DISSERTATION[1]

Statistical Properties of Sequences: q-Analysis and Distribution Functions

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Ich erkläre, dass ich diese Arbeit selbst verfasst, alle verwendeten Quellen zitiert und mich keiner unerlaubten Hilfsmittel bedient habe.

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Introduction

In the present thesis we investigate certain statistical distributions and their limit laws and statistical properties of sequences of real numbers. In particular we deal with certain q-analogues of the binomial distribution, with sequences of real numbers which are uniformly distributed modulo 1, and with the relationships between the limit distribution functions of a given sequence of real numbers and the limit distribution functions of the block sequence associated to the given sequence. We proceed as follows.

In Chapter 1 we establish limit theorems for certain q-binomial distributions. For the classical binomial distribution it is very well known that this distribution converges to the Poisson distribution (if we fix the mean) and to the normal distribution. Arising from the study of basic hypergeometric series, i.e., q-analogues of hypergeometric series that converge to the classical hypergeometric series as $q \to 1$, many q-analogues of the binomial distribution were introduced. Basic hypergeometric series have been studied since the 18th century (the starting point was Euler's investigation of the generating function of the number of partitions of a positive integer n into positive integers in 1748), but they are still an active field of research today due to many recent publications on orthogonal q-polynomials and q-distributions related to these series. The q-calculus has a wide range of applications, especially in combinatorics, number theory, approximation theory and computer science, but also in physics and biology. For instance, a q-analogue of the exponential function is the generating function of the number of partitions of a positive integer, the q-binomial coefficient counts the number of k-dimensional subspaces of an n-dimensional vector space over a field with q elements and counts the number of lattice paths in the plane from the origin to a given point taking the area below the path into account. Moreover, basic hypergeometric techniques can be used to prove that the number of different representations of an integer n as sum of two squares equals four times the difference between the number of positive divisors of n congruent 1 modulo 4 and congruent 3 modulo 4. In approximation theory, the study of the q-Bernstein operator has become very attractive.
But applications of the q-calculus are not restricted only to mathematics, in particular some q-distributions appear in models of specific processes in physics, biology and mathematical economy: for example, Kemp's q-binomial distribution can be used to describe the dichotomy between parasites on hosts with and without open wounds. Consider a large population of fish and a population of parasites, say leeches. A leech slits an opening in the skin of the fish, consumes blood, remains passive for a while before seeking a new site. Under some additional assumptions the stationary distribution of the number of those leeches which are located on a fish parasited for the first time is Kemp-q-binomially distributed.
The q-deformed binomial distribution arose from the study of the q-quantum harmonic oscillator in physics, but it became very attractive to mathematicians since it appears as kernel of the q-Bernstein polynomials, which are important in the approximation theory. The Euler distribution (a q-analogue of the Poisson distribution) was introduced as a q-Poisson energy distribution in the theory of the quantum harmonic oscillator. Moreover, the Euler distribution and a second analogue of the Poisson distribution, the Heine distribution, were found to be feasible prior

distributions for the number of undiscovered sources of oil.
Our goal is to extend the convergence results of the classical binomial distribution to different q-analogues, in particular we investigate besides the two q-binomial-distributions mentioned above the Rogers-Szegö- and the Stieltjes-Wigert distribution. In Section 1.1 we give an introduction to the q-calculus and the q-distributions under consideration. Afterwards we study in Sections 1.3–1.5 the various q-binomial distributions.

Chapter 2 is devoted to the study of the distribution of sequences of real numbers. Section 2.1 deals with a classical topic in number theory, namely with uniformly distributed sequences. Multidimensional extensions of such sequences play an important role in Quasi-Monte Carlo integration. Evaluating a high-dimensional integral is very extensive, so a fruitful alternative is the following one: We choose randomly points x_i and approximate the integral by the arithmetic mean of the values of the integrand evaluated at the points x_i. The accuracy of this method depends on the behaviour of the points x_i, i.e., on the discrepancy of the sequence formed by these points. In practice, one chooses deterministically constructed Quasi-Monte Carlo points instead of random points. Such high-dimensional integrals occur e.g. in financial modelling: An insurance with a process of claims S_t (this is modelled by a sum of independently identically distributed variables) wants to pay a dividend to its shareholders or its clients, and decides to proceed in the following way: Dividends are paid whenever the free reserve of the insurance company reaches a given barrier. The problems of computing the expectation of the amount of dividends that are paid invokes a high-dimensional integral and this can be solved by the ideas described above.
In the present thesis we consider the following problem: It is well known that the sequence $(\{n\alpha\})_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 for all irrational α and more generally, those α such that the sequence $(\{n_k\alpha\})_{k\in\mathbb{N}}$ is uniformly distributed form a set of Lebesgue measure 1 if (n_k) is a sequence of distinct integers. On the other hand, Goldstern et al. showed that in the sense of Baire this set is very small provided the sequence (n_k) grows fast enough. More precisely, they showed that the set of those α such that $(n_k\alpha)$ is uniformly distributed modulo 1 is of first category if (n_k) grows exponentially. We establish generalisations of this and related results to different types of multisequences, for example to sequences in $\mathbb{R}^d$ and in particular to sequences with multidimensional indices. In the latter case we study three different concepts of uniform distribution.
In Section 2.2 we consider the following setup: Given a sequence of real numbers (x_n) in the interval $[0, 1)$ we can associate to this sequence in a very natural way a sequence of step distribution functions F_n. Moreover, we can divide our original sequence into blocks of increasing length and associate to each block a step distribution function G_n. We are now interested in the relationship between the accumulation points of the sequences (F_n) and (G_n). Indeed, it is possible to construct the accumulation points of the sequence (F_n) from the accumulation points of the sequence (G_n) by taking certain convex combinations.

Graz, April 2010 Martin Zeiner

Contents

Chapter 1

Limit Theorems for certain q-Binomial Distributions

The aim of this chapter is to study sequences of random variables X_n which are q-binomially distributed. In fact, there are many q-binomial distributions related to basic hypergeometric series, but we will focus on Kemp's q-binomial distribution, the q-deformed binomial distribution, the Rogers-Szegö distribution and the Stieltjes-Wigert distribution. In particular we are interested in analogues of the convergence of the classical binomial distribution to the Poisson and the normal distribution. We proceed as follows: In Section 1.1 we give an elementary introduction to the q-calculus including the q-factorial, q-Pochhammer symbol, q-binomial coefficient, basic hypergeometric series and q-exponential functions. Moreover, we give some analogues of classical orthogonal polynomials. Section 1.2 contains definitions and properties of the q-distributions under consideration. In Sections 1.3–1.5 we investigate sequences X_n of q-binomially distributed random variables. We start with Kemp's q-binomial distribution, where we establish analogues of the convergence of the binomial distribution with constant mean to the Poisson distribution and of the convergence to the normal distribution. More precise, we show that these limits are either Heine or discrete normal, depending on the choice of the parameters. Section 1.4 is devoted to the study of the q-deformed binomial distribution. Again, an analogue of the convergence of the binomial distribution to the Poisson distribution holds. The limit laws are the Heine distribution and (truncated) exponential distributions. To study the limits of the Rogers-Szegö and the Stieltjes-Wigert distribution we introduce in Section 1.5 a parameter family of q-binomial distributions which contains besides the Rogers-Szegö- and the Stieltjes-Wigert the Kemp distribution too. We will show that the limit relations obtained for the latter distribution extend to this family.

1.1 The q-calculus

As mentioned above, the present section will give an introduction to the q-calculus. We define analogues of the classical factorial, the binomial coefficient, the hypergeometric series, the binomial theorem and deduce two analogues of the exponential function. Afterwards we define some q-orthogonal polynomials which are closely related to the q-distributions we will investigate in this chapter.

1.1.1 Basic definitions and relations

Throughout this chapter we use the notation of Gasper and Rahman [21]. For $q \neq 1$ the q-number $[z]_q$ of any complex number z is defined as

$$[z]_q := \frac{1-q^z}{1-q}.$$

For any nonnegative integer n we introduce the q-shifted factorial $(z;q)_n$ and the q-factorial $[n]_q!$ by

$$(z;q)_n := \prod_{i=0}^{n-1}(1-zq^i) \qquad \text{and} \qquad [n]_q! := \prod_{i=1}^{n}[i]_q.$$

The q-binomial coefficients (or Gaussian polynomials) are - similar to the classical binomial coefficient - defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

These coefficients are indeed polynomials in q, which can be easily seen using the recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q q^k + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q. \tag{1.1}$$

For the q-shifted factorial the following inversion formula holds:

$$(z;q)_n = (z^{-1};q^{-1})_n(-z)^n q^{\binom{n}{2}}. \tag{1.2}$$

Moreover, the following expansion is valid:

$$(z;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-z)^k q^{k(k-1)/2}.$$

Now we define the q-analogue of hypergeometric series:

$${}_r\phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n(b_1;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n.$$

If one of the a_i is of the form q^{-m}, $m \in \mathbb{N}$, then this series terminates. Whenever we deal with nonterminating basic hypergeometric series, we will assume that $|q| < 1$. If $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$. Moreover, this series converges absolutely too if $|q| > 1$ and $|z| < |b_1 \cdots b_s|/|a_1 \cdots a_r|$. A nonterminating series diverges for $z \neq 0$ if $0 < |q| < 1$ and $r > s+1$, and if $|q| > 1$ and $|z| < |b_1 \cdots b_s|/|a_1 \cdots a_r|$.

Note that the basic hypergeometric series has the property that if we replace z by z/a_r and let $a_r \to \infty$, then the resulting series is of the same form with r replaced by $(r-1)$.

For ${}_1\phi_0$-series we have the following representation as products

$${}_1\phi_0(a;-;q,z) = \sum_{n=}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1, \ |q| < 1. \tag{1.3}$$

This identity is a q-analogue of the binomial theorem and was first derived by Cauchy (1843) and Heine (1847). To see this, let us set

$$h_a(z) := \sum_{n=}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \quad |z| < 1, \ |q| < 1$$

and compute the difference

$$\begin{aligned}
h_a(z) - h_{aq}(z) &= \sum_{n=1}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n \\
&= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} \left(1 - a - (1 - aq^n)\right) z^n \\
&= -a \sum_{n=1}^{\infty} \frac{(1-q^n)(aq;q)_{n-1}}{(q;q)_n} z^n \\
&= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_{n-1}} z^n = -az h_{aq}(z),
\end{aligned}$$

which gives

$$h_{aq}(z) = h_a(z) \frac{1}{1-az}. \tag{1.4}$$

Moreover, we compute the difference

$$\begin{aligned}
h_a(z) - h_a(qz) &= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n} (z^n - q^n z^n) \\
&= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_{n-1}} z^n = \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}}{(q;q)_n} z^{n+1} \\
&= (1-a) z h_{aq}(z).
\end{aligned}$$

Using (1.4) this yields

$$h_a(z) = \frac{1-az}{1-z} h_a(qz).$$

Iterating this relation $(n-1)$ times and then letting $n \to \infty$ we obtain

$$\begin{aligned}
h_a(z) &= \frac{(az;q)_n)}{(z;q)_n} h_a(qz) \\
&= \frac{(az;q)_\infty}{(z;q)_\infty} h_a(0) = \frac{(az;q)_\infty}{(z;q)_\infty},
\end{aligned}$$

since $q^n \to 0$ and $h_a(0) = 1$, which completes the proof of (1.3).

Now we define two q-analogues of the exponential function. Setting $a = 0$ in (1.3) we get

$$e_q(z) := {}_1\phi_0(0; -; q, z) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1.$$

Since the product gives an analytic continuation of the function defined by the basic hypergeometric series to $\mathbb{C} \setminus \{q^{-i} : \ i = 0, 1, 2, \dots\}$, we will always have this in mind when writing $e_q(z)$. The second q-analogue can be obtained from (1.3) by replacing z by $-z/a$ and then letting $a \to \infty$:

$${}_1\phi_0(q; -, q; -z/a) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (-1)^n \frac{z^n}{a^n} = \frac{(-z;q)_\infty}{(-z/a;q)_\infty}.$$

Taking the limit we obtain (since $(-z/a;q)_\infty \to 1$ and by the remark above)

$$E_q(z) := {}_0\phi_0(-;-;q,-z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = (-z;q)_\infty. \tag{1.5}$$

Obviously, we have $e_q(z)E_q(-z) = 1$. As $q \to 1$, $e_q((1-q)z) \to e^z$ and $E_q((1-q)z) \to e^z$.

For applications of the q-calculus (in particular to the enumeration of integer partitions) we refer to Andrews [3, 4], Andrews and Eriksson [6] and Gasper and Rahman [21].

1.1.2 q-orthogonal polynomials

In the following we give the definition and basic properties of three families of q-orthogonal polynomials which are closely related to the q-distributions we are interested in. For a general theory to orthogonal polynomials see Szegö [57], for more information about hypergeometric orthogonal polynomials and their q-analogues and for details of the following polynomials we refer to the encyclopedic report by Koekoek and Swarttouw [42] and the references therein.

q-Krawtchouk polynomials

The q-Krawtchouk polynomials are given by

$$\begin{aligned} K_n(q^{-x};p,N;q) &= {}_3\phi_2(q^{-n},q^{-x},-pq^n;q^{-N},0;q,q) \\ &= \frac{(q^{x-N};q)_n}{(q^{-N};q)_n q^{nx}} {}_2\phi_1(q^{-n},q^{-x};q^{N-x-n+1};q,-pq^{n+N+1}), \quad n = 0,\dots,N. \end{aligned}$$

They fulfill the recurrence relation

$$-(1-q^{-x})K_n(q^{-x}) = A_n K_{n+1}(q^{-x}) - (A_n + C_n)K_n(q^{-x}) + C_n K_{n-1}(q^{-x}),$$

where

$$K_n(q^{-x}) := K_n(q^{-x};p,N;q)$$

and

$$\begin{cases} A_n = \frac{(1-q^{n-N})(1+pq^n)}{(1+pq^{2n})(1+pq^{2n+1})} \\ C_n = -pq^{2n-N-1}\frac{(1+pq^{n+N})(1-q^n)}{(1+pq^{2n-1})(1+pq^{2n})} \end{cases}.$$

Moreover, the following orthogonality relation holds:

$$\begin{aligned} &\sum_{x=0}^{N} \frac{(q^{-N};q)_x}{(q;q)_x}(-p)^{-x} K_m(q^{-x};p,N;q)K_n(q^{-x};p,N;q) \\ &= \frac{(q;q)_n(-pq^{N+1};q)_n}{(-p;q)_n(q^{-N};q)_n} \frac{1+p}{1+pq^{2n}} (-pq;q)_N p^{-N} q^{-(N+1)N/2} \left(-pq^{-N}\right)^n q^{n^2} \delta_{mn}, \quad p > 0. \end{aligned}$$

Indeed, the weight function in the above relation is Kemp's q-binomial distribution (see Section 1.2.1).

q-Charlier polynomials

The q-Charlier polynomials are defined as

$$\begin{aligned} C_n(q^{-x};a;q) &= {}_2\phi_1\left(q^{-n},q^{-x};0;q,-\frac{q^{n+1}}{a}\right) \\ &= (-a^{-1}q;q)_n \cdot {}_1\phi_1\left(q^{-n};-a^{-1}q;q,-\frac{q^{n+1-x}}{a}\right). \end{aligned}$$

We have the recurrence relation (with $C_n(q^{-x}) := C_n(q^{-x};a;q)$)

$$\begin{aligned} q^{2n+1}(1-q^{-x})C_n(q^{-x}) = aC_{n+1}(q^{-x}) &- \left(a+q(1-q^n)(a+q^n)\right)C_n(q^{-x}) \\ &+ q(1-q^n)(a+q^n)C_{n-1}(q^{-x}). \end{aligned}$$

The orthogonality relation for these polynomials is

$$\sum_{x=0}^{\infty} \frac{a^x}{(q;q)_x} q^{x(x-1)/2} C_m(q^{-x})C_n(q^{-x}) = q^{-n}(-a;q)_\infty(-a^{-1}q;q)_n(q;q)_n\delta_{mn}, \quad a>0.$$

The weights in this relation are the probabilities of the Heine distribution (Section 1.2.2).

Stieltjes-Wigert polynomials

The Stieltjes-Wigert polynomials are defined as

$$S_n(x;q) = \frac{1}{(q;q)_n}\,{}_1\phi_1(q^{-n};0;q,-q^{n+1}x)$$

and fulfill the recurrence relation

$$-q^{2n+1}xS_n(x;q) = (1-q^{n+1})S_{n+1}(x;q) - (1+q-q^{n+1})S_n(x;q) + qS_{n-1}(x;q).$$

They are orthogonal with respect to the discrete normal distribution (see [13]). The Stieltjes-Wigert polynomials are the probability generating function of the Stieltjes-Wigert distribution (Section 1.2.1).

1.2 q-Distributions

In this section we present some q-analogues of classical discrete probability distributions, in particular analogues of the binomial distribution and the Poisson distribution. The distributions we are interested in are the Kemp q-binomial distribution, the q-deformed binomial distribution, the Rogers-Szegö and the Stieltjes-Wigert distribution as analogues of the binomial distribution and the Euler and the Heine distribution as analogues of the Poisson distribution. Indeed, there exist much more analogues of the binomial distribution and of classical discrete distributions in general, but we won't need them in the following, so we don't discuss them here and refer to the relevant literature: An overview of q-distributions can be found in Johnson, Kemp and Kotz [34] and in Kupershmidt [44]. By replacing the exponential function by its q-analogue $e_q(z)$, Li and Kai [45] found analogues of continuous distribution functions.

Before we turn to the q-distributions, we want to recall a few important definitions and concepts of the classical probability theory. For a random variable X defined on the natural numbers with $p_n := \mathbb{P}(X = n)$ the rth moment of X is given by

$$\mathbb{E}(X^r) = \sum_{n=0}^{\infty} n^r p_n.$$

The mean and the variance of X are $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathbb{E}(X-\mu)^2 = \mathbb{E}X^2 - \mu^2$ respectively. The factorial moment $m_{(k)}$ is defined as $m_{(k)} := \mathbb{E}(X(X-1)\cdots(X-k+1))$. The binomial distribution $B(n,p)$ is given by $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$. Moreover, we have $\mu = np$, $\sigma^2 = np(1-p)$ and $m_{(k)} = n(n-1)\cdots(n-k+1)p^k$. For the Poisson distribution $P(\lambda)$ we have $\mathbb{P}(X = n) = \lambda^n/n! e^{-\lambda}$ and $\mu = \sigma^2 = \lambda$ and $m_{(k)} = \lambda^k$. The discrete normal distribution is defined by

$$\mathbb{P}(X = x) = \frac{q^{-x\alpha} q^{x^2/2}}{\sum_{k=-\infty}^{\infty} q^{-k\alpha} q^{k^2/2}} \quad \alpha \in \mathbb{R}, \quad x \in \mathbb{Z}, \quad 0 < q < 1.$$

As an analogue to the classical (factorial) moments one can consider the q-(factorial)-moments

$$\mathbb{E}([X]^r) = \sum_{n=0}^{\infty} [n]_q^r p_n \qquad \text{and} \qquad \mathbb{E}([X]_{r,q}) = \sum_{n=r}^{\infty} \frac{[n]_q!}{[n-r]_q!} p_n,$$

respectively. Studying these moments instead of the classical ones often leads to very simple formulas (see below). For the relationship between factorial moments and q-factorial moments we refer to Charalambides and Papadatos [12].

1.2.1 q-binomial distributions

Kemp's q-binomial distribution

The first q-analogue of the binomial distribution we consider is Kemp's q-binomial distribution $KB(n, \theta, q)$ defined by

$$\mathbb{P}(X_{KB} = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \frac{\theta^x q^{x(x-1)/2}}{(-\theta; q)_n}, \qquad 0 \le x \le n,\ 0 < \theta,\ 0 < q < 1. \tag{1.6}$$

It was introduced by Kemp and Kemp [35]. In the following, we sum up some properties of this distribution. Details can be found in Charalambides and Papadatos [12], Jing [32], Johnson, Kemp, and Kotz [34], Kemp [38, 39], Kemp and Kemp [35] and Kemp and Newton [40].

In the limit $q \to 1$, the Kemp distribution $KB(n, \theta, q)$ converges to a binomial distribution:

$$KB(n, \theta, q) \to B\left(n, \frac{\theta}{1+\theta}\right).$$

For $n \to \infty$ we obtain a Heine distribution $H(\theta)$ (see Section 1.2.2). Kemp's q binomial distribution is log-concave, i.e.,

$$\frac{\mathbb{P}(X = x+1)}{\mathbb{P}(X = x)} > \frac{\mathbb{P}(X = x+1)}{\mathbb{P}(X = x+1)}, \quad x = 0, \ldots, n-2,$$

and thus unimodal. Using basic hypergeometric series its probability generating function can be written as

$$G_{KB}(z) = \frac{{}_1\phi_0(0; -; q, -q^n \theta z)}{{}_1\phi_0(0; -; q, -q^n \theta)}.$$

Using (1.3) we can rewrite this as

$$G_{KB}(z) = \prod_{j=0}^{n-1} \frac{1+\theta q^j z}{1+\theta q^j}.$$

This formula immediately implies that the random variable $X \sim KB(n, \theta, q)$ can be represented as the sum of n independent Bernoulli trials X_i with probability of success equals $\frac{\theta q^i}{1+\theta q^i}$, which leads to the expressions

$$\mu = \sum_{i=0}^{n-1} \frac{\theta q^i}{1+\theta q^i} \qquad \text{and} \qquad \sigma^2 = \sum_{i=0}^{n-1} \frac{\theta q^i}{(1+\theta q^i)^2} \tag{1.7}$$

for the mean and the variance. Furthermore, the random variable $n - X_{KB}$ has the law $KB(n, \theta^{-1}q^{1-n}, q)$.

Kemp and Newton [40] gave an application of this distribution in biology. It can be used to describe the dichotomy between parasites on hosts with and without open wounds. Consider a population of fish and a population of parasites (e.g. ectoparasitic leeches) of fixed size N. A leech slits an opening in the skin of the fish, consumes a large amount of blood, and remains passive for a while before actively seeking a new site, either on the same fish or on another fish never previously parasited. If a fish which has never previously been parasited is available then one of the active leeches will transfer to it instead of relocating on its existing host. For simplicity we may assume that no fish has more than one leech attached to it. The leeches located on fish being parasited for the first time we call type A leeches, the leeches on fish with open wounds type B leeches. Let the probability that a leech is passive be given by q; consequently the probability that it is active is $(1-q)$. Moreover, assume that a active leech is able to move to a fish never previously parasited with probability $\rho/(1+\rho)$. Thus, given x leeches of type A, the birth- and death-rates for type A leeches are proportional to $q^x(1-q^{N-x})\rho/(1+\rho)$ (all type A leeches must be passive, there must be at least on active type B leech and a fish never previously parasited must be available) and $(1-q^x)/(1+\rho)$ (there has to be at least one active type A leech, but no fish without wound is free). Hence the stationary distribution of the number of type A leeches has the law $KB(N, \rho, q)$.

The q-polynomials which are orthogonal with respect to this distribution are the q-Krawtchouk polynomials (see Section 1.1.2).

The q-factorial moments are [12]

$$E([X_n]_{r,q}) = \frac{[n]_{r,q} q^{r(r-1)/2} \theta^r}{(-\theta; q)_r}.$$

We note in passing that Kemp [39] deduced the following characterisation result for the KB distribution from a theorem of Rao and Shanbhag [54]: The distribution given by (1.6) with $\theta = \lambda/\mu$ is the distribution of $U|(U+V=n)$, where U and V are independent iff U and V have a Heine distribution and an Euler distribution with parameters λ and μ, respectively (for the Heine and Euler distribution see Section 1.2.2).

The q-deformed binomial distribution

Another q-analogue of the binomial distribution is the q-deformed binomial distribution $QD(n, \tau, q)$, which was introduced by Jing [32] in connection with the q-deformed boson oscillator and by

Chung et al. [14]. Its probabilities are given by

$$\mathbb{P}(X_{QD} = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \tau^x (\tau; q)_{n-x}, \qquad 0 \le x \le n,\ 0 \le \tau \le 1,\ 0 < q < 1, \tag{1.8}$$

This distribution was studied by many authors and has applications in physics as well as in approximation theory due to the q-Bernstein polynomials and the q-Bernstein operator.

In the limit $q \to 1$ the q-deformed binomial distribution with parameter (n, τ, q) reduces to the binomial distribution with parameters (n, τ). The limit $n \to \infty$ of random variables $X_n \sim QD(n, \tau, q)$ leads to an Euler distribution with parameter $\lambda = \tau$. If we denote the probabilities (1.8) by $p_n(x, \tau)$, then the following recurrence relation holds (see Videnskii [59, Section 3]):

$$p_n(x, \tau) = \tau p_{n-1}(x - 1, \tau) + (1 - \tau) p_{n-1}(x, q\tau). \tag{1.9}$$

For $\tau \le q$ this distribution is logconcave and hence unimodal. Similar to the characterisation of Kemp's q-binomial distribution, Kemp [39] characterised the q-deformed binomial distribution as the distribution of $U|(U + V = n)$, where U is an Euler variable with parameters $(\eta\tau, q)$ (see Section 1.2.2) and V is an independent q-negative binomial variable with parameters (η, τ, q) (for details see [39]). For this distribution the q-mean and the q-variance are given by the simple formulas

$$\mathbb{E}([X]) = [n]_q \tau \qquad \text{and} \qquad \mathbb{E}\left([X - \mathbb{E}([X])]^2\right) = [n]_q \tau (1 - \tau).$$

For details we refer to Jing [32], Jing and Fan [33], Kemp [38, 39], the encyclopedic book Johnson, Kemp and Kotz [34], and to Charalambides [11]. Chung et al. [14], Kupershmidt [44] and Il'inski [28] gave representations of the q-deformed binomial distribution as a sum of dependent and not identically distributed random variables, for example, according to [14] we choose a sequence of n Bernoulli X_i trials starting with $\mathbb{P}(X_1 = 1) = \tau$ and $\mathbb{P}(X_1 = 0) = 1 - \tau$. The following Bernoulli variables are given by $\mathbb{P}(X_i = 1|X_{i-1} = 1) = \mathbb{P}(X_{i-1} = 1)$ and $\mathbb{P}(X_i = 1|X_{i-1} = 0) = q\mathbb{P}(X_{i-1} = 1)$. Then the sum of these n Bernoulli trials has the law $QD(n, \tau, q)$.

As mentioned above the q-deformed binomial distribution and the Euler distribution appear in particular both in physics ([10, 14, 32, 33]) and in approximation theory. The q-Bernstein polynomials of order n are defined by

$$B_n(f(t), q; x) = \sum_{r=0}^{n} f\left(\frac{[r]_q}{[n]_q}\right) \begin{bmatrix} n \\ r \end{bmatrix}_q x^r (x; q)_{n-r},$$

where f is a continuous function on the interval $[0, 1]$. There exists a vast literature on these polynomials, closely related to the distributions under consideration are e.g. [11, 29, 47, 50, 51, 59].

The Rogers-Szegö-distribution

Another q-analogue of the binomial distribution introduced by Kemp [38] is the Rogers-Szegö-distribution (RS) which probabilities are given by

$$\mathbb{P}(X_{RS} = x) = C_{RS} \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x, \qquad 0 \le x \le n,\ 0 < \theta,$$

where C_{RS} is a normalising constant. For $q \to 1$ this distribution tends to a binomial distribution with parameter $\frac{\theta}{1+\theta}$. The probability generating function is

$$G_{RS}(z) = \frac{{}_2\phi_0(q^{-n}, 0; -; q, q^n\theta z)}{{}_2\phi_0(q^{-n}, 0; -; q, q^n\theta)};$$

this is a Rogers-Szegö-polynomial (see Andrews [3], Ismail [30]). Reversing this distribution gives a distribution of the same form. Moreover, it is logconcave and strongly unimodal. The RS-distribution can be characterised as the distribution of $U|(U+V=n)$, where U and V are both Euler random variables. In the limit $n \to \infty$ the RS-distribution converges for $\theta < 1$ to an Euler distribution with parameter θ.

The Stieltjes-Wigert-distribution

The Stieltjes-Wigert-distribution $SW(n,\theta,q)$ (also introduced in Kemp [38]) is defined as

$$\mathbb{P}(X_{SW} = x) = C_{SW} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{x(x-1)}\theta^x, \qquad 0 \le x \le n,\ 0 < \theta,$$

where C_{SW} is a normalising constant. It is a q-analogue of the binomial distribution since $SW(n,\theta,q) \to B(n,\theta/(1+\theta)$ as $q \to 1$. The name Stieltjes-Wigert is appropriate since the probability generating function

$$G_{SW}(z) = \frac{{}_1\phi_1(q^{-n}; 0; q, q^n\theta z)}{{}_1\phi_1(q^{-n}; 0; q, q^n\theta)}$$

is a Stieltjes-Wigert-polynomial (see Section 1.1.2). As above, reversing the SW-distribution does not change the nature of the distribution. Moreover, it is logconcave and strongly unimodal, too. Choosing both U and V as Heine random variables leads to a SW-distribution as the law of $U|(U+V=n)$.

1.2.2 q-Poisson distributions

In this section we present two q-analogues of the Poisson distribution, namely the Euler and the Heine distribution. We note in passing, that a third q-analogue, the pseudo-Euler distribution, was introduced in [36]. The Euler distribution $E(\lambda, q)$ with parameter λ is defined by

$$\mathbb{P}(X_E = x) = \frac{\lambda^x}{(q;q)_x}(\lambda;q)_\infty = \frac{\lambda^x}{(q;q)_x}E_q(-\lambda), \qquad 0 < q < 1,\ 0 < \lambda < 1,$$

and was introduced by Biedenhahn [10] as a q-Poisson energy distribution in the theory of the quantum harmonic oscillator, and by Benkherouf and Bather [9] as a feasible prior distribution for the number of undiscovered sources of oil. The probability generating function equals

$$G_E(z) = \frac{{}_1\phi_0(0; -; q; \lambda z)}{{}_1\phi_0(0; -; q; \lambda)} = \prod_{j=0}^{\infty} \frac{1-\lambda q^j}{1-\lambda q^j z};$$

here we used (1.3). Moreover, we have

$$\mu = \sum_{x=0}^{\infty} \frac{\lambda q^x}{1-\lambda q^x} \qquad \text{and} \qquad \sigma^2 = \sum_{x=0}^{\infty} \frac{\lambda q^x}{(1-\lambda q^x)^2}.$$

The q-factorial moments are

$$E([X]_{r,q}) = \lambda^r.$$

The Poisson family of distribution is characterised by the mean-variance equality; Charalambides and Papadatos [12] obtained an analogous characterisation for this distribution. As limit of the q-deformed binomial distribution, it plays an important role in approximation theory.

The probabilities of the Heine distribution $H(\theta)$ (introduced by [9] too) are given by

$$\mathbb{P}(X_H = x) = \frac{q^{x(x-1)/2}\theta^x}{(q;q)_x} e_q(-\theta), \qquad x \geq 0,\ 0 < q < 1,\ 0 \leq \theta.$$

For the Heine distribution we have

$$G_H(z) = \frac{{}_0\phi_0(-;-;q,-\theta z)}{{}_0\phi_0(-;-;q,-\theta)} = \prod_{j=0}^{\infty} \frac{1+\theta q^j z}{1+\theta q^j}$$

by (1.5) and

$$\mu = \sum_{x=0}^{\infty} \frac{\theta q^x}{1+\theta q^x} \qquad \text{and} \qquad \sigma^2 = \sum_{x=0}^{\infty} \frac{\theta q^x}{(1+\theta q^x)^2}.$$

The q-mean equals $p/(1-q)$ and the q-variance equals $p(1-p)/(1-q)$. Additionally, the q-factorial moments are

$$E([X]_{r,q}) = \frac{q^{r(r-1)/2}\theta^r}{(-\theta(1-q);q)_r}.$$

The q-polynomials orthogonal with respect to this distribution are the q-Charlier polynomials (see Section 1.1.2).

As $q \to 1$ we have $E((1-q)\lambda, q) \to P(\lambda)$ for $q \to 1$, $H((1-q)\theta) \to P(\theta)$, where $P(\theta)$ denotes the Poisson distribution with parameter θ. Both the Heine and Euler distribution are unimodal. The Euler distribution is infinitely divisible, whereas the Heine distribution is not.

For details, further properties and applications of these distributions we refer to Johnson, Kemp and Kotz [34], Benkherouf and Bather [9], Biedenharn [10], Charalambides and Papadatos [12], Jing [32], Jing and Fan [33], Kemp [36, 37, 39], Kupershmidt [44] and Ostrovska [50, 51].

1.3 Kemp's q-binomial distribution

In this section we establish convergence properties of Kemp's q-binomial distribution (see Gerhold and Zeiner [22]). The main object of interest are sequences $(X_n)_{n\in\mathbb{N}}$ with $X_n \sim KB(n, \theta_n, q)$ ($\theta_n \geq 0$). As noted above, for fixed parameter sequences $\theta_n = \theta$ we obtain a Heine distribution $H(\theta)$ as the limit law of X_n. So we investigate sequences with non-constant parameter. We will start with the case of convergent parameter sequences in Section 1.3.1. Due to continuity arguments the limit is again Heine. Considering in particular sequences X_n with constant mean yields an q-analogue of the convergence of the classical binomial distribution with constant mean to the Poisson distribution. In Section 1.3.2 we treat parameter sequences $\theta_n \to \infty$. Here the limit law depends on the growth of θ_n. For fast growing parameter sequence we obtain - using the reversing property of Kemp's distribution - a Heine distribution. The main part of this section is dedicated to the study of slowly growing parameter sequences, i.e. sequences θ_n of the form $\theta_n = q^{-f(n)}$, where $f(n)$ and $(n - f(n))$ both tend to infinity, as $n \to \infty$. We will see that this leads to a discrete normal distribution. Moreover, we deduce from the convergence results of Kemp's q-binomial distribution convergence properties of (q)-orthogonal polynomials. The involved polynomials are the q-Krawtchouk, the q-Charlier, the Stieltjes-Wigert and the Krawtchouk polynomials.

1.3.1 Convergent Parameter

As noted above we consider in the present section sequences of random variables X_n with $X_n \sim X_{KB}(n, \theta_n(q), q)$ and will provide convergence results for sequences $\theta_n(q)$ which tend to a limit as $n \to \infty$. In particular this includes the case of sequences X_n with constant mean.

We will need the following simple continuity argument.

Lemma 1.3.1. *Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with limit $\theta \geq -1$. Then*

$$\lim_{n\to\infty} \prod_{i=0}^{n-1} \left(1 + \theta_n q^i\right) = E_q(\theta).$$

Proof: For small $\epsilon > 0$ and n large enough, we have

$$\prod_{i=0}^{n-1} \left(1 + (\theta - \epsilon)\, q^i\right) \leq \prod_{i=0}^{n-1} \left(1 + \theta_n q^i\right) \leq \prod_{i=0}^{n-1} \left(1 + (\theta + \epsilon)\, q^i\right),$$

hence

$$\begin{aligned} E_q(\theta - \epsilon) &= \lim_{n\to\infty} \prod_{i=0}^{n-1} \left(1 + (\theta - \epsilon)\, q^i\right) \leq \liminf_{n\to\infty} \prod_{i=0}^{n-1} \left(1 + \theta_n q^i\right) \\ &\leq \limsup_{n\to\infty} \prod_{i=0}^{n-1} \left(1 + \theta_n q^i\right) \leq \lim_{n\to\infty} \prod_{i=0}^{n-1} \left(1 + (\theta + \epsilon)\, q^i\right) \\ &= E_q(\theta + \epsilon). \end{aligned}$$

Since E_q is continuous, the lemma follows. ⌀

Now we can establish our first convergence result which is a mild generalisation of the convergence of Kemp's distribution with constant parameter to the Heine distribution.

Proposition 1.3.2. *Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with limit $\theta \geq 0$. Then the sequence of Kemp's q-binomial distributions $X_{KB}(n, \theta_n, q)$ converges for $n \to \infty$ to a Heine distribution $H(\theta)$.*

Proof. The proof is an easy application of Lemma 1.3.1:

$$\begin{aligned} \mathbb{P}(X_n = x) &= \begin{bmatrix} n \\ x \end{bmatrix}_q (\theta_n)^x \frac{q^{x(x-1)/2}}{\prod_{i=0}^{n-1} \left(1 + \theta_n q^i\right)} \\ &\to \frac{q^{x(x-1)/2} (\theta)^x}{(q;q)_x} e_q(-\theta). \end{aligned}$$

⌀

Example 1.3.3. Let λ be a real number with $0 < \lambda < n$, and put $\theta_n(q) = \lambda/[n - \lambda]_q$. Then the sequence of Kemp's q-binomial distributions $X_{KB}(n, \theta_n(q), q)$ converges for $n \to \infty$ to a Heine distribution $H((1-q)\lambda)$. Thus the following diagram is commutative:

$$\begin{array}{ccc} X_{KB}(n, \theta_n(q), q) & \xrightarrow{n\to\infty} & H((1-q)\lambda) \\ {\scriptstyle q\to 1}\downarrow & & \downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\lambda}{n}\right) & \xrightarrow[n\to\infty]{} & P(\lambda) \end{array}$$

Theorem 1.3.2 yields limit relations for orthogonal polynomials. As noted above, the orthogonal polynomials for Kemp's q-binomial, the Heine, and the binomial distribution are, respectively, the q-Krawtchouk, the q-Charlier, and the Krawtchouk polynomials.

Corollary 1.3.4.

(i) Let θ_n be as in the preceding Theorem. The q-Krawtchouk polynomial $K_k(q^{-x}; q^{-n}\theta_n^{-1}, n; q)$ converges for $n \to \infty$ to the q-Charlier polynomial $C_k(q^{-x}; \theta; q)$.

(ii) For the parameter sequence $\theta_n(q) = \lambda/[n-\lambda]q$, the polynomial $K_k(q^{-x}; q^{-n}\theta_n(q)^{-1}, n; q)$ converges to the Krawtchouk polynomial $K_k(x; \lambda/n, n)$, as $q \to 1$.

For our next result, we note the following elementary fact, which is an immediate consequence of [26, Lemma 1.1.1].

Lemma 1.3.5. *Let $f_n(x)$, $n \in \mathbb{N}$, be a sequence of continuous functions that are increasing in x, and suppose that for each n there is a unique solution x_n of $f_n(x) = 0$. Moreover, assume f_n converges pointwise to a limit f with a unique solution $\hat{x}$ of $f(x) = 0$. Then $(x_n)_{n\in\mathbb{N}}$ converges to $\hat{x}$.*

Our second convergence property is an analogue of the convergence of the classical binomial distribution with constant mean to the Poisson distribution.

Theorem 1.3.6. *Fix $\mu > 0$ and choose the parameter $\theta_n = \theta_n(\mu, q)$ of Kemp's q-binomial distribution such that $\mu_n = \mu$. Then we have*

(i) The sequence $KB(n, \theta_n, q)$ converges for $n \to \infty$ to a Heine distribution $H(\theta)$, where $\theta = \lim_{n\to\infty} \theta_n$.

(ii) For fixed n, $KB(n, \theta_n, q)$ tends to a binomial distribution $B\left(n, \frac{\mu}{n}\right)$ in the limit $q \to 1$.

(iii) For $q \to 1$, the Heine distribution $H(\theta)$ converges to a Poisson distribution with parameter μ.

So we obtain the following commutative diagram:

$$\begin{array}{ccc} KB(n, \theta_n(\mu, q), q) & \xrightarrow{n\to\infty} & H(\theta(\mu, q)) \\ {\scriptstyle q\to 1}\big\downarrow & & \big\downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\mu}{n}\right) & \xrightarrow[n\to\infty]{} & P(\mu) \end{array}$$

Proof: First we check that for given μ, q and large n, there is a unique θ_n such that $\mu_n(\theta_n, q) = \mu$. The function $\mu_n(\theta, q)$ is strictly increasing in θ and $\mu_n(0, q) = 0$. Since

$$\mu_n(q^{-n+1}, q) \geq \sum_{i=0}^{n-1} \frac{q^{i-n+1}}{2q^{i-n+1}} = \frac{n}{2}$$

and $\mu_n(\theta, q)$ is continuous in θ, there exists a unique solution θ_n of $\mu_n(\theta, q) = \mu$ for each $n \geq 2\mu$. Applying Lemma 1.3.5 shows that $\lim_{n\to\infty} \theta_n = \theta$, with θ the unique solution of $\mu_\infty(\theta, q) = \mu$. Thus $KB(n, \theta_n, q) \to H(\theta)$ by Proposition 1.3.2.

Again by Lemma 1.3.5 we get $\theta_n \to \frac{\mu}{n-\mu}$ for $q \to 1$. Hence $KB(n, \theta_n, q) \to B\left(n, \frac{\mu}{n}\right)$.

It remains to check that $\theta/(1-q)$ converges to μ for $q \to 1$, which yields $H(\theta) \to P(\mu)$. The value $\theta/(1-q)$ is the unique solution of $\mu_\infty((1-q)\theta, q) = \mu$. Moreover, $\lim_{q\to 1} \mu_\infty((1-q)\theta, q) = \theta$, because $H((1-q)\theta) \to P(\theta)$. Thus we can again apply Lemma 1.3.5 ❦

Analogously to Corollary 1.3.4, Theorem 1.3.6 implies the following result about (q-)Krawtchouk polynomials.

Corollary 1.3.7. *Let $\theta_n(q)$ and $\theta(q)$ be as in Theorem 1.3.6. For $q \to 1$, the q-Krawtchouk polynomial $K_k(q^{-x}; q^{-n}\theta_n(q)^{-1}, n; q)$ converges to the Krawtchouk polynomial $K_k(x; \mu/n, n)$.*

1.3.2 Increasing Parameter

After the study of sequences $X_n \sim KB(n, \theta_n, q)$ with convergent parameter sequence we turn to sequences $\theta_n \to \infty$, as $n \to \infty$. If we consider fast growing parameter sequences in the sense that $\theta_n = q^{-n-g(n)}$ with $g(n) \to \infty$ or convergent we obtain the corresponding limit distribution easily:

Corollary 1.3.8. *Let $X_n \sim KB(n, q^{-n-g(n)}, q)$.*

(i) If $g(n)$ converges to a limit g_0, then the distribution of $n - X_n$ tends to the Heine distribution $H(q^{1+g_0})$ as $n \to \infty$.

(ii) If $g(n) \to \infty$ for $n \to \infty$, then the distribution of $n - X_n$ tends to the point measure δ_0 as $n \to \infty$.

Proof: As remarked in Section 1.2.1, $n - X_n \sim KB(n, \tau, q)$ with $\tau = q^{g(n)+1}$. Applying Proposition 1.3.2 yields the result. ❦

It follows from the preceding corollary (i) that the q-Krawtchouk polynomials converge to the alternative q-Charlier polynomials, which is a known result [42, (4.15.1)].

In the following we consider parameter sequences $\theta_n = q^{-f(n)}$ with $f(n) \to \infty$ and $n - f(n) \to \infty$ for $n \to \infty$. These assumptions on $f(n)$ will be in force throughout the section. Theorems 1.3.10 and 1.3.11 and Lemmas 1.3.12–1.3.14 are devoted to the asymptotic behaviour of the sequence (μ_n) of means. As they tend to infinity, we will normalise our sequence of random variables to $(X_n - \mu_n)/\sigma_n$. Still, this sequence does not converge in distribution without further assumptions on $f(n)$. A fruitful way to proceed is to pick subsequences along which the fractional part $\{f(n)\}$ is constant. Theorem 1.3.15 shows that this induces convergence to discrete normal distributions.

To investigate the sequence of means, we begin by providing an elementary estimate for the variance.

Lemma 1.3.9. *If $\theta_n = q^{-f(n)}$ then the sequence of variances satisfies $\sigma_n^2 \le 2/(1-q)$.*

Proof: By (1.7), the variance σ_n^2 equals

$$\begin{aligned}
\sum_{i=0}^{n} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2} &= \sum_{i=0}^{\lfloor f(n)\rfloor} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2} + \sum_{i=\lfloor f(n)\rfloor+1}^{n} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2} \\
&= \sum_{i=0}^{\lfloor f(n)\rfloor} \frac{q^{-\{f(n)\}-i}}{(1+q^{-\{f(n)\}-i})^2} + \sum_{i=0}^{n-\lfloor f(n)\rfloor-1} \frac{q^{i+1-\{f(n)\}}}{(1+q^{i+1-\{f(n)\}})^2} \qquad (1.10)\\
&< \sum_{i=0}^{\infty} \frac{1}{q^{-\{f(n)\}-i}} + \sum_{i=0}^{\infty} q^{i+1-\{f(n)\}} \\
&\le \sum_{i=0}^{\infty} q^i + \sum_{i=0}^{\infty} q^i = \frac{2}{1-q}.
\end{aligned}$$

The following theorem is our first result about the sequence of means in the case of a slowly increasing parameter. It does not reveal the behaviour of the O(1)-term as clearly as Theorem 1.3.11, but will be useful later on (Lemma 1.3.13).

Theorem 1.3.10. *Let $X_n \sim KB(n, \theta_n, q)$ with $\theta_n = q^{-f(n)}$. Then, for $n \to \infty$,*

$$\mu_n = f(n) + c(\{f(n)\}, q) + \mathrm{o}(1), \qquad (1.11)$$

where

$$c(\{f(n)\}, q) := 1 - \frac{1}{1+q^{-\{f(n)\}}} - \{f(n)\} - \sum_{\ell\ge 0} \frac{1}{1+q^{-\ell-\{f(n)\}-1}} + \sum_{\ell\ge 0} \frac{1}{1+q^{-\ell+\{f(n)\}-1}} = \mathrm{O}(1).$$

Proof: We start from

$$\mu_n = \sum_{i=0}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} = \sum_{i=0}^{n-1} \frac{1}{1+q^{f(n)-i}} \qquad (1.12)$$

and split the sum into two parts (w.l.o.g. $f(n) < n$). Expanding the denominator as a geometric series and changing the order of summation yields

$$\begin{aligned}
\sum_{i=0}^{\lfloor f(n)\rfloor-1} \frac{1}{1+q^{f(n)-i}} &= \sum_{i=0}^{\lfloor f(n)\rfloor-1} \sum_{\ell\ge 0} (-1)^\ell q^{\ell(f(n)-i)} \\
&= \sum_{\ell\ge 0} (-1)^\ell q^{\ell f(n)} \sum_{i=0}^{\lfloor f(n)\rfloor-1} q^{-\ell i}.
\end{aligned}$$

For $\ell = 0$ we obtain $\lfloor f(n)\rfloor$, and evaluating the inner sum leads to

$$= \lfloor f(n)\rfloor + \sum_{\ell\ge 1} (-1)^\ell q^{\ell f(n)} \frac{1-q^{-\ell\lfloor f(n)\rfloor}}{1-q^{-\ell}}.$$

The first term of the fraction gives the O-Term:

$$= \lfloor f(n)\rfloor - \sum_{\ell\ge 1} \frac{(-1)^\ell q^{\ell\{f(n)\}}}{1-q^{-\ell}} + \mathrm{O}\left(q^{f(n)}\right),$$

an by expanding the fraction with $-q^\ell$ we get

$$= \lfloor f(n) \rfloor + \sum_{\ell \geq 1} \frac{q^\ell (-1)^\ell q^{\ell \{f(n)\}}}{1-q^\ell} + \mathrm{O}\left(q^{f(n)}\right).$$

The fraction can be written as a geometric series again, and by changing the order of summation and evaluating the inner sum we obtain

$$\begin{aligned}
&= \lfloor f(n) \rfloor + \sum_{\ell \geq 1} q^\ell (-1)^\ell q^{\ell \{f(n)\}} \sum_{j \geq 0} q^{\ell j} + \mathrm{O}\left(q^{f(n)}\right) \\
&= \lfloor f(n) \rfloor + \sum_{j \geq 0} \sum_{\ell \geq 1} \left(-q^{j+1+\{f(n)\}}\right)^\ell + \mathrm{O}\left(q^{f(n)}\right) \\
&= \lfloor f(n) \rfloor + \sum_{j \geq 0} \frac{-q^{j+1+\{f(n)\}}}{1+q^{j+1+\{f(n)\}}} + \mathrm{O}\left(q^{f(n)}\right).
\end{aligned}$$

Thus we have

$$\sum_{i=0}^{\lfloor f(n) \rfloor -1} \frac{1}{1+q^{f(n)-i}} = \lfloor f(n) \rfloor - \sum_{j \geq 0} \frac{1}{1+q^{-j-1-\{f(n)\}}} + \mathrm{O}\left(q^{f(n)}\right).$$

For the upper portion of the sum, we find

$$\begin{aligned}
\sum_{i=\lfloor f(n) \rfloor +1}^{n-1} \frac{1}{1+q^{f(n)-i}} &= \sum_{i=\lfloor f(n) \rfloor +1}^{\infty} \frac{1}{1+q^{f(n)-i}} + \mathrm{O}\left(q^{n-f(n)}\right) \\
&= \sum_{i=0}^{\infty} \frac{1}{1+q^{\{f(n)\}-i-1}} + \mathrm{O}\left(q^{n-f(n)}\right),
\end{aligned}$$

since

$$\sum_{i=n}^{\infty} \frac{1}{1+q^{f(n)-i}} = \sum_{i=0}^{\infty} \frac{1}{1+q^{f(n)-n-i}} \leq \sum_{i=0}^{\infty} \frac{1}{q^{f(n)-n-i}} = q^{n-f(n)} \frac{1}{1-q}.$$

Adding the term for $i = \lfloor f(n) \rfloor$ yields the lemma.

In the limit $q \to 1$, the term $c(\{f(n)\}, q)$ tends to $\frac{1}{2}$. To see this, apply the Euler-Maclaurin formula (see [5]) to

$$f(x) = \frac{1}{1+q^{-x-b}}$$

with $b > 0$, which yields

$$\sum_{\ell \geq 0} f(\ell) = \int_0^\infty f(x) \mathrm{d}x + \frac{f(0)}{2} + \frac{1}{12} f'(x)\Big|_{x=0}^{\infty} + R_2 \tag{1.13}$$

with

$$R_2 = -\frac{1}{2} \int_0^\infty \left(\{x\}^2 - \{x\} + \frac{1}{6} \right) f''(x) \mathrm{d}x.$$

Since

$$f''(x) = \frac{(\log q)^2 q^{x+b} \left(1 - q^{x+b}\right)}{\left(1+q^{x+b}\right)^3}$$

does not change sign, we have

$$|R_2| \leq \frac{1}{12}\int_0^\infty |f''(x)|\mathrm{d}x = \frac{1}{12}\int_0^\infty f''(x)\mathrm{d}x$$
$$= -(\log q)q^{-b}(1+q^{-b})^{-2} = \mathrm{o}(1), \qquad q \to 1.$$

The first integral in (1.13) is

$$\int_0^\infty f(x)\mathrm{d}x = -\frac{\log\left(1+q^b\right)}{\log q} = \frac{\log 2}{1-q} - \frac{\log 2 + b}{2} + \mathrm{O}\left(1-q\right), \qquad q \to 1.$$

So we have

$$\sum_{\ell \geq 0} f(\ell) = \frac{\log 2}{1-q} - \frac{\log 2 + b}{2} + \frac{1}{4} + \mathrm{o}(1), \qquad q \to 1.$$

Application to the sums appearing in $c(\{f(n)\}, q)$ gives

$$c(\{f(n)\}, q) = \frac{1}{2} - \{f(n)\} + \{f(n)\} + \mathrm{o}(1), \qquad q \to 1.$$

Note that for $q \to 1$ the error term in the representation (1.11) for μ_n increases. This is why the limits for $q \to 1$ and $n \to \infty$ can't be exchanged. Indeed, it is clear from (1.12) that μ_n tends to $n/2$ for $q \to 1$.

The following theorem represents the O(1)-term from Theorem 1.3.10 as a Fourier series, which shows that it is a $\frac{1}{2}$-periodic function of $f(n)$. Indeed, the stated asymptotic (with respect to n) follows directly from a general result given in [17], but we will also need the behaviour (w.r.t. q) of the error term, thus we do the following detailed analysis.

Theorem 1.3.11. *Let $X_n \sim KB(n, \theta_n, q)$ with $\theta_n = q^{-f(n)}$. Then, as $n \to \infty$,*

$$\mu_n = f(n) + \frac{1}{2} + \sum_{k>0} \frac{2\pi \sin(2kf(n)\pi)}{\log q \sinh\left(\frac{2k\pi^2}{\log q}\right)} + \mathrm{O}\left(q^{\min(f(n)/2, n-f(n))}\right). \tag{1.14}$$

Proof: We write

$$\mu_n = \sum_{i=0}^{n-1} \frac{1}{1+q^{f(n)-i}} = \sum_{i=0}^{\infty} \frac{1}{1+q^{f(n)-i}} + \mathrm{O}\left(q^{n-f(n)}\right)$$

and apply the Mellin transformation [17, 18] to

$$h(t) = \sum_{i=0}^{\infty} \frac{1}{1+tq^{-i}}.$$

By the linearity of the Mellin transformation $\mathcal{M}$ and the properties $\mathcal{M}\left(\frac{1}{1+t}\right) = \frac{\pi}{\sin \pi s}$ and $\mathcal{M}h(\alpha t)(s) = \alpha^{-s}\mathcal{M}(h)(s)$, we see that

$$\mathcal{M}(h)(s) = \sum_{i=0}^{\infty} \left(q^{-i}\right)^{-s} \frac{\pi}{\sin \pi s} = \frac{1}{1-q^s}\frac{\pi}{\sin \pi s}.$$

Exchanging $\mathcal{M}$ and the sum is permitted by the monotone convergence theorem. From the inverse transformation formula we get

$$h\left(q^{f(n)}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q^{-f(n)s} \frac{1}{1-q^s} \frac{\pi}{\sin \pi s} \mathrm{d}s \tag{1.15}$$

for $c \in (0,1)$. To evaluate this integral, we choose the integration contour $\gamma_k = \gamma_{k,1} \cup \gamma_{k,2} \cup \gamma_{k,3} \cup \gamma_{k,4}$ with

$$\begin{aligned}
\gamma_{k,1} &= \left\{ s \mid s = \frac{1}{2} + iv \ : \ -T_k \le v \le T_k \right\}, \\
\gamma_{k,2} &= \left\{ s \mid s = u + iT_k : \ -\frac{1}{2} \le u \le \frac{1}{2} \right\}, \\
\gamma_{k,3} &= \left\{ s \mid s = -\frac{1}{2} + iv \ : \ -T_k \le v \le T_k \right\}, \\
\gamma_{k,4} &= \left\{ s \mid s = u - iT_k : \ -\frac{1}{2} \le u \le \frac{1}{2} \right\},
\end{aligned}$$

where $T_k = \frac{2\pi}{\log q}\left(k + \frac{1}{4}\right)$. Then

$$h\left(q^{f(n)}\right) = \lim_{k\to\infty} \frac{1}{2\pi i} \int_{\gamma_{k,1}} = -\lim_{k\to\infty} \Big(\frac{1}{2\pi i} \int_{\gamma_{k,2}} + \frac{1}{2\pi i} \int_{\gamma_{k,3}} + \frac{1}{2\pi i} \int_{\gamma_{k,4}} + \sum \text{residues}\Big),$$

since the integral on the left side exists. Now we estimate the integrals on the right side.

$$\begin{aligned}
\left| \int_{\gamma_{k,3}} q^{-f(n)s} \frac{1}{1-q^s} \frac{\pi}{\sin \pi s} \mathrm{d}s \right| &= \left| \int_{-T_k}^{T_k} q^{-f(n)(-\frac{1}{2}+iv)} \frac{1}{1-q^{-\frac{1}{2}+iv}} \frac{\pi}{\sin(\pi(-\frac{1}{2}+iv))} \mathrm{d}v \right| \\
&\le \pi q^{\frac{f(n)}{2}} \int_{-\infty}^{\infty} \frac{1}{\left|1-q^{-\frac{1}{2}+iv}\right|} \frac{1}{\left|\sin(\pi(-\frac{1}{2}+iv))\right|} \mathrm{d}v \\
&\le \pi q^{\frac{f(n)}{2}} \frac{1}{1-q^{-\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\sin^2 \frac{\pi}{2} + \sinh^2 \pi v}} \mathrm{d}v \\
&= q^{\frac{f(n)}{2}} \frac{\pi}{1-q^{-\frac{1}{2}}}
\end{aligned}$$

$$\begin{aligned}
\left|\int\limits_{\gamma_{k,2}} q^{-f(n)s}\frac{1}{1-q^s}\frac{\pi}{\sin\pi s}\mathrm{d}s\right| &= \left|\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} q^{-f(n)(u+iT_k)}\frac{1}{1-q^{u+iT_k}}\frac{\pi}{\sin(\pi(u+iT_k))}\mathrm{d}u\right| \\
&\le \pi\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} q^{-f(n)u}\frac{1}{|1-q^{u+iT_k}|}\frac{1}{\sqrt{\sin^2\pi u+\sinh^2\pi T_k}}\mathrm{d}u \\
&\le \pi q^{-\frac{1}{2}f(n)}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{q^u\,|\sin(T_k\log q)|}\frac{1}{\sqrt{\sinh^2\pi T_k}}\mathrm{d}u \\
&\le \frac{\pi q^{-\frac{1}{2}f(n)}}{\sinh\pi T_k}\int\limits_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{q^u}\mathrm{d}u \overset{k\to\infty}{\longrightarrow} 0
\end{aligned}$$

The integral over $\gamma_{k,4}$ is treated similarly. Now let us compute the residues: $\frac{1}{1-q^s}$ has simple poles at $z_k := \frac{2\pi ik}{\log q}$, and $\frac{1}{\sin\pi s}$ has a simple pole at 0. First we consider the residue at z_k for $k \neq 0$:

$$\begin{aligned}
\lim_{z\to z_k}(z-z_k)q^{-f(n)z}\frac{1}{1-q^z}\frac{\pi}{\sin\pi z} &= q^{-f(n)\frac{2\pi ik}{\log q}}\frac{\pi}{\sin\left(\frac{2\pi ik}{\log q}\pi\right)}\lim_{z\to z_k}\frac{z-z_k}{1-q^z} \\
&= e^{-f(n)2\pi ik}\frac{\pi}{i\sinh\left(\frac{2\pi^2 k}{\log q}\right)}\frac{1}{-\log q}.
\end{aligned}$$

The sum extended over the residues at the poles z_k, $k \neq 0$, therefore equals

$$\sum_{k\neq 0}\frac{i\pi e^{-2if(n)k\pi}}{\log q\sinh\left(\frac{2k\pi^2}{\log q}\right)}.$$

Putting together the summands k and $-k$,

$$\begin{aligned}
e^{-2if(n)k\pi}-e^{2if(n)k\pi} &= \cos(-2if(n)k\pi)+i\sin(-2f(n)k\pi)-\cos(2f(n)k\pi)-i\sin(2f(n)k\pi) \\
&= -2i\sin(2f(n)k\pi),
\end{aligned}$$

we obtain

$$\sum_{k>0}\frac{2\pi\sin(2kf(n)\pi)}{\log q\sinh\left(\frac{2k\pi^2}{\log q}\right)}.$$

Finally, by the expansions

$$\begin{aligned}
q^{-f(n)s} &= 1-f(n)\log q\ s+\mathrm{O}(s^2) \\
\frac{1}{1-q^s} &= -\frac{1}{\log q\ s}+\frac{1}{2}+\mathrm{O}(s) \\
\frac{\pi}{\sin\pi s} &= \frac{1}{s}+\mathrm{O}(s),
\end{aligned}$$

the residue at $z_0 = 0$ is $f(n)+\frac{1}{2}$. ✐

It is worthwhile to evaluate the sum in (1.14) in the limit $q \to 0$. First note that, if n is fixed and $q \to 0$, then (1.12) easily yields

$$\mu_n \to \begin{cases} f(n) + 1 - \{f(n)\} & \text{if } \{f(n)\} > 0 \\ f(n) + \frac{1}{2} & \text{if } \{f(n)\} = 0 \end{cases}.$$

Moreover, the O-term in (1.14) is o(1) for $q \to 0$, as follows readily from the estimates in the proof of Theorem 1.3.11. These two facts combined imply

$$\lim_{q\to 0} \sum_{k>0} \frac{2\pi \sin(2kf(n)\pi)}{\log q \sinh\left(\frac{2k\pi^2}{\log q}\right)} = \begin{cases} \frac{1}{2} - \{f(n)\} & \text{if } \{f(n)\} > 0 \\ 0 & \text{if } \{f(n)\} = 0 \end{cases}.$$

Note that in the special case $f(n) = \alpha n$ with positive α, the summands tend to the summands of the Fourier series of $\frac{1}{2} - \{f(n)\}$, if $\{\alpha n\} > 0$.

The following three lemmas complete our analysis of the means μ_n and prepare for the main result of this section, Theorem 1.3.15.

Lemma 1.3.12. *If we choose a subsequence (n_k) such that $\{f(n_k)\} = \beta$ constant, then:*

(a) For $k \to \infty$

$$\mu_{n_k} = f(n_k) + c(\beta, q) + \mathrm{o}(1),$$

where $c(\beta, q)$ is a constant depending on β and q.

(b) (i) $c(0, q) = c(1/2, q) = 1/2$

(ii) $c(\beta, q) + c(-\beta, q) = 1$

Proof. Use (1.14) and simple properties of sin. ∎

Lemma 1.3.13. *Set $\beta = \{f(n)\}$. Then*

$$\lfloor c(\beta, q) + \beta \rfloor = \begin{cases} 0 & 0 \le \beta < 1/2 \\ 1 & 1/2 \le \beta < 1 \end{cases}.$$

Proof. We define

$$\hat{c}(\{f(n)\}, q) := c(\{f(n)\}, q) - 1 + \{f(n)\}.$$

By Theorem 1.3.10, $\hat{c}(\beta, q)$ is strictly increasing in β. Therefore we have for $0 \le \beta < 1/2$

$$\hat{c}(0, q) = -\frac{1}{2} \le \hat{c}(\beta, q) < \hat{c}(1/2, q) = 0.$$

Thus

$$\frac{1}{2} - \beta \le c(\beta, q) < 1 - \beta \qquad \text{and} \qquad \frac{1}{2} \le c(\beta, q) + \beta < 1. \tag{1.16}$$

Similarly, we get for $1/2 \le \beta < 1$

$$1 - \beta \le c(\beta, q) < \frac{1}{2} \qquad \text{and} \qquad 1 \le c(\beta, q) + \beta < \frac{1}{2} + \beta < \frac{3}{2}. \tag{1.17}$$

∎

Lemma 1.3.14.

(i) If $\beta \neq \frac{1}{2}$, then $f(n) + c(\beta, q) \notin \mathbb{Z}$. Thus

$$\lfloor \mu_n \rfloor = \lfloor f(n) + c(\beta, q) \rfloor = \lfloor f(n) \rfloor + \lfloor \beta + c(\beta, q) \rfloor.$$

(ii) For $\beta = \frac{1}{2}$,

$$\mu_n > f(n) + \frac{1}{2} \quad \text{if} \;\; 2f(n) \leq n-1 \qquad \text{and} \qquad \mu_n < f(n) + \frac{1}{2} \quad \text{if} \;\; 2f(n) \geq n.$$

Thus

$$\lfloor \mu_n \rfloor = f(n) + \frac{1}{2} \quad \text{if} \;\; 2f(n) \leq n-1 \qquad \text{and} \qquad \lceil \mu_n \rceil = f(n) + \frac{1}{2} \quad \text{if} \;\; 2f(n) \geq n.$$

Proof: (i): From (1.16) we get for $0 \leq \beta < 1/2$ by adding $f(n)$ and subtracting β

$$\lfloor f(n) \rfloor + \frac{1}{2} < f(n) + c(\beta, q) < \lfloor f(n) \rfloor + 1.$$

The case $1/2 < \beta < 1$ can be treated similarly.
(ii): Assume $2f(n) \leq n-1$ first. Then

$$\begin{aligned}
\sum_{i=0}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} &= \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} + \sum_{i=f(n)+\frac{1}{2}}^{2f(n)} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} + \sum_{2f(n)+1}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} \\
&= \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{-i-\frac{1}{2}}}{1+q^{-i-\frac{1}{2}}} + \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i+\frac{1}{2}}}{1+q^{i+\frac{1}{2}}} + \mathrm{o}(1) \\
&= f(n) + \frac{1}{2} + \mathrm{o}(1).
\end{aligned}$$

We used $\frac{q}{1+q} + \frac{q^{-1}}{1+q^{-1}} = 1$; the o(1)-term is non-negative (and vanishes only for $2f(n) = n-1$). If $2f(n) \geq n$, then the third sum vanishes and the second sum just runs up to $n - 1 < 2f(n)$, so $\mu_n < f(n) + \frac{1}{2}$.

Note that similarly to the proof of (ii) we can prove the properties of $c(\beta, q)$ in Lemma 1.3.12 (b). We can now state and prove the main result of this section. It shows that the limit distribution of the normalised X_n is discrete normal.

Theorem 1.3.15. *Let $(n_k)_{k\in\mathbb{N}}$ be an increasing sequence of natural numbers and $X_{n_k} \sim KB(n_k, \theta_{n_k}, q)$ with $\theta_{n_k} = q^{-f(n_k)}$ and $\{f(n_k)\} = \beta$ constant. (Recall that we assume $f(n) \to \infty$ and $n - f(n) \to \infty$ throughout the present section.) Then $(X_{n_k} - \mu_{n_k})/\sigma_{n_k}$ converges for $k \to \infty$ to a limit X, with*

$$\mathbb{P}\left(X = -(\beta + c)\frac{1}{\sigma} + \frac{1}{\sigma}x\right) = e_q(q)e_q(-q^{\beta})e_q(-q^{1-\beta})q^{(x-1)(x-2\beta)/2}, \qquad x \in \mathbb{Z}, \tag{1.18}$$

where $c = c(\beta, q)$ is the constant from Lemma 1.3.12 and $\sigma = \lim_{k\to\infty} \sigma_{n_k}$. The distribution of X is symmetric iff $\beta = 0$ or $\beta = 1/2$.

Proof: For simplicity we write in the following n instead of n_k. By (1.10), we see that the σ_n converge. First we consider the case $\beta \neq 1/2$:

$$\begin{aligned}\mathbb{P}(X_n = \lfloor\mu_n\rfloor + x) &= \begin{bmatrix} n \\ \lfloor\mu_n\rfloor + x\end{bmatrix}_q \frac{\theta_n^{\lfloor\mu_n\rfloor+x} q^{(\lfloor\mu_n\rfloor+x)(\lfloor\mu_n\rfloor+x-1)/2}}{\prod_{i=0}^{n-1}(1+\theta_n q^i)} \\ &= \begin{bmatrix} n \\ \lfloor\mu_n\rfloor + x\end{bmatrix}_q \frac{q^{-(\lfloor\mu_n\rfloor+x)f(n)+(\lfloor\mu_n\rfloor+x)(\lfloor\mu_n\rfloor+x-1)/2}}{\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right)}. \end{aligned} \tag{1.19}$$

The product in the denominator equals

$$\begin{aligned}\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right) &= \prod_{i=0}^{\lfloor f(n)\rfloor}\left(1+\frac{q^i}{q^{f(n)}}\right)\prod_{i=\lfloor f(n)\rfloor+1}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right) \\ &= q^{-f(n)(\lfloor f(n)\rfloor+1)+(\lfloor f(n)\rfloor+1)\lfloor f(n)\rfloor/2}\prod_{i=0}^{\lfloor f(n)\rfloor}\left(q^{f(n)-i}+1\right) \\ &\quad\times\prod_{i=0}^{n-\lfloor f(n)\rfloor-2}\left(1+q^{i+\lfloor f(n)\rfloor-f(n)+1}\right) \\ &= q^{-f(n)(\lfloor f(n)\rfloor+1)+(\lfloor f(n)\rfloor+1)\lfloor f(n)\rfloor/2}\prod_{i=0}^{\lfloor f(n)\rfloor}\left(q^{f(n)-\lfloor f(n)\rfloor}q^{\lfloor f(n)\rfloor-i}+1\right) \\ &\quad\times\prod_{i=0}^{n-\lfloor f(n)\rfloor-2}\left(1+q^i q^{\lfloor f(n)\rfloor-f(n)+1}\right) \\ &= q^{-f(n)(\lfloor f(n)\rfloor+1)+(\lfloor f(n)\rfloor+1)\lfloor f(n)\rfloor/2}\left(-q^{\beta};q\right)_{\lfloor f(n)\rfloor+1}\left(-q^{-\beta+1};q\right)_{n-\lfloor f(n)\rfloor-2}. \end{aligned} \tag{1.20}$$

The second equality uses the easy relation (1.2). The last two terms in (1.20) tend to $e_q\left(-q^{\beta}\right)$ and $e_q\left(-q^{-\beta+1}\right)$. The q-binomial coefficient in (1.19) tends to $e_q(q)$. The exponent of q resulting from (1.19) and (1.20) leads is

$$\begin{aligned}&-(\lfloor\mu_n\rfloor+x)f(n)+\frac{1}{2}(\lfloor\mu_n\rfloor+x)(\lfloor\mu_n\rfloor+x-1)+f(n)\left(\lfloor f(n)\rfloor+1\right)-\frac{1}{2}\left(\lfloor f(n)\rfloor+1\right)\lfloor f(n)\rfloor \\ &\quad= \left(\lfloor f(n)\rfloor+\lfloor\beta+c\rfloor+x\right)\left(\lfloor f(n)\rfloor+\lfloor\beta+c\rfloor-1+x\right)/2 \\ &\qquad -\left(\lfloor f(n)\rfloor+\lfloor\beta+c\rfloor+x\right)f(n)+f(n)\left(\lfloor f(n)\rfloor+1\right)-\left(\lfloor f(n)\rfloor+1\right)\lfloor f(n)\rfloor/2 \\ &\quad= \frac{1}{2}(x-1+\delta)(\delta-2f(n)+2\lfloor f(n)\rfloor+x) \\ &\quad= \frac{1}{2}(x-1+\delta)(\delta-2\beta+x),\end{aligned}$$

where $c = c(\beta, q)$ and

$$\delta = \lfloor\beta+c\rfloor = \begin{cases} 0 & \beta < 1/2 \\ 1 & \beta > 1/2\end{cases}$$

by Lemma 1.3.13. Putting things together, we obtain

$$\mathbb{P}(X_n = \lfloor\mu_n\rfloor + x) \to e_q(q)e_q\left(-q^{\beta}\right)e_q\left(-q^{-\beta+1}\right)q^{\frac{(\delta+x-1)(\delta+x-2\beta)}{2}}.$$

By normalising X_n we get (1.18). The distribution of X is symmetric iff

$$\begin{aligned} & -(\beta+c-\lfloor\beta+c\rfloor) = -(\beta+c-\lfloor\beta+c\rfloor)+1 \\ \iff \quad & \beta+c-\lfloor\beta+c\rfloor = \frac{1}{2}. \end{aligned}$$

This is true for $\beta = 0$ by Lemma 1.3.12 (b) (i). For $0 < \beta < \frac{1}{2}$ we have $\lfloor\beta + c\rfloor = 0$ by Lemma 1.3.13. But then we must have $\beta + c = \frac{1}{2}$, which would contradict (1.16) (since equality only holds for $\beta = 0$). For $\beta > \frac{1}{2}$ we must have $\beta + c = \frac{3}{2}$ by Lemma 1.3.13, but this would be a contradiction to (1.17).

For $\beta = 1/2$ define

$$H(\mu_n) := \begin{cases} \lfloor\mu_n\rfloor & \text{if } 2f(n) \le n-1 \\ \lceil\mu_n\rceil & \text{if } 2f(n) \ge n \end{cases}.$$

Then

$$\mathbb{P}\left(X_n = H(\mu_n)+x\right) = \begin{bmatrix} n \\ H(\mu_n)+x \end{bmatrix}_q \frac{q^{-(H(\mu_n)+x)f(n)+(H(\mu_n)+x)(H(\mu_n)+x-1)/2}}{\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right)}.$$

The q-binomial-coefficient tends to $e_q(q)$, and the product can be transformed as above. This time the exponent of q equals

$$\begin{aligned} -(H(\mu_n)+x)f(n) &+ (H(\mu_n)+x)(H(\mu_n)+x-1)/2 + f(n)\left(\lfloor f(n)\rfloor+1\right) \\ & - \left(\lfloor f(n)\rfloor+1\right)\lfloor f(n)\rfloor/2 \\ &= -(f(n)+\frac{1}{2}+x)f(n) + (f(n)+\frac{1}{2}+x)(f(n)-\frac{1}{2}+x)/2 \\ & \quad + f(n)\left(f(n)-\frac{1}{2}+1\right) - \left(f(n)-\frac{1}{2}+1\right)\left(f(n)-\frac{1}{2}\right)/2 \\ &= \frac{x^2}{2}. \end{aligned}$$

So we have

$$\mathbb{P}\left(X_n = H(\mu_n)+x\right) \to e_q(q)e_q\left(-q^{\frac{1}{2}}\right)^2 q^{\frac{x^2}{2}}.$$

Normalising X_n yields (1.18). ✐

So the limit distributions in the preceding theorem are normalised discrete normal distributions with parameters

$$\begin{array}{ll} \alpha = \frac{1}{2}+\beta & \text{if } \beta < \frac{1}{2} \\ \alpha = -\frac{1}{2}+\beta & \text{if } \beta > \frac{1}{2} \\ \alpha = 0 & \text{if } \beta = \frac{1}{2} \end{array}.$$

For $q \to 1$, they converge to the standard normal distribution [56]. Therefore, as in Proposition 1.3.2 and Theorem 1.3.6, the limits $q \to 1$ and $n \to \infty$ can be exchanged. Indeed, for $q \to 1$, the distribution of X_n in Theorem 1.3.15 tends to the binomial distribution $B(n, \frac{1}{2})$. The latter converges to the standard normal distribution after normalisation.

Again, the convergence of the distributions in Theorem 1.3.15 yields a convergence property of the corresponding orthogonal polynomials. The orthogonal polynomials for the discrete normal distribution are the Stieltjes-Wigert polynomials $S_k(x;q)$ [13, 42].

Corollary 1.3.16. *Let x be a real number, and $f(n)$ as usual. Then the q-Krawtchouk polynomial $K_k(q^{-x-f(n)+o(1)}; q^{f(n)-n}, n; q)$ tends to $(q;q)_k \times S_k(q^{-x};q)$ as $n \to \infty$.*

As above, a direct proof of the corollary easily follows from the series representations of the polynomials.

1.4 The q-deformed binomial distribution

The next distribution we study is the q-deformed binomial distribution (see Zeiner [63]). As mentioned in the introduction we are interested in the behaviour of sequences of random variables $X_n \sim QD(n, \tau_n, q)$. For fixed parameter $\tau_n = \tau$ we obtain an Euler distribution as the limit law. This still remains true if we consider non-constant parameter sequences τ_n which have a limit in $[0, 1)$. In particular we can provide a q-analogue of the convergence of the classical binomial distribution with constant mean to the Poisson distribution. This is done in Section 1.4.1. Afterwards we investigate in Section 1.4.2 sequences τ_n which tend to 1. Here the limiting distribution depends on the limit of τ_n^n and can be degenerate, truncated exponential or exponential. Note that all these distributions are independent of the choice of q.

1.4.1 Parameter sequences with limit < 1

In the present section we study sequences of random variables $X_n \sim QD(n, \tau_n, q)$, where the parameters τ_n converge to a limit $c \in [0, 1)$. In particular we prove a q-analogue of the convergence of the classical binomial distribution with constant mean to the Poisson distribution.

As noted above the sequence converges in the case of constant parameters $\tau_n = \tau$ to an Euler distribution with parameter τ. The following proposition is a mild generalisation of this fact and shows that the Euler distribution is the limit distribution for every convergent parameter sequence τ_n with limit in $[0, 1)$.

Proposition 1.4.1. *Let $X_n \sim QD(n, \tau_n, q)$. Then, for $n \to \infty$,*

$$X_n \to E(\tau, q)$$

if $\tau_n \to \tau$ and $0 \leq \tau < 1$.

Proof: Note that

$$\mathbb{P}(X_n = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \tau_n^x \prod_{i=0}^{n-x} (1 - \tau_n q^i).$$

The q-binomial coefficient tends to $1/(q;q)_x$. For the product apply the dominated convergence theorem to its logarithm to see that it converges to $E_q(-\tau)$. ✎

We are now interested in special choices of the parameters τ_n such that the limit $X(q)$ of the sequence $X_n(q)$ converges to a Poisson distribution for $q \to 1$. From the previous theorem we conclude immediately

Corollary 1.4.2. *Let $X_n \sim QD(n, \tau_n(q), q)$ with $\tau_n(q) \to \frac{\lambda}{n}$ for $q \to 1$ and $\tau_n(q) \to \tau(q)$ for $n \to \infty$ with the additional property $\frac{\tau(q)}{1-q} \to \lambda$ in the limit $q \to 1$ (recall that we assume $\tau(q) < 1$ in this section). Then the following diagram is commutative:*

$$\begin{array}{ccc} QD(n, \tau_n, q) & \xrightarrow{n\to\infty} & E(\tau(q), q) \\ {\scriptstyle q\to 1}\downarrow & & \downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\lambda}{n}\right) & \xrightarrow[n\to\infty]{} & P(\lambda) \end{array}$$

One very natural way to choose the parameters is to set $\tau_n = \frac{\lambda}{[n]_q}$.

Our next goal is to establish a convergence result which is analogous to the convergence of the classical binomial distribution with constant mean to a Poisson distribution and reduces in the limit $q \to 1$ to that theorem. For this purpose we start with an elementary fact.

Lemma 1.4.3. *Let $f_n(x)$, $n \in \mathbb{N}$, be a sequence of continuous functions which converges pointwise to a continuous limit $f(x)$. Assume that for each n the function $f_n(x)$ has a single root $\hat{x}_n$, and $f(x)$ has a single root $\hat{x}$, and that $f(y)f(z) < 0$ for $y < \hat{x}$ and $z > \hat{x}$. Then $\hat{x}_n \to \hat{x}$.*

Proof: W.l.o.g. we may assume that $f(z) > 0$ for $z > \hat{x}$. For given $\varepsilon > 0$ choose a $\delta(\varepsilon) < \min(f(\hat{x}+\varepsilon), -f(\hat{x}-\varepsilon))$. Then there exists an $N = N(\delta(\varepsilon))$ such that for all $n \geq N$ we have $|f_n(\hat{x}+\varepsilon) - f(\hat{x}+\varepsilon)| < \delta(\varepsilon)$. Therefore $f_n(\hat{x}+\varepsilon) > 0$. Moreover there exists an $M = M(\delta(\varepsilon))$ such that for all $n \geq M$ we have $|f_n(\hat{x}-\varepsilon) - f(\hat{x}-\varepsilon)| < \delta(\varepsilon)$. Therefore $f_n(\hat{x}-\varepsilon) < 0$. Hence, by continuity, for all $n \geq \max(N, M)$ we have $|\hat{x} - \hat{x}_n| < 2\varepsilon$.

The essential key to apply this lemma is the following representation of the means $\mu_n(\tau, q)$, which allows us to extract important properties of the means easily.

Lemma 1.4.4. *The means $\mu_n(\tau, q)$ have the representation*

$$\mu_n(\tau, q) = \sum_{j=1}^{n} (q;q)_{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau^j.$$

Proof. We proceed by induction. For $n = 1$ this is obviously true. Now suppose that the statement is true for $n-1$. In order to calculate $\mu_n(\tau, q)$ we use the recurrence relation (1.9). Hence we have

$$\mu_n(\tau, q) = \sum_{x=1}^{n} x p_n(x, \tau) = \tau \sum_{x=1}^{n} x p_{n-1}(x-1, \tau) + (1-\tau) \sum_{x=1}^{n-1} x p_{n-1}(x, q\tau).$$

Shifting the summation index in the first sum, splitting this sum and using the induction hypothesis yields

$$\begin{aligned} \mu_n(\tau, q) = \tau \sum_{j=1}^{n-1} (q;q)_{j-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q \tau^j + \sum_{x=0}^{n-1} \tau p_{n-1}(x, \tau) \\ + (1-\tau) \sum_{j=1}^{n-1} (q;q)_{j-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q \tau^j q^j. \end{aligned}$$

The second sum reduces to τ. Collecting powers of τ gives

$$\begin{aligned} \mu_n(\tau, q) &= \tau \left(1 + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q\right) \\ &\quad + \sum_{j=2}^{n} \left((q;q)_{j-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q q^j + (q;q)_{j-2} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q (1-q^{j-1}) \right) \tau^j. \end{aligned}$$

Consequently the desired result follows by the recurrence relation (1.1) for the q-binomial coefficients.

Remark 1.4.5. An alternative way to prove this lemma is to use Kemp's [38, p. 300] representation of the probability generating function, to differentiate and to manipulate the sum.

Using the monotonicity of the q-binomial coefficients in n we immediately get

Proposition 1.4.6. *The means $\mu_n(\tau, q)$ are strictly increasing in n (for $\tau > 0$) and τ.*

Now we turn to the convergence result:

Theorem 1.4.7. *Fix $\mu > 0$ and choose the parameter $\tau_n = \tau_n(q, \mu)$ of the q-deformed binomial distribution such that $\mu_n = \mu$. Then we have*

(i) The sequence $QD(n, \tau_n, q)$ converges for $n \to \infty$ to an Euler distribution $E(\tau, q)$, where $\tau = \lim_{n\to\infty} \tau_n$.

(ii) For fixed n, $QD(n, \tau_n, q)$ tends to a binomial distribution $B(n, \frac{\mu}{n})$ in the limit $q \to 1$.

(iii) For $q \to 1$, the Euler distribution $E(\tau, q)$ converges to a Poisson distribution with parameter μ.

So we obtain the following commutative diagram:

$$\begin{array}{ccc} QD(n, \tau_n(q), q) & \xrightarrow{n\to\infty} & E(\tau(q), q) \\ {\scriptstyle q\to 1}\downarrow & & \downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\mu}{n}\right) & \xrightarrow[n\to\infty]{} & P(\mu) \end{array}$$

Proof. First we check that for given μ, q and large n there exists a unique τ_n with $\mu_n(\tau_n, q) = \mu$. The function $\mu_n(\tau, q)$ is continuous and strictly increasing in n and τ by the previous theorem. Moreover, we have $\lim_{\tau\to 0} \mu_n(\tau, q) = 0$. Choosing τ_n in such a way that $\tau_n \to 1$, then $\mu_n(\tau_n, q)$ becomes arbitrarily large. Consequently there is a unique solution of $\mu_n(\tau, q) = \mu$. By Lemma 1.4.3 the sequence τ_n converges to a limit τ where τ is the unique solution of $\mu_E(\tau, q) = \mu$, where $\mu_E(\tau, q)$ is the mean of an Euler-distribution with parameters τ and q. This mean can be written as

$$\mu_E(\tau, q) = \sum_{i=0}^{\infty} \frac{q^i \tau}{1 - q^i \tau},$$

see [36] or take the limit $n \to \infty$ (using the dominated convergence theorem) in Lemma 1.4.4 and manipulate the sum (i.e., expand the denominator as a geometric series and change the order of summation).

Again by Lemma 1.4.3 we get that $\tau_n \to \mu/n$. It remains to check that $\tau/(1-q)$ converges to μ in the limit $q \to 1$. But this is again a consequence of Lemma 1.4.3 since $\tau/(1-q)$ is the unique solution of $\mu_E((1-q)\tau, q) = \mu$ and $\mu_E((1-q)\tau, q)$ tends to τ for $q \to 1$.

1.4.2 Parameter sequences with limit 1

In this section we investigate sequences X_n of random variables where X_n is $QD(n, \tau_n, q)$-distributed and the parameters τ_n converge to 1. The behaviour of the sequence X_n depends on the growth rate of τ_n. For this purpose we will distinguish three cases: Firstly we examine the case $\tau_n^n \to 1$, where it will turn out that the limit distribution is degenerate. Then we study the case $\tau_n^n \to c$ with $0 < c < 1$. Here the limit distribution will depend only on c and is a truncated exponential distribution. Finally we turn to the case $\tau_n^{f(n)} \to c$ where $0 < c < 1$ and $f(n) = o(n)$; this will lead to an exponential distribution.

Consider sequences of random variables $X_n \sim QD(n, \tau_n, q)$ with $\tau_n \to 1$ and additionally $\tau_n^n \to 1$ first. Then we have the following theorem:

Theorem 1.4.8. *Let $X_n \sim QD(n, \tau_n, q)$ with $\tau_n \to 1$ and $\tau_n^n \to 1$. Then $n - X_n$ converges to the point measure at* 0.

Proof. The probability that $Y_n = n - X_n$ is equal to 0 is given by

$$\mathbb{P}(Y_n = 0) = \tau_n^n$$

which converges to 1 by assumption.

Now let us investigate sequences $X_n \sim QD(n, \tau_n, q)$ where $\tau_n \to 1$ and $\tau_n^n \to c$ for a $c \in (0, 1)$. Before we can establish the distribution of the limit of such a sequence, we start with several lemmas which allow us to compute the asymptotic behaviour of certain sums of probabilities of $QD(n, \tau_n, q)$-distributed random variables and their means and variances.

The first lemma is an analogue to Lemma 1.4.4 and gives an alternative representation of the variance:

Lemma 1.4.9. *The second moment of $X_n(\tau, q)$ can be written as*

$$\sum_{x=1}^{n} x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q \tau^x (\tau; q)_{n-x} = \sum_{j=1}^{n} {}_n a_j \tau^j$$

with

$${}_n a_j = \begin{bmatrix} n \\ j \end{bmatrix}_q (q; q)_{j-1} \left(1 + 2 \sum_{i=1}^{j-1} \frac{1}{1 - q^i} \right).$$

Proof. We prove this by induction. The case $n = 1$ is obvious. To compute σ_n^2 we use the recurrence (1.9) again and shift the summation index. This gives

$$V_n := \sum_{x=1}^{n} x^2 p_n(x, \tau) = \tau \sum_{x=0}^{n-1} (x^2 + 2x + 1) p_{n-1}(x, \tau) + (1 - \tau) \sum_{x=1}^{n} x^2 p_{n-1}(x, q\tau).$$

By splitting sums and by using Lemma 1.4.4 and the induction hypothesis we find

$$V_n = \tau \sum_{j=1}^{n-1} {}_{n-1}a_j \tau^j + 2\tau \sum_{j=1}^{n-1} (q; q)_{j-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q \tau^j + \tau + (1 - \tau) \sum_{j=1}^{n-1} {}_{n-1}a_j q^j \tau^j.$$

Collecting powers of τ yields

$$V_n = \tau \left(1 + \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \right) + \sum_{j=2}^{n} \left({}_{n-1}a_{j-1} (1 - q^{j-1}) + 2 \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q (q; q)_{j-2} + {}_{n-1}a_j q^j \right) \tau^j.$$

The first term gives $\begin{bmatrix} n \\ 1 \end{bmatrix}_q \tau$ and the coefficient of τ^j in the sum equals

$$\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q (q; q)_{j-2} \left(1 + 2 \sum_{i=1}^{j-2} \frac{1}{1 - q^i} \right) \left[1 - q^{j-1} \right] + 2 \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q (q; q)_{j-2} + \begin{bmatrix} n-1 \\ j \end{bmatrix}_q (q; q)_{j-1} \left(1 + 2 \sum_{i=1}^{j-1} \frac{1}{1 - q^i} \right) q^j,$$

which implies the statement by using the recurrence relation of the q-binomial coefficients again.

The next three lemmas are devoted to the asymptotic behaviour of sums of powers of θ_n, where $0 < \theta_n < 1$ and $\theta_n \to 1$.

Lemma 1.4.10. *If $f(n) \to \infty$ for $n \to \infty$ and $\theta_n \le 1$ such that $\theta_n^{f(n)} \to c$ with $0 < c < 1$, then*

$$\sum_{i=0}^{\infty} \theta_n^i \sim \frac{-f(n)}{\log c}, \qquad n \to \infty.$$

Proof. Since only a finite number of $\theta_n = 1$ is possible, we assume w.l.o.g. that $\theta_n < 1$ and obtain

$$\sum_{i=0}^{\infty} \theta_n^i = \frac{1}{1-\theta_n} \sim -\frac{1}{\log \theta_n}$$

using the substitution $\theta_n = 1 + x_n$ in the elementary equivalence

$$\log(1+x) \sim x, \quad x \to 0. \tag{1.21}$$

Since $f(n)\log\theta_n \sim \log c$, the statement follows. ✎

Lemma 1.4.11. *For $\theta_n \le 1$ and $\theta_n \to 1$, $\theta_n^{f(n)} \to c$ ($c \in (0,1)$) and $g(n)/f(n) \sim \beta$, $g(n) \le n$ we have*

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \theta_n^i \sim \frac{c^\beta - 1}{\log c} f(n)$$

and

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i \sim e_q(q) \frac{c^\beta - 1}{\log c} f(n)$$

as $n \to \infty$.

Proof. We rewrite the first sum as

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \theta_n^i = \frac{1 - \theta_n^{\lfloor g(n) \rfloor + 1}}{1 - \theta_n}.$$

The growth of the denominator is given in Lemma 1.4.10, and the numerator tends to $1 - c^\beta$, since $\theta_n^{\lfloor g(n) \rfloor} = \theta_n^{g(n) - \{g(n)\}} \to c^\beta$ because of $\theta_n \to 1$.

To get the asymptotic of the second sum we write

$$\sum_{i=0}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i = \sum_{i=0}^{\lfloor \sqrt{g(n)} \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i + \sum_{\lfloor \sqrt{g(n)} \rfloor + 1}^{\lfloor g(n) - \sqrt{g(n)} \rfloor - 1} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i + \sum_{\lfloor g(n) - \sqrt{g(n)} \rfloor}^{\lfloor g(n) \rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i.$$

The first and the third sum on the right-hand side are $\mathcal{O}(\sqrt{g(n)})$ and therefore asymptotically negligible. The second sum is bounded by

$$\frac{(q;q)_n}{(q;q)_{\lfloor \sqrt{g(n)} \rfloor + 1} (q;q)_{n - \lfloor \sqrt{g(n)} \rfloor - 1}} \sum_{\lfloor \sqrt{g(n)} \rfloor + 1}^{\lfloor g(n) - \sqrt{g(n)} \rfloor - 1} \theta_n^i \le \sum_{\lfloor \sqrt{g(n)} \rfloor + 1}^{\lfloor g(n) - \sqrt{g(n)} \rfloor - 1} \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i$$

$$\le \frac{(q;q)_n}{(q;q)_{\lfloor n/2 \rfloor}^2} \sum_{\lfloor \sqrt{g(n)} \rfloor + 1}^{\lfloor g(n) - \sqrt{g(n)} \rfloor - 1} \theta_n^i.$$

By the first part of this lemma the lower and the upper bound have the asserted asymptotic. ✎

Lemma 1.4.12. *If $\theta_n \le 1$ and $\theta_n \to 1$ with $\theta_n^n \to c$ for $0 < c < 1$, then*

$$\sum_{i=0}^{n} i\theta_n^i \sim \frac{1-c+c\log c}{\log^2 c} n^2$$

and

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q i\theta_n^i \sim e_q(q)\frac{1-c+c\log c}{\log^2 c} n^2$$

as $n \to \infty$.

Proof: To estimate the first sum we use Lemma 1.4.10 again and the identity

$$\sum_{i=0}^{n} it^i = \frac{t(1-t^n-nt^n(1-t))}{(1-t)^2}.$$

Hence, setting $t = \theta_n$,

$$\sum_{i=0}^{n} i\theta_n^i \sim (1-c-n\theta_n^n(1-\theta_n))\frac{n^2}{\log^2 c} \sim (1-c+c\log c)\frac{n^2}{\log^2 c}.$$

Here we used that under the assumption $\theta_n^n \to c$ we have $(1-\theta_n)n \to -\log c$. This can easily be seen from the equivalence (1.21). The asymptotic for the sum with the q-binomial coefficient is obtained as in Lemma 1.4.11.

Now we are ready to establish the essential key in proving the convergence result: we give the asymptotic behaviour of sums of probabilities and the means and variances of $QD(n, \tau_n, q)$-distributed random variables.

Lemma 1.4.13. *Let X_n be $QD(n, \tau_n, q)$-distributed and denote by $\mu_n(\tau_n, q)$ and $\sigma_n^2(\tau_n, q)$ the corresponding mean and variance. If $\tau_n \to 1$ and $\tau_n^n \to c$ with $0 < c < 1$ and $f(n) \sim \beta n$, $f(n) < n$, then*

$$\sum_{x=0}^{\lfloor f(n)\rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} \sim 1 - c^\beta,$$

$$\mu_n(\tau_n, q) \sim \frac{c-1}{\log c} n,$$

$$\sigma_n^2(\tau_n, q) \sim \frac{1+2c\log c - c^2}{(\log c)^2} n^2,$$

as $n \to \infty$.

Proof: We start with the first assertion. Since $f(n) < n$ we can write

$$S_n := \sum_{x=0}^{\lfloor f(n)\rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} = (1-\tau_n) \sum_{x=0}^{\lfloor f(n)\rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q \prod_{i=1}^{n-x-1} (1-\tau_n q^i).$$

The summands are bounded by $e_q(q)^2$, hence

$$S_n \sim (1-\tau_n) \sum_{x=\lfloor\sqrt{n}\rfloor}^{\lfloor f(n)\rfloor - \lfloor\sqrt{n}\rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q \prod_{i=1}^{n-x-1} (1-\tau_n q^i) =: \hat{S}_n.$$

Estimating the product and using again the boundedness of the summands yields

$$\begin{aligned}\hat{S}_n &\leq (1-\tau_n)(\tau_n;q)_{n-\lfloor f(n)\rfloor+\lfloor\sqrt{n}\rfloor-1}\sum_{x=\lfloor\sqrt{n}\rfloor}^{\lfloor f(n)\rfloor-\lfloor\sqrt{n}\rfloor}\tau_n^x\begin{bmatrix}n\\x\end{bmatrix}_q\\ &\sim (1-\tau_n)(\tau_n;q)_{n-\lfloor f(n)\rfloor+\lfloor\sqrt{n}\rfloor-1}\sum_{x=0}^{n}\tau_n^x\begin{bmatrix}n\\x\end{bmatrix}_q =: \hat{\hat{S}}_n.\end{aligned}$$

As in the proof of Proposition 1.4.1 and with use of Lemma 1.4.11 (with $g(n) := f(n)$ and $f(n) := n$) we obtain

$$\hat{\hat{S}}_n \sim (1-\tau_n)\frac{1}{e_q(q)}e_q(q)\frac{c^\beta-1}{\log c}n \sim 1-c^\beta.$$

In an analogous way we find a lower bound of $\hat{S}_n$ that is asymptotically equivalent to $1-c^\beta$.

Now we prove the second proposition of the lemma: Use Lemma 1.4.4, easy estimates of the q-Pochhammer symbol and the asymptotics given in Lemma 1.4.11 to obtain

$$\mu_n(\tau_n,q) \leq \sum_{j=1}^{\lfloor\sqrt{n}\rfloor}\frac{(q;q)_n}{(q;q)^2_{\lceil n/2\rceil}} + (q;q)_{\lfloor\sqrt{n}\rfloor}\sum_{j=\lfloor\sqrt{n}\rfloor}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j \sim \frac{1}{e_q(q)}e_q(q)\frac{c-1}{\log c}n$$

and

$$\mu_n(\tau_n,q) \geq (q;q)_n\sum_{j=1}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j \sim \frac{c-1}{\log c}n.$$

Similarly we proceed for the second moments of $X_n(\tau_n,q)$ and estimate with use of Lemma 1.4.12

$$\begin{aligned}\mathbb{E}(X_n^2) &\geq \sum_{j=1}^{n}(q;q)_{j-1}\left(1+2(j-1)\right)\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j\\ &\geq 2(q;q)_n\sum_{j=1}^{n}(j-1)\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j \sim 2\frac{1-c+c\log c}{(\log c)^2}n^2.\end{aligned}$$

To bound the second moment from above we split the sum into two parts

$$\mathbb{E}(X_n^2) \leq \sum_{j=1}^{\lfloor\sqrt{n}\rfloor}\frac{(q;q)_n}{(q;q)^2_{\lceil n/2\rceil}}\left(1+\frac{2n}{1-q}\right) + \sum_{j=\lfloor\sqrt{n}\rfloor}^{n}(q;q)_{j-1}\left(1+2\sum_{i=1}^{j-1}\frac{1}{1-q^{j-i}}\right)\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j.$$

The first sum is $\mathrm{o}(n^2)$, and splitting the inner sum in the second term we obtain

$$\begin{aligned}\mathbb{E}(X_n^2) = o(n^2) &+ \sum_{j=\lfloor\sqrt{n}\rfloor}^{n}(q;q)_{j-1}\left(1+2\sum_{i=\lfloor\sqrt{j}\rfloor}^{j-1}\frac{1}{1-q^{j-i}}\right)\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j\\ &+ \sum_{j=\lfloor\sqrt{n}\rfloor}^{n}(q;q)_{j-1}\left(1+2\sum_{i=1}^{\lfloor\sqrt{j}\rfloor}\frac{1}{1-q^{j-i}}\right)\begin{bmatrix}n\\j\end{bmatrix}_q\tau_n^j.\end{aligned}$$

Here the first sum is $o(n^2)$ again and easy estimates of the second term yield

$$\begin{aligned}\mathbb{E}(X_n^2) &\le o(n^2) + 2(q;q)_{\lfloor\sqrt{n}\rfloor} \sum_{j=\lfloor\sqrt{n}\rfloor}^{n} j \frac{1}{1-q^{j-\lfloor\sqrt{j}\rfloor-1}} \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \\ &\le o(n^2) + 2(q;q)_{\lfloor\sqrt{n}\rfloor} \frac{1}{1-q^{n-\sqrt{n}-1}} \sum_{j=1}^{n} j \begin{bmatrix} n \\ j \end{bmatrix}_q \tau_n^j \\ &\sim 2\frac{1-c+c\log c}{(\log c)^2} n^2.\end{aligned}$$

Thus

$$\mathbb{E}(X_n^2(\tau_n,q)) \sim 2\frac{1-c+c\log c}{(\log c)^2} n^2.$$

Hence

$$\begin{aligned}\sigma_n^2(\tau_n,q) = \mathbb{E}(X_n^2(\tau_n,q)) - \mu_n(\tau,q)^2 &\sim \left(2\frac{1-c+c\log c}{(\log c)^2} - \left(\frac{c-1}{\log c}\right)^2\right) n^2 \\ &\sim \frac{1+2c\log c - c^2}{(\log c)^2} n^2,\end{aligned}$$

which completes the proof. ✐

After this analysis of the means and variances it is now easy to obtain the limiting distribution of the sequence X_n.

Theorem 1.4.14. *Let $Y_n \sim QD(n,q,\tau_n)$ with $\tau_n \to 1$ and $\tau_n^n \to c$ with $0 < c < 1$. Then the sequence of the normalised random variables $X_n = (Y_n - \mu_n)/\sigma_n$ converges to a limit X with*

$$\mathbb{P}(X \le x) = 1 - e^{c-1} e^{-\sqrt{1+2c\log c - c^2}\,x}$$

for

$$x \in \left[-\frac{1-c}{\sqrt{1+2c\log c-c^2}}, \frac{c-\log c-1}{\sqrt{1+2c\log c - c^2}}\right)$$

and

$$\mathbb{P}(X \le x) = 1 \qquad \text{for} \qquad x = \frac{c-\log c-1}{\sqrt{1+2c\log c-c^2}}.$$

Proof. The support of X is given by

$$\left[\lim_{n\to\infty} -\frac{\mu_n(\tau_n,q)}{\sigma_n(\tau_n,q)}, \lim_{n\to\infty} \frac{n-\mu_n(\tau_n,q)}{\sigma_n(\tau_n,q)}\right].$$

Using Lemma 1.4.13 the stated support follows immediately.

Computing the distribution function of X yields with use of Lemma 1.4.13

$$\mathbb{P}(X_n \le x) = \sum_{0\le y\le \sigma_n x+\mu_n} \tau_n^y \begin{bmatrix} n \\ y \end{bmatrix}_q (\tau_n;q)_{n-y} \sim 1 - c^\alpha$$

with

$$\alpha = \frac{\sqrt{1+2c\log c-c^2}}{-\log c} x + \frac{c-1}{\log c}$$

for

$$x < \frac{c - \log c - 1}{\sqrt{1 + 2c\log c - c^2}}.$$

Simplifying c^α yields the theorem. ✐

Now we turn to the third case which treats sequences of random variables $X_n \sim QD(n, \tau_n, q)$ where $\tau_n \to 1$ and $\tau_n^{f(n)} \to c$ for a $c \in (0,1)$ and $f(n) = o(n)$. This case is very similar to the previous one, and so we start with an analogue of Lemma 1.4.12.

Lemma 1.4.15. *Let $f(n) \to \infty$, $f(n) = o(n)$, $\theta_n^{f(n)} \to c$ with $0 < c < 1$. Then*

$$\sum_{i=0}^{n} i\theta_n^i \sim \frac{f(n)^2}{\log^2 c} \qquad \text{and} \qquad \sum_{i=0}^{n} i \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i \sim e_q(q)\frac{f(n)^2}{\log^2 c}$$

as $n \to \infty$.

Proof. Follows from the proof of Lemma 1.4.12 and observe that $n\theta_n^n(1-\theta_n)$ tends to zero. ✐

Following the proof of Lemma 1.4.13 and using Lemma 1.4.15 instead of Lemma 1.4.12 we obtain

Lemma 1.4.16. *If $\tau_n \to 1$ and $\tau_n^{f(n)} \to c$ with $0 < c < 1$ and $f(n) = o(n)$, $g(n) \sim \beta f(n)$, then*

$$\sum_{x=0}^{\lfloor g(n) \rfloor} \tau_n^x \begin{bmatrix} n \\ x \end{bmatrix}_q (\tau_n; q)_{n-x} \sim 1 - c^\beta,$$

$$\mu_n(\tau_n, q) \sim \frac{-f(n)}{\log c},$$

$$\sigma_n^2(\tau_n, q) \sim \frac{f(n)^2}{(\log c)^2},$$

as $n \to \infty$.

As an immediate consequence we get the distribution of the limit of X_n, which is an exponential distribution and is again independent of q.

Theorem 1.4.17. *Let $Y_n \sim QD(n, q, \tau_n)$ with $\tau_n \to 1$ and $\tau_n^{f(n)} \to c$ with $0 < c < 1$ and $f(n) = o(n)$. Then the sequence of the normalised random variables $X_n = (Y_n - \mu_n)/\sigma_n$ converges to a normalised exponential distribution with parameter 1, i.e.*

$$\mathbb{P}(X \leq x) = 1 - e^{-x-1}, \qquad x \geq -1.$$

Proof. Lemma 1.4.16 yields immediately that the support of the limit distribution is $[-1, \infty)$. Computing the distribution function gives

$$\mathbb{P}(X \leq x) = \sum_{0 \leq y \leq \sigma_n x + \mu_n} \tau_n^y \begin{bmatrix} n \\ y \end{bmatrix}_q (\tau_n; q)_{n-y} \sim 1 - c^{\frac{x+1}{-\log c}} = 1 - e^{-x-1}.$$ ✐

Comparing this result with Theorem 1.4.14 we see that this corresponds to taking the limit $c \to 0$.

1.5 A family of q-binomial distributions

So far we studied convergence properties of Kemp's distribution and of the q-deformed binomial distribution. For the other two q-binomial distributions mentioned in Section 1.2.1, the Stieltjes-Wigert- and the Rogers-Szegö-distribution, we could ask the same questions. But before attacking these problems recall the definitions of Kemp's q-deformed binomial distribution, the Stieltjes-Wigert- and the Rogers-Szegö-distribution. They all are of the form

$$\mathbb{P}(X = x) = \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta^x}{\sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2} \theta^y}, \qquad x = 0, \ldots, n,\ 0 < \theta,\ 0 \le \alpha; \tag{1.22}$$

for $\alpha = 0$ this is the RS-distribution, $\alpha = \frac{1}{2}$ gives a $KB(n, \theta q^{1/2}, q)$-distribution and $\alpha = 1$ a $SW(n, \theta q, q)$-distribution. This gives rise to the following definition: We say a random variable X is $\mathcal{B}(\alpha, \theta, n, q)$-distributed iff its probabilities are given by (1.22). The present section is devoted to the study of this family $\mathcal{B}$ of q-binomial distributions and generalises some of the results obtained for Kemp's distribution in Section 1.3. The following investigations are a little bit more involved than that for Kemp's distribution since we used the special formulas (1.7) for the mean and variance before.

In order to state all our results we need besides this family of q-binomial distribution a family of q-analogues of the Poisson distribution as well. We call a random variable X $\mathcal{P}(\alpha, \theta, q)$-distributed iff it satisfies

$$\mathbb{P}(X = x) = \frac{q^{\alpha x^2} \theta^x}{(q, q)_x} \frac{1}{E_q^{2\alpha}(\theta)}, \qquad 0 \le x,$$

where $0 < \theta < 1$ if $\alpha = 0$, and $0 < \theta$ if $\alpha > 0$, and E_q^{α} is a q-analogue of the exponential function (which was introduced by Floreanini et al. [19] and studied by Atakishiyev [7] and also appears in Cigler [15]) defined by

$$E_q^{\alpha}(z) = \sum_{x \ge 0} \frac{q^{\frac{\alpha}{2} x^2}}{(q, q)_x} z^x, \tag{1.23}$$

since $E_q^{\alpha}((1-q)z) \to e^z$. For $\alpha = 0$ we obtain the Euler distribution, and $\alpha = \frac{1}{2}$ gives a $H(\theta q^{1/2})$-distribution. The sum in (1.23) has a different behaviour for $\alpha = 0$ and $\alpha > 0$: In the case $\alpha = 0$ it is convergent only for $0 \le |z| < 1$, but for $\alpha > 0$ it converges for all $z \in \mathbb{C}$. This is why we restricted the parameter θ in the definition of our q-Poisson family. Consequently there is a big difference in the behaviour of the RS-distribution and the other members of this q-binomial-family. So we will often distinguish between $\alpha = 0$ and $\alpha > 0$ in the convergence results.

We are now ready to start our investigations: In Section 1.5.1 we study basic properties of the family $\mathcal{B}$. We show that they are indeed q-analogues of the binomial distribution, converge to the family $\mathcal{P}$ as $n \to \infty$ if the parameter θ is fixed and that they are logconcave. Moreover, we give characterisation theorems and random walk models. Finally we study the behaviour of the means in dependence on n, θ and α.

In Sections 1.5.2 and 1.5.3 we investigate sequences of random variables X_n with $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$. In particular we show that there are analogues to the convergence of the classical binomial distribution to the Poisson distribution and the normal distribution, and that the limits $q \to 1$ and $n \to \infty$ can be exchanged. Section 1.5.2 deals with convergent parameter sequences, in particular with the case of constant parameter and constant mean, and contains a detailed analysis of the behaviour of the RS-distribution in the limit $\theta_n \to 1$. Section 1.5.3 is devoted to the study of parameter sequences θ_n that tend to infinity (in the case $\alpha > 0$). If the parameter grows fast the limit is obtained by using the reversing property and reducing the problem in this way to a convergent parameter sequence. Afterwards we examine slowly increasing parameter sequences which will lead to a discrete normal distribution as the limit law.

1.5.1 Properties of the Family $\mathcal{B}$

As noted above we study basic properties of our family $\mathcal{B}$. We show that it is in fact a q-analogue of the binomial distribution and logconcave. These properties hold for the family $\mathcal{P}$ too. Then we give a characterisation of a $\mathcal{B}(\alpha, \theta, n, q)$-distribution and a random walk model for $\mathcal{B}$ and then we turn to the study of the behaviour of the mean of a $\mathcal{B}(\alpha, \theta, n, q)$-distribution in dependence on n, θ and α. In the present section we always allow $\alpha \geq 0$.

The following two theorems show that our families $\mathcal{B}$ and $\mathcal{P}$ tend to the classical binomial and Poisson distribution. This generalises the results for the Kemp-, SW-, RS-, Heine, and Euler distributions.

Theorem 1.5.1. *For $q \to 1$ we have*

$$\mathcal{B}(\alpha, \theta, n, q) \to B\left(n, \frac{\theta}{1+\theta}\right).$$

Proof. By definition,

$$\begin{aligned}
\mathbb{P}(X = x) &= \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta^x}{\sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2} \theta^y} \\
&\to \frac{\binom{n}{x} \theta^x}{\sum_{y=0}^{n} \binom{n}{y} \theta^y} = \frac{\binom{n}{x} \theta^x}{(1+\theta)^n} \\
&= \binom{n}{x} \left(\frac{\theta}{1+\theta}\right)^x \left(\frac{1}{1+\theta}\right)^{n-x}.
\end{aligned}$$

Theorem 1.5.2. *In the limit $q \to 1$*

$$\mathcal{P}(\alpha, (1-q)\theta, q) \to P(\theta).$$

Proof. By definition

$$\mathbb{P}(X = x) = \frac{q^{\alpha x^2} (1-q)^x \theta^x}{(q, q)_x} \frac{1}{E_q^{2\alpha}((1-q)\theta)} \to \frac{\theta^x}{x!} \exp(-\theta).$$

Kemp showed in [38] that the RS-, SW-, and Kemp-distribution are logconcave, i.e.

$$\Delta(x) := \frac{\mathbb{P}(X = x+1)}{\mathbb{P}(X = x)} - \frac{\mathbb{P}(X = x+2)}{\mathbb{P}(X = x+1)} > 0$$

for $x = 0, \ldots, n-2$. We now generalise this to

Theorem 1.5.3. *$\mathcal{B}(\alpha, \theta, n, q)$ is logconcave.*

Proof: We have

$$\begin{aligned}\Delta(x) &= \frac{q^{\alpha(x+1)^2}\theta^{x+1}(q,q)_x(q,q)_{n-x}}{(q,q)_{x+1}(q,q)_{n-x-1}q^{\alpha x^2}\theta^x} - \frac{q^{\alpha(x+2)^2}\theta^{x+2}(q,q)_{x+1}(q,q)_{n-x-1}}{(q,q)_{x+2}(q,q)_{n-x-2}q^{\alpha(x+1)^2}\theta^{x+1}} \\ &= \theta\left(\frac{q^{2\alpha x+\alpha}\left(1-q^{n-x}\right)}{1-q^{x+1}} - \frac{\left(1-q^{n-x-1}\right)q^{2\alpha x+3\alpha}}{1-q^{x+2}}\right) \\ &= \theta q^{2\alpha x+\alpha}\left(\frac{1-q^{x+2}-q^{n-x}+q^{n+2}-\left(1-q^{n-x-1}-q^{x+1}+q^n\right)q^{2\alpha}}{\left(1-q^{x+1}\right)\left(1-q^{x+2}\right)}\right).\end{aligned}$$

For $\alpha = 0$ we have $\Delta(x) > 0$ by [38], and the numerator is increasing in α, since

$$1-q^{n-x-1}-q^{x+1}+q^n = q^{n-x-1}\left(q^{x+1}-1\right)-\left(q^{x+1}-1\right) > 0$$

for $x < n-1$.

In the same way we obtain

Theorem 1.5.4. *$\mathcal{P}(\alpha,\theta,q)$ is logconcave.*

For the Heine- and Euler-distribution this property was proven by Kemp [36].

In [39] Kemp characterised some q-analogues of the binomial distribution as the conditional distribution of $U|(U+V=m)$ where U and V are independent. We can characterise our family $\mathcal{B}$ in an analogous way and generalise some of Kemp's results.

Theorem 1.5.5. *A $\mathcal{B}(\alpha,\theta/\lambda,m,q)$-distribution is the distribution of $U|(U+V=m)$, where U and V are independent, iff U has a $\mathcal{P}(\alpha,\beta,\theta)$-distribution and V has an Euler-distribution with parameter λ.*

Proof: The proof runs along the same lines as the proofs in [39]: If U and V have the postulated distributions, then

$$\begin{aligned}\mathbb{P}(U=u|U+V=n) &= C\frac{\theta^u q^{\alpha u^2}}{(q,q)_u}\frac{\lambda^{m-u}}{(q,q)_{m-u}} \\ &= C\frac{\lambda^m}{(q,q)_u(q,q)_{m-u}}\left(\frac{\theta}{\lambda}\right)^u q^{\alpha u^2}.\end{aligned}$$

To prove the other implication, we need the following theorem ([53]):

Let X and Y be independent discrete random variables and $c(x,x+y) = \mathbb{P}(X=x|X+Y=x+y)$. If

$$\frac{c(x+y,x+y)c(0,y)}{c(x,x+y)c(y,y)} = \frac{h(x+y)}{h(x)h(y)},$$

where h is a nonnegative function, then

$$f(x) = f(0)h(x)e^{ax}, \qquad g(y) = g(0)k(y)e^{ay}$$

where a is an arbitrary parameter and

$$0 < f(x) = \mathbb{P}(X=x), \quad 0 < g(y) = \mathbb{P}(Y=y), \quad k(y) = \frac{h(y)c(0,y)}{c(y,y)}.$$

Here we have

$$\frac{c(u+v,u+v)c(0,v)}{c(u,u+v)c(u,v)} = \frac{\left(\frac{\theta}{\lambda}\right)^{u+v} q^{\alpha(u+v)^2}}{\frac{(q,q)_{u+v}}{(q,q)_v(q,q)_u}\left(\frac{\theta}{\lambda}\right)^u q^{\alpha u^2}\left(\frac{\theta}{\lambda}\right)^v q^{\alpha v^2}} = \frac{h(u+v)}{h(u)h(v)},$$

where

$$h(u) = \frac{q^{\alpha u^2}}{(q,q)_u}.$$

Thus $k(v) = (\theta/\lambda)^v/(q,q)_v$ and

$$\mathbb{P}(U=u) = C_1 \frac{q^{\alpha u^2} e^{au}}{(q,q)_u},$$
$$\mathbb{P}(V=v) = C_2 \left(\frac{\theta e^a}{\lambda}\right)^v \frac{1}{(q,q)_v}$$

yielding a $\mathcal{P}(\alpha, e^a, q)$-distribution and an Euler distribution.

We now give a random-walk-model for the family $\mathcal{B}$ (the models for the Kemp-, RS-, and SW-distribution given in [38] are special cases of this model). Let a_x and b_x denote the probabilities to move up and down and choose

$$a_x = c\gamma q^{2\alpha x}\left(1-q^{n-x}\right) \qquad \text{and} \qquad b_x = c(1-q^x)$$

for $x = 0,\ldots,n$. Then $\mathcal{B}(\alpha, \gamma q^{-\alpha}, n, q)$ is a stationary distribution. To see this, note that for a stationary distribution we must have

$$\mathbb{P}(X=x) = \mathbb{P}(X=x)(1-a_x-b_x) + \mathbb{P}(X=x+1)b_{x+1} + \mathbb{P}(X=x-1)a_{x-1}.$$

So we have to show that $\Delta(x) := -\mathbb{P}(X=x)(a_x+b_x)+\mathbb{P}(X=x+1)b_{x+1}+\mathbb{P}(X=x-1)a_{x-1} = 0$ if $X \sim \mathcal{B}(\alpha,\gamma q^{-\alpha}, n, q)$. For $1 \le x \le n-1$ we have

$$\begin{aligned}\Delta(x) = C\Bigg(&-\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2}\gamma^x q^{-\alpha x}\left(c(1-q^x) + c\gamma q^{2\alpha x}(1-q^{n-x})\right) + \\ &+ \begin{bmatrix} n \\ x-1 \end{bmatrix}_q q^{\alpha(x-1)^2}\gamma^{x-1}q^{-\alpha(x-1)}c\gamma q^{2\alpha(x-1)}(1-q^{n-x+1}) + \\ &+ \begin{bmatrix} n \\ x+1 \end{bmatrix}_q q^{\alpha(x+1)^2}\gamma^{x+1}q^{-\alpha(x+1)}c(1-q^{x+1})\Bigg).\end{aligned}$$

Using the relation

$$\begin{bmatrix} n \\ x-1 \end{bmatrix}_q \left(1-q^{n-x+1}\right) = \begin{bmatrix} n \\ x \end{bmatrix}_q (1-q^x)$$

we obtain that the terms with γ^x and γ^{x+1} vanish. Similarly $\Delta(0)$ and $\Delta(n)$ can be treated.

Let us denote by $\mu_n(\alpha,\theta,q)$ the mean of a random variable $X \sim \mathcal{B}(\alpha,\theta,n,q)$. The following lemmas are devoted to the behaviour of $\mu_n(\alpha,\theta,q)$ in dependence on n, α and θ. The first result shows that the means are increasing in n.

Lemma 1.5.6. *For all $\alpha \ge 0$ $\mu_n(\alpha,\theta,q)$ is increasing in n.*

Proof. For $0 \le x < y \le n$ we have

$$q^{-x} < q^{-y}$$

and therefore

$$\begin{aligned} q^{n+1-x} &< q^{n+1-y}, \\ 1 - q^{n+1-x} &> 1 - q^{n+1-y}, \\ \frac{1}{1-q^{n+1-x}} &< \frac{1}{1-q^{n+1-y}} \end{aligned}$$

This is equivalent to

$$\begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q (y-x) < \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q (y-x)$$

and

$$\begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q y + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q x < \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q x + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q y.$$

Multiplication with $\theta^{x+y} q^{\alpha(x^2+y^2)}$ yields

$$\begin{aligned} &\begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q y\theta^{x+y} q^{\alpha(x^2+y^2)} + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q x\theta^{x+y} q^{\alpha(x^2+y^2)} \\ &\quad < \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q x\theta^{x+y} q^{\alpha(x^2+y^2)} + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q y\theta^{x+y} q^{\alpha(x^2+y^2)}. \end{aligned}$$

Now we sum over all pairs (x, y) with $x < y$:

$$\sum_{\substack{x,y=0 \\ x \ne y}}^{n} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \begin{bmatrix} n \\ y \end{bmatrix}_q y\theta^y q^{\alpha y^2} < \sum_{\substack{x,y=0 \\ x \ne y}}^{n} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} x \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}.$$

By adding the terms for $x = y$ and an extra-sum we get

$$\begin{aligned} &\theta^{n+1} q^{\alpha(n+1)^2} \sum_{y=0}^{n} y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} + \sum_{x=0}^{n} \sum_{y=0}^{n} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \begin{bmatrix} n \\ y \end{bmatrix}_q y\theta^y q^{\alpha y^2} \\ &\quad < (n+1)\theta^{n+1} q^{\alpha(n+1)^2} \sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} + \sum_{x=0}^{n} \sum_{y=0}^{n} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} x \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}. \end{aligned}$$

This can be written as

$$\sum_{x=0}^{n+1} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^{n} y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} < \sum_{x=0}^{n+1} x \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2},$$

and so we have

$$\frac{\sum_{y=0}^{n} y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}}{\sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}} < \frac{\sum_{x=0}^{n+1} x \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}{\sum_{x=0}^{n+1} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}.$$

The means are increasing in the parameter θ too:

Lemma 1.5.7. *$\mu_n(\alpha, \theta, q)$ is increasing in θ for all $\alpha \ge 0$.*

Proof. We show that $\frac{\partial}{\partial\theta}\mu_n(\alpha,\theta,q) > 0$. Differentiating gives

$$\frac{\partial}{\partial\theta}\left(\frac{\sum_{x=0}^n x\left[\begin{smallmatrix}n\\x\end{smallmatrix}\right]_q \theta^x q^{\alpha x^2}}{\sum_{x=0}^n \left[\begin{smallmatrix}n\\x\end{smallmatrix}\right]_q \theta^x q^{\alpha x^2}}\right) =$$

$$= \frac{\sum_{x=0}^n \left[\begin{smallmatrix}n\\x\end{smallmatrix}\right]_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y^2 \left[\begin{smallmatrix}n\\y\end{smallmatrix}\right]_q \theta^{y-1} q^{\alpha y^2} - \sum_{x=0}^n x\left[\begin{smallmatrix}n\\x\end{smallmatrix}\right]_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y\left[\begin{smallmatrix}n\\y\end{smallmatrix}\right]_q \theta^{y-1} q^{\alpha y^2}}{\left(\sum_{x=0}^n \left[\begin{smallmatrix}n\\x\end{smallmatrix}\right]_q \theta^x q^{\alpha x^2}\right)^2}$$

Thus it suffices to show that

$$\left(\sum_{x=1}^n x\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}\theta^{x-1}\right)^2 < \sum_{x=0}^n \begin{bmatrix}n\\x\end{bmatrix}_q \theta^{x-1} q^{\alpha x^2} \sum_{y=0}^n y^2 \begin{bmatrix}n\\y\end{bmatrix}_q \theta^{y-1} q^{\alpha y^2}.$$

The left-hand side can be written as

$$\sum_{x=1}^n x^2 \begin{bmatrix}n\\x\end{bmatrix}_q^2 q^{2\alpha x^2}\theta^{2(x-1)} + \sum_{\substack{x,y=0\\x\neq y}}^n xy \begin{bmatrix}n\\x\end{bmatrix}_q \begin{bmatrix}n\\y\end{bmatrix}_q q^{\alpha(x^2+y^2)}\theta^{x+y-2} =: A_1 + B_1$$

and the right-hand side as

$$\sum_{x=1}^n x^2 \begin{bmatrix}n\\x\end{bmatrix}_q^2 q^{2\alpha x^2}\theta^{2(x-1)} + \sum_{\substack{x,y=0\\x\neq y}}^n x^2 \begin{bmatrix}n\\x\end{bmatrix}_q \begin{bmatrix}n\\y\end{bmatrix}_q q^{\alpha(x^2+y^2)}\theta^{x+y-2} =: A_2 + B_2.$$

Since $A_1 = A_2$, it suffices to show that $B_1 < B_2$. For this purpose we consider the pairs (x,y) and (y,x) with $x < y$: In B_1 we have the term

$$2xy\begin{bmatrix}n\\x\end{bmatrix}_q \begin{bmatrix}n\\y\end{bmatrix}_q q^{\alpha(x^2+y^2)}\theta^{x+y-2} \tag{1.24}$$

and in B_2

$$\begin{bmatrix}n\\x\end{bmatrix}_q \begin{bmatrix}n\\y\end{bmatrix}_q q^{\alpha(x^2+y^2)}\theta^{x+y-2}(x^2+y^2). \tag{1.25}$$

Since $2xy < x^2 + y^2$ for $x \neq y$, we have (1.24) < (1.25) and so $B_1 < B_2$. ∎

For α the situation is a little bit different:

Lemma 1.5.8. *$\mu_n(\alpha,\theta,q)$ is decreasing in α if $\alpha \in (0,1]$ and increasing in α if $\alpha \geq 1$.*

Proof. Assume $\alpha > 1$ (in the same way we can treat the case $0 < \alpha < 1$). We show that $\frac{\partial}{\partial\alpha}\mu_n(\alpha,\theta,q) > 0$. This is equivalent to

$$\sum_{x=0}^n \begin{bmatrix}n\\x\end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y^3 \begin{bmatrix}n\\y\end{bmatrix}_q \theta^y q^{\alpha y^2} \log\alpha > \sum_{x=0}^n x\begin{bmatrix}n\\x\end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^n y^2 \begin{bmatrix}n\\y\end{bmatrix}_q \theta^y q^{\alpha y^2}\log\alpha.$$

So it is sufficient to show that

$$\sum_{\substack{x,y=0\\x\neq y}}^n \begin{bmatrix}n\\x\end{bmatrix}_q \begin{bmatrix}n\\y\end{bmatrix}_q q^{\alpha(x^2+y^2)}\theta^{x+y}y^3\log\alpha > \sum_{\substack{x,y=0\\x\neq y}}^n \begin{bmatrix}n\\x\end{bmatrix}_q \begin{bmatrix}n\\y\end{bmatrix}_q q^{\alpha(x^2+y^2)}\theta^{x+y}xy^2\log\alpha.$$

Considering the pairs (x,y) and (y,x), it is sufficient that $x^3 + y^3 > xy^2 + yx^2$. This is fulfilled because this can be written as $(y^2 - x^2)(y - x) = (y + x)(y - x)^2 > 0$. ∎

Finally we show that our family $\mathcal{B}$ is closed under reversing, i.e. $n - X$ has the same form as X.

Theorem 1.5.9. *If $X \sim \mathcal{B}(\alpha, \theta, n, q)$ then $n - X \sim \mathcal{B}(\alpha, \theta^{-1}q^{-2\alpha n}, n, q)$.*

Proof. We compute

$$\begin{aligned}\mathbb{P}(n - X = x) &= \frac{\begin{bmatrix} n \\ n-x \end{bmatrix}_q q^{\alpha(n-x)^2}\theta^{n-x}}{\sum_{y=0}^{n}\begin{bmatrix} n \\ n-y \end{bmatrix}_q q^{\alpha(n-y)^2}\theta^{n-y}} \\ &= \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha(n^2-2nx+x^2)}\theta^{-x}}{\sum_{y=0}^{n}\begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(n^2-2ny+y^2)}\theta^{-y}} \\ &= \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2}\theta^{-x}q^{-2\alpha nx}}{\sum_{y=0}^{n}\begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2}\theta^{-y}q^{-2\alpha ny}}.\end{aligned}$$

1.5.2 Convergent Parameter

In this section we consider sequences $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$ where the parameter sequence θ_n tends to a limit as $n \to \infty$. This will lead to the family $\mathcal{P}$ as limit law. In particular we prove that the convergence of the classical binomial distribution with constant mean has a q-analogue. But in the case $\alpha = 0$ and $\theta_n \to 1$ these results fail. In this case we obtain - depending on the limit of θ_n - a uniform distribution or exponential-like distributions. In the following we need the two auxiliary results below.

Lemma 1.5.10. *For $\alpha > 0$ we have for all $z \in \mathbb{C}$*

$$\sum_{x=0}^{n}\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} z^x \to E_q^{2\alpha}(z), \qquad n \to \infty.$$

For $\alpha = 0$ this holds for $|z| < 1$.

Proof. We estimate the difference

$$\left|\sum_{x=0}^{\infty}\frac{q^{\alpha x^2}}{(q,q)_x}z^x - \sum_{x=0}^{n}\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} z^x\right| \le \sum_{x=n+1}^{\infty}\frac{q^{\alpha x^2}}{(q,q)_x}|z|^x + \sum_{x=1}^{n} q^{\alpha x^2}|z|^x \left|\begin{bmatrix} n \\ x \end{bmatrix}_q - \frac{1}{(q,q)_x}\right|.$$

Estimating in the first sum the q-shifted factorial by the q-exponential function yields

$$\le e_q(q)\sum_{x=n+1}^{\infty}(q^{\alpha n}|z|)^x + \sum_{x=1}^{n}\frac{q^{\alpha x^2}}{(q,q)_x}|z|^x\left(1 - \prod_{i=1}^{x}(1 - q^{n-i+1})\right);$$

the same estimate we use for the second sum, split it and compute the first sum to obtain

$$\begin{aligned}&\le e_q(q)\left(\frac{(q^{\alpha n}|z|)^{n+1}}{1 - q^{\alpha n}|z|} + \sum_{x=1}^{\lfloor \frac{n}{2}\rfloor} q^{\alpha x^2}|z|^x\left(1 - \prod_{i=1}^{x}(1 - q^{n-i+1})\right)\right. \\ &\qquad \left. + \sum_{x=\lfloor \frac{n}{2}\rfloor}^{n} q^{\alpha x^2}|z|^x\left(1 - \prod_{i=1}^{x}(1 - q^{n-i+1})\right)\right).\end{aligned}$$

The first term is obviously o(1). Estimating the products gives

$$\leq e_q(q)\left(\mathrm{o}(1)+\left(1-\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}(1-q^{n-i})\right)\sum_{x=1}^{\infty}q^{\alpha x^2}|z|^x+\sum_{x=\lfloor\frac{n}{2}\rfloor}^{\infty}q^{\alpha x^2}|z|^x\right)$$

and further

$$\leq e_q(q)\left(\mathrm{o}(1)+\left(1-\left(1-q^{\lfloor\frac{n}{2}\rfloor}\right)^{\lfloor\frac{n}{2}\rfloor}\right)\sum_{x=1}^{\infty}q^{\alpha x^2}|z|^x+\sum_{x=\lfloor\frac{n}{2}\rfloor}^{\infty}\left(q^{\alpha\lfloor\frac{n}{2}\rfloor}|z|\right)^x\right);$$

the latter sum is o(1) as before, thus

$$=\mathrm{o}(1)+\mathrm{O}(n^2q^n)\sum_{x=1}^{\infty}q^{\alpha x^2}|z|^x=\mathrm{o}(1).$$

Lemma 1.5.11. *Assume $\alpha>0$ and let (θ_n) be a sequence of real numbers with limit $\theta\geq 0$. Then*

$$\lim_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}\theta_n^x=E_q^{2\alpha}(\theta).$$

If $\theta<1$, this holds for $\alpha=0$ as well.

Proof. For small $\epsilon>0$ and n large enough we have

$$\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}(\theta-\varepsilon)^x\leq\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}\theta_n^x\leq\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}(\theta+\varepsilon)^x,$$

hence, with use of Lemma 1.5.10,

$$\begin{aligned}E_q^{2\alpha}(\theta-\varepsilon)&=\lim_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}(\theta-\varepsilon)^x\leq\liminf_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}\theta_n^x\\&\leq\limsup_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}\theta_n^x\leq\lim_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}(\theta+\varepsilon)^x\\&=E_q^{2\alpha}(\theta+\varepsilon).\end{aligned}$$

By continuity of $E_q^{2\alpha}$, the lemma follows.

The first result is a generalisation of the fact that the Kemp's distribution converges to the Heine distribution.

Proposition 1.5.12. *If $X_n\sim\mathcal{B}(\alpha,\theta_n,n,q)$, $\alpha>0$, then for $n\to\infty$*

$$X_n\to\mathcal{P}(\alpha,\theta,q),$$

if $\theta_n\to\theta$. This still remains true in the case $\alpha=0$ and $\theta<1$.

Proof. This follows immediately from the fact that

$$\begin{bmatrix}n\\x\end{bmatrix}_q\to\frac{1}{(q,q)_x}$$

for $n\to\infty$ and from Lemma 1.5.11.

In the case $\alpha = 0$ and $\theta > 1$ the situation is slightly different:

Proposition 1.5.13. *If $X_n \sim \mathcal{B}(0, \theta_n, n, q)$, then for $n \to \infty$, if $\theta_n \to \theta > 1$,*

$$n - X_n \to \mathcal{P}\left(0, \tfrac{1}{\theta}, q\right),$$

which is an Euler distribution.

Proof. Define $Y_n = n - X_n$. Then

$$\mathbb{P}(Y_n = x) = \frac{\left[{n \atop x}\right]_q \theta_n^{n-x}}{\sum_{y=0}^{n} \left[{n \atop y}\right]_q \theta_n^{n-y}} = \frac{\left[{n \atop x}\right]_q \theta_n^{-x}}{\sum_{y=0}^{n} \left[{n \atop y}\right]_q \theta_n^{-y}} \to \frac{\theta^{-x}}{(q,q)_x} \frac{1}{\sum_{y=0}^{n} \frac{1}{(q,q)_y} \theta^{-y}}$$

by Lemma 1.5.11. ∎

In particular we are interested in sequences X_n such that the limits $q \to 1$ and $n \to \infty$ can be exchanged. The propositions above immediately yield

Corollary 1.5.14. *For each $\alpha > 0$ let $X_n \sim \mathcal{B}(\alpha, \theta_n(q), n, q)$ with $\theta_n(q) \to \theta$. Additionally assume that $\theta_n(q) \to \lambda/n$ and $\theta/(1-q) \to \lambda$ as $q \to 1$. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{B}(\alpha, \theta_n(q), n, q) & \xrightarrow{n\to\infty} & \mathcal{P}(\alpha, (1-q)\theta, q) \\ {\scriptstyle q\to 1}\downarrow & & \downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\lambda}{n}\right) & \xrightarrow[n\to\infty]{} & P(\lambda) \end{array}$$

One very natural way to choose the parameter sequence is to set $\theta_n(q) = \frac{\lambda}{[n-\lambda]_q}$, $\lambda > 0$.

The convergence $\mathcal{B}(\alpha, \theta_n(q), n, q) \to \mathcal{P}(\alpha, (1-q)\theta, q)$ still remains true for $\alpha = 0$ if we require $(1-q)\theta < 1$. Moreover, the commutative diagram remains correct for given $\lambda > 0$, if we restrict q to values $\geq \max(0, 1 - \frac{1}{\lambda})$.

The next result is a q-analogue of the classical convergence of the binomial distribution with constant mean to the Poisson distribution.

Theorem 1.5.15. *Fix $\mu > 0$ and $\alpha > 0$. Consider a sequence of random variables $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$ with parameter sequence $\theta_n = \theta_n(q, \mu)$ chosen such that the means μ_n of X_n are equal to μ. Then we have*

(i) The sequence X_n converges to the limit law $\mathcal{P}(\alpha, \theta, q)$, where θ is the limit of the sequence θ_n.

(ii) As $q \to 1$, X_n tends to a binomial distribution with parameters n and μ/n.

(iii) In the limit $q \to 1$, $\mathcal{P}(\alpha, \theta(q, \mu), q)$ converges to a Poisson distribution with parameter μ.

Thus the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{B}(\alpha, \theta_n(q, \mu), n, q) & \xrightarrow{n\to\infty} & \mathcal{P}(\alpha, \theta(q, \mu), q) \\ {\scriptstyle q\to 1}\downarrow & & \downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\mu}{n}\right) & \xrightarrow[n\to\infty]{} & P(\mu) \end{array}$$

Proof. First we check, that for given μ, q and large n there is a unique $\theta_n(q)$, such that $\mu_n(\theta_n(q), q) = \mu$. The function $\mu_n(\theta, q)$ is continuous and increasing in θ (see Lemma 1.5.7). Moreover $\lim_{\theta\to 0}\mu_n(\theta, q) = 0$. From Corollary 1.5.25 we see that for sufficiently large n and suitable θ_n, $\mu_n(\theta_n, q) \geq \frac{n}{2}$. Consequently there exists a unique solution $\theta_n(q)$ of $\mu_n(\theta, q) = \mu$. By Lemma 1.4.3, $\theta_n(q)$ converges to a limit $\theta(q)$, where $\theta(q)$ is the unique solution of $\mu_\infty(\theta, q) = \mu$. Hence $\mathcal{B}(\alpha, \theta_n(q), n, q) \to \mathcal{P}(\alpha, \theta(q), q)$ by Lemma 1.5.11.

Again by Lemma 1.4.3 we get that $\theta_n(q) \to \frac{\mu}{n-\mu}$ for $q \to 1$ and so $\frac{\theta_n(q)}{1+\theta_n(q)} \to \frac{\mu}{n}$. Consequently $\mathcal{B}(\alpha, \theta_n(q), n, q) \to B\left(n, \frac{\mu}{n}\right)$.

It remains to check that $\theta(q)/(1-q)$ converges to μ for $q \to 1$ (then $\mathcal{P}(\alpha, \theta(q), q) \to P(\mu)$). The value $\theta(q)/(1-q)$ is the unique solution of $\mu_\infty((1-q)\theta, q) = \mu$. Moreover, $\mu_\infty((1-q)\theta, q)$ converges pointwise to θ for $q \to 1$, so we can apply Lemma 1.4.3.

In the case $\alpha = 0$ an analogous result holds for X_n or $n - X_n$ depending on the values of the parameters, i.e., if $\theta(q,\mu) < 1$ then the theorem holds for the sequence X_n, and if $\theta(q,\mu) > 1$ then this is true for $n - X_n$.

Now we turn our attention to the case $\alpha = 0$. To finish the analysis of the RS-distribution we consider $\theta_n \to 1$. It is worthwhile to point out that the limit distributions only depend on the growth rate of the parameter sequences and are independent of q. This is why we will distinguish three cases in dependence on the speed of the convergence of the parameters θ_n to the limit 1. First we will provide a result of fast growing θ_n. In order to do so we start with an auxiliary result.

Lemma 1.5.16. *If $f(n) \leq n$, $\theta_n \leq 1$ and $f(n) \to \infty$ and $\theta_n^{f(n)} \to 1$ for $n \to \infty$, then for $k \in \mathbb{N}$*

$$\sum_{0\leq i\leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i \sim e_q(q)\frac{f(n)^{k+1}}{k+1}, \qquad n \to \infty.$$

Proof. Write

$$\sum_{0\leq i\leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i = \sum_{i=0}^{\lfloor\sqrt{f(n)}\rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i + \sum_{i=\lfloor\sqrt{f(n)}\rfloor+1}^{f(n)-\lfloor\sqrt{f(n)}\rfloor-1} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i$$
$$+ \sum_{n-\lfloor\sqrt{n}\rfloor\leq i\leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i.$$

The first and the third term on the right-hand side can be estimated by

$$(\sqrt{f(n)}+1)f(n)^k \frac{(q,q)_n}{(q,q)_{\lfloor n/2\rfloor}(q,q)_{n-\lfloor n/2\rfloor}}$$

and are therefore negligible. The middle term can be bounded by

$$\frac{(q,q)_n}{(q,q)_{\lfloor\sqrt{f(n)}\rfloor+1}(q,q)_{n-\lfloor\sqrt{f(n)}\rfloor-1}}\theta^{f(n)} \sum_{\lfloor\sqrt{f(n)}\rfloor+1}^{f(n)-\lfloor\sqrt{f(n)}\rfloor-1} i^k \leq \sum_{\lfloor\sqrt{f(n)}\rfloor+1}^{f(n)-\lfloor\sqrt{f(n)}\rfloor-1} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k$$
$$\leq \frac{(q,q)_n}{(q,q)_{\lfloor n/2\rfloor}(q,q)_{n-\lfloor n/2\rfloor}} \sum_{\lfloor\sqrt{f(n)}\rfloor+1}^{f(n)-\lfloor\sqrt{f(n)}\rfloor-1} i^k$$

and has the asserted asymptotic.

This lemma implies that under the assumption $\theta_n^n \to 1$ the limit law is uniform on the interval $[-\sqrt{3}, \sqrt{3}]$.

Theorem 1.5.17. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \le 1$ and $\theta_n^n \to 1$, then $(X_n - \mu_n)/\sigma_n$ converges for $n \to \infty$ to the uniform distribution on the interval $[-\sqrt{3}, \sqrt{3}]$.*

Proof. We start with an asymptotic of the means and the variances. By Lemma 1.5.16 we have

$$\mu_n = \frac{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q i\theta_n^i}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} \sim \frac{e_q(q)\frac{n^2}{2}}{e_q(q)n} = \frac{n}{2}$$

and

$$\sigma_n^2 = \frac{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q i^2\theta_n^i}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} - \mu_n^2 \sim \frac{n^2}{3} - \frac{n^2}{4} = \frac{n^2}{12}.$$

From these two fact one can easily see that the support of the limiting distribution is

$$\lim_{n\to\infty} [-\mu_n/\sigma_n, (n-\mu_n)/\sigma_n] = [-\sqrt{3}, \sqrt{3}].$$

Now we compute

$$\begin{aligned} \mathbb{P}(X \le x) &= \lim_{n\to\infty} \sum_{-\frac{\mu_n}{\sigma_n} \le \frac{k-\mu_n}{\sigma_n} \le x} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} = \lim_{n\to\infty} \frac{1}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q \theta_n^i} \sum_{0 \le k \le \sigma_n x + \mu_n} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k \\ &= \lim_{n\to\infty} \frac{1}{e_q(q)n} e_q(q)(\sigma_n x + \mu_n) = \lim_{n\to\infty} \frac{1}{n}\left(\frac{n}{2\sqrt{3}}x + \frac{n}{2}\right) = \frac{1}{2\sqrt{3}}x + \frac{1}{2}, \end{aligned}$$

which is the distribution function of the uniform distribution on $[-\sqrt{3}, \sqrt{3}]$. ✐

Using the fact that a $RS(n, \theta, q)$-distribution corresponds to a $(n - RS(n, 1/\theta, q))$-distribution or following the above proofs we immediately get the following corollary:

Corollary 1.5.18. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \ge 1$ and $\theta_n^n \to 1$, then $(\mu_n - X_n)/\sigma_n$ and $(X_n - \mu_n)/\sigma_n$ converge for $n \to \infty$ to the uniform distribution on the interval $[-\sqrt{3}, \sqrt{3}]$.*

Now we turn to the case that $\theta_n^n \to c$ with $0 < c < 1$. For this purpose we start with the following lemma, which supplements Lemmas 1.4.11 and 1.4.12 and is crucial for the analysis of the variances.

Lemma 1.5.19. *For $\theta_n \le 1$ and $\theta_n \to 1$, $\theta_n^n \to c$ with $0 < c < 1$ and $f(n)/n \sim \beta > 0$ we have*

$$\sum_{i=0}^n i^2\theta_n^i \sim -2\frac{1 - c + c\log c - \frac{1}{2}c\log^2 c}{\log^3 c} n^3$$

as $n \to \infty$.

$$\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q i^2\theta_n^i \sim -2e_q(q)\frac{1 - c + c\log c - \frac{1}{2}c\log^2 c}{\log^3 c} n^3$$

as $n \to \infty$.

Proof. Using

$$\sum_{i=0}^{n} i^2 t^i = \frac{t(-1-t+t^n+2nt^n(1-t)+n^2t^n(1-t)^2+t^{n+1})}{(t-1)^3}$$

we obtain for the first sum

$$\sum_{i=0}^{n} i^2\theta_n^i \sim (-2+2c-2c\log c+c\log^2 c)\frac{n^3}{\log^3 c},$$

The second sum follows immediately as in Lemma 1.4.11. ∎

Now we are able to establish the convergence result in this case.

Theorem 1.5.20. *If $X_n \sim RS(n,\theta_n,q)$ with $\theta_n \le 1$, $\theta_n \to 1$ and $\theta_n^n \to c$ with $0<c<1$, then $(X_n-\mu_n)/\sigma_n$ converges to a limit distribution X with*

$$\mathbb{P}(X \le x) = \frac{c^{\alpha(c,x)}-1}{c-1},$$

where

$$\alpha(c,x) = \frac{\sqrt{(c-1)^2-c\log^2 c}}{(c-1)\log c}x + \frac{1-c+c\log c}{(c-1)\log c}$$

and $x \in [-\gamma_1,\gamma_2]$ with

$$\gamma_1 = \frac{1-c+c\log c}{\sqrt{(c-1)^2-c\log^2 c}} \qquad \text{and} \qquad \gamma_2 = \frac{c-1-\log c}{\sqrt{(c-1)^2-c\log^2 c}}.$$

Proof. Using Lemmas 1.4.11, 1.4.12 and 1.5.19 we get for the means μ_n

$$\mu_n = \frac{\sum_{i=0}^n i\theta^i \begin{bmatrix} n \\ i \end{bmatrix}_q}{\sum_{i=0}^n \theta^i \begin{bmatrix} n \\ i \end{bmatrix}_q} \sim \frac{(1-c+c\log c)n^2}{\log^2 c}\frac{\log c}{(c-1)n} = \frac{1-c+c\log c}{(c-1)\log c}n$$

and for the variances σ_n^2

$$\begin{aligned}\sigma_n^2 &= \frac{\sum_{i=0}^n i^2\theta^i \begin{bmatrix} n \\ i \end{bmatrix}_q}{\sum_{i=0}^n \theta^i \begin{bmatrix} n \\ i \end{bmatrix}_q} - \mu_n^2 \\ &\sim \frac{-2(1-c+c\log c-\frac{1}{2}c\log^3 c)n^3}{\log^3 c}\frac{\log c}{(c-1)n} - \frac{(1-c+c\log c)^2}{(c-1)^2\log^2 c}n^2 \\ &= \frac{c^2+1-2c-c\log^2 c}{(c-1)^2\log^2 c}n^2.\end{aligned}$$

As an immediate consequence we get that the support of the limit distribution

$$[-\gamma_1,\gamma_2] = \lim_{n\to\infty}[-\mu_n/\sigma_n,(n-\mu_n)/\sigma_n]$$

is as stated in the theorem. Now we compute the distribution function of X:

$$\mathbb{P}(X\le x) = \lim_{n\to\infty}\sum_{\frac{k-\mu_n}{\sigma_n}\le x}\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q\theta_n^k}{\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q\theta_n^i} = \lim_{n\to\infty}\frac{1}{e_q(q)\frac{c-1}{\log c}n}\sum_{k\le\sigma_n x+\mu_n}\begin{bmatrix} n \\ k \end{bmatrix}_q\theta_n^k.$$

Since $\sigma_n x + \mu_n \sim n\alpha(c,x)$ we have further

$$\mathbb{P}(X \le x) = \lim_{n\to\infty} \frac{1}{e_q(q)\frac{c-1}{\log c} n} \sum_{k \le n\alpha(c,x)} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^q$$

$$= \lim_{n\to\infty} \frac{\frac{c^{\alpha(c,x)}-1}{\log c} n e_q(q)}{e_q(q)\frac{c-1}{\log c} n} = \frac{c^{\alpha(c,x)}-1}{c-1},$$

what completes the proof of this theorem. ∅

Again we get as an immediate consequence

Corollary 1.5.21. *If $X_n \sim RS(n,\theta_n,q)$ with $\theta_n \ge 1$, $\theta_n \to 1$ and $\theta_n \to \tilde{c}$ with $1 < \tilde{c} < \infty$, then $(\mu_n - X_n)/\sigma_n$ and $(X_n - \mu_n)/\sigma_n$ converge to a limit X, whose distribution is given in Theorem 1.5.20 with $c = 1/\tilde{c}$ resp. $\tilde{c}$.*

Finally we study the case that $\theta_n^{f(n)} \to c$ with $0 < c < 1$ and $f(n) = o(n)$. The analysis of this case is very similar to that of the previous case. So we start again with a lemma which is useful to find the asymptotic behaviour of the means and variances.

Lemma 1.5.22. *Let $f(n) \to \infty$ for $n \to \infty$, $\frac{f(n)}{n} \to 0$, $\theta_n^{f(n)} \to c$ with $0 < c < 1$ and. Then*

$$\sum_{i=0}^{n} i^2 \theta_n^i \sim \frac{f(n)^3}{\log^3 c}.$$

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q i^2 \theta_n^i \sim e_q(q) \frac{f(n)^3}{\log^3 c}.$$

Proof. Similar to Lemma 1.4.15 ∅

The following theorem shows that in this case the limiting distribution is an exponential distribution.

Theorem 1.5.23. *If $X_n \sim RS(n,\theta_n,q)$ with $\theta_n \le 1$, $\theta_n \to 1$, $\theta_n^{f(n)} \to c$ with $f(n) = o(n)$ and $0 < c < 1$, then $(X_n - \mu_n)/\sigma_n$ converges to a normalised exponential distribution with parameter $\lambda = 1$, i.e.*

$$\mathbb{P}(X \le x) = 1 - e^{-x-1}, \quad x \ge -1.$$

Proof. From Lemmas 1.4.11, 1.4.15 and 1.5.22 we get

$$\mu_n \sim \frac{-f(n)}{\log c} \qquad \text{and} \qquad \sigma_n^2 \sim \frac{2f(n)^2}{\log^2 c} - \frac{f(n)^2}{\log^2 c} = \frac{f(n)^2}{\log^2 c}.$$

Computing the distribution function of the limit distribution yields

$$\mathbb{P}(X \le x) = \lim_{n\to\infty} \sum_{k \le \sigma_n x + \mu_n} \frac{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q \theta_n^k}{\sum_{i=0}^n \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q \theta_n^i}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{-e_q(q)}{\log c} n} \sum_{k \le \frac{-f(n)}{\log c} x + \frac{-f(n)}{\log c}} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k$$

$$= \lim_{n\to\infty} \frac{1 - c^{\frac{1+x}{-\log c}}}{\log c} f(n) e_q(q) \frac{\log c}{e_q(q) f(n)}$$

$$= 1 - c^{\frac{1+x}{-\log c}} = 1 - e^{-x-1}.$$

∅

Corollary 1.5.24. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \geq 1$, $\theta_n \to 1$, $\theta_n^{f(n)} \to c$ with $f(n) = o(n)$ and $1 < c < \infty$, then $(\mu_n - X_n)/\sigma_n$ converges to a normalised exponential distribution with parameter $\lambda = 1$.*

1.5.3 Increasing Parameter

Now we turn our attention to sequences of random variables X_n with $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$, where the parameter sequence $\theta_n = \theta_n(q)$ tends to infinity. We start with fast growing parameters θ_n, i.e., $\theta_n = q^{-2\alpha n - g(n)}$ with $g(n)$ convergent or $g(n) \to \infty$. Due to the reversing property 1.5.9 we conclude immediately from Lemma 1.5.11:

Corollary 1.5.25. *Let $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$ with $\theta_n = q^{-2\alpha n - g(n)}$.*

(i) If $g(n) \to \gamma$ then for $\alpha > 0$ we have $n - X_n \to \mathcal{P}(\alpha, q^{-\gamma}, q)$.

(ii) If $g(n) \to \infty$ then for all $\alpha \geq 0$ we have $n - X_n \to \delta_0$.

Now we consider parameter sequences $\theta_n(q) = q^{-f(n)}$ with $f(n) \to \infty$ and $2\alpha n - f(n) \to \infty$ for $n \to \infty$ and $\alpha > 0$. These assumptions will be on force throughout the section. We will prove in Theorem 1.5.31 that a suitable chosen subsequence of the normalised sequence of random variables X_n converges to a discrete normal distribution. Theorem 1.5.26 and Lemmas 1.5.27 and 1.5.28 are devoted to the asymptotic behaviour of the sequence (μ_n) of means. Afterwards we study the sequence (σ_n) of variances in Lemmas 1.5.29 and 1.5.30 and then we establish the convergence result.

To simplify notation, we define

$$\Sigma_1(z) := \sum_{x=0}^{\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor} z \begin{bmatrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{bmatrix}_q q^{\alpha(a+x)^2}, \qquad \Sigma_1 := \Sigma_1(1),$$

$$\Sigma_2(z) := \sum_{x=0}^{n-\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - 1} z \begin{bmatrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{bmatrix}_q q^{\alpha(a-(x+1))^2}, \qquad \Sigma_2 := \Sigma_2(1),$$

$$\Sigma_1^\infty(z) := \sum_{x=0}^{\infty} z q^{\alpha(a+x)^2}, \qquad \Sigma_1^\infty := \Sigma_1^\infty(1),$$

$$\Sigma_2^\infty(z) := \sum_{x=0}^{\infty} z q^{\alpha(a-(x+1))^2}, \qquad \Sigma_2^\infty := \Sigma_2^\infty(1),$$

where $a = \left\{ \frac{f(n)}{2\alpha} \right\}$.

Now we turn to the study of the sequence of means.

Lemma 1.5.26. *For $n \to \infty$*

$$\mu_n = \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + c(a, \alpha, q) + \mathrm{o}(1),$$

where

$$c(a, \alpha, q) = \frac{\sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^{\infty} \left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2} \right)}.$$

Proof. We have to study the behaviour of

$$\frac{\sum_{x=0}^{n} x \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}{\sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}.$$

For this purpose we expand the fraction by $q^{\frac{f(n)^2}{4\alpha}}$ and analyse the denominator D and the numerator N separately.

$$D = \sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2 - f(n)x + \frac{f(n)^2}{4\alpha}} = \sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\frac{(-2\alpha x + f(n))^2}{4\alpha}};$$

splitting the sum into two parts gives

$$= \sum_{x=0}^{\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\frac{(-2\alpha x + f(n))^2}{4\alpha}} + \sum_{x=\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + 1}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\frac{(-2\alpha x + f(n))^2}{4\alpha}}.$$

By reversing the order of summation in the first sum and shifting the summation index in the second sum we obtain

$$= \sum_{x=0}^{\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor} \begin{bmatrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{bmatrix}_q q^{\frac{\left(-2\alpha \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + f(n) + 2\alpha x\right)^2}{4\alpha}}$$
$$+ \sum_{x=0}^{n - \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - 1} \begin{bmatrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{bmatrix}_q q^{\frac{\left(-2\alpha \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - 2\alpha - 2\alpha x + f(n)\right)^2}{4\alpha}};$$

simplifying the exponents of q leads to

$$= \sum_{x=0}^{\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor} \begin{bmatrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{bmatrix}_q q^{\alpha(a+x)^2}$$
$$+ \sum_{x=0}^{n - \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - 1} \begin{bmatrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{bmatrix}_q q^{\alpha(a-(x+1))^2}.$$

This tends to

$$e_q(q) \left(\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=0}^{\infty} q^{\alpha(a-(x+1))^2} \right) =: \gamma \tag{1.26}$$

since we can bound the first sum as follows:

$$e_q(q) \sum_{x=0}^{\infty} q^{\alpha(a+x)^2} \geq \sum_{x=0}^{\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor} \begin{bmatrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{bmatrix}_q q^{\alpha(a+x)^2}$$
$$\geq \sum_{x=0}^{\frac{1}{2}\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor} \begin{bmatrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{bmatrix}_q q^{\alpha(a+x)^2},$$

estimating the q-binomial coefficient yields

$$\geq \frac{\left(1-q^{n-\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1}\right)^{\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1}}{(q,q)_{\frac{1}{2}\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor}} \sum_{x=0}^{\frac{1}{2}\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor} q^{\alpha(a+x)^2}$$
$$\to e_q(q)\sum_{x=0}^{\infty} q^{\alpha(a+x)^2}.$$

Here we used that

$$\left(1-q^{n-\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1}\right)^{\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1} = 1+\mathcal{O}\left(\left(\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1\right) nq^{n-\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1}\right).$$

Similar arguments hold for the second sum. Now we turn to the numerator N.

$$N=\sum_{x=0}^{n} x \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2 - f(n)x + \frac{f(n)^2}{4\alpha}},$$

we split the sum again, reverse the order of summation resp. shift the summation index and get

$$= \sum_{x=0}^{\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor} \left(\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor - x\right) \begin{bmatrix} n \\ \lfloor\frac{f(n)}{2\alpha}\rfloor - x \end{bmatrix}_q q^{\alpha(a+x)^2} +$$
$$+ \sum_{x=0}^{n-\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor-1} \left(x+\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1\right) \begin{bmatrix} n \\ x+\lfloor\frac{f(n)}{2\alpha}\rfloor+1 \end{bmatrix}_q q^{\alpha(a-(x+1))^2}.$$

Using the same arguments as above yields

$$= \left\lfloor\frac{f(n)}{2\alpha}\right\rfloor \gamma - e_q(q)\left(\Sigma_1^\infty(x) - \Sigma_2^\infty(x)\right) + e_q(q)\Sigma_2^\infty + \mathrm{o}(1). \tag{1.27}$$

Combining (1.26) and (1.27) we obtain

$$\mu_n = \left\lfloor\frac{f(n)}{2\alpha}\right\rfloor + \frac{\sum_{x=0}^\infty q^{\alpha(a-(x+1))^2} - \sum_{x=0}^\infty x\left(q^{\alpha(a+x)^2} - q^{\alpha(a-(x+1))^2}\right)}{\sum_{x=0}^\infty \left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2}\right)} + o(1).$$

Simplifying the fraction yields the theorem. ✐

Now we provide an estimate for the O(1)-term in the preceding theorem.

Lemma 1.5.27. *Let $c(a,\alpha,q)$ be defined as in Theorem 1.5.26. Then*

(i) $0 \leq c(a,\alpha,q) < 1$,

(ii) $c(a,\alpha,q) = 0 \Leftrightarrow a = 0$,

(iii) $c(a,\alpha,q) + c(1-a,\alpha,q) = 1$.

Proof. Since for all $x \geq 0$

$$q^{\alpha(-a+x)^2} \geq q^{\alpha(a+x)^2}, \tag{1.28}$$

$0 \leq c(a,\alpha,q)$. Moreover, $c(a,\alpha,q) = 0$ iff in (1.28) equality holds for all $x \geq 1$. But this is the case iff $(x-a)^2 = (x+a)^2$ for all x. So $c(a,\alpha,q) = 0$ iff $a = 0$. For (i) it remains to show that

$$\sum_{x=1}^{\infty} x\left(q^{\alpha(-a+x)^2} - q^{\alpha(a+x)^2}\right) < \sum_{x=1}^{\infty}\left(q^{\alpha(-a+x)^2} + q^{\alpha(a+x)^2}\right) + q^{\alpha a^2}.$$

We can rewrite this as

$$\sum_{x=1}^{\infty}(x-1)q^{\alpha(x-a)^2} - \sum_{x=0}^{\infty}(x+1)q^{\alpha(x+a)^2} < 0.$$

The left-hand side is increasing in a, and for $a = 1$ we have

$$\sum_{x=1}^{\infty}(x-1)q^{\alpha(x-1)^2} - \sum_{x=0}^{\infty}(x+1)q^{\alpha(x+1)^2} = 0.$$

Since $0 \leq a < 1$, (ii) follows.

To see (iii), note that the denominators of $c(a,\alpha,q)$ and $c(1-a,\alpha,q)$ are invariant under the substitution $a \mapsto 1-a$. Then add the numerators:

$$\sum_{x=1}^{\infty} x\left(q^{\alpha(1-a-x)^2} - q^{\alpha(1-a+x)^2}\right) + \sum_{x=1}^{\infty} x\left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2}\right),$$

by splitting the first sum and shifting the summation index we obtain

$$\begin{aligned}
&= \sum_{x=0}^{\infty}(x+1)q^{\alpha(a+x)^2} - \sum_{x=2}^{\infty}(x-1)q^{\alpha(a-x)^2} + \sum_{x=1}^{\infty} x\left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2}\right)\\
&= q^{\alpha a^2} + \sum_{x=1}^{\infty} q^{\alpha(a+x)^2} + q^{\alpha(a-1)^2} + \sum_{x=2}^{\infty} q^{\alpha(a-x)^2}\\
&= \sum_{x=0}^{\infty}\left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2}\right),
\end{aligned}$$

which is exactly the denominator of $c(a,\alpha,q)$. ⍉

Lemma 1.5.28. *Let $c(a,\alpha,q)$ be defined as in Theorem 1.5.26.*

(i) If $a > 0$, then $\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + c(a,\alpha,q) \notin \mathbb{Z}$.

(ii) If $a = 0$, then

$$\mu_n \begin{cases} \geq \frac{f(n)}{2\alpha} & \text{if } n \geq \frac{f(n)}{\alpha} \\ < \frac{f(n)}{2\alpha} & \text{if } n < \frac{f(n)}{\alpha} \end{cases}.$$

Proof. Lemma 1.5.27 implies (i). To see (ii) we use

$$\mu_n = \frac{f(n)}{2\alpha} + \frac{\sum_{x=1}^{n-\frac{f(n)}{2\alpha}} x \begin{bmatrix} n \\ x+\frac{f(n)}{2\alpha} \end{bmatrix}_q q^{\alpha x^2} - \sum_{x=1}^{\frac{f(n)}{2\alpha}} x \begin{bmatrix} n \\ \frac{f(n)}{2\alpha}-x \end{bmatrix}_q q^{\alpha x^2}}{\sum_{x=0}^{\frac{f(n)}{2\alpha}} \begin{bmatrix} n \\ \frac{f(n)}{2\alpha}-x \end{bmatrix}_q q^{\alpha x^2} + \sum_{x=1}^{n-\frac{f(n)}{2\alpha}} \begin{bmatrix} n \\ x+\frac{f(n)}{2\alpha} \end{bmatrix}_q q^{\alpha x^2}}.$$

Now consider the case $n \geq \frac{f(n)}{\alpha}$: We have to prove that

$$\sum_{x=1}^{n-\frac{f(n)}{2\alpha}} x \begin{bmatrix} n \\ x+\frac{f(n)}{2\alpha} \end{bmatrix}_q q^{\alpha x^2} \geq \sum_{x=1}^{\frac{f(n)}{2\alpha}} x \begin{bmatrix} n \\ \frac{f(n)}{2\alpha}-x \end{bmatrix}_q q^{\alpha x^2}.$$

For all $1 \leq x \leq \frac{f(n)}{2\alpha}$, the term on the right-hand side is less than or equal to the corresponding term on the left-hand side (there are enough terms on the left-hand side by our assumption), i.e.,

$$\begin{bmatrix} n \\ x+\frac{f(n)}{2\alpha} \end{bmatrix}_q \geq \begin{bmatrix} n \\ \frac{f(n)}{2\alpha}-x \end{bmatrix}_q,$$

since

$$\begin{aligned}
& \begin{bmatrix} n \\ x+\frac{f(n)}{2\alpha} \end{bmatrix}_q \geq \begin{bmatrix} n \\ \frac{f(n)}{2\alpha}-x \end{bmatrix}_q \\
\iff \quad & \frac{1}{(q,q)_{x+\frac{f(n)}{2\alpha}}(q,q)_{n-x-\frac{f(n)}{2\alpha}}} \geq \frac{1}{(q,q)_{f-\frac{f(n)}{2\alpha}}(q,q)_{n-\frac{f(n)}{2\alpha}+x}} \\
\iff \quad & \left(1-q^{n-x-\frac{f(n)}{2\alpha}+1}\right)\cdots\left(1-q^{n-\frac{f(n)}{2\alpha}+x}\right) \geq \left(1-q^{\frac{f(n)}{2\alpha}-x+1}\right)\cdots\left(1-q^{x+\frac{f(n)}{2\alpha}}\right) \\
\iff \quad & n-x-\frac{f(n)}{2\alpha}+1+i \geq \frac{f(n)}{2\alpha}-x+1+i \qquad 0 \leq i \leq 2x-1 \\
\iff \quad & n-\frac{f(n)}{2\alpha} \geq \frac{f(n)}{2\alpha}.
\end{aligned}$$

The case $n < \frac{f(n)}{4\alpha}$ can be treated similarly.

In Section 1.3.2 we studied the behaviour of the means of Kemp's q-binomial distribution in the limit $q \to 0$ and $c(a,\alpha,q)$ in the limit $q \to 1$. We will do the same here now. First we will show that for $q \to 0$

$$\mu_n \to \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + \begin{cases} 0 & \text{if } 0 \leq a < \frac{1}{2} \\ \frac{1}{2} & \text{if } a = \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < a < 1 \end{cases}.$$

For this purpose we estimate $c(a,\alpha,q)$:

$$\begin{aligned}
c(a,\alpha,q) &= \frac{q^{\alpha(1-a)^2}+2q^{\alpha(2-a)^2}+\sum_{x=3}^{\infty} xq^{\alpha(a-x)^2}-\sum_{x=1}^{\infty} xq^{\alpha(a+x)^2}}{q^{\alpha a^2}+q^{\alpha(a-1)^2}+\sum_{x=1}^{\infty} q^{\alpha(a+x)^2}+\sum_{x=2} q^{\alpha(a-x)^2}} \\
&\leq \frac{q^{\alpha(1-a)^2}+2q^{\alpha(2-a)^2}+\sum_{x=3}^{\infty} xq^{\alpha(1-x)^2}-\sum_{x=1}^{\infty} xq^{\alpha(1+x)^2}}{q^{\alpha a^2}+q^{\alpha(a-1)^2}+\sum_{x=1}^{\infty} q^{\alpha(1+x)^2}+\sum_{x=2}^{\infty} q^{\alpha x^2}} \\
&= \frac{q^{\alpha(1-a)^2}+2q^{\alpha(2-a)^2}+2\sum_{x=2}^{\infty} q^{\alpha x^2}}{q^{\alpha a^2}+q^{\alpha(a-1)^2}+2\sum_{x=2}^{\infty} q^{\alpha x^2}} \\
&= \frac{1+2q^{\alpha(3-2a)}+2q^{-\alpha(1-a)^2}\sum_{x=2}^{\infty} q^{\alpha x^2}}{q^{\alpha(-1+2a)}+1+2q^{-\alpha(1-a)^2}\sum_{x=2}^{\infty} q^{\alpha x^2}}.
\end{aligned}$$

For $a \in [0, \frac{1}{2})$ we have $2a-1 < 0$ and therefore the denominator tends to infinity while the numerator goes to 1. Consequently $c(a,\alpha,q) \to 0$. Lemma 1.5.27 (iii) implies that $c(a,\alpha,q) \to 1$ if $a \in (\frac{1}{2}, 1)$ and $c(\frac{1}{2},\alpha,q) = \frac{1}{2}$. Moreover, from the estimates in the proof of Theorem 1.5.26 we get easily that the o(1)-term vanishes in the limit $q \to 0$.

In the limit $q \to 1$ we have $c(a, \alpha, q) \to a$. To see this, apply the Euler-Maclaurin formula to

$$f^+(x) = q^{Ax^2+Bx} \qquad \text{and} \qquad g^+(x) = xq^{Ax^2+Bx}$$

first, which yields

$$\sum_{\ell \geq 0} f^+(\ell) = I_f^+ + \frac{f^+(0)}{2} + R_f^+$$

with

$$I_f^+ = \int_0^\infty f^+(x)\mathrm{d}x \qquad \text{and} \qquad R_f^+ = \int_0^\infty \left(x - \lfloor x \rfloor - \frac{1}{2}\right) f^{+'}(x)\mathrm{d}x.$$

Computing I_f^+ gives

$$I_f^+ = \frac{\sqrt{\pi} q^{-\frac{B^2}{4A}} \left(1 + \operatorname{erf}\left(\frac{B \log q}{2\sqrt{-A \log q}}\right)\right)}{2\sqrt{-A \log q}},$$

where $\operatorname{erf}(z)$ denotes the error-function. Similarly, we get for $g^+(x)$

$$\sum_{\ell \geq 0} g^+(\ell) = I_g^+ + \frac{g^+(0)}{2} + R_g^+$$

with

$$I_g^+ = \int_0^\infty g^+(x)\mathrm{d}x \qquad \text{and} \qquad R_g^+ = \int_0^\infty \left(x - \lfloor x \rfloor - \frac{1}{2}\right) g^{+'}(x)\mathrm{d}x.$$

Computing I_g^+ gives

$$I_g^+ = \frac{-B\sqrt{\pi} q^{-\frac{B^2}{4A}} \left(1 + \operatorname{erf}\left(\frac{B \log q}{2\sqrt{-A \log q}}\right)\right)}{4A\sqrt{-A \log q}} - \frac{1}{2}\frac{1}{A \log q}.$$

In an analogous way we treat the functions

$$f^-(x) = q^{Ax^2-Bx} \qquad \text{and} \qquad g^-(x) = xq^{Ax^2-Bx}.$$

Note that $f^-(0) = f^+(0) = \frac{1}{2}$ and $g^-(0) = g^+(0) = 0$. Putting things together we obtain with $A = \alpha$ and $B = 2\alpha a$

$$\begin{aligned} c(a, \alpha, q) &= \frac{I_g^- + R_g^- - I_g^+ - R_g^+}{I_f^+ + I_f^- + R_f^+ + R_f^- + q^{-a^2}} \\ &= \frac{\frac{B\sqrt{\pi} q^{-\frac{B^2}{4A}}}{2A\sqrt{-A \log q}} + R_g^- - R_g^+}{\frac{\sqrt{\pi} q^{-\frac{B^2}{4A}}}{\sqrt{-A \log q}} + R_f^- + R_f^+ + q^{-a^2}} \\ &= \frac{a + q^{\frac{B^2}{4A}} \frac{\sqrt{-A \log q}}{\sqrt{\pi}} \left(R_g^- - R_g^+\right)}{1 + q^{\frac{B^2}{4A}} \frac{\sqrt{-A \log q}}{\sqrt{\pi}} \left(R_f^- + R_f^+ + q^{-a^2}\right)}. \end{aligned}$$

Thus it remains to show that $\sqrt{-\log q}(R_g^- - R_g^+)$ and $\sqrt{-\log q}(R_f^- + R_f^+)$ both tend to 0. We have

$$R_f^+ = \int_0^\infty \left(x - \lfloor x \rfloor - \frac{1}{2}\right) q^{Ax^2+Bx} \log q (2Ax + B)\mathrm{d}x.$$

The integral

$$J_1 := B \int_0^\infty \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} \mathrm{d}x$$

is bounded uniformly for all $q \in [0,1)$, since q^{Ax^2+Bx} is decreasing in x:

$$\begin{aligned}
-\frac{1}{4} &= -\frac{1}{4} + \sum_{i=0}^{\infty} 0 \\
&\leq \int_0^{\frac{1}{2}} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} \mathrm{d}x + \sum_{i=0}^{\infty} \int_{\frac{1}{2}+i}^{\frac{1}{2}+i+1} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} \mathrm{d}x \\
&= J_1 \\
&= \int_0^1 \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} \mathrm{d}x + \sum_{i=1}^{\infty} \int_i^{i+1} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} \mathrm{d}x \\
&\leq 0 + \sum_{i=1}^{\infty} 0 = 0.
\end{aligned}$$

Thus $(-\log q)^{3/2} J_1 \to 0$. With the same idea we want to estimate

$$J_2 := 2A \int_0^\infty \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} x \mathrm{d}x.$$

Unfortunately $h(x) := q^{Ax^2+Bx} x$ must not be decreasing in x for $x \geq 0$. Differentiating gives

$$h'(x) = q^{Ax^2+Bx} \left(1 + (2Ax^2 + Bx) \log q \right).$$

Obvious $h'(0) > 0$ and $\lim_{x \to -\infty} h'(x) = \lim_{x \to \infty} h'(x) = -\infty$ since $\log q < 0$. Consequently there exists a single positive root r of $h'(x)$. For q near at 1 we have $r \leq 1/\sqrt{-A \log q}$ since

$$\begin{aligned}
h' \left(\frac{1}{\sqrt{-A \log q}} \right) &= 1 + \log q \left(-\frac{2A}{A \log q} + \frac{B}{\sqrt{-A \log q}} \right) \\
&= 1 - 2 + \frac{B\sqrt{-\log q}}{\sqrt{A}} < 0.
\end{aligned}$$

Thus $h(x)$ is decreasing for $x \geq r$. Split J_2 into integrals over $[0, \lceil r \rceil]$ and $[\lceil r \rceil, \infty)$. The second integral is bounded by same arguments as above. The first integral can be estimated trivially by

$$2A \left| \int_0^{\lceil r \rceil} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} x \mathrm{d}x \right| \leq A \int_0^{\lceil r \rceil} x \leq A \lceil r \rceil^2.$$

Therefore $\sqrt{-\log q} R_f^+ \to 0$ for $q \to 1$. Analogously we get $\sqrt{-\log q} R_f^- \to 0$. In order to show that the term with

$$R_g^+ = \int_0^\infty \left(x - \lfloor x \rfloor - \frac{1}{2} \right) \left(q^{Ax^2+Bx} + x q^{Ax^2+Bx} \log q (2Ax + B) \right) \mathrm{d}x$$

vanishes, it remains to consider the integral

$$J_3 := \int_0^\infty \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} q x^2 \mathrm{d}x.$$

Again we compute where $H(x) := x^2 q^{Ax^2+Bx}$ is decreasing. We have

$$H'(x) = q^{Ax^2+Bx} \left(2x + (2Ax^3 + Bx^2) \log q \right)$$

and therefore $\lim_{x\to-\infty} H'(x) = +\infty$, $\lim_{x\to\infty} H'(x) = -\infty$ and $H'(0) = 0$. Since $H''(0) > 0$, there exists a single positive root s of $H'(x)$. Moreover $s \le 1/\sqrt{-A\log q}$ since

$$\begin{aligned} H'\left(\frac{1}{\sqrt{-A\log q}} \right) &= \frac{2}{\sqrt{-A\log q}} + \log q \left(\frac{2a}{(-A\log q)^{3/2}} - \frac{B}{A\log q} \right) \\ &= \frac{2}{\sqrt{-A\log q}} - \frac{2}{\sqrt{-A\log q}} - \frac{B}{A} \le 0. \end{aligned}$$

Thus $H(x)$ is decreasing for $x \ge s$. Split the integral into integrals over $[0, \lfloor s \rfloor]$, $[\lfloor s \rfloor, \lceil s \rceil]$ and $[\lceil s \rceil, \infty)$. The third integral is bounded as above. The second integral is trivially bounded by $\frac{1}{2}\lceil s \rceil^2$. And the first integral - the increasing part - we estimate with the same ideas as for the decreasing part:

$$\begin{aligned} 0 &\le \sum_{i=0}^{\lfloor s \rfloor -1} \int_i^{i+1} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} q x^2 \mathrm{d}x \\ &= \int_0^{\lfloor s \rfloor} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} x^2 \mathrm{d}x \\ &= \int_0^{\frac{1}{2}} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} x^2 \mathrm{d}x + \sum_{i=0}^{\lfloor s \rfloor -2} \int_{\frac{1}{2}+i}^{\frac{1}{2}+i+1} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} x^2 \mathrm{d}x \\ &\quad + \int_{\lfloor s \rfloor - \frac{1}{2}}^{\lfloor s \rfloor} \left(x - \lfloor x \rfloor - \frac{1}{2} \right) q^{Ax^2+Bx} x^2 \mathrm{d}x \\ &\le 0 + \sum_{i=0}^{\lfloor s \rfloor -2} 0 + \frac{1}{4} \lfloor s \rfloor^2 \end{aligned}$$

Therefore $\sqrt{-\log q} R_g^+ \to 0$ for $q \to 1$. In a similar way we find $\sqrt{-\log q} R_g^- \to 0$.

After this analysis of the means, we turn our attention to the sequence of variances.

Lemma 1.5.29. *For $n \to \infty$ we have*

$$\sigma_n^2 = \phi(a, \alpha, q) - c(a, \alpha, q)^2 + \mathrm{o}(1),$$

where

$$\phi(a, \alpha, q) := \frac{e_q(q)}{\gamma} \sum_{x=1}^\infty x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right).$$

Proof. By definition we have

$$\mathbb{E}\left(X_n^2\right) = \frac{\sum_{x=0}^{n} x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}{\sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}.$$

Now we proceed as in the proof of Theorem 1.5.26 and study the numerator $\tilde{N}$ after expansion by $q^{\frac{f(n)^2}{4\alpha}}$.

$$\tilde{N} = \sum_{x=0}^{n} x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2 - f(n)x + \frac{f(n)^2}{4\alpha}};$$

we split the sum and reverse the order of summation resp. shift the summation index and get

$$\begin{aligned} &= \sum_{x=0}^{\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor} \left(\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - x \right)^2 \begin{bmatrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{bmatrix}_q q^{\alpha(a+x)^2} + \\ &\quad + \sum_{x=0}^{n - \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor - 1} \left(x + \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + 1 \right)^2 \begin{bmatrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{bmatrix}_q q^{\alpha(a-(x+1))^2}, \end{aligned}$$

which can be written as

$$\begin{aligned} &= \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \Sigma_1 - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1(x) + \Sigma_1(x^2) + \Sigma_2(x^2) + \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \Sigma_2 + \Sigma_2 \\ &\quad + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2(x) + 2\Sigma_2(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2. \end{aligned}$$

Using similar arguments as above yields

$$\begin{aligned} &= \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \gamma + e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) \right. \\ &\quad \left. + \Sigma_1^\infty(x^2) + \Sigma_2^\infty(x^2) + \Sigma_2^\infty + 2\Sigma_2^\infty(x) \right) + \mathrm{o}(1). \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}\left(X_n^2\right) &= \frac{1}{\gamma} \left(\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \gamma + e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) \right) \right) \\ &\quad + \phi(a, \alpha, q). \end{aligned}$$

Since

$$\begin{aligned} \mu_n^2 &= \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 - \frac{e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty \right)}{\gamma} \\ &\quad + c(a, \alpha, q)^2 + \mathrm{o}(1), \end{aligned}$$

we obtain

$$\sigma_n^2 = \phi(a, \alpha, q) - c(a, \alpha, q)^2 + \mathrm{o}(1).$$

❦

Lemma 1.5.30.

$$\phi(a,\alpha,q) > c(a,\alpha,q)^2.$$

Proof. We have to show that

$$\frac{\sum_{x=1}^{\infty} x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2}\right)}{\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}} > \left(\frac{\sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2}\right)}{\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}}\right)^2.$$

A sufficient condition for this is

$$\frac{\sum_{x=1}^{\infty} x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2}\right)}{\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}} > \left(\frac{\sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2}\right)}{\sum_{x=0}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}}\right)^2.$$

We show that

$$\left(\sum_{x=1}^{\infty} x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2}\right)\right)\left(\sum_{x=1}^{\infty} q^{\alpha(a+x)^2} + \sum_{x=1}^{\infty} q^{\alpha(a-x)^2}\right) \geq$$
$$\geq \left(\sum_{x=1}^{\infty} x \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2}\right)\right)^2.$$

Expanding gives

$$\begin{aligned}
&\sum_{x,y=1}^{\infty} x^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} \\
&\quad + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} \\
&\geq \sum_{x,y=1}^{\infty} x q^{\alpha(x+a)^2} y q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x q^{\alpha(x-a)^2} y q^{\alpha(y-a)^2} \\
&\quad + 2\sum_{x,y=1}^{\infty} x q^{\alpha(x+a)^2} y q^{\alpha(y-a)^2}.
\end{aligned}$$

Now we consider the pairs (x,y) and (y,x) again and obtain

$$\begin{aligned}
&x^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + y^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + x^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} + y^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} \\
&\quad + x^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} + y^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} + x^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} + y^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} \\
&\quad \geq 2xy q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + 2xy q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} + 2xy q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} \\
&\qquad + 2xy q^{\alpha(x-a)^2} q^{\alpha(y+a)^2}.
\end{aligned}$$

This is true since $x^2 + y^2 \geq 2xy$. ∎

Now we are able to establish the next convergence result. For this purpose recall that $c(a,\alpha,q)$ and $\phi(a,\alpha,q)$ depend on the fractional part of $\frac{f(n)}{2\alpha}$. Since convergent variances and convergent fractional parts of means are required for convergence to a discrete distribution, we will choose a subsequence (n_k) of (n) such that $\{\frac{f(n)}{2\alpha}\}$ remains constant.

Theorem 1.5.31. *Let (n_k) be an increasing sequence of natural numbers and $X_{n_k} \sim \mathcal{B}(\alpha, q^{-f(n_k)}, n_k, q)$ such that $\{\frac{f(n)}{2\alpha}\} = a$ constant. Recall that we always assume $f(n) \to \infty$, $2\alpha n - f(n) \to \infty$ and $\alpha > 0$. Then $(X_{n_k} - \mu_{n_k})/\sigma_{n_k}$ converges for $k \to \infty$ to a normalised discrete normal distribution, i.e., they converge to a limit X with*

$$\mathbb{P}\left(X = \frac{1}{\sigma}\left(x - c(a, \alpha, q)\right)\right) = \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2}},$$

where $\sigma = \lim_{k\to\infty} \sigma_{n_k}$ and $c(a, \alpha, q)$ is defined as in Theorem 1.5.26..

Proof. For simplicity we write in the following n instead of n_k. First we note that Lemmas 1.5.29 and 1.5.30 imply that the sequence of variances (σ_n^2) converges since $\{\frac{f(n)}{2\alpha}\}$ is constant by assumption. We define

$$H(\mu_n) := \begin{cases} \lfloor \mu_n \rfloor & \text{if } a > 0 \\ \lfloor \mu_n \rfloor & \text{if } a = 0, n \geq \frac{f(n)}{4\alpha} \\ \lceil \mu_n \rceil & \text{if } a = 0, n < \frac{f(n)}{4\alpha} \end{cases}.$$

Since $H(\mu_n) = \frac{f(n)}{2\alpha} - a$, we have

$$\begin{aligned} \mathbb{P}\left(X_n = H(\mu_n) + x\right) &= \begin{bmatrix} n \\ \frac{f(n)}{2\alpha} - a + x \end{bmatrix}_q \frac{q^{\left(\frac{f(n)}{2\alpha} - a + x\right)^2 - f(n)\left(\frac{f(n)}{2\alpha} - a + x\right)}}{\sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2 - f(n) y}} \\ &= \begin{bmatrix} n \\ \frac{f(n)}{2\alpha} - a + x \end{bmatrix}_q \frac{q^{-\frac{f(n)^2}{4\alpha} + \alpha(x-a)^2}}{\sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2 - f(n) y}} \\ &\to e_q(q) \frac{q^{\alpha(x-a)^2}}{e_q(q) \sum_{x=0}^{\infty} \left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2}\right)} \\ &= \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(a+x)^2}} \\ &= \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2}}. \end{aligned}$$

By normalising we get the theorem. ∎

For $\alpha = \frac{1}{2}$ this theorem reduces to the convergence property of Kemp's binomial distribution established in Section 1.3.2

Using Jacobi's Triple Product we can rewrite the infinite sum as

$$\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2} = q^{\alpha a^2} \left(q^{2\alpha}, q\right)_\infty \left(-q^{\alpha - 2\alpha a}, q\right)_\infty \left(-q^{\alpha + 2\alpha a}, q\right)_\infty.$$

In the limit $q \to 1$ these discrete normal distributions converge to the standard normal distribution, see [56].

Chapter 2

Distribution of Sequences

So far we studied sequences of random variables and their limit laws. In this chapter we investigate again sequences, but sequences of real numbers and the induced sequences of measures and distribution functions.

The first topic we are interested in is a very classical one. We study sequences of real numbers (x_n) which are uniformly distributed modulo 1, i.e., the finite measures induced by the sequence (x_n) converge to the Lebesgue measure (for details see below). In [25] Goldstern, Winkler and Schmeling found out that for a given sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers with certain growth conditions the set of those $\alpha \in [0,1)$ such that the sequence $(n_k\alpha)_{k\in\mathbb{N}}$ is uniformly distributed modulo 1 is of first Baire's category. Our goal is to extend this and related results to multisequences, in particular to sequences with multidimensional indices. This is done in Section 2.1.

In Section 2.2 we investigate the relationship between the limit distribution functions of a given sequence $(x_n)_{n\in\mathbb{N}}$ and the limit distribution functions of the corresponding block sequence, i.e., we divide the given sequence into blocks $x_{\frac{n(n-1)}{2}+1}, \dots, x_{\frac{n(n+1)}{2}}$, associate to this block a step distribution function and consider the limits of this sequence. It is possible - under certain conditions - to obtain the limit distribution functions of the original sequence (x_n) from the limits of the block sequence by taking convex combinations of the limits of the block sequence.

2.1 Baire results of multisequences

This section is devoted to the generalisation of Baire results about sequences of real numbers (see Tichy and Zeiner [58]). A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers is called uniformly distributed (u.d.) modulo 1, if for every pair a, b of real numbers with $0 \leq a < b \leq 1$ the following condition holds:

$$\lim_{n\to\infty} \frac{A([a,b), n, \mathbf{x})}{n} = b - a, \tag{2.1}$$

where $A(I, n, \mathbf{x})$ is the number of elements x_i, $i \leq n$ with $x_i \in I$ for an interval I. For a general theory of uniform distribution we refer to Kuipers and Niederreiter [43] and Drmota and Tichy [16].

In [25] Goldstern, Schmeling and Winkler studied the size (in the sense of Baire) of the set

$$\mathcal{U} := \{\alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is uniformly distributed mod } 1\}$$

for a given sequence $\mathbf{n} = (n_j)_{j\in\mathbb{N}}$ of natural numbers; the size of this set depends on the growth rate of the sequence $\mathbf{n}$. In particular they showed that $\mathcal{U}$ is meager if $\mathbf{n}$ grows exponentially (for theory about Baire categories see Oxtoby [52]). By Ajtai, Havas, Komlós [2] this condition cannot be weakened.

Moreover it was proven in [25] that the set

$$\mathcal{V} := \{\alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is maldistributed}\}$$

is residual if $\mathbf{n}$ grows very fast (for the precise statement we refer to [25]).

The aim of this section (which is closely related to the very recent work of Winkler [61]) is to generalise these results in different ways. After summing up some important facts about uniform distribution and Baire categories in Section 2.1.1 we consider in Section 2.1.2 for a given sequence $(\mathbf{n}_j)_{j\in\mathbb{N}}$ of r-dimensional vectors of nonnegative integers and an r-dimensional vector $\boldsymbol{\alpha}$ of real numbers the sequence $(\mathbf{n}_j\boldsymbol{\alpha})_{j\in\mathbb{N}}$, where $\mathbf{n}_j\boldsymbol{\alpha}$ means the scalar product of two vectors. Afterwards we investigate in Section 2.1.3 uniform distribution in $\mathbb{R}^d$, i.e., for a d-dimensional sequence $(\mathbf{n}_j)$ and a d-dimensional vector $\boldsymbol{\alpha}$ we consider the sequence $(\mathbf{n}_j\boldsymbol{\alpha})_{j\in\mathbb{N}}$, where $\mathbf{n}_j\boldsymbol{\alpha}$ means the Hadamard product of two vectors.

Section 2.1.4 is devoted to the generalisation of elementary properties of uniform distribution of sequences to uniform distribution of nets. Afterwards we extend in Section 2.1.5 the characterisation of the set of limit measures of a sequence (see Winkler [62]) to a special kind of nets over $\mathbb{N}^d$. Finally, we turn in Section 2.1.6 to $\mathbf{n}\alpha$-sequences with multidimensional indices. Besides the classical notion of uniform distribution of such sequences (see Kuipers and Niederreiter [43]) we study the $(s_1, \ldots, s_d)$-uniform distribution (see Kirschenhofer and Tichy [41]) and introduce a new concept of uniform distribution modulo 1, which is inspired by Aistleitner [1]. In most of these cases it turns out that the known results for the classical case remain true in these generalised settings.

2.1.1 Preliminaries

For those who are not familiar with the notion of uniform distribution modulo 1 and Baire categories we give the essential definitions and facts below. For details and further reading we refer for uniform distribution to Kuipers and Niederreiter [43] and Drmota and Tichy [16], for Baire categories to Oxtoby [52].

Uniform Distribution of Sequences

Let $c_{[a,b)}$ denote the characteristic function of an interval $[a,b) \subseteq [0,1)$, then (2.1) can be rewritten in the following form:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{[a,b)}(\{x_n\}) = \int_0^1 c_{[a,b)}(x)\mathrm{d}x.$$

Applying an approximation technique leads to the following criteria.

Theorem 2.1.1.

(i) The sequence (x_n) is u.d. mod 1 iff for every real-valued continuous function f defined on $[0,1]$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\mathrm{d}x. \tag{2.2}$$

(ii) The sequence (x_n) is u.d. mod 1 iff for every Riemann-integrable function f defined on $[0,1]$ equation (2.2) holds.

(iii) The sequence (x_n) is u.d. mod 1 iff for every complex-valued continuous function f on $\mathbb{R}$ with period 1 we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\mathrm{d}x.$$

The functions f of the form $f(x) = e^{2\pi i h x}$ ($h \in \mathbb{Z}\setminus\{0\}$) satisfy the conditions of the above theorem (iii). Indeed, they are already sufficient to determine the u.d. mod 1 of a sequence:

Theorem 2.1.2 (Weyl-Criterion)*.* The sequence (x_n) is u.d. mod 1 iff for all $h \in \mathbb{Z}\setminus\{0\}$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0.$$

Example 2.1.3*.* Let θ be an irrational number. The the sequence $(n\theta)_{n\in\mathbb{N}}$ is u.d. mod 1. This follows from the Weyl-Criterion and the inequality

$$\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h n\theta}\right| = \frac{\left|e^{2\pi i h N\theta}-1\right|}{N\left|e^{2\pi i h\theta}-1\right|} \le \frac{2}{N|\sin(\pi h\theta)|}$$

for integers $h \neq 0$.

Example 2.1.4*.* Choosing $\theta = e$ in the example above gives that the sequence (ne) is u.d. mod 1. But the subsequence $(n!e)$ has 0 as the only limit point. To see this, observe that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{e^\alpha}{(n+1)!}, \qquad 0 < \alpha < 1$$

implies that $n!e = k + e^\alpha/(n+1)$ for an integer k. Thus $\{n!e\} = e^\alpha/(n+1) < e/(n+1) \to 0$.

Example 2.1.5*.* Let (a_n) be a given sequence of distinct integers. Then the sequence $(a_n x)$ is u.d. mod 1 for almost all real numbers x.

Uniform Distribution of Double Sequences

We can extend the concept given above in the following way: Instead of sequences $(x_n)_{n\in\mathbb{N}}$ we consider double sequences (x_{jk}), $j\in\mathbb{N}$, $k\in\mathbb{N}$. Then a double sequence is said to be u.d. mod 1 if for any a and b such that $0 \le a < b \le 1$,

$$\lim_{M,N\to\infty} \frac{A([a,b);M,N)}{MN} = b - a,$$

where $A([a,b);M,N)$ is the number of x_{jk}, $1 \le j \le M$, $1 \le k \le N$, for which $a \le \{x_{jk}\} < b$. As for classical sequences we have the following criteria.

Theorem 2.1.6*.*

(i) The double sequence (x_{jk}) is u.d. mod 1 iff for every Riemann-integrable function f on $[0,1]$ we have

$$\lim_{M,N\to\infty} \frac{1}{MN}\sum_{j=1}^{M}\sum_{k=1}^{N} f(\{x_{jk}\}) = \int_0^1 f(x)\mathrm{d}x.$$

(ii) The double sequence (x_{jk}) is u.d. mod 1 iff

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{j=1}^{M} \sum_{k=1}^{N} e^{2\pi i h x_{jk}} = 0$$

for all integers $h \neq 0$.

Example 2.1.7. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\alpha \in \mathbb{R}$. Then $(j\theta + k\alpha)$ is u.d. mod 1.

Uniform Distribution in $\mathbb{R}^d$

Let $(\mathbf{x}_n)_{n\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{R}^d$. For a set $E \subseteq [0,1)^d$, let $A(E;N)$ denote the number of points $\{\mathbf{x}_n\}$, $1 \le n \le N$, that lie in E. The sequence $(\mathbf{x}_n)$ is said to be u.d. mod 1 in $\mathbb{R}^d$ if

$$\lim_{N\to\infty} \frac{A([\mathbf{a},\mathbf{b});N)}{N} = \prod_{j=1}^{d} (b_j - a_j)$$

for all intervals $[\mathbf{a},\mathbf{b}) \subseteq [0,1)^s$. Here $\mathbf{a} = (a_1,\dots,a_d)$, $\mathbf{b} = (b_1,\dots,b_d)$ and $[\mathbf{a},\mathbf{b}) = [a_1,b_1) \times \cdots \times [a_d,b_d)$. As in the case $d = 1$ we have the following criteria.

Theorem 2.1.8.

(i) A sequence $(\mathbf{x}_n)$ is u.d. mod 1 in $\mathbb{R}^d$ iff for every continuous complex-valued function f on $[0,1]^d$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{\mathbf{x}_n\}) = \int_{[0,1]^d} f(\mathbf{x})\mathrm{d}\mathbf{x}.$$

(ii) A sequence $(\mathbf{x}_n)$ is u.d. mod 1 in $\mathbb{R}^d$ iff for every lattice point $\mathbf{h} \in \mathbb{Z}^d$, $\mathbf{h} \neq \mathbf{0}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i \langle \mathbf{h},\mathbf{x}_n\rangle} = 0,$$

where $\langle \cdot,\cdot \rangle$ denotes the standard inner product.

Theorem 2.1.9. A sequence $(\mathbf{x}_n)$ is u.d. mod 1 in $\mathbb{R}^d$ iff for every lattice point $\mathbf{h} \in \mathbb{Z}^d$, $\mathbf{h} \neq \mathbf{0}$, the sequence of real numbers $(\langle \mathbf{h},\mathbf{x}_n\rangle)$ is u.d. mod 1.

Example 2.1.10. Let $\boldsymbol{\theta} = (\theta_1,\dots,\theta_d)$ be a vector of real numbers such that $1,\theta_1,\dots,\theta_d$ are linearly independent over the rationals, then the sequence $(n\boldsymbol{\theta}) = ((n\theta_1,\dots,n\theta_d))$ is u.d. mod 1 in $\mathbb{R}^d$. To see this, note that for all $\mathbf{h} \in \mathbb{Z}^d$, $\mathbf{h} \neq \mathbf{0}$, the real number $\langle \mathbf{h},\boldsymbol{\theta}\rangle$ is irrational, so Theorem 2.1.9 can be applied.

Baire categories

In a topological space X a subset A is *dense* (in X), if the closure of A equals X, i.e., if every non-empty open set contains at least one point of A. A set A is *nowhere dense* if the interior of its closure is empty, that is, if for every non-empty open set G there is a non-empty open set H contained in $G \setminus A$. A set is of *first category* or *meager* if it can be represented as a countable union of nowhere dense sets, otherwise it is of *second category*. Clearly, a subset of a nowhere dense set is nowhere dense, and the union of finitely many nowhere dense sets is nowhere dense.

Moreover, the subset of a meager set is meager and the union of a countable family of sets of first category is of first category.

A topological space X is called a *Baire space* if every non-empty open set in X is of second category, or equivalently, if the complement of every meager set is dense. In a Baire space, the complement of any set of first category is called a *residual* set. By the Theorem of Baire ([52, Theorem 1.3]), $\mathbb{R}^k$ is a Baire space.

If $E \subseteq X \times Y$ and $x \in X$ the set E_x is defined as

$$E_x = \{y : \ (x, y) \in E\}.$$

Theorem 2.1.11 (Kuratovski-Ulam). If E is a plane set of first category, then E_x is a linear set of first category for all x except a set of first category. If E is a nowhere dense subset of the plane $X \times Y$, then E_x is a nowhere dense subset of Y for all x except a set of first category in X.

With this theorem we can prove the following theorem:

Theorem 2.1.12. A product set $A \times B$ is meager in $X \times Y$ iff at least one of the sets A or B is meager.

Proof. If G is a dense open subset of X, then $G \times Y$ is a dense open subset of $X \times Y$. Hence $A \times B$ is nowhere dense in $X \times Y$ whenever A is nowhere dense in X. Since $(\bigcup A_i) \times B) = \bigcup (A_i \times B)$, it follows that $A \times B$ is meager whenever A is meager. The same argument applies to B.

Conversely, if $A \times B$ is meager and A is not, then by the preceding theorem there exists a point x in A such that $(A \times B)_x$ is of first category. Since $(A \times B)_x = B$ for all x in A, B is of first category.

2.1.2 Vectors

In this section let $(\mathbf{n}_j)_{j \in \mathbb{N}}$ be a sequence of r-dimensional vectors of nonnegative integers, i.e.

$$\mathbf{n}_j = (n_{j,1}, \ldots, n_{j,r}) \qquad \text{with} \qquad n_{j,i} \in \mathbb{N},$$

and let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ denote an r-dimensional vector of real numbers $0 \leq \alpha_i \leq 1$, $i = 1, \ldots, r$. We are now interested in the distribution of the sequence

$$\mathbf{n}\boldsymbol{\alpha} := (\mathbf{n}_j \boldsymbol{\alpha})_{j \in \mathbb{N}} \qquad \text{with} \qquad \mathbf{n}_j \boldsymbol{\alpha} = \sum_{i=1}^{r} n_{j,i} \alpha_i.$$

Note that $\mathbf{n}\boldsymbol{\alpha}$ is a one-dimensional sequence of real numbers. To study the size in the sense of Baire of the set

$$\mathcal{U} = \{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^r : \ (\mathbf{n}_j \boldsymbol{\alpha})_{j \in \mathbb{N}} \text{ is uniformly distributed mod } 1\}$$

we follow Goldstern, Schmeling, Winkler [25]. For this purpose we generalise the definition of ε-mixing sequences of functions:

Definition 2.1.13. A sequence of functions $f_i : [0, 1)^r \to [0, 1)$ is called ε-mixing in $(\delta_1, \ldots, \delta_r)$ if for all sequences of intervals $J_1, J_2, \ldots$ of length ε and for all cuboids J' of size $\delta_1 \times \cdots \times \delta_r$ and for all $k \geq 0$

$$J' \cap \bigcap_{i=1}^{k} f_i^{-1}(J_i)$$

contains an inner point.

To proceed further we need a criterion when a sequence of functions is ε-mixing in $(\delta_1, \ldots, \delta_r)$:

Lemma 2.1.14. *Let $f_j : [0,1)^r \to [0,1)$ be the function mapping $\boldsymbol{\alpha}$ to $\mathbf{n}_j\boldsymbol{\alpha}$ modulo 1, where $(\mathbf{n}_j)_{j\in\mathbb{N}}$ is a sequence of r-dimensional vectors of nonnegative integers satisfying*

1. $n_{j+1,s} > \frac{4}{\varepsilon} n_{j,s}$ *for all j and*
2. $n_{0,s} > \frac{\varepsilon}{2\delta_s}$

for a fixed $s \in \{1, \ldots, r\}$. Then $(f_1, f_2, \ldots)$ is ε-mixing in $(\delta_1, \ldots, \delta_r)$.

Proof. For simplicity we just prove the case $r = 2$. Let $\mathbf{n}_j = (m_j, n_j)$ and assume that the conditions (1) and (2) hold for the sequence of n_j. We will show (by induction on k) that each set

$$J' \cap \bigcap_{i=1}^{k} f_i^{-1}(J_i)$$

contains a cuboid of size $c_k \times \frac{\varepsilon}{2n_k}$ with $c_k > 0$. This is true for $k = 0$, since $\delta_2 > \frac{\varepsilon}{2n_0}$ and $c_0 := \delta_1 > 0$.

Consider $k > 0$. Note that $f_k^{-1}(J_k)$ is a union of stripes of height $\frac{\varepsilon}{n_k}$ and distance $\frac{1-\varepsilon}{n_k}$ (see Figure 2.1). By induction hypothesis, the set $J' \cap \bigcap_{i=1}^{k-1} f_i^{-1}(J_i)$ contains a cuboid I of size

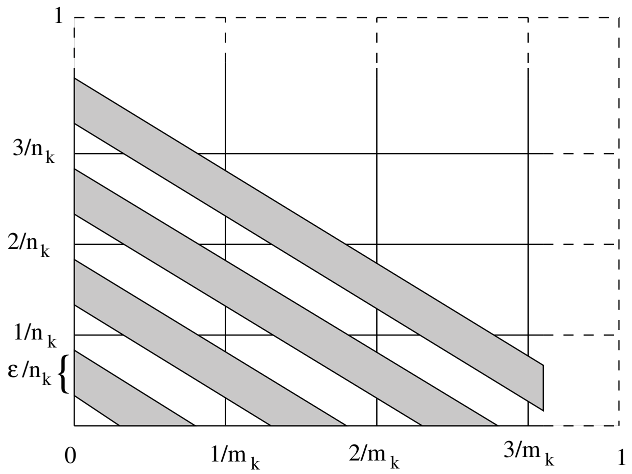

Figure 2.1:

$c_{k-1} \times \frac{\varepsilon}{2n_{k-1}}$. Since $\frac{\varepsilon}{2n_{k-1}} > \frac{2}{n_k}$, I crosses one stripe - say S - of height $\frac{\varepsilon}{n_k}$. Thus $I \cap S$ contains a cuboid I' of size $c_k \times \frac{\varepsilon}{2n_k}$ for some $c_k \le c_{k-1}$ (see Figure 2.2).

To be able to state the theorem, we need the following definitions.

Definition 2.1.15. For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers we define the measures $\mu_{\mathbf{x},n}$ by

$$\mu_{\mathbf{x},n} = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i},$$

where δ_x denotes the point measure in x. The set of accumulation points of the sequence $(\mu_{\mathbf{x},n})_{n\in\mathbb{N}}$ is denoted by $M(\mathbf{x})$ and is called the set of limit measures of the sequence $\mathbf{x}$.

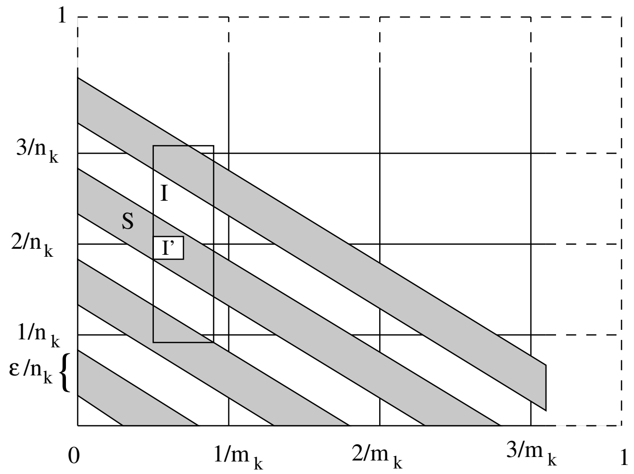

Figure 2.2:

Definition 2.1.16. For any sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and any interval I we define $\overline{\mu}_{\mathbf{x}}(I)$ by

$$\overline{\mu}_{\mathbf{x}}(I) := \sup\{\mu(I) : \mu \in M(\mathbf{x})\}.$$

Now we can establish the theorem, which shows that the set of r-dimensional real vectors $\boldsymbol{\alpha}$, such that $\mathbf{n}\boldsymbol{\alpha}$ is uniformly distributed mod 1 is meager, if at least one component of the $\mathbf{n}_j$ grows exponentially.

Theorem 2.1.17. *Let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of r-dimensional vectors of nonnegative integers and assume $q := \liminf_j(n_{j,s+1}/n_{j,s}) > 1$ for an $s \in \{1,\dots,r\}$. Then the set*

$$\mathcal{U} := \{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^r : \ (\mathbf{n}_j\boldsymbol{\alpha})_{j\in\mathbb{N}} \textit{ is uniformly distributed mod 1}\}$$

is meager.

Moreover: There is a number $Q > 0$ such that for all intervals I the set

$$\{\boldsymbol{\alpha} : \ \overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I) > \frac{Q}{-\log\lambda(I)}\}$$

is residual.

Before proving this theorem we state the following fact. It is completely analogous to the one-dimensional case. For details we refer to [25].

Definition 2.1.18. For an open cuboid I and a Borel set B we write $I \Vdash B$ for "$B \cap I$ is residual in I" or equivalently "$I \setminus B$ is meager".

Fact 2.1.19. Let I be an open cuboid.

1. If B_n is a Borel set for every $n \in \{0,1,2,\dots\}$ and $I \cap \bigcup_n B_n$ is residual in I, then there is some open nonempty cuboid $J \subseteq I$ and some n such that B_n is residual in J, i.e.,

$$I \Vdash \bigcup_{n\in\mathbb{N}} B_n \Rightarrow \exists J \subseteq I \ \exists n \in \mathbb{N} : J \Vdash B_n.$$

2. If B_n is a Borel set for every $n \in \{0, 1, 2, \dots\}$, then $I \cap \bigcap_n B_n$ is residual in I iff each $I \cap B_n$ is residual in B_n:
$$I \Vdash \bigcap_{n\in\mathbb{N}} B_n \Leftrightarrow \forall n \in \mathbb{N} : I \Vdash B_n.$$

3. If B is a Borel set then $B \cap I$ is not residual in I iff there is some open cuboid $J \subseteq I$ such that B is meager in J:
$$I \nVdash B \Leftrightarrow \exists J \subseteq I : J \Vdash B^C,$$
where B^C denotes the complement of B.

Proof of the theorem. The proof is completely analogous to the proof of Theorem 2.4 in [25], we have just to adapt the choice of Q and notation.
Choose $Q > 0$ so small that $(\frac{1}{4Q} - 1) - \log 4 > 1$.
Without loss of generality we may assume $\frac{n_{j+1,s}}{n_{j,s}} > q$ for all k.
Let $\varepsilon := \lambda(I)$. Since $\overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}} > \frac{Q}{-\log\lambda(I)}$ will be trivially true for large intervals if we choose Q small enough, we may assume $\varepsilon < \frac{1}{q}$, so $(-\log\varepsilon) > 1$; here log always denotes the logarithm to base q. Hence $(-\log\varepsilon)(\frac{1}{4Q} - 1) - \log 4 > 1$, thus the interval
$$(\log 4 - \log\varepsilon, -\frac{1}{4Q}\log\varepsilon)$$
has length > 1. Let c be an integer in this interval. Then

- $q^c > \frac{4}{\varepsilon}$.
- $\frac{1}{2c} > \frac{2Q}{-\log\varepsilon}$.

Now suppose that the theorem is false. Since the set $\{\boldsymbol{\alpha} : \overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I) > \frac{Q}{-\log\varepsilon}\}$ is a Borel set and not residual, its complement is residual in I', for some open cuboid I':
$$I' \Vdash \left\{\boldsymbol{\alpha} : \overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I) \le \frac{Q}{-\log\varepsilon}\right\}.$$
Since $\overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I) \ge \limsup_{n\to\infty} \mu_{\mathbf{n}\boldsymbol{\alpha},n}$ the set $\{\boldsymbol{\alpha} : \overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I) \le \frac{Q}{-\log\varepsilon}\}$ is contained in the set
$$\left\{\boldsymbol{\alpha} : \exists m\ \forall N \ge m : \mu_{\mathbf{n}\boldsymbol{\alpha},N}(I) \le \frac{2Q}{-\log\varepsilon}\right\}.$$
Denote the set $\{j < N : \mathbf{n}_j\boldsymbol{\alpha} \in I\}$ by $Z_N(\boldsymbol{\alpha})$. So $\mu_{\mathbf{n}\boldsymbol{\alpha},N}(I) = \frac{\#Z_N(\boldsymbol{\alpha})}{N}$. Therefore
$$I' \Vdash \bigcup_m \bigcap_{N\ge m} \left\{\boldsymbol{\alpha} : \frac{\#Z_N(\boldsymbol{\alpha})}{N} \le \frac{2Q}{-\log\varepsilon}\right\}.$$
So, by Fact 2.1.19, we can find an open cuboid $J \subseteq I'$ and a k^* such that
$$J \Vdash \bigcap_{N\ge k^*} \left\{\boldsymbol{\alpha} : \frac{\#Z_N(\boldsymbol{\alpha})}{N} \le \frac{2Q}{-\log\varepsilon}\right\},$$
or equivalently, for all $N \ge k^*$:
$$J \Vdash \left\{\boldsymbol{\alpha} : \frac{\#Z_N(\boldsymbol{\alpha})}{N} \le \frac{2Q}{-\log\varepsilon}\right\}. \tag{2.3}$$

Let $\delta_i := \lambda_i(J)$, where $\lambda_i(J)$ is the length of the edge in the i-th dimension. Without loss of generality we assume $n_{k^*c,s} > \frac{\varepsilon}{2\delta_s}$ (otherwise we just increase k^*). Now consider the functions $f_{k^*c}, f_{(k^*+1)c}, \dots, f_{(2k^*-1)c}$, defined as in Lemma 2.1.14. Since

$$\frac{n_{(k^*+i+1)c,s}}{n_{(k^*+i)c,s}} \geq q^c > \frac{4}{\varepsilon}$$

and $n_{k^*c,s} > \frac{\varepsilon}{\delta_s}$, these functions are ε-mixing in $(\delta_1, \dots, \delta_r)$ by Lemma 2.1.14. So there is an open cuboid $K \subseteq J$ such that for all $\boldsymbol{\alpha} \in K$ and all $i \in \{0, \dots, k^*\}$:

$$\boldsymbol{\alpha} \in f^{-1}_{(k^*+i)c}(I) \qquad \text{i.e.} \qquad \mathbf{n}_{(k^*+i)c}\boldsymbol{\alpha} \in I.$$

Hence for all $\boldsymbol{\alpha} \in K$

$$\#Z_{2k^*c}(\boldsymbol{\alpha}) = \#\{i < 2k^*c : \ \mathbf{n}_i\boldsymbol{\alpha} \in I\} \geq \#\{k^*c, (k^*+1)c, \dots, (2k^*-1)c\} = k^*.$$

So for all $\boldsymbol{\alpha} \in K$

$$\frac{\#Z_{2k^*c}(\boldsymbol{\alpha})}{2k^*c} \geq \frac{1}{2c}. \tag{2.4}$$

Since $\frac{1}{2c} > \frac{2Q}{-\log\varepsilon}$ and $K \subseteq J$, (2.3) with $N := 2k^*c$ implies

$$K \Vdash \left\{ \boldsymbol{\alpha} : \ \frac{\#Z_{2k^*c}(\boldsymbol{\alpha})}{2k^*c} \leq \frac{1}{2c} \right\}. \tag{2.5}$$

Now consider the set $\{\boldsymbol{\alpha} : \ \frac{\#Z_{2k^*c}(\boldsymbol{\alpha})}{2k^*c} < \frac{1}{2c}\} \cap K$. By (2.4), this set is empty, but by (2.5) it is residual in K, which is a contradiction.

Note that the above theorem still remains true if we require instead of

$$q := \liminf_j (n_{j,s+1}/n_{j,s}) > 1$$

for an $s \in \{1, \dots, r\}$ only that there exists an $s \in \{1, \dots, r\}$ and a constant C with

$$\left|\{j : \ 2^r \leq n_{j,s} < 2^{r+1}\}\right| \leq C \quad \forall r.$$

Then you can choose each $c := 2C\lceil 2 - \log_2 \varepsilon \rceil$-th term to obtain a growth of factor $4/\varepsilon$, and Q has to be chosen so small, that $\frac{1}{2c} > \frac{2Q}{-\log\varepsilon}$. Indeed, one can use instead of the base 2 in the above condition any number $K > 1$, but we will state all theorems in terms of the base 2 throughout this section.

Remark 2.1.20. With the same argument as above, Theorem 2.4 in [25] holds also for sequences $(n_j)_{j\in\mathbb{N}}$ with

$$\left|\{j : \ 2^r \leq n_j < 2^{r+1}\}\right| \leq C \quad \forall r.$$

So far we gave sufficient conditions that $\mathbf{n}\boldsymbol{\alpha}$ is u.d. mod 1 only for $\boldsymbol{\alpha}$ in a set of first category. If we weaken the growth condition in the following way, there will be sequences $\mathbf{n}$, such that $\mathbf{n}\boldsymbol{\alpha}$ is u.d. mod 1 for $\boldsymbol{\alpha}$ in a set of second category. For this purpose we start with an extension of a result due to Ajtai, Havas, Komlós [2].

Lemma 2.1.21. *Given any r sequences $(\varepsilon_{j,k})_{j\in\mathbb{N}}$, $1 \leq k \leq r$, $\varepsilon_{j,k} \geq 0$, $\lim_{j\to\infty} \varepsilon_{j,k} = 0$ for all k, there is a sequence of r-dimensional vectors of nonnegative integers $(\mathbf{n}_j)_{j\in\mathbb{N}}$ with*

$$\frac{n_{j+1,k}}{n_{j,k}} > 1 + \varepsilon_{j,k} \qquad 1 \leq k \leq r$$

such that for all $\boldsymbol{\alpha}$ with $\sum_{i=1}^r \alpha_i \notin \mathbb{Q}$ the sequence $\mathbf{n}\boldsymbol{\alpha}$ is u.d. mod 1.

Proof. Set $\varepsilon_j := \max\{\varepsilon_{j,k} :\ 1 \le k \le r\}$. Then, by [2, Lemma 1], there exists a sequence $(n_j)_{j\in\mathbb{N}}$ with $n_{j+1}/n_j > 1+\varepsilon_j$ such that $n_j\alpha$ is u.d. mod 1 for all irrational α. Define $\mathbf{n}_j := (n_j, \ldots, n_j)$. Then

$$\mathbf{n}_j\boldsymbol{\alpha} = \sum_{i=1}^{r} n_j\alpha_i = n_j \sum_{i=1}^{r} \alpha_i = n_j\alpha'$$

with irrational α'.

To get a statement in Baire's categories, we need a lemma which tells us that the set of d-dimensional real vectors, whose entries are linearly independent over $\mathbb{Q}$, is residual in $\mathbb{R}^d$.

Lemma 2.1.22. *The set*

$$\mathcal{I} := \{\boldsymbol{\alpha} :\ 1, \alpha_1, \ldots, \alpha_d \text{ are linearly independent over } \mathbb{Q}\}$$

is residual, and hence of second category, in $\mathbb{R}^d$.

Proof. Note that

$$\mathcal{I} = \mathbb{R}^d \setminus \bigcup_{\substack{a_0,\ldots,a_d\in\mathbb{Q}\\(a_0,\ldots,a_d)\neq(0,\ldots,0)}} S(a_1, \ldots a_d)$$

where

$$S(a_1, \ldots a_d) = \{\alpha : a_0 + a_1\alpha_1 + \cdots + a_d\alpha_d = 0\}.$$

Since $S(a_1, \ldots a_d)$ is a subspace of dimension smaller than d, all these sets $S(a_1, \ldots a_d)$ are nowhere dense. Therefore

$$\bigcup_{\substack{a_0,\ldots,a_d\in\mathbb{Q}\\(a_0,\ldots,a_d)\neq(0,\ldots,0)}} S(a_1, \ldots a_d)$$

is of first category.

Consequently we have

Theorem 2.1.23. *Given any r sequences $(\varepsilon_{j,k})_{j\in\mathbb{N}}$, $1 \le k \le r$, $\varepsilon_{j,k} \ge 0$, $\lim_{j\to\infty} \varepsilon_{j,k} = 0$ for all k, there is a sequence of r-dimensional vectors of nonnegative integers $(\mathbf{n}_j)_{j\in\mathbb{N}}$ with*

$$\frac{n_{j+1,k}}{n_{j,k}} > 1 + \varepsilon_{j,k} \qquad 1 \le k \le r$$

such that the set

$$\{\boldsymbol{\alpha} :\ \mathbf{n}\boldsymbol{\alpha} \text{ is u.d. mod 1}\}$$

is residual.

In [25] Goldstern, Schmeling and Winkler also proved that if the sequence $(n_j)_{j\in\mathbb{N}}$ grows very fast (i.e., if $\lim_{j\to\infty} n_{j+1}/n_j = \infty$), then the set of those α, for which $\mathbf{n}\alpha$ is maldistributed, is residual. A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is called maldistributed, iff the set $M(\mathbf{x})$ is the whole set of Borel probability measures on $[0, 1]$. It is as easy as the modification of the proof of [25, Theorem 2.4] to the proof of Theorem 2.1.17 to obtain a generalisation of [25, Theorem 2.6]:

Theorem 2.1.24. *Let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of r-dimensional vectors of nonnegative integers and assume that there is a $s \in \{1, \ldots, r\}$ such that $\lim_{k\to\infty} n_{s,k+1}/n_{s,k} = \infty$. Then the set*

$$\{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^r :\ \mathbf{n}\boldsymbol{\alpha} \text{ is maldistributed}\}$$

is residual.

2.1.3 $\mathbf{n}\boldsymbol{\alpha}$-sequences in $\mathbb{R}^d$

In this section we investigate uniform distribution in $\mathbb{R}^d$. For a sequence $(\mathbf{n}_j)_{j\in\mathbb{N}}$ of d-dimensional vectors of nonnegative integers and a d-dimensional vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ of real numbers we are interested in the sequence

$$\mathbf{n}\boldsymbol{\alpha} := (n_{j,1}\alpha_1, \dots, n_{j,d}\alpha_d)_{j\in\mathbb{N}}.$$

To obtain results for such sequences we use the connection between uniform distribution modulo 1 in $[0,1]^d$ and uniform distribution in $[0,1]$. As in the previous section our first theorem shows that the set of $\boldsymbol{\alpha}$ such that $\mathbf{n}\boldsymbol{\alpha}$ is u.d. is meager if at least one component of the $\mathbf{n}_j$ grows exponentially.

Theorem 2.1.25. *Let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of d-dimensional vectors of nonnegative integers and assume that there exists an $s \in \{1, \dots, d\}$ and a constant C with*

$$\left|\left\{j : \; 2^r \le n_{j,s} < 2^{r+1}\right\}\right| \le C \quad \forall r.$$

Then

$$\mathcal{A} := \left\{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^d : \; \mathbf{n}\boldsymbol{\alpha} \text{ is u.d. mod 1 in } \mathbb{R}^d\right\}$$

is meager.

Proof. By Theorem 2.1.9, uniform distribution of $\mathbf{n}\boldsymbol{\theta}$ implies that each component $\mathbf{n}_i\theta_i := (n_{j,i}\theta_i)_{j\in\mathbb{N}}$, $1 \le i \le d$ is u.d. mod 1, especially $\mathbf{n}_s\theta_s$ is u.d. mod 1. Therefore

$$\mathcal{A} \subseteq \mathbb{R} \times \cdots \times \mathbb{R} \times \mathcal{A}_s \times \mathbb{R} \times \cdots \times \mathbb{R},$$

where

$$\mathcal{A}_s := \{\theta : \; \mathbf{n}_s\theta \text{ u.d. mod } 1\}.$$

By Remark 2.1.20, the set $\mathcal{A}_s$ is meager. Hence, by Theorem 2.1.12, $\mathcal{A}$ is meager. ∎

As before the growth condition in the theorem above cannot be weakened:

Lemma 2.1.26. *Given any d sequences $(\varepsilon_{j,k})_{j\in\mathbb{N}}$, $1 \le k \le d$, $\varepsilon_{j,k} \ge 0$, $\lim_{j\to\infty} \varepsilon_{j,k} = 0$ for all k, there is a sequence of d-dimensional vectors of nonnegative integers $(\mathbf{n}_j)_{j\in\mathbb{N}}$ with*

$$\frac{n_{j+1,k}}{n_{j,k}} > 1 + \varepsilon_{j,k} \qquad 1 \le k \le d$$

such that for all $\boldsymbol{\alpha}$ with $1, \alpha_1, \dots, \alpha_d$ linearly independent over $\mathbb{Q}$ the sequence

$$(n_{j,1}\alpha_1, \dots, n_{j,d}\alpha_d)_{j\in\mathbb{N}}$$

is u.d. mod 1 in $\mathbb{R}^d$.

Proof. Set $\varepsilon_j := \max\{\varepsilon_{j,k} : \; 1 \le k \le d\}$. Then, by [2, Lemma 1], there exists a sequence $(n_j)_{j\in\mathbb{N}}$ with $n_{j+1}/n_j > 1+\varepsilon_j$ such that $n_j\alpha$ is u.d. mod 1 for all irrational α. Define $\mathbf{n}_j := (n_j, \dots, n_j)$. By Theorem 2.1.9 we have to show that for all $\mathbf{h} \in \mathbb{Z}^d$, $\mathbf{h} \ne 0$ the sequence $\langle \mathbf{h}, n_j\boldsymbol{\alpha}\rangle$ is u.d. mod 1 for all $\boldsymbol{\alpha}$ with $1, \alpha_1, \dots, \alpha_d$ linearly independent over $\mathbb{Q}$. This is true since

$$\langle \mathbf{h}, n_j\boldsymbol{\alpha}\rangle = \sum_{i=1}^{d} h_i n_j \alpha_i = n_j \sum_{i=1}^{d} h_i\alpha_i = n_j\alpha'$$

with $\alpha' \in \mathbb{R} \setminus \mathbb{Q}$. ∎

Using Lemma 2.1.22 we get as an immediate consequence

Theorem 2.1.27. *Given any d sequences $(\varepsilon_{j,k})_{j\in\mathbb{N}}$, $1 \le k \le d$, $\varepsilon_{j,k} \ge 0$, $\lim_{j\to\infty} \varepsilon_{j,k} = 0$ for all k, there is a sequence of d-dimensional vectors of nonnegative integers $(\mathbf{n}_j)_{j\in\mathbb{N}}$ with*

$$\frac{n_{j+1,k}}{n_{j,k}} > 1 + \varepsilon_{j,k} \qquad 1 \le k \le d$$

such that the set

$$\left\{\boldsymbol{\alpha} : \ (n_{j,1}\alpha_1, \dots, n_{j,d}\alpha_d)_{j\in\mathbb{N}} \text{ is u.d. mod 1 in } \mathbb{R}^d\right\}$$

is residual.

Again using Theorem 2.1.12 we obtain for fast growing sequences $(\mathbf{n}_j)$:

Theorem 2.1.28. *Let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of d-dimensional vectors of nonnegative integers and assume $\lim_{k\to\infty} n_{t,k+1}/n_{t,k} = \infty$ for all $t \in \{1, \dots, d\}$, then the set*

$$\{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^d : \ \mathbf{n}\boldsymbol{\alpha} \text{ is maldistributed}\}$$

is residual.

We can combine the ideas of this and the previous section: Consider a sequence of $d\times r$-matrices of nonnegative integers

$$(N_j)_{j\in\mathbb{N}} \qquad \text{with} \qquad N_j = (n^j_{ik}), \ i = 1, \dots, d, k = 1, \dots, r.$$

We are now interested in the distribution of the sequence $\mathbf{N}\boldsymbol{\alpha} := (N_j\boldsymbol{\alpha})_{j\in\mathbb{N}}$, where $N_j\boldsymbol{\alpha}$ means the classical matrix-vector-product. Same argumentation as in the proof of Theorem 2.1.25 yields

Theorem 2.1.29. *Let $(N_j)_{j\in\mathbb{N}}$ be a sequence of $d \times r$-matrices of nonnegative integers and assume that there exist $s \in \{1, \dots, d\}$, $t \in \{1, \dots, r\}$ and a constant C with*

$$\left|\left\{j : 2^r \le n^j_{st} < 2^{r+1}\right\}\right| \le C \quad \forall r.$$

Then the set

$$\mathcal{A} := \left\{\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) : \ \mathbf{N}\boldsymbol{\alpha} \text{ is u.d. mod 1 in } \mathbb{R}^d\right\}$$

is meager.

2.1.4 Uniform distribution of nets

In this section we define uniform distribution of nets of elements of a locally compact Hausdorff space and give a list of some elementary properties which generalise the results for classical sequences given in Bauer [8], Helmberg [27], Kuipers and Niederreiter [43] and Winkler [62]. The proofs are analogous to the ones of the case of one-dimensional sequences, so we omit them and just state the theorems. As explained in the following such nets induce nets of certain discrete probability measures. Uniform distribution properties of nets of general probability measures on locally compact groups were studied in Gerl [23] and Maxones and Rindler [48, 49]. A special kind of nets are sequences indexed by d-dimensional vectors in $\mathbb{N}^d$. Such sequences of random variables also appear in probability theory, see eg. Jacod and Shiryaev [31]. For an introduction to nets we refer to Willard [60].

Throughout this and the following section let $X \neq \emptyset$ be a locally compact Hausdorff space with countable topology base. Moreover, let $\mathcal{M}(X)$ be the compact sets of nonnegative finite Borel measures with $\mu(X) = 1$ if X is compact and $\mu(X) \leq 1$ if X is not compact, equipped with the topology of weak convergence. On $\mathcal{M}(X)$ we use the metric given in [62]. Furthermore let Λ (equipped with two relations $(\leq_1, \leq_2)$) be a countable directed set (w.r.t. both relations) with the additional property that for all $\lambda \in \Lambda$ the sets $\mathcal{V}_i(\lambda) := \{\nu : \nu \leq_i \lambda\}$ $(i = 1, 2)$ is finite. Moreover, assume $|\{\lambda : \; |\mathcal{V}_1(\lambda)|\}| = o(n^\alpha)$ $(\alpha \in \mathbb{R})$ as $n \to \infty$.

For a net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ of elements in X and a function $f \in \mathcal{K}(X)$, the space of all continuous real-valued functions on X whose support is compact, we define the net $\boldsymbol{\mu}_f = (\mu_{\lambda,f})_{\lambda \in \Lambda}$ by

$$\mu_{\lambda,f} = \frac{1}{|\mathcal{V}_1(\lambda)|} \sum_{\ell \leq_1 \lambda} f(x_\ell). \tag{2.6}$$

If the nets $\boldsymbol{\mu}_f$ converges (w.r.t. the relation $\leq_2$) to the integral

$$\int_X f \mathrm{d}\mu$$

for all $f \in \mathcal{K}(X)$ then we say $\mathbf{x}$ is μ-*uniformly distributed* (μ-u.d.) in X.

Now we give some basic properties:

(i) If $\mathscr{V}$ is a class of functions from $\mathcal{K}(X)$ such that $\mathrm{sp}(\mathscr{V})$ is dense in $\mathcal{K}(X)$, then $\mathscr{V}$ is convergence-determining with respect to any μ in X.

(ii) If $\mathrm{sp}(\mathscr{V})$ is a subalgebra of $\mathcal{K}(X)$ that separates points and vanishes nowhere, then $\mathscr{V}$ is a convergence-determining class with respect to any μ in X.

(iii) The net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ is μ-u.d. in X iff the nets $\mathbf{y}^M = (y_\lambda^M)_{\lambda \in \Lambda}$ defined by

$$y_\lambda^M = \frac{A(M;\lambda)}{|\mathcal{V}_1(\lambda)|}$$

converge (w.r.t. $\leq_2$) to $\mu(M)$ for all compact μ-continuity sets $M \subseteq X$. Here $A(M;\lambda) = \sum_{\ell \leq_1 \lambda} \mathbf{1}_M(x_\ell)$.

(iv) In a locally compact Hausdorff space X with countable space, there exists a countable convergence-determining class of real-valued continuous functions with compact support with respect to any $\mu \in \mathcal{M}(X)$.

(v) Let S be the set of all μ-u.d. sequences in X, viewed as a subset of $X_\Lambda := X_{\prod_{\lambda \in \Lambda}}$. Then $\mu_\infty(S) = 1$.

(vi) If X contains more than one element, then the set S from the above theorem is a set of first category in X_Λ.

(vii) The set S is everywhere dense in X_Λ.

Generalising the concept of uniform distribution we introduce the set $M(\mathbf{x})$, the set of limit measures of the net $\mathbf{x}$, as the set of cluster points of the net (w.r.t. $\leq_2$) $\boldsymbol{\mu} = (\mu_\lambda)_{\lambda \in \Lambda}$ of induced measures defined by

$$\mu_\lambda = \frac{1}{|\mathcal{V}_1(\lambda)|} \sum_{\ell \leq_1 \lambda} \delta_{x_\ell}. \tag{2.7}$$

If $M(\mathbf{x}) = \{\mu\}$ ($\mu \in \mathcal{M}(X)$), then this net is μ-u.d. in X. If $M(\mathbf{x}) = \mathcal{M}(X)$ we say $\mathbf{x}$ is *maldistributed* in X.

As in the classical case (see [62]) only very few (in a topological sense) nets are μ-u.d. Moreover, almost all nets are maldistributed. We have:
The typical situation in the sense of Baire is $M(\mathbf{x}) = \mathcal{M}(X)$, i.e., the set

$$Y = \{\mathbf{x} \in X_\Lambda | \; M(\mathbf{x}) = \mathcal{M}(X)\} \subseteq X_\Lambda$$

is residual.

At the end of this section we define two notions of uniform distribution on $\Lambda = \mathbb{N}^d$, which we will use in the following. First let $\mathbb{N}^d$ be equipped with the relations $(\leq_1, \leq_2) = (\leq, \leq)$ defined by $\mathbf{x} \leq \mathbf{y}$ iff $x_i \leq y_i$ $(1 \leq i \leq d)$. The second concept is to introduce the relation $\leq_s$ defined by $\mathbf{x} \leq_s \mathbf{y}$ iff $|\mathbf{x}| \leq |\mathbf{y}|$, where $|\mathbf{x}| := \prod_{i=1}^d x_i$ and to consider $(\mathbb{N}^d, \leq, \leq_s)$. A μ-u.d. net w.r.t. to this relation on $\mathbb{N}^d$ we will call strongly uniformly distributed (s.u.d.). The set of limit measure we denote by $M_s(\mathbf{x})$. The first concept is in accord with Kuipers and Niederreiter [43], the second concept is motivated by Aistleitner [1], who studied the discrepancy of sequences with multidimensional indices.

2.1.5 Characterisation of $M(\mathbf{x})$ and distribution of subnets for a special kind of nets on $\mathbb{N}^d$

This section is devoted to the generalisation of the characterisation of the sets of limit measures given in Winkler [62, Theorem 3.1] to nets defined on $\Lambda = (\mathbb{N}^d, \leq, \leq)$ (see Section 2.1.4).

To simplify notation we introduce some operations on multidimensional indices. For an index $\mathbf{i} = (i_1, \ldots, i_d)$ we define

$$\mathbf{i} + c = (i_1 + c, \ldots, i_d + c),$$
$$\mathbf{i} \bmod c = (i_1 \bmod c, \ldots, i_d \bmod c).$$

Furthermore, we define the index-sets

$$I[\mathbf{i}, \mathbf{j}] := \{\mathbf{k} : \mathbf{k} \geq \mathbf{i} \text{ and } \exists \ell : \; k_\ell \leq j_\ell\},$$
$$I[\mathbf{i}, \mathbf{j}) := \{\mathbf{k} : \mathbf{k} \geq \mathbf{i} \text{ and } \exists \ell : \; k_\ell < j_\ell\} = I[\mathbf{i}, \mathbf{j} - 1],$$
$$I(\mathbf{i}, \mathbf{j}] := \{\mathbf{k} : \mathbf{k} > \mathbf{i} \text{ and } \exists \ell : \; k_\ell \leq j_\ell\} = I[\mathbf{i} + 1, \mathbf{j}].$$

A sequence of the form $\mathbf{x} = (x_\mathbf{i})_{\mathbf{i} \in I[\mathbf{1}, \mathbf{N}]}$ we call an angle-sequence, and by the periodic continuation of an angle-sequence by a finite sequence $\mathbf{y} = (y_\mathbf{i})_{\mathbf{1} \leq \mathbf{i} \leq \mathbf{N}_1}$ we mean the sequence $\mathbf{x}' = (x'_\mathbf{i})_{\mathbf{i} \in \mathbb{N}^d}$ defined by

$$x'_\mathbf{i} = \begin{cases} x_\mathbf{i} & \text{if } \mathbf{i} \in I[\mathbf{1}, \mathbf{N}] \\ y^k_{\mathbf{i} - N \bmod N_1} & \text{if } \mathbf{i} > \mathbf{N} \end{cases} .$$

Here we assume $\mathbf{N} = (N, \ldots, N)$ and $\mathbf{N}_1 = (N_1, \ldots, N_1)$. In fact, we could define the above construction for arbitrary indices $\mathbf{N}$ and $\mathbf{N}_1$, but in the following we will just need this definition.

Example 2.1.30. In two dimensions the periodic continuation of $\mathbf{x}$ with period $\mathbf{y}$ looks like the following:

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \iddots \\ \vdots & \mathbf{y} & \mathbf{y} & \mathbf{y} & \cdots \\ \vdots & \mathbf{y} & \mathbf{y} & \mathbf{y} & \cdots \\ \vdots & \mathbf{y} & \mathbf{y} & \mathbf{y} & \cdots \\ \mathbf{x} & \cdots & \cdots & \cdots & \cdots \end{array}$$

The following lemma generalises [62, Section 2] and gives some properties of the set $M(\mathbf{x})$.

Lemma 2.1.31. *For all sequences $\mathbf{x} \in X^{\omega\times\cdots\times\omega}$ the set $M(\mathbf{x})$ has the following properties: $M(\mathbf{x})$ is*

(i) *nonempty,*

(ii) *contained in $M(X)$,*

(iii) *closed (hence compact),*

(iv) *connected.*

Proof. (*i*): $M(\mathbf{x}) \neq \emptyset$ since every net in a compact space has a convergent subnet and all $\lambda_{\mathbf{N}} \in \mathcal{M}(X)$.

(*ii*): The proof is completely analogous to the proof in [62], you just have to take the multidimensional limit.

(*iii*): $M(\mathbf{x})$ is the set of cluster points of the net $\mathbf{y} = (y_{\mathbf{N}})_{\mathbf{N}\in\mathbb{N}^k}$, and the set of cluster points of any net in any topological space is closed.

(*iv*): Assume that $M = M(\mathbf{x})$ is not connected. Therefore there are nonempty disjoint closed subsets M_1, $M_2 \subseteq M$ with $M = M_1 \cup M_2$. Since compact Hausdorff spaces are normal, we can find open sets O_i and V_i, $i = 1, 2$, in $M(X)$ satisfying

$$M_i \subseteq O_i \subseteq \overline{O}_i \subseteq V_i, \quad i = 1, 2, \quad \text{and} \quad V_1 \cap V_2 = \emptyset.$$

Thus the closures of the O_i are compact and disjoint. This yields that they have positive distance

$$d(\overline{O}_1, \overline{O}_2) = \inf_{\mu_i \in O_i} d(\mu_1, \mu_2) = \varepsilon > 0.$$

Now consider the compact set $L = \mathcal{M}(X) \setminus O_1 \setminus O_2$. Since both M_1 and M_2 contain cluster points of Λ, the net has to be infinitely many times in O_1 as well as in O_2 for all tails $(\lambda_{\mathbf{N}})_{\mathbf{N}\geq\mathbf{n}}$ with $\mathbf{n} \in \mathbb{N}^d$. Observe that $d(\lambda_{\mathbf{N}}, \lambda_{\mathbf{N}+1}) \leq c/(N+1)$, where $\mathbf{N} = (N, \ldots, N)$. Thus the distance of subsequent members in the diagonal of Λ gets arbitrarily small, say less than ε. This means that Λ has to intersect L infinitely many times for all tails $(\lambda_{\mathbf{N}})_{\mathbf{N}\geq\mathbf{n}}$ with $\mathbf{n} \in \mathbb{N}^d$. Since L is compact, there must be a cluster point of Λ in L, but we also have

$$L \cap M = L \cap (M_1 \cup M_2) \subseteq (L \cap O_1) \cup (L \cap O_2) = \emptyset,$$

which is a contradiction.

The parts (i), (ii), and (iii) of the lemma above are valid for arbitrary nets as considered in Section 2.1.4, whereas part (iv) fails in general. We give the following example: Let $\mathbf{x} = (x_{n,m})_{(n,m)\in\mathbb{N}^2}$ be the net defined by

$$x_{(n,m)} = \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m > 0 \end{cases}$$

and consider the pair of relations $(\leq, \leq_s)$. Then the set of limit measures is the set

$$M(\mathbf{x}) = \{\lambda : \ \lambda(0) = \tfrac{1}{n}, \ \lambda(1) = 1 - \tfrac{1}{n}\} \cup \{\delta_1\}.$$

Now we turn to the main result of this section. The proof uses two lemmas which we will present afterwards. With the definitions given in Section 2.1.4 we have

Theorem 2.1.32. *Let X be a locally compact Hausdorff space with countable topological base and $\mathbf{x} = (x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ a net. Then:*

1. *Every $M(\mathbf{x})$ is a nonempty, closed (hence compact) and connected subset of $\mathcal{M}(X)$.*
2. *Let $M \subseteq \mathcal{M}(X)$ be nonempty, compact and connected. Then there is a net $\mathbf{x} \in X^{\omega\times\cdots\times\omega}$ with $M(\mathbf{x}) = M$.*

Proof. (1) see Lemma 2.1.31

(2) By Lemma 2.1.33 there exists a net $(\mu_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ in M whose set of cluster points equals M and with the additional property that $\lim_{\mathbf{k}\to\infty} \varepsilon_{\mathbf{k}} = 0$ with a monotonically nonincreasing sequence of $\varepsilon_{\mathbf{k}} > d_{\mathbf{k}}$ where $d_{\mathbf{k}}$ is the maximum of the distances of $\mu_{\mathbf{k}}$ to its successors (see Lemma 2.1.33). Now we construct a sequence $\mathbf{x} = (x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ such that the induced sequence of the $\lambda_{\mathbf{N}}$ approximates the $\mu_{\mathbf{k}}$ in the following sense: There are indices $N_1 < N_2 < \cdots$ such that $d(\mu, \lambda_{\mathbf{N}}(\mathbf{x})) < 2\varepsilon_{\mathbf{k}}$ for all $\mathbf{N} \in I[\mathbf{N}_k, \mathbf{N}_{k+1})$ where $\mathbf{N}_j = (N_j, \ldots, N_j)$. Then the relation $M = M(\mathbf{x})$ is an immediate consequence.

To construct such a sequence we take a finite sequence $\mathbf{x}_0 = (x^0_{\mathbf{i}})_{\mathbf{1}\leq\mathbf{i}\leq\mathbf{N}_0}$ such that $d(\lambda_{\mathbf{N}_0}(\mathbf{x}_0), \mu_1) < \varepsilon_1$ (the existence of $\mathbf{x}_0$ is guaranteed by Section 2.1.4). Consider the sequence $\mathbf{x}_1 = (\mathbf{x}^1_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ with $x^1_{\mathbf{i}} = x_{\mathbf{i} \bmod N_0}$ such that there is a number N_1 with $d(\lambda_{\mathbf{N}}(\mathbf{x}_1), \mu_1) < \varepsilon_1$ for all $\mathbf{N} \geq \mathbf{N}_1$. Now we proceed by induction:

For arbitrary $k \geq 1$ assume that there is an angle-sequence $\mathbf{x}_k = (x^k_{\mathbf{i}})_{\mathbf{i}\in I[\mathbf{1},\mathbf{N}_k]}$ with the following properties:

(1) $d(\lambda_{\mathbf{N}_k}(\mathbf{x}_k), \mu_k) < \varepsilon_k$.

(2) There is a finite sequence $\mathbf{y}_k = (y^k_{\mathbf{i}})_{\mathbf{1}\leq\mathbf{i}\leq(K,\ldots,K)}$ such that for the periodic continuation $\mathbf{x}^c_k$ of $\mathbf{x}_k$ with period $\mathbf{y}_k$ we have $d(\lambda_{\mathbf{N}}(\mathbf{x}^c_k), \mu_k) < \varepsilon_k$ for all $\mathbf{N} \geq \mathbf{N}_k$.

By Lemma 2.1.35 there is an angle-sequence $\mathbf{x}' = (x'_{\mathbf{i}})_{\mathbf{i}\in I(\mathbf{N}_k,\mathbf{N}_{k+1}]}$ such that for the angle-sequence $\mathbf{x}_{k+1} = (x^{k+1}_{\mathbf{i}})_{\mathbf{i}\in I[\mathbf{1},\mathbf{N}_{k+1}]}$ defined by

$$x^{k+1}_{\mathbf{i}} = \begin{cases} x^k_{\mathbf{i}} & \text{if } \mathbf{i} \in I[1, \mathbf{N}_k] \\ x'_{\mathbf{i}} & \text{if } \mathbf{i} \in I(\mathbf{N}_k, \mathbf{N}_{k+1}] \end{cases}$$

the following conditions hold:

(i) If $\mathbf{N} \in I[\mathbf{N}_k, \mathbf{N}_{k+1})$, then there is a point μ on the linear connection between μ_k and μ_{k+1} with $d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}), \mu) < C\varepsilon_k$, where C is a constant depending only on the dimension d.

(ii) There is a finite sequence $\mathbf{y}_{k+1} = (\mathbf{y}^{k+1}_{\mathbf{i}})_{\mathbf{1}\leq\mathbf{i}\leq(K',\ldots,K')}$ such that for the periodic continuation of $\mathbf{x}_{k+1}$ with $\mathbf{y}_{k+1}$, which we denote by $\mathbf{x}$, we have $d(\lambda_{\mathbf{N}}(\mathbf{x}), \mu_{k+1}) < \varepsilon_{k+1}$ for all $\mathbf{N} \geq \mathbf{N}_{k+1}$.

Then the limit sequence $\lim_{k\to\infty} \mathbf{x}_{k+1}$, generated by the above induction, has the desired properties.

Lemma 2.1.33. *Let M be a nonempty closed and connected subset of $\mathcal{M}(X)$. Then there is a net $(\mu_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ in M, whose set of cluster points equals M and with the additional properties* $\lim_{\mathbf{k}\to\infty} d_{\mathbf{k}} = 0$, *where $d_{\mathbf{k}}$ is the maximum of the distances of $\mu_{\mathbf{k}}$ to its successors, i.e.* $d_{\mathbf{k}} = \max_{\mathbf{k}'\in I_{\mathbf{k}}} d(\mu_{\mathbf{k}}, \mu_{\mathbf{k}'})$, *where* $I_{\mathbf{k}} = \{(K_1, \ldots, K_d) : K_i \in \{k_i, k_i{+}1\}, i = 1, \ldots, d\}$ *if* $\mathbf{k} = (k_1, \ldots k_d)$, *and $\mu_{\mathbf{k}'} = \mu_{(k,k,\ldots,k)}$, where $\mathbf{k}'$ runs over all indices which coincide with $(k, \ldots, k)$ in at least one coordinate.*

Example 2.1.34. In two dimensions such a net has the following form:

$$\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_4 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_3 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_2 & \mu_2 & \mu_2 & \cdots \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & \mu_1 & \cdots \end{array}$$

Proof. By [62, Lemma 3.3], there exists a sequence $(\mu_k)_{k\in\mathbb{N}}$ in M whose set of accumulation points equals M and with $\lim_{k\to\infty} d(\mu_k,\mu_{k+1}) = 0$. The net determined by $\mu_{(k,\dots,k)} = \mu_k$ has the desired properties.

Lemma 2.1.35. *Let μ_k, $\mu_{k+1} \in \mathcal{M}(X)$ and $\varepsilon_k > \varepsilon_{k+1} > 0$ be given. Assume that $\mathbf{x}_k = (x_{\mathbf{i}}^k)_{\mathbf{i}\in I[\mathbf{1},\mathbf{N}_k]}$ with $\mathbf{N}_k = (N_k,\dots,N_k)$ is an angle-sequence with the following properties:*

1. *$d(\lambda_{\mathbf{N}_k}(\mathbf{x}_k),\mu_k) < \varepsilon_k$.*
2. *There exists a finite sequence $\mathbf{y}_k = (y_{\mathbf{i}}^k)_{\mathbf{1}\le\mathbf{i}\le(K,\dots,K)}$ such that for the periodic continuation $\mathbf{x}_k^c$ of $\mathbf{x}_k$ with period $\mathbf{y}_k$ the following property holds: $d(\lambda_{\mathbf{n}}(\mathbf{x}_k^c),\mu_k) < \varepsilon_k$ for all $\mathbf{n} \ge \mathbf{N}_k$.*

Then there is an angle-sequence $\mathbf{x}' = (x'_{\mathbf{i}})_{\mathbf{i}\in I(\mathbf{N}_k,\mathbf{N}_{k+1}]}$ such that for the angle-sequence $\mathbf{x}_{k+1} = (x_{\mathbf{i}}^{k+1})_{\mathbf{i}\in I[\mathbf{1},\mathbf{N}_{k+1}]}$ defined by

$$x_{\mathbf{i}}^{k+1} = \begin{cases} x_{\mathbf{i}}^k & \text{if } \mathbf{i}\in I[1,\mathbf{N}_k] \\ x'_{\mathbf{i}} & \text{if } \mathbf{i}\in I(\mathbf{N}_k,\mathbf{N}_{k+1}] \end{cases}$$

the following conditions hold:

(i) If $\mathbf{n}\in I[\mathbf{N}_k,\mathbf{N}_{k+1})$ then there is a point μ on the linear connection between μ_k and μ_{k+1} with $d(\lambda_{\mathbf{n}}(\mathbf{x}_{k+1}),\mu) < C\varepsilon_k$, where the constant C depends only on the dimension d.

(ii) There is a finite sequence $\mathbf{y}_{k+1} = (y_{\mathbf{i}}^{k+1})_{\mathbf{1}\le\mathbf{i}\le(K',\dots,K')}$ such that for the sequence $\mathbf{x}$, which denotes the periodic continuation of $\mathbf{x}_{k+1}$ with period $\mathbf{y}_{k+1}$, we have

$$d(\lambda_{\mathbf{n}}(\mathbf{x}),\mu_{k+1}) < \varepsilon_{k+1}$$

for all $\mathbf{n}\ge\mathbf{N}_{k+1}$.

Proof. By Section 2.1.4 there is a sequence $\mathbf{y}$ with limit distribution μ_{k+1}. Take the initial part $\mathbf{y}_{k+1} = (y_{\mathbf{i}}^{k+1})_{\mathbf{1}\le\mathbf{i}\le(K',\dots,K')}$ in such a way that the induced measure $\lambda = \lambda_{(K',\dots,K')}(\mathbf{y}_{k+1})$ satisfies

$$d(\lambda,\mu_{k+1}) < \varepsilon_{k+1} < \varepsilon_k$$

and $K|K'$. Consider the angle-sequence $\mathbf{x}_{k+1} = (x_{\mathbf{i}}^{k+1})_{\mathbf{i}\in I[\mathbf{1},\mathbf{N}_{k+1}]}$ constructed in the following way:

$$x_{\mathbf{i}}^{k+1} = \begin{cases} x_{\mathbf{i}}^k & \text{if } \mathbf{i}\in I[\mathbf{1},\mathbf{N}_k] \\ y^k_{\mathbf{i}-N_k \bmod K} & \text{if } \mathbf{i}\in I(\mathbf{N}_k,\mathbf{N}_k+m_kK] \\ y_{\mathbf{i}-(N_k+m_kK) \bmod K'} & \text{if } \mathbf{i}\in I(\mathbf{N}_k+m_kK,\mathbf{N}_{k+1}] \end{cases},$$

where $\mathbf{N}_{k+1} = (N_{k+1},\dots,N_{k+1})$ and $N_{k+1} = N_k + m_kK + m_{k+1}K'$ with suitable chosen m_k and m_{k+1}; this is done below. Let $\mathbf{x}$ denote the sequence obtained from $\mathbf{x}_{k+1}$ by periodic continuation with period $\mathbf{y}_{k+1}$. We first prove the second statement of the lemma. Given m_k, we can choose

m_{k+1} large enough that $d(\lambda_{\mathbf{n}}(\mathbf{x}), \mu_{k+1}) < \varepsilon_{k+1}$ for all $\mathbf{n} \geq \mathbf{N}_{k+1}$ since for $\mathbf{n} = (n_1, \ldots, n_d)$ with $n_i = N_k + m_k K + m_{k+1} K' + s_i K' + c_i$ with $0 \leq c_i < K'$ and $s_i \geq 0$ we have

$$\begin{aligned} d(\lambda_{\mathbf{N}}(\mathbf{x}), \mu_{k+1}) &= d\left(\frac{1}{|\mathbf{n}|}\left(K'^d \prod_{i=1}^d (m_{k+1} + s_i)\lambda + \sum\right), \mu_{k+1}\right) \\ &\leq \frac{1}{|\mathbf{n}|} K'^d \prod_{i=1}^d (m_{k+1} + s_i) d(\lambda, \mu_{k+1}) \\ &\quad + \frac{|\mathbf{n}| - K'^d \prod_{i=1}^d (m_{k+1} + s_i)}{|\mathbf{n}|} \\ &= 1 - (1 - d(\lambda, \mu_{k+1})) \frac{1}{|\mathbf{n}|} K'^d \prod_{i=1}^d (m_{k+1} + s_i), \end{aligned}$$

where $\sum$ is a sum over $|\mathbf{n}| - K'^d \prod_{i=1}^d (m_{k+1} + s_i)$ indicator functions. For m_{k+1} so large that

$$\frac{1}{|\mathbf{n}|} K'^d \prod_{i=1}^d (m_{k+1} + s_i) > \frac{1 - \varepsilon_{k+1}}{1 - d(\lambda, \mu_{k+1})},$$

the statement is true.

Now we turn to the first assertion. Firstly we give a detailed proof of the two-dimensional case, afterwards we prove the general case. Indeed, the general case uses the same idea as the two-dimensional case, but it is not necessary (and in higher dimension also very awful to write things down) to be so accurate as we are in the two-dimensional case, but this accuracy will be very helpful to understand what's going on.

So we have to show that for all $\mathbf{n} \in I[\mathbf{N}_k, \mathbf{N}_{k+1})$ we have $d(\lambda_{\mathbf{n}}(\mathbf{x}_{k+1}), \mu) < C\varepsilon_k$ for some μ on the linear connection between μ_k and μ_{k+1}. For $\mathbf{n} \in I[\mathbf{N}, \mathbf{N}_k + m_k K]$ this is true by assumption. Now consider a point $\mathbf{N} = (N_1, N_2) = (N_k + m_k K + sK' + d, N_k + m_k K + tK' + e)$ with $0 \leq s, t \leq m_{k+1}$ and $0 \leq d, e < K'$. We can write $\lambda_{\mathbf{N}}$ as

$$\begin{aligned} \mathbf{N}\lambda_{\mathbf{N}} &= N_{k+1}^2 \frac{st}{m_{k+1}^2} \lambda_{\mathbf{N}_{k+1}} \\ &\quad - (N_k + m_k K)(N_k + m_k K + m_{k+1} K') \frac{st}{m_{k+1}^2} \lambda_{(N_k + m_k K, N_{k+1})} \\ &\quad - (N_k + m_k K)(N_k + m_k K + m_{k+1} K') \frac{st}{m_{k+1}^2} \lambda_{(N_{k+1}, N_k + m_k K)} \\ &\quad + (N_k + m_k K)(N_k + m_k K + tK' + e) \lambda_{(N_k + m_k K, N_2)} \\ &\quad + (N_k + m_k K)(N_k + m_k K + sK' + d) \lambda_{(N_1, N_k + m_k K)} \\ &\quad + \left(\frac{st}{m_{k+1}^2} - 1\right)(N_k + m_k K)^2 \lambda_{(N_k + m_k K, N_k + m_k K)} \\ &\quad + \sum \\ &=: a_1 \lambda_1 + \sum_{i=2}^6 a_i \lambda_i + \sum, \end{aligned}$$

where $\sum$ is a sum over $de + esK' + dtK'$ indicator functions. The first term is needed to count the indicator functions induced by the complete $\mathbf{y}_{k+1}$-blocks. In $\lambda_{\mathbf{N}_{k+1}}$ we have m_{k+1}^2 such blocks and we need st blocks, so we multiply with $\frac{st}{m_{k+1}^2}$. But with this measure we count too many indicator functions, namely those in the areas $A = I((1, N_2), (N_k + m_k K, N_{k+1})]$ and $B = I((N_1, 1), (N_{k+1}, N_k + m_k K)]$ (see Figure 2.3). This error is corrected by subtracting the terms with the measures $\lambda_{(N_k+m_kK, N_{k+1})}$ and $\lambda_{(N_{k+1}, N_k+m_kK)}$. But now we have eliminated all contributions from the areas $I((1, N_k + m_k K), (N_k + m_k K, N_2)]$ and $I((N_k + m_k K, 1), (N_1, N_k + m_k K)]$ too, so we add the terms with $\lambda_{(N_k+m_kK, N_2)}$ and $\lambda_{(N_1, N_k+m_kK)}$. Last we correct the contribution of $I[\mathbf{1}, (N_k + m_k K, N_k + m_k K)]$. The measure $\sum$ contains all the indicator functions from the incomplete $\mathbf{y}_{k+1}$-blocks.

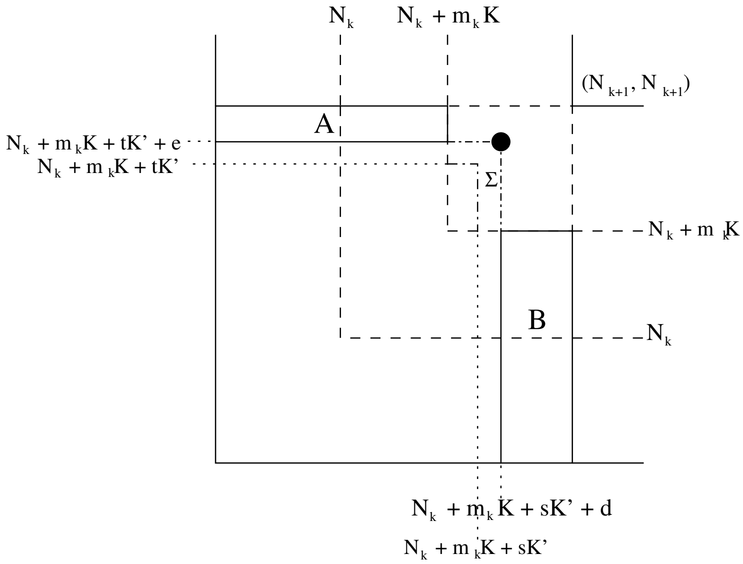

Figure 2.3:

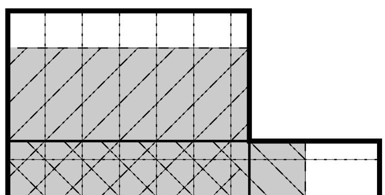

Figure 2.4:

Figure 2.4 illustrates this procedure (except the error $\sum$): We have the thick-bordered area and want to construct the grey area, using only rectangles starting in the origin. So we subtract the vertical and horizontal dotted areas first, then we have to add their intersection, the thick-bordered square again. Afterwards we add the diagonally lined areas and correct the error we made by subtracting their intersections, again thick-bordered square.

Now define $\mu := \frac{1}{|\mathbf{N}|}\left(a_1\mu_{k+1} + \mu_k\left(\sum_{i=2}^{6} a_i + de + dtK' + esK'\right)\right)$. Then μ is on linear connection between μ_k and μ_{k+1} and we have

$$\begin{aligned}d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}),\mu) &\leq \frac{a_1 d}{N_1N_2}(\lambda_1(\mathbf{x}_{k+1}),\mu_{k+1}) + \frac{1}{N_1N_2}\sum_{i=2}^{6}|a_i|d(\lambda_i(\mathbf{x}_{k+1}),\mu_k)\\ &\quad + 2\frac{de+dtK'+esK'}{N_1N_2}\\ &< 6\varepsilon_k + 2\frac{2K'^2(s+t+1)}{N_1N_2}\end{aligned}$$

Now we reduce the fraction by $\max(s,t)$, hence

$$d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}),\mu) < 6\varepsilon_k + 4K'^2\frac{3}{K'(N_k+m_kK)} < 7\varepsilon_k$$

if m_k is chosen large enough.

In a similar way we can decompose $\lambda_{\mathbf{N}}$ with $\mathbf{N}=(N_1,N_2)=(N_k+m_kK+sK'+d, N_k+m_kK+tK'+e)$, $s<m_{k+1}$, $t>m_{k+1}$ and $0\leq d,e<K'$ (the case $t<m_{k+1}$, $s>m_{k+1}$ is symmetric) into

$$\begin{aligned}N_1N_2\lambda_{\mathbf{N}} = N_{k+1}&(N_k+m_kK+(t+1)K')\frac{st}{m_{k+1}(t+1)}\lambda_{(N_{k+1},N_k+m_kK+(t+1)K')}\\ &-\frac{st}{m_{k+1}(t+1)}(N_k+m_kK)(N_k+m_kK+(t+1)K')\lambda_{(N_k+m_kK,N_k+m_kK+(t+1)K')}\\ &-\frac{st}{m_{k+1}(t+1)}N_{k+1}(N_k+m_kK)\lambda_{(N_{k+1},N_k+m_kK)}\\ &+(N_k+m_kK)N_2\lambda_{(N_k+m_kK,N_2)}\\ &+N_1(N_k+m_kK)\lambda_{(N_1,N_k+m_kK)}\\ &+\left(\frac{st}{m_{k+1}(t+1)}-1\right)(N_k+m_kK)^2\lambda_{(N_k+m_kK,N_k+m_kK)}\\ &+\sum,\end{aligned}$$

where $\sum$ is a sum over $de+esK'+dtK'$ indicator functions and obtain that for suitable chosen μ on the linear connection between μ_k and μ_{k+1}

$$d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}),\mu) < 7\varepsilon_k.$$

Now we turn to the general case. Therefore consider a point $\mathbf{N}=(N_1,\ldots,N_d)$ with $N_i = N_k+m_kK+s_iK'+d_i$ and $s_i<m_{k+1}$ and $0\leq d_i<K'$ first. Then we can write

$$|\mathbf{N}|\lambda_{\mathbf{N}} = |\mathbf{N}_{k+1}|\frac{\prod_{i=1}^d s_i}{m_{k+1}^d}\lambda_{\mathbf{N}_{k+1}} + \sum_{i=1}^{T}a_i\lambda_{\mathbf{n}_i} + \sum,$$

where $\mathbf{n}_i$ is of the form $\mathbf{n}_i=(n_1,\ldots n_d)$ with all $n_i\in\{N_{k+1},N_k+m_kK\}$ or all $n_i\in\{N_i,N_k+m_kK\}$ but not all $n_i=N_i$. The coefficients $a_i=v_i|\mathbf{n}_i|c_i$ with $v_i\in\{1,-1\}$ and $c_i\in\{1,\frac{\prod_{i=1}^d s_i}{m_{k+1}^d}\}$. The above formula is true, since after taking $\lambda_{\mathbf{N}_{k+1}}$, we have to subtract the error we made. Therefore we subtract the $\lambda_{\mathbf{n}_i}$ with exactly one $n_i=N_k+m_kK$ and for all the other $j\neq i$ with $n_j=N_{k+1}$; there are $p_1=d$ such measures. Each two of them have an intersection, so we have the correct this, which leads to $p_2=\binom{d}{2}$ summands (each such index has exactly two entries N_k+m_kK). Each of them have again an intersection (now there are $p_3=\binom{p_2}{2}$ of them) and so on ($p_{i+1} = \binom{p_i}{2}$). After d steps this procedure must end. Afterwards we start adding the terms with those $\mathbf{n}_i$ with exactly one entry equals $N_k + m_k K$ and the other entries equal N_i. There are p_1 of them. Then we correct the intersections again and so on. Last we add the term due to the non-complete $\mathbf{y}_k$ blocks, this is denoted by $\sum$ and is a sum over

$$S := 1 + \sum_{j=1}^{d} \sum_{\substack{A \subseteq \{1,\ldots,d\} \\ |A|=j}} K'^{d-j} \prod_{p \in A} d_p \prod_{q \in \{1,\ldots,d\} \setminus A} s_q$$

indicator functions.

Hence $T \leq 2 \sum_{i=1}^{d} p_i < F(d)$, where $F(d)$ is a constant only depending on the dimension d. Taking

$$\mu = \frac{1}{\mathbf{N}} |\mathbf{N}_{k+1}| \frac{\prod_{i=1}^{d} s_i}{m_{k+1}^{d}} \mu_{k+1} + \frac{1}{\mathbf{N}} \left(\sum_{i=1}^{T} a_i + S \right) \mu_k$$

we find that

$$d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}), \mu) \leq F(d)\varepsilon_k + \frac{2S}{|\mathbf{N}|}.$$

By reducing the fraction on the right-hand side by the product of the $(d-1)$ greatest s_i and estimating $d_i \leq K'$ we see

$$d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}), \mu) \leq F(d)\varepsilon_k + 2 \frac{2^d K'^d}{K'^{d-1}(N_k + m_k K)}.$$

So we have to choose m_k in such a way that the fraction becomes small. A similar construction holds for the other points $\mathbf{N} \in I[\mathbf{N}_k + m_k K, \mathbf{N}_{k+1})$.

After this characterisation of $M(\mathbf{x})$ we will study the distribution of certain subnets of a given net and generalise results due to Goldstern, Winkler and Schmeling [24]. We study subnets as studied in Losert and Tichy [46]: Choose d sequences $\mathbf{a}_1, \ldots, \mathbf{a}_d \in \{0,1\}^{\mathbb{N}}$ and define $\mathbf{a} = (a_{\mathbf{n}})_{n \in \mathbb{N}^d}$ by

$$a_{(n_1,\ldots,n_d)} = \prod_{i=1}^{d} \mathbf{a}_{i,n_i}.$$

Then the subnet $\mathbf{ax}$ of $\mathbf{x}$ is the net obtained by taking those elements $x_{\mathbf{n}}$ for which $a_{\mathbf{n}} = 1$ and using the given relation $\leq$.

The next theorem is a consequence of Theorem 2.1.32 and generalises [24, Theorem 1.2]:

Theorem 2.1.36. *Let $\mathbf{x} \in X^{\mathbb{N}^d}$ and $M \subseteq \mathcal{M}(X)$. Then there exists a subsequence $\mathbf{ax}$ with $M(\mathbf{ax}) = M$ iff M is closed and connected with $\emptyset \neq M \subseteq \mathcal{M}(A(\mathbf{x}))$, where $A(\mathbf{x})$ is the set of cluster points of the net $\mathbf{x}$.*

Proof. This proof runs along the same lines as the one in [24]: First assume $M = M(\mathbf{ax})$. Using Lemma 2.1.31 we get that M is nonempty, closed and connected. It remains to show that $M \subseteq \mathcal{M}(A(\mathbf{x}))$. For this purpose it suffices to show that every $x \in X \setminus A(\mathbf{x})$ has a neighbourhood U with $\lim_{\mathbf{N}\to\infty} \mu_{\mathbf{N},\mathbf{ax}}(U) = 0$. To see this, take a neighbourhood U with compact closure $\overline{U}$ and with $\overline{U} \cap A(\mathbf{x}) = \emptyset$. If $x_{\mathbf{n}} \in \overline{U}$ for an infinite increasing sequence of indices $\mathbf{n}_1 < \mathbf{n}_2 < \cdots$, $\overline{U}$ would contain a cluster point of $\mathbf{x}$, which is a contradiction. Hence $x_{\mathbf{n}} \notin U$ for all $\mathbf{n} \geq \mathbf{N}_0$. Thus

$$\lim_{\mathbf{N}\to\infty} \mu_{\mathbf{N},\mathbf{ax}}(U) \leq \lim_{\mathbf{N}\to\infty} \frac{|\mathbf{N}_0|}{|\mathbf{N}|} = 0.$$

The other direction is completely analogous to [24].

Similarly to [24, Theorem 1.3] we get that a typical subsequence of a given sequence is maldistributed in $A(\mathbf{x})$:

Theorem 2.1.37. *$M(\mathbf{a}\mathbf{x}) = \mathcal{M}(A(\mathbf{x}))$ holds for all $\mathbf{a} \in R$ from a residual set $R \subseteq [0,1)^d$.*

2.1.6 $\mathbf{n}\alpha$-nets over $\mathbb{N}^d$

In this section we specialise on $\mathbf{n}\alpha$-nets over $\mathbb{N}^d$, i.e., we consider $X = [0,1)$ and μ the Lebesgue-measure. Besides the two notions of uniform distribution mod 1 according to Section 2.1.4 we consider the $(s_1,\ldots,s_d)$-u.d. (see Kirschenhofer and Tichy [41]). After some elementary properties and examples of these three concepts we turn to the generalisation of results given in Goldstern, Schmeling and Winkler [25] and Ajtai, Havas and Komlós [2].

In Section 2.1.4 we introduced two special notions of uniform distribution. In the context of this section we call a net $\mathbf{x}$ uniformly distributed mod 1 iff for any a and b with $0 \le a < b \le 1$,

$$\lim_{N_1,\ldots,N_d\to\infty} \frac{A([a,b);\mathbf{N})}{|\mathbf{N}|} = b - a,$$

where $A([a,b);\mathbf{N})$ is the number of $x_{\mathbf{k}}$, $\mathbf{1} \le \mathbf{k} \le \mathbf{N}$ with $a \le \{x_{\mathbf{k}}\} < b$.

This definition is a direct extension of uniform distribution in the case $d = 2$ given by Kuipers and Niederreiter [43] and a special case of the concept studied in Losert and Tichy [46]. Following [43] one gets immediately the theorems given below.

Theorem 2.1.38. *The sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is u.d. mod 1 iff for every Riemann-integrable function f on $[0,1]$*

$$\lim_{N_1,\ldots,N_d\to\infty} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} f(\{x_{\mathbf{k}}\}) = \int_0^1 f(x)\mathrm{d}x,$$

where $\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} = \sum_{\mathbf{k}:\mathbf{1}\le\mathbf{k}\le\mathbf{N}}$.

Theorem 2.1.39. *The sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is u.d. mod 1 iff*

$$\lim_{N_1,\ldots,N_d\to\infty} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} e^{2\pi i h x_{\mathbf{k}}} = 0$$

for all integers $h \neq 0$.

Moreover, a net $\mathbf{x} = (x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is said to be strongly uniformly distributed (s.u.d.) mod 1 iff for any a and b with $0 \le a < b \le 1$,

$$\lim_{|\mathbf{N}|\to\infty} \frac{A([a,b);\mathbf{N})}{|\mathbf{N}|} = b - a.$$

Here $\lim_{|\mathbf{N}|\to\infty} f(\mathbf{N}) = f$ means that $\forall\varepsilon > 0\ \exists N \in \mathbb{N} : \forall\mathbf{N}$ with $|\mathbf{N}| \ge N : |f(\mathbf{N}) - f| < \varepsilon$.

The following theorems hold:

Theorem 2.1.40. *The sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is s.u.d. mod 1 iff for every Riemann-integrable function f on $[0,1]$*

$$\lim_{|\mathbf{N}|\to\infty} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} f(\{x_{\mathbf{k}}\}) = \int_0^1 f(x)\mathrm{d}x.$$

Theorem 2.1.41. *The sequence* $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ *is s.u.d. mod 1 iff*

$$\lim_{|\mathbf{N}|\to\infty} \frac{1}{|\mathbf{N}|} \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} e^{2\pi i h x_{\mathbf{k}}} = 0$$

for all integers $h \neq 0$.

As in the one-dimensional case, a sequence with multidimensional sequences is strongly uniformly distributed modulo 1 if and only if the multidimensional discrepancy introduced by Aistleitner [1] tends to 0.

Clearly, strong uniform distribution implies uniform distribution. The converse is not true: Consider the double sequence $\mathbf{x}$ defined by $x_{j,k} = j\theta$ with θ irrational. Then this sequence is u.d. mod 1 (this follows easily from Theorem 2.1.39), but not s.u.d., since this sequence is constant for fixed k. Thus $\mathbf{x}$ is not s.u.d. mod 1 by the following theorem:

Theorem 2.1.42. *Let* $x_{\mathbf{k}}$ *be s.u.d. mod 1. Then all "one-dimensional sequences", i.e., sequences* $(x_{(k_1,\dots,k_j,\dots,k_d)})_{k_j\in\mathbb{N}}$ *with fixed* k_s, $s \neq j$, *are u.d. mod 1.*

Proof. By the criterion of Weyl, we have to show that

$$\lim_{k_j\to\infty} \frac{1}{k_j} \sum_{n=1}^{k_j} e^{2\pi i h x_{(k_1,\dots,k_{j-1},n,k_{j+1},\dots,k_d)}} = 0 \tag{2.8}$$

for all $k_s \in \mathbb{N}$, $s \neq j$ and $h \in \mathbb{Z}\setminus\{0\}$. We use induction. From Theorem 2.1.41 we get readily that (2.8) holds for $k_s = 1$, $s \neq j$ for all integers $h \neq 0$ and all j. Assume that (2.8) holds for all $\mathbf{k}'_j := (k'_1,\dots,k'_{j-1},k'_{j+1},\dots,k'_d) < (k_1,\dots,k_{j-1},k_{j+1},\dots,k_d) := \mathbf{k}_j$. Again by Theorem 2.1.41 we have

$$\varepsilon > \left| \frac{1}{|\mathbf{k}|} \sum_{\mathbf{k}'_j\le\mathbf{k}_j} \sum_{n=1}^{k_j} e^{2\pi i h x_{(k'_1,\dots,k'_{j-1},n,k'_{j+1},\dots,k'_d)}} \right| \tag{2.9}$$

for k_j big enough. Hence

$$\varepsilon > \frac{1}{|\mathbf{k}|} \left| \left| \sum_{n=1}^{k_j} e^{2\pi i h x_{(k_1,\dots,k_{j-1},n,k_{j+1},\dots,k_d)}} \right| - \left| \sum_{n=1}^{k_j} \sum_{\mathbf{1}\le\mathbf{k}'_j<\mathbf{k}_j} e^{2\pi i h x_{(k'_1,\dots,k'_{j-1},n,k'_{j+1},\dots,k'_d)}} \right| \right|.$$

The second term on the right-hand side tends to 0 by (2.9). Thus

$$\frac{1}{|\mathbf{k}|} \left| \sum_{n=1}^{k_j} e^{2\pi i h x_{(k_1,\dots,k_{j-1},n,k_{j+1},\dots,k_d)}} \right| \to 0$$

for $k_j \to \infty$. Therefore (2.8) holds. ✐

We give an example of a sequence which is s.u.d. mod 1. This sequence can be seen as a generalisation of the one-dimensional sequence $(n\theta)_{n\in\mathbb{N}}$. This sequence is u.d. mod 1 for all irrational θ. Choose now $n_{\mathbf{k}} = \sum_{i=1}^{d} k_i - (d-1)$. In two dimensions this sequence is

⋮	⋮	⋮	⋮	⋮	⋰
4	5	6	7	8	…
3	4	5	6	7	…
2	3	4	5	6	…
1	2	3	4	5	…

We will prove now that this sequence is s.u.d. mod 1 for all irrational α. The proof is similar to the proof of the one-dimensional case (see [43]).We have to show that

$$\lim_{|\mathbf{n}|\to\infty} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} e^{2\pi i h\alpha \sum_{s=1}^{d} k_s} = 0$$

for all integers $h \neq 0$ and irrational α. Here we assume $n_1 \geq n_2 \geq \cdots \geq n_d$. Therefore $n_1 \to \infty$. With $S(\mathbf{n}) = \sum_{i=1}^{s} n_i$ we have

$$\begin{aligned}
\frac{1}{|\mathbf{n}|}\left|\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} e^{2\pi i h\alpha \sum_{s=1}^{d} k_s}\right| &= \frac{1}{|\mathbf{n}|}\left|\sum_{\mathbf{k}=\mathbf{1}}^{(n_2,\dots n_d)} \sum_{j=S(\mathbf{k})}^{S(\mathbf{n})-S(\mathbf{k})} e^{2\pi i h\alpha j}\right| \\
&= \frac{1}{|\mathbf{n}|}\left|\sum_{\mathbf{k}=\mathbf{1}}^{(n_2,\dots n_d)} \frac{e^{2\pi i h\alpha(S(\mathbf{n})-S(\mathbf{k})+1)} - e^{2\pi i h\alpha S(\mathbf{k})}}{1-e^{2\pi i h\alpha}}\right| \\
&\leq \frac{1}{|\mathbf{n}|} \frac{2\prod_{s=2}^{d} n_s}{|1-e^{2\pi i h\alpha}|} \\
&= \frac{1}{n_1} \frac{1}{|1-e^{2\pi i h\alpha}|} \to 0.
\end{aligned}$$

We give another example: In the one-dimensional case the sequence $(\{k!e\})_{k\in\mathbb{N}}$ has 0 as the only limit point (see [43]). Consider now the sequence $n_{\mathbf{k}} = (S(\mathbf{k}) - d + 1)! =: \mathbf{k}!$. Then

$$\mathbf{k}!e = A + \frac{e^{\alpha}}{S(\mathbf{k}) - d + 1}, \qquad 0 < \alpha < 1,\ A \in \mathbb{N}.$$

Thus $\{\mathbf{k}!e\} = e^{\alpha}/(S(\mathbf{k}) - d + 1) \to 0$ in the first sense. Therefore it is not u.d. mod 1 (and hence not s.u.d. mod 1).

The third concept is the $(s_1, \dots, s_d)$-uniform distribution introduced by Kirschenhofer and Tichy [41]. According to the definitions above we gave an equivalent definition to that one stated in [41].

Definition 2.1.43. A sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is $(s_1, \dots, s_d)$-u.d. iff for all $a_{i_1\dots i_d}$ and $b_{i_1\dots i_d}$ with $0 \leq a_{i_1\dots i_d} < b_{i_1\dots i_d} \leq 1$ and $1 \leq i_j \leq s_j$ for $1 \leq j \leq d$

$$\begin{aligned}
\lim_{N_1,\dots,N_d\to\infty} \prod_{i=1}^{d} \binom{N_i}{s_i}^{-1} A\left([a_{11\dots 1}, b_{11\dots 1}), \dots, [a_{s_1\dots s_d}, b_{s_1\dots s_d}); N_1, \dots, N_d; s_1, \dots s_d\right) \\
= \prod_{i=1}^{d} \prod_{j_i=1}^{s_i} b_{j_1\dots j_d} - a_{j_1\dots j_d}
\end{aligned}$$

where $A\left([a_{11\dots 1}, b_{11\dots 1}), \dots, [a_{s_1\dots s_d}, b_{s_1\dots s_d}); N_1, \dots, N_d; s_1, \dots s_d\right)$ is the number of $(s_1 \cdots s_d)$-tuples $(x_{i_{1_1},\dots i_{d_1}}, \dots x_{i_{1_{s_1}},\dots i_{d_{s_d}}})$ with $1 \leq i_{j_1} < \cdots < i_{j_{s_d}} \leq N_j$ for all $1 \leq j \leq d$ in $[a_{11\dots 1}, b_{11\dots 1}) \times \cdots \times [a_{s_1\dots s_d}, b_{s_1\dots s_d})$.

As in [43] we have that the set S of $(s_1, \dots, s_d)$-u.d. sequences is everywhere dense in $X^{\omega\times\cdots\times\omega}$. By [41], $(s_1, \dots, s_d)$-uniform distribution implies uniform distribution. Thus from Section 2.1.4 we conclude: If X contains more than one element, then the set S of $(s_1, \dots, s_d) - \mu$-u.d. sequences is a set of first category in $X^{\omega\times\cdots\times\omega}$.

After these examples and elementary properties of uniform distribution of sequences with multidimensional indices, we turn to the generalisation of [25, Theorem 2.4] for these cases. For the sake of completeness we mention that in [55] Šalát proved that for a sequence $(n_k)_{k\in\mathbb{N}}$ with $n_k = \prod_{j=1}^{k} q_j$, where $(q_j)_{j\in\mathbb{N}}$ is a sequence of integers greater than 1, then the set $\mathcal{U} := \{\alpha \in \mathbb{R} : (n_k\alpha) \text{ is u.d. mod } 1\}$ is meager. By modifying the proof slightly, we get

Theorem 2.1.44. *Let $(q_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ be a sequence of integers greater than 1. Put*

$$a_{\mathbf{n}} = \prod_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} q_{\mathbf{k}}, \qquad \mathbf{n} \in \mathbb{N}^d.$$

Then the set

$$\mathcal{U} := \{\alpha \in \mathbb{R} : (n_{\mathbf{k}}\alpha) \text{ is u.d. mod } 1\}$$

is meager. Consequently the sets

$$\mathcal{U}' := \{\alpha \in \mathbb{R} : (n_{\mathbf{k}}\alpha) \text{ is s.u.d. mod } 1\}$$

and

$$\mathcal{U}'' := \{\alpha \in \mathbb{R} : (n_{\mathbf{k}}\alpha) \text{ is } (s_1, \ldots, s_d)\text{-u.d.}\}$$

are meager.

Now we turn to the stronger result. We will generalise [25, Theorem 2.4]. For this purpose we will follow [25] again. Recall the definitions of Section 2.1.4. Moreover, let λ denote the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.

We start with an elementary property.

Theorem 2.1.45. *Given a sequence $\mathbf{x} = (x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ we have $\emptyset \neq M(\mathbf{x}) \subseteq M_s(\mathbf{x})$.*

Now we can establish the main result of this section.

Theorem 2.1.46. *Let $\mathbf{n} = (n_{\mathbf{k}})_{\mathbf{k}\in\mathbf{N}^d}$ be a sequence of nonnegative integers and assume that there exists a constant Q such that*

$$\#\{\mathbf{k} : 2^r \leq n_{\mathbf{k}} < 2^{r+1}\} \leq Q, \qquad \forall r = 0, 1, 2, \ldots.$$

Then the set

$$\mathcal{U} := \{\alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is uniformly distributed w.r.t. } \lambda\}$$

is meager. Moreover, there is a number $P > 0$ such that for all intervals I the set

$$\{\alpha : \overline{\mu}_{\mathbf{n}\alpha}(I) > \frac{P}{-\log\lambda(I)}\}$$

is residual (here $\overline{\mu}_{\mathbf{n}\alpha}$ is defined analogously to Definition 2.1.16).

Consequently, the sets

$$\mathcal{U}' := \{\alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is s.u.d. mod } 1\}$$

and

$$\mathcal{U}'' := \{\alpha \in \mathbb{R}/\mathbb{Z} : (n_{\mathbf{k}}\alpha) \text{ is } (s_1, \ldots, s_d)\text{-u.d.}\}$$

are meager.

Before proving the theorem we note the following lemma:

Lemma 2.1.47. *Assume that $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is a sequence of positive integers with the property that whenever you choose $T+1$ elements $n_{\mathbf{k}_1} \le \dots \le n_{\mathbf{k}_{T+1}}$ you know that $n_{\mathbf{k}_{T+1}}/n_{\mathbf{k}_1} > U$. Then in a cuboid with $X \ge T+1$ elements there are $D := \lfloor \frac{X-1}{T} \rfloor + 1$ elements $n_{\mathbf{k}'_1}, \dots, n_{\mathbf{k}'_D}$ such that $n_{\mathbf{k}'_{i+1}}/n_{\mathbf{k}'_i} > U$ for $i = 1, \dots, D-1$.*

Proof. Let $n_{\mathbf{k}_1} \le n_{\mathbf{k}_2} \le \dots \le n_{\mathbf{k}_X}$ be a sorting of the X elements in the cuboid and choose the elements with indices $\mathbf{k}_1, \mathbf{k}_{T+1}, \dots, \mathbf{k}_{\lfloor (X-1)/T \rfloor T+1}$.

Proof of the theorem. Choose $P > 0$ so small, that

$$\left(\frac{1}{2^{d+1}PQ} - 1\right) - 1 > 1$$

and assume $\lambda(I) =: \varepsilon < \frac{1}{2}$. Then there exists an integer c in the interval

$$\left(1 - \log\varepsilon, -\frac{1}{2^{d+1}PQ}\log\varepsilon\right).$$

Thus $\frac{1}{2^{d+1}Qc} > \frac{2P}{-\log\varepsilon}$ and $2^c > 2/\varepsilon$. Again we assume that the theorem is false. Since the set $\{\alpha :\ \overline{\mu}_{\mathbf{n}\alpha}(I) > \frac{P}{-\log\varepsilon}\}$ is a Borel set and not residual, its complement is residual in I, for some open interval I:

$$I \Vdash \left\{\alpha :\ \overline{\mu}_{\mathbf{n}\alpha}(I) \le \frac{P}{-\log\varepsilon}\right\}.$$

As in Section 2.1.2 the set $\{\alpha :\ \overline{\mu}_{\mathbf{n}\alpha}(I) \le \frac{P}{-\log\varepsilon}\}$ is contained in the set

$$\left\{\alpha :\ \exists\mathbf{m}\ \forall\mathbf{N} \ge \mathbf{m} :\ \mu_{\mathbf{n}\alpha,\mathbf{N}}(I) \le \frac{2P}{-\log\varepsilon}\right\}.$$

Denote the set $\{\mathbf{j} \le \mathbf{N} :\ n_{\mathbf{j}}\alpha \in I\}$ by $Z_{\mathbf{N}}(\alpha)$. So $\mu_{\mathbf{n}\alpha,\mathbf{N}}(I) = \frac{\#Z_{\mathbf{N}}(\alpha)}{|\mathbf{N}|}$. Therefore

$$I \Vdash \bigcup_{\mathbf{m}} \bigcap_{\mathbf{N}\ge\mathbf{m}} \left\{\alpha :\ \frac{\#Z_{\mathbf{N}}(\alpha)}{|\mathbf{N}|} \le \frac{2P}{-\log\varepsilon}\right\}.$$

So, by Fact 2.1.19, we can find an open interval $J \subseteq I$ and a $\mathbf{k}^*$ such that

$$J \Vdash \bigcap_{\mathbf{N}\ge\mathbf{k}^*} \left\{\alpha :\ \frac{\#Z_{\mathbf{N}}(\alpha)}{|\mathbf{N}|} \le \frac{2P}{-\log\varepsilon}\right\},$$

or equivalently, for all $\mathbf{N} \ge \mathbf{k}^*$:

$$J \Vdash \left\{\alpha :\ \frac{\#Z_{\mathbf{N}}(\alpha)}{|\mathbf{N}|} \le \frac{2P}{-\log\varepsilon}\right\}. \tag{2.10}$$

Let $\delta := \lambda(J)$. Without loss of generality we assume $\mathbf{k}^* = (k, k, \dots, k)$ and $n_{\mathbf{k}} > \varepsilon/\delta$ for all $\mathbf{k} \ge (kc, \dots, kc)$. Now consider the cuboid starting at $(kc, \dots, kc)$ and ending at $(kc(2Q+1), \dots, kc(2Q+1)) =: \mathbf{K}$. Then, by Lemma 2.1.47 with $U = 2/\varepsilon$, $T = 2Qc$ and $X = (2Qkc+1)^d$, there are at least

$$\left\lfloor \frac{(2Qkc+1)^d - 1}{2Qc} \right\rfloor + 1 \ge \frac{2^d Q^d k^d c^d}{2Qc} =: D$$

elements $n_{\mathbf{k}_1}, \ldots, n_{\mathbf{k}_D}$ with $n_{\mathbf{k}_{i+1}}/n_{\mathbf{k}_i} > 2/\varepsilon$ for $i = 1, \ldots, D-1$. Thus the corresponding functions are ε-mixing in δ by [25, Lemma 2.13]. So there is an open interval $K \subseteq J$ such that for all $\alpha \in K$

$$\#Z_{\mathbf{K}}(\alpha) = \#\{\mathbf{j} \leq \mathbf{K} : \ \mathbf{n}_i \boldsymbol{\alpha} \in I\} \geq D.$$

Thus for all $\boldsymbol{\alpha} \in K$

$$\frac{\#Z_{\mathbf{K}}(\alpha)}{|\mathbf{K}|} = \frac{\#Z_{\mathbf{K}}(\alpha)}{k^d c^d (2Q+1)^d} \geq \frac{D}{k^d c^d 4^d Q^d} = \frac{1}{2^{d+1} Qc}. \tag{2.11}$$

Since $\frac{1}{2^{d+1}Qc} > \frac{2P}{-\log \varepsilon}$ and $K \subseteq J$, (2.10) with $\mathbf{N} := \mathbf{K}$ implies

$$K \Vdash \left\{ \alpha : \ \frac{\#Z_{\mathbf{K}}(\alpha)}{|\mathbf{K}|} \leq \frac{1}{2^{d+1} Qc} \right\}. \tag{2.12}$$

Now consider the set $\{\alpha : \ \frac{\#Z_{\mathbf{K}}(\alpha)}{|\mathbf{K}|} < \frac{1}{2^{d+1}Qc}\} \cap K$. By (2.11), this set is empty, but by (2.12) it is residual in K, which is a contradiction.

To obtain the extension of [25, Theorem 2.6], the theorem about the fast growing sequences, we call - in analogy to the classical case - a sequence with multidimensional indices maldistributed in $[0, 1]$, if $M(\mathbf{x}) = \mathscr{P}$.

Theorem 2.1.48. *Let* $\mathbf{n} = (n_{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}^d}$ *be a sequence of nonnegative integers and assume that there are* $R, Q \in \mathbb{N}$*, such that*

$$Q_r := \{\mathbf{k} : \ 2^r \leq n_{\mathbf{k}} < 2^{r+1}\} \leq Q \qquad \forall \ r = 0, 1, 2, \ldots,$$

and that $Q_r \leq 1$ *for all* $r \geq R$*. Moreover, let* $(r_j)_{j \in \mathbb{N}}$ *be the sequence of those indices* r_j *with* $Q_{r_j} > 0$*. Define a sequence* $(\tilde{r}_j)_{j \in \mathbb{N}}$ *by* $\tilde{r}_j = r_j - r_{j-1}$ $(j \geq 0)$ *and* $\tilde{r}_0 = 0$*. Suppose* $\tilde{r}_j \to \infty$*. Then the set*

$$\{\alpha \in \mathbb{R}/\mathbb{Z} : \ \mathbf{n}\alpha \textit{ is maldistributed}\}$$

is residual. Consequently, the set

$$\{\alpha \in \mathbb{R}/\mathbb{Z} : \ \mathbf{n}\alpha \textit{ is strongly maldistributed}\}$$

is residual.

Proof. We follow [25] and adapt the notation. With similar arguments it suffices to show that for each list $\vec{e}$ and each η the set

$$\{\alpha : \text{ for all tails there is an index } \mathbf{N} \text{ such that } \mu_{\mathbf{n}\alpha, \mathbf{N}} \in M_{\vec{e}, \eta}\} \tag{2.13}$$

is residual. Now assume that this fails. Therefore we can find a nonempty interval I, an index $\mathbf{N}_0$, a sequence $\vec{e} = (e_0, \ldots, e_{\ell-1})$ of natural numbers and an $\eta \in \mathbb{R}$ with

$$I \Vdash \{\alpha : \ \forall \mathbf{N} \geq \mathbf{N}_0 : \mu_{\mathbf{n}\alpha, \mathbf{N}} \notin M_{\vec{e}, \eta}\}.$$

W.l.o.g. we assume $\mathbf{N}_0 = (n_0, \ldots, n_0) > (\frac{d}{\eta}, \ldots, \frac{d}{\eta})$, that $e := \sum e_i$ divides $|\mathbf{N}_0|$, $n_{\mathbf{N}_0} > \frac{1}{\lambda(I)}$ and that $\frac{n_{\mathbf{k}'}}{n_{\mathbf{k}}} > 2\ell$ if $n_{\mathbf{k}'} > n_{\mathbf{k}}$.

Choose a sequence of intervals $(I_{\mathbf{j}} : 1 \leq \mathbf{j} \leq \mathbf{N}_0^2)$ where $\mathbf{N}_0^2 = (n_0^2, \ldots, n_0^2)$ such that for all $0 \leq i \leq \ell - 1$ we have

$$|\{\mathbf{j} : 1 \leq \mathbf{j} \leq \mathbf{N}_0^2, I_j = [\frac{i}{\ell}, \frac{i+1}{\ell})\}| = \frac{e_i}{e} |\mathbf{N}_0|^2.$$

So each interval I_j has length $\frac{1}{\ell}$. Let $f_{\mathbf{j}}(x) = n_{\mathbf{j}}x$ for $\mathbf{N}_0 \le \mathbf{j} \le \mathbf{N}_0^2$. Let (f_j) be a sorting of these functions such that $n_{j+1}/n_j > 2\ell$. Then, by [25, Lemma 2.13], the f_j are $\frac{1}{\ell}$-mixing in $\lambda(I)$, i.e., we can find an interval

$$J \subseteq I \cap \bigcap_j f_j^{-1}(I_j).$$

We will show that $\mu_{\mathbf{n}\alpha,\mathbf{N}_0^2} \in M_{\vec{e},\eta}$ for all $\alpha \in J$, which is a contradiction to (2.13). Indeed, if $\alpha \in J$, then for all j we have $f_j(\alpha) \in I_j$. Consequently (writing $O(1)$ for a quantity between -1 and 1) we obtain

$$\mu_{\mathbf{n}\alpha,\mathbf{N}_0^2}([\frac{i}{\ell}, \frac{i+1}{\ell})) = \frac{1}{|\mathbf{N}_0|^2}\left(\frac{e_i}{e}|\mathbf{N}_0^2| + O(1)\cdot d \cdot n_0^{2(d-1)+1}\right) = \frac{e_i}{e} + \frac{dO(1)}{n_0},$$

so $\mu_{\mathbf{n}\alpha,\mathbf{N}_0^2} \in M_{\vec{e},\eta}$, since $\frac{d}{n_0} < \eta$.

Replacing in [2] N by $\mathbf{N}$ and the one-dimensional limits by the multidimensional limits, we get immediately

Theorem 2.1.49.
Given any sequence $(\varepsilon_{i_1,\ldots,i_j,\ldots,i_d}^{i_1,\ldots,i_j+1,\ldots,i_d})_{i_1,\ldots,i_d\in\mathbb{N},j=1,\ldots,d}$ *with* $\varepsilon_{i_1,\ldots,i_j,\ldots,i_d}^{i_1,\ldots,i_j+1,\ldots,i_d} \to 0$ *in the classical (strong) sense, there is a sequence* $n_{\mathbf{k}}$ *of positive integers with*

$$\frac{n_{k_1,\ldots,k_j+1,\ldots,k_d}}{n_{k_1,\ldots,k_j,\ldots,k_d}} > 1 + \varepsilon_{k_1,\ldots,k_j,\ldots,k_d}^{k_1,\ldots,k_j+1,\ldots,k_d}$$

such that for any irrational α *the sequence* $\mathbf{n}\alpha$ *is (strongly) uniformly distributed mod 1.*

2.2 Block-sequences

In the present section we study the relationship between the set of distribution functions of a sequence $(x_n)_{n\in\mathbb{N}}$ and the set of distribution functions of the block sequence induced by (x_n). By *distribution function* we mean any nondecreasing function $g: [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$. Any two distribution functions coinciding at all points of continuity are identified. For a given finite sequence $T = (t_1, \ldots, t_N)$ with $t_n \in [0,1)$ and $x \in [0,1]$ we denote by

$$A([0,x);T) = |\{n \le N : t_n < x\}|$$

the number of elements t_n of the sequence T which are less than x. The sequence T induces the *step distribution function* of T defined by

$$F_T(x) = \frac{A([0,x);T)}{N}$$

for $x \in [0,1]$. Let T_n be a sequence of finite sequences (blocks) in $[0,1)$. The set

$$G(T_n) = \left\{\lim_{k\to\infty} F_{T_{n_k}} : n_k \in \mathbb{N}\right\}$$

is called *the set of distribution functions of the block sequence* T_n. If $(t_n)_{n\in\mathbb{N}}$ is an infinite sequence in $[0,1)$ then the set $G(t_n) := G(T_n)$ with $T_N = (t_n)_{1\le n\le N}$ is called the set of distribution functions of the sequence t_n. Moreover, consider the block sequence U_n defined by

$$U_n = \left\{t_{\frac{(n-1)n}{2}+1}, \cdots, t_{\frac{n(n+1)}{2}}\right\}.$$

Our main object of interest is the relationship between set sets $G(t_n)$ and $G(U_n)$. Indeed, it is possible to obtain $G(t_n)$ from $G(U_n)$ by taking certain convex combinations of functions from $G(U_n)$ if there exist sets $A_1, A_2, \ldots, A_k \subseteq \mathbb{N}$ ($k \in \mathbb{N}$ or $k = \infty$) with certain conditions on the (lower and upper) densities of the sets A_i and if we have convergence of F_{U_n} along these sets, i.e., if the limits

$$\lim_{n\to\infty,\ n\in A_i} F_{U_n}$$

exist for all i. We will proceed as follows: First we examine the case where the sets A_i have densities, afterwards we will relax these conditions.

Let us shortly recall the concept of density of a subset A of the positive integers. The lower and the upper density are defined as

$$\underline{d}(A) = \liminf \frac{|A \cap [1,n]|}{n} \qquad \text{and}$$
$$\bar{d}(A) = \limsup \frac{|A \cap [1,n]|}{n},$$

respectively. If $\underline{d}(A) = \bar{d}(A)$ this value is called the density $d(A)$ of the set A.

In the following we need the lemma below, which follows directly from a general result by Fuchs and Guiliano Antonini [20], but we will give the easy proof here.

Lemma 2.2.1. *Let $A \subseteq \mathbb{N}$ with $d(A) = \alpha$. Then we have*

$$\lim_{n\to\infty} \frac{2}{n(n+1)} \sum_{\substack{1\le i\le n\\ i\in A}} i = \lim_{n\to\infty} \frac{2}{n(n+1)} \sum_{\substack{\sqrt{n}\le i\le n\\ i\in A}} i = \alpha.$$

Proof. Using Abel's partial summation formula, we can rewrite the sum as

$$\frac{2}{n(n+1)} \sum_{\substack{1\le i\le n\\ i\in A}} i = \frac{2}{n(n+1)} A_n n - \frac{2}{n(n+1)} \sum_{i=1}^{n-1} A_i,$$

where $A_i = |A \cap [1,i]|$. Obviously, the first term tends to 2α. Now consider the second term and split the sum:

$$\frac{2}{n(n+1)} \sum_{i=1}^{n-1} A_i = \frac{2}{n(n+1)} \sum_{1\le i\le\sqrt{n}} A_i + \frac{2}{n(n+1)} \sum_{\sqrt{n}<i\le n-1} A_i. \tag{2.14}$$

Since $A_i \le i$, we can bound the first term on the right-hand side by

$$\frac{2}{n(n+1)} \frac{\sqrt{n}(\sqrt{n}+1)}{2},$$

and this converges to 0. By assumption, $\alpha - \varepsilon \le A_n/n \le \alpha + \varepsilon$ holds for all $n \ge N(\varepsilon)$. Hence, for n large enough,

$$(\alpha-\varepsilon) \sum_{\sqrt{n}<i\le n-1} i \le \sum_{\sqrt{n}<i\le n-1} A_i \le (\alpha+\varepsilon) \sum_{\sqrt{n}<i\le n-1} i.$$

Thus

$$\frac{2}{n(n+1)}(\alpha-\varepsilon)\frac{n(n-1) - \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor+1)}{2}$$
$$\le \frac{2}{n(n+1)} \sum_{\sqrt{n}<i\le n-1} A_i \le \frac{2}{n(n+1)}(\alpha+\varepsilon)\frac{n(n-1) - \lfloor\sqrt{n}\rfloor(\lfloor\sqrt{n}\rfloor+1)}{2}.$$

Consequently the second term in (2.14) tends to α. So we obtain the limit by $2\alpha - \alpha = \alpha$. ∎

The following results show that the set $G(U_n)$ determines the set of distribution function completely:

Theorem 2.2.2. *Let $\{A_i\}_{i\in I}$ be a countable family of subsets of the natural numbers with $A_i \cap A_j = \emptyset$ for $i \neq j$. Assume that their densities exist, denote them by $d(A_i) = \alpha_i$, and $\sum_{i\in I} \alpha_i = 1$. Moreover, assume*

$$\lim_{\substack{n\to\infty \\ n\in A_i}} F(U_n, x) = f_i \quad \forall i.$$

Then

$$G(T_n) = \left\{ \sum_{i\in I} \alpha_i f_i \right\}.$$

Proof. Let $\varepsilon > 0$. Choose $s \in \mathbb{N}$ such that

$$d\left(\bigcup_{i=1}^{s} A_i \right) > 1 - \frac{\varepsilon}{2}.$$

We define $k_j := j(j+1)/2$. Let $j \in \mathbb{N}$ be the number with $k_j \leq n < k_{j+1}$. Then we have

$$F(T_n) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i};$$

split the sum at k_j to obtain

$$= \frac{1}{n} \frac{k_j}{k_j} \sum_{i=1}^{k_j} \delta_{x_i} + \frac{1}{n} \sum_{i=k_j+1}^{n} \delta_{x_i}$$

and use the definition of the step distribution functions $F(U_i)$ which yields

$$= \frac{k_j}{n} \sum_{i=1}^{j} \frac{k_i - k_{i-1}}{k_j} F(U_i) + \frac{1}{n} \sum_{i=k_j+1}^{n} \delta_{x_i}.$$

Now split the range of the summation index of the first sum into three parts: The first part contains the sets $A_1, \ldots, A_s$, the second part the remaining sets A_i and the third one those indices which are in any of the sets A_i:

$$\begin{aligned} &= \sum_{\ell=1}^{s} \frac{k_j}{n} \sum_{\substack{1\leq i\leq j \\ i\in A_\ell}} \frac{k_i - k_{i-1}}{k_j} F(U_i) + \sum_{\ell=s+1}^{\infty} \frac{k_j}{n} \sum_{\substack{1\leq i\leq j \\ i\in A_\ell}} \frac{k_i - k_{i-1}}{k_j} F(U_i) \\ &\quad + \frac{k_j}{n} \sum_{\substack{1\leq i\leq j \\ i\in B}} \frac{k_i - k_{i-1}}{k_j} F(U_i) + \frac{1}{n} \sum_{i=k_j+1}^{n} \delta_{x_i}, \end{aligned}$$

where $B = \mathbb{N} \setminus \bigcup_{i=1}^{\infty} A_i$. Thus

$$\begin{aligned} \left| F(T_n) - \sum_{\ell=1}^{s} \alpha_\ell f_\ell \right| &\leq \frac{k_j}{n} \sum_{\ell=1}^{s} \left| \sum_{\substack{1\leq i\leq j \\ i\in A_\ell}} \frac{k_i - k_{i-1}}{k_j} F(U_i) - \alpha_\ell f_\ell \right| + \frac{k_j}{n} \sum_{\ell=s+1}^{\infty} \sum_{\substack{1\leq i\leq j \\ i\in A_\ell}} \frac{k_i - k_{i-1}}{k_j} \\ &\quad + \frac{k_j}{n} \sum_{\substack{1\leq i\leq j \\ i\in B}} \frac{k_i - k_{i-1}}{k_j} + \frac{n - k_j}{n}. \end{aligned}$$

For the fourth term observe that $k_j/n \to 1$ since $k_j/n \le 1$ and

$$\frac{k_j}{n} \ge \frac{k_j}{k_{j+1}} = \frac{j(j+1)}{(j+1)(j+2)} \to 1.$$

Hence

$$\frac{n-k_j}{n} \le \varepsilon$$

for sufficiently large j (say for all $j > J_1(\varepsilon)$).

To bound the second and third term, we note that our choice of s implies that there exists $J_2(\varepsilon)$ such that

$$\frac{1}{j}\left|\bigcup_{i=1}^{s} A_i \cap [1,j]\right| > 1-\varepsilon$$

for all $j > J_2(\varepsilon)$. Consequently

$$\frac{1}{j}\left|\mathbb{N}\setminus\bigcup_{i=1}^{s} A_i \cap [1,j]\right| < \varepsilon.$$

Thus the second and third term can be bounded from above by

$$\begin{aligned}
\frac{2}{j(j+1)}\sum_{i=\lfloor j(1-\varepsilon)\rfloor+1}^{j} i &\le \frac{2}{j(j+1)}\left(\frac{j(j+1)}{2} - \frac{(\lfloor j(1-\varepsilon)\rfloor+1)\lfloor j(1-\varepsilon)\rfloor}{2}\right)\\
&\le \frac{1}{j^2}\big(j^2+j-j(1-\varepsilon)(j(1-\varepsilon)-1)\big)\\
&= \frac{1}{j^2}\big(j^2+j-j^2(1-\varepsilon)^2+j(1-\varepsilon)\big)\\
&= \frac{1}{j^2}\big(j^2(2\varepsilon-\varepsilon^2)+j(2-\varepsilon)\big)\\
&\le 2\varepsilon+\frac{2}{j} < 3\varepsilon
\end{aligned}$$

for j large enough (say for all $j > J_3(\varepsilon)$).

Split the first term into two parts:

$$\sum_{\ell=1}^{s}\left|\sum_{\substack{1\le i\le\sqrt{j}\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j}F(U_i)\right| + \sum_{\ell=1}^{s}\left|\sum_{\substack{\sqrt{j}\le i\le j\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j}F(U_i)-\alpha_\ell f_\ell\right|.$$

Here the first sum can be estimated by

$$\frac{2}{j^2}\sum_{1\le i\le\sqrt{j}} i \le \frac{\sqrt{j}(\sqrt{j}+1)}{j^2} < \varepsilon$$

if $j > J_4(\varepsilon)$. In the second term consider the inner sum. By assumption, we find $J_5(\delta)$ such that

$$(f_\ell-\delta)\sum_{\substack{\sqrt{j}\le i\le j\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j} \le \sum_{\substack{\sqrt{j}\le i\le j\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j}F(U_i) \le (f_\ell+\delta)\sum_{\substack{\sqrt{j}\le i\le j\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j},$$

for all $j > J_5(\delta)$. Hence, by Lemma 2.2.1,

$$(f_\ell - \delta)(\alpha_\ell - \delta) \le \sum_{\substack{\sqrt{j} \le i \le j \\ i \in A_\ell}} \frac{k_i - k_{i-1}}{k_j} F(U_i) \le (f_\ell + \delta)(\alpha_\ell + \delta)$$

for sufficiently large j (say for all $j > J_6(\delta)$). Thus the second term can be bounded by

$$\sum_{\ell=1}^{s} \delta(f_\ell + \alpha_\ell) + \delta^2 \le 3s\delta.$$

Choose $\delta = \delta(s, \varepsilon) < \varepsilon/(3s)$, so this term is $< \varepsilon$. Putting things together we obtain

$$\left| F(T_n) - \sum_{\ell=1}^{s} \alpha_\ell f_\ell \right| \le 6\varepsilon$$

if $j > \max\{J_1(\varepsilon), J_2(\varepsilon), J_3(\varepsilon), J_4(\varepsilon), J_5(\varepsilon, s), J_6(\varepsilon, s)\}$. If ε tends to 0 we obtain the desired limit.

Our next goal is to generalise the preceding theorem partially. We start with a lemma which generalises Lemma 2.2.1.

Lemma 2.2.3. *Let $A \subseteq \mathbb{N}$ with $\underline{d}(A) = \alpha$ and $\bar{d}(A) = \beta$. Then we have*

$$\frac{\alpha^2}{\beta} \le \liminf_{n\to\infty} \frac{2}{n(n+1)} \sum_{\substack{1 \le i \le n \\ i \in A}} i = \liminf_{n\to\infty} \frac{2}{n(n+1)} \sum_{\substack{\sqrt{n} \le i \le n \\ i \in A}} i \le \alpha$$

and

$$\beta \le \limsup_{n\to\infty} \frac{2}{n(n+1)} \sum_{\substack{1 \le i \le n \\ i \in A}} i = \limsup_{n\to\infty} \frac{2}{n(n+1)} \sum_{\substack{\sqrt{n} \le i \le n \\ i \in A}} i \le 1 - \frac{(1-\beta)^2}{1-\alpha}.$$

Proof. Using Abel's partial summation formula, we can rewrite the sum as

$$\frac{2}{n(n+1)} \sum_{\substack{1 \le i \le n \\ i \in A}} i = \frac{2}{n(n+1)} A_n n - \frac{2}{n(n+1)} \sum_{i=1}^{n-1} A_i, \tag{2.15}$$

where $A_i = |A \cap [1, i]|$. Consider the second term and split the sum:

$$\frac{2}{n(n+1)} \sum_{i=1}^{n-1} A_i = \frac{2}{n(n+1)} \sum_{1 \le i \le \sqrt{n}} A_i + \frac{2}{n(n+1)} \sum_{\sqrt{n} < i \le n-1} A_i. \tag{2.16}$$

Since $A_i \le i$, we can bound the first term on the right-hand side by

$$\frac{2}{n(n+1)} \frac{\sqrt{n}(\sqrt{n}+1)}{2},$$

and this tends to 0. Thus we can study instead of (2.15) the sum

$$S_n := \frac{2}{n(n+1)} A_n n - \frac{2}{n(n+1)} \sum_{\sqrt{n} < i \le n-1} A_i. \tag{2.17}$$

Firstly, we show the lower bound of the lim sup: Choose a subsequence n_k with the property that A_{n_k}/n_k converges to β. Hence, for each $\varepsilon > 0$,

$$\limsup_{n\to\infty} S_n \geq \lim_{k\to\infty} \frac{2n_k}{n_k+1}\frac{A_{n_k}}{n_k} - \lim_{n\to\infty} \frac{2}{n(n+1)} \sum_{\sqrt{n}<i\leq n-1} i(\beta+\varepsilon).$$

Simplifying the sum yields

$$\limsup S_n \geq 2\beta - \lim_{n\to\infty} \frac{2}{n(n+1)}(\beta+\varepsilon)\frac{n(n-1)}{2} = \beta - \varepsilon$$

Since ε was arbitrary we get $\limsup S_n \geq \beta$. Similarly, we find $\liminf S_n \leq \alpha$. After these easy estimates we want to establish the two other bounds. For this purpose we define $S_n(c)$ as the sum in (2.17) with $A_n/n = c$. By assumption there exists $N(\varepsilon)$ such that

$$A_n/n \in [\alpha-\varepsilon, \beta+\varepsilon] \;\forall n \geq N(\varepsilon). \tag{2.18}$$

Assume $n \geq N^2(\varepsilon)$ in the following. To prove the lower bound of the lim inf note that the sum on the right-hand side of (2.17) is large iff the A_i are large. So we have to choose the A_i as big as possible. The trivial bound $A_i/i \leq \beta+\varepsilon$ for all i would not lead to the desired result, so we have to be more careful. A fruitful way for choosing the A_i is the choice

$$A_i = A_n \quad \text{for} \quad J \leq i \leq n, \tag{2.19}$$

where J is the smallest number such that the condition (2.19) does not violate condition (2.18). For the remaining A_i we take the trivial bound $A_i/i = \beta+\varepsilon$ $(i < J)$. Roughly speaking this means that A contains no element between J and n and each $(\beta+\varepsilon)n$-th element before. To compute J we conclude from (2.19) and (2.18) that

$$\frac{A_J}{J} = \frac{A_n}{n}\frac{n}{J} = \frac{cn}{J} \leq \beta+\varepsilon.$$

Thus J is the smallest number such that the inequality above holds. Hence we have to choose

$$J = \left\lceil \frac{cn}{\beta+\varepsilon} \right\rceil.$$

So we can rewrite $S_n(c)$ as

$$S_n(c) = 2\frac{nc}{n+1} - \frac{2}{n(n+1)} \sum_{\sqrt{n}<i<J} A_i - \frac{2}{n(n+1)} \sum_{J\leq i\leq n-1} A_i.$$

Using (2.19) and the bound $\beta+\varepsilon$ we get

$$\geq 2\frac{nc}{n+1} - \frac{2}{n(n+1)} \sum_{1\leq i<J} (\beta+\varepsilon)i - \frac{2}{n(n+1)} \sum_{J\leq i\leq n-1} A_n$$

and simplifying the sums yields

$$= 2\frac{nc}{n+1} - \frac{2}{n(n+1)}\frac{J(J-1)}{2}(\beta+\varepsilon) - \frac{2c}{(n+1)}(n-J).$$

Now plug in the definition of J

$$\geq 2\frac{nc}{n+1} - \frac{1}{n(n+1)}\frac{cn}{\beta+\varepsilon}\left(\frac{cn}{\beta+\varepsilon}+1\right)(\beta+\varepsilon) - \frac{2c}{(n+1)}\left(n - \frac{cn}{\beta+\varepsilon}\right)$$

and simplify the resulting term to obtain

$$\begin{aligned}
&= \frac{n}{n+1}\left(2c - \frac{c^2}{(\beta+\varepsilon)^2}(\beta+\varepsilon) - 2c + \frac{2c^2}{\beta+\varepsilon}\right) - \frac{c}{n+1} \\
&= \frac{n}{n+1}\frac{c^2}{\beta+\varepsilon} - \frac{c}{n+1} \geq \frac{c^2}{\beta+\varepsilon} - \frac{2+\varepsilon}{n+1}.
\end{aligned}$$

Since the bound above is increasing in c, this implies $\liminf S_n \geq \frac{\alpha^2}{\beta}$.
To bound S_n from above we proceed in a very similar way, but now A has to contain all elements $K, \ldots, n$ from some number K on, i.e.,

$$A_i = A_n - (n-i) \qquad K \leq i \leq n.$$

Having in mind that $A_i/i \geq \alpha - \varepsilon$ we find

$$K = \left\lceil \frac{n(c-1)}{\alpha - \varepsilon - 1} \right\rceil.$$

Using $1/(n(n+1)) \leq 1/n^2$ and splitting $S_n(c)$ as above we obtain

$$S_n(c) \leq 2c - \frac{2}{n^2}\sum_{\sqrt{n}<i<K} A_i - \frac{2}{n^2}\sum_{K\leq i\leq n-1} A_i$$

which can be estimated by using the bounds on A_i

$$\leq 2c - \frac{2}{n^2}\sum_{\sqrt{n}<i<K}(\alpha-\varepsilon)i - \frac{2}{n^2}\sum_{K\leq i\leq n-1}(A_n - n + i)$$

and leads after simplifying the sums to

$$\begin{aligned}
&\leq 2c - \frac{2}{n^2}(n-K)(A_n - n) - \frac{2}{n^2}\left(\frac{(n-1)n}{2} - \frac{(K-1)K}{2}\right) \\
&\qquad - \frac{2}{n^2}(\alpha-\varepsilon)\left(\frac{(K-1)K}{2} - \frac{\lceil\sqrt{n}\rceil(\lceil\sqrt{n}\rceil+1)}{2}\right).
\end{aligned}$$

Using the definition of K gives

$$\begin{aligned}
&\leq 2c - 2\left(1 - \frac{c-1}{\alpha-\varepsilon-1}\right)(c-1) - \frac{n-1}{n} + \frac{1}{n^2}\frac{n(c-1)}{\alpha-\varepsilon-1}\left(\frac{n(c-1)}{\alpha-\varepsilon-1}+1\right) \\
&\qquad - \frac{\alpha-\varepsilon}{n^2}\frac{n(c-1)}{\alpha-\varepsilon-1}\left(\frac{n(c-1)}{\alpha-\varepsilon-1}-1\right) + \frac{2}{n}.
\end{aligned}$$

Simplifying yields

$$S_n(c) \leq 1 + \frac{(c-1)^2}{\alpha-\varepsilon-1} + \frac{1}{n}\frac{4}{1-\alpha},$$

which is increasing in c, implies the upper bound, and completes the proof. ✎

The bounds in this lemma can't be improved, since there exist examples which reach these bounds.

Example 2.2.4. Clearly, if we choose a set A with density a (i.e., $\alpha = \beta = a$) then the considered sum equals a.

Example 2.2.5. But we can also give an example with $\alpha \neq \beta$ that reaches the upper bound for the lim inf and the lower bound of the lim sup: Let in the following $n_k = 2^{2^k}$. Roughly speaking the idea is to take every $1/\alpha$-th element between n_{2k} and n_{2k+1} and every $1/\beta$-th element between n_{2k+1} and n_{2k+2}. To formalise this take two sets $\mathcal{A}$ and $\mathcal{B}$ with densities α and β respectively. Define the set A by

$$A \cap [n_{2k}+1, n_{2k+1}] = \mathcal{A} \cap [n_{2k}+1, n_{2k+1}]$$
$$A \cap [n_{2k+1}+1, n_{2k+2}] = \mathcal{B} \cap [n_{2k+1}+1, n_{2k+2}]$$

To check $\underline{d}(A) = \alpha$ and $\bar{d}(A) = \beta$ note that for $n_{2k} < n \leq n_{2k+1}$ we have

$$d_n(A) = \frac{|A \cap [1,n]|}{n} = \frac{|\mathcal{B} \cap [1, n_{2k}]|}{n_{2k} - \bar{n}} \frac{n_{2k} - \bar{n}}{n} + \frac{|\mathcal{A} \cap [1,n]|}{\bar{n} + n - n_{2k}} \frac{\bar{n} + n - n_{2k}}{n},$$

where $\bar{n} = \sum_{i=1}^{k} n_{2i-1} - n_{2i-2}$. If n is large enough we can bound $d_n(A)$ by

$$(\beta - \varepsilon)t + (\alpha - \varepsilon)(1-t) \leq d_n(A) \leq (\beta + \varepsilon)t + (\alpha + \varepsilon)(1-t)$$

with $t = (n_{2k} - \bar{n})/n$. Obviously, $0 \leq t \leq 1$, and for $n = n_{2k}$ t approaches 1, whereas for $n = n_{2k+1}$ t tends to 0. A similar computation holds for $n_{2k+1} < n \leq n_{2k+2}$. Thus A has the given lower and upper density. Now we want to show that for this A the sum (2.17) reaches the stated bounds. For this purpose consider

$$S_n := \frac{2n|A \cap [1,n]|}{n(n+1)} - \frac{2}{n(n+1)} \sum_{\sqrt{n} < i \leq n-1} |A \cap [1,i]|$$

for $n_{2k} < n \leq n_{2k+1}$. The condition on n implies $n_{2k-1} < \sqrt{n}$. To compute the bounds of the limit of S_n we use for the first term the computation above (for $d_n(A)$), split the sum into parts (where the summations run over $n \leq n_{2k}$ and $n_{2k} < n$, respectively) and apply the decomposition of A into $\mathcal{A}'$ and $\mathcal{B}'$. This yields

$$\begin{aligned} S_n &= 2\beta t + 2\alpha(1-t) + \mathrm{O}(\varepsilon) \\ &- \frac{2}{n^2} \sum_{\sqrt{n} \leq i \leq n_{2k}} \left(\frac{|\mathcal{A} \cap [1, n_{2k-1}]|}{n_{2k-1} - \hat{n}} (n_{2k-1} - \hat{n}) + \frac{|\mathcal{B} \cap [1,i]|}{\hat{n} + i - n_{2k-1}} (\hat{n} + i - n_{2k-1}) \right) \\ &- \frac{2}{n^2} \sum_{n_{2k} < i \leq n-1} \left(\frac{|\mathcal{B} \cap [1, n_{2k}]|}{n_{2k} - \bar{n}} (n_{2k} - \bar{n}) + \frac{|\mathcal{A} \cap [1,i]|}{\bar{n} + i - n_{2k}} (\bar{n} + i - n_{2k}) \right), \end{aligned}$$

where $\hat{n} = \sum_{i=2}^{k} n_{2i-2} - n_{2i-3}$ and $\bar{n} = \sum_{i=1}^{k} n_{2i-1} - n_{2i-2}$. The quotients above are in $[\alpha - \varepsilon, \alpha + \varepsilon]$ and $[\beta - \varepsilon, \beta + \varepsilon]$ for n large enough. Hence

$$\begin{aligned} S_n &= 2\beta t + 2\alpha(1-t) + \mathrm{O}(\varepsilon) - \frac{2}{n^2} \alpha (n_{2k-1} - \hat{n})(n_{2k} - \sqrt{n}) \\ &- \frac{2}{n^2} \beta \left((\hat{n} - n_{2k-1})(n_{2k} - \sqrt{n}) + \frac{n_{2k}^2}{2} - \frac{n}{2} \right) \\ &- \frac{2}{n^2} \beta (n_{2k} - \bar{n})(n - n_{2k}) - \frac{2}{n^2} \alpha \left((\bar{n} - n_{2k})(n - n_{2k}) + \frac{n^2}{2} - \frac{n_2 k^2}{2} \right) + \mathrm{o}(n). \end{aligned}$$

To proceed further, observe that the last term in the first line tends to 0, since $n_{2k-1}/n_{2k} \to 0$. Moreover, from the guys in the brackets in the second line only the term $n_{2k}^2/2$ gives a contribution to the limit. Using the definition of t we rewrite S_n as

$$\begin{aligned} S_n = 2\beta t + 2\alpha(1-t) + 2\beta t\frac{n-n_{2k}}{n} + 2\alpha t\frac{n-n_{2k}}{n} \\ - \alpha\frac{n^2-n_{2k}^2}{n^2} - \beta\frac{n_2k^2}{n^2} + \mathrm{o}(n) + \mathrm{O}(\varepsilon). \end{aligned}$$

Simplifying yields

$$S_n = \beta\frac{n_{2k}^2 - 2n_{2k}\bar{n}}{n^2} + \alpha\frac{n^2 - n_{2k}^2 + 2n_{2k}\bar{n}}{n^2} + \mathrm{o}(n) + \mathrm{O}(\varepsilon).$$

Using the fact that $\bar{n}/n \to 0$ we obtain

$$S_n = \beta s + \alpha(1-s) + +\mathrm{o}(n) + \mathrm{O}(\varepsilon)$$

with $s = n_{2k}^2/n^2$. Since s reaches 0 for $n = n_{2k+1}$ and 1 for $n = n_{2k}+1$ we get the desired bounds if $n_{2k} < n \leq n_{2k+1}$. Similarly the case $n_{2k-1} < n \leq n_{2k}$ can be treated. Thus the upper and lower limit of S_n are established.

Example 2.2.6. Now we give an example which shows that the lower bound of the lim inf can not be improved. Let A be any set with density $d(A) = \beta$, $J_k = \lceil \alpha n_{k+1}/\beta \rceil$ and $\hat{J}_k = \lfloor (1-\alpha)n_k/(1-\beta) \rfloor$. Define the set $\mathcal{A}$ by

$$\begin{aligned} \mathcal{A} \cap [\hat{J}_k, J_k] &= A \cap [\hat{J}_k, J_k], \\ \mathcal{A} \cap [n_k+1, \hat{J}_k - 1] &= [n_k+1, \hat{J}_k - 1], \\ \mathcal{A} \cap [J_k+1, n_{k+1}] &= \emptyset. \end{aligned}$$

Now we check $\underline{d}(\mathcal{A}) = \alpha$ and $\bar{d}(\mathcal{A}) = \beta$. For this purpose we show that $d_{J_k}(\mathcal{A}) \to \beta$:

$$d_{J_k}(\mathcal{A}) = \frac{\mathcal{A} \cap [1, J_k]}{J_k} = \underbrace{\frac{\mathcal{A} \cap [1, \hat{J}_k - 1]}{\hat{J}_k - 1}}_{\leq 1} \underbrace{\frac{\hat{J}_k - 1}{J_k}}_{\to 0} + \underbrace{\frac{\mathcal{A} \cap [\hat{J}_k, J_k]}{J_k - \hat{J}_k + 1}}_{\to \beta} \underbrace{\frac{J_k - \hat{J}_k + 1}{J_k}}_{\to 1}.$$

Moreover, we have $d_{n_{k+1}}(\mathcal{A}) \to \alpha$, since

$$\begin{aligned} d_{n_{k+1}}(\mathcal{A}) &= \frac{\mathcal{A} \cap [1, n_{k+1}]}{n_{k+1}} \\ &= \underbrace{\frac{\mathcal{A} \cap [1, \hat{J}_k - 1]}{\hat{J}_k - 1}}_{\leq 1} \underbrace{\frac{\hat{J}_k - 1}{n_{k+1}}}_{\to 0} + \underbrace{\frac{\mathcal{A} \cap [\hat{J}_k, J_k]}{J_k - \hat{J}_k + 1}}_{\to \beta} \underbrace{\frac{J_k - \hat{J}_k + 1}{n_{k+1}}}_{\to \alpha/\beta} + \underbrace{\frac{\mathcal{A} \cap [J_k + 1, n_{k+1}]}{n_{k+1}}}_{=0}. \end{aligned}$$

Consequently, $d_{\hat{J}_k}(\mathcal{A}) \to \beta$:

$$d_{\hat{J}_k}(\mathcal{A}) = \frac{\mathcal{A} \cap [1, \hat{J}_k]}{\hat{J}_k} = \underbrace{\frac{\mathcal{A} \cap [1, n_k]}{n_k - 1}}_{\to \alpha} \underbrace{\frac{n_k - 1}{\hat{J}_k}}_{\to (1-\beta)/(1-\alpha)} + \underbrace{\frac{\mathcal{A} \cap [n_k + 1, \hat{J}_k]}{\hat{J}_k - n_k}}_{=1} \underbrace{\frac{\hat{J}_k - n_k}{\hat{J}_k}}_{\to 1-(1-\beta)/(1-\alpha)}.$$

Finally we show that for given $\varepsilon > 0$ we can find K such that for all $k \geq K$ and $\hat{J}_k < n < J_k$ we have $d_n(\mathcal{A}) \leq \beta + 3\varepsilon$:

$$\begin{aligned} d_n(\mathcal{A}) &= \frac{\mathcal{A} \cap [1,n]}{n} = \frac{\mathcal{A} \cap [1,\hat{J}_k]}{\hat{J}_k} \frac{\hat{J}_k}{n} + \frac{\mathcal{A} \cap [\hat{J}_k + 1, n]}{n} \\ &\leq (\beta + \varepsilon)\frac{\hat{J}_k}{n} + \frac{\mathcal{A} \cap [1,n]}{n} - \frac{\mathcal{A} \cap [1,\hat{J}_k]}{\hat{J}_k}\frac{\hat{J}_k}{n} \\ &\leq (\beta + \varepsilon)\frac{\hat{J}_k}{n} + (\beta + \varepsilon) - (\beta - \varepsilon)\frac{\hat{J}_k}{n} \leq \beta + 3\varepsilon, \end{aligned}$$

if k (and consequently n) is large enough. Now we compute very similarly to the proof of Lemma 2.2.3

$$\begin{aligned} S_{n_k} &= \frac{2}{n_k(n_k+1)} A_{n_k} n_k - \frac{2}{n_k(n_k+1)} \sum_{\hat{J}_k \leq i \leq n_k - 1} A_i + \mathrm{o}(1) \\ &= 2\alpha - \frac{2}{n_k(n_k+1)} \left(\sum_{\hat{J}_k \leq i \leq J_k} A_i + \sum_{J_k < i \leq n_k - 1} A_i \right) + \mathrm{o}(1) \\ &= 2\alpha - \frac{2}{n_k(n_k+1)} \frac{(J_k+1)J_k}{2}\beta - \frac{2}{n_k(n_k+1)}(n_k - J_k) + \mathrm{o}(1) \\ &= 2\alpha - \frac{\alpha^2}{\beta} - 2\alpha + 2\frac{\alpha^2}{\beta} + \mathrm{o}(1) = \alpha + \mathrm{o}(1). \end{aligned}$$

Thus the lim inf is established. In a similar way we can construct examples which give the upper bound of the lim sup.

Now we are able to establish the following theorem which generalises Theorem 2.2.2 and shows that even under the relaxed conditions on the sets A_i the set of distribution functions $G(t_n)$ can be obtained from the set $G(U_n)$. Let $s \in \mathbb{N}$, denote $H = \{(x_1, \ldots, x_s) \in \mathbb{R} : \sum_{i=1}^s x_i = 1\}$ and $\pi_i : \mathbb{R}^s \to \mathbb{R}$ the i-th projection.

Theorem 2.2.7. *Let $A_1, \ldots, A_s \subseteq \mathbb{N}$ be such that $d(\bigcup_{i=0}^s A_i) = 1$. Denote their lower and upper densities by $\underline{d}(A_i) = \alpha_i$ and $\overline{d}(A) = \beta_i > 0$, respectively. Moreover, assume*

$$\lim_{\substack{n \to \infty \\ n \in A_i}} F(U_n, x) = f_i \quad \forall i.$$

Then there exists a connected closed subset P of H such that

(a) $[\alpha_i, \beta_i] \subseteq \pi_i(P) \subseteq [\frac{\alpha_i^2}{\beta_i}, 1 - \frac{(1-\beta_i)^2}{1-\alpha_i}]$ *and*

(b) $G(T_n) = \{\sum_{i=1}^s c_i f_i : (c_1, \ldots, c_s) \in P\}$.

Proof. Let $\varepsilon > 0$. We proceed as in the proof of Theorem 2.2.2. Let $j \in \mathbb{N}$ be the number with $k_j \leq n < k_{j+1}$. Then we have

$$F(T_n) = \sum_{\ell=1}^{s} \frac{k_j}{n} \sum_{\substack{1 \leq i \leq j \\ i \in A_\ell}} \frac{k_i - k_{i-1}}{k_j} F(U_i) + \frac{k_j}{n} \sum_{\substack{1 \leq i \leq j \\ i \in B}} \frac{k_i - k_{i-1}}{k_j} F(U_i) + \frac{1}{n} \sum_{i=k_j+1}^{n} \delta_{x_i},$$

where $B = \mathbb{N} \setminus \bigcup_{i=1}^s A_i$. The second and third term tend to 0 as above.

Now turn to the first term and split it into two parts:

$$\sum_{\ell=1}^{s}\sum_{\substack{1\le i\le\sqrt{j}\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j}F(U_i)+\sum_{\ell=1}^{s}\sum_{\substack{\sqrt{j}\le i\le j\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j}F(U_i).$$

Here the first sum can be estimated by

$$\frac{2}{j^2}\sum_{1\le i\le\sqrt{j}} i\le\frac{\sqrt{j}(\sqrt{j}+1)}{j^2}<\varepsilon$$

if j large enough. Hence the accumulation points of the sequence $F(T_n)$ are the same as the accumulation points of

$$S:=\sum_{\ell=1}^{s}\frac{k_j}{n}\sum_{\substack{\sqrt{j}\le i\le j\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j}F(U_i).$$

Since $k_j/n\to 1$ and $F(U_i)\to f_\ell$ we have

$$S=\sum_{\ell=1}^{s}f_\ell\sum_{\substack{\sqrt{j}\le i\le j\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j}+\mathrm{o}(1).$$

Using the estimates of Lemma 2.2.3 we conclude that there exists a set $P\subseteq\mathbb{R}^s$ with $\pi_i(P)\subseteq[\frac{\alpha_i^2}{\beta_i},1-\frac{(1-\beta_i)^2}{1-\alpha_i}]$ and the property (b) of the theorem. Since for each index ℓ the sum $\sum_{\substack{\sqrt{j}\le i\le j\\ i\in A_\ell}}\frac{k_i-k_{i-1}}{k_j}$ runs through the interval $[\alpha_\ell,\beta_\ell]$ by Lemma 2.2.3, property (a) is proven. Since $G(T_n)$ is completely determined by P, P must be connected and closed.

As in Lemma 2.2.3, the bounds can not be improved.

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