\}$. By using a duality argument we get

$$\|\nabla_\Gamma \phi\|_{\mathbf{H}_\perp^{-1/2}(\operatorname{curl}_\Gamma,\Gamma)} = \sup_{0 \neq \boldsymbol{\lambda} \in \mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} \frac{\langle \nabla_\Gamma \phi, \boldsymbol{\lambda}\rangle_\Gamma}{\|\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}} = \sup_{0 \neq \boldsymbol{\lambda} \in \mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} \frac{\langle \phi, \operatorname{div}_\Gamma \boldsymbol{\lambda}\rangle_\Gamma}{\|\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}}$$

This yields the inf-sup condition

$$\sup_{0 \neq \boldsymbol{\lambda} \in \mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} \frac{\langle \phi, \operatorname{div}_\Gamma \boldsymbol{\lambda}\rangle_\Gamma}{\|\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}} \geq c\|\phi\|_{H^{1/2}(\Gamma)}$$

for all $\phi \in H_{**}^{1/2}(\Gamma)$. By setting $\psi = V_\kappa^{-1}\phi$ and using the ellipticity of V_κ we get the estimate

$$\|\phi\|_{H^{1/2}(\Gamma)} = \|V_\kappa \psi\|_{H^{1/2}(\Gamma)} = \sup_{0 \neq p \in H^{-1/2}(\Gamma)} \frac{\langle V_\kappa \psi, p\rangle_\Gamma}{\|p\|_{H^{-1/2}(\Gamma)}} \geq \frac{\langle V_\kappa \psi, \psi\rangle_\Gamma}{\|\psi\|_{H^{-1/2}(\Gamma)}} \geq c_1^V \|\psi\|_{H^{-1/2}(\Gamma)}$$

which finishes the proof. □

Theorem 4.33. *For $\kappa = 0$ there exists a unique solution of the variational problem: Find $(\boldsymbol{\lambda}, \phi) \in \mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma) \times H_{**}^{-1/2}(\Gamma)$ such that*

$$\langle A_0 \boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma + \langle \nabla V_0 \phi, \boldsymbol{\mu}\rangle_\Gamma = \langle \boldsymbol{F}, \boldsymbol{\mu}\rangle_\Gamma \tag{4.67}$$

$$\langle \psi, V_0 \operatorname{div}_\Gamma \boldsymbol{\lambda}\rangle_\Gamma = 0 \tag{4.68}$$

holds for all $(\boldsymbol{\mu}, \psi) \in \mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma) \times H_{**}^{-1/2}(\Gamma)$.

Proof. The above varational form can be written in the scheme

$$a(\boldsymbol{\lambda}, \boldsymbol{\mu}) + b(\boldsymbol{\mu}, \phi) = f(\boldsymbol{\mu}), \tag{4.69}$$

$$b(\boldsymbol{\lambda}, \psi) = 0 \tag{4.70}$$

with $a(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \langle A_0\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma$ and $b(\boldsymbol{\mu}, \phi) = \langle \nabla_\Gamma V_0 \phi, \boldsymbol{\mu}\rangle_\Gamma$. The space

$$V_0 = \{\boldsymbol{\mu} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \ : b(\boldsymbol{\mu}, \phi) = 0 \quad \forall \phi \in H^{-1/2}(\Gamma)\}$$

can be identified with the space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$ since we have

$$b(\boldsymbol{\mu}, \phi) = \langle \phi, V_0 \,\mathrm{div}_\Gamma \boldsymbol{\mu}\rangle_\Gamma$$

and V_0 is elliptic. The inf-sup condition from Lemma 4.32 together with the ellipticity

$$\langle A_0\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\Gamma \geq c\|\boldsymbol{\lambda}\|^2_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)}, \quad \forall \boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma), \tag{4.71}$$

give us the unique solvability of the variational problem by applying Theorem 3.4. □

Theorem 4.34. *For $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ there exists a unique solution of the variational problem:*
Find $(\boldsymbol{\lambda}, \phi) \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\langle A_\kappa\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma + \langle \nabla V_\kappa \phi, \boldsymbol{\mu}\rangle_\Gamma = \langle \boldsymbol{F}, \boldsymbol{\mu}\rangle_\Gamma, \tag{4.72}$$

$$\langle \psi, V_\kappa \,\mathrm{div}_\Gamma \boldsymbol{\lambda}\rangle_\Gamma + \kappa^2 \langle \psi, V_\kappa \phi\rangle_\Gamma = 0 \tag{4.73}$$

holds for all $(\boldsymbol{\mu}, \psi) \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma)$.

Proof. In the case $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ the operator $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ defines a bijective mapping, and therefore we can conclude from (4.73) that $\mathrm{div}_\Gamma \boldsymbol{\lambda} = -\kappa^2 \phi$ holds. Inserting this into (4.72) results in the variational problem

$$\langle A_\kappa\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma - \frac{1}{\kappa^2}\langle \nabla_\Gamma V_\kappa \,\mathrm{div}_\Gamma \lambda, \boldsymbol{\mu}\rangle_\Gamma = \langle (\frac{1}{2}I - C_\kappa)\mathbf{F}, \boldsymbol{\mu}\rangle_\Gamma \quad \forall \boldsymbol{\mu} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma). \tag{4.74}$$

From the ellipticity of the operator

$$S_\kappa = A_\kappa - \frac{1}{\kappa^2}\nabla_\Gamma \circ V_\kappa \circ \mathrm{div}_\Gamma : \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$$

we can finally deduce the unique solvability of the variational problem (4.72)-(4.73). □

4.6.2 Prescribing the Neumann trace

In this section we will show that the above stabilized boundary integral equation can also be used when considering the Neumann boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}(x) + \kappa^2\mathbf{E}(x) = 0, \quad x \in \Omega, \quad \boldsymbol{\gamma}_N^{\mathrm{int}}\mathbf{E}(x) = \mathbf{F}(x), \quad x \in \Gamma.$$

If we insert the given boundary data into the second boundary integral equation (4.54) we are left with solving

$$(N_\kappa \mathbf{u})(x) = (\frac{1}{2}I - B_\kappa)\mathbf{F}(x), \quad x \in \Gamma. \tag{4.75}$$

Let us first look at the low frequency behaviour of N_κ: If we let κ tend to zero we get

$$\langle N_0\mathbf{u}, \mathbf{v}\rangle_\Gamma = \langle V_0 \mathrm{curl}_\Gamma \mathbf{u}, \mathrm{curl}_\Gamma \mathbf{v}\rangle_\Gamma, \quad \forall \mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma). \tag{4.76}$$

In contrast to the Maxwell single layer potential S_κ the Maxwell hypersingular operator N_κ is defined for $\kappa = 0$, however the resulting operator is not invertible. In fact, every function $\mathbf{u} = \nabla_\Gamma \phi$ lies in the kernel of N_0. This results in the fact that N_κ is also ill-conditioned if we let κ tend to zero. We will show how we can use the stabilized ansatz from the previous section for the hypersingular operator N_κ.
Due to Remark 4.26 we have $\langle N_\kappa \mathbf{u}, \mathbf{v}\rangle_\Gamma = \kappa^2\langle S_\kappa \mathbf{R}u, \mathbf{R}v\rangle_\Gamma$, which motivates the choice

$$\boldsymbol{\lambda}(x) = \kappa^2 \mathbf{R}\boldsymbol{\gamma}_t^{\mathrm{int}}\mathbf{E}(x), \quad x \in \Gamma. \tag{4.77}$$

Hence the boundary integral equation (4.75) can be rewritten as the variational problem:
Find $\boldsymbol{\lambda} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ such that

$$\langle S_\kappa \boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma = \langle(\frac{1}{2}I - B_\kappa)\mathbf{F}, \mathbf{R}\boldsymbol{\mu}\rangle_\Gamma$$

holds for all $\boldsymbol{\mu} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.
For the operator S_κ we can now apply the stablization as considered in the previous section and obtain the following variational problem:
Find $(\boldsymbol{\lambda}, \phi) \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \langle A_\kappa \boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma + \langle \nabla_\Gamma V_\kappa \phi, \boldsymbol{\mu}\rangle_\Gamma &= \langle(\frac{1}{2}I - B_\kappa)\mathbf{F}, \mathbf{R}\boldsymbol{\mu}\rangle_\Gamma, \\ \langle V_\kappa \,\mathrm{div}_\Gamma \,\boldsymbol{\lambda}, \psi\rangle_\Gamma + \kappa^2\langle V_\kappa \phi, \psi\rangle_\Gamma &= 0 \end{aligned}$$

holds for all $(\boldsymbol{\mu}, \psi) \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma)$.

Thus we have found a boundary integral formulation for the Neumann boundary value problem, which is stable for $\kappa \to 0$. The tangential trace of $\mathbf{E}$ can finally be obtained by computing

$$\boldsymbol{\gamma}_t^{\mathrm{int}}\mathbf{E} = -\frac{1}{\kappa^2}\mathbf{R}\boldsymbol{\lambda}, \quad \text{on } \Gamma.$$

5 BOUNDARY INTEGRAL EQUATIONS FOR MAXWELL TRANSMISSION PROBLEMS

In this chapter we focus on the solution of transmission problems arising from applications in electromagnetic engineering by using boundary element methods. The model application under consideration can be described by the following setting: We have a given bounded conducting object, outside of this object we place a coil in which a time-harmonic current is induced. The time harmonic current in the coil generates a time harmonic magnetic field $\mathbf{B}_p$, which induces eddy currents inside the conducting object. We will assume that the conducting object has linear and isotropic material properties and piecewise constant material parameters, i.e. σ, ε and μ are piecewise constant. An application for such a setting is the forward problem of Magnetic Induction Tomography as described in Section 2.3. Another application arises from the simulation of transformers in industrial applications. For such problems the boundary element method is very suitable since we can describe the solution of a transmission problem in the whole space $\mathbb{R}^3$ by boundary potentials which are defined only on the boundary of the conducting domain.

For this type of problems boundary integral formulations have been derived using a collocation method for an indirect ansatz in [61], a boundary integral formulation using the Galerkin method can be found in [32] based on the idea of symmetric coupling [18]. In both papers, only domains with constant material parameters were considered. Here we will also consider structures with piecewise constant conductivities, permittivities and permeabilities. A boundary integral formulation for problems with piecewise constant material properties has also been considered in [9] in a slightly different setting. Another possibility to solve this problem would be a FEM-BEM coupling, as it has been done in [31].

In the first part of this chapter we are going to introduce a formulation for the transmission problem based on the electric field $\mathbf{E}$. In the next section we will consider the $\mathbf{H}$-field formulation, where we derive a boundary element formulation for the eddy current transmission problem which is formulated in terms of the magnetic field intensity $\mathbf{H}$. For a conducting object, which has constant material properties, both formulations, which are based on a direct ansatz were also derived in [32]. Here we extend those formulations to domains with piecewise constant coefficients. For the $\mathbf{H}$-field we are going to introduce a new formulation for the transmission problem which is based on an indirect approach. For the collocation method this has been done in [61], here we are going to derive a formulation for the Galerkin method. In the last section we will derive a boundary integral formulation for the reduced model.

5.1 The Stratton-Chu Representation Formula for the Electric Field E and the Magnetic Field Intensity H

In Section 2 we showed that if there are no impressed currents, i.e. $\mathbf{j}_i = 0$, the electric field $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$ are both governed by the second order partial differential equations

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}(x) + i\omega\mu(\sigma + i\omega\varepsilon)\mathbf{E}(x) = 0, \quad x \in \mathbb{R}^3,$$
$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{H}(x) + i\omega\mu(\sigma + i\omega\varepsilon)\mathbf{H}(x) = 0, \quad x \in \mathbb{R}^3,$$

which correspond to the electromagnetic wave equation. We now set

$$\kappa := \sqrt{i\omega\mu(\sigma + i\omega\varepsilon)},$$

and assume that the material parameters σ, ε and μ are either constant or piecewise constant. Hence in a domain Ω with constant κ the fields $\mathbf{E}$ and $\mathbf{H}$ can be represented by the Stratton-Chu representation formula (4.32). By using Maxwell's equations (2.3) and (2.4) we have the relations

$$\mathbf{curl}\,\mathbf{E}(x) = -i\omega\mu\mathbf{H}(x), \quad \mathbf{curl}\,\mathbf{H}(x) = i\omega\tilde{\varepsilon}\mathbf{E}(x), \tag{5.1}$$

where we have set $\tilde{\varepsilon} := \frac{\sigma}{i\omega} + \varepsilon$. This gives the following relations for the traces

$$-i\omega\mu\boldsymbol{\gamma}_\times\mathbf{H} = \boldsymbol{\gamma}_N\mathbf{E}, \quad i\omega\tilde{\varepsilon}\boldsymbol{\gamma}_\times\mathbf{E} = \boldsymbol{\gamma}_N\mathbf{H}, \quad \text{on } \Gamma = \partial\Omega.$$

By using these relations we can find the following representation formula for the electric field $\mathbf{E}$ and the magnetic field intensity $\mathbf{H}$

$$\mathbf{E}(x) = -i\omega\mu\boldsymbol{\Psi}^\kappa_{\mathrm{SL}}(\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{H})(x) - \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}(\mathbf{R}\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{E})(x) + \nabla\Psi^\kappa_{\mathrm{SL}}(\gamma^{\mathrm{int}}_n\mathbf{E})(x), \quad x \in \Omega, \tag{5.2}$$
$$\mathbf{H}(x) = i\omega\tilde{\varepsilon}\boldsymbol{\Psi}^\kappa_{\mathrm{SL}}(\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{E})(x) - \boldsymbol{\Psi}^\kappa_{\mathrm{DL}}(\mathbf{R}\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{H})(x) + \nabla\Psi^\kappa_{\mathrm{SL}}(\gamma^{\mathrm{int}}_n\mathbf{H})(x), \quad x \in \Omega. \tag{5.3}$$

We observe that here the relevant traces are now $\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{E}$, $\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{H}$, $\gamma^{\mathrm{int}}_n\mathbf{E}$ and $\gamma^{\mathrm{int}}_n\mathbf{H}$ instead of $\boldsymbol{\gamma}^{\mathrm{int}}_t\mathbf{E}, \boldsymbol{\gamma}^{\mathrm{int}}_N\mathbf{E}, \gamma^{\mathrm{int}}_n\mathbf{E}$ or $\boldsymbol{\gamma}^{\mathrm{int}}_t\mathbf{H}, \boldsymbol{\gamma}^{\mathrm{int}}_N\mathbf{H}$ and $\gamma^{\mathrm{int}}_n\mathbf{H}$. In addition to the representation formulae (5.2) and (5.3) we have the relations

$$\mathrm{div}_\Gamma(\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{E}) = -i\omega\mu\gamma^{\mathrm{int}}_n\mathbf{H}, \quad \mathrm{div}_\Gamma(\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{H}) = i\omega\tilde{\varepsilon}\gamma^{\mathrm{int}}_n\mathbf{E}, \quad \text{on } \Gamma. \tag{5.4}$$

In terms of the **E**-**H** formulation we can also find an equivalent to the Steklov-Poincaré operator, i.e.

$$\mathbf{R}\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{H} = T_\kappa(\boldsymbol{\gamma}^{\mathrm{int}}_\times\mathbf{E}), \tag{5.5}$$

which maps the twisted tangential trace of the electric field to the twisted tangential trace of the magnetic field intensity. For this boundary integral operator we can find the following representation:

$$T_\kappa = -i\omega\tilde{\varepsilon} S_\kappa + \frac{1}{i\omega\mu}(\frac{1}{2}I + C_\kappa)\mathbf{R}S_\kappa^{-1}(\frac{1}{2}I - C_\kappa)\mathbf{R}. \tag{5.6}$$

Note that in our case, where $\kappa = \sqrt{i\omega\mu(\sigma + i\omega\varepsilon)}$ we have $\Re(\kappa) > 0$ when $\sigma \neq 0$. Thus the ellipticity results for the operators S_κ and N_κ from the previous chapter hold. Consequently the operators S_κ and N_κ are invertible.
In our setting we have $\kappa = 0$ in the exterior domain Ω^c, hence we have the following representation for the secondary electric field $\mathbf{E}_s$ and for the magnetic field intensity $\mathbf{H}_s$:

$$\mathbf{E}(x) = i\omega\mu\mathbf{\Psi}_{\mathrm{SL}}^0(\boldsymbol{\gamma}_\times^{\mathrm{ext}}\mathbf{H}_s)(x) + \mathbf{\Psi}_{\mathrm{DL}}^\kappa(\mathbf{R}\boldsymbol{\gamma}_\times^{\mathrm{ext}}\mathbf{E}_s)(x) + \nabla\Psi_{\mathrm{SL}}^0(\gamma_n^{\mathrm{ext}}\mathbf{E}_s)(x),\ x \in \Omega^c, \tag{5.7}$$

$$\mathbf{H}(x) = -\mathbf{\Psi}_{\mathrm{DL}}^0(\mathbf{R}\boldsymbol{\gamma}_\times^{\mathrm{ext}}\mathbf{H}_s)(x) - \nabla\Psi_{\mathrm{SL}}^0(\gamma_n^{\mathrm{ext}}\mathbf{H}_s)(x), \quad x \in \Omega^c, \tag{5.8}$$

with the additional relations

$$\mathrm{div}_\Gamma(\boldsymbol{\gamma}_\times^{\mathrm{ext}}\mathbf{E}_s)(x) = -i\omega\mu\gamma_n^{\mathrm{ext}}\mathbf{H}_s(x), \quad \mathrm{div}_\Gamma(\boldsymbol{\gamma}_\times^{\mathrm{ext}}\mathbf{H}_s)(x) = 0, \quad x \in \Gamma. \tag{5.9}$$

5.2 The Eddy Current Model

From now on we assume that we have given a non-overlapping domain decomposition of the conducting domain Ω, see Figure 5.1:

Definition 5.1. *We call $\{\Omega_1, \Omega_2, \ldots, \Omega_M\}$ a non-overlapping domain decomposition of the domain Ω if*

$$\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \textit{for } i \neq j, \quad \Gamma_i = \partial\Omega_i \tag{5.10}$$

holds. The coupling boundaries are denoted by $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$, the skeleton Γ_s is defined by

$$\Gamma_s = \bigcup_{i,j} \Gamma_{ij} \cup \Gamma. \tag{5.11}$$

We assume that the material parameters are piecewise constant with respect to this domain decomposition, i.e.

$$\mu(x) = \mu_i, \quad \varepsilon(x) = \varepsilon_i, \quad \sigma(x) = \sigma_i, \quad \kappa(x) = \kappa_i = \sqrt{i\omega\mu_i(\sigma_i + i\omega\varepsilon_i)}, \quad x \in \Omega_i.$$

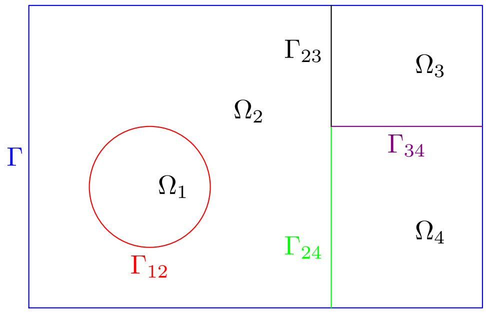


Figure 5.1: Example for a domain decomposition.

5.2.1 A direct boundary integral formulation based on the E-field

In what follows we will derive a boundary integral formulation for the eddy current problem introduced in Section 2.3 and which is formulated in terms of the electric field $\mathbf{E}$. For this formulation we follow the approach in [32], and for the domain decomposition method we follow the ideas in [9].
Let us assume that we have given a conducting domain Ω with the domain decomposition as given in Definition 5.1. To make the eddy current formulation suitable for a boundary element formulation, we introduce the following splitting in the non-conducting domain:

$$\mathbf{E}(x) = \mathbf{E}_s(x) + \mathbf{E}_p(x), \quad x \in \Omega^c, \tag{5.12}$$

where $\mathbf{E}_p$ is the primary electric field as defined in (2.33). Then we conclude the following transmission problem:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}(x) + \kappa_i^2 \mathbf{E}(x) = 0, \quad x \in \Omega_i, \tag{5.13}$$
$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_s(x) = 0, \quad x \in \Omega^c, \tag{5.14}$$
$$\operatorname{div}\mathbf{E}_s(x) = 0, \quad x \in \Omega^c, \tag{5.15}$$

with $\kappa_i = \sqrt{i\omega\mu_i(\sigma_i + i\omega\varepsilon_i)}$. Moreover, we assume that $\mathbf{E}_s$ satisfies suitable radiation conditions. From the Stratton-Chu representation formula we get

$$\mathbf{E}(x) = \mathbf{\Psi}_{\mathrm{SL}}^{\kappa}(\gamma_N^i \mathbf{E})(x) + \mathbf{\Psi}_{\mathrm{DL}}^{\kappa}(\gamma_N^i \mathbf{E})(x) + \nabla\Psi_{\mathrm{SL}}^{\kappa}(\gamma_n^i \mathbf{E})(x), \quad x \in \Omega_i, \tag{5.16}$$
$$\mathbf{E}_s(x) = -\mathbf{\Psi}_{\mathrm{SL}}^{0}(\gamma_N^{\mathrm{ext}} \mathbf{E}_s)(x) - \mathbf{\Psi}_{\mathrm{DL}}^{0}(\gamma_t^{\mathrm{ext}} \mathbf{E}_s)(x) - \nabla\Psi_{\mathrm{SL}}^{0}(\gamma_n^{\mathrm{ext}} \mathbf{E}_s)(x), \quad x \in \Omega^c. \tag{5.17}$$

The unknowns in this formulation are the tangential and the Neumann and normal traces of the electric fields $\mathbf{E}$ and $\mathbf{E}_s$.

Remark 5.2. *The eddy current model can also be formulated in terms of the magnetic field intensity* $\mathbf{H}$, *the traces of* $\mathbf{H}$ *and* $\mathbf{E}$ *are linked via*

$$\gamma_\times^{\mathrm{int}} \boldsymbol{H} = -\frac{1}{i\omega\mu}\gamma_N^{\mathrm{int}} \boldsymbol{E}, \quad \gamma_N^{\mathrm{int}} \boldsymbol{H} = (\sigma + i\omega\varepsilon)\gamma_\times^{\mathrm{int}} \boldsymbol{E}, \quad \gamma_\times^{\mathrm{ext}} \boldsymbol{H} = -\frac{1}{i\omega\mu}\gamma_N^{\mathrm{ext}} \boldsymbol{E}.$$

Note that in the eddy current model the exterior traces $\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E}$ and $\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{H}$ are not connected any more!

The Steklov-Poincaré operator as introduced in Section 4.5.1 maps the tangential trace to the Neumann trace $\boldsymbol{\gamma}_N^i\mathbf{E}$ of the local electric field $\mathbf{E}$ in Ω_i:

$$\boldsymbol{\gamma}_N^i\mathbf{E} = \mathcal{S}_\kappa^i\boldsymbol{\gamma}_t^i\mathbf{E} = \left[N_\kappa^i + \left(\frac{1}{2}I + B_\kappa^i\right)(S_k^i)^{-1}\left(\frac{1}{2}I - C_\kappa^i\right)\right]\boldsymbol{\gamma}_t^i\mathbf{E}, \tag{5.18}$$

where the boundary integral operators $N_\kappa^i, C_\kappa^i, S_\kappa^i, B_\kappa^i$ are defined on the boundary Γ_i with the parameter κ_i. For ease in the notation we skip the index i in κ_i and write $N_\kappa^i, C_\kappa^i, S_\kappa^i, B_\kappa^i$ instead of $N_{\kappa_i}^i, C_{\kappa_i}^i, S_{\kappa_i}^i, B_{\kappa_i}^i$. The transmission conditions read

$$\boldsymbol{\gamma}_t^i\mathbf{E} - \boldsymbol{\gamma}_t^j\mathbf{E} = 0, \quad \frac{1}{\mu_i}\boldsymbol{\gamma}_N^i\mathbf{E} + \frac{1}{\mu_j}\boldsymbol{\gamma}_N^j\mathbf{E} = 0 \quad \text{on } \Gamma_{ij} \tag{5.19}$$

in the interior domain. On the transmission boundary between the air-domain and the conducting domain we have the transmission conditions

$$\boldsymbol{\gamma}_t^i\mathbf{E} + \boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E}_s = \boldsymbol{\gamma}_t\mathbf{E}_p, \quad \frac{1}{\mu_i}\boldsymbol{\gamma}_N^i\mathbf{E} + \frac{1}{\mu_0}\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}_s = \frac{1}{\mu_0}\boldsymbol{\gamma}_N\mathbf{E}_p \quad \text{on } \Gamma_i \cap \Gamma, \tag{5.20}$$

which motivate the choice of $\mathbf{u} = \boldsymbol{\gamma}_t\mathbf{E} \in \mathbf{H}_\perp^{-1/2}(\text{curl}_{\Gamma_s}, \Gamma_s)$ as a global unknown on the skeleton Γ_s. This gives rise to the variational problem

$$\sum_{i=1}^M \frac{1}{\mu_i}\langle \mathcal{S}_\kappa^i\mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} + \frac{1}{\mu_0}\langle \boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}_s, \mathbf{v}|_\Gamma\rangle_\Gamma = \frac{1}{\mu_0}\langle \boldsymbol{\gamma}_N\mathbf{E}_p, \mathbf{v}|_\Gamma\rangle_\Gamma, \tag{5.21}$$

for all test functions $\mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\text{curl}_{\Gamma_s}, \Gamma_s)$. As a next step we introduce the local functions $\boldsymbol{\lambda}_i \in \mathbf{H}_\parallel^{-1/2}(\text{div}_{\Gamma_i}, \Gamma_i)$ and set

$$\boldsymbol{\lambda}_i := \frac{1}{\mu_i}(S_\kappa^i)^{-1}\left(\frac{1}{2}I - C_\kappa^i\right)\mathbf{u}|_{\Gamma_i} \quad \text{for } i = 1, \ldots, M.$$

Using the representation (5.18) of the Steklov-Poincaré operator $\mathcal{S}_\kappa^i$ we obtain the following variational equation

$$\sum_{i=1}^M \frac{1}{\mu_i}\langle N_\kappa^i\mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} + \sum_{i=1}^M \frac{1}{\mu_i}\langle(\frac{1}{2}I + B_\kappa^i)\boldsymbol{\lambda}_i, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} + \frac{1}{\mu_0}\langle \boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}_s, \mathbf{v}|_\Gamma\rangle_\Gamma$$
$$= \frac{1}{\mu_0}\langle \boldsymbol{\gamma}_N\mathbf{E}_p, \mathbf{v}|_\Gamma\rangle_\Gamma,$$

for all test functions $\mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma_s}, \Gamma_s)$. For the exterior Neumann trace $\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s$ we have derived the following boundary integral equation in (4.62):

$$\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s = (\frac{1}{2}I - B_0)\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s - N_0\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s.$$

We set

$$\boldsymbol{\lambda}_0 := -\frac{1}{\mu_0}\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s + \frac{1}{\mu_0}\boldsymbol{\gamma}_N\mathbf{E}_p$$

and $\Gamma_0 := \Gamma$ and we use the transmission condition (5.20) to conclude the variational equation

$$\sum_{i=0}^{M}\frac{1}{\mu_i}\langle N_\kappa^i\mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} + \sum_{i=1}^{M}\langle B_\kappa^i\boldsymbol{\lambda}_i, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i}$$
$$= \frac{1}{\mu_0}\langle N_0\boldsymbol{\gamma}_t^{\mathrm{ext}}\mathbf{E}_p, \mathbf{v}|_\Gamma\rangle_\Gamma + \frac{1}{\mu_0}\langle(\frac{1}{2}I + B_0)\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_p, \mathbf{v}|_\Gamma\rangle_\Gamma,$$

for all $\mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma_s}, \Gamma_s)$. The exterior traces $\boldsymbol{\gamma}_t^{\mathrm{ext}}\mathbf{E}_s$ and $\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s$ satisfy the boundary integral equation derived in (4.61):

$$A_0\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s(x) + (\frac{1}{2}I + C_0)\boldsymbol{\gamma}_t^{\mathrm{ext}}\mathbf{E}_s(x) + \nabla_\Gamma V_0\gamma_n^{\mathrm{ext}}\mathbf{E}_s(x) = 0, \quad x \in \Gamma. \tag{5.22}$$

Due to Remark 4.45 we have that the function $\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s$ is an element of the space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$, therefore we have $\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$. Thus, if we test equation (5.22) with a test function in the space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$, the term $\nabla_\Gamma V_0\gamma_n^{\mathrm{ext}}\mathbf{E}_s(x)$ vanishes. Hence we get

$$\langle A_0\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s, \boldsymbol{\mu}_0\rangle_\Gamma + \langle(\frac{1}{2}I + C_0)\boldsymbol{\gamma}_t^{\mathrm{ext}}\mathbf{E}_s, \boldsymbol{\mu}_0\rangle_\Gamma = 0. \tag{5.23}$$

As a next step we insert $\boldsymbol{\gamma}_N^{\mathrm{ext}}\mathbf{E}_s = -\mu_0\boldsymbol{\lambda}_0 + \boldsymbol{\gamma}_N\mathbf{E}_p$ in (5.23). Since we know that the primary field E_p satisfies

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}_p(x) = 0, \quad x \in \Omega,$$

we also conclude that $\mathrm{div}_\Gamma\,\mathbf{E}_p = 0$, from which we can deduce that $\boldsymbol{\lambda}_0$ is also an element of the space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$. This leads us to the following variational equation to find $\boldsymbol{\lambda}_0 \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$

$$\mu_0\langle A_0\boldsymbol{\lambda}_0, \boldsymbol{\mu}_0\rangle_\Gamma + \langle(\frac{1}{2}I + C_0)\mathbf{u}|_\Gamma, \boldsymbol{\mu}_0\rangle_\Gamma = \langle(\frac{1}{2}I + C_0)\boldsymbol{\gamma}_t\mathbf{E}_p, \boldsymbol{\mu}_0\rangle_\Gamma + \langle A_0\boldsymbol{\gamma}_N\mathbf{E}_p, \boldsymbol{\mu}_0\rangle_\Gamma \tag{5.24}$$

for all test functions $\boldsymbol{\mu}_0 \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$. For $\boldsymbol{\lambda}_i$ we have the variational equations

$$\mu_i \langle S^i_\kappa \boldsymbol{\lambda}_i, \boldsymbol{\mu}_i\rangle + \langle(-\frac{1}{2}I + C^i_\kappa)\mathbf{u}|_{\Gamma_i}, \boldsymbol{\mu}_i\rangle = 0, \quad \forall \boldsymbol{\mu}_i \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma), \tag{5.25}$$

for $i = 1, \ldots, M$. Due to the transmission condition (5.20) we have that $\mathrm{div}_\Gamma \boldsymbol{\lambda}_i(x) = 0$ for $x \in \Gamma_i \cap \Gamma$, hence we define the space

$$\mathbf{H}_\|^{-1/2}(\mathrm{div}_{\Gamma_i \cap \Gamma} 0, \Gamma_i) := \{\boldsymbol{\mu} \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_{\Gamma_i}, \Gamma_i), \mathrm{div}_{\Gamma \cap \Gamma_i} \boldsymbol{\mu} = 0\},$$

in which we seek the unknowns $\{\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_M\}$. Adding (5.24) and (5.25) up and using the transmission conditions the variational problem (5.21) becomes:
Find $\boldsymbol{\lambda}_i \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_{\Gamma_i \cap \Gamma} 0, \Gamma_i)$ for $i = 0, \ldots, M$ and $\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_{\Gamma_s}, \Gamma_s)$ such that

$$\sum_{i=0}^{M} \left(\frac{1}{\mu_i} \langle N^i_\kappa \mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} + \langle B^i_\kappa \boldsymbol{\lambda}_i, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} \right) = f(\mathbf{v}|_\Gamma), \tag{5.26}$$

$$\sum_{i=0}^{M} \left(\langle C^i_\kappa \mathbf{u}|_{\Gamma_i}, \boldsymbol{\mu}_i\rangle_{\Gamma_i} + \mu_i \langle S^i_\kappa \boldsymbol{\lambda}_i, \boldsymbol{\mu}_i\rangle_{\Gamma_i} \right) = g(\boldsymbol{\mu}_0) \tag{5.27}$$

holds for all $\boldsymbol{\mu}_i \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_{\Gamma_i \cap \Gamma} 0, \Gamma_i), i = 0, \ldots, M$ and $\mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_{\Gamma_s}, \Gamma_s)$. For ease in the notation we have set $S^0_0 := A_0$. The right hand side is given by

$$f(\mathbf{v}|_\Gamma) = \frac{1}{\mu_0} \langle (\frac{1}{2}I + B_0)\boldsymbol{\gamma}_N \mathbf{E}_p, \mathbf{v}|_\Gamma\rangle_\Gamma + \frac{1}{\mu_0}\langle N_0 \boldsymbol{\gamma}_t \mathbf{E}_p, \mathbf{v}|_\Gamma\rangle_\Gamma, \tag{5.28}$$

$$g(\boldsymbol{\mu}_0) = \langle A_0 \boldsymbol{\gamma}_N \mathbf{E}_p, \boldsymbol{\mu}_0\rangle_\Gamma + \langle (\frac{1}{2}I + C_0)\boldsymbol{\gamma}_t \mathbf{E}_p, \boldsymbol{\mu}_0\rangle_\Gamma. \tag{5.29}$$

Theorem 5.3. *The associated bi-linear form*

$$a(\mathbf{u}, \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_M; \mathbf{v}, \boldsymbol{\mu}_0, \ldots, \boldsymbol{\mu}_M) := \sum_{i=0}^{M} \left(\frac{1}{\mu_i} \langle N^i_\kappa \mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} + \langle B^i_\kappa \boldsymbol{\lambda}_i, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} \right) + \sum_{i=0}^{M} \left(\langle \boldsymbol{\mu}_i, C^i_\kappa \mathbf{u}|_{\Gamma_i}\rangle_{\Gamma_i} + \mu_i \langle \boldsymbol{\mu}_i, S^i_\kappa \boldsymbol{\lambda}_i\rangle_{\Gamma_i} \right)$$

is $\mathbf{H}_\|^{-1/2}(\mathrm{div}_{\Gamma_i \cap \Gamma} 0, \Gamma_i) \times \boldsymbol{H}_\perp^{-1/2}(\mathit{curl}_{\Gamma_s}, \Gamma_s)$*-elliptic.*

Proof. Using Theorem 4.27 and Lemma 4.30 we easily obtain

$$|a(\mathbf{u}, \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_M; \mathbf{u}, \boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_M)| = \left| \sum_{i=0}^{M} \left(\frac{1}{\mu_i} \langle N^i_\kappa \mathbf{u}|_{\Gamma_i}, \mathbf{u}|_{\Gamma_i}\rangle_{\Gamma_i} + \mu_i \langle \boldsymbol{\lambda}_i, S^i_\kappa \boldsymbol{\lambda}_i\rangle_{\Gamma_i} \right) \right| \geq c_1 \|\mathbf{u}\|^2_{\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_{\Gamma_s}, \Gamma_s)} + c_2 \sum_{i=0}^{M} \|\boldsymbol{\lambda}_i\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_{\Gamma_i}, \Gamma_i)}$$

for all $\mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_{\Gamma_s}, \Gamma_s)$ and $\boldsymbol{\lambda}_i \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_{\Gamma_i}, \Gamma_i), i = 1, \ldots, M$. □

Remark 5.4. *If we assume that $\kappa(x) = \kappa$ for $x \in \Omega$, i.e. κ is constant in Ω, then we get that $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_0 =: \frac{1}{\mu_0}\boldsymbol{\lambda}$. Inserting this information into the above variational problem results in the variational problem which was derived in [32]:*
Find $(\boldsymbol{\lambda}, \mathbf{u}) \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma) \times \mathbf{H}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$ such that

$$\langle (\frac{1}{\mu_r} N_\kappa + N_0)\mathbf{u}, \mathbf{v}\rangle_\Gamma + \langle (B_\kappa + B_0)\boldsymbol{\lambda}, \mathbf{v}\rangle_\Gamma = f(\mathbf{v}), \tag{5.30}$$

$$\langle (C_\kappa + C_0)\mathbf{u}, \boldsymbol{\mu}\rangle_\Gamma + \langle (\mu_r S_\kappa + S_0)\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma = g(\boldsymbol{\mu}), \tag{5.31}$$

holds for all $(\boldsymbol{\mu}, \mathbf{v}) \in \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma) \times \mathbf{H}_{\perp}^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$. The right hand side is given by

$$f(\mathbf{v}) = \langle (\frac{1}{2} I + B_0)\boldsymbol{\gamma}_N \mathbf{E}_p, \mathbf{v}\rangle_\Gamma + \langle N_0 \boldsymbol{\gamma}_t \mathbf{E}_p, \mathbf{v}\rangle_\Gamma,$$

$$g(\boldsymbol{\mu}) = \langle S_0 \boldsymbol{\gamma}_N \mathbf{E}_p, \boldsymbol{\mu}\rangle_\Gamma + \langle (\frac{1}{2} I + C_0)\boldsymbol{\gamma}_t \mathbf{E}_p, \boldsymbol{\mu}\rangle_\Gamma$$

and we have set $\mu_r = \frac{\mu}{\mu_0}$.

Remark 5.5. *Note that so far this method is also valid for mulitple connected domains Ω. The difficulty when dealing with non-simple connected domains lies in the discretization of the space $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$.*

Approximation of the space $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_{\Gamma_i \cap \Gamma} 0, \Gamma_i)$

For the implementation of the above presented method we have to deal with the question how to incorporate the space $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_{\Gamma_i \cap \Gamma} 0, \Gamma_i)$. Here we present two methods for an approximation of this space. One method makes use of the representation of functions $\boldsymbol{\lambda}$ with $\operatorname{div}_\Gamma \boldsymbol{\lambda} = 0$ as surface curl of $H^{1/2}(\Gamma)$ functions, i.e. $\boldsymbol{\lambda} = \operatorname{curl}_\Gamma \phi$ with $\phi \in H^{1/2}(\Gamma)$. The other enforces the condition $\operatorname{div}_\Gamma \boldsymbol{\lambda} = 0$ on $\Gamma \cap \Gamma_i$ by using a Lagrange multiplier. For simplicity we will now assume that the material parameters, i.e. σ, ε, μ are constant in Ω.

Method 1:
In Method 1 we will make use of an explicit representation of the space $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma 0, \Gamma)$ in the case of simply connected domains. For simplicity we will now assume that κ is constant in Ω. For such type of problems we can use the formulation (5.30)-(5.31). Furthermore we assume that we consider a simple connected domain, then we can make use of the fact that $\operatorname{div}_\Gamma \boldsymbol{\lambda} = 0$ implies $\boldsymbol{\lambda} = \mathbf{curl}_\Gamma \phi$, which is a result of the Hodge decomposition of the space $\mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ (see e.g. [15]). This

leads to the variational formulation:
Find $(\mathbf{u}, \phi) \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times H^{1/2}(\Gamma)$ such that

$$\langle (\frac{1}{\mu} N_\kappa + \frac{1}{\mu_0} N_0)\mathbf{u}, \mathbf{v}\rangle_\Gamma + \langle (B_\kappa + B_0)\mathbf{curl}_\Gamma \phi, \mathbf{v}\rangle_\Gamma = f(\mathbf{v}),$$
$$\langle (C_\kappa + C_0)\mathbf{u}, \mathbf{curl}_\Gamma \psi\rangle_\Gamma + \langle (\mu A_\kappa + \mu_0 A_0)\mathbf{curl}_\Gamma \phi, \mathbf{curl}_\Gamma \psi\rangle_\Gamma = g(\mathbf{curl}_\Gamma \psi)$$

holds for all $(\mathbf{v}, \psi) \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \times H^{1/2}(\Gamma)$.
A treatment of this approach when dealing with mulitple connected domains can be found in [32]. As it has been also pointed out in [9], this approach has to be modified if $\mathrm{div}_\Gamma\, \boldsymbol{\lambda} = 0$ is required on a part of the boundary, i.e. when we have given a conducting domain with piecewise constant κ.

Method 2:
For the second method we enforce the condition $\mathrm{div}_\Gamma\, \boldsymbol{\lambda} = 0$ on $\Gamma \cap \Gamma_i$ by introducing a Lagrange mulitplier ϕ_i on every subboundary Γ_i. This leads to the following variational problem, which has a block skew symmetric structure:
Find $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma_s}, \Gamma_s), \boldsymbol{\lambda}_i, \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_i}, \Gamma_i)$ and $\phi_i \in H^{-1/2}(\Gamma_i \cap \Gamma)$ for $i = 1, \ldots M$ such that

$$\sum_{i=0}^{M} \left(\frac{1}{\mu_i} \langle N_\kappa^i \mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} + \langle B_\kappa^i \boldsymbol{\lambda}_i, \mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i} \right) = f(\mathbf{v}|_\Gamma), \tag{5.32}$$

$$\sum_{i=0}^{M} \left(\langle C_\kappa^i \mathbf{u}|_{\Gamma_i}, \boldsymbol{\mu}_i\rangle_{\Gamma_i} + \mu_i \langle S_\kappa^i \boldsymbol{\lambda}_i, \boldsymbol{\mu}_i\rangle_{\Gamma_i} + \langle \nabla_\Gamma V_\kappa^i \phi_i, \boldsymbol{\mu}_i\rangle_{\Gamma\cap\Gamma_i} \right) = g(\boldsymbol{\mu}_0), \tag{5.33}$$

$$\sum_{i=0}^{M} \left(\langle V_\kappa^i \,\mathrm{div}_\Gamma\, \boldsymbol{\lambda}_i, \psi_i\rangle_{\Gamma\cap\Gamma_i} + \langle 1, \phi_i\rangle_{\Gamma\cap\Gamma_i} \langle 1, \psi_i\rangle_{\Gamma\cap\Gamma_i} \right) = 0 \tag{5.34}$$

holds for all $\mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_{\Gamma_s}, \Gamma_s)$, $\boldsymbol{\mu}_i \in \mathbf{H}_{\parallel}^{-1/2}(\mathrm{div}_{\Gamma_i}, \Gamma_i), \psi_i \in H^{-1/2}(\Gamma_i \cap \Gamma)$ for $i = 1, \ldots, M$. Here the stabilization $\sum_{i=0}^{M} \langle 1, \phi\rangle_\Gamma \langle 1, \psi\rangle_\Gamma$ was added.
The unqiue solvability of this formulation was shown in [9]. This approach is also valid in the case of multiple connceted domains, when adding the stabilization

$$\sum_{i=1}^{N_{cc}} \langle 1, \phi\rangle_{\tilde{\Gamma}_i} \langle 1, \psi\rangle_{\tilde{\Gamma}_i},$$

where N_{cc} denotes the number of connected components.

Remark 5.6. *The advantage of Method 1 is that the resulting discrete system is smaller than the one in the second approach, moreover it holds*

$$\langle A_0 \mathbf{curl}_\Gamma \phi, \mathbf{curl}_\Gamma \psi\rangle_\Gamma = \langle D_0 \phi, \psi\rangle_\Gamma, \tag{5.35}$$

so in the second diagonal block of the system we have an operator, which corresponds to the hypersingular operator of the Laplace operator, for which suitable preconditioners already exist. Furthermore the system matrix of Method 1 is as examples in the next Section show, better conditioned. So using Method 1 is advisable if problems with a domain with constant material parameters are considered.

Determining the voltage in a coil

In applications the voltage in a coil, which is located in the non-conducting domain, needs to be evaluated, i.e. the expression

$$v = -i\omega \int\limits_{\mathcal{C}} \mathbf{B} \cdot n(x) ds_x$$

has to be computed. Out of the solution of the variational problem (5.26)-(5.27) we obtain the interior traces $\boldsymbol{\gamma}_t^i \mathbf{E}$ and $\boldsymbol{\gamma}_N^i \mathbf{E}$. By using the transmission condition (5.20) we can compute the traces $\boldsymbol{\gamma}_t^{\mathrm{ext}} \mathbf{E}_s$ and $\boldsymbol{\gamma}_N^{\mathrm{ext}} \mathbf{E}_s$. We will now show how we can compute the voltage v by using the traces $\boldsymbol{\gamma}_t^{\mathrm{ext}} \mathbf{E}_s$ and $\boldsymbol{\gamma}_N^{\mathrm{ext}} \mathbf{E}_s$.
We apply Stokes theorem and insert the representation formula for $\mathbf{E}_s$ in the non-conducting domain Ω^c (5.17):

$$\begin{aligned} v &= \int\limits_{\partial\mathcal{C}} \mathbf{E}_s(x) \cdot \tau ds = - \int\limits_{\partial\mathcal{C}} \boldsymbol{\Psi}_{\mathrm{SL}}^0 (\boldsymbol{\gamma}_N^{\mathrm{ext}} \mathbf{E}_s)(x) \cdot \tau ds - \int\limits_{\partial\mathcal{C}} \boldsymbol{\Psi}_{\mathrm{DL}}^0 (\boldsymbol{\gamma}_t^{\mathrm{ext}} \mathbf{E}_s)(x) \cdot \tau ds \\ &= - \int\limits_{\partial\mathcal{C}} \int\limits_{\Gamma} U_0^*(x,y) \boldsymbol{\gamma}_N^{\mathrm{ext}} \mathbf{E}_s(y) ds_y \cdot \tau ds - \int\limits_{\partial\mathcal{C}} \mathbf{curl}_x \int\limits_{\Gamma} U_0^*(x,y) \boldsymbol{\gamma}_t^{\mathrm{ext}} \mathbf{E}_s(y) ds_y \cdot \tau ds. \end{aligned}$$

Note that the gradient part $\nabla \Psi_{\mathrm{SL}}^0 (\gamma_n^{\mathrm{ext}} \mathbf{E}_s)$ in the representation formula (5.17) drops out when integrating over a closed line. By recalling the formula for the computation of the primary fields from Section 2.3:

$$\mathbf{E}_p(x) = -i\omega\mu_0 \int\limits_{\partial\mathcal{C}} U_0^*(x,y) \tau ds_y, \quad \mathbf{H}_p(x) = \mathbf{curl}_x \int\limits_{\partial\mathcal{C}} U_0^*(x,y) \tau ds_y$$

and by interchanging the order of integration we get the following formula for the computation of the voltage v in the coil $\mathcal{C}$

$$\begin{aligned} v &= - \int\limits_{\Gamma} \boldsymbol{\gamma}_N^{\mathrm{ext}} \mathbf{E}_s(y) \cdot \int\limits_{\partial\mathcal{C}} U^*(x,y) \tau ds ds_y - \int\limits_{\Gamma} \boldsymbol{\gamma}_t^{\mathrm{ext}} \mathbf{E}_s(y) \cdot \mathbf{curl}_x \int\limits_{\partial\mathcal{C}} U^*(x,y) \tau ds ds_y \\ &= \frac{1}{i\omega\mu_0} \int\limits_{\Gamma} \boldsymbol{\gamma}_N^{\mathrm{ext}} \mathbf{E}_s(y) \cdot \mathbf{E}_p(y) ds_y - \int\limits_{\Gamma} \boldsymbol{\gamma}_t^{\mathrm{ext}} \mathbf{E}_s(y) \cdot \mathbf{H}_p(y) ds_y \\ &= \frac{1}{i\omega\mu_0} (\langle \boldsymbol{\gamma}_N^{\mathrm{ext}} \mathbf{E}_s, \boldsymbol{\gamma}_t \mathbf{E}_p \rangle_\Gamma + \langle \boldsymbol{\gamma}_t^{\mathrm{ext}} \mathbf{E}_s, \boldsymbol{\gamma}_N \mathbf{E}_p \rangle)_\Gamma . \end{aligned}$$

Thus the voltage v in a coil $\mathcal{C}$ can be evaluated by computing an inner product of the traces $\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E}$ and $\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}$ with the primary field of the coil $\mathcal{C}$.

Remark 5.7. *The magnetic field* $\mathbf{B}$ *in the exterior domain can also be evaluated by using the traces* $\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E}_s$ *and* $\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}_s$*:*

$$\mathbf{B}(x) = \mu_0\mathbf{H}(x) = \frac{1}{i\omega}\boldsymbol{\Psi}_{\text{DL}}^0(\mathbf{R}\boldsymbol{\gamma}_N^{\text{ext}}\mathbf{E}_s)(x) + \frac{1}{i\omega}\nabla\Psi_{\text{SL}}^0(\text{curl}_\Gamma\boldsymbol{\gamma}_t^{\text{ext}}\mathbf{E}_s)(x), \quad x \in \Omega^c. \tag{5.36}$$

This formula can be obtained by combining (5.8) with (5.1).

5.2.2 A direct boundary integral formulation based on the H-field

In this section we consider a formulation for the solution of the eddy current problem, in which the unknowns are expressed using the magnetic field intensity $\mathbf{H}$. We extend the approach, which has been proposed in [31] to problems with piecewise constant material properties. In contrast to [31] we will only consider simply connected domains here. The $\mathbf{H}$-field formulation differs from the $\mathbf{E}$-formulation in such a way that in the exterior domain the $\mathbf{H}$-field can be written as a gradient field, hence we only have to solve a potential equation in the exterior domain.
Let us assume that we have given a non-overlapping domain decomposition as in Definition 5.1, then in the interior domains the $\mathbf{H}$-field is governed by

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{H}(x) + \kappa_i^2\mathbf{H}(x) = 0, \quad x \in \Omega_i, \quad i = 1, \dots, M. \tag{5.37}$$

In the exterior domain we introduce the decomposition

$$\mathbf{H}(x) = \mathbf{H}_s(x) + \mathbf{H}_p(x), \quad x \in \Omega^c, \tag{5.38}$$

with the primary magnetic field intensity $\mathbf{H}_p$ as defined in (2.33). With this decomposition we get $\mathbf{curl}\,\mathbf{H}_s(x) = 0$ and thus $\mathbf{H}_s$ can be written as the gradient of a scalar function, which satisfies the potential equation due to Gauss law (2.2):

$$\mathbf{H}_s(x) = \nabla\phi(x), \quad \Delta\phi(x) = 0, \quad x \in \Omega^c. \tag{5.39}$$

In the interior domains the magnetic field intensity $\mathbf{H}$ can be described by the Stratton-Chu representation formula (4.32) as derived in the previous chapter:

$$\mathbf{H}(x) = \boldsymbol{\Psi}_{\text{SL}}^\kappa(\boldsymbol{\gamma}_N^i\mathbf{H})(x) + \boldsymbol{\Psi}_{\text{DL}}^\kappa(\boldsymbol{\gamma}_N^i\mathbf{H})(x) + \nabla\Psi_{\text{SL}}^\kappa(\gamma_n^i\mathbf{H})(x), \quad x \in \Omega_i,$$

For the exterior domain we have

$$\nabla\phi(x) = -\nabla\Psi_{\text{SL}}^0(\gamma_1^{\text{ext}}\phi)(x) + \nabla\Psi_{\text{DL}}^0(\gamma_0^{\text{ext}}\phi)(x), \quad x \in \Omega^c.$$

For the interior domain the following transmission conditions hold:

$$\boldsymbol{\gamma}_t^i \mathbf{H} - \boldsymbol{\gamma}_t^j \mathbf{H} = 0, \quad \frac{1}{\tilde{\varepsilon}_i}\boldsymbol{\gamma}_N^i \mathbf{H} + \frac{1}{\tilde{\varepsilon}_j}\boldsymbol{\gamma}_N^j \mathbf{H} = 0 \quad \text{on } \Gamma_{ij}, \tag{5.40}$$

and for the transmission boundary between the conducting and the non-conducting domain we have

$$\boldsymbol{\gamma}_t^i \mathbf{H} + \nabla_\Gamma \phi = \boldsymbol{\gamma}_t \mathbf{H}_p, \quad \mu_i \gamma_n^i \mathbf{H} + \mu_0 \gamma_1^{\text{int}} \phi = \mu_0 \gamma_n \mathbf{H}_p \quad \text{on } \Gamma_i \cup \Gamma. \tag{5.41}$$

In Section 4.5.1 we have introduced the Steklov-Poincaré operator, which maps the tangential trace of the magnetic field intensity to its Neumann trace:

$$\boldsymbol{\gamma}_N^i \mathbf{H} = \mathcal{S}_\kappa^i \boldsymbol{\gamma}_t^i \mathbf{H} = \left[N_\kappa^i + (\frac{1}{2}I + B_\kappa^i)(S_\kappa^i)^{-1}(\frac{1}{2}I - C_\kappa^i) \right] \boldsymbol{\gamma}_t^i \mathbf{H}, \quad \text{on } \Gamma_i. \tag{5.42}$$

By summing up over all boundaries Γ_i and inserting the transmission conditions for the interior tangential traces (5.40) we get

$$\sum_{i=1}^M \frac{1}{\tilde{\varepsilon}_i} \mathcal{S}_\kappa^i \boldsymbol{\gamma}_t^i \mathbf{H} = \sum_{i=1}^M \frac{1}{\tilde{\varepsilon}_i} \boldsymbol{\gamma}_N^i \mathbf{H}|_{\Gamma_i} = \sum_{i=1}^M \frac{1}{\tilde{\varepsilon}_i} \boldsymbol{\gamma}_N^i \mathbf{H}|_{\Gamma \cap \Gamma_i}.$$

Due to (5.40) we can choose the tangential trace $\mathbf{u} = \boldsymbol{\gamma}_t \mathbf{H} \in \mathbf{H}_\perp^{-1/2}(\text{curl}_{\Gamma_s}, \Gamma_s)$ as global unknown defined on the skeleton Γ_s, this gives the following integral equation:

$$\sum_{i=1}^M \frac{1}{\tilde{\varepsilon}_i} \mathcal{S}_\kappa^i \mathbf{u}|_{\Gamma_i} = \sum_{i=1}^M \frac{1}{\tilde{\varepsilon}_i} \boldsymbol{\gamma}_N^i \mathbf{H}|_{\Gamma \cap \Gamma_i}. \tag{5.43}$$

To be able to incorporate the transmission conditions (5.41) we introduce the space

$$\mathcal{V}_0 = \{(\mathbf{v}, \psi) \in \mathbf{H}_\perp^{-1/2}(\text{curl}_{\Gamma_s}, \Gamma_s) \times H^{1/2}(\Gamma) : \ \mathbf{v}|_\Gamma = \nabla_\Gamma \psi\}.$$

Testing (5.43) with functions in the space $\mathcal{V}_0$ gives

$$\begin{aligned}
\sum_{i=1}^M \frac{1}{\tilde{\varepsilon}_i} \langle \mathcal{S}_\kappa^i \mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i} \rangle_{\Gamma_i} &= \sum_{i=1}^M \frac{1}{\tilde{\varepsilon}_i} \langle \boldsymbol{\gamma}_N^i \mathbf{H}|_{\Gamma \cap \Gamma_i}, \nabla_\Gamma \psi|_{\Gamma \cap \Gamma_i} \rangle_{\Gamma \cap \Gamma_i} \\
&= -\sum_{i=1}^M \frac{1}{\tilde{\varepsilon}_i} \langle \text{div}_\Gamma\, \boldsymbol{\gamma}_N^i \mathbf{H}|_{\Gamma \cap \Gamma_i}, \psi|_{\Gamma \cap \Gamma_i} \rangle_{\Gamma \cap \Gamma_i} \\
&= \sum_{i=1}^M \frac{\kappa_i^2}{\tilde{\varepsilon}_i} \langle \gamma_n^i \mathbf{H}|_{\Gamma \cap \Gamma_i}, \psi|_{\Gamma \cap \Gamma_i} \rangle_{\Gamma \cap \Gamma_i} \\
&= -\omega^2 \sum_{i=1}^M \mu_i \langle \gamma_n^i \mathbf{H}|_{\Gamma \cap \Gamma_i}, \psi|_{\Gamma \cap \Gamma_i} \rangle,
\end{aligned}$$

where we have used the identity $\operatorname{div}_\Gamma \boldsymbol{\gamma}_t^i \mathbf{H} = -\kappa_i^2 \gamma_n^i \mathbf{H}$. By inserting the transmission condition (5.41) we obtain the following variational equation

$$\sum_{i=1}^{M} \frac{1}{\tilde{\varepsilon}_i} \langle \mathcal{S}^i_{\kappa_i} \mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i} \rangle_{\Gamma_i} = \omega^2 \mu_0 \langle \gamma_1^{\text{ext}} \phi, \psi \rangle_\Gamma - \omega^2 \mu_0 \langle \gamma_n \mathbf{H}_p, \psi \rangle_\Gamma$$

for a pair of test functions $(\mathbf{v}, \psi) \in \mathcal{V}_0$. Using the Steklov-Poincaré operator for the scalar Laplace equation (4.21) we can write

$$\sum_{i=1}^{M} \frac{1}{\tilde{\varepsilon}_i} \langle \mathcal{S}^i_{\kappa_i} \mathbf{u}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i} \rangle - \omega^2 \mu_0 \langle \mathcal{S}_0^{\text{ext}} \phi, \psi \rangle_\Gamma = -\omega^2 \mu_0 \langle \gamma_n \mathbf{H}_p, \psi \rangle_\Gamma$$

for $(\mathbf{v}, \psi) \in \mathcal{V}_0$. By introducing the new auxiliary unknowns

$$\boldsymbol{\lambda}_i = \frac{1}{\tilde{\varepsilon}_i} (S^i_\kappa)^{-1} (\frac{1}{2} I - C^i_\kappa) \mathbf{u}|_{\Gamma_i} \quad \text{for } i = 1, \dots, M \tag{5.44}$$

and

$$t = -\omega^2 \mu_0 V_0^{-1} (\frac{1}{2} I - K_0) \phi \tag{5.45}$$

we get the variational equation

$$\sum_{i=1}^{M} \frac{1}{\tilde{\varepsilon}_i} \langle N^i_\kappa \tilde{\mathbf{u}}|_{\Gamma_i}, \mathbf{v}|_{\Gamma_i} \rangle_{\Gamma_i} + \omega^2 \mu_0 \langle D_0 \phi, \psi \rangle_\Gamma + \sum_{i=1}^{M} \langle (\frac{1}{2} I + B^i_\kappa) \boldsymbol{\lambda}_i, \mathbf{v}|_{\Gamma_i} \rangle_{\Gamma_i} - \langle (\frac{1}{2} I - K_0') t, \psi \rangle_\Gamma$$
$$= -\omega^2 \mu_0 \langle \gamma_n \mathbf{H}_p, \psi \rangle_\Gamma + \sum_{i=1}^{M} \frac{1}{\tilde{\varepsilon}_i} \langle N^i_\kappa \boldsymbol{\gamma}_t \mathbf{H}_p, \mathbf{v}|_{\Gamma_i} \rangle_{\Gamma_i},$$

for $(\mathbf{v}, \psi) \in \mathcal{V}_0$. We have set $\tilde{\mathbf{u}}(x) = \mathbf{u}(x) - \boldsymbol{\gamma}_t \mathbf{H}_p(x)$ for $x \in \Omega$. Due to the transmission conditions (5.41) we have then $(\tilde{\mathbf{u}}, \phi) \in \mathcal{V}_0$. Adding this with the equations for the unknowns $\boldsymbol{\lambda}_i$ and t we obtain the variational problem:
Find $((\tilde{\mathbf{u}}, \phi), \boldsymbol{\lambda}_i, t) \in \mathcal{V}_0 \times \mathbf{H}_{\|}^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \times H^{-1/2}(\Gamma)$ such that

$$n((\tilde{\mathbf{u}}, \phi), (\mathbf{v}, \psi)) + \sum_{i=1}^{M} \langle (\frac{1}{2} I + B^i_\kappa) \boldsymbol{\lambda}_i, \mathbf{v}|_{\Gamma_i} \rangle_{\Gamma_i} - \langle (\frac{1}{2} I - K_0') t, \psi \rangle_\Gamma = f(\mathbf{v}), \tag{5.46}$$

$$\sum_{i=1}^{M} \langle (-\frac{1}{2} I + C^i_\kappa) \mathbf{u}|_{\Gamma_i}, \boldsymbol{\mu}_i \rangle_{\Gamma_i} + \sum_{i=1}^{M} \langle \tilde{\varepsilon}_i S^i_\kappa \boldsymbol{\lambda}_i, \boldsymbol{\mu}_i \rangle_{\Gamma_i} = \sum_{i=1}^{M} g(\boldsymbol{\mu}_i), \tag{5.47}$$

$$\langle (\frac{1}{2} I - K_0) \phi, p \rangle_\Gamma + \frac{1}{\omega^2 \mu_0} \langle V_0 t, p \rangle_\Gamma = 0 \tag{5.48}$$

is satisfied for all $((\mathbf{v},\psi),\boldsymbol{\mu}_i,p)\in\mathcal{V}_0\times\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\Gamma)$. The bi-linear form $n((\tilde{\mathbf{u}},\phi),(\mathbf{v},\psi))$ is given by

$$n((\tilde{\mathbf{u}},\phi),(\mathbf{v},\psi))=\sum_{i=1}^{M}\frac{1}{\tilde{\varepsilon}_i}\langle N_\kappa^i\tilde{\mathbf{u}}|_{\Gamma_i},\mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i}+\omega^2\mu_0\langle D_0\phi,\psi\rangle_\Gamma \tag{5.49}$$

and the right hand side is defined by

$$\begin{aligned}f(\mathbf{v})&=-\omega^2\mu_0\langle\gamma_n\mathbf{H}_p,\psi\rangle_\Gamma+\sum_{i=1}^{M}\frac{1}{\tilde{\varepsilon}_i}\langle N_\kappa^i\boldsymbol{\gamma}_t\mathbf{H}_p,\mathbf{v}|_{\Gamma_i}\rangle_{\Gamma_i},\\ g(\boldsymbol{\mu}_i)&=\langle(-\frac{1}{2}I+C_\kappa^i)\boldsymbol{\gamma}_t\mathbf{H}_p,\boldsymbol{\mu}_i\rangle_{\Gamma_i}.\end{aligned}$$

Proposition 5.8. *The variational problem (5.46)-(5.48) admits a unique solution in the space* $\mathcal{V}_0\times\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\Gamma)$.

The proof of Proposition 5.8 follows from the block-skew symmetric structure of the variational problem (5.46)-(5.48) and from the ellipticity of the operators N_κ, D_0, S_κ and V_0. Let us now consider the simple case that κ is constant in Ω, this means that we have only one transmission boundary Γ between the conducting and the non-conducting domains. We can replace $\tilde{u}$ by $\nabla_\Gamma\phi$ and skip the space $\mathcal{V}_0$. This setting leads us to the following variational problem, which was also derived in [31]: Find $(\phi,\boldsymbol{\lambda},t)\in H^{1/2}(\Gamma)\times\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\Gamma)$ such that

$$n(\phi,\psi)+\langle(\frac{1}{2}I+B_\kappa)\boldsymbol{\lambda},\nabla_\Gamma\psi\rangle_\Gamma-\langle(\frac{1}{2}I-K_0')t,\psi\rangle_\Gamma=f(\psi), \tag{5.50}$$

$$\langle(-\frac{1}{2}I+C_\kappa)\nabla_\Gamma\phi,\boldsymbol{\mu}\rangle_\Gamma+\tilde{\varepsilon}\langle S_\kappa\boldsymbol{\lambda},\boldsymbol{\mu}\rangle_\Gamma=g(\boldsymbol{\mu}), \tag{5.51}$$

$$\langle(\frac{1}{2}I-K_0)\phi,p\rangle_\Gamma+\frac{1}{\omega^2\mu_0}\langle V_0t,p\rangle_\Gamma=0 \tag{5.52}$$

is satisfied for all $(\psi,\boldsymbol{\mu},p)\in H^{1/2}(\Gamma)\times\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)\times H^{-1/2}(\Gamma)$ with

$$n(\phi,\psi)=\frac{1}{\tilde{\varepsilon}}\langle N_\kappa\nabla_\Gamma\phi,\nabla_\Gamma\psi\rangle_\Gamma+\frac{1}{\omega^2\mu_0}\langle D_0\phi,\psi\rangle_\Gamma,$$

and the right hand side

$$f(\psi)=\frac{1}{\tilde{\varepsilon}}\langle N_\kappa\gamma_t\mathbf{H}_p,\nabla_\Gamma\psi\rangle_\Gamma-\omega^2\mu_0\langle\gamma_n\mathbf{H}_p,\psi\rangle_\Gamma,\quad g(\boldsymbol{\mu})=\langle(-\frac{1}{2}I+C_\kappa)\gamma_t\mathbf{H}_p,\boldsymbol{\mu}\rangle_\Gamma.$$

5.2.3 An indirect boundary integral formulation based on the H-field

For the case that κ is constant in the conducting domain Ω we are going to present an indirect boundary integral method for the $\mathbf{H}$-ϕ formulation. Such an approach has also been considered in [61] for a collocation boundary element method. We will introduce an indirect formulation, which is also suited for the Galerkin method. The advantage of the indirect approach compared to the direct approach in Section 5.2.2, is that it leads to a smaller system of boundary integral equations, also less boundary integral operators are involved, which means that in a boundary element method less matrices have to be set up.

We consider the same setting as in the previous section, this means that in the conducting domain Ω the magnetic field intensity is governed by the electromagnetic wave equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{H}(x) + \kappa^2\mathbf{H}(x) = 0, \quad x \in \Omega. \tag{5.53}$$

In the exterior domain Ω^c we have $\mathbf{curl}\,\mathbf{H}_s(x) = 0$ and thus

$$\mathbf{H}_s(x) = \nabla\Phi(x), \quad \Delta\Phi(x) = 0, \quad x \in \Omega^c. \tag{5.54}$$

For the interior domain we make a single layer potential ansatz

$$\mathbf{H}(x) = \mathbf{\Psi}^{\kappa}_{\mathrm{SL}}(\boldsymbol{\lambda}) - \frac{1}{\kappa^2}\nabla\Psi^{\kappa}_{\mathrm{SL}}(\mathrm{div}_\Gamma(\boldsymbol{\lambda})), \quad x \in \Omega,$$

and for the exterior domain we set

$$\Phi(x) = -\Psi^0_{\mathrm{DL}}(\phi), \quad x \in \Omega^c,$$

from which we obtain

$$\nabla\Phi(x) = -\nabla\Psi^0_{\mathrm{DL}}(\phi)(x) = \mathbf{\Psi}^0_{\mathrm{DL}}(\nabla_\Gamma\phi)(x), \quad x \in \Omega^c$$

by using Lemma 4.17. By applying the interior tangential and normal trace to $\mathbf{H}$ we get

$$\begin{aligned}
\boldsymbol{\gamma}_t^{\mathrm{int}}\mathbf{H}(x) &= S_\kappa(\boldsymbol{\lambda})(x),\\
\gamma_n^{\mathrm{int}}\mathbf{H}(x) &= n(x)\cdot A_\kappa(\boldsymbol{\lambda})(x) - \frac{1}{\kappa^2}(\frac{1}{2}I + K'_\kappa)(\mathrm{div}_\Gamma\,\boldsymbol{\lambda})(x), \quad x \in \Gamma,
\end{aligned}$$

and applying the exterior tangential and normal trace to $\nabla\phi$ gives

$$\boldsymbol{\gamma}_t^{\mathrm{ext}}\mathbf{H}_s(x) = (-\frac{1}{2}I + C_0)\nabla_\Gamma\phi(x), \quad \gamma_n^{\mathrm{ext}}\mathbf{H}_s = (D_0\phi)(x), \quad x \in \Gamma.$$

Inserting the transmission conditions (5.41),

$$\boldsymbol{\gamma}_t^{\mathrm{int}}\mathbf{H}(x) = \nabla_\Gamma\Phi(x) + \boldsymbol{\gamma}_t\mathbf{H}_p(x), \quad \mu\gamma_n^{\mathrm{int}}\mathbf{H}(x) = \mu_0\gamma_1^{\mathrm{ext}}\Phi(x) + \mu_0\gamma_n\mathbf{H}_p(x), \quad x \in \Gamma$$

gives the following variational problem:
Find $(\boldsymbol{\lambda}, \phi) \in \mathbf{H}_{\|}^{-1/2}(\text{div}_\Gamma, \Gamma) \times H^{1/2}(\Gamma)$ such that

$$\langle S_\kappa \boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma + \langle (-\frac{1}{2}I + C_0)\nabla_\Gamma \phi, \boldsymbol{\mu}\rangle_\Gamma = \langle \gamma_t \mathbf{H}_p, \boldsymbol{\mu}\rangle_\Gamma,$$

$$\mu\langle n \cdot A_\kappa(\boldsymbol{\lambda}), \psi\rangle_\Gamma - \frac{1}{\kappa^2}\langle (\frac{1}{2}I + K'_\kappa)(\text{div}_\Gamma \boldsymbol{\lambda}), \psi\rangle_\Gamma - \mu_0\langle D_0\phi, \psi\rangle_\Gamma = \mu_0\langle \gamma_n \mathbf{H}_p, \psi\rangle_\Gamma$$

holds for all $(\boldsymbol{\mu}, \psi) \in \mathbf{H}_{\|}^{-1/2}(\text{div}_\Gamma, \Gamma) \times H^{1/2}(\Gamma)$.
We are now going to rewrite the above variational problem, for this we need the following lemma:

Lemma 5.9. *For every* $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\text{div}_\Gamma, \Gamma)$ *and for* $\kappa \neq 0$ *we have the identity*

$$\gamma_n^{\text{int}}\left[\boldsymbol{\Psi}_{\text{SL}}^\kappa(\boldsymbol{\lambda}) - \frac{1}{\kappa^2}\nabla\Psi_{\text{SL}}^\kappa(\text{div}_\Gamma(\boldsymbol{\lambda}))\right] = n(x)\cdot A_\kappa(\boldsymbol{\lambda})(x) - \frac{1}{\kappa^2}(\frac{1}{2}I + K'_\kappa)\,\text{div}_\Gamma \boldsymbol{\lambda}(x)$$
$$= \frac{1}{\kappa^2}\,\text{div}_\Gamma(\frac{1}{2}I + B_\kappa)\boldsymbol{\lambda}(x), \quad x \in \Gamma.$$

Proof. Applying the exterior Dirichlet trace $\boldsymbol{\gamma}_t^{\text{ext}}$ to identity (4.17) in Lemma (4.17) and testing it with a function $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\text{div}_\Gamma, \Gamma)$ we obtain the follwing relation

$$\frac{1}{\kappa^2}\langle (-\frac{1}{2}I + C_\kappa)\nabla_\Gamma\psi, \boldsymbol{\lambda}\rangle_\Gamma = -\langle A_\kappa(n\psi), \boldsymbol{\lambda}\rangle_\Gamma - \frac{1}{\kappa^2}\langle \nabla_\Gamma(\frac{1}{2}I + K_\kappa)\psi, \boldsymbol{\lambda}\rangle_\Gamma,$$

which holds for any $\psi \in H^{1/2}(\Gamma)$. Using Theorem 4.23 we conclude that

$$\frac{1}{\kappa^2}\langle \psi, \text{div}_\Gamma(\frac{1}{2}I + B_\kappa)\boldsymbol{\lambda}\rangle_\Gamma = \langle \psi, n\cdot A_\kappa\boldsymbol{\lambda}\rangle_\Gamma - \frac{1}{\kappa^2}\langle \psi, (\frac{1}{2}I + K'_\kappa)\,\text{div}_\Gamma \boldsymbol{\lambda}\rangle_\Gamma$$

holds for all $\psi \in H^{1/2}(\Gamma)$, which finishes the proof. □

With this auxiliary result we can now rewrite the variational problem as:
Find $(\boldsymbol{\lambda}, \sigma) \in \mathbf{H}_{\|}^{-1/2}(\text{div}_\Gamma, \Gamma) \times H^{1/2}(\Gamma)$ such that

$$\langle S_\kappa \boldsymbol{\lambda}, \boldsymbol{\mu}\rangle_\Gamma + \langle (-\frac{1}{2}I + C_0)\nabla_\Gamma\phi, \boldsymbol{\mu}\rangle_\Gamma = \langle \gamma_t \mathbf{H}_p, \boldsymbol{\mu}\rangle_\Gamma, \tag{5.55}$$

$$-\langle \boldsymbol{\lambda}, (-\frac{1}{2}I + C_\kappa)\nabla_\Gamma\psi\rangle_\Gamma + \frac{\kappa^2}{\mu_r}\langle D_0\phi, \psi\rangle_\Gamma = \frac{\kappa^2}{\mu_r}\langle \gamma_n \mathbf{H}_p, \psi\rangle_\Gamma \tag{5.56}$$

holds for all $(\boldsymbol{\mu}, \psi) \in \mathbf{H}_{\|}^{-1/2}(\text{div}_\Gamma, \Gamma) \times H^{1/2}(\Gamma)$.

Remark 5.10. *In contrast to the direct boundary integral formulations (5.30)-(5.31) and (5.50)-(5.52) this formulation is non symmetric. Moreover the variational problem (5.55)-(5.56) does not directly give rise to an elliptic bilinear form. To be able to make a statement about the solvability of the variatonal problem (5.55)-(5.56) we introduce the splitting*

$$\langle \mathbf{j}, (-\frac{1}{2}I + C_\kappa)\nabla_\Gamma \psi\rangle_\Gamma = \langle \mathbf{j}, (-\frac{1}{2}I + C_0)\nabla_\Gamma \psi\rangle_\Gamma + \langle \mathbf{j}, (C_\kappa - C_0)\nabla_\Gamma \psi\rangle_\Gamma.$$

Since $C_\kappa - C_0 : \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *is a compact operator (see [71]), we get that the corresponding operator of the variational problem is a Fredholm operator of index zero. This means that if we are able to prove the injectivity of the variational problem, the unique solvability of the variational problem (5.55)-(5.56) follows.*

5.3 The Reduced Model

In this section we derive a boundary element formulation for the reduced model, which was derived in Section 2.3.2. As already shown, the reduced model requires only the solution of a Neumann type boundary value problem for the Laplace equation

$$\begin{aligned} -\nabla \cdot [\kappa(x)\nabla\widetilde{\phi}(x)] &= i\omega\nabla \cdot [\kappa(x)\mathbf{A}_p(x)] \quad \text{for } x \in \Omega, \\ \kappa(x)\frac{\partial\widetilde{\phi}(x)}{\partial n(x)} &= -i\omega\kappa(x)\mathbf{A}_p(x)\cdot n(x) \quad \text{for } x \in \Gamma. \end{aligned}$$

In addition we choose the scaling condition

$$\int_\Gamma \widetilde{\phi}(x) ds_x = 0.$$

The variational formulation of this Neumann boundary value problem is to find $\widetilde{\phi} \in H^1(\Omega)$ such that

$$\int_\Omega \kappa(x)\nabla\widetilde{\phi}(x)\cdot\nabla\psi(x)dx + \int_\Gamma \widetilde{\phi}(x)ds_x \int_\Gamma \psi(x)ds_x = i\omega\int_\Omega \kappa(x)\mathbf{A}_p(x)\cdot\nabla\psi(x)\,dx \quad (5.57)$$

for all $\psi \in H^1(\Omega)$. For a piecewise constant conductivity $\kappa(x)$ we consider the non-overlapping domain decomposition (5.10) and (5.11). Instead of the global Neumann boundary value problem (2.51) and (2.52) we now consider the local boundary value problems

$$-\kappa_i\Delta\widetilde{\phi} = 0 \quad \text{for } x \in \Omega_i, \qquad (5.58)$$

$$\kappa_i\frac{\partial\widetilde{\phi}(x)}{\partial n_i(x)} = -i\omega\kappa_i\mathbf{A}_p(x)\cdot n_i(x) \quad \text{for } x \in \Gamma_i \cap \Gamma. \qquad (5.59)$$

together with the transmission boundary conditions, see (2.39),

$$\kappa_i \frac{\partial \widetilde{\phi}(x)}{\partial n_i(x)} + \kappa_j \frac{\partial \widetilde{\phi}(x)}{\partial n_j(x)} = -i\omega\kappa_i \mathbf{A}_p(x) \cdot n_i(x) - i\omega\kappa_j \mathbf{A}_p(x) \cdot n_j(x) \quad \text{for } x \in \Gamma_i \cap \Gamma_j.$$

Thus we can rewrite the variational formulation (5.57) as

$$\sum_{i=1}^{p} \int_{\Gamma_i} \kappa_i \frac{\partial \widetilde{\phi}(x)}{\partial n_i(x)} \psi(x) ds_x + \int_{\Gamma} \widetilde{\phi}(x) ds_x \int_{\Gamma} \psi(x) ds_x = -\sum_{i=1}^{p} i\omega \int_{\Gamma_i} \kappa_i [\mathbf{A}_p(x) \cdot n_i(x)] \psi(x) ds_x.$$

For the solution of the local partial differential equation in (5.58) we use the local Dirichlet to Neumann map

$$\frac{\partial \widetilde{\phi}(x)}{\partial n(x)} = (\mathcal{S}_0^i \widetilde{\phi})(x) \quad \text{for } x \in \Gamma_i = \partial\Omega_i,$$

where $\mathcal{S}_0^i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is the associated Steklov–Poincaré operator [65]. Let $H^{1/2}(\Gamma_S) := H^1(\Omega)_{|\Gamma_S}$ be the skeleton trace space of $H^1(\Omega)$. We then have to solve a variational problem to find $\widetilde{\phi} \in H^{1/2}(\Gamma_S)$ such that

$$\sum_{i=1}^{p} \kappa_i \int_{\Gamma_i} (\mathcal{S}_0^i \widetilde{\phi})(x) \psi(x) ds_x + \int_{\Gamma} \widetilde{\phi}(x) ds_x \int_{\Gamma} \psi(x) ds_x = -\sum_{i=1}^{p} i\omega \int_{\Gamma_i} \kappa_i [\mathbf{A}_p(x) \cdot n_i(x)] \psi(x) ds_x \quad (5.60)$$

is satisfied for all $\psi \in H^{1/2}(\Gamma)$. Since the bilinear form in the variational formulation (5.60) is bounded and $H^{1/2}(\Gamma_S)$–elliptic, see, e.g. [36], unique solvability of the variational formulation (5.60) follows. To describe the application of the local Steklov–Poincaré operators which are involved in the variational formulation (5.60) we use the symmetric boundary integral operator representation as derived in Section 4.1.3

$$(S_0 \widetilde{\phi}_{|\Gamma_i})(x) = \left[D_0 + (\frac{1}{2} I + K_0') V_0^{-1} (\frac{1}{2} I + K_0) \right] \widetilde{\phi}_{|\Gamma_i}(x) \quad \text{for } x \in \Gamma. \quad (5.61)$$

5.3.1 Determining the voltage in a coil

The quantity, which is measured in Magnetic Induction Tomography is the voltage in the receiver coil $\mathcal{C}$, i.e.,

$$v := -i\omega \int_{\mathcal{C}} \mathbf{B}_s(x) \cdot n(x) ds_x.$$

Hence we need to evaluate

$$\mathbf{B}_s(x) = \nabla \times \mathbf{A}_s(x) = \frac{\mu_0}{4\pi} \int_{\Omega} \kappa(y) \nabla_x \frac{1}{|x-y|} \times [i\omega \mathbf{A}_p(y) + \nabla_y \phi(y)] dy \quad \text{for } x \in \mathcal{C}. \tag{5.62}$$

By using integration by parts, and by using the symmetry of the fundamental solution

$$\nabla_x \frac{1}{|x-y|} = -\nabla_y \frac{1}{|x-y|},$$

the volume integral in (5.62) can be reformulated as

$$\begin{aligned} \mathbf{B}_s(x) &= \frac{\mu_0}{4\pi} \sum_{i=1}^{N} \kappa_i \left(i\omega \int_{\Omega_i} \nabla_x \frac{1}{|x-y|} \times \mathbf{A}_p(y) dy - \int_{\Omega_i} \nabla_y \frac{1}{|x-y|} \times \nabla_y \phi(y)) dy \right) \\ &= \frac{\mu_0}{4\pi} \sum_{i=1}^{N} \kappa_i \left(i\omega \int_{\Omega_i} \nabla_x \frac{1}{|x-y|} \times \mathbf{A}_p(y) dy - \int_{\Gamma_i} \frac{\nabla_y \phi(y) \times n(y)}{|x-y|} ds_y \right). \end{aligned}$$

For the solution of this problem by using the boundary integral formulation as derived above, this representation of $\mathbf{B}_s$ is very suitable since we only have to evaluate the surface curl of the scalar potential ϕ on the boundary Γ. If we would use the representation (5.62) we would have to evaluate the gradient of ϕ inside the domain, e.g. on a finite element volume mesh, which would be computationally more expensive.

6 BOUNDARY ELEMENT METHODS

In this section we deal with the discretization of the boundary integral equations discussed in Sections 4 and 5. We introduce boundary elements spaces for the discretization of the boundary spaces $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. The derivation of those spaces is based on the existing results on finite elements for the space $\mathbf{H}(\mathbf{curl}; \Omega)$. The classical discrete trial and test spaces for $\mathbf{H}(\mathbf{curl}; \Omega)$ have been introduced by Jean-Claude Nédélec (cf. [50, 51]). Applying the trace operators to these basis functions give us the lowest order boundary element basis functions. In the engineering community those basis functions are also called Rao-Wilton-Glisson basis functions (see [57]).
Based on the introduced boundary element spaces we are going to formulate boundary element methods for the solution of boundary value problems for the electromagnetic wave equation. We are going to consider the case when κ is small and introduce a boundary element formulation for the approach as given in Section 4.6.1. We are going to illustrate the effect that this approach is stable when κ tends to zero in numerical examples.
For a numerical solution of the eddy current problem we are going to deal with the discretization of the boundary integral formulations as derived in Section 5. We deduce the discrete variatonal problems for the $\mathbf{E}$-field and $\mathbf{H}$-field formulations and we compare the different approaches. At the end of this section we derive a boundary element method for the reduced model.

6.1 Discrete Trial and Test Spaces

In this section we are going to introduce discrete trial and test spaces on the boundary Γ. For this we assume that we have a given shape regular and conforming triangular boundary element mesh $\Gamma_h = \{\tau_i\}_{i=1}^N$, which satisfies

$$\Gamma = \bigcup_{i=1}^{N} \overline{\tau_i}.$$

The area of a boundary element τ_i is denoted by

$$\Delta_i = \int_{\tau_i} ds_x$$

and the local mesh width by

$$h_i = \sqrt{\Delta_i}.$$

The nodes of the boundary element mesh Γ_h are denoted by $\{x_1, x_2, \ldots, x_{N^n}\}$.

6.1.1 Basis Functions

On the boundary element mesh Γ_h we introduce the discrete basis function spaces

$$\begin{aligned} S_h^0(\Gamma) &= \text{span}\,\{\phi_1^0, \phi_2^0, \ldots, \phi_N^0\} \subset H^{-1/2}(\Gamma), \\ S_h^1(\Gamma) &= \text{span}\,\{\phi_1^1, \phi_2^1, \ldots, \phi_{N^n}^1\} \subset H^{1/2}(\Gamma), \end{aligned}$$

where $S_h^0(\Gamma)$ denotes the space of piecewise constant basis functions

$$\phi_i^0(x) = \begin{cases} 1, & x \in \tau_i, \\ 0, & \text{else}, \end{cases}$$

which are associated to the elements of the boundary mesh Γ_h. The space $S_h^1(\Gamma)$ denotes the space of piecewise nodal basis functions

$$\phi_i^1(x) = \begin{cases} 1, & x = x_i, \\ 0, & x = x_j, j \neq i, \\ \text{linear}, & \text{else}. \end{cases}$$

The basis functions ϕ_i^1 are associated to the nodes of the boundary element mesh. The discrete spaces $S_h^0(\Gamma)$ and $S_h^1(\Gamma)$ satisfy the following approximation properties:

Theorem 6.1. *Let Γ be sufficiently smooth and u be in $H^s(\Gamma)$ for some $s \in [\sigma, 2]$ with $\sigma \in [0, 1]$. Then we have the approximation property*

$$\inf_{v_h \in S_h^1(\Gamma)} \|u - v_h\|_{H^\sigma(\Gamma)} \leq c h^{s-\sigma} |u|_{H^s(\Gamma)}.$$

Theorem 6.2. *Let u be in $H^s(\Gamma)$ for some $s \in [\sigma, 1]$ with $\sigma \in [-1, 0]$. Then we have the approximation property*

$$\inf_{v_h \in S_h^0(\Gamma)} \|u - v_h\|_{H^\sigma(\Gamma)} \leq c h^{s-\sigma} |u|_{H^s(\Gamma)}.$$

These results and the corresponding proofs can be found in [66].
As a next step we introduce a boundary element discretization for the spaces $\mathbf{H}_{\|}^{-1/2}(\text{div}_\Gamma, \Gamma)$ and $\mathbf{H}_{\perp}^{-1/2}(\text{curl}_\Gamma, \Gamma)$, which are the trace spaces of the space $\mathbf{H}(\mathbf{curl}; \Omega)$. The basis functions for the space $\mathbf{H}(\mathbf{curl}; \Omega)$ are associated to the edges of a finite

element mesh. Let us consider the edge, which is spanned by the nodes x_i and x_j. The lowest order Nédélec basis function associated with the edge $\{x_i, x_j\}$ is defined by

$$\mathbf{U}_{ij}(x) = \phi_i^1(x)\nabla\phi_j^1 - \phi_i^1(x)\nabla\phi_i^1,$$

where ϕ_i^1 denotes the nodal finite element basis function for the node x_i. The lowest order basis function for the space $\mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma, \Gamma)$ on the triangle τ_l with the nodes $\{x_i, x_j, x_k\}$ associated to the edge $\{x_i, x_j\}$ can be obtained by applying the trace operator $\boldsymbol{\gamma}_t$ to the lowest order Nedelec basis function $\mathbf{U}_{ij}$. This gives us the lowest order basis function

$$\mathbf{u}_{ij}(x) = \boldsymbol{\gamma}_t\mathbf{U}_{ij}(x) = \phi_i^1(x)\nabla_\Gamma\phi_j^1 - \phi_j^1(x)\nabla_\Gamma\phi_i^1, \quad x \in \tau_l.$$

By inserting the definition of the nodal basis function $\phi_i^1(x)$ we get the representation

$$\mathbf{u}_{ij}(x) = n(x) \times \frac{x - x_k}{\Delta_l}, \quad x \in \tau_l,$$

where n denotes the normal vector corresponding to the boundary element $\{x_i, x_j, x_k\}$. The lowest order basis functions for the space $\mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma, \Gamma)$ can be obtained by applying the operator $\mathbf{R}$ to the function $\mathbf{u}_{ij}(x)$,

$$\boldsymbol{\lambda}_{ij}(x) = \boldsymbol{\gamma}_\times\mathbf{U}_{ij}(x) = (\phi_i^1(x)\nabla_\Gamma\phi_j^1 - \phi_j^1(x)\nabla_\Gamma\phi_i^1) \times n(x), \quad x \in \tau_l.$$

Hence we obtain the representation

$$\boldsymbol{\lambda}_{ij}(x) = \frac{x - x_k}{\Delta_l}, \quad x \in \tau_l$$

for the lowest order basis functions for the space $\mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma, \Gamma)$. We observe that each basis function is now associated with an edge in the boundary element mesh, thus we introduce the set of basis functions for a boundary element mesh Γ_h:

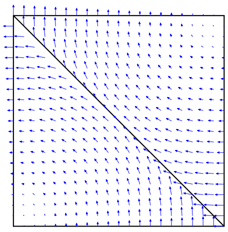
$$\mathcal{F}_h(\Gamma) = \text{span}\,\{\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{N^e}\} \subset \mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma, \Gamma),$$
$$\mathcal{E}_h(\Gamma) = \text{span}\,\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{N^e}\} \subset \mathbf{H}_\perp^{-1/2}(\text{curl}_\Gamma, \Gamma),$$

where N^e denotes the number of edges in the boundary element mesh. The spaces $\mathcal{F}_h(\Gamma)$ and $\mathcal{E}_h(\Gamma)$ satisfy the following approximation properties:

Theorem 6.3 (Approximation Property)**.** *For any functions $\boldsymbol{\lambda}, \mathbf{u} \in \boldsymbol{H}_t^s(\Gamma)$ with $\text{div}_\Gamma\, \boldsymbol{\lambda} \in H^s(\Gamma)$ and $\text{curl}_\Gamma \mathbf{u} \in H^s(\Gamma)$ for some $0 \le s \le 1$, there holds the approximation property*

$$\inf_{\boldsymbol{\lambda}_h \in \mathcal{F}_h(\Gamma)} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma, \Gamma)} \le ch^{\min(\frac{3}{2}-\varepsilon, s+\frac{1}{2}-\varepsilon, 1+s^*, s+s^*)} \left(\|\boldsymbol{\lambda}\|_{\boldsymbol{H}^s(\Gamma)} + \|\,\text{div}_\Gamma\, \boldsymbol{\lambda}\|_{H^s(\Gamma)}\right)$$
$$\inf_{\mathbf{u}_h \in \mathcal{E}_h(\Gamma)} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}_\parallel^{-1/2}(\text{div}_\Gamma, \Gamma)} \le ch^{\min(\frac{3}{2}-\varepsilon, s+\frac{1}{2}-\varepsilon, 1+s^*, s+s^*)} \left(\|\mathbf{u}\|_{\boldsymbol{H}^s(\Gamma)} + \|\text{curl}_\Gamma \mathbf{u}\|_{H^s(\Gamma)}\right)$$

for any $\varepsilon > 0$. The constant s^* depends on the regularity of the domain.



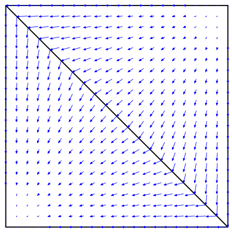


Figure 6.1: Sketch of lowest order basis functions for $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

Now we are able to discretize the boundary integral formulations as derived in Chapter 4 and 5. We will start with the analysis of a simple boundary value problem, where we prescribe the tangential trace.

6.2 Boundary Element Methods for the Electromagnetic Wave Equation

We consider the boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}(x) + \kappa^2 \mathbf{E}(x) = 0, \quad x \in \Omega, \quad \boldsymbol{\gamma}_t^{\mathrm{int}} \mathbf{E}(x) = \mathbf{F}(x), \quad x \in \Gamma. \tag{6.1}$$

For this type of boundary value problems we derived in Section 4.5 the boundary integral equation

$$S_\kappa \boldsymbol{\gamma}_N^{\mathrm{int}} \mathbf{E}(x) = (\frac{1}{2} I - C_\kappa)\mathbf{F}(x), \quad x \in \Gamma.$$

In what follows we assume that we have given a representation of the prescribed tangential trace $\mathbf{F}$ by the basis functions in $\mathcal{F}_h(\Gamma)$:

$$\mathbf{F}(x) \approx \mathbf{F}_h(x) = \sum_{i=1}^{N^e} F_i \mathbf{u}_i(x).$$

Then we can formulate the discrete variational problem:
Find $\boldsymbol{\lambda}_h \in \mathcal{F}_h(\Gamma)$ such that

$$\langle S_\kappa \boldsymbol{\lambda}_h, \boldsymbol{\mu}_h \rangle_\Gamma = \langle (\frac{1}{2} M_h - C_{\kappa,h}) \mathbf{F}_h, \boldsymbol{\mu}_h \rangle_\Gamma \tag{6.2}$$

holds for all $\boldsymbol{\mu}_h \in \mathcal{F}_h(\Gamma)$. The discrete variational problem can be reformulated as a linear system of equations

$$S_{\kappa,h}\underline{\lambda} = (\frac{1}{2}M_h - C_\kappa)\underline{F}, \tag{6.3}$$

with the matrices

$$S_{\kappa,h}[i,j] = \langle S_\kappa \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_i\rangle_\Gamma = \langle A_\kappa \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_i\rangle_\Gamma + \frac{1}{\kappa^2}\langle V_\kappa \operatorname{div}_\Gamma \boldsymbol{\lambda}_j, \operatorname{div}_\Gamma \boldsymbol{\lambda}_i\rangle_\Gamma, \quad i,j = 1,\ldots,N^e$$

and

$$C_{\kappa,h}[i,j] = \langle C_\kappa \mathbf{u}_j, \boldsymbol{\lambda}_i\rangle_\Gamma, \quad M_h[i,j] = \langle \mathbf{u}_j, \boldsymbol{\lambda}_i\rangle_\Gamma, \quad i,j = 1,\ldots,N^e.$$

For $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$ we know that S_κ is $\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$-elliptic, thus we immediately conclude the unique solvability of the discrete variational problem and the quasi optimal estimate

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} \le \inf_{\boldsymbol{\mu}_h \in \mathcal{F}_h(\Gamma)} \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)}$$

by using Cea's lemma. Together with the approximation property given in Theorem 6.3 we get an error estimate in dependence on the mesh width h.
In what follows we present some numerical examples: In the Tables 6.1 and 6.2 we give the number of GMRES-iterations and the error in the boundary data and the relative error in the point evaluation for some $x^* \in \Omega$:

$$\text{error}_1 = \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathbf{L}^2(\Gamma)}/\|\boldsymbol{\lambda}\|_{\mathbf{L}^2(\Gamma)}, \quad \text{error}_2 = |\mathbf{E}(x^*) - \mathbf{E}_h(x^*)|/|\mathbf{E}(x^*)|. \tag{6.4}$$

As a reference solution we choose the function

$$\mathbf{E}(x) = \nabla U^*_\kappa(x, x_s) \times x. \tag{6.5}$$

As a solver for the linear system we use the GMRES method [58], all linear systems are solved with the relative precision 10^{-8}. In Table 6.1 we have computed the approximate solution to the boundary value problem (4.64) by solving the discrete variational problem (6.3) for $\Omega = B_1(0)$ and $\kappa = 1 + i$. In Table 6.2 we find the results for $\Omega = (0, 0.5)^3$ and $\kappa = 1 + i$ with the reference solution (6.5).

6.2.1 A stabilization for the single layer potential for small κ

We are now going to consider the case when κ is small. Such problems have been as well considered in [5,16,42,45,69,73], where the low frequency stabilization is based on a decomposition of the space $\mathcal{F}_h(\Gamma)$ into two parts, where one part consists of

edges	It	error$_1$	eoc	error$_2$	eoc
120	29	1.215e-1	-	1.916e-2	-
480	65	6.284e-2	1.0	4.468e-3	2.1
1920	116	3.174e-2	1.0	1.085e-3	2.0

Table 6.1: Results for the domain $\Omega = B_1(0)$ with $\kappa = 1 + i$, the evaluation point was chosen as $x^* = (0.2, 0.2, 0.2)^\top$. The reference solution is given by (6.5) with the source point $x_s = (1.3, 1.0, 1.1)^\top$.

edges	It	error$_1$	eoc	error$_2$	eoc
36	11	7.623e-02	-	4.324e-03	-
114	51	3.328e-02	1.2	1.342e-03	1.7
576	115	1.617e-02	1.0	2.958e-04	2.2
2304	216	7.986e-03	1.0	7.277e-05	2.0
9216	394	4.029e-03	1.0	1.845e-05	2.0

Table 6.2: Results for the domain $\Omega = (0, 0.5)^3$ with $\kappa = 1 + i$, the evaluation point was chosen as $x^* = (0.2, 0.2, 0.2)^\top$. The reference solution is given by (6.5) with the source point $x_s = (1.3, 1.0, 1.1)^\top$.

solenoidal functions. These solenoidal functions are also referred to as 'loop-currents', the decomposition is therefore considered as 'loop-star' or 'loop-tree' decomposition. Let us now take a closer look at the behaviour of the single layer matrix $S_{\kappa,h}$ when considering small κ. The single layer matrix is composed by two matrices,

$$S_{\kappa,h} = A_{\kappa,h} + \frac{1}{\kappa^2} W_{\kappa,h},$$

where

$$A_{\kappa,h}[i,j] = \langle A_\kappa \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_i \rangle_\Gamma, \quad W_{\kappa,h}[i,j] = \langle V_\kappa \operatorname{div}_\Gamma \boldsymbol{\lambda}_j, \operatorname{div}_\Gamma \boldsymbol{\lambda}_i \rangle_\Gamma.$$

It is obvious that when κ is small the second part of the single layer matrix is dominating. The matrix W has a large kernel. For simple connected domains he kernel can be described very well:

$$\ker(W_{\kappa,h}) = \operatorname{curl}_\Gamma(S_h^1(\Gamma)).$$

This effect is illustrated in Figure 6.2, where we see a plot of the eigenvalues of the boundary element matrices $A_{\kappa,h}, V_{\kappa,h}, S_{\kappa,h}$ and $F_{\kappa,h}$. We observe that for small κ the spectrum of the matrix $S_{\kappa,h}$ almost coincides with the spectrum of the matrix

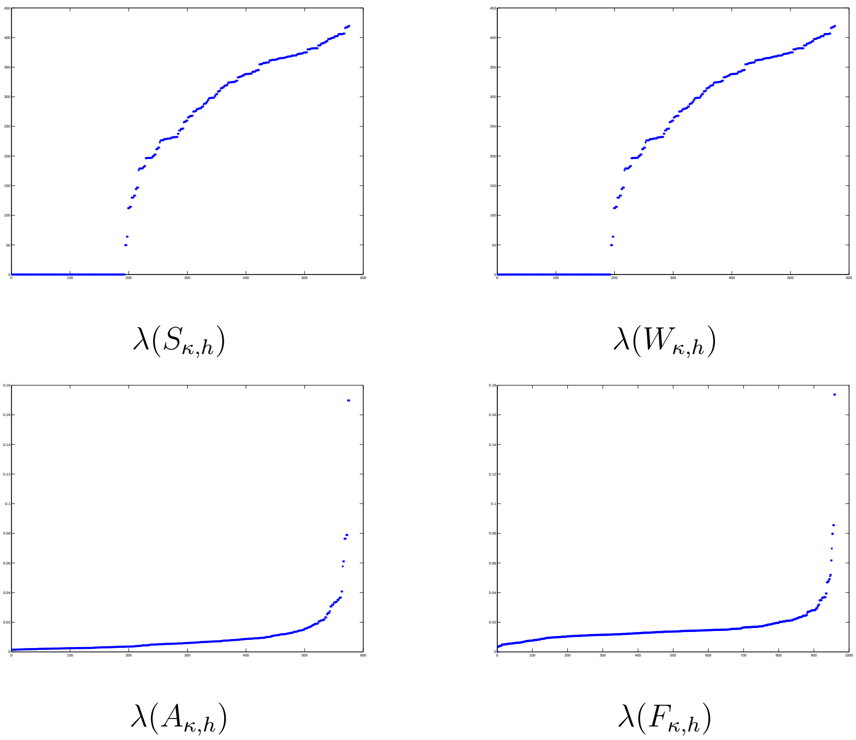


Figure 6.2: Eigenvalues of the matrices $A_{\kappa,h}, W_{\kappa,h}, S_{\kappa,h}$ and $F_{\kappa,h}$ for $\kappa = 0.1(1+i)$ for $\Omega = (0, 0.5)^3$ with 384 boundary elements.

$V_{\kappa,h}$. The matrix has a kernel, whose size corresponds to the number of nodes of the boundary element mesh minus one. This kernel influences the condition number of the matrix $S_{\kappa,h}$. In Figure 6.2 we also see that the spectrum of the matrix $F_{\kappa,h}$ is not affected by the kernel of the matrix $V_{\kappa,h}$.
To find a remedy for this problem let us recall the stabilized ansatz from Section 4.6.1:

$$\begin{pmatrix} A_\kappa & \nabla_\Gamma V_\kappa \\ V_\kappa \operatorname{div}_\Gamma & \kappa^2 V_\kappa \end{pmatrix} \begin{pmatrix} \gamma_N^{\text{int}} \mathbf{E} \\ \gamma_n^{\text{int}} \mathbf{E} \end{pmatrix} = \begin{pmatrix} (\frac{1}{2} I - C_\kappa) \mathbf{F} \\ 0 \end{pmatrix}. \tag{6.6}$$

The bi-linear form of the stabilized system (6.6) is not elliptic, therefore we cannot apply Cea's lemma, but we can use the theorem of Brezzi for the discrete case [10]. An important result, which is needed is the following discrete inf-sup condition:

Lemma 6.4. *For $\kappa = 0$ or $\kappa \in \mathbb{C}$ with $\Re(\kappa) > 0$, there exists a constant $c > 0$ such that*

$$\sup_{0 \neq \boldsymbol{\lambda}_h \in \mathcal{F}_h(\Gamma)} \frac{\langle V_\kappa \operatorname{div}_\Gamma \boldsymbol{\lambda}_h, \phi_h \rangle_\Gamma}{\|\boldsymbol{\lambda}\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma, \Gamma)}} \geq c \|\phi_h\|_{H^{-1/2}(\Gamma)}, \quad \forall \phi_h \in S_h^0(\Gamma). \tag{6.7}$$

For a proof of Lemma 6.4 we refer to [9, 31]. With the discrete inf-sup we are now ready to prove the stability of the discrete variational problem and an a priori error bound:

Proposition 6.5. *For* $\kappa \in \mathbb{C}$ *with* $\Re(\kappa) > 0$ *the discrete variational problem: Find* $(\boldsymbol{\lambda}_h, \phi_h) \in \mathcal{F}_h(\Gamma) \times S_h^0(\Gamma)$ *such that*

$$\langle A_\kappa \boldsymbol{\lambda}_h, \boldsymbol{\mu}_h\rangle_\Gamma + \langle V_\kappa \phi_h, \operatorname{div}_\Gamma \boldsymbol{\mu}\rangle_\Gamma = \langle(\frac{1}{2}I + C_\kappa)\boldsymbol{F}_h, \boldsymbol{\mu}_h\rangle_\Gamma \tag{6.8}$$

$$\langle V_\kappa \operatorname{div}_\Gamma \boldsymbol{\lambda}_h, \psi_h\rangle_\Gamma + \kappa^2 \langle V_\kappa \phi_h, \psi_h\rangle_\Gamma = 0 \tag{6.9}$$

is satisfied for all $(\boldsymbol{\mu}_h, \psi_h) \in \mathcal{F}_h(\Gamma) \times S_h^0(\Gamma)$*, has a unique solution, and we have the quasi optimal estimate*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} + \|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \leq$$
$$c\left(\inf_{\boldsymbol{\mu}_h \in \mathcal{F}_h(\Gamma)} \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{\mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma,\Gamma)} + \inf_{\eta_h \in S_h^0(\Gamma)} \|\phi - \eta_h\|_{H^{-1/2}(\Gamma)}\right).$$

Proof. For the proof we are going to apply Proposition 2.11 in [10]. For this proposition it is required that the bi-linear forms

$$\langle V_\kappa ., .\rangle_\Gamma : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{C}$$
$$\langle A_\kappa ., .\rangle_\Gamma : \mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \times \mathbf{H}_\|^{-1/2}(\operatorname{div}_\Gamma, \Gamma) \to \mathbb{C}$$

are positive definite. In the case of complex valued bi-linear forms this means that its real part is greater or equal zero. The bi-linear form $\langle V_\kappa ., .\rangle_\Gamma$ this is the case due to Lemma 4.30. The bi-linear form $\langle A_\kappa ., .\rangle_\Gamma$ is also positive definite since we have that

$$\Re(\langle A_\kappa \boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle_\Gamma = \sum_{i=1}^{3} \langle V_\kappa \lambda_i, \lambda_i\rangle_\Gamma.$$

The proof follows then by applying Proposition 2.11 in [10] together with Lemma 6.4. □

For the case $\kappa = 0$ we have a similar result:

Proposition 6.6. *The discrete variational problem: Find* $(\boldsymbol{\lambda}_h, \phi_h) \in \mathcal{F}_h(\Gamma) \times (S_h^0(\Gamma) \cap H_{**}^{-1/2}(\Gamma))$ *such that*

$$\langle A_0 \boldsymbol{\lambda}_h, \boldsymbol{\mu}_h\rangle_\Gamma + \langle V_0 \phi_h, \operatorname{div}_\Gamma \boldsymbol{\mu}\rangle_\Gamma = \langle(\frac{1}{2}I + C_0)\boldsymbol{F}_h, \boldsymbol{\mu}_h\rangle_\Gamma \tag{6.10}$$

$$\langle V_0 \operatorname{div}_\Gamma \boldsymbol{\lambda}_h, \psi_h\rangle_\Gamma = 0 \tag{6.11}$$

is satisfied for all $(\boldsymbol{\mu}_h, \psi_h) \in \mathcal{F}_h(\Gamma) \times (S_h^0(\Gamma) \cap H_{**}^{-1/2}(\Gamma))$*, has a unique solution, and we have the quasi optimal estimate*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} + \|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \leq$$
$$c\left(\inf_{\boldsymbol{\mu}_h \in \mathcal{F}_h(\Gamma)} \|\boldsymbol{\lambda} - \boldsymbol{\mu}_h\|_{\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} + \inf_{\eta_h \in S_h^0(\Gamma)} \|\phi - \eta_h\|_{H^{-1/2}(\Gamma)}\right).$$

Since the discrete inf-sup condition (6.4) is also fulfilled for the case $\kappa = 0$, the proof follows by applying the discrete version of Brezzi's theorem given in [10]. Together with the approximation properties from Theorem 6.2 and 6.3 we can obtain an error bound in dependence of the mesh width h. The convergence order, which we get from this error bound depends on the regularity of the solution. The regularity is influenced by the properties of the domain, in particular solutions for domains with edges and corners can have very little regularity, results on the regularity of solutions of the eddy current problem can be found in [20, 21].
The discrete variational problem (6.8)-(6.9) can be rewritten as the following linear system

$$\underbrace{\begin{pmatrix} A_{\kappa,h} & B_{\kappa,h} \\ B_{\kappa,h}^\top & \kappa^2 V_{\kappa,h} \end{pmatrix}}_{F_{\kappa,h}} \begin{pmatrix} \underline{\lambda} \\ \underline{\phi} \end{pmatrix} = \begin{pmatrix} (-\frac{1}{2}M_h + C_{\kappa,h})\underline{F} \\ 0 \end{pmatrix},$$

where the system matrix $F_{\kappa,h}$ is invertible due to Proposition 6.5. In what follows we present numerical examples, which illustrate the theoretical results for the behaviour of the stabilized ansatz for small κ. In Table 6.3 we have noted the number of GMRES iterations for the stabilized ansatz and for the standard boundary integral equation (6.3). We observe that for the stabilized ansatz the number of GMRES iterations is significantly lower, for the third and fourth level the GMRES method for the single layer potential S_κ did not converge anymore.

6.3 Eddy Current Model

In this section we deal with the numerical solution of eddy current problems using boundary element methods. We are going to discretize the boundary integral formulations for the eddy current model as introduced in Section 5. We compare those formulations with respect to the conditioning of the system matrix. At first we are going to consider the **E**-field formulation:

edges	It	It S_κ	error$_1$	eoc	error$_2$	eoc
120	46	82	3.325e-01	-	5.167e-02	-
480	65	178	1.850e-01	0.8	7.970e-03	2.7
1920	87	not conv.	9.528e-02	1.0	1.853e-03	2.1
7680	138	not conv.	4.820e-02	1.0	4.566e-04	2.0

Table 6.3: Results for the stabilized system for a ball $\Omega = B_1(0)$ with $\kappa = -0.01 - 0.01i$, the source point $x_s = (1.3, 1.0, 1.1)^\top$, and the evaluation point $x^* = (0.2, 0.2, 0.2)^\top$ and the analytic solution (6.5). In the second column we find the GMRES-Iteration numbers for the stabilized system, in the third column we find the GMRES-Iteration numbers for the Maxwell single layer potential S_κ.

edges	It	It S_κ	error$_1$	eoc	error$_2$	eoc
36	23	11	8.049e-01	-	1.005e+00	-
114	66	39	4.266e-01	0.9	2.687e-01	1.9
576	89	72	2.135e-01	1.0	5.975e-02	2.2
2304	122	129	1.058e-01	1.0	1.404e-02	2.1
9216	198	243	5.275e-02	1.0	3.452e-03	2.0

Table 6.4: Results for the stabilized system for a cube $\Omega = (0.5, 0.5, 0.5)^3$ with $\kappa = -5 - 5i$, the source point $x_s = (1.3, 1.0, 1.1)^\top$, and the evaluation point $x^* = (0.2, 0.2, 0.2)^\top$ and the analytic solution (6.5). In the second column we find the GMRES-Iteration numbers for the stabilized system, in the third column we find the GMRES-Iteration numbers for the Maxwell single layer potential S_κ.

6.3.1 A boundary element method based on the E-field

The discretization of the continuous variational problem (5.26)-(5.31) using the boundary element spaces $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ leads to the following discrete variational problem:
Find $\boldsymbol{\lambda}_{i,h} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_{\Gamma_i \cap \Gamma}\, 0, \Gamma_i) \cap \mathcal{F}_h(\Gamma_i)$ and $\mathbf{u}_h \in \mathcal{E}_h(\Gamma)$ such that

$$\sum_{i=0}^{M} \left(\frac{1}{\mu_i} \langle N_\kappa^i \mathbf{u}_h|_{\Gamma_i}, \mathbf{v}_h|_{\Gamma_i} \rangle_{\Gamma_i} + \langle B_\kappa^i \boldsymbol{\lambda}_{i,h}, \mathbf{v}_h|_{\Gamma_i} \rangle_{\Gamma_i} \right) = f(\mathbf{v}_h|_\Gamma), \tag{6.12}$$

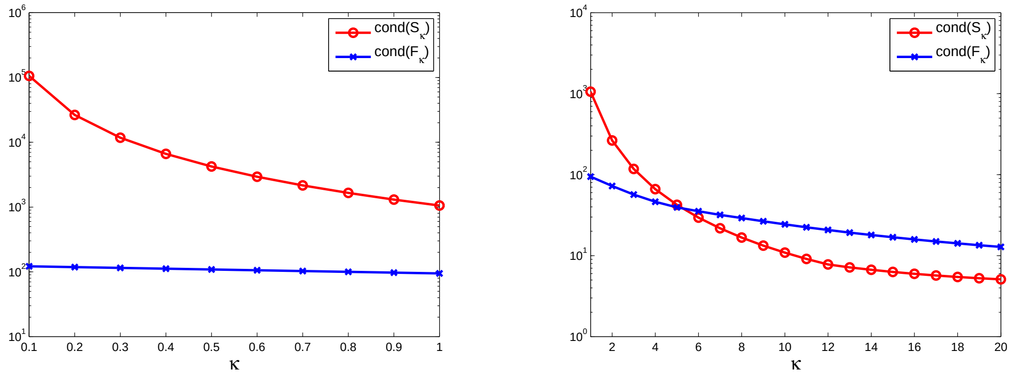


Figure 6.3: Plot of the condition number against κ of the matrices $S_{\kappa,h}$ and $F_{\kappa,h}$ for $\Omega = (0, 0.5)^3$ with a surface mesh with 384 boundary elements and 576 edges.

	stabilized ansatz			direct ansatz	
κ	It	$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{L^2(\Gamma)}$	$\|t - t_h\|_{L^2(\Gamma)}$	It	$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{L^2(\Gamma)}$
i	98	8.389e-02	3.511e-02	135	8.389e-02
1e-1i	98	9.814e-02	4.119e-02	196	3.910e+03
1e-2i	98	9.841e-02	4.132e-02	243	4.097e+02
1e-3i	98	9.841e-02	4.132e-02	301	6.003e+09
1e-4i	98	9.841e-02	4.132e-02	-	-

Table 6.5: Results for the stabilized and direct ansatz for $\Omega = (0, 0.5)^3$ and a discretization containing 576 edges and 384 elements. The wave number κ varies from i to $1e - 4i$.

$$\sum_{i=0}^{M} \left(\langle C_\kappa^i \mathbf{u}_h|_{\Gamma_i}, \boldsymbol{\mu}_{i,h} \rangle_{\Gamma_i} + \mu_i \langle S_\kappa^i \boldsymbol{\lambda}_{i,h}, \boldsymbol{\mu}_{i,h} \rangle_{\Gamma_i} \right) = g(\boldsymbol{\mu}_{0,h}) \tag{6.13}$$

is satisfied for all $\boldsymbol{\mu}_{i,h} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma_i \cap \Gamma} 0, \Gamma_i) \cap \mathcal{F}_h(\Gamma_i)$ and $\mathbf{v}_h \in \mathcal{E}_h(\Gamma)$.
The right hand side functionals are given by (5.28)-(5.29). The unique solvability of the discrete variational problem follows immediately from the ellipticity of the continuous variational problem by using Cea's lemma. For the implementation of the presented boundary element method we need to discretize the space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_{\Gamma_i \cap \Gamma} 0, \Gamma_i) \cap \mathcal{F}_h$. In Chapter 5 we gave two possibilities for the implementation, we will now discuss the realization in a boundary element method:

Method 1:
In Method 1 we will incorporate the condition $\mathrm{div}_\Gamma \boldsymbol{\lambda}_h = 0$ by an explicit represen-

tation of $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma) \cap \mathcal{F}_h(\Gamma)$:
Let us assume that Ω is simply connected and that the material parameter κ is constant in Ω, i.e. we have only one subdomain Ω. Then we have that

$$\mathcal{F}_h(\Gamma) \cap \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma) = \mathbf{curl}_\Gamma(S_h^1(\Gamma)) \tag{6.14}$$

holds and thus $\boldsymbol{\lambda} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$ can be represented as

$$\boldsymbol{\lambda}_h(x) = \mathbf{curl}_\Gamma \sum_{i=1}^{M} \alpha_i \phi_i^1(x).$$

This means that the degrees of freedom are now given by the nodes of the surface mesh and not by the edges of the surface mesh. In the engineering literature the functions in the space $\mathbf{curl}_\Gamma(S_h^1(\Gamma))$ are also known as loop currents. The discrete variational problem reads:
Find $(\mathbf{u}_h, \phi_h) \in \mathcal{E}_h(\Gamma) \times S_h^1(\Gamma)$ such that

$$\begin{aligned} \langle (\frac{1}{\mu_r} N_\kappa + N_0)\mathbf{u}_h, \mathbf{v}_h \rangle_\Gamma + \langle (B_\kappa + B_0)\mathbf{curl}_\Gamma \phi_h, \mathbf{v}_h \rangle_\Gamma &= f(\mathbf{v}_h), \\ \langle (C_\kappa + C_0)\mathbf{u}_h, \mathbf{curl}_\Gamma \psi_h \rangle_\Gamma + \langle (\mu_r A_\kappa + A_0)\mathbf{curl}_\Gamma \phi_h, \mathbf{curl}_\Gamma \psi_h \rangle_\Gamma &= g(\mathbf{curl}_\Gamma \psi_h) \end{aligned}$$

holds for all $(\mathbf{v}_h, \psi_h) \in \mathcal{E}_h(\Gamma) \times S_h^1(\Gamma)$ with the right hand side given by

$$\begin{aligned} f(\mathbf{v}_h) &= \langle (\frac{1}{2} I + B_0)\boldsymbol{\gamma}_N \mathbf{E}_p, \mathbf{v}_h \rangle_\Gamma + \langle N_0 \boldsymbol{\gamma}_t \mathbf{E}_p, \mathbf{v}_h \rangle_\Gamma, \\ g(\mathbf{curl}_\Gamma \psi_h) &= \langle S_0 \boldsymbol{\gamma}_N \mathbf{E}_p, \mathbf{curl}_\Gamma \psi_h \rangle_\Gamma + \langle (\frac{1}{2} I + C_0)\boldsymbol{\gamma}_t \mathbf{E}_p, \mathbf{curl}_\Gamma \psi_h \rangle_\Gamma. \end{aligned}$$

The corresponding linear system then reads

$$\begin{pmatrix} \frac{1}{\mu_r}(N_{\kappa,h} + N_{0,h}) & -(\tilde{C}_{\kappa,h} + \tilde{C}_{0,h})^\top \\ \tilde{C}_{\kappa,h} + \tilde{C}_{0,h} & \mu_r(\tilde{D}_{\kappa,h} + D_{0,h}) \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{\phi} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix} \tag{6.15}$$

with the matrices

$$\begin{aligned} \tilde{C}_{0,h}[i,j] &= \langle C_0 \mathbf{u}_j, \mathbf{curl}_\Gamma \phi_i^1 \rangle_\Gamma,\ i = 1, \ldots, N^n, j = 1, \ldots N^e, \\ \tilde{C}_{\kappa,h}[i,j] &= \langle C_\kappa \mathbf{u}_j, \mathbf{curl}_\Gamma \phi_i^1 \rangle_\Gamma,\ i = 1, \ldots, N^n,\ j = 1, \ldots N^e, \end{aligned}$$

and

$$D_{0,h}[i,j] = \langle A_0 \mathbf{curl}_\Gamma \phi_j^1, \phi_i^1 \rangle_\Gamma, \quad D_{\kappa,h}[i,j] = \langle A_\kappa \mathbf{curl}_\Gamma \phi_j^1, \phi_i^1 \rangle_\Gamma, \quad i,j = 1, \ldots, N^n.$$

The matrix $D_{0,h}$ corresponds to the boundary element matrix of the hypersingular operator for the Laplace equation.

The drawback of this method is that it is difficult to implement if the unknown $\boldsymbol{\lambda}$ is required to be solenoidal only on a part of the surface mesh. This would be the case if we deal with a conducting domain with piecewise constant material properties. This approach can also be extended to the case of multiple connected domains, however in this case we have to have some knowledge about the topology of the conducting domain and we have to introduce 'cutting surfaces' (see [32]).

Method 2:
To enforce the condition $\operatorname{div}_\Gamma \boldsymbol{\lambda}_h = 0$ we follow Method 2 as described in Section 5.2.1. At first we assume that κ is constant in Ω. The discretization of (5.32)-(5.34) leads to a system of the type

$$\begin{pmatrix} \mu_r(A_{\kappa,h}+A_{0,h}) & C_{\kappa,h}+C_{0,h} & -\nabla_\Gamma(G_{\kappa,h}+G_{0,h}) \\ -(C_{\kappa,h}+C_{0,h})^\top & \frac{1}{\mu_r}(N_{\kappa,h}+N_{0,h}) & 0 \\ (G_{\kappa,h}+G_{0,h})^\top & 0 & S \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \\ 0 \end{pmatrix}$$

with the matrices

$$G_{\kappa,h}[i,j] = \langle \nabla_\Gamma V_\kappa \phi_j^0, \boldsymbol{\lambda}_i \rangle_\Gamma, \quad G_{0,h}[i,j] = \langle \nabla_\Gamma V_0 \phi_j^0, \boldsymbol{\lambda}_i \rangle_\Gamma$$

and the stabilization

$$S[i,j] = \langle 1, \phi_i^1 \rangle_\Gamma \langle 1, \phi_j^1 \rangle_\Gamma .$$

Let us now consider the case that κ is piecewise constant on a domain decomposition (5.1) of Ω. In this case the linear system is given by

$$\begin{pmatrix} \frac{1}{\mu_r}N_{\kappa,h} & -[C_h^0]^\top & -[C_h^1]^\top & \dots & -[C_h^M]^\top & \\ C_h^0 & \mu_r A_h^0 & -G_h^0 & & & \\ & [G_h^0]^\top & S_h^0 & & & \\ C_h^1 & & \mu_r A_h^1 & -G_h^1 & & \\ & & [G_h^1]^\top & S_h^1 & & \\ \vdots & & & \ddots & & \\ C_h^M & & & & \mu_r A_h^M & -G_h^M \\ & & & & [G_h^M]^\top & S_h^M \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{\lambda}_1 \\ \underline{p}_1 \\ \underline{\lambda}_2 \\ \underline{p}_2 \\ \vdots \\ \underline{\lambda}_M \\ \underline{p}_M \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

For the implementation we need an operator, which maps the local degrees of freedom of $\mathbf{u}_h|_{\Gamma_i}$ to the global vector $\mathbf{u}_h$.
This approach has been followed by Breuer in his thesis [9]. The advantage is that not simply connected domains can be treated in an easy way, the drawback is that this approach results in a larger linear system, which needs more GMRES-iterations as examples show (cf. Table 6.7).

6.3.2 A boundary element method based on the H-field

The **H**-field formulation leads to the discrete variational problem:
Find $((\mathbf{u}_h, \phi_h), \boldsymbol{\lambda}_{i,h}, t_h) \in \mathcal{V}_{0,h} \times \mathcal{E}_h(\Gamma_i) \times S_h^0(\Gamma)$ such that

$$n((\mathbf{u}_h, \phi_h), (\mathbf{v}_h, \psi_h)) + \sum_{i=1}^{M} \langle (\frac{1}{2} I + B_\kappa^i) \boldsymbol{\lambda}_{i,h}, \mathbf{v}_h|_{\Gamma_i} \rangle_\Gamma - \langle (\frac{1}{2} I - K_0') t_h, \psi_h \rangle_\Gamma = f(\mathbf{v}_h), \tag{6.16}$$

$$\sum_{i=1}^{M} \langle (-\frac{1}{2} I + C_\kappa^i) \mathbf{u}_h|_{\Gamma_i}, \boldsymbol{\mu}_{i,h} \rangle_\Gamma + \sum_{i=1}^{M} \langle \tilde{\varepsilon}_i S_\kappa^i \boldsymbol{\lambda}_{i,h}, \boldsymbol{\mu}_{i,h} \rangle_\Gamma = \sum_{i=1}^{M} g(\boldsymbol{\mu}_{i,h}), \tag{6.17}$$

$$\langle (\frac{1}{2} I - K_0) \phi_h, p_h \rangle_\Gamma + \frac{1}{\omega^2 \mu_0} \langle V_0 t_h, p_h \rangle_\Gamma = 0 \tag{6.18}$$

holds for all $((\mathbf{v}_h, \psi_h), \boldsymbol{\mu}_{i,h}, p_h) \in \mathcal{V}_{0,h} \times \mathcal{E}_h(\Gamma_i) \times S_h^0(\Gamma)$ with

$$n((\tilde{\mathbf{u}}_h, \phi_h), (\mathbf{v}_h, \psi_h)) = \sum_{i=1}^{M} \frac{1}{\tilde{\varepsilon}_i} \langle N_\kappa^i \tilde{\mathbf{u}}_h|_{\Gamma_i}, \mathbf{v}_h|_{\Gamma_i} \rangle_{\Gamma_i} + \omega^2 \mu_0 \langle D_0 \phi_h, \psi_h \rangle_\Gamma.$$

The discrete space $\mathcal{V}_{0,h}$ is defined by

$$\mathcal{V}_{0,h} = \{ (\mathbf{v}_h, \psi_h) \in \mathcal{F}_h(\Gamma_s) \times S_h^1(\Gamma) : \ \mathbf{v}_h|_\Gamma = \nabla_\Gamma \psi_h \}. \tag{6.19}$$

This means that on the outer boundary Γ the degrees of freedom of $\mathbf{u}$ can be represented as nodal degrees of freedom.
The discrete variational formulation for the direct **H**-field formulation leads to a linear system of equations, which has a two-fold saddle point structure. Let us assume that κ is constant in Ω, in this case we can replace the unknown $\mathbf{u}$ by $\nabla_\Gamma \phi$. The discrete variational problem (6.16)-(6.18) then leads to the linear system

$$\begin{pmatrix} \omega^2(\mu D_{\kappa,h} + \mu_0 D_{0,h}) & \frac{1}{2} M_h^\top - C_{\kappa,h}^\top & -\frac{1}{2} M_h + K_{0,h}^\top \\ -\frac{1}{2} M_h + C_{\kappa,h} & \tilde{\varepsilon} S_{\kappa,h} & 0 \\ \frac{1}{2} M_h - K_{0,h} & 0 & \frac{1}{\omega^2 \mu_0} V_{0,h} \end{pmatrix} \begin{pmatrix} \underline{\phi} \\ \underline{\lambda} \\ \underline{t} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \\ \underline{0} \end{pmatrix}$$

with the matrices

$$D_{\kappa,h}[i,j] = \langle A_\kappa \mathbf{curl}_\Gamma \phi_j^1, \mathbf{curl}_\Gamma \phi_i^1 \rangle_\Gamma,$$
$$D_{0,h}[i,j] = \langle A_0 \mathbf{curl}_\Gamma \phi_j^1, \mathbf{curl}_\Gamma \phi_i^1 \rangle_\Gamma, \quad i,j = 1, \ldots, N^n$$

and

$$K_{0,h}[i,j] = \langle K_0 \phi_j^1, \phi_i^0 \rangle_\Gamma, \quad i = 1, \ldots, N, j = 1, \ldots, N^n.$$

6.3.3 An indirect boundary element method based on the H-field

In Chapter 5 we have presented an indirect formulation, which is based on the **H**-field, for the case that all material parameters are constant in the conducting domain. We now replace the continuous variational problem (5.55)-(5.56) by the following discrete variational problem:
Find $(\boldsymbol{\lambda}, \sigma) \in \mathcal{F}_h(\Gamma) \times S_h^1(\Gamma)$ such that

$$\langle S_\kappa \boldsymbol{\lambda}_h, \boldsymbol{\mu}_h\rangle_\Gamma + \langle(-\frac{1}{2}I + C_0)\nabla_\Gamma \phi_h, \boldsymbol{\mu}_h\rangle_\Gamma = \langle \gamma_t \mathbf{H}_p, \boldsymbol{\mu}\rangle_\Gamma \quad (6.20)$$

$$-\langle \boldsymbol{\lambda}_h, (-\frac{1}{2}I + C_\kappa)\nabla_\Gamma \psi_h\rangle_\Gamma + \frac{\kappa^2}{\mu_r}\langle D_0 \phi_h, \psi_h\rangle_\Gamma = \frac{\kappa^2}{\mu_r}\langle \gamma_n \mathbf{H}_p, \psi\rangle_\Gamma \quad (6.21)$$

is satisfied for all $(\boldsymbol{\mu}_h, \psi_h) \in \mathcal{F}_h(\Gamma) \times S_h^1(\Gamma)$.
The corresponding linear system of equations has the following saddle point structure

$$\begin{pmatrix} S_{\kappa,h} & C_{0,h} \\ C_{\kappa,h}^\top & \frac{\kappa^2}{\mu_r} D_{0,h} \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \underline{\phi} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}$$

with the matrices

$$S_{\kappa,h}[i,j] = \langle S_\kappa \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_i\rangle_\Gamma, \quad i,j = 1,\dots,N^e$$
$$C_{\kappa,h}[i,j] = \langle(-\frac{1}{2}I + C_\kappa)\nabla_\Gamma \phi_j^1, \boldsymbol{\lambda}_i\rangle_\Gamma, \quad i = 1,\dots,N^e, j = 1,\dots,N^n$$
$$D_{0,h}[i,j] = \langle D_0 \phi_j^1, \phi_i^1\rangle_\Gamma, \quad i,j = 1,\dots,N^n$$

and the right hand side vectors

$$\underline{f}[i] = \langle \boldsymbol{\gamma}_t \mathbf{H}_p, \boldsymbol{\lambda}_i\rangle, \quad i = 1,\dots,N^e$$
$$\underline{g}[i] = \frac{\kappa^2}{\mu_r}\langle \gamma_n \mathbf{H}_p, \phi_i^1\rangle, \quad i = 1,\dots N^n.$$

As this is an indirect approach, the discrete solution of the linear system $\boldsymbol{\lambda}_h$ and ϕ_h have no physical meaning in general. However, by inserting those functions into the representation formula, we can compute the magnetic field inside the conducting domain

$$\mathbf{B}(x) = \mu \boldsymbol{\Psi}_{\mathrm{SL}}^\kappa(\boldsymbol{\lambda}_h)(x), \quad x \in \Omega,$$

and in the non-conducting domain

$$\mathbf{B}_s(x) = \mu_0 \nabla \Psi_{\mathrm{DL}}^0(\phi_h)(x), \quad x \in \Omega^c.$$

The gradient of the double layer potential can be computed by applying the Maxwell double layer potential to the surface gradient of ϕ_h due to Lemma 4.17, hence we get

$$\mathbf{B}_s(x) = \mu_0 \boldsymbol{\Psi}_{\mathrm{DL}}^0(\nabla_\Gamma \phi_h)(x), \quad x \in \Omega^c.$$

6.3.4 Numerical examples

We will present two numerical examples. In the first numerical example we consider a conducting ball with a wire around it. In the wire we have an impressed time harmonic current, which generates the primary magnetic field. For this particular problem there exists an analytic solution of the eddy current model, which was derived in [54]. The electric field is then given by

$$\mathbf{E}(x) = \mathbf{E}_\phi(r,\theta)\mathbf{e}_\phi \tag{6.22}$$

where $\mathbf{E}_\phi(r,\theta)$ is given by

$$\mathbf{E}_\phi(r,\theta) = \begin{cases} \sum_{n=0}^{\infty} \mathbf{E}_n^{\mathrm{ext}} r^{-1/2} J_{2n+\frac{3}{2}}(ir\sqrt{ik}) P_{2n+1}^1(\cos(\theta)), & r \le a, \\ \sum_{n=0}^{\infty} \mathbf{E}_n^{\mathrm{int}} r^{-2n-2} P_{2n+1}^1(\cos(\theta)) - \mathbf{E}_n r^{2n+1} P_{2n+1}^1(\cos(\theta)), & r > a. \end{cases}$$

The constants can be computed by

$$\mathbf{E}_n = \frac{\mu_0(-1)^n(2n-1)!!}{(2b)^{2n+2}(n+1)!}, \quad \mathbf{E}_n^{\mathrm{int}} = \frac{\mathbf{E}_n^{\mathrm{ext}} a^{-2n-\frac{3}{2}} - \mathbf{E}_n a^{-2n-\frac{3}{2}}}{J_{2n+\frac{3}{2}}(ia\sqrt{ik})},$$

$$\mathbf{E}_n^{\mathrm{ext}} = \mathbf{E}_n a^{4n+3} \frac{ai\sqrt{ik}\frac{J_{2n+\frac{1}{2}}(ia\sqrt{ik})}{J_{2n+\frac{3}{2}}(ia\sqrt{ik})} - 2n - 1 - \mu_r(2n+2)}{ai\sqrt{ik}\frac{J_{2n+\frac{1}{2}}(ia\sqrt{ik})}{J_{2n+\frac{3}{2}}(ia\sqrt{ik})} - 2n - 1 + \mu_r(2n+1)},$$

where a denotes the radius of the sphere, b stands for the radius of the coil. We compare the boundary element solution with the analytic solution.
In the second example we consider a conducting plate, in the front of this plate we place a coil with an impressed time harmonic current, which generates the primary magentic field. For this problem setting we don't have an analytic solution at hand, we will compare the number of GMRES iterations of the presentend boundary element formulations.

Conducting Sphere

Let us now look at the first example. The conducting domain is the ball $\Omega = B_1(0)$. Around the sphere we have a wire, the center of the wire is in the origin and the radius of the wire is 1.5. For the following numerical example we assume $\sigma = 0.1$ and $\mu = \mu_0$. In Table 6.6 we give the GMRES-iterations and the error as defined in (6.4). We observe that for low conductivities, i.e. for small κ the method tends to

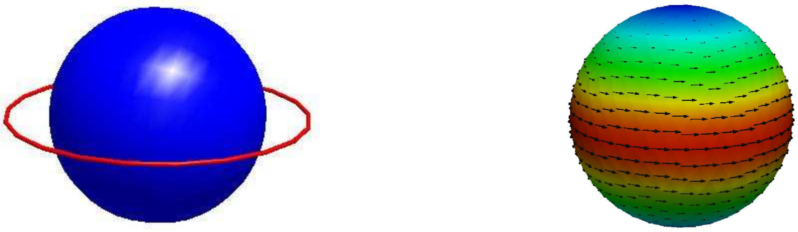


Figure 6.4: Conducting ball with a wire.

	$f = 1e6$		$f = 1e4$	
N	It	error	It	error
80	52	8.983e-2	58	5.801e-1
320	133	4.350e-2	166	5.773e-1
1280	329	2.150e-2	-	-

Table 6.6: Results for the E-field formulation - Method1 for $\Omega = B_1(0)$, for the case $f = 1e6$ we have $\kappa = 0.63(1+i)$ and for $f = 1e4$ we have $\kappa = 0.063(1+i)$.

be instable as the results do not converge.

We have now derived three formulations for the eddy current model, one was based on the **E**-field, the two other formulations were based on the **H**-field. In the **E**-field formulation we have two possibilities to deal with the space $\mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$, in one method we enforced the condition $\mathrm{div}_\Gamma \boldsymbol{\gamma}_N^{\mathrm{ext}} \mathbf{E} = 0$ by using Lagrange multipliers, in the other method we used an explicit representation of the space $\mathcal{F}_h(\Gamma) \cap \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma 0, \Gamma)$. In Table 6.7 we compare the formulations for the **E**-field and for the **H**-field.

	E-field Method 1		**E**-field Method 2		**H**-field	
edges	It	error_1	It	error_1	It	error_1
120	27	1.137e-01	101	1.138e-1	93	1.227e-01
489	68	5.656e-02	379	5.650e-2	256	5.676e-02
1920	164	2.823e-02	924	2.872e-2	657	2.823e-02

Table 6.7: Examples for the Benchmark problem, $\kappa = -5 - 5i$

Conducting Plate

Let us now consider Example 2, a conducting plate. In front of the conducting plate we place a coil with the center in $(0, 0.5, 1.5)^\top$ and the normal vector $(0, 0, 1)^\top$, the radius of the coil is $r = 0.2$ (cf. Figure 6.5). The conductivity of the plate is $\sigma_{\text{Plate}} = 1e7S/m$ and the frequency is 50 Hz. We start with a boundary element discretization with 798 boundary elements. Table 6.8 shows the GMRES iteration

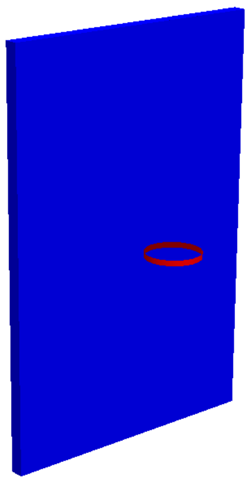
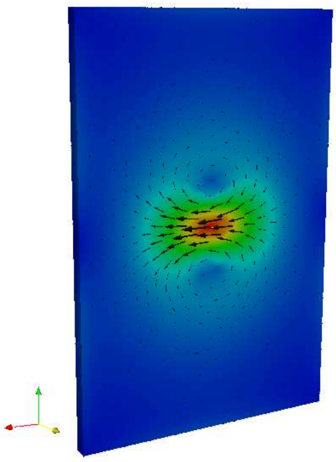


Figure 6.5: Conducting plate with coil.

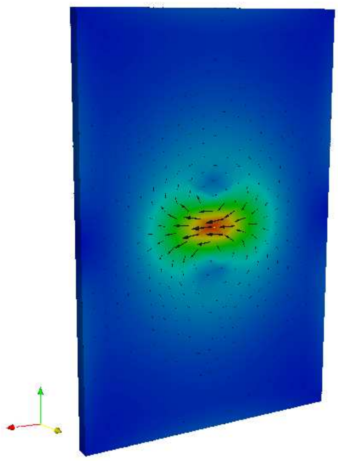
	E-field Method1	**E**-field Method2	**H**-field	**H**-field indirect
edges	It	It	It	It
1197	102	1028	658	314
4788	117	2092	1827	464

Table 6.8: GMRES Iterations for the solution of the eddy current problem for the conducting plate.

numbers for the solution of the linear systems for the conducting plate. It shows that for the **E**-field formulation Method 1 needs significantly less iterations than Method 2.



$\Re(\boldsymbol{\gamma}_t^{\text{int}}\mathbf{E})$



$\Re(\boldsymbol{\gamma}_N^{\text{int}}\mathbf{E})$

Figure 6.6: Real part of the tangential and Neumann trace of the electric field **E**.

6.4 The Reduced Model

For a symmetric boundary element discretization of the variational formulation (5.60) we introduce a sequence of admissible boundary element meshes $\Gamma_{i,h}$ for the domain decomposition (5.1) with a globally quasi uniform mesh size h. By $S_h^1(\Gamma_i) = S_h^1(\Gamma_s)_{|\Gamma_i}$ we denote the localized boundary element space of local basis functions ϕ_i^1, and by $\underline{\phi}_k = A_k\underline{\phi}$ we describe the localization of the global degrees of freedom. The symmetric boundary element approximation of the variational problem (5.60) results in the linear system, see, e.g. [65],

$$\sum_{k=1}^{p} \kappa_k A_k^\top S_{k,h} A_k \underline{\phi} = -i\omega \sum_{k=1}^{p} \kappa_k A_k^\top \underline{f}_k, \tag{6.23}$$

where

$$S_{k,h} = D_{k,h} + (\frac{1}{2}M_{k,h}^\top + K_{k,h}^\top)V_{k,h}^{-1}(\frac{1}{2}M_{k,h} + K_{k,h})$$

are the discrete Steklov–Poincaré operators. Note that

$$D_{k,h}[j,i] = \langle D_k\phi_i^1, \phi_j^1\rangle_{\Gamma_k}, \quad V_{k,h}[\ell,m] = \langle V_k\phi_m^0, \phi_\ell^0\rangle_{\Gamma_k},$$
$$K_{k,h}[\ell,i] = \langle K_k\phi_i^1, \phi_\ell^0\rangle_{\Gamma_k}, \quad M_{k,h}[\ell,i] = \langle \phi_i^1, \phi_\ell^0\rangle_{\Gamma_k}$$

are local boundary element matrices. Moreover, the right hand side in (6.23) is given locally as

$$f_{k,j} = \int_{\Gamma_k} [\mathbf{A}_k(x)\cdot n_k]\phi_j^1(x)ds_x.$$

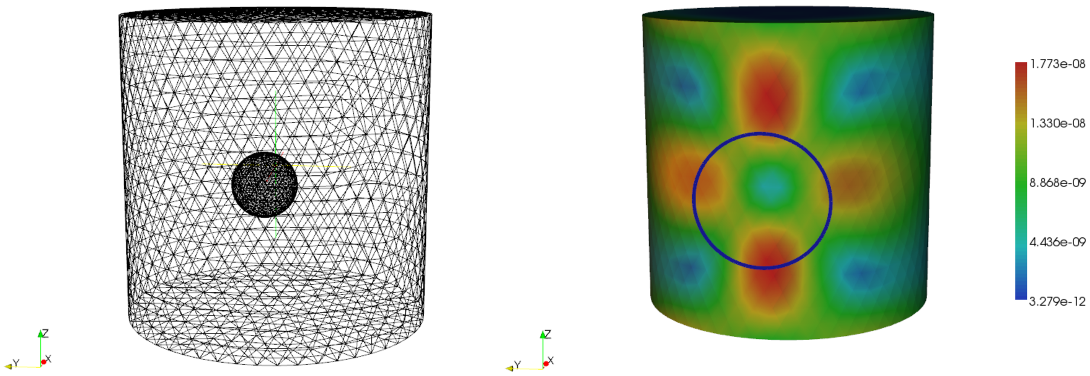
The stability and error analysis of the symmetric boundary element discretization of the variational problem (5.60) is well established, see, e.g. [65], and the references given therein.

6.4.1 Numerical results

As conducting domain we first consider the cylinder

$$\Omega = \{x \in \mathbb{R}^3,\, x_1^2 + x_2^2 < 0.1,\, 0 < x_3 < 0.2\},$$

where the transmitting coil is modeled as a current loop of radius 0.04 which is centered at $(-1.4, 0, 1)^\top$, see Fig. 6.7. The vector normal to the current loop points into the direction of the x_1–axis, i.e., $n = (1, 0, 0)^\top$. Inside the cylinder we place a ball with radius $r = 0.02$, whose center lies in the point $(-0.06, 0, 0.1)^\top$.

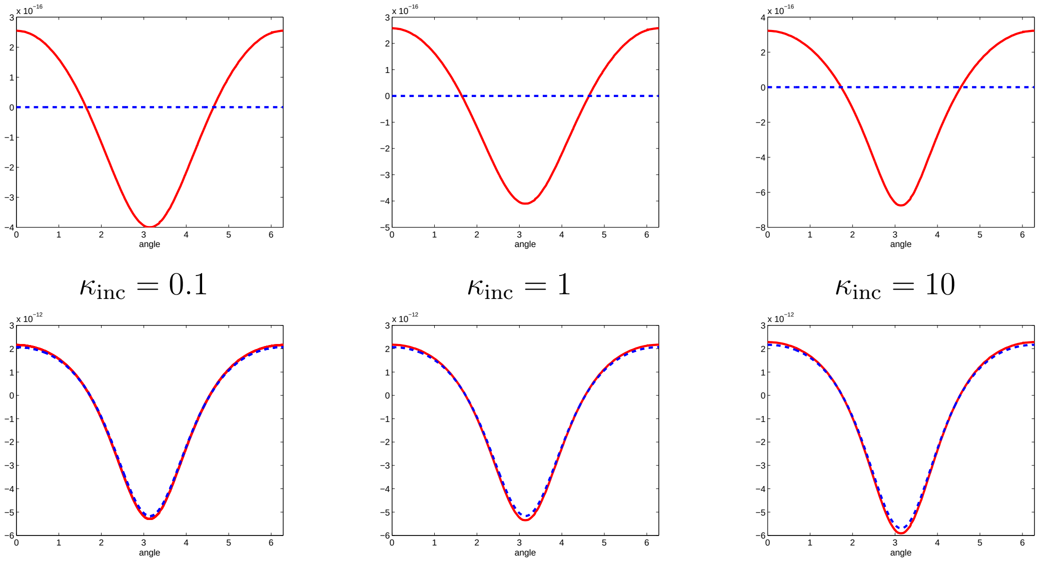


Cylinder mesh $\qquad\qquad$ $|n \times (\boldsymbol{E}|_\Gamma \times n)|]$

Figure 6.7: Mesh of the cylinder and the magnitude of the tangential electric field on Γ.

The background conductivity of the cylinder is $\kappa = 0.1$, and the conductivity of the inscribed ball is κ_{inc}. Fig. 6.7 shows the magnitude of the electric field $|n \times (\mathbf{E}|_\Gamma \times n)|$ for $\kappa_{\text{inc}} = 0.1$. In Fig. 6.8 we give a comparison of the reduced model with the full eddy current. For this we plot the real and imaginary part of the normal component of the magnetic field $\mathbf{B}(x) \cdot n(x)$ along a circle around the cylinder for the frequency $f = 100\text{kHz}$, and for varying conductivities $\kappa_{\text{inc}} \in \{0.1, 1, 10\}$. For the reduced model $\mathbf{B}(x) \cdot n(x)$ was computed by using the boundary element approach as described in the previous section. The solution of the full eddy current problem was computed by using the finite element software packages Netgen [62] and NGSolve [1].

For the reduced model we have $\Re(\mathbf{B}(x) \cdot n(x)) = 0$, while for the full eddy current model $\Re(\mathbf{B}(x) \cdot n(x))$ is comparable small. For the imaginary part we obtain a



$\kappa_{\text{inc}} = 0.1$ $\kappa_{\text{inc}} = 1$ $\kappa_{\text{inc}} = 10$

$\kappa_{\text{inc}} = 0.1$ $\kappa_{\text{inc}} = 1$ $\kappa_{\text{inc}} = 10$

Figure 6.8: Real (upper row) and imaginary (lower row) parts of $\mathbf{B}(x) \cdot n(x)$, $f = 10^5$.

very good coincidence between the solution of the reduced and of the full model. Indeed, in Fig. 6.9 we give a plot of the error and of the relative error in $x = (-0.141, -0.141, 0.15)^\top$ between the normal magnetic field computed with the full eddy current model and the reduced model in the case $\kappa_{\text{inc}} = 0.1$, and for a frequency range from 100kHz up to 1GHz.

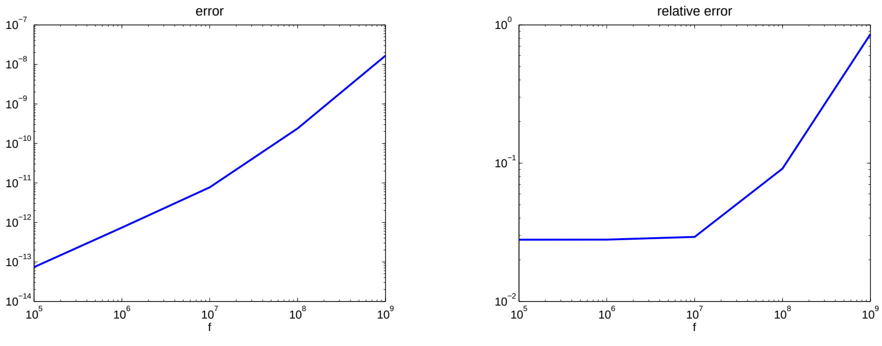
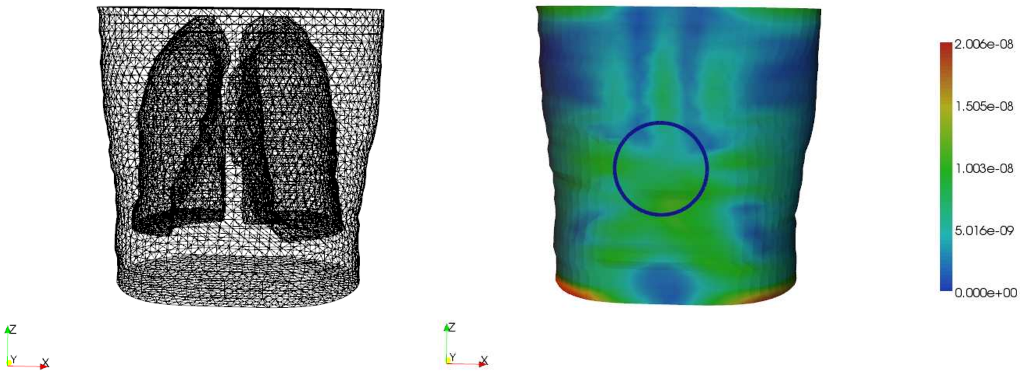


Figure 6.9: Absolute and relative pointwise error for different frequencies.

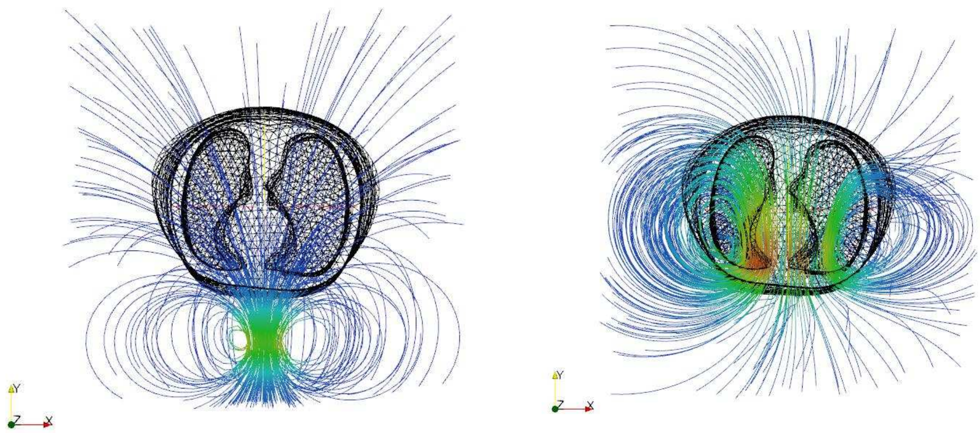
Based on the above results we conclude that the reduced modell describes an appropriate approximation of the full eddy current model as used in magnetic induction tomography models.

In a second example we consider the model of a human thorax with two lungs, see Fig. 6.10. The volume mesh consists of 83514 volume elements and 15641 volume nodes, while the boundary element mesh consists of 13076 boundary elements and 7548 boundary nodes. The background conductivtiy of the thorax was set to the conductivity of a muscle at 100kHz, i.e., $\kappa_{\text{muscle}} = 0.3618\text{S/m}$, while the conductivity of the lungs at 100kHz is $\kappa_{\text{lung}} = 0.2716\text{S/m}$. The center of the transmitting coil was placed in the point $(0, -0.2, 0)^\top$, the normal vector of the coil is given by $(0, 1, 0)^\top$, and its radius is 0.05. In Fig. 6.10 we plot the magnitude of the tangential trace of the electric field, i.e. $|n \times (\boldsymbol{E}|_\Gamma \times n)|$. The field lines of the primary magnetic field $\mathbf{B}_p$ of the secondary magnetic field $\mathbf{B}_s$ are given in Fig. 6.11.



Lung mesh $|n \times (\boldsymbol{E}|_\Gamma \times n)|$

Figure 6.10: Mesh of the thorax and lungs and the magnitude of the tangential electric field.



Primary field $\mathbf{B}_p$ Secondary field $\mathbf{B}_s$

Figure 6.11: Field lines of the primary and secondary magnetic fields.

7 THE INVERSE PROBLEM OF MAGNETIC INDUCTION TOMOGRAPHY

In this last chapter we deal with the inverse problem of Magnetic Induction Tomography. The forward problem of Magnetic Induction Tomography was described in Section 2.3. In principle there are two ways of dealing with the inverse problem. Either we can see the inverse problem as a parameter identification problem, where we get the distribution of certain parameters such as the complex conductivity in the case of MIT out of the solution of the inverse problem. The shape and position of objects inside the domain have to be reconstructed from the parameter distribution in a postprocessing step. The solution of the inverse problem using the parameter identification approach in combination with the finite element method was done in [29, 35].
The second method is to view the inverse problem as a shape reconstruction problem, where we reconstruct the shape of a certain object in the inverse problem solution procedure. For the shape reconstruction approach we have to assume that the material parameters of the object, which we want to reconstruct, are constant. For the shape reconstruction approach the boundary element method is suited very well, since no volume mesh is needed in the inverse solution process.
In this thesis we will concentrate on the shape reconstruction approach. Dealing with the inverse problem as a shape reconstruction problem requires the minimization of a cost functional, for the minimization we need to find a representation for the associated shape derivative [23, 64]. We define the cost functional for Magnetic Induction Tomography and compute its shape derivative for the reduced model.

7.1 Shape Reconstruction

Our aim is to reconstruct the shape of a hidden domain Ω_1 inside a domain Ω with a given, fixed boundary Γ. We denote the boundary of Ω_1 by Σ. We set $\Omega_0 = \Omega \setminus \overline{\Omega_1}$. We further assume that the material parameters σ, ε and μ are constant in Ω_0 and Ω_1 (cf. Figure 7.1). To be able to perform a shape sensitivity analysis we need to specify how to describe the perturbation of the domain Ω_1. There are several possibilities to do this, here we will use the velocity method, where the deformation of a domain is given by a velocity field $\mathbf{V}$ (cf. [41]). Let us now render more precisely how such a deformation looks like:
For a given Lipschitz continuous vector field $\mathbf{V} : \mathbb{R}^3 \to \mathbb{R}^3$, and for $\tau > 0$ we define

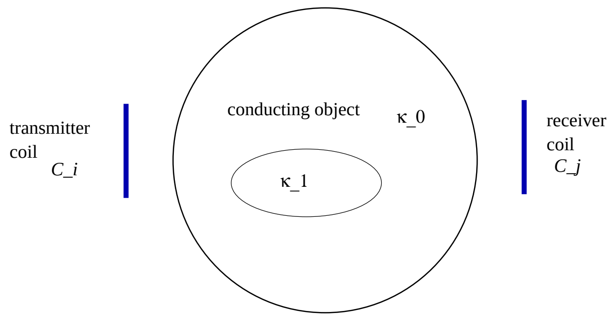


Figure 7.1: Setting for the shape reconstruction for Magnetic Induction Tomography

the family of transformations $T_t(\mathbf{V})(X) = x(t, X)$ by a system of ordinary differential equations:

$$\frac{d}{dx}x(t, X) = \mathbf{V}(x(t, x)), \quad 0 < t < \tau, \quad x(0, X) = X. \tag{7.1}$$

Using these transformations, we can now define the perturbed domain $\Omega_{1,t} = T_t(\mathbf{V})(\Omega_1)$, which enables us to define Eulerian semiderivative of a given functional:

Definition 7.1. *For* $\mathbf{V} \in \mathcal{C}^k(\mathbb{R}^3, \mathbb{R}^3)$ *with* $k \geq 0$ *the Eulerian semiderivative is defined by*

$$dJ(\Omega_1; \mathbf{V}) := \lim_{t \searrow 0} \frac{J(\Omega_{1,t}) - J(\Omega_1)}{t}. \tag{7.2}$$

The notion of the Eulerian semiderivative enables us to give a definition of the shape differentiability of a functional:

Definition 7.2. *The functional* $\mathcal{J}(\Omega_1)$ *is shape differentiable at* Ω_1 *if its Eulerian semiderivative exists for all* $\mathbf{V} \in \mathcal{C}^k(\mathbb{R}^3, \mathbb{R}^3)$ *with* $k \geq 0$ *and*

$$\mathbf{V} \to d\mathcal{J}(\Omega_1, \mathbf{V}) \tag{7.3}$$

defines a linear and continuous mapping from $\mathcal{C}^k(\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}$.

Let us now return to the Magnetic Induction Tomography setting. We denote the measured voltage in the receiver coil by v_δ. The voltage, which is obtained by the solution of the forward problem is given by the functional $\mathcal{J}(\Omega_{1,t})$. The aim when solving the inverse problem is then to minimize the functional

$$r(\Omega_{1,t}) := |\mathcal{J}(\Omega_{1,t}) - v_\delta| \to \min. \tag{7.4}$$

For the minimization of the functional $r(\Omega_{1,t})$ we use a descent method, i.e. we choose a direction $\mathbf{V}^*$ such that

$$dr(\Omega_1; \mathbf{V}^*) = \Re(d\mathcal{J}(\Omega_1; \mathbf{V}^*)(\mathcal{J}(\Omega_{1,t}) - v_\delta)) < 0.$$

Later we formulate the forward model for the shape reconstruction setting. In the following section we compute the shape derivatives for the reduced model, where we use the adjoint variable technique.

7.2 Forward Model

In this section we define the forward map for the reduced model. We assume that outside of the conducting object Ω we have two coils, a transmitting coil $\mathcal{C}_i$ and a receiver coil $\mathcal{C}_j$. The measurement in the receiver coil is given by

$$v = -i\omega \int_{\mathcal{C}_j} \mathbf{B}_s(x) \cdot n(x) ds_x = -i\omega \int_{\partial\mathcal{C}_j} \mathbf{A}_s(x) ds_x.$$

Due to (2.54) we can compute $\mathbf{A}_s$ by evaluating the following integral:

$$\mathbf{A}_s(x) = \frac{\mu_0}{4\pi} \int_{\Omega} \kappa(y) \frac{\nabla\phi(y) - \mathbf{E}_p^i(y)}{|x-y|} dy, \tag{7.5}$$

where $\mathbf{E}_p^i$ is the primary electric field produced by the transmitter coil $\mathcal{C}_i$ in free space. Hence it is independent of Ω and κ and can be computed by (2.33). Thus the voltage v can be computed by the formula:

$$v = -i\omega \int_{\partial\mathcal{C}_j} \tau(x) \cdot \mathbf{A}_s(x) ds_x = -i\omega \int_{\partial\mathcal{C}_j} \frac{\mu_0}{4\pi} \tau(x) \cdot \int_{\Omega} \kappa(y) \frac{\nabla\phi(y) - \mathbf{E}_p^i(y)}{|x-y|} dy ds_x. \tag{7.6}$$

By interchanging the order of integration we get

$$v = -i\omega \int_{\Omega} \kappa(y)(\nabla\phi(y) - \mathbf{E}_p^i(y)) \frac{\mu_0}{4\pi} \int_{\partial\mathcal{C}_j} \frac{\tau(x)}{|x-y|} ds_x dy. \tag{7.7}$$

Inserting the representation

$$\mathbf{E}_p^j(y) = -i\omega \frac{\mu_0}{4\pi} \int_{\partial\mathcal{C}_j} \frac{\tau(x)}{|x-y|} ds_x \tag{7.8}$$

for the primary field $\mathbf{E}_p^j$ we obtain

$$v = \int_\Omega \kappa(y)(\nabla\phi(y) - \mathbf{E}_p^i(y))\mathbf{E}_p^j(y)dy. \tag{7.9}$$

Let us now consider the shape reconstruction setting: We decompose the conducting domain Ω into two domains Ω_0 and Ω_1 with two different conductivities, see Figure 7.2:

$$\overline{\Omega} = \overline{\Omega}_0 \cup \overline{\Omega}_1, \quad \Omega_0 \cap \Omega_1 = \emptyset \quad \kappa(x) = \begin{cases} \kappa_0, & x \in \Omega_0, \\ \kappa_1, & x \in \Omega_1. \end{cases}$$

We denote the outer boundary by $\Gamma = \partial\Omega$ and the transmission boundary by

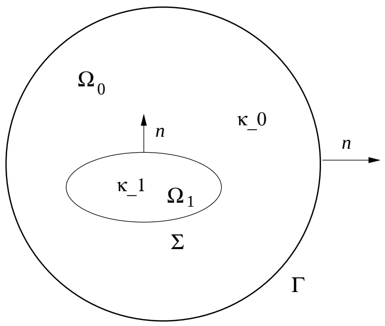


Figure 7.2: Fixed domain Ω with subdomains Ω_0 and Ω_1

$\Sigma = \partial\Omega_1$. With n_0 we denote the exterior normal vector of Ω_0 and n_1 the exterior normal vector of Ω_1. We set $n = n_0$ on Γ and $n = n_1 = -n_0$ on Σ as depicted. We also set

$$\phi(x) = \begin{cases} \phi_0(x), & x \in \Omega_0, \\ \phi_1(x), & x \in \Omega_1. \end{cases}$$

From (2.51)-(2.52) we conclude that ϕ_0 and ϕ_1 satisfy the following transmission problem

$$\begin{aligned} -\Delta\phi_0(x) &= 0, \quad x \in \Omega_0, && (7.10)\\ -\Delta\phi_1(x) &= 0, \quad x \in \Omega_1, && (7.11)\\ \phi_0(x) - \phi_1(x) &= 0, \quad x \in \Sigma, && (7.12)\\ \kappa_0\left(\frac{\partial\phi_0(x)}{\partial n(x)} - \mathbf{E}_p^i(x)\cdot n(x)\right) &= \kappa_1\left(\frac{\partial\phi_1(x)}{\partial n(x)} - \mathbf{E}_p^i(x)\cdot n(x)\right), \quad x \in \Sigma, && (7.13)\\ \kappa_0\left(\frac{\partial\phi_0(x)}{\partial n(x)} - \mathbf{E}_p^i(x)\cdot n\right) &= 0, \quad x \in \Gamma. && (7.14) \end{aligned}$$

7.3 Shape Sensitivity Analysis

As we have discussed the forward model in the previous section we can now state a formula for the shape functional $\mathcal{J}(\Omega_{1,t})$:

$$\mathcal{J}(\Omega_{1,t}) = \kappa_0 \int\limits_{\Omega_{0,t}} (\nabla\phi_{0,t}(y) - \mathbf{E}_p^i(y))\mathbf{E}_p^j(y)dy + \kappa_1 \int\limits_{\Omega_{1,t}} (\nabla\phi_{1,t}(y) - \mathbf{E}_p^i(y))\mathbf{E}_p^j(y)dy. \quad (7.15)$$

The functions ϕ_0 and ϕ_1 are defined as the solution of the transmission problem (7.12)-(7.22), hence ϕ_0 and ϕ_1 both depend on the shape of the domain Ω. We introduce the derivatives of $\phi_0 = \phi_0(\Omega)$ and $\phi_1 = \phi_1(\Omega)$ by

$$\phi_0' := d\phi_0(\Omega; \mathbf{V}), \quad \phi_1' := d\phi_1(\Omega; \mathbf{V}).$$

From (7.10) we deduce that the shape derivatives ϕ_0' and ϕ_1' also satisfy

$$-\Delta\phi_0'(x) = 0, \quad x \in \Omega_0 \quad (7.16)$$

$$-\Delta\phi_1'(x) = 0, \quad x \in \Omega_1. \quad (7.17)$$

For computing the shape derivative $\mathcal{J}'(0)\mathbf{V} := d\mathcal{J}(\Omega; \mathbf{V})$ we use Reynold's transport theorem. We obtain $\mathcal{J}'(0)\mathbf{V} = \mathcal{J}_1 + \mathcal{J}_2$ with

$$\begin{aligned}
\mathcal{J}'(0)\mathbf{V} &= \kappa_0 \int\limits_{\Omega_0} \nabla\phi_0'(x)\mathbf{E}_p^j(x)dx + \kappa_1 \int\limits_{\Omega_1} \nabla\phi_1'(x)\mathbf{E}_p^j(x)dx \\
&+ \kappa_0 \int\limits_{\partial\Omega_0} (\nabla\phi_0(x) - \mathbf{E}_p^i(x))\mathbf{E}_p^j(x)(\mathbf{V}(x)\cdot n_0(x))ds_y \\
&+ \kappa_1 \int\limits_{\partial\Omega_1} (\nabla\phi_1(x) - \mathbf{E}_p^i(x))\mathbf{E}_p^j(x)(\mathbf{V}(x)\cdot n_1(x))ds_x.
\end{aligned}$$

We are now going to derive a different representation for the shape derivative $\mathcal{J}'(0)\mathbf{V}$, for this we split $\mathcal{J}'(0)\mathbf{V}$ into two parts, which we will treat seperately:

$$\mathcal{J}_1 := \kappa_0 \int\limits_{\Omega_0} \nabla\phi_0'(x)\mathbf{E}_p^j(x)dx + \kappa_1 \int\limits_{\Omega_1} \nabla\phi_1'(x)\mathbf{E}_p^j(x)dx$$

and

$$\begin{aligned}
\mathcal{J}_2 &:= \kappa_0 \int\limits_{\partial\Omega_0} (\nabla\phi_0(x) - \mathbf{E}_p^i(x))\mathbf{E}_p^j(x)(\mathbf{V}(x)\cdot n_0(x))ds_x \\
&+ \kappa_1 \int\limits_{\partial\Omega_1} (\nabla\phi_1(x) - \mathbf{E}_p^i(x))\mathbf{E}_p^j(x)(\mathbf{V}(x)\cdot n_1(x))ds_x.
\end{aligned}$$

At first we are going to reformulate $\mathcal{J}_1$. Using Gauss' formula we obtain

$$\begin{aligned}
\mathcal{J}_1 &= \kappa_0 \int_{\Omega_0} \nabla \phi_0'(x) \mathbf{E}_p^j(x) dx + \kappa_1 \int_{\Omega_1} \nabla \phi_1'(x) \mathbf{E}_p^j(x) dx \\
&= \kappa_0 \int_{\partial\Omega_0} \phi_0'(x) \mathbf{E}_p^j(x) \cdot n_0(x) ds_x + \kappa_1 \int_{\partial\Omega_1} \phi_1'(x) \mathbf{E}_p^j(x) \cdot n_1(x) ds_x \\
&- \kappa_0 \int_{\Omega_0} \phi_0'(x) \operatorname{div} \mathbf{E}_p^j(x) dx - \kappa_1 \int_{\Omega_1} \phi_1'(x) \operatorname{div} \mathbf{E}_p^j(x) dx.
\end{aligned}$$

As a next step we introduce the adjoint transmission problem

$$\Delta \psi_0(x) = \operatorname{div} \mathbf{E}_p^j(x), \quad x \in \Omega_0, \tag{7.18}$$

$$\Delta \psi_1(x) = \operatorname{div} \mathbf{E}_p^j(x), \quad x \in \Omega_1, \tag{7.19}$$

$$\psi_0(x) - \psi_1(x) = 0, \quad x \in \Sigma, \tag{7.20}$$

$$\kappa_0 \left(\frac{\partial \psi_0(x)}{\partial n(x)} - \mathbf{E}_p^j(x) \cdot n(x) \right) = \kappa_1 \left(\frac{\partial \psi_1(x)}{\partial n(x)} - \mathbf{E}_p^j(x) \cdot n(x) \right), \quad x \in \Sigma, \tag{7.21}$$

$$\kappa_0 \left(\frac{\partial \psi_0(x)}{\partial n(x)} - \mathbf{E}_p^j(x) \cdot n(x) \right) = 0, \quad x \in \Gamma. \tag{7.22}$$

Using (7.18)-(7.19) we obtain the following expression for $\mathcal{J}_1$:

$$\begin{aligned}
\mathcal{J}_1 &= \kappa_0 \int_{\partial\Omega_0} \phi_0'(x) \mathbf{E}_p^j(x) \cdot n_0(x) ds_x + \kappa_1 \int_{\partial\Omega_1} \phi_1'(x) \mathbf{E}_p^j(x) \cdot n_1(x) ds_x \\
&- \kappa_0 \int_{\Omega_0} \phi_0'(x) \Delta \psi_0(x) dx - \kappa_1 \int_{\Omega_1} \phi_1'(x) \Delta \psi_1(x) dx.
\end{aligned}$$

Applying Green's second formula gives

$$\begin{aligned}
\mathcal{J}_1 &= \kappa_0 \int_{\partial\Omega_0} \phi_0'(x) \mathbf{E}_p^j(x) \cdot n_0(x) ds_x + \kappa_1 \int_{\partial\Omega_1} \phi_1'(x) \mathbf{E}_p^j(x) \cdot n_1(x) ds_x \\
&- \kappa_0 \int_{\Omega_0} \Delta \phi_0'(x) \psi_0(x) dx - \kappa_1 \int_{\Omega_1} \Delta \phi_1'(x) \psi_1(x) dx \\
&+ \kappa_0 \int_{\partial\Omega_0} \frac{\partial \phi_0'(x)}{\partial n_0(x)} \psi_0(x) ds_x - \kappa_0 \int_{\partial\Omega_0} \phi_0'(x) \frac{\partial \psi_0(x)}{\partial n_0(x)} ds_x \\
&+ \kappa_1 \int_{\partial\Omega_1} \frac{\partial \phi_1'(x)}{\partial n_1(x)} \psi_1(x) ds_x - \kappa_1 \int_{\partial\Omega_1} \phi_1'(x) \frac{\partial \psi_1(x)}{\partial n_1(x)} ds_x.
\end{aligned}$$

We further make use of the fact that the shape derivatives ϕ_0' and ϕ_1' satisfy the Laplace equation (7.16)-(7.17), this results in the following relation:

$$\begin{aligned}\mathcal{J}_1 &= \kappa_0 \int\limits_{\partial\Omega_0} \phi_0'(x) \left(\mathbf{E}_p^j(x) \cdot n_0(x) - \frac{\partial \psi_0(x)}{\partial n_0(x)} \right) ds_x \\ &+ \kappa_1 \int\limits_{\partial\Omega_1} \phi_1'(x) \left(\mathbf{E}_p^j(x) \cdot n_1(x) - \frac{\partial \psi_1(x)}{\partial n_1(x)} \right) ds_x \\ &+ \kappa_0 \int\limits_{\partial\Omega_0} \frac{\partial \phi_0'(x)}{\partial n_0(x)} \psi_0(x) ds_x + \kappa_1 \int\limits_{\partial\Omega_1} \frac{\partial \phi_1'(x)}{\partial n_1(x)} \psi_1(x) ds_x.\end{aligned}$$

With the transmission conditions (7.20) -(7.22) we then obtain

$$\begin{aligned}\mathcal{J}_1 = \kappa_0 &\int\limits_{\Sigma} (\phi_1'(x) - \phi_0'(x)) \left(\mathbf{E}_p^j(x) \cdot n(x) - \frac{\partial \psi_0(x)}{\partial n(x)} \right) ds_x \\ + &\int\limits_{\Sigma} \left(\kappa_1 \frac{\partial \phi_1'(x)}{\partial n(x)} - \kappa_0 \frac{\partial \phi_0'(x)}{\partial n(x)} \right) \psi_0(x) ds_x.\end{aligned} \tag{7.23}$$

To be able to reformulate (7.23), we have to derive transmission conditions for ϕ_0' and ϕ_1'. Computing the shape derivative of the transmission condition (7.12) gives us

$$\phi_1'(x) - \phi_0'(x) = -\frac{[\kappa]}{\kappa_1} \left(\frac{\partial \phi_0(x)}{\partial n(x)} - \mathbf{E}_p^i(x) \cdot n(x) \right) (\mathbf{V}(x) \cdot n(x)), \quad x \in \Sigma, \tag{7.24}$$

where $[\kappa] = \kappa_1 - \kappa_0$ denotes the jump of κ at the transmission boundary Σ. Since $\mathbf{V}$ is zero on the fixed outer boundary Γ we obtain

$$\kappa_0 \frac{\partial \phi_0'(x)}{\partial n(x)} = 0, \quad x \in \Gamma. \tag{7.25}$$

For the derivation of the shape derivative of the Neumann transmission condition (7.13) we need the following lemma:

Lemma 7.3. *Let $u : \Omega \to \mathbb{C}$ be a function, which satisfies*

$$\Delta u(x) = 0, \quad x \in \Omega, \quad (\nabla u(x) - \mathbf{F}(x)) \cdot n(x) = 0, \quad x \in \Gamma,$$

for a given function $\mathbf{F}$ with $\operatorname{div} \mathbf{F}(x) = 0$ for $x \in \Omega$. Then the normal derivative of u' is given by

$$\frac{\partial u'(x)}{\partial n} = \operatorname{div}_\Gamma \left[(\nabla_\Gamma u(x) - \mathbf{F}(x))(\mathbf{V}(x) \cdot n(x)) \right] - H(x)(\mathbf{F}(x) \cdot n(x))(\mathbf{V}(x) \cdot n(x)),$$

where $H(x)$ denotes the curvature in the point x on the boundary Γ.

Proof. Using the chain rule and (7.1) we obtain

$$
\begin{aligned}
0 &= \frac{d}{dt}\left[(\nabla u_t(T_t(x)) - \mathbf{F}(T_t(x))) \cdot n_t(T_t(x))\right]_{t=0} \\
&= \frac{\partial u'(x)}{\partial n(x)} + [\nabla(\nabla u(x) - \mathbf{F}(x))\mathbf{V}(x)] \cdot n(x) + (\nabla u(x) - \mathbf{F}(x)) \cdot (\nabla n(x)\mathbf{V}(x) + n'(x)),
\end{aligned}
$$

where $u' = du(\Omega; \mathbf{V})$ denotes the shape derivative of u. For the shape derivative of the normal vector $n(x)$ we have the following representation (see [41])

$$
n'(x) = -(\nabla n(x) n(x))(\mathbf{V}(x) \cdot n(x)) - \nabla_\Gamma(\mathbf{V}(x) \cdot n(x)).
$$

Moreover we introduce the decomposition of $\mathbf{V} = \mathbf{V}_\Gamma + (\mathbf{V} \cdot n)n$ into its tangential and normal parts, this leads to

$$
\nabla n \mathbf{V} + n' = \nabla n \mathbf{V}_\Gamma - \nabla_\Gamma(\mathbf{V} \cdot n),
$$

hence we obtain

$$
\begin{aligned}
\frac{\partial u'(x)}{\partial n} = &- [\nabla(\nabla u(x) - F(x))\mathbf{V}(x)] \cdot n(x) \\
&- (\nabla u(x) - \mathbf{F}(x)) \cdot (\nabla n(x)\mathbf{V}_\Gamma(x) - \nabla_\Gamma(\mathbf{V}(x) \cdot n(x))).
\end{aligned}
$$

As a next step we set

$$
L := -\nabla(\nabla u(x) - \mathbf{F}(x))\mathbf{V}(x) \cdot n(x) - (\nabla u(x) - \mathbf{F}(x)) \cdot \nabla n(x)\mathbf{V}_\Gamma(x) = L_n + L_\Gamma.
$$

Inserting the decomposition $\mathbf{V} = \mathbf{V}_\Gamma + (\mathbf{V} \cdot n)n$ we get $L = L_n + L_\Gamma$, where

$$
\begin{aligned}
L_\Gamma &:= -[\nabla(\nabla u(x) - \mathbf{F}(x))\mathbf{V}_\Gamma(x)] \cdot n(x) - (\nabla u(x) - \mathbf{F}(x)) \cdot \nabla n(x)\mathbf{V}_\Gamma(x), \\
L_n &:= -[(\nabla(\nabla u(x) - \mathbf{F}(x))n(x)) \cdot n(x)]\,(\mathbf{V} \cdot n(x)).
\end{aligned}
$$

Because the expression $(\nabla u(x) - \mathbf{F}(x)) \cdot n(x)$ is zero on the boundary Γ, we deduce that its gradient is also zero on the boundary, i.e.

$$
\nabla\left[(\nabla u(x) - \mathbf{F}(x)) \cdot n(x)\right] \cdot \mathbf{V}_\Gamma(x) = 0, \quad \text{on } \Gamma. \tag{7.26}
$$

This yields $L_\Gamma = 0$ and thus

$$
\begin{aligned}
\frac{\partial u'(x)}{\partial n(x)} =& (\nabla u(x) - \mathbf{F}(x)) \cdot \nabla_\Gamma(\mathbf{V}(x) \cdot n(x))) \\
&-(\nabla(\nabla u(x) - F(x))n(x) \cdot n(x))(\mathbf{V}(x) \cdot n(x)).
\end{aligned}
$$

With using $\Delta u = 0$ we get

$$
\nabla(\nabla u(x) n(x)) \cdot n(x) = -\Delta_\Gamma u(x) - H(x)\frac{\partial u(x)}{\partial n(x)}
$$

Out of div $\mathbf{F}(x) = 0$ it follows that

$$\nabla(\mathbf{F}(x)n(x)) \cdot n(x) = -\mathrm{div}_\Gamma \mathbf{F}(x)$$

holds and hence we finally obtain

$$\begin{aligned}\frac{\partial u'}{\partial n} &= (\nabla_\Gamma u - \mathbf{F}) \cdot \nabla_\Gamma(\mathbf{V} \cdot n)) + \mathrm{div}_\Gamma(\nabla_\Gamma u - \mathbf{F})(\mathbf{V} \cdot n) - H(\mathbf{F} \cdot n)(\mathbf{V} \cdot n) \\ &= \mathrm{div}_\Gamma((\nabla_\Gamma u - \mathbf{F})(\mathbf{V} \cdot n)) - H(\mathbf{F} \cdot n)(\mathbf{V} \cdot n).\end{aligned}$$

□

For the derivation of the shape derivative of the transmission condition (7.27) we use lemma 7.3 by setting

$$u = \kappa_1 \phi_1 - \kappa_0 \phi_0, \quad \mathbf{F} = -[\kappa]\mathbf{E}_p^j.$$

Due to the transmission condition (7.12) we have that $\nabla_\Gamma \phi_1 = \nabla_\Gamma \phi_0$ and thus it follows

$$\nabla_\Gamma u = \kappa_1 \nabla_\Gamma \phi_1 - \kappa_0 \nabla_\Gamma \phi_0 = -[\kappa]\nabla_\Gamma \phi_0.$$

With this information we obtain the following Neumann transmission condition for the normal derivatives of ϕ_0' and ϕ_1':

$$\begin{aligned}\kappa_1 \frac{\partial \phi_1'(x)}{\partial n(x)} - \kappa_0 \frac{\partial \phi_0'(x)}{\partial n(x)} &= -[\kappa]\mathrm{div}_\Gamma \left[(\nabla_\Gamma \phi_0(x) - \mathbf{E}_p^j(x))(\mathbf{V}(x) \cdot n(x))\right] \\ &\quad + [\kappa](H(x)\mathbf{E}_p^j(x) \cdot n(x))(\mathbf{V}(x) \cdot n(x)), \quad x \in \Sigma.\end{aligned} \tag{7.27}$$

Inserting the transmission condition (7.24)-(7.25) into (7.23) yields

$$\begin{aligned}\mathcal{J}_1 &= -\frac{\kappa_0}{\kappa_1}[\kappa]\int_\Sigma \left(\frac{\partial \phi_0(x)}{\partial n(x)} - \mathbf{E}_p^i(x) \cdot n(x)\right)\left(\frac{\partial \psi_0(x)}{\partial n(x)} - \mathbf{E}_p^j(x) \cdot n(x)\right)(\mathbf{V}(x) \cdot n(x))ds_x \\ &- [\kappa]\int_\Sigma \mathrm{div}_\Gamma \left[(\nabla_\Gamma \phi_0(x) - \mathbf{E}_p^i(x))(\mathbf{V}(x) \cdot n(x))\right]\psi_0(x)ds_x \\ &+ [\kappa]\int_\Sigma H(x)(\mathbf{E}_p^i(x) \cdot n)(\mathbf{V}(x) \cdot n(x))\psi_0(x)ds_x \\ &= -\frac{\kappa_0}{\kappa_1}[\kappa]\int_\Sigma \left(\frac{\partial \phi_0(x)}{\partial n(x)} - \mathbf{E}_p^i(x) \cdot n(x)\right)\left(\frac{\partial \psi_0(x)}{\partial n(x)} - \mathbf{E}_p^j(x) \cdot n(x)\right)(\mathbf{V}(x) \cdot n(x))ds_x \\ &+ [\kappa]\int_\Sigma (\nabla_\Gamma \phi_0(x) - \mathbf{E}_p^i(x))(\mathbf{V}(x) \cdot n(x))\nabla_\Gamma \psi_0(x)ds_x.\end{aligned}$$

Let us now take a look at the expression $\mathcal{J}_2$. Since on the fixed boundary Γ the velocity field is zero, i.e. $\mathbf{V} = 0$, we get

$$\mathcal{J}_2 = -\kappa_0 \int_\Sigma (\nabla \phi_0(x) - \mathbf{E}_p^i(x)))\mathbf{E}_p^j(x)(\mathbf{V}(x) \cdot n(x))ds_x$$
$$+ \kappa_1 \int_\Sigma (\nabla \phi_1(x) - \mathbf{E}_p^i(x))\mathbf{E}_p^j(x)(\mathbf{V}(x) \cdot n(x))ds_x.$$

Combining this with the transmission conditions (7.12) and (7.13) leads us to

$$\mathcal{J}_2 = -[\kappa] \int_\Sigma (\nabla_\Gamma \phi_0(x) - \gamma_t \mathbf{E}_p^i(x))\gamma_t \mathbf{E}_p^j(x)(\mathbf{V}(x) \cdot n(x))ds_x, \tag{7.28}$$

where the tangential part of a vector is defined as $\gamma_t \mathbf{E} = \mathbf{E} - (\mathbf{E} \cdot n)n$. By summing up $\mathcal{J}_1$ and $\mathcal{J}_2$ we finally obtain the following formula for the shape derivative:

$$\mathcal{J}'(0) = -\frac{\kappa_0}{\kappa_1}[\kappa] \int_\Sigma \left(\frac{\partial \phi_0(x)}{\partial n(x)} - \mathbf{E}_p^i(x) \cdot n(x) \right) \left(\frac{\partial \psi_0(x)}{\partial n(x)} - \mathbf{E}_p^j(x) \cdot n(x) \right) (\mathbf{V} \cdot n(x))ds_x$$
$$+ [\kappa] \int_\Sigma (\nabla_\Gamma \phi_0(x) - \gamma_t \mathbf{E}_p^i(x))(\nabla_\Gamma \psi_0(x) - \gamma_t \mathbf{E}_p^j(x))(\mathbf{V}(x) \cdot n(x))ds_x. \tag{7.29}$$

A descent direction to minimize the shape functional $r(\Omega_{1,t})$ is thus given by

$$\mathbf{V}^*(x) = -[\kappa](\mathcal{J}(\Omega_{1,t}) - v_\delta) \left[-\frac{\kappa_0}{\kappa_1} \left(\frac{\partial \phi_0(x)}{\partial n(x)} - \mathbf{E}_p^i(x) \cdot n(x) \right) \left(\frac{\partial \psi_0(x)}{\partial n(x)} - \mathbf{E}_p^j(x) \cdot n(x) \right) \right.$$
$$\left. + (\nabla_\Gamma \phi_0(x) - \gamma_t \mathbf{E}_p^i(x))(\nabla_\Gamma \psi_0(x) - \gamma_t \mathbf{E}_p^j(x)) \right] n(x), \; x \in \Sigma. \tag{7.30}$$

7.4 The Level Set Method

For the solution of the inverse problem of Magnetic Induction Tomography we need to choose a method how to represent the transmission boundary Σ. One possibility would be to represent Σ by a parametrization [40]. Another method for representing a domain and its boundary is given by the Level Set Method ([53]). In the Level Set Method the transmission boundary Σ_t is described by a level set function $\varphi(x,t)$: $\Omega_1 \times \mathbb{R}^+ \to \mathbb{R}$:

$$\Sigma_t = \{x : \varphi(x,t) = 0\}.$$

The level set function φ satisfies the partial differential equation

$$\frac{\partial}{\partial t}\varphi(t,x) + \mathbf{V} \cdot \nabla_x \varphi(t,x) = 0.$$

For the Magnetic Induction Tomography the direction $\mathbf{V}$ can be computed by the formula (7.30).
To solve the inverse problem of Magnetic Induction Tomography we would proceed as showed by the following algorithm:
Choose an initial level set function φ^0 and determine the initial transmission boundary Σ^0.

1. Solve the transmission problem (7.10)-(7.14) for ϕ_0^k and ϕ_1^k.
2. Solve the adjoint transmission problem (7.18)-(7.22) for ψ_0^k and ψ_1^k.
3. Compute the descent direction $\mathbf{V}^k(x)$ by using formula (7.30).
4. Update the level set function φ by solving

$$\frac{\partial}{\partial t}\varphi(t,x) + \mathbf{V}^k(x) \cdot \nabla_x \varphi(t,x) = 0.$$

5. Determine the transmission boundary Σ^k.
6. Check if convergence is reached

$$|\mathcal{J}(\Omega_{1,t}) - v_\delta| < \varepsilon.$$

   If the convergence is not reached go to 1.

An implementation of the level set method has not been done in this thesis, however using a Level Set Method in combination with the solution of the forward problem with the boundary element method seems promising.

8 CONCLUSIONS AND OPEN PROBLEMS

8.1 Conclusions

In this thesis we investigated the simulation of eddy current problems using boundary element methods. Taking the Maxwell equations as a starting point we derived mathematical models for the eddy current problem. We introduced a representation formula for electric and magnetic fields in or outside a conducting domain. We further derived integral equations for the electric and magnetic fields. We presented a new stabilized formulation for the solution of boundary value problems, when considering small wave numbers. We illustrated this effect by numerical examples.

For the eddy current transmission problem we introduced different formulations. We presented two formulations, which are based on a direct boundary element method, one was formulated in terms of the electric field $\mathbf{E}$, the other one was formulated using the magnetic field intensity $\mathbf{H}$. In addition we deduced an indirect formulation for the magnetic field intensity $\mathbf{H}$. We introduced a discretization for the derived boundary integral formulations and gave numerical examples.

In the end we dealt with the inverse problem of Magnetic Induction Tomography using the shape reconstruction approach. We computed shape derivatives for a reduced model.

8.2 Open Problems and Possible Further Work

The simulation of eddy current problems in industrial applications requires a fast and robust solver. The bottleneck when using the boundary element method to simulate the eddy current problem is the setup time and the memory consumption of the boundary element matrices. To decrease the memory requirements and to speed up the setup time for the matrices a fast boundary element method can to be implemented, e.g. a $\mathcal{H}$ matrix structure in combination with the ACA method can be used for the boundary element matrices [6, 30]. Another possibility would be to use the Fast Multipole method [27].

For the solution of the linear systems a good preconditioner has to be implemented.

As seen in Chapter 6 the eddy current transmission problem requires the solution of a block system. When using an iterative solver a good balance of the blocks in the linear system is crucial. A possibility for the preconditioning of the block systems would be an implementation of the Bramble-Pasciak transformation [8].

From a theoretical point of view the proof of the unique solvability of the variational problem (5.55)-(5.56), which resulted from an indirect formulation for the eddy current problem based on the magnetic field intensity $\mathbf{H}$, is still open.

In industrial applications, e.g. the simulation of transformers one also has to deal with nonlinear materials. To be able to cope such type of problems a coupling of the finite element method and the boundary element method is advantageous (cf. [31]).

In the last chapter of the thesis we presented a strategy for solving the inverse problem of Magnetic Induction Tomography by using a shape reconstruction approach. A numerical implementation of this strategy has not been done so far. A possibility for an implementation would be to combine the level set method with the boundary element method presented in Chapter 6.

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