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Combined Generalized Newton Approach with Boundary Element Methods for Contact Problems

DISSERTATION

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CONTENTS

Abstract

In this thesis semi-smooth Newton methods and boundary element methods are developed and analyzed for the solution of contact problems in linear elasticity of Yukawa type and of quasistatic contact problems. First we consider the 2-D Signorini contact problem in linear elasticity of Yukawa type with Coulomb friction. This is approximated by a sequence of contact problems with given friction known as Tresca problems. This leads to a constrained non-differentiable minimization problem where the solvability is in general problematic. But, by utilizing the Fenchel duality theory, the dual formulation in terms of contact stresses turns out to be a quadratic optimization problem with a smooth functional. The regularization of the dual problem motivated by the augmented Lagrangian is suitable for the application of the generalized Newton method. Applying the boundary integral equation method, the problem is reduced to the boundary curve. The corresponding boundary integral equations are approximated by using a Galerkin method with the help of B-splines on the boundary curve (BEM). This yields an algebraic system of linear equations involving dense matrices but which are partly circulant. The associated entries of circulant matrices are computed explicitly and efficiently. Additionally, the circulant block structure enables us to develop some preconditioning matrices for the iterative solution of the linear systems at each Newton step. The methods are carried over to the Coulomb friction problem by means of a fixed point approach. Second, the above methods are extended to a discrete quasistatic contact problem. In particular, the analysis of the algorithm is presented and some numerical examples, which show a remarkable efficiency and reliability of the semi-smooth Newton method, are given.

Zusammenfassung

In dieser Arbeit werden Newton–Methoden mit Randelementmethoden für Kontaktprobleme mit Reibung in der linearen Elastizität und für quasistatische Kontaktprobleme entwickelt und analysiert.

Zuerst werden zweidimensionale Signorini–Kontaktprobleme der linearen Elastizitätstheorie vom Yukawa–Typ mit Coulomb–Reibung betrachtet. Diese werden durch eine Folge von Kontaktproblemen mit gegebener Reibung, bekannt als Tresca–Probleme, approximiert. Dies führt auf ein nicht differenzierbares Minimierungsproblem mit Nebenbedingungen, wobei die Lösbarkeit im allgemeinen problematisch ist. Aber unter Verwendung der Fenchel–Dualitätstheorie stellt sich die duale Formulierung mittels Kontaktspannungen als ein quadratisches Optimierungsproblem mit einem glatten Funktional heraus. Die Regularisierung des dualen Problems, motiviert durch das Konzept des augmentierten Lagrange–Funktionals, ist geeignet für die Anwendung der verallgemeinerten Newton–Methode. Durch den Einsatz der Randelementmethode wird das Problem auf die Randkurve reduziert. Die entsprechenden Randintegralgleichungen werden mittels einer Galerkin–Methode mit B–Splines auf der Randkurve approximiert. Dies führt auf ein lineares Gleichungssystem vollbesetzter Matrizen, welche jedoch teilweise zirkulant sind. Die Einträge der zirkulanten Matrizen werden explizit und effizient berechnet. Zusätzlich ermöglicht die zirkulante Blockstruktur die Entwicklung von Vorkonditionierern zur effizienten iterativen Lösung der linearen Gleichungssysteme jedes Newton–Schritts. Die Methoden werden mittels einer Fixpunktiteration auf Probleme mit Coulomb–Reibung übertragen.

Weiters werden die beschriebenen Methoden für ein diskretes quasistatisches Kontaktproblem erweitert. Insbesondere werden die Analyse des Algorithmus und einige numerische Beispiele, welche eine beachtliche Effizienz und Zuverlässigkeit der verallgemeinerten Newton–Methode zeigen, präsentiert.

Dedication

To my mother Weyeno Justine.

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Table of symbols

The notations, the mathematical symbols and some common abbreviations used throughout this thesis are listed below.

Symbol		**Definition**
$\mathbb{Z}$	$\equiv$	set of integers,
$\mathbb{Z}_+$	$\equiv$	set of positive integers,
$\mathbb{R}$	$\equiv$	set of real numbers,
$\overline{\mathbb{R}}$	$\equiv$	$\mathbb{R} \cup \{-\infty, +\infty\}$,
Ω	$\equiv$	open set in $\mathbb{R}^d$, $d = 2, 3$
$\overline{\Omega}$	$\equiv$	closure of the set Ω
$\Gamma := \partial\Omega$	$\equiv$	boundary of Ω,
d	$\equiv$	dimension of the domain
Γ_D	$\equiv$	Dirichlet's boundary,
Γ_N	$\equiv$	Neumann's boundary,
Γ_C	$\equiv$	possible contact boundary,
$\underline{n}$	$\equiv$	unit outer normal to Γ,
I_A	$\equiv$	indicator function of a set A,
f^*	$\equiv$	convex conjugate function of f,
$\partial f(x)$	$\equiv$	set of subgradients at x,
$\dfrac{\partial(.)}{\partial x}$	$\equiv$	partial derivative of $(.)$,
α	$\equiv$	$(\alpha_1, ..., \alpha_n)$,
$\|\alpha\|$	$\equiv$	$\alpha_1 + ... + \alpha_n$ and $\alpha_i \in \mathbb{Z}_+$,
D^α	$\equiv$	$\dfrac{\partial^{\|\alpha\|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$,
$\|.\|$	$\equiv$	Euclidean norm,
$\\|\cdot\\|_X$	$\equiv$	the norm in the space X,
$(.,.)_X$	$\equiv$	the inner product in the space X,
$a(.,.)$	$\equiv$	bilinear form,
$\langle .,. \rangle$	$\equiv$	duality pairing,
∇ , grad	$\equiv$	gradient,
div	$\equiv$	divergence,
$\triangle$	$\equiv$	Laplacian,
u	$\equiv$	unknown function,
$\dot{u}, \ddot{u}$	$\equiv$	first and second time derivative of u,
$\underline{u}$	$\equiv$	displacement field,

$\underline{u}_0$, $\underline{u}_1$	$\equiv$	prescribed initial data for $\underline{u}$ and $\dot{\underline{u}}$,
$\underline{u}_t$	$\equiv$	tangential displacement,
u_n	$\equiv$	normal displacement,
$\eta(\underline{u})$	$\equiv$	strain tensor,
$\varepsilon(\underline{u})$	$\equiv$	linearized strain tensor,
$\tau(\underline{u})$	$\equiv$	first Piola-Kirchoff stress tensor,
$\sigma(\underline{u})$	$\equiv$	Cauchy stress tensor,
$\sigma_t(\underline{u})$	$\equiv$	tangential stress,
$\sigma_n(\underline{u})$	$\equiv$	normal stress,
λ	$\equiv$	Lagrange multiplier,
λ, μ	$\equiv$	Lamé moduli,
ν	$\equiv$	Poisson ratio,
E	$\equiv$	Young modulus,
$\mathbf{E} := 0.57721566...$	$\equiv$	Euler-Mascheroni constant,
g	$\equiv$	given friction,
g_D	$\equiv$	prescribed Dirichlet datum,
g_N	$\equiv$	prescribed Neumann datum,
$\delta_0(.)$	$\equiv$	Dirac delta,
$U^*(x,y)$	$\equiv$	fundamental solution,
x,y	$\equiv$	points in global coordinates,
V	$\equiv$	single layer operator,
K	$\equiv$	double layer operator,
K'	$\equiv$	adjoint double layer operator,
D	$\equiv$	hypersingular integral operator,
N_0, N_1	$\equiv$	Newton potentials,
S	$\equiv$	Steklov-Poincaré operator,
$\tilde{S}$	$\equiv$	approximate Steklov-Poincaré operator,
$\tilde{S}_h$	$\equiv$	Galerkin discretization of $\tilde{S}$,
V_h	$\equiv$	Galerkin discretization of the single layer operator,
K_h	$\equiv$	Galerkin discretization of the double layer operator,
D_h	$\equiv$	Galerkin discretization of the hypersingular operator,
M_h	$\equiv$	mass matrix,
$\phi^{(j)}$	$\equiv$	B-spline of order j,
$\mathbb{H}_N^j := Span(\phi_1^{(j)},...,\phi_N^{(j)})$	$\equiv$	N-dimensional subspace of one-periodic functions,
I_n	$\equiv$	modified Bessel function of first kind,
K_n	$\equiv$	modified Bessel function of second kind,
$\mathcal{A}_{ij}$	$\equiv$	boundary integral operator,
$\mathcal{M}(.)$	$\equiv$	Günter derivative of $(.)$,
A^{-1}	$\equiv$	inverse of A,
$A^\top$	$\equiv$	transpose of A,
trA	$\equiv$	trace of A,

$A[i,j]$	$\equiv$	entry (i,j) of matrix A,
$\mathcal{F}$	$\equiv$	the friction coefficient,
$\mathfrak{F}$	$\equiv$	Fourier transform,
F, Q	$\equiv$	matrix of the discrete Fourier transform,
F^*	$\equiv$	complex conjugate of the discrete Fourier transform F,
$meas(.)$	$\equiv$	positive measure of $(.)$,
C	$\equiv$	tensor of elastic moduli,
$\underline{f}(x,t)$	$\equiv$	body force,
$\underline{s}_n$	$\equiv$	surface force,
$t_l := l\delta t$	$\equiv$	time point on time grid,
δt	$\equiv$	size of time step,
h	$\equiv$	mesh size,
ρ	$\equiv$	mass density,
R	$\equiv$	radius of the domain,
$\mathbf{d}$	$\equiv$	the gap between the rigid foundation and the body,
$a \cdot b$	$\equiv$	scalar product of vectors a and b,
$a \times b$	$\equiv$	cross product of vectors a and b,
$a \otimes b$	$\equiv$	tensor product of vectors a and b,
$i.e.$	$\equiv$	that is,
$a.e.$	$\equiv$	almost everywhere,
$e.g.$	$\equiv$	example given.

1 INTRODUCTION

Contact problems are abundant in daily life and play a very important role in engineering structures and systems, for example in the design of machines and metal forming etc. This motivates the development of a comprehensive wellposed mathematical theory based on fundamental physical principles and numerical algorithms that can predict reliably and efficiently the evolution of the contact process in different situations and under various conditions.

Numerous studies deal with the widespread Coulomb friction law introduced in the eighteenth century which takes into account the possibility of slip and stick on the contact area. This model is generally coupled with a contact law, and very often one considers the Signorini contact condition. Although simple in its formulation, the Coulomb friction law shows great mathematical difficulties which have not permit a complete understanding of the model yet. In the simple case of elastostatics only existence results for a small friction coefficient have been obtained, see, e.g. [27, 62, 85], as well as some examples of nonuniqueness of solutions for large friction coefficients [44, 45].

In the quasistatic case, that is when the inertial forces can be neglected, the first result was proved by Andersson in [5] by using an incremental approach with the contact law described by the normal compliance. Cocu, Pratt and Raous [21] proved the existence of a solution for a nonlocal friction law. Andersson in [7], and Rocca and Cocu in [95, 96] extended this result to the Signorini contact condition with local friction law. A similar result was obtained in [22] by considering friction and adhesion. Eck, Steinbach and Wendland [30] obtained the existence of solutions for the Signorini contact condition and a local friction law by using a symmetric boundary element formulation combined with a penalty method and a regularization technique.

The dynamic case, that is when the inertial terms are considered, a certain viscous damping of the material is needed in order to establish existence results for a general multi-dimensional domain [30]. The unique solvability for a viscoelastic material with normal compliance was proved in [55, 82]. The existence of solutions for a frictional problem with normal compliance for a viscoplastic material can be found in [74]; and for the frictionless case in [9], and when wear is considered on the contacting surfaces in [76]. Further, when the Signorini contact condition and the nonlocal friction law are considered, the existence for the viscoelastic material is established in [20, 73]. A recent substantial regularity result for dynamic frictionless contact problems with normal compliance was obtained in [77, 78]. Eck in [29], Jarušek in [63] and Kuttler in [75] discussed the solvability for a viscoelastic material where the Signorini contact conditions were given in terms of the displacement velocity. But, this model turns out to be realistic only for a short time interval and for a vanishing initial gap between the body's boundary and the obstacle. This obviously limits the applicability of these results to various types of problems. However, very often mathematicians and engineers used the normal compliance contact condition as regularization or approximation of the Signorini contact condition, which is an idealization and describes a perfectly rigid surface. The Signorini condition is easy to write and mathematically elegant, but seems not to describe well the real contact. Indeed, this leads to low regularity on the solutions for a dynamic viscoelastic material [29], and makes proving unique solvability almost impossible [73]. Therefore, the existence of solutions for purely elastodynamic contact problems (hyperbolic problems) coupled with a Signorini contact condition in the displacement and the local Coulomb friction law up to now is still an open problem. However, few recent results for the frictionless case can be found in [13, 14].

Hence, in this work we first consider the contact problem in linear elastostatics of Yukawa type. This is a static contact problem obtained after an implicit time discretization of the contact problem in linear elastodynamics. For the sake of mathematical completeness we utilize the Signorini contact condition together with the local Coulomb friction law. Note that this problem is inherently nonlinear making the modeling, analysis and numerical realization truly challenging. Moreover, a closed form of solutions is up to now not known. Therefore, fast and reliable numerical techniques for the derivation of approximate solutions are extremely important.

In literature the mostly used numerical methods to determine approximate solutions are the finite element methods (FEM). With respect to the variational formulation of the problem in appropriate function spaces, approximate solutions are searched in finite dimensional subspaces. The main idea consists to decompose the computational domain into finite subdomains called finite elements, on which finite dimensional subspaces are defined and in which solutions are approximated by polynomials. This leads in general to algebraic systems of equations involving sparse matrices which can enable the application of efficient tools [71, 98]. But, the accuracy of approximate solutions on the boundary curve requires a very fine mesh or a higher order of polynomials on the boundary which leads to a higher number of degrees of freedom.

An alternative approach to determine numerical solutions of contact problems can be the boundary element methods (BEM) [30] which are especially suitable since the nonlinearities of the problem appear only on the boundaries of the contacting bodies. The main idea is to transform equivalently by using the Green formula the domain variational equation to a boundary variational equation with a symmetric representation of the Steklov-Poincaré operator. This operator maps a given boundary displacement to the corresponding boundary stress of the solution of the homogeneous elasticity equations and which can be expressed in terms of suitable boundary integral operators. In comparison to finite element methods in which the whole computational domain needs to be discretized, for the boundary integral formulation one needs to discretize only the boundary surface of the considered domain. The discretized problem turns out to be an algebraic system of equations having fully dense, but symmetric and positive definite matrices. This symmetry property can enable the application of efficient iterative processes to determine the solutions.

In this thesis a coupling strategy of a boundary integral formulation with a semi-smooth Newton method is developed and analyzed for contact problems. Related approaches can be found in the work by Kunisch and Stadler [71] for the contact problem or in [47, 58–61] for applications to optimal control and obstacle problems. The approach we take here consists to write the problem as a sequence of contact problems with given friction known as Tresca problem. The variational formulation of the Tresca problem results into a variational inequality of second kind, see, e.g., [29, 35, 67, 68]. This leads to a minimization problem with a non-differentiable functional which can be problematic. The Fenchel duality theory [31] enables us to transform the non-differentiable problem (primal problem) into a constrained optimization problem of a smooth functional (dual problem). We additionally derive the extremality conditions which characterize solutions of the dual and primal problems. But due to the lack of regularity of the underlying function space, a regularization technique inspired from the augmented Lagrangian enables us to work in an adequate function space setting where a superlinear convergence of the infinite dimensional version of a generalized Newton method can be obtained [71, 98]. The methods are carried over to the Coulomb friction problems by means of a fixed point approach. Second, the above theories are extended to discrete quasistatic contact problems.

This work is organized as follows:

- The frequently used results are provided in chapter 2.
- In chapter 3 we first present the mathematical model for an elastic body undergoing small deformations. Secondly, we derive the contact condition and friction law for the contact between an elastic body and a rigid foundation, and present an implicit time discretization of the problem.
- In chapter 4 and chapter 5, the boundary integral formulation of the scalar Yukawa problem and the linear elastostatic problem of Yukawa type are presented and analyzed respectively.
- In chapter 6, the two-dimensional contact problem in linear elasticity of Yukawa type is investigated and the boundary integral formulation of the problem is established. Generalized Newton methods for the solution of Tresca problems are analyzed, the results of this problem are combined with a fixed point approach to determine the solutions of the Coulomb friction problem. Finally, the boundary element (BEM) discretization is presented.
- In chapter 7, a quasistatic contact problem is considered, the existence proof by means of a penalty method is shown, and the coupling strategy of a boundary integral formulation with a semi-smooth Newton method is developed and analyzed for the determination of its approximate solutions.

- In chapter 8, some numerical results are presented. Firstly, numerical examples for a non-homogeneous Dirichlet problem, a mixed Yukawa problem, and a non-homogeneous Dirichlet elasticity problem of Yukawa type are considered and results of boundary element approximations are presented. These results confirm the theoretical error estimates. Secondly, a numerical example of contact problem without friction is investigated, and a superlinear convergence and the monotone behavior of the semi-smooth Newton algorithm are observed. Similar results are obtained for numerical examples of a Tresca problem. The combined semi-smooth Newton method with a fixed point algorithm is successfully tested for the Coulomb problem. The above algorithms are successfully extended to a discrete quasistatic problem.
- We end with some conclusions and comments on open problems and future work.

2 PRELIMINARIES

The goal of this chapter is to present some results and definitions that are relevant for the mathematical formulation and the analysis of contact problems in linear elasticity. These results are essentially from convex analysis, functional analysis, generalized derivatives in function spaces, and circulant matrices. The results we present in this chapter will be recalled in subsequent chapters at appropriate time and in a more elaborated form as needed.

2.1 Function spaces

The use of function spaces is essential for the mathematical formulation of contact problems. In this section we confine our attention only to their definitions and describe some aspects of them which are sufficient for the understanding of the solution of the problems. For detailed investigations on their properties we refer the reader to the books by Adams [2], Lions and Magenes [80], Duvaut and Lions [25], Hsiao and Wendland [53], and McLean [83].

2.1.1 Sobolev spaces in the domain

Definition 2.1. *Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open set, let m be a nonnegative integer, and let $1 \leq p \leq \infty$. The Sobolev space $W_p^m(\Omega)$ is defined by*

$$W_p^m(\Omega) := \{v \in L_p(\Omega) : D^\alpha v \in L_p(\Omega), \quad \textit{for} \quad |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{Z}_+^d$, $|\alpha| = \alpha_1 + ... + \alpha_d$, and $D^\alpha u(x) := \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} .... \partial x_d^{\alpha_d}} u(x)$ are the distributional partial derivatives.

The *Sobolev space* $W_p^m(\Omega)$ is equipped with the following norm

$$\|u\|_{W_p^m(\Omega)} := \begin{cases} \left(\sum\limits_{|\alpha| \leq m} \|D^\alpha u\|_{L_p(\Omega)}^p \right)^{1/p} & \text{if} \quad 1 \leq p < \infty, \\ \max\limits_{|\alpha| \leq m} \|D^\alpha u\|_{L_\infty(\Omega)} & \text{if} \quad p = \infty. \end{cases}$$

Remark 2.1. *For $p=2$ the Sobolev space $W_2^m(\Omega)$ is a Hilbert space endowed with the inner product*

$$(u,v)_m := \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(x)\overline{D^\alpha v(x)}dx,$$

see [2, p. 61].

Now and onward we consider $p=2$. The above definition of Hilbert spaces can be extended for arbitrary $s>0$.

Definition 2.2. *Let $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ be an open set, let $s := m+\delta$ with $m \in \mathbb{Z}_+$ and $\delta \in (0,1)$. The Sobolev space $W_2^s(\Omega)$ defined by*

$$W_2^s(\Omega) := \{v \in W_2^m(\Omega) : |D^\alpha v|_{\delta,\Omega} < \infty, \quad \textit{for} \quad |\alpha| = m\},$$

where

$$|v|_{\delta,\Omega} := \left(\int_\Omega \int_\Omega \frac{|v(x)-v(y)|^2}{|x-y|^{d+2\delta}} dxdy \right)^{1/2},$$

is a Hilbert space endowed with the inner product

$$(u,v)_{W_2^s(\Omega)} := (u,v)_{W_2^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{(D^\alpha u(x) - D^\alpha u(y))\overline{(D^\alpha v(x) - D^\alpha v(y))}}{|x-y|^{d+2\delta}} dxdy,$$

and with the associated norm

$$\|u\|_{W_2^s(\Omega)} := \left(\|u\|^2_{W_2^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{d+2\delta}} dxdy \right)^{1/2},$$

known as Sobolev-Slobodeckii norm, see [100] and references in there.

Next we introduce *Sobolev spaces* $H^s(\Omega)$ which can be equivalent to the above *Sobolev spaces* $W_2^s(\Omega)$ under some regularity assumptions on Ω. To this end let us first introduce the space $\mathbb{S}(\mathbb{R}^d)$ of rapidly decreasing functions.

Definition 2.3.

$$\mathbb{S}(\mathbb{R}^d) := \{ f \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} |x^\beta D^\alpha f(x)| < \infty, \quad \textit{for all multi-indices } \alpha \textit{ and } \beta \}.$$

Further, we set $\mathbb{S}^*(\mathbb{R}^d)$ the space of tempered distributions or the space of continuous linear maps defined on $\mathbb{S}(\mathbb{R}^d)$, see [2, p. 251]. By using the Fourier transform

$$\mathfrak{F}u(\xi) := \widehat{u}(\xi) = \int\limits_{\mathbb{R}^d} e^{-i2\pi x\cdot\xi} u(x)dx$$

for $u \in L_1(\mathbb{R}^d)$, we define the *Sobolev space* $H^s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ as follows

$$H^s(\mathbb{R}^d) := \{u \in \mathbf{S}^*(\mathbb{R}^d) : (1+|\xi|^2)^{s/2}\widehat{u}(\xi) \in L_2(\mathbb{R}^d)\},$$

see [25, p. 42]. We have then $W_2^s(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$, see [100, Theorem 2.14]. We can define the *Sobolev spaces* $H^s(\Omega)$ as follows.

Definition 2.4. *Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be an open and bounded set, and let $s \in \mathbb{R}$, then*

$$H^s(\Omega) := \{v = \tilde{v}|_\Omega : \tilde{v} \in H^s(\mathbb{R}^d)\},$$

equipped with the norm

$$\|v\|_{H^s(\Omega)} := \inf_{\tilde{v}\in H^s(\mathbb{R}^d), \tilde{v}|_\Omega = v} \|\tilde{v}\|_{H^s(\mathbb{R}^d)}.$$

In addition,

$$\begin{aligned}\tilde{H}^s(\Omega) &:= \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^d)}},\\ H_0^s(\Omega) &:= \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}.\end{aligned}$$

Furthermore, for Lipschitz domains we have the following relations between *Sobolev spaces*.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a Lipschitz domain. For $s \geq 0$ we have*

$$\begin{aligned}W_2^s(\Omega) &= H^s(\Omega),\\ \tilde{H}^s(\Omega) &\subset H_0^s(\Omega),\\ \tilde{H}^s(\Omega) &= H_0^s(\Omega) \text{ for } s \notin \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\right\}.\end{aligned}$$

Moreover, $\tilde{H}^s(\Omega) = [H^{-s}(\Omega)]^*$, $H^s(\Omega) = [\tilde{H}^{-s}(\Omega)]^*$ for all $s \in \mathbb{R}$.

Proof. see [100, p. 33]. ☐

An important property of *Sobolev spaces* is that they can be embedded in several other spaces. Moreover, these embeddings are sometimes compact. The most important one is the Rellich-Kondrachov theorem.

Theorem 2.2. ***(Rellich-Kondrachov)*** *Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a Lipschitz domain and let $m \in \mathbb{Z}_+$. Then there hold the following compact embeddings*

$$H^m(\Omega) \hookrightarrow H^j(\Omega) \quad \text{for } j \leq m,$$
$$H_0^m(\Omega) \hookrightarrow H_0^j(\Omega) \quad \text{for } j \leq m,$$
$$H^m(\Omega) \hookrightarrow C^j(\overline{\Omega}) \quad \text{for } m \geq \frac{d}{2} + j.$$

Proof. See [2, Theorem 6.2]. □

See [80, Corollary 9.1] for an almost similar result for $s \in \mathbb{R}$ and $s > 0$. But there the embedding is only continuous.

2.1.2 Sobolev spaces on the boundary

In this section we assume that $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ is at least a Lipschitz domain and further we set $\Gamma := \partial\Omega$ to be its boundary.
The spaces $L_2(\Gamma)$ and $H^s(\Gamma)$ for $s \in (0,1)$ are defined respectively as the closure of the set $C^0(\Gamma)$ as follows

$$L_2(\Gamma) := \overline{C^0(\Gamma)}^{\|.\|_{L_2(\Gamma)}},$$
$$H^s(\Gamma) := \overline{C^0(\Gamma)}^{\|.\|_{H^s(\Gamma)}}.$$

In addition, the spaces $L_2(\Gamma)$ and $H^s(\Gamma)$ for $s \in (0,1)$ are Hilbert spaces endowed with the inner products

$$(u,\mathrm{v})_{L_2(\Gamma)} := \int\limits_{\Gamma} u(x)\overline{\mathrm{v}(x)}ds_x,$$
$$(u,\mathrm{v})_{H^s(\Gamma)} := (u,\mathrm{v})_{L_2(\Gamma)} + \int\limits_{\Gamma}\int\limits_{\Gamma} \frac{[u(x)-u(y)]\overline{[\mathrm{v}(x)-\mathrm{v}(y)]}}{|x-y|^{d-1+2s}}ds_x ds_y$$

respectively, and the associated norms are defined by

$$\|u\|_{L_2(\Gamma)} := \left(\int\limits_{\Gamma} |u(x)|^2 ds_x\right)^{1/2},$$
$$\|u\|_{H^s(\Gamma)} := \left(\|u\|_{L_2(\Gamma)}^2 + \int\limits_{\Gamma}\int\limits_{\Gamma} \frac{|u(x)-u(y)|^2}{|x-y|^{d-1+2s}}ds_x ds_y\right)^{1/2}$$

respectively. $\|.\|_{H^s(\Gamma)}$ is the *Sobolev-Slobodeckii* norm, see [53, p. 172]. The above definitions for the *Sobolev spaces* on the boundary can also be extended for the case of $s > 1$, but this requires stronger regularity assumptions for the boundary, which is more than the Lipschitz property, for example of class $C^{m,k}$ with $s \leq m+k$, see, e.g. [53]. For $s < 0$ the *Sobolev spaces* $H^s(\Gamma)$ are defined as the dual spaces of $H^{-s}(\Gamma)$ with respect to the $L_2(\Gamma)$-inner product, that is

$$H^s(\Gamma) := [H^{-s}(\Gamma)]^*,$$

with the associated norm given by

$$\|t\|_{H^s(\Gamma)} := \sup_{0 \neq v \in H^{-s}(\Gamma)} \frac{|\langle t, \mathrm{v} \rangle_\Gamma|}{\|\mathrm{v}\|_{H^{-s}(\Gamma)}}.$$

Let Γ_0 be an open subset of a sufficient smooth boundary $\Gamma := \partial\Omega$. For $s \geq 0$ we introduce the following *Sobolev spaces*

$$\begin{aligned} H^s(\Gamma_0) &:= \{\mathrm{v} = \tilde{\mathrm{v}}|_{\Gamma_0} : \tilde{\mathrm{v}} \in H^s(\Gamma)\}, \\ \tilde{H}^s(\Gamma_0) &:= \{\mathrm{v} = \tilde{\mathrm{v}}|_{\Gamma_0} : \tilde{\mathrm{v}} \in H^s(\Gamma), \mathrm{supp}\ \tilde{\mathrm{v}} \subset \Gamma_0\}, \end{aligned}$$

where the norm

$$\|\mathrm{v}\|_{H^s(\Gamma_0)} := \inf_{\tilde{\mathrm{v}} \in H^s(\Gamma) : \tilde{\mathrm{v}}|_{\Gamma_0} = \mathrm{v}} \|\tilde{\mathrm{v}}\|_{H^s(\Gamma)}.$$

For $s < 0$ the above *Sobolev spaces* are defined by duality as follows

$$H^s(\Gamma_0) := [\tilde{H}^{-s}(\Gamma_0)]^*, \quad \tilde{H}^s(\Gamma_0) := [H^{-s}(\Gamma_0)]^*.$$

If we assume that the boundary Γ is closed and piecewise smooth, i.e.

$$\Gamma = \cup_{i=1}^J \overline{\Gamma}_i, \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \text{for} \quad i \neq j,$$

then for $s > 0$ the *Sobolev space* $H^s_{pw}(\Gamma)$ is defined by

$$H^s_{pw}(\Gamma) := \{\mathrm{v} \in L_2(\Gamma) : \mathrm{v}|_{\Gamma_i} \in H^s(\Gamma_i),\ i = 1, \ldots, J\}$$

with the norm

$$\|\mathrm{v}\|_{H^s_{pw}(\Gamma)} := \left(\sum_{i=1}^J \|\mathrm{v}|_{\Gamma_i}\|^2_{H^s(\Gamma_i)} \right)^{1/2},$$

while for $s < 0$ we have

$$H^s_{pw}(\Gamma) := \prod_{i=1}^J \tilde{H}^s(\Gamma_i)$$

with the associated norm

$$\|\mathrm{v}\|_{H^s_{pw}(\Gamma)} := \sum_{i=1}^J \|\mathrm{v}|_{\Gamma_i}\|_{\tilde{H}^s(\Gamma_i)}.$$

Further we have the following result.

Lemma 2.1. *If* $w \in H^s_{pw}(\Gamma)$ *for* $s < 0$ *then* $\|w\|_{H^s(\Gamma)} \leq \|w\|_{H^s_{pw}(\Gamma)}$.

Proof. See [100, Lemma 2.20]. □

Trace theorem

To solve partial differential equations it is necessary to have boundary values, but *Sobolev spaces* are in general subspaces of L_p in which boundary values make no sense. However, for *Sobolev spaces* having higher oder it is possible to define the boundary values, (see e.g. Theorem 2.2).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be an open and bounded domain. The interior trace operator $\gamma_0^{int} : C^\infty(\overline{\Omega}) \to C^\infty(\Gamma)$ is defined by*

$$\gamma_0^{int} u := u|_\Gamma.$$

If Ω is of class $C^{k-1,1}$, γ_0^{int} is then extended by continuity to a continuous linear mapping

$$\gamma_0^{int} : H^s(\Omega) \to H^{s-1/2}(\Gamma) \quad \text{for } \frac{1}{2} < s \leq k.$$

Moreover, this extension is surjective and has a continuous right inverse mapping

$$\mathcal{E} : H^{s-1/2}(\Gamma) \to H^s(\Omega).$$

Proof. See [80, Theorem 9.4]. □

If Ω is bounded and Lipschitz, that is $k = 1$, the above result remains then valid for $s \in (\frac{1}{2}, \frac{3}{2})$, see [100, Theorem 2.21].

Remark 2.2. *As a consequence of the surjectivity we have*

$$\|g\|_{H^s(\Gamma),\gamma_0} := \inf_{v \in H^{s+1/2}(\Omega),\gamma_0^{int} v = g} \|v\|_{H^{s+1/2}(\Omega)} \quad \text{for } s \in (0,1).$$

In what follows we are going to define suitable *Sobolev spaces* to represent trace operators on the contact boundary. Since the variables are of d-dimensional type, we define the *Sobolev space*

$$\mathbf{H}^1(\Omega) := \prod_{j=1}^{d} H^1(\Omega).$$

In a similar way we define $\mathbf{L}_2(\Omega)$, $\mathbf{H}^{\frac{1}{2}}(\Gamma)$, etc. We introduce

$$\mathbb{V} := \{\underline{v} \in \mathbf{H}^1(\Omega) : \gamma_0^{int} \underline{v} = 0 \quad \text{on } \Gamma_D\}$$

as the set of admissible deformations, where $\Gamma_D \subset \Gamma$ denotes the set with given Dirichlet data. Let us denote by $\Sigma := \text{int}(\Gamma \setminus \Gamma_D)$ the interior of $\Gamma \setminus \Gamma_D$, by $\Gamma_C \subset \Sigma$ the nonempty open region of a possible contact, and by $\Gamma_N := \mathrm{int}(\Sigma \setminus \Gamma_C)$ the set with given Neumann conditions. Note that if the trace operator is considered from $\mathbb{V}$ to $\mathbf{H}^{\frac{1}{2}}(\Sigma)$ it is not surjective. To avoid this inconvenience, let us assume that $\overline{\Gamma}_C \subset \Sigma$, and that $\partial\Gamma_C$, $\partial\Sigma \subset \Gamma$ are smooth and let us define a closed subspace of $\mathbf{H}^{\frac{1}{2}}(\Sigma)$ by

$$\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma) := \{\underline{\xi} \in \mathbf{L}_2(\Sigma) : \exists \underline{\mathrm{v}} \in \mathbb{V}, \gamma_0^{int}\underline{\mathrm{v}}|_\Sigma = \underline{\xi}\} = \tilde{\mathbf{H}}^{1/2}(\Sigma).$$

For an equivalent way to define $\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)$, see [25, 67]. In addition, we suppose that the boundary Γ is more regular, that is of class $C^{1,1}$. Then a unit outward normal vector to Γ, $\underline{n} = (n_1, ..., n_d)^\top$ exists almost everywhere on Γ and is Lipschitz continuous, see [67, Theorem 5.4]. As a consequence we can decompose an element of $\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)$ into normal and tangential components, which enable us to define the surjective, continuous and linear normal trace operator

$$\gamma_N : \mathbb{V} \to H^{\frac{1}{2}}_{00}(\Sigma)$$

by $\gamma_N\underline{\mathrm{v}} := (\gamma_0^{int}\underline{\mathrm{v}})^\top\underline{n} := \mathrm{v}_n$, and the corresponding tangential trace operator

$$\gamma_T : \mathbb{V} \to \mathbf{H}^{\frac{1}{2}}_{T00}(\Sigma) := \{\underline{\mathrm{v}} \in \mathbf{H}^{\frac{1}{2}}_{00}(\Sigma) : \gamma_N\underline{\mathrm{v}} = 0\}$$

by $\gamma_T\underline{\mathrm{v}} := \underline{\mathrm{v}} - (\gamma_N\underline{\mathrm{v}})\underline{n} := \underline{\mathrm{v}}_t$, which is also linear, continuous and surjective, see [67, p. 88]. Since $\overline{\Gamma}_C \subset \Sigma$ is smooth, we have

$$\mathbf{H}^{\frac{1}{2}}(\Gamma_C) := \{\underline{\mathrm{v}} = \underline{\tilde{\mathrm{v}}}|_{\Gamma_C} : \underline{\tilde{\mathrm{v}}} \in \mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)\}.$$

Therefore, the trace operators on the contact boundary Γ_C are defined as the restriction of mappings γ_N and γ_T on Γ_C as follows

$$\begin{aligned} \gamma_{N_c} &:= \gamma_N|_{\Gamma_C} : \mathbb{V} \to H^{\frac{1}{2}}(\Gamma_C), \\ \gamma_{T_c} &:= \gamma_T|_{\Gamma_C} : \mathbb{V} \to \mathbf{H}^{\frac{1}{2}}_T(\Gamma_C) := \{\underline{\mathrm{v}} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_C) : \gamma_{N_c}\underline{\mathrm{v}} = 0\}, \end{aligned}$$

which are also linear, continuous and surjective. In the sequel, if no confusion occurs, we will set $\gamma_{N_c}\underline{\mathrm{v}} := \mathrm{v}_n$ and $\gamma_{T_c}\underline{\mathrm{v}} := \underline{\mathrm{v}}_t$. On the other hand, since we are also interested in time-dependent problems, the following is a short overview on the spaces of time-dependent functions.

2.1.3 Spaces of time-dependent functions

Let X be a Banach space and X^* its topological dual, and let T be a positive number.

Definition 2.5. *Let $m = 0, 1, 2, ...,$ and let $v^{(k)}$ be the k^{th} derivative with respect to t. The Banach space $C^m([0,T],X)$ is defined by*

$$C^m([0,T],X) := \{v : [0,T] \to X : v^{(k)} \text{ is continuous from } [0,T] \text{ to } X \text{ for } k \leq m\}.$$

The Banach space $C^m([0,T],X)$ is equipped with the norm

$$\|\mathrm{v}\|_{C^m([0,T],X)} := \sum_{k=0}^{m} \max_{0 \leq t \leq T} \|\mathrm{v}^{(k)}(t)\|_X.$$

Definition 2.6. *For $1 \leq p < \infty$, $L_p(0,T;X)$ is the space of all measurable functions v from $[0,T]$ to X for which*

$$\left(\int_0^T \|v(t)\|_X^p dt \right)^{1/p} < \infty.$$

For $1 \leq p < \infty$, $L_p(0,T;X)$ is a Banach space equipped with the norm

$$\|\mathrm{v}\|_{L_p(0,T;X)} := \left(\int_0^T \|\mathrm{v}(t)\|_X^p dt \right)^{1/p}.$$

Remark 2.3.

(i) For $p = \infty$, the space $L_\infty(0,T;X)$ is the set of all measurable functions v from $[0,T]$ to X that are essentially bounded. Moreover, $L_\infty(0,T;X)$ is also a Banach space equipped with the following norm

$$\|v\|_{L_\infty(0,T;X)} := ess \sup_{0 \leq t \leq T} \|v(t)\|_X.$$

(ii) For $1 < p < \infty$, the topological dual space of $L_p(0,T;X)$ is defined by

$$[L_p(0,T;X)]^* = L_q(0,T;X^*) \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1.$$

Definition 2.7. *For $m \geq 0$ integer, the Banach space $H^m(0,T;X)$ is defined by*

$$H^m(0,T;X) := \{f \in L_2(0,T;X) : f^{(i)} \in L_2(0,T;X), \quad i \leq m\},$$

where $f^{(i)} := \dfrac{\partial^i}{\partial t^i} f$ are partial derivatives with respect to t.

The Banach space $H^m(0,T;X)$ is equipped with the norm

$$\|f\|_{H^m(0,T;X)} = \left(\sum_{i=0}^{m} \|f^{(i)}\|^2_{L_2(0,T;X)} \right)^{\frac{1}{2}}.$$

Remark 2.4. *If X is a Hilbert space with inner product $(.,.)_X$, then, see [40]:*

(i) $L_2(0,T;X)$ is a Hilbert space with the inner product

$$(u,v)_{L_2(0,T;X)} = \int_0^T (u(t),v(t))_X dt.$$

(ii) Also $H^m(0,T;X)$ is a Hilbert space equipped with the following inner product

$$(u,v)_{H^m(0,T;X)} = \int_0^T \sum_{i=0}^{m} (u^{(i)}(t),v^{(i)}(t))_X dt.$$

We also have the following embedding result.

Theorem 2.4. *The embedding $H^1(0,T;X) \hookrightarrow C([0,T];X)$ is continuous, that is there exists a constant $c > 0$ such that*

$$\|v\|_{C([0,T];X)} \leq c\|v\|_{H^1(0,T;X)} \quad \textit{for all } v \in H^1(0,T;X).$$

Proof. See [40]. □

As a consequence of the above result, for v $\in H^1(0,T;X)$, v(0) is understood in the sense of the embedding $H^1(0,T;X) \hookrightarrow C([0,T];X)$.

2.2 Convex analysis and duality theory

In this section we present some results from convex analysis and duality theory that will be used in the following chapters. We first define convex sets and convex functions, state the orthogonal projection onto convex sets and its characterization. Further, we define convex conjugate functions and the subdifferential of a convex function. We end this section with the Fenchel duality theorem. For more detailed informations regarding this section we refer the reader to the following books [11, 31, 40, 68].

2.2.1 Convex sets and convex functions

Let X be a given Banach and reflexive space (Hilbert space), and X^* its topological dual space.

Definition 2.8.

(a) *Let K be a subset of X. K is said to be convex if for any x, $y \in K$ and for any $\theta \in (0,1)$, $\theta x + (1-\theta)y \in K$.*

(b) *Consider a function $f : X \to \overline{\mathbb{R}}$, $\left(\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}\right)$. f is called a convex function if (and only if)*

$$f((1-\theta)x_1 + \theta x_2) \leq (1-\theta) f(x_1) + \theta f(x_2) \quad \textit{for all} \;\; x_1, x_2 \in X \;\; \textit{and} \;\; \theta \in (0,1).$$

2.2.2 Projections onto convex sets

Let H be a Hilbert space and let $K \subset H$ be a nonempty, closed and convex set. Then there exists a unique mapping $P : H \to K$ such that

$$\|\mathrm{v} - P\mathrm{v}\|_H := \inf_{u \in K} \|\mathrm{v} - u\|_H \quad \text{for all } \mathrm{v} \in H.$$

The mapping P is called the orthogonal projection of H onto K. Furtheremore, the well-known characterization of P is given by:

Theorem 2.5. *The mapping P is characterized by: for all $v \in H$*

$$\langle v - Pv, u - Pv \rangle_{H \times H} \leq 0 \quad \textit{for all } u \in K.$$

In addition, for all $v_1, v_2 \in H$ the following holds:

$$\|Pv_1 - Pv_2\|_H \leq \|v_1 - v_2\|_H.$$

Proof. See [11, p. 79] or [68, p. 9]. □

2.2.3 Convex conjugate functions and the subdifferential

We start this paragraph by recalling some useful definitions. Let us consider a function $f : X \to \overline{\mathbb{R}}$.

Definition 2.9.

(a) *The set defined by*

$$dom(f) = \{x \in X : f(x) < \infty\}$$

is called the effective domain of f.

(b) *The function f is said to be proper if $dom(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$.*

(c) *f is said to be lower semi-continuous (l.s.c) at x_0 if and only if $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$ holds for any sequence (x_n), $x_n \in X$, satisfying $\lim_{n\to\infty} x_n = x_0$.*

(d) *f is said to be weakly lower semi-continuous (weakly l.s.c) at x_0 if and only if $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$ holds for any sequence (x_n), $x_n \in X$, satisfying $x_n \rightharpoonup x_0$ as $n \to \infty$.*

Note that $x_n \rightharpoonup x_0$ means that $x_n \to x_0$ weakly.

Theorem 2.6. *Let $f : X \to \overline{\mathbb{R}}$ be a convex functional. If f is lower semi-continuous (l.s.c), then f is weakly lower semi-continuous.*

Proof. See [40]. ☐

Definition 2.10. *A function $g : X \to [0,\infty]$ is called a gauge if*

(i) *$g(x) \geq 0$ for all $x \in X$,*

(ii) *$g(0) = 0$,*

(iii) *g is convex, positively homogeneous, and lower semi-continuous.*

Definition 2.11. *Let $A \subset X$. The indicator function I_A of the set A is defined by*

$$I_A(x) = \begin{cases} 0 & \text{if} \quad x \in A, \\ \infty & \text{if} \quad x \notin A. \end{cases}$$

Further, we define the conjugate of a convex function.

Definition 2.12. *Let $f : X \to \overline{\mathbb{R}}$ be a convex functional, and let X^* be the topological dual of X. The convex conjugate function $f^* : X^* \to \overline{\mathbb{R}}$ is defined by*

$$f^*(x^*) = \sup_{x\in X} \{\langle x^*, x\rangle - f(x)\}.$$

Convex conjugate functions are useful tools in optimization problems. In general, these functions permit to derive the so-called dual problems for given optimization problems, which often allow to gain a deeper insight into the problem structure, see [11, 31]. In addition, the convex conjugate function is closely related to the subdifferential of a convex function, which will be given in the next step.

Definition 2.13. *Let $f : X \to \overline{\mathbb{R}}$ be a proper and convex function, then $x^* \in X^*$ is said to be a subgradient of f at $x \in dom(f)$ if*

$$f(y) - f(x) \geq \langle x^*, y - x\rangle \quad \text{for all } y \in X.$$

Moreover, the set of all the subgradients of f at x is called subdifferential and denoted by $\partial f(x)$, where

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle \quad \text{for all } y \in X\}.$$

Note that f is subdifferentiable at x if $\partial f(x) \neq \emptyset$. Moreover, the set $\partial f(x)$ is convex and closed, see [31, p.21]. If $f \in C^1(\mathbb{R}^d)$ and convex, then $\partial f(x) = \nabla f(x)$ for all $x \in \mathbb{R}^d$. Conversely, if f is continuous, finite and the set $\partial f(x)$ has only one element at x then $f \in C^1(\mathbb{R}^d)$, see [31, p.22].
We now have a direct consequence of the above definition, that is the role of subdifferentiation in optimization problems:

$$f(x) = \min_{y \in X} f(y) \quad \text{if and only if } 0 \in \partial f(x).$$

Lemma 2.2. *Let X be a Banach and reflexive space and let the mapping $g : X \to [0,\infty]$ be a gauge. Then its convex conjugate $g^* : X^* \to [0,\infty]$ is defined by*

$$g^*(x^*) = \begin{cases} 0 & \text{if} \quad x^* \in \partial g(0), \\ \infty & \quad \text{else}. \end{cases}$$

Proof. See [40, Lemma 4.2]. □

Further, we derive the dual formulation of an optimization problem and state the Fenchel duality theorem that characterizes the relation between the primal and dual problem.

2.2.4 Fenchel duality theory

In this section we present the Fenchel duality theory in general Banach spaces. For the detailed proof of our statement we refer the reader to the famous text book by Ekeland and Témam [31].

Let X and Y be Banach spaces and let X^* and Y^* be their topological duals respectively. Furthermore, let Λ be a linear and bounded mapping from X to Y and let $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$, be convex, proper and lower semi-continuous. Let us now consider the following optimization problem, the so-called primal problem

$$(\mathcal{P}) \qquad \inf_{x\in X}[f(x)+g(\Lambda x)].$$

The corresponding dual problem to $(\mathcal{P})$ is defined by

$$(\mathcal{P}^*) \qquad \sup_{y^*\in Y^*}[-f^*(-\Lambda' y^*)-g^*(y^*)],$$

where $f^* : X^* \to \overline{\mathbb{R}}$ and $g^* : Y^* \to \overline{\mathbb{R}}$ denote the convex conjugates of f and g respectively, while $\Lambda' \in \mathcal{L}(Y^*,X^*)$ is the adjoint operator to Λ, see, e.g. [11, p.11] or [31, p.59].

Theorem 2.7. *Let us assume that X is a reflexive Banach space and that there exists $x_0 \in X$ such that $f(x_0) < \infty$ and $g(\Lambda x_0) < \infty$, and g is continuous at Λx_0. Furthermore, we suppose that $f(x)+g(\Lambda x) \to \infty$ for $\|x\| \to \infty$. Then the problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ admit (at least) one solution and*

$$\inf_{x\in X}[f(x)+g(\Lambda x)] = \sup_{y^*\in Y^*}[-f^*(-\Lambda' y^*)-g^*(y^*)].$$

Proof. See [11, p.11] or [31, Theorem 4.2]. □

The next result states the extremality conditions that relate solutions of the primal and the dual problems.

Theorem 2.8. *If $(\mathcal{P})$ and $(\mathcal{P}^*)$ admit solutions and if*

$$\inf_{x\in X}[f(x)+g(\Lambda x)] = \sup_{y^*\in Y^*}[-f^*(-\Lambda' y^*)-g^*(y^*)], \tag{2.1}$$

and this number is finite, then all solutions $\bar{x}$ of $(\mathcal{P})$ and all solutions $\bar{y}^*$ of $(\mathcal{P}^*)$ are related by the so-called extremality conditions

$$\begin{aligned} -\Lambda'\bar{y}^* &\in \partial f(\bar{x}), \\ \bar{y}^* &\in \partial g(\Lambda\bar{x}). \end{aligned} \tag{2.2}$$

Conversely, if $\bar{x} \in X$ and $\bar{y}^* \in Y^*$ satisfy (2.2), then $\bar{x}$ is a solution of $(\mathcal{P})$, and $\bar{y}^*$ a solution of $(\mathcal{P}^*)$ and (2.1) holds.

Proof. See [31, Proposition 2.4]. □

2.3 Generalized differentiability in function spaces

In this section we present a summary of some useful results about the Newton differentiability in Banach spaces that will be essential for the methods and analysis developed in the sequel. In the next paragraph we comment about the methods, further, we give some definitions and some important theorems.

2.3.1 Semi-smooth Newton methods

The application of semi-smooth Newton methods for non-differentiable operators in finite dimensions have been studied for decades [32, 33, 88, 90, 91]. Recently, that concept was developed in infinite dimension spaces (Banach spaces), see, e.g., [17,47,70,101]. We will prefer in this work instead of the terminology “slant differentiability in the neighborhood” as used in [17, 47] the name “Newton differentiability” as in [101]. It is shown in [47] that the primal-dual active set strategy can be interpreted as a certain application of the semi-smooth Newton method to nonlinear complementarity functions. The primal-dual active set strategy has been successfully applied to optimal control problems, see [46,58,60], and more recently to contact and Signorini problems [48, 54, 61, 71, 98, 99].

2.3.2 Definition and properties

In this section we define the Newton differentiability as presented in [47] and derive some inherent results relevant to our work. Let X, Y and Z be Banach spaces and let $F : D \subset X \to Y$ be a nonlinear mapping with D an open domain.

Definition 2.14. *The mapping F is said Newton differentiable on the open set $U \subset D$ if there exists a mapping $G : U \to \mathcal{L}(X,Z)$ such that*

$$\lim_{h\to 0} \frac{1}{\|h\|} \|F(x+h) - F(x) - G(x+h)h\| = 0 \quad \textit{for all } x \in U.$$

The mapping G is called a generalized derivative. Moreover, G is not uniquely defined, see, e.g. [17]. We now introduce the Newton derivative of certain functions that will be frequently used in this work. To this end let X be a space of real functions defined on Ω or Γ and let $\max(0,y)$ and $\min(0,y)$ be the pointwise max- and min-operators respectively. As candidates for the Newton derivatives we introduce

$$G_{\max}(y)(x) = \begin{cases} 1 & \text{if} \quad y(x) \geq 0, \\ 0 & \text{if} \quad y(x) < 0; \end{cases} \qquad G_{\min}(y)(x) = \begin{cases} 1 & \text{if} \quad y(x) \leq 0, \\ 0 & \text{if} \quad y(x) > 0. \end{cases}$$

We then have the following result:

Theorem 2.9. *The mappings* $\max(0,.) : L_q(\Omega) \to L_p(\Omega)$ *and* $\min(0,.) : L_q(\Omega) \to L_p(\Omega)$ *with* $1 \le p < q < \infty$ *are Newton differentiable on* $L_q(\Omega)$ *with the generalized derivatives* $G_{\max}$ *and* $G_{\min}$*, respectively.*

Proof. See [47]. □

Note that the above result holds for $p < q$. It is shown in [47] that $G_{\max}$ and $G_{\min}$ can not serve as generalized derivatives for $\max(0,y)$ and $\min(0,y)$ respectively if $p \ge q$.

We now focus on the application of the iterative process for the generalized Newton methods to solve a possible nonsmooth equation $F(x) = 0$. Based on the above differentiability concept, the solution of the problem is given by the following algorithm

$$x^{k+1} = x^k - G(x^k)^{-1} F(x^k), \quad k \ge 0, \tag{2.3}$$

where G is a generalized derivative as defined above. The next theorem shows the superlinear convergence of the above algorithm, that is

$$\lim_{k\to\infty} \frac{\|x_{k+1} - \bar{x}\|}{\|x_k - \bar{x}\|} = 0,$$

where $\bar{x}$ is the solution of the equation $F(x) = 0$. For the proof one can see [17, 47].

Theorem 2.10. *Let us assume that* $\bar{x} \in D$ *solves* $F(x) = 0$ *and that* F *is Newton differentiable in an open neighborhood* U *of* $\bar{x}$ *and that the set* $\{\|G(x)^{-1}\| : x \in U\}$ *is bounded. Then the Newton iteration (2.3) converges superlinear to* $\bar{x}$ *provided that* $\|x^0 - \bar{x}\|$ *is sufficiently small.*

Next we turn to one useful chain rule for Newton differentiability which will be frequently used in this work.

Theorem 2.11. *(Chain rule) Let* $F_1 : Y \to X$ *be an affine mapping with* $F_1 y := By + b$, $B \in \mathcal{L}(Y,X)$, $b \in X$*, and let us assume that* $F : X \to Z$ *is Newton differentiable on the open subset* $U \subset D$ *with generalized derivative* G*. If* $F_1^{-1}(U)$ *is nonempty, then* $F \circ F_1$ *is Newton differentiable on* $F_1^{-1}(U)$ *with the generalized derivative given by* $G(By+b)B \in \mathcal{L}(Y,Z)$*, for* $y \in F_1^{-1}(U)$.

Proof. See [59]. □

2.4 Circulant matrices

A circulant matrix is a special kind of Toeplitz matrix [38]. Circulant matrices can play an important role in investigating the characteristics of discrete boundary integral operators. In addition, it can also be of great interest for solving linear systems arising from the Galerkin discretization of boundary integral equations. For reference purposes, we point the reader to the elegant treatment given in [92–94] and the monograph [23] devoted to the subject.

2.4.1 Definition and characteristics of circulant matrices

Definition 2.15. *A matrix $B \in \mathbb{R}^{n\times n}$ with elements $(b_{k,l})$ is called a circulant matrix if*

$$b_{k+1,l+1} = b_{k,l} \quad for \quad k,l = 1,\ldots,n-1,$$

$$b_{k+1,1} = b_{k,n} \quad for \quad k = 1,\ldots,n-1.$$

It is clear that every circulant matrix is determined uniquely by its first row (or first column) unambiguously.

Remark 2.5. *The $n\times n$-matrix*

$$J = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

is the simplest circulant matrix and can be used to define the basis of circulant matrices. Furthermore, we have

$$J^n = I, \ \ where \ \ I \ \ is \ the \ identity \ matrix.$$

From the above definition it is easy to show the following result.

Proposition 2.1. *Every circulant matrix B can be represented as*

$$B = \sum_{l=1}^{n} b_{1l} J^{l-1}. \tag{2.4}$$

The formula (2.4) makes easy the examination of the spectrum of the matrix *B*.

Lemma 2.3. *The eigenvalues and eigenvectors of J are given respectively by*

$$w_k = e^{i\frac{2\pi}{n}(k-1)}, \quad k = 1,...,n,$$

$$f^{(k)} = \begin{pmatrix} w_k^0 \\ w_k^1 \\ . \\ . \\ w_k^{n-1} \end{pmatrix}, \quad k = 1,...,n$$

with $i^2 = -1$.

Proof. We have $det(J - wI) = (-1)^n w^n + (-1)^{n+1}$. Then the eigenvalues of J are the n^{th} roots of unity that are the complex solutions of the equation

$$w^n = 1.$$

On the other hand, the eigenvectors associated to the eigenvalues w_k are obtained by a direct computation. □

Definition 2.16. *A matrix* $F = (f^{(1)},..., f^{(n)})$ *which columns are made from the eigenvectors* $f^{(k)}$, $k = 1,...,n$, *is called the matrix of the discrete Fourier transform.*

The matrix F is complex and symmetric (i.e. $F = F^\top$), and

$$FF^* = F^*F = nI,$$

where F^* is the complex conjugate of the matrix F.

Proposition 2.2. *Every circulant matrix* $B = (b_{lk})$ *can be written as follows*

$$B = n^{-1}F\Lambda F^*$$

with

$$\Lambda = diag(\lambda_1,...,\lambda_n) = diag(Fb),$$

where $(b_{11},...,b_{1n}) = b^\top$ *is the first row of the matrix B.*

In addition, if B is a symmetric and circulant matrix, we then obtain

$$B = n^{-1}Q\Lambda Q,$$

where $Q = Re(F) + Im(F)$ and $\Lambda = diag(\lambda_1,...,\lambda_n)$, with $Q^2 = nI$, see, e.g., [15,92]. Note that a circulant matrix B is diagonalizable by the matrix F of the discrete Fourier transform. In addition, if B is invertible its inverse is then easily computed.

2.4.2 Application of circulant matrices

Given a system of linear equations

$$Bx = r, \tag{2.5}$$

where B is a circulant square matrix of size n and $r = (r_1, \ldots, r_n)^\top$. We can write the system of linear equations (2.5) as a circular convolution as follows

$$c * x = r,$$

where c is the first column of the circulant matrix B. Furthermore, the vectors c, x and r can be cyclically extended in each direction so that by using the results of the circular convolution theorem, we can apply the discrete Fourier transform to transform the cyclic convolution into a component-wise multiplication as follows

$$F(c * x) := F(c)F(x) = F(r).$$

Hence,

$$x = F^{-1}\left(\frac{F(r)}{F(c)}\right).$$

This algorithm is much faster than the standard Gaussian elimination, especially if a Fast Fourier transform is used [15, 16, 43, 94].

3 CONTACT PROBLEMS IN LINEAR ELASTODYNAMICS AND TIME DISCRETIZATION

In this chapter we introduce some basic equations that will be solved later by some appropriate numerical methods. Here we confine our attention basically on the linear elastodynamics model problem with its static correspondence, the elastostatic system for the static case and the contact problem in linear elastodynamics. Further, the governing equations of the considered physical problems are derived under the assumption that the changes in the state variable such as the displacement field and the strain tensor are infinitesimal in such a way that the resulting equations are linear. Moreover, a Cartesian coordinate system will be used for the spatial description. The chapter is organized as follows: In the first section the governing equations of linear elastodynamics are derived. Several details as kinematics and some balance laws are discussed. In the second and third section we derive the Signorini contact condition and the Coulomb friction law respectively and present the contact problem in linear elastodynamics. In section 4 we focus on the semi-discretization of the contact problem.

3.1 Linear elastodynamics

In this section basic ideas for the derivation of the governing equations of linear elastodynamics are summarized. Details of the derivation can be found in any textbook on the basic theory of elasticity. Special treatment of the elastodynamics model problem is detailed in the book by W. Han and D. Reddy [40] and [19, 102] for the linear elasticity case.

Kinematics

Let us consider a homogeneous elastic body which occupies at time $t = 0$ a region $\Omega \subset \mathbb{R}^d$ ($d = 2,3$) in its undeformed state (*reference configuration*). Subjected to dynamic forces the body moves and deforms, so that at time $t > 0$ it occupies a new region Ω_t, called the *current configuration*, see Figure 3.1. Therefore, a material particle initially at position x will be located at position $y(t,x)$ at time $t > 0$ which is called the deformation.

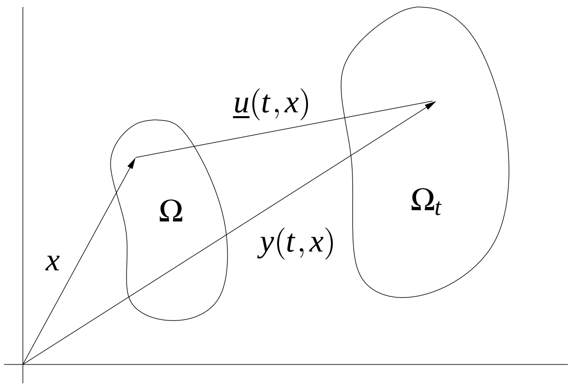


Figure 3.1: Undeformed and current configuration.

Obviously we have $y(0,x) = x$. Further, $x \in \Omega$ is called the Lagrangian coordinate while $y \in \Omega_t$ is the Eulerian coordinate. y can be written in component form as follows

$$y_i = y_i(t, x_1, \ldots, x_d), \;\; 1 \le i \le d \;\text{ for } x \in \Omega \text{ and for } t \in [0,T].$$

For further investigation we assume y to be differentiable, locally invertible and orientation-preserving, that is the Jacobian $J(t,x)$ satisfies

$$J(t,x) = \det\left(\frac{\partial y_i}{\partial x_j}\right)(t,x) > 0 \;\text{ for all }\; x \in \Omega,\; t \ge 0. \tag{3.1}$$

It can be suitable to introduce the *displacement* vector $\underline{u}$ by

$$\underline{u}(t,x) = y(t,x) - x$$

as the unknown primary variable. But this may not be enough to describe the complete deformation of the body, since it can not give any information about the deformation angle of particles. Therefore, we introduce as new variable the so-called *strain tensor* which is used to measure the deformation. To derive this quantity, let us consider a point $x \in \Omega$ and let Δx and δx be two vectors describing two fibers of material particles starting from x. The fiber Δx is mapped to the fiber $\Delta y = y(t, x+\Delta x) - y(t,x)$ in Ω_t. In the same way the fiber δx corresponds to the fiber $\delta y = y(t, x+\delta x) - y(t,x)$ in Ω_t. Since we assume y to be differentiable, a Taylor expansion of $y(t, x+\Delta x)$ in the neighborhood of x yields

$$y(t, x+\Delta x) = y(t,x) + \nabla y \Delta x + o(|\Delta x|).$$

By using $\nabla y = I + \nabla \underline{u}(x)$, where I is the $d \times d$-identity tensor, one obtains

$$\Delta y = y(t, x+\Delta x) - y(t,x) = \Delta x + \nabla \underline{u} \Delta x + o(|\Delta x|).$$

Likewise, we have

$$\delta y = y(t, x+\delta x) - y(t,x) = \delta x + \nabla \underline{u}\delta x + o(|\delta x|).$$

Therefore,

$$\Delta y \cdot \delta y - \Delta x \cdot \delta x = (\nabla \underline{u}\Delta x)\cdot \delta x + (\nabla \underline{u}\delta x)\cdot \Delta x + (\nabla \underline{u}\Delta x)\cdot(\nabla \underline{u}\delta x) + o(|\Delta x|^2 + |\delta x|^2). \quad (3.2)$$

Let us set now $h = \max\{|\Delta x|, |\delta x|\}$, $n = \Delta x/h$ and $m = \delta x/h$; n and m are assumed to be fixed vectors. Further, if we divide both sides of (3.2) by h^2 and take the limit when $h \to 0$ we then obtain

$$\lim_{h\to 0} \frac{\Delta y \cdot \delta y - \Delta x \cdot \delta x}{h^2} = 2n \cdot \eta(\underline{u}) \cdot m, \quad (3.3)$$

where η is the *strain tensor* associated to the displacement field $\underline{u}$ defined by

$$\eta(\underline{u}) = \frac{1}{2}[\nabla \underline{u} + (\nabla \underline{u})^\top + (\nabla \underline{u})^\top \nabla \underline{u}]. \quad (3.4)$$

Notice that if the body deforms as rigid body $\Delta y \cdot \delta y - \Delta x \cdot \delta x = 0$ which implies that $\eta(\underline{u}) = 0$ too. Moreover, under the assumption that the body undergoes infinitesimal deformations, that is the displacement gradient $\nabla \underline{u}$ is small enough, the nonlinear term in (3.4) can be neglected and we obtain the so-called *linearized strain tensor* defined by

$$\varepsilon(\underline{u}) = \frac{1}{2}[\nabla \underline{u} + (\nabla \underline{u})^\top]. \quad (3.5)$$

We now move to the investigation of the consequences of the applied forces on material bodies. Let us set ω to be an arbitrary subset of Ω which is mapped to ω_t in Ω_t, and let us denote the mass density and the velocity by $\tilde{\rho}(t,y)$ and $\underline{v}(t,y) = \frac{\partial}{\partial t} y(t,x) = \frac{\partial}{\partial t}\underline{u}(t,x)$ respectively. Further, we consider the following balance laws.

Conservation of mass

The principle of conservation of mass is the postulate that the mass of a fixed set of particles does not change in time, that is

$$\int_{\omega_t} \tilde{\rho}(t,y)dy = \int_{\omega} \tilde{\rho}(0,x)dx = \text{constant} \quad \text{for all } t.$$

By using the Reynolds theorem one obtains

$$0 = \frac{d}{dt}\int_{\omega_t} \tilde{\rho}(t,y)dy \equiv \int_{\omega_t} \left[\frac{\partial}{\partial t}\tilde{\rho}(t,y) + \mathrm{div}_y\,(\underline{v}(t,y)\tilde{\rho}(t,y))\right] dy \quad \text{for all } \omega_t,$$

which implies

$$\frac{\partial}{\partial t}\tilde{\rho}(t,y) + \mathrm{div}_y\,(\underline{v}(t,y)\tilde{\rho}(t,y)) = 0 \quad (3.6)$$

since ω_t is arbitrary. This is known as continuity equation.

Balance of linear momentum

Let $\tilde{f}(t,y)$ and $\underline{s}_n(t,y)$ be a volume force and the Cauchy stress vector respectively. The postulate of balance of linear momentum is the statement that the rate of change of linear momentum of a fixed mass of the body is equal to the sum of acting forces, that is

$$\frac{d}{dt}\int_{\omega_t} \tilde{\rho}(t,y)\mathrm{v}_i(t,y)dy = \int_{\omega_t} \tilde{f}_i(t,y)dy + \int_{\partial\omega_t} (\underline{s}_n)_i(t,y)ds_y \quad \text{for } i = 1,\ldots,d.$$

By using the Reynolds theorem and the conservation of mass (3.6) one obtains

$$\int_{\omega_t} \tilde{\rho}(t,y)\frac{d}{dt}\mathrm{v}_i(t,y)dy = \int_{\omega_t} \tilde{f}_i(t,y)dy + \int_{\partial\omega_t} (\underline{s}_n)_i(t,y)ds_y \quad \text{for } i = 1,\ldots,d.$$

In addition, we have $\underline{s}_n(t,y) = \sigma(t,y)\underline{n}_y$, where $\sigma(t,y)$ and $\underline{n}_y$ denote the Cauchy stress tensor and the unit outer normal vector to the boundary $\partial\omega_t$ respectively. Thus, by using the Stokes theorem we obtain

$$\int_{\partial\omega_t} (\underline{s}_n)_i(t,y)ds_y = \int_{\partial\omega_t} \sigma_i(t,y)\underline{n}_y ds_y = \int_{\omega_t} \mathrm{div}_y\,(\sigma_i(t,y))dy.$$

Taking this into the above equation one obtains

$$\tilde{\rho}(t,y)\frac{d}{dt}\underline{\mathrm{v}}(t,y) = \underline{\tilde{f}}(t,y) + \mathrm{div}_y\,(\sigma(t,y)) \tag{3.7}$$

since ω_t is arbitrary.

Balance of angular momentum

The total moment acting on ω_t is equal to the rate of change of the angular momentum of ω_t given by

$$\frac{d}{dt}\int_{\omega_t} y\times\tilde{\rho}(t,y)\underline{\mathrm{v}}(t,y)dy = \int_{\omega_t} y\times\underline{\tilde{f}}(t,y)dy + \int_{\partial\omega_t} y\times\underline{s}_n(t,y)ds_y. \tag{3.8}$$

This implies that the Cauchy stress tensor $\sigma(t,y)$ is symmetric, that is

$$\sigma(t,y) = [\sigma(t,y)]^{\top}. \tag{3.9}$$

By using the first Piola transformation given by

$$\tau(t,x) = J(t,x)\sigma(t,y)(D_x y)^{-\top} = J(t,x)\sigma(t,y(t,x))(I+\nabla\underline{u})^{-\top}, \tag{3.10}$$

the equation of motion (3.7) can be rewritten in Lagrangian coordinates as follows

$$\rho(t,x)\frac{\partial^2}{\partial t^2}\underline{u}(t,x) = \underline{f}(t,x) + \mathrm{div}_x\,(\tau(t,x)), \tag{3.11}$$

where the volume force density

$$\underline{f}(t,x) = J(t,x)\underline{\tilde{f}}(t,y) \tag{3.12}$$

and the mass density

$$\rho(t,x) = J(t,x)\tilde{\rho}(t,y). \tag{3.13}$$

Under the assumption that the body undergoes an infinitesimal deformation we have

$$\sigma(\underline{u}) \approx \tau(\underline{u}),$$

further the distinction between the current and reference configuration is ignored and the equation of motion (3.11) takes the form

$$\rho\underline{\ddot{u}}(t,x) - \mathrm{div}\,\sigma(\underline{u}(t,x)) = \underline{f}(t,x), \tag{3.14}$$

where $\underline{\ddot{u}}(t,x) := \frac{\partial^2}{\partial t^2}\underline{u}(t,x)$. The stress tensor is then given by the generalized Hooke's law

$$\sigma(\underline{u}) = C\varepsilon(\underline{u}), \tag{3.15}$$

where C is a fourth order tensor (independent of time) and satisfies the following symmetry properties

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{3.16}$$

In addition, we assume C to be elliptic, that is

$$C_{ijkl}\xi_{ij}\xi_{kl} \geq c_0\xi_{ij}\xi_{ij}, \tag{3.17}$$

where c_0 is a positive constant and ξ is any symmetric second order tensor ($\xi_{ij} = \xi_{ji}$). If we assume the material to be homogeneous and isotropic (that is, its response to a force does not depend of its orientation), the fourth order tensor C is then defined by two parameters as follows

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{3.18}$$

where δ_{ij} is the Kronecker symbol, and λ and μ are the so-called Lamé constants. Hooke's law is then given as follows

$$\sigma(\underline{u}) = \lambda(tr\varepsilon(\underline{u}))I + 2\mu\varepsilon(\underline{u}), \tag{3.19}$$

where $tr()$ represents the trace of the tensor, and I the $d \times d$ -identity tensor. For further investigation we assume the material to be not incompressible, and specify the dimension of the space d. For the three dimensional problem the Lamé constants are given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}, \tag{3.20}$$

where $E > 0$ and $\nu \in (0, 1/2)$ denote the Young modulus and the Poisson ratio respectively.

Plane elasticity

To describe the two-dimensional elasticity problem there exists two different methods, the plain strain approach and the plain stress approach. In the first approach the components of the strain tensor $e_{ij}(\underline{u},x)$ defined by

$$e_{ij}(\underline{u},x) \equiv \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.21}$$

depend only on the two first space coordinates (x_1,x_2) and all components in the third direction vanish, that is

$$\begin{aligned} e_{ij}(\underline{u},x_1,x_2,x_3) &= e_{ij}(\underline{u},x_1,x_2) \quad \text{for} \quad i,j=1,2, \\ e_{i3}(\underline{u},x) &= e_{3i}(\underline{u},x) = 0 \quad \text{for} \quad i,j=1,2,3. \end{aligned}$$

In addition, Hooke's law is given in this case by

$$\begin{aligned} \sigma_{ij}(\underline{u}) &= \lambda\delta_{ij}(e_{11}(\underline{u}) + e_{22}(\underline{u})) + 2\mu e_{ij}(\underline{u}) \quad \text{for} \quad i,j=1,2, \\ \sigma_{i3}(\underline{u}) &= \sigma_{3i}(\underline{u}) = 0 \quad \text{for} \quad i,j=1,2, \\ \sigma_{33}(\underline{u}) &= \lambda(e_{11}(\underline{u}) + e_{22}(\underline{u})), \end{aligned}$$

where λ and μ are the Lamé constants given in (3.20). In the second approach, the components $\sigma_{11}(\underline{u})$, $\sigma_{12}(\underline{u})$ and $\sigma_{22}(\underline{u})$ of the stress only depend on variables x_1 and x_2 and $\sigma_{13}(\underline{u}) = \sigma_{23}(\underline{u}) = \sigma_{33}(\underline{u}) = 0$. Applying Hooke's law we obtain

$$\begin{aligned} e_{3i}(\underline{u}) &= e_{i,3}(\underline{u}) = 0 \quad \text{for} \quad i=1,2, \\ e_{33}(\underline{u}) &= -\frac{\nu}{E}(\sigma_{11}(\underline{u}) + \sigma_{22}(\underline{u})). \end{aligned}$$

Additionally we have, see,e.g. [26, 100]

$$\begin{aligned} \sigma_{11}(\underline{u}) &= \frac{E}{(1+\nu)(1-\nu)}e_{11}(\underline{u}) + \frac{E\nu}{(1+\nu)(1-\nu)}e_{22}(\underline{u}), \\ \sigma_{22}(\underline{u}) &= \frac{E\nu}{(1+\nu)(1-\nu)}e_{11}(\underline{u}) + \frac{E}{(1+\nu)(1-\nu)}e_{22}(\underline{u}), \\ \sigma_{12}(\underline{u}) &= \frac{E}{(1+\nu)}e_{12}(\underline{u}), \end{aligned}$$

The initial boundary value problem of linear elasticity reads: Find the displacement field $\underline{u} \in \mathbb{R}^d$ for $(t,x) \in (0,T) \times \Omega$ such that
the *motion equation*

$$\rho\ddot{\underline{u}}(t,x) - \text{div}\,(\sigma(\underline{u}(t,x))) = \underline{f}(t,x) \quad \text{in } (0,T) \times \Omega, \tag{3.22}$$

the *elastic constitutive law*

$$\sigma(\underline{u}) = C\varepsilon(\underline{u}) \tag{3.23}$$

with the *strain-displacement*

$$\varepsilon(\underline{u}) = \frac{1}{2}(\nabla\underline{u} + (\nabla\underline{u})^\top), \tag{3.24}$$

the *boundary conditions*

$$\underline{u}(t,x) = \underline{g}_D(t,x) \text{ on } (0,T)\times\Gamma_D \text{ and } \sigma(\underline{u}(t,x))\underline{n}(t,x) = \underline{g}_N(t,x) \text{ on } (0,T)\times\Gamma_N, \tag{3.25}$$

the *initial conditions*

$$\underline{u}(0,x) = \underline{u}_0(x) \text{ and } \underline{\dot{u}}(0,x) = \underline{u}_1(x),\ x\in\Omega, \tag{3.26}$$

are satisfied with functions $\underline{g}_D$ and $\underline{g}_N$ which represent the given displacement and the traction on the Dirichlet boundary Γ_D and the Neumann boundary Γ_N respectively, where $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$. Notice that, if we assume the data to be independent of time the initial boundary value problem (3.22)-(3.26) becomes then a boundary value problem where we have to find the displacement field $\underline{u}(x)$ for $x \in \Omega$ such that the following relations are satisfied
the *equilibrium equation*

$$\text{div}\,(\sigma(\underline{u}(x))) + \underline{f}(x) = 0 \quad \text{in } \Omega, \tag{3.27}$$

the *elastic constitutive law*

$$\sigma(\underline{u}) = C\varepsilon(\underline{u}) \tag{3.28}$$

with the *strain-displacement*

$$\varepsilon(\underline{u}) = \frac{1}{2}(\nabla\underline{u} + (\nabla\underline{u})^\top), \tag{3.29}$$

the *boundary conditions*

$$\underline{u}(x) = \underline{g}_D(x) \text{ on } \Gamma_D \text{ and } \sigma(\underline{u}(x))\underline{n}(x) = \underline{g}_N(x) \text{ on } \Gamma_N. \tag{3.30}$$

Remark 3.1. *The space $\mathcal{R}$ of rigid motion for the elastostatics model problem is of finite dimension and its basis is defined as follows, see [100]:*

$$2D: \ \mathcal{R} = Span\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-x_2\\x_1\end{pmatrix}\right\}, \tag{3.31}$$

$$3D: \ \mathcal{R} = Span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-x_2\\x_1\\0\end{pmatrix}, \begin{pmatrix}x_3\\0\\-x_1\end{pmatrix}, \begin{pmatrix}0\\-x_3\\x_2\end{pmatrix}\right\}. \tag{3.32}$$

We can easily check that any member of the spaces (3.31) and (3.32) produces a zero strain, that is

$$\varepsilon(\underline{u}) = 0 \ \ \textit{for all } \underline{u} \in \mathcal{R}.$$

Before we proceed, let us introduce some usuful results that will be frequently used in this work. If we suppose the measure of the Dirichlet boundary Γ_D to be strictly positive, the space of test functions is then given by:

$$\mathbb{V} = \{\underline{\mathrm{v}} \in \mathbf{H}^1(\Omega) : \gamma_0^{int} \underline{\mathrm{v}} = 0 \quad \text{on} \quad \Gamma_D\},$$

where $\mathbf{H}^1(\Omega)$ is the Sobolev space defined by:

$$\mathbf{H}^1(\Omega) := (H^1(\Omega))^d$$

and

$$\gamma_0^{int} : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$$

is the trace operator. In addition, if we suppose the material to be homogeneous and isotropic we then obtain the following results:

Lemma 3.1. *For $\underline{w} \in \boldsymbol{H}^1(\Omega)$ we have*

$$A(\underline{w},\underline{w}) \geq \frac{E}{1+\nu} \int\limits_{\Omega} \sum_{i,j=1}^{d} (\varepsilon_{ij}(\underline{w}))^2 dx,$$

where $A(.,.) : \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \to \mathbb{R}$ is a bilinear form defined by

$$A(\underline{w},\underline{v}) = \int\limits_{\Omega} \sigma(\underline{w}) : \varepsilon(\underline{v}) dx.$$

Proof. This follows immediately from Hooke's law (3.19), i.e.

$$\begin{aligned} A(\underline{w},\underline{w}) &= 2\mu \int\limits_{\Omega} \sum_{i,j=1}^{d} (\varepsilon_{ij}(\underline{w}))^2 dx + \lambda \int\limits_{\Omega} (\operatorname{div} \underline{w})^2 dx \\ &\geq 2\mu \int\limits_{\Omega} \sum_{i,j=1}^{d} (\varepsilon_{ij}(\underline{w}))^2 dx = \frac{E}{1+\nu} \int\limits_{\Omega} \sum_{i,j=1}^{d} (\varepsilon_{ij}(\underline{w}))^2 dx. \end{aligned}$$

□

By using Lemma 3.1 we can show that $A(\underline{w},\underline{w})$ is a norm on $\mathbb{V}$. Indeed, note that because Γ_D has a positive measure,

$$\mathcal{R} \cap \mathbb{V} = \{0\},$$

where $\mathcal{R}$ denotes the space of all rigid motions given above. Consequently, we have then

$$\forall \underline{w} \in \mathbb{V}, \quad A(\underline{w},\underline{w}) = 0 \Rightarrow \varepsilon(\underline{w}) = 0 \Leftrightarrow \underline{w} \in \mathcal{R} \Rightarrow \underline{w} = 0.$$

Moreover, it can be shown that this norm is equivalent to the Hilbert norm in $\mathbf{H}^1(\Omega)$. This is a direct consequence of Korn's second inequality, see [25, 100].

Theorem 3.1. ***(Korn's Second Inequality).*** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with piecewise smooth boundary $\Gamma := \partial\Omega$. There exists a constant $c > 0$ (dependent on Ω) such that*

$$\int_\Omega \sum_{i,j=1}^{d} (\varepsilon_{ij}(\underline{w}))^2 dx + \|\underline{w}\|^2_{\boldsymbol{L}_2(\Omega)} \geq c\|\underline{w}\|^2_{\boldsymbol{H}^1(\Omega)} \quad \text{for all } \underline{w} \in \boldsymbol{H}^1(\Omega).$$

3.2 Contact and Signorini conditions

In this section we recall the setting of a unilateral contact condition (Signorini contact conditions) for linear elasticity. To this end, we consider a deformable body occupying in its reference configuration a domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$, as in the previous section, with boundary $\Gamma := \partial\Omega$ which is divided into three disjoint subsets Γ_D, Γ_N and Γ_C. We assume that the displacements, the stresses, and other functions used in this section are defined pointwise. We consider furthermore a rigid foundation with a surface denoted by Γ_F. In addition, we assume that the contact surface Γ_C and the rigid foundation are defined parametrically by

$$\begin{aligned} x_d &= \Phi(\tilde{x}),\ x = (x_1,\ldots,x_d) \in \Gamma_C, \\ x_d &= \Psi(\tilde{x}),\ x = (x_1,\ldots,x_d) \in \Gamma_F, \end{aligned} \tag{3.33}$$

where $\tilde{x} = (x_1,\ldots,x_{d-1})$. In addition, Φ and Ψ are assumed to be sufficiently smooth functions. For convenience, we assume that Γ_C lies above the rigid foundation Γ_F as shown in Figure 3.2, that is

$$\Phi(\tilde{x}) \geq \Psi(\tilde{x}).$$

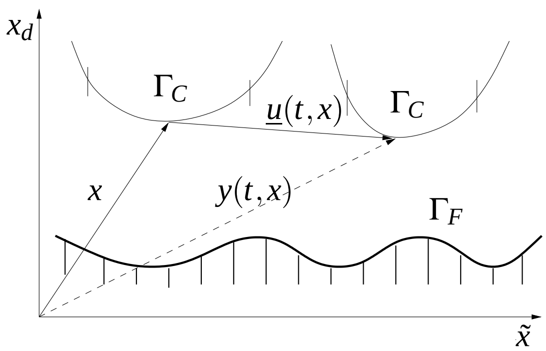


Figure 3.2: Unilateral contact.

Let $x = (\tilde{x}, \Phi(\tilde{x}))$ be the coordinate labels of a particle on the contact surface Γ_C in the reference configuration. After a certain deformation of the body, the particle x occupies at time t a new position denoted by y. Since the particle still lies on Γ_C, its deformation must satisfy

$$y_i = x_i + u_i(\tilde{x}, \Phi(\tilde{x})), \ i = 1, \ldots, d-1, \tag{3.34}$$

and

$$y_d := x_d + u_d(\tilde{x}, \Phi(\tilde{x})) = \Phi(\tilde{x}) + u_d(\tilde{x}, \Phi(\tilde{x})) \geq \Psi(\tilde{y}), \tag{3.35}$$

where $\tilde{y} := (y_1, \ldots, y_{d-1})$ and $\underline{u} := (u_1, \ldots, u_d)$ is the displacement field in Ω. The inequality (3.35) is the kinematic contact condition for finite displacements. From (3.34) we obtain

$$x_i = y_i - u_i, \ i = 1, \ldots, d-1.$$

Taking this into (3.35) yields

$$\Phi(y_1 - u_1, \ldots, y_{d-1} - u_{d-1}) + u_d \geq \Psi(\tilde{y}). \tag{3.36}$$

Moreover, if we suppose that the body is displaced to its current configuration by a small amount u_i for $i = 1, \ldots d$, then an expansion of Φ about $y_i = x_i + u_i$ yields

$$\frac{\partial}{\partial y_1}\Phi(\tilde{y})u_1 + \ldots + \frac{\partial}{\partial y_{d-1}}\Phi(\tilde{y})u_{d-1} - u_d \leq \Phi(\tilde{y}) - \Psi(\tilde{y}). \tag{3.37}$$

The unit outward normal vector to Γ_C at point $(y_1, \ldots, y_d)$ in the current configuration is given by

$$\underline{n}(y) = \left(\frac{\partial \Phi}{\partial y_1}, \ldots, \frac{\partial \Phi}{\partial y_{d-1}}, -1\right) \Big/ \sqrt{\left(\frac{\partial \Phi}{\partial y_1}\right)^2 + \ldots + \left(\frac{\partial \Phi}{\partial y_{d-1}}\right)^2 + 1}.$$

Thus, dividing both sides of inequality (3.37) by $\sqrt{\left(\frac{\partial \Phi}{\partial y_1}\right)^2 + \ldots + \left(\frac{\partial \Phi}{\partial y_{d-1}}\right)^2 + 1}$ we obtain the condition

$$\underline{n}(y) \cdot \underline{u}(y) \leq \mathbf{d}(y), \quad y = (y_1, \ldots, y_d) \in \Gamma_C, \tag{3.38}$$

where $\mathbf{d}(y)$ is the gap between the body and the rigid foundation in the current reference given by

$$\mathbf{d}(y) = \frac{\Phi(\tilde{y}) - \Psi(\tilde{y})}{\sqrt{\left(\frac{\partial \Phi}{\partial y_1}\right)^2 + \ldots + \left(\frac{\partial \Phi}{\partial y_{d-1}}\right)^2 + 1}}, \quad y_i = x_i + u_i(x).$$

Under the assumption of infinitesimal deformations the distinction between the reference and current configurations is ignored and (3.38) can be approximated by

$$\underline{n}(x) \cdot \underline{u}(x) - \mathbf{d}(x) \leq 0, \quad x \in \Gamma_C, \tag{3.39}$$

where $\underline{n}(x)$ and $\mathbf{d}(x)$ are the unit outward normal vector at x to Γ_C and the gap between the foundation Γ_F and the contact boundary Γ_C in the reference configuration respectively (see [67]). For convenience we will use in the sequel and onward $u_n(x) = \underline{n}(x) \cdot \underline{u}(x)$ for all $x \in \Gamma_C$.

Boundary stress relation

Taking into account the relation (3.39) there arises relations between the displacement field $\underline{u}$ and the boundary stress $\sigma(\underline{u})\underline{n}$ on Γ_C. Therefore, we first decompose the boundary stress on Γ_C into its normal and tangential components as follows

$$\sigma(\underline{u})\underline{n} := \sigma_n(\underline{u})\underline{n} + \sigma_t(\underline{u}), \tag{3.40}$$

where $\sigma_n(\underline{u})$ is a scalar given by $\sigma_n(\underline{u}) := \underline{n} \cdot (\sigma(\underline{u})\underline{n})$ and $\sigma_t(\underline{u}) := \sigma(\underline{u})\underline{n} - \sigma_n(\underline{u})\underline{n}$ is the tangential component on Γ_C.

Normal stress relation

The relation (3.39) indicates when the body and rigid foundation are in contact. Indeed, the body and the rigid foundation are said to be in contact if the initial gap is equal to the normal displacement, that is $u_n = \mathbf{d}$ on Γ_C, otherwise we have $u_n < \mathbf{d}$. Moreover, there is a transfer of forces between the body and the rigid foundation when they are in contact, otherwise no transfer occurs. These forces exert pressure only in the normal direction, and they do not depend directly on the material properties. All these statements can be formulated mathematically as follows:

$$\begin{aligned} u_n &< \mathbf{d} \Rightarrow \sigma_n(\underline{u}) = 0, \\ u_n &= \mathbf{d} \Rightarrow \sigma_n(\underline{u}) < 0. \end{aligned} \tag{3.41}$$

The above relation (3.41) can be written again in the following form

$$u_n \leq \mathbf{d} \text{ and } \sigma_n(\underline{u}) \leq 0 \text{ and } \sigma_n(\underline{u})(u_n - \mathbf{d}) = 0, \tag{3.42}$$

which is called the classical Signorini formulation. On the other hand, (3.42) is equivalent to

$$u_n \leq \mathbf{d},\ \sigma_n(\underline{u})(\mathrm{v}_n - u_n) \geq 0,\ \forall \mathrm{v}_n \leq \mathbf{d}. \tag{3.43}$$

Indeed, we have

$$\begin{aligned} \sigma_n(\underline{u})(\mathrm{v}_n - u_n) &= \sigma_n(\underline{u})(\mathrm{v}_n - \mathbf{d}) + \sigma_n(\underline{u})(\mathbf{d} - u_n), \\ &= \sigma_n(\underline{u})(\mathrm{v}_n - \mathbf{d}) + 0, \text{ see } (3.42), \\ &= \sigma_n(\underline{u})(\mathrm{v}_n - \mathbf{d}) \geq 0, \end{aligned}$$

since $\sigma_n(\underline{u}) \leq 0$ and $\mathrm{v}_n - \mathbf{d} \leq 0$. Conversely, let us suppose that (3.43) holds. If $u_n = \mathbf{d}$, then

$$\sigma_n(\underline{u})(\mathrm{v}_n - u_n) = \sigma_n(\underline{u})(\mathrm{v}_n - \mathbf{d}) \geq 0\ \ \forall \mathrm{v}_n \leq \mathbf{d},$$

which implies that $\sigma_n(\underline{u}) \leq 0$ since $\mathrm{v}_n - \mathbf{d} \leq 0$. Further, if $u_n < \mathbf{d}$ and $\mathrm{v}_n = \mathbf{d}$, then

$$\sigma_n(\underline{u})(\mathrm{v}_n - u_n) = \sigma_n(\underline{u})(\mathbf{d} - u_n) \geq 0,$$

which yields $\sigma_n(\underline{u})(\mathbf{d} - u_n) = 0$, and $\sigma_n(\underline{u}) = 0$. The relation (3.43) is called the variational formulation of the Signorini contact condition (3.42).

3.3 Friction law (Coulomb friction law)

In this work the Coulomb law will be used. It is the historically oldest and simplest phenomenological friction law, and it states that the sliding of the body depends on the proportionality of the tangential force F_t to the normal force F_n with which the body is pressed perpendicularly against the rigid foundation (see Figure 3.3).

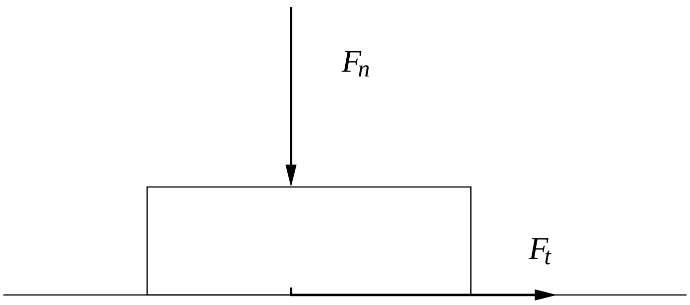


Figure 3.3: Tangential force.

In general, the friction coefficient depends upon different parameters like the surface roughness, the relative sliding velocity $\underline{\dot{u}}_t$ between the contacting bodies, the contact normal pressure P_n or the temperature. One of such friction laws which incorporates the sliding velocity can be written as follows

$$\mathcal{F}(|\underline{\dot{u}}_t|) = \mathcal{F}_D + (\mathcal{F}_S - \mathcal{F}_D)e^{-\beta|\underline{\dot{u}}_t|}, \tag{3.44}$$

where $\mathcal{F}_S$ and $\mathcal{F}_D$ represent the static and the dynamic friction coefficients respectively, and where β is a constitutive parameter which describes how fast the static coefficient approaches the dynamic one (see [102]). Therefore, to take into account the difference occurring in practice between static and dynamic friction coefficients at least approximately, the friction coefficient must be a function of the sliding velocity. Thus the friction law for a deformable body can take the form

$$\begin{aligned} \underline{\dot{u}}_t &= 0, \Rightarrow |\sigma_t(\underline{u})| \leq \mathcal{F}(0)|\sigma_n(\underline{u})|, \\ \underline{\dot{u}}_t &\neq 0, \Rightarrow \sigma_t(\underline{u}) = -\mathcal{F}(|\underline{\dot{u}}_t|)|\sigma_n(\underline{u})|\frac{\underline{\dot{u}}_t}{|\underline{\dot{u}}_t|}. \end{aligned} \tag{3.45}$$

This law describes the dependence of the tangential stress upon the normal stress and the sliding velocity: at the *sticking* point x (no sliding) we have $\underline{\dot{u}}_t = 0$ and the Euclidean norm of the tangential stress is bounded by the product of the friction coefficient and the magnitude of the normal stress. On the other hand, if the body slides at a point x, the tangential stress is then equal in magnitude to the product of the friction coefficient and the magnitude of the normal stress. In addition, this opposes the sliding velocity.

Remark 3.2. *If the surface roughness is not too large or not too smooth and if the sliding velocity is neither too large nor too small, the friction coefficient can then be chosen constant.*

Proposition 3.1. *The relation (3.45) is equivalent to*

$$\sigma_t(\underline{u}) \cdot (\underline{v}_t - \underline{\dot{u}}_t) + \mathcal{F}(|\underline{\dot{u}}_t|)|\sigma_n(\underline{u})|(|\underline{v}_t| - |\underline{\dot{u}}_t|) \geq 0, \ \forall \underline{v}_t. \tag{3.46}$$

Proof.

(i) Let us suppose that (3.45) holds and show (3.46).
For $\underline{\dot{u}}_t = 0$ we have $|\sigma_t(\underline{u})| \leq \mathcal{F}(0)|\sigma_n(\underline{u})|$ and

$$\begin{aligned} \sigma_t(\underline{u}) \cdot (\underline{\mathrm{v}}_t - \underline{\dot{u}}_t) + \mathcal{F}(|\underline{\dot{u}}_t|)|\sigma_n(\underline{u})|(|\underline{\mathrm{v}}_t| - |\underline{\dot{u}}_t|) &= \sigma_t(\underline{u}) \cdot \underline{\mathrm{v}}_t + \mathcal{F}(0)|\sigma_n(\underline{u})|\ |\underline{\mathrm{v}}_t| \\ &\geq (-|\sigma_t(\underline{u})| + \mathcal{F}(0)|\sigma_n(\underline{u})|)|\underline{\mathrm{v}}_t| \geq 0. \end{aligned}$$

For $\underline{\dot{u}}_t \neq 0$ we have $\sigma_t(\underline{u}) = -\mathcal{F}(|\underline{\dot{u}}_t|)|\sigma_n(\underline{u})|\frac{\underline{\dot{u}}_t}{|\underline{\dot{u}}_t|}$ and

$$\begin{aligned} \sigma_t(\underline{u}) \cdot (\underline{\mathrm{v}}_t - \underline{\dot{u}}_t) + \mathcal{F}(|\underline{\dot{u}}_t|)|\sigma_n(\underline{u})|(|\underline{\mathrm{v}}_t| - |\underline{\dot{u}}_t|) &= \mathcal{F}(|\underline{\dot{u}}_t|)|\sigma_n(\underline{u})| \left(-\frac{\underline{\dot{u}}_t}{|\underline{\dot{u}}_t|}\underline{\mathrm{v}}_t + |\underline{\mathrm{v}}_t| \right) \\ &\geq (-|\underline{\mathrm{v}}_t| + |\underline{\mathrm{v}}_t|) = 0. \end{aligned}$$

(ii) On the other hand, let us assume that (3.46) holds and show (3.45).
For $\underline{\dot{u}}_t = 0$, (3.46) yields

$$\sigma_t(\underline{u}) \cdot \underline{\mathrm{v}}_t + \mathcal{F}(0)|\sigma_n(\underline{u})|\ |\underline{\mathrm{v}}_t| \geq 0 \ \forall \underline{\mathrm{v}}_t,$$

that is $-\sigma_t(\underline{u}) \cdot \underline{\mathrm{v}}_t \leq \mathcal{F}(0)|\sigma_n(\underline{u})|\ |\underline{\mathrm{v}}_t|$. Further, if we set $\underline{\mathrm{v}}_t = -\sigma_t(\underline{u})$, with $\sigma_t(\underline{u}) \neq 0$ we then obtain $|\sigma_t(\underline{u})| \leq \mathcal{F}(0)|\sigma_n(\underline{u})|$. The case $\sigma_t(\underline{u}) = 0$ follows immediately.
For $\underline{\dot{u}}_t \neq 0$, if we set $\underline{\mathrm{v}}_t = 2\underline{\dot{u}}_t$ and $\underline{\mathrm{v}}_t = \frac{1}{2}\underline{\dot{u}}_t$ respectively into (3.46), this yields

$$\sigma_t(\underline{u}) \cdot \underline{\dot{u}}_t + \mathcal{F}(|\underline{\dot{u}}_t|)|\sigma_n(\underline{u})|\ |\underline{\dot{u}}_t| = 0. \tag{3.47}$$

This gives then, $\sigma_t(\underline{u}) = -\mathcal{F}(|\underline{\dot{u}}_t|)|\sigma_n(\underline{u})|\frac{\underline{\dot{u}}_t}{|\underline{\dot{u}}_t|}$.

□

The relation (3.46) is called the variational formulation of the Coulomb friction law (3.45). After the presentation of the linear elastodynamics governing equations and a closer examination of the contact condition and the friction law, we can formulate now the contact problem in linear elastodynamics, which reads: Find the displacement field $\underline{u}(t,x) : (0,T) \times \Omega \to \mathbb{R}^d$ such that

$$\begin{aligned} \rho \ddot{\underline{u}} - \operatorname{div}(\sigma(\underline{u})) &= \underline{f} \quad \text{in} \quad (0,T) \times \Omega, && (3.48)\\ \underline{u} &= \underline{g}_D \quad \text{on} \quad (0,T) \times \Gamma_D, && (3.49)\\ \sigma(\underline{u})\underline{n} &= \underline{g}_N \quad \text{on} \quad (0,T) \times \Gamma_N, && (3.50) \end{aligned}$$

$$u_n \leq \mathbf{d}, \quad \sigma_n(\underline{u}) \leq 0, \quad \sigma_n(\underline{u})(u_n - \mathbf{d}) = 0 \quad \text{on} \quad (0,T) \times \Gamma_C, \qquad (3.51)$$

$$\left.\begin{array}{l} \dot{\underline{u}}_t = 0 \Rightarrow |\sigma_t(\underline{u})| < \mathcal{F}(0)|\sigma_n(\underline{u})|, \\ \dot{\underline{u}}_t \neq 0 \Rightarrow \sigma_t(\underline{u}) = -\mathcal{F}(|\dot{\underline{u}}_t|)|\sigma_n(\underline{u})|\frac{\dot{\underline{u}}_t}{|\dot{\underline{u}}_t|}, \\ |\dot{\underline{u}}_t|(|\sigma_t(\underline{u})| - \mathcal{F}(|\dot{\underline{u}}_t|)|\sigma_n(\underline{u})|) = 0, \end{array}\right\} \quad \text{on} \quad (0,T) \times \Gamma_C, \qquad (3.52)$$

$$\underline{u}(0,x) = \underline{u}_0(x), \quad \dot{\underline{u}}(0,x) = \underline{u}_1(x) \quad \text{in} \quad \Omega, \qquad (3.53)$$

are satisfied.

The system of equations (3.48)-(3.53) is the general formulation of the contact problem in linear elastodynamics with Coulomb friction. But, unfortunately up to now there is no general proof concerning the existence and uniqueness of its solutions. However, very few results about existence of solutions for the frictionless case can be found in [13, 14] and a special one dimensional frictional case in [8]. Thus, in this work we will base our study on the static problem derived from the dynamic problem by an implicit time discretization and the quasistatic problem, i.e. the case where the body is assumed to be deformed very slowly so that the inertial forces are neglected in equation (3.48).

3.4 Backward time discretization

In general, the analytical solution of (3.48)-(3.53) is not available and for this reason a numerical approximation is required. Furthermore, initial boundary value problems of hyperbolic type appear so frequently in many areas of application. Therefore, methods have been developed especially to approximate them. Although many of such methods exist [65, 97], here only the well-known backward Euler scheme [26, 29] is used. At first, the time interval of interest, denoted by $(0,T)$ is subdivided into L sub-intervals of equal size δt, thereby establishing the time grid

$$t_l = l\delta t \quad \text{for } l = 0, \ldots, L, \qquad (3.54)$$

with $\delta t := \frac{T}{L}$. Note that the constant time step δt is taken only for simplicity, the method itself does not require this. Further, the approximation of the unknown $\underline{u}$, its first and second derivatives are given by

$$\begin{aligned} \underline{u}(t_l) &\approx \underline{u}^l,\ \underline{\dot{u}}(t_l) \approx \frac{\delta \underline{u}^l}{\delta t} := \frac{\underline{u}^l - \underline{u}^{l-1}}{\delta t}, \\ \underline{\ddot{u}}(t_l) &\approx \frac{\delta^2 \underline{u}^l}{(\delta t)^2} := \frac{\frac{\delta \underline{u}^l}{\delta t} - \frac{\delta \underline{u}^{l-1}}{\delta t}}{\delta t} := \frac{\underline{u}^l - 2\underline{u}^{l-1} + \underline{u}^{l-2}}{(\delta t)^2} \end{aligned} \tag{3.55}$$

respectively. Taking (3.55) into (3.48)-(3.53) yields the following recursive problem.
For $l = 1, \ldots, L$, find $\underline{u}^l$ such that

$$\underline{u}^0 := \underline{u}_0(x), \quad \frac{\delta \underline{u}^0}{\delta t} := \underline{u}_1(x) \quad \text{in} \quad \Omega, \tag{3.56}$$

$$s^2 \underline{u}^l - \text{div}\,(\sigma(\underline{u}^l)) = \underline{F}^l \quad \text{in} \quad \Omega, \tag{3.57}$$

$$\underline{u}^l = \underline{g}_D^l \quad \text{on} \quad \Gamma_D, \tag{3.58}$$

$$\sigma(\underline{u}^l)\underline{n} = \underline{g}_N^l \quad \text{on} \quad \Gamma_N, \tag{3.59}$$

$$u_n^l \leq \mathbf{d}^l, \quad \sigma_n(\underline{u}^l) \leq 0, \quad \sigma_n(\underline{u}^l)(u_n^l - \mathbf{d}^l) = 0 \quad \text{on} \quad \Gamma_C, \tag{3.60}$$

$$\left.\begin{array}{l} (\delta \underline{u}^l)_t = 0 \Rightarrow |\sigma_t(\underline{u}^l)| < \mathcal{F}(0)|\sigma_n(\underline{u}^l)|, \\ (\delta \underline{u}^l)_t \neq 0 \Rightarrow \sigma_t(\underline{u}^l) = -\mathcal{F}\left(\frac{|(\delta \underline{u}^l)_t|}{\delta t}\right)|\sigma_n(\underline{u}^l)|\frac{(\delta \underline{u}^l)_t}{|(\delta \underline{u}^l)_t|}, \\ |(\delta \underline{u}^l)_t|\left(|\sigma_t(\underline{u}^l)| - \mathcal{F}\left(\frac{|(\delta \underline{u}^l)_t|}{\delta t}\right)|\sigma_n(\underline{u}^l)|\right) = 0, \end{array}\right\} \quad \text{on} \quad \Gamma_C, \tag{3.61}$$

are satisfied with $\underline{F}^l$, $\underline{g}_D^l$, $\underline{g}_N^l$, $\mathbf{d}^l$, s and $\underline{u}^{l-1}$ which are all given data to the problem (3.56)-(3.61). In addition, we have

$$\underline{F}^l = \begin{cases} \underline{f}^1 + s^2 \underline{u}_0 + \dfrac{\rho}{\delta t} \underline{u}_1 & \text{if} \quad l = 1, \\ \underline{f}^l + s^2(2\underline{u}^{l-1} - \underline{u}^{l-2}) & \text{if} \quad l \geq 2, \end{cases}$$

and

$$\delta \underline{u}^l = \underline{u}^l - \underline{u}^{l-1}, \quad s^2 = \frac{\rho}{(\delta t)^2}.$$

Note that the recursive problem (3.56)-(3.61) is a static contact problem in linear elastostatics of Yukawa type. The conditions (3.61) define a form of Coulomb friction law for elastostatics of Yukawa type. Moreover, (3.60)-(3.61) asserts that if contact takes place at a point on Γ_C, no sliding of the point occurs if the magnitude of the tangential stress $|\sigma_t(\underline{u}^l)|$ is less than the magnitude of the normal stress $|\sigma_n(\underline{u}^l)|$ times the friction coefficient $\mathcal{F}$, whereas sliding occurs if the magnitude of the tangential stress $|\sigma_t(\underline{u}^l)|$ reaches a critical value and the motion opposes the tangential stress $\sigma_t(\underline{u}^l)$.

4 BEM FORMULATIONS FOR SCALAR MIXED BOUNDARY VALUE PROBLEM OF YUKAWA TYPE

This chapter is devoted to the development and analysis of a boundary element method for the solution of the scalar Yukawa problem. Yukawa problems have many important applications, for example it generally arises after an implicit time discretization of the time dependent heat equation, time dependent diffusion equation or time dependent elasticity equations etc. Therefore, reliable and efficient numerical algorithms for the solution of the scalar Yukawa equation can be of great use in many different areas of solid mechanics. Here we consider the nonhomogeneous mixed boundary value model in a two-dimensional simply connected domain Ω (in particular a two-dimensional disc). Applying the boundary integral equation method, the partial differential equations are reduced equivalently to boundary integral equations on the boundary curve [100]. Due to the shape of the domain, the boundary (circle) can easily be represented by a one-periodical parametrization [92,94]. Making use of this parametrization the eigensystems of the boundary integral operators can be derived [3,4,69], which are crucial results for the computation of eigenvalues of the discrete operators. The boundary integral equations are approximated by using Galerkin method with the help of B-splines as basis functions. This yields an equivalent algebraic linear system involving dense matrices, but having circulant property [15,16,23]. The entries of those matrices are computed explicitly and efficiently. Furthermore, the circulant property enables us to use the discrete Fourier matrix as preconditioner within an iterative solver or the fast Fourier transform (FFT) as direct solver [43, 84, 92–94]. However, the boundary element formulation (BEM) loses at a first glance its attractiveness due to the fact that the equation is nonhomogeneous which requires an integration over the whole domain.

During the past two decades, much effort has been devoted to dealing with this issue in the BEM community. One of the most widely used methods in engineering is the dual reciprocity method (DRM) introduced by Nardini and Brebbia in 1982 [89]. This method transfers the domain integrals to boundary integrals. The main idea is to approximate the right hand side, for example by radial basis functions (RBF) [37], which help to determine a particular solution of the partial differential equation. Furthermore, particular solutions can be computed by finite difference methods or by finite element methods [64] by embedding the domain into an auxiliary domain and solving the nonhomogeneous equation with homogeneous Dirichlet boundary conditions. The recent development for the evaluation of the Newton potential is the fast multipole method [81, 87, 100]. The main idea of this technique is based on the multipole expansion of the fundamental solution in the far field of the evaluation point. In [87], the computational domain is divided into the far and near field so that in the near field the evaluation is done by using a standard collocation approach, while in the far field the multipole expansion is used. Additionally, an error analysis is given. In this chapter we present and analyze a new approach for an efficient evaluation of the Newton potential in the boundary element method of the mixed boundary value problem by using the Steklov-Poincaré operator. The technique we present here is based on a special mesh discretization of the domain (disc). First, at each level of refinement the disc is split into M rings in an adaptive way. Second, a uniform mesh is defined on each ring in such a way that the mesh on the external ring has the same size as the boundary mesh. Further, on each ring the right hand side is approximated by piecewise constant functions. This enables us to write the Newton potential vector on each ring in terms of a matrix-vector multiplication. Moreover, the FFT can be used to speed up this process due to the circulant property of the matrices on each ring [43, 84, 92–94]. The chapter is organized as follows: In section 1 we describe the considered boundary value problem, establish the boundary integral equations and derive the eigensystems of the operators involved. In section 2 we are interested in the standard Galerkin procedure for the boundary integral equations formulated in section 1 with the help of one-periodical B-splines. The focus of section 3 is the evaluation of the Newton potential by using the method we described above and the presentation of the numerical errors analysis.

4.1 Model Problem and Boundary Integral Formulation

In this section we precisely state the model problem. Furthermore, by using Green's formula we establish the representation formulae, derive the boundary integral equations and the computation of the eigensystem associated to the boundary integral operators.

4.1.1 Problem statement

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain (in particular a disc in $\mathbb{R}^2$) with boundary $\partial\Omega := \Gamma$ divided into two mutually disjoint parts Γ_D and Γ_N, that is $\Gamma := \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. On Γ_D the boundary value is prescribed and on Γ_N the flux is given. Furthermore, we assume $meas(\Gamma_D) > 0$. We find u (in a suitable space) satisfying the following equations

$$\alpha^2 u(x) - \Delta u(x) = f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^2, \tag{4.1}$$

$$u(x) = g_D(x) \quad \text{for } x \in \Gamma_D, \tag{4.2}$$

$$\frac{\partial u}{\partial n_x}(x) = g_N(x) \quad \text{for } x \in \Gamma_N, \tag{4.3}$$

where Δ denotes the Laplacian, and $\alpha > 0$ is the so called generalized wave number and n_x represents the outer unit normal to the boundary.

We now move to the Green formula of equation (4.1), to this end we assume that the solution u of (4.1) is sufficiently smooth, further let v be an arbitrary test function. The first Green identity is then

$$\int_\Omega [\alpha^2 u(y) - \Delta u(y)]\mathrm{v}(y)dy = \int_\Omega [\alpha^2 u(y)\mathrm{v}(y) + \nabla u(y)\cdot\nabla \mathrm{v}(y)]dy - \int_\Gamma \gamma_1^{int} u(y)\gamma_0^{int}\mathrm{v}(y)ds_y, \tag{4.4}$$

where in this particular case $\gamma_1^{int} u(y) := \frac{\partial u}{\partial n_y}(y) := \nabla u(y)\cdot n_y$. Further, if we interchange u and v in (4.4) we then obtain

$$\int_\Omega [\alpha^2 \mathrm{v}(y) - \Delta \mathrm{v}(y)]u(y)dy = \int_\Omega [\alpha^2 \mathrm{v}(y)u(y) + \nabla \mathrm{v}(y)\cdot\nabla u(y)]dy - \int_\Gamma \gamma_1^{int} \mathrm{v}(y)\gamma_0^{int}u(y)ds_y. \tag{4.5}$$

Finally, subtracting (4.4) from (4.5) results in the second Green identity

$$\int_\Omega [\alpha^2 \mathrm{v}(y) - \Delta \mathrm{v}(y)]u(y)dy = \int_\Gamma \gamma_1^{int} u(y)\gamma_0^{int}\mathrm{v}(y)ds_y - \int_\Gamma \gamma_1^{int} \mathrm{v}(y)\gamma_0^{int}u(y)ds_y + \int_\Omega f(y)\mathrm{v}(y)dy. \tag{4.6}$$

Further, if there exists for any $x \in \Omega$ a function $\mathrm{v}(y) := U^*(x,y)$, such that

$$\int_\Omega [\alpha^2 U^*(x,y) - \Delta_y U^*(x,y)]u(y)dy = u(x), \tag{4.7}$$

then inserting (4.7) into (4.6) the solution u of the partial differential (4.1) is given by the so-called representation formula for $x \in \Omega$

$$u(x) = \int_\Gamma U^*(x,y)\gamma_{1,y}^{int}u(y)ds_y - \int_\Gamma \gamma_{1,y}^{int}U^*(x,y)\gamma_0^{int}u(y)ds_y + \int_\Omega U^*(x,y)f(y)dy. \tag{4.8}$$

In (4.7) and (4.8), the operators subscripts denote that the operators have to be applied with regards to their respective index onto the according quantities. We can notice from the representation formula (4.8) that any solution of the partial differential equation (4.1) can be described if the Cauchy data $\{\gamma_0^{int}u(x), \gamma_1^{int}u(x)\}$ for $x \in \Gamma$ are known. Due to

$$u(x) = \int_{\mathbb{R}^d} \delta_0(y-x)u(y)dy \quad \text{for } x \in \mathbb{R}^d \ (d = 2,3), \tag{4.9}$$

where δ_0 is the Dirac delta, the range of integration in (4.7) can be extended to be in d-dimensional space since Ω is a subset of $\mathbb{R}^d$. Hence equating (4.9) and (4.7) yields the following equation

$$\alpha^2 U^*(x,y) - \Delta_y U^*(x,y) = \delta_0(y-x) \quad \text{for } x,y \in \mathbb{R}^d \tag{4.10}$$

in the distributional sense. Any solution $U^*(x,y)$ of (4.10) is called a fundamental solution of the partial differential equation (4.1).

Note that the existence of a fundamental solution $U^*(x,y)$ is needed to establish the representation formula (4.8), which is necessary to derive appropriate boundary integral equations to find the complete Cauchy data. Since the Yukawa operator is invariant with respect to translations and rotations, we can find the fundamental solution as $U^*(x,y) = \mathrm{v}(z)$ where $z := y - x$, see [100] which yields

$$\alpha^2 \mathrm{v}(z) - \Delta \mathrm{v}(z) = \delta_0(z) \quad \text{for } z \in \mathbb{R}^d. \tag{4.11}$$

We then obtain

$$\mathrm{v}(z) := U^*(x,y) = \begin{cases} \frac{1}{2\pi} K_0(\alpha|x-y|) & \text{if} \quad d = 2, \\ \dfrac{1}{4\pi} \dfrac{e^{-\alpha|x-y|}}{|x-y|} & \text{if} \quad d = 3, \end{cases} \tag{4.12}$$

with

$$\begin{aligned} K_0(r) &= (\ln 2 - \mathbf{E} - \ln r) I_0(r) + \sum_{k=1}^{\infty} \left[\left(\sum_{j=1}^{k} \frac{1}{j} \right) \frac{1}{(k!)^2} \left(\frac{r}{2} \right)^{2k} \right], \\ I_0(r) &= 1 + \sum_{k=1}^{\infty} \frac{1}{(k!)^2} \left(\frac{r}{2} \right)^{2k}, \end{aligned}$$

and

$$\mathbf{E} = \lim_{n \to \infty} \left[\sum_{j=1}^{n} \frac{1}{j} - \ln n \right] \approx 0.57721566490...$$

$\mathbf{E}$ represents the so-called Euler-Mascheroni constant, see, e.g., ([1, 100]) while I_0 and K_0 denote the first and the second kind of modified Bessel functions respectively [1]. The derivation of the fundamental solution (4.12) follows as in the case of the Helmholtz equation, see [100, p.105] we just have to replace the wave number by $i\alpha$ and used the relations between modified Bessel functions and Bessel functions [1] which yields (4.12). Having derived a fundamental solution the focus of the next section will be on the establishment of appropriate boundary integral equations.

4.1.2 Boundary integral equations

The representation formula (4.8) states that u is determined uniquely for a point $x \in \Omega$ if the complete Cauchy data $\{\gamma_0^{int} u, \gamma_1^{int} u\}$ and the source f are known. But, it turns out that $\{\gamma_0^{int} u, \gamma_1^{int} u\}$ are given only partially on the boundary Γ that is $\gamma_0^{int} u|_{\Gamma_D} := g_D$ and

$\gamma_1^{int} u|_{\Gamma_N} := g_N$. Therefore, we have to determine $\gamma_0^{int} u|_{\Gamma_N}$ and $\gamma_1^{int} u|_{\Gamma_D}$. By proceeding as in [100], we first apply the trace operator γ_0^{int} to (4.8) which leads to

$$\gamma_0^{int} u(x) = (V\gamma_1^{int} u)(x) - \left(-\frac{1}{2}I + K\right)\gamma_0^{int} u(x) + (N_0 f)(x) \quad \text{for all} \quad x \in \Gamma. \tag{4.13}$$

Further, by applying the conormal derivative γ_1^{int} again to (4.8) yields

$$\gamma_1^{int} u(x) = \left(\frac{1}{2}I + K'\right)\gamma_1^{int} u(x) + (D\gamma_0^{int} u)(x) + (N_1 f)(x) \quad \text{for all} \quad x \in \Gamma. \tag{4.14}$$

In (4.13) and (4.14) $\gamma_1^{int} u(x)$ and I denote the conormal derivative of u, i.e. $\dfrac{\partial u}{\partial n_x}(x)$ and the identity respectively, and $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the single layer operator, $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ the double layer potential, the adjoint double layer potential $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, the hypersingular boundary integral operator $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, and $N_0 : \tilde{H}^{-1}(\Omega) \to H^{1/2}(\Gamma)$ and $N_1 : \tilde{H}^{-1}(\Omega) \to H^{-1/2}(\Gamma)$ are the Newton potentials or volume potentials. Note that the above representations are considered on a smooth boundary $\Gamma = \partial\Omega$, i.e. at least differentiable. In addition, these boundary integral operators are linear, bounded and defined as follows:

$$\begin{aligned}
(Vt)(x) &:= \int_\Gamma U^*(x,y)t(y)ds_y \quad \text{for } x \in \Gamma,\\
(Ku)(x) &:= \int_\Gamma \gamma_{1,y}^{int} U^*(x,y)u(y)ds_y \quad \text{for } x \in \Gamma,\\
(K't)(x) &:= \int_\Gamma \gamma_{1,x}^{int} U^*(x,y)t(y)ds_y \quad \text{for } x \in \Gamma,\\
(Du)(x) &:= -\gamma_{1,x}^{int} \int_\Gamma \gamma_{1,y}^{int} U^*(x,y)u(y)ds_y \quad \text{for } x \in \Gamma,\\
(N_0 f)(x) &:= \int_\Omega U^*(x,y)f(y)dy \quad \text{for } x \in \Gamma,\\
(N_1 f)(x) &:= \gamma_{1,x}^{int} \int_\Omega U^*(x,y)f(y)dy \quad \text{for } x \in \Gamma.
\end{aligned}$$

The integral representations of V and D are understood as weakly singular and as hypersingular boundary integral respectively, while the integrals for K and K' are in general Cauchy singular integrals [53, 83].

Now, since all boundary integral operators in (4.13) and (4.14) are well defined, the boundary integral equations can be written in matricial form as follows

$$\begin{pmatrix} \gamma_0^{int} u \\ \gamma_1^{int} u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \gamma_0^{int} u \\ \gamma_1^{int} u \end{pmatrix} + \begin{pmatrix} N_0 f \\ N_1 f \end{pmatrix}, \tag{4.15}$$

where

$$\mathcal{C} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix}$$

is the so-called Calderón projection, and satisfies the identity $\mathcal{C} = \mathcal{C}^2$ by what useful relations between integral operators are gained, see [100]. Furthermore, the single layer and the hypersingular integral operators have the following properties.

Lemma 4.1. *$V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are self-adjoint, positive definite and elliptic on $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ respectively, i.e.*

$$\langle V\tau, \tau\rangle_\Gamma \geq c_1^V \parallel \tau \parallel^2_{H^{-1/2}(\Gamma)} \quad \textit{for all} \quad \tau \in H^{-1/2}(\Gamma) \quad \textit{with} \quad c_1^V > 0$$

and

$$\langle Du, u\rangle_\Gamma \geq c_1^D \parallel u \parallel^2_{H^{1/2}(\Gamma)} \quad \textit{for all} \quad u \in H^{1/2}(\Gamma) \quad \textit{with} \quad c_1^D > 0.$$

Proof. See [100] for the Laplace operator and use the properties of the modified Bessel functions for $\alpha > 0$ [1]. □

Since the single layer integral operator V is $H^{-1/2}(\Gamma)$-elliptic, the unique solvability of the boundary integral equation (4.13) with respect to the conormal derivative follows immediately. Therefore, the conormal derivative $t := \gamma_1^{int} u$ is given by

$$t := \gamma_1^{int} u = V^{-1}\left(\frac{1}{2}I + K\right)u - V^{-1}N_0 f. \tag{4.16}$$

But, by using the representation (4.16) in a Galerkin discretization may lead to a non-symmetric discrete matrix for the symmetric Dirichlet to Neumann map. To avoid such an inconvenience, let us use the symmetric representation which is obtained by inserting the representation (4.16) into the second boundary integral equation (4.14):

$$\begin{aligned} t &= \left(\frac{1}{2}I + K'\right)t + Du + N_1 f \\ &= \left[\left(\frac{1}{2}I + K'\right)V^{-1}\left(\frac{1}{2}I + K\right) + D\right]u + \left[N_1 - \left(\frac{1}{2}I + K'\right)V^{-1}N_0\right]f. \end{aligned} \tag{4.17}$$

In the representations (4.16) and (4.17) if we set $f = 0$, we then obtain two representations of the Dirichlet to Neumann map in the case of homogeneous boundary values problem, known as the Steklov-Poincaré operator. Furthermore, the one obtained from (4.16) is non-symmetric while the one obtained from (4.17) is symmetric and given by

$$S := \left(\frac{1}{2}I + K'\right)V^{-1}\left(\frac{1}{2}I + K\right) + D, \tag{4.18}$$

and satisfies the following properties.

Theorem 4.1. *If Ω is a bounded domain of class C^∞, then for all $s \in \mathbb{R}$, the Steklov-Poincaré operator $S : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ is a linear and continuous map. Moreover, it is self-adjoint, positive definite and elliptic, that is satisfiying the following inequality*

$$\langle Su, u \rangle_\Gamma \geq c_1^D \parallel u \parallel^2_{H^{1/2}(\Gamma)} \quad \textit{for all} \quad u \in H^{1/2}(\Gamma) \quad \textit{with} \quad c_1^D > 0.$$

Proof. By using the $H^{1/2}(\Gamma)$-ellipticity of the inverse single layer operator V^{-1} we obtain

$$\langle Su, u \rangle_\Gamma = \langle V^{-1}\left(\frac{1}{2}I + K\right)u, \left(\frac{1}{2}I + K\right)u \rangle_\Gamma + \langle Du, u \rangle_\Gamma \geq \langle Du, u \rangle_\Gamma$$

for all $u \in H^{1/2}(\Gamma)$. Therefore, the Steklov-Poincaré operator S admits the same ellipticity estimate as the hypersingular boundary integral operator D. □

Furthermore, by equating (4.16) and (4.17) one obtains

$$-V^{-1}N_0 f = \left[N_1 - \left(\frac{1}{2}I + K'\right)V^{-1}N_0\right] f =: -Nf \tag{4.19}$$

and

$$N_1 f = \left(-\frac{1}{2}I + K'\right)V^{-1}N_0 f$$

As a consequence of (4.19), the conormal derivative can take the following form

$$t := \gamma_1^{int} u = Su - Nf. \tag{4.20}$$

Let us now return to our problem where we have to find the unknown Dirichlet datum $\gamma_0 u(x)$ for $x \in \Gamma_N$ and the Neumann datum $\gamma_1 u(x)$ for $x \in \Gamma_D$. There exists a wide range of different boundary integral formulation to solve the mixed boundary value problem (4.1)-(4.3). However, the most frequently used in literature are the symmetric formulation, and the formulation with the help of the Dirichlet to Neumann map. In the first formulation one has to use the boundary integral equation (4.13) for $x \in \Gamma_D$ while (4.14) is considered for $x \in \Gamma_N$, see, e.g. [100] for details. Here we consider the formulation by using the Dirichlet to Neumann map. Therefore, the problem reads: find $\gamma_0^{int} u \in H^{1/2}(\Gamma)$ such that

$$\gamma_0^{int} u(x) = g_D(x) \ \text{ for } \ x \in \Gamma_D, \tag{4.21}$$

$$\gamma_1^{int} u(x) := (S\gamma_0^{int} u)(x) - (Nf)(x) = g_N(x) \ \text{ for } \ x \in \Gamma_N. \tag{4.22}$$

Let $\tilde{g}_D \in H^{1/2}(\Gamma)$ be a suitable extension of the given Dirichlet datum $g_D \in H^{1/2}(\Gamma_D)$ and satisfying $\tilde{g}_D(x) = g_D(x)$ for $x \in \Gamma_D$. Then we have to find $\hat{u} := \gamma_0^{int} u - \tilde{g}_D \in \tilde{H}^{1/2}(\Gamma_N)$ such that

$$\langle S\hat{u}, \mathrm{v} \rangle_{\Gamma_N} = \langle g_N + Nf - S\tilde{g}_D, \mathrm{v} \rangle_{\Gamma_N} \tag{4.23}$$

is satisfied for all v $\in \tilde{H}^{1/2}(\Gamma_N)$. Since the Steklov-Poincaré operator $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bounded and $\tilde{H}^{1/2}(\Gamma_N)$-elliptic, see [100, p.149], the unique solvability of (4.23) is therefore shown.

Before moving to the Galerkin discretization of the problem (4.21)-(4.22), we determine the eigensystems of different operators involved in the equation.

4.1.3 Eigensystems of integral operators

In general it is not possible to obtain the eigenfunctions of boundary integral operators for arbitrary boundaries [3]. In the particular case of a two-dimensional circular domain $\Omega = B_R(O)$ of radius R and centered at the origin it is possible to give an explicit representation of the eigenfunctions of the single and double layer boundary integral operators, and the hypersingular boundary integral operator as well. The parametrization of $\Gamma := \partial B_R(O)$ is given by

$$\Gamma := \{x \in \mathbb{R}^2 : x(\tau) = R \begin{pmatrix} \cos 2\pi\tau \\ \sin 2\pi\tau \end{pmatrix}, 0 \le \tau < 1\}. \tag{4.24}$$

By using the parametrization (4.24) the boundary integral operators can be written as follows:

$$\begin{aligned}
(Vt)(\tau) &= R \int_0^1 K_0(2R\alpha|\sin\pi(\tau - s)|)t(s)ds, \\
(Ku)(\tau) &= -\alpha R \int_0^1 K_1(2R\alpha|\sin\pi(\tau - s)|)|\sin\pi(\tau - s)|u(s)ds,
\end{aligned}$$

$$(Du)(\tau) = -\alpha^2 \int_0^1 \left[RK_0(2R\alpha|\sin\pi(\tau - s)|)\sin^2\pi(\tau - s) + \frac{K_1(2R\alpha|\sin\pi(\tau - s)|)}{2\alpha R \sin\pi(\tau - s)} \right] u(s)ds.$$

Lemma 4.2. *Let Γ be given as in (4.24). The Fourier functions*

$$v_n(s) = e^{\mp i2\pi ns} \quad \textit{for } n = 0, 1, \ldots.$$

are eigenfunctions of the single layer boundary integral operator V, of the double layer boundary integral operator K, and of the hypersingular boundary integral operator D, associated to the eigenvalues

$$\begin{aligned}
\lambda_{V,n} &= RK_n(\alpha R)I_n(\alpha R), \\
\lambda_{K,n} &= -\frac{1}{2} + (\alpha R)I_n'(\alpha R)K_n(\alpha R) \\
&= \frac{1}{2} + (\alpha R)I_n(\alpha R)K_n'(\alpha R), \\
\lambda_{D,n} &= -(\alpha R)^2 I_n'(\alpha R)K_n'(\alpha R)
\end{aligned}$$

for $n = 0, 1, \ldots.$ respectively, where I_n and K_n denote the first kind and the second kind of modified Bessel functions respectively, while I_n' and K_n' represent the first derivative of the modified Bessel functions I_n and K_n respectively.

Proof. The derivation of the eigenvalues follows as in the case of the Helmholtz equation, see, e.g., [3, 4, 69], by using the wave number $k = i\alpha$. In addition, we used the relations between modified Bessel functions and Bessel functions [1] to obtain the desired result. □

Remark 4.1. *Note that the Fourier functions v_n are also eigenfunctions to the double layer integral operator $\frac{1}{2}I + K$ and the Steklov-Poincaré operator S associated to the eigenvalues*

$$\lambda_{\frac{1}{2}I+K,n} = (\alpha R) I_n'(\alpha R) K_n(\alpha R),$$

$$\lambda_{S,n} = \frac{\lambda^2_{\frac{1}{2}I+K,n}}{\lambda_{V,n}} + \lambda_{D,n} = \frac{\alpha^2 R (I_n'(\alpha R))^2 K_n(\alpha R)}{I_n(\alpha R)} - (\alpha R)^2 I_n'(\alpha R) K_n'(\alpha R)$$

for $n = 0, 1, \ldots$ respectively.

4.2 Boundary element methods for the mixed boundary value problem

In this section we describe the standard Galerkin boundary element method to solve the boundary integral equation (4.23) numerically with the help of one-periodic B-splines of order $\nu \geq 0$. First, we consider the one-periodic parametrization of the boundary Γ as given in (4.24), further we divide the interval $[0, 1)$ into $N > \nu + 1$ subintervals of mesh size $h = 1/N$ and define as follows

$$[0, 1) = \bigcup_{l=1}^{N} [s_l, s_{l+1}) \quad \text{with } s_l = (l-1)h \quad \text{for } l = 1, \ldots, N+1.$$

Moreover, we introduce the N-dimensional subspace $\mathbb{H}_N^\nu$ of one-periodic functions, see, e.g., [92, 94], i.e.,

$$\mathbb{H}_N^\nu = Span(\phi_1^{(\nu)}(s), \ldots, \phi_N^{(\nu)}(s)),$$

where $\phi_k^{(\nu)}(s)$, $k = 1, \ldots, N$ are the B-splines of order ν defined by the following recurrence formulae

$$\phi_1^{(0)}(s) = \begin{cases} 1 & -h/2 \leq s < h/2, \\ 0 & -1/2 \leq s < -h/2, \quad h/2 \leq s < 1/2, \end{cases} \tag{4.25}$$

$$\phi_1^{(\nu)}(s) = \frac{1}{h}\int_{-1/2}^{1/2} \phi_1^{(\nu-1)}(\tau)\phi_1^{(0)}(s-\tau)d\tau \quad \text{for } \nu = 1,2,..., \tag{4.26}$$

and

$$\phi_k^{(\nu)}(s) = \phi_1^{(\nu)}(s-(k-1)h) \quad \text{for } k = 1,2,...,N, \quad \phi_k^{(\nu)}(s+m) = \phi_k^{(\nu)}(s) \quad \text{for } m \in \mathbb{Z}.$$

Further, $\phi_1^{(\nu)}$ is an even function, i.e.

$$\phi_1^{(\nu)}(-s) = \phi_1^{(\nu)}(s) \quad \text{for all } s \in [-1/2, 1/2].$$

Indeed, the proof is done by induction for $\nu = 0,1,....$ For $\nu = 0$ we can easily see from (4.25) that $\phi_1^{(0)}$ is even. Let us assume that $\phi_1^{(\nu-1)}$ is even, further we want to show that $\phi_1^{(\nu)}$ is even too. From (4.26) we have

$$\begin{aligned}
\phi_1^{(\nu)}(-t) &= \frac{1}{h}\int_{-1/2}^{1/2} \phi_1^{(\nu-1)}(\tau)\phi_1^{(0)}(-t-\tau)d\tau \\
&= \frac{1}{h}\int_{-1/2}^{1/2} \phi_1^{(\nu-1)}(\tau)\phi_1^{(0)}(t+\tau)d\tau, \ (\phi_1^{(0)} \text{ even}) \\
&= -\frac{1}{h}\int_{1/2}^{-1/2} \phi_1^{(\nu-1)}(-u)\phi_1^{(0)}(t-u)du, \ (\text{change of variable } u = -\tau) \\
&= \frac{1}{h}\int_{-1/2}^{1/2} \phi_1^{(\nu-1)}(u)\phi_1^{(0)}(t-u)du, \ (\phi_1^{(\nu-1)} \text{ even}) \\
&= \phi_1^{(\nu)}(t).
\end{aligned}$$

In addition, we will use the Fourier series representation of the basis functions $\phi_l^{(\nu)}$, i.e.,

$$\phi_l^{(\nu)}(t) = \sum_{k\in\mathbb{Z}} c_l^{\nu}(k)e^{i2\pi kt}, \quad l = 1,...,N, \quad \nu = 0,1,...,$$

where the Fourier coefficients are given by [92]

$$c_1^{\nu}(k) = \begin{cases} h & \text{if} \quad k = 0, \\ \dfrac{\sin^{\nu+1}(\pi kh)}{h^{\nu}(\pi k)^{\nu+1}} & \text{if} \quad k \neq 0, \end{cases}$$

and

$$c_l^\nu(k) = c_1^\nu(k) e^{-i2\pi k(l-1)h} \quad \text{for } l = 1, \ldots, N.$$

Note that we have also

$$c_1^\nu(-k) = c_1^\nu(k) \quad \text{for all } k \in \mathbb{Z}.$$

The Galerkin boundary element formulation of the boundary integral equation (4.23) is to find for $M \le N$, $\hat{u}_h(x) = \sum_{i=1}^{M} \hat{u}_i \phi_i^{(1)}(x) \in \mathbb{H}_M^1 \cap \tilde{H}^{1/2}(\Gamma_N)$ such that

$$\langle S\hat{u}_h, \mathrm{v}_h\rangle_{\Gamma_N} = \langle g_N - S\tilde{g}_D, \mathrm{v}_h\rangle_{\Gamma_N} + \langle Nf, \mathrm{v}_h\rangle_{\Gamma_N} \quad \text{for all } \mathrm{v}_h \in \mathbb{H}_M^1 \cap \tilde{H}^{1/2}(\Gamma_N). \tag{4.27}$$

By using Theorem 4.1 and the Lax-Milgram lemma we conclude the unique solvability of the Galerkin formulation (4.27). In addition, Cea's lemma and the approximation property of $\mathbb{H}_M^1$ [100, Theorem 10.9] yield the following error estimate

$$\|\hat{u} - \hat{u}_h\|_{H^{1/2}(\Gamma)} \le ch^{3/2} |u|_{H^2(\Gamma)}, \tag{4.28}$$

when assuming that $u \in H^2(\Gamma)$.

The boundary element formulation (4.27) is equivalent to the algebraic system of linear equations

$$S_h \underline{\hat{u}} = \underline{f}_1 + \underline{f}_2, \tag{4.29}$$

where the stiffness matrix is defined by

$$S_h[i,j] = \langle S\phi_j^{(1)}, \phi_i^{(1)}\rangle_{\Gamma_N} \quad \text{for } i,j = 1, \ldots, M,$$

and the right hand side

$$\underline{f}_{1i} = \langle g_N - S\tilde{g}_D, \phi_i^{(1)}\rangle_{\Gamma_N} \text{ and } \underline{f}_{2i} = \langle Nf, \phi_i^{(1)}\rangle_{\Gamma_N} \quad \text{for } i = 1, \ldots, M.$$

Note that $Nf = V^{-1}N_0 f$.

Note that for this particular case, the Galerkin discretization of the Steklov-Poincaré operator can be carried out explicitly by means of its eigenfunctions defined in Remark 4.1. This is only possible for the scalar Yukawa case. But, since our main interest is in the elasticity case, we will define and use in this chapter a symmetric approximation of the continuous Steklov-Poincaré operator S. To this end let us first define the Galerkin discretizations of the boundary integral operators V, K, K' and D as follows:

$$\begin{aligned} V_h[i,j] &:= \langle V\phi_j^{(0)}, \phi_i^{(0)}\rangle_\Gamma, \quad D_h[l,k] := \langle D\phi_k^{(1)}, \phi_l^{(1)}\rangle_{\Gamma_N}, \\ K_h[i,k] &:= \langle K\phi_k^{(1)}, \phi_i^{(0)}\rangle_\Gamma, \quad K_h' := K_h^\top \end{aligned}$$

for $i,j = 1, \ldots, N$ and $k,l = 1, \ldots, M$. These matrices can be computed explicitly and in an efficient way by using the following results.

Lemma 4.3. *The discrete single and double layer integral operators V_h and $\hat{K}_h = \frac{1}{2}M_h + K_h$, and the discrete hypersingular integral operator D_h are circulant matrices. Moreover, V_h and D_h are symmetric and positive definite. In addition, their eigenvalues are given respectively by*

$$\lambda_{V_{h,j}} = \begin{cases} hRI_0(\alpha R)K_0(\alpha R) & \text{for} \quad j=1, \\ hR\left(\frac{\sin\pi s}{\pi}\right)^{2\nu+2}\left(\sum_{k=0}^{+\infty}\frac{I_{(j-1+kn)}(\alpha R)K_{(j-1+kn)}(\alpha R)}{(k+s)^{2\nu+2}} \right. & \text{for} \quad j=2,\dots,N, \\ \qquad \left. + \sum_{k=1}^{+\infty}\frac{I_{(1-j+kn)}(\alpha R)K_{(1-j+kn)}(\alpha R)}{(k-s)^{2\nu+2}}\right) & \end{cases}$$

$$\lambda_{\hat{K}_{h,j}} = \begin{cases} h(\alpha R)I_1(\alpha R)K_0(\alpha R) & \text{for} \quad j=1, \\ h(\alpha R)\left(\frac{\sin\pi s}{\pi}\right)^{\nu+\mu+2}\left(\sum_{k=0}^{+\infty}\frac{I'_{(j-1+kn)}(\alpha R)K_{(j-1+kn)}(\alpha R)}{(k+s)^{\nu+\mu+2}}(-1)^{k(\nu+\mu)} \right. & \text{for} \quad j=2,\dots,N, \\ \qquad \left. + \sum_{k=1}^{+\infty}\frac{I'_{(1-j+kn)}(\alpha R)K_{(1-j+kn)}(\alpha R)}{(k-s)^{\nu+\mu+2}}(-1)^{(k+1)(\nu+\mu)}\right) & \end{cases}$$

$$\lambda_{D_{h,j}} = \begin{cases} h(\alpha R)^2 I_1(\alpha R)K_1(\alpha R) & \text{for} \quad j=1, \\ -h(\alpha R)^2\left(\frac{\sin\pi s}{\pi}\right)^{2\mu+2}\left(\sum_{k=0}^{+\infty}\frac{I'_{(j-1+kn)}(\alpha R)K'_{(j-1+kn)}(\alpha R)}{(k+s)^{2\mu+2}} \right. & \text{for} \quad j=2,\dots,N, \\ \qquad \left. + \sum_{k=1}^{+\infty}\frac{I'_{(1-j+kn)}(\alpha R)K'_{(1-j+kn)}(\alpha R)}{(k-s)^{2\mu+2}}\right) & \end{cases}$$

for $s=(j-1)/N$. I_n and K_n represent the first and second kind of modified Bessel functions respectively while I'_n and K'_n represent their first derivative respectively.

Proof. Since the kernel functions of all boundary integral operators depend only on the difference $\tau - s$, the Galerkin discretization of these operators results in circulant matrices. In addition, the symmetry of V_h and D_h follows immediately from the symmetry of the kernel functions. Further, by using the eigenvalues of the boundary integral operators as given in Lemma 4.2 and by using [92, Lemma 3.6] we can compute the eigenvalues of all discrete boundary integral operators. Moreover, from the properties of the modified Bessel functions [1] it follows, that all eigenvalues of V_h and D_h are positive. □

The most important property of the circulant matrices [15, 23, 92, 94] is that they are diagonalizable and easily invertible if the inverse exists,

$$V_h = \frac{1}{N} Q D_V Q, \quad \hat{K}_h = \frac{1}{N} F D_{\hat{K}} F^*, \quad D_h = \frac{1}{N} Q D_D Q,$$

where $F \in \mathbb{C}^{N\times N}$ and $Q \in \mathbb{R}^{N\times N}$ are the Fourier matrices given by

$$F[k,l] = e^{i2\pi(k-1)(l-1)h}, \quad Q[k,l] = \cos[2\pi(k-1)(l-1)h] + \sin[2\pi(k-1)(l-1)h]$$

for $k,l = 1,\dots,N$ and D_V, $D_{\hat{K}}$ as well as D_D are diagonal matrices which are defined by the eigenvalues of V_h, $\hat{K}_h$ and D_h respectively.

The Steklov-Poincaré operator can now be approximated by:

$$\tilde{S}_h := (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h) + D_h, \tag{4.30}$$

where $M_h[i,k] := \langle \phi_k^{(1)}, \phi_i^{(0)} \rangle_\Gamma$ is the mass matrix. Remark that from Lemma 4.3 the entries of the discrete approximate Steklov-Poincaré operator $\tilde{S}_h$ can be computed explicitly and exactly. Moreover, its eigenvalues are given by

$$\lambda_{\tilde{S}_{h,j}} := \frac{\lambda^2_{\hat{K}_{h,j}}}{\lambda_{V_{h,j}}} + \lambda_{D_{h,j}} \quad \text{for} \quad j = 1,\dots,N.$$

The first modified discretization of boundary integral equation (4.23) can be written as follows

$$\tilde{S}_h \underline{\tilde{u}} = \underline{f}_1 + \underline{f}_2. \tag{4.31}$$

Note that for some given function $\mathrm{v} \in H^{1/2}(\Gamma)$ the action of the symmetric Steklov-Poincaré operator S on v is given by

$$S\mathrm{v} := D\mathrm{v} + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\mathrm{v} = D\mathrm{v} + (\frac{1}{2}I + K')\psi, \tag{4.32}$$

where ψ is an intermediate function which satisfies $\psi := V^{-1}(\frac{1}{2}I + K)\mathrm{v} \in H^{-1/2}(\Gamma)$, i.e. ψ is the unique solution of the variational problem

$$\langle V\psi, \tau \rangle_\Gamma = \langle (\frac{1}{2}I + K)\mathrm{v}, \tau \rangle_\Gamma \ \text{ for all } \ \tau \in H^{-1/2}(\Gamma). \tag{4.33}$$

The approximate representation (4.30) of the Steklov-Poincaré operator can be interpreted as the Galerkin discretization of the operator $\tilde{S}$ which is defined by

$$\tilde{S}\mathrm{v} = D\mathrm{v} + (\frac{1}{2}I + K')\psi_h, \tag{4.34}$$

where the auxiliary function $\psi_h \in \mathbb{H}_N^0$ is the solution of the Galerkin discretization of (4.33) given by

$$\langle V\psi_h, \phi_i^{(0)}\rangle_\Gamma = \langle (\frac{1}{2}I + K)\mathrm{v}, \phi_i^{(0)}\rangle_\Gamma \quad \text{for} \quad i = 1, \dots, N. \tag{4.35}$$

Note that the modification of the approximate Steklov-Poincaré operator also changes the computed solution of the problem. Therefore, it is required to analyze the corresponding additional error. To this end the properties of the approximate Steklov-Poincaré operator are required.

Lemma 4.4. *The approximate Steklov-Poincaré operator $\tilde{S} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is linear and bounded, and $\tilde{H}^{1/2}(\Gamma_N)$-elliptic, that is satisfying*

$$\langle \tilde{S}v, v\rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)} \text{ for all } v \in \tilde{H}^{1/2}(\Gamma_N). \tag{4.36}$$

Moreover, it satisfies the error estimate

$$\|(S - \tilde{S})v\|_{H^{-1/2}(\Gamma)} \leq c \inf_{\tau_h \in \mathbb{H}_N^0} \|Sv - \tau_h\|_{H^{-1/2}(\Gamma)}. \tag{4.37}$$

Proof. We have from the definition of $\tilde{S}$

$$\begin{aligned} \|\tilde{S}\mathrm{v}\|_{H^{-1/2}(\Gamma)} &= \|D\mathrm{v} + (\frac{1}{2}I + K')\psi_h\|_{H^{-1/2}(\Gamma)} \\ &\leq c_2^D \|\mathrm{v}\|_{H^{1/2}(\Gamma)} + (\frac{1}{2} + c_2^K)\|\psi_h\|_{H^{-1/2}(\Gamma)}. \end{aligned} \tag{4.38}$$

On the other hand, using the ellipticity of the single layer operator V we obtain

$$\begin{aligned} c_1^V \|\psi_h\|^2_{H^{-1/2}(\Gamma)} &\leq \langle V\psi_h, \psi_h\rangle_\Gamma = \langle (\frac{1}{2}I + K)\mathrm{v}, \psi_h\rangle_\Gamma \\ &\leq (\frac{1}{2} + c_2^K)\|\mathrm{v}\|_{H^{1/2}(\Gamma)} \|\psi_h\|_{H^{-1/2}(\Gamma)}. \end{aligned}$$

This yields,

$$\|\psi_h\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^V}(\frac{1}{2} + c_2^K)\|\mathrm{v}\|_{H^{1/2}(\Gamma)}.$$

Taking this into (4.38) the boundedness of the approximate Steklov-Poincaré operator $\tilde{S}$ is shown. The $\tilde{H}^{1/2}(\Gamma_N)$-ellipticity is obtained straight from the $H^{-1/2}(\Gamma)$-ellipticity of the single layer operator V and from the $\tilde{H}^{1/2}(\Gamma_N)$-ellipticity of the hypersingular operator D as follows

$$\begin{aligned} \langle \tilde{S}\mathrm{v}, \mathrm{v}\rangle_\Gamma &= \langle D\mathrm{v}, \mathrm{v}\rangle_\Gamma + \langle (\frac{1}{2}I + K')\psi_h, \mathrm{v}\rangle_\Gamma \\ &= \langle D\mathrm{v}, \mathrm{v}\rangle_\Gamma + \langle \psi_h, (\frac{1}{2}I + K)\mathrm{v}\rangle_\Gamma \\ &= \langle D\mathrm{v}, \mathrm{v}\rangle_\Gamma + \langle V\psi_h, \psi_h\rangle_\Gamma \geq c_1^D \|\mathrm{v}\|^2_{H^{1/2}(\Gamma)}. \end{aligned}$$

Finally, by taking the difference, we obtain

$$(S-\tilde{S})\mathrm{v} = (\frac{1}{2}I + K')(\psi - \psi_h),$$

which yields

$$\|(S-\tilde{S})\mathrm{v}\|_{H^{-1/2}(\Gamma)} = \|(\frac{1}{2}I + K')(\psi - \psi_h)\|_{H^{-1/2}(\Gamma)} \leq (\frac{1}{2} + c_2^K)\|\psi - \psi_h\|_{H^{-1/2}(\Gamma)},$$

and by the use of Cea's lemma the proof is completed. □

The Galerkin boundary element formulation of the first modified problem is to find $\tilde{u}_h \in \mathbb{H}_M^1 \cap \tilde{H}^{1/2}(\Gamma_N)$ such that

$$\langle \tilde{S}\tilde{u}_h, \phi_i^{(1)} \rangle_{\Gamma_N} = \langle g_N + Nf - \tilde{S}\tilde{g}_D, \phi_i^{(1)} \rangle_{\Gamma_N} \quad \text{for } i = 1, \ldots, M. \tag{4.39}$$

From Lemma 4.4 we conclude the unique solvability and the stability of the Galerkin formulation (4.39). Moreover, error estimates in the case where we assume that no modification occurs in the right hand side are given as follows:

Lemma 4.5. *If $u \in H^2(\Gamma)$ and $Su \in H^1_{pw}(\Gamma)$ we then obtain the error estimate*

$$\|\hat{u} - \tilde{u}_h\|_{H^{1/2}(\Gamma)} \leq ch^{3/2}\left[\|u\|_{H^2(\Gamma)} + \|Su\|_{H^1_{pw}(\Gamma)}\right].$$

Proof. By using the Strang lemma, see, e.g., [100, p.192] we obtain

$$\|\hat{u} - \tilde{u}_h\|_{H^{1/2}(\Gamma)} \leq c\left[\inf_{\mathrm{v}_h \in \mathbb{H}_M^1 \cap \tilde{H}^{1/2}(\Gamma_N)} \|\hat{u} - \mathrm{v}_h\|_{H^{1/2}(\Gamma)} + \|(\tilde{S} - S)\hat{u}\|_{H^{-1/2}(\Gamma)}\right],$$

further by using Lemma 4.4 we obtain

$$\|\hat{u} - \tilde{u}_h\|_{H^{1/2}(\Gamma)} \leq c\left[\inf_{\mathrm{v}_h \in \mathbb{H}_M^1 \cap \tilde{H}^{1/2}(\Gamma_N)} \|\hat{u} - \mathrm{v}_h\|_{H^{1/2}(\Gamma)} + \inf_{\tau_h \in \mathbb{H}_N^0} \|S\hat{u} - \tau_h\|_{H^{-1/2}(\Gamma)}\right].$$

Finally, by using the approximation property of the spaces $\mathbb{H}_M^1$ and $\mathbb{H}_N^0$ with the assumption that $u \in H^2(\Gamma)$ and $Su \in H^1_{pw}(\Gamma)$ the lemma is shown. □

The L^2 estimate of the error can be given, see, e.g., [100] by

$$\|\hat{u} - \tilde{u}_h\|_{L^2(\Gamma)} \leq ch^2\left[\|u\|_{H^2(\Gamma)} + \|Su\|_{H^1_{pw}(\Gamma)}\right].$$

It remains to describe the evaluation of the vector $\underline{f}_2$ which results from the Newton potential $N_0 f$. This will be done again by using circulant matrices which are efficient for a matrix-vector multiplication, and the memory storage. For $l = 1,\dots,M$ we have

$$\underline{f}_2[l] = \langle Nf, \phi_l^{(1)}\rangle_{\Gamma_N} := \langle V^{-1}N_0 f, \phi_l^{(1)}\rangle_{\Gamma_N} = \langle w, \phi_l^{(1)}\rangle_{\Gamma_N}, \tag{4.40}$$

where $w = V^{-1}N_0 f \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle Vw, z\rangle_\Gamma = \langle N_0 f, z\rangle_\Gamma \text{ for all } z \in H^{-1/2}(\Gamma), \tag{4.41}$$

with the Galerkin boundary element formulation which reads: find $w_h \in \mathbb{H}_N^0$ such that

$$\langle Vw_h, z_h\rangle_\Gamma = \langle N_0 f, z_h\rangle_\Gamma \text{ for all } z_h \in \mathbb{H}_N^0. \tag{4.42}$$

By using standard arguments, see for example [100], we conclude the unique solvability of (4.42), as well as the error estimates

$$\|w - w_h\|_{H^{-1/2}(\Gamma)} \le ch^{3/2}|w|_{H^1_{pw}(\Gamma)} \tag{4.43}$$

and

$$\|w - w_h\|_{L_2(\Gamma)} \le ch|w|_{H^1_{pw}(\Gamma)} \tag{4.44}$$

for $w \in H^1_{pw}(\Gamma)$.

Note that in order to compute the vector $\underline{f}_2$ we need to determine w which is the unique solution of the variational equation (4.41) with the Newton potential $N_0 f$ as the right hand side. In particular, we have to compute for $i = 1,\dots,N$

$$N_0 f[i] = \int_\Gamma \phi_i^{(0)}(x) \int_\Omega U^*(x,y) f(y) dy ds_x. \tag{4.45}$$

4.3 Evaluation of Newton potential

To compute the discrete Newton potential, the order of integration is interchanged first. In particular, for a B-spline of order zero $\phi_i^{(0)}$ we have

$$N_0 f[i] = \int_\Omega f(y) \int_{\tau_i} U^*(x,y) ds_x dy.$$

Further, a special volume mesh is done in the domain Ω (see *Appendix*) for the detailed procedure. First, the domain Ω is split into rings, and on each ring suitable meshes are constructed in an adaptive way. For simplicity, we assume the mesh size of the volume elements near the boundary to be equal to the mesh size of the boundary elements. Additionally, from the ring close to the boundary to the inner rings the mesh size is reduced with a certain rate (see *Appendix*). We then obtain

$$N_0 f[i] = \sum_{j=1}^{M_r} \sum_{k=1}^{N} \int_{T_{jk}} f(y) \int_{\tau_i} U^*(x,y) ds_x dy, \tag{4.46}$$

where M_r and N are the number of rings and the number of elements on each ring respectively. Note that N is also the number of boundary elements and $\overline{\Omega} = \cup_{j=1}^{M_r} \cup_{k=1}^{N} T_{jk}$. Then, an approximation of (4.46) can be given by

$$\tilde{N}_0 f[i] = \sum_{j=1}^{M_r} \sum_{k=1}^{N} |T_{jk}| f(y_{jk}) \int_{\tau_i} U^*(x, y_{jk}) ds_x, \tag{4.47}$$

where $|T_{jk}|$ and y_{jk} are the volume and the center of mass of the element T_{jk} respectively. Note that the remaining boundary integral corresponds to the discretization of the single layer potential by using the collocation method which can be computed easily. From (4.47) the vector $\tilde{N}_0 f$ of the approximate Newton potential can be written in matricial form as follows

$$\underline{\tilde{N}}_0 f := \sum_{j=1}^{M_r} A_j \underline{f}_j, \tag{4.48}$$

where for $j = 1, \ldots, M_r$, $\underline{f}_j$ and A_j represent the piecewise constant approximation of the function f and the matrix obtained by computing the remaining boundary integral on the j^{th} ring respectively. Moreover, we have for $j = 1, \ldots, M_r$

$$\underline{f}_j[k] := |T_{jk}| f(y_{jk}) \quad \text{for } k = 1, \ldots, N \tag{4.49}$$

and

$$A_j[i,k] := \int_{\tau_i} U^*(x, y_{jk}) ds_x \quad \text{for } i,k = 1, \ldots, N. \tag{4.50}$$

Since the meshes on the boundary and on each ring are uniform, (i.e. the same size) and since the fundamental solution U^* is invariant with respect to rotations, we obtain for $j = 1, \ldots, M_r$

$$A_j[i+1, k+1] = A_j[i,k] \quad \text{for } i,k = 1, \ldots, N, \tag{4.51}$$

which means that, for $j = 1, \ldots, M_r$ the matrices A_j are circulant. This reduces the effort to generate the matrix A_j from quadratic to linear, i.e. from $O(N^2)$ to $O(N)$ as well as the matrix-vector multiplication effort from $O(N^2)$ to $O(N \log(N))$, see, e.g. [92].

The variational problem (4.42) can be replaced by the unique solvable perturbed problem

$$\langle V \tilde{w}_h, z_h \rangle_\Gamma = \langle N_0 f_h, z_h \rangle_\Gamma \quad \text{for all } z_h \in \mathbb{H}_N^0. \tag{4.52}$$

The Galerkin formulation (4.52) is equivalent to a linear system of algebraic equations,

$$V_h\underline{\tilde{w}} = \underline{\tilde{N}_0 f}, \tag{4.53}$$

where V_h and $\underline{\tilde{N}_0 f}$ are the Galerkin stiffness matrix of the single layer integral operator and the numerical approximation of the Newton potential as given in (4.48) respectively. In addition, the system of linear equations (4.53) can be solved efficiently by performing the FFT, see for example [23,92,94]. Moreover, the approximate errors are given as follows

Lemma 4.6. *For $w \in H^1_{pw}(\Gamma)$ and $f \in H^1_{pw}(\Omega)$ then the error estimates are given by*

$$\|w - \tilde{w}_h\|_{H^{-1/2}(\Gamma)} \le c_1 h^{3/2}|w|_{H^1_{pw}(\Gamma)} + c_2 h|f|_{H^1(\Omega)} \tag{4.54}$$

and

$$\|w - \tilde{w}_h\|_{L_2(\Gamma)} \le c_1 h|w|_{H^1_{pw}(\Gamma)} + c_2 h^{1/2}|f|_{H^1(\Omega)}. \tag{4.55}$$

Proof. By using the triangle inequality we have

$$\|w - \tilde{w}_h\|_{H^{-1/2}(\Gamma)} \le \|w - w_h\|_{H^{-1/2}(\Gamma)} + \|w_h - \tilde{w}_h\|_{H^{-1/2}(\Gamma)}.$$

Remember that, $w_h \in \mathbb{H}^0_N$ is the unique solution of the Galerkin variational problem

$$\langle V w_h, z_h\rangle_\Gamma = \langle N_0 f, z_h\rangle_\Gamma \text{ for all } z_h \in \mathbb{H}^0_N, \tag{4.56}$$

while $\tilde{w}_h \in \mathbb{H}^0_N$ is the unique solution of the pertubed problem

$$\langle V \tilde{w}_h, z_h\rangle_\Gamma = \langle N_0 f_h, z_h\rangle_\Gamma \text{ for all } z_h \in \mathbb{H}^0_N. \tag{4.57}$$

If we subtract (4.57) from (4.56), set $z_h := w_h - \tilde{w}_h$ and use the ellipticity of the single layer boundary integral operator V we then obtain

$$\|w_h - \tilde{w}_h\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^V}\|N_0(f - f_h)\|_{H^{1/2}(\Gamma)}.$$

Further, if we use the continuity of N_0 and the property of the interpolation operator $f_h := I_h f$, see [100], we then obtain

$$\|N_0(f - f_h)\|_{H^{1/2}(\Gamma)} \le c_2^N\|f - f_h\|_{\tilde{H}^{-1}(\Omega)} \le c_2^N\|f - f_h\|_{L_2(\Omega)} \le ch|f|_{H^1(\Omega)},$$

that is

$$\|w_h - \tilde{w}_h\|_{H^{-1/2}(\Gamma)} \le c_2 h|f|_{H^1(\Omega)}. \tag{4.58}$$

Remark that instead of the interpolation, one may use the L_2-projection of the function f, that is $f_h = Q_h f$ this leads then to

$$\|N_0(f - f_h)\|_{H^{1/2}(\Gamma)} \le c_2^N\|f - f_h\|_{\tilde{H}^{-1}(\Omega)} \le ch^2|f|_{H^1(\Omega)},$$

but the computation of $Q_h f$ would require the integration of f in Ω. By using the estimates (4.43) and (4.58) the proof of (4.54) is shown. On the other hand, the inverse inequality in $\mathbb{H}^0_N$ yields,

$$\|w_h - \tilde{w}_h\|_{L_2(\Gamma)} \le c_I h^{-1/2} \|w_h - \tilde{w}_h\|_{H^{-1/2}(\Gamma)} \le c_2 h^{1/2} |f|_{H^1(\Omega)}. \tag{4.59}$$

Further, if we use the triangle inequality with respect to the L_2 norm, and the estimates (4.44) and (4.59) the proof of (4.55) is then shown. ☐

Remember that $\hat{u} \in \tilde{H}^{1/2}(\Gamma_N)$ is the unique solution of the variational formulation

$$\langle S\hat{u}, \mathrm{v}\rangle_{\Gamma_N} = \langle g_N - S\tilde{g}_D + w, \mathrm{v}\rangle_{\Gamma_N} \quad \text{for all } \mathrm{v} \in \tilde{H}^{1/2}(\Gamma_N), \tag{4.60}$$

$\tilde{u}_h \in \mathbb{H}^1_M \cap \tilde{H}^{1/2}(\Gamma_N)$ is the unique solution of the first modified Galerkin variational problem

$$\langle \tilde{S}\tilde{u}_h, \mathrm{v}_h\rangle_{\Gamma_N} = \langle g_N - \tilde{S}\tilde{g}_D + w, \mathrm{v}_h\rangle_{\Gamma_N} \quad \text{for all } \mathrm{v}_h \in \mathbb{H}^1_M \cap \tilde{H}^{1/2}(\Gamma_N), \tag{4.61}$$

$\tilde{\tilde{u}}_h \in \mathbb{H}^1_M \cap \tilde{H}^{1/2}(\Gamma_N)$ is the unique solution of the second modified Galerkin variational problem

$$\langle \tilde{S}\tilde{\tilde{u}}_h, \mathrm{v}_h\rangle_{\Gamma_N} = \langle g_N - \tilde{S}\tilde{g}_D + \tilde{w}_h, \mathrm{v}_h\rangle_{\Gamma_N} \quad \text{for all } \mathrm{v}_h \in \mathbb{H}^1_M \cap \tilde{H}^{1/2}(\Gamma_N), \tag{4.62}$$

and finally $w := Nf = V^{-1}N_0 f$. Therefore the final error estimate is given by:

Lemma 4.7. *If $u \in H^2(\Gamma)$, $Su \in H^1_{pw}(\Gamma)$, $w := Nf = V^{-1}N_0 f \in H^1_{pw}(\Gamma)$ and $f \in H^1(\Omega)$ then the following estimate holds*

$$\|\hat{u} - \tilde{\tilde{u}}_h\|_{H^{1/2}(\Gamma)} \le c h^{3/2} \left(\|u\|_{H^2(\Gamma)} + |Su|_{H^1_{pw}(\Gamma)} + |w|_{H^1_{pw}(\Gamma)} \right) + c_2 h |f|_{H^1(\Omega)}. \tag{4.63}$$

Proof. By using the triangle inequality we have

$$\|\hat{u} - \tilde{\tilde{u}}_h\|_{H^{1/2}(\Gamma)} \le \|\hat{u} - \tilde{u}_h\|_{H^{1/2}(\Gamma)} + \|\tilde{u}_h - \tilde{\tilde{u}}_h\|_{H^{1/2}(\Gamma)}.$$

On the other hand, if we subtract (4.62) from (4.61), and set $\mathrm{v}_h = \tilde{u}_h - \tilde{\tilde{u}}_h$ the ellipticity of $\tilde{S}$ yields then

$$\|\tilde{u}_h - \tilde{\tilde{u}}_h\|_{H^{1/2}(\Gamma)} \le c \|w - \tilde{w}_h\|_{H^{-1/2}(\Gamma)}. \tag{4.64}$$

Finally, if we use (4.64), (4.54) and Lemma 4.5 the lemma is then proved. ☐

If the complete Dirichlet datum $u_h := \tilde{\tilde{u}}_h + \tilde{g}_D$ is computed, the approximate Neumann datum $\tilde{\tilde{t}}_h \in \mathbb{H}_N^0$ can then be obtained by solving the Dirichlet boundary value problem, this satisfies the following error estimates

$$\|t - \tilde{\tilde{t}}_h\|_{H^{-1/2}(\Gamma)} \leq ch^{3/2}\left(\|u\|_{H^2(\Gamma)} + |Su|_{H^1_{pw}(\Gamma)} + |w|_{H^1_{pw}(\Gamma)}\right) + c_2 h|f|_{H^1(\Omega)} \tag{4.65}$$

and

$$\|t - \tilde{\tilde{t}}_h\|_{L_2(\Gamma)} \leq c_1 h\left(\|u\|_{H^2(\Gamma)} + |Su|_{H^1_{pw}(\Gamma)} + |w|_{H^1_{pw}(\Gamma)}\right) + c_2 h^{1/2}|f|_{H^1(\Omega)}, \tag{4.66}$$

when assuming that $u \in H^2(\Gamma)$, $Su \in H^1_{pw}(\Gamma)$, $w := Nf = V^{-1}N_0 f \in H^1_{pw}(\Gamma)$ and $f \in H^1(\Omega)$.

5 BOUNDARY INTEGRAL FORMULATIONS FOR YUKAWA TYPE LINEAR ELASTICITY PROBLEMS

This chapter is dedicated to the development of some prerequisities for the formulation of the contact problems in linear elasticity of Yukawa type. To this end, we consider a general linear elasticity model problem of Yukawa type in a two-dimensional simply connected domain Ω. We first derive the fundamental solution [51, 52, 79] which is essential to establish a representation formula and which enables us to formulate boundary integral equations. Moreover, we define the boundary integral operators, and analyze their properties and characteristics.

The chapter is organized as follows: in section 1 we formulate boundary integral equations and derive the eigensystems of some operators related to the single layer integral operator. The main focus of section 2 is the regularization of the double layer and the hypersingular integral operators [65,66,86], while section 3 is concerned with the evaluation of the vector related to the Newton potential.

5.1 Model Problem and Boundary Integral Formulation

In this section we state the model problem in a general form. By using Green's formula we derive representation formulae and compute the fundamental solution. Further, we derive boundary integral equations and the eigensystems associated to the boundary integral operators.

5.1.1 Model problem

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain (in particular a disc in $\mathbb{R}^2$) as already introduced in chapter 4. Further, we consider the system of linear elastostatics of Yukawa type,

$$s^2\underline{u}(x) - \mu\Delta\underline{u}(x) - (\lambda+\mu)\,\text{grad div}\,\underline{u}(x) = \underline{f}(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^2, \tag{5.1}$$

where Δ denotes the Laplacian, s^2 may come from the time discretization,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}$$

are the Lamé constants with $E > 0$ and $\nu \in (0, 1/2)$ which denote Young's modulus and the Poisson ratio respectively.

If we multiply (5.1) by $\underline{\mathrm{v}}$, integrate over Ω and do the integration by parts, we then obtain

$$\int_{\Omega} \sum_{i=1}^{2} f_i(x)\mathrm{v}_i(x)dx + \int_{\Gamma} \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}(\underline{u})n_j\mathrm{v}_i ds_x = \int_{\Omega} \sum_{i=1}^{2} \left(s^2 u_i \mathrm{v}_i + \sum_{j=1}^{2} \sigma_{ij}(\underline{u}) \frac{\partial}{\partial x_j}\mathrm{v}_i \right) dx, \quad (5.2)$$

where

$$\sigma(\underline{u}) := \lambda tr(\varepsilon(\underline{u}))I + 2\mu\varepsilon(\underline{u})$$

and

$$\varepsilon(\underline{u}) := \frac{1}{2}(\nabla \underline{u} + (\nabla \underline{u})^{\top})$$

are the stress and the strain tensor respectively. By using the symmetry of the stress tensor, i.e. $\sigma_{ij}(\underline{u}) = \sigma_{ji}(\underline{u})$, one can write (5.2) as follows

$$\begin{aligned} \int_{\Omega} \sum_{i=1}^{2} f_i(x)\mathrm{v}_i(x)dx + \int_{\Gamma} \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}(\underline{u})n_j\mathrm{v}_i ds_x &= \int_{\Omega} \sum_{i=1}^{2} \left(s^2 u_i \mathrm{v}_i + \sum_{j=1}^{2} \sigma_{ij}(\underline{u})\varepsilon_{ij}(\underline{\mathrm{v}}) \right) dx, \\ &= a(\underline{u}, \underline{\mathrm{v}}) \end{aligned} \quad (5.3)$$

which is the first Green (or Betti's) formula. On the other hand, we have

$$a(\underline{\mathrm{v}}, \underline{u}) = \int_{\Omega} \sum_{i=1}^{2} \left(s^2 \mathrm{v}_i - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\sigma_{ij}(\underline{\mathrm{v}}) \right) u_i dx + \int_{\Gamma} \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}(\underline{\mathrm{v}})n_j u_i ds_x. \quad (5.4)$$

Since the bilinear form $a(.,.)$ is symmetric, by equating (5.3) and (5.4) we obtain the second Green (or Betti's) formula

$$\begin{aligned} \int_{\Omega} \sum_{i=1}^{2} \left(s^2 \mathrm{v}_i - \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\sigma_{ij}(\underline{\mathrm{v}}) \right) u_i dx &= \int_{\Gamma} \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}(\underline{u})n_j\mathrm{v}_i ds_x - \int_{\Gamma} \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}(\underline{\mathrm{v}})n_j u_i ds_x \\ &+ \int_{\Omega} \sum_{i=1}^{2} f_i(x)\mathrm{v}_i(x)dx. \end{aligned} \quad (5.5)$$

As in chapter 4, to derive a representation formula for the components u_k, we have to choose $\underline{\mathrm{v}} = \underline{U}_k^*(x,y)$ for $x \in \Omega$ such that

$$\int_{\Omega} \sum_{i=1}^{2} \left(s^2 \mathrm{v}_i(y) - \sum_{j=1}^{2} \frac{\partial}{\partial y_j}\sigma_{ij}(\underline{U}_k^*(x,y),y) \right) u_i(y)dy = u_k(x) \quad \text{for } k = 1,2. \quad (5.6)$$

Further, if we set $\underline{e}^k \in \mathbb{R}^2$ the unit vector such that $e_l^k = \delta_{kl}$ for $k,l = 1,2$ where δ_{kl} is the Kronecker symbol, and by using the argument that the fundamental solution depends only on the distance between the two points x and y, we can set $z := y - x$ and the above equations lead then to the partial differential equations

$$s^2\underline{U}_k^*(z) - \mu\Delta_z\underline{U}_k^*(z) - (\lambda+\mu)\ \mathrm{grad}_z\ \mathrm{div}_z\ \underline{U}_k^*(z) = \delta_0(z)\underline{e}^k \quad \text{for } z \in \mathbb{R}^2,\ k = 1,2, \tag{5.7}$$

where δ_0 is the Dirac function. By making the ansatz

$$\underline{U}_k^*(z) := \Delta[\psi(z)\underline{e}^k] + \alpha\ \mathrm{grad\ div}\ [\psi(z)\underline{e}^k] + \beta[\psi(z)\underline{e}^k], \tag{5.8}$$

substituting (5.8) into (5.7) this yields for $z \in \mathbb{R}^2$

$$\begin{aligned} - \quad & \mu\Delta^2[\psi(z)\underline{e}^k] - [\alpha\mu + \alpha(\lambda+\mu) + (\lambda+\mu)]\Delta\ \mathrm{grad\ div}\ [\psi(z)\underline{e}^k] + s^2\beta[\psi(z)\underline{e}^k] \\ + \quad & [s^2\alpha - \beta(\lambda+\mu)]\ \mathrm{grad\ div}\ [\psi(z)\underline{e}^k] + (s^2 - \mu\beta)\Delta[\psi(z)\underline{e}^k] = \delta_0(z)\underline{e}^k. \end{aligned} \tag{5.9}$$

In addition, if one sets

$$\alpha = -\frac{\lambda+\mu}{\lambda+2\mu}, \quad \beta = -\frac{s^2}{\lambda+2\mu},$$

one obtains then

$$(-\Delta + k_2^2)(\Delta - k_1^2)\psi(z) = \frac{1}{\mu}\delta_0(z) \quad \text{for } z \in \mathbb{R}^2 \tag{5.10}$$

with

$$k_1^2 \ = \ \frac{s^2}{\lambda+2\mu}, \quad k_2^2 = \frac{s^2}{\mu}.$$

The equation (5.10) is equivalent to

$$(-\Delta + k_2^2)\varphi(z) = \frac{1}{\mu}\delta_0(z), \quad (\Delta - k_1^2)\psi(z) = \varphi(z) \quad \text{for } z \in \mathbb{R}^2. \tag{5.11}$$

From the fundamental solution of the scalar Yukawa problem as considered in chapter 4 we obtain

$$\varphi(z) = \frac{1}{\mu}\frac{1}{2\pi}K_0(k_2|z|). \tag{5.12}$$

Further, by utilizing polar coordinates we then obtain, see [100]

$$\psi(z) = \frac{1}{2\pi\mu}\frac{1}{k_2^2 - k_1^2}[K_0(k_2|z|) - K_0(k_1|z|)]. \tag{5.13}$$

Taking ψ into (5.8), one obtains $\underline{U}_k^*$ for $k = 1,2$ and therefore the fundamental solution $U^*(x,y) := (\underline{U}_1^*, \underline{U}_2^*)$ is found with the components defined for $i,j = 1,2$ by

$$\begin{aligned} U_{ij}^*(x,y) = \frac{1}{2\pi s^2}\Bigg\{ & \left(k_2^2K_0(k_2|z|) + \frac{1}{|z|}[k_2K_1(k_2|z|) - k_1K_1(k_1|z|)]\right)\delta_{ij} \\ -\frac{z_iz_j}{|z|^2} & \left(\left[k_2^2K_0(k_2|z|) - k_1^2K_0(k_1|z|)\right] + \frac{2}{|z|}[k_2K_1(k_2|z|) - k_1K_1(k_1|z|)]\right)\Bigg\}. \end{aligned} \tag{5.14}$$

Note that any solution $\underline{u}$ of (5.1) is given by the representation formula for $x \in \Omega$

$$\underline{u}(x) = \int_\Gamma U^*(x,y)\gamma_{1,y}^{int}\underline{u}(y)ds_y - \int_\Gamma \gamma_{1,y}^{int}U^*(x,y)\gamma_{0,y}^{int}\underline{u}(y)ds_y + \int_\Omega U^*(x,y)\underline{f}(y)dy, \quad (5.15)$$

where the boundary operator $\gamma_{1,y}^{int}(.)$ is the boundary stress operator with respect to the variable y defined by

$$\gamma_{1,y}^{int}(.) := T_y(.) := \lambda \text{ div}_y\,(.)\,\underline{n}(y) + 2\mu\,\frac{\partial}{\partial n_y}(.) + \mu\,\underline{n}(y)\times \text{ curl}_y\,(.) \quad \text{for } y \in \Gamma,$$

where $\underline{n}(y)$, $y \in \Gamma$ is the outer unit normal vector at y. Note that for a two-dimensional vector $\underline{\mathrm{v}} = (\mathrm{v}_1, \mathrm{v}_2)^\top$ we have

$$\underline{n}(y)\times \text{ curl}_y\,\underline{\mathrm{v}} = (n_2\frac{\partial \mathrm{v}_2}{\partial y_1} - n_2\frac{\partial \mathrm{v}_1}{\partial y_2}, n_1\frac{\partial \mathrm{v}_1}{\partial y_2} - n_1\frac{\partial \mathrm{v}_2}{\partial y_1})^\top.$$

5.1.2 Boundary integral operators

We know that $\underline{u}$ is uniquely determined for $x \in \Omega$ just by its boundary data $\{\gamma_{0,y}^{int}\underline{u}, \gamma_{1,y}^{int}\underline{u}\}$ and the sources $\underline{f}$. To find the complete Cauchy data, we first apply the trace operator to the representation formula (5.15) which leads to

$$\gamma_0^{int}\underline{u}(x) = (V\gamma_1^{int}\underline{u})(x) - \left(-\frac{1}{2}I + K\right)\gamma_0^{int}\underline{u}(x) + (N_0\underline{f})(x) \quad \text{for all} \quad x \in \Gamma. \quad (5.16)$$

Second, if we apply the boundary stress operator $T_y := \gamma_1^{int}$ again to (5.15), we then obtain

$$\gamma_1^{int}\underline{u}(x) = \left(\frac{1}{2}I + K'\right)\gamma_1^{int}\underline{u}(x) + (D\gamma_0^{int}\underline{u})(x) + (N_1\underline{f})(x) \quad \text{for all} \quad x \in \Gamma. \quad (5.17)$$

Let us define the Sobolev space $\mathbf{H}^{1/2}(\Gamma)$ by

$$\mathbf{H}^{1/2}(\Gamma) := \prod_{j=1}^{2} H^{1/2}(\Gamma).$$

In a similar way $\mathbf{H}^{-1/2}(\Gamma)$, $\tilde{\mathbf{H}}^{-1}(\Omega)$ and $\mathbf{L}_2(\Gamma)$ are defined.

Note that all the mapping properties of the boundary integral operators as shown in chapter 4 for the scalar Yukawa problem can be transfered to the linear elasticity problem of Yukawa type. We have then that $V : \mathbf{H}^{-1/2}(\Gamma) \to \mathbf{H}^{1/2}(\Gamma)$ is the single layer operator, $K : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{H}^{1/2}(\Gamma)$ is the double layer potential, the adjoint double layer potential

$K' : \mathbf{H}^{-1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\Gamma)$, the hypersingular boundary integral operator $D : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\Gamma)$, and $N_0 : \tilde{\mathbf{H}}^{-1}(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$ and $N_1 : \tilde{\mathbf{H}}^{-1}(\Omega) \to \mathbf{H}^{-1/2}(\Gamma)$ are the Newton potentials or volume potentials. These results are also obtained in the case the boundary is at least Lipschitz. Moreover, they are all linear, bounded and defined by

$$\begin{aligned}
(V\underline{t})(x) &:= \int_\Gamma U^*(x,y)\underline{t}(y)ds_y \quad \text{for } x \in \Gamma,\\
(K\underline{u})(x) &:= \int_\Gamma [T_y U^*(x,y)]^\top \underline{u}(y)ds_y \quad \text{for } x \in \Gamma,\\
(K'\underline{t})(x) &:= \int_\Gamma T_x U^*(x,y)\underline{t}(y)ds_y \quad \text{for } x \in \Gamma,\\
(D\underline{u})(x) &:= -T_x \int_\Gamma [T_y U^*(x,y)]^\top \underline{u}(y)ds_y \quad \text{for } x \in \Gamma,\\
(N_0\underline{f})(x) &:= \int_\Omega U^*(x,y)\underline{f}(y)dy \quad \text{for } x \in \Gamma,\\
(N_1\underline{f})(x) &:= T_x \int_\Omega U^*(x,y)\underline{f}(y)dy \quad \text{for } x \in \Gamma.
\end{aligned}$$

In addition, the integral representations of V and D are understood as weakly singular and as hypersingular boundary integral respectively, while the integrals for K and K' are in general Cauchy singular integrals. Note that as for the scalar Yukawa problem, Lemma 4.1 remains true, i.e. the single layer operator V and the hypersingular integral operator D are self-adjoint, positive definite and satisfy the following properties

$$\langle V\underline{\tau},\underline{\tau}\rangle_\Gamma \geq c_1^V \|\underline{\tau}\|^2_{\mathbf{H}^{-1/2}(\Gamma)} \quad \text{for all} \quad \underline{\tau} \in \mathbf{H}^{-1/2}(\Gamma) \quad \text{with} \quad c_1^V > 0$$

and

$$\langle D\underline{u},\underline{u}\rangle_\Gamma \geq c_1^D \|\underline{u}\|^2_{\mathbf{H}^{1/2}(\Gamma)} \quad \text{for all} \quad \underline{u} \in \mathbf{H}^{1/2}(\Gamma) \quad \text{with} \quad c_1^D > 0$$

respectively.

5.1.3 Eigensystems of boundary integral operators

In this section we are interested to derive the eigensystems of some boundary integral operators related to the single layer integral operator V. To this end we consider in particular a two-dimensional circular domain $\Omega := B_R(O)$ of radius R and centered at the origin as introduced in chapter 4. Further, we assume a one-periodic parametrization of the boundary $\Gamma := \partial\Omega$ given by

$$\Gamma := \{x \in \mathbb{R}^2 : x(\tau) = R\begin{pmatrix}\cos 2\pi\tau\\ \sin 2\pi\tau\end{pmatrix}, 0 \leq \tau < 1\}. \tag{5.18}$$

Remember that the single layer integral operator V is defined by

$$(V\underline{v})(x) = \int_\Gamma U^*(x,y)\underline{v}(y)ds_y. \tag{5.19}$$

Remark that the single layer integral operator V can be written in matrix form as follows

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

so that we have $V_{ii} = V_{ii}^1 + V_{ii}^2$, where V_{ii}^1 are the principal parts of the integral operators V_{ii} for $i = 1,2$. By using the parametrization (5.18) we obtain then for $r_i := x_i - y_i$, $i = 1,2$, and $r := |x-y|$

$$\begin{aligned}(V_{11}^1 v_1)(x) &= \frac{1}{2\pi s^2}\int_\Gamma \left[k_2^2 K_0(k_2 r) + \frac{[k_2K_1(k_2r) - k_1K_1(k_1r)]}{r}\right] v_1(y)ds_y \\ &= \frac{R}{s^2}\int_0^1 \Big[k_2^2 K_0(2k_2R|\sin\pi(\tau-t)|) \\ &+ \frac{[k_2K_1(2k_2R|\sin\pi(\tau-t)|) - k_1K_1(2k_1R|\sin\pi(\tau-t)|)]}{2R|\sin\pi(\tau-t)|}\Big] v_1(t)dt,\end{aligned}$$

$$\begin{aligned}(V_{11}^2 v_1)(x) &= -\frac{1}{2\pi s^2}\int_\Gamma \left[[k_2^2K_0(k_2r) - k_1^2K_0(k_1r)] + \frac{2[k_2K_1(k_2r) - k_1K_1(k_1r)]}{r}\right]\frac{r_1^2}{r^2} v_1(y)ds_y \\ &= -\frac{R}{s^2}\int_0^1 \Big[[k_2^2K_0(2k_2R|\sin\pi(\tau-t)|) - k_1^2K_0(2k_1R|\sin\pi(\tau-t)|)] \\ &+ \frac{[k_2K_1(2k_2R|\sin\pi(\tau-t)|) - k_1K_1(2k_1R|\sin\pi(\tau-t)|)]}{R|\sin\pi(\tau-t)|}\Big]\sin^2\pi(\tau+t)v_1(t)dt,\end{aligned}$$

$$\begin{aligned}(V_{12} v_2)(x) &= -\frac{1}{2\pi s^2}\int_\Gamma \left[[k_2^2K_0(k_2r) - k_1^2K_0(k_1r)] + \frac{2[k_2K_1(k_2r) - k_1K_1(k_1r)]}{r}\right]\frac{r_1 r_2}{r^2} v_2(y)ds_y \\ &= \frac{R}{2s^2}\int_0^1 \Big[[k_2^2K_0(2k_2R|\sin\pi(\tau-t)|) - k_1^2K_0(2k_1R|\sin\pi(\tau-t)|)] \\ &+ \frac{[k_2K_1(2k_2R|\sin\pi(\tau-t)|) - k_1K_1(2k_1R|\sin\pi(\tau-t)|)]}{R|\sin\pi(\tau-t)|}\Big]\sin 2\pi(\tau+t)v_2(t)dt \\ &= (V_{21}v_2)(x),\end{aligned}$$

$$
\begin{aligned}
(V_{22}^1 \mathrm{v}_2)(x) &= \frac{1}{2\pi s^2} \int_\Gamma \left[k_1^2 K_0(k_1 r) - \frac{[k_2 K_1(k_2 r) - k_1 K_1(k_1 r)]}{r} \right] \mathrm{v}_2(y) ds_y \\
&= \frac{R}{s^2} \int_0^1 \left[k_1^2 K_0(2k_1 R|\sin\pi(\tau - t)|) \right. \\
&- \left. \frac{[k_2 K_1(2k_2 R|\sin\pi(\tau-t)|) - k_1 K_1(2k_1 R|\sin\pi(\tau-t)|)]}{2R|\sin\pi(\tau-t)|} \right] \mathrm{v}_2(t) dt, \\
(V_{22}^2 \mathrm{v}_2)(x) &= -(V_{11}^2 \mathrm{v}_2)(x).
\end{aligned}
$$

Proposition 5.1. *Let Γ be given as in (5.18). The Fourier functions*

$$v_n(t) = e^{\mp i2\pi nt} \quad \text{for } n = 0, 1, \ldots$$

are the eigenfunctions of the boundary integral operators V_{11}^1 and V_{22}^1 respectively, i.e.

$$
\begin{aligned}
(V_{11}^1 v_n)(\tau) &= \frac{1}{2s^2} \{\Lambda_n(k_1,k_2) + 2\Gamma_{n,n-1,n+1}(k_1,k_2)\} v_n(\tau), \\
(V_{22}^1 v_n)(\tau) &= \frac{1}{2s^2} \{\Lambda_n(k_1,k_2) - 2\Gamma_{n,n-1,n+1}(k_1,k_2)\} v_n(\tau),
\end{aligned}
$$

where

$$\Gamma_{m,n,k}(.) = (\lambda_m^D(k_1) - \lambda_m^D(k_2)) + \frac{1}{4} \left[(k_2^2 \lambda_n^V(k_2) - k_1^2 \lambda_n^V(k_1)) + (k_2^2 \lambda_k^V(k_2) - k_1^2 \lambda_k^V(k_1)) \right],$$

and

$$\Lambda_n(k_1,k_2) = [k_1^2 \lambda_k^V(k_1) + k_2^2 \lambda_k^V(k_2)]$$

for $k_1^2 = \dfrac{s^2}{\lambda + 2\mu}$, $k_2^2 = \dfrac{s^2}{\mu}$. In addition, the following relations hold

$$\Gamma_{m,n,k}(k_1,k_2) = \Gamma_{-m,-n,-k}(k_1,k_2) = \Gamma_{m,k,n}(k_1,k_2), \;\; \Lambda_n(k_1,k_2) = \Lambda_{-n}(k_1,k_2),$$

where λ_n^V and λ_n^D for $n = 0, 1, \ldots$ stand for the eigenvalues of the single layer integral operator and the hypersingular integral operator for the scalar Yukawa problem respectively as given in chapter 4.

Proof. Note that the eigenvalues of the first part of the operators V_{11}^1 and V_{22}^1 can be computed easily by using the eigenvalues of the scalar Yukawa problem (see Lemma 4.2). The second part remains a bit challenging. To overcome this difficulty we rewrite this term by utilizing the hypersingular boundary integral operator for the scalar Yukawa problem. To this end, let us first define the boundary operators $\mathcal{B}_j$ for $j = 1, 2$ by

$$(\mathcal{B}_j v_n)(\tau) = \frac{Rk_j}{s^2} \int_0^1 \frac{K_1(2k_j R|\sin\pi(\tau-t)|)}{2R|\sin\pi(\tau-t)|} v_n(t) dt.$$

From the hypersingular boundary integral operator for the scalar Yukawa problem (see section 4.1.3) it follows immediately that $\mathcal{B}_j$ can be written as follows:

$$(\mathcal{B}_j v_n)(\tau) = -(D_{k_j} v_n)(\tau) - k_j^2 R \int_0^1 K_0(2k_j R|\sin \pi(\tau - t)|) \sin^2 \pi(\tau - t) v_n(t) dt \quad \text{for } j = 1, 2,$$

where D_{k_j} is the hypersingular boundary integral operator for the scalar Yukawa problem, with the wave number $\alpha = k_j$. Finally, by a direct computation we obtain

$$(\mathcal{B}_j v_n)(\tau) = -\left(\lambda_n^D(k_j) - \frac{k_j^2}{4} [\lambda_{n-1}^V(k_j) - 2\lambda_n^V(k_j) + \lambda_{n+1}^V(k_j)] \right) v_n(\tau) \quad \text{for } j = 1, 2,$$

which ends the proof. □

On the other hand, the double layer and the hypersingular boundary integral operators given in the section above are not suitable for any numerical treatment. This is due to the higher singularity of their kernel. Therefore, these need to be regularized in order to reduce the singularity at least to a weak one.

5.2 Regularization of the double layer and the hypersingular operators

The regularization we present here is based on the concept of Günter derivatives [72] and the integration by parts as given by Nédélec in the case of the Laplace equation as well as for the Helmholtz problem [86]. Our presentation is inspired from the work done by L. Kielhorn in the three-dimensional viscoelastodynamics problem, therefore readers can consult [65, 66] for details. Similar results can be found in [39, 72, 100] for the linear elasticity problem. In this section, we will mostly confine our attention on the regularization of the double layer operator. The case of the hypersingular operator is done in a similar way.

Indeed, we know that the boundary stress for any vector $\underline{v}$ is given by

$$(T_y \, \underline{v})^\top = \lambda \text{ div } \underline{v} \, \underline{n}(y) + 2\mu \frac{\partial \underline{v}}{\partial \underline{n}} + \mu \, \underline{n}(y) \times \text{ curl } \underline{v}, \tag{5.20}$$

where $\underline{n}$ denotes the outer normal. Further, the so-called Günter derivative is given by

$$\mathcal{M}(\partial_y, \underline{n}(y))(.) = \frac{\partial(.)}{\partial \underline{n}} - \underline{n}(y) \text{ div } (.) + \underline{n}(y) \times \text{ curl } (.). \tag{5.21}$$

$\mathcal{M}(\partial_y, \underline{n}(y))(.)$ can be written in matrix form as follows

$$\mathcal{M}(\partial_y, \underline{n}(y)) = \nabla_y \otimes \underline{n}(y) - \underline{n}(y) \otimes \nabla_y = \begin{pmatrix} 0 & n_2 \frac{\partial}{\partial y_1} - n_1 \frac{\partial}{\partial y_2} \\ n_1 \frac{\partial}{\partial y_2} - n_2 \frac{\partial}{\partial y_1} & 0 \end{pmatrix},$$

where $\otimes$ stands here for a tensor product. If we use (5.21) into (5.20) we then obtain

$$(T_y\, \underline{v})^\top = 2\mu\ \mathcal{M}\underline{v} + (\lambda + 2\mu)\ \mathrm{div}\ \underline{v}\ \underline{n}(y) - \mu\ \underline{n}(y) \times\ \mathrm{curl}\ \underline{v}. \tag{5.22}$$

We need to rearrange the last two terms of the formula (5.22), i.e., the terms with div and curl respectively. To this end, we split the fundamental solution U^* into two parts as follows

$$U^*_{ij}(x,y) = \frac{1}{\mu}\left[\Delta\chi\delta_{ij} - \frac{\lambda+\mu}{\lambda+2\mu}\partial_{ij}\chi\right] - \frac{1}{\mu}k_1^2\chi\delta_{ij}, \tag{5.23}$$

where δ_{ij} represents the Kronecker symbol and

$$\chi(r) = \frac{1}{2\pi}\frac{1}{k_1^2 - k_2^2}\left[K_0(k_1 r) - K_0(k_2 r)\right]$$

is a regular function, in particular a solution of the partial differential equation given by

$$(\Delta - k_2^2)(\Delta - k_1^2)\chi = \delta(r),$$

where $\delta(r)$ represents the Dirac delta function. Moreover, we have

$$(\Delta - k_1^2)\chi = \varphi_2(r) \equiv \frac{1}{2\pi}K_0(k_2 r) \quad \text{and} \quad (\Delta - k_2^2)\chi = \varphi_1(r) \equiv \frac{1}{2\pi}K_0(k_1 r),$$

where $\varphi_1(r)$ and $\varphi_2(r)$ are fundamental solutions of scalar Yukawa problems for $\alpha \equiv k_1$ and $\alpha \equiv k_2$ respectively. By utilizing (5.22) we compute the boundary stress of each part of (5.23) and by adding them we obtain

$$\begin{aligned}(T_y\, U^*)^\top &= 2\mu\ (\mathcal{M}U^*)^\top + \mathcal{M}(\Delta\chi) + \frac{\partial}{\partial \underline{n}(y)}\Delta\chi\mathbf{I} \\ &- k_2^2\ \nabla\chi \otimes \underline{n}(y) + k_1^2\ \underline{n}(y) \otimes \nabla\chi - k_1^2\ \frac{\partial\chi}{\partial \underline{n}(y)}\mathbf{I},\end{aligned}$$

rearranging we obtain

$$(T_y\, U^*)^\top = 2\mu\ (\mathcal{M}U^*)^\top + \mathcal{M}(\Delta\chi - k_1^2\chi) + \frac{\partial}{\partial \underline{n}(y)}(\Delta\chi - k_1^2\chi)\mathbf{I} + (k_1^2 - k_2^2)\ \nabla\chi \otimes \underline{n}(y),$$

that is

$$(T_y\, U^*)^\top = 2\mu\ (\mathcal{M}U^*)^\top + \mathcal{M}\varphi_2 + \frac{\partial\varphi_2}{\partial \underline{n}(y)}\mathbf{I} + (k_1^2 - k_2^2)\ \nabla\chi \otimes \underline{n}(y).$$

Further, if the parametrization (5.18) of the boundary Γ is used one obtains then

$$(T_y\, U^*)^\top = \frac{\mu}{\pi R}\frac{d}{dt}\begin{pmatrix} -U^*_{12} & U^*_{11} \\ -U^*_{22} & U^*_{12} \end{pmatrix} + \frac{1}{2\pi R}\frac{d}{dt}\begin{pmatrix} 0 & -\varphi_2 \\ \varphi_2 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial \varphi_2}{\partial \underline{n}(y)} & 0 \\ 0 & \frac{\partial \varphi_2}{\partial \underline{n}(y)} \end{pmatrix}$$
$$+(k_1^2-k_2^2)\begin{pmatrix} n_1\frac{\partial \chi}{\partial y_1} & n_2\frac{\partial \chi}{\partial y_1} \\ n_1\frac{\partial \chi}{\partial y_2} & n_2\frac{\partial \chi}{\partial y_2} \end{pmatrix},$$

further reordering yields

$$(T_y\, U^*)^\top = \frac{\mu}{\pi R}\frac{d}{dt}\begin{pmatrix} -U^*_{12} & U^*_{11} \\ -U^*_{22} & U^*_{12} \end{pmatrix} + \frac{1}{2\pi R}\frac{d}{dt}\begin{pmatrix} 0 & -\varphi_2 \\ \varphi_1 & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial \varphi_1}{\partial \underline{n}(y)} & 0 \\ 0 & \frac{\partial \varphi_2}{\partial \underline{n}(y)} \end{pmatrix}$$
$$+n_2(y)\begin{pmatrix} -\frac{\partial}{\partial y_2}(\varphi_1-\varphi_2) & \frac{\partial}{\partial y_1}(\varphi_1-\varphi_2) \\ \frac{\partial}{\partial y_1}(\varphi_1-\varphi_2) & \frac{\partial}{\partial y_2}(\varphi_1-\varphi_2) \end{pmatrix}.$$

By utilizing integration by parts, the double layer integral operator can be written as follows

$$\begin{aligned}
(\mathbf{K}\underline{u})(x) &= \int_\Gamma [T_y\, U^*(x,y)]^\top\, \underline{u}(y) ds_y \\
&= 2\mu \int_0^1 \begin{pmatrix} U^*_{12} & -U^*_{11} \\ U^*_{22} & -U^*_{12} \end{pmatrix} \frac{d\underline{u}}{dt} dt + \int_0^1 \begin{pmatrix} 0 & \varphi_2 \\ -\varphi_1 & 0 \end{pmatrix} \frac{d\underline{u}}{dt} dt \\
&+ 2\pi R \int_0^1 \begin{pmatrix} \frac{\partial \varphi_1}{\partial \underline{n}(y(t))} & 0 \\ 0 & \frac{\partial \varphi_2}{\partial \underline{n}(y(t))} \end{pmatrix} \underline{u}(y(t)) dt \\
&+ 2\pi R \int_0^1 n_2(y(t)) \begin{pmatrix} -\frac{\partial}{\partial y_2}(\varphi_1-\varphi_2) & \frac{\partial}{\partial y_1}(\varphi_1-\varphi_2) \\ \frac{\partial}{\partial y_1}(\varphi_1-\varphi_2) & \frac{\partial}{\partial y_2}(\varphi_1-\varphi_2) \end{pmatrix} \underline{u}(y(t)) dt.
\end{aligned}$$

Remark 5.1. *We notice that the first block of the above relation can be derived from the single layer operator of linear elasticity. This will enable us to compute that block matrix for the double layer matrix in an efficient way in terms of time since this is already stored. The second and the third block can be computed from the single layer and the double layer of the scalar Yukawa problem respectively whose eigenvalues are already computed in chapter 4, see Lemma 4.3, we just have to set the wave number* $\alpha = k_i$, $i = 1,2$. *The fourth block is symmetric and has the same diagonal blocks, and moreover the functions involved are regular, therefore only two block matrices will be computed here by the help of Gauss quadratures.*

On the other hand, by following the same idea as above, the bilinear form of the hypersingular integral operator can be written as follows

$$\begin{aligned}
\langle D\underline{u},\underline{v}\rangle_\Gamma &= -\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot\left[\frac{\partial^2\varphi_2}{\partial\underline{n}(y)\partial\underline{n}(x)}I\right]\cdot\underline{u}(y)ds_yds_x\\
&-\mu\int_\Gamma\int_\Gamma (\mathcal{M}_x\cdot\underline{v}(x))\cdot[4\mu U^*]\cdot(\mathcal{M}_y\cdot\underline{u}(y))\,ds_yds_x\\
&+\mu\int_\Gamma\int_\Gamma (\mathcal{M}_x\cdot\underline{v}(x))\cdot[2\varphi_1 I]\cdot(\mathcal{M}_y\cdot\underline{u}(y))\,ds_yds_x\\
&-\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot[\mathcal{M}_y\cdot(\mathcal{M}_x\varphi_2)]\cdot\underline{u}(y)ds_yds_x\\
&+\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot\left[(k_2^2-2k_1^2)\varphi_1\underline{n}(x)\otimes\underline{n}(y)\right]\cdot\underline{u}(y)ds_yds_x\\
&+\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot\left[k_2^2\varphi_2\underline{n}(y)\otimes\underline{n}(x)\right]\cdot\underline{u}(y)ds_yds_x\\
&+\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot\left[2(k_1^2-k_2^2)\,(\nabla_y\nabla_y\chi)\,\underline{n}(y)\cdot\underline{n}(x)\right]\cdot\underline{u}(y)ds_yds_x,
\end{aligned}$$

where $\underline{n}$ denotes the unit outer normal vector on Γ, see, e.g. [65,66] for details. In addition, for the two-dimensional case we have

$$-\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot[\mathcal{M}_y\cdot(\mathcal{M}_x\varphi_2)]\cdot\underline{u}(y)ds_yds_x=\mu\int_\Gamma\int_\Gamma(\mathcal{M}_x\cdot\underline{v}(x))\cdot[\varphi_2 I]\cdot(\mathcal{M}_y\cdot\underline{u}(y))\,ds_yds_x,$$

which yields

$$\begin{aligned}
\langle D\underline{u},\underline{v}\rangle_\Gamma &= -\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot\left[\frac{\partial^2\varphi_2}{\partial\underline{n}(y)\partial\underline{n}(x)}I\right]\cdot\underline{u}(y)ds_yds_x\\
&-\mu\int_\Gamma\int_\Gamma (\mathcal{M}_x\cdot\underline{v}(x))\cdot[4\mu U^*-(2\varphi_1+\varphi_2)I]\cdot(\mathcal{M}_y\cdot\underline{u}(y))\,ds_yds_x\\
&+\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot\left[(k_2^2-2k_1^2)\varphi_1\underline{n}(x)\otimes\underline{n}(y)\right]\cdot\underline{u}(y)ds_yds_x\\
&+\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot\left[k_2^2\varphi_2\underline{n}(y)\otimes\underline{n}(x)\right]\cdot\underline{u}(y)ds_yds_x\\
&+\mu\int_\Gamma\int_\Gamma \underline{v}(x)\cdot\left[2(k_1^2-k_2^2)\,(\nabla_y\nabla_y\chi)\,\underline{n}(y)\cdot\underline{n}(x)\right]\cdot\underline{u}(y)ds_yds_x.
\end{aligned}$$

In the next step we want to show that the integrand in the last integral can be split in such a way that the singular part is evaluated by using the eigenvalues of the hypersingular operator for the scalar Yukawa problem and the regular part by a standard Gauss quadratures. Indeed, remember that

$$\chi(r) = \frac{1}{2\pi}\frac{1}{k_1^2-k_2^2}[K_0(k_1 r) - K_0(k_2 r)].$$

Therefore we have

$$(k_1^2-k_2^2)\,(\nabla_y\nabla_y\chi) = \frac{1}{2\pi}\begin{bmatrix} \frac{\partial^2}{\partial y_1^2}[K_0(k_1 r) - K_0(k_2 r)] & \frac{\partial^2}{\partial y_1 \partial y_2}[K_0(k_1 r) - K_0(k_2 r)] \\ \frac{\partial^2}{\partial y_1 \partial y_2}[K_0(k_1 r) - K_0(k_2 r)] & \frac{\partial^2}{\partial y_2^2}[K_0(k_1 r) - K_0(k_2 r)] \end{bmatrix},$$

where $r = |x-y|$,

$$K_0(r) = (\ln 2 - \mathbf{E} - \ln r)I_0(r) + \sum_{k=1}^{\infty}\left[\left(\sum_{j=1}^{k}\frac{1}{j}\right)\frac{1}{(k!)^2}\left(\frac{r}{2}\right)^{2k}\right]$$

with $I_0(r) = 1 + \sum_{k=1}^{\infty}\frac{1}{(k!)^2}\left(\frac{r}{2}\right)^{2k}$, and $\mathbf{E} = \lim_{n\to\infty}\left[\sum_{j=1}^{n}\frac{1}{j} - \ln n\right] \approx 0.57721566490...$,

and

$$K_1(r) := -K_0'(r) := \frac{1}{r} + \sum_{k=1}^{\infty}\left[\frac{1}{2} + k\left(\mathbf{E} - \ln 2 + \ln r - \sum_{j=1}^{k}\frac{1}{j}\right)\right]\frac{1}{(k!)^2}\left(\frac{r}{2}\right)^{2k-1}.$$

Utilize the following recurrence relation between the modified Bessel functions, see [1]

$$K_n'(x) = -K_{n-1}(x) - \frac{n}{x}K_n(x),$$

one obtains for $r_i = x_i - y_i,\ i = 1,2,$

$$\begin{aligned} \frac{\partial^2 K_0(k_j r)}{\partial y_i^2} &= \left[k_j^2 K_0(k_j r) + \frac{2k_j}{r}K_1(k_j r)\right]\frac{r_i^2}{r^2} - \frac{k_j}{r}K_1(k_j r), \\ \frac{\partial^2 K_0(k_j r)}{\partial y_i \partial y_j} &= \left[k_j^2 K_0(k_j r) + \frac{2k_j}{r}K_1(k_j r)\right]\frac{r_i r_j}{r^2},\ i \neq j. \end{aligned}$$

Remember that $\frac{1}{2\pi}K_0(k_j r)$ is a fundamental solution for the scalar Yukawa problem with a positive wave number k_j . The associated hypersingular integral operator is then defined for $\mathbf{r} = x - y$ by

$$\begin{aligned}\left(D_{k_j}\mathrm{v}\right)(x) &= -\frac{1}{2\pi}\int\limits_\Gamma \frac{\partial}{\partial \underline{n}(x)}\frac{\partial}{\partial \underline{n}(y)}K_0(k_j r)\mathrm{v}(y)ds_y \\ &= \frac{1}{2\pi}\int\limits_\Gamma \left[k_j^2 K_0(k_j r) + \frac{2k_j}{r}K_1(k_j r)\right]\frac{(\mathbf{r},\underline{n}(y))(\mathbf{r},\underline{n}(x))}{r^2}\mathrm{v}(y)ds_y \\ &\quad -\frac{k_j}{2\pi}\int\limits_\Gamma K_1(k_j r)\frac{\underline{n}(y)\cdot\underline{n}(x)}{r}\mathrm{v}(y)ds_y.\end{aligned}$$

We then obtain

$$\begin{aligned}&-\frac{k_j}{2\pi}\int\limits_\Gamma K_1(k_j r)\frac{\underline{n}(y)\cdot\underline{n}(x)}{r}\mathrm{v}(y)ds_y \\ &= \left(D_{k_j}\mathrm{v}\right)(x) - \frac{1}{2\pi}\int\limits_\Gamma \left[k_j^2 K_0(k_j r) + \frac{2k_j}{r}K_1(k_j r)\right]\frac{(\mathbf{r},\underline{n}(y))(\mathbf{r},\underline{n}(x))}{r^2}\mathrm{v}(y)ds_y.\end{aligned}$$

This yields

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_\Gamma \frac{\partial^2 K_0(k_j r)}{\partial y_i^2}\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y \\ &= \frac{1}{2\pi}\int\limits_\Gamma \left[k_j^2 K_0(k_j r) + \frac{2k_j}{r}K_1(k_j r)\right]\frac{r_i^2}{r^2}\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y \\ &-\frac{k_j}{2\pi}\int\limits_\Gamma K_1(k_j r)\frac{\underline{n}(y)\cdot\underline{n}(x)}{r}\mathrm{v}(y)ds_y, \\ &= \frac{1}{2\pi}\int\limits_\Gamma \left[k_j^2 K_0(k_j r) + \frac{2k_j}{r}K_1(k_j r)\right]\frac{r_i^2}{r^2}\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y \\ &-\frac{1}{2\pi}\int\limits_\Gamma \left[k_j^2 K_0(k_j r) + \frac{2k_j}{r}K_1(k_j r)\right]\frac{(\mathbf{r},\underline{n}(y))(\mathbf{r},\underline{n}(x))}{r^2}\mathrm{v}(y)ds_y + \left(D_{k_j}\mathrm{v}\right)(x).\end{aligned} \tag{5.24}$$

We also have

$$\begin{aligned}&\frac{1}{2\pi}\int\limits_\Gamma \frac{\partial^2 K_0(k_j r)}{\partial y_i \partial y_j}\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y \\ &= \frac{1}{2\pi}\int\limits_\Gamma \left[k_j^2 K_0(k_j r) + \frac{2k_j}{r}K_1(k_j r)\right]\frac{r_i r_j}{r^2}\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y.\end{aligned} \tag{5.25}$$

By utilizing (5.24) we obtain

$$\frac{1}{2\pi}\int_{\Gamma}\frac{\partial^2}{\partial y_i^2}\left[K_0(k_1r)-K_0(k_2r)\right]\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y$$
$$=\frac{1}{2\pi}\int_{\Gamma}\left[k_1^2K_0(k_1r)+\frac{2k_1}{r}K_1(k_1r)-k_2^2K_0(k_2r)-\frac{2k_2}{r}K_1(k_2r)\right]\frac{r_i^2}{r^2}\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y$$
$$-\frac{1}{2\pi}\int_{\Gamma}\left[k_1^2K_0(k_1r)+\frac{2k_1}{r}K_1(k_1r)-k_2^2K_0(k_2r)-\frac{2k_2}{r}K_1(k_2r)\right]\frac{(\mathbf{r},\underline{n}(y))(\mathbf{r},\underline{n}(x))}{r^2}\mathrm{v}(y)ds_y$$
$$+\left(D_{k_1}\mathrm{v}\right)(x)-\left(D_{k_2}\mathrm{v}\right)(x),$$

and (5.25) yields,

$$\frac{1}{2\pi}\int_{\Gamma}\frac{\partial^2}{\partial y_i\partial y_j}\left[K_0(k_1r)-K_0(k_2r)\right]\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y=$$
$$\frac{1}{2\pi}\int_{\Gamma}\left[k_1^2K_0(k_1r)+\frac{2k_1}{r}K_1(k_1r)-k_2^2K_0(k_2r)-\frac{2k_2}{r}K_1(k_2r)\right]\frac{r_ir_j}{r^2}\underline{n}(y)\cdot\underline{n}(x)\mathrm{v}(y)ds_y.$$

Note that

$$\frac{2k_j}{r}K_1(k_jr)=\frac{2}{r^2}+k_j^2[\mathbf{E}-\ln 2-\frac{1}{2}+\ln(k_jr)]$$
$$+k_j^2\sum_{k=2}^{\infty}\left[\frac{1}{2}+k\left(\mathbf{E}-\ln 2+\ln(k_jr)-\sum_{l=1}^{k}\frac{1}{l}\right)\right]\frac{1}{(k!)^2}\left(\frac{k_jr}{2}\right)^{2k-2}.$$

This yields then

$$k_j^2K_0(k_jr)+\frac{2k_j}{r}K_1(k_jr)=-\frac{k_j^2}{2}+\frac{2}{r^2}$$
$$+k_j^2\sum_{k=1}^{\infty}\left[\ln 2-\mathbf{E}-\ln(k_jr)+\sum_{l=1}^{k}\frac{1}{l}\right]\frac{1}{(k!)^2}\left(\frac{k_jr}{2}\right)^{2k}$$
$$+k_j^2\sum_{k=2}^{\infty}\left[\frac{1}{2}+k\left(\mathbf{E}-\ln 2+\ln(k_jr)-\sum_{l=1}^{k}\frac{1}{l}\right)\right]\frac{1}{(k!)^2}\left(\frac{k_jr}{2}\right)^{2k-2}. \tag{5.26}$$

The relation (5.26) shows that in the neighborhood of zero we have

$$k_1^2K_0(k_1r)+\frac{2k_1}{r}K_1(k_1r)-k_2^2K_0(k_2r)-\frac{2k_2}{r}K_1(k_2r)=\frac{k_2^2-k_1^2}{2}+O(r). \tag{5.27}$$

Note that, as in the case of the double layer integral operator, the regularization of the hypersingular integral operator above shows that the evaluation can be done easily via the single layer integral operator of the elasticity problem, and of the single layer and hypersingular integral operator of the scalar Yukawa problem.

5.3 Evaluation of the Newton potential

In this section our interest is to compute the vector related to the Newton potential $N_0\underline{f}$, i.e.

$$(N_0\underline{f})_l[i] = \sum_{m=1}^{2} \int_{\Gamma} \phi_i^{(0)}(x) \int_{\Omega} U_{lm}^*(x,y) f_m(y) dy ds_x \quad \text{for } i = 1,\dots,N;\ l = 1,2,$$

where $\phi_i^{(0)}$ for $i = 1,\dots,N$ are the B-splines of order zero. First we interchange the order of integration as follows

$$(N_0\underline{f})_l[i] = \sum_{m=1}^{2} \int_{\Omega} f_m(y) \int_{\tau_i} U_{lm}^*(x,y) ds_x dy \quad \text{for } i = 1,\dots,N;\ l = 1,2. \tag{5.28}$$

Note that the main assumptions for the computation of (5.28) are as for chapter 4, but for the reader convenience we briefly repeat them here.

Let $\Omega = B_R(c)$ be a two-dimensional circular domain centered at c with radius R. First, Ω is divided into M_r rings, second on each ring N suitable meshes are constructed (see Appendix). Therefore, (5.28) can be written as follows

$$(N_0\underline{f})_l[i] = \sum_{j=1}^{M_r} \sum_{k=1}^{N} \sum_{m=1}^{2} \int_{T_{jk}} f_m(y) \int_{\tau_i} U_{l,m}^*(x,y) ds_x dy \quad \text{for } i = 1,\dots,N;\ l = 1,2, \tag{5.29}$$

where T_{jk} is the k^{th} element on the j^{th} ring, and $\overline{\Omega} = \cup_{j=1}^{M_r} \cup_{k=1}^{N} T_{jk}$. Furthermore, T_{jk} is an isoparametric triangle or an isoparametric quadrangle whether it is on the last inner ring or others rings. Further, the fundamental solution can be written as follows

$$U^* = \begin{pmatrix} U_{11}^{*1} & 0 \\ 0 & U_{22}^{*1} \end{pmatrix} + \begin{pmatrix} U_{11}^{*2} & U_{12}^* \\ U_{12}^* & -U_{11}^{*2} \end{pmatrix},$$

where for $r = |x-y|$, $r_i = x_i - y_i$, $i = 1,2$

$$\begin{aligned}
U_{11}^{*1}(x,y) &= \frac{1}{2\pi s^2}\left[k_2^2 K_0(k_2 r) + \frac{1}{r}[k_2 K_1(k_2 r) - k_1 K_1(k_1 r)]\right], \\
U_{11}^{*2}(x,y) &= -\frac{1}{2\pi s^2}\frac{r_1^2}{r^2}\left[[k_2^2 K_0(k_2 r) - k_1^2 K_0(k_1 r)] + \frac{2}{r}[k_2 K_1(k_2 r) - k_1 K_1(k_1 r)]\right], \\
U_{12}^{*}(x,y) &= -\frac{1}{2\pi s^2}\frac{r_1 r_2}{r^2}\left[[k_2^2 K_0(k_2 r) - k_1^2 K_0(k_1 r)] + \frac{2}{r}[k_2 K_1(k_2 r) - k_1 K_1(k_1 r)]\right], \\
U_{22}^{*1}(x,y) &= \frac{1}{2\pi s^2}\left[k_1^2 K_0(k_1 r) - \frac{1}{r}[k_2 K_1(k_2 r) - k_1 K_1(k_1 r)]\right].
\end{aligned}$$

If we approximate functions f_1 and f_2 on each ring by piecewise constant functions respectively, the approximation of the vector $\underline{N}_0\underline{f}$ denoted by $\underline{\tilde{N}}_0\underline{f}$ can then be written in the matrix form as follows

$$\underline{\tilde{N}}_0\underline{f} = \sum_{j=1}^{M_r}\left[\begin{pmatrix} A_{11}^{1j} & 0 \\ 0 & A_{22}^{1j}\end{pmatrix} + \begin{pmatrix} A_{11}^{2j} & A_{12}^{j} \\ A_{12}^{j} & -A_{11}^{2j}\end{pmatrix}\right]\begin{pmatrix}\underline{f}_1^j \\ \underline{f}_2^j\end{pmatrix},$$

where for $j = 1,\ldots,M_r$

$$\begin{aligned} A_{ll}^{1j}[i,k] &= \int_{\tau_i} U_{ll}^{*1}(x,y_{jk})ds_x \quad \text{for } i,k = 1,\ldots,N,\; l = 1,2, \\ A_{11}^{2j}[i,k] &= \int_{\tau_i} U_{11}^{*2}(x,y_{jk})ds_x \quad \text{for } i,k = 1,\ldots,N, \\ A_{12}^{j}[i,k] &= \int_{\tau_i} U_{12}^{*}(x,y_{jk})ds_x \quad \text{for } i,k = 1,\ldots,N \end{aligned}$$

and

$$\underline{f}_l^j[k] = |T_{jk}|f_l(y_{jk}) \quad \text{for } k = 1,\ldots,N,\; l = 1,2,$$

where $|T_{jk}|$ and y_{jk} represent the volume and the center of mass of the element T_{jk} respectively. Note that for $j = 1,\ldots,M_r$ the matrices A_{11}^{1j} and A_{22}^{1j} are derived from the boundary integral operators V_{11}^1 and V_{22}^1 respectively. Hence, A_{11}^{1j} and A_{22}^{1j} are circulant matrices, see Proposition 5.1. Finally, by using the Fast Fourier transform (FFT) we obtain

$$\underline{\tilde{N}}_0\underline{f} = \sum_{j=1}^{M_r}\begin{pmatrix} F^{-1}\left(F(\underline{C}_1^j)F(\underline{f}_1^j)\right) + A_{11}^{2j}\underline{f}_1^j + A_{12}^{j}\underline{f}_2^j \\ F^{-1}\left(F(\underline{C}_2^j)F(\underline{f}_2^j)\right) + A_{12}^{j}\underline{f}_1^j - A_{11}^{2j}\underline{f}_2^j\end{pmatrix},$$

where for $j = 1,\ldots,M_r$, $\underline{C}_1^j$ and $\underline{C}_2^j$ are the first columns of the circulant matrices A_{11}^{1j} and A_{22}^{1j} respectively.

Remark 5.2. *Note that on each ring the approximate Newton potential vector $\underline{\tilde{N}}_0\underline{f}$ is given in terms of a matrix-vector multiplication like for the scalar Yukawa problem presented in chapter 4, but the matrix in this case has a block structure, the first block is made from circulant matrices A_{11}^{1j} and A_{22}^{1j}, therefore the matrix-vector multiplication of this first part can be speed up by applying the Fast Fourier transform (FFT), while the second block has a symmetric structure, this permits us to compute only two matrices A_{11}^{2j} and A_{12}^{j}. Moreover, the error estimate for the approximation is given as in chapter 4 by*

$$\|N_0(\underline{f} - \underline{f}_h)\|_{H^{1/2}(\Gamma)} \le c_2^N\|\underline{f} - \underline{f}_h\|_{\tilde{\boldsymbol{H}}^{-1}(\Omega)} \le c_2^N\|\underline{f} - \underline{f}_h\|_{\boldsymbol{L}_2(\Omega)} \le ch|\underline{f}|_{\boldsymbol{H}^1(\Omega)},$$

where $\underline{f}_h := I_h\underline{f}$ is the nodal interpolation of $\underline{f}$.

6 TWO-DIMENSIONAL CONTACT PROBLEMS IN LINEAR ELASTOSTASTICS OF YUKAWA TYPE

The previous chapters were dedicated to the formulation of adequate boundary value problems as well as the setting of suitable function spaces and the development of useful results to ease the formulation and analysis of contact problems. Thus, the goal of this chapter is to develop and to analyze algorithms for the solution of contact problems with friction in linear elastostatics of Yukawa type. The contact problems with friction, even in the view of their long history, are still one of the most challenging subjects in mechanics with many open questions. This is due to the non-monotone and non-compact character of the friction functional which does not permit the application of standard results from nonlinear functional analysis [29, 67].

In literature, the most frequently used friction laws are the Tresca and Coulomb law. The variational formulation of the first one leads to a classical variational inequality of second kind which is equivalent to an optimization problem and where the solvability can be established easily [35, 40, 68], while the second one leads to a quasivariational inequality [29, 67]. This makes proving theoretical results for Coulomb friction problem difficult. However, the first existence proof for the Coulomb friction problem was established by Nesčas, Jarušek and Haslinger in (1980) [85] for the very special problem of a two-dimensional infinite strip provided the friction coefficient was sufficiently small. To overcome mathematical difficulties related to the Coulomb friction problem, regularized versions such as a nonlocal or a normal compliance friction law were considered in [67]. In [27–30, 67] the authors used another technique based on a simultaneous *penalization* of unilateral conditions and a *regularization* of the frictional term. This technique is powerful from the theoretical point of view but not very convenient for computations. Indeed, after a discretization one obtains a system of nonlinear algebraic equations which depends on two small parameters. It turns out that the computational process depends strongly on their choice [30]. The fixed point technique is preferred nowadays as a basis for the numerical realization of the contact problems with Coulomb friction. A possible way to determine fixed points is to express the corresponding weak formulation in the form of a generalized equation which can be solved by methods of non-smooth optimization [18]. Another way which is commonly used towards the solution of Coulomb frictional contact problems is to define the solution as a fixed point of a sequence of solutions to the Tresca problem [24, 27, 28, 30, 41, 42, 44, 67].

The Tresca friction law is frequently used instead of the Coulomb friction law because it is simpler to analyze, see [10, 44, 67]. Therefore, an efficient numerical algorithm for contact problems with Coulomb friction is based on fast and reliable algorithms for contact problems with Tresca friction. In addition, the simplicity of Tresca problems highly motivates the development of faster solvers [71, 98]. Some works concerned with this are [24, 41]. The authors use a dual formulation of the problem and quadratic programming methods with proportioning and projections for the solution of the discrete 3D Tresca frictional contact problem. But, note that the above methods are rather very slow and are in general of first order or converge at a linear rate. A different idea is followed in [71, 98], where a second order semi-smooth Newton method and an augmented Lagrangian approach were proposed for the solutions of 2D contact problems with Tresca friction. This method is generalized to 3D in [54].

In contrast to the frictionless case, where we have to derive only the normal deformations, here the tangential deformations are needed as well. Therefore, besides the unilateral contact conditions which can be handled easily via projection techniques, the variational formulation of contact problems with Tresca friction leads to a variational inequality of second kind [67, 68], but this turns out to be a constrained non-differentiable minimization problem (primal problem) which is problematic. Therefore, a more subtle approach appropriately dealing with generalized derivatives is necessary to overcome this difficulty [47, 48]. The key ingredient for these steps is a dual regularization strategy which enables us, on one hand, to get uniqueness of the dual variable and, on the other hand, to work in an adequate function space setting where the superlinearity of a generalized Newton method can be obtained. Therefore, by the help of the Fenchel duality theorem [31, 49, 71] the corresponding dual problem is obtained, which is a constrained maximization problem involving a differentiable operator. Whenever, the problems (primal and dual) are seen from optimizational point of view, instead of considering only the first order necessary optimality conditions, which are usually the starting points of analysis, we derive and consider as well the extremality conditions which characterize solutions of the dual and primal problems. In addition, an important aspect of this approach is that the extremality conditions can be written as a variational formulation in terms of nonlinear and non differentiable complementarity functions [71, 98, 99]. But, a regularization of the dual problem, motivated by the augmented Lagrangian [34, 36, 56, 57, 98] turns those complementarity functions in the extremality conditions to be Newton differentiable [47, 71, 98]. Therefore, this can motivate the application of the semi-smooth Newton method in function spaces, see, e.g., [47, 58–61] for their applications to optimal control and obstacle problems. An immediate consequence, is that the resulting algorithm in 2-D turns out to be equivalent to an active set strategy and is observed to converge in numerical practice regardless of the initialization and the mesh [47, 50, 59, 98].

Due to the fact that the nonlinearities of the problem lie on the boundaries of the contacting bodies, this can motivate the application of boundary integral equation methods [30, 100]. This reduces the extremality conditions to the boundary curve. The boundary integral equations involved in the extremality conditions are approximated by using Galerkin method with the help of B-splines on the boundary curve (BEM) [30, 100]. This yields an equivalent algebraic system of linear equations with few unknowns but involving dense matrices and having block structure with circulant properties [23, 92, 94] in the particular case of a circular domain. The associated matrix entries are computed partly explicitly and efficiently. Additionally, the circulant block structure enables us to develop preconditioning matrices for a conjugate gradient method. The combined semi-smooth Newton method with the boundary element method are then carried over to the Coulomb friction problem by means of a fixed point approach. The chapter is organized as follows: In section 1 the Signorini contact problem with Coulomb friction is stated and its variational formulation in a framework of Hilbert spaces is given. Section 2 is concerned with the boundary integral formulation of the problem, the introduction of the Tresca problem and the analysis of the wellposedness is established, while section 3 is concerned with the derivation of the dual problem and of the extremality conditions for the Tresca problem. In section 4 we derive the regularized dual and primal problems for the Tresca problem and present the semi-smooth Newton approach. In section 5 we present the regularized contact problem with Coulomb friction and establish the existence proof. Section 6 is concerned with the BEM discretization.

6.1 Signorini contact problems with Coulomb friction

In this section we present the problem in its strong formulation, state all necessary ingredients needed for the variational formulation. Further, we discuss some mathematical difficulties inherent in it.

6.1.1 Presentation of the problem

For the problem setting, we assume a deformable body to occupy an open and bounded domain Ω of $\mathbb{R}^2$ with boundary $\Gamma := \partial\Omega$ divided into three disjoint subsets, namely the Dirichlet part Γ_D, where we assume the body to be fixed, Γ_N is the Neumann part with a given traction, and Γ_C is the part where a possible contact may occur. For convenience we assume that the rigid foundation lies below Γ_C as it is shown in Figure 6.1.

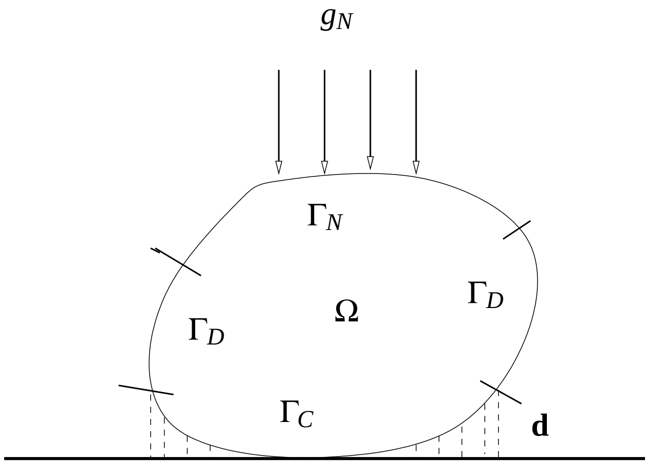


Figure 6.1: Deformable body with rigid foundation.

Governing Equations

Using the results from chapter 3 the strong formulation of the contact problem in linear elastostatics of Yukawa type can be stated as follows: Find the displacement $\underline{u}$ such that

$$s^2\underline{u} - \text{div}\,(\sigma(\underline{u})) = \underline{f} \quad \text{in} \quad \Omega, \tag{6.1}$$

$$\underline{u} = 0 \quad \text{on} \quad \Gamma_D, \tag{6.2}$$

$$\sigma(\underline{u})\underline{n} = \underline{g}_N \quad \text{on} \quad \Gamma_N, \tag{6.3}$$

$$u_n \leq \mathbf{d}, \quad \sigma_n(\underline{u}) \leq 0, \quad \sigma_n(\underline{u})(u_n - \mathbf{d}) = 0 \quad \text{on} \quad \Gamma_C, \tag{6.4}$$

$$\left.\begin{array}{l} \underline{u}_t - \underline{w}_t = 0 \Rightarrow |\sigma_t(\underline{u})| < \mathcal{F}|\sigma_n(\underline{u})| \\ \underline{u}_t - \underline{w}_t \neq 0 \Rightarrow \sigma_t(\underline{u}) = -\mathcal{F}|\sigma_n(\underline{u})|\dfrac{\underline{u}_t - \underline{w}_t}{|\underline{u}_t - \underline{w}_t|} \\ |\underline{u}_t - \underline{w}_t|(|\sigma_t(\underline{u})| - \mathcal{F}|\sigma_n(\underline{u})|) = 0 \end{array}\right\} \quad \text{on} \quad \Gamma_C, \tag{6.5}$$

where $\underline{n}$ is the unit outward normal vector along the boundary Γ, $\sigma_n(\underline{u})$ and $\sigma_t(\underline{u})$ represent the normal and the tangential stresses along the boundary Γ_C respectively, while u_n and $\underline{u}_t$ are respectively the normal and the tangential displacements along the boundary Γ_C. The function $\underline{w} \in \mathbf{H}^{1/2}(\Gamma_C)$ and the scalar s arise from the time discretization as described in chapter 3, there $\underline{w} = \underline{u}^{l-1}$ and $s^2 = \dfrac{\rho}{(\delta t)^2}$. Moreover, we assume the material to have an isotropic behavior, i.e. Hooke's law yields

$$\sigma(\underline{u}) := \mathbb{C}\varepsilon(\underline{u}) = (\lambda tr(\varepsilon(\underline{u}))I + 2\mu\varepsilon(\underline{u})),$$

where $\varepsilon(\underline{u}) = \frac{1}{2}(\nabla\underline{u} + (\nabla\underline{u})^\top)$ denotes the linearized strain tensor, I is the 2×2-identity tensor, $tr(.)$ the trace of a tensor. Further, λ and μ are the Lamé constants given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)},$$

where $E > 0$ and $\nu \in (0, 1/2)$ denote Young's modulus and the Poisson ratio respectively.

Before we derive the variational formulation, let us define the Sobolev space

$$\mathbf{H}^1(\Omega) := \left(H^1(\Omega)\right)^2.$$

In a similar way we define $\mathbf{L}_2(\Gamma)$, $\mathbf{H}^{1/2}(\Gamma)$ and so on. We then introduce

$$\mathbb{V} = \{\underline{\mathrm{v}} \in \mathbf{H}^1(\Omega) : \gamma_0^{int}\underline{\mathrm{v}} = 0 \quad \text{on} \quad \Gamma_D\},$$

where

$$\gamma_0^{int} : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma)$$

is the trace operator. For the reader's convenience we recall the two linear, continuous and surjective trace operators on Γ_C defined in chapter 2:

$$\gamma_{Nc} : \mathbb{V} \to H^{1/2}(\Gamma_C), \quad \underline{\mathrm{v}} \mapsto \gamma_{Nc}\underline{\mathrm{v}} \equiv (\gamma_0^{int}\underline{\mathrm{v}}|_{\Gamma_C}) \cdot \underline{n}$$

and

$$\gamma_{Tc} : \mathbb{V} \to \mathbf{H}_T^{1/2}(\Gamma_C), \quad \underline{v} \mapsto \gamma_{Tc}\underline{\mathrm{v}} \equiv \gamma_0^{int}\underline{\mathrm{v}}|_{\Gamma_C} - (\gamma_{Nc}\underline{\mathrm{v}})\underline{n}$$

with

$$\mathbf{H}_T^{1/2}(\Gamma_C) = \{\underline{\mathrm{v}} \in \mathbf{H}^{1/2}(\Gamma_C) : \gamma_{Nc}\underline{\mathrm{v}} = 0\}.$$

In the sequel and onward, we set for $\underline{\mathrm{v}} \in \mathbb{V}$, $\gamma_{Nc}\underline{\mathrm{v}} \equiv \mathrm{v}_n$ and $\gamma_{Tc}\underline{\mathrm{v}} \equiv \underline{\mathrm{v}}_t$. By following [98], we assume that the friction coefficient $\mathcal{F}$ does not depend on the solution and

$$\mathcal{F} \in L^\infty(\Gamma_C).$$

Moreover, we assume that $\mathcal{F}$ is positive, and belongs to the space of factors on $H^{1/2}(\Gamma_C)$, that is the mapping defined by

$$H^{1/2}(\Gamma_C) \ni \lambda \mapsto \mathcal{F}\lambda \in H^{1/2}(\Gamma_C),$$

is welldefined and bounded, see [98, p. 85].

6.1.2 Variational formulation

The set of admissible displacements is defined by:

$$\mathbb{K} = \{\underline{\mathrm{v}} \in \mathbb{V} : \mathrm{v}_n \leq \mathbf{d} \quad \text{on} \quad \Gamma_C\}.$$

The crucial point to establish the variational formulation lies on the derivation of the variational formulation for the nonlinear contact conditions and the Coulomb friction law. However, it can be easily proved that the following inequality holds, see (3.43),

$$u_n \leq \mathbf{d}, \;\; \sigma_n(\underline{u})(\mathrm{v}_n - u_n) \geq 0, \quad \forall \underline{\mathrm{v}} \in \mathbb{K}. \tag{6.6}$$

On the other hand, the variational formulation of the Coulomb friction law (6.5) is given by

$$\sigma_t(\underline{u}) \cdot (\underline{\mathrm{v}}_t - \underline{u}_t) + \mathcal{F}|\sigma_n(\underline{u})|(|\underline{\mathrm{v}}_t - \underline{w}_t| - |\underline{u}_t - \underline{w}_t|) \geq 0 \quad \text{for all } \underline{\mathrm{v}}_t \quad \text{orthogonal to } \underline{n}. \tag{6.7}$$

The proof of this statement is given in Proposition 3.1. Next, if we multiply the equilibrium equation (6.1) by $(\underline{\mathrm{v}} - \underline{u})$, utilize the Green formula and boundary conditions we obtain then

$$\begin{aligned}\int\limits_{\Omega} \left[s^2 \underline{u} \cdot (\underline{\mathrm{v}} - \underline{u}) + \sigma(\underline{u}) : \varepsilon(\underline{\mathrm{v}} - \underline{u})\right] dx - \int\limits_{\Omega} \underline{f} \cdot (\underline{\mathrm{v}} - \underline{u}) dx &= \int\limits_{\Gamma} \sigma(\underline{u})\underline{n} \cdot (\underline{\mathrm{v}} - \underline{u}) ds_x \\ &= \int\limits_{\Gamma_N} \underline{g}_N \cdot (\underline{\mathrm{v}} - \underline{u}) ds_x + \int\limits_{\Gamma_C} \sigma(\underline{u})\underline{n} \cdot (\underline{\mathrm{v}} - \underline{u}) ds_x.\end{aligned} \tag{6.8}$$

The last integral on Γ_C in the right hand side of (6.8) can be split into normal and tangential components as follows

$$\int\limits_{\Gamma_C} \sigma(\underline{u})\underline{n} \cdot (\underline{\mathrm{v}} - \underline{u}) ds_x = \int\limits_{\Gamma_C} \left[\sigma_n(\underline{u})(\mathrm{v}_n - u_n) + \sigma_t(\underline{u}) \cdot (\underline{\mathrm{v}}_t - \underline{u}_t)\right] ds_x. \tag{6.9}$$

If we substitute (6.9) into (6.8), add $\int\limits_{\Gamma_C} \mathcal{F}|\sigma_n(\underline{u})|(|\underline{\mathrm{v}}_t - \underline{w}_t| - |\underline{u}_t - \underline{w}_t|) ds_x$ in both sides of (6.8) and further utilize (6.6) and (6.7), we then obtain the following variational inequality: Find $\underline{u} \in \mathbb{K}$ such that

$$a(\underline{u}, \underline{\mathrm{v}} - \underline{u}) + j(\underline{u}, \underline{\mathrm{v}}) - j(\underline{u}, \underline{u}) \geq \mathcal{L}(\underline{\mathrm{v}} - \underline{u}) \quad \text{for all } \underline{\mathrm{v}} \in \mathbb{K}, \tag{6.10}$$

where for all $\underline{u}, \underline{\mathrm{v}} \in \mathbb{K}$

$$\begin{aligned} a(\underline{u}, \underline{\mathrm{v}}) &= \int\limits_{\Omega} [s^2 \underline{u} \cdot \underline{\mathrm{v}} + \sigma(\underline{u}) : \varepsilon(\underline{\mathrm{v}})] dx, \\ j(\underline{u}, \underline{\mathrm{v}}) &= \int\limits_{\Gamma_C} \mathcal{F}|\sigma_n(\underline{u})| \; |\underline{\mathrm{v}}_t - \underline{w}_t| ds_x, \\ \mathcal{L}(\underline{\mathrm{v}}) &= \int\limits_{\Omega} \underline{f} \cdot \underline{\mathrm{v}} dx + \int\limits_{\Gamma_N} \underline{g}_N \cdot \underline{\mathrm{v}} ds_x. \end{aligned}$$

The equivalence of the two problems (6.1)-(6.5) and (6.10) is given by the following result.

Theorem 6.1. *If $\underline{u}$ is a sufficiently smooth solution of (6.10), then $\underline{u}$ satisfies (6.1)-(6.5). Conversely, any solution of (6.1)-(6.5) satisfies (6.10).*

Proof. See [67, Theorem 10.1]. □

Unfortunately, there are major mathematical difficulties inherent in the variational inequality (6.10). On the one hand, the functional

$$j(\underline{u},\underline{u}) = \int_{\Gamma_C} \mathcal{F}|\sigma_n(\underline{u})| \; |\underline{u}_t - \underline{w}_t| ds_x$$

is not monotone. In addition, it is neither convex nor differentiable, and can not be analyzed by the standard theories from nonlinear functional analysis. Thus, the question of existence of solutions to the general problem (6.10) is still open. However, there are available existence proofs for some very special cases in which the contact pressures happen to be very smooth, see [85]. To overcome these difficulties, two approaches are frequently used in literature, the first one is the nonlocal or the normal compliance friction law [67], and the second one, which is of our interest, is a reduced version of (6.10) that represents a contact problem in which it is assumed that the contact pressure is known. This is also known as contact problem with Tresca friction law [24, 41, 42, 71]. Before we introduce that, let us reformulate (6.10) by using boundary integral operators.

6.2 Boundary integral formulation

Since the unknowns we are interested in deriving lie on the boundary it can be beneficial or attractive to reformulate (6.10) by using the boundary element method. This is motivated by its better approximation of data on the boundary in comparison to the traditional finite element method [30, 100]. Therefore, if the Cauchy data $\gamma_0^{int}\underline{u}$ and $\gamma_1^{int}\underline{u} \equiv \sigma(\underline{u})\underline{n}$ are known, then the solution of the differential equations (6.1) is given by the so-called representation formula for $x \in \Omega$,

$$\underline{u}(x) = \int_{\Gamma} [U^*(x,y)]\gamma_1^{int}\underline{u}(y)ds_y - \int_{\Gamma} [T_y U^*(x,y)]\gamma_0^{int}\underline{u}(y)ds_y + \int_{\Omega} [U^*(x,y)]\underline{f}(y)dy, \quad (6.11)$$

where U^* is the fundamental solution of linear elastostatics of Yukawa type given by (5.14). Our goal in this section is to write the boundary stress in terms of the symmetric Steklov-Poincaré operator. To this end, by following the same procedure as given in section 4.1.2, we then obtain

$$\sigma(\underline{u})\underline{n} := \gamma_1^{int}\underline{u} = S\gamma_0^{int}\underline{u} - \mathbf{N}\underline{f} \tag{6.12}$$

where the operator $\mathbf{N}\underline{f}$ is given by

$$-V^{-1}N_0\underline{f} = \left[N_1 - \left(\frac{1}{2}I + K'\right)V^{-1}N_0\right]\underline{f} =: -\mathbf{N}\underline{f},$$

and S is the symmetric Steklov-Poincaré operator, i.e. it maps a given boundary displacement to the corresponding boundary stress of the solution of the homogeneous elasticity equations. Furthermore, it is defined by

$$S := \left(\frac{1}{2}I + K'\right)V^{-1}\left(\frac{1}{2}I + K\right) + D$$

with V, K, K' and D which represent the single layer integral operator, the double layer integral operator, the adjoint double layer integral operator and the hypersingular integral operator respectively. N_0 and N_1 are the Newton potentials. All these operators are given in section 5.1.2. In addition, as for the scalar Yukawa problem, Theorem 4.1 remains valid here, i.e. the Steklov-Poincaré operator S is bounded, self-adjoint, positive definite and satisfying

$$\langle S\underline{u}, \underline{u}\rangle_\Gamma \geq c_1^D \|\underline{u}\|^2_{\mathbf{H}^{1/2}(\Gamma)} \quad \text{for all } \underline{u} \in \mathbf{H}^{1/2}(\Gamma)$$

with the same constant of ellipticity c_1^D as in the case of the hypersingular integral operator. Next, by using the Green formula, we then obtain

$$\begin{aligned} a(\underline{u}, \underline{\mathrm{v}} - \underline{u}) - \int_\Omega \underline{f}\cdot(\underline{\mathrm{v}} - \underline{u})dx &= \int_\Gamma \sigma(\underline{u})\underline{n}\cdot(\underline{\mathrm{v}} - \underline{u})ds_x, \\ &= \int_\Gamma (S\underline{u} - \mathbf{N}\underline{f})\cdot(\underline{\mathrm{v}} - \underline{u})ds_x. \end{aligned} \tag{6.13}$$

By substituting (6.13) into (6.10), we then obtain the boundary variational inequality: Find $\underline{u} \in \mathcal{K}$ such that

$$\int_\Gamma S\underline{u}\cdot(\underline{\mathrm{v}} - \underline{u})ds_x + \int_{\Gamma_C} \mathcal{F}|\sigma_n(\underline{u})|(|\underline{\mathrm{v}}_t - \underline{w}_t| - |\underline{u}_t - \underline{w}_t|)ds_x \geq L(\underline{\mathrm{v}} - \underline{u}) \text{ for all } \underline{\mathrm{v}} \in \mathcal{K} \tag{6.14}$$

with

$$L(\underline{\mathrm{v}}) = \int_\Gamma \mathbf{N}\underline{f}\cdot\underline{\mathrm{v}}ds_x + \int_{\Gamma_N} \underline{g}_N\cdot\underline{\mathrm{v}}ds_x,$$

and where the set $\mathcal{K}$ is defined by

$$\mathcal{K} := \{\underline{\mathrm{v}} \in \mathcal{V} : \mathrm{v}_n \leq \mathbf{d}\}$$

with $\mathcal{V}$ given by

$$\mathcal{V} := \{\underline{\mathrm{v}} \in \mathbf{H}^{1/2}(\Gamma) : \gamma_0^{int}\underline{\mathrm{v}} = 0 \text{ on } \Gamma_D\}.$$

Proposition 6.1. *The domain variational formulation (6.10) and the boundary variational formulation (6.14) are equivalent.*

Proof. If $\underline{u}$ is a solution of (6.10), the trace $\gamma_0^{int}\underline{u}$ is then a solution of the boundary variational formulation (6.14). On the other hand, if $\gamma_0^{int}\underline{u}$ is a solution (6.14), the boundary stress is then given by the relation (6.12) $\sigma(\underline{u})\underline{n} := \gamma_1^{int}\underline{u} = S\gamma_0^{int}\underline{u} - \mathbf{N}\underline{f}$. Hence, having the Cauchy data $\gamma_0^{int}\underline{u}$ and $\gamma_1^{int}\underline{u}$ the solution of the domain variational (6.10) is given by the representation formula (6.11). □

After suitable reformulation of the domain variational formulation (6.10) to the boundary variational problem (6.14), the next paragraph is concerned about the Tresca problem.

6.2.1 Signorini contact problem with Tresca friction

As we have mentioned above, the contact problem with Tresca friction is the contact problem with given friction. This is widely accepted in practice and it is known to be not only manageable from the mathematical and computational point of view but can form as well a step in an iterative process for obtaining numerical solutions to the general problem (6.14) when such solutions exist [29, 30, 42, 67], see the next section. In addition, contact problems with Tresca friction can be associated to an optimization problem for which standard a priori estimates would guarantee existence or uniqueness of a solution. To give the weak formulation of the problem with Tresca friction we assume that the friction is given and set

$$|\sigma_n(\underline{u})| \equiv g \in \mathcal{K}^* := \{g \in H^{-1/2}(\Gamma_C) : \langle g, \mathrm{v}\rangle_{\Gamma_C} \geq 0 \ \ \forall \mathrm{v} \in H^{1/2}(\Gamma_C) \quad \text{with } \ \mathrm{v} \geq 0\}.$$

The variational formulation is then given by: Find $\underline{u} \in \mathcal{K}$ such that

$$\langle S\underline{u}, \underline{\mathrm{v}} - \underline{u}\rangle_\Gamma + j_g(\underline{\mathrm{v}}) - j_g(\underline{u}) \geq L(\underline{\mathrm{v}} - \underline{u}) \quad \text{for all } \ \underline{\mathrm{v}} \in \mathcal{K}, \tag{6.15}$$

where

$$j_g(\underline{\mathrm{v}}) = \int_{\Gamma_C} \mathcal{F} g |\underline{\mathrm{v}}_t - \underline{w}_t| ds_x.$$

The variational inequality (6.15) is equivalent to the minimization problem

$$\min_{\underline{\mathrm{v}} \in \mathcal{K}} J(\underline{\mathrm{v}}), \qquad\qquad (\mathcal{P}) \tag{6.16}$$

where $J(\underline{\mathrm{v}}) = J_0(\underline{\mathrm{v}}) + j_g(\underline{\mathrm{v}})$ with $J_0(\underline{\mathrm{v}}) := \frac{1}{2}\langle S\underline{\mathrm{v}}, \underline{\mathrm{v}}\rangle_\Gamma - L(\underline{\mathrm{v}})$. Indeed, let us assume that $\underline{u}$ is a solution of the minimization problem (6.16), note that the functional J_0 is Gateaux differentiable and convex whereas j_g is not differentiable at $\underline{v} = \underline{w}$ but, since j_g is convex we have

$$\lim_{\alpha>0} \frac{j_g(\underline{u}+\alpha(\underline{v}-\underline{u})) - j_g(\underline{u})}{\alpha} \leq j_g(\underline{v}) - j_g(\underline{u}) \ \ \forall\alpha \in (0,1), \ \ \forall \underline{v} \in \mathcal{K}. \tag{6.17}$$

We have then

$$\begin{aligned} 0 &\leq \lim_{\alpha>0} \frac{J(\underline{u}+\alpha(\underline{v}-\underline{u})) - J(\underline{u})}{\alpha} \ \ \text{(since } \underline{u} \text{ is a minimizer of } J) \\ &= \lim_{\alpha>0} \frac{J_0(\underline{u}+\alpha(\underline{v}-\underline{u})) - J_0(\underline{u})}{\alpha} + \lim_{\alpha>0} \frac{j_g(\underline{u}+\alpha(\underline{v}-\underline{u})) - j_g(\underline{u})}{\alpha} \\ &\leq \langle DJ_0(\underline{u}), \underline{v}-\underline{u}\rangle + j_g(\underline{v}) - j_g(\underline{u}) \ \ \text{(by using (6.17))} \\ &= \langle S\underline{u}, \underline{v}-\underline{u}\rangle_\Gamma - L(\underline{v}-\underline{u}) + j_g(\underline{v}) - j_g(\underline{u}). \end{aligned}$$

Hence $\underline{u}$ solves (6.15). On the other hand, let $\underline{u}$ be a solution of the boundary variational problem (6.15). Since J_0 is also convex applying the relation (6.17) yields for all $\underline{v} \in \mathcal{K}$

$$\langle S\underline{u}, \underline{v}-\underline{u}\rangle_\Gamma - L(\underline{v}-\underline{u}) \ := \ \lim_{\alpha>0} \frac{J_0(\underline{u}+\alpha(\underline{v}-\underline{u})) - J_0(\underline{u})}{\alpha} \leq J_0(\underline{v}) - J_0(\underline{u}). \tag{6.18}$$

Finally, by adding $j_g(\underline{v}) - j_g(\underline{u})$ in both sides of inequality (6.18) and by using (6.15) end the proof.

Some abstract results for variational inequalities

In this paragraph, we will limit ourselves to an abstract elliptic variational inequality of the second kind which is the main interest of this section. Nevertheless, we will derive an elliptic variational inequality of the first kind as a particular case of the second kind. In addition, some results concerning the existence and the uniqueness of their solutions are presented.

In the following V will denote a real Hilbert space, V^* its topological dual space with the duality pairing $\langle .,.\rangle_{V^*\times V}$. The norm on V will be denoted by $\|.\|_V$. Let $A : V \to V^*$ be a linear and bounded operator, A is said to be V-elliptic if there exists a positive constant α such that $\langle A\mathrm{v}, \mathrm{v}\rangle_{V^*\times V} \geq \alpha\|\mathrm{v}\|_V^2$ for all $\mathrm{v} \in V$. Let $L : V \to \mathbb{R}$ be a linear and bounded functional, K a closed convex and nonempty subset of V, and let $j(.) : V \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex lower semi-continuous (l.s.c) and proper functional.

Definition 6.1. *The elliptic variational inequality to find $u \in V$ such that*

$$\langle Au, v-u\rangle_{V^*\times V} + j(v) - j(u) \geq L(v-u) \quad \textit{for all } v \in V \tag{6.19}$$

is said to be of second kind.

In addition, if A is symmetric, i.e. $\langle Au, \mathrm{v}\rangle_{V^*\times V} = \langle A\mathrm{v}, u\rangle_{V^*\times V}$, the variational inequality (6.19) is then equivalent to a minimization problem

$$\min_{\mathrm{v}\in V} \mathcal{J}(\mathrm{v}), \tag{6.20}$$

where $\mathcal{J} : V \to \overline{\mathbb{R}}$ is a functional defined by:

$$\mathcal{J}(\mathrm{v}) := \frac{1}{2}\langle A\mathrm{v}, \mathrm{v}\rangle_{V^*\times V} + j(\mathrm{v}) - L(\mathrm{v}). \tag{6.21}$$

If $j = I_K$, where I_K is the indicator function of K defined by

$$I_K(\mathrm{v}) = \begin{cases} 0 & \text{if} \quad \mathrm{v} \in K, \\ \infty & \text{elsewhere,} \end{cases}$$

the variational inequality (6.19) is then equivalent to

$$\langle Au, \mathrm{v} - u\rangle_{V^*\times V} \geq L(\mathrm{v} - u) \quad \text{for all } \mathrm{v} \in K \tag{6.22}$$

and it is said to be of first kind. If A is symmetric, (6.22) is then equivalent to a minimization problem

$$\min_{\mathrm{v}\in K} \mathcal{J}(\mathrm{v}), \tag{6.23}$$

where $\mathcal{J} : V \to \mathbb{R}$ is a quadratic functional (Ritz functional) defined by

$$\mathcal{J}(\mathrm{v}) := \frac{1}{2}\langle A\mathrm{v}, \mathrm{v}\rangle_{V^*\times V} - L(\mathrm{v}). \tag{6.24}$$

Theorem 6.2. ***(Lions and Stampacchia)*** *The elliptic variational inequalities of first kind and of second kind (6.22) and (6.19) respectively have a unique solution.*

Proof. See [29, 35, 40, 67, 68]. ☐

Note that if we set

$$j(\mathrm{v}) = \begin{cases} j_g(\mathrm{v}) & \text{if} \quad \mathrm{v} \in \mathcal{K}, \\ \infty & \text{elsewhere,} \end{cases}$$

and $A = S$ the existence of a unique solution of the Tresca problem (6.15) follows then from Theorem 6.2. Moreover, the unique solution $\underline{u}_g = \underline{u} \in \mathcal{K}$ depends continuously on the given friction g and satisfies the a priori estimate

$$\|\underline{u}\|_{\mathbf{H}^{1/2}(\Gamma)} \leq c_0,$$

where the constant c_0 is independent of the given stress g, see, e.g. [29, p.30].

6.2.2 Weak formulation of the contact problem with Coulomb friction

After showing the unique solvability of the contact problem with Tresca friction, the question that one may ask is the relation between the solution $\underline{u}_g$ of the Tresca problem (6.15) and a solution $\underline{u}$ of the contact problem with Coulomb friction (6.14). To answer this question we consider the following mapping

$$\Phi : \mathcal{K}^* \to \mathcal{K}^*$$

defined by $\Phi(g) := |\sigma_n(\underline{u}_g)|$. Note that Φ is welldefined since for a given $g \in \mathcal{K}^*$ the Tresca problem (6.15) has a unique solution. It is then natural that $\underline{u}_g$ is a weak solution of the contact problem with Coulomb friction (6.14) if and only if $|\sigma_n(\underline{u}_g)|$ is a fixed point of the mapping Φ. In [29, 42, 85] much effort was done to fulfill some requirements that guarantee the existence of a fixed point of Φ, i.e. the existence of a solution of the contact problem with Coulomb friction. In general it is said that the contact problem with Coulomb friction (6.14) has a weak solution if the friction coefficient $\mathcal{F}$ is sufficiently small. To be more precise, it is shown in [29, 30] that a weak solution to the Coulomb frictional contact problem exists if for all $x \in \Gamma_C$,

$$\mathcal{F}(x) < \begin{cases} \frac{\sqrt{3-4\nu}}{2-2\nu} & \text{if} \quad d = 2, \\ \\ \sqrt{\frac{3-4\nu}{4-4\nu}} & \text{if} \quad d = 3, \end{cases} \tag{6.25}$$

where ν denotes the Poisson ratio and d the dimension of Ω.

The authors in [28–30] used another technique based on a simultaneous penalization of unilateral conditions and a regularization of the frictional term. This turns out to be powerful from the theoretical point of view but not very attractive for computations. Indeed, after a discretization one obtains a system of nonlinear algebraic equations which depends on two small parameters. Moreover, the efficiency of the algorithm strongly depends on their choice [30].

We can remark that the development of fast algorithms for contact problems with Coulomb friction depends on the efficiency of the solvers for the Tresca problem. But, since the functional j_g in (6.15) is non-differentiable this becomes problematic. Therefore, a more subtle approach appropriately dealing with generalized derivatives becomes necessary for the variational inequality (6.15), see, e.g., [17, 47, 70, 101]. The fixed point technique presented above is rather slow and is typically of first order or converges at a linear rate. In [71, 98, 99] and [46, 57, 58, 60] generalized Newton methods which are in general of second order methods were proposed for contact problems or for optimal control problems respectively. The key ingredient for these steps is a dual regularization.

6.3 Dual problem and extremality conditions for the Tresca problem

The main tools of this section will be about the convex analysis [31] presented in chapter 2, but for the reader's convenience let us briefly repeat them here. We now recall the Fenchel duality theorem in infinite dimensional spaces in a form that is convenient for our work.

In the following, X and Y will denote Banach spaces, X^* and Y^* their topological duals respectively. Let $\Lambda : X \to Y$ be a linear and bounded operator, i.e., $\Lambda \in \mathcal{L}(X,Y)$, and let $\mathbb{F} : X \to \overline{\mathbb{R}}$ and $\mathbb{G} : Y \to \overline{\mathbb{R}}$ be convex, proper and lower semi-continuous (l.s.c) functionals. Moreover, there exists $\mathrm{v}_0 \in X$ such that $\mathbb{F}(\mathrm{v}_0) < \infty$, $\mathbb{G}(\Lambda \mathrm{v}_0) < \infty$. Then we have

$$\inf_{\mathrm{v}\in X} [\mathbb{F}(\mathrm{v}) + \mathbb{G}(\Lambda \mathrm{v})] = \sup_{q\in Y^*} \left[-\mathbb{F}^*(-\Lambda' q) - \mathbb{G}^*(q)\right], \tag{6.26}$$

where $\Lambda' \in \mathcal{L}(Y^*, X^*)$ is the adjoint of Λ, $\mathbb{F}^*$ and $\mathbb{G}^*$ denote the Fenchel convex conjugate for the functionals $\mathbb{F}$ and $\mathbb{G}$ respectively, defined by

$$\mathbb{F}^*(\mathrm{v}^*) = \sup_{\mathrm{v}\in X} [\langle \mathrm{v}, \mathrm{v}^* \rangle_{X,X^*} - \mathbb{F}(\mathrm{v})].$$

The left and the right hand sides of (6.26) are called primal and dual problems respectively. In addition, if $\mathrm{v} \in X$ and $q \in Y^*$ are the solutions of the primal and dual problems respectively, they are then characterized by the extremality conditions

$$\begin{aligned} -\Lambda' q &\in \partial \mathbb{F}(\mathrm{v}), \\ q &\in \partial \mathbb{G}(\Lambda \mathrm{v}), \end{aligned} \tag{6.27}$$

where $\partial \mathbb{F}$ and $\partial \mathbb{G}$ denote the subdifferentials of the convex functionals $\mathbb{F}$ and $\mathbb{G}$ respectively, defined by

$$\partial \mathbb{F}(\mathrm{v}) = \{\mathrm{v}^* \in X^* : \mathbb{F}(u) \geq \mathbb{F}(\mathrm{v}) + \langle u - \mathrm{v}, \mathrm{v}^* \rangle_{X,X^*} \quad \forall u \in X\}.$$

If we set now $X := \mathcal{V}$ and $Y := \mathcal{V} \times \mathbf{H}^{1/2}(\Gamma_C)$, we have then $\Lambda \in \mathcal{L}(\mathcal{V}, \mathcal{V} \times \mathbf{H}^{1/2}(\Gamma_C))$ and defined by

$$\Lambda(\underline{\mathrm{v}}) := (\underline{\mathrm{v}}, \underline{\mathrm{v}}_t).$$

Further, we use $\mathbb{G} : \mathcal{V} \times \mathbf{H}^{1/2}(\Gamma_C) \to \mathbb{R}$, which is defined by

$$\mathbb{G}(\underline{\mathrm{v}}, \underline{\mathrm{v}}_t) := \frac{1}{2} \langle S\underline{\mathrm{v}}, \underline{\mathrm{v}} \rangle + j_g(\underline{\mathrm{v}}),$$

where $j_g(\underline{\mathrm{v}}) := \int\limits_{\Gamma_C} \mathcal{F} g |\underline{\mathrm{v}}_t - \underline{w}_t| ds_x$ and $\mathbb{F} : \mathcal{V} \to \overline{\mathbb{R}}$ is defined by

$$\mathbb{F}(\underline{\mathrm{v}}) := \begin{cases} -L(\underline{\mathrm{v}}) & \text{if} \quad \underline{\mathrm{v}} \in \mathcal{K}, \\ \infty & \text{else.} \end{cases}$$

It can be easily verified that Λ, $\mathbb{F}$ and $\mathbb{G}$ satisfy the above properties of the Fenchel duality theorem. Therefore, the minimization problem (6.16) can be written as follows

$$\min_{\underline{v}\in\mathcal{V}} \left[\mathbb{F}(\underline{v}) + \mathbb{G}(\Lambda\underline{v})\right], \tag{6.28}$$

with the corresponding dual problem

$$\sup_{(p,\underline{\mu})\in\mathcal{V}^*\times\mathbf{H}^{-1/2}(\Gamma_C)} \left[-\mathbb{F}^*(-\Lambda'(p,\underline{\mu})) - \mathbb{G}^*(p,\underline{\mu})\right]. \tag{6.29}$$

Next, we are going to evaluate the above convex conjugates

$$\begin{aligned}
\mathbb{G}^*(p,\underline{\mu}) &= \sup_{(\underline{v},\underline{v}_t)\in\mathcal{V}\times\mathbf{H}^{1/2}(\Gamma_C)} \left[\langle p,\underline{v}\rangle_\Gamma + \langle\underline{\mu},\underline{v}_t\rangle_{\Gamma_C} - \mathbb{G}(\underline{v},\underline{v}_t)\right] \\
&= \sup_{(\underline{v},\underline{v}_t)\in\mathcal{V}\times\mathbf{H}^{1/2}(\Gamma_C)} \left[\langle p,\underline{v}\rangle_\Gamma - \frac{1}{2}\langle S\underline{v},\underline{v}\rangle_\Gamma + \langle\underline{\mu},\underline{v}_t\rangle_{\Gamma_C} - \langle\mathcal{F}g,|\underline{v}_t-\underline{w}_t|\rangle_{\Gamma_C}\right] \\
&= \sup_{\underline{v}\in\mathcal{V}} \left[\langle p,\underline{v}\rangle_\Gamma - \frac{1}{2}\langle S\underline{v},\underline{v}\rangle_\Gamma\right] + \sup_{\underline{v}_t\in\mathbf{H}^{1/2}(\Gamma_C)} \left[\langle\underline{\mu},\underline{v}_t\rangle_{\Gamma_C} - \langle\mathcal{F}g,|\underline{v}_t-\underline{w}_t|\rangle_{\Gamma_C}\right] \\
&= \sup_{\underline{v}\in\mathcal{V}} \left[\langle p,\underline{v}\rangle_\Gamma - \frac{1}{2}\langle S\underline{v},\underline{v}\rangle_\Gamma\right] + \sup_{\underline{v}_t\in\mathbf{H}^{1/2}(\Gamma_C)} \left[\langle\underline{\mu},\underline{v}_t-\underline{w}_t\rangle_{\Gamma_C} - j_g(\underline{v})\right] \\
&\qquad + \sup_{\underline{v}_t\in\mathbf{H}^{1/2}(\Gamma_C)} \langle\underline{\mu},\underline{w}_t\rangle_{\Gamma_C} \\
&= \sup_{\underline{v}\in\mathcal{V}} \left[\langle p,\underline{v}\rangle_\Gamma - \frac{1}{2}\langle S\underline{v},\underline{v}\rangle_\Gamma\right] + j_g^*(\underline{\mu}) + \sup_{\underline{v}_t\in\mathbf{H}^{1/2}(\Gamma_C)} \langle\underline{\mu},\underline{w}_t\rangle_{\Gamma_C},
\end{aligned}$$

where j_g^* is the convex conjugate of the functional j_g. Since j_g is a gauge, Lemma 2.2 yields

$$j_g^*(\underline{\mu}) := \begin{cases} 0 & \text{if} \quad \langle\mathcal{F}g,|\underline{v}_t-\underline{w}_t|\rangle_{\Gamma_C} - \langle\underline{\mu},\underline{v}_t-\underline{w}_t\rangle_{\Gamma_C} \geq 0 \text{ for all } \underline{v}_t \in \mathbf{H}^{1/2}(\Gamma_C), \\ \infty & \text{else.} \end{cases}$$

Further, the first order necessary condition for the convex optimization yields $p := S\underline{v}$, and we then obtain

$$\sup_{\underline{v}\in\mathcal{V}} \left[\langle p,\underline{v}\rangle_\Gamma - \frac{1}{2}\langle S\underline{v},\underline{v}\rangle_\Gamma\right] := \frac{1}{2}\langle S^{-1}p,p\rangle_\Gamma.$$

Hence, the convex conjugate of $\mathbb{G}$ is given by

$$\mathbb{G}^*(p,\underline{\mu}) := \begin{cases} \frac{1}{2}\langle S^{-1}p,p\rangle_\Gamma + \langle\underline{\mu},\underline{w}_t\rangle_{\Gamma_C} & \text{if} \quad \langle\mathcal{F}g,|\underline{v}_t-\underline{w}_t|\rangle_{\Gamma_C} - \langle\underline{\mu},\underline{v}_t-\underline{w}_t\rangle_{\Gamma_C} \geq 0 \quad \forall\underline{v}_t. \\ \infty & \text{else.} \end{cases}$$

On the other hand, we have

$$
\begin{aligned}
\mathbb{F}^*(-\Lambda'(p,\underline{\mu})) &= \sup_{\underline{\mathrm{v}}\in\mathcal{V}}\Big[\langle -\Lambda'(p,\underline{\mu}),\underline{\mathrm{v}}\rangle_\Gamma - \mathbb{F}(\underline{\mathrm{v}})\Big]\\
&= \sup_{\underline{\mathrm{v}}\in\mathcal{V}}\Big[\langle -(p,\underline{\mu}),\Lambda(\underline{\mathrm{v}})\rangle_\Gamma + L(\underline{\mathrm{v}})\Big]\\
&= \sup_{\underline{\mathrm{v}}\in\mathcal{V}}\Big[-\langle p,\underline{\mathrm{v}}\rangle_\Gamma - \langle \underline{\mu},\underline{\mathrm{v}}_t\rangle_{\Gamma_C} + L(\underline{\mathrm{v}})\Big]\\
&= \sup_{\underline{\mathrm{v}}\in\mathcal{K}}\Big[-\langle p-\mathbf{N}\underline{f},\underline{\mathrm{v}}\rangle_\Gamma - \langle \underline{\mu},\underline{\mathrm{v}}_t\rangle_{\Gamma_C} + \langle \underline{g}_N,\underline{\mathrm{v}}\rangle_{\Gamma_N}\Big]\\
&= \sup_{\underline{\mathrm{v}}\in\mathcal{K}}\Big[-\langle p_n-(\mathbf{N}\underline{f})_n,\mathrm{v}_n\rangle_{\Gamma_C} - \langle p_t-(\mathbf{N}\underline{f})_t+\underline{\mu},\underline{\mathrm{v}}_t\rangle_{\Gamma_C}\\
&\qquad + \langle -(p-\mathbf{N}\underline{f})+\underline{g}_N,\underline{\mathrm{v}}\rangle_{\Gamma_N}\Big]
\end{aligned}
$$

and $\mathbb{F}^*(-\Lambda'(p,\underline{\mu}))$ is equal to ∞ unless

$$p-\mathbf{N}\underline{f}=\underline{g}_N \quad a.e. \quad \text{on} \quad \Gamma_N \text{ and } \underline{\mu}+p_t-(\mathbf{N}\underline{f})_t=0 \text{ in } \mathbf{H}^{-1/2}(\Gamma_C). \tag{6.30}$$

Further, for

$$p_n-(\mathbf{N}\underline{f})_n\le 0, \text{ in } H^{-1/2}(\Gamma_C), \tag{6.31}$$

we then obtain

$$\mathbb{F}^*(-\Lambda'(p,\underline{\mu}))=\begin{cases} -\langle p_n-(\mathbf{N}\underline{f})_n,\mathbf{d}\rangle_{\Gamma_C} & \text{if} \quad (6.30) \text{ and } (6.31) \text{ hold,}\\ \infty & \text{else.}\end{cases}$$

Hence, the dual problem is given as follows

$$\sup_{\substack{(p,\underline{\mu})\in\mathcal{V}^*\times\mathbf{H}^{-1/2}(\Gamma_C),\\ \text{s.t.} \quad (6.30) \text{ and } (6.31),\\ \langle \mathcal{F}g,|\underline{\mathrm{v}}_t-\underline{w}_t|\rangle_{\Gamma_C}-\langle\underline{\mu},\underline{\mathrm{v}}_t-\underline{w}_t\rangle_{\Gamma_C}\ge 0\\ \forall \underline{\mathrm{v}}_t\in\mathbf{H}^{1/2}(\Gamma_C).}} \Big[-\frac{1}{2}\langle S^{-1}p,p\rangle_\Gamma - \langle\underline{\mu},\underline{w}_t\rangle_{\Gamma_C} + \langle p_n-(\mathbf{N}\underline{f})_n,\mathbf{d}\rangle_{\Gamma_C}\Big] \tag{$\mathcal{P}^*$}$$

Note that the cost functional in the dual problem $(\mathcal{P}^*)$ is differentiable. The existence of a solution follows from the standard theorem of duality theory, see Theorem 2.7. Moreover, the uniform convexity guarantees the uniqueness.

The next theorem summarizes the uniquely characterization between the unique solution $\underline{u}\in\mathcal{K}$ of the primal problem (6.16) and the unique solution $(p,\underline{\mu})\in\mathcal{V}^*\times\mathbf{H}^{-1/2}(\Gamma_C)$ of the dual problem $(\mathcal{P}^*)$ in general for $g\in H_+^{-1/2}(\Gamma_C)$, where

$$H_+^{-1/2}(\Gamma_C):=\{g\in H^{-1/2}(\Gamma_C):\langle g,\mathrm{v}\rangle_{\Gamma_C}\ge 0 \;\text{ for all } \mathrm{v}\in H^{1/2}(\Gamma_C) \;\text{ with } \mathrm{v}\ge 0\}.$$

Theorem 6.3. *The solution* $\underline{u} \in \mathcal{K}$ *of the primal problem (6.16) and the solution* $(p, \underline{\mu}) \in \mathcal{V}^* \times \boldsymbol{H}^{-1/2}(\Gamma_C)$ *of the dual problem* $(\mathcal{P}^*)$ *are characterized by* $S\underline{u} := p$ *and by the existence of a multiplier* $\lambda \in H^{-1/2}(\Gamma_C)$ *such that*

$$\begin{aligned}
\langle S\underline{u}, \underline{v}\rangle_\Gamma - L(\underline{v}) + \langle \underline{\mu}, \underline{v}_t\rangle_{\Gamma_C} + \langle \lambda, v_n\rangle_{\Gamma_C} &= 0 \quad \textit{for all} \quad \underline{v} \in \mathcal{V}, && (6.32)\\
\langle \lambda, v_n\rangle_{\Gamma_C} &\leq 0 \quad \textit{for all} \quad v_n \leq 0, && (6.33)\\
\langle \lambda, u_n - \boldsymbol{d}\rangle_{\Gamma_C} &= 0, && (6.34)\\
\langle \mathcal{F}g, |\underline{v}_t - \underline{w}_t|\rangle_{\Gamma_C} - \langle \underline{\mu}, \underline{v}_t - \underline{w}_t\rangle_{\Gamma_C} &\geq 0 \quad \textit{for all} \quad \underline{v}_t \in \boldsymbol{H}^{1/2}(\Gamma_C), && (6.35)\\
\langle \mathcal{F}g, |\underline{u}_t - \underline{w}_t|\rangle_{\Gamma_C} - \langle \underline{\mu}, \underline{u}_t - \underline{w}_t\rangle_{\Gamma_C} &= 0. && (6.36)
\end{aligned}$$

Proof. Since the primal problem $(\mathcal{P})$ and the dual problem $(\mathcal{P}^*)$ have each a unique solution, the extremal relations are given by $(p, \underline{\mu}) \in \partial\mathbb{G}(\Lambda\underline{u})$ and $-\Lambda'(p, \underline{\mu}) \in \partial\mathbb{F}(\underline{u})$, see Theorem 2.8. Indeed, $(p, \underline{\mu}) \in \partial\mathbb{G}(\Lambda\underline{u})$ implies

$$\mathbb{G}(\Lambda\underline{u}) - \mathbb{G}(\Lambda\underline{\mathrm{v}}) \leq \langle p, \underline{u} - \underline{\mathrm{v}}\rangle_\Gamma + \langle \underline{\mu}, \underline{u}_t - \underline{\mathrm{v}}_t\rangle_{\Gamma_C},$$

that is for all $\underline{\mathrm{v}} \in \mathcal{V}$

$$\frac{1}{2}\langle S\underline{u}, \underline{u}\rangle_\Gamma - \frac{1}{2}\langle S\underline{\mathrm{v}}, \underline{\mathrm{v}}\rangle_\Gamma + \langle \mathcal{F}g, |\underline{u}_t - \underline{w}_t| - |\underline{\mathrm{v}}_t - \underline{w}_t|\rangle_{\Gamma_C} \leq \langle p, \underline{u} - \underline{\mathrm{v}}\rangle_\Gamma + \langle \underline{\mu}, \underline{u}_t - \underline{\mathrm{v}}_t\rangle_{\Gamma_C}. \quad (6.37)$$

In the above inequality let us assume that $\underline{\mathrm{v}}_t = \underline{u}_t$, we then obtain

$$\frac{1}{2}\langle S\underline{u}, \underline{u}\rangle_\Gamma - \langle p, \underline{u}\rangle_\Gamma \leq \frac{1}{2}\langle S\underline{\mathrm{v}}, \underline{\mathrm{v}}\rangle_\Gamma - \langle p, \underline{\mathrm{v}}\rangle_\Gamma.$$

We can notice that $\underline{u}$ is the minimizer of the above functional in $\{\underline{\mathrm{v}} \in \mathcal{V} : \underline{\mathrm{v}}_t = \underline{u}_t\}$. Moreover, the first order necessary condition for the above minimization problem yields

$$p := S\underline{u}. \quad (6.38)$$

On the other hand, $-\Lambda'(p, \underline{\mu}) \in \partial\mathbb{F}(\underline{u})$ implies

$$\begin{aligned}
\mathbb{F}(\underline{u}) - \mathbb{F}(\underline{\mathrm{v}}) &\leq \langle -\Lambda'(p, \underline{\mu}), \underline{u} - \underline{\mathrm{v}}\rangle_\Gamma \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}\\
&= \langle -p, \underline{u} - \underline{\mathrm{v}}\rangle_\Gamma + \langle -\underline{\mu}, \underline{u}_t - \underline{\mathrm{v}}_t\rangle_{\Gamma_C} \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V},
\end{aligned}$$

which yields

$$\langle p, \underline{\mathrm{v}} - \underline{u}\rangle_\Gamma - L(\underline{\mathrm{v}} - \underline{u}) + \langle \underline{\mu}, \underline{\mathrm{v}}_t - \underline{u}_t\rangle_{\Gamma_C} \geq 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}. \quad (6.39)$$

By using (6.38) we obtain

$$\langle S\underline{u}, \underline{\mathrm{v}} - \underline{u}\rangle_\Gamma - L(\underline{\mathrm{v}} - \underline{u}) + \langle \underline{\mu}, \underline{\mathrm{v}}_t - \underline{u}_t\rangle_{\Gamma_C} \geq 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}.$$

Further, if we set $\underline{\mathrm{v}} = \underline{\mathrm{v}} - \underline{u}$, we then obtain

$$\langle S\underline{u}, \underline{\mathrm{v}}\rangle_\Gamma - L(\underline{\mathrm{v}}) + \langle \underline{\mu}, \underline{\mathrm{v}}_t\rangle_{\Gamma_C} \geq 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V},$$

which yields

$$\langle S\underline{u}, \underline{\mathrm{v}}\rangle_\Gamma - L(\underline{\mathrm{v}}) + \langle \underline{\mu}, \underline{\mathrm{v}}_t\rangle_{\Gamma_C} + \langle \lambda, \mathrm{v}_n\rangle_{\Gamma_C} = 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}, \tag{6.40}$$

for some $\lambda \in H^{-1/2}(\Gamma_C)$ such that

$$\langle \lambda, \mathrm{v}_n\rangle_{\Gamma_C} \leq 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}, \tag{6.41}$$

which shows (6.32) and (6.33).

Let us substituting now (6.38) into (6.37), this yields

$$\frac{1}{2}\langle S(\underline{u}-\underline{\mathrm{v}}), \underline{u}-\underline{\mathrm{v}}\rangle_\Gamma - \langle \mathcal{F}g, |\underline{u}_t - \underline{w}_t| - |\underline{\mathrm{v}}_t - \underline{w}_t|\rangle_{\Gamma_C} + \langle \underline{\mu}, \underline{u}_t - \underline{\mathrm{v}}_t\rangle_{\Gamma_C} \geq 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}. \tag{6.42}$$

If we set $\underline{\mathrm{v}} := \underline{u} + t(\underline{\mathrm{v}}^* - \underline{u})$ into (6.42) for $t \in (0,1)$ and for an arbitrary $\underline{\mathrm{v}}^*$ in $\mathcal{V}$, further, use the positivity of $\mathcal{F}g$ and the convexity of the Euclidean norm $|.|$, we then obtain

$$\frac{t^2}{2}\langle S(\underline{u}-\underline{\mathrm{v}}^*), \underline{u}-\underline{\mathrm{v}}^*\rangle_\Gamma - t\langle \mathcal{F}g, |\underline{u}_t - \underline{w}_t| - |\underline{\mathrm{v}}_t^* - \underline{w}_t|\rangle_{\Gamma_C} + t\langle \underline{\mu}, \underline{u}_t - \underline{\mathrm{v}}_t^*\rangle_{\Gamma_C} \geq 0. \tag{6.43}$$

If we divide (6.43) by t and let $t \to 0$, we then obtain

$$\langle \mathcal{F}g, |\underline{u}_t - \underline{w}_t|\rangle_{\Gamma_C} - \langle \underline{\mu}, \underline{u}_t - \underline{w}_t\rangle_{\Gamma_C} \leq \langle \mathcal{F}g, |\underline{\mathrm{v}}_t^* - \underline{w}_t|\rangle_{\Gamma_C} - \langle \underline{\mu}, \underline{\mathrm{v}}_t^* - \underline{w}_t\rangle_{\Gamma_C} \quad \text{for all} \quad \underline{\mathrm{v}}^* \in \mathcal{V}. \tag{6.44}$$

If we choose $\underline{\mathrm{v}}^*$ such that $\underline{\mathrm{v}}_t^* = \underline{w}_t$, (6.44) yields then

$$\langle \mathcal{F}g, |\underline{u}_t - \underline{w}_t|\rangle_{\Gamma_C} - \langle \underline{\mu}, \underline{u}_t - \underline{w}_t\rangle_{\Gamma_C} \leq 0. \tag{6.45}$$

By using (6.45), and the following inequality from the constrained dual problem $(\mathcal{P}^*)$ given by

$$\langle \mathcal{F}g, |\underline{\mathrm{v}}_t - \underline{w}_t|\rangle_{\Gamma_C} - \langle \underline{\mu}, \underline{\mathrm{v}}_t - \underline{w}_t\rangle_{\Gamma_C} \geq 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}, \tag{6.46}$$

we obtain

$$\langle \mathcal{F}g, |\underline{u}_t - \underline{w}_t|\rangle_{\Gamma_C} - \langle \underline{\mu}, \underline{u}_t - \underline{w}_t\rangle_{\Gamma_C} = 0. \tag{6.47}$$

□

Remark 6.1.

- *By using (6.30), (6.38) and (6.12), we obtain*

$$\underline{\mu} = -[p_t - (\boldsymbol{N}\underline{f})_t] := -[(S\underline{u} - \boldsymbol{N}\underline{f})_t] \equiv -[\sigma_t(\underline{u})], \tag{6.48}$$

that is $\underline{\mu}$ is the negative tangential stress.

- *On the other hand, if we use (6.32), the definition of the linear map L, (6.38) and (6.12) we then show*

$$\lambda = -[p_n - (\boldsymbol{N}\underline{f})_n] := -[(S\underline{u} - \boldsymbol{N}\underline{f})_n] \equiv -\sigma_n(\underline{u}), \tag{6.49}$$

that is λ is the negative normal boundary stress.

- *If $\lambda \in L^1(\Gamma_C)$ and $\underline{u} \in \mathcal{K}$, the constraints (6.33) and (6.34) can then be written equivalently in terms of a complementarity function as follows*

$$\lambda = \max(0, \lambda + c(u_n - \boldsymbol{d})) \quad \textit{for all} \quad c > 0, \tag{6.50}$$

see, e.g., [59, 61, 71, 98, 99].

Next we investigate the influence of the regularity of the given friction g on the dual problem and the associated extremality conditions.

Assumption 6.1. *We suppose that $g \in L^2_+(\Gamma_C) = \{f \in L_2(\Gamma_C) : f \geq 0\}$.*

Under Assumption 6.1 the space $\mathbf{H}^{1/2}(\Gamma_C)$ in the definition of $\mathbb{G}$ and Λ can be replaced by $\mathbf{L}_2(\Gamma_C)$. Since $\mathbf{L}_2(\Gamma_C)$ is identified with its dual, the duality product between $\mathbf{H}^{1/2}(\Gamma_C)$ and $\mathbf{H}^{-1/2}(\Gamma_C)$ can be replaced by the $\mathbf{L}_2(\Gamma_C)$-scalar products. This shows that the Lagrange multiplier $\underline{\mu}$ belonging to the non-differentiability of the cost functional is more regular, that is $\underline{\mu} \in \mathbf{L}_2(\Gamma_C)$. Moreover, constraints (6.35) and (6.36) of the extremality conditions take the following forms.

Proposition 6.2. *For $g \in L^2_+(\Gamma_C)$ the constraints (6.35) and (6.36) are equivalent to*

$$|\underline{\mu}| \leq \mathcal{F}g \text{ a.e. on } \Gamma_C, \tag{6.51}$$

and

$$\begin{cases} \underline{u}_t - \underline{w}_t = 0 & \textit{or} \\ \underline{u}_t - \underline{w}_t \neq 0 & \textit{and } \underline{\mu} = \mathcal{F}g \dfrac{\underline{u}_t - \underline{w}_t}{|\underline{u}_t - \underline{w}_t|}, \end{cases} \tag{6.52}$$

respectively.

Proof. If $\underline{\mu} \in \mathbf{L}_2(\Gamma_C)$, (6.35) becomes then

$$(\mathcal{F}g, |\underline{v}_t - \underline{w}_t|)_{\Gamma_C} - (\underline{\mu}, \underline{v}_t - \underline{w}_t)_{\Gamma_C} \geq 0 \text{ for all } \underline{v}_t \in \mathbf{L}_2(\Gamma_C). \tag{6.53}$$

Therefore, we have to verify that (6.53) is equivalent to (6.51). Indeed, from the inequality

$$\begin{aligned} (\mathcal{F}g, |\underline{v}_t - \underline{w}_t|)_{\Gamma_C} - (\underline{\mu}, \underline{v}_t - \underline{w}_t)_{\Gamma_C} &\geq (\mathcal{F}g, |\underline{v}_t - \underline{w}_t|)_{\Gamma_C} - (|\underline{\mu}|, |\underline{v}_t - \underline{w}_t|)_{\Gamma_C} \\ &= (\mathcal{F}g - |\underline{\mu}|, |\underline{v}_t - \underline{w}_t|)_{\Gamma_C} \text{ for all } \underline{v}_t \in \mathbf{L}^2(\Gamma_C), \end{aligned}$$

(6.51) implies (6.53). Conversely, we assume that (6.51) does not hold, that is

$$\mathbb{S} := \{x \in \Gamma_C : \mathcal{F}g - |\underline{\mu}| < 0 \text{ a.e.}\}$$

has a positive measure. Further, we choose $\underline{\mathrm{v}}^* \in \mathbf{L}_2(\Gamma_C)$ defined by

$$\underline{\mathrm{v}}^*(x) := \begin{cases} \underline{\mu}(x) + \underline{w}_t(x) & \text{on } \mathbb{S}, \\ \underline{w}_t(x) & \text{on } \Gamma_C \setminus \mathbb{S}. \end{cases}$$

This leads to

$$\begin{aligned} (\mathcal{F}g, |\underline{\mathrm{v}}^* - \underline{w}_t|)_{\Gamma_C} - (\underline{\mu}, \underline{\mathrm{v}}^* - \underline{w}_t)_{\Gamma_C} &= (\mathcal{F}g - |\underline{\mu}|, |\underline{\mu}|)_{\mathbb{S}} \\ &= -(\mathcal{F}g - |\underline{\mu}|, \mathcal{F}g - |\underline{\mu}|)_{\mathbb{S}} + (\min(0, \mathcal{F}g - |\underline{\mu}|), \mathcal{F}g)_{\mathbb{S}} \\ &\leq -\int_{\mathbb{S}} (\mathcal{F}g - |\underline{\mu}|)^2 ds_x < 0, \end{aligned}$$

which contradicts (6.53). The proof of the equivalence of (6.36) and (6.52) is done in a similar way, one can check [98] for details. □

Proposition 6.3. *Under the above assumptions, (6.51) and (6.52) are equivalent to*

$$\mathcal{F}g(\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t) - \max(\mathcal{F}g\sigma, |\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t|)\underline{\mu} = 0 \quad \textit{for all } \sigma > 0. \tag{6.54}$$

Proof. From (6.54) it follows that

$$\underline{\mu} = \mathcal{F}g \frac{\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t}{\max(\mathcal{F}g\sigma, |\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t|)},$$

which immediately implies (6.51). Further, to prove (6.52) we distinguish two cases as follows

$$\mathcal{F}g\sigma \geq |\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t| \ \text{ and } \ \mathcal{F}g\sigma < |\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t|.$$

In the first case, we obtain from (6.54) that $\underline{u}_t - \underline{w}_t = 0$ which is the upper condition of (6.52). In the second case, we obtain $\underline{u}_t - \underline{w}_t \neq 0$, otherwise we get $\mathcal{F}g < |\underline{\mu}|$ which is a contradiction. Furthermore, from (6.54) we have

$$\mathcal{F}g(\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t) = |\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t|\underline{\mu}, \tag{6.55}$$

and this yields

$$\mathcal{F}g(u_t - \underline{w}_t) = (|\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t| - \mathcal{F}g\sigma)\underline{\mu} = \beta\underline{\mu}, \tag{6.56}$$

where $\beta = |\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t| - \mathcal{F}g\sigma > 0$. Thus, by considering the norms of the two expressions (6.55) and (6.56) yields $\beta = |\underline{u}_t - \underline{w}_t|$. Therefore, we have shown that (6.54) implies (6.51) and (6.52). The vice versa can be checked easily, we just have to distinguish two cases for (6.52). □

By utilizing the condition $S\underline{u} := p$ and (6.49) into the cost functional of the dual problem $(\mathcal{P}^*)$ together with the simplification (6.51), we obtain the following dual problem by transforming sup to min

$$\begin{cases} -\min\limits_{\substack{\lambda\geq 0 \text{ in } H^{-1/2}(\Gamma_C),\\ |\underline{\mu}|\leq \mathcal{F}g \text{ a.e. on } \Gamma_C}} \left[\frac{1}{2}\langle S\underline{u}_{\lambda,\mu}, \underline{u}_{\lambda,\mu}\rangle_\Gamma + \langle \underline{\mu}, \underline{w}_t\rangle_{\Gamma_C} + \langle \lambda, \mathbf{d}\rangle_{\Gamma_C}\right], \\ \text{where } \underline{u}_{\lambda,\mu} \text{ satisfies} \\ \langle S\underline{u}_{\lambda,\mu}, \underline{\mathrm{v}}\rangle_\Gamma - L(\underline{\mathrm{v}}) + \langle \underline{\mu}, \underline{\mathrm{v}}_t\rangle_{\Gamma_C} + \langle \lambda, \mathrm{v}_n\rangle_{\Gamma_C} = 0 \text{ for all } \underline{\mathrm{v}} \in \mathcal{V}, \end{cases} \tag{6.57}$$

where the primal variable $\underline{u}_{\lambda,\mu}$ appears here as an auxiliary variable, since it is determined for given λ and $\underline{\mu}$. In addition, the extremality conditions (6.32)-(6.36) can be written as a system of nonlinear equations by the help of complementarity functions as follows

Proposition 6.4. *The extremality conditions (6.32)-(6.36) can be written equivalently as*

$$\langle S\underline{u}, \underline{v}\rangle_\Gamma - L(\underline{v}) + \langle \underline{\mu}, \underline{v}_t\rangle_{\Gamma_C} + \langle \lambda, v_n\rangle_{\Gamma_C} = 0 \quad \textit{for all} \quad \underline{v} \in \mathcal{V}, \tag{6.58}$$

$$\lambda - \max(0, \lambda + c(u_n - \boldsymbol{d})) = 0 \quad \textit{for all} \quad c > 0, \tag{6.59}$$

$$\mathcal{F}g(\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t) - \max(\mathcal{F}g\sigma, |\sigma\underline{\mu} + \underline{u}_t - \underline{w}_t|)\underline{\mu} = 0 \quad \textit{for all } \sigma > 0. \tag{6.60}$$

Proof. Use (6.50) and *Propositions 6.2,6.3* . □

Next we are going to solve the minimization problem (6.57). It is well known that solving an optimization problem is nothing else than solving the optimality conditions, that is solving the extremality conditions (6.58)-(6.60), see Theorem 2.8. But, the system of equations (6.58)-(6.60) is nonlinear. Therefore, the first attempt which comes in mind will be to use the Newton method. But, it turns out that the $\max(.,.)$ function is not differentiable in the usual sense, which makes the application of the traditional Newton method impossible. Nevertheless, the $\max(.,.)$ function is Newton differentiable under certain constraints, see [17, 47, 101]. This motivates the application of generalized Newton method (semi-smooth Newton method) for the system of equations (6.58)-(6.60), see, e.g., [48, 61, 71, 98, 99].

We can remark from Theorem 2.9 that the max-operator is Newton differentiable from $L_p(\Gamma)$ to $L_2(\Gamma)$ if $p > 2$. But, for the system of equations (6.58)-(6.60) if we consider $\underline{u}$, $\underline{\mu}$ and λ to be independent variables, we can then obtain the necessary smoothing requirement for the semi-smooth Newton methods for the variable $\underline{u}$ due to the trace theorem. But we lack the smoothing property with respect to the variables $\underline{\mu}$ and λ respectively. To overcome this difficulty, a regularization technique inspired from the augmented Lagrangian is necessary, see, e.g., [58, 59, 61, 71, 98, 99].

6.4 Regularization of dual and primal Tresca problems

The regularization technique we present in this section is motivated from the augmented Lagrangian [61, 71, 98, 99] and can enable us to apply a semi-smooth Newton method on the system of optimality conditions (6.58)-(6.60). To this end, we assume λ to be more regular, that is $\lambda \in L_2(\Gamma_C)$, further we define for $\gamma_1, \gamma_2 > 0$ and for given $\widehat{\lambda} \in L_2(\Gamma_C)$, $\underline{\widehat{\mu}} \in \mathbf{L}_2(\Gamma_C)$ the regularized version of the dual problem by

$$(\mathcal{P}^*_{\gamma_1,\gamma_2}) \begin{cases} -\min\limits_{\substack{L_2(\Gamma_C)\ni\lambda\geq 0,\\ |\underline{\mu}|\leq \mathcal{F}g \text{ a.e. on } \Gamma_C}} J^*_{\gamma_1,\gamma_2}(\lambda,\underline{\mu}) := \Big[\frac{1}{2}\langle S\underline{u}_{\lambda,\mu},\underline{u}_{\lambda,\mu}\rangle_\Gamma + (\underline{\mu},\underline{w}_t)_{\Gamma_C} + (\lambda,\mathbf{d})_{\Gamma_C} \\ \qquad + \frac{1}{2\gamma_1}\|\lambda-\widehat{\lambda}\|^2_{\Gamma_C} + \frac{1}{2\gamma_2}\|\underline{\mu}-\underline{\widehat{\mu}}\|^2_{\Gamma_C} \\ \qquad - \frac{1}{2\gamma_1}\|\widehat{\lambda}\|^2_{\Gamma_C} - \frac{1}{2\gamma_2}\|\underline{\widehat{\mu}}\|^2_{\Gamma_C}\Big], \\ \text{where } \underline{u}_{\lambda,\mu} \text{ satisfies} \\ \langle S\underline{u}_{\lambda,\mu},\underline{\mathrm{v}}\rangle_\Gamma - L(\underline{\mathrm{v}}) + (\lambda,\mathrm{v}_n)_{\Gamma_C} + (\underline{\mu},\underline{\mathrm{v}}_t)_{\Gamma_C} = 0 \quad \text{for all } \underline{\mathrm{v}}\in\mathcal{V}. \end{cases}$$

The corresponding regularized primal problem to $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ is an unconstrained minimization problem defined by

$$(\mathcal{P}_{\gamma_1,\gamma_2}) \qquad \min_{\underline{u}\in\mathcal{V}} J_{\gamma_1,\gamma_2}(\underline{u}) := \Big[\frac{1}{2}\langle S\underline{u},\underline{u}\rangle_\Gamma - L(\underline{u}) + \frac{1}{2\gamma_1}\|\max(0,\widehat{\lambda}+\gamma_1(u_n-\mathbf{d}))\|^2_{\Gamma_C} + \frac{1}{\gamma_2}\int_{\Gamma_C}\mathcal{F}gh((\underline{u}_t-\underline{w}_t)(x),\underline{\widehat{\mu}}(x))ds_x\Big],$$

where $h(,) : \mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ is the local regularization defined by

$$h(x,y) := \begin{cases} \|\gamma_2 x+y\| - \frac{1}{2}\mathcal{F}g & \text{if } \|\gamma_2 x+y\|\geq \mathcal{F}g, \\ \frac{1}{2\mathcal{F}g}\|\gamma_2 x+y\|^2 & \text{if } \|\gamma_2 x+y\| < \mathcal{F}g. \end{cases}$$

This can be verified by using the Fenchel duality theory, we refer to [98, p. 22] for the detailed derivation. Note that the regularization turns the primal problem into the unconstrained minimization of a differentiable functional, while the dual problem is still the constrained minimization of a quadratic functional. Moreover, both $(\mathcal{P}_{\gamma_1,\gamma_2})$ and $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ admit unique solutions $\underline{u}_{\gamma_1,\gamma_2}$ and $(\lambda_{\gamma_1,\gamma_2},\underline{\mu}_{\gamma_1,\gamma_2})$ respectively, see [29, p.163] and [67, p.37].

From now onward for the sake of the clarity of the presentation, all variables of the regularized problem will be marked only by γ instead of γ_1,γ_2. We then obtain the following results.

Lemma 6.1. *The solutions $\underline{u}_\gamma$ and $(\lambda_\gamma, \underline{\mu}_\gamma)$ of the regularized problems $(\mathcal{P}_{\gamma_1,\gamma_2})$ and $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ respectively, are characterized by the extremality conditions*

$$\begin{aligned}
(S\underline{u}_\gamma, \underline{v})_\Gamma - L(\underline{v}) + (\underline{\mu}_\gamma, \underline{v}_t)_{\Gamma_C} + (\lambda_\gamma, v_n)_{\Gamma_C} &= 0 \quad \forall \underline{v} \in \mathcal{V}, && (6.61)\\
\lambda_\gamma - \max(0, \widehat{\lambda} + \gamma_1((u_n)_\gamma - \boldsymbol{d})) &= 0 \quad \text{on} \quad \Gamma_C, && (6.62)\\
\mathcal{F}g(\gamma_2(\underline{u}_t - \underline{w}_t)_\gamma + \underline{\widehat{\mu}}) - \max(\mathcal{F}g, \|\gamma_2(\underline{u}_t - \underline{w}_t)_\gamma + \underline{\widehat{\mu}}\|)\underline{\mu}_\gamma &= 0 \quad \text{on} \quad \Gamma_C. && (6.63)
\end{aligned}$$

Proof. See [98]. □

6.4.1 Convergence of the regularized Tresca problem

In this section we investigate the convergence of the solutions of the regularized problems to the solution of the original problem, that is we show that $(\underline{u}_\gamma, \lambda_\gamma, \underline{\mu}_\gamma)$ converge to $(\underline{u}, \lambda, \underline{\mu})$ as γ_1, γ_2 go to infinity.

Theorem 6.4. *For all $\underline{\widehat{\mu}} \in \boldsymbol{L}_2(\Gamma_C)$, $\widehat{\lambda} \in L_2(\Gamma_C)$ and for a given friction $g \in L_2(\Gamma_C)$. We have that $\underline{u}_\gamma \to \underline{u}$ strongly in $\boldsymbol{H}^{1/2}(\Gamma)$ and $(\lambda_\gamma, \underline{\mu}_\gamma) \to (\lambda, \underline{\mu})$ weakly in $H^{-1/2}(\Gamma_C) \times \boldsymbol{L}_2(\Gamma_C)$ as $\gamma_1, \gamma_2 \to \infty$.*

Proof. The proof presented here is similar to those presented in [59, 98]. Recall that, both $(\underline{u}_\gamma, \lambda_\gamma, \underline{\mu}_\gamma)$ and $(\underline{u}, \lambda, \underline{\mu})$ satisfy equation (6.58). In addition, $(\underline{u}, \lambda)$ satisfy (6.59) and $(\underline{u}_\gamma, \lambda_\gamma)$ satisfy

$$\lambda_\gamma = \max(0, \widehat{\lambda} + \gamma_1((u_n)_\gamma - \mathbf{d})). \qquad (6.64)$$

By setting $\underline{v} := \underline{u}_\gamma - \underline{u}$ in (6.61) this yields

$$(S\underline{u}_\gamma, (\underline{u}_\gamma - \underline{u}))_\Gamma - L(\underline{u}_\gamma - \underline{u}) + (\lambda_\gamma, (u_\gamma - u)_n)_{\Gamma_C} + (\underline{\mu}_\gamma, (\underline{u}_\gamma - \underline{u})_t)_{\Gamma_C} = 0. \qquad (6.65)$$

Let us estimate

$$\begin{aligned}
(\lambda_\gamma, (u_\gamma - u)_n)_{\Gamma_C} &= (\lambda_\gamma, (u_\gamma)_n - \mathbf{d})_{\Gamma_C} - (\lambda_\gamma, u_n - \mathbf{d})_{\Gamma_C}\\
&\geq \frac{1}{\gamma_1}(\lambda_\gamma, \widehat{\lambda} + \gamma_1((u_\gamma)_n - \mathbf{d}))_{\Gamma_C} - \frac{1}{\gamma_1}(\lambda_\gamma, \widehat{\lambda})_{\Gamma_C},
\end{aligned}$$

where $(\lambda_\gamma, u_n - \mathbf{d})_{\Gamma_C} \leq 0$ was used, (it is obtained from (6.64) and $u_n - \mathbf{d} \leq 0$). Then we have

$$\begin{aligned}
(\lambda_\gamma, (u_\gamma - u)_n)_{\Gamma_C} &\geq \frac{1}{\gamma_1}(\lambda_\gamma, \max(0, \widehat{\lambda} + \gamma_1((u_\gamma)_n - \mathbf{d})))_{\Gamma_C} - \frac{1}{\gamma_1}(\lambda_\gamma, \widehat{\lambda})_{\Gamma_C} \\
&= \frac{1}{\gamma_1}\|\lambda_\gamma\|^2_{\Gamma_C} - \frac{1}{\gamma_1}(\lambda_\gamma, \widehat{\lambda})_{\Gamma_C} \qquad (6.66) \\
&= \frac{1}{2\gamma_1}\|\lambda_\gamma - \widehat{\lambda}\|^2_{\Gamma_C} + \frac{1}{2\gamma_1}\|\lambda_\gamma\|^2_{\Gamma_C} - \frac{1}{2\gamma_1}\|\widehat{\lambda}\|^2_{\Gamma_C} \\
&\geq -\frac{1}{2\gamma_1}\|\widehat{\lambda}\|^2_{\Gamma_C}. \qquad (6.67)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(\underline{\mu}_\gamma, (\underline{u}_\gamma - \underline{u})_t)_{\Gamma_C} &\leq (\mathcal{F}g, |(\underline{u}_\gamma - \underline{u})_t|)_{\Gamma_C} \\
&\leq c_1\|\mathcal{F}g\|_{\Gamma_C}\|\underline{u}_\gamma - \underline{u}\|_{\mathbf{H}^{1/2}(\Gamma_C)}. \qquad (6.68)
\end{aligned}$$

By using (6.66) and (6.68) into (6.65) we obtain

$$\begin{aligned}
(S\underline{u}_\gamma, \underline{u}_\gamma)_{\Gamma_C} + \frac{1}{\gamma_1}\|\lambda_\gamma\|^2_{\Gamma_C} &\leq (S\underline{u}_\gamma, \underline{u})_{\Gamma_C} + \frac{1}{\gamma_1}(\lambda_\gamma, \widehat{\lambda})_{\Gamma_C} + L(\underline{u}_\gamma - \underline{u}) \\
&+ c_1\|\mathcal{F}g\|_{\Gamma_C}\|\underline{u}_\gamma - \underline{u}\|_{\mathbf{H}^{1/2}(\Gamma_C)}. \qquad (6.69)
\end{aligned}$$

Now by using the ellipticity (with constant $c_1^D > 0$), the continuity (with constant $c_2^S > 0$) of S and the continuity of L we obtain

$$c_1^D\|\underline{u}_\gamma\|^2_{\mathbf{H}^{1/2}(\Gamma)} + \frac{1}{\gamma_1}\|\lambda_\gamma\|^2_{\Gamma_C} \leq c_2^S\|\underline{u}_\gamma\|_{\mathbf{H}^{1/2}(\Gamma)}\|\underline{u}\|_{\mathbf{H}^{1/2}(\Gamma)} +$$
$$\left(\|L\|_{H^{-1/2}(\Gamma)} + c_1\|\mathcal{F}g\|_{\Gamma_C}\right)\|\underline{u}_\gamma - \underline{u}\|_{\mathbf{H}^{1/2}(\Gamma)} + \frac{1}{\gamma_1}\|\lambda_\gamma\|_{\Gamma_C}\|\widehat{\lambda}\|_{\Gamma_C}.$$

Finally, we obtain that

$$c_1^D\|\underline{u}_\gamma\|_{\mathbf{H}^{1/2}(\Gamma)} + \frac{1}{\gamma_1}\|\lambda_\gamma\|_{\Gamma_C}$$

is uniformly bounded with respect to $\gamma_1 \geq 1$. Therefore, $\underline{u}_\gamma$ is bounded in $\mathbf{H}^{1/2}(\Gamma)$ and λ_γ in $H^{-1/2}(\Gamma_C)$. As consequence there exist $(\overline{u}, \overline{\lambda}) \in \mathbf{H}^{1/2}(\Gamma) \times H^{-1/2}(\Gamma_C)$ and a sequence γ_k with $\lim_{k\to\infty} \gamma_k = \infty$ such that

$$\underline{u}_{\gamma_k} \rightharpoonup \overline{u} \quad \text{weakly in} \quad \mathbf{H}^{1/2}(\Gamma) \quad \text{and} \quad \lambda_{\gamma_k} \rightharpoonup \overline{\lambda} \quad \text{weakly in} \quad H^{-1/2}(\Gamma_C), \qquad (6.70)$$

where the last convergence follows from (6.61) where we set $\underline{\mathrm{v}}_t = 0$ for all $\underline{\mathrm{v}} \in \mathbf{H}^{1/2}(\Gamma)$.

On the other hand, since $|\underline{\mu}_{\gamma}| \leq \mathcal{F}g$ almost everywhere on Γ_C for all $\gamma_1, \gamma_2 > 0$, there exists $\overline{\mu} \in \mathbf{L}_2(\Gamma)$ and a subsequence γ_{k_l} of γ_k, such that

$$\underline{\mu}_{\gamma_{k_l}} \rightharpoonup \overline{\mu} \quad \text{weakly in} \quad \mathbf{L}_2(\Gamma_C). \tag{6.71}$$

Furthermore, since the set $\{\underline{\mathrm{v}} \in \mathbf{L}_2(\Gamma_C) : |\underline{\mathrm{v}}| \leq \mathcal{F}g\}$ is convex and closed, therefore it is weakly closed and we have $|\overline{\mu}| \leq \mathcal{F}g$ almost everywhere. In the sequel let us drop the subscript k and k_l on γ. Then we have from the definition of λ_γ

$$\frac{1}{\gamma_1}\|\lambda_\gamma\|^2_{\Gamma_C} = \gamma_1 \| \max(0, \frac{1}{\gamma_1}\widehat{\lambda} + (u_\gamma)_n - \mathbf{d})\|^2_{\Gamma_C}. \tag{6.72}$$

This yields,

$$\| \max(0, \frac{1}{\gamma_1}\widehat{\lambda} + (u_\gamma)_n - \mathbf{d})\|^2_{\Gamma_C} \to 0 \ \text{ as } \gamma_1 \to \infty, \tag{6.73}$$

since $\frac{1}{\gamma_1}\|\lambda_\gamma\|^2_{\Gamma_C}$ is uniformly bounded with respect to γ_1. On the other hand, $(u_\gamma)_n$ converges to $(\overline{u})_n$ almost everywhere on Γ_C, since $H^{1/2}(\Gamma_C)$ is embedded compactly into $L_2(\Gamma_C)$ and (6.73) implies then $(\overline{u})_n - \mathbf{d} \leq 0$ almost everywhere on Γ_C. Now if we subtract (6.61) from (6.58) and set $\underline{\mathrm{v}} = \underline{u}_\gamma - \underline{u}$ we then obtain

$$(S(\underline{u}_\gamma - \underline{u}), (\underline{u}_\gamma - \underline{u}))_\Gamma = -\langle \lambda_\gamma - \lambda, (u_\gamma - u)_n \rangle_{\Gamma_C} - (\underline{\mu}_\gamma - \underline{\mu}, (\underline{u}_\gamma - \underline{u})_t)_{\Gamma_C}. \tag{6.74}$$

Let us estimate the term

$$-(\underline{\mu}_\gamma - \underline{\mu}, (\underline{u}_\gamma - \underline{u})_t)_{\Gamma_C} = (\underline{\mu}_\gamma - \underline{\mu}, \underline{u}_t - \underline{w}_t)_{\Gamma_C} - (\underline{\mu}_\gamma - \underline{\mu}, (\underline{u}_\gamma)_t - \underline{w}_t)_{\Gamma_C}. \tag{6.75}$$

We obtain from (6.36) that

$$\begin{aligned}(\underline{\mu}_\gamma - \underline{\mu}, \underline{u}_t - \underline{w}_t)_{\Gamma_C} &= (\underline{\mu}_\gamma, \underline{u}_t - \underline{w}_t)_{\Gamma_C} - (\underline{\mu}, \underline{u}_t - \underline{w}_t)_{\Gamma_C} \\ &\leq (|\underline{\mu}_\gamma| - \mathcal{F}g, |\underline{u}_t - \underline{w}_t|)_{\Gamma_C} \leq 0.\end{aligned}$$

Note that for the estimate of the second term of (6.75) we consider two cases, the first one is

$$|\gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}}| \geq \mathcal{F}g \ \text{ which implies } \underline{\mu}_\gamma = \mathcal{F}g \frac{\gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}}}{|\gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}}|}.$$

We have then,

$$\begin{aligned}(\underline{\mu} - \underline{\mu}_\gamma)^\top((\underline{u}_\gamma)_t - \underline{w}_t) &= \frac{1}{\gamma_2}\left(\underline{\mu} - \frac{\mathcal{F}g}{|\gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}}|}(\gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}})\right)^\top \\ &\qquad \left(\gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}} - \underline{\widehat{\mu}}\right) \\ &\leq \frac{1}{\gamma_2}(\mathcal{F}g + |\underline{\mu}|)|\underline{\widehat{\mu}}|.\end{aligned}$$

For the second case we have

$$|\gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}}| < \mathcal{F}g \ \text{ which implies } \underline{\mu}_\gamma = \gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}}.$$

This yields then

$$\begin{aligned}(\underline{\mu} - \underline{\mu}_\gamma)^\top((\underline{u}_\gamma)_t - \underline{w}_t) &= (\underline{\mu} - \gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) - \underline{\widehat{\mu}})^\top((\underline{u}_\gamma)_t - \underline{w}_t)\\ &= -\gamma_2|(\underline{u}_\gamma)_t - \underline{w}_t|^2 + \frac{1}{\gamma_2}(\underline{\mu} - \underline{\widehat{\mu}})^\top(\gamma_2((\underline{u}_\gamma)_t - \underline{w}_t) + \underline{\widehat{\mu}} - \underline{\widehat{\mu}})\\ &\leq \frac{1}{\gamma_2}|\underline{\mu} - \underline{\widehat{\mu}}|(\mathcal{F}g + |\underline{\widehat{\mu}}|).\end{aligned}$$

Hence, combining the above estimates we obtain

$$-(\underline{\mu}_\gamma - \underline{\mu}, (\underline{u}_\gamma - \underline{u})_t)_{\Gamma_C} \leq \frac{1}{\gamma_2}K(\underline{\mu}, \underline{\widehat{\mu}}), \tag{6.76}$$

where $K(\underline{\mu}, \underline{\widehat{\mu}})$ is independent of γ_1, γ_2. By utilizing the ellipticity of S, (6.67) and (6.76) into (6.74) yield

$$\begin{aligned}0 &\leq \limsup_{\gamma_1,\gamma_2\to\infty} c_1^D \|\underline{u}_\gamma - \underline{u}\|^2_{\mathbf{H}^{1/2}(\Gamma)} \leq \lim_{\gamma_1,\gamma_2\to\infty}\left[\langle\lambda, (u_\gamma - u)_n\rangle_{\Gamma_C} + \frac{1}{\gamma_2}K(\mu, \widehat{\mu}) + \frac{1}{2\gamma_1}\|\widehat{\lambda}\|^2_{\Gamma_C}\right]\\ &= \lim_{\gamma_1,\gamma_2\to\infty}\left[\langle\lambda, (u_\gamma)_n - \mathbf{d}\rangle_{\Gamma_C} - \langle\lambda, u_n - \mathbf{d}\rangle_{\Gamma_C}\right]\\ &= \lim_{\gamma_1,\gamma_2\to\infty}\langle\lambda, (u_\gamma)_n - \mathbf{d}\rangle_{\Gamma_C} \quad (\langle\lambda, u_n - \mathbf{d}\rangle_{\Gamma_C} = 0)\\ &= \langle\lambda, (\overline{u})_n - \mathbf{d}\rangle_{\Gamma_C} \leq 0 \quad (\text{from } (6.73)).\end{aligned}$$

It follows from the above estimate that $\underline{u}_\gamma \to \underline{u}$ strongly in $\mathbf{H}^{1/2}(\Gamma)$ and we have $\underline{u} = \overline{u}$. Passing now to the limit in

$$(S\underline{u}_\gamma, \underline{\mathrm{v}})_\Gamma - L(\underline{\mathrm{v}}) + (\lambda_\gamma, \mathrm{v}_n)_{\Gamma_C} + (\underline{\mu}_\gamma, \underline{\mathrm{v}}_t)_{\Gamma_C} = 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V},$$

this yields

$$(S\underline{u}, \underline{\mathrm{v}})_\Gamma - L(\underline{\mathrm{v}}) + (\overline{\lambda}, \mathrm{v}_n)_{\Gamma_C} + (\overline{\mu}, \underline{\mathrm{v}}_t)_{\Gamma_C} = 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}. \tag{6.77}$$

Comparing (6.77) and (6.58) yields $\overline{\lambda} = \lambda$ and $\overline{\mu} = \underline{\mu}$. Hence, every sequence γ_n with $\gamma_n \to \infty$ for $n \to \infty$ contains a subsequence $\gamma_{n_{k_l}}$ such that

$$\underline{u}_{\gamma_{n_{k_l}}} \to \underline{u} \quad \text{in} \quad \mathbf{H}^{1/2}(\Gamma), \quad \lambda_{\gamma_{n_{k_l}}} \rightharpoonup \lambda \quad \text{in} \quad H^{-1/2}(\Gamma_C) \quad \text{and} \quad \underline{\mu}_{\gamma_{n_{k_l}}} \rightharpoonup \underline{\mu} \quad \text{in} \quad \mathbf{L}_2(\Gamma_C).$$

Due to the uniqueness of the solution variables $(\underline{u}, \lambda, \underline{\mu})$, this implies that, the whole family $\{(\underline{u}_\gamma, \lambda_\gamma, \underline{\mu}_\gamma)\}$ converges. □

Linearization of the extremality conditions

In this paragraph we are interested to linearize the equation (6.63),

$$\mathcal{F}g(\gamma_2((\underline{u}_t)_\gamma - \underline{w}_t) + \underline{\widehat{\mu}}) - \max(\mathcal{F}g, |\gamma_2((\underline{u}_t)_\gamma - \underline{w}_t) + \underline{\widehat{\mu}}|)\underline{\mu}_\gamma = 0 \quad \text{on} \quad \Gamma_C, \tag{6.78}$$

in such a way that it is suitable for the application of a semi-smooth Newton method. In the particular case of a two-dimensional domain Ω the expression (6.78) can be significantly simplified. Indeed, we can eliminate the Euclidean norm and the inner product between $(\underline{u}_t)_\gamma$ and $\underline{\mu}_\gamma$. In plane elasticity, all the tangential variables can be written as follows

$$(\underline{u}_t)_\gamma = (u_t)_\gamma \mathbf{t}, \; \underline{\mu}_\gamma = \mu_\gamma \mathbf{t}, \quad \underline{\widehat{\mu}} = \widehat{\mu}\mathbf{t}, \text{ and } \underline{w}_t = w_t \mathbf{t}$$

where $\mathbf{t}$ denotes the unit outward tangential vector rotated in the mathematically positive direction, $(u_t)_\gamma, w_t \in H^{1/2}(\Gamma_C)$, and $\mu_\gamma, \widehat{\mu} \in L_2(\Gamma_C)$. By using this, (6.78) can be written again

$$\mathcal{F}g(\gamma_2((u_t)_\gamma - w_t) + \widehat{\mu}) - \max(\mathcal{F}g, |\gamma_2((u_t)_\gamma - w_t) + \widehat{\mu}|)\mu_\gamma = 0, \tag{6.79}$$

where the symbol $|.|$ here stands for the absolute value. Hence, (6.79) can be written equivalently as follows

$$\begin{aligned} \gamma_2((u_t)_\gamma - w_t) + \widehat{\mu} - \mu_\gamma - \max(0, \gamma_2((u_t)_\gamma - w_t) + \widehat{\mu} - \mathcal{F}g) \\ - \min(0, \gamma_2((u_t)_\gamma - w_t) + \widehat{\mu} + \mathcal{F}g) = 0. \end{aligned} \tag{6.80}$$

Indeed, this is easily shown if we distinguish the cases

$$|\gamma_2((u_t)_\gamma - w_t) + \widehat{\mu}| \leq \mathcal{F}g \quad \text{and} \quad \gamma_2((u_t)_\gamma - w_t) + \widehat{\mu} \begin{cases} \geq \mathcal{F}g, \\ \leq -\mathcal{F}g. \end{cases}$$

Furthermore, if we introduce the Lagrange multiplier $\xi_\gamma \in L_2(\Gamma_C)$ associated to the constraint $|\mu_\gamma| \leq \mathcal{F}g$, (6.80) is then equivalent to

$$\begin{cases} \gamma_2(\xi_\gamma - (u_t)_\gamma + w_t) + \mu_\gamma - \widehat{\mu} = 0, \\ \xi_\gamma - \max(0, \xi_\gamma + \sigma(\mu_\gamma - \mathcal{F}g)) - \min(0, \xi_\gamma + \sigma(\mu_\gamma + \mathcal{F}g)) = 0 \end{cases} \tag{6.81}$$

for arbitrary $\sigma > 0$. Note that (6.80) is obtained from (6.81) by setting $\sigma = \frac{1}{\gamma_2}$ and substituting the above line of (6.81) into the lower. Hence, the linearized extremality conditions for the regularized problem are written as

$$(S\underline{u}_\gamma, \underline{\mathrm{v}})_\Gamma - L(\mathrm{v}) + (\mu_\gamma, \mathrm{v}_t)_{\Gamma_C} + (\lambda_\gamma, \mathrm{v}_n)_{\Gamma_C} = 0 \quad \text{for all} \quad \mathrm{v} \in \mathcal{V}, \tag{6.82}$$

$$\lambda_\gamma - \max(0, \widehat{\lambda} + \gamma_1((u_n)_\gamma - \mathbf{d})) = 0 \quad \text{on} \quad \Gamma_C, \tag{6.83}$$

$$\begin{cases} \gamma_2(\xi_\gamma - (u_t)_\gamma + w_t) + \mu_\gamma - \widehat{\mu} = 0, \\ \xi_\gamma - \max(0, \xi_\gamma + \sigma(\mu_\gamma - \mathcal{F}g)) - \min(0, \xi_\gamma + \sigma(\mu_\gamma + \mathcal{F}g)) = 0. \end{cases} \tag{6.84}$$

6.4.2 The semi-smooth approach for the linearized extremality conditions

In this section, for the sake of clarity of our presentation we drop the subscript γ on the variables $\underline{u}$, λ and μ. Considering in (6.83) and (6.84) the variable $\underline{u}$ as a function of λ and μ, we observe that u_n and u_t are smoother than λ and μ. This property is necessary for the semi-smoothness of the max-function and min-function (see Theorem 2.9). In the original problem (6.59) and (6.60) due to the explicit appearance of λ and μ inside the max-function we could not expect the smoothness required for the Newton differentiability. We now focus on the presentation of the generalized Newton algorithm for the solutions $(\underline{u}, \lambda, \mu) \in \mathbf{H}^{1/2}(\Gamma) \times L_2(\Gamma_C) \times L_2(\Gamma_C)$ of the regularized problem (6.82)-(6.84). Since (6.82) is linear, the derivation of the algorithm for this equation is obvious. Therefore, we confine our attention here on the second and third equations (6.83) and (6.84) respectively. To this end, let us define the mapping $F : L_2(\Gamma_C) \times L_2(\Gamma_C) \to L_2(\Gamma_C) \times L_2(\Gamma_C)$ such that

$$F(\lambda,\mu) := \begin{pmatrix} \lambda - \max(0, \widehat{\lambda} + \gamma_1(u_n(\lambda,\mu) - \mathbf{d})) \\ \gamma_2(u_t(\lambda,\mu) - w_t) + \widehat{\mu} - \mu - \max(0, \gamma_2(u_t(\lambda,\mu) - w_t) + \widehat{\mu} - \mathcal{F}g) \\ - \min(0, \gamma_2(u_t(\lambda,\mu) - w_t) + \widehat{\mu} + \mathcal{F}g) \end{pmatrix}, \tag{6.85}$$

where $\underline{u}(\lambda,\mu) \in \mathbf{H}^{1/2}(\Gamma)$ is the unique solution of (6.82) for given $\lambda, \mu \in L_2(\Gamma_C)$. We have $u_n(\lambda,\mu) \in H^{1/2}(\Gamma_C)$ and $u_t(\lambda,\mu) \in H^{1/2}(\Gamma_C)$. In addition, $H^{1/2}(\Gamma_C)$ imbeds continuously and compactly into $L_q(\Gamma_C)$ for all $q < \infty$ when $\Omega \subset \mathbb{R}^2$. Thus, $u_n(\lambda,\mu) \in L_q(\Gamma_C)$, and $u_t(\lambda,\mu) \in L_q(\Gamma_C)$ for some $q > 2$, and we obtain the requirement for Newton differentiability of the max-function and min-function (see Theorem 2.9). The generalized derivative of F is defined as follows

$$G(\lambda,\mu)\begin{pmatrix} \delta\lambda \\ \delta\mu \end{pmatrix} := \begin{pmatrix} \delta\lambda - \gamma_1 \chi_{A_C} \delta u_n(\lambda,\mu) \\ -\delta\mu + \gamma_2 \delta u_t(\lambda,\mu) - \gamma_2 \chi_{A_{F+}} \delta u_t(\lambda,\mu) - \gamma_2 \chi_{A_{F-}} \delta u_t(\lambda,\mu) \end{pmatrix}, \tag{6.86}$$

where χ_{A_C}, $\chi_{A_{F+}}$ and $\chi_{A_{F-}}$ are the characteristic functions of the sets

$$A_C := \{x \in \Gamma_C : \widehat{\lambda}(x) + \gamma_1(u_n(\lambda,\mu) - \mathbf{d})(x) > 0\},$$
$$A_{F+} := \{x \in \Gamma_C : \widehat{\mu}(x) + (\gamma_2(u_t(\lambda,\mu) - w_t) - \mathcal{F}g)(x) > 0\},$$

and

$$A_{F-} := \{x \in \Gamma_C : \widehat{\mu}(x) + (\gamma_2(u_t(\lambda,\mu) - w_t) + \mathcal{F}g)(x) < 0\},$$

respectively. Thus, the Newton step is then

$$G(\lambda^k, \mu^k)(\delta\lambda, \delta\mu)^\top = -F(\lambda^k, \mu^k), \tag{6.87}$$

which yields

$$\lambda^{k+1} = \widehat{\lambda} + \gamma_1(u_n^{k+1} - \mathbf{d}) \;\text{ on }\; A_C^{k+1},\; \lambda^{k+1} = 0 \;\text{ on }\; I_C^{k+1},$$
$$\mu^{k+1} = \mathcal{F}g \;\text{ on }\; A_{F+}^{k+1},\; \mu^{k+1} = -\mathcal{F}g \;\text{ on }\; A_{F-}^{k+1},$$
$$\mu^{k+1} - \widehat{\mu} - \gamma_2(u_t^{k+1} - w_t) = 0 \;\text{ on }\; I_{F-}^{k+1},$$

where

$$\begin{aligned}
A_C^{k+1} &:= \{x \in \Gamma_C : \widehat{\lambda}(x) + \gamma_1(u_n^k(x) - \mathbf{d}(x)) > 0\}, \quad I_C^{k+1} := \Gamma_C \setminus A_C^{k+1}, \\
A_{F+}^{k+1} &:= \{x \in \Gamma_C : \widehat{\mu}(x) + (\gamma_2(u_t^k - w_t) - \mathcal{F}g)(x) > 0\}, \\
A_{F-}^{k+1} &:= \{x \in \Gamma_C : \widehat{\mu}(x) + (\gamma_2(u_t^k - w_t) + \mathcal{F}g)(x) < 0\}, \\
I_F^{k+1} &:= \Gamma_C \setminus (A_{F+}^{k+1} \cup A_{F-}^{k+1}),
\end{aligned}$$

and $(\lambda^{k+1}, \mu^{k+1})$, $\underline{u}(\lambda^{k+1}, \mu^{k+1}) := \underline{u}^{k+1}$ are solutions to

$$(S\underline{u}^{k+1}, \underline{\mathrm{v}})_\Gamma - L(\underline{\mathrm{v}}) + (\lambda^{k+1}, \mathrm{v}_n)_{\Gamma_C} + (\mu^{k+1}, \mathrm{v}_t)_{\Gamma_C} = 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}. \tag{6.88}$$

The semi-smooth approach for the system of equations (6.82)-(6.84) turns out to be the active set strategy algorithm if we set $\sigma = \frac{1}{\gamma_2}$, see, e.g., [61,71,98], and given as follows:

Algorithm : (SSN)

(1) Choose $(\xi^0, \lambda^0, \mu^0, \underline{u}^0) \in L_2(\Gamma_C) \times L_2(\Gamma_C) \times L_2(\Gamma_C) \times \mathbf{H}^{1/2}(\Gamma)$ satisfying (6.82), $\sigma > 0$ and set $k := 0$.

(2) Determine active and inactive sets

$$\begin{aligned}
A_C^{k+1} &:= \{x \in \Gamma_C : \widehat{\lambda}(x) + \gamma_1(u_n^k(x) - \mathbf{d}(x)) > 0\}, \\
I_C^{k+1} &:= \Gamma_C \setminus A_C^{k+1}, \\
A_{F+}^{k+1} &:= \{x \in \Gamma_C : \xi^k(x) + \sigma(\mu^k - \mathcal{F}g)(x) > 0\}, \\
A_{F-}^{k+1} &:= \{x \in \Gamma_C : \xi^k(x) + \sigma(\mu^k + \mathcal{F}g)(x) < 0\}, \\
I_F^{k+1} &:= \Gamma_C \setminus (A_{F+}^{k+1} \cup A_{F-}^{k+1}).
\end{aligned}$$

(3) for $k \geq 1$ if $A_C^{k+1} = A_C^k$, $A_{F+}^{k+1} = A_{F+}^k$ and $A_{F-}^{k+1} = A_{F-}^k$ stop, else

(4) solve

$$\begin{aligned}
(S\underline{u}^{k+1}, \underline{\mathrm{v}})_\Gamma - L(\underline{\mathrm{v}}) + (\lambda^{k+1}, \mathrm{v}_n)_{\Gamma_C} + (\mu^{k+1}, \mathrm{v}_t)_{\Gamma_C} = 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}, \\
\lambda^{k+1} = \widehat{\lambda} + \gamma_1(u_n^{k+1} - \mathbf{d}) \ \text{ on } \ A_C^{k+1}, \ \lambda^{k+1} = 0 \ \text{ on } \ I_C^{k+1}, \\
\mu^{k+1} = \mathcal{F}g \ \text{ on } \ A_{F+}^{k+1}, \ \mu^{k+1} = -\mathcal{F}g \ \text{ on } \ A_{F-}^{k+1}, \\
\mu^{k+1} - \widehat{\mu} - \gamma_2(u_t^{k+1} - w_t) = 0 \ \text{ on } \ I_F^{k+1},
\end{aligned}$$

(5) set

$$\xi^{k+1} := \begin{cases} u_t^{k+1} - w_t + \frac{1}{\gamma_2}(\widehat{\mu} + \mathcal{F}g) \ \text{ on } \ A_{F-}^{k+1}, \\ u_t^{k+1} - w_t + \frac{1}{\gamma_2}(\widehat{\mu} - \mathcal{F}g) \ \text{ on } \ A_{F+}^{k+1}, \\ 0 \ \text{ on } \ I_F^{k+1}, \end{cases}$$

and set $k := k+1$ and go to Step (2).

Remark that the system at Step (4) is uniquely solvable, since it is the necessary and sufficient optimality condition for the following minimization problem

$$\min_{\substack{\lambda=0 \text{ on } I_C^{k+1}, \\ \mu=\mathcal{F}g \text{ on } A_{F+}^{k+1},\ \mu=-\mathcal{F}g \text{ on } A_{F-}^{k+1}}} J^*_{\gamma_1,\gamma_2}(\lambda,\mu),$$

which has a unique solution. The advantage of this algorithm is that it solves both for contact and friction simultaneously.

The properties of the semi-smooth Newton method or equivalently of the primal dual active set strategy are analyzed next.

Proposition 6.5. *If the algorithm (SSN) stops, that is $A_C^{k+1} = A_C^k$, $A_{F+}^{k+1} = A_{F+}^k$ and $A_{F-}^{k+1} = A_{F-}^k$, then $\underline{u}^k$ is the solution to $(\mathcal{P}_{\gamma_1,\gamma_2})$ and (λ^k,μ^k) are solutions to $(\mathcal{P}^*_{\gamma_1,\gamma_2})$.*

Proof. Note that all the iterates $(\underline{u}^k,\lambda^k,\mu^k)$ satisfy (6.82). If $A_C^{k+1} = A_C^k$, $A_{F+}^{k+1} = A_{F+}^k$ and $A_{F-}^{k+1} = A_{F-}^k$ we then obtain from the uniqueness of the solution for the system at Step (4) that $\underline{u}^{k+1} = \underline{u}^k$, $\lambda^{k+1} = \lambda^k$ and $\mu^{k+1} = \mu^k$. It follows from $\lambda^{k+1} = \lambda^k$ that $\lambda^k > 0$ on $A_C^{k+1} = A_C^k$ and $\lambda^k = 0$ on $I_C^{k+1} = I_C^k$ and hence

$$\lambda^k = \max(0,\lambda^k) = \max(0,\widehat{\lambda} + \gamma_1(u_n^k - \mathbf{d})),$$

this shows that $(\underline{u}^k,\lambda^k)$ also satisfy (6.83). On the other hand, $\mu^{k+1} = \mu^k$ yields $\mu^k = \mathcal{F}g$ on $A_{F+}^{k+1} = A_{F+}^k$, $\mu^k = -\mathcal{F}g$ on $A_{F-}^{k+1} = A_{F-}^k$, $\mu^k - \widehat{\mu} - \gamma_2(u_t^k - w_t) = 0$ on $I_F^{k+1} = I_F^k$ and

$$\xi^k := \begin{cases} u_t^k - w_t + \frac{1}{\gamma_2}(\widehat{\mu} + \mathcal{F}g) \text{ on } A_{F-}^{k+1} = A_{F-}^k, \\ u_t^k - w_t + \frac{1}{\gamma_2}(\widehat{\mu} - \mathcal{F}g) \text{ on } A_{F+}^{k+1} = A_{F+}^k, \\ 0 \text{ on } I_F^{k+1} = I_F^k, \end{cases}$$

which yield the complementarity condition (6.84). □

On the other hand, the local superlinear convergence of the algorithm is ensured by the following results:

Theorem 6.5. *If there exists a constant $g_0 > 0$ such that $\mathcal{F}g \geq g_0$, further for all $\gamma_1,\gamma_2 > 0$ if $\sigma \geq \frac{1}{\gamma_2}$, and if $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_C}$ and $\|\mu^0 - \mu_\gamma\|_{\Gamma_C}$ are sufficiently small. Then for all $\widehat{\lambda} \in L_2(\Gamma_C)$ and $\widehat{\mu} \in L_2(\Gamma_C)$, $(\underline{u}^k,\lambda^k,\mu^k,\xi^k)$ converge to $(\underline{u}_\gamma,\lambda_\gamma,\mu_\gamma,\xi_\gamma)$ superlinearly in $\boldsymbol{H}^{1/2}(\Gamma)\times L_2(\Gamma_C)\times L_2(\Gamma_C)\times L_2(\Gamma_C)$.*

Proof. See [98, Theorem 5.8]. □

6.5 Regularization of contact problems with Coulomb friction

The regularized contact problem with Coulomb friction presented in this section is similar to the one given in [28–30]. Moreover, the regularized problems with given friction correspond to $(\mathcal{P}_{\gamma_1,\gamma_2})$ and $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ respectively. The variational problem is: Find $\underline{u} \in \mathcal{V}$ such that

$$(S\underline{u}, \underline{\mathrm{v}} - \underline{u})_\Gamma + (\max(0, \widehat{\lambda} + \gamma_1(u_n - \mathbf{d})), (\mathrm{v}_n - u_n))_{\Gamma_C} - L(\underline{\mathrm{v}} - \underline{u}) + \int_{\Gamma_C} \mathcal{F} \max(0, \widehat{\lambda} + \gamma_1(u_n - \mathbf{d}))(h((\underline{\mathrm{v}}_t - \underline{w}_t), \underline{\widehat{\mu}}) - h((\underline{u}_t - \underline{w}_t), \underline{\widehat{\mu}}))ds_x \geq 0 \ \forall \underline{\mathrm{v}} \in \mathcal{V}, \quad (6.89)$$

where $h(,) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the local regularization defined by

$$h(x,y) := \begin{cases} |x + \frac{y}{\gamma_2}| - \frac{1}{2\gamma_2}\mathcal{F}g & \text{if} \quad |x + \frac{y}{\gamma_2}| \geq \frac{\mathcal{F}g}{\gamma_2}, \\ \frac{\gamma_2}{2\mathcal{F}g}|x + \frac{y}{\gamma_2}|^2 & \text{if} \quad |x + \frac{y}{\gamma_2}| < \frac{\mathcal{F}g}{\gamma_2}. \end{cases}$$

Furthermore, the normal contact stress is given by $\lambda_\gamma = \max(0, \widehat{\lambda} + \gamma_1(u_n - \mathbf{d}))$. Due to the fact that $h(.,.)$ is differentiable with respect to the first variable if we set the test function in (6.89) $\tilde{\underline{\mathrm{v}}} := \underline{u} \pm \alpha \underline{\mathrm{v}}$, further divide (6.89) by α and let $\alpha \to 0$, we then obtain the variational equality: Find $\underline{u} \in \mathcal{V}$ such that for all $\underline{\mathrm{v}} \in \mathcal{V}$

$$(S\underline{u}, \underline{\mathrm{v}})_\Gamma + (\max(0, \widehat{\lambda} + \gamma_1(u_n - \mathbf{d})), \mathrm{v}_n)_{\Gamma_C} + \int_{\Gamma_C} \mathcal{F} \max(0, \widehat{\lambda} + \gamma_1(u_n - \mathbf{d})) Dh((\underline{u}_t - \underline{w}_t), \underline{\widehat{\mu}})\underline{\mathrm{v}}_t ds_x = L(\underline{\mathrm{v}}), \quad (6.90)$$

where $Dh(.,.)$ is the derivative of h with respect to the first variable. On the other hand, we have

$$\begin{aligned} Dh((\underline{u}_t - \underline{w}_t), \underline{\widehat{\mu}})(\underline{\mathrm{v}}_t - \underline{u}_t) &:= \lim_{\alpha \to 0} \frac{h((\underline{u}_t - \underline{w}_t) + \alpha(\underline{\mathrm{v}}_t - \underline{u}_t), \underline{\widehat{\mu}}) - h((\underline{u}_t - \underline{w}_t), \underline{\widehat{\mu}})}{\alpha} \\ &\leq h((\underline{\mathrm{v}}_t - \underline{w}_t), \underline{\widehat{\mu}}) - h((\underline{u}_t - \underline{w}_t), \underline{\widehat{\mu}}), \end{aligned} \quad (6.91)$$

since h is convex. Conversely, if we set the test function in (6.90) $\tilde{\underline{\mathrm{v}}} := \underline{\mathrm{v}} - \underline{u}$ and utilize the relation (6.91), we then obtain the variational inequality (6.89) and the following result.

Lemma 6.2. *The variational inequality (6.89) is equivalent to the variational equality (6.90).*

6.5.1 Existence of solutions for the regularized contact problem with Coulomb friction

Solutions of the regularized contact problem with Coulomb friction (6.89) or (6.90) can be obtained similarly as in the original contact problem in section 2 by the means of a sequence of regularized Tresca friction problems. To this end let us introduce the cone of non-negative L_2-functions

$$L^2_+(\Gamma_C) := \{g \in L_2(\Gamma_C) : g \geq 0 \text{ a.e.}\}$$

and define the mapping $\Phi_\gamma : L^2_+(\Gamma_C) \to L^2_+(\Gamma_C)$ by $\Phi_\gamma(g) = \lambda_\gamma$, where

$$\lambda_\gamma = \max(0, \widehat{\lambda} + \gamma_1((u_\gamma)_n - \mathbf{d})),$$

and $\underline{u}_\gamma$ is the unique solution of the regularized contact problem with given friction g. Naturally, a fixed point of Φ_γ solves (6.89) or equivalently (6.90). Furthermore, the mapping Φ_γ can be written as the composition of three mappings as follows: $\Phi_\gamma := \Upsilon \circ \Theta \circ \Psi_\gamma$, where $\Psi_\gamma : L^2_+(\Gamma_C) \to \mathbf{H}^{1/2}(\Gamma)$ is defined by $\Psi_\gamma(g) = \underline{u}_\gamma$, $\Theta : \mathbf{H}^{1/2}(\Gamma) \to L_2(\Gamma_C)$ is defined by $\Theta(\underline{u}_\gamma) = (u_\gamma)_n$, and $\Upsilon : L_2(\Gamma_C) \to L^2_+(\Gamma_C)$ is defined by $\Upsilon((u_\gamma)_n) = \max(0, \widehat{\lambda} + \gamma_1((u_\gamma)_n - \mathbf{d}))$. We then obtain the following results:

Lemma 6.3. *For all $\gamma_1, \gamma_2 > 0$, and $\widehat{\lambda} \in L_2(\Gamma_C)$, $\underline{\widehat{\mu}} \in \boldsymbol{L}_2(\Gamma_C)$ the mapping Ψ_γ defined above is Lipschitz-continuous with constant*

$$\mathcal{L} = \frac{\|\mathcal{F}\|_\infty c_1}{c_1^D}, \tag{6.92}$$

where $\|\mathcal{F}\|_\infty$ denotes the essential supremum of $\mathcal{F}$, c_1^D the ellipticity constant of S. Moreover, $\mathcal{L}$ is independent of the regularized parameters γ_1, γ_2.

Proof. Let us fix $\gamma_1, \gamma_2 > 0$ and choose the given frictions $g_1, g_2 \in L^2_+(\Gamma_C)$. Further, let $(\underline{u}_1, \lambda_1, \underline{\mu}_1)$ and $(\underline{u}_2, \lambda_2, \underline{\mu}_2)$ be solutions associated to g_1 and g_2 respectively. Taking $(\underline{u}_1, \lambda_1, \underline{\mu}_1)$ and $(\underline{u}_2, \lambda_2, \underline{\mu}_2)$ respectively into (6.61), subtracting and setting $\underline{v} := \underline{u}_1 - \underline{u}_2$ yields

$$(S(\underline{u}_1 - \underline{u}_2), \underline{u}_1 - \underline{u}_2)_\Gamma + (\underline{\mu}_1 - \underline{\mu}_2, (\underline{u}_1 - \underline{u}_2)_t)_{\Gamma_C} + (\lambda_1 - \lambda_2, (u_1 - u_2)_n)_{\Gamma_C} = 0. \tag{6.93}$$

Since $\lambda = \max(0, \widehat{\lambda} + \gamma_1(u_n - \mathbf{d}))$, we obtain

$$(\lambda_1 - \lambda_2, (u_1 - u_2)_n)_{\Gamma_C} = \frac{1}{\gamma_1}\Big(\lambda_1 - \lambda_2, (\widehat{\lambda} + \gamma_1((u_1)_n - \mathbf{d})) - (\widehat{\lambda} + \gamma_1((u_2)_n - \mathbf{d}))\Big)_{\Gamma_C} \geq 0.$$

As immediate consequence we obtain

$$(S(\underline{u}_1 - \underline{u}_2), \underline{u}_1 - \underline{u}_2)_\Gamma \leq (\underline{\mu}_1 - \underline{\mu}_2, (\underline{u}_2 - \underline{u}_1)_t)_{\Gamma_C}. \tag{6.94}$$

Furthermore, we have

$$\begin{aligned}(\underline{\mu}_1 - \underline{\mu}_2, (\underline{u}_2 - \underline{u}_1)_t)_{\Gamma_C} &\leq \|\mathcal{F}(g_1 - g_2)\|_{\Gamma_C} \|(\underline{u}_1 - \underline{u}_2)_t\|_{\Gamma_C} \\ &\leq c_1 \|\mathcal{F}\|_\infty \|g_1 - g_2\|_{\Gamma_C} \|\underline{u}_1 - \underline{u}_2\|_{\mathbf{H}^{1/2}(\Gamma)},\end{aligned}$$

see [98, p.118]. By using this and the ellipticity of S the lemma is proved. □

Lemma 6.4. *For all $\gamma_1, \gamma_2 > 0$, and $\widehat{\lambda} \in L_2(\Gamma_C)$, $\widehat{\mu} \in L_2(\Gamma_C)$ the mapping Φ_γ defined above is compact and Lipschitz-continuous with the constant*

$$\mathcal{L} = \frac{\gamma_1 c}{c_1^D} \|\mathcal{F}\|_\infty.$$

Proof. Since we have for all $\eta_1, \eta_2 \in L_2(\Gamma_C)$,

$$\|\max(0, \widehat{\lambda} + \gamma_1(\eta_1 - \mathbf{d})) - \max(0, \widehat{\lambda} + \gamma_1(\eta_2 - \mathbf{d}))\|_{\Gamma_C} \leq \gamma_1 \|\eta_1 - \eta_2\|_{\Gamma_C}, \qquad (6.95)$$

the mapping Υ is Lipschitz continuous with constant γ_1. Further, it is easy to check that the mapping Θ is compact and linear. Therefore, it is Lipschitz continuous with a constant c_2. By using Lemma 6.3, and the fact that Θ and Υ are Lipschitz continuous we conclude that $\Phi_\gamma := \Upsilon \circ \Theta \circ \Psi_\gamma$ is Lipschitz continuous with the constant

$$\mathcal{L} := \frac{\gamma_1 c_1 c_2}{c_1^D} \|\mathcal{F}\|_\infty.$$

Since Θ is compact, the composition of Θ and Ψ_γ is compact too. Further, from inequality (6.95), any L_2-convergent sequence remains L_2-convergent under the mapping Υ which ends the proof. □

We can easily obtain the existence of the fixed point to Φ_γ as follows.

Proposition 6.6. *The mapping Φ_γ admits at least one fixed point. Further, if $\|\mathcal{F}\|_\infty$ is sufficiently small, the fixed point is then unique.*

Proof. We apply the Leray-Schauder fixed point theorem [12] to the mapping $\Phi_\gamma : L_+^2(\Gamma_C) \to L_+^2(\Gamma_C)$. By using Lemma 6.4 it suffices to show that λ is bounded in $L_2(\Gamma_C)$ independently of g. This is clear if one takes in account the dual problem $(\mathcal{P}^*_{\gamma_1,\gamma_2})$. Indeed,

$$\min_{\lambda \geq 0, |\underline{\mu}| \leq \mathcal{F}g} J^*_{\gamma_1,\gamma_2}(\lambda, \underline{\mu}) \leq \min_{\lambda \geq 0} J^*_{\gamma_1,\gamma_2}(\lambda, 0) < \infty.$$

Note that the second minimization problem leads to a contact problem without friction, since $\underline{\mu} = 0$, and it admits a solution independent from $\mathcal{F}g$. Thus, the Leray-Schauder fixed point theorem yields the existence of the fixed point for the mapping Φ_γ. Furthermore, if the friction coefficient $\mathcal{F}$ satisfies

$$\mathcal{L} := \frac{\gamma_1 c_1 c_2}{c_1^D} \|\mathcal{F}\|_\infty < 1,$$

the mapping Φ_γ is then a contraction and has a unique fixed point. Therefore, the regularized contact problem with Coulomb friction has a unique solution. □

6.5.2 Algorithm for the solution of the regularized Coulomb friction problem (RCF)

The fixed point idea presented in Proposition 6.6 can be exploited for the numerical implementation for the solution of the regularized Coulomb friction problem. Similar idea are frequently used by a sequence of Tresca friction problems toward the solution of the Coulomb friction problem, see, e.g., [29, 30, 41]. The fixed point algorithm can be presented as follows:

Algorithm : (RCF-FP)

(1) Choose $\gamma_1, \gamma_2 > 0$, $\widehat{\lambda}$ and $\underline{\widehat{\mu}}$. Initialize $g^0 \in L^2_+(\Gamma_C)$, and set $m := 0$.

(2) Determine the solution $(\lambda^m, \underline{\mu}^m)$ to problem $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ with given friction g^m.

(3) Update $g^{m+1} := \lambda^m$, $m := m+1$ and, unless an appropriate stopping criterion is met, go to Step (2).

Theorem 6.6. *Provided that $\|\mathcal{F}\|_\infty$ is sufficiently small, the Algorithm (RCF-FP) converges regardless of the initialization.*

Proof. The proof follows immediately from the fact that if $\|\mathcal{F}\|_\infty$ is sufficiently small, the mapping Φ_γ is then a contraction and the fixed point is unique. □

6.6 BEM discretization

In this section we describe the standard Galerkin procedure to solve the system of equations at Step (4) of the active set strategy algorithm numerically with the help of one-periodic B-splines of order $\nu \geq 0$. The description of the Galerkin procedure in this section is the same we presented in chapter 4, we then refer the reader to section 4.2.

But, note that the Steklov-Poincaré operator defined as follows

$$S := (\frac{1}{2}I + \mathbf{K}')\mathbf{V}^{-1}(\frac{1}{2}I + \mathbf{K}) + \mathbf{D} : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\Gamma) \tag{6.96}$$

is not suitable for a Galerkin discretization. This is due to the explicit appearance of the inverse single layer operator, which is not available explicitly. As an alternative we define a symmetric approximation of the continuous Steklov-Poincaré operator as follows, for some given function $\underline{v} \in \mathbf{H}^{1/2}(\Gamma)$

$$S\underline{v} = (\frac{1}{2}I + \mathbf{K}')\mathbf{V}^{-1}(\frac{1}{2}I + \mathbf{K})\underline{v} + \mathbf{D}\underline{v} = (\frac{1}{2}I + \mathbf{K}')\underline{w} + \mathbf{D}\underline{v},$$

where $\underline{w} \in \mathbf{H}^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle \mathbf{V}\underline{w}, \underline{\tau} \rangle = \langle (\frac{1}{2}I + \mathbf{K})\underline{v}, \underline{\tau} \rangle, \forall \underline{\tau} \in \mathbf{H}^{-1/2}(\Gamma).$$

The associated Galerkin variational formulation is: find $\underline{w}_h \in \left(\mathrm{Span}\{\phi_k^0\}_{k=1}^N\right)^2 \subset \mathbf{H}^{-1/2}(\Gamma)$ such that

$$\langle \mathbf{V}\underline{w}_h, \underline{\tau}_h \rangle = \langle (\frac{1}{2}I + \mathbf{K})\underline{v}, \underline{\tau}_h \rangle, \forall \underline{\tau}_h \in \left(\mathrm{Span}\{\phi_k^0\}_{k=1}^N\right)^2,$$

where ϕ_k^0 represent the B-splines basis functions of order zero. Therefore the approximation of the Steklov-Poincaré operator can be given by

$$\tilde{S}\underline{v} = \mathbf{D}\underline{v} + (\frac{1}{2}I + \mathbf{K}')\underline{w}_h.$$

Note that the results of Lemma 4.4, for the scalar Yukawa problem are still valid here, i.e. the approximation $\tilde{S}$ of the Steklov-Poincaré operator is spectrally equivalent to S in the sense that for all $\underline{v} \in \mathbf{H}^{1/2}(\Gamma)$ we have:

$$c_1^D \|\underline{v}\|^2_{\mathbf{H}^{1/2}(\Gamma)} \le \langle \tilde{S}\underline{v}, \underline{v} \rangle \le c_2^{\tilde{S}} \|\underline{v}\|^2_{\mathbf{H}^{1/2}(\Gamma)}$$

and furthermore, we have the consistency estimate:

$$\|(\tilde{S} - S)\underline{v}\|_{\mathbf{H}^{-1/2}(\Gamma)} \le ch^{\alpha+1/2}\|S\underline{v}\|_{\mathbf{H}^{\alpha}(\Gamma)} \quad \text{for all} \quad \underline{v} \in \mathcal{V}_\alpha,$$

where $\mathcal{V}_\alpha := \{\underline{v} \in \mathbf{H}^\alpha(\Gamma) : \underline{v} = 0 \quad \text{on} \quad \Gamma_D\}$ and $\frac{1}{2} \le \alpha \le 1$. The Galerkin discretization of the approximated Steklov-Poincaré operator $\tilde{S}$ is now given by

$$\tilde{S}_h := (\frac{1}{2}\mathbf{M}_h^\top + \mathbf{K}_h^\top)\mathbf{V}_h^{-1}(\frac{1}{2}\mathbf{M}_h + \mathbf{K}_h) + \mathbf{D}_h, \tag{6.97}$$

where $\mathbf{M}_h$ is the so-called mass matrix [30], given by

$$\mathbf{M}_h = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix},$$

where

$$M = \left(\int_\Gamma \phi_j^1 \cdot \phi_i^0 ds \right)_{i,j=1}.$$

On the other hand, the single layer operator can be written in matrix form as follows

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix}.$$

Therefore, the matrix of the discrete single layer operator $\mathbf{V}_h := \mathbb{A}$ is a square matrix of size $2N$ where N is the number of points of discretization. Moreover, $\mathbb{A}$ has the following block structure

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_{11} & \mathbb{A}_{12} \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix},$$

where

$$\mathbb{A}_{kl}[i,j] = \left\langle \mathbf{V}_{kl}\phi_j^\nu, \phi_i^\nu \right\rangle_\Gamma \quad \text{for } i,j = 1,...,N; \quad k,l = 1,2.$$

By utilizing the one-periodic parametrization of the boundary Γ, the definitions of the boundary integral operators $\mathbf{V}_{kl}$, $k,l = 1,2$ given in section 5.1.3 and Proposition 5.1 the entries of the block structure matrices $\mathbb{A}_{kl}$ for $k,l = 1,2$ are given by the following proposition.

Proposition 6.7. *The matrices $\mathbb{A}_{kl}$ for $k,l = 1,2$ can be written:*

$$\begin{aligned} \mathbb{A}_{11}[i,j] &= \mathbb{A}_0[i,j] + \mathbb{A}_c[i,j], \quad \mathbb{A}_{12}[i,j] \equiv \mathbb{A}_{21}[i,j] = \mathbb{A}_s[i,j], \\ \mathbb{A}_{22}[i,j] &= \mathbb{A}_0[i,j] - \mathbb{A}_c[i,j], \end{aligned}$$

where

$$\begin{aligned} \mathbb{A}_0[i,j] &= \frac{1}{s^2}\left\{ \frac{1}{2}\Lambda_0(c_1^{(\nu)}(0))^2 + \sum_{p>0}^{\infty} \Lambda_p(c_1^{(\nu)}(p))^2 \cos 2\pi[p(j-i)h] \right\}, \\ \mathbb{A}_c[i,j] &= \frac{1}{s^2}\left\{ \Gamma_{0,1,1}(c_1^{(\nu)}(0))^2 + 2\sum_{p>0}^{\infty} \Gamma_{p,p-1,p+1}(c_1^{(\nu)}(p))^2 \cos[2\pi p(j-i)h] \right\} \\ &+ \mathbb{A}_{11}^2[i,j], \end{aligned}$$

and $c_1^{(\nu)}(.)$ are the Fourier coefficients defined in section 4.2.

Remark 6.2.

- *The matrix $\mathbb{A}_0$ and the first part of the matrix $\mathbb{A}_c$ are symmetric and circulant [23, 92, 94].*

- *The matrices* $\mathbb{A}_{11}^2$ *and* $\mathbb{A}_s$ *are also symmetric, and are derived from the regular part of the operator* $\boldsymbol{V}_{11}$ *and of the operator* $\boldsymbol{V}_{12}$ *respectively by*

$$\mathbb{A}_{11}^2[i,j] = \left\langle \boldsymbol{V}_{11}^2 \phi_j^\nu, \phi_i^\nu \right\rangle_\Gamma, \quad \mathbb{A}_s[i,j] = \left\langle \boldsymbol{V}_{12} \phi_j^\nu, \phi_i^\nu \right\rangle_\Gamma \quad \textit{for } i,j = 1,\dots N.$$

Hence, they can be computed by utilizing a Gauss quadrature.

Moreover, the eigenvalues of the circulant matrix $\mathbb{A}_0$ are given as follows:

Proposition 6.8. *The matrix* $\mathbb{A}_0$ *is circulant, symmetric and positive definite, and its eigenvalues are given by:*

$$\lambda_m = \begin{cases} \frac{h}{2s^2}\Lambda_0(k_1,k_2) & \textit{if} \quad m=1, \\ \frac{h}{2s^2}\left(\frac{\sin\pi s'}{\pi}\right)^{2\nu+2}\left[\sum_{p=0}^{\infty}\frac{\Lambda_{(m+pN-1)}(k_1,k_2)}{(p+s')^{2\nu+2}} + \sum_{p=1}^{\infty}\frac{\Lambda_{(pN+1-m)}(k_1,k_2)}{(p-s')^{2\nu+2}}\right] \\ s' = \frac{m-1}{N}, & \textit{if} \quad m = 2,\dots,N, \end{cases}$$

where

$$\Lambda_n(k_1,k_2) = k_1^2\lambda_n^V(k_1) + k_2^2\lambda_n^V(k_2) \quad \textit{and} \quad \lambda_n^V(k_l) = RI_n(k_lR)K_n(k_lR), \quad l=1,2.$$

Moreover, $\mathbb{A}_0$ *is diagonalizable and can be written as follows*

$$\mathbb{A}_0 = \frac{1}{N} Q\Lambda Q,$$

where $\Lambda = diag(\lambda_1,\dots,\lambda_N)$*, and* Q *is the so-called discrete Fourier matrix defined as follows*

$$Q[i,j] = \cos[2\pi(i-1)(j-1)h] + \sin[2\pi(i-1)(j-1)h] \quad \textit{for } i,j = 1,\dots,N$$

and satisfies $Q \cdot Q = NI$*, where* I *is the* $N \times N$*-identity matrix.*

Proof. The proof is similar to Lemma 4.3. □

Finally, the matrices $\mathbf{K}_h$ and $\mathbf{D}_h$ of discrete double layer and hypersingular operators are computed via the regularization technique as given in section 2 of chapter 5 respectively.

7 TWO-DIMENSIONAL QUASISTATIC CONTACT PROBLEMS

In this chapter we consider a quasistatic elastic body in contact with a rigid foundation. That is when the volume and surface forces are applied so slowly that the inertial forces can be neglected. The problems of this kind were first treated by Andersson [5] by using an incremental approach and by considering the contact law in terms of normal compliance. Cocu, Pratt and Raous [21] proved the existence of a solution for a nonlocal friction law. But, by using the so-called Signorini condition of the non-penetration and the Coulomb friction law one encounters considerable mathematical problems and very few results are known for this case without any form of regularization of the friction or any relaxation of the impenetrability condition. The first existence proofs of this approach were given by Andersson [7] by the penalization of the Signorini condition and the regularization of the friction term. Similar results were obtained in [95, 96]. Eck, Steinbach and Wendland [30] obtained the existence of solutions by using a symmetric boundary element method combined with a penalty method.

Here, for the simplicity of the model, we consider the Signorini contact condition together with the local Coulomb friction law. The weak formulation of the problem is given in terms of a variational inequality. By penalizing the Signorini contact condition and performing the time discretization we obtain a discrete variational inequality. Further, by using a certain smoothing of the friction law this yields an equivalent variational equality. Results on the convergence of the solutions of the approximate problems to a solution of the quasistatic contact problem with Coulomb friction are available in [7, 29].

Hence, the aim of this chapter is to develop and analyze efficient and reliable algorithms for the approximate solutions of the discrete problems. But, since the nonlinearities of the problem appear only on the boundary of the contacting bodies, we then transform equivalently with the help of Green's formula the domain variational equation to a boundary variational equation by using a symmetric representation of the Steklov-Poincaré operator. If the Galerkin method is used for the discretization of the boundary integral equations, we then obtain a system of linear equations with a symmetric stiffness matrix. This can motivate the application of efficient solution strategies. The development of efficient algorithm to determine the numerical solutions of these discrete boundary variational equalities strongly depends in a fast and reliable algorithm for solutions of the Coulomb frictional contact problem at each time step. But, the discrete contact problem at each time step is similar to the static problem we treated in chapter 6 and this can motivate the application of all theories we have developed there. Therefore, instead of the Coulomb friction law at each time step, we consider a sequence of contact problems with given friction known as the Tresca problem, since this law is simple to analyze. In addition, the solution of contact problem with Coulomb friction at each time step can be defined as a fixed point of a sequence of solutions to the Tresca problem. This approach was used for the first time in [62, 85]. Hence, the crucial requirement to obtain an efficient and reliable algorithm for the Coulomb frictional contact problem at each time step lies in the fast numerical algorithm to determine the solution of the Tresca problem. The approach taken here is to large extent based on writting the Tresca problem under consideration as optimization problem. Further, by using the Fenchel duality theory [31] we derive the dual problem. Whenever, the problems are seen from the optimizational point of view, instead of just using the first order necessary optimality conditions, which are usually the starting points of analysis, we also use alternately the primal and the dual formulation of the problem for our investigation. Another important aspect of this work is to write the complementarity conditions in terms of non-smooth $\max(.,.)$ and $\min(.,.)$ operators which allows the application of the generalized Newton method in infinite-dimensional function spaces [71,98]. The chapter is organized as follows, in section 1 the Signorini quasistatic contact problem with Coulomb friction is stated in is strong formulation and its variational formulation in a framework of Hilbert spaces. In section 2 we discussed the existence of the penalized problem via a discrete problem, while section 3 is concerned with the boundary element formulation of the problem. In section 4 we present the algorithm for the discrete problem and analysis. Finally, in section 5 we present a BEM discretization for a special two-dimensional circular domain.

7.1 Quasistatic contact problems with Coulomb friction

In this section we state the problem of determining the deformation of a quasistatic linear elastic body subject to a frictional contact. We start by giving the strong formulation of the problem, state all necessary ingredients needed for the variational formulation.

7.1.1 Presentation of the problem

The main assumptions in this section are as for the chapter 6, but for reader's convenience let us recall them again here. We assume that, the body occupies an open and bounded domain Ω of $\mathbb{R}^2$ with $C^{1,1}$ boundary $\Gamma := \partial\Omega$ divided into three disjoint subsets Γ_D, Γ_N and Γ_C. We suppose the body to be fixed on Γ_D, on Γ_N the boundary tractions are given while Γ_C is a potential contact part. The strong formulation of the quasistatic contact problem with Coulomb friction state as follows: Find a displacement field $\underline{u} : (0,T) \times \Omega \to \mathbb{R}^2$ with

$T > 0$ such that the following relations are satisfied:

$$-\text{div}\,(\sigma(\underline{u})) = \underline{f} \quad \text{in} \quad (0,T)\times\Omega, \tag{7.1}$$

$$\underline{u} = 0 \quad \text{on} \quad (0,T)\times\Gamma_D, \tag{7.2}$$

$$\sigma(\underline{u})\underline{n} = \underline{g}_N \quad \text{on} \quad (0,T)\times\Gamma_N, \tag{7.3}$$

$$u_n \le \mathbf{d}, \quad \sigma_n(\underline{u}) \le 0, \quad \sigma_n(\underline{u})(u_n - \mathbf{d}) = 0 \quad \text{on} \quad (0,T)\times\Gamma_C, \tag{7.4}$$

$$\left.\begin{array}{l} \underline{\dot{u}}_t = 0 \Rightarrow |\sigma_t(\underline{u})| < \mathcal{F}|\sigma_n(\underline{u})|, \\ \underline{\dot{u}}_t \neq 0 \Rightarrow \sigma_t(\underline{u}) = -\mathcal{F}|\sigma_n(\underline{u})|\dfrac{\underline{\dot{u}}_t}{|\underline{\dot{u}}_t|}, \\ |\underline{\dot{u}}_t|(|\sigma_t(\underline{u})| - \mathcal{F}|\sigma_n(\underline{u})|) = 0, \end{array}\right\} \quad \text{on} \quad (0,T)\times\Gamma_C, \tag{7.5}$$

$$\underline{u}(0,x) = \underline{u}_0(x) \quad \text{in} \quad \Omega, \tag{7.6}$$

where $\underline{n}$ is the unit outward normal vector along the boundary Γ, $\sigma_n(\underline{u})$ and $\sigma_t(\underline{u})$ represent the normal and tangential stresses along the boundary Γ_C respectively. On the other hand, $u_n := \underline{u}\cdot\underline{n}$ and $\underline{\dot{u}}_t$ represent the normal displacement and the tangential velocity on Γ_C respectively. $\mathcal{F}$ is the friction coefficient where we suppose to be solution independent and $\underline{u}_0$ is the initial displacement. We assume the material to be homogeneous and isotropic so that Hooke's law is applied. Let us set the space of admissible displacements by:

$$\mathbb{K} = \{\underline{\mathrm{v}} \in \mathbb{V} : \mathrm{v}_n \le \mathbf{d} \quad \text{on} \quad \Gamma_C\},$$

where

$$\mathbb{V} = \{\underline{\mathrm{v}} \in \mathbf{H}^1(\Omega) : \gamma_0^{int}\underline{\mathrm{v}} = 0 \quad \text{on} \quad \Gamma_D\}.$$

Further, let us assume that $\underline{f} \in H^1((0,T);\mathbb{V}^*)$ and $\underline{g}_N \in H^1((0,T);(\mathbf{H}^{1/2}(\Gamma_N))^*)$, where $\mathbb{V}^*$ and $(\mathbf{H}^{1/2}(\Gamma_N))^*$ denote the topological dual of spaces $\mathbb{V}$ and $\mathbf{H}^{1/2}(\Gamma_N)$ respectively.

7.1.2 Variational formulation

The variational formulation of the contact condition (7.4) is given by

$$u_n \le \mathbf{d}, \;\; \sigma_n(\underline{u})(\mathrm{v}_n - u_n) \ge 0 \quad \text{for all } \underline{\mathrm{v}} \in \mathbb{K}, \tag{7.7}$$

while the variational formulation of the Coulomb friction law (7.5) is given by

$$\sigma_t(\underline{u})\cdot(\underline{\mathrm{v}}_t^* - \underline{\dot{u}}_t) + \mathcal{F}|\sigma_n(\underline{u})|(|\underline{\mathrm{v}}_t^*| - |\underline{\dot{u}}_t|) \ge 0 \quad \text{for all } \underline{\mathrm{v}}_t^* \quad \text{orthogonal to } \underline{n}. \tag{7.8}$$

The proofs are given in (3.43) and in Proposition 3.1 respectively. By using the usual procedure, we first multiply the differential equation (7.1) by $(\underline{\mathrm{v}} - \underline{u})$, second utilize integration by parts and the boundary conditions yields

$$\int_\Omega \sigma(\underline{u}) : \varepsilon(\underline{\mathrm{v}} - \underline{u})dx - \int_\Omega \underline{f}\cdot(\underline{\mathrm{v}} - \underline{u})dx - \int_{\Gamma_N} \underline{g}_N\cdot(\underline{\mathrm{v}} - \underline{u})ds_x = \int_{\Gamma_C} \sigma(\underline{u})\underline{n}\cdot(\underline{\mathrm{v}} - \underline{u})ds_x. \tag{7.9}$$

If we split the integral in the right hand side of (7.9) into normal and tangential components we then obtain

$$\int_{\Gamma_C} \sigma(\underline{u})\underline{n}\cdot(\underline{v}-\underline{u})ds_x := \int_{\Gamma_C} [\sigma_n(\underline{u})(v_n-u_n)+\sigma_t(\underline{u})\cdot((\underline{v}_t-\underline{u}_t+\dot{\underline{u}}_t)-\dot{\underline{u}}_t)]\,ds_x. \quad (7.10)$$

If we substitute (7.10) into (7.9), add $\int_{\Gamma_C} \mathcal{F}|\sigma_n(\underline{u})|(|\underline{v}_t-\underline{u}_t+\dot{\underline{u}}_t|-|\dot{\underline{u}}_t|)ds_x$ in both sides of (7.9) further employ $\underline{v}^* \equiv \underline{v}-\underline{u}+\dot{\underline{u}}$, and utilize (7.7) and (7.8), the variational formulation of the quasistatic contact problem is then given as follows: Find a function $\underline{u} \in H^1((0,T);\mathbb{V})$ with $\underline{u}(t,.) \in \mathbb{K}$ for almost every $t \in (0,T)$ such that for all $\underline{v} \in \mathbb{K}$ there holds

$$A(\underline{u},\underline{v}-\underline{u})+\int_{\Gamma_C} \mathcal{F}|\sigma_n(\underline{u})|(|\underline{v}_t-\underline{u}_t+\dot{\underline{u}}_t|-|\dot{\underline{u}}_t|)ds_x \geq \mathcal{L}(\underline{v}-\underline{u}), \quad (7.11)$$

where the bilinear form $A(.,.) : \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ is defined by

$$A(\underline{u},\underline{v}) = \int_{\Omega} \sigma(\underline{u}) : \varepsilon(\underline{v})dx$$

and the linear functional $\mathcal{L} : \mathbb{V} \to \mathbb{R}$ given by

$$\mathcal{L}(\underline{v}) = \int_{\Omega} \underline{f}\cdot\underline{v}dx + \int_{\Gamma_N} \underline{g}_N\cdot\underline{v}ds_x.$$

Before we show the existence of a solution of the variational formulation (7.11), note that the bilinear form $A(.,.)$ defined a norm in $\mathbb{V}$ and this norm is equivalent to the Hilbert norm in $\mathbf{H}^1(\Omega)$, see Lemma 3.1 and Theorem 3.1 respectively. The existence proof of solutions for the variational problem (7.11) is a cumbersome task. The fact is that the friction functional is neither convex nor differentiable, and, therefore, can not be analyzed by the results from nonlinear functional analysis. In order to treat this problem, we use the penalty method. This approach was used to investigate the solvability of the frictional contact problem in [7, 29]. We then replace the normal stress on the contact part by

$$-\sigma_n(\underline{u}) = \max(0,\widehat{\lambda}+\gamma_1(u_n-\mathbf{d})) \quad (7.12)$$

for $\gamma_1 > 0$ and for given $\widehat{\lambda} \in L_2(\Gamma_C)$. Note that more general relations of the type $-\sigma_n(\underline{u}) = \phi_\epsilon(u_n-\mathbf{d})$ with $\epsilon > 0$ and small are possible too. Such functions are used in the normal compliance model to describe contact condition, see, e.g., [5, 6] where $\phi_\epsilon(x) = (\max(0,x))^p$ with $p \geq 1$. The resulting penalized problem is then obtained from the variational inequality (7.11) by replacing the set of admissible functions $\mathbb{K}$ by $\mathbb{V}$, adding the penalty functional

$$\int_{\Gamma_C} \max(0,\widehat{\lambda}+\gamma_1(u_n-\mathbf{d}))(v_n-u_n)ds_x$$

to the left-hand side, and by substituting $|\sigma_n(\underline{u})|$ in the friction term by $\max(0,\widehat{\lambda}+\gamma_1(u_n - \mathbf{d}))$. Further, if we replace the test function by $\underline{\mathrm{v}}^* = \underline{\mathrm{v}} + \underline{u} - \underline{\dot{u}}$ we then obtain the following variational inequality.
Find a function $\underline{u} \in H^1((0,T);\mathbb{V})$ such that for all $\underline{\mathrm{v}} \in \mathbb{V}$ there holds

$$A(\underline{u},\underline{\mathrm{v}}-\underline{\dot{u}}) + \int\limits_{\Gamma_C} \max(0,\widehat{\lambda}+\gamma_1(u_n-\mathbf{d}))(\mathrm{v}_n-\dot{u}_n)ds_x$$
$$+\int\limits_{\Gamma_C} \mathcal{F}\max(0,\widehat{\lambda}+\gamma_1(u_n-\mathbf{d}))(|\underline{\mathrm{v}}_t|-|\underline{\dot{u}}_t|)ds_x \geq \mathcal{L}(\underline{\mathrm{v}}-\underline{\dot{u}}). \tag{7.13}$$

7.2 Existence of solutions for penalized problem

The solvability of (7.13) will be proved by using a time discretization. Let us consider a partition of the time interval $(0,T)$ with uniform time steps $\delta t := T/L$. Further for $l = 0,\ldots,L$, we set $t_l := l\delta t$, $\underline{u}^l$ an approximation for $\underline{u}(t_l)$ and $\delta\underline{u}^l := \underline{u}^l - \underline{u}^{l-1}$ the time difference operator. We then obtain the time discretized problem from (7.13) by replacing $\underline{u}$ with $\underline{u}^l$ and $\underline{\dot{u}}^l$ with $\delta\underline{u}^l/\delta t$. Next if we multiply the result by the time step δt we then obtain the following variational inequality:
Find a function $\underline{u}^l \in \mathbb{V}$ such that for all $\underline{\mathrm{v}} \in \mathbb{V}$ there holds

$$A(\underline{u}^l,\underline{\mathrm{v}}-\delta\underline{u}^l) + \int\limits_{\Gamma_C} \max(0,\widehat{\lambda}+\gamma_1(u_n^l-\mathbf{d}))(\mathrm{v}_n-\delta u_n^l)ds_x$$
$$+\int\limits_{\Gamma_C} \mathcal{F}\max(0,\widehat{\lambda}+\gamma_1(u_n^l-\mathbf{d}))(|\underline{\mathrm{v}}_t|-|\delta\underline{u}_t^l|)ds_x \geq \mathcal{L}^l(\underline{\mathrm{v}}-\delta\underline{u}^l). \tag{7.14}$$

In order to obtain a variational equation it is necessary to smooth the friction functional by replacing the non-differentiable terms $|\delta\underline{u}_t^l|$ and $|\underline{\mathrm{v}}_t|$ by differentiable approximations. To this end for $\gamma_2 > 0$ we define a local convex and differentiable regularization $h(,) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that for some y, $|h(x,y) - |x|| \leq \frac{C(y)}{\gamma_2}$ with $C(y)$ a constant independent of γ_2 but dependent of y. The smoothing penalized problem reads then for $\gamma_1,\gamma_2 > 0$ and for given $\widehat{\lambda} \in L_2(\Gamma_C)$, $\underline{\widehat{\mu}} \in \mathbf{L}_2(\Gamma_C)$: Find a function $\underline{u}^l \in \mathbb{V}$ such that for all $\underline{\mathrm{v}} \in \mathbb{V}$ there holds

$$A(\underline{u}^l,\underline{\mathrm{v}}-\delta\underline{u}^l) + \int\limits_{\Gamma_C} \max(0,\widehat{\lambda}+\gamma_1(u_n^l-\mathbf{d}))(\mathrm{v}_n-\delta u_n^l)ds_x$$
$$+\int\limits_{\Gamma_C} \mathcal{F}\max(0,\widehat{\lambda}+\gamma_1(u_n^l-\mathbf{d}))(h(\underline{\mathrm{v}}_t,\underline{\widehat{\mu}})-h(\delta\underline{u}_t^l,\underline{\widehat{\mu}}))ds_x \geq \mathcal{L}^l(\underline{\mathrm{v}}-\delta\underline{u}^l). \tag{7.15}$$

Due to the fact that $h(.,.)$ is differentiable if we set the test function in (7.15) $\underline{\mathrm{v}}^* := \delta\underline{u}^l \pm \alpha\underline{\mathrm{v}}$, next divide the result by $\alpha \in (0,1)$ and let $\alpha \to 0$, we then obtain the variational equality: Find $\underline{u}^l \in \mathbb{V}$ such that for all $\underline{\mathrm{v}} \in \mathbb{V}$ there holds

$$
\begin{aligned}
& A(\underline{u}^l, \underline{\mathrm{v}}) + \int_{\Gamma_C} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d}))\mathrm{v}_n ds_x \\
& + \int_{\Gamma_C} \mathcal{F} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d})) Dh(\delta\underline{u}_t^l, \underline{\widehat{\mu}})\underline{\mathrm{v}}_t ds_x = \mathcal{L}^l(\underline{\mathrm{v}}),
\end{aligned} \tag{7.16}
$$

where $Dh(.,.)$ is the derivative of h with respect to the first variable. Conversely, we have

$$
\begin{aligned}
Dh(\delta\underline{u}_t^l, \underline{\widehat{\mu}})(\underline{\mathrm{v}}_t - \delta\underline{u}_t^l) &:= \lim_{\alpha \to 0} \frac{h(\delta\underline{u}_t^l + \alpha(\underline{\mathrm{v}}_t - \delta\underline{u}_t^l), \underline{\widehat{\mu}}) - h(\delta\underline{u}_t^l, \underline{\widehat{\mu}})}{\alpha} \\
&\leq h(\underline{\mathrm{v}}_t, \underline{\widehat{\mu}}) - h(\delta\underline{u}_t^l, \underline{\widehat{\mu}}),
\end{aligned} \tag{7.17}
$$

since h is convex, if we set the test function in (7.16) $\underline{\mathrm{v}}^* := \underline{\mathrm{v}} - \delta\underline{u}^l$ and utilize the relation (7.17), we then obtain the variational inequality (7.15) and the following result.

Lemma 7.1. *The variational inequality (7.15) is equivalent to the variational equality (7.16).*

Problems (7.14) and (7.16) have the same forms as the corresponding versions of the static contact problem. The existence of a solution to (7.16) is proved as in the static case, see, e.g. chapter 6. An a priori estimate for its solution $\underline{u}^l$ has been investigated in [7,29]. There the following results have been obtained.

Assumption 7.1. *Let the domain Ω be bounded and connected, let its boundary Γ be Lipschitz and be composed of the closures of the mutually disjoint parts Γ_D, Γ_N and Γ_C which are open with respect to the surface topology, and let meas $\Gamma_D > 0$ with $\Gamma_C \in C^k$, $k > 2$. The bilinear form $A(.,.)$ is symmetric, bounded and $\mathbb{V}$-elliptic. Let $\underline{f}^l \in \mathbb{V}^*$, $\underline{g}_N^l \in \boldsymbol{H}^{-1/2}(\Gamma_N)$, $\boldsymbol{d} \in H^{1/2}(\Gamma_C)$ with $\boldsymbol{d} \geq 0$ a.e. on Γ_C. For a set $\Omega_C \subset \Omega$ satisfying $\Gamma_C \subset \partial\Omega_C$ there holds $\underline{f}^l \in \boldsymbol{H}^{-1/2}(\Omega_C)$. The norms of these functions in the corresponding spaces are independent of l, i.e. uniformly bounded. The coefficient of friction $\mathcal{F}$ shall be non-negative with its support in a set $\Gamma_{\mathcal{F}} \subset \Gamma_C$ having a positive distance to $\Gamma \setminus \Gamma_C$. Moreover, $\mathcal{F}$ shall be bounded by*

$$
\|\mathcal{F}\|_{L_\infty(\Gamma_{\mathcal{F}})} < C_{\mathcal{F}}
$$

where the admissible constant $C_{\mathcal{F}}$ is given in [29, p. 208].

Theorem 7.1. *Under the Assumption 7.1, for every* $l \in \{1,\dots,L\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, *and for given* $\widehat{\lambda} \in L_2(\Gamma_C)$, $\underline{\widehat{\mu}} \in \boldsymbol{L}_2(\Gamma_C)$ *the semi-discrete problem (7.16) has a solution* $\underline{u}^l$ *which satisfies*

$$\|\underline{u}^l\|_{\boldsymbol{H}^1(\Omega)} \leq c_1 \tag{7.18}$$

and

$$\|\underline{u}^l\|_{\boldsymbol{H}^1(\Gamma_{\mathcal{F}})} + \|\sigma_n(\underline{u}^l)\|_{L_2(\Gamma_{\mathcal{F}})} \leq c_2 \tag{7.19}$$

with constants c_1 *and* c_2 *independent of* l, γ_1 *and* γ_2. *For every fixed* l, γ_1 *there exists a sequence* $\gamma_{2,k} \to \infty$ *such that a corresponding sequence* $\underline{u}^l_k$ *of solutions to (7.16) converges in* $\boldsymbol{H}^1(\Omega)$ *to a solution* $\underline{u}^l$ *of the penalized problem (7.14). This solution also satisfies the a priori estimates (7.18) and (7.19).*

The a priori estimate (7.18) for $\underline{u}^l$ is a crucial step to show the existence of a solution to the penalized problem (7.13).

Theorem 7.2. *Let the Assumption 7.1 be valid, and let* $\underline{f} \in H^1((0,T);\mathbb{V}^*)$ *and* $\underline{g}_N \in H^1((0,T);(\boldsymbol{H}^{-1/2}(\Gamma_N))^*)$. *If the coefficient of friction is solution independent and satisfies*

$$\|\mathcal{F}\|_{H^{-1/2}(\Gamma_{\mathcal{F}})\to H^{-1/2}(\Gamma_{\mathcal{F}})} < C_{\mathcal{F}}, \tag{7.20}$$

where the norm of $\mathcal{F}$

$$\|\mathcal{F}\|_{H^{-1/2}(\Gamma_{\mathcal{F}})\to H^{-1/2}(\Gamma_{\mathcal{F}})} := \sup_{\substack{v\in H^{-1/2}(\Gamma_{\mathcal{F}})\\ \|v\|_{H^{-1/2}(\Gamma_{\mathcal{F}})}\leq 1}} \|\mathcal{F}v\|_{H^{-1/2}(\Gamma_{\mathcal{F}})}$$

is a Sobolev multiplier norm. Then there exists a sequence of time steps $\delta t_k \to 0$ *and a corresponding sequence of solutions* $\underline{u}_{L_k}$ *of the time-discretized problem (7.14) such that their extensions denoted again by* $\underline{u}_{L_k}$ *converge strongly in* $L_2((0,T);\boldsymbol{H}^1(\Omega))$ *to a solution of the penalized quasistatic problem (7.13). In addition, for some* $\widehat{\lambda} \in L_2(\Gamma_C)$ *the following a priori estimate holds*

$$\begin{aligned}\|\underline{u}\|_{H^1((0,T);\boldsymbol{H}^1(\Omega))} + \|\underline{u}\|_{L_\infty((0,T);\boldsymbol{H}^1(\Gamma_{\mathcal{F}}))} + \|\max(0,\widehat{\lambda}+\gamma_1(u_n-\boldsymbol{d}))\|_{L_\infty((0,T);L_2(\Gamma_{\mathcal{F}}))} \\ + \|\max(0,\widehat{\lambda}+\gamma_1(u_n-\boldsymbol{d}))\|_{H^1((0,T);(H_0^{1/2}(\Gamma_C))^*)} \leq c\end{aligned} \tag{7.21}$$

with a constant c independent of the penalty parameter and the space $H_0^{1/2}(\Gamma_C)$ *define by*

$$H_0^{1/2}(\Gamma_C) := \{\underline{v} \in \boldsymbol{H}^{1/2}(\Gamma) : \underline{v} = 0 \ on \ \Gamma_D\}.$$

Proof. See [29, p. 215]. □

After showing that the discrete problem (7.14) converges to the continuous problem (7.13) we then focus our investigation on the solvability of (7.14) by the help of a semi-smooth Newton approach. But, since the nonlinearities of the problem appear only on the boundaries of the contacting bodies it can be suitable to couple this approach with the boundary element method [29, 100].

7.3 Boundary element formulation

In this section, by proceeding as in section 6.2, the boundary stress is given by

$$\sigma(\underline{u})\underline{n} := \gamma_1^{int}\underline{u} = S\gamma_0^{int}\underline{u} - \mathbf{N}\underline{f}, \tag{7.22}$$

where the symmetric Steklov-Poincaré operator S and the Newton potential $\mathbf{N}\underline{f}$ are defined as in section 6.2. But, note that in this section the fundamental solution of the differential equation (7.1) is given by the Kelvin tensor as follows:

$$U_{ij}^*(x,y)\frac{1}{4\pi(d-1)}.\frac{1}{E}\frac{1+\nu}{1-\nu}\left[(3-4\nu)E_0(x,y)\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^d}\right]$$

with

$$E_0(x,y) = \begin{cases} -\log|x-y| & \text{if} \quad d=2, \\ \dfrac{1}{|x-y|} & \text{if} \quad d=3. \end{cases}$$

Moreover, the symmetric Steklov-Poincaré operator satisfies the following properties.

Lemma 7.2. *Let Γ be bounded and Lipschitz. For all $r \in (-\frac{1}{2}, \frac{1}{2})$ the symmetric Steklov-Poincaré operator $S : \boldsymbol{H}^{1/2+r}(\Gamma) \to \boldsymbol{H}^{-1/2+r}(\Gamma)$ is a linear and continuous mapping. Moreover, it is self-adjoint, positive semidefinite and satisfying*

$$\langle S\underline{u}, \underline{u}\rangle_\Gamma \geq c_1^D \|\underline{u}\|^2_{\boldsymbol{H}^{1/2}(\Gamma)}, \quad \forall \underline{u} \in \boldsymbol{H}^{1/2}(\Gamma) \setminus Ker(D),$$

where $Ker(D)$ and c_1^D represent the kernel and the constant of ellipticity of the hypersingular integral operator D respectively.

Proof. See [30]. □

In order to derive boundary integral formulations of the time discrete contact problem we use the Green formula in terms of the Steklov-Poincaré operator

$$\begin{aligned} A(\underline{u},\underline{\mathrm{v}}) - \int_{\Omega} \underline{f} \cdot \underline{\mathrm{v}} dx &= \int_{\Gamma} \sigma(\underline{u})\underline{n} \cdot \underline{\mathrm{v}} ds_x, \\ &= \int_{\Gamma} (S\underline{u} - \mathbf{N}\underline{f}) \cdot \underline{\mathrm{v}} ds_x. \end{aligned} \tag{7.23}$$

By using the relation (7.23) into (7.14), with the test function $\underline{\mathrm{v}}^* = \underline{\mathrm{v}} - \underline{u}^{l-1}$, the discrete variational inequality (7.14) can be written as the following boundary variational inequality: Find $\underline{u}^l \in \mathcal{V} := \{\underline{\mathrm{v}} \in \mathbf{H}^{1/2}(\Gamma) : \gamma_0^{int}\underline{\mathrm{v}} = 0 \;\text{ on }\; \Gamma_D\}$ such that for all $\underline{\mathrm{v}} \in \mathcal{V}$

$$\begin{aligned} &\int_{\Gamma} S\underline{u}^l \cdot (\underline{\mathrm{v}} - \underline{u}^l) ds_x + \int_{\Gamma_C} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d}))(\mathrm{v}_n - u_n^l) ds_x \\ &+ \int_{\Gamma_C} \mathcal{F} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d}))(|\delta\underline{\mathrm{v}}_t| - |\delta\underline{u}_t^l|) ds_x \geq \\ &\int_{\Gamma} \mathbf{N}\underline{f}^l \cdot (\underline{\mathrm{v}} - \underline{u}^l) ds_x + \int_{\Gamma_N} \underline{g}_N^l \cdot (\underline{\mathrm{v}} - \underline{u}^l) ds_x \end{aligned} \tag{7.24}$$

with $\delta\underline{\mathrm{v}}_t = \underline{\mathrm{v}}_t - \underline{u}_t^{l-1}$. In a similar way the variational equality (7.16) can be transformed equivalently to the following boundary variational equality:
Find $\underline{u}^l \in \mathcal{V} := \{\underline{\mathrm{v}} \in \mathbf{H}^{1/2}(\Gamma) : \underline{\mathrm{v}} = 0 \;\text{ on }\; \Gamma_D\}$ such that there holds

$$\begin{aligned} &\int_{\Gamma} S\underline{u}^l \cdot \underline{\mathrm{v}} ds_x + \int_{\Gamma_C} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d}))\mathrm{v}_n ds_x \\ &+ \int_{\Gamma_C} \mathcal{F} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d})) Dh(\delta\underline{u}_t^l, \underline{\widehat{\mu}}) \cdot \underline{\mathrm{v}}_t ds_x = \\ &\int_{\Gamma} \mathbf{N}\underline{f}^l \cdot \underline{\mathrm{v}} ds_x + \int_{\Gamma_N} \underline{g}_N^l \cdot \underline{\mathrm{v}} ds_x \quad \text{for all } \underline{\mathrm{v}} \in \mathcal{V}. \end{aligned} \tag{7.25}$$

Remark 7.1. *The boundary variational formulations (7.24) and (7.25) are equivalent to the domain variational problems (7.14) and (7.16) respectively. This is valid in the following sense: if $\underline{u}^l$ is a solution of (7.14), then $\gamma_0\underline{u}^l$ is a solution of the boundary variational problem (7.24). Conversely, if $\gamma_0\underline{u}^l$ is a solution of (7.24), the boundary traction is given by $\sigma(\underline{u}^l)\underline{n} := S\gamma_0\underline{u}^l - \mathbf{N}\underline{f}^l$ and the solution $\underline{u}^l$ of the domain variational inequality (7.14) is then given by a representation formula, see, e.g. (6.11). The same relation is valid for the variational equations (7.16) and (7.25).*

From now onward for convenience we will use $\underline{u}^l$ instead $\gamma_0\underline{u}^l$. As an immediate consequence of the above remark we have the following result.

Lemma 7.3. *Under the Assumption 7.1, for every $l \in \{1,\dots,L\}$, $\gamma_1 > 0$, $\gamma_2 > 0$, and for given $\widehat{\lambda} \in L_2(\Gamma_C)$, $\underline{\widehat{\mu}} \in \boldsymbol{L}_2(\Gamma_C)$, the semi-discrete problem (7.25) has a solution $\underline{u}^l$ which satisfies*

$$\|\underline{u}^l\|_{\boldsymbol{H}^{1/2}(\Gamma)} + \|\underline{u}^l\|_{\boldsymbol{H}^1(\Gamma_{\mathcal{F}})} + \|\sigma_n(\underline{u}^l)\|_{L_2(\Gamma_{\mathcal{F}})} \leq C \tag{7.26}$$

with a constant C independent of l, γ_1 and γ_2. For every fixed l, γ_1 there exists a sequence $\gamma_{2,k} \to \infty$ such that a corresponding sequence $\underline{u}^l_k$ of solutions to (7.25) converges in $\boldsymbol{H}^{1/2}(\Gamma)$ to a solution $\underline{u}^l$ of the penalized problem (7.24). This solution also satisfies the a priori estimate (7.26).

Proof. Use Theorem 7.1 with the trace theorem. □

After showing the existence of solutions of the discrete problem (7.25), in the next section we are going to present the method to determine these solutions.

7.4 Algorithms for the regularized discrete problem

The development of an efficient numerical algorithm to determine solutions of the regularized discrete problem (7.25) relies on a fast and reliable algorithm for the solution of the Coulomb frictional contact problem at each time step. But it turns out that the friction functional at each time step is non-monotone. This makes both a theoretical analysis as well as an efficient numerical realization truly challenging. Therefore, instead of the Coulomb law at each time step we consider the frequently used Tresca friction law, since this law is simple to analyze. Moreover, the solution of the Coulomb frictional contact problem at each time step can be defined as a fixed point of a sequence of solutions for the Tresca problem. Thus, the crucial requirement to obtain a fast and reliable algorithm for the Coulomb frictional contact problem at each time step strongly depends on the efficiency of the numerical algorithm to determine the solution of the Tresca problem. The approach we use here consists to write the Tresca problem equivalently to a minimization problem. Further, by using the Fenchel duality theory [31] we derive the dual problem. Whenever the problems are seen from the optimizational point of view, instead of just using the first order necessary optimality conditions, which are usually the starting points of analysis, we also consider the extremality conditions which relate the solutions of primal and dual problems. Another important aspect in this section is to write the complementarity conditions in terms of non-smooth $\max(.,.)$ and $\min(.,.)$ operators which allow the application of the generalized Newton method [71, 98]. Therefore, for fixed $l \in \{1,\dots,L\}$ we consider the problem: for $\gamma_1 > 0$, $\gamma_2 > 0$ and for some $\widehat{\lambda} \in L_2(\Gamma_C)$, $\underline{\widehat{\mu}} \in \mathbf{L}_2(\Gamma_C)$, find

$\underline{u}^l \in \mathcal{V}$ such that there holds

$$\int_\Gamma S\underline{u}^l \cdot \underline{\mathrm{v}} ds_x + \int_{\Gamma_C} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d}))\mathrm{v}_n ds_x$$
$$+ \int_{\Gamma_C} \mathcal{F} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d}))Dh(\delta \underline{u}_t^l, \underline{\widehat{\mu}}) \cdot \underline{\mathrm{v}}_t ds_x = \tag{7.27}$$
$$\int_\Gamma \mathbf{N}\underline{f}^l \cdot \underline{\mathrm{v}} ds_x + \int_{\Gamma_N} \underline{g}_N^l \cdot \underline{\mathrm{v}} ds_x \quad \text{for all } \underline{\mathrm{v}} \in \mathcal{V}.$$

Since the friction functional in (7.27) is non-monotone, we consider the contact problem with given friction, the so-called Tresca problem, i.e. for given $g^l \in L^2_+(\Gamma_C)$ with

$$L^2_+(\Gamma_C) := \{ f \in L_2(\Gamma_C) : f \geq 0 \ \text{ a.e.} \quad \text{on} \quad \Gamma_C \},$$

we set $g^l = \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d}))$ in the friction functional which yields: for $\gamma_1 > 0$, $\gamma_2 > 0$ and for some $\widehat{\lambda} \in L_2(\Gamma_C)$, $\underline{\widehat{\mu}} \in \mathbf{L}_2(\Gamma_C)$, find $\underline{u}^l \in \mathcal{V}$ such that there holds

$$\int_\Gamma S\underline{u}^l \cdot \underline{\mathrm{v}} ds_x + \int_{\Gamma_C} \max(0, \widehat{\lambda} + \gamma_1(u_n^l - \mathbf{d}))\mathrm{v}_n ds_x + \int_{\Gamma_C} \mathcal{F} g^l Dh(\delta \underline{u}_t^l, \underline{\widehat{\mu}}) \cdot \underline{\mathrm{v}}_t ds_x =$$
$$\int_\Gamma \mathbf{N}\underline{f}^l \cdot \underline{\mathrm{v}} ds_x + \int_{\Gamma_N} \underline{g}_N^l \cdot \underline{\mathrm{v}} ds_x \quad \text{for all } \underline{\mathrm{v}} \in \mathcal{V} \tag{7.28}$$

with

$$h(x,y) := \begin{cases} |x + \frac{y}{\gamma_2}| - \frac{1}{2\gamma_2}\mathcal{F}g^l & \text{if} \quad |x + \frac{y}{\gamma_2}| \geq \frac{\mathcal{F}g^l}{\gamma_2}, \\ \frac{\gamma_2}{2\mathcal{F}g^l}|x + \frac{y}{\gamma_2}|^2 & \text{if} \quad |x + \frac{y}{\gamma_2}| < \frac{\mathcal{F}g^l}{\gamma_2}. \end{cases}$$

The problem (7.28) is equivalent to the minimization problem

$$(\mathcal{P}_{\gamma_1,\gamma_2}) \quad \min_{\underline{\mathrm{v}} \in \mathcal{V}} J_{\gamma_1,\gamma_2}(\underline{\mathrm{v}}) := \Bigg[\frac{1}{2} \langle S\underline{\mathrm{v}}, \underline{\mathrm{v}} \rangle_\Gamma - L^l(\underline{\mathrm{v}}) + \frac{1}{2\gamma_1} \| \max(0, \widehat{\lambda} + \gamma_1(\mathrm{v}_n - \mathbf{d})) \|^2_{\Gamma_C}$$
$$+ \int_{\Gamma_C} \mathcal{F} g^l h((\underline{\mathrm{v}}_t - \underline{w}_t)(x), \underline{\widehat{\mu}}(x)) ds_x \Bigg] \tag{7.29}$$

with

$$L^l(\underline{\mathrm{v}}) = \int_\Gamma \mathbf{N}\underline{f}^l \cdot \underline{\mathrm{v}} ds_x + \int_{\Gamma_N} \underline{g}_N^l \cdot \underline{\mathrm{v}} ds_x$$

and $\underline{w} := \underline{u}^{l-1}$. Indeed, let $\underline{u}^l$ be a solution of the optimization problem (7.29). Since the functional $J_{\gamma_1,\gamma_2}(.)$ is differentiable, by performing the limits

$$0 \le \lim_{\alpha \to 0} \frac{J_{\gamma_1,\gamma_2}(\underline{u}^l \pm \alpha \underline{\mathrm{v}}) - J_{\gamma_1,\gamma_2}(\underline{u}^l)}{\alpha} \quad \text{for } \underline{\mathrm{v}} \in \mathcal{V} \text{ and } \alpha \in (0,1),$$

we show that $\underline{u}^l$ is a solution of the variational equality (7.28). On the other hand, let us suppose that $\underline{u}^l$ is a solution of the variational equality (7.28). Since $J_{\gamma_1,\gamma_2}(.)$ is differentiable and convex, if we set the test function in (7.28) $\underline{\mathrm{v}} := \underline{\mathrm{v}}^* - \underline{u}^l$ with $\underline{\mathrm{v}}^* \in \mathcal{V}$ and use the relation

$$\langle DJ_{\gamma_1,\gamma_2}(\underline{u}^l), \underline{\mathrm{v}}^* - \underline{u}^l \rangle_\Gamma \le J_{\gamma_1,\gamma_2}(\underline{\mathrm{v}}^*) - J_{\gamma_1,\gamma_2}(\underline{u}^l),$$

we then show that $\underline{u}^l$ is a minimizer of the functional $J_{\gamma_1,\gamma_2}(.)$. With this we then conclude the equivalence of problems (7.28) and (7.29). Next we have to show the existence of a solution to (7.28). To this end, we decompose the functional as follows $J_{\gamma_1,\gamma_2}(\underline{\mathrm{v}}) := J(\underline{\mathrm{v}}) + j_{g^l}(\underline{\mathrm{v}})$ with

$$\begin{aligned} J(\underline{\mathrm{v}}) &= \frac{1}{2}\langle S\underline{\mathrm{v}}, \underline{\mathrm{v}} \rangle_\Gamma - L^l(\underline{\mathrm{v}}), \\ j_{g^l}(\underline{\mathrm{v}}) &= \frac{1}{2\gamma_1}\|\max(0, \widehat{\lambda} + \gamma_1(\mathrm{v}_n - \mathbf{d}))\|^2_{\Gamma_C} + \int_{\Gamma_C} \mathcal{F} g^l h((\underline{\mathrm{v}}_t - \underline{w}_t)(x).\underline{\widehat{\mu}}(x)) ds_x. \end{aligned}$$

We can easily check that $J_{\gamma_1,\gamma_2} := J + j_{g^l} : \mathcal{V} \to \mathbb{R}$ is strictly convex and continuous, thus from Theorem 2.6 it is weakly lower semi-continuous. Since the Steklov-Poincaré operator is elliptic and the linear map L^l is bounded, the functional $J(.)$ is then coercive. This then shows that the functional $J_{\gamma_1,\gamma_2} := J + j_{g^l}$ is coercive since $j_{g^l}(\underline{\mathrm{v}}) \ge 0$ for all $\underline{\mathrm{v}} \in \mathcal{V}$. According to [29, Theorem 1.5.3] the variational equality (7.28) has a unique solution $\underline{u}^l$. A solution of the Coulomb problem (7.27) can be defined as a fixed point of the following mapping $\Phi_\gamma : L^2_+(\Gamma_C) \to L^2_+(\Gamma_C)$ defined by $\Phi_\gamma(g^l) = \lambda^l_\gamma$, where $\lambda^l_\gamma = \max(0, \widehat{\lambda} + \gamma_1(u^l_n - \mathbf{d}))$ and $\underline{u}^l$ is the unique solution of the variational equality (7.28) with given friction g^l. The mapping Φ_γ is well defined since for a given friction g^l, $\underline{u}^l$ is unique. By Proposition 6.6, the mapping Φ_γ has a fixed point and therefore, the contact problem with Coulomb friction (7.27) has a solution. After showing the existence of solutions for the Coulomb frictional problem (7.27) and for the Tresca problem (7.28), next we present and analyze algorithms to determine these solutions. We start by the Tresca problem (7.28) which is equivalent to the minimization problem (7.29). But instead to consider only the first order optimality conditions, which are usually the starting points of the analysis, we consider here the corresponding dual problem to the primal problem (7.29) defined by: for

$\gamma_1, \gamma_2 > 0$ and for given $\widehat{\lambda} \in L_2(\Gamma_C)$, $\underline{\widehat{\mu}} \in \mathbf{L}_2(\Gamma_C)$

$$(\mathcal{P}^*_{\gamma_1,\gamma_2}) \begin{cases} -\min\limits_{\substack{L_2(\Gamma_C)\ni\lambda\geq 0,\\ |\underline{\mu}|\leq \mathcal{F}g^l \text{ a.e. on } \Gamma_C}} J^*_{\gamma_1,\gamma_2}(\lambda,\underline{\mu}) := \Big[\frac{1}{2}\langle S\underline{u}_{\lambda,\mu},\underline{u}_{\lambda,\mu}\rangle_\Gamma + (\underline{\mu},\underline{w}_t)_{\Gamma_C} + (\lambda,\mathbf{d})_{\Gamma_C} \\ \qquad + \frac{1}{2\gamma_1}\|\lambda-\widehat{\lambda}\|^2_{\Gamma_C} + \frac{1}{2\gamma_2}\|\underline{\mu}-\underline{\widehat{\mu}}\|^2_{\Gamma_C} \\ \qquad - \frac{1}{2\gamma_1}\|\widehat{\lambda}\|^2_{\Gamma_C} - \frac{1}{2\gamma_2}\|\underline{\widehat{\mu}}\|^2_{\Gamma_C}\Big], \\ \text{where } \underline{u}_{\lambda,\mu} \text{ satisfies} \\ \langle S\underline{u}_{\lambda,\mu},\underline{\mathrm{v}}\rangle_\Gamma - L^l(\underline{\mathrm{v}}) + (\lambda,\mathrm{v}_n)_{\Gamma_C} + (\underline{\mu},\underline{\mathrm{v}}_t)_{\Gamma_C} = 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V}. \end{cases}$$

The dual problem $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ has also a unique solution $(\lambda^l,\underline{\mu}^l) \in L_2(\Gamma_C)\times\mathbf{L}_2(\Gamma_C)$, see [67, p.37]. Next, if we consider the two-dimensional case, the extremality conditions which relate the solutions of the primal problem $(\mathcal{P}_{\gamma_1,\gamma_2})$ and the solutions of the dual problem $(\mathcal{P}^*_{\gamma_1,\gamma_2})$, are then given by:

$$\begin{aligned} (S\underline{u},\underline{\mathrm{v}})_\Gamma - L^l(\mathrm{v}) + (\mu,\mathrm{v}_t)_{\Gamma_C} + (\lambda,\mathrm{v}_n)_{\Gamma_C} &= 0 \quad \text{for all} \quad \underline{\mathrm{v}}\in\mathcal{V}, &(7.30)\\ \lambda - \max(0,\widehat{\lambda}+\gamma_1(u_n-\mathbf{d})) &= 0 \quad \text{on} \quad \Gamma_C, &(7.31)\end{aligned}$$

$$\begin{cases} \gamma_2(\xi - u_t + w_t) + \mu - \widehat{\mu} = 0, \\ \xi - \max(0,\xi+\sigma(\mu-\mathcal{F}g^l)) - \min(0,\xi+\sigma(\mu+\mathcal{F}g^l)) = 0, \end{cases} \qquad (7.32)$$

where $u_t \in H^{1/2}(\Gamma_C)$ and $\mu \in L_2(\Gamma_C)$, see section 6.4.1. Moreover, the system of nonlinear equations (7.30)-(7.32) is suitable for the application of the generalized Newton method and the algorithm is given as follows:

Algorithm : (SSN)

(1) Choose $(\xi^0,\lambda^0,\mu^0,\underline{u}^0) \in L_2(\Gamma_C)\times L_2(\Gamma_C)\times L_2(\Gamma_C)\times\mathbf{H}^{1/2}(\Gamma)$ satisfying (7.30), $\sigma > 0$ and set $k := 0$.

(2) Determine active and inactive sets

$$\begin{aligned} A_C^{k+1} &:= \{x\in\Gamma_C : \widehat{\lambda}(x)+\gamma_1(u_n^k(x)-\mathbf{d}(x)) > 0\},\\ I_C^{k+1} &:= \Gamma_C\setminus A_C^{k+1},\\ A_{F+}^{k+1} &:= \{x\in\Gamma_C : \xi^k(x)+\sigma(\mu^k-\mathcal{F}g^l)(x) > 0\},\\ A_{F-}^{k+1} &:= \{x\in\Gamma_C : \xi^k(x)+\sigma(\mu^k+\mathcal{F}g^l)(x) < 0\},\\ I_F^{k+1} &:= \Gamma_C\setminus(A_{F+}^{k+1}\cup A_{F-}^{k+1}).\end{aligned}$$

(3) for $k\geq 1$ if $A_C^{k+1} = A_C^k$, $A_{F+}^{k+1} = A_{F+}^k$ and $A_{F-}^{k+1} = A_{F-}^k$ stop, else

(4) solve

$$
\begin{aligned}
(S\underline{u}^{k+1},\underline{\mathrm{v}})_\Gamma - L^l(\underline{\mathrm{v}}) + (\lambda^{k+1},\mathrm{v}_n)_{\Gamma_C} + (\mu^{k+1},\mathrm{v}_t)_{\Gamma_C} = 0 \quad \text{for all} \quad \underline{\mathrm{v}} \in \mathcal{V},\\
\lambda^{k+1} = \widehat{\lambda} + \gamma_1(u_n^{k+1} - \mathbf{d}) \ \text{ on } \ A_C^{k+1},\ \lambda^{k+1} = 0 \ \text{ on } \ I_C^{k+1},\\
\mu^{k+1} = \mathcal{F}g^l \ \text{ on } \ A_{F+}^{k+1},\ \mu^{k+1} = -\mathcal{F}g^l \ \text{ on } \ A_{F-}^{k+1},\\
\mu^{k+1} - \widehat{\mu} - \gamma_2(u_t^{k+1} - w_t) = 0 \ \text{ on } \ I_F^{k+1}.
\end{aligned}
$$

(5) set

$$
\xi^{k+1} := \begin{cases} u_t^{k+1} - w_t + \frac{1}{\gamma_2}(\widehat{\mu} + \mathcal{F}g^l) \ \text{ on } \ A_{F-}^{k+1},\\[2ex] u_t^{k+1} - w_t + \frac{1}{\gamma_2}(\widehat{\mu} - \mathcal{F}g^l) \ \text{ on } \ A_{F+}^{k+1},\\[2ex] 0 \ \text{ on } \ I_F^{k+1}, \end{cases}
$$

and set $k := k+1$ and go to Step (2).

Note that the system at Step (4) is uniquely solvable, since it is the necessary and sufficient optimality condition for the following minimization problem

$$
\min_{\substack{\lambda=0 \ \text{on} \ I_C^{k+1},\\ \mu=\mathcal{F}g^l \ \text{on} \ A_{F+}^{k+1},\ \mu=-\mathcal{F}g^l \ \text{on} \ A_{F-}^{k+1}}} J^*_{\gamma_1,\gamma_2}(\lambda,\mu),
$$

which has a unique solution. The advantages of this algorithm are that it is of second order and the convergence is locally superlinear, see Theorem 6.5. In addition, it solves both for contact and friction problem together. On the other hand, since the solution of the Coulomb problem (7.27) is a fixed point of the mapping Φ_γ defined above, the algorithm to determine this solution can be defined as follows:

Algorithm : (RCF-FP)

(1) Choose $\gamma_1,\gamma_2 > 0$, $\widehat{\lambda}$ and $\underline{\widehat{\mu}}$. Initialize $g_0^l \in L^2_+(\Gamma_C)$, and set $m := 0$.

(2) Determine the solution $(\lambda_m, \underline{\mu}_m)$ to problem $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ with given friction g_m^l.

(3) Update $g_{m+1}^l := \lambda_m$, $m := m+1$ and, unless an appropriate stopping criterion is met, go to Step (2).

Provided that $\|\mathcal{F}\|_\infty$ is sufficiently small, the Algorithm (RCF-FP) converges regardless of the initialization since the mapping Φ_γ is a contraction, see Theorem 6.6. Note that we determine $(\lambda_m, \underline{\mu}_m)$ at Step (2) of algorithm (RCF-FP) by performing the semi-smooth Newton algorithm (SSN). By Lemma 7.3 the discrete problem (7.25) has a solution and an algorithm to determine this solution can be given by:

Algorithm : (RSDP)

(1) Compute $\tilde{S}_h$ and set the time step $l = 0$.

(2) Start of time step l and compute the linear mapping L^l.

(3) Choose $\gamma_1, \gamma_2 > 0$, $\widehat{\lambda}$ and $\underline{\widehat{\mu}}$. Initialize $g_0^l \in L_+^2(\Gamma_C)$, and set $m := 0$.

(4) Determine the solutions $(\lambda_m^l, \underline{\mu}_m^l)$ and $\underline{u}_m^l$ to problems $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ and $(\mathcal{P}_{\gamma_1,\gamma_2})$ respectively with given friction g_m^l.

(5) (a) If $\dfrac{\|\lambda_m^l - g_m^l\|_{\Gamma_C}}{\|\lambda_m^l\|_{\Gamma_C}} < Tol$

- (a1) and if $l = l_{Max} := L$ stop,
- (a2) else update $g_0^{l+1} := \lambda^l$, compute L^{l+1}, set $l = l+1$ and go to Step (3).

(5) (b) Else, if $\dfrac{\|\lambda_m^l - g_m^l\|_{\Gamma_C}}{\|\lambda_m^l\|_{\Gamma_C}} > Tol$, update $g_{m+1}^l := \lambda_m^l$, $m := m+1$ and go to Step (4).

Note that $(\lambda^l, \underline{\mu}^l)$ and $\underline{u}^l$ are solutions of $(\mathcal{P}^*_{\gamma_1,\gamma_2})$ and $(\mathcal{P}_{\gamma_1,\gamma_2})$ respectively at time step l. In addition, the Step (4) of the algorithm (RSDP) is also realized by the semi-smooth Newton algorithm (SSN).

7.5 BEM discretization

In this section we apply a Galerkin procedure to solve the system of equations at Step (4) of the active set strategy algorithm (SSN) for a particular case of a two-dimensional circular domain $\Omega := B_R(0)$ of radius R and centered at the origin. It turns out that this approach is the same as described in section 6.6. We then refer the reader there for details. Moreover, since the Steklov-Poincaré operator defined by

$$S := (\frac{1}{2}I + \mathbf{K}')\mathbf{V}^{-1}(\frac{1}{2}I + \mathbf{K}) + \mathbf{D} : \mathbf{H}^{1/2}(\Gamma) \to \mathbf{H}^{-1/2}(\Gamma) \tag{7.33}$$

is not suitable for a Galerkin discretization, by following the same idea as in section 6.6, we then define a symmetric approximation of the continuous Steklov-Poincaré operator denoted by $\tilde{S}$. Note that the results of Lemma 4.4 for the scalar Yukawa problem are still valid here, i.e $\tilde{S}$ is spectrally equivalent to S in the sense that for all $\underline{v} \in \mathbf{H}^{1/2}(\Gamma)$ we have

$$c_1^D \|\underline{v}\|^2_{\mathbf{H}^{1/2}(\Gamma)} \le \langle \tilde{S}\underline{v}, \underline{v} \rangle \le c_2^{\tilde{S}} \|\underline{v}\|^2_{\mathbf{H}^{1/2}(\Gamma)}$$

and in addition, we have the consistency estimate:

$$\|(\tilde{S} - S)\underline{v}\|_{\mathbf{H}^{-1/2}(\Gamma)} \le c h^{\eta+1/2} \|S\underline{v}\|_{\mathbf{H}^{\eta}(\Gamma)} \quad \text{for all} \quad \underline{v} \in \mathcal{V}_\eta,$$

where $\mathcal{V}_\eta := \{\underline{\mathrm{v}} \in \mathbf{H}^\eta(\Gamma) : \underline{\mathrm{v}} = 0 \quad \text{on} \quad \Gamma_D\}$ and $\frac{1}{2} \leq \eta \leq 1$. The Galerkin discretization of the approximated Steklov-Poincaré operator $\tilde{S}$ is then given by

$$\tilde{S}_h := (\frac{1}{2}\mathbf{M}_h^\top + \mathbf{K}_h^\top)\mathbf{V}_h^{-1}(\frac{1}{2}\mathbf{M}_h + \mathbf{K}_h) + \mathbf{D}_h. \tag{7.34}$$

Note that $\mathbf{M}_h$ is the mass matrix given in section 6.6. On the other hand, if the fundamental solution of the differential equation (7.1) is written as follows

$$U^*(x,y) = \alpha \log|x-y|\mathbf{I} + \beta \frac{(x-y)(x-y)^\top}{|x-y|^2},$$

where

$$\beta = \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}, \quad \alpha = \beta(4\nu - 3),$$

$\mathbf{I}$ is a 2×2-identity tensor, $E > 0$ and $\nu \in (0, 1/2)$ the Young modulus and the Poisson ratio respectively, the single layer operator can then be written in matrix form as follows

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

In addition, if we use the one-periodic parametrization of the boundary Γ, the boundary integral operators V_{kl} for $k,l = 1,2$ take then the following forms

$$\begin{aligned}
(V_{11}\mathrm{v}_1)(x) &= \int_\Gamma \left(\alpha \log|x-y| + \beta \frac{(x_1-y_1)^2}{|x-y|^2}\right) \mathrm{v}_1(y) ds_y, \\
&= 2\pi R \int_0^1 \left(\alpha \log(2R|\sin\pi(\tau - t)|) + \beta \sin^2\pi(\tau + t)\right) \mathrm{v}_1(t) dt \\
(V_{12}\mathrm{v}_2)(x) &\equiv (V_{21}\mathrm{v}_2)(x) = \int_\Gamma \beta \frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2} \mathrm{v}_2(y) ds_y, \\
&= -2\pi R\beta \int_0^1 \sin\pi(\tau+t)\cos\pi(\tau+t)\mathrm{v}_2(t)dt, \\
(V_{22}\mathrm{v}_2)(x) &= \int_\Gamma \left(\alpha \log|x-y| + \beta \frac{(x_2-y_2)^2}{|x-y|^2}\right) \mathrm{v}_2(y) ds_y, \\
&= 2\pi R \int_0^1 \left(\alpha \log(2R|\sin\pi(\tau - t)|) + \beta \cos^2\pi(\tau + t)\right) \mathrm{v}_2(t) dt.
\end{aligned}$$

By using the above results the matrix of the discrete single layer operator $V_h := A$ is a square matrix of size $2N$ where N is the number of points of discretization. In addition, A

has the following block structure

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where the entries of matrices A_{kl} for $k,l = 1,2$ are given by the following lemma.

Lemma 7.4. *The entries of matrices A_{11}, A_{12} and A_{22} are given by*

$$\begin{aligned}
A_{11}[i,j] &= 2\pi R\alpha \log(R)h^2 - 2\pi R\alpha \sum_{k>0} \frac{(c_1^{(0)}(k))^2}{|k|} \cos[2\pi k(i-j)h] \\
&\quad + \pi R\beta h^2 + \frac{R\beta}{4\pi}\left[\cos[2\pi(a_{ij}+2h)] - 2\cos[2\pi(a_{ij}+h)] + \cos(2\pi a_{ij})\right], \\
A_{12}[i,j] \equiv A_{21}[i,j] &= \frac{R\beta}{4\pi}\left[\sin[2\pi(a_{ij}+2h)] - 2\sin[2\pi(a_{ij}+h)] + \sin(2\pi a_{ij})\right], \\
A_{22}[i,j] &= 2\pi R\alpha \log(R)h^2 - 2\pi R\alpha \sum_{k>0} \frac{(c_1^{(0)}(k))^2}{|k|} \cos[2\pi k(i-j)h] \\
&\quad + \pi R\beta h^2 - \frac{R\beta}{4\pi}\left[\cos[2\pi(a_{ij}+2h)] - 2\cos[2\pi(a_{ij}+h)] + \cos(2\pi a_{ij})\right]
\end{aligned}$$

with $a_{ij} = \tau_i + t_j$, τ_i and t_j are parameters of elements number i and j respectively.

Proof. Use $A_{kl}[i,j] = \langle V_{kl}\phi_j^0, \phi_i^0\rangle_\Gamma$ for $i,j = 1,\ldots,N$; $k,l = 1,2$ and the Fourier representation of the B-spline ϕ_i^0. □

From the above lemma the matrices A_{11}, A_{12} and A_{22} can be written as follows

$$A_{11} = A_0 + A_c,\ A_{12} = A_s,\ A_{22} = A_0 - A_c,$$

where

$$\begin{aligned}
A_0[i,j] &= 2\pi R(\alpha \log(R) + \frac{1}{2}\beta)h^2 - 2\pi R\alpha \sum_{k>0} \frac{(c_1^{(0)}(k))^2}{|k|} \cos[2\pi k(i-j)h], \\
A_c[i,j] &= \frac{R\beta}{4\pi}\left[\cos[2\pi(a_{ij}+2h)] - 2\cos[2\pi(a_{ij}+h)] + \cos(2\pi a_{ij})\right], \\
A_s[i,j] &= \frac{R\beta}{4\pi}\left[\sin[2\pi(a_{ij}+2h)] - 2\sin[2\pi(a_{ij}+h)] + \sin(2\pi a_{ij})\right].
\end{aligned}$$

Lemma 7.5. *The matrix A_0 is circulant, and if the radius of the disc $R < \exp(\frac{1}{6-8\nu})$ it is then positive definite. Moreover, its eigenvalues are given by:*

$$\lambda_m = \begin{cases} 2\pi R(\alpha \log(R) + \frac{1}{2}\beta)h & m = 1, \\ -\pi R\alpha h^2 \left(\frac{\sin(\pi s)}{\pi}\right)^{2q+2} \sum_{k=0}^{\infty} \left[\frac{1}{(k+s)^{(2q+3)}} + \frac{1}{(k+1-s)^{(2q+3)}}\right] & m \geq 2, \\ s = \frac{m-1}{N}, \quad h = \frac{1}{N}, & \end{cases}$$

where q is the order of the B-spline ϕ^q. Hence, the matrix A_0 is diagonalizable and can be written as follows

$$A_0 = \frac{1}{N} Q \Lambda Q,$$

where $\Lambda = diag(\lambda_1, \ldots, \lambda_N)$, and Q is the so-called discrete Fourier matrix defined as follows

$$Q[i,j] = \cos[2\pi(i-1)(j-1)h] + \sin[2\pi(i-1)(j-1)h], \quad i,j = 1,\ldots,N$$

and satisfies $Q \cdot Q = NI$, where I is the identity matrix.

Proof. As Lemma 4.3. □

We can easily check that the matrices A_0, A_c and A_s are all symmetric and the entries can be computed explicitly and exactly.

Proposition 7.1. *There hold*

$$A_c Q_j = \begin{cases} \gamma \frac{N}{2} \mathcal{B} Q_2 & for \quad j = 2, \\ \gamma \frac{N}{2} \mathcal{B} Q_N & for \quad j = N, \\ 0 & else, \end{cases}$$

$$A_s Q_j = \begin{cases} \gamma \frac{N}{2} \mathcal{B} Q_N & for \quad j = 2, \\ -\gamma \frac{N}{2} \mathcal{B} Q_2 & for \quad j = N, \\ 0 & else, \end{cases}$$

where Q_j is the j^{th} column of discrete Fourier matrix Q , $\gamma = -\pi R \beta (c_1^q(1))^2$ and $\mathcal{B}$ is the matrix defined by

$$\mathcal{B} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \\ . & . & . & \ldots & . \\ . & . & 1 & \ldots & . \\ 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

which were verified by experiments.

From Proposition 7.1 we obtain immediately that

$$\begin{aligned} A_c Q &= \gamma \frac{N}{2} \mathcal{B} Q \overline{D}_c, \\ A_s Q &= \gamma \frac{N}{2} \mathcal{B} Q \overline{D}_s, \end{aligned} \tag{7.35}$$

where

$$\overline{D}_c = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & . \\ . & . & . & \dots & . \\ 0 & . & . & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad \overline{D}_s = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & -1 \\ . & . & . & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix}.$$

Both matrices $\overline{D}_c$ and $\overline{D}_s$ have only two entries different from zero. If we multiply both the left and right hand side of (7.35) by Q, and use the property $Q \cdot Q = NI$, we then obtain

$$\begin{aligned} A_c &= \frac{\gamma}{2} \mathcal{B} Q \overline{D}_c Q, \\ A_s &= \frac{\gamma}{2} \mathcal{B} Q \overline{D}_s Q. \end{aligned} \tag{7.36}$$

Since the matrices A_c and A_s are symmetric, if we first transpose both sides of (7.36) and multiply the result by $Q \cdot Q$, we then obtain

$$\begin{aligned} A_c &= \frac{1}{N} \frac{\gamma}{2} Q (\overline{D}_c Q \mathcal{B} Q) Q, \\ A_s &= \frac{1}{N} \frac{\gamma}{2} Q (\overline{D}_s^{\top} Q \mathcal{B} Q) Q. \end{aligned} \tag{7.37}$$

Further computations show that

$$D_c = (\overline{D}_c Q \mathcal{B} Q) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & -b_N & 0 & \dots & a_N \\ 0 & 0 & 0 & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & . & \dots & 0 \\ 0 & a_N & 0 & \dots & b_N \end{pmatrix},$$

$$D_s = (\overline{D}_s^{\top} Q \mathcal{B} Q) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & a_N & 0 & \dots & b_N \\ . & . & . & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \\ 0 & b_N & 0 & \dots & -a_N \end{pmatrix},$$

where

$$a_N = \sum_{j=1}^{N} Q_2[j]Q_N[N-j+1] = \sum_{j=1}^{N} Q_N[j]Q_2[N-j+1],$$
$$b_N = \sum_{j=1}^{N} Q_N[j]Q_N[N-j+1] = -\sum_{j=1}^{N} Q_2[j]Q_2[N-j+1],$$

and Q_j is the j^{th} column of the discrete Fourier matrix Q. Therefore, the matrices A_c and A_s can be written as follows

$$A_c = \frac{1}{N}\frac{\gamma}{2}QD_cQ, \quad A_s = \frac{1}{N}\frac{\gamma}{2}QD_sQ. \tag{7.38}$$

Remark 7.2. *We can notice that the matrices D_c and D_s are symmetric and have at most four entries different from zero. Furthermore, numeric computations show that*

- *for $N = 4$, $a_N = 0$ and $b_N = 4$,*
- *for $N = 8$, $a_N = b_N = 5.65685$,*
- *for $N > 8$, $a_N \to N$ and $b_N \to 2\pi$.*

By using Lemma 7.5 and (7.38), the matrix A of the discrete single layer operator can be written as follows

$$A = \frac{1}{N}\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}\begin{pmatrix} \Lambda + \frac{\gamma}{2}D_c & \frac{\gamma}{2}D_s \\ \frac{\gamma}{2}D_s & \Lambda - \frac{\gamma}{2}D_c \end{pmatrix}\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix},$$

where $\Lambda := diag(\lambda_1, ..., \lambda_N)$ is the diagonal matrix defined by the eigenvalues of the circulant matrix A_0 and $\gamma = -\pi R\beta(c_1^q(1))^2$ with $c_1^q(1)$ which is the Fourier coefficient given in section 4.2. We remark that

$$\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$$

can be utilized as a preconditioning matrix for the matrix A of the discrete single layer operator. Moreover, if we set

$$\mathcal{K} = \begin{pmatrix} \Lambda + \frac{\gamma}{2}D_c & \frac{\gamma}{2}D_s \\ \frac{\gamma}{2}D_s & \Lambda - \frac{\gamma}{2}D_c \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_3 \end{pmatrix},$$

with $A_1 = \Lambda + \frac{\gamma}{2}D_c$, $A_2 = \frac{\gamma}{2}D_s$ and $A_3 = \Lambda - \frac{\gamma}{2}D_c$, the matrix $\mathcal{K}$ can then be written again as follows

$$\mathcal{K} = \begin{pmatrix} I & 0 \\ A_2A_1^{-1} & I \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & \mathcal{S} \end{pmatrix}\begin{pmatrix} I & A_1^{-1}A_2 \\ 0 & I \end{pmatrix},$$

where $\mathcal{S} := A_3 - A_2 A_1^{-1} A_2$ and I is the $(N \times N)$-identity matrix. Since the matrices A_1, A_2, A_3 and $\mathcal{S}$ are sparse, the inverse of the matrix $\mathcal{K}$ can be computed easily and it is given by

$$\mathcal{K}^{-1} = \begin{pmatrix} I & -A_1^{-1}A_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0 \\ 0 & \mathcal{S}^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_2 A_1^{-1} & I \end{pmatrix},$$

where

$$B_1 = -A_1^{-1}A_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & -(ab+Bc) & 0 & \dots & -bB+ac \\ . & . & . & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \\ 0 & -(ac+Bd_1) & 0 & \dots & -Bc+ad_1 \end{pmatrix},$$

$$B_2 = A_1^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & 0 & ..0 & 0 \\ 0 & b & 0 & \dots & c \\ 0 & 0 & \frac{1}{\lambda_3} & ..0 & . \\ . & . & . & \dots & . \\ 0 & 0 & . & ..\frac{1}{\lambda_{N-1}} & 0 \\ 0 & c & 0 & \dots & d_1 \end{pmatrix},$$

$$B_3 = \mathcal{S}^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & 0 & ..0 & 0 \\ 0 & b_1 & 0 & \dots & c_1 \\ 0 & 0 & \frac{1}{\lambda_3} & ..0 & . \\ . & . & . & \dots & . \\ 0 & 0 & . & ..\frac{1}{\lambda_{N-1}} & 0 \\ 0 & c_1 & 0 & \dots & d_2 \end{pmatrix},$$

$$B_4 = -A_2 A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & -(ab+Bc) & 0 & \dots & -(ac+Bd_1) \\ . & . & . & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 \\ 0 & ac-Bb & 0 & \dots & -Bc+ad_1 \end{pmatrix}$$

with

$$\begin{aligned} a &= \frac{\gamma}{2}a_N, \quad B = \frac{\gamma}{2}b_N, \quad l_2 = \lambda_2 - B, \quad l_N = \lambda_2 + B, \\ b &= \frac{l_N}{l_2 l_N - a^2}, \quad c = -\frac{a}{l_2 l_N - a^2}, \quad d_1 = \frac{l_2}{l_2 l_N - a^2}, \\ L_2 &= l_N - (a^2 b + 2aBc + B^2 d_1), \quad L_N = l_2 - B^2 b + 2aBc - a^2 d_1, \\ D &= a[B(d_1 - b) - 1] + c(a^2 - B^2), \\ b_1 &= \frac{L_N}{L_2 L_N - D^2}, \quad c_1 = -\frac{D}{L_2 L_N - D^2}, \quad d_2 = \frac{L_2}{L_2 L_N - D^2}. \end{aligned}$$

Hence, the inverse of the matrix A of the discrete single layer is then given by

$$V_h^{-1} \equiv A^{-1} = \frac{1}{N} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} I & B_1 \\ 0 & I \end{pmatrix} \begin{pmatrix} B_2 & 0 \\ 0 & B_3 \end{pmatrix} \begin{pmatrix} I & 0 \\ B_4 & I \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}.$$

On the other hand, the matrices $\mathbf{K}_h$ and $\mathbf{D}_h$ of discrete double layer and hypersingular operators are computed respectively by performing a regularization technique and integration by parts, see, e.g., [39, 65, 66, 72, 100]. We have then

$$(T_y U^*)^\top = 2\mu (\mathcal{M}_y U^*)^\top + \mathcal{M}_y(\triangle_y \chi) + \frac{\partial}{\partial \underline{n}(y)} \triangle \chi \mathbf{I} \tag{7.39}$$

with $\mathbf{I}$ the 2×2-identity matrix, and

$$\mathcal{M}_y = \begin{pmatrix} 0 & n_2 \frac{\partial}{\partial y_1} - n_1 \frac{\partial}{\partial y_2} \\ n_1 \frac{\partial}{\partial y_2} - n_2 \frac{\partial}{\partial y_1} & 0 \end{pmatrix},$$

$$\chi(r) = -\frac{1}{8\pi} r^2 \log(r) \quad \text{with } r = |y - x|,$$

as well as

$$\Delta_y \chi := \psi(r) = -\frac{1}{2\pi} \log(r).$$

We remark that ψ is the fundamental solution of the Laplace equation. By using the one-periodic parametrization of the boundary Γ we obtain then

$$y_1(t) = R\cos 2\pi t, \quad y_2(t) = R\sin 2\pi t, \quad t \in [0,1),$$

which yields

$$(T_y U^*)^\top = \frac{\mu}{\pi R} \frac{d}{dt} \begin{pmatrix} -U_{12}^* & U_{11}^* \\ -U_{22}^* & U_{12}^* \end{pmatrix} + \frac{1}{2\pi R} \frac{d}{dt} \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial \psi}{\partial \underline{n}(y)} & 0 \\ 0 & \frac{\partial \psi}{\partial \underline{n}(y)} \end{pmatrix}. \tag{7.40}$$

By utilizing integration by parts, the double layer integral operator can then be written as follows

$$\begin{aligned} (\mathbf{K}\underline{u})(x) &= \int_\Gamma [T_y\, U^*(x,y)]^\top \, \underline{u}(y) ds_y \\ &= 2\mu \int_0^1 \begin{pmatrix} U_{12}^* & -U_{11}^* \\ U_{22}^* & -U_{12}^* \end{pmatrix} \frac{d\underline{u}}{dt} dt + \int_0^1 \begin{pmatrix} 0 & \psi \\ -\psi & 0 \end{pmatrix} \frac{d\underline{u}}{dt} dt \\ &+ 2\pi R \int_0^1 \begin{pmatrix} \frac{\partial \psi}{\partial \underline{n}(y(t))} & 0 \\ 0 & \frac{\partial \psi}{\partial \underline{n}(y(t))} \end{pmatrix} \underline{u}(y(t)) dt. \end{aligned} \tag{7.41}$$

In a similar way, the bilinear form of the hypersingular integral operator can be written as follows

$$
\begin{aligned}
\langle \mathbf{D}\underline{u}, \underline{v} \rangle_\Gamma &= -\mu \int_\Gamma \int_\Gamma \underline{v}(x) \cdot \left[\frac{\partial^2 \psi}{\partial \underline{n}(y) \partial \underline{n}(x)} \mathbf{I} \right] \underline{u}(y) ds_y ds_x \\
&\quad -\mu \int_\Gamma \int_\Gamma (\mathcal{M}_x \underline{v}(x)) \cdot [4\mu U^* - 3\psi \mathbf{I}] (\mathcal{M}_y \underline{u}(y)) ds_y ds_x.
\end{aligned} \tag{7.42}
$$

From (7.41) it can be seen that the matrix $\mathbf{K}_h$ of the discrete double layer operator can be computed by using the matrix $A = V_h$ of the discrete single layer operator of the linear elasticity equation, and the matrices of the discrete single layer operator and the discrete double layer operator for the Laplace equation respectively; while from (7.42) the matrix $\mathbf{D}_h$ of the discrete hypersingular operator can be realized by using also the matrix $A = V_h$, and the matrices of the discrete single layer operator and the discrete hypersingular operator for the Laplace equation respectively.

Lemma 7.6. *Let $\Omega := B_R(0)$ be a two-dimensional circular domain and let us consider a one-periodic parametrization of its boundary Γ. The matrices of the discrete single layer operator, the discrete double layer operator and the discrete hypersingular operator for the Laplace equation are all circulant and given respectively as follows:*

$$
\begin{aligned}
V_\Delta[i,j] &:= -R \int_0^1 \int_0^1 \phi_j^{(q)}(t) \log|2R \sin \pi(t-\tau)| \phi_i^{(q)}(\tau) dt d\tau, \\
K_\Delta[i,j] &:= \frac{1}{2} \int_0^1 \phi_i^{(q)}(t) \phi_j^{(p)}(t) dt - \frac{1}{2} \int_0^1 \int_0^1 \phi_j^{(p)}(t) \phi_i^{(q)}(\tau) dt d\tau, \\
D_\Delta[i,j] &:= -\frac{1}{4R} \int_0^1 \int_0^1 \phi_j^{(p)}(t) \frac{1}{\sin^2 \pi(t-\tau)} \phi_i^{(p)}(\tau) dt d\tau, \quad i,j = 1,\dots,N.
\end{aligned}
$$

Moreover, their eigenvalues are given by

$$\lambda_m^{V_\Delta} = \begin{cases} -R\log(R)h & m=1, \\ \frac{1}{2}Rh^2\left(\frac{\sin(\pi s)}{\pi}\right)^{2q+2}\sum_{k=0}^{\infty}\left[\frac{1}{(k+s)^{(2q+3)}}+\frac{1}{(k+1-s)^{(2q+3)}}\right] & m\geq 2, \\ s=\frac{m-1}{N}, \quad h=\frac{1}{N}, \end{cases}$$

$$\lambda_m^{K_\Delta} = \begin{cases} 0 & m=1, \\ \frac{1}{2}h\left(\frac{\sin(\pi s)}{\pi}\right)^{q+p+2}\sum_{k=0}^{\infty}\left[\frac{(-1)^{k(q+p)}}{(k+s)^{(q+p+2)}}+\frac{(-1)^{k(q+p)}}{(k+1-s)^{(q+p+2)}}\right] & m\geq 2, \\ s=\frac{m-1}{N}, \quad h=\frac{1}{N}, \end{cases}$$

$$\lambda_m^{D_\Delta} = \begin{cases} 0 & m=1, \\ \frac{1}{2R}\left(\frac{\sin(\pi s)}{\pi}\right)^{2p+2}\sum_{k=0}^{\infty}\left[\frac{1}{(k+s)^{(2p+1)}}+\frac{1}{(k+1-s)^{(2p+1)}}\right] & m\geq 2, \\ s=\frac{m-1}{N}, \quad h=\frac{1}{N}, \end{cases}$$

respectively.

Proof. See [92]. □

Remark 7.3. *By using Lemma 7.4 and Lemma 7.6, the matrix $\boldsymbol{K}_h$ of the discrete double layer operator and the matrix $\boldsymbol{D}_h$ of the discrete hypersingular operator can be computed exactly and efficiently.*

8 NUMERICAL EXAMPLES

In this chapter Galerkin boundary element methods (BEM) developed in previous chapters are applied to some numerical examples. In the first part, we consider the domain $\Omega := B_R(c)$ to be a two-dimensional disc of radius $R = 1.0$ and centered at $c = (1.0, 1.0)$. For the boundary element discretization the trial functions for the approximation of the unknown function u or the displacement $\underline{u}$ will be B-splines of order one $(\phi_i^{(1)})$, while the Neumann data will be approximated by B-splines of order zero $(\phi_i^{(0)})$.

The chapter is organized as follows, in the first section we present some numerical results for scalar Yukawa problems, we consider in particular the non-homogeneous Dirichlet and the mixed boundary value problems. The second section will be about the non-homogeneous Dirichlet linear elasticity problem of Yukawa type, while the third section will be about the application of the combined generalized Newton method with BEM for the solution of contact problems in linear elastostatics of Yukawa type. Finally, the combined generalized Newton method with BEM will be applied to quasistatic contact problems.

8.1 Numerical results for scalar Yukawa problems

Here both the non-homogeneous Dirichlet and the non-homogeneous mixed boundary value problem are considered. For the mixed boundary value problem given in (4.1)-(4.3), that is

$$\alpha^2 u(x) - \Delta u(x) = f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^2, \tag{8.1}$$

$$u(x) = g_D(x) \quad \text{for } x \in \Gamma_D, \tag{8.2}$$

$$\frac{\partial u}{\partial n_x}(x) = g_N(x) \quad \text{for } x \in \Gamma_N, \tag{8.3}$$

we assume that the boundary is divided into two disjoint parts such that $\Gamma := \Gamma_D \cup \Gamma_N$, see Figure 8.1.

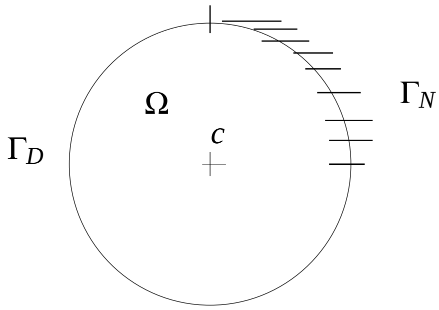


Figure 8.1: Disc of center c and radius R.

Further we set $\alpha = 1.0$, the test solution $u(x) = x_1^2 + x_2^2$, the function $f(x) = -4 + x_1^2 + x_2^2$, $g_D(x) = x_1^2 + x_2^2$ and $g_N(x) = \frac{\partial u}{\partial n(x)}$. The boundary element discretization of the non-homogeneous Dirichlet problem is equivalent to the following algebraic system of linear equations

$$V_h \underline{t} = (\frac{1}{2} M_h + K_h) \underline{g} - \underline{\tilde{N}_0} f, \tag{8.4}$$

where V_h and K_h are the matrices of the Galerkin discretization of the single layer and double layer operators respectively, while M_h is the mass matrix, $\underline{\tilde{N}_0} f$ is the vector computed from the Newton potential, see Appendix, $\underline{g}$ here is the piecewise linear interpolation of the Dirichlet data, but one can also utilize the L_2-projection (see e.g. [100]). Since V_h is symmetric, positive definite and circulant [15, 16, 92], the system (8.4) is solved efficiently by performing the Fast Fourier transform (FFT). In addition, the pointwise evaluation is done at the point $\hat{x} = (1.0, 1.0)^\top$, see Table 8.1 for results.

refinement		Approx.Sol., ptwise error		L_2 Error, eoc., CPU time		
Level	N	$u_h(\hat{x})$	$\lvert u(\hat{x}) - u_h(\hat{x}) \rvert$	$\lVert t - t_h \rVert_{L^2(\Gamma)}$	eoc	Tot.time (sec)
3	32	2.00121	0.00120914	0.284098		0.01
4	64	2.0003	0.000303272	0.142071	0.99978	0.03
5	128	2.00008	7.57068e-05	0.0710384	0.99994	0.13
6	256	2.00002	1.88993e-05	0.0355196	0.99998	0.53
7	512	2.0	4.72057e-06	0.0177598	1	1.98
8	1024	2.0	1.17956e-06	0.00887992	1.00000	8.24
Theory		2.0			1	

Table 8.1: The approximate solution u_h, the pointwise error, the L_2 error of the conormal derivative, the order of convergence and the CPU time for the Dirichlet problem by using the FFT.

On the other hand, the non-homogeneous mixed boundary value problem is equivalent to the linear system

$$\tilde{S}_h \underline{u} = \underline{f}, \tag{8.5}$$

where $\tilde{S}_h = D_h + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h)$. Since the Galerkin discretization of the approximate Steklov-Poincaré operator $\tilde{S}_h$ is symmetric and positive definite we solve the linear system (8.5) by using the preconditioned conjugate gradient algorithm, with the preconditioning matrix

$$C_D := \underline{M}\overline{V}^{-1}\underline{M},$$

see [100], where $\underline{M}$ and $\overline{V}$ are the mass matrix and the discrete single layer matrix respectively, define by B-splines of order one as follows:

$$\underline{M}[i,j] := \langle \phi_j^{(1)}, \phi_i^{(1)} \rangle_\Gamma \quad \overline{V}[i,j] := \langle V\phi_j^{(1)}, \phi_i^{(1)} \rangle_\Gamma, \ \ i,j = 1,...,n.$$

Moreover, $\overline{V}$ is circulant symmetric and positive definite and its eigenvalues are given by Lemma 4.3 where we have to set $\nu = 1$. Therefore, $\overline{V}$ is diagonalizable and has the form

$$\overline{V} = \frac{1}{n} Q\Lambda^{(1)}Q.$$

In addition, the action of $\tilde{S}_h$ on any vector $\underline{p}$ is given in two steps as follows

$$\tilde{S}_h\underline{p} := D_h\underline{p} + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h)\underline{p} = D_h\underline{p} + (\frac{1}{2}M_h^\top + K_h^\top)\underline{w},$$

where $\underline{w}$ is the unique solution of the linear system

$$V_h\underline{w} = (\frac{1}{2}M_h + K_h)\underline{p}, \tag{8.6}$$

note that (8.6) is solved efficiently by utilizing the Fast Fourier transform (FFT), see Table 8.2 below for the results.

N	Iterations	L_2 Error u, eoc.		L_2 Error $\frac{\partial u}{\partial n}$, eoc., CPU time		
	Precond. CG	$\|u - u_h\|_{L^2(\Gamma)}$	eoc	$\|t - t_h\|_{L^2(\Gamma)}$	eoc	Tot. time (sec)
32	04	0.0122801		0.270957		0.00
64	08	0.00304241	2.0130	0.13543	1.0001	0.02
128	12	0.00075676	2.0073	0.067736	1.0000	0.08
256	12	0.000188689	2.0038	0.0338679	1.0000	0.34
512	12	4.71083e-05	2.0020	0.0169339	1.0000	1.36
1024	12	1.1769e-05	2.0010	0.00846697	1.0000	5.71
Theory			2		1	

Table 8.2: The L_2 error of the solution u, the L_2 error of the conormal derivative, the order of convergence and the CPU time for the mixed boundary value problem by using the preconditioned CG method.

The results presented in Table 8.1 are in agreement with the order of convergence for a Dirichlet problem, see, e.g. [100], while the Table 8.2 confirms the error estimates we obtained in (4.63) and (4.66). In Table 8.3 below we compare the number of iterations for the CG method, and preconditioned CG method and give the number of rings we used for the computation of the Newton potential for the Dirichlet and mixed boundary value problem.

refinement		Nb. Ite. for CG and Precond. CG methods, and Nb. rings		
Level	N	Preconditioned CG	CG without Precond.	Nb. rings
3	32	04	04	08
4	64	08	08	16
5	128	12	16	32
6	256	12	20	64
7	512	12	28	128
8	1024	12	42	256
9	2048	12	61	512

Table 8.3: Comparison between the number of iterations for the CG and preconditioned CG algorithms and the number of rings.

8.2 Numerical results for linear elasticity of Yukawa type

Here we consider the non-homogeneous Dirichlet problem and the domain Ω is still as above a disc of center $c = (1.0, 1.0)$ and radius $R := 1.0$,

$$s^2\underline{u}(x) - \mu\Delta\underline{u}(x) - (\lambda+\mu)\text{ grad div }\underline{u}(x) = \underline{f}(x) \quad \text{for } x \in \Omega, \tag{8.7}$$

$$\underline{u}(x) = \underline{g}(x) \quad \text{for } x \text{ on } \Gamma. \tag{8.8}$$

Our goal here is to investigate the reliability of the algorithm we presented in section 3 of chapter 5 for the computation of the Newton potential. To this end we set
$\lambda := 115.3846$, $\mu := 76.9231$ and $s := 10.0$. To test the algorithm, we utilize the functions

$$\underline{u}(x) = \begin{pmatrix} x_1^2 + x_2 \\ x_2^2 - x_1 \end{pmatrix}, \quad \underline{f}(x) = \begin{pmatrix} -2\lambda - 4\mu + s^2(x_1^2 + x_2) \\ -2\lambda - 4\mu + s^2(x_2^2 - x_1) \end{pmatrix}, \quad \underline{g}(x) = \begin{pmatrix} x_1^2 + x_2 \\ x_2^2 - x_1 \end{pmatrix}.$$

The boundary element discretization of the non-homogeneous Dirichlet problem (8.7)-(8.8) yields the following algebraic system of linear equations

$$V_h\underline{t} = (\frac{1}{2}M_h + K_h)\underline{g} - \underline{\tilde{N}_0}\, f, \tag{8.9}$$

where V_h is the matrix of the discrete single layer operator with size $2N$. Since V_h is symmetric and positive definite we solve (8.9) by using the CG method and the solutions are given in Table 8.4 and Table 8.5 below.

refinement		Nb. rings, Nb. Ite		L_2 Error, and eoc	
Level	N	Nb.rings	Nb. Ite.	$\|t - t_h\|_{L^2(\Gamma)}$	eoc
3	32	08	08	137.499	
4	64	16	08	68.7058	1.0009
5	128	32	07	34.3466	1.0003
6	256	64	06	17.1724	1.0001
7	512	128	06	8.58611	1.0000
8	1024	256	06	4.29304	1.0000
Theory					1

Table 8.4: The Number of rings and iterations, the L_2 error for the boundary stress and the order of convergence for the Dirichlet problem by using the CG method.

The pointwise evaluations are carried out at the point $\hat{x} = (1.0, 1.0)^\top$, and the results are given in the Table 8.5.

refinement		Approx.sol. u_{1h}, pt-wise error		Approx.sol. u_{2h}, pt-wise error	
Level	N	$u_{1h}(\hat{x})$	$\|u_1(\hat{x}) - u_{1h}(\hat{x})\|$	$u_{2h}(\hat{x})$	$\|u_2(\hat{x}) - u_{2h}(\hat{x})\|$
3	32	1.9984	0.00160395	-0.000769031	0.000769031
4	64	1.99964	0.000363557	-0.000149874	0.000149874
5	128	1.99991	8.62953e-05	-3.22676e-05	3.22676e-05
6	256	1.99998	2.09998e-05	-7.41494e-06	7.41494e-06
7	512	1.99999	5.178e-06	-1.77191e-06	1.77191e-06
8	1024	2.0	1.28549e-06	-4.32717e-07	4.32717e-07
Theory		2.0		0.0	

Table 8.5: The pointwise evaluation of the approximate solutions and the pointwise error for the Dirichlet problem.

Note that in Table 8.4 we observe a linear convergence of the boundary stress while in Table 8.5 we obtain a factor of four for the pointwise evaluation which are expected from the theory.

8.3 Numerical results for the linear elastostatic contact problems of Yukawa type

In this section we present the feasibility of the primal-dual active set strategy, and the fixed point algorithm we developed in chapter 6. These results are presented in three parts as follows, in the first part we present some numerical examples for the frictionless contact problem, the second part is concerned with the contact problem with given friction, the so-called Tresca problem, and the last part will be on the contact problem with Coulomb friction via the fixed point concept. For all tests, we use the normal to the contact boundary of the body and the normalized gap function

$$\mathbf{d}(x) = x_2 / \sqrt{\left(\frac{\partial \Phi(x)}{\partial x_1}\right)^2 + 1}.$$

Additionally, we assume that the elastic body occupies a disc centered at $(1.0, 1.0)$ with radius $R = 1.0$ and its boundary Γ is divided into three mutually disjoint parts Γ_D where we assume the body to be fixed in the horizontal direction, Γ_N where the traction is given and Γ_C the potential contact part, see Figure 8.2. For the material properties we choose the Young modulus $E = 200.0$, the Poisson ratio $\nu = 0.30$. For the boundary conditions we assume the following Dirichlet conditions on both sides of the disc $u_1 = 0.0$, but a vertical load is applied on the top given by $t = -20$ furthermore we set $\underline{f} \equiv 0$, and $s := 10.0$. Remark, that the Dirichlet boundary condition different in x_1 and x_2 direction is still valid for the model.

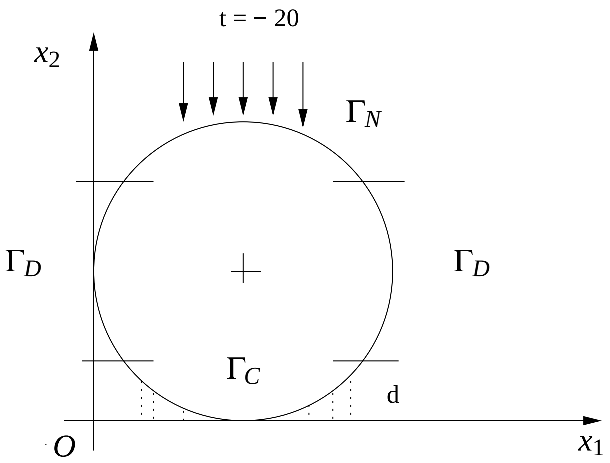
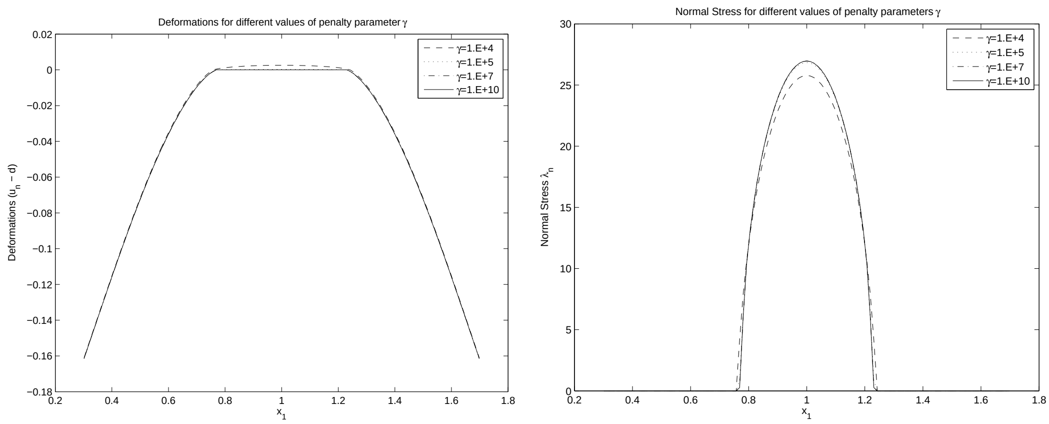


Figure 8.2: Geometry for numerical test.

In this section we will always initialize our algorithm with the solution of the contact problem where we have assumed that a symmetric part of the contact boundary is already in contact. Furthermore, the system at Step (4) of the semi-smooth Newton method will be solved by utilizing the CG and the preconditioned CG methods.

8.3.1 Numerical example for the frictionless contact problem

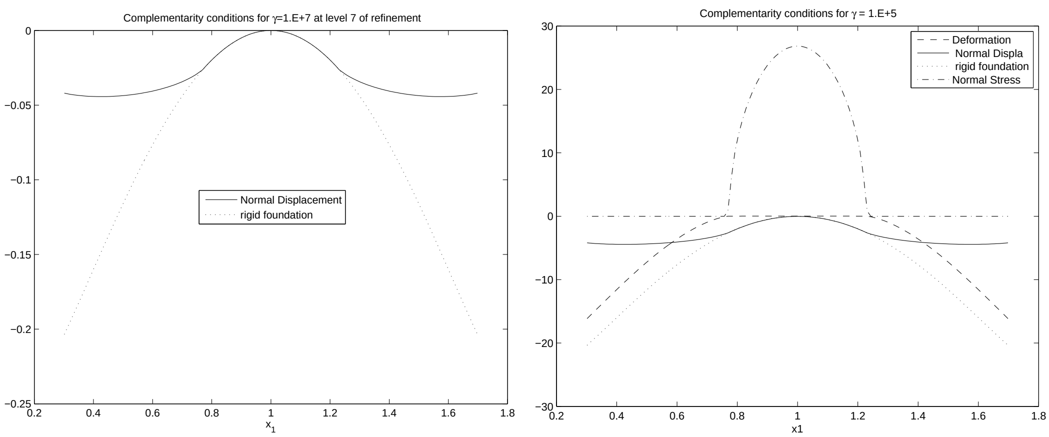
In this part we assume the friction to be negligible, that is $\mu_\gamma = 0.0$. Further, for the initialization of the algorithm we set $\widehat{\lambda} = 0.0$ and all the graphics are done at level 8 of refinement. The semi-smooth Newton algorithm always converges after a few number of iterations. We first investigate the influence of the penalty parameter γ_1 on the solutions.



(a) (b)

Figure 8.3: (a) Deformations, and (b) Stresses for different values of γ_1.

In Figure 8.3 we can observe that the solutions strongly depend on the penalty parameter γ_1. Furthermore, for $\gamma_1 = 10^{+4}$, although the algorithm converges, we have a small penetration, see Figure 8.3 (a), but for $\gamma_1 \geq 10^{+5}$ we can observe a quite nice resolution as expected from the theory.



(a) (b)

Figure 8.4: (a) Non penetrability, and (b) Complementarity conditions.

The above Figure 8.4 represents the non penetrability conditions for $\gamma_1 \geq 10^{+7}$ and the complementarity conditions for $\gamma_1 = 10^{+5}$ both at the level 7 of refinement. Furthermore, in the Figure 8.4 (b) the normal displacement, deformation and the gap (rigid foundation) are multiplied by 100. Next, we comment about the performance of the semi-smooth Newton method and CG methods. Table 8.6 and Table 8.7 show the number of iterations needed for the algorithm to converge at level 5, 6, 7 and 8 for different values of the penalty parameter γ_1, when the CG without preconditioner and preconditioned CG methods are applied on the system at Step (4).

refinement		γ_1			
Level	N	10^4	10^5	10^7	10^8
5	128	3/ 49	3/ 52	3/ 52	3/ 53
6	256	3/ 69	4/ 79	4/ 79	4/ 65
7	512	5/ 89	4/ 114	4/ 107	4/ 95
8	1024	5/ 115	6/ 158	6/ 156	6/ 140

Table 8.6: Number of iterations with respect to γ_1 and the level of refinement, the first number is the number of iterations for the Newton method and the second for the CG without preconditioner.

refinement		γ_1			
Level	N	10^4	10^5	10^7	10^8
5	128	3/ 29	3/ 36	3/ 44	3/ 48
6	256	3/ 33	4/ 46	4/ 61	4/ 62
7	512	5/ 32	4/ 54	4/ 83	4/ 85
8	1024	5/ 32	6/ 66	6/ 110	6/ 125

Table 8.7: Number of iterations with respect to γ_1 and the level of refinement, the first number is the number of iterations for the Newton method and the second for the preconditioned CG.

We observe that the semi-smooth Newton method converges for few iterations, and in addition, the number of iterations depends very little on the penalty parameter γ_1, and stay even constant for $\gamma_1 \geq 10^{+5}$, see Table 8.6 and Table 8.7. But, we can notice the influence of the penalty parameter γ_1 on the number of iterations for the CG methods. When the preconditioned CG is used we observe a considerable reduction of the number of iterations for the small parameters while the reduction is quite little for bigger parameters. On the other hand, the semi-smooth Newton method depends also quite little on the mesh refinement.

We now investigate the superlinear convergence of the semi-smooth Newton method, the Table 8.8 below represents the values

$$q_\lambda^k := \frac{\|\lambda_\gamma - \lambda^{k+1}\|}{\|\lambda_\gamma - \lambda^k\|} \quad \text{for} \quad k = 1,2\ldots \tag{8.10}$$

at level 8 of refinement.

γ_1	Superlinear convergence variables			
	q_λ^1	q_λ^2	q_λ^3	q_λ^4
10^5	0.4606	0.4183	0.2126	0.0
10^7	0.4676	0.4296	0.2178	0.0
10^8	0.4705	0.4367	0.2258	0.0

Table 8.8: Variables for superlinear convergence of λ_γ.

We observe that q^k decreases close to the solution λ_γ indicating the local superlinear convergence of the semi-smooth Newton method. The Table 8.8 motivates the application of a continuation procedure with respect to the refinement level (or nested iteration strategy), that is one solves the problem at the coarse level, and utilizes the solution as the initialization for the fine level. As can be seen from Table 8.9 below this strategy reduces the number of iterations and makes the semi-smooth Newton method almost mesh-independent.

refinement		γ_1		
Level	N	10^5	10^7	10^8
5	128	2/36	2/44	2/48
6	256	3/46	3/61	3/62
7	512	3/53	3/83	3/85
8	1024	4/66	4/110	4/120

Table 8.9: The first number is number of iterations for the Newton method and the second is the number of iterations for preconditioned CG method.

In addition, we observe also a monotone behavior of the semi-smooth Newton method, that is the size of the active set decreases in every iteration ($A_C^k \supset A_C^{k+1}$), see Table 8.10.

refinement		size of active set ($\gamma_1 = 10^5, 10^7, 10^8, 10^{10}$)						
Level	N	0	1	2	3	4	5	6
5	128	17	13	11	11			
6	256	33	25	21	19	19		
7	512	65	51	43	39	39		
8	1024	129	101	87	81	77	75	75

Table 8.10: Size of active set at each iteration of the Newton method.

Furthermore, we notice that the size of the active set does not depend upon the penalty parameter γ_1 as we observe in the case of the contact problem in linear elastostatics.

8.3.2 Numerical examples for contact with Tresca friction

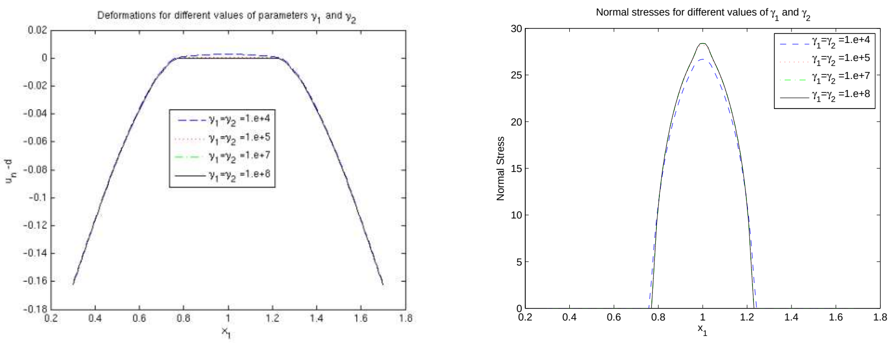
In contrast to the above subsection, here the tangential stress is too large to be neglected. Further, we assume it to be known and all other assumptions are still valid in this subsection.

Example 8.1. *For this first example, we consider the given friction*

$$g(x) = 50exp(-20(x_1 - 1.0)^2)$$

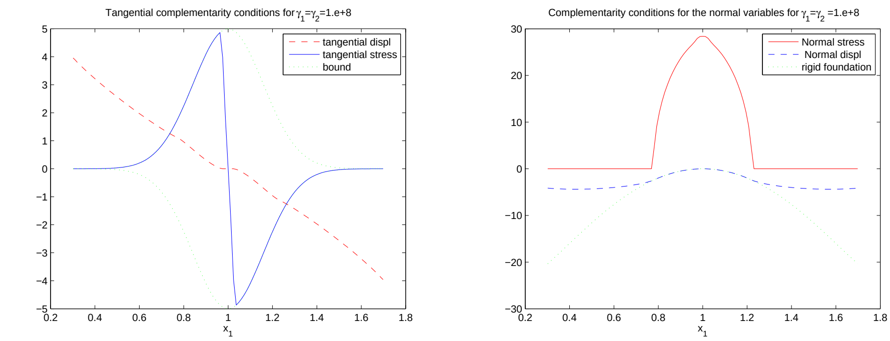
and the friction coefficient $\mathcal{F} = 0.10$. *Here we first investigate the influence of the parameters* γ_1, γ_2 *on the convergence of the solutions. Second, we discuss the performance of our algorithms with respect to* γ_1, γ_2 *and report on the number of iterations of the semi-smooth Newton method and CG method.*

The following figures show the depicted solutions we obtain at the level 7 of refinement.



(a) (b)

Figure 8.5: (a) Deformations, and (b) Stresses for different values of γ_1 and γ_2.



(a) (b)

Figure 8.6: Complementarity conditions for $\gamma_1 = \gamma_2 = 10^{+8}$.

Results for Example 8.1

We now summarize the results of example 1. In Figure 8.5 (a) there are depicted the deformations, i.e. $u_n - \mathbf{d}$ for different values γ_1, γ_2. Although the semi-smooth Newton algorithm stop for $\gamma_1 = \gamma_2 = 10^4$ after a few iterations, i.e. 05, we can observe that there is a small penetration for that case, see Figure 8.5 (a). But for $\gamma_1, \gamma_2 \geq 10^5$ the algorithm converges and we can observe a quite nice resolution when γ_1, γ_2 increase. In addition, for $\gamma_1 = \gamma_2 \geq 10^7$ no change occurs and the solution is obtained, see Figure 8.5 (a) and (b). On the other hand, for $\gamma_1 = \gamma_2 = 10^8$ the complementarity conditions are depicted in Figure 8.6 (a) and (b). The dual variable μ_γ (solid), the corresponding bounds $\pm\mathcal{F}g$ (dotted) and the tangential displacement (multiplied by 10^2, dashed) are presented in Figure 8.6 (a), while the normal complementarity conditions are given in Figure 8.6 (b) with the variable λ_γ (solid), rigid foundation, i.e. $-\mathbf{d}$ (multiplied by 10^2, dotted) and the negative normal displacement, i.e. $-u_n$ (multiplied by 10^2, dashed). One can observe from the graphs in Figure 8.6, (a) and (b) that the complementarity conditions hold. Remember that the active sets (A_{F+}, A_{F-}) for friction correspond to parts of the boundary where there is sliding in the tangential direction while the inactive set I_F corresponds to sticking regions, that are sets where $(u_\gamma)_t \cong 0$.
Let us now comment about the performance of our algorithm when the preconditioned CG and CG without preconditioner are used. The following tables show the number of iterations needed to reach the solutions, at level 6 and level 7 of refinement, and for different values of parameters γ_1, γ_2.

γ_2	γ_1				
	10^4	10^5	10^7	10^8	10^{10}
10^4	5/ 68	5 / 85	5/ 81	5/ 72	5/ 57
10^5	4/ 84	4/ 80	4/ 90	4/ 81	4/61
10^7	4/ 101	4/ 110	4/ 74	4/ 87	4/70
10^8	4/ 107	4/ 116	4/ 100	4/ 66	4/ 70
10^{10}	4/ 116	4/ 127	4/ 113	4/ 95	4/ 51

Table 8.11: The first number represents the number of iterations for the Newton method, the second for the CG method with respect to γ_1 and γ_2 at level 6.

γ_2	γ_1				
	10^4	10^5	10^7	10^8	10^{10}
10^4	5/ 90	5/120	5/117	5/104	5/80
10^5	5/114	5/114	5/137	5/122	5/97
10^7	5/138	5/170	5/108	5/133	5/105
10^8	5/144	5/182	5/152	5/97	5/107
10^{10}	5/154	5/195	5/170	5/146	5/73

Table 8.12: The first number represents the number of iterations for the Newton method, the second for the CG method with respect to γ_1 and γ_2 at level 7.

When the preconditioned CG method is used we obtain the following results:

γ_2	γ_1				
	10^4	10^5	10^7	10^8	10^{10}
10^4	5/ 34	5 / 47	5/ 68	5/ 72	5/ 66
10^5	4/ 58	4/ 50	4/ 75	4/ 80	4/70
10^7	4/ 58	4/ 74	4/ 66	4/ 78	4/81
10^8	4/ 68	4/ 86	4/ 84	4/ 68	4/ 81
10^{10}	4/ 83	4/ 109	4/ 118	4/ 114	4/ 63

Table 8.13: The first number represents the number of iterations for the Newton method, the second for the preconditioned CG method with respect to γ_1 and γ_2 at level 6.

γ_2	γ_1				
	10^4	10^5	10^7	10^8	10^{10}
10^4	5/ 34	5/57	5/95	5/102	5/90
10^5	5/69	5/56	5/107	5/115	5/106
10^7	5/69	5/96	5/88	5/119	5/134
10^8	5/80	5/119	5/114	5/90	5/130
10^{10}	5/104	5/159	5/188	5/172	5/89

Table 8.14: The first number represents the number of iterations for the Newton method, the second for the preconditioned CG method with respect to γ_1 and γ_2 at level 7.

One can observe from Tables 8.11-8.14 that the number of iterations for the semi-smooth Newton approach does not depend on the parameters γ_1 and γ_2. Nevertheless, this has a little dependency on the mesh grid. For the CG methods we observe that the number of iterations is less when $\gamma_1 = \gamma_2$, further, the number of iterations for the preconditioned CG are reduced when $\gamma_1, \gamma_2 \leq 10^8$.

Remark 8.1. *To obtain convergence of the algorithm independently of the initialization, we notice that we have to choose the parameter σ (parameter for the friction active sets) large enough, that is $\sigma = 1$ (which we use for our computation). By setting $\sigma = \frac{1}{\gamma_2}$ as suggested by the interpretation of the algorithm as infinite-dimensional semi-smooth Newton approach, we encounter some difficulties related to the convergence of the algorithm. For example, on a coarse level our algorithm converges but fails at a fine level. To overcome this problem one may initialize the algorithm as suggested in [71,98]. Therefore, by using $\sigma = 1$, the method converges for all initializations and furthermore, we observe locally fast convergence.*

In addition, as in the case of pure contact without friction, we also observe a monotone decreasing for the active sets of contact condition ($A_C^k \supset A_C^{k+1}$) during the iteration process, see Table 8.15 below. But, we do not observe the same behavior for the active sets of friction condition. Therefore, this example is said to be strongly contact dominant [98].

refinement		size of active set A_C ($\gamma_1 = \gamma_2 = 10^5, 10^7, 10^8, 10^{10}$)					
Level	N	0	1	2	3	4	5
6	256	33	25	21	19	19	
7	512	65	51	43	39	37	37

Table 8.15: Size of active set at each iteration of the Newton method.

We now turn to the investigation of the local superlinear convergence of the algorithm, the following table represents the values of

$$q_\lambda^k := \left(\frac{(S(u_\gamma - u^{k+1}), u_\gamma - u^{k+1})}{(S(u_\gamma - u^k), u_\gamma - u^k)} \right)^{1/2} + \frac{\|\lambda_\gamma - \lambda^{k+1}\|_{\Gamma_C}}{\|\lambda_\gamma - \lambda^k\|_{\Gamma_C}} + \frac{\|\mu_\gamma - \mu^{k+1}\|_{\Gamma_C}}{\|\mu_\gamma - \mu^k\|_{\Gamma_C}} \tag{8.11}$$

for $k = 1, 2 \ldots.$

$\gamma_1 = \gamma_2$	Superlinear convergence variables (level 7)		
	q_λ^1	q_λ^2	q_λ^3
10^7	2.3681	0.88112	0.0
10^8	2.3681	0.88154	0.0
10^{10}	2.3682	0.88102	0.0

Table 8.16: Variables for superlinear convergence for different parameters $\gamma_1 = \gamma_2$.

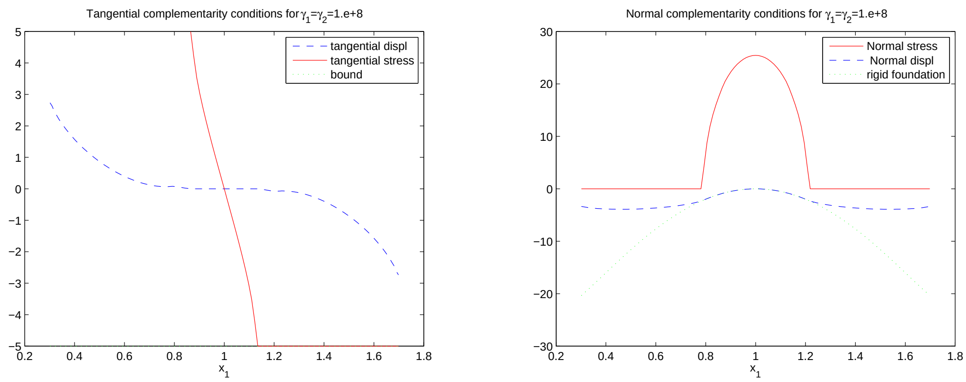
We also observe a decreasing of q^k close to the solution $(u_\gamma, \lambda_\gamma, \mu_\gamma)$ indicating the local superlinear convergence of the algorithm. Therefore, this suggests to utilize the nested iteration strategy, that is the continuation procedure with respect to the refinement level where one solves the problem at the coarse level, and uses the solutions as the initialization for the fine level. As one can notice in the Table 8.17, this process reduces the number of iterations. In addition, this makes the algorithm almost mesh independent.

refinement		$\gamma_1 = \gamma_2$			
Level	N	10^5	10^7	10^8	10^{10}
5	128	3	3	3	3
6	256	3	3	3	3
7	512	4	4	4	4

Table 8.17: Number of iterations for the nested process.

Example 8.2. *The aim of this example is to investigate the influence of the given friction on the deformation, the stress and on the performance of the algorithm, that is the number of iterations. To this end, we keep all the boundary conditions, the friction coefficient* $\mathcal{F} = 0.10$ *and the initialization the same as for the example 1. But, we set the given friction* $g(x) = 50$.

The following Figure 8.7 represents the results we obtained at the level 7 of refinement and after 6 iterations.



(a) (b)

Figure 8.7: Complementarity conditions for $\gamma_1 = \gamma_2 = 10^{+8}$.

Results for Example 8.2

For this example we observe the same behavior as in the previous case, however, we notice that for various regularization parameters the algorithm converges for few iterations. But, some more iterations are needed than in example 1, for example the semi-smooth Newton method converges after 05 and 06 iterations at level 6 and 7 of refinement respectively. We observe that the dual variable μ_γ and the tangential displacement really depend on the given friction g, this can be seen by comparing Figure 8.6 (a) and Figure 8.7 (a). In the last figure we can see that the size of the inactive set of friction is larger compared to the first one.

8.3.3 Numerical examples for contact problem with Coulomb friction

Example 8.3. *Here we use the same data and initialization as in the example above and set*

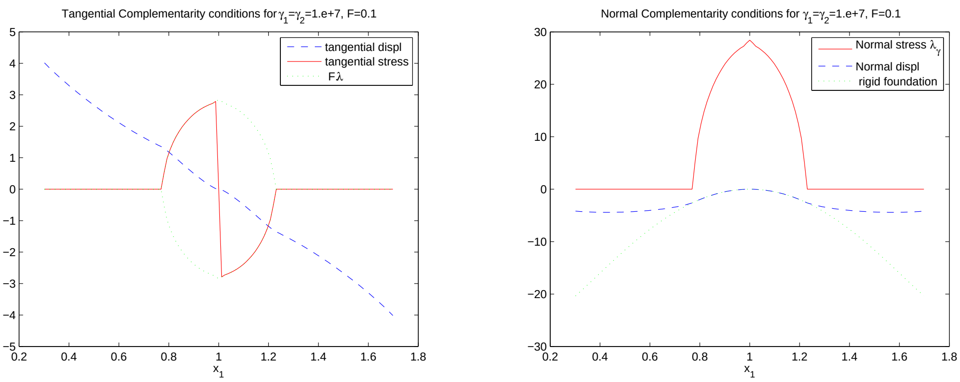
$$g^0(x) = 50exp(-20(x_1 - 1.0)^2).$$

Further, we report on the performance of algorithm (RCF-FP) and the influence of the friction coefficient on the solutions variables. To this end, we then examine the convergence of the algorithm for $\mathcal{F} = 0.1, 0.5, 1.0$. *The outer iteration (i.e., the fixed point iteration) is terminated if the following tolerance is reached*

$$Tol := \frac{\|g^m - g^{m-1}\|_{\Gamma_C}}{\|g^m\|_{\Gamma_C}} \leq \epsilon, \tag{8.12}$$

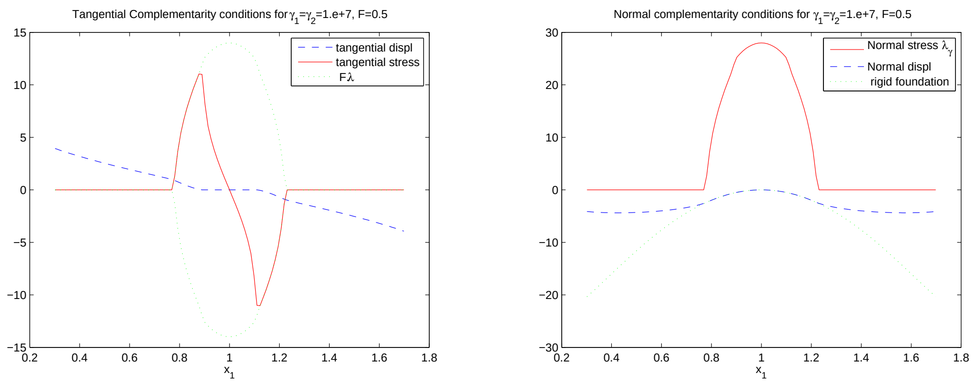
where $\epsilon := 10^{-7}$.

The solution variables are obtained after few outer iterations and are shown in Figures 8.8, 8.9 and 8.10 for $\gamma_1 = \gamma_2 = 10^7$ respectively.



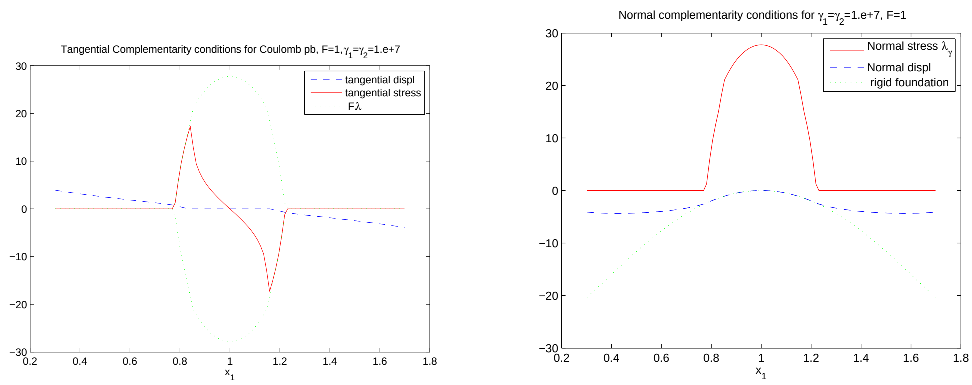
(a) (b)

Figure 8.8: Complementarity conditions for $\gamma_1 = \gamma_2 = 10^{+7}$ and $\mathcal{F} = 0.1$.



(a) (b)

Figure 8.9: Complementarity conditions for $\gamma_1 = \gamma_2 = 10^{+7}$ and $\mathcal{F} = 0.5$.



(a) (b)

Figure 8.10: Complementarity conditions for $\gamma_1 = \gamma_2 = 10^{+7}$ and $\mathcal{F} = 1.0$.

Results for Example 8.3

In Figures 8.8, 8.9, 8.10 (a) we have the dual variable μ_γ (solid), the bounds $\mathcal{F}\lambda_\gamma$ (dotted) and the tangential displacement u_t (multiplied by 10^2, dashed), while in Figures 8.8, 8.9, 8.10 (b) the normal stress λ_γ (solid), the normal displacement $-u_n$ (multiplied by 10^2, dashed) and the rigid foundation $-\mathbf{d}$ (multiplied by 10^2, dotted). The first observation we can notice out of these figures is that the complementarity conditions expected from theory hold. Second, we notice that the dual variable μ_γ depends on the bounds $\mathcal{F}\lambda_\gamma$, that is the friction coefficient as in the case of the Tresca problem. Further, we observe that the sticking zone or the friction inactive zone increases with the friction coefficient which is expected from physics.

Let us now comment about the performance of the fixed point algorithm, we notice that the algorithm converges quite fast (few iterations) for small friction coefficient but the number of iterations increases as the friction coefficient becomes bigger, see the table below.

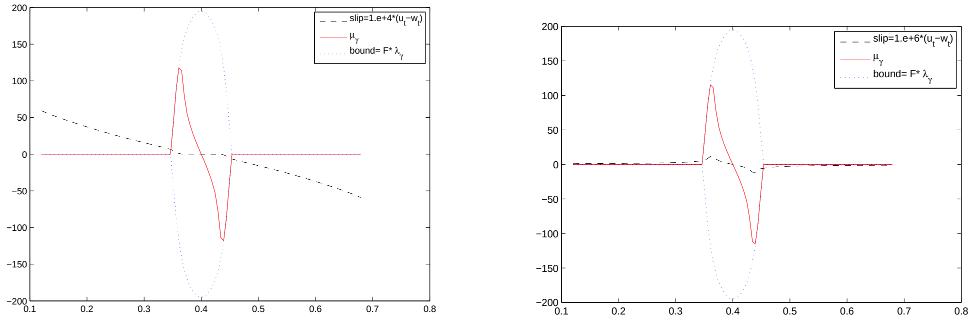
refinement		Coef of friction $\mathcal{F}$, $\gamma_1 = \gamma_2 = 10^7, 10^8$			
Level	N	0.1	0.5	0.7	1.0
6	256	5	8	9	11
7	512	5	9	9	11

Table 8.18: Number of fixed point iterations for different coefficient of friction.

8.4 Numerical examples for quasistatic contact problems with Coulomb friction

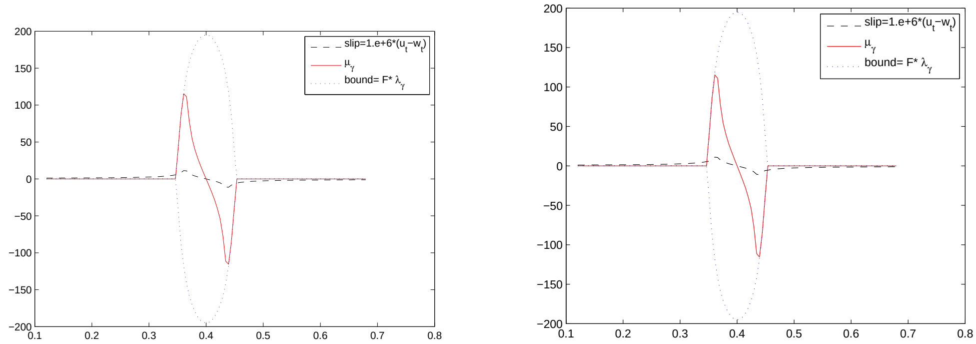
We consider the domain $\Omega := B_R(c)$ to be a two-dimensional disc of radius $R = 0.4$ and centered at $c = (0.4, 0.4)$, see Figure 8.2. In this section, to avoid the rigid motion to occur, the semi-smooth Newton algorithm will be initialized with a solution of a contact problem where we have assumed a symmetric part of the contact boundary Γ_C to be inactive, i.e. in I_F. In addition, we assume the Young modulus $E = 5000$, the Poisson ratio $\nu = 0.4$ and the coefficient of friction $\mathcal{F} = 0.5$. We set the volume force $\underline{f} = 0.0$, assume the body to be fixed in the horizontal direction, i.e. $u_1 = 0.0$ on Γ_D and set $\widehat{\lambda} = \widehat{\mu} = 0.0$.

Example 8.4. *For the first example we consider the Neumann data to be constant, i.e.* $\underline{g}_N^\top = (0.0, -50.0)$ *in addition, we use a uniform time step* $\delta t = 0.1$. *The problem is solved on a uniform grid with 512 nodes, the solutions are obtained after few iterations. In Figures 8.11,8.12 and Figures 8.13, 8.14 the tangential and normal complementarity conditions are depicted respectively for four time steps and for* $\gamma_1 = \gamma_2 = 10^{+7}$.



(a) (b)

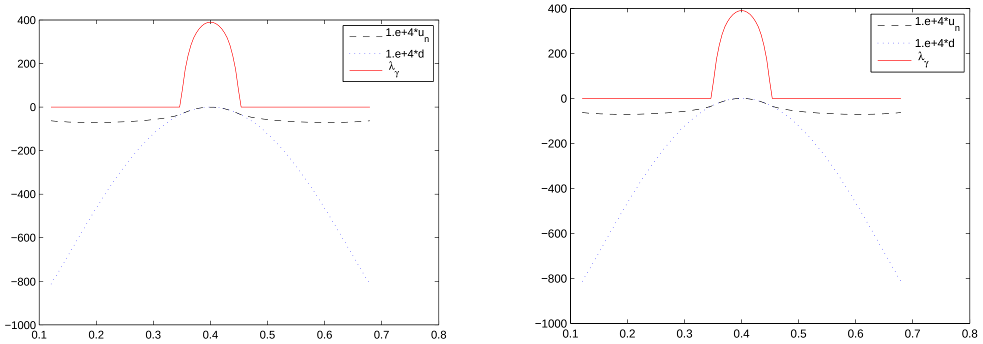
Figure 8.11: Tangential complementarity conditions first and second time step.



(a) (b)

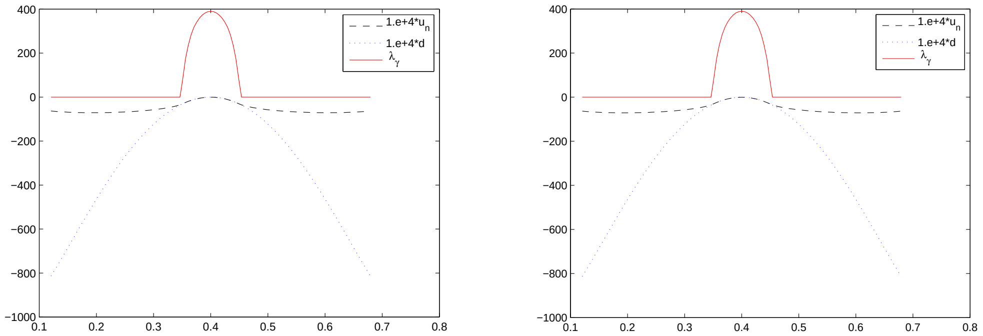
Figure 8.12: Tangential complementarity conditions third and fourth time step.

In Figures 8.11-8.12 the tangential stresse μ_γ is given in solid line, the dashed represents the slip velocity $u_t - w_t$ which is multiplied by 10^4 in the first time step and by 10^6 in other time steps, while the dotted represents the bound, i.e. $\mathcal{F}\lambda_\gamma$.



(a) (b)

Figure 8.13: Normal complementarity conditions first and second time step.

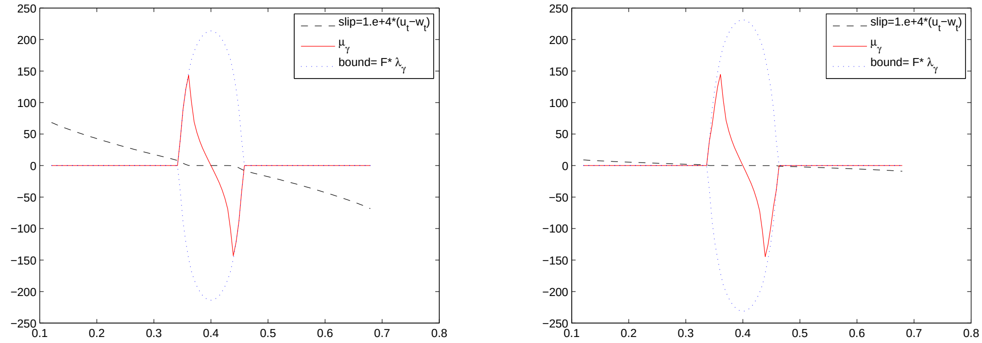


(a) (b)

Figure 8.14: Normal complementarity conditions third and fourth time step.

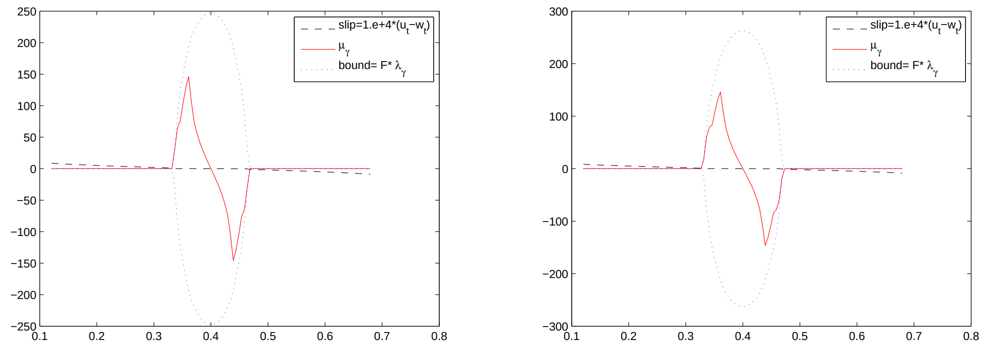
In Figures 8.13-8.14 the normal stresse λ_γ is given solid line, the dashed represents the normal displacement $-u_n$ which is multiplied by 10^4, while the dotted represents the gap, i.e. $-\mathbf{d}$.

Example 8.5. *For the second example we consider the same assumptions as in Example 8.4, but we assume the Neumann data to be time dependent* $\underline{g}_N^\top = (0.0, -100.0t - 50.0)$ *again the solutions are depicted in the Figures 8.15-8.18 for four time steps.*



(a) (b)

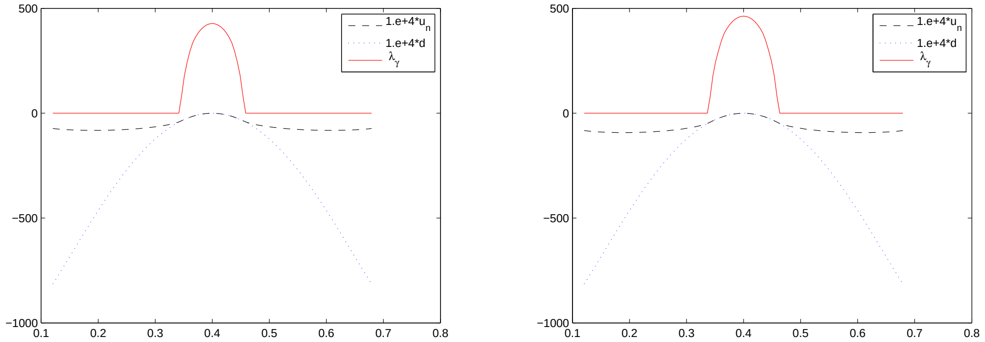
Figure 8.15: Tangential complementarity conditions first and second time step.



(a) (b)

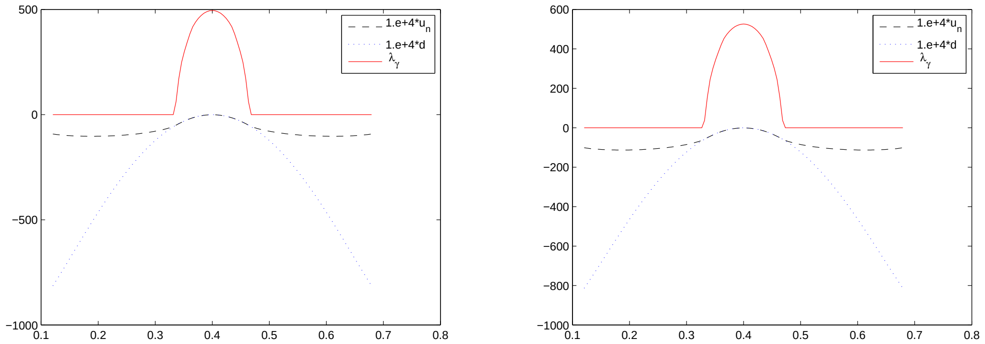
Figure 8.16: Tangential complementarity conditions third and fourth time step.

Again the tangential stresse μ_γ is given in solid line, the dashed represents the slip velocity $u_t - w_t$ which is multiplied by 10^4, while the dotted represents the bound, i.e. $\mathcal{F}\lambda_\gamma$. We can also notice that, this value increases from one time step to another which is due to the increasing of the magnitude of the Neumann datum $\underline{g}_N$ at each time step.



(a) (b)

Figure 8.17: Normal complementarity conditions first and second time step.



(a) (b)

Figure 8.18: Normal complementarity conditions third and fourth time step.

From Figures 8.17-8.18, due to the increasing of the magnitude of the Neumann datum $\underline{g}_N$, we observe that the normal stress λ_γ increases from one time step to another one. Note that the complementarity conditions hold at each time step. Further, we observe that our algorithm is faster from one time step to another, that is the number of iterations for the fixed point approach reduces from one time step to the next one which is the advantage of the semi-smooth Newton method.

	uniform time step				
	1	2	3	4	5
Iterations	7	4	4	4	3

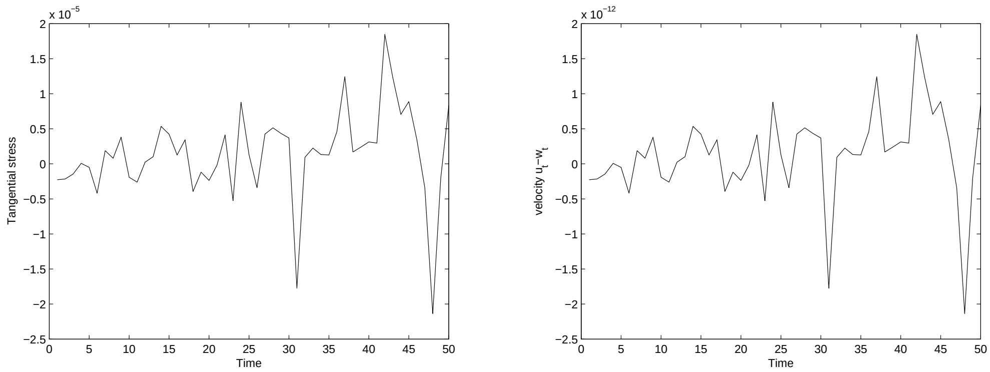
Table 8.19: Number of iterations for the fixed point for uniform time step.

Instead of the uniform time step if we consider a non-uniform time step so that this reduces from one time step to the next this number of iterations reduces then considerably. For example for the time discretization defined by $t_{n+1} := t_n + (\frac{1}{2})^n \delta t$, where n represents the number of step we obtain

	uniform time step				
	1	2	3	4	5
Iterations	7	4	2	2	1

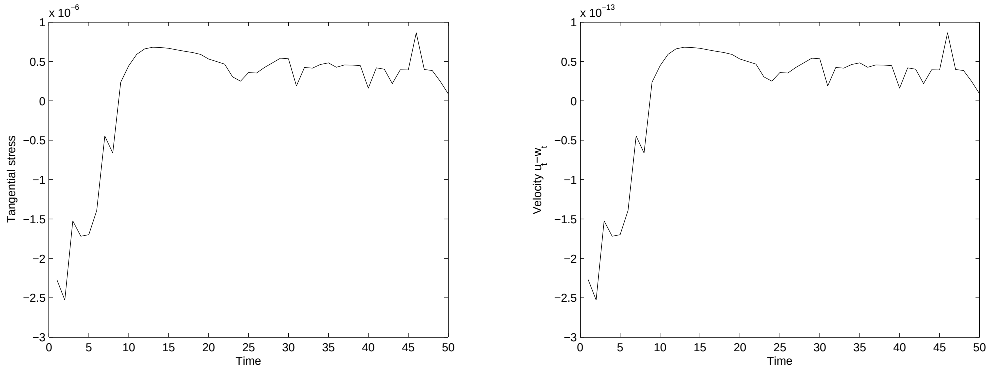
Table 8.20: Number of iterations for the fixed point for non-uniform time step.

Further, for 50 time steps we compute the tangential stresses, the slip velocity, the normal stresses and the normal displacement in the lowest point of the disc for $\underline{g}_N^\top = (0.0, -100.0t - 50.0)$. The results are depicted in the figures below:



(a) (b)

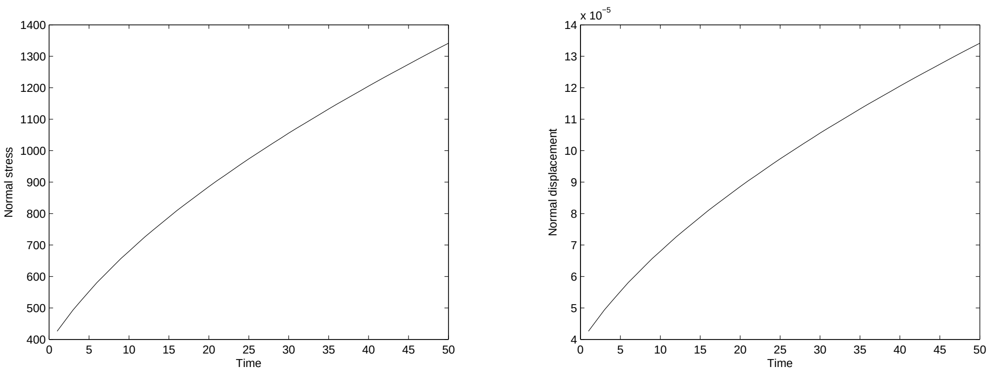
Figure 8.19: Tangential stress (a) and the slip velocity (b) for uniform time step.



(a) (b)

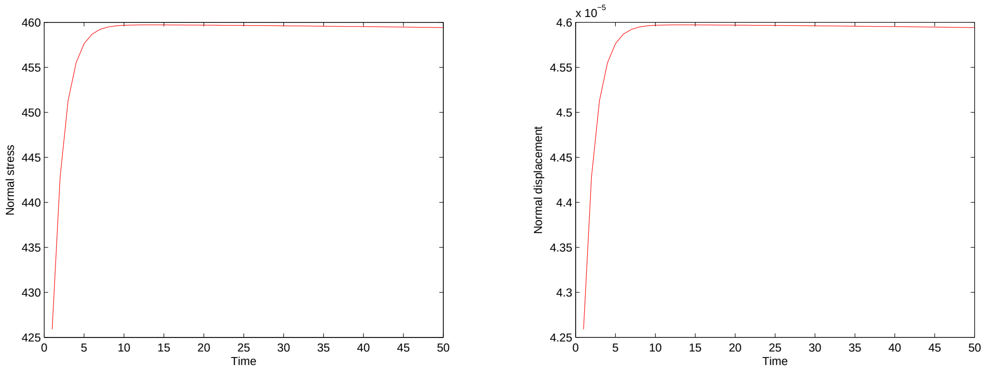
Figure 8.20: Tangential stress (a) and the slip velocity (b) for non-uniform time step.

The tangential stress and slip velocity at the lowest point of the disc compute by using the uniform time discretization Figure 8.19 are very noisy. But, Figure 8.20 shows that the tangential stress and slip velocity at the lowest point of the disc are more regular than the first one.



(a) (b)

Figure 8.21: Normal stress (a) and the normal displacement (b) for uniform time step.



(a) (b)

Figure 8.22: Normal stress (a) and the normal displacement (b) for non-uniform time step.

Figures 8.21, 8.22 show that the normal stress and the normal displacement in the lowest point of the disc are more regular with respect to time.

Note that in this section for the solution of the linear subproblems at Step (4) of the semi-smooth Newton method, a preconditioned conjugate gradient method were used. As a preconditioner for the Steklov-Poincaré operator we employ the discretization $\overline{V}$ of the single layer operator as in equation (7.5). In order to realize the multiplication of the coefficient matrix $\tilde{S}_h$ with a vector, a linear equation like (7.6) must be solved. This is realized by using the discrete Fourier matrix

$$\begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}$$

as a preconditioner for the matrix V_h of the discrete single layer operator or by using directly the inverse of the matrix V_h, see section 7.5.

9 CONCLUSIONS

In this chapter we review the main contributions of this work. Our goal in this thesis was to develop and analyze efficient and reliable numerical methods for the solution of contact problems in linear elasticity of Yukawa type and quasistatic contact problems.

After an introductory chapter (chapter 2) on various results necessary for the analysis and the simulation of the problem, we focus in chapter 3 on the foundation of the mechanics of continua including the derivation of the state equations of linear elasticity under the assumption of infinitesimal deformations. Further, general nonlinear contact conditions and the Coulomb friction law for an elastic body coming into contact with a rigid foundation were given.

In chapter 4 and chapter 5 the scalar Yukawa problem and the static elasticity problem of Yukawa type are considered respectively. Further, we give the boundary integral formulations of both problems and derive eigenfunctions of some operators related to the problems.

In chapter 6 which is the backbone of this thesis, the application of the new combined boundary integral methods and the semi-smooth Newton methods to the contact problems in linear elasticity of Yukawa type with Coulomb friction have been analyzed. The approach taken here consists to approximate the problem by a sequence of auxiliary problems, the so-called Tresca problems. Note that each Tresca problem is equivalent to a non-differentiable minimization problem (primal problem). But, by using the Fenchel duality theorem [31] the non-differentiable minimization problem is transformed into an inequality constrained maximization of a smooth functional (dual problem). Instead of using only the first order necessary conditions of the optimization problem, which are usually the starting points of the analysis, we consider as well the extremality conditions which characterize the solutions of primal and dual problems for our investigation. Another important aspect of this work is the introduction of non-linear max- and min- operators which enable us to write the complementarity conditions as non-linear operator equations. But, due to the lack of regularity of function spaces the regularization of the dual and primal problems motivated by the augmented Lagrangian turns out to be suitable for the application of the generalized Newton method. We then show that the solutions of the regularized problems converge to the solutions of the original problems as the parameters of the regularization tend to infinity. Instead of the original problems we then consider a sequence of regularized problems for the analysis of our algorithm. Additionally, we prove the existence of a fixed point for the sequence of regularized problems, that is the existence of a solution for the regularized Coulomb friction problem. The uniqueness is shown provided the friction coefficient is sufficiently small.

In chapter 7 we extend the theories we developed in chapter 6 for the resolution of a quasistatic contact problem with Coulomb friction.

In chapter 8 some comprehensive numerical examples are carried out for our algorithms. The combined semi-smooth Newton approach and the boundary element methods turns out to be suitable for the contact with and without friction in 2D. This yields a remarkable efficiency and reliability, the algorithms always detect the solution after few iterations (usually 3-6). In addition, we investigated in our numerical tests the dependence of our algorithms on the regularization parameters and the mesh. Furthermore, when the nested iteration principles is used we can also confirm the superlinear convergence of the semi-smooth Newton algorithm.

In the future the algorithm will be extended into two different directions: on the one hand to time dependent problems with wear calculation, and on the other hand to more realistic three-dimensional problems.

A APPENDIX

A.1 Computation of Newton potential

In this section we present the detailed procedure for the evaluation of the vector related to the Newton potential $N_0 f$

$$N_0 f[i] = \int_{\Gamma} \phi_i^{(0)}(x) \int_{\Omega} f(y) U^*(x,y) dy ds_x \quad \text{for } i = 1,\ldots,N,$$

assuming the B-spline of order zero $\phi_i^{(0)}$ for basis functions.

First, we interchange the order of integration as follows

$$N_0 f[i] = \int_{\Omega} f(y) \int_{\tau_i} U^*(x,y) ds_x dy \quad \text{for } i = 1,\ldots,N. \tag{A.1}$$

Here, we consider the particular case of a two-dimensional circular domain $\Omega = B_R(c)$ of radius R and centered at the point c.

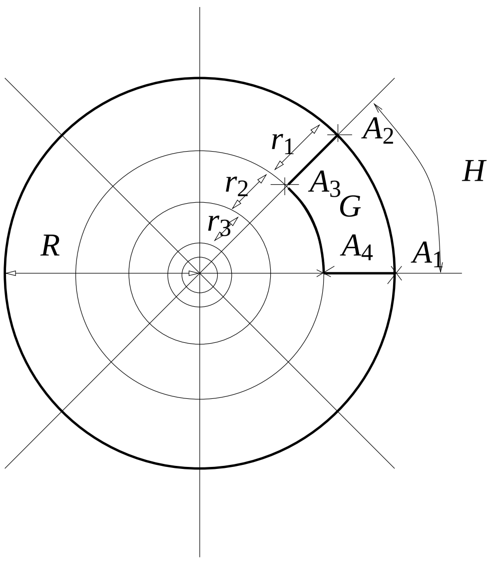


Figure A.1: Volume and boundary meshes.

Note that an efficient evaluation of (A.1) requires a suitable discretization of the domain Ω and the boundary $\Gamma := \partial\Omega$. To this end, we start by dividing the domain into rings. This is done in such a way that the thickness of the ring near the boundary is equal to the mesh size at the boundary, that is $r_1 = H$. Further, the thickness of the inner ring is reduced at the rate $q < 1$ when we are leaving from the boundary to an inner point, i.e. $r_{j+1} = qr_j$ for $j = 1,\dots,(M_r - 1)$, where M_r is the number of ring. Next, a uniform mesh is constructed on each ring in such a way that elements on the ring near the boundary match with boundary elements, see Figure A.1.

The next step consists to find the values of q and M_r for a given value of a boundary mesh size $H = \frac{2\pi R}{N}$, where N is the number of elements on the boundary as well as in each ring. Note that

$$R := \sum_{j=1}^{M_r} r_j = \sum_{j=0}^{(M_r-1)} q^j H = H\frac{1-q^{M_r}}{1-q} \quad \text{for } q \neq 1,$$

which yields the following equation

$$q^{M_r} - \frac{N}{2\pi}q + \frac{N-2\pi}{2\pi} = 0 \quad \text{with } q \neq 1. \tag{A.2}$$

The equation (A.2) can be solved by performing the Newton Raphson method for given N and M_r, the results are given in Table A.1 below.

Level of refinement	N	minimum value of M_r	q
0	04	impossible	impossible
1	08	02	0.27324
2	16	03	0.840328
3	32	06	0.934003
4	64	11	0.9845
5	128	21	0.996951
6	256	41	0.999686
7	512	82	0.999845
8	1024	163	0.999998
9	2048	326	0.999999
10	4096	653	0.999995

Table A.1: Values of q for given values of N and M_r.

Remark that in the table above the values of M_r are the minimum values for which the equation (A.2) is solvable. In addition, for any value of M_r greater than its minimum value the equation (A.2) is still solvable.

Having all the necessary data we can proceed to the evaluation of (A.1). Then we have, for $i = 1,...,N$,

$$N_0 f[i] = \int_{\Omega} f(y) \int_{\tau_i} U^*(x,y) ds_x dy = \sum_{j=1}^{M_r} \sum_{k=1}^{N} \int_{T_{jk}} f(y) \int_{\tau_i} U^*(x,y) ds_x dy, \tag{A.3}$$

where T_{jk} is an isoparametric quadrangle, or isoparametric triangle when we are on the last inner ring, see Figure A.1. Next, f is approximated on each ring by a piece-wise constant functions, we obtain therefore

$$\tilde{N}_0 f[i] = \sum_{j=1}^{M_r} \sum_{k=1}^{N} |T_{jk}| f(y_{jk}) \int_{\tau_i} U^*(x, y_{jk}) ds_x \quad \text{for } i = 1,...,N, \tag{A.4}$$

where y_{jk} and $|T_{jk}|$ represent the center of mass and the volume of element T_{jk} respectively. Note that, (A.4) can be written in a matrix form as follows,

$$\underline{\tilde{N}}_0 f := \sum_{j=1}^{M_r} A_j \underline{f}_j, \tag{A.5}$$

where for $j = 1,...,M_r$, matrices A_j and vectors $\underline{f}_j$ are given by

$$\underline{f}_j[k] := |T_{jk}| f(y_{jk}) \quad \text{for } k = 1,...,N \tag{A.6}$$

and

$$A_j[i,k] := \int_{\tau_i} U^*(x, y_{jk}) ds_x \quad \text{for } i,k = 1,...,N, \tag{A.7}$$

respectively. On the other, we have

$$A_j[i+1, k+1] := \int_{\tau_{i+1}} U^*(x, y_{jk+1}) ds_x. \tag{A.8}$$

We remark that the boundary element τ_{i+1} and the center of mass y_{jk+1} are the images of the boundary element τ_i and the center of mass y_{jk} respectively by the affine rotation $Rot(c)$ with the center at c the center of the disc and the associated matrix given by

$$\overrightarrow{Rot}(c) = \begin{pmatrix} \cos 2\beta & -\sin 2\beta \\ \sin 2\beta & \cos 2\beta \end{pmatrix}, \tag{A.9}$$

where $\beta = \frac{\pi}{N}$, and N the number of boundary element as well as in each ring. Since the fundamental solution U^* is invariant with respect to rotations, we then obtain for $j = 1,...,M_r$

$$A_j[i+1, k+1] := \int_{\tau_{i+1}} U^*(x, y_{jk+1}) ds_x = \int_{\tau_i} U^*(x, y_{jk}) ds_x = A_j[i,k] \tag{A.10}$$

for $i,k=1,\ldots,N$. This implies that the matrices A_j are all circulant [23].

Note that since the matrices A_j are circulant [23,92–94], we do not need to compute all the entries. The first row (or first column) is enough to determine the matrix. Further, we have to show how to compute the center of mass of each element which is essential to perform the computation of the matrix A_j and the vector $\underline{f}_j$. To this end, we consider for a simple illustration a circular sector centered at c, with central angle 2β (in radians), and radius r and having only two elements, see Figure A.2.

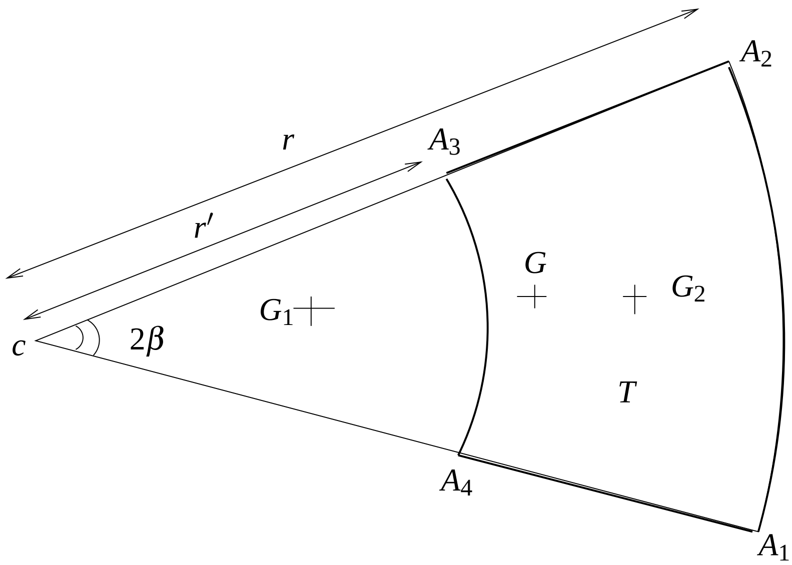


Figure A.2: The center of mass.

The coordinates of the center of mass G of the circular sector (cA_1A_2) are given by

$$\overrightarrow{cG} = \frac{r}{3\beta}\begin{pmatrix}\sin 2\beta\\(1-\cos 2\beta)\end{pmatrix}. \tag{A.11}$$

In a similar way the coordinates of the center of mass G_1 of the element (cA_4A_3) are given by

$$\overrightarrow{cG_1} = \frac{r'}{3\beta}\begin{pmatrix}\sin 2\beta\\(1-\cos 2\beta)\end{pmatrix}. \tag{A.12}$$

By utilizing the relation

$$m_1\overrightarrow{GG_1} + m_2\overrightarrow{GG_2} = \overrightarrow{O},$$

where m_1 and m_2 represent the masses of the element (cA_4A_3) and of the element $(A_4A_1A_2A_3)$ respectively. By using the fact that the circular sector is homogeneous we obtain then the coordinates of the center of mass G_2 of element $(A_1A_2A_3A_4)$

$$\overrightarrow{cG_2} = \frac{r^2 + rr' + (r')^2}{3\beta(r+r')} \begin{pmatrix} \sin 2\beta \\ (1-\cos 2\beta) \end{pmatrix}. \tag{A.13}$$

We also remark that to compute the entries of the vector $\underline{f}_j$ and the entries of the matrix A_j, we need only to compute the center of masses of elements in the first circular sector of the domain, the others are computed by applying the affine rotation $Rot_k(c)$ with center c and the associated matrix given by

$$\overrightarrow{Rot}_k(c) = \begin{pmatrix} \cos 2k\beta & -\sin 2k\beta \\ \sin 2k\beta & \cos 2k\beta \end{pmatrix} \quad \text{for } k = 0, \ldots, (N-1), \tag{A.14}$$

where $k+1$ is the position of element in the ring.

Note that since the matrices A_j for $j = 1, \ldots, M_r$ are circulant the vector $\underline{\tilde{N}}_0 f$ in (A.5) can be written as the circular convolution as follows

$$\underline{\tilde{N}}_0 f := \sum_{j=1}^{M_r} A_j \underline{f}_j = \sum_{j=1}^{M_r} \underline{C}_j * \underline{f}_j, \tag{A.15}$$

where $\underline{C}_j$ is the first column of matrix A_j. By applying the discrete Fourier transform F on (A.15) and by using its linearity this is transformed into component-wise multiplication as follows

$$F(\underline{\tilde{N}}_0 f) = \sum_{j=1}^{M_r} F(\underline{C}_j) F(\underline{f}_j). \tag{A.16}$$

Further, if we apply the inverse discrete Fourier transform on (A.16) and use again the linearity we then obtain

$$\underline{\tilde{N}}_0 f = \sum_{j=1}^{M_r} F^{-1}(F(\underline{C}_j) F(\underline{f}_j)). \tag{A.17}$$

REFERENCES

[1] M. Abramowitz and Irene A. Stegun. *Handbook of mathematical functions, with Formulas, Graphs, and Mathematical Tables*. Dover Publications, New York, 1970.

[2] R. A. Adams. *Sobolev spaces*. Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[3] S. Amini. On the choice of the coupling parameter in boundary integral formulations of the exterior acoustic problem. *Appl. Anal.*, 35(1-4):75–92, 1990.

[4] S. Amini. Boundary integral solution of the exterior Helmholtz problem. *Comput. Mech.*, 13(1-2):2–11, 1993. International Symposium on Boundary Element Methods (Boulder, CO, 1992).

[5] L. E. Andersson. A quasistatic frictional problem with normal compliance. *Nonlinear Anal.*, 16(4):347–369, 1991.

[6] L. E. Andersson. A quasistatic frictional problem with a normal compliance penalization term. *Nonlinear Anal.*, 37(6, Ser. A: Theory Methods):689–705, 1999.

[7] L. E. Andersson. Existence results for quasistatic contact problems with Coulomb friction. *Appl. Math. Optim.*, 42(2):169–202, 2000.

[8] K. T. Andrews, M. Shillor, and S. Wright. On the dynamic vibrations of an elastic beam in frictional contact with a rigid obstacle. *J. Elasticity*, 42(1):1–30, 1996.

[9] Y. Ayyad and M. Sofonea. Analysis of two dynamic frictionless contact problems for elastic-visco-plastic materials. *Electron. J. Differential Equations*, pages No. 55, 17 pp., 2007.

[10] P. Bisegna, F. Lebon, and F. Maceri. Relaxation procedures for solving signorini-Coulomb contact problems. *Adv. in Engineering Software*, 35(1):595–600, 2004.

[11] H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[12] F. E. Browder. On a generalization of the Schauder fixed point theorem. *Duke Math. J.*, 26:291–303, 1959.

[13] M. T. Cao and P. Quintela. Existence of solutions for a dynamic Signorini's contact problem. *C. R. Math. Acad. Sci. Paris*, 343(5):355–360, 2006.

[14] M. T. Cao-Rial, P. Quintela, and C. Moreno. Numerical solution of a time-dependent Signorini contact problem. *Discrete Contin. Dyn. Syst.*, (Dynamical Systems and Differential Equations. Proceedings of the 6th AIMS International Conference, suppl.):201–211, 2007.

[15] C. Y. Chao. A remark on symmetric circulant matrices. *Linear Algebra Appl.*, 103:133–148, 1988.

[16] M. K. Chen. On the solution of circulant linear systems. *SIAM J. Numer. Anal.*, 24(3):668–683, 1987.

[17] X. Chen, Z. Nashed, and L. Qi. Smoothing methods and semismooth methods for nondifferentiable operator equations. *SIAM J. Numer. Anal.*, 38(4):1200–1216, 2000.

[18] P. W. Christensen and J. S. Pang. Frictional contact algorithms based on semismooth Newton methods. In *Reformulation: nonsmooth, piecewise smooth, semismooth and smoothing methods (Lausanne, 1997)*, volume 22 of *Appl. Optim.*, pages 81–116. Kluwer Acad. Publ., Dordrecht, 1999.

[19] P. G. Ciarlet. *Mathematical elasticity. Vol. III*, volume 29 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 2000. Theory of shells.

[20] M. Cocu. Existence of solutions of a dynamic Signorini's problem with nonlocal friction in viscoelasticity. *Z. Angew. Math. Phys.*, 53(6):1099–1109, 2002.

[21] M. Cocu, E. Pratt, and M. Raous. Formulation and approximation of quasistatic frictional contact. *Internat. J. Engrg. Sci.*, 34(7):783–798, 1996.

[22] M. Cocu and R. Rocca. Existence results for unilateral quasistatic contact problems with friction and adhesion. *M2AN Math. Model. Numer. Anal.*, 34(5):981–1001, 2000.

[23] P. J. Davis. *Circulant matrices*. Wiley, New York, 1979.

[24] Z. Dostál, J. Haslinger, and R. Kučera. Implementation of the fixed point method in contact problems with Coulomb friction based on a dual splitting type technique. In *Proceedings of the 9th International Congress on Computational and Applied Mathematics (Leuven, 2000)*, volume 140, pages 245–256, 2002.

[25] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics*. Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[26] C. Eck. *Existenz und Regularität der Lösungen für Kontaktprobleme mit Reibung*. Dissertation, Mathematisches Institut A, Universität Stuttgart, 1996.

[27] C. Eck and J. Jarušek. Existence results for the static contact problem with Coulomb friction. *Math. Models Methods Appl. Sci.*, 8(3):445–468, 1998.

[28] C. Eck and J. Jarušek. Existence results for the semicoercive static contact problem with Coulomb friction. *Nonlinear Anal.*, 42(6, Ser. A: Theory Methods):961–976, 2000.

[29] C. Eck, J. Jarušek, and M. Krbec. *Unilateral contact problems*, volume 270 of *Pure and Applied Mathematics (Boca Raton)*. Chapman & Hall/CRC, Boca Raton, FL, 2005. Variational methods and existence theorems.

[30] C. Eck, O. Steinbach, and W. L. Wendland. A symmetric boundary element method for contact problems with friction. *Math. Comput. Simulation*, 50(1-4):43–61, 1999. Modelling '98 (Prague).

[31] I. Ekeland and R. Témam. *Convex analysis and variational problems*, volume 28 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[32] F. Facchinei and J.-S. Pang. *Finite-dimensional variational inequalities and complementarity problems. Vol. I*. Springer Series in Operations Research. Springer-Verlag, New York, 2003.

[33] F. Facchinei and J.-S. Pang. *Finite-dimensional variational inequalities and complementarity problems. Vol. II*. Springer Series in Operations Research. Springer-Verlag, New York, 2003.

[34] M. Fortin and R. Glowinski. *Augmented Lagrangian methods*, volume 15 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1983. Applications to the numerical solution of boundary value problems.

[35] R. Glowinski. *Numerical methods for nonlinear variational problems*. Springer Series in Computational Physics. Springer-Verlag, New York, 1984.

[36] R. Glowinski and P. Le Tallec. *Augmented Lagrangian and operator-splitting methods in nonlinear mechanics*, volume 9 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.

[37] M. A. Golberg and C. S. Chen. The theory of radial basis functions applied to bem for inhomogeneous partial differential equations. *Bound. Elem. Comm.*, 5:57–61, 1994.

[38] R. M. Gray. Toeplitz and circulant matrices: A review. 2001, http://ee.stanford.edu/ gray/toeplitz.pdf.

[39] H. D. Han. The boundary integro-differential equations of three-dimensional Neumann problem in linear elasticity. *Numer. Math.*, 68:269–281, 1994.

[40] W. Han and B. D. Reddy. *Plasticity*, volume 9 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, 1999. Mathematical theory and numerical analysis.

[41] J. Haslinger, R. Kučera, and Z. Dostál. An algorithm for the numerical realization of 3D contact problems with Coulomb friction. In *Proceedings of the 10th International Congress on Computational and Applied Mathematics (ICCAM-2002)*, volume 164/165, pages 387–408, 2004.

[42] J. Haslinger and O. Vlach. Approximation and numerical realization of 2D contact problems with Coulomb friction and a solution-dependent coefficient of friction. *J. Comput. Appl. Math.*, 197(2):421–436, 2006.

[43] P. Henrici. Fast fourier methods in computational complex analysis. *SIAM*, 21:481–527, 1979.

[44] P. Hild. An example of nonuniqueness for the continuous static unilateral contact model with Coulomb friction. *C. R. Math. Acad. Sci. Paris*, 337(10):685–688, 2003.

[45] P. Hild. Non-unique slipping in the Coulomb friction model in two-dimensional linear elasticity. *Quart. J. Mech. Appl. Math.*, 57(2):225–235, 2004.

[46] M. Hintermüller. A primal-dual active set algorithm for bilaterally control constrained optimal control problems. *Quart. Appl. Math.*, 61(1):131–160, 2003.

[47] M. Hintermüller, K. Ito, and K. Kunisch. The primal-dual active set strategy as a semismooth Newton method. *SIAM J. Optim.*, 13(3):865–888, 2003.

[48] M. Hintermüller, V. Kovtunenko, and K. Kunisch. Semismooth Newton methods for a class of unilaterally constrained variational problems. *Adv. Math. Sci. Appl.*, 14(2):513–535, 2004.

[49] M. Hintermüller and K. Kunisch. Total bounded variation regularization as a bilaterally constrained optimization problem. *SIAM J. Appl. Math.*, 64(4):1311–1333, 2004.

[50] M. Hintermüller and M. Ulbrich. A mesh-independence result for semismooth Newton methods. *Math. Program.*, 101(1, Ser. B):151–184, 2004.

[51] L. Hörmander. *The analysis of linear partial differential operators. I*, volume 256 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.

[52] L. Hörmander. *The analysis of linear partial differential operators. II*, volume 257 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1983. Differential operators with constant coefficients.

[53] G. C. Hsiao and W. L. Wendland. *Boundary integral equations*, volume 164 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 2008.

[54] S. Hüeber, G. Stadler, and B. I. Wohlmuth. A primal-dual active set algorithm for three-dimensional contact problems with Coulomb friction. *SIAM J. Sci. Comput.*, 30(2):572–596, 2008.

[55] I. R. Ionescu and Q. L. Nguyen. Dynamic contact problems with slip-dependent friction in viscoelasticity. *Int. J. Appl. Math. Comput. Sci.*, 12(1):71–80, 2002. Mathematical modelling and numerical analysis in solid mechanics.

[56] K. Ito and K. Kunisch. Augmented Lagrangian methods for nonsmooth, convex optimization in Hilbert spaces. *Nonlinear Anal.*, 41(5-6, Ser. A: Theory Methods):591–616, 2000.

[57] K. Ito and K. Kunisch. Optimal control of elliptic variational inequalities. *Appl. Math. Optim.*, 41(3):343–364, 2000.

[58] K. Ito and K. Kunisch. Semi-smooth Newton methods for state-constrained optimal control problems. *Systems Control Lett.*, 50(3):221–228, 2003.

[59] K. Ito and K. Kunisch. Semi-smooth Newton methods for variational inequalities of the first kind. *M2AN Math. Model. Numer. Anal.*, 37(1):41–62, 2003.

[60] K. Ito and K. Kunisch. Optimal control of obstacle problems by H^1-obstacles. *Appl. Math. Optim.*, 56(1):1–17, 2007.

[61] K. Ito and K. Kunisch. Semi-smooth Newton methods for the Signorini problem. *Appl. Math.*, 53(5):455–468, 2008.

[62] J. Jarušek. Contact problems with bounded friction coercive case. *Czechoslovak Math. J.*, 33(108)(2):237–261, 1983.

[63] J. Jarušek and C. Eck. Dynamic contact problems with small Coulomb friction for viscoelastic bodies. Existence of solutions. *Math. Models Methods Appl. Sci.*, 9(1):11–34, 1999.

[64] M. Jung and O. Steinbach. A finite element-boundary element algorithm for inhomogeneous boundary value problems. *Computing*, 68(1):1–17, 2002.

[65] L. Kielhorn. *A Time-Domain Symmetric Galerkin BEM for Viscoelastodynamics*. Verlag der Technischen Universität, Graz, 2009. PhD thesis, Graz University of Technology.

[66] L. Kielhorn and M. Schanz. Convolution quadrature method-based symmetric Galerkin boundary element method for 3-d elastodynamics. *Internat. J. Numer. Methods Engrg.*, 76(11):1724–1746, 2008.

[67] N. Kikuchi and J. T. Oden. *Contact problems in elasticity: a study of variational inequalities and finite element methods*, volume 8 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988.

[68] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, volume 88 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1980.

[69] R. Kress. Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering. *Quart. J. Mech. Appl. Math.*, 38(2):323–341, 1985.

[70] B. Kummer. Generalized Newton and NCP-methods: convergence, regularity, actions. *Discuss. Math. Differ. Incl. Control Optim.*, 20(2):209–244, 2000. German-Polish Conference on Optimization—Methods and Applications (Żagań, 1999).

[71] K. Kunisch and G. Stadler. Generalized Newton methods for the 2D-Signorini contact problem with friction in function space. *M2AN Math. Model. Numer. Anal.*, 39(4):827–854, 2005.

[72] V. D. Kupradze. *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*, volume 25 of *of Applied Mathematics and Mechanics*. North-Holland, Amsterdam, 1979.

[73] K. Kuttler and Meir Shillor. Dynamic contact with Signorini's condition and slip rate dependent friction. *Electron. J. Differential Equations*, pages No. 83, 21 pp., 2004.

[74] K. L. Kuttler. Dynamic frictional contact for elastic viscoplastic material. *Electron. J. Differential Equations*, pages No. 75, 20 pp., 2007.

[75] K. L. Kuttler and M. Shillor. Set-valued pseudomonotone maps and degenerate evolution inclusions. *Commun. Contemp. Math.*, 1(1):87–123, 1999.

[76] K. L. Kuttler and M. Shillor. Dynamic contact with normal compliance wear and discontinuous friction coefficient. *SIAM J. Math. Anal.*, 34(1):1–27, 2002.

[77] K. L. Kuttler and M. Shillor. Regularity of solutions to a dynamic frictionless contact problem with normal compliance. *Nonlinear Anal.*, 59(7):1063–1075, 2004.

[78] K. L. Kuttler, M. Shillor, and J. R. Fernández. Existence for a thermoviscoelastic beam model of brakes. *Nonlinear Anal. Real World Appl.*, 5(5):857–880, 2004.

[79] P. K. Kythe. *Fundamental solutions for differential operators and applications*. Birkhäuser Boston Inc., Boston, MA, 1996.

[80] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. II*. Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 182.

[81] Y. Liu, L. Shen, and M. Bapat. Development of the fast multipole boundary method for acoustic wave problems. *Recent Advances in Bound. Elem. Meth. Vol*, pages 287–303, 2009.

[82] J. A. C. Martins and J. T. Oden. Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws. *Nonlinear Anal.*, 11(3):407–428, 1987.

[83] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[84] A. Meyer and S. Rjasanow. An effective direct solution method for certain boundary element equations in 3d. *Math. Methods Appl. Sci.*, 13:43–53, 1990.

[85] J. Nečas, J. Jarušek, and J. Haslinger. On the solution of the variational inequality to the Signorini problem with small friction. *Boll. Un. Mat. Ital. B (5)*, 17(2):796–811, 1980.

[86] J.-C. Nédélec. Integral equations with nonintegrable kernels. *Integral Equations Operator Theory*, 5(4):562–572, 1982.

[87] G. Of, O. Steinbach, and P. Urthaler. Fast evaluation of volume potentials in boundary element methods. *SIAM J. Sci. Comp.*, 32(2):585–602, 2010.

[88] J.-S. Pang and L. Q. Qi. Nonsmooth equations: motivation and algorithms. *SIAM J. Optim.*, 3(3):443–465, 1993.

[89] P. W. Partridge, C. A. Brebbia, and L. C. Wrobel. *The dual reciprocity boundary element method*. International Series on Computational Engineering. Computational Mechanics Publications, Southampton, 1992.

[90] L. Qi and J. Sun. A nonsmooth version of Newton's method. *Math. Program.*, 58(3):353–367, 1993.

[91] L. Q. Qi. Convergence analysis of some algorithms for solving nonsmooth equations. *Math. Oper. Res.*, 18(1):227–244, 1993.

[92] S. Rjasanow. *Vorkonditionierte iterative Auflösung von Randelementgleichungen für die Dirichlet-Aufgabe*. Wissenschaftliche Schriftenreihe der Technischen Universität, Chemnitz, 1990.

[93] S. Rjasanow. Effective algorithms with circulant-block matrices. *Linear Algebra Appl.*, 202:55–69, 1994.

[94] S. Rjasanow. Optimal preconditioner for boundary element formulation of the Dirichlet problem in elasticity. *Math. Methods Appl. Sci.*, 18(8):603–613, 1995.

[95] R. Rocca and M. Cocou. Numerical analysis of quasi-static unilateral contact problems with local friction. *SIAM J. Numer. Anal.*, 39(4):1324–1342, 2001.

[96] R. Rocca and M. Cocu. Existence and approximation of a solution to quasistatic Signorini problem with local friction. *Internat. J. Engrg. Sci.*, 39(11):1233–1255, 2001.

[97] T. Rüberg. *Non-conforming Coupling of Finite and Boundary Element Methods in Time Domain*. Graz, 2007. PhD thesis, Graz University of Technology.

[98] G. Stadler. *Infinite-Dimension Semi-Smooth Newton and Augmented Lagrangian Methods for friction and Contact Problems in Elasticity*. Graz, 2004. PhD thesis, Karl-Franzens University Graz.

[99] G. Stadler. Path-following and augmented Lagrangian methods for contact problems in linear elasticity. *J. Comput. Appl. Math.*, 203(2):533–547, 2007.

[100] O. Steinbach. *Numerical approximation methods for elliptic boundary value problems*. Springer, New York, 2008. Finite and boundary elements.

[101] M. Ulbrich. Semismooth newton methods for operator equations in function spaces. *SIAM J. on Optimization*, 13(3):805–842, 2002.

[102] P. Wriggers and T. A. Laursen, editors. *Computational contact mechanics*, volume 498 of *CISM Courses and Lectures*. Springer, NewYork, Vienna, 2007.

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I, Laurent TCHOUALAG, declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitely marked all material which has been quotes either literally or by content from the used sources.

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