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Boundary Element Methods for Stokes Dirichlet Control Problems

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Dedication

To my wife, my kids and all those Pakistanies who are more talented than me but by stoke of luck are not writing theses lines. I want to make a special tribute to my parents. My mother who died during my study in Austria. Although I was with her in last days of her life but she was not able to talk to me due to her brain hamrage. She looked after me more than any of her kid. My father who always showed faith in me and will always be proud of my little achievements in the studies. He could very easily ask me to help him in his business, to support the large family, but due to my health and interests in the studies, he did it all alone. I owe them a lot.

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CONTENTS

Abstract

In this thesis we discuss the application of boundary element methods for the solution of Dirichlet boundary control problems subject to the Stokes equations with box constraints on the control. Starting with a model problem of the Stokes equations and a quadratic cost functional, we set up the optimality system. The solutions of both the primal and adjoint boundary value problems are given by representation formulae, where the state variable enters the adjoint problem as volume density. By applying integration by parts to the representation formula of the adjoint problem we express this Newton potential as surface potentials. This results in a system of boundary integral equations which is related to the Bi-Stokes equations. We analyze boundary integral representations which are based on the use of single and double layer potentials, and which requires some additional assumptions to ensure stability of the discrete scheme. A numerical example is considered and the results of a boundary element approximation for the Dirichlet control problem for the Stokes system are presented which are in conformity with the theoretical findings. Finally we discuss the suitable control space and comment on the choice of $H^{1/2}(\Gamma)$ instead of the more prevalent $L_2(\Gamma)$.

Zusammenfassung

In dieser Arbeit diskutieren wir die Anwendung von Randelementmethoden für die Lösung eines Dirichlet–Kontrollproblems mit einem quadratischen Kostenfunktional für das Stokes–System. Die Minimierung des reduzierten Kostenfunktionals wird durch die Lösung einer Variationsungleichung beschrieben. Die Lösungen der primalen und adjungierten Randwertprobleme können durch Darstellungsformeln angegeben werden. Dabei geht der unbekannte Zustand des primalen Problems als Dichte eine Volumenpotentials in die Darstellungsformel für die Lösung des adjungierten Problems ein. Durch partielle Integration kann dieses Volumenpotential auf Oberflächenpotentiale zurückgeführt werden, welche die Fundamentallösung eines Bi–Stokes Differentialoperators enthalten. Die Analysis des resultierenden Optimalitätssystems beruht dann auf den Eigenschaften der auftretenden Randintegraloperatoren. Für die Stabilität der zugehörigen Randelementdiskretisierung ist eine Bedingung an die verwendeten Ansatzräume zu fordern, welche auch den Nachweis optimaler Fehlerabschätzungen ermöglicht. Ein numerisches Beispiel bestätigt die theoretischen Ergebnisse. Abschliessend wird die Wahl des Funktionenraumes für die Kontrolle diskutiert, insbesondere die in dieser Arbeit diskutierte $H^{1/2}(\Gamma)$–Kontrolle mit der in Anwendungen üblicherweise verwendeten $L_2(\Gamma)$–Kontrolle.

1 INTRODUCTION

Flow control has had a long history since Ludwig Prandtl's early experiments. In 1904, at the third international mathematics congress held in Heidelberg Germany, a 10 minutes presentation of a little–known physicist was enough to revolutionize the understanding and analysis of fluid dynamics by his idea of boundary layer–a thin region near the surface of an object moving through a fluid. He explains that the frictional effects were experienced only in the boundary layer, outside which the flow was essentially an inviscid flow that had been studied for the previous two centuries. Before Prandtl there were much confusion about the role of viscosity in a fluid motion, but after Prandl's paper the picture was made clear by the notion of boundary layer [2].

During the past years many scientists, mathematicians and engineers have given considerable attention to the problems of an active control of fluid flows, see, e.g., [14, 29, 39]. This interest is motivated by a number of potential applications such as the control of flow separation, the adjustment of mixing patterns in chemical rectors to increase the reactor performance, or combustion, noise suppression, fluid–structure interaction problems and super maneuverable aircrafts.

The increased concern for performance and efficiency issues has led to the incorporation of mathematical optimization routines into engineering design techniques. For example, the use of heat sensitive components in spacecraft raises the question of designing systems which are energy efficient and yet maintain feasible operating temperatures. These design problems may be modeled by optimal control problems with state constraints. The control of fluid flow is very important in achieving the desired design objectives to optimize the performance of the systems that exploit the fluid motion.

Optimal control problems with a Dirichlet boundary control play a vital role, for example, in the context of computational fluid mechanics., see, e.g., [23, 29], and the references given therein. In [29], the cost functional $\mathcal{J}(\mathbf{u},\mathbf{z}) = F(\mathbf{u})$ is the domain integral over the strain tensor of the velocity field $\mathbf{u}$ satisfying the steady state Navier–Stokes equations with a Dirichlet boundary condition $\mathbf{u} = \mathbf{z} \in \mathcal{U} \subset H^{1/2}(\Gamma)$. A similar minimization problem is considered in [23]. In both cases, the cost functional $\mathcal{J}(\mathbf{u},\mathbf{z})$ describes an energy in $H^1(\Omega)$, or equivalently, in the Sobolev trace space $H^{1/2}(\Gamma)$.

It is quite common among the control community to take $L_2(\Gamma)$ as control space instead of $H^{1/2}(\Gamma)$. The main difference appears in the optimality condition. In particular, as in the case of the present study, the use of the inverse Stokes single layer potential maps the Dirichlet control to the traction of the adjoint variable. This also accounts for a higher regularity of the control in related optimality condition. It turned out that the use of $H^{1/2}(\Gamma)$ as control space preserves the proper mapping properties. Instead, if we use $L_2(\Gamma)$ as control space, the two different types of boundary data are identified with each other [53, 54].

In [54], a finite element approach for Laplace Dirichlet boundary control problems was considered, where the energy norm was induced by the so–called Steklov–Poincaré operator which realizes the Dirichlet to Neumann map and the control was considered in a closed and convex subspace of the energy space $H^{1/2}(\Gamma)$. The stability and error analysis presented for the discretization of the resulting variational inequality is based on the mapping properties of the solution operators related to the primal and adjoint boundary value problems and their finite element approximations.

As solving Dirichlet boundary control problems requires the unknown function–the control, to be found on the boundary of the computational domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, the use of boundary integral equations seems to be a natural choice [8], but to the best of our knowledge, there are only few results known on the use of boundary integral equations to solve optimal boundary control problems, see, e.g., [16, 74] for problems with point observations. Although, in the present work we describe the Dirichlet control problem for the Stokes system, however, this technique can be applied to any partial differential equation provided the fundamental solution is known.

Recently in [53], the application of boundary element methods for the solution of Dirichlet boundary control problems subject to the Poisson equation with box constraints on the control was studied. The solutions of both the primal and adjoint boundary value problems are given by representation formulae, where the state enters the adjoint problem as volume density. The authors apply integration by parts to the representation formula of the adjoint problem to avoid the related volume potential. This results in a system of boundary integral equations which is related to the Bi-Laplacian. For the related Dirichlet to Neumann map, they analyze two different boundary integral representations. The first one is based on the use of single and double layer potentials only, but requires some additional assumptions to ensure stability of the discrete scheme. As a second approach, the authors have considered a symmetric formulation which is based on the use of the Calderon projector and which is stable for standard boundary element discretizations. For both methods, they have proved stability and related error estimates and have presented both finite element and boundary element discretization results. Numerical results show that all different approaches behave almost similar.

This thesis is organized as follows. In chapter 2 we briefly describe the fundamental concepts relevant to this work. We start with the introduction to boundary element methods followed by a discussion on the most improtant function spaces required for the weak formulation for Stokes Dirichlet boundary value problem. Boundary integral formulations and the Galerkin discretization with standard error estimates forms the major part of this chapter.

Chapter 3 begins with a short introduction to optimal control problems. We then present a model problem for the Stokes Dirichlet control problems. After introducing the solution operator and its adjoint, we formulate the reduced cost functional to be minimized. It turns out that the said minimization is equivalent to solving a variational inequality whose unique solvability follows from standard arguments. The chapter ends by setting up an optimality system consisting of the primal problem–the realization of the solution operator $\mathcal{S}$, the adjoint problem–another Stokes system, and the realization of the adjoint operator $\mathcal{S}^*$ with the primal variable appearing on the right hand side, and an elliptic variational inequality of the first kind.

The boundary integral formulation of the primal and adjoint problems and the unique solvability of the coupled system is covered in chapter 4. The solutions of both the primal and adjoint boundary value problems are given by representation formulae, where the state variable appears in the adjoint problem as volume density. We express the said volume integral as surface potential so as to represent the control variable $\mathbf{z}$. For this purpose we apply integration by parts to the representation formula of the adjoint problem. This gives rise to a system of boundary integral equations which is related to the, so–called, Bi–Stokes equations. We introduce the standard boundary integral operators for Stokes and for Bi–Stokes. We end this chapter by describing the relationship among the different boundary integral operators mentioned above. These relations are useful in proving the unique solvability of the proposed formulation.

Galerkin boundary element approximations of the Stokes Dirichlet control problem are analyzed in chapter 5. Due to the composition of the resulting boundary integral operator, an additional approximation is introduced. In a similar way, an approximation of the right hand side of the boundary integral operator equation is also discussed. We use two different meshes, one of size H to approximate the control $\mathbf{z}$ and another of size h to approximate the two tractions $\mathbf{t}$ and $\mathbf{q}$ which are related to the primal and adjoint velocities, respectively. We prove a necessary condition on the two meshes so that the resulting approximate operator is elliptic. Finally for the non–constrained case we set up a Schur complement system to find the unknown control $\mathbf{z}$ on the Dirichlet boundary.

In chapter 6, a numerical example for the Stokes Dirichlet control problem is considered and results of a boundary element approximation are presented. The numerical results of the boundary element approximations confirm the theoretical error estimates. Finally we comment on the choice of the control space $H^{1/2}(\Gamma)$ instead of $L_2(\Gamma)$.

The final chapter 7 deals with some conclusions and a short outlook. The purpose of the study is described along with some future work. The thesis also contains two appendices. Appendix *A* deals with the computation of the kernel function for the Bi–Stokes system and the corresponding traction function. Appendix *B* gives a short introduction to variational inequalities along with existence and uniqueness results of the solutions.

2 BOUNDARY ELEMENT METHODS

The boundary element methods (BEM) are numerical methods for solving boundary integral equations, based on a discretization procedure. The aim is to find approximate solutions of boundary value problems. We need to discretize the boundary only, instead of a volume discretization, and as such it is an important alternative to the prevailing domain methods such as the finite difference method (FDM) and the finite element method (FEM). Boundary element methods can compute solution for any interior point $x \in \Omega$. The method is highly accurate and can also handle exterior boundary value problems. For the general references to boundary integral equations and boundary element methods, see [37, 49, 63].

The basic idea of the method is to transform the original partial differential equation or the system of partial differential equations, that define a given physical problem, into an equivalent boundary integral equation or system of boundary integral equations with the help of corresponding Green's formula (such formulations is given a name known as direct method) or in terms of continuous distributions of singular solutions of the PDE(s) over the boundaries of the problem domain (indirect method). In case of a direct formulation the unknowns are the Cauchy data on the boundary whereas in the indirect case the unknowns are the surface densities of the singular solutions. In this way, the obtained integral equation satisfy the governing field equation exactly and one seeks to satisfy the imposed boundary conditions approximately.

The issue of stability and convergence is of the utmost importance when solving the boundary integral equations numerically. The Galerkin methods and the collocation method are the two most popular ones. The latter is widely used, especially in the engineering community. These methods provide, at a first glance, an easier practical implementation compared with the Galerkin methods. In contrast the Galerkin methods which perfectly fit to the variational formulation of the boundary integral equations. The theoretical study of the Galerkin methods is now complete and provides a powerful theoretical background for boundary element methods. However, the stability and convergence theory for collocation methods is available only for particular two and three–dimensional problems. Furthermore, the error analysis of the collocation methods when assuming their stability, shows that the rate of convergence for computing the solutions of the PDE by the Galerkin methods is higher, when assuming that the solution is smooth enough.

Irrespective of the numerical method, collocation or Galerkin, applied to boundary integral equations, this leads to a linear system of algebraic equations. The matrix of this system is in general dense, i.e., almost all of its entries are different from zero and therefore, have to be stored in computer memory. This is a restriction, not only to a large problem size, but also the costs of the computation of the matrix entries are considerably high. It is a main disadvantage of the boundary element methods compared with finite element methods which leads to sparse matrices. Hence, there is a need for so called fast boundary element methods in order to reduce the memory requirements and the costs of the computations. Several techniques are available for this purpose as the fast multipole method [26], the adaptive cross approximation [7, 57], panel clustering [31], or hierarchical matrices [6, 30]. Although we are not going to discuss fast boundary element methods in this thesis, we have presented few references for the sake of completeness.

2.1 Function spaces

In this section we discuss briefly the most important function spaces which are needed for the weak formulation of the Stokes boundary value problem. The main reference for this section are the text books [37] and [49]. For a further overview on the used Sobolev spaces in the domain and on the boundary, see, for example, [1, 63, 66].

Definition 2.1.1. *Let Ω be an open subset of $\mathbb{R}^d (d = 2, 3)$. For $k \in \mathbb{N}_0$ the Sobolev space $W_2^k(\Omega)$ is defined as*

$$W_2^k(\Omega) := \{u \in L_2(\Omega) : \partial^\alpha u \in L_2(\Omega) \text{ for } |\alpha| \leq k\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ is a multi–index, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, is the length of the multi–index, and $\partial^\alpha u(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_1}} u(x)$ are to be understood as weak partial derivatives.

The Sobolev space $W_2^k(\Omega)$ equipped with the norm

$$\|u\|_{W_2^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u(x)|^2 dx \right)^{\frac{1}{2}}$$

forms a Hilbert space with inner product

$$(u, v)_{W_2^k(\Omega)} := \sum_{|\alpha| \leq k} \int_\Omega \partial^\alpha u(x) \partial^\alpha v(x) dx,$$

for details, see [49, page 75]. We can extend the definition of Sobolev spaces $W_2^s(\Omega)$ for any arbitrary $s > 0$.

Definition 2.1.2. *Let Ω be an open subset of $\mathbb{R}^d$. For $s = k + \mu$ with $k \in \mathbb{N}_0$ and $\mu \in (0, 1)$, the Sobolev space $W_2^s(\Omega)$ is defined as*

$$W_2^s(\Omega) := \left\{ u \in W_2^k(\Omega) : |\partial^\alpha u|_{\mu,\Omega} < \infty \text{ for } |\alpha| = k \right\},$$

where the Sobolev–Slobodeckii semi–norm $|\cdot|_{\mu,\Omega}$ is given by

$$|v|_{\mu,\Omega} := \left(\int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x - y|^{d+2\mu}} dx dy \right)^{\frac{1}{2}}.$$

The Sobolev space $W_2^s(\Omega)$ for $s = k+\mu$ with $k \in \mathbb{N}_0$ and $\mu \in (0,1)$ is equipped with the norm

$$\|u\|_{W_2^s(\Omega)} := \left(\|u\|^2_{W_2^k(\Omega)} + \sum_{|\alpha|=k} |\partial^\alpha u|^2_{\mu,\Omega} \right)^{\frac{1}{2}} .$$

Again, $W_2^s(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,v)_{W_2^s(\Omega)} := (u,v)_{W_2^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{[\partial^\alpha u(x) - \partial^\alpha u(y)]\,[\partial^\alpha v(x) - \partial^\alpha v(y)]}{|x-y|^{d+2\mu}} dxdy,$$

see [49] for details.

So far we have considered the Sobolev spaces on a non–empty subset $\Omega \subset \mathbb{R}^d$, in order to relate the above defined Sobolev spaces with each other, we need some regularity assumptions for $\Gamma := \partial\Omega$. First of all we consider the set

$$\Omega = \left\{ x = (x', x_d) \in \mathbb{R}^d : x_d < f(x') \text{ for all } x' \in \mathbb{R}^{d-1} \right\}, \tag{2.1}$$

where $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is a bounded function which is k times differentiable, and where the derivatives $\partial^\alpha f$ with $|\alpha| = k$ satisfy

$$|\partial^\alpha f(x') - \partial^\alpha f(y')| \leq M|x'-y'|^\mu \quad \text{for all } x', y' \in \mathbb{R}^{d-1}$$

with some $\mu \in [0,1]$. Such a set Ω as defined in (2.1) is called $C^{k,\mu}$ hypograph [49].

Definition 2.1.3. *An open set $\Omega \subset \mathbb{R}^d$ is called a $C^{k,\mu}$ domain if its boundary Γ is compact and if there exist finite families $\{W_j\}$ and $\{\Omega_j\}$ with the following properties:*

1. *The family $\{W_j\}$ is a finite open cover of Γ.*
2. *Each Ω_j can be transformed to a $C^{k,\mu}$ hypograph by a rigid motion.*
3. *For each j the equality $W_j \cap \Omega = W_j \cap \Omega_j$ is satisfied.*

If Ω is a $C^{k,\mu}$ domain, then the boundary can be parameterized by k times differentiable functions. Therefore we call the boundary of a $C^{k,\mu}$ domain k times differentiable. If this property is only locally satisfied, then we call the boundary piecewise smooth.

A $C^{0,1}$ domain is called a Lipschitz domain. For instance, any polygonal bounded domain in $\mathbb{R}^2$ and any domain in $\mathbb{R}^3$ which is bounded by a polyhedron is a Lipschitz domain. Note that a Lipschitz domain may be unbounded. For example, if Ω is a bounded Lipschitz domain, then its complement $\mathbb{R}^d \setminus \overline{\Omega}$ is also a Lipschitz domain.

Sobolev spaces on the boundary

In the following we assume that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain. The L_2 norm on the boundary $\Gamma = \partial\Omega$ is defined by

$$\|u\|_{L_2(\Gamma)} := \left(\int_\Gamma |u(x)|^2 ds_x \right)^{\frac{1}{2}}.$$

For $s \in (0,1)$ the Sobolev-Slobodeckii norm is defined by

$$\|u\|_{H^s(\Gamma)} := \left(\|u\|^2_{L_2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(x)-u(y)|^2}{|x-y|^{d-1+2s}} ds_x ds_y \right)^{\frac{1}{2}}.$$

Definition 2.1.4. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain with boundary $\Gamma = \partial\Omega$. The spaces $L_2(\Gamma)$ and $H^s(\Gamma)$ are defined as the closures*

$$\begin{aligned} L_2(\Gamma) &:= \overline{C^0(\Gamma)}^{\|\cdot\|_{L_2(\Gamma)}}, \\ H^s(\Gamma) &:= \overline{C^0(\Gamma)}^{\|\cdot\|_{H^s(\Gamma)}} \quad \textit{for } s \in (0,1). \end{aligned}$$

The spaces $L_2(\Gamma)$ and $H^s(\Gamma)$ for $s \in (0,1)$ are Hilbert spaces equipped with the inner products

$$\begin{aligned} \langle u,v \rangle_{L_2(\Gamma)} &:= \int_\Gamma u(x)v(x) ds_x, \\ \langle u,v \rangle_{H^s(\Gamma)} &:= \langle u,v \rangle_{L_2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{[u(x)-u(y)]\,[v(x)-v(y)]}{|x-y|^{d-1+2s}} ds_x ds_y \quad \text{for } s \in (0,1), \end{aligned}$$

see [37, page 172].

The Sobolev spaces $H^s(\Gamma)$ can also be defined for the case $s \geq 1$, see, e.g., [37, Section 4.2]. This requires, for $s > 1$, strong regularity assumption for the boundary Γ than merely a Lipschitz property, i.e., the boundary must be of the class $C^{k,\kappa}$ and $s \leq k + \kappa$. For definitions and more details see [37, Section 4.2].

Sobolev spaces with negative indices

For negative indices $s < 0$ the Sobolev spaces $H^s(\Gamma)$ are defined via duality, i.e., $H^{-s}(\Gamma) = [H^s(\Gamma)]^*$ with the norm

$$\|t\|_{H^s(\Gamma)} := \sup_{0 \neq u \in H^{-s}(\Gamma)} \frac{|\langle u,t \rangle_{L_2(\Gamma)}|}{\|u\|_{H^{-s}(\Gamma)}}, \tag{2.2}$$

see [37, page 175].

Sobolev spaces on a part of the boundary Γ

For an open subset $\Gamma_0 \subset \Gamma$ and for a sufficiently smooth boundary Γ we define Sobolev spaces for $s \geq 0$,

$$\begin{aligned} H^s(\Gamma_0) &:= \left\{ v = \widetilde{v}_{|\Gamma_0} : \widetilde{v} \in H^s(\Gamma) \right\}, \\ \widetilde{H}^s(\Gamma_0) &:= \left\{ v = \widetilde{v}_{|\Gamma_0} : \widetilde{v} \in H^s(\Gamma),\ \operatorname{supp} \widetilde{\mathbf{v}} \subset \Gamma_0 \right\}, \end{aligned}$$

with the norm

$$\|v\|_{H^s(\Gamma_0)} := \inf_{\widetilde{v} \in H^s(\Gamma) : \widetilde{v}_{|\Gamma_0} = v} \|\widetilde{v}\|_{H^s(\Gamma)} .$$

For $s < 0$ the Sobolev spaces are defined as dual spaces

$$H^s(\Gamma_0) := \left[\widetilde{H}^{-s}(\Gamma_0)\right]^* \quad \text{and} \quad \widetilde{H}^s(\Gamma_0) := \left[H^{-s}(\Gamma_0)\right]^* . \tag{2.3}$$

Let us now consider the case of Γ to be closed and piecewise smooth,

$$\Gamma = \bigcup_{i=1}^{J} \overline{\Gamma}_i, \quad \Gamma_i \cap \Gamma_j = \phi \quad \text{for } i \neq j.$$

The Sobolev space $H^s_{pw}(\Gamma)$ for $s > 0$ is defined as

$$H^s_{pw}(\Gamma) := \left\{ v \in L_2(\Gamma) : v_{|\Gamma_i} \in H^s(\Gamma_i), i = 1, \ldots, J \right\},$$

with the norm

$$\|v\|_{H^s_{pw}(\Gamma)} := \left(\sum_{i=1}^{J} \left\| v_{|\Gamma_i} \right\|^2_{H^s(\Gamma_i)} \right)^{\frac{1}{2}} .$$

For $s < 0$ we define

$$H^s_{pw}(\Gamma) := \prod_{i=1}^{J} \widetilde{H}^s(\Gamma_i),$$

with the norm

$$\|w\|_{H^s_{pw}(\Gamma)} := \sum_{i=1}^{J} \left\| w_{|\Gamma_i} \right\|_{\widetilde{H}^s(\Gamma_i)} .$$

Lemma 2.1.1. *For $w \in H^s_{pw}(\Gamma)$ and $s < 0$ there hold*

$$\|w\|_{H^s(\Gamma)} \leq \|w\|_{H^s_{pw}(\Gamma)} .$$

Proof. See [63, Lemma 2.20]. □

Remark 2.1.1. *If Ω is a Lipschitz domain then for all above definitions and statements regarding the Sobolev spaces on the subsets of the boundary Γ we have to assume that $|s| \leq 1$. For the validity of the results for the case $|s| > 1$ stronger regularity conditions for Γ needs to be assumed, see [37, Section 4.3].*

Trace operators and normal derivatives

The trace operators are used to relate the Sobolev spaces in a domain Ω to the Sobolev spaces on its boundary $\Gamma = \partial\Omega$.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Define the interior trace operator or the interior trace operator of order zero, $\gamma_0^{int} : C^\infty(\overline{\Omega}) \to C^\infty(\Gamma)$ by*

$$\gamma_0^{int} u := u|_\Gamma.$$

If Ω is a $C^{k-1,1}$ domain then the operator γ_0^{int} has a unique extension to a bounded linear operator

$$\gamma_0^{int} : H^s(\Omega) \to H^{s-1/2}(\Gamma) \quad \text{for } \frac{1}{2} < s \leq k. \tag{2.4}$$

This extension has a continuous right inverse $\mathcal{E} : H^{s-1/2}(\Gamma) \to H^s(\Omega)$.

Proof. See [49, Theorem 3.37]. □

If Ω is a bounded Lipschitz domain, i.e., $k = 1$, then (2.4) implies that the interior trace operator is a continuous linear mapping

$$\gamma_0^{int} : H^s(\Omega) \to H^{s-1/2}(\Gamma) \quad \text{for } \frac{1}{2} < s \leq 1.$$

This result can be extended to $\frac{1}{2} < s < \frac{3}{2}$, see [49, Theorem 3.38].

2.2 Stokes problem

In the next four sections we very briefly describe the boundary element method for the Dirichlet boundary value problem for the Stokes system. We begin with a short overview of the Stokes system followed by a boundary integral formulation and the Galerkin discretization. The main reference for these sections include [37, 44, 56, 63].

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$. The linearized and stationary equations of an incompressible viscous fluid are modeled by the Stokes system consisting of the equations in the form

$$-\mu\Delta\mathbf{u}(x) + \nabla p(x) = \mathbf{f}(x), \quad \nabla \cdot \mathbf{u}(x) = 0, \quad \text{for } x \in \Omega, \tag{2.5}$$

i.e., the linearized stationary form of the full Navier–Stokes equations. Here μ is the dynamic viscosity of the fluid, $(\mathbf{u}, p)$ are the two unknowns, representing the velocity and the pressure of the fluid respectively and $\mathbf{f}(x)$ corresponds to a given forcing term. The model is used to describes slow viscous flow. The Stokes problem enters in biological applications, not for large–scale blood flow in the heart but for small–scale movements in capillaries. For a viscous flow with a given fluid density $\widetilde{\rho}$, one introduces

$$\nu := \frac{\mu}{\widetilde{\rho}} \gg 1,$$

which is usually known as the kinematic viscosity of the fluid. Note that the methods of solving the linear and stationary Stokes problems play an important part, since the non–linear and non–stationary problems can be reduced to linear and stationary ones by means of perturbations and time–stepping procedures, see, e.g., [22, 35, 65] for details.

Dirichlet boundary value problem

As model problem we consider the Dirichlet boundary value problem for the Stokes system

$$\begin{aligned} -\mu\Delta\mathbf{u}(x) + \nabla p(x) &= \mathbf{f}(x), \quad x \in \Omega, &(2.6)\\ \nabla \cdot \mathbf{u}(x) &= 0, \quad x \in \Omega, &(2.7)\\ \mathbf{u}(x) &= \mathbf{g}(x), \quad x \in \Gamma. \end{aligned}$$

A compatibility condition is obtained from (2.7) by integrating over the domain Ω and using the Gauss integral formula, i.e.,

$$\int_\Omega \nabla \cdot \mathbf{u}(x) dx = \int_\Gamma [\mathbf{n}(x)]^\top \mathbf{u}(x) ds_x = \int_\Gamma [\mathbf{n}(x)]^\top \mathbf{g}(x) ds_x = 0, \tag{2.8}$$

where $\mathbf{n}(x)$ is the outer normal vector which is defined for allmost all x $\in \Gamma$. Thus the given Dirichlet datum has to satisfy this compatibility condition. Hence we introduce the following subspace

$$H_*^{1/2}(\Gamma) := \left\{ \mathbf{v}(x) \in H^{1/2}(\Gamma) : \int_\Gamma [\mathbf{n}(x)]^\top \mathbf{v}(x) ds_x = 0 \right\}.$$

Due to the presence of the term "$\nabla p(x)$", the pressure p is unique only up to an additive constant. Thus if $p(x)$ satisfies the partial differential equation then $p(x) + \alpha$, where $\alpha \in \mathbb{R}$, also satisfies the PDE. However, as for the Neumann boundary value problem for the Laplace equation, see [63, Section 7.2], we can use an appropriate scaling condition to fix this constant, e.g.,

$$\int_\Omega p(x) dx = 0.$$

Green's first formula

In order to find Green's first formula for the Stokes system, we multiply (2.6) with some test function $\mathbf{v}(x)$, integrate over the domain Ω both sides and apply integration by parts to obtain

$$\mu \int_\Omega \sum_{i,j=1}^d e_{ij}(\mathbf{u},x) e_{ij}(\mathbf{v},x) dx - \mu \int_\Omega \operatorname{div} \mathbf{u}(x) \operatorname{div} \mathbf{v}(x) dx - \int_\Omega p(x) \operatorname{div} \mathbf{v}(x) dx$$
$$+ \int_\Gamma \sum_{i=1}^d t_i(\mathbf{u},p) v_i(x) ds_x = \int_\Omega [\mathbf{v}(x)]^\top \mathbf{f}(x) dx,$$

where

$$t_i(\mathbf{u},p)(x) := -[p(x) + \operatorname{div} \mathbf{u}(x)] n_i(x) + 2\mu \sum_{j=1}^d e_{ij}(\mathbf{u},x) n_j(x)$$

is the co–normal derivative representing the boundary stress defined for almost all $x \in \Gamma$ and $i = i, \ldots, d$. For divergence–free functions $\mathbf{u}$ satisfying $\nabla \cdot \mathbf{u} = 0$ we obtain, as for the system of linear elastostatics, the following alternative representation

$$\mathbf{t}(\mathbf{u},p) = -p(x)\mathbf{n}(x) + 2\mu \frac{\partial}{\partial n_x} \mathbf{u}(x) + \mu \mathbf{n}(x) \times \operatorname{curl} \mathbf{u}(x). \tag{2.9}$$

The symmetric bilinear form is given by

$$a(\mathbf{u},\mathbf{v}) = a(\mathbf{v},\mathbf{u}) = 2\mu \int_\Omega \sum_{i,j=1}^d e_{ij}(\mathbf{u},x) e_{ij}(\mathbf{v},x) dx - \mu \int_\Omega \operatorname{div} \mathbf{u}(x) \operatorname{div} \mathbf{v}(x) dx, \tag{2.10}$$

so the Green's first formula takes the form

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega p(x) \operatorname{div} \mathbf{v}(x) dx + \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + \langle \mathbf{t}(\mathbf{u},p), \gamma_0^{int} \mathbf{v} \rangle_\Gamma. \tag{2.11}$$

Here we have introduced the strain tensor, the stress tensor and the boundary stress which are defined as follows

$$\begin{aligned} e_{ij}(\mathbf{u}) &= \frac{1}{2} \left(\frac{\partial}{\partial x_j} u_i + \frac{\partial}{\partial x_i} u_j \right), \\ \sigma_{ij}(\mathbf{u},p) &= -\delta_{ij} p + 2\mu e_{ij}(\mathbf{u}), \\ t_i(\mathbf{u},p)(x) &= \sum_{j=1}^d \sigma_{ij}(\mathbf{u},p) n_j(x) \quad \text{for } i = 1, \ldots, d. \end{aligned} \tag{2.12}$$

Green's second formula

From Green's first formula (2.11) and by using the symmetry of the bilinear form $a(\cdot,\cdot)$, we can derive Green's second formula

$$\int_\Omega \sum_{i=1}^d \left[-\mu\Delta v_i(y) + \frac{\partial}{\partial y_i} q(y)\right] u_i(y)dy + \int_\Omega p(y)\ \mathrm{div}\ \mathbf{v}(y)dy = \int_\Gamma \mathbf{v}(y)^\top \mathbf{t}(\mathbf{u},p)ds_y$$
$$- \int_\Gamma \mathbf{u}(y)^\top \mathbf{t}(\mathbf{v},q)ds_y + \int_\Omega \mathbf{f}(y)^\top \mathbf{v}(y)dy. \tag{2.13}$$

Fundamental solution and representation formulae

A solution of the partial differential equations (2.6)–(2.7) with a Dirac impulse as the right hand side is defined to be the fundamental solution. A prior knowledge of the fundamental solution of the underlaying partial differential operator is essential to use boundary integral equation methods to describe solutions of boundary value problems. At first glance this seems to be a draw back of boundary element methods, but there is a large class of partial differential operators for which the existence of fundamental solutions can be ensured even with the class of partial differential operators with constant coefficients [17, 34, 50], the explicit computation can be a formidable task in general, see, e.g., [43, 55].

To find a representation formula for $u_k(x)$ for $k = 1,\ldots,d$, we need to find the fundamental solution for the Stokes system which is defined by a pair of distributions $\mathbf{v}^k(x,y)$ and $q^k(x,y)$ satisfying

$$\begin{aligned} -\mu\Delta_x \mathbf{v}^k(x,y) + \nabla_x q^k(x,y) &= \delta(x,y)\mathbf{e}^k, \\ \mathrm{div}\ _x \mathbf{v}^k(x,y) &= 0, \end{aligned}$$

where $\mathbf{e}^k$ denotes the unit vector along the x_k-axis, $k = 1,\ldots,d$. By using the Fourier transform we can compute the fundamental solution explicitly, see [44] for details.

For $d = 2$ the matrix valued fundamental solution is

$$\begin{aligned} \mathbf{v}_\ell^k &= U^*_{k\ell}(x,y) = \frac{1}{4\pi}\frac{1}{\mu}\left[-\log|x-y|\delta_{k\ell} + \frac{(y_k - x_k)(y_\ell - x_\ell)}{|x-y|^2}\right], \\ q^k &= \mathbf{Q}^*_k(x,y) = \frac{1}{2\pi}\frac{y_k - x_k}{|x-y|^2}, \quad with \quad k,\ell = 1,2. \end{aligned} \tag{2.14}$$

For $d = 3$

$$\begin{aligned} U^*_{k\ell}(x,y) &= \frac{1}{8\pi}\frac{1}{\mu}\left[\frac{\delta_{k\ell}}{|x-y|} + \frac{(y_k - x_k)(y_\ell - x_\ell)}{|x-y|^3}\right], \\ \mathbf{Q}^*_k(x,y) &= \frac{1}{4\pi}\frac{(y_k - x_k)}{|x-y|^3}, \quad with \quad k,\ell = 1,2,3. \end{aligned}$$

Recall that the fundamental solution for the two–dimensional linear elasticity system is given by [63, page 98]

$$U^*_{k\ell}(x,y) = \frac{1}{4\pi}\frac{1}{E}\frac{1+\nu}{1-\nu}\left[(4\nu-3)\log|x-y|\delta_{k\ell} + \frac{(y_k-x_k)(y_\ell-x_\ell)}{|x-y|^2}\right]. \tag{2.15}$$

Remark 2.2.1. *On comparing* (2.14) *with* (2.15) *we found that for* $\nu = \frac{1}{2}$ *and* $E = 3\mu$*, the fundamental solution for linear elasticity coincides with the fundamental solution of the Stokes system provided the material is incompressible. A similar observation can be made for the case* $d = 3$ *as well.*

Inserting the fundamental solutions in (2.13) we get a representation formula for the velocity

$$\begin{aligned} u_k(x) &= \int_\Gamma \mathbf{U}^*_k(x,y)^\top \mathbf{t}(\mathbf{u}(y),p(y))\,ds_y - \int_\Gamma \mathbf{u}(y)^\top \mathbf{t}(\mathbf{U}^*_k(x,y),Q^*_k(x,y))ds_y \\ &\quad + \int_\Omega \mathbf{f}(y)^\top \mathbf{U}^*_k(x,y)dy, \quad \text{for } x\in\Omega, \quad (k=1,\dots,d). \end{aligned} \tag{2.16}$$

Analogously we can get the representation formula for the pressure, for $x \in \Omega$

$$p(x) = \int_\Gamma \mathbf{Q}^*_k(x,y)^\top \mathbf{t}(y)ds_y - 2\mu\int_\Gamma \mathbf{u}(y)^\top \frac{\partial}{\partial n_y}\mathbf{Q}^*_k(x,y)ds_y + \int_\Omega \mathbf{f}(y)^\top \mathbf{Q}^*_k(x,y)dy. \tag{2.17}$$

Note that the representation for $p(x)$ is unique up to an additive constant.

The co–normal derivative $\mathbf{T}^*_k(x,y)$ is defined via (2.9), for almost all $y \in \Gamma$, by

$$\begin{aligned} \mathbf{T}^*_k(x,y) &= \mathbf{t}(\mathbf{U}^*_k(x,y),Q^*_k(x,y)), \\ &= -Q^*_k(x,y)\mathbf{n}(y) + 2\mu\frac{\partial}{\partial n_y}\mathbf{U}^*_k(x,y) + \mu\mathbf{n}(y)\times \operatorname{curl}\mathbf{U}^*_k(x,y). \end{aligned}$$

Remark 2.2.2. *It can be shown easily, see [63, pages 7,14], that the boundary stress represented by the co–normal derivative of the fundamental solution of the Stokes system also coincides with the boundary stress of the fundamental solution of linear elasticity when choosing the material parameters* $E = 3\mu$ *and* $\nu = \frac{1}{2}$.

2.3 Hydrodynamic potentials and boundary integral equations

The representation of the velocity for the inhomogeneous Stokes system (2.5) is obtained from (2.16)

$$\mathbf{u}(\widetilde{x}) = (\widetilde{V}\mathbf{t})(\widetilde{x}) - (W\mathbf{g})(\widetilde{x}) + (\widetilde{N}_0\mathbf{f})(\widetilde{x}), \quad \text{for } \widetilde{x}\in\Omega. \tag{2.18}$$

Here $\widetilde{V}$, W and $\widetilde{N}_0$ are the single layer, double layer and the Newton potentials for the Stokes system respectively which are defined as, for $\widetilde{x} \in \Omega$

$$(\widetilde{V}\mathbf{t})(\widetilde{x}) : = \int_{\Gamma} \mathbf{U}^*(\widetilde{x},y)^\top \mathbf{t}(y) ds_y, \tag{2.19}$$

$$\begin{aligned} (W\mathbf{u})(\widetilde{x}) : &= \int_{\Gamma} \mathbf{u}(y)^\top \mathbf{T}^*(\widetilde{x},y) ds_y, \\ (\widetilde{N}_0\mathbf{f})(\widetilde{x}) : &= \int_{\Omega} \mathbf{f}(y)^\top \mathbf{U}^*(\widetilde{x},y) dy. \end{aligned} \tag{2.20}$$

Applying the trace operator γ_0 and the boundary stress operator $\mathbf{t}(\cdot,\cdot)$ to these potentials yields

$$\begin{aligned} \gamma_0^{int}\widetilde{V}\psi &= V\psi \\ \mathbf{t}(\widetilde{V}\psi) &= (\frac{1}{2}I + K')\psi \\ \gamma_0^{int}(W\phi) &= (-\frac{1}{2}I + K)\phi, \\ \mathbf{t}(W\phi) &= -D\phi. \end{aligned}$$

almost everywhere on Γ, with the single layer potential operator $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, the double layer potential operator $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, the adjoint double layer potential operator $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and the hypersingular boundary integral operator $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. These boundary integral operators are continuous linear operators and admit the following representations

$$\begin{aligned} (V\mathbf{t})_j(x) &= \int_{\Gamma} \sum_{i=1}^{d} \mathbf{U}^*_{ij}(x,y)\mathbf{t}_i(y) ds_y \quad \text{for } x \in \Gamma, \\ (K\phi)_j(x) &= \int_{\Gamma} \sum_{i=1}^{d} \mathbf{T}^*_{ij}(x,y)\phi_i(y) ds_y \quad \text{for } x \in \Gamma, \\ (K'\mathbf{t})_j(x) &= \int_{\Gamma} \sum_{i=1}^{d} \mathbf{T}^*_{ji}(x,y)\mathbf{t}_i(y) ds_y \quad \text{for } x \in \Gamma, \\ (D\phi)_j(x) &= \text{p.f.} \int_{\Gamma} \mathbf{D}_{j\ell}(x,y)\phi_\ell(y) ds_y \quad \text{for } x \in \Gamma, \end{aligned}$$

where $\mathbf{T}^*_{ij}(x,y)$, $\mathbf{D}_{j\ell}(x,y)$ are the kernel functions for the double layer potential and hypersingular operator respectively, and p.f. denotes Hadamard's finite part integral–the natural regularization of homogeneous distributions and of the hypersingular boundary integral operators. In the following lemma we describe the mapping properties of the these hydrodynamic potentials. The basic idea of the proof is to use the close relationship of these potentials with that of the Lamé system.

Lemma 2.3.1. *Let* Γ *be the boundary of a Lipschitz domain. If* $\sigma \in \mathbb{R}$, $|\sigma| < 1/2$, *then the following basic boundary integral operators, as defined above, are continuous linear mappings.*

$$\begin{aligned} V &: H^{-1/2+\sigma}(\Gamma) \to H^{1/2+\sigma}(\Gamma), \\ K &: H^{1/2+\sigma}(\Gamma) \to H^{1/2+\sigma}(\Gamma), \\ K' &: H^{-1/2+\sigma}(\Gamma) \to H^{-1/2+\sigma}(\Gamma), \\ D &: H^{1/2+\sigma}(\Gamma) \to H^{-1/2+\sigma}(\Gamma). \end{aligned}$$

Proof. See [37, Lemma 5.6.4] or [41, Section 6]. □

In equation (2.18) the boundary charges are the Cauchy data $\mathbf{g}(x) = \mathbf{u}(x)|_\Gamma$ and $\mathbf{t}(x) = \mathbf{t}(\mathbf{u}, p)|_\Gamma$ of the solution of the Stokes system

$$-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega. \tag{2.21}$$

Since the pressure p will be completely determined once the Cauchy data are known, in what follows, it is sufficient to consider the boundary integral equations for the velocity only. Nevertheless, we need the representation formula for p implicitly when we deal with the stress operator.

Applying the trace operator and the boundary stress operator to both sides of the representation formula (2.16), we obtain the overdetermined system of boundary integral equations

$$\begin{aligned} \mathbf{u}(x) &= (\tfrac{1}{2}I - K)\mathbf{u}(x) + (V\mathbf{t})(x) + (N_0\mathbf{f})(x) \qquad \text{for } x \in \Gamma, \\ \mathbf{t}(x) &= (D\mathbf{u})(x) + (\tfrac{1}{2}I + K')\mathbf{t}(x) + (N_1\mathbf{f})(x) \qquad \text{for } x \in \Gamma. \end{aligned}$$

Hence, the Calderón projector can be written in operator matrix form as

$$\mathcal{C} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix}$$

Here V, K, K' and D are the four basic boundary integral operators for the Stokes flow. It may be noted that the Calderón projector for Stokes has the same form as that of the Laplacian, see [63, page 137]. As for the Laplacian the Dirichlet problem can be solved by using the weakly singular boundary integral equation

$$(V\mathbf{t})(x) = (\frac{1}{2}I + K)\mathbf{u}(x) - (N_0\mathbf{f})(x) \quad \text{for } x \in \Gamma. \tag{2.22}$$

Inserting the Dirichlet data in (2.22) we get

$$(V\mathbf{t})(x) = (\frac{1}{2}I + K)\mathbf{g}(x) - (N_0\mathbf{f})(x) \quad \text{for } x \in \Gamma. \tag{2.23}$$

To ensure the solvability of (2.23), we have to consider the mapping properties of the single layer potential V. Since the boundary integral equation (2.22) holds for any pair $(\mathbf{u}^*, p^*)$ satisfying the Stokes system (2.21), when choosing $u^* = 0$ and $p^* =$ constant we get

$$(V\mathbf{t}^*)(x) = 0 \quad \text{for all } x \in \Gamma,$$

in particular, when using (2.12) with $p = -1$ we found that

$$\mathbf{t}^*(x) = \mathbf{n}(x) \quad \text{for all } x \in \Gamma$$

is an eigenfunction yielding a zero eigenvalue of the single layer potential. Thus we note that the boundary integral equation (2.23) is uniquely solvable modulo $\mathbf{t}^*$ (if $\mathbf{t}$ is solution then $\mathbf{t} + \alpha\mathbf{t}^*$, for $\alpha \in \mathbb{R}$, is also a solution) and we have to consider (2.23) in an appropriate factor space.

Lemma 2.3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Then $\mathbf{t}^* \in H^{-1/2}(\Gamma)$.*

Proof. See [56, Proposition 2.1]. □

Let

$$(V^L t)(x) = \int_\Gamma U_L^*(x,y) t(y) ds_y \quad \text{for } x \in \Gamma$$

denotes the single layer potential associated with the Laplace operator with $U_L^*(x,y)$ as fundamental solution. Note that $V^L\colon H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded and $H^{-1/2}(\Gamma)$-elliptic, for $d = 2$ we assume diam $\Omega < 1$, for the proof of ellipticity, see [63, Theorem 6.23].

We now define the factor space

$$H_*^{-1/2}(\Gamma) := \left\{ \mathbf{t} \in H^{-1/2}(\Gamma) : \langle \mathbf{t}, \mathbf{t}^* \rangle_{V^L} := \sum_{i=1}^{d} (V^L t_i, t_i^*)_{L_2(\Gamma)} = 0 \right\}. \tag{2.24}$$

Remark 2.3.1. *For the Stokes single layer potential V, we have $V : H_*^{-1/2}(\Gamma) \to H_*^{1/2}(\Gamma)$.*

Note that in literature, e.g., Reference [70] the factor space $H_*^{-1/2}(\Gamma)$ is often defined with respect to the L_2 inner product. It seems that (2.24) is a more suitable choice giving an optimal error control.

To ensure the existence of at least one solution of (2.23) we need to have the solvability condition for the right hand side and hence we have the following lemma.

Lemma 2.3.3. *Let* $\mathbf{b} := (\frac{1}{2}I + K)\mathbf{g}(x) - N_0\mathbf{f}$ *be the right hand side of the boundary integral equation* $V\mathbf{t} = \mathbf{b}$. *Assume further* $\mathbf{g} \in H_*^{1/2}(\Gamma)$, *then there holds the solvability condition*

$$\langle \mathbf{b}, \mathbf{t}^* \rangle_\Gamma = 0. \tag{2.25}$$

Proof. For the inhomogeneous Stokes system (2.5) the second boundary integral equation is

$$(D\mathbf{u})(x) = (\frac{1}{2}I - K')\mathbf{t}(x) - N_1\mathbf{f}(x) \quad \text{for } x \in \Gamma. \tag{2.26}$$

Since (2.5) holds for any pair of solutions $(\mathbf{u}, p)$, in particular for $\mathbf{u} = 0$ and $p = -1$, we get from (2.5), $\mathbf{f} = \mathbf{0}$ and $\mathbf{t} = \mathbf{t}^* = \mathbf{n}$. Hence equation (2.26) becomes

$$\mathbf{0} = (\frac{1}{2}I - K')\mathbf{t}^*(x) \quad \text{for } x \in \Gamma, \tag{2.27}$$

and therefore

$$\begin{aligned} \langle \mathbf{b}, \mathbf{t}^* \rangle_\Gamma &= \langle (\frac{1}{2}I + K)\mathbf{g} - N_0\mathbf{f}, \mathbf{t}^* \rangle_\Gamma \\ &= \langle \mathbf{g}, \mathbf{t}^* \rangle_\Gamma - \langle \mathbf{g}, (\frac{1}{2}I - K')\mathbf{t}^* \rangle_\Gamma - \langle N_0\mathbf{f}, \mathbf{t}^* \rangle_\Gamma. \end{aligned}$$

The first term on the right hand side vanishes due to the solvability condition (2.8), the second term is zero due to (2.27). We need to to show that the third term is also zero. For this we use the fact that the Newton potential is the generalized solution of the inhomogeneous Stokes system (2.5). Let $\mathbf{u}_f = N_0\mathbf{f}$. Then

$$\langle N_0\mathbf{f}, \mathbf{t}^* \rangle_\Gamma = \langle \mathbf{u}_f, \mathbf{n} \rangle_\Gamma = \int_\Gamma \mathbf{u}_f^\top \mathbf{n} \, ds_x = \int_\Omega \nabla \cdot \mathbf{u}_f \, dx = 0,$$

since

$$\nabla_x \cdot \mathbf{u}_f = \nabla_x \cdot N_0\mathbf{f} = \nabla_x \cdot \int_\Omega U^*(x,y)\mathbf{f}(y)dy = \int_\Omega \nabla_x \cdot U^*(x,y)\mathbf{f}(y)dy = 0.$$

Hence the assertion follows. □

We now establish an ellipticity estimate of the single layer potential for Stokes through the following lemma.

Lemma 2.3.4. *Let* $\Omega \subset \mathbb{R}^d \, (d = 2, 3)$ *be a bounded simply connected domain with Lipschitz boundary* $\Gamma := \partial\Omega$. *Then, the single layer potential* V *is* $H_*^{-1/2}(\Gamma)$*–elliptic, i.e.,*

$$\langle V\mathbf{w}, \mathbf{w} \rangle_\Gamma \geq c_1^V \|\mathbf{w}\|^2_{H^{-1/2}(\Gamma)} \quad \textit{for all } \mathbf{w} \in H_*^{-1/2}(\Gamma). \tag{2.28}$$

Proof. See [15]. □

Remark 2.3.2. *For the two dimensional case an appropriate scaling condition for the domain* Ω *is needed to ensure* (2.28)*, see [36, 69]. For the single layer potential of the Bi-Laplace this was considered in Reference [13]; then Airy's stress function yields corresponding scaling for the elastostatics problem and therefore for the Stokes system.*

Remark 2.3.3. *Note that here we have the ellipticity of* V *in* $H_*^{-1/2}(\Gamma)$ *and not in the full space* $H^{-1/2}(\Gamma)$*. In the next section we will define a stabilized version of the single layer potential which will ensure the ellipticity in the full* $H^{-1/2}(\Gamma)$ *space.*

Variational formulation of the boundary integral equation

The variational formulation of the boundary integral equation (2.23) reads.

Find $\mathbf{t} \in H_*^{-1/2}(\Gamma)$ such that

$$\langle V\mathbf{t}, \tau\rangle_\Gamma = \langle (\frac{1}{2}I + K)\mathbf{g} - N_0\mathbf{f}, \tau\rangle_\Gamma \quad \text{for all } \tau \in H_*^{-1/2}(\Gamma). \tag{2.29}$$

Since the single layer potential V is $H_*^{-1/2}(\Gamma)$–elliptic and bounded, the unique solvability of (2.29) follows from the lemma of Lax–Milgram [63, Theorem 3.4] and by assuming the solvability condition (2.8).

The trial space $H_*^{-1/2}(\Gamma)$ may not be suitable for a conformal Galerkin approach, so we will derive a formulation which allows a standard Galerkin discretization. This can be done as in mixed finite element methods, see [10] for details. The constraints in $H_*^{-1/2}(\Gamma)$ are formulated as side condition to get a saddle point problem.

Find $(\mathbf{t}, \omega) \in H^{-1/2} \times \mathbb{R}$ such that

$$\begin{aligned} &\langle V\mathbf{t}, \tau\rangle_\Gamma + \omega\langle \tau, \mathbf{t}^*\rangle_{V^L} = \langle \mathbf{b}, \tau\rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma), \\ &\langle \mathbf{t}, \mathbf{t}^*\rangle_{V^L} = 0. \end{aligned} \tag{2.30}$$

We now establish the unique solvability of (2.30) with the help of the following theorem.

Theorem 2.3.1. *Let* $\Omega \subset \mathbb{R}^d$ *for* $d = 2$ *or* 3 *be a simply connected domain with Lipschitz boundary* Γ*. Let the solvability condition* (2.8) *be also valid. Then there exists a unique solution* $(\mathbf{t}, \omega) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ *of* (2.30) *satisfying*

$$\|\mathbf{t}\|_{H^{-1/2}(\Gamma)} \leq c\|\mathbf{g}\|_{H^{1/2}(\Gamma)},$$

in particular, we have $\mathbf{t} \in H_*^{-1/2}(\Gamma)$ *and* $\omega = 0$.

Proof. See [56, Theorem 2.1]. □

The Galerkin discretization of saddle point problems can be done in the standard way requiring some discrete stability or Babuška–Brezzi–Ladyshenskaya (BBL) condition [10]. Further the resulting system of equations is symmetric but indefinite. However due to $\omega = 0$, instead of (2.30) we can consider a modified saddle point problem.

Find $(\mathbf{t}, \omega) \in H^{-1/2} \times \mathbb{R}$ such that

$$\begin{aligned} &\langle V\mathbf{t}, \tau\rangle_\Gamma + \omega\langle \tau, \mathbf{t}^*\rangle_{V^L} = \langle \mathbf{b}, \tau\rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma), \\ &\langle \mathbf{t}, \mathbf{t}^*\rangle_{V^L} - \omega = 0. \end{aligned} \tag{2.31}$$

Now we can eliminate ω and we end up with a modified variational problem.

Find $\mathbf{t} \in H^{-1/2}(\Gamma)$ such that

$$\langle V\mathbf{t}, \tau\rangle_\Gamma + \langle \mathbf{t}, \mathbf{t}^*\rangle_{V^L}\langle \tau, \mathbf{t}^*\rangle_{V^L} = \langle \mathbf{b}, \tau\rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma). \tag{2.32}$$

Since the bilinear form in (2.32) is $H^{-1/2}(\Gamma)$–elliptic, there exists a unique solution $\mathbf{t} \in H^{-1/2}(\Gamma)$. Moreover, if we choose the test function $\tau = \mathbf{t}^*$ we find $\mathbf{t} \in H_*^{-1/2}(\Gamma)$. Hence the variational formulations (2.29), (2.30) and (2.32) are equivalent.

Remark 2.3.4. *Since a bilinear form defines an operator and conversely an operator induces a bilinear form, see [63, page 42] we can define*

$$\langle \widehat{V}\mathbf{t}, \tau\rangle_\Gamma :\, = \langle V\mathbf{t}, \tau\rangle_\Gamma + \langle \mathbf{t}, \mathbf{t}^*\rangle_{V^L}\langle \tau, \mathbf{t}^*\rangle_{V^L} \quad \textit{for } \mathbf{t}, \tau \in H^{-1/2}(\Gamma). \tag{2.33}$$

Here $\widehat{V}$ is, so called, the stabilized single layer potential.

Then the variational formulation of the boundary integral equation becomes to find $\mathbf{t} \in H^{-1/2}(\Gamma)$:

$$\langle \widehat{V}\mathbf{t}, \tau\rangle_{L_2(\Gamma)} = \langle (\frac{1}{2}I + K)\mathbf{g} - N_0\mathbf{f}, \tau\rangle_{L_2(\Gamma)} \quad \text{for all } \tau \in H^{-1/2}(\Gamma). \tag{2.34}$$

For further works on the Stokes system, we refer to [15, 18, 27, 41, 44, 48, 60, 67, 70].

2.4 Boundary elements

Recall that we have assumed that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain with piecewise smooth boundary $\Gamma = \partial\Omega$. We consider a family $\{\Gamma_h\}$ of decompositions of the boundary Γ,

$$\Gamma_h = \bigcup_{\ell=1}^{n_h} \overline{\tau}_\ell, \tag{2.35}$$

with the boundary elements τ_ℓ. For the present discussion we restrict ourselves to plane triangles for the choice of the boundary elements. The errors which may occur by this approximation of the boundary Γ are not considered here. For an overview and analysis of these errors, we refer to [52]. The decomposition, as defined in (2.35), is called admissible if two neighboring boundary elements share either a node or an edge. We define the local mesh size of a boundary element τ_ℓ

$$h_\ell := \left(\int_{\tau_\ell} ds_x \right)^{\frac{1}{2}},$$

and the global mesh sizes of a boundary decomposition Γ_h

$$h = h_{\max} := \max_{\ell=1,\dots,n_h} h_\ell, \quad h_{\min} := \min_{\ell=1,\dots,n_h} h_\ell.$$

The diameter of a boundary element τ_ℓ is defined by

$$d_\ell := \sup_{x,y\in\tau_\ell} |x-y|.$$

We assume that the family $\{\Gamma_h\}$ is uniformly shape regular, i.e., there exists a constant $c > 0$ which is independent of the boundary decomposition such that

$$d_\ell \le c h_\ell \quad \text{for all } \ell = 1, \dots, n_h.$$

For the Galerkin discretization of the boundary integral operators for the Dirichlet Stokes problems we consider finite dimensional trial spaces with respect to the boundary decompositions Γ_h. A conforming trial space of $H^{-1/2}(\Gamma)$ is $S_h^0(\Gamma)$, the space of piecewise constant functions. We use $\left\{\psi_\ell^h\right\}_{\ell=1}^{n_h}$ as basis functions of $S_h^0(\Gamma)$ with respect to the boundary decomposition Γ_h, where ψ_ℓ^h is constant one on the boundary element τ_ℓ and zero elsewhere. The space $S_h^1(\Gamma)$ of continuous piecewise linear functions is a conforming trial space of $H^{1/2}(\Gamma)$. We use nodal basis functions $\left\{\phi_j^h\right\}_{j=1}^{m_h}$ for $S_h^1(\Gamma)$, where the vertices of the boundary decomposition Γ_h are the nodes. Let $\left\{x_j\right\}_{j=1}^{m_h}$ be the set of vertices of Γ_h, then the basis functions of $S_h^1(\Gamma)$ are given by

$$\phi_j^1(x) = \begin{cases} 1 & \text{if } x = x_j, \\ 0 & \text{if } x = x_i \neq x_j, \\ \text{piecewise linear} & \text{elsewhere,} \end{cases}$$

for $j = 1, \dots, m_h$. Note, in order to use vector valued test functions we use upper case Greek letters Ψ and Φ for the piecewise constant and piecewise linear ones, respectively, i.e.,

$$\Psi_\ell^0(x) = \begin{pmatrix} \psi_\ell^0(x) \\ 0 \end{pmatrix}, \quad \Psi_{n_h+\ell}^0(x) = \begin{pmatrix} 0 \\ \psi_\ell^0(x) \end{pmatrix}, \quad \ell = 1, \dots, n_h,$$

and

$$\Phi_j^1(x) = \begin{pmatrix} \phi_j^0(x) \\ 0 \end{pmatrix}, \quad \Phi_{m_h+j}^0(x) = \begin{pmatrix} 0 \\ \phi_j^0(x) \end{pmatrix}, \quad j = 1, \dots, m_h.$$

The trial spaces $S_h^0(\Gamma)$ and $S_h^1(\Gamma)$ have the following approximation properties.

Theorem 2.4.1. *Let $\sigma \in [-1,0]$. For $\mathbf{u} \in H^s(\Gamma)$ with some $s \in [0,1]$ there holds the approximation property of $S_h^0(\Gamma)$*

$$\inf_{\mathbf{v}_h \in S_h^0(\Gamma)} \|\mathbf{u}-\mathbf{v}_h\|_{H^\sigma(\Gamma)} \leq ch^{s-\sigma} \|\mathbf{u}\|_{H^s(\Gamma)},$$

with some constant $c > 0$.

Proof. See, e.g., [58, page 252], [63, Section 10.2]. □

Theorem 2.4.2. *Let $\Gamma = \partial\Omega$ be sufficiently smooth. For $\sigma \in [0,1]$ and for some $s \in [0,2]$, $s \geq \sigma$, we assume $\mathbf{u} \in H^s(\Gamma)$. Then there holds the approximation property of $S_h^1(\Gamma)$*

$$\inf_{\mathbf{v}_h \in S_h^1(\Gamma)} \|\mathbf{u}-\mathbf{v}_h\|_{H^\sigma(\Gamma)} \leq ch^{s-\sigma} \|\mathbf{u}\|_{H^s(\Gamma)}.$$

Proof. See, [63, Theorem 10.9]. □

Similar approximation properties also hold for an open part Γ_i of Γ, see [57, Theorem 2.1, Theorem 2.3].

2.5 Boundary element methods for Dirichlet Stokes problem

Let $[S_h^0(\Gamma)]^2$ be some finite dimensional trial space spanned by, e.g., piecewise constant basis functions. The Galerkin variational formulation of (2.32) reads :

Find $\mathbf{t}_h \in [S_h^0(\Gamma)]^2$ such that

$$\langle V\mathbf{t}_h, \tau_h\rangle_\Gamma + \langle \mathbf{t}_h, \mathbf{t}^*\rangle_{V^L}\langle \tau_h, \mathbf{t}^*\rangle_{V^L} = \langle (\frac{1}{2}I+K)\mathbf{g} - N_0\mathbf{f}, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in [S_h^0(\Gamma)]^2,$$

$$\text{or} \quad \langle \widehat{V}\mathbf{t}_h, \tau_h\rangle_\Gamma = \langle (\frac{1}{2}I+K)\mathbf{g} - N_0\mathbf{f}, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in [S_h^0(\Gamma)]^2. \tag{2.36}$$

If we take $\tau = \tau_h$ in (2.34) and subtract it from (2.36), we get the Galerkin orthogonality.

$$\langle \widehat{V}(\mathbf{t}-\mathbf{t}_h), \tau_h\rangle_\Gamma = 0 \quad \text{for all } \tau_h \in [S_h^0(\Gamma)]^2. \tag{2.37}$$

Since the bilinear form in (2.36) is $H^{-1/2}(\Gamma)$–elliptic, hence by the Lax–Milgram lemma, there exists a unique solution $\mathbf{t}_h \in [S_h^0(\Gamma)]^2$ satisfying the quasi–optimal error estimate, i.e., Cea's lemma, see [63, Theorem 8.1]

$$\|\mathbf{t}-\mathbf{t}_h\|_{H^{-1/2}(\Gamma)} \leq c \inf_{\tau_h \in [S_h^0(\Gamma)]^2} \|\mathbf{t}-\tau_h\|_{H^{-1/2}(\Gamma)}.$$

Applying the approximation property of the piecewise constant trial space, Theorem 2.4.1, and when assuming $\mathbf{t} \in H^1_{pw}(\Gamma)$, we end up with the following estimate

$$\|\mathbf{t}-\mathbf{t}_h\|_{H^{-1/2}(\Gamma)} \leq ch^{3/2} \|\mathbf{t}\|_{H^1_{pw}(\Gamma)}.$$

The solution of the boundary integral equation (2.36) gives us an approximate solution $\mathbf{t}_h$, so now we have the complete Cauchy data, i.e., we are given the Dirichlet data $\mathbf{g}$, and we have computed the approximate Neumann datum $\mathbf{t}_h$, and we can obtain an approximate solution of the boundary value problem from the representation formula. If $\widetilde{u}_k(\widetilde{x})$ is the approximate solution for $\widetilde{x} \in \Omega$, then the corresponding representation formula reads

$$\widetilde{u}_k(\widetilde{x}) = (\widetilde{V}\mathbf{t}_h)(\widetilde{x}) - (W\mathbf{g})(\widetilde{x}) \quad \text{for } \widetilde{x} \in \Omega.$$

Also the representation formula for the continuous function is

$$u_k(\widetilde{x}) = (\widetilde{V}\mathbf{t})(\widetilde{x}) - (W\mathbf{g})(\widetilde{x}) \quad \text{for } \widetilde{x} \in \Omega.$$

The above discussion leads to the following theorem.

Theorem 2.5.1. *The pointwise error of the approximate solution can be estimated by*

$$|u_k(\widetilde{x}) - \widetilde{u}_k(\widetilde{x})| \le ch^3 \, \|\mathbf{U}_k^*(\widetilde{x},.)\|_{H^2(\Gamma)} \, |\mathbf{t}|_{H^1_{pw}(\Gamma)}.$$

Proof. Since

$$u_k(\widetilde{x}) \;=\; \int_\Gamma \mathbf{U}_k^*(\widetilde{x},y)^\top \mathbf{t}(y) ds_y - \int_\Gamma \mathbf{u}_k(y)^\top \mathbf{T}_k^*(\widetilde{x},y) ds_y \quad \text{for } \widetilde{x} \in \Omega,$$

also

$$\widetilde{u}_k(\widetilde{x}) \;=\; \int_\Gamma \mathbf{U}_k^*(\widetilde{x},y)^\top \mathbf{t}_h(y) ds_y - \int_\Gamma \mathbf{u}_k(y)^\top \mathbf{T}_k^*(\widetilde{x},y) ds_y \quad \text{for } \widetilde{x} \in \Omega,$$

and subtracting the last two equations and using a duality argument, we have

$$\begin{aligned} |u_k(\widetilde{x}) - \widetilde{u}_k(\widetilde{x})| &= \left| \int_\Gamma \mathbf{U}_k^*(\widetilde{x},y)^\top \left[\mathbf{t}(y) - \mathbf{t}_h(y)\right] ds_y \right|, \\ &\le \|\mathbf{U}_k^*(\widetilde{x},\cdot)\|_{H^{-\sigma}(\Gamma)} \, \|\mathbf{t} - \mathbf{t}_h\|_{H^\sigma(\Gamma)} \,. \end{aligned}$$

As $\widetilde{x} \in \Omega$ and $y \in \Gamma$, so $\mathbf{U}_k^*(\widetilde{x},y)$ is infinitely many times continuously differentiable, i.e., $\mathbf{U}_k^*(\widetilde{x},\cdot) \in H^{-\sigma}(\Gamma)$ for any $\sigma \in \mathbb{R}$. So for the best possible error estimate of the pointwise error, we need to have an error estimate of $\|\mathbf{t} - \mathbf{t}_h\|_{H^\sigma(\Gamma)}$ for a minimal $\sigma \in \mathbb{R}$. The next theorem (Aubin–Nitsche trick) will give us

$$\|\mathbf{t} - \mathbf{t}_h\|_{H^{-2}(\Gamma)} \le ch^3 \, \|\mathbf{t}\|_{H^1_{pw}(\Gamma)} \,.$$

□

Theorem 2.5.2. (Aubin–Nitsche trick) *Let* $\mathbf{t} \in H^s_{pw}(\Gamma)$, *for some* $s \in [-1/2, 1]$ *be the solution of the boundary integral equation* (2.34), *let* $\mathbf{t}_h \in S_h^0(\Gamma)$ *be the corresponding Galerkin approximation. Let* $\widehat{V} : H^{-1-\sigma}(\Gamma) \to H^{-\sigma}(\Gamma)$, *the stabilized single layer potential, be continuous and bijective for* $-2 \le \sigma \le -1/2$. *Then there holds the error estimate*

$$\|\mathbf{t} - \mathbf{t}_h\|_{H^\sigma(\Gamma)} \le ch^{s-\sigma} \, \|\mathbf{t}\|_{H^s_{pw}(\Gamma)} \,.$$

Proof. See [63, Theorem 12.3]. □

Equivalent system of linear equations

Recall the Galerkin variational formulation (2.36) for the boundary integral equation reads: Find $\mathbf{t}_h \in S_h^0(\Gamma)$ such that

$$\langle V\mathbf{t}_h, \tau_h\rangle_\Gamma + \langle \mathbf{t}_h, \mathbf{t}^*\rangle_{V^L}\langle \tau_h, \mathbf{t}^*\rangle_{V^L} = \langle (\frac{1}{2}I + K)\mathbf{g} - N_0\mathbf{f}, \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma).$$

In other words,

$$\begin{aligned} \left(V_h + \mathbf{a}\cdot\mathbf{a}^\top\right) &= \mathbf{f}, \\ V_h[i,j] &= \langle V\Psi_j^0, \Psi_i^0\rangle_\Gamma, \\ a[i] = \langle \Psi_i^0, t^*\rangle_{V^L}, \quad f[i] &= \langle (\frac{1}{2}I + K)\mathbf{g} - N_0\mathbf{f}, \Psi_i^0\rangle_\Gamma, \end{aligned}$$

where $i, j = 1, \dots, dn_h$.

Let us define $\widehat{V}_h := V_h + \mathbf{a}\cdot\mathbf{a}^\top$, which is symmetric and positive definite. To bound the spectral condition number of the stiffness matrix $\widehat{V}_h$, we state the following lemma.

Lemma 2.5.1. *For all $\mathbf{w} \in \mathbb{R}^{dn_h} \leftrightarrow \mathbf{w}_h \in [S_h^0(\Gamma)]^d$ and a globally quasi uniform boundary discretization , there hold the spectral equivalence inequalities*

$$c_1 h^d \|\mathbf{w}\|_2^2 \le \left(\widehat{V}_h\mathbf{w}, \mathbf{w}\right) \le c_2 h^{d-1}\|\mathbf{w}\|_2^2,$$

with some positive constants $c_i, i = 1, 2$.

Proof. See [63, page 269]. □

The approximation of right hand side

In practical computations, the given Dirichlet datum $\mathbf{g}$ has to be approximated by a function $\mathbf{g}_h \in S_h^1(\Gamma)$ where $S_h^1(\Gamma) \subset H^{1/2}(\Gamma)$ is some trial space, e.g., of piecewise linear basis functions. For example we may consider the piecewise linear interpolation

$$\mathbf{g}_{k,h}(x) = \sum_{i=1}^{m_h} \mathbf{g}_k(x_i)\phi_i^1(x), \quad k = 1, 2.$$

Lemma 2.5.2. *Let $\mathbf{v} \in H^s(\Gamma)$, $s \in \left[\frac{d-1}{2}, 2\right]$, i.e., $\mathbf{v}$ is continuous and a sufficiently smooth boundary Γ be given. Let $I_h\colon H^s(\Gamma) \to S_h^1(\Gamma)$ be the linear interpolation operator, i.e., $I_h\mathbf{v}(x_k) = \mathbf{v}(x_k)$ for $k = 1, \dots, d_{m_h}$. Then there holds the error estimate*

$$\|\mathbf{v} - I_h\mathbf{v}\|_{H^\sigma(\Gamma)} \le ch^{s-\sigma}|\mathbf{v}|_{H^s(\Gamma)} \quad \text{for } 0 \le \sigma \le min\{1, s\}. \tag{2.38}$$

Proof. See [63, page 241]. □

We note that on a boundary element τ_k, a piecewise linear function $\mathbf{v}_h(x)$ is described by the values $\mathbf{v}_h(x_{k_i})$ in the nodes x_{k_i}, hence instead of (2.36) we consider the perturbed variational formulation which reads:
Find $\widetilde{\mathbf{t}}_h \in S_h^0(\Gamma)$ such that

$$\langle \widehat{V}_h \widetilde{\mathbf{t}}_h, \tau_h \rangle_\Gamma = \langle (\frac{1}{2}I + K)\mathbf{g}_h, \tau_h \rangle_\Gamma - \langle N_0 \mathbf{f}, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma), \tag{2.39}$$

and an equivalent perturbed system of linear equations is

$$\begin{aligned} \widehat{V}_h \widetilde{\mathbf{t}}_h &= \widehat{\mathbf{f}}, \quad \text{where} \\ \widehat{\mathbf{f}} = \left(\frac{1}{2}M_h + K_h\right)\mathbf{g} - N_0\mathbf{f} \Leftrightarrow \widetilde{f}[\ell] &= \sum_{i=1}^{dm_h} g_i \langle (\frac{1}{2}I + K)\Phi_i^1, \Psi_\ell^0 \rangle_\Gamma - \sum_{i=1}^{dm_h} \langle N_0\mathbf{f}, \Psi_\ell^0 \rangle_\Gamma, \\ M_h[\ell, i] = \langle \Phi_i^1, \Psi_\ell^0 \rangle; \quad K_h[\ell, i] &= \langle K\Phi_i^1, \Psi_\ell^0 \rangle, \end{aligned}$$

for $i = 1, \ldots, dm_h$ and $\ell = 1, \ldots, dn_h$.

Due to the Strang lemma, see [63, Theorem 8.2], we get the following error estimate

$$\left\| \mathbf{t} - \widetilde{\mathbf{t}}_h \right\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^V} \left\{ c_2^V \inf_{\tau_h \in X_h} \|\mathbf{t} - \tau_h\|_{H^{-1/2}(\Gamma)} + c_2^W \|\mathbf{g} - \mathbf{g}_h\|_{H^{1/2}(\Gamma)} \right\},$$

using Theorem 2.4.1, we end up with

$$\left\| \mathbf{t} - \widetilde{\mathbf{t}}_h \right\|_{H^{-1/2}(\Gamma)} \le c_1 h^{s+1/2} |\mathbf{t}|_{H^s_{pw}(\Gamma)} + c_2^W \|\mathbf{g} - \mathbf{g}_h\|_{H^{1/2}(\Gamma)}. \tag{2.40}$$

L_2 Galerkin projection

To prove a more general error estimate, we now consider projection operators which are defined by some variational problem.

Lemma 2.5.3. *For* $\mathbf{u} \in L_2(\Gamma)$ *the* L_2 *projection* $Q_h\mathbf{u} \in S_h^1(\Gamma)$ *is defined by the unique solution of the variational formulation*

$$\langle Q_h \mathbf{u}, \mathbf{v}_h \rangle_{L_2(\Gamma)} = \langle \mathbf{u}, \mathbf{v}_h \rangle_{L_2(\Gamma)} \quad \textit{for all } \mathbf{v}_h \in S_h^1(\Gamma),$$

there holds the error estimate for $\mathbf{u} \in H^s(\Gamma)$, $s \in [0, 2]$,

$$\|\mathbf{u} - Q_h \mathbf{u}\|_{L_2(\Gamma)} \le c h^s |\mathbf{u}|_{H^s(\Gamma)}. \tag{2.41}$$

Proof. See [63, Theorem 10.2]. ☐

Remark 2.5.1. *The above estimate can be extended to*

$$\|\mathbf{u}-Q_h\mathbf{u}\|_{H^\sigma(\Gamma)} \quad \le \quad ch^{s-\sigma}|\mathbf{u}|_{H^s(\Gamma)},$$

where $s \in [0,2]$, $-1 \le \sigma \le min\{1,s\}$ *and* $Q_h\colon H^{1/2}(\Gamma) \to S_h^1(\Gamma) \subset H^{1/2}(\Gamma)$ *is a bounded operator*

$$\|Q_h\mathbf{v}\|_{H^{1/2}(\Gamma)} \le \|\mathbf{v}\|_{H^{1/2}(\Gamma)}\,,$$

see, [63]. As we have defined the L_2 projection (consider it as H^0 projection), we can accordingly define a H^σ projection. We state the following lemma.

Lemma 2.5.4. *For* $\mathbf{u} \in H^\sigma(\Gamma)$ *and* $\sigma \in (0,1]$, *we define the* H^σ *projection* $Q_h^\sigma\colon H^\sigma(\Gamma) \to S_h^1(\Gamma)$ *as the unique solution of the variational problem*

$$\langle Q_h^\sigma\mathbf{u},\mathbf{v}_h\rangle_{H^\sigma(\Gamma)} = \langle \mathbf{u},\mathbf{v}_h\rangle_{H^\sigma(\Gamma)} \quad \textit{for all } \mathbf{v}_h \in S_h^1(\Gamma), \tag{2.42}$$

there hold the error estimate, for $\mathbf{u} \in H^\sigma(\Gamma)$ *and* $s \in [\sigma,2]$

$$\|\mathbf{u}-Q_h^\sigma\mathbf{u}\|_{H^\sigma(\Gamma)} \quad \le \quad ch^{s-\sigma}|\mathbf{u}|_{H^s(\Gamma)}.$$

Proof. See [63, Section 12.1]. □

We are now interested to investigate the effect of the approximation of right the hand side on the error estimate of the perturbed solution. For this we state the following lemma.

Theorem 2.5.3. *For* $\mathbf{t} \in H^s_{pw}(\Gamma)$, $s \in [-1/2,1]$ *and* $\mathbf{g} \in H^\sigma(\Gamma)$, $\sigma \in [1/2,2]$. *Let* $Q_h\mathbf{g}$ *be the* L_2 *projection and* $\sigma \in [\frac{d-1}{2},2]$ *for the interpolation* $I_h\mathbf{g}$. *Then there holds the error estimate for the perturbed solution*

$$\left\|\mathbf{t}-\widetilde{\mathbf{t}}_h\right\|_{H^{-1/2}(\Gamma)} \le c_1h^{s+1/2}|\mathbf{t}|_{H^s_{pw}(\Gamma)} + c_2h^{\sigma-1/2}|\mathbf{g}|_{H^\sigma(\Gamma)}.$$

Proof. Applying the Strang lemma on the perturbed variational formulation and then using the approximation property of piecewise constant basis function, (see (2.40)),

$$\left\|\mathbf{t}-\widetilde{\mathbf{t}}_h\right\|_{H^{-1/2}(\Gamma)} \le c_1h^{s+1/2}\,\|\mathbf{t}\|_{H^s_{pw}(\Gamma)} \; + c_2^W\,\|\mathbf{g}-\mathbf{g}_h\|_{H^{1/2}(\Gamma)}\,,$$

the assertion follows by applying estimate (2.38) on the second term on the right hand side in case of $I_h\mathbf{g}$ and applying Lemma 2.5.3 and Remark 2.5.1 in the case of the L_2 projection $Q_h\mathbf{g}$. □

Remark 2.5.2. *Let us consider the optimal case, i.e., $s=1$ and $\sigma=2$,*

$$\left\|\mathbf{t}-\widetilde{\mathbf{t}}_h\right\|_{H^{-1/2}(\Gamma)} \le ch^{3/2}\left\{|\mathbf{t}|_{H^1_{pw}(\Gamma)}+|\mathbf{g}|_{H^2(\Gamma)}\right\}.$$

Note that we get the same order of convergence as in the non–perturbed case. This means that an approximation of the right hand side, either by $I_h\mathbf{g}$ or by $Q_h\mathbf{g}$, has no effect on the order of convergence of the approximate solution of the Dirichlet boundary value problem in the energy norm, but in more negative Sobolev norms.

Theorem 2.5.4. (Aubin–Nitsche trick) *For some $s\in[-\frac{1}{2},1]$ let $\mathbf{t}\in H^s_{pw}(\Gamma)$ be the solution of the boundary integral equation* (2.34)*, $\widetilde{\mathbf{t}}_h\in S^0_h(\Gamma)$ be the solution of the perturbed variational problem* (2.39)*. Let $\widehat{V}: H^{-1-\sigma}(\Gamma)\to H^{-\sigma}(\Gamma)$, the stabilized single layer potential be continuous and bijective for $\sigma\in[-2,-\frac{1}{2}]$, and let the double layer potential $\frac{1}{2}I+K\colon H^{1+\sigma}(\Gamma)\to H^{1+\sigma}(\Gamma)$ be bounded. For $\mathbf{g}\in H^\rho(\Gamma)$, $\rho\in(1,2]$. Then there holds the error estimate*

$$\left\|\mathbf{t}-\widetilde{\mathbf{t}}_h\right\|_{H^\sigma(\Gamma)} \le c_1h^{s-\sigma}|\mathbf{t}|_{H^s_{pw}(\Gamma)}+c_2h^{\rho-\sigma-1}|\mathbf{g}|_{H^\rho(\Gamma)},$$

where $\sigma>-1$ for $\mathbf{g}_h=I_h\mathbf{g}$ and $\sigma>-2$ for $\mathbf{g}_h=Q_h\mathbf{g}$.

Proof. See [63, Theorem 12.7]. ☐

Effect on the pointwise error by an approximation of the right hand side

Let $\widehat{u}_k(x)$ be the approximation of $u_k(\widetilde{x})$, for $k=1,2$.

$$u_k(\widetilde{x}) = \int_\Gamma \mathbf{U}^*_k(\widetilde{x},y)^\top\mathbf{t}(y)ds_y-\int_\Gamma \mathbf{g}(y)^\top\mathbf{T}^*_k(\widetilde{x},y)ds_y \quad \text{for } \widetilde{x}\in\Omega,$$

also

$$\widehat{u}_k(\widetilde{x}) = \int_\Gamma \mathbf{U}^*_k(\widetilde{x},y)^\top\mathbf{t}_h(y)ds_y-\int_\Gamma \mathbf{g}_h(y)^\top\mathbf{T}^*_k(\widetilde{x},y)ds_y \quad \text{for } \widetilde{x}\in\Omega,$$

and subtracting the last two equations and using a duality argument, we have

$$\begin{aligned}|u_k(\widetilde{x})-\widehat{u}_k(\widetilde{x})| &\le \left|\int_\Gamma \mathbf{U}^*_k(\widetilde{x},y)^\top\left[\mathbf{t}(y)-\widetilde{\mathbf{t}}_h(y)\right]ds_y\right|+\left|\int_\Gamma \mathbf{T}^*_k(\widetilde{x},y)^\top\left[\mathbf{g}(y)-\mathbf{g}_h(y)\right]ds_y\right|,\\ &\le \|\mathbf{U}^*_k(\widetilde{x},\cdot)\|_{H^{-\sigma}(\Gamma)}\left\|\mathbf{t}-\widetilde{\mathbf{t}}_h\right\|_{H^\sigma(\Gamma)}+\|\mathbf{T}^*_k(\widetilde{x},\cdot)\|_{H^{-\mu}(\Gamma)}\|\mathbf{g}-\mathbf{g}_h\|_{H^\mu(\Gamma)}.\end{aligned}$$

As $\widetilde{x}\in\Omega$ and $y\in\Gamma$, so $\mathbf{U}^*_k(\widetilde{x},y)$ and $\mathbf{T}^*_k(\widetilde{x},y)$ are infinitely many times continuously differentiable, i.e., $\mathbf{U}^*_k(\widetilde{x},\cdot)\in H^{-\sigma}(\Gamma)$,$\mathbf{T}^*_k(\widetilde{x},\cdot)\in H^{-\mu}(\Gamma)$ for any $\sigma,\mu\in\mathbb{R}$. So for the best possible estimate of the pointwise error, we need to have $s=1,\rho=2$ and $\sigma=-1$ or -2

depending upon $\mathbf{g}_h = I_h\mathbf{g}$ or $\mathbf{g}_h = Q_h\mathbf{g}$ respectively. We finally end up with following two estimates

$$\begin{aligned} |u_k(\widetilde{x}) - \widehat{u}_k(\widetilde{x})| &\leq ch^2 \left\{ \|\mathbf{t}\|_{H^1_{pw}(\Gamma)} + \|\mathbf{g}\|_{H^2(\Gamma)} \right\} \quad \text{for} \quad \mathbf{g}_h = I_h\mathbf{g}, \\ &\leq ch^3 \left\{ |\mathbf{t}|_{H^1_{pw}(\Gamma)} + |\mathbf{g}|_{H^2(\Gamma)} \right\} \quad \text{for} \quad \mathbf{g}_h = Q_h\mathbf{g}. \end{aligned}$$

Remark 2.5.3. *For the pointwise error, we gain one order of convergence if we approximate the right hand side by using the L_2 projection.*

3 OPTIMAL CONTROL FOR STOKES PROBLEM

In this chapter we start with a brief introduction to the optimal control problems and their solutions. The main references for this section are [28, 33, 68]. We then discuss an elliptic optimal control model problem with a quadratic cost functional and the Stokes system as linear constraint. The control is on the Dirichlet boundary which lie in a closed and convex subset of $H^{1/2}(\Gamma)$ and also satisfies a certain compatibility condition. The goal is to reformulate the boundary control problem into an equivalent system of boundary integral equations. With the help of the state to control mapping $\mathcal{S}$, we formulate the reduced cost functional in the control. The adjoint problem is obtained by the realization of the adjoint operator $\mathcal{S}^*$. We end this chapter by setting up the optimality system consisting of the primal problem, the adjoint problem and a coupling condition which turns out to be an elliptic variational inequality of the first kind.

3.1 An introduction to optimal control

An optimal control problem is an optimization problem in which the variable to be optimized admits a natural splitting into a state variable and a control variable. Thus we can say that all optimal control problems are optimization problems but not all optimization problems are optimal control problems.

The basic ingredients of an optimal control problem

An optimal control problem consists of the following components.

- State variables.
- Control variables or design parameters.
- An objective or a cost functional.
- Constraints (restrictions) which the state and control variables are required to satisfy.

Then an optimal control problem is:

Find the state and the control variables that minimizes (or maximizes) the objective functional subject to the requirements that the constraints are satisfied.

It may be noted that we are going to consider only the class of optimization problem which requires the differentiability of the cost functional and of the constraints. In other words we are not going to discuss the so called 'Non–smooth Optimization'. Further we are going to present the text within the context of fluid mechanics. Now we define and explain the basic components of an optimal control problem.

Definition 3.1.1. *In the fluid mechanics setting, variables describing the mechanical and thermodynamical behavior are called* ***state variables****, e.g., one or more of the velocity, velocity potential, pressure, density, temperature and internal energy can be regarded as state variable.*

Definition 3.1.2. *One or more of the data specified that serve to determine the state variables are termed as* ***control variables or design parameter****, e.g., heat flux, temperature at the wall, an inflow mass flow rate, parameters that determine the shape of the boundary.*

Definition 3.1.3. *The functional describing the ultimate goal of the optimization problem is called* ***objective or performance functional****, e.g., we may be interested to see how close is the velocity field to a given target velocity field, or the size of the drag or the lift, or temperature variations.*

Definition 3.1.4. *The restrictions or the conditions that the state and control variables must abide are called the* ***constraints****, e.g., the main constraints governing the flow equations are Naiver–Stokes, or Euler, or potential flow equations. We also encounter the side constraints such as minimum lift, or minimum volume, or maximum power requirements.*

The structure of flow optimal control problems

The flow control optimization problem has usually three components. First we have an *objective*, a reason why we want to control the flow. There may be various *objectives* of interest in applications, e.g., flow matching, drag minimization, lift enhancement, preventing separation, preventing transition to turbulence, deterring temperature variations, enhancing mixing, deterring mixing, etc. Mathematically, such an object is expressed as cost, or objective, or performance functional.

Next one has *control* or design parameters at one's disposal in order to meet the objective. One has boundary value control such as injection or suction of the fluid and heating or cooling or temperature control, one could have distributed control such as heating sources or magnetic fields, one could have shape control such as leading or trailing edge flaps, movable wall rudders, propeller pitch, surface roughness, or domain design; or, one could have combinations of numbers of controls or design parameters. Mathematically, control or design parameters are expressed in terms of unknown data in the mathematical specification of the problem.

Finally one has *constraints* that determines what type of flow one is interested in and that place direct or indirect limits on candidate optimizer. One must decide, e.g., what type of fluid model is adequate for the application in mind assuming that flow is potential flow, an inviscid flow, a viscous flow, a compressible or incompressible flow, stationery flow, a time dependent flow, etc. Mathematically the type of flow is expressed in terms of a specific set of partial differential equations. One may also impose side constraints motivated by practical necessities, for example, one may want to minimize the drag on an airfoil subject to the lift and/or the volume being grater than a specific value. One then puts together the three ingredients in an optimization problem by seeking the optimal state and the control that satisfy the constraints and minimize the objective functional.

An abstract optimal control problem

If ϕ denotes the state variables, g the control variable or design parameter, $\mathcal{J}(\phi,g)$ the cost or the objective functional, and $F(\phi,g)=0$ are constraints. Then the optimization problem reads.

Find the control g and state ϕ such that $\mathcal{J}(\phi,g)$ is minimized subject to $F(\phi,g)=0$.

It may be noted that many objective functionals, we see in practise, do not explicitly depend on the design parameters which leads to unbounded optimal control problems. In such a situation, we must somehow limit the size of the control. This can be achieved in two ways:

- Restrict the size of an admissible control so that they are sought for within a bounded set, e.g., look for an optimal control such that, for some suitable norm $\|\cdot\|$ and a constant κ

$$\|\cdot\| \quad \leq \quad \kappa.$$

- Penalize the objective functional , i.e., instead of minimizing a given functional $\mathcal{E}(\phi)$ that depends only on the state variables, minimize, for some suitable norm $\|\cdot\|$ and constants σ and β, the functional

$$\mathcal{J}(\phi,g)=\mathcal{E}(\phi)+\sigma\,\|\,g\,\|^{\beta}\,.$$

Note that in the first case we are just imposing the side constraints on the optimization problem. This we are not going to discuss here. In the second case, we have changed the problem, i.e., the minimizer of $\mathcal{J}$ are not, in general, minimizers of the given functional $\mathcal{E}$. However, in many if not most practical settings, limiting the size of the control through penalization is easier to implement than through placing the explicit bound on the control variables.

Solution of optimal control and optimization problems

There are three approaches to solve optimal control and optimization problems. These are

- one–shot, or adjoint, or co–state, or Lagrange multiplier methods,

- optimization methods based on the sensitivity equations and,
- optimization methods based on adjoint equations.

3.2 Unconstrained optimization problem

Our aim is to set up an optimality system which consists of the state equation, the adjoint equation and an optimality condition. The resulting optimality system then can be solved by some suitable method. We set up the optimality system for our abstract example by a Lagrange multiplier method. This method recasts the constrained optimization problem as unconstrained optimization problem. We introduce the Lagrange multiplier or adjoint variable or co–state variable ξ and the Lagrangian functional

$$\mathcal{L}(\phi, g, \xi) = \mathcal{J}(\phi, g) - \xi^* F(\phi, g),$$

where $\xi^* F$ can be viewed as inner product or duality pairing. Then we pose the following unconstrained optimization problem:
Find controls g, states ϕ and co–states ξ such that $\mathcal{L}(\phi, g, \xi)$ is rendered stationary.
The first order necessary conditions then give the optimality system from which we can find the optimal states and design parameters.

$$\frac{\delta \mathcal{L}}{\delta \xi} = 0 \Rightarrow \text{state equation,} \tag{3.1}$$

$$\frac{\delta \mathcal{L}}{\delta \phi} = 0 \Rightarrow \text{adjoint or co–state equation,} \tag{3.2}$$

$$\frac{\delta \mathcal{L}}{\delta g} = 0 \Rightarrow \text{optimality condition.} \tag{3.3}$$

Here we consider the case that each argument of $\mathcal{L}(\phi, g, \xi)$ is independent of each other. This was not true in the original optimization problem involving $\mathcal{J}(\phi, g)$ since the argument ϕ and g were constrained to satisfy $F(\phi, g) = 0$ and thus could not be chosen independently. The equation (3.1) can be written in an alternative way as

$$\lim_{\varepsilon \to 0} \left(\frac{\mathcal{L}(\phi, g, \xi + \varepsilon \widetilde{\xi}) - \mathcal{L}(\phi, g, \xi)}{\varepsilon} \right) = 0,$$

where the variation $\widetilde{\xi}$ in the Lagrange multiplier ξ is arbitrary. Substituting for $\mathcal{L}$, we have

$$\lim_{\varepsilon \to 0} \left[\frac{\mathcal{J}(\phi, g) - (\xi + \varepsilon \widetilde{\xi})^* F(\phi, g) - (\mathcal{J}(\phi, g) - \xi^* F(\phi, g))}{\varepsilon} \right] = 0,$$

$$\widetilde{\xi}^* F(\phi, g) = 0.$$

Since the variation $\widetilde{\xi}$ in the Lagrange multiplier ξ is arbitrary, we recover the state equation $F(\phi,g)=0$. Setting the first variation of $\mathcal{L}$ with respect to the state ϕ equal to zero is equivalent to the condition

$$\lim_{\varepsilon\to 0}\left(\frac{\mathcal{L}(\phi+\varepsilon\widetilde{\phi},g,\xi)-\mathcal{L}(\phi,g,\xi)}{\varepsilon}\right)=0,$$

where the variation $\widetilde{\phi}$ in the state ϕ is arbitrary. Again substituting for $\mathcal{L}$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\left[\frac{\mathcal{J}(\phi+\varepsilon\widetilde{\phi},g)-\xi^{*}F(\phi+\varepsilon\widetilde{\phi},g)-(\mathcal{J}(\phi,g)-\xi^{*}F(\phi,g))}{\varepsilon}\right] &= 0,\\ \lim_{\varepsilon\to 0}\left[\frac{\mathcal{J}(\phi+\varepsilon\widetilde{\phi},g)-\mathcal{J}(\phi,g)}{\varepsilon}-\frac{\xi^{*}\left(F(\phi+\varepsilon\widetilde{\phi},g)-F(\phi,g)\right)}{\varepsilon}\right] &= 0. \qquad (3.4)\end{aligned}$$

We assume the functionals and variables to be sufficiently smooth and introduce the Taylor series expansion as

$$\mathcal{J}(\phi+\varepsilon\widetilde{\phi},g)=\mathcal{J}(\phi,g)+\varepsilon\left(\frac{\partial\mathcal{J}}{\partial\phi}|_{(\phi,g)}\right)\widetilde{\phi}+\mathcal{O}(\varepsilon^{2}),$$

and a similar expansion for $F(\phi+\varepsilon\widetilde{\phi},g)$, so equation (3.4) takes the form

$$\lim_{\varepsilon\to 0}\left[\left(\frac{\partial\mathcal{J}}{\partial\phi}|_{(\phi,g)}\right)\widetilde{\phi}-\xi^{*}\left(\frac{\partial F}{\partial\phi}|_{(\phi,g)}\widetilde{\phi}\right)+\mathcal{O}(\varepsilon)\right]=0,$$

$$\left(\frac{\partial\mathcal{J}}{\partial\phi}|_{(\phi,g)}\right)\widetilde{\phi}-\xi^{*}\left(\frac{\partial F}{\partial\phi}|_{(\phi,g)}\widetilde{\phi}\right)=0,$$

$$\widetilde{\phi}^{*}\left(\left(\frac{\partial\mathcal{J}}{\partial\phi}|_{(\phi,g)}\right)^{*}-\left(\frac{\partial F}{\partial\phi}|_{(\phi,g)}\right)^{*}\xi\right)=0,$$

where $\left(\frac{\partial\mathcal{J}}{\partial\phi}|_{(\phi,g)}\right)^{*}$ denotes the adjoint operator of $\left(\frac{\partial\mathcal{J}}{\partial\phi}|_{(\phi,g)}\right)$. Since the variation $\widetilde{\phi}$ in the state ϕ is arbitrary, we get the adjoint equation

$$\left(\frac{\partial F}{\partial\phi}|_{(\phi,g)}\right)^{*}\xi=\left(\frac{\partial\mathcal{J}}{\partial\phi}|_{(\phi,g)}\right)^{*}.$$

Finally, setting the first variation of $\mathcal{L}$ with respect to g equal to zero and proceeding in a similar manner as above we get the following optimality condition

$$\left(\frac{\partial F}{\partial g}|_{(\phi,g)}\right)^{*}\xi=\left(\frac{\partial\mathcal{J}}{\partial g}|_{(\phi,g)}\right)^{*}.$$

Hence the optimality system is

$$\begin{aligned}
F(\phi,g) &= 0 && \Rightarrow \text{state equation},\\
\left(\frac{\partial F}{\partial \phi}|_{(\phi,g)}\right)^{*}\xi &= \left(\frac{\partial \mathcal{J}}{\partial \phi}|_{(\phi,g)}\right)^{*} && \Rightarrow \text{adjoint equation},\\
\left(\frac{\partial F}{\partial g}|_{(\phi,g)}\right)^{*}\xi &= \left(\frac{\partial \mathcal{J}}{\partial g}|_{(\phi,g)}\right)^{*} && \Rightarrow \text{optimality condition}.
\end{aligned}$$

Note that if the state system is expensive to solve, then this coupled system is an even more formidable. However, if we can solve the coupled optimality system with the help of some computational methods, then optimal states and controls can be obtained without an optimization iteration. Due to this specific reason, such an approach is sometimes called a one–shot method for optimization.

3.3 Constrained optimization problem

In the previous section we have setup the optimality system by a Lagrange multiplier method which recasts the optimization problem with equality constraints as an unconstrained optimization problem. Now we consider the case where we have in addition inequality constraints, i.e., box constraints for the control. But before that we discuss some basic definitions and results which can be found in most standard textbooks on functional analysis, e.g., [1, 73].

Definition 3.3.1 (Dual space)**.** *The space of all continuous linear functionals on* $(U,\|\cdot\|_U)$*, denoted by* U^**, is called the dual space of* U.

Observe that $U^* = \mathcal{L}(U,\mathbb{R})$. The associated norm is given by

$$\|f\|_{U^*} = \sup_{\|u\|_U=1} |f(u)|.$$

Moreover, since $\mathbb{R}$ is a complete space, the dual space U^* is always a Banach space.

Theorem 3.3.1 (Riesz representation theorem)**.** *Let* $(H,(\cdot,\cdot)_H)$ *be a real Hilbert space. Then for any continuous linear functional* $F \in H^*$ *there exists a uniquely determined* $f \in H$ *such that* $\|F\|_{H^*} = \|f\|_H$ *and*

$$F(v) = (f,v)_H \qquad \text{for all } v \in H.$$

Proof. See [63]. □

Definition 3.3.2 (Bidual and reflexive spaces)**.** *Let U denote a real Banach space with associated dual space U^*. We fix an arbitrary $u \in U$, let f vary over U^*, and consider the mapping $F_u : U \to \mathbb{R}$ induced by u,*

$$F_u : f \to f(u).$$

Clearly, F_u is linear, and its continuity is a consequence of the simple estimate

$$|F_u(f)| = |f(u)| \leq \|u\|_U \|f\|_{U^*}.$$

Hence, the functional F_u induced by u belongs to the dual space $(U^*)^* =: U^{**}$ of U^*. Since the mapping $u \to F_u$ turns out to be injective, we may identify u with F_u, thereby interpreting $u \in U$ as an element of U^{**}.

The space U^{**} is called the bidual space of U. In light of the above identification, it is always true that $U \subset U^{**}$. The mapping $u \to F_u$ from U into U^{**} is called the canonical embedding or canonical mapping. If this mapping is surjective, i.e., if $U = U^{**}$, then U is called a reflexive space. In the case of reflexive spaces, taking the dual twice leads back to the original space. In particular, we infer from the Riesz representation theorem that Hilbert spaces are always reflexive.

Definition 3.3.3 (Weak convergence)**.** *Let U be a real Banach space. We say that a sequence $\{u_n\}_{n=1}^{\infty} \subset U$ converges weakly to some $u \in U$ if*

$$\lim_{n\to\infty} f(u_n) = f(u) \quad \text{for all } f \in U^*.$$

We denote weak convergence by the symbol $\rightharpoonup$, i.e., we write $u_n \rightharpoonup n$ as $n \to \infty$. The limit u is uniquely determined and is called the weak limit of the sequence.

Remark 3.3.1. *If a sequence $\{u_n\}_{n=1}^{\infty} \subset U$ converges strongly (that is, with respect to the norm of U) to some $u \in U$, then it also converges weakly to u, i.e.,*

$$u_n \to u \Rightarrow u_n \rightharpoonup u \text{ as } n \to \infty.$$

Definition 3.3.4 (Weakly sequentially continuous mappings)**.** *Let U and V denote real Banach spaces. A mapping $F : U \to V$ is said to be weakly sequentially continuous if the following holds: whenever a sequence $\{u_n\}_{n=1}^{\infty} \subset U$ converges weakly in U to some $u \in U$, its image $\{F(u_n)\}_{n=1}^{\infty} \subset V$ converges weakly to $F(u)$ in V,i.e.,*

$$u_n \rightharpoonup u \Rightarrow F(u_n) \rightharpoonup F(u) \text{ as } n \to \infty.$$

Definition 3.3.5 (Weakly sequentially closed/compact set)**.** *Let M be a subset of a real Banach space U. We say that M is weakly sequentially closed if the limit of every weakly convergent sequence $\{u_n\}_{n=1}^{\infty} \subset M$ lies in M. We say that M is weakly sequentially relatively compact if every sequence M contains a weakly convergent subsequence; if, in addition, M is weakly sequentially closed, then M is said to be weakly sequentially compact.*

Remark 3.3.2. *Every strongly convergent sequence also converges weakly. The converse is not true in general.*

Every weakly sequentially closed set is also (strongly) closed; however, not every strongly closed set must be weakly sequentially closed.

Theorem 3.3.2. *Every bounded subset of a reflexive Banach space is weakly sequentially relatively compact.*

Proof. See [42] or [72]. □

Definition 3.3.6 (Convex set)**.** *A subset C of a real Banach space U is said to be convex if for any pair $u,v \in C$ and any $\lambda \in [0,1]$ the convex combination $\lambda u + (1-\lambda)v$ also lies in C.*

Definition 3.3.7 (Convex and strictly convex functionals)**.** *Let U be a real Banach space and $C \subset U$. A functional $f : C \to \mathbb{R}$ is said to be convex if*

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v) \quad \text{for all } \lambda \in [0,1] \text{ and for all } u,v \in C. \tag{3.5}$$

The functional f is said to be strictly convex if the above inequality (3.5) *holds with $<$ in place of $\leq$, whenever $u \neq v$ and $\lambda \in (0,1)$.*

Theorem 3.3.3. *Every convex and closed subset of a Banach space is weakly sequentially closed. If the space is reflexive and the set is in addition bounded, then it is weakly sequentially compact.*

Proof. The first assertion of the theorem is an easy consequence of Mazur's theorem, which states that the weak limit of a weakly convergent sequence is at the same time the strong limit of a sequence consisting of suitable convex combinations of the terms of the sequence. This part of the assertion is already true in normed spaces; see [5] and [71]. The second assertion follows from Theorem 3.3.2. □

Theorem 3.3.4. *Every continuous and convex functional $f : U \to \mathbb{R}$ on a Banach space U is weakly lower semicontinuous, i.e., for any sequence $\{u_n\}_{n=1}^{\infty} \subset U$ such that $u_n \rightharpoonup n$ as $n \to \infty$ we have*

$$\liminf_{n\to\infty} f(u_n) \geq f(u).$$

Proof. See [5] or [72]. □

Remark 3.3.3. *In the literature, the notions of weak compactness and weak closedness in the sense of the weak topology are often used in place of weak sequential compactness and weak sequential closedness, respectively. It should be noted, however, that in reflexive Banach spaces the two concepts are equivalent [12].*

Theorem 3.3.5. *Let U and V are two real Banach spaces. A linear operator $A : U \to V$ is continuous if and only if it is bounded.*

Proof. See [73]. □

Reduced minimization problem

We can define the solution operator for a given control problem as

$$u = S(z),$$

where z is the control and u is the associated state. By using this control to state operator S, the general minimization problem can be written as following

$$\mathcal{J}(z) := \frac{1}{2}\|S(z) - \overline{u}\|_U^2 + \frac{\rho}{2}\|z\|_Z^2. \quad \text{for } z \in Z,$$

which is the reduced quadratic minimization problem in the control space Z. We can use Theorem 3.3.4 to establish weakly lower semicontinuity of the functional $\mathcal{J}(z)$, but still we need to prove the continuity and the convexity of this functional. To this end we state the following lemma.

Lemma 3.3.1. *Let $(Z, \|\cdot\|_Z)$ and $(U, \|\cdot\|_U)$ denote two real Hilbert spaces, let $S : Z \to U$ be a continuous linear operator and $\overline{u} \in U$. Then the quadratic functional*

$$\mathcal{J}(z) := \frac{1}{2}\|S(z) - \overline{u}\|_U^2 + \frac{\rho}{2}\|z\|_Z^2. \quad \text{for } z \in Z$$

is convex for $\rho \geq 0$ and strictly convex for $\rho > 0$.

Proof. The given functional can be written as

$$\mathcal{J}(z) := \frac{1}{2}\|S(z) - \overline{u}\|_U^2 + \frac{\rho}{2}\|z\|_Z^2 = f_1(z) + f_2(z).$$

It is easy to show that $f_1(z)$ is convex for $\rho \geq 0$ and $f_2(z)$ is strictly convex for $\rho > 0$. Hence the sum $\mathcal{J}(z)$ is convex for $\rho \geq 0$ and strictly convex for $\rho > 0$. This completes the proof. ☐

Remark 3.3.4. *If for the bounded linear operator S with $S(u) = 0$, we get $u = 0$, then the functional $\mathcal{J}$ is also strictly convex for $\rho \geq 0$.*

Theorem 3.3.6. *Let $(Z, \|\cdot\|_Z)$ and $(U, \|\cdot\|_U)$ denote two real Hilbert spaces, and let a non–empty, closed, bounded, and convex set $Z_{ad} \subset Z$, as well as some $\overline{u} \in U$ and a constant $\rho \geq 0$ be given. Moreover, let $S : Z \to U$ be a continuous linear operator. Then the quadratic Hilbert space optimization*

$$\min_{z \in Z_{ad}} \mathcal{J}(z) := \frac{1}{2}\|S(z) - \overline{u}\|_U^2 + \frac{\rho}{2}\|z\|_Z^2,$$

admits an optimal solution $\overline{z}$. If $\rho > 0$ or S is injective, then the solution is uniquely determined.

Proof. See [68, Theorem 2.14]. ☐

Definition 3.3.8 (Gâteaux derivatives)**.** *Let* $(U, \|\cdot\|_U)$ *and* $(V, \|\cdot\|_V)$ *are real Banach spaces. Let* $\mathcal{U}$ *be a non–empty and an open subset of* U. *Let* F *denotes a mapping from* $\mathcal{U}$ *into* V. *Suppose that the first variation* $\delta F(u,h)$ *at* $u \in \mathcal{U}$ *exists, and suppose there exists a continuous linear operator* $A : U \to V$ *such that*

$$\delta F(u,h) := \lim_{t \downarrow 0} \frac{1}{t}\left(F(u+th) - F(u)\right) = Ah \quad \text{for all } h \in U.$$

Then F *is said to be Gâteaux differentiable at* u, *and* A *is referred to as the Gâteaux derivative of* F *at* u. *We write* $A = F'(u)$.

It follows from the definition that Gâteaux derivatives can be determined as directional derivatives. Note also that in the case where $V = \mathbb{R}$, i.e., if a functional $f : U \to \mathbb{R}$ is Gâteaux differentiable at a point $u \in \mathcal{U}$, then $f'(u)$ is an element of the dual space U^*.

Definition 3.3.9 (Fréchet derivatives)**.** *Let* $(U, \|\cdot\|_U)$ *and* $(V, \|\cdot\|_V)$ *are two real Banach spaces. Let* $\mathcal{U}$ *be a non–empty and an open subset of* U. *A mapping* $F : \mathcal{U} \subset U \to V$ *is said to be Fréchet differentiable at* $u \in U$ *if there exists an operator* $A \in \mathcal{L}(U,V)$ *and a mapping* $r(u,\cdot) : U \to V$ *with the following properties: for* $h \in U$ *such that* $u + h \in \mathcal{U}$, *we have*

$$F(u+h) = F(u) + Ah + r(u,h),$$

where the so–called remainder r *satisfies the condition*

$$\frac{\|r(u,h)\|_V}{\|h\|_U} \to 0 \quad \text{as } \|h\|_U \to 0.$$

The operator A *is called the Fréchet derivative of* F *at* u *and we write* $A = F'(u)$. *If* A *is Fréchet differentiable at every point* $u \in \mathcal{U}$, *then* A *is said to be Fréchet differentiable in* $\mathcal{U}$.

Remark 3.3.5. *In order to avoid confusion, we denote the Gâteaux derivative of* F *by* $F'_G(u)$ *and not by* $F'(u)$. *If the Fréchet derivative exists, then so does the Gâteaux derivative and we have* $F'_G(u) = F'(u)$. *The converse is false, in general.*

Definition 3.3.10 (Hilbert space adjoint)**.** *Let real Hilbert spaces* $(U, (\cdot,\cdot)_U)$ *and* $(V, (\cdot,\cdot)_V)$ *as well as an operator* $A \in \mathcal{L}(U,V)$ *be given. An operator* $A^\star$ *is called the Hilbert space adjoint or adjoint of* A *if*

$$(v, Au)_V = (A^\star v, u)_U \quad \text{for all } u \in U, \quad \text{for all } v \in V.$$

Quadratic optimization in Hilbert spaces

In order to prove the existence of optimal controls, we transformed the control problems under investigation into a reduced quadratic optimization problem in terms of z, namely

$$\mathcal{J}(z) := \frac{1}{2}\|S(z) - \overline{u}\|_U^2 + \frac{\rho}{2}\|z\|_Z^2. \tag{3.6}$$

For the minimization problem (3.6), the following fundamental result can be applied. It is the key to the derivation of first–order necessary optimality conditions in the presence of control constraints.

Lemma 3.3.2. *Let Z be a real Banach space, $M \subset Z$ is a given convex set and $f : M \to \mathbb{R}$ is real–valued Gâteaux differentiable functional on M. If $\bar{z} \in M$ is a solution to the problem*

$$f(\bar{z}) = \min_{z \in M} f(z) \tag{3.7}$$

then it solves the variational inequality

$$f'_G(\bar{z})(z - \bar{z}) \geq 0 \quad \text{for all } z \in M. \tag{3.8}$$

Conversely, if $\bar{z} \in M$ solves the variational inequality (3.8) *and f is convex, then $\bar{z}$ is a solution to the minimization problem* (3.7).

Proof. See [68, page 63]. □

Theorem 3.3.7. *Suppose that real Hilbert spaces Z and H, a non–empty and convex set $Z_{ad} \subset Z$, some $\bar{u} \in U$, and a constant $\rho \geq 0$ are given. Moreover, let $S : Z \to U$ denote a continuous linear operator. Then $\bar{z} \in Z_{ad}$ is a solution to the minimization problem* (3.6) *if and only if $\bar{z}$ solves the variational inequality*

$$(S^{\star}(S\bar{z} - \bar{u}) + \rho \bar{z}, w - \bar{z})_Z \geq 0 \quad \text{for all } w \in Z_{ad}. \tag{3.9}$$

Proof. The gradient of the functional (3.6) is given by ([68, page 60])

$$f'(\bar{z}) = S^{\star}(S\bar{z} - \bar{u}) + \rho \bar{z}$$

The assertion is thus a direct consequence of Lemma 3.3.2. □

In many instances it is advantageous to write the variational inequality (3.9) in the equivalent form

$$(S\bar{z} - \bar{u}, Sw - S\bar{z})_U + \rho(\bar{z}, w - \bar{z})_Z \geq 0 \quad \text{for all } w \in Z_{ad}.$$

which avoids the adjoint operator $S^{\star}$.

3.4 Dirichlet Boundary control for Stokes Problem

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded domain with Lipschitz boundary $\Gamma := \partial\Omega$. The boundary control of Stokes flows can be stated as following boundary control model problem.

$$\text{Minimize} \quad \mathcal{J}(\mathbf{u}, \mathbf{z}) = \underbrace{\frac{1}{2}\int_{\Omega} [\mathbf{u}(x) - \bar{\mathbf{u}}(x)]^2 dx}_{tracking} + \underbrace{\frac{\rho}{2}\|\mathbf{z}\|^2_{H^{1/2}(\Gamma)}}_{control\ cost}, \tag{3.10}$$

subject to the elliptic boundary value problem (state equations)

$$\begin{aligned} -\mu\Delta\mathbf{u}(x)+\nabla p(x) &= \mathbf{f}(x) \quad \text{for } x\in\Omega,\\ \nabla\cdot\mathbf{u}(x) &= 0 \quad \text{for } x\in\Omega, \\ \gamma_0^{int}\mathbf{u}(x) &= \mathbf{z}(x) \quad \text{for } x\in\Gamma, \end{aligned} \tag{3.11}$$

where $\mathbf{u}\in H^1(\Omega)$ is the velocity field (the state variable), p is the pressure, and the control $\mathbf{z}$ satisfies the box constraints

$$\mathbf{z}\in\mathcal{U}:=\left\{\mathbf{w}\in H_*^{1/2}(\Gamma)\colon \mathbf{g}_a(x)\le\mathbf{w}(x)\le\mathbf{g}_b(x)\right\}. \tag{3.12}$$

Note that $\mathbf{z}\in\mathcal{U}\subset H_*^{1/2}(\Gamma)$ satisfies the solvability condition (2.8). We assume $\mathbf{f}\in L_2(\Omega)$, $\overline{\mathbf{u}}\in L_2(\Omega)$ is the given target, $\rho\in\mathbb{R}_+$ is a fixed parameter which penalizes the cost of the control or some regularization parameter and $\mathbf{g}_a,\mathbf{g}_b\in H_*^{1/2}(\Gamma)$. The aim is to find the velocity $\mathbf{u}$ as close as possible to a given (the desired state) target velocity field $\overline{\mathbf{u}}$.

Optimal control problems, such as above, belong to a class of PDE constrained optimization problems, which appear in many areas of science and engineering, e.g., see [33]. Since the control variable is on the boundary, the idea is to reformulate the boundary control problem into an equivalent system of boundary integral equations.

Suppose there exists an equivalent norm $\|\cdot\|_{\mathcal{A}}$ in $H_*^{1/2}(\Gamma)$ which is induced by an elliptic, self–adjoint and bounded operator $\mathcal{A}\colon H_*^{1/2}(\Gamma)\to H_*^{-1/2}(\Gamma)$, i.e.,

$$\langle\mathcal{A}\mathbf{w},\mathbf{w}\rangle_\Gamma\ge\gamma_1^{\mathcal{A}}\|\mathbf{w}\|^2_{H_*^{1/2}(\Gamma)};\quad \|\mathcal{A}\mathbf{w}\|_{H_*^{-1/2}(\Gamma)}\le\gamma_2^{\mathcal{A}}\|\mathbf{w}\|_{H_*^{1/2}(\Gamma)}\quad \text{for all } w\in H_*^{1/2}(\Gamma).$$

For example the operator $\mathcal{A}$ can be the inverse single layer potential for Stokes.

3.5 The solution of the Stokes system

To find a particular solution with homogeneous Dirichlet boundary condition we first consider the system

$$\begin{aligned} -\Delta\mathbf{u}_f(x)+\nabla p_f(x) &= \mathbf{f}(x) \quad \text{for } x\in\Omega,\\ \nabla\cdot\mathbf{u}_f(x) &= 0 \quad \text{for } x\in\Omega, \\ \gamma_0^{int}\mathbf{u}_f(x) &= \mathbf{0} \quad \text{for } x\in\Gamma. \end{aligned}$$

It remains to consider the homogeneous Dirichlet boundary value problem

$$\begin{aligned} -\Delta\mathbf{u}_z(x)+\nabla p_z(x) &= \mathbf{0} \quad \text{for } x\in\Omega,\\ \nabla\cdot\mathbf{u}_z(x) &= 0 \quad \text{for } x\in\Omega, \\ \gamma_0^{int}\mathbf{u}_z(x) &= \mathbf{z}(x) \quad \text{for } x\in\Gamma. \end{aligned} \tag{3.13}$$

From (3.13) we can say that the solution $\mathbf{u}_z$, of the homogeneous Dirichlet boundary value problem defines a control to state map $\mathcal{S}\colon H_*^{1/2}(\Gamma)\to H^1(\Omega)\subset L_2(\Omega)$ as $\mathbf{u}_z=\mathcal{S}\mathbf{z}$.

If $\mathbf{u}_f$ is the particular solution of the homogeneous Dirichlet boundary value problem then the solution of (3.11) is defined as $\mathbf{u} = \mathbf{u}_z + \mathbf{u}_f$ or $\mathbf{u} = \mathcal{S}\mathbf{z} + \mathbf{u}_f$.

The functional to be minimized is

$$\begin{aligned} \mathcal{J}(\mathbf{u},\mathbf{z}) &= \frac{1}{2}\int_\Omega [\mathbf{u}(x) - \overline{\mathbf{u}}(x)]^2 dx + \frac{\rho}{2}\langle \mathcal{A}\mathbf{z},\mathbf{z}\rangle_\Gamma \\ &= \frac{1}{2}\langle \mathbf{u} - \overline{\mathbf{u}}, \mathbf{u} - \overline{\mathbf{u}}\rangle_\Omega + \frac{\rho}{2}\langle \mathcal{A}\mathbf{z},\mathbf{z}\rangle_\Gamma, \end{aligned}$$

by using $\mathbf{u} = \mathcal{S}\mathbf{z} + \mathbf{u}_f$ we have

$$\begin{aligned} \widetilde{\mathcal{J}}(\mathbf{z}) &= \frac{1}{2}\langle \mathcal{S}\mathbf{z} + (\mathbf{u}_f - \overline{\mathbf{u}}), \mathcal{S}\mathbf{z} + (\mathbf{u}_f - \overline{\mathbf{u}})\rangle_\Gamma + \frac{\rho}{2}\langle \mathcal{A}\mathbf{z},\mathbf{z}\rangle_\Gamma \\ &= \frac{1}{2}\langle \mathcal{S}\mathbf{z}, \mathcal{S}\mathbf{z}\rangle_\Gamma + \frac{1}{2}\langle \mathcal{S}\mathbf{z}, \mathbf{u}_f - \overline{\mathbf{u}}\rangle_\Gamma + \frac{1}{2}\langle \mathbf{u}_f - \overline{\mathbf{u}}, \mathcal{S}\mathbf{z}\rangle_\Gamma + \frac{1}{2}\langle \mathbf{u}_f - \overline{\mathbf{u}}, \mathbf{u}_f - \overline{\mathbf{u}}\rangle_\Omega \\ &\quad + \frac{\rho}{2}\langle \mathcal{A}\mathbf{z},\mathbf{z}\rangle_\Gamma \\ &= \frac{1}{2}\langle \mathcal{S}^*\mathcal{S}\mathbf{z},\mathbf{z}\rangle_\Gamma + \langle \mathbf{u}_f - \overline{\mathbf{u}}, \mathcal{S}\mathbf{z}\rangle_\Gamma + \frac{1}{2}\left\|\mathbf{u}_f - \overline{\mathbf{u}}\right\|_\Omega^2 + \frac{\rho}{2}\langle \mathcal{A}\mathbf{z},\mathbf{z}\rangle_\Gamma \\ &= \frac{1}{2}\langle \mathcal{S}^*\mathcal{S}\mathbf{z},\mathbf{z}\rangle_\Gamma + \langle \mathcal{S}^*(\mathbf{u}_f - \overline{\mathbf{u}}),\mathbf{z}\rangle_\Gamma + \frac{1}{2}\left\|\mathbf{u}_f - \overline{\mathbf{u}}\right\|_\Omega^2 + \frac{\rho}{2}\langle \mathcal{A}\mathbf{z},\mathbf{z}\rangle_\Gamma. \end{aligned} \tag{3.14}$$

We now consider the problem to find the minimizer $\mathbf{z} \in \mathcal{U} \subset H_*^{1/2}(\Gamma)$ of the reduced cost functional (3.14). To characterize this minimiser $\mathbf{z} \in \mathcal{U}$, we introduce a self adjoint, bounded and $H_*^{1/2}(\Gamma)$–elliptic operator (later on we will prove these properties of such an operator, say T_ρ, involving boundary integral operators).

$$T_\rho :\, = \rho\mathcal{A} + \mathcal{S}^*\mathcal{S} :\, H_*^{1/2}(\Gamma) \to H_*^{-1/2}(\Gamma) \tag{3.15}$$

satisfying, see, e.g., [53]

$$\langle T_\rho \mathbf{z},\mathbf{z}\rangle_\Gamma \geq c_1^{T_\rho} \|\mathbf{z}\|^2_{H_*^{1/2}(\Gamma)}; \quad \left\|T_\rho \mathbf{z}\right\|_{H_*^{-1/2}(\Gamma)} \leq c_2^{T_\rho} \|\mathbf{z}\|_{H_*^{1/2}(\Gamma)} \quad \text{for all } \mathbf{z} \in H_*^{1/2}(\Gamma).$$

Let $\mathcal{S}^* :\, L_2(\Omega) \to H_*^{-1/2}(\Gamma)$ be the adjoint operator of $\mathcal{S} :\, H_*^{1/2}(\Gamma) \to L_2(\Omega)$, i.e.,

$$\langle \mathcal{S}^*\psi, \phi\rangle_\Gamma = \langle \psi, \mathcal{S}\phi\rangle_\Omega = \int_\Omega \psi(x)(\mathcal{S}\phi)(x)dx \quad \text{for all } \psi \in L_2(\Omega), \text{ for all } \phi \in H_*^{1/2}(\Gamma).$$

Moreover we define

$$\mathbf{g} := \mathcal{S}^*(\overline{\mathbf{u}} - \mathbf{u}_f) \in H_*^{-1/2}(\Gamma). \tag{3.16}$$

As the set $\mathcal{U} \subset H_*^{1/2}(\Gamma)$ is closed and convex, and since T_ρ is self adjoint and $H_*^{1/2}(\Gamma)$–elliptic, the minimization of the reduced cost functional (3.14) is equivalent to solving the variational inequality to find $\mathbf{z} \in \mathcal{U}$ such that

$$\langle T_\rho \mathbf{z}, \mathbf{w} - \mathbf{z}\rangle_\Gamma \geq \langle \mathbf{g}, \mathbf{w} - \mathbf{z}\rangle_\Gamma \quad \text{for all } \mathbf{w} \in \mathcal{U}. \tag{3.17}$$

Since (3.17) is an elliptic variational inequality of the first kind, we can use standard arguments as given, e.g., in [9, 24, 40, 45, 46] to establish the unique solvability of the variational inequality (3.17). For a short overview of variational inequalities and the relevant theorems, see Appendix B.

Note that the variational inequality (3.17) can also be written in the following way

$$\begin{aligned}
\langle T_\rho \mathbf{z} - \mathbf{g}, \mathbf{w} - \mathbf{z} \rangle_\Gamma &\geq 0, \\
\langle \rho \mathcal{A} \mathbf{z} + \mathcal{S}^* \mathcal{S} \mathbf{z} - \mathcal{S}^* (\overline{\mathbf{u}} - \mathbf{u}_f), \mathbf{w} - \mathbf{z} \rangle_\Gamma &\geq 0, \\
\langle \rho \mathcal{A} \mathbf{z} + \mathcal{S}^* (\mathcal{S} \mathbf{z} - \overline{\mathbf{u}} + \mathbf{u}_f), \mathbf{w} - \mathbf{z} \rangle_\Gamma &\geq 0, \\
\langle \rho \mathcal{A} \mathbf{z} + \mathcal{S}^* (\mathbf{u} - \overline{\mathbf{u}}), \mathbf{w} - \mathbf{z} \rangle_\Gamma &\geq 0, \\
\langle \rho \mathcal{A} \mathbf{z} + \tau, \mathbf{w} - \mathbf{z} \rangle_\Gamma &\geq 0, \qquad (3.18)
\end{aligned}$$

where $\tau = \mathcal{S}^*(\mathbf{u} - \overline{\mathbf{u}})$.

3.6 Realization of the adjoint operator

We want to see which boundary value problem corresponds to the application of the adjoint operator $\mathcal{S}^* : L_2(\Omega) \to H_*^{-1/2}(\Gamma)$. For $\tau = \mathcal{S}^*(\mathbf{u} - \overline{\mathbf{u}}) \in H_*^{-1/2}(\Gamma)$ and for $\mathbf{g} \in H_*^{1/2}(\Gamma)$, we consider the following duality pairing

$$\langle \tau, \mathbf{g} \rangle_\Gamma = \langle \mathcal{S}^*(\mathbf{u} - \overline{\mathbf{u}}), \mathbf{g} \rangle_\Gamma = \langle \mathbf{u} - \overline{\mathbf{u}}, \mathcal{S} \mathbf{g} \rangle_\Omega = \langle \mathbf{u} - \overline{\mathbf{u}}, \mathbf{m} \rangle_\Omega, \qquad (3.19)$$

where $\mathbf{m} = \mathcal{S}\mathbf{g}$, i.e.,

$$\begin{aligned}
-\mu \Delta \mathbf{m}(x) + \nabla p_m(x) &= \mathbf{0} && \text{for } x \in \Omega, \\
\nabla \cdot \mathbf{m}(x) &= 0 && \text{for } x \in \Omega, \qquad (3.20) \\
\gamma_0^{int} \mathbf{m}(x) &= \mathbf{g}(x) && \text{for } x \in \Gamma.
\end{aligned}$$

Consider the general forward problem (later adjoint problem)

$$-\mu \Delta \mathbf{w}(x) + \nabla r(x) = \mathbf{f}(x), \quad \nabla \cdot \mathbf{w}(x) = 0 \quad x \in \Omega, \quad \gamma_0^{int} \mathbf{w}(x) = 0, \quad x \in \Gamma.$$

Using Green's first formula we have

$$a(\mathbf{w}, \mathbf{v}) = \int_\Omega r(x) \operatorname{div} \mathbf{v}(x) dx + \langle \mathbf{f}, \mathbf{v} \rangle_\Omega + \langle \mathbf{t}(\mathbf{w}, r), \gamma_0^{int} \mathbf{v} \rangle_\Gamma.$$

In particular for $\mathbf{v} = \mathbf{m}$ we obtain

$$\begin{aligned}
a(\mathbf{w}, \mathbf{m}) &= \int_\Omega r(x) \underbrace{\operatorname{div} \mathbf{m}(x)}_{=0} dx + \langle \mathbf{f}, \mathbf{m} \rangle_\Omega + \langle \mathbf{t}(\mathbf{w}, r), \gamma_0^{int} \mathbf{m} \rangle_\Gamma, \\
a(\mathbf{w}, \mathbf{m}) &= \langle \mathbf{f}, \mathbf{m} \rangle_\Omega + \langle \mathbf{t}(\mathbf{w}, r), \gamma_0^{int} \mathbf{m} \rangle_\Gamma.
\end{aligned}$$

Take $\mathbf{f} = \mathbf{u} - \overline{\mathbf{u}}$, this gives

$$a(\mathbf{w},\mathbf{m}) - \langle \mathbf{t}(\mathbf{w},r),\mathbf{g}\rangle_\Gamma = \langle \mathbf{u} - \overline{\mathbf{u}},\mathbf{m}\rangle_\Omega. \tag{3.21}$$

From (3.19) and (3.21) we can write

$$\langle \tau,\mathbf{g}\rangle_\Gamma = a(\mathbf{w},\mathbf{m}) - \langle \mathbf{t}(\mathbf{w},r),\mathbf{g}\rangle_\Gamma. \tag{3.22}$$

Now the symmetry of the bilinear form $a(\cdot,\cdot)$ and the Green's first formula gives us

$$a(\mathbf{w},\mathbf{m}) = a(\mathbf{m},\mathbf{w}) \;=\; \int_\Omega p_m(x)\underbrace{\operatorname{div}\mathbf{w}(x)}_{=0}dx + \langle \mathbf{t}(\mathbf{m},p_m),\underbrace{\gamma_0^{int}\mathbf{w}}_{=0}\rangle_\Gamma,$$

so (3.22) becomes

$$\begin{aligned}\langle \tau,\mathbf{g}\rangle_\Gamma &= -\langle \mathbf{t}(\mathbf{w},r),\mathbf{g}\rangle_\Gamma \quad \text{for all } \mathbf{g}\in H_*^{1/2}(\Gamma),\\ \tau &= -\mathbf{t}(\mathbf{w},r) =: -\mathbf{q}.\end{aligned}$$

Hence the adjoint problem takes the form

$$\begin{aligned}-\mu\Delta\mathbf{w}(x) + \nabla r(x) &= \mathbf{u} - \overline{\mathbf{u}} &&\text{for } x\in\Omega,\\ \nabla\cdot\mathbf{w}(x) &= 0 &&\text{for } x\in\Omega,\\ \gamma_0^{int}\mathbf{w}(x) &= 0 &&\text{for } x\in\Gamma.\end{aligned}$$

3.7 Optimality (Karush–Kuhn–Tucker) system

For the solution of the Dirichlet boundary control problem (3.10)–(3.11) we therefore obtain the following system. Find $\mathbf{z}\in\mathcal{U}\subset H_*^{1/2}(\Gamma)$:

Primal Problem

$$\begin{aligned}-\mu\Delta\mathbf{u}(x) + \nabla p(x) &= \mathbf{f}(x) &&\text{for } x\in\Omega,\\ \nabla\cdot\mathbf{u}(x) &= 0 &&\text{for } x\in\Omega,\\ \gamma_0^{int}\mathbf{u}(x) &= \mathbf{z}(x) &&\text{for } x\in\Gamma.\end{aligned} \tag{3.23}$$

Adjoint Problem

$$\begin{aligned}-\mu\Delta\mathbf{w}(x) + \nabla r(x) &= \mathbf{u} - \overline{\mathbf{u}} &&\text{for } x\in\Omega,\\ \nabla\cdot\mathbf{w}(x) &= 0 &&\text{for } x\in\Omega,\\ \gamma_0^{int}\mathbf{w}(x) &= 0 &&\text{for } x\in\Gamma.\end{aligned} \tag{3.24}$$

Variational inequality

$$\begin{aligned}\langle T_\rho\mathbf{z},\mathbf{v}-\mathbf{z}\rangle_\Gamma &\geq \langle \mathbf{g},\mathbf{v}-\mathbf{z}\rangle_\Gamma \quad \text{or}\\ \langle \rho\mathcal{A}\mathbf{z}+\tau,\mathbf{v}-\mathbf{z}\rangle_\Gamma &\geq 0 \quad \text{for all } \mathbf{v}\in\mathcal{U}, \quad \text{where } \tau = -\mathbf{t}(\mathbf{w},r).\end{aligned}$$

Remark 3.7.1. *Since the unknown control* $\mathbf{z} \in \mathcal{U} \subset H_*^{1/2}(\Gamma)$ *is considered on the boundary* $\Gamma = \partial\Omega$, *the use of boundary integral equations to solve both the primal and the adjoint boundary value problem seems to be the natural one. In what follows we will describe and analyze boundary element methods to solve the variational inequality* (3.17) *numerically. This will be based on the use of appropriate boundary integral operators representation of* T_ρ *and* $\mathbf{g}$ *as introduced above.*

4 BOUNDARY INTEGRAL EQATIONS FOR THE PRIMAL AND THE ADJOINT PROBLEMS

This chapter deals with the formulation of boundary integral equations to solve the primal and the dual problems. It turns out that the representation formula for the adjoint problem involves the primal variable as a volume density. We need to express this as a surface potential so as to represent the control variable. This computation leads to the Bi–Stokes system. The detailed calculation of the kernel function, i.e., the fundamental solution $\mathbf{V}^*(x,y)$ for the Bi–Stokes system is given in Appendix A.

In order to prove the unique solvability of the proposed operator equation we need relations among the boundary integral operators. For this we setup a Calderón projection and discuss its properties. By using the invertibility of the single layer potential $\widehat{V}$ of the Stokes system we formulate the operator T_ρ and the corresponding right hand side in terms of boundary integral operators. We end this chapter by proving the unique solvability of the proposed formulation.

4.1 Boundary integral equations for the primal problem

For simplicity we take $\mu = 1$ and recall the primal problem is

$$\begin{aligned} -\Delta \mathbf{u}(x) + \nabla p(x) &= \mathbf{f}(x) &&\text{for } x \in \Omega, \\ \nabla \cdot \mathbf{u}(x) &= 0 &&\text{for } x \in \Omega, \\ \gamma_0^{int} \mathbf{u}(x) &= \mathbf{z}(x) &&\text{for } x \in \Gamma. \end{aligned}$$

The corresponding representation formula, for $x \in \Omega$ and $k = 1, \ldots, d$, is

$$u_k(x) = \int_\Gamma \mathbf{U}_k^*(x,y)^\top \mathbf{t}(\mathbf{u}(y), p(y))\, ds_y - \int_\Gamma \mathbf{u}(y)^\top \mathbf{T}_k^*(x,y) ds_y + \int_\Omega \mathbf{f}(y)^\top \mathbf{U}_k^*(x,y) dy.$$

Applying the Dirichlet trace operator to the above representation formula we get the boundary integral equation for the primal problem

$$(\widehat{V}\mathbf{t})(\mathbf{u}, p) := (\widehat{V}\mathbf{t})(x) = (\frac{1}{2}I + K)\mathbf{z}(x) - N_0 \mathbf{f}(x) \quad \text{for } x \in \Gamma, \tag{4.1}$$

where $\widehat{V}$ and K are the boundary integral operators as defined in (2.33) and (2.20) respectively, and N_0 is the related Newton potential.

4.2 Boundary integral equations for the adjoint problem

Recall the adjoint problem is

$$\begin{aligned} -\Delta \mathbf{w}(x) + \nabla r(x) &= \mathbf{u} - \overline{\mathbf{u}} && \text{for } x \in \Omega, \\ \nabla \cdot \mathbf{w}(x) &= 0 && \text{for } x \in \Omega, \\ \gamma_0^{int} \mathbf{w}(x) &= 0 && \text{for } x \in \Gamma. \end{aligned}$$

The corresponding representation formula, for $x \in \Omega$ and $k = 1, \dots, d$, is

$$w_k(x) = \int_\Gamma \mathbf{U}_k^*(x,y)^\top \mathbf{t}(\mathbf{w}(y), r(y)) ds_y - \int_\Gamma \underbrace{\mathbf{w}(y)^\top}_{=0} \mathbf{T}_k^*(x,y) ds_y + \int_\Omega (\mathbf{u}(\mathbf{y}) - \overline{\mathbf{u}}(\mathbf{y}))^\top \mathbf{U}_k^*(x,y) dy,$$

and therefore

$$w_k(x) = \int_\Gamma \mathbf{U}_k^*(x,y)^\top \mathbf{t}(\mathbf{w}(y), r(y)) ds_y + \int_\Omega \mathbf{u}(y)^\top \mathbf{U}_k^*(x,y) dy - \int_\Omega \overline{\mathbf{u}}(y)^\top \mathbf{U}_k^*(x,y) dy. \quad (4.2)$$

Again by applying the Dirichlet trace operator, we have the first boundary integral equation for the adjoint problem,

$$(\widehat{V}\mathbf{t})(\mathbf{w}, r) := (\widehat{V}\mathbf{q})(x) = (N_0 \overline{\mathbf{u}})(x) - (N_0 \mathbf{u})(x) \quad \text{for } x \in \Gamma. \quad (4.3)$$

Remark 4.2.1. *While the boundary integral equation for the primal problem* (4.1) *can be used to determine the unknown Neumann datum* $\mathbf{t} \in H_*^{-1/2}(\Gamma)$*, the corresponding traction for the adjoint Dirichlet boundary value problem is given as the solution of the boundary integral equation* (4.3)*. Then by using* $\tau = -\mathbf{q}$ *the control* $\mathbf{z} \in H_*^{1/2}(\Gamma)$ *is determined from the variational inequality* (3.18)*. However , since the solution* $\mathbf{u}$ *of the primal Dirichlet boundary value problem enters the volume potential* $N_0\mathbf{u}$ *in the boundary integral equation of the adjoint problem, so in this case we have to solve a coupled system of domain and boundary integral equations, which still would require some domain mesh. Instead we will describe a system of only boundary integral equations to solve the adjoint boundary value problem. In the next section we describe how to express this volume density as surface potential.*

4.3 Expressing the volume potential as a surface potential

For smooth $(\mathbf{u}, p)$ and $(\mathbf{v}, q)$, interchange $\mathbf{u}$ and $\mathbf{v}$ in the Green's first formula (2.11), with $\nabla \cdot \mathbf{v} = 0$, to obtain

$$a(\mathbf{v}, \mathbf{u}) = \int_\Omega q(x) \operatorname{div} \mathbf{u}(x) dx + \langle -\Delta \mathbf{v} + \nabla q, \mathbf{u} \rangle_\Omega + \langle \mathbf{t}(\mathbf{v}, q), \gamma_0^{int} \mathbf{u} \rangle_\Gamma.$$

From the symmetry of the bilinear form (2.10) we have

$$\langle -\Delta \mathbf{u} + \nabla p, \mathbf{v}\rangle_\Omega + \langle \mathbf{t}(\mathbf{u},p), \gamma_0^{int}\mathbf{v}\rangle_\Gamma = \langle -\Delta \mathbf{v} + \nabla q, \mathbf{u}\rangle_\Omega + \langle \mathbf{t}(\mathbf{v},q), \gamma_0^{int}\mathbf{u}\rangle_\Gamma \tag{4.4}$$

by requiring

$$\begin{aligned} -\Delta \mathbf{V}_k^*(x,y) + \nabla \mathbf{Q}_k^*(x,y) &= \mathbf{U}_k^*(x,y), \\ \nabla \cdot \mathbf{V}_k^*(x,y) &= 0 \end{aligned} \tag{4.5}$$

with $k = 1,2$. Here $\mathbf{V}^*(x,y)$ can be calculated by considering the case of linear elastostatics and by choosing a suitable ansatz, i.e.,

$$\mathbf{V}^*(x,y) = \begin{pmatrix} V_{11}^*(x,y) & V_{12}^*(x,y) \\ V_{21}^*(x,y) & V_{22}^*(x,y) \end{pmatrix},$$

where

$$\begin{aligned} V_{11}^*(x,y) &= \frac{4\left\{(y_1-x_1)^2 + 3(y_2-x_2)^2\right\}\log|x-y| - 7(y_1-x_1)^2 - 17(y_2-x_2)^2}{128\pi}, \\ V_{12}^*(x,y) &= V_{21}^*(x,y) = -\frac{(y_1-x_1)(y_2-x_2)\left\{(4\log|x-y| - 5\right\}}{64\pi}, \\ V_{22}^*(x,y) &= \frac{4\left\{3(y_1-x_1)^2 + (y_2-x_2)^2\right\}\log|x-y| - 17(y_1-x_1)^2 - 7(y_2-x_2)^2}{128\pi}. \end{aligned}$$

For the detailed computation of $\mathbf{V}^*(x,y)$ see Appendix A.

Now the expression for the volume potential in terms of surface potentials is given by

$$\int_\Omega \mathbf{u}(y)^\top \mathbf{U}_k^*(x,y)dy = \langle \mathbf{t}(\mathbf{u},p), \gamma_0^{int}\mathbf{v}\rangle_\Gamma - \langle \mathbf{t}(\mathbf{v},q), \gamma_0^{int}\mathbf{u}\rangle_\Gamma + \int_\Omega \mathbf{f}(y)^\top \mathbf{V}_k^*(x,y)dy, \tag{4.6}$$

where the remaining volume potential includes given functions only.

Using (4.6) in (4.2), we obtain, so called, the modified representation formula for the adjoint problem

$$\begin{aligned} w_k(x) &= \int_\Gamma \mathbf{U}_k^*(x,y)^\top \mathbf{t}(\mathbf{w}(y), r(y))ds_y + \int_\Gamma \mathbf{V}_k^*(x,y)^\top \mathbf{t}(\mathbf{u}(y), p(y))ds_y \\ &\quad - \int_\Gamma \mathbf{u}(y)^\top \mathbf{T}_k^*(x,y)ds_y + \int_\Omega \mathbf{f}(y)^\top \mathbf{V}_k^*(x,y)dy - \int_\Omega \overline{\mathbf{u}}(\mathbf{y})^\top \mathbf{U}_k^*(x,y)dy. \end{aligned} \tag{4.7}$$

We can now formulate the first boundary integral equation and a hypersingular boundary integral equation for the adjoint problem by applying the trace operator γ_0^{int} and the boundary stress operator (2.12) to the representation formula (4.7)

$$(\widehat{V}\mathbf{q})(x) = (K_1\mathbf{z})(x) - (V_1\mathbf{t})(x) - (M_0\mathbf{f})(x) + (N_0\overline{\mathbf{u}})(x) \quad \text{for } x \in \Gamma, \tag{4.8}$$

$$\begin{aligned} \mathbf{q}(x) &= (\tfrac{1}{2}I + K')\mathbf{q}(x) + (K_1'\mathbf{t})(x) + (D_1\mathbf{z})(x) \\ &\quad + (M_1\mathbf{f})(x) - (N_1\overline{\mathbf{u}})(x) \qquad \text{for } x \in \Gamma. \end{aligned} \tag{4.9}$$

We have introduced the four basic boundary integral operators for the Bi–Stokes system. V_1, K_1, K_1' and D_1 are the single layer, the double layer, the adjoint double layer and the hypersingular boundary integrals operators respectively for the system represented by (4.5), where

$$(V_1\mathbf{t})(x) = \int_\Gamma \mathbf{V}^*(x,y)^\top \mathbf{t}(\mathbf{u}(y), p(y))ds_y \quad \text{for } x \in \Gamma$$

is the Bi–Stokes single layer potential operator $V_1 : H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)$. Moreover,

$$(K_1\mathbf{u})(x) = \int_\Gamma \mathbf{T}_y(\mathbf{V}^*(x,y))^\top \mathbf{u}(y)ds_y \quad \text{for } x \in \Gamma$$

is the Bi–Stokes double layer potential operator $K_1 : H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)$. Also

$$(K_1'\mathbf{u})(x) = \int_\Gamma \mathbf{T}_x(\mathbf{V}^*(x,y))^\top \mathbf{u}(y)ds_y \quad \text{for } x \in \Gamma$$

is the adjoint Bi–Stokes double layer potential operator $K_1' : H^{-3/2}(\Gamma) \to H^{1/2}(\Gamma)$ and

$$(D_1\mathbf{u})(x) = -\mathbf{T}_x \left\{ \int_\Gamma (\mathbf{T}'(V_k^*(x,y))^\top \right\} \mathbf{u}(y)\mathrm{d}s_y \quad \text{for } x \in \Gamma$$

is the Bi–Stokes hypersingular operator $D_1 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ In addition, we have introduced another Newton potential which is related to the fundamental solution of Bi–Stokes,

$$(M_0\mathbf{f})(x) = \int_\Omega \mathbf{f}(y)^\top \mathbf{V}^*(x,y)dy \quad \text{for } x \in \Gamma.$$

Remark 4.3.1. *We have enough regularity of $V^*(x,y)$ so that K_1 and K_1' do not involve any jump terms.*

The modified representation formula for the inhomogeneous adjoint problem is now given by

$$\begin{aligned} w_k(x) \;=\; & \int_\Gamma \mathbf{U}_k^*(x,y)^\top \mathbf{t}(\mathbf{w}(y), r(y))ds_y + \int_\Gamma \mathbf{V}_k^*(x,y)^\top \mathbf{t}(\mathbf{u}(y), p(y))ds_y - \int_\Gamma \mathbf{u}(y)^\top \mathbf{T}_k^*(x,y)ds_y \\ & - \int_\Omega \mathbf{w}(y)^\top \mathbf{T}_k^*(x,y)ds_y + \int_\Omega \mathbf{f}(y)^\top \mathbf{V}_k^*(x,y)dy - \int_\Omega \overline{\mathbf{u}}(\mathbf{y})^\top \mathbf{U}_k^*(x,y)dy. \end{aligned} \tag{4.10}$$

For the four basic boundary integral operators of the Bi-Stokes system (4.5), the following hold.

Lemma 4.3.1. *The single layer potential* $V_1 : H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)$ *is bounded, i.e.,*

$$\|V_1\mathbf{t}\|_{H^{3/2}(\Gamma)} \le c_2^{V_1}\,\|\mathbf{t}\|_{H^{-3/2}(\Gamma)} \qquad \text{for all } \mathbf{t} \in H^{-3/2}(\Gamma). \tag{4.11}$$

Lemma 4.3.2. *The double layer potential* $K_1 : H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)$ *is bounded, i.e.,*

$$\|K_1\mathbf{z}\|_{H^{3/2}(\Gamma)} \le c_2^{K_1}\,\|\mathbf{z}\|_{H^{-1/2}(\Gamma)} \qquad \text{for all } \mathbf{z} \in H^{-1/2}(\Gamma). \tag{4.12}$$

Lemma 4.3.3. *The adjoint double layer potential* $K_1' : H^{-3/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is bounded, i.e.,*

$$\left\|K_1'\mathbf{t}\right\|_{H^{1/2}(\Gamma)} \le c_2^{K_1'}\,\|\mathbf{t}\|_{H^{-3/2}(\Gamma)} \qquad \text{for all } \mathbf{t} \in H^{-3/2}(\Gamma).$$

Lemma 4.3.4. *The hypersingular boundary integral operator* $D_1 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is bounded, i.e.,*

$$\|D_1\mathbf{z}\|_{H^{1/2}(\Gamma)} \le c_2^{D_1}\,\|\mathbf{z}\|_{H^{-1/2}(\Gamma)} \qquad \text{for all } \mathbf{z} \in H^{-1/2}(\Gamma).$$

Proof. See [37]. □

4.4 Calderón projector

Applying the trace operator γ_0^{int} and the boundary stress operator (2.12) to both sides of equations (4.10) and (2.16), we end up with the following overdetermined system of boundary integral equations. For $x \in \Gamma$

$$\begin{aligned}
\gamma_0^{int}\mathbf{w}(x) &= (\tfrac{1}{2}I - K)\gamma_0^{int}\mathbf{w}(x) + (\widehat{V}\mathbf{q})(x) - (K_1\gamma_0^{int}\mathbf{u})(x) + (V_1\mathbf{t})(x) - (N_0\overline{\mathbf{u}})(x) + (M_0\mathbf{f})(x),\\
\mathbf{q}(x) &= (D\gamma_0^{int}\mathbf{w})(x) + (\tfrac{1}{2}I + K')\mathbf{q}(x) + (D_1\gamma_0^{int}\mathbf{u})(x) + (K_1'\mathbf{t})(x) - (N_1\overline{\mathbf{u}})(x) + (M_1\mathbf{f})(x),\\
\gamma_0^{int}\mathbf{u}(x) &= (\widehat{V}\mathbf{t})(x) + (\tfrac{1}{2}I - K)\gamma_0^{int}\mathbf{u}(x) + (N_0\mathbf{f})(x),\\
\mathbf{t}(x) &= (\tfrac{1}{2}I + K')\mathbf{t}(x) + (D\gamma_0^{int}\mathbf{u})(x) + (N_1\mathbf{f})(x).
\end{aligned}$$

The above system of boundary integral equations can be written in a compact way, including the so called Calderón projection $\mathcal{C}$,

$$\begin{pmatrix} \gamma_0^{int}\mathbf{w} \\ \mathbf{q} \\ \gamma_0^{int}\mathbf{u} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & \widehat{V} & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & D_1 & K_1' \\ 0 & 0 & \frac{1}{2}I - K & \widehat{V} \\ 0 & 0 & D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \gamma_0^{int}\mathbf{w} \\ \mathbf{q} \\ \gamma_0^{int}\mathbf{u} \\ \mathbf{t} \end{pmatrix} + \begin{pmatrix} M_0\mathbf{f} - N_0\overline{\mathbf{u}} \\ M_1\mathbf{f} - N_1\overline{\mathbf{u}} \\ N_0\mathbf{f} \\ N_1\mathbf{f} \end{pmatrix}. \tag{4.13}$$

Here we state the following lemma.

Lemma 4.4.1. *If*

$$\mathcal{C} = \begin{pmatrix} \frac{1}{2}I - K & \widehat{V} & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & D_1 & K_1' \\ 0 & 0 & \frac{1}{2}I - K & \widehat{V} \\ 0 & 0 & D & \frac{1}{2}I + K' \end{pmatrix},$$

then $\mathcal{C} = \mathcal{C}^2$, i.e., $\mathcal{C}$ is a projection.

Proof. Let $\phi, \widetilde{\phi} \in H^{1/2}(\Gamma)$ and $\psi, \widetilde{\psi} \in H^{-1/2}(\Gamma)$ are arbitrary, but fixed. For $\widetilde{x} \in \Omega$, define

$$\begin{aligned} \mathbf{w}(\widetilde{x}) &:= (\widetilde{\widehat{V}}\psi)(\widetilde{x}) - (W\phi)(\widetilde{x}) + (\widetilde{V}_1\widetilde{\psi})(\widetilde{x}) - (K_1\widetilde{\phi})(\widetilde{x}), \\ \mathbf{u}(\widetilde{x}) &:= (\widetilde{\widehat{V}}\widetilde{\psi})(\widetilde{x}) - (W\widetilde{\phi})(\widetilde{x}). \end{aligned}$$

Apply the Dirichlet trace operator γ_0^{int} and the boundary stress operator (2.12), we have the following system. For $x \in \Gamma$

$$\begin{aligned} \gamma_0^{int}\mathbf{w}(x) &= (\frac{1}{2}I - K)\phi(x) + (\widehat{V}\psi)(x) - (K_1\widetilde{\phi})(x) + (V_1\widetilde{\psi})(x), \\ \mathbf{q}(x) &= (\frac{1}{2}I + K')\psi(x) + (D\phi)(x) + (D_1\widetilde{\phi})(x) + (K_1'\widetilde{\psi})(x), \\ \gamma_0^{int}\mathbf{u}(x) &= (\frac{1}{2}I - K)\widetilde{\phi}(x) + (\widehat{V}\widetilde{\psi})(x), \\ \mathbf{t}(x) &= (D\widetilde{\phi})(x) + (\frac{1}{2}I + K')\widetilde{\psi}(x), \end{aligned}$$

or

$$\begin{pmatrix} \gamma_0^{int}\mathbf{w}(x) \\ \mathbf{q}(x) \\ \gamma_0^{int}\mathbf{u}(x) \\ \mathbf{t}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & \widehat{V} & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & D_1 & K_1' \\ 0 & 0 & \frac{1}{2}I - K & \widehat{V} \\ 0 & 0 & D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \phi(x) \\ \psi(x) \\ \widetilde{\phi}(x) \\ \widetilde{\psi}(x) \end{pmatrix}. \tag{4.14}$$

Also the homogeneous form of equation (4.13) is

$$\begin{pmatrix} \gamma_0^{int}\mathbf{w}(x) \\ \mathbf{q}(x) \\ \gamma_0^{int}\mathbf{u}(x) \\ \mathbf{t}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & \widehat{V} & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & D_1 & K_1' \\ 0 & 0 & \frac{1}{2}I - K & \widehat{V} \\ 0 & 0 & D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \gamma_0^{int}\mathbf{w}(x) \\ \mathbf{q}(x) \\ \gamma_0^{int}\mathbf{u}(x) \\ \mathbf{t}(x) \end{pmatrix}, \tag{4.15}$$

using (4.14) on both sides of (4.15), we have

$$\mathcal{C} \begin{pmatrix} \phi(x) \\ \psi(x) \\ \widetilde{\phi}(x) \\ \widetilde{\psi}(x) \end{pmatrix} = \mathcal{C}^2 \begin{pmatrix} \phi(x) \\ \psi(x) \\ \widetilde{\phi}(x) \\ \widetilde{\psi}(x) \end{pmatrix}.$$

and therefore $\mathcal{C} = \mathcal{C}^2$ holds. Hence the lemma is proved. □

As a direct consequence of the Calderón projection, we state the following lemma.

Lemma 4.4.2. *For all boundary integral operators, there hold the relations*

$$\widehat{V}K' = K\widehat{V}, \quad K'D = DK, \quad \widehat{V}D = \frac{1}{4}I - K^2, \quad D\widehat{V} = \frac{1}{4}I - K'^2, \tag{4.16}$$

$$\begin{aligned} K_1\widehat{V} - V_1K' &= \widehat{V}K_1' - KV_1, &(4.17)\\ K'D_1 - D_1K &= DK_1 - K_1'D, &(4.18)\\ DV_1 + D_1\widehat{V} + K'K_1' + K_1'K' &= 0, &(4.19)\\ \widehat{V}D_1 + V_1D + KK_1 + K_1K &= 0. &(4.20)\end{aligned}$$

Proof.

$$Let \quad \mathcal{C} = \begin{pmatrix} \frac{1}{2}I - K & \widehat{V} & -K_1 & V_1 \\ D & \frac{1}{2}I + K' & D_1 & K_1' \\ 0 & 0 & \frac{1}{2}I - K & \widehat{V} \\ 0 & 0 & D & \frac{1}{2}I + K' \end{pmatrix} = \begin{pmatrix} W & T \\ O & W \end{pmatrix},$$

where

$$W = \begin{pmatrix} \frac{1}{2}I - K & \widehat{V} \\ D & \frac{1}{2}I + K' \end{pmatrix}; \quad T = \begin{pmatrix} -K_1 & V_1 \\ D_1 & K_1' \end{pmatrix} \quad \text{and} \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\mathcal{C}$ is a projection, i.e., $\mathcal{C} = \mathcal{C}^2$, so

$$\begin{pmatrix} W & T \\ O & W \end{pmatrix} = \begin{pmatrix} W & T \\ O & W \end{pmatrix}\begin{pmatrix} W & T \\ O & W \end{pmatrix} = \begin{pmatrix} W^2 & WT + TW \\ O & W^2 \end{pmatrix}.$$

Now, $W = W^2$ gives

$$\begin{pmatrix} \frac{1}{2}I - K & \widehat{V} \\ D & \frac{1}{2}I + K' \end{pmatrix} = \begin{pmatrix} (\frac{1}{2}I - K)^2 + \widehat{V}D & \frac{1}{2}\widehat{V} - K\widehat{V} + \frac{1}{2}\widehat{V} + \widehat{V}K' \\ \frac{1}{2}D - DK + \frac{1}{2}D + K'D & D\widehat{V} + (\frac{1}{2}I + K')^2 \end{pmatrix}.$$

Equating the entry in the first row on both sides, we have

$$\begin{aligned} \frac{1}{2}I - K &= (\frac{1}{2}I - K)^2 + \widehat{V}D, \\ \widehat{V}D &= \frac{1}{4}I - K^2 = (\frac{1}{2}I - K)(\frac{1}{2}I + K). \end{aligned}$$

Similarly on equating the entries in the first row and second column on both sides, we have

$$\begin{aligned} \widehat{V} &= \frac{1}{2}\widehat{V} - \widehat{V}K + \frac{1}{2}\widehat{V} + \widehat{V}K', \\ \widehat{V}K' &= K\widehat{V}, \\ \widehat{V}^{-1}\widehat{V}K' &= \widehat{V}^{-1}K\widehat{V}, \\ K' &= \widehat{V}^{-1}K\widehat{V}, \\ K'\widehat{V}^{-1} &= \widehat{V}^{-1}K\widehat{V}\widehat{V}^{-1}, \\ K'\widehat{V}^{-1} &= \widehat{V}^{-1}K, \end{aligned}$$

the other two results can be proved by equating the corresponding entries on both sides.

Again the last four results can be proved by taking $T = WT + TW$ and equating the corresponding entries, i.e.,

$$\begin{aligned} -K_1 = -K_1 + KK_1 + \widehat{V}D_1 + K_1K + V_1D, \\ \widehat{V}D_1 + V_1D + KK_1 + K_1K = 0. \end{aligned}$$

Similarly, the other three results can be proved easily. □

Remark 4.4.1. *All results presented in lemma 4.4.2 are well known for the Laplace operator, see, e.g., [63] and for the Bi–Laplace operator, see also [59].*

4.5 Non–symmetric form of the coupled problem

Formulation

The first boundary integral equations for the primal problem and that of the adjoint problem is the following system

$$\begin{aligned} (\widehat{V}\mathbf{t})(x) &= (\frac{1}{2}I + K)\mathbf{z}(x) - (N_0\mathbf{f})(x) \quad \text{for } x \in \Gamma, \\ (\widehat{V}\mathbf{q})(x) &= (K_1\mathbf{z})(x) - (V_1\mathbf{t})(x) - (M_0\mathbf{f})(x) + (N_0\overline{\mathbf{u}})(x) \quad \text{for } x \in \Gamma. \end{aligned} \tag{4.21}$$

As the single layer potential $\widehat{V}$ is invertible, we have from the first equation of system (4.21)

$$\mathbf{t} = \widehat{V}^{-1}(\frac{1}{2}I + K)\mathbf{z} - \widehat{V}^{-1}N_0\mathbf{f}, \tag{4.22}$$

by the same argument as above, the second equation of the system (4.21) gives

$$\begin{aligned} \mathbf{q} &= \widehat{V}^{-1}K_1\mathbf{z} - \widehat{V}^{-1}V_1\mathbf{t} - \widehat{V}^{-1}M_0\mathbf{f} + \widehat{V}^{-1}N_0\overline{\mathbf{u}} \\ &= \widehat{V}^{-1}K_1\mathbf{z} - \widehat{V}^{-1}V_1\left[\widehat{V}^{-1}(\frac{1}{2}I + K)\mathbf{z} - \widehat{V}^{-1}N_0\mathbf{f}\right] - \widehat{V}^{-1}M_0\mathbf{f} + \widehat{V}^{-1}N_0\overline{\mathbf{u}}, \end{aligned}$$

and therefore,

$$\mathbf{q} = \widehat{V}^{-1}K_1\mathbf{z} - \widehat{V}^{-1}V_1\widehat{V}^{-1}(\frac{1}{2}I + K)\mathbf{z} + \widehat{V}^{-1}V_1\widehat{V}^{-1}N_0\mathbf{f} - \widehat{V}^{-1}M_0\mathbf{f} + \widehat{V}^{-1}N_0\overline{\mathbf{u}}. \tag{4.23}$$

Using $\mathbf{q} = \rho\mathcal{A}\mathbf{z}$ and rearranging the terms, we have

$$\underbrace{\rho\mathcal{A} - \widehat{V}^{-1}K_1\mathbf{z} + \widehat{V}^{-1}V_1\widehat{V}^{-1}(\frac{1}{2}I + K)}_{T_\rho}\mathbf{z} = \underbrace{\widehat{V}^{-1}V_1\widehat{V}^{-1}N_0\mathbf{f} - \widehat{V}^{-1}M_0\mathbf{f} + \widehat{V}^{-1}N_0\overline{\mathbf{u}}}_{\mathbf{g}}.$$

By replacing $\tau = -\mathbf{q}$ in (3.18) we therefore obtain a boundary integral representation of T_ρ as defined in (3.15),

$$T_\rho := \rho\mathcal{A} - \widehat{V}^{-1}K_1 + \widehat{V}^{-1}V_1\widehat{V}^{-1}(\frac{1}{2}I + K), \tag{4.24}$$

and the corresponding right hand side as defined in (3.16), is

$$\mathbf{g} := \widehat{V}^{-1}V_1\widehat{V}^{-1}N_0\mathbf{f} - \widehat{V}^{-1}M_0\mathbf{f} + \widehat{V}^{-1}N_0\bar{\mathbf{u}}. \tag{4.25}$$

Unique solvability

In order to prove the ellipticity of the boundary integral operator T_ρ as defined in (4.24), we need to prove the following result.

Lemma 4.5.1. *For any* $\mathbf{t} \in H_*^{-1/2}(\Gamma)$ *there holds the equality*

$$\|\widetilde{V}\mathbf{t}\|^2_{L_2(\Omega)} = \langle V_1(\frac{1}{2}I + K')\mathbf{t}, \mathbf{t}\rangle_\Gamma - \langle K_1\widehat{V}\mathbf{t}, \mathbf{t}\rangle_\Gamma,$$

where

$$(\widetilde{V}\mathbf{t})(x) = \int_\Gamma \mathbf{U}^*(x,y)^\top \mathbf{t}(y)ds_y \quad \textit{for } x \in \Omega.$$

Proof. First note that $\widehat{V}\mathbf{t} = V\mathbf{t}$ for $\mathbf{t} \in H_*^{-1/2}(\Gamma)$. For the homogeneous Stokes system, the equation (4.4) reads

$$\langle \mathbf{t}(\mathbf{u},p), \gamma_0^{int}\mathbf{v}\rangle_\Gamma = \langle -\Delta\mathbf{v} + \nabla q, \mathbf{u}\rangle_\Omega + \langle \mathbf{t}(\mathbf{v},q), \gamma_0^{int}\mathbf{u}\rangle_\Gamma. \tag{4.26}$$

Note that (See [63], page 100)

$$\begin{aligned} \mathbf{T}_k^*(x,y) : &= \gamma_{1,y}^{int}\mathbf{V}_k^*(x,y), \\ &= \lambda\,\mathrm{div}_y\mathbf{V}_k^*(x,y)\mathbf{n}(y) + 2\mu\frac{\partial}{\partial n_y}\mathbf{V}_k^*(x,y) + \mu\mathbf{n}(y)\times \mathrm{Curl}_y\mathbf{V}_k^*(x,y). \end{aligned}$$

For $d = 2$, we make use of the following relation

$$\begin{aligned} \mathrm{div}_y\mathbf{V}_k^*(x,y) &= \frac{1}{E}\frac{1+\nu}{1-\nu}2(2\nu-1)\frac{1}{4\pi}\frac{y_2-x_2}{|x-y|^2} \\ &= \frac{1}{E}\frac{1+\nu}{1-\nu}2(2\nu-1)\mathbf{Q}_k^*(x,y) \end{aligned}$$

On using the divergence free property of the Bi–Stokes system (4.5), we find $\mathbf{Q}_k^*(x,y) = 0$, so (4.5) becomes

$$-\Delta\mathbf{V}_k^*(x,y) = \mathbf{U}_k^*(x,y), \quad \nabla\cdot\mathbf{V}_k^*(x,y) = 0.$$

Further for $\mathbf{v} = \widetilde{V}_1\mathbf{t}$ we see

$$\begin{aligned}
-\Delta_x \mathbf{v}(x) + \nabla q(x) &= -\Delta_x(\widetilde{V}_1\mathbf{t}(x)) + 0, \\
&= -\Delta_x \int_\Gamma \mathbf{V}_k^*(x,y)^\top \mathbf{t}(y) ds_y, \\
&= -\int_\Gamma \Delta_x \mathbf{V}_k^*(x,y)^\top \mathbf{t}(y) ds_y, \quad (x \in \Omega, y \in \Gamma) \\
&= \int_\Gamma \mathbf{U}_k^*(x,y)^\top \mathbf{t}(y) ds_y = (\widetilde{V}\mathbf{t})(x) \quad \text{for } x \in \Omega.
\end{aligned}$$

Since the single layer potential is a solution of the homogeneous Stokes system, so for $\mathbf{u} = \widetilde{V}\mathbf{t}$ and $\mathbf{v} = \widetilde{V}_1\mathbf{t}$ the equation (4.26) reads

$$\begin{aligned}
\langle \widetilde{V}\mathbf{t}, \widetilde{V}\mathbf{t}\rangle_\Omega &= \int_\Gamma \left[\frac{1}{2}\mathbf{t}(y) + (K'\mathbf{t})(y)\right](V_1\mathbf{t})(y) ds_y - \int_\Gamma (K_1'\mathbf{t})(y)(\widehat{V}\mathbf{t})(y) ds_y, \\
\|\widetilde{V}\mathbf{t}\|^2_{L_2(\Omega)} &= \langle \frac{1}{2}\mathbf{t} + K'\mathbf{t}, V_1\mathbf{t}\rangle_\Gamma - \langle K_1'\mathbf{t}, \widehat{V}\mathbf{t}\rangle_\Gamma, \\
\|\widetilde{V}\mathbf{t}\|^2_{L_2(\Omega)} &= \langle V_1(\frac{1}{2}I + K')\mathbf{t}, \mathbf{t}\rangle_\Gamma - \langle \mathbf{t}, K_1\widehat{V}\mathbf{t}\rangle_\Gamma,
\end{aligned}$$

which proves the desired result. □

Now we are able to state the mapping properties of the boundary integral operator T_ρ as defined in (4.24), see also the properties of T_ρ as defined in (3.15)

Theorem 4.5.1. *The composed boundary integral operator*

$$T_\rho := \rho\mathcal{A} - \widehat{V}^{-1}K_1 + \widehat{V}^{-1}V_1\widehat{V}^{-1}(\frac{1}{2}I + K) : H_*^{1/2}(\Gamma) \to H_*^{-1/2}(\Gamma)$$

is self–adjoint, bounded and $H_*^{1/2}(\Gamma)$–elliptic, i.e.,

$$\langle T_\rho \mathbf{z}, \mathbf{z}\rangle \geq c_1^{T_\rho}\|\mathbf{z}\|^2_{H_*^{1/2}(\Gamma)} \quad \textit{for all } \mathbf{z} \in H_*^{1/2}(\Gamma).$$

Proof. We note that the mapping properties of the composed boundary integral operator $T_\rho : H_*^{1/2}(\Gamma) \to H_*^{-1/2}(\Gamma)$ will follow from the boundedness of all used boundary integral operators. Further, we have the compact embedding of the $H_*^{3/2}(\Gamma)$ in $H_*^{1/2}(\Gamma)$.

Next we will show the self–adjointness of T_ρ. We note that $\widehat{V}, \widehat{V}^{-1}$ and V_1 are all self adjoint. For $\mathbf{u}, \mathbf{v} \in H_*^{1/2}(\Gamma)$ we have

$$\begin{aligned}
\langle T_\rho \mathbf{u}, \mathbf{v}\rangle_\Gamma &= \langle \rho\mathcal{A}\mathbf{u}, \mathbf{v}\rangle_\Gamma - \langle \widehat{V}^{-1}K_1\mathbf{u}, \mathbf{v}\rangle_\Gamma + \frac{1}{2}\langle \widehat{V}^{-1}V_1\widehat{V}^{-1}\mathbf{u}, \mathbf{v}\rangle_\Gamma + \langle \widehat{V}^{-1}V_1\widehat{V}^{-1}K\mathbf{u}, \mathbf{v}\rangle_\Gamma \\
&= \langle \mathbf{u}, \rho\mathcal{A}\mathbf{v}\rangle_\Gamma - \langle \mathbf{u}, K_1'\widehat{V}^{-1}\mathbf{v}\rangle_\Gamma + \frac{1}{2}\langle \mathbf{u}, \widehat{V}^{-1}V_1\widehat{V}^{-1}\mathbf{v}\rangle_\Gamma + \langle \mathbf{u}, K'\widehat{V}^{-1}V_1\widehat{V}^{-1}\mathbf{v}\rangle_\Gamma \\
&= \langle \mathbf{u}, \rho\mathcal{A}\mathbf{v}\rangle_\Gamma + \frac{1}{2}\langle \mathbf{u}, \widehat{V}^{-1}V_1\widehat{V}^{-1}\mathbf{v}\rangle_\Gamma + \langle \mathbf{u}, \left(K'\widehat{V}^{-1}V_1\widehat{V}^{-1} - K_1'\widehat{V}^{-1}\right)\mathbf{v}\rangle_\Gamma.
\end{aligned}$$

Now on using the relations (4.16) and (4.17)

$$\begin{aligned}
K'\widehat{V}^{-1}V_1\widehat{V}^{-1} - K_1'\widehat{V}^{-1} &= \widehat{V}^{-1}KV_1\widehat{V}^{-1} - K_1'\widehat{V}^{-1} \\
&= \widehat{V}^{-1}\left[KV_1 - \widehat{V}K_1'\right]\widehat{V}^{-1} \\
&= \widehat{V}^{-1}\left[V_1K' - K_1\widehat{V}\right]\widehat{V}^{-1} \\
&= \widehat{V}^{-1}V_1K'\widehat{V}^{-1} - \widehat{V}^{-1}K_1\widehat{V}\widehat{V}^{-1} \\
&= \widehat{V}^{-1}V_1K'\widehat{V}^{-1} - \widehat{V}^{-1}K_1.
\end{aligned}$$

We finally get

$$\begin{aligned}
\langle T_\rho \mathbf{u}, \mathbf{v}\rangle_\Gamma &= \langle \mathbf{u}, \rho\mathcal{A}\mathbf{v}\rangle_\Gamma + \frac{1}{2}\langle \mathbf{u}, \widehat{V}^{-1}V_1\widehat{V}^{-1}\mathbf{v}\rangle_\Gamma + \langle \mathbf{u}, \left(\widehat{V}^{-1}V_1K'\widehat{V}^{-1} - \widehat{V}^{-1}K_1\right)\mathbf{v}\rangle_\Gamma \\
&= \langle \mathbf{u}, \rho\mathcal{A}\mathbf{v}\rangle_\Gamma + \langle \mathbf{u}, \left(\widehat{V}^{-1}V_1\widehat{V}^{-1}(\frac{1}{2}I + K) - \widehat{V}^{-1}K_1\right)\mathbf{v}\rangle_\Gamma \\
&= \langle \mathbf{u}, [\rho\mathcal{A} + \widehat{V}^{-1}V_1\widehat{V}^{-1}(\frac{1}{2}I + K) - \widehat{V}^{-1}K_1]\mathbf{v}\rangle_\Gamma, \\
\langle T_\rho \mathbf{u}, \mathbf{v}\rangle_\Gamma &= \langle \mathbf{u}, T_\rho \mathbf{v}\rangle_\Gamma.
\end{aligned}$$

Hence T_ρ is self–adjoint. Moreover, for $\mathbf{z} \in H_*^{1/2}(\Gamma)$ we have, by using (4.16), $\mathbf{t} = \widehat{V}^{-1}\mathbf{z}$, and by Lemma 4.5.1

$$\begin{aligned}
\langle T_\rho \mathbf{z}, \mathbf{z}\rangle_\Gamma &= \langle \rho\mathcal{A}\mathbf{z}, \mathbf{z}\rangle_\Gamma + \langle \widehat{V}^{-1}V_1\widehat{V}^{-1}(\frac{1}{2}I + K)\mathbf{z}, \mathbf{z}\rangle_\Gamma - \langle \widehat{V}^{-1}K_1\mathbf{z}, \mathbf{z}\rangle_\Gamma \\
&= \langle \rho\mathcal{A}\mathbf{z}, \mathbf{z}\rangle_\Gamma + \langle V_1\widehat{V}^{-1}(\frac{1}{2}I + K)\mathbf{z}, \widehat{V}^{-1}\mathbf{z}\rangle_\Gamma - \langle K_1\mathbf{z}, \widehat{V}^{-1}\mathbf{z}\rangle_\Gamma \\
&= \langle \rho\mathcal{A}\mathbf{z}, \mathbf{z}\rangle_\Gamma + \langle V_1(\frac{1}{2}I + K')\widehat{V}^{-1}\mathbf{z}, \widehat{V}^{-1}\mathbf{z}\rangle_\Gamma - \langle K_1\widehat{V}\widehat{V}^{-1}\mathbf{z}, \widehat{V}^{-1}\mathbf{z}\rangle_\Gamma \\
&= \langle \rho\mathcal{A}\mathbf{z}, \mathbf{z}\rangle_\Gamma + \langle V_1(\frac{1}{2}I + K')\mathbf{t}, \mathbf{t}\rangle_\Gamma - \langle K_1\widehat{V}\mathbf{t}, \mathbf{t}\rangle_\Gamma \\
&= \langle \rho\mathcal{A}\mathbf{z}, \mathbf{z}\rangle_\Gamma + \left\|\widetilde{V}\mathbf{t}\right\|^2_{L_2(\Omega)}, \\
\langle T_\rho \mathbf{z}, \mathbf{z}\rangle_\Gamma &\geq \rho\,\|\mathbf{z}\|^2_{\mathcal{A}},
\end{aligned}$$

i.e., the $H_*^{1/2}(\Gamma)$–ellipticity of T_ρ, due to the fact that $\|.\|_{\mathcal{A}}$ defines an equivalent norm in $H_*^{1/2}(\Gamma)$. Hence the theorem is proved. □

5 BOUNDARY ELEMENT METHODS FOR OPTIMAL CONTROL

In this chapter we describe a Galerkin boundary element discretization of the boundary integral equations. Due to the composition of the boundary integral operator T_ρ as defined in (4.24), we are unable to apply a direct boundary element discretization. Instead we introduce a suitable boundary element approximation of the operator. Necessary estimates for the boundedness and the error of the approximated operator follow next. We also discuss the boundary element approximation of the right hand side along with the error estimate caused by this approximation. Finally we discuss the perturbed Galerkin variational formulation for the non–symmetric case and setup the Schur complement system in the control variable for this perturbed system. We use two different meshes to discretize the control $\mathbf{z}$ and the tractions $\mathbf{t}$ and $\mathbf{q}$. We describe the necessary condition on the two meshes so that the resulting approximate Schur complement system is positive definite.

5.1 Galerkin boundary element discretization

Let

$$S_H^1(\Gamma) = \operatorname{span}\{\Phi_i\}_{i=1}^M \subset H^{1/2}(\Gamma),$$

be some boundary element space, i.e., piecewise linear and continuous basis functions Φ_i which are defined with respect to a globally quasi–uniform and shape regular boundary element mesh of size H. Define the discrete convex set

$$\mathcal{U}_H := \left\{ \mathbf{w}_H \in S_H^1(\Gamma) \colon \mathbf{g}_a(x_i) \leq \mathbf{w}_H(x_i) \leq \mathbf{g}_b(x_i) \text{ for all nodes } x_i \in \Gamma \text{ with } \int_\Gamma \mathbf{n}^\top \mathbf{w}_H ds_x = 0 \right\}.$$

The Galerkin discretization of the variational inequality (3.17) is to find $\mathbf{z}_H \in \mathcal{U}_H$ such that

$$\langle T_\rho \mathbf{z}_H, \mathbf{w}_H - \mathbf{z}_H \rangle_\Gamma \geq \langle \mathbf{g}, \mathbf{w}_H - \mathbf{z}_H \rangle_\Gamma \quad \text{for all } \mathbf{w}_H \in \mathcal{U}_H. \tag{5.1}$$

Theorem 5.1.1. *Let $\mathbf{z} \in \mathcal{U}$ and $\mathbf{z}_H \in \mathcal{U}_H$ be the unique solutions of the variational inequalities* (3.17) *and* (5.1)*, respectively. If we assume $\mathbf{z}, \mathbf{g}_a, \mathbf{g}_b \in H^s(\Gamma)$ for some $s \in [1/2, 2]$, then there hold the error estimates*

$$\|\mathbf{z} - \mathbf{z}_H\|_{H^{1/2}(\Gamma)} \leq cH^{s-1/2} \|\mathbf{z}\|_{H^s(\Gamma)}, \tag{5.2}$$

and

$$\|\mathbf{z} - \mathbf{z}_H\|_{L_2(\Gamma)} \leq cH^s \|\mathbf{z}\|_{H^s(\Gamma)}. \tag{5.3}$$

Proof. The error estimate (5.2) in the energy norm follows from the general abstract theory as presented, e.g., in [11, 21], see also [24]. The error estimate (5.3) follows from the Aubin–Nitsche trick for variational inequalities, see [51] for the case $\mathcal{U}_H \subset \mathcal{U}$, and [64] for the more general case $\mathcal{U}_H \not\subset \mathcal{U}$. □

Although the error estimates (5.2) and (5.3) seem to be optimal, the operator T_ρ, as considered in the variational inequality (5.1), does not allow a practical implementation. This would require the discretization of the operator T_ρ as defined in (4.24), which is not possible in general due the presence of an inverse operator. Hence, instead of (5.1) we need to consider a perturbed variational inequality to find $\widetilde{\mathbf{z}}_H \in \mathcal{U}_H$ such that

$$\langle \widetilde{T}_\rho \widetilde{\mathbf{z}}_H, \mathbf{w}_H - \widetilde{\mathbf{z}}_H \rangle_\Gamma \geq \langle \widetilde{\mathbf{g}}, \mathbf{w}_H - \widetilde{\mathbf{z}}_H \rangle_\Gamma \quad \text{for all } \mathbf{w}_H \in \mathcal{U}_H, \tag{5.4}$$

where $\widetilde{T}_\rho$ and $\widetilde{\mathbf{g}}$ are appropriate approximations of T_ρ and $\mathbf{g}$, respectively. The following theorem, see, e.g., [53], presents an abstract consistency result, which will later be used to analyze the boundary element approximation of both the primal and adjoint boundary value problems.

Theorem 5.1.2. *Let $\widetilde{T}_\rho : H_*^{1/2}(\Gamma) \to H_*^{-1/2}(\Gamma)$ be a bounded and $S_H^1(\Gamma)$–elliptic approximation of T_ρ satisfying*

$$\langle \widetilde{T}_\rho \mathbf{z}_H, \mathbf{z}_H \rangle_\Gamma \geq c_1^{\widetilde{T}_\rho} \|\mathbf{z}_H\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } \mathbf{z}_H \in S_H^1(\Gamma),$$

and

$$\left\| \widetilde{T}_\rho \mathbf{z} \right\|_{H^{-1/2}(\Gamma)} \leq c_2^{\widetilde{T}_\rho} \|\mathbf{z}\|_{H^{1/2}(\Gamma)} \quad \textit{for all } \mathbf{z} \in H_*^{1/2}(\Gamma).$$

Let $\widetilde{\mathbf{g}} \in H_*^{-1/2}(\Gamma)$ be some approximation of $\mathbf{g}$. For the unique solution $\widetilde{\mathbf{z}}_H \in \mathcal{U}_H$ of the perturbed variational inequality (5.4) *there holds the error estimate*

$$\|\mathbf{z} - \widetilde{\mathbf{z}}_H\|_{H^{1/2}(\Gamma)} \leq c_1 \|\mathbf{z} - \mathbf{z}_H\|_{H^{1/2}(\Gamma)} + c_2 \left[\left\| (T_\rho - \widetilde{T}_\rho)\mathbf{z} \right\|_{H^{-1/2}(\Gamma)} + \|\mathbf{g} - \widetilde{\mathbf{g}}\|_{H^{-1/2}(\Gamma)} \right] \tag{5.5}$$

where $\mathbf{z}_H \in \mathcal{U}_H$ is the unique solution of the discrete variational inequality (5.1).

5.2 Boundary element approximation of $\mathbf{T}_\rho$

For an arbitrary but fixed $\mathbf{z} \in H_*^{1/2}(\Gamma)$, the application of $T_\rho \mathbf{z}$ reads

$$\begin{aligned} T_\rho \mathbf{z} &= \rho \mathcal{A} \mathbf{z} + \underbrace{\widehat{V}^{-1} V_1 \underbrace{\widehat{V}^{-1} (\tfrac{1}{2} I + K)}_{\mathbf{t}_z} \mathbf{z} - \widehat{V}^{-1} K_1 \mathbf{z}}_{\mathbf{q}_z}, \\ T_\rho \mathbf{z} &= \rho \mathcal{A} \mathbf{z} + \mathbf{q}_z, \end{aligned}$$

where $\mathbf{q}_z, \mathbf{t}_z \in H_*^{-1/2}(\Gamma)$ are the unique solutions of the boundary integral equations

$$\widehat{V}\mathbf{q}_z = V_1\mathbf{t}_z - K_1\mathbf{z}, \quad \widehat{V}\mathbf{t}_z = (\frac{1}{2}I + K)\mathbf{z}. \tag{5.6}$$

Now, to define a Galerkin approximation of (5.6), let

$$S_h^0(\Gamma) = \text{span}\{\Psi_k\}_{k=1}^N \subset H^{-1/2}(\Gamma)$$

be some boundary element space, e.g., of piecewise constant basis functions Ψ_i which are defined with respect to a second globally quasi–uniform and shape regular boundary element mesh of size h. Now, $\mathbf{t}_{z,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin formulation

$$\langle \widehat{V}\mathbf{t}_{z,h}, \tau_h \rangle = \langle (\frac{1}{2}I + K)\mathbf{z}, \tau_h \rangle \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.7}$$

Moreover, $\widetilde{\mathbf{q}}_{\mathbf{z},\mathbf{h}} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin formulation

$$\langle \widehat{V}\widetilde{\mathbf{q}}_{z,h}, \tau_h \rangle = \langle V_1\mathbf{t}_{z,h} - K_1\mathbf{z}, \tau_h \rangle \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.8}$$

Hence we can define an approximation $\widetilde{T}_\rho$ of the operator T_ρ by

$$\widetilde{T}_\rho \mathbf{z} : = \rho \mathcal{A}\mathbf{z} + \widetilde{\mathbf{q}}_{z,h}. \tag{5.9}$$

Now we describe some properties of the approximate operator $\widetilde{T}_\rho$, as defined in (5.9).

Lemma 5.2.1. *The approximate operator*

$$\widetilde{T}_\rho : H_*^{1/2}(\Gamma) \to H_*^{-1/2}(\Gamma)$$

as defined in (5.9) *is bounded, i.e.,*

$$\|\widetilde{T}_\rho \mathbf{z}\|_{H^{-1/2}(\Gamma)} \le c_2^{\widetilde{T}_\rho} \|\mathbf{z}\|_{H^{1/2}(\Gamma)} \quad \textit{for all } \mathbf{z} \in H_*^{1/2}(\Gamma).$$

Proof. Take $\tau_h = \mathbf{t}_{z,h}$ in (5.7), we have

$$\langle \widehat{V}\mathbf{t}_{z,h}, \mathbf{t}_{z,h} \rangle = \langle (\frac{1}{2}I + K)\mathbf{z}, \mathbf{t}_{z,h} \rangle,$$

further use the $H^{-1/2}(\Gamma)$–ellipticity of the single layer potential $\widehat{V}$ and the boundedness of the double layer potential, see [63, page 72 ff]

$$\|\mathbf{t}_{z,h}\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^{\widehat{V}}} \|(\frac{1}{2}I + K)\mathbf{z}\|_{H^{1/2}(\Gamma)} \le \frac{c_2^K}{c_1^{\widehat{V}}} \|\mathbf{z}\|_{H^{1/2}(\Gamma)}.$$

Again from (5.8), take $\tau_h = \widetilde{\mathbf{q}}_{z,h}$, and use the same argument as above , we have

$$\|\widetilde{\mathbf{q}}_{z,h}\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^{\widehat{V}}} \|V_1\mathbf{t}_{z,h} - K_1\mathbf{z}\|_{H^{1/2}(\Gamma)},$$

use the triangle inequality and the fact $H^{3/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$

$$\|\widetilde{\mathbf{q}}_{z,h}\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^{\widehat{V}}} \left\{ \|V_1 \mathbf{t}_{z,h}\|_{H^{3/2}(\Gamma)} + \|K_1 \mathbf{z}\|_{H^{3/2}(\Gamma)} \right\}. \tag{5.10}$$

Now using (4.11) and (4.12) in (5.10), we have

$$\|\widetilde{\mathbf{q}}_{z,h}\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^{\widehat{V}}} \left\{ c_2^{V_1} \|\mathbf{t}_{z,h}\|_{H^{-3/2}(\Gamma)} + c_2^{K_1} \|\mathbf{z}\|_{H^{-1/2}(\Gamma)} \right\}.$$

The assertion now follows from $H^{1/2}(\Gamma) \subset H^{-1/2}(\Gamma)$ and $H^{-1/2}(\Gamma) \subset H^{-3/2}(\Gamma)$. □

Lemma 5.2.2. *There holds the error estimate*

$$\left\| T_\rho \mathbf{z} - \widetilde{T}_\rho \mathbf{z} \right\|_{H^{-1/2}(\Gamma)} \leq \frac{c_2^{\widehat{V}}}{c_1^{\widehat{V}}} \inf_{\tau_H \in S_h^0} \|\mathbf{q}_z - \tau_h\|_{H^{-1/2}(\Gamma)} + \frac{c_2^{V_1}}{c_1^{\widehat{V}}} \|\mathbf{t}_z - \mathbf{t}_{z,h}\|_{H^{-3/2}(\Gamma)}, \tag{5.11}$$

where $T_\rho : H_*^{1/2}(\Gamma) \to H_*^{-1/2}(\Gamma)$ *is given in* (4.24) *and* $\widetilde{T}_\rho$ *is defined by* (5.9), $\mathbf{q}_z, \mathbf{t}_z \in H_*^{-1/2}(\Gamma)$ *are defined in* (5.6) *and* $\mathbf{t}_{z,h} \in S_h^0(\Gamma)$ *is the unique solution of the Galerkin variational problem* (5.7).

Proof. Let $\mathbf{z} \in H_*^{1/2}(\Gamma)$ be arbitrary but a fixedly chosen element of the control space. Then by definition

$$\begin{aligned} T_\rho \mathbf{z} &= \rho \mathcal{A} \mathbf{z} + \mathbf{q}_z, \\ \widetilde{T}_\rho \mathbf{z} &= \rho \mathcal{A} \mathbf{z} + \widetilde{\mathbf{q}}_{\mathbf{z},\mathbf{h}}, \end{aligned}$$

and therefore

$$\left\| T_\rho \mathbf{z} - \widetilde{T}_\rho \mathbf{z} \right\|_{H^{-1/2}(\Gamma)} = \left\| \mathbf{q}_z - \widetilde{\mathbf{q}}_{\mathbf{z},\mathbf{h}} \right\|_{H^{-1/2}(\Gamma)}.$$

Also

$$\mathbf{q}_z = \widehat{V}^{-1} \left[V_1 \mathbf{t}_z - K_1 \mathbf{z} \right], \quad \mathbf{t}_z = \widehat{V}^{-1} (\frac{1}{2} I + K) \mathbf{z},$$

in particular $\mathbf{t}_z \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle \widehat{V} \mathbf{t}_z, \tau \rangle = \langle (\frac{1}{2} I + K) \mathbf{z}, \tau \rangle \quad \text{for all } \tau \in H^{-1/2}(\Gamma),$$

and $\mathbf{q}_z \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle \widehat{V} \mathbf{q}_z, \tau \rangle = \langle V_1 \mathbf{t}_z, \tau \rangle - \langle K_1 \mathbf{z}, \tau \rangle \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

By using the definition of $\widetilde{T}_\rho \mathbf{z}$, we note that $\widetilde{\mathbf{q}}_{z,h}$ is the unique solution of the variational formulation

$$\langle \widehat{V}\widetilde{\mathbf{q}}_{z,h}\tau_h\rangle = \langle V_1 t_{z,h}\tau\rangle - \langle K_1\mathbf{z},\tau_h\rangle \quad \text{for all } \tau_h \in S_h^0(\Gamma),$$

and $\mathbf{t}_{z,h}$ is the unique solution of the variational problem

$$\langle \widehat{V}\mathbf{t}_{z,h}\tau_h\rangle = \langle (\frac{1}{2}I + K)\mathbf{z},\tau_h\rangle \quad \text{for all } \tau_h \in S_h^0(\Gamma).$$

By applying Cea's lemma, we first get the error estimate

$$\|\mathbf{t}_z - \mathbf{t}_{z,h}\|_{H^{-1/2}(\Gamma)} \leq \frac{c_2^{\widehat{V}}}{c_1^{\widehat{V}}} \inf_{\tau_h \in S_h^0(\Gamma)} \|\mathbf{t}_z - \tau_h\|_{H^{-1/2}(\Gamma)}.$$

Let us further define $\mathbf{q}_{z,h} \in S_h^0(\Gamma)$ as the unique solution of the variational problem

$$\langle \widehat{V}\mathbf{q}_{z,h},\tau_h\rangle_\Gamma = \langle V_1\mathbf{t}_z,\tau_h\rangle - \langle K_1\mathbf{z},\tau_h\rangle \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.12}$$

Again by using Cea's lemma, we have

$$\|\mathbf{q}_z - \mathbf{q}_{z,h}\|_{H^{-1/2}(\Gamma)} \leq \frac{c_2^{\widehat{V}}}{c_1^{\widehat{V}}} \inf_{\tau_h \in S_h^0(\Gamma)} \|\mathbf{q}_z - \tau_h\|_{H^{-1/2}(\Gamma)}. \tag{5.13}$$

We obtain a perturbed Galerkin orthogonality after subtracting (5.8) from (5.12)

$$\langle \widehat{V}(\mathbf{q}_{z,h} - \widetilde{\mathbf{q}}_{z,h}),\tau_h\rangle_\Gamma = \langle V_1(\mathbf{t}_z - \mathbf{t}_{z,h}),\tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma).$$

From this perturbed Galerkin orthogonality, we further conclude the stability estimate

$$\begin{aligned}
\|\mathbf{q}_{z,h} - \widetilde{\mathbf{q}}_{z,h}\|_{H^{-1/2}(\Gamma)} &\leq \frac{1}{c_1^{\widehat{V}}}\|V_1(\mathbf{t}_z - \mathbf{t}_{z,h}\|_{H^{1/2}(\Gamma)} \\
&\leq \frac{1}{c_1^{\widehat{V}}}\|V_1(\mathbf{t}_z - \mathbf{t}_{z,h}\|_{H^{3/2}(\Gamma)} \\
&\leq \frac{c_2^{V_1}}{c_1^{\widehat{V}}}\|\mathbf{t}_z - \mathbf{t}_{z,h}\|_{H^{-3/2}(\Gamma)}.
\end{aligned} \tag{5.14}$$

Here we have used the fact $H^{3/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$ and the boundedness of V_1. Now by using the triangle inequality, the estimates (5.13) and (5.14), we have

$$\begin{aligned}
\|\mathbf{q}_z - \widetilde{\mathbf{q}}_{z,h}\|_{H^{-1/2}(\Gamma)} &= \|(\mathbf{q}_z - \mathbf{q}_{z,h}) - (\widetilde{\mathbf{q}}_{z,h} - \mathbf{q}_{z,h})\|_{H^{-1/2}(\Gamma)} \\
&\leq \|\mathbf{q}_z - \mathbf{q}_{z,h}\|_{H^{-1/2}(\Gamma)} + \|\widetilde{\mathbf{q}}_{z,h} - \mathbf{q}_{z,h}\|_{H^{-1/2}(\Gamma)} \\
&\leq \frac{c_2^{\widehat{V}}}{c_1^{\widehat{V}}} \inf_{\tau_h \in S_h^0(\Gamma)} \|\mathbf{q}_z - \tau_h\|_{H^{-1/2}(\Gamma)} + \frac{c_2^{V_1}}{c_1^{\widehat{V}}}\|\mathbf{t}_z - \mathbf{t}_{z,h}\|_{H^{-3/2}(\Gamma)}.
\end{aligned}$$

Hence we get the required estimate. □

Remark 5.2.1. *By using the approximation property of the trial space* $S_h^0(\Gamma)$ *and the Aubin–Nitsche trick, we can conclude the following error estimate from* (5.11), *when assuming some regularity on* $\mathbf{q}_z$ *and* $\mathbf{t}_z$

Corollary 5.2.1. *We assume* $\mathbf{q}_z, \mathbf{t}_z \in H^s_{pw}(\Gamma)$ *for some* $s \in [0,1]$. *Then there holds the error estimate*

$$\|T_\rho \mathbf{z} - \widetilde{T}_\rho \mathbf{z}\|_{H^{-1/2}(\Gamma)} \leq c_1 h^{s+1/2} \|\mathbf{q}_z\|_{H^s_{pw}(\Gamma)} + c_2 h^{s+3/2} \|\mathbf{t}_z\|_{H^s_{pw}(\Gamma)}. \tag{5.15}$$

5.3 Boundary element approximation of the right hand side

As we have defined an approximation of the operator T_ρ, on similar lines we can also define an approximation of the right hand side $\mathbf{g}$ as defined in (4.25),

$$\mathbf{g} := \widehat{V}^{-1} N_0 \overline{\mathbf{u}} - \widehat{V}^{-1} M_0 \mathbf{f} + \widehat{V}^{-1} V_1 \widehat{V}^{-1} N_0 \mathbf{f}$$

In particular, $\mathbf{g} \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle \widehat{V}\mathbf{g}, \tau \rangle_\Gamma = \langle N_0 \overline{\mathbf{u}} - M_0 \mathbf{f} + V_1 \widehat{V}^{-1} N_0 \mathbf{f}, \tau \rangle_\Gamma = \langle N_0 \overline{\mathbf{u}} - M_0 \mathbf{f}, \tau \rangle_\Gamma + \langle V_1 \mathbf{t}_f, \tau \rangle_\Gamma$$

for all $\tau \in H^{-1/2}(\Gamma)$, where $\mathbf{t}_f = \widehat{V}^{-1} N_0 \mathbf{f} \in H^{-1/2}(\Gamma)$ solves the variational problem

$$\langle \widehat{V} \mathbf{t}_f, \tau \rangle_\Gamma = \langle N_0 \mathbf{f}, \tau \rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

Hence we define the boundary element approximation $\widetilde{\mathbf{g}}_\mathbf{h} \in S_h^0(\Gamma)$ as the unique solution of the Galerkin variational problem

$$\langle \widehat{V} \widetilde{\mathbf{g}}_\mathbf{h}, \tau_h \rangle_\Gamma = \langle N_0 \overline{\mathbf{u}} - M_0 \mathbf{f} + V_1 \widehat{V}^{-1} N_0 \mathbf{f}, \tau_h \rangle_\Gamma = \langle N_0 \overline{\mathbf{u}} - M_0 \mathbf{f}, \tau_h \rangle_\Gamma + \langle V_1 \mathbf{t}_{f,h}, \tau_h \rangle_\Gamma \tag{5.16}$$

for all $\tau_h \in S_h^0(\Gamma)$, where $\mathbf{t}_{f,h} \in S_h^0(\Gamma)$ is the unique solution of the Galerkin problem

$$\langle \widehat{V} \mathbf{t}_{f,h}, \tau_h \rangle_\Gamma = \langle N_0 \mathbf{f}, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.17}$$

Lemma 5.3.1. *Let* $\mathbf{g}$ *be the right hand side as defined by* (4.25) *and let* $\widetilde{\mathbf{g}}_\mathbf{h}$ *be the boundary element approximation as defined in* (5.16). *Then there holds the error estimate*

$$\|\mathbf{g} - \widetilde{\mathbf{g}}_h\|_{H^{-1/2}(\Gamma)} \leq \frac{c_2^{\widehat{V}}}{c_1^{\widehat{V}}} \inf_{\tau_h \in S_h^0(\Gamma)} \|\mathbf{g} - \tau_h\|_{H^{-1/2}(\Gamma)} + \frac{c_2^{V_1}}{c_1^{\widehat{V}}} \|\mathbf{t}_f - \mathbf{t}_{f,h}\|_{H^{-3/2}(\Gamma)}. \tag{5.18}$$

Proof. In addition to (5.16), let us consider the Galerkin formulation to find $\mathbf{g}_h \in S_h^0(\Gamma)$ such that

$$\langle \widehat{V} \mathbf{g}_h, \tau_h \rangle_\Gamma = \langle N_0 \overline{u} - M_0 \mathbf{f}, \tau_h \rangle_\Gamma + \langle V_1 \mathbf{t}_f, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.19}$$

Again by using Cea's lemma we have

$$\|\mathbf{g}-\mathbf{g}_h\|_{H^{-1/2}(\Gamma)} \le \frac{c_2^{\widehat{V}}}{c_1^{\widehat{V}}} \inf_{\tau_h \in S_h^0(\Gamma)} \|\mathbf{g}-\tau_h\|_{H^{-1/2}(\Gamma)}. \tag{5.20}$$

We obtain a perturbed Galerkin orthogonality after subtracting (5.16) from (5.19), i.e.,

$$\langle \widehat{V}(\mathbf{g}_h-\widetilde{\mathbf{g}}_h), \tau_h\rangle_\Gamma = \langle V_1(\mathbf{t}_f-\mathbf{t}_{f,h}), \tau_h\rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma).$$

For $\tau_h = \mathbf{g}_h - \widetilde{\mathbf{g}}_h$, using the $H_*^{-1/2}(\Gamma)$–ellipticity of $\widehat{V}$, the boundedness estimate of V_1 and the fact $H^{3/2}(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$, we get

$$\begin{aligned}
\|\mathbf{g}_h-\widetilde{\mathbf{g}}_h\|_{H^{-1/2}(\Gamma)} &\le \frac{1}{c_1^{\widehat{V}}}\|V_1(\mathbf{t_f}-\mathbf{t}_{f,h})\|_{H^{1/2}(\Gamma)}, \\
&\le \frac{1}{c_1^{\widehat{V}}}\|V_1(\mathbf{t}_f-\mathbf{t}_{f,h})\|_{H^{3/2}(\Gamma)}, \\
&\le \frac{c_2^{V_1}}{c_1^{\widehat{V}}}\|\mathbf{t_f}-\mathbf{t}_{f,h}\|_{H^{-3/2}(\Gamma)}.
\end{aligned} \tag{5.21}$$

By the triangle inequality we have

$$\|\mathbf{g}-\widetilde{\mathbf{g}}_h\|_{H^{-1/2}(\Gamma)} \le \|\mathbf{g}-\mathbf{g}_h\|_{H^{-1/2}(\Gamma)} + \|\widetilde{\mathbf{g}}_h-\mathbf{g}_h\|_{H^{-1/2}(\Gamma)}\,. \tag{5.22}$$

By using (5.20) and (5.21) in (5.22), the assertion follows. □

Remark 5.3.1. *By using the approximation property of the trial space* $S_h^0(\Gamma)$ *and the Aubin–Nitsche trick, we can conclude the following error estimate from* (5.18) *when assuming some regularity on* $\mathbf{g}$ *and* $\mathbf{t}_f$*.*

Corollary 5.3.1. *We assume* $\mathbf{g}\,,\mathbf{t}_f \in H^s_{pw}(\Gamma)$ *for some* $s \in [0,1]$*. Then there holds the error estimate*

$$\|\mathbf{g}-\widetilde{\mathbf{g}}_h\|_{H^{-1/2}(\Gamma)} \le c_1 h^{s+1/2}\|\mathbf{g}\|_{H^s_{pw}(\Gamma)} + c_2 h^{s+3/2}\|\mathbf{t}_f\|_{H^s_{pw}(\Gamma)}. \tag{5.23}$$

5.4 Approximate variational inequality

We consider the variational inequality (3.18) with $\tau = -\mathbf{q}$ to find $\mathbf{z} \in \mathcal{U}$ such that

$$\langle \rho\mathcal{A}\mathbf{z}-\mathbf{q}, \mathbf{w}-\mathbf{z}\rangle_\Gamma \ge 0 \quad \text{for all } \mathbf{w} \in \mathcal{U}, \tag{5.24}$$

where $\mathbf{q} \in H_*^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(\widehat{V}\mathbf{q})(x) = (K_1\mathbf{z})(x) - (V_1\mathbf{t})(x) - (M_0\mathbf{f})(x) + (N_0\overline{\mathbf{u}})(x) \quad \text{for } x \in \Gamma,$$

and $\mathbf{t} \in H_*^{-1/2}(\Gamma)$ is the unique solution of the boundary integral equation

$$(\widehat{V}\mathbf{t})(x) = (\frac{1}{2}I + K)\mathbf{z}(x) - (N_0\mathbf{f})(x) \quad \text{for } x \in \Gamma.$$

The Galerkin boundary element approximation of the variational inequality (5.24), and therefore the boundary element discretization of the perturbed variational inequality (5.4) is to find $\widetilde{\mathbf{z}}_H \in \mathcal{U}_H$ such that

$$\langle \rho \mathcal{A}\widetilde{\mathbf{z}}_H - \mathbf{q}_h, \mathbf{w}_H - \widetilde{\mathbf{z}}_H \rangle_\Gamma \geq 0 \quad \text{for all } \mathbf{w}_H \in \mathcal{U}_H, \tag{5.25}$$

where $\mathbf{q}_h \in S_h^0(\Gamma)$ is the unique solution of the Galerkin formulation

$$\langle \widehat{V}\mathbf{q}_h, \tau_h \rangle_\Gamma = \langle K_1\widetilde{\mathbf{z}}_H - V_1\mathbf{t}_h - M_0\mathbf{f} + N_0\overline{\mathbf{u}}, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma), \tag{5.26}$$

and $\mathbf{t}_h \in S_h^0(\Gamma)$ solves

$$\langle \widehat{V}\mathbf{t}_h, \tau_h \rangle_\Gamma = \langle (\frac{1}{2}I + K)\widetilde{\mathbf{z}}_H - N_0\mathbf{f}, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in S_h^0(\Gamma). \tag{5.27}$$

The Galerkin formulation (5.26) is equivalent to the linear system

$$\widehat{V}_h\underline{\mathbf{q}} = K_{1,h}\underline{\mathbf{z}} - V_{1,h}\underline{\mathbf{t}} + \underline{\mathbf{f}}_1,$$

and (5.27) is equivalent to

$$\widehat{V}_h\underline{\mathbf{t}} = (\frac{1}{2}M_h + K_h)\underline{\widetilde{\mathbf{z}}} - \underline{\mathbf{f}}_2,$$

where

$$\begin{aligned} \widehat{V}_h[\ell,k] &= \langle \widehat{V}\Psi_k, \Psi_\ell \rangle_\Gamma, \quad K_h[\ell,i] = \langle K\Phi_i, \Psi_\ell \rangle_\Gamma, \quad V_{1,h}[\ell,k] = \langle V_1\Psi_k, \Psi_\ell \rangle_\Gamma, \\ K_{1,h}[\ell,i] &= \langle K_1\Phi_i, \Psi_\ell \rangle_\Gamma, \quad \mathcal{A}_H[j,i] = \langle \mathcal{A}\Phi_i, \Phi_j \rangle_\Gamma, \quad M_h[\ell,i] = \langle \Phi_i, \Psi_\ell \rangle_\Gamma, \end{aligned}$$

and

$$\mathbf{f}_{1,\ell} = \langle N_0\overline{u} - M_0\mathbf{f}, \Psi_\ell \rangle_\Gamma, \quad \mathbf{f}_{2,\ell} = -\langle N_0\mathbf{f}, \Psi_\ell \rangle_\Gamma,$$

for $k, \ell = 1, \ldots, 2m_h$ and $i, j = 1, \ldots, 2n_h$. Recall that we have used vector valued piecewise linear basis functions $\mathbf{\Phi}_i$ and piecewise constant basis functions $\mathbf{\Psi}_k$.

The matrix representation of the variational inequality (5.25) is then given by the discrete variational inequality

$$(\rho\mathcal{A}_H\widetilde{\mathbf{z}} - M_h^\top\underline{\mathbf{q}}, \mathbf{w} - \widetilde{\mathbf{z}}) \geq 0 \quad \text{for all } \mathbf{w} \in \mathbb{R}^M \leftrightarrow \mathbf{w}_H \in \mathcal{U}_H,$$

or

$$(\widetilde{T}_{\rho,H}\widetilde{\mathbf{z}} - \widetilde{\mathbf{g}}, \mathbf{w} - \widetilde{\mathbf{z}}) \geq 0 \quad \text{for all } \mathbf{w} \in \mathbb{R}^M \leftrightarrow \mathbf{w}_H \in \mathcal{U}_H, \tag{5.28}$$

where

$$\widetilde{T}_{\rho,H} = \rho \mathcal{A}_H + M_h^\top \widehat{V}_h^{-1} V_{1,h} \widehat{V}_h^{-1} (\frac{1}{2} M_h + K_h) - M_h^\top \widehat{V}_h^{-1} K_{1,h} \tag{5.29}$$

defines a non–symmetric Galerkin boundary element approximation of the self–adjoint boundary integral operator T_ρ as defined in (4.24). Moreover,

$$\widetilde{\mathbf{g}} = M_h^\top \widehat{V}_h^{-1} (\underline{\mathbf{f}}_1 - V_{1,h} \widehat{V}_h^{-1} \underline{\mathbf{f}}_2)$$

is a boundary element approximation of $\mathbf{g}$ as defined in (4.25), where

$$\underline{\mathbf{f}}_1 := N_0 \overline{\mathbf{u}} - M_0 \mathbf{f}, \quad \underline{\mathbf{f}}_2 := -N_0 \mathbf{f}.$$

Theorem 5.4.1. *The approximate Schur complement $\widetilde{T}_{\rho,H}$ as defined in* (5.29) *is positive definite,i.e.,*

$$\left(\widetilde{T}_{\rho,H} \mathbf{z}, \mathbf{z}\right) \geq \frac{1}{2} c_1^{\widetilde{T}_\rho} \|\mathbf{z}_H\|^2_{H^{1/2}(\Gamma)},$$

for all $\mathbf{z} \in \mathbb{R}^M \leftrightarrow \mathbf{z}_H \in S_H^1(\Gamma)$, if $h \leq c_0 H$ is sufficiently small.

Proof. For an arbitrary, but fixed $\mathbf{z} \in \mathbb{R}^M$, let $\mathbf{z}_H \in S_H^1(\Gamma)$ be the associated boundary element function. Then we have

$$\begin{aligned}
\left(\widetilde{T}_{\rho,H} \mathbf{z}, \mathbf{z}\right) &= \langle \widetilde{T}_\rho \mathbf{z}_H, \mathbf{z}_H \rangle_\Gamma = \langle (T_\rho - (T_\rho - \widetilde{T}_\rho)) \mathbf{z}_H, \mathbf{z}_H \rangle_\Gamma, \\
&= \langle T_\rho \mathbf{z}_H, \mathbf{z}_H \rangle_\Gamma - \langle (T_\rho - \widetilde{T}_\rho)) \mathbf{z}_H, \mathbf{z}_H \rangle_\Gamma, \\
&\geq c_1^{T_\rho} \|\mathbf{z}_H\|^2_{H^{1/2}(\Gamma)} - \left\| (T_\rho - \widetilde{T}_\rho) \mathbf{z}_H \right\|_{H^{-1/2}(\Gamma)} \|\mathbf{z}_H\|_{H^{1/2}(\Gamma)}.
\end{aligned}$$

We note that $\mathbf{z}_H \in H^1(\Gamma)$ as $\mathbf{z}_H \in S_H^1(\Gamma)$ is a continuous function. Hence we find from equations (5.7) and (5.8) that both $\mathbf{t}_{z_H}$ and $\mathbf{q}_{z_H}$ are in $L_2(\Gamma)$. Therefore we can apply the error estimate (5.15) for $s = 0$ to obtain

$$\|T_\rho \mathbf{z}_H - \widetilde{T}_\rho \mathbf{z}_H\|_{H^{-1/2}(\Gamma)} \leq c_1 h^{1/2} \|\mathbf{q}_{\mathbf{z}_H}\|_{L_2(\Gamma)} + c_2 h^{3/2} \|\mathbf{t}_{\mathbf{z}_H}\|_{L_2(\Gamma)} \leq c_3 h^{1/2} \|\mathbf{z}_H\|_{H^1(\Gamma)}.$$

Now by applying the inverse inequality for $S_H^1(\Gamma)$ we get

$$\|\mathbf{z}_H\|_{H^1(\Gamma)} \leq c_I H^{-1/2} \|\mathbf{z}_H\|_{H^{1/2}(\Gamma)},$$

so we get

$$\left\| T_\rho \mathbf{z}_H - \widetilde{T}_\rho \mathbf{z}_H \right\|_{H^{-1/2}(\Gamma)} \leq c_3 c_I \left(\frac{h}{H}\right)^{1/2} \|\mathbf{z}_H\|_{H^{1/2}(\Gamma)}.$$

Hence, we finally get

$$\begin{aligned}
\left(\widetilde{T}_{\rho,H} \mathbf{z}_H, \mathbf{z}_H\right) &\geq c_1^{T_\rho} \|\mathbf{z}_H\|^2_{H^{1/2}(\Gamma)} - c_3 c_I \left(\frac{h}{H}\right)^{1/2} \|\mathbf{z}_H\|^2_{H^{1/2}(\Gamma)}, \\
&\geq \left[c_1^{T_\rho} - c_3 c_I \left(\frac{h}{H}\right)^{1/2} \right] \|\mathbf{z}_H\|^2_{H^{1/2}(\Gamma)} \geq \frac{1}{2} c_1^{T_\rho} \|\mathbf{z}_H\|^2_{H^{1/2}(\Gamma)},
\end{aligned}$$

if $c_3 c_I \left(\frac{h}{H}\right)^{1/2} \leq \frac{1}{2} c_1^{T_\rho}$ is satisfied. □

Now we are in a position to apply Theorem 5.4.1 to ensure the unique solvability of the perturbed variational inequality (5.25), and to derive related error estimates.

Corollary 5.4.1. *When combining the error estimate* (5.5) *with the approximation property of the ansatz space* $S_H^1(\Gamma)$*, and with the error estimates* (5.15) *and* (5.23) *, we finally obtain the error estimate*

$$\begin{aligned}\|\mathbf{z}-\widetilde{\mathbf{z}}_H\|_{H^{1/2}(\Gamma)} \leq c_1 H^{s+1/2}\|\mathbf{z}\|_{H^{1+s}(\Gamma)} \quad + \quad & c_2 h^{s+1/2}\|\mathbf{q}_z\|_{H^s_{pw}(\Gamma)} + c_3 h^{s+3/2}\|\mathbf{t}_z\|_{H^s_{pw}(\Gamma)} \\ & + c_4 h^{s+1/2}\|\mathbf{g}\|_{H^s_{pw}(\Gamma)} + c_5 h^{s+3/2}\|\mathbf{t}_f\|_{H^s_{pw}(\Gamma)},\end{aligned}$$

when assuming $\mathbf{z}\in H^{1+s}(\Gamma)$ *and* $\mathbf{q}_z, \mathbf{t}_z, \mathbf{g}, \mathbf{t}_f \in H^s_{pw}(\Gamma)$ *for some* $s\in[0,1]$*. For* $h \leq c_0 H$ *we therefore obtain the error estimate*

$$\|\mathbf{z}-\widetilde{\mathbf{z}}_H\|_{H^{1/2}_*(\Gamma)} \leq c(\mathbf{z},\overline{\mathbf{u}},\mathbf{f})H^{s+1/2}. \tag{5.30}$$

Further, we are able to derive an error estimate in $L_2(\Gamma)$*, i.e.,*

$$\|\mathbf{z}-\widetilde{\mathbf{z}}_H\|_{L_2(\Gamma)} \leq c(\mathbf{z},\overline{\mathbf{u}},\mathbf{f})H^{s+1}, \tag{5.31}$$

when applying the Aubin–Nitsche trick.

In the particular case of a non–constrained minimization problem, instead of the discrete variational inequality (5.28) we have to solve the linear system

$$\widetilde{T}_{\rho,H}\underline{\widetilde{\mathbf{z}}} = \underline{\widetilde{\mathbf{g}}},$$

which is equivalent to the system

$$\begin{pmatrix} V_{1,h} & \widehat{V}_h & -K_{1,h} \\ \widehat{V}_h & 0 & -(\frac{1}{2}M_h + K_h) \\ 0 & -M_h^\top & \rho\mathcal{A}_H \end{pmatrix} \begin{pmatrix} \underline{\mathbf{t}} \\ \underline{\mathbf{q}} \\ \underline{\widetilde{\mathbf{z}}} \end{pmatrix} = \begin{pmatrix} N_0\overline{\mathbf{u}} - M_0\mathbf{f} \\ -N_0\mathbf{f} \\ \mathbf{0} \end{pmatrix}. \tag{5.32}$$

Remark 5.4.1. *The error estimates* (5.30) *and* (5.31) *provide optimal convergence rates when approximating the control* $\mathbf{z}$ *by using piecewise linear basis functions. However, we have to assume* $h \leq c_0 H$ *to ensure the unique solvability of the perturbed Galerkin variational inequality* (5.25)*, where the constant* c_0 *is in general unknown. Moreover, the matrix* $\widetilde{T}_{\rho,H}$ *defines a non–symmetric approximation of a self–adjoint operator* T_ρ*. As an alternative to the non–symmetric approach presented in this work one can use the symmetric boundary element method which is stable without an additional constraints on the choice of the boundary element trial space. In contrast, the use of the non–symmetric formulation does not require the use of hypersingular boundary integral operators, further it allows us the use of collocation instead of Galerkin.*

6 NUMERICAL EXAMPLES

In this chapter we present some numerical results for the boundary element approximation of the Dirichlet boundary control for the Stokes problem (3.10)–(3.12). The numerical results presented confirm the theoretical findings. Consider a square domain $\Omega = (0,0.5)^2 \subset \mathbb{R}^2$. For simplicity take both $\rho = 1$ and $\mu = 1$. We take a pre–described solution $(\mathbf{w}, r)$ of the adjoint problem (3.24) where the pressure $r = 0$ and $\mathbf{w}$, which is divergence free and vanishes on the boundary, is given by

$$\mathbf{w}(x,y) = \left[(x-2x^2)^2(y-2y^2)(4y-1), (x-2x^2)(y-2y^2)^2(1-4x)\right]^\top .$$

For the boundary element discretization we introduce a uniform triangulation of the boundary $\Gamma = \partial\Omega$ on several levels. Since the minimiser of the cost functional (3.10) is not known in advance in this case, we use the boundary element solution of the 9–th refinement level as reference solution. The boundary element discretization is done by using the trial space $S_h^0(\Gamma)$ of piecewise constant basis functions, and $S_h^1(\Gamma)$ of piecewise linear and continuous functions. We use two different boundary element meshes, one to approximate the control $\mathbf{z}$, of size H, by a piecewise linear approximation, and another space of piecewise constant approximations for the tractions $\mathbf{t}$ and $\mathbf{q}$ with size h. Note that we have $h = H/2$ in this case, and therefore by Theorem 5.4.1, we may ensure $S_h^1(\Gamma)$–ellipticity of the non–symmetric boundary element approximation. Further, the numerical example shows stability. In Table 6.1, we have presented the errors for the control $\mathbf{z}$ and the traction

level	#ite(BiCGStab)	$\|\mathbf{z}-\mathbf{z}_h\|_{L_2(\Gamma)}$	eoc	$\|\mathbf{q}-\mathbf{q}_h\|_{L_2(\Gamma)}$	eoc
0	1	4.03E-04	–	7.89E-03	–
1	2	5.71E-05	2.82	3.62E-03	1.12
2	4	2.39E-05	1.26	1.78E-03	1.03
3	6	5.92E-06	2.01	8.83E-04	1.01
4	7	1.41E-06	2.07	4.41E-04	1
5	12	3.34E-07	2.08	2.20E-04	1
6	17	8.03E-08	2.06	1.10E-04	1
7	25	1.97E-08	2.03	5.50E-05	1
8	38	5.45E-09	1.86	2.75E-05	1
Theory			2		1

Table 6.1: Convergence analysis for the Dirichlet control and the traction.

$\mathbf{q}$ in the $L_2(\Gamma)$ norm along with their estimated order of convergence (eoc). The result for the control $\mathbf{z}$ corresponds to the error estimate (5.31) of the non–symmetric boundary element approximation.

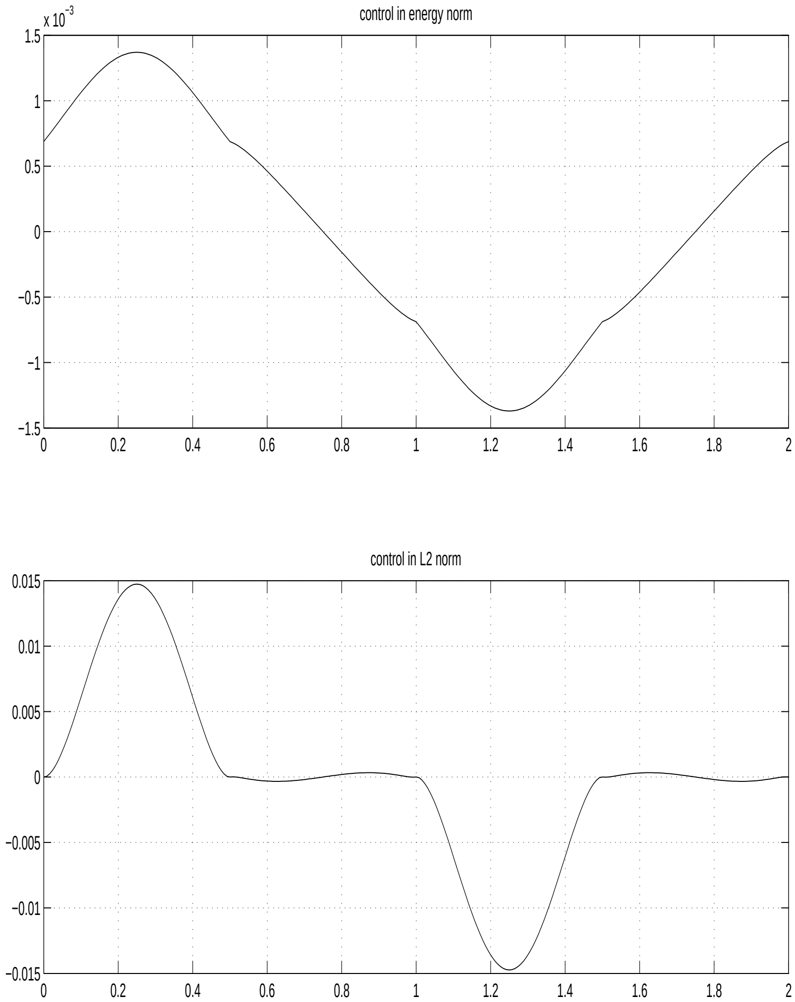

Figure 6.1: *Plot of the control* $\mathbf{z}$ *in* $H^{1/2}(\Gamma)$ *and* $L_2(\Gamma)$ *settings.*

In Figure 6.1 we have plotted the control when considering both the energy norm and the L_2 norm. It is clear from the figure that when using the L_2 norm the control $\mathbf{z}$ is zero on the corner points independent of the value of the target function $\overline{\mathbf{u}}$. This is not seen in the energy norm setting as the use of the inverse single layer potential $\widehat{V}^{-1}$ preserves the mapping properties. This also justifies the use of $H^{1/2}(\Gamma)$ as the control space instead of $L_2(\Gamma)$.

The fact which we have highlighted graphically can also be verified analytically. As already pointed out that the effect of choice of the control space can be seen in the optimality condition, so we start from the optimality condition for the non–constrained case

$$\tau + \rho \mathcal{A}\mathbf{z} = 0.$$

For $\tau = -\mathbf{q}$ and $\mathcal{A} = I$, we have

$$-\mathbf{q} + \rho \mathbf{z} = 0.$$

By definition, see [37, Section 2.3],

$$\mathbf{q} := T'(\mathbf{w}) = \mathbf{t}(\mathbf{w}, -r) = r\,\mathbf{n} + \mu(\nabla \mathbf{w} + \nabla \mathbf{w}^\top)\mathbf{n}. \tag{6.1}$$

Since $\mathbf{w} = 0$ on Γ, this means $(\nabla \mathbf{w})\mathbf{n} = \mathbf{0}$ in the each corner point. So (6.1) takes the form

$$\mathbf{q} = \rho \mathbf{z} = r\,\mathbf{n}.$$

As $\mathbf{z}$ is continuous and since we consider the case of a square domain, we can write

$$\rho \mathbf{z} = r \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \rho \mathbf{z} = r \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$z_1 = 0 \quad \text{and} \quad z_2 = 0,$$
$$\text{and therefore } \mathbf{z} = 0 \text{ at each corner point.}$$

In the case of an arbitrary domain with a corner between two normal vectors $\mathbf{n}^1$ and $\mathbf{n}^2$, we can write

$$\rho \mathbf{z} = r\mathbf{n}^1 \quad \text{and} \quad \rho \mathbf{z} = r\mathbf{n}^2,$$
$$\rho \mathbf{z} \cdot \mathbf{n}^1 = r \quad \text{and} \quad \rho \mathbf{z} \cdot \mathbf{n}^2 = r,$$
$$\text{and therefore } \rho \mathbf{z} \cdot (\mathbf{n}^1 - \mathbf{n}^2) = 0, \text{ at each corner point,}$$

which gives a necessary condition for the control $\mathbf{z}$ to vanish at each corner point of an arbitrary domain.

One last comment on the choice of the control space. The use of $L_2(\Gamma)$ as control space is quite common in the engineering community instead of $H^{1/2}(\Gamma)$. The effect is clearly visible in the optimality condition in a way that mapping properties are lost and in fact the two pieces of boundary data are identified with each other. Instead, if we use $H^{1/2}(\Gamma)$ as control space it not only preserves the corresponding mapping properties from Dirichlet control to the traction of the adjoint variable but also accounts for some higher regularity of the control in the related optimality condition.

In Table 6.2, we have computed the L_2 distance of the state $\widetilde{\mathbf{u}}$ from the desired state $\overline{\mathbf{u}}$ for the different refinement levels in the energy and the L_2 settings. We have found that the values in both the cases are very close to each other. Since $\overline{\mathbf{u}}$ is fixed so the approximate state $\widetilde{\mathbf{u}}$ tends to some exact state $\mathbf{u}$.

$\|\widetilde{\mathbf{u}}-\overline{\mathbf{u}}\|_{L_2(\Omega)}$		
Level	$\mathcal{A}=\widehat{V}^{-1}$	$\mathcal{A}=I$
0	0.0630781	0.0617194
1	0.0617038	0.0591016
2	0.0617217	0.0590576
3	0.0617214	0.0590387
4	0.0617213	0.0590373
5	0.0617213	0.0590372
6	0.0617213	0.0590372

Table 6.2: The L_2 distance of the tracking term.

7 CONCLUSION AND OUTLOOK

In this final chapter we, very briefly, give few conclusions of this work along with some future outlook.

7.1 Conclusions

In this thesis, we have shown that one can use boundary element methods to solve Dirichlet boundary control problems for the Stokes system. The numerical results presented are in conformity with the theoretical findings. The clear advantage of the boundary element methods lies in the fact, that only the boundary of the computational domain needs to be discretized. For given smooth data we can prove, with respect to the used lowest order trial spaces, the best possible order of convergence for the boundary element approximation for the control $\mathbf{z}$. In addition to that, optimal control problems subject to partial differential equations as constraints in an unbounded exterior domain can be treated in a similar way.

7.2 Future work

Symmetric formulation

As already pointed out in Remark 5.4.1, we have presented a non–symmetric formulation which requires some condition on the two meshes. Further the matrix $\widetilde{T}_{\rho,H}$ defines a non–symmetric approximation of a self–adjoint operator T_ρ. We can derive a symmetric formulation which is stable witout any further condition on the two meshes. For this we can use a second boundary integral equation for the adjoint boundary value problem to obtain an alternative representation for the traction $\mathbf{q}$ and therefore the adjoint operator $\mathcal{S}^*$ [53].

Preconditioning and fast methods

The main goal of this study was to solve an optimal control problem for Stokes by boundary element methods and as such we have shown the stability of the method and presented some error analysis. Further research can be done for an efficient solution of the resulting discrete system. Special emphasis will be on efficient solution methods for solving the discrete variational inequalities. This can be done by constructing efficient preconditioners, as well as by the use of fast boundary element methods.

A APPENDIX

In this appendix we give the detailed computation of the kernel function for the Bi–Stokes system represented by (4.5).We also give the boundary traction for this kernel function.

Computation of $V^*(x,y)$

The system of linear elastostatics is

$$-\mu\Delta\mathbf{u} - (\lambda+\mu)\text{grad div }\mathbf{u}(x) = \mathbf{f}(x),$$

Let us make following ansatz

$$\mathbf{u}(x) = \Delta\mathbf{w} - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\text{grad div }\mathbf{w}(x).$$

By this transformation and on simplification we get

$$-\mu\Delta^2\mathbf{w}(x) = \mathbf{f}(x).$$

If $\mathbf{U}_1^*(x,y)$ is the fundamental solution of linear elastostatics, then we must have

$$-\mu\Delta_y\mathbf{U}_1^*(x,y) - (\lambda+\mu)\text{grad}_y\text{ div}_y\,\mathbf{U}_1^*(x,y) = \delta_0(y-x)\mathbf{e}_1,$$

analogously, we make the following ansatz

$$\mathbf{U}_1^*(x,y) = \Delta_y\mathbf{W}_1(x,y) - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\text{grad}_y\text{ div}_y\,\mathbf{W}_1(x,y),$$

where

$$\mathbf{W}_1(x,y) = \begin{pmatrix} W_{11}(x,y) \\ W_{12}(x,y) \end{pmatrix}^\top; \quad \mathbf{U}_1^*(x,y) = \begin{pmatrix} U_{11}^*(x,y) \\ U_{12}^*(x,y) \end{pmatrix}^\top.$$

Again by this transformation, we end up having the following Bi–Laplace equation

$$-\mu\Delta^2\mathbf{W}_1(x,y) = \delta_0(y-x)\mathbf{e}_1 = \delta_0(y-x)\begin{pmatrix}1\\0\end{pmatrix}.$$

Equating the components on both sides gives the following system

$$\begin{aligned} -\mu\Delta^2 W_{11}(x,y) &= \delta_0(y-x), \\ -\mu\Delta^2 W_{12}(x,y) &= 0. \end{aligned}$$

Now consider

$$-\mu\Delta^2 W_{11}(x,y) = \delta_0(y-x)$$

which is the Bi–Laplace equation whose solution, with special choice of the constant, is given by

$$\begin{aligned} W_{11}(x,y) &= -\frac{1}{8\pi\mu}|x-y|^2\left[\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right], \\ W_{12}(x,y) &= 0. \end{aligned}$$

Also

$$\begin{aligned} \Delta_y W_{11} &= \frac{\partial^2}{\partial y_1^2}W_{11}(x,y) + \frac{\partial^2}{\partial y_2^2}W_{11}(x,y), \\ &= -\frac{1}{2\pi\mu}\left[\log|x-y| - \frac{(\lambda+\mu)}{(2\lambda+6\mu)}\right]. \end{aligned}$$

The fundamental solution is given by

$$\begin{aligned} U_{11}(x,y) &= \Delta_y W_{11}(x,y) - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial}{\partial y_1}\,\mathrm{div}_y\,\mathbf{W}_1(x,y), \\ &= \frac{1}{4\pi\mu}\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)\left[-\frac{\lambda+3\mu}{\lambda+\mu}\log|x-y| + \frac{(y_1-x_1)^2}{|x-y|^2}\right]. \\ U_{12}(x,y) &= \Delta_y W_{12}(x,y) - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial}{\partial y_2}\,\mathrm{div}_y\,\mathbf{W}_1(x,y), \\ &= \frac{1}{4\pi\mu}\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)\frac{(y_1-x_1)(y_2-x_2)}{|x-y|^2}. \end{aligned}$$

The system of linear elastostatics is

$$-\mu\Delta_y\mathbf{V}_1^*(x,y) - (\lambda+\mu)\mathrm{grad}_y\,\mathrm{div}_y\,\mathbf{V}_1^*(x,y) = \mathbf{U}_1^*(x,y),$$

on making the following ansatz we have

$$\mathbf{V}_1^*(x,y) = \Delta\mathbf{Z}_1(x,y) - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\mathrm{grad}_y\,\mathrm{div}_y\,\mathbf{Z}_1(x,y).$$

By this transformation we get

$$-\mu\Delta^2\mathbf{Z}_1(x,y) = \mathbf{U}_1^*(x,y),$$

which results in the form of the following system

$$\begin{aligned} -\mu\Delta_y^2 Z_{11}(x,y) &= U_{11}(x,y), \\ -\mu\Delta_y^2 Z_{12}(x,y) &= U_{12}(x,y). \end{aligned}$$

First we consider

$$\begin{aligned}
-\mu\Delta_y^2 Z_{12}(x,y) &= U_{12}(x,y), \\
&= -\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial}{\partial y_2}\,\mathrm{div}_y\,\mathbf{W}_1(x,y), \\
&= -\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial}{\partial y_2}\frac{\partial}{\partial y_1}W_{11}(x,y), \\
&= \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{1}{\mu}\frac{1}{8\pi}\frac{\partial}{\partial y_2}\frac{\partial}{\partial y_1}\left[|x-y|^2\{\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right].
\end{aligned}$$

With the ansatz

$$Z_{12}(x,y) = -\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{1}{\mu^2}\frac{1}{8\pi}\underbrace{\frac{\partial}{\partial y_2}\frac{\partial}{\partial y_1}}_{D}S(x,y),$$

we obtain

$$\begin{aligned}
\Delta_y^2 DS(x,y) &= D\left[|x-y|^2\{\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\}\right], \\
D\Delta_y^2 S(x,y) &= D\left[|x-y|^2\{\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\}\right], \\
\Delta_y^2 S(x,y) &= |x-y|^2\left[\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right].
\end{aligned}$$

We still have to solve

$$\Delta_y^2 S(x,y) = \Delta_y(\underbrace{\Delta_y S(x,y)}_{T(x,y)}) = |x-y|^2\left[\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right],$$

$$\Delta_y(\underbrace{\Delta_y S(x,y)}_{T(x,y)}) = |x-y|^2\left[\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right],$$

so the resulting system is

$$\Delta_y T(x,y) = |x-y|^2\left[\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right] \Longleftrightarrow \Delta_y S(x,y) = T(x,y).$$

Consider first

$$\Delta_y T(x,y) = |x-y|^2\left[\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right],$$

using the polar form of the Laplace equation we have

$$\begin{aligned}
\frac{1}{r}\frac{d}{dr}\left[r\frac{dT}{dr}\right] &= r^2\left[\log r - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right], \\
\frac{d}{dr}\left[r\frac{dT}{dr}\right] &= r^3\left[\log r - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right].
\end{aligned}$$

Separating the variables and integration two times, we get

$$\begin{aligned} r\frac{dT}{dr} &= \frac{1}{4}\left[r^4(\log r - \frac{3\lambda+7\mu}{2\lambda+6\mu}) - \frac{1}{4}r^4\right], \\ \frac{dT}{dr} &= \frac{1}{4}\left[r^3(\log r - \frac{3\lambda+7\mu}{2\lambda+6\mu}) - \frac{1}{4}r^3\right], \\ T(r) &= \frac{1}{16}\left[r^4(\log r - \frac{3\lambda+7\mu}{2\lambda+6\mu}) - \frac{1}{2}r^4\right]. \end{aligned}$$

Separating the variables and integrating once again, we get

$$\begin{aligned} \Delta_y S(x,y) = T(x,y) &= \frac{1}{16}\left[r^4(\log r - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}) - \frac{1}{2}r^4\right], \\ \Delta_y S(x,y) &= \frac{1}{16}\left[r^4(\log r - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}) - \frac{1}{2}r^4\right], \\ \frac{1}{r}\frac{d}{dr}\left[r\frac{dS}{dr}\right] &= \frac{1}{16}\left[r^4(\log r - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}) - \frac{1}{2}r^4\right], \\ \frac{d}{dr}\left[r\frac{dS}{dr}\right] &= \frac{1}{16}\left[r^5(\log r - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}) - \frac{1}{2}r^5\right], \\ \Rightarrow S(x,y) &= \frac{1}{576}\left[|x-y|^6 \log|x-y| - \frac{7\lambda+18\mu}{3\lambda+9\mu}|x-y|^6\right]. \end{aligned}$$

Now

$$\begin{aligned} \Delta_y S(x,y) &= \frac{\partial^2}{\partial y_1^2}S(x,y) + \frac{\partial^2}{\partial y_2^2}S(x,y), \\ \Delta_y S(x,y) &= \frac{|x-y|^4}{16}\left[\log|x-y| - \frac{(2\lambda+5\mu)}{(\lambda+3\mu)}\right]. \end{aligned}$$

As

$$\begin{aligned} Z_{12}(x,y) &= -\frac{1}{\mu^2}\frac{1}{8\pi}\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial^2}{\partial y_2 \partial y_1}S(x,y), \\ Z_{12}(x,y) &= -\frac{1}{2304\pi\mu^2}\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)(y_1-x_1)(y_2-x_2)|x-y|^2\left[12\log|x-y| - \frac{23\lambda+57\mu}{\lambda+3\mu}\right]. \end{aligned}$$

Now the solution of

$$
\begin{aligned}
-\mu\Delta_y^2 Z_{11}(x,y) &= U_{11}(x,y),\\
&= \Delta_y W_{11}(x,y) - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial}{\partial y_1}\,\mathrm{div}_y\,\mathbf{W}_1(x,y),\\
&= \Delta_y W_{11}(x,y) - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial^2}{\partial y_1^2}W_{11}(x,y),\\
&= \left[\Delta_y - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial^2}{\partial y_1^2}\right]W_{11}(x,y),\\
&= -\frac{1}{8\pi\mu}\left[\Delta_y - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial^2}{\partial y_1^2}\right]|x-y|^2(\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)},
\end{aligned}
$$

is determined by the ansatz

$$
Z_{11}(x,y) = \frac{1}{8\pi\mu^2}\left[\Delta_y - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial^2}{\partial y_1^2}\right]S(x,y).
$$

This gives us

$$
\begin{aligned}
\Delta_y^2 DS(x,y) &= D\left[|x-y|^2\{\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\}\right],\\
D\Delta_y^2 S(x,y) &= D\left[|x-y|^2\{\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\}\right],\\
\Delta_y^2 S(x,y) &= |x-y|^2\left[\log|x-y| - \frac{(3\lambda+7\mu)}{(2\lambda+6\mu)}\right].
\end{aligned}
$$

The solution of this Bi–Laplace equation is already obtained

$$
S(x,y) = \frac{1}{576}\left[|x-y|^6\log|x-y| - \frac{7\lambda+18\mu}{3\lambda+9\mu}|x-y|^6\right].
$$

Now

$$
\begin{aligned}
Z_{11}(x,y) &= \frac{1}{8\pi\mu^2}\left[\Delta_y - \frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{\partial^2}{\partial y_1^2}\right]S(x,y),\\
&= \frac{1}{8\pi\mu^2}\left[\left(\frac{\mu}{\lambda+2\mu}\right)\frac{\partial^2}{\partial y_1^2}S(x,y) + \frac{\partial^2}{\partial y_2^2}S(x,y)\right].
\end{aligned}
$$

Inserting the values of the second order partial derivatives and on simplification, we have

$$
\begin{aligned}
= \frac{|x-y|}{4608\pi\mu^2(\lambda+2\mu)(\lambda+3\mu)}\Bigg[&6(\lambda+3\mu)\left\{(\lambda+7\mu)(y_1-x_1)^2 + (5\lambda+11\mu)(y_2-x_2)^2\right\}\log|x-y|\\
&-a(y_1-x_1)^2 - b(y_2-x_2)^2\Bigg],
\end{aligned}
$$

where

$$a = 13\lambda^2 + 118\mu\lambda + 213\mu^2,$$
$$b = 59\lambda^2 + 278\mu\lambda + 327\mu^2.$$

Recall

$$\mathbf{V}_1^*(x,y) = \Delta \mathbf{Z}_1(x,y) - \frac{(\lambda+\mu)}{(\lambda+2\mu)} \mathrm{grad}_y\, \mathrm{div}_y\, \mathbf{Z}_1(x,y),$$
$$V_{11}^*(x,y) = \Delta Z_{11} - \frac{(\lambda+\mu)}{(\lambda+2\mu)} \left(\frac{\partial^2}{\partial y_1^2} Z_{11} + \frac{\partial^2}{\partial y_2^2} Z_{12} \right),$$
$$= \left(\frac{\mu}{\lambda+2\mu} \right) \frac{\partial^2}{\partial y_1^2} Z_{11} + \frac{\partial^2}{\partial y_2^2} Z_{11} - \frac{(\lambda+\mu)}{(\lambda+2\mu)} \frac{\partial^2}{\partial y_1 \partial y_2} Z_{12}.$$

Replacing the partial derivatives and on simplification

$$V_{11}^* = -\frac{c(y_1-x_1)^2 + d(y_2-x_2)^2 - 4(\lambda+3\mu)\left\{e(y_1-x_1)^2 + f(y_2-x_2)^2\right\} \log|x-y|}{128\pi\mu^2(\lambda+2\mu)^2(\lambda+3\mu)},$$

analogously we can show that

$$V_{22}^* = -\frac{c(y_2-x_2)^2 + d(y_1-x_1)^2 - 4(\lambda+3\mu)\left\{e(y_2-x_2)^2 + f(y_1-x_1)^2\right\} \log|x-y|}{128\pi\mu^2(\lambda+2\mu)^2(\lambda+3\mu)},$$

where

$$\begin{aligned} c &= 7\lambda^3 + 45\mu\lambda^2 + 113\mu^2\lambda + 107\mu^3, \\ d &= 17\lambda^3 + 107\mu\lambda^2 + 231\mu^2\lambda + 173\mu^3, \\ e &= \lambda^2 + 4\mu\lambda + 7\mu^2, \\ f &= 3\lambda^2 + 12\mu\lambda + 13\mu^2. \end{aligned}$$

Again,

$$V_{12}^*(x,y) = \Delta Z_{12} - \frac{(\lambda+\mu)}{(\lambda+2\mu)} \left(\frac{\partial^2}{\partial y_2 \partial y_1} Z_{11} + \frac{\partial^2}{\partial y_2^2} Z_{12} \right),$$
$$= \frac{\partial^2}{\partial y_1^2} Z_{12} + \left(\frac{\mu}{\lambda+2\mu} \right) \frac{\partial^2}{\partial y_2^2} Z_{12} - \frac{(\lambda+\mu)}{(\lambda+2\mu)} \frac{\partial^2}{\partial y_2 \partial y_1} Z_{11},$$

so inserting the corresponding partial derivatives and simplifying, we have

$$V_{12}^*(x,y) = -\frac{(y_1-x_1)(y_2-x_2)(\lambda+\mu)\{-5\lambda - 11\mu + 4(\lambda+3\mu)\log|x-y|\}}{64\pi\mu^2(\lambda+2\mu)^2}.$$

In a similar manner we can show

$$V_{21}^*(x,y) = -\frac{(y_1-x_1)(y_2-x_2)(\lambda+\mu)\{-5\lambda - 11\mu + 4(\lambda+3\mu)\log|x-y|\}}{64\pi\mu^2(\lambda+2\mu)^2}.$$

The Stokes case will follow by taking the limits, $\lambda \to \infty, \mu \to 1$, i.e.,

$$\mathbf{V}^* = \begin{pmatrix} V_{11}^*(x,y) & V_{12}^*(x,y) \\ V_{21}^*(x,y) & V_{22}^*(x,y) \end{pmatrix},$$

where

$$\begin{aligned} V_{11}^*(x,y) &= \frac{4\left\{(y_1-x_1)^2+3(y_2-x_2)^2\right\}\log|x-y|-7(y_1-x_1)^2-17(y_2-x_2)^2}{128\pi}, \\ V_{12}^*(x,y) &= \widetilde{V}_{21}^*(x,y) = -\frac{(y_1-x_1)(y_2-x_2)\left\{(4\log|x-y|-5\right\}}{64\pi}, \\ V_{22}^*(x,y) &= \frac{4\left\{3(y_1-x_1)^2+(y_2-x_2)^2\right\}\log|x-y|-17(y_1-x_1)^2-7(y_2-x_2)^2}{128\pi}. \end{aligned}$$

Further it is easy to verify that $\mathbf{V}^*(x,y)$ satisfy the divergence free property, i.e.,

$$\begin{aligned} \frac{\partial}{\partial y_1}V_{11}^*(x,y) + \frac{\partial}{\partial y_2}V_{12}^*(x,y) = 0, \\ \frac{\partial}{\partial y_1}V_{21}^*(x,y) + \frac{\partial}{\partial y_2}V_{22}^*(x,y) = 0. \end{aligned}$$

Computation of boundary stress

The boundary stress $\mathbf{T}_k^*(x,y)$ of the fundmental solution $V_k^*(x,y)$ is given for allmost all $y \in \Gamma$ as

$$\begin{aligned} \mathbf{T}_k^*(x,y) :&= \gamma_{1,y}^{int}\mathbf{V}_k^*(x,y), \\ &= \lambda\, \mathrm{div}_y \mathbf{V}_k^*(x,y)\mathbf{n}(y) + 2\mu\frac{\partial}{\partial n_y}\mathbf{V}_k^*(x,y) + \mu\mathbf{n}(y) \times \mathrm{curl}_y \mathbf{V}_k^*(x,y). \end{aligned}$$

Since $\mathbf{V}_k^*(x,y)$ is divergence free and if for simplicity we take $\mu = 1$, we get

$$\mathbf{T}_k^*(x,y) = 2\frac{\partial}{\partial n_y}\mathbf{V}_k^*(x,y) + \mathbf{n}(y) \times \mathrm{curl}_y \mathbf{V}_k^*(x,y), \tag{A.1}$$

with $k = 1,2$.

If we define, for a two dimensional vector field $\mathbf{v}$, the rotation as

$$\begin{aligned} \mathrm{curl}_y \mathbf{v}(y) :&= \frac{\partial}{\partial y_1}v_2(y) - \frac{\partial}{\partial y_2}v_1(y) \quad \text{and if we declare} \\ \mathbf{v} \times \alpha :&= \alpha \begin{pmatrix} v_2(y) \\ -v_1(y) \end{pmatrix}, \end{aligned}$$

where $\mathbf{v} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}^1$. By using such a representation we can write

$$\mathbf{n}(y) \times \mathrm{curl}_y \mathbf{V}_k^*(x,y) = \mathrm{curl}_y \mathbf{V}_k^*(x,y) \begin{pmatrix} n_2(y) \\ -n_1(y) \end{pmatrix}.$$

So we need to compute $\mathrm{curl}_y \mathbf{V}_1^*(x,y)$ and $\mathrm{curl}_y \mathbf{V}_2^*(x,y)$.

$$\begin{aligned} \mathrm{curl}_y \mathbf{V}_1^*(x,y) &= \frac{\partial}{\partial y_1} V_{12}^*(y) - \frac{\partial}{\partial y_2} V_{11}^*(y), \\ &= -\frac{(y_2 - x_2)(\log|x-y| - 1)}{4\pi}, \end{aligned}$$

so

$$\mathbf{n}(y) \times \mathrm{curl}_y \mathbf{V}_1^*(x,y) = -\frac{(y_2 - x_2)(\log|x-y| - 1)}{4\pi} \begin{pmatrix} n_2(y) \\ -n_1(y) \end{pmatrix}. \tag{A.2}$$

Also

$$\frac{\partial}{\partial n_y} \mathbf{V}_1^*(x,y) := \begin{pmatrix} \frac{\partial}{\partial n_y} V_{11}^*(x,y) \\ \frac{\partial}{\partial n_y} V_{12}^*(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial y_1} V_{11}^*(x,y) n_1(y) + \frac{\partial}{\partial y_2} V_{11}^*(x,y) n_2(y) \\ \frac{\partial}{\partial y_1} V_{12}^*(x,y) n_1(y) + \frac{\partial}{\partial y_2} V_{12}^*(x,y) n_2(y) \end{pmatrix}, \tag{A.3}$$

so using (A.2) and (A.3) in (A.1), we have

$$\begin{aligned} \mathbf{T}_1^*(x,y) &= 2 \begin{pmatrix} \frac{\partial}{\partial y_1} V_{11}^*(x,y) n_1(y) + \frac{\partial}{\partial y_2} V_{11}^*(x,y) n_2(y) \\ \frac{\partial}{\partial y_1} V_{12}^*(x,y) n_1(y) + \frac{\partial}{\partial y_2} V_{12}^*(x,y) n_2(y) \end{pmatrix} + \begin{pmatrix} -\frac{(y_2-x_2)(\log|x-y|-1)}{4\pi} n_2(y) \\ \frac{(y_2-x_2)(\log|x-y|-1)}{4\pi} n_1(y) \end{pmatrix}, \\ &= \begin{pmatrix} 2\frac{\partial}{\partial y_1} V_{11}^*(x,y) n_1(y) + \left\{ 2\frac{\partial}{\partial y_2} V_{11}^*(x,y) - \frac{(y_2-x_2)(\log|x-y|-1)}{4\pi} \right\} n_2(y) \\ \left\{ 2\frac{\partial}{\partial y_1} V_{12}^*(x,y) + \frac{(y_2-x_2)(\log|x-y|-1)}{4\pi} \right\} n_1(y) + 2\frac{\partial}{\partial y_2} V_{12}^*(x,y) n_2(y) \end{pmatrix}. \end{aligned}$$

On inserting the partial derivatives and doing similar computations for the case $k = 2$, we finally get the boundary traction operator corresponding to the Bi–Stokes fundamental solution

$$T^*(x,y) = \frac{1}{32\pi|x-y|^2} \begin{pmatrix} T_{11}^*(x,y) & T_{12}^*(x,y) \\ T_{21}^*(x,y) & T_{22}^*(x,y) \end{pmatrix},$$

where

$$\begin{aligned} T_{11}^*(x,y) &= (y_1 - x_1)\left\{4|x-y|^2 \log|x-y| - 5(y_1-x_1)^2 - (y_2-x_2)^2\right\} n_1(y) \\ &+ (y_2 - x_2)\left\{4|x-y|^2 \log|x-y| - 7(y_1-x_1)^2 - 3(y_2-x_2)^2\right\} n_2(y), \\ T_{12}^*(x,y) &= (y_2 - x_2)\left\{4|x-y|^2 \log|x-y| - 7(y_1-x_1)^2 - 3(y_2-x_2)^2\right\} n_1(y) \\ &- (y_1 - x_1)\left\{4|x-y|^2 \log|x-y| - 5(y_1-x_1)^2 - (y_2-x_2)^2\right\} n_2(y), \\ T_{21}^*(x,y) &= -(y_2 - x_2)\left\{4|x-y|^2 \log|x-y| - (y_1-x_1)^2 - 5(y_2-x_2)^2\right\} n_1(y) \\ &+ (y_1 - x_1)\left\{4|x-y|^2 \log|x-y| - 3(y_1-x_1)^2 - 7(y_2-x_2)^2\right\} n_2(y), \\ T_{22}^*(x,y) &= (y_1 - x_1)\left\{4|x-y|^2 \log|x-y| - 3(y_1-x_1)^2 - 7(y_2-x_2)^2\right\} n_1(y) \\ &+ (y_2 - x_2)\left\{4|x-y|^2 \log|x-y| - (y_1-x_1)^2 - 5(y_2-x_2)^2\right\} n_2(y). \end{aligned}$$

B APPENDIX

In this appendix we present some introduction to variational inequalities along with existence and uniqueness results of solutions. The main references for this section are [19, 20, 40].

Variational inequalities

The theory of variational inequalities has its roots in the calculus of variations associated with the minimization of infinite–dimensional functionals. The study of the subject began in the 1960's with the seminal work of Guido Stampacchia and Philip Hartman, who used the variational inequality as an analytic tool for studying free boundary problems defined by nonlinear partial differential operators arising from unilateral problems in elasticity and plasticity theory and in mechanics.

Some of the earliest papers on variational inequalities are [32,46,47,61,62]. In [61] the first theorem of existence and uniqueness of the solution of variational inequalities was proved. The books by Baiocchi and Capelo [3] and Kinderlehrer and Stampacchia [40] provide a thorough introduction to the application of variational inequalities in infinite–dimensional function spaces; see also [4]. The lecture notes [38] treat complementarity problems in abstract spaces. The book by Glowinski, Lions, and Tremoliere [25] is among the earliest references to give a detailed numerical treatment of such variational inequalities. There is a huge literature on the subject of infinite–dimensional variational inequalities and related problems. For the study of the finite dimensional variational inequalities and complementarity problems see [19, 20].

In the following, let H be a real Hilbert space H^* its dual with duality pairing $\langle \cdot, \cdot \rangle_{H^* \times H}$. The norm on H will be denoted by $\|\cdot\|_H$ and the dual norm by $\|\cdot\|_{H^*}$. Let $a : H^* \times H \to \mathbb{R}$ be a (real) bilinear form which defines an operator $A : H \to H^*$

$$\langle A\mathbf{u}, \mathbf{v} \rangle_{H^* \times H} = a(\mathbf{u}, \mathbf{v}).$$

Definition B.0.1. *Let $\mathbb{K}$ be a given nonempty, closed and convex subset of H and $\mathbf{f} \in H^*$, then the variational inequality problem is to find $\mathbf{u} \in \mathbb{K}$ such that*

$$\langle A\mathbf{u}, \mathbf{v} - \mathbf{u} \rangle_{H^* \times H} \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle_{H^* \times H} \quad \textit{for all } \mathbf{v} \in \mathbb{K}. \tag{B.1}$$

Lemma B.0.1. *Let* $\mathbb{K}$ *be a closed convex subset of a Hilbert space* H*. Then for each* $\mathbf{x} \in H$ *there is a unique* $\mathbf{y} \in \mathbb{K}$ *such that*

$$\|\mathbf{x}-\mathbf{y}\| = \inf_{\eta \in \mathbb{K}} \|\mathbf{x}-\eta\|. \tag{B.2}$$

The point $\mathbf{y}$ *satisfying* (B.2) *is called the projection of* $\mathbf{x}$ *on* $\mathbb{K}$ *and we write it as*

$$\mathbf{y} := \text{Proj}_{\mathbb{K}} \mathbf{x}.$$

Proof. See [40, Lemma 2.1]. □

Next we characterize the projection by the following theorem.

Theorem B.0.1. *Let* $\mathbb{K}$ *be a closed convex subset of a Hilbert space* H*. Then* $\mathbf{y} = \text{Proj}_{\mathbb{K}} \mathbf{x}$*, the projection of* $\mathbf{x}$ *on* $\mathbb{K}$*, if and only if*

$$\mathbf{y} \in \mathbb{K} : \quad \langle \mathbf{y}, \eta - \mathbf{y} \rangle_{H \times H} \geq \langle \mathbf{x}, \eta - \mathbf{y} \rangle_{H \times H} \quad \textit{for all } \eta \in \mathbb{K}.$$

Proof. See [40, Theorem 2.3]. □

Corollary B.0.1. *Let* $\mathbb{K}$ *be a closed convex subset of a Hilbert space* H*. Then the projection operator* $\text{Proj}_{\mathbb{K}}$ *is nonexpensive, i.e.,*

$$\|\text{Proj}_{\mathbb{K}} \mathbf{x} - \text{Proj}_{\mathbb{K}} \mathbf{x}^*\| \leq \|\mathbf{x}-\mathbf{x}^*\| \quad \textit{for all } \mathbf{x}, \mathbf{x}^* \in H.$$

Proof. See [40, Corollary 2.4]. □

To prove the existence and the uniqueness of the solution of the variational inequality (B.1) first we give few definitions followed by a theorem.

Definition B.0.2. *A bilinear form is symmetric if*

$$a(\mathbf{u}, \mathbf{v}) = a(\mathbf{v}, \mathbf{u}) \quad \textit{for all } \mathbf{u}, \mathbf{v} \in H.$$

Definition B.0.3. *A bilinear form* $a(\mathbf{u}, \mathbf{v})$ *is coercive on* H *if there exists a positive constant* α *such that*

$$a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|^2 \quad \textit{for all } \mathbf{v} \in H.$$

Theorem B.0.2. *Let* $a(\mathbf{u}, \mathbf{v})$ *be a coercive bilinear form on* H*,* $\mathbb{K} \subset H$ *closed and convex and* $\mathbf{f} \in H^*$*. Then there exists a unique solution of the variational inequality* (B.1)*. In addition if* $\mathbf{f}$ *is Lipschitz, i.e., if* $\mathbf{u}_1, \mathbf{u}_2$ *are two solution of the variational inequality* (B.1) *corresponding to* $\mathbf{f}_1, \mathbf{f}_2 \in H^*$*, then*

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_H \leq (1/\alpha) \|\mathbf{f}_1 - \mathbf{f}_2\|_{H^*}.$$

Proof. See [40, Theorem 2.1]. □

We present another characterization of the solution of the variational inequality (B.1). If the operator A is symmetric then the problem (B.1) is equivalent to the following minimization problem. Let

$$\mathbf{u} = \underset{\mathbf{v}\in\mathbb{K}}{\operatorname{argMin}}\, \mathcal{J}(\mathbf{v}),$$

where $\mathcal{J} : H \to \mathbb{R}$ is the quadratic functional

$$\mathcal{J}(\mathbf{v}) = \frac{1}{2}\langle A\mathbf{v}, \mathbf{v}\rangle_{H^*\times H} - \langle \mathbf{f}, \mathbf{v}\rangle_{H^*\times H}\,.$$

Then $\mathbf{u}$ solves the variational inequality (B.1), if and only if it minimizes $\mathcal{J}(\mathbf{v})$ over $\mathbb{K}$.

Proof. See [40]. □

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I, Muhammad YUSSOUF, declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

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