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Lamplighter Random Walks and Entropy-Sensitivity of Languages

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STATUTORY DECLARATION

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly marked all material which has been quotes either literally or by content from the used sources.

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Introduction

The main purpose of this thesis is to study the interplay between geometric properties of infinite graphs and analytic and probabilistic objects such as transition operators, harmonic functions and random walks on these graphs.

Suppose we are given a random walk (X_n) on a graph G . There are many questions regarding its behaviour, as the discrete parameter n goes to infinity. Standard questions of this kind are: will the random walk visit some given vertex infinitely many times, or will it leave any bounded set after a finite time with probability one? In the first case the random walk is called *recurrent* and in the second *transient*.

For a transient random walk, there are several problems one is interested in: for instance to study its *convergence* (in a sense to be specified), to describe the *bounded harmonic functions* for the random walk, to describe its *Poisson boundary*, or to study the parameter of exponential decay of the transition probabilities of the random walk (*spectral radius*).

In the *first part* of the thesis we deal with similar problems in the context of random walks on the so-called *lamplighter graphs*, which are *wreath products* of graphs. Random walks on such graphs will be called *lamplighter random walks*. All walks we consider will be transient and irreducible throughout this part. The convergence and the Poisson boundary of *lamplighter random walks* is studied for different underlying graphs, and the used methods are mostly of a geometrical nature. Most of the results presented here are published in SAVA [Sav10].

In the *second part* of the thesis we consider Markov chains on directed, labelled graphs. With such graphs we associate in a natural way a class of infinite languages (sets of labels of paths in the graph) and we study the *growth sensitivity* (or *entropy sensitivity*) of these languages using Markov chains. The growth sensitivity of a language is a well-known and studied topic in group theory and symbolic dynamics. Under suitable general assumptions on the graphs, we prove that the associated languages are growth sensitive, by using Markov chains with forbidden transitions. This part of the thesis is mainly based on the paper by HUSS, SAVA AND WOESS [HSW10].

Overview

In *Chapter 1* a general introduction to the theory of Markov chains, graphs, random walks over graphs and groups, and their properties like nearest neighbour type, transience/recurrence, spectral radius, rate of escape, is given. For a Markov chain with transition matrix P over an infinite state space \mathbf{G} we shall either use the notation (\mathbf{G}, P) or the sequence (X_n) of \mathbf{G} -valued random variables, depending on the context.

Part I

In *Chapter 2* we introduce a class of random walks on wreath products $\mathbb{Z}_2 \wr \mathbf{G}$ of graphs \mathbb{Z}_2 and \mathbf{G} , where \mathbf{G} is an infinite, connected, transitive graph and $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ represents a finite set with two elements, which encodes the state of lamps. The intuitive interpretation is as follows. One considers the base graph \mathbf{G} , and at each vertex of \mathbf{G} there is a lamp which can be switched on or off. If one defines the set

$$\mathcal{C} = \{\eta : \mathbf{G} \rightarrow \mathbb{Z}_2, \eta \text{ finitely supported}\}$$

of finitely supported lamp configurations, then the *wreath product* $\mathbb{Z}_2 \wr \mathbf{G}$ is the graph with vertex set $\mathcal{C} \times \mathbf{G}$, and adjacency relation

$$(\eta, x) \sim (\eta', x') :\Leftrightarrow \begin{cases} x \sim x' & \text{in } \mathbf{G} \text{ and } \eta = \eta', \\ x = x' & \text{in } \mathbf{G} \text{ and } \eta \triangle \eta' = \{x\}. \end{cases}$$

The space $\mathbb{Z}_2 \wr \mathbf{G}$ is called the *lamplighter graph*, and consists of pairs (η, x) where η is a finitely supported configuration of lamps and x a vertex in \mathbf{G} . Let P be a transition matrix on $\mathbb{Z}_2 \wr \mathbf{G}$ whose entries are of the form

$$p((\eta, x), (\eta', x')),$$

and (Z_n) the random walk with transitions in P . This walk will be called the *lamplighter random walk (LRW)*, and it can be written as $Z_n = (\eta_n, X_n)$, where η_n is the random configuration of lamps and X_n is the random position in \mathbf{G} at time n . Then (X_n) defines a random walk on \mathbf{G} , called the *base random walk*, whose transition matrix $P_{\mathbf{G}}$ is given by the projection of P on the base graph \mathbf{G} . We shall assume that (X_n) is transient, and this implies transience of the lamplighter random walk (Z_n) .

The first part of the thesis is devoted to the description of the behaviour of (Z_n) , as n goes to infinity, and the Poisson boundary is the main object of study in this part. As we shall later see, the behaviour at infinity depends on the base random walk (X_n) and on the geometry of the transitive base graph \mathbf{G} .

Let $\text{AUT}(\mathbf{G})$ be the group of all automorphisms of the graph \mathbf{G} , and Γ a closed subgroup of $\text{AUT}(\mathbf{G})$. Also, let $\partial\mathbf{G}$ be a general boundary at infinity of \mathbf{G} , such that the action of Γ on \mathbf{G} extends to this boundary, and such that (X_n) converges to $\partial\mathbf{G}$. Then, in *Chapter 3* the convergence of (Z_n) to a “natural” boundary Π of the lamplighter graph $\mathbb{Z}_2 \wr \mathbf{G}$ is proved. The boundary Π is defined as

$$\Pi = \bigcup_{\mathbf{u} \in \partial\mathbf{G}} \mathcal{C}_{\mathbf{u}} \times \{\mathbf{u}\},$$

where $\mathcal{C}_{\mathbf{u}}$ consists of all configurations ζ which are either finitely supported, or infinitely supported with \mathbf{u} being the only accumulation point of $\text{supp}(\zeta)$. Note that $\text{supp}(\zeta)$ is the subset of \mathbf{G} where the lamps are on. In this settings, we can prove the following.

Theorem 3.2.3. *Let (Z_n) be an irreducible and homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathbf{G}$. Then there exists a Π -valued random variable Z_∞ , such that $Z_n \rightarrow Z_\infty$ almost surely and the distribution of Z_∞ is a continuous measure on Π .*

We remark that the limit random variable Z_∞ is a pair of the form (η_∞, X_∞) , where η_∞ is the limit configuration of lamps, which is not necessary finitely supported, and X_∞ is the limit of the base random walk (X_n) on \mathbf{G} .

Chapter 4 starts with preliminaries and definitions on the Poisson boundary of a random walk. Most of them are due to KAIMANOVICH AND VERSHIK [KV83], and KAIMANOVICH [Kai00]. In the most general formulation, it represents the space of ergodic components of the time shift in the path space. It is a measure theoretical space, which gives a representation of the bounded harmonic functions for the respective random walk in terms of the *Poisson formula*.

The Poisson boundary can also be described using purely geometric approaches: for instance the *Strip Criterion* and the *Ray Criterion*, developed by KAIMANOVICH [Kai00]. Based on the Strip Criterion (Theorem 4.1.4), we describe the so-called *Half Space Method* for lamplighter random walks. It requires the existence of a strip $\mathfrak{s}(\mathbf{u}, \mathbf{v}) \subset \mathbf{G}$, with $\mathbf{u}, \mathbf{v} \in \partial\mathbf{G}$, which has the properties requested by the Strip Criterion. If such a strip \mathfrak{s} exists for the base random walk (X_n) on \mathbf{G} , then the *Half Space Method* explains how to construct a “bigger” strip S as a subset of $\mathbb{Z}_2 \wr \mathbf{G}$, which satisfies again the conditions of the Strip Criterion. This method requires that the state space \mathbf{G} can be split into “half spaces” \mathbf{G}_+ and \mathbf{G}_- .

Under suitable assumptions on the transitive base graph \mathbf{G} and on the random walk (X_n) on it, we can prove the following for lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathbf{G}$, with $Z_n = (\eta_n, X_n)$.

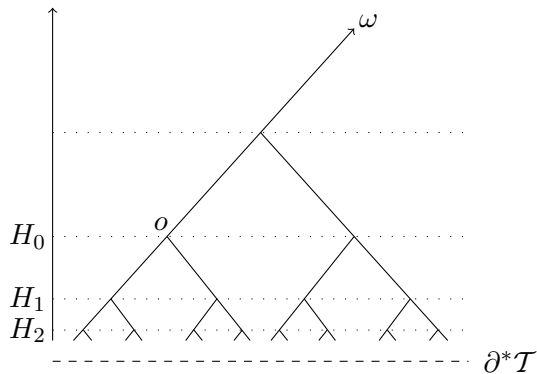
Theorem 4.2.1. *Let (Z_n) be an irreducible, homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathbf{G}$. If ν_∞ is the limit distribution of (Z_n) on Π , then the measure space (Π, ν_∞) is the Poisson boundary of (Z_n) .*

In the remaining chapters of the first part, the base graph \mathbf{G} is replaced by the following: a graph with infinitely many ends in *Chapter 5*, a hyperbolic graph in *Chapter 6*, and the Euclidean lattice \mathbb{Z}^d in *Chapter 7*. For LRW on $\mathbb{Z}_2 \wr \mathbf{G}$, the convergence and the Poisson boundary will be described as an application of Theorem 3.2.3 and Theorem 4.2.1.

In *Chapter 5* we let \mathbf{G} be a graph with infinitely many ends and its boundary $\partial\mathbf{G}$ be the space of its ends. Like before, $\Gamma \subset \text{AUT}(\mathbf{G})$ is a group which acts transitively on \mathbf{G} . Two cases should be distinguished: when Γ fixes no end in $\partial\mathbf{G}$, and when Γ fixes one end in $\partial\mathbf{G}$, which then has to be unique. In the first case, Theorem 3.2.3 and Theorem 4.2.1 can be adapted in order to prove the convergence and to describe the Poisson boundary of LRW (Z_n) on $\mathbb{Z}_2 \wr \mathbf{G}$. The conditions required in the *Half Space Method* are satisfied for graphs with infinitely many ends and random walks on them. The construction of the “small” strip \mathfrak{s} as a subset of \mathbf{G} is based on the theory of cuts and structure trees of a graph.

The second case, when Γ fixes an end of \mathbf{G} , is more interesting. The graph \mathbf{G} can be viewed as an oriented tree \mathcal{T} with a fixed end ω , like below.

The behaviour at infinity of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$, depends on the *modular drift* $\delta(P_{\mathcal{T}})$ of the base random walk (X_n) on the oriented tree \mathcal{T} . By $P_{\mathcal{T}}$, we denote the transition matrix of (X_n) . We emphasize that the case $\delta(P_{\mathcal{T}}) = 0$ is the most difficult and interesting one, which is studied in *Section 5.4*.



For this special case, the correspondence with a random walk on \mathbb{Z} is used. If we set

$$\Pi = \Pi^* \cup \omega^*,$$

where

$$\Pi^* = \bigcup_{u \in \partial^*\mathcal{T}} \mathcal{C}_u \times \{u\} \text{ and } \omega^* = \{(\zeta, \omega) : \zeta \in \mathcal{C}_\omega\},$$

then we can prove the following.

Summary of Theorems 5.3.7, 5.3.9 and 5.4.3. *Let (Z_n) be an irreducible and homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathcal{T}$, where \mathcal{T} is an homogeneous tree and Γ a subgroup of $\text{AUT}(\mathcal{T})$, which acts transitively on \mathcal{T} and fixes one end $\omega \in \partial\mathcal{T}$. Then*

- (a) *If $\delta(P_{\mathcal{T}}) > 0$, then there exists a Π^* -valued random variable Z_∞ , such that $Z_n \rightarrow Z_\infty$, almost surely. If ν_∞ is the limit distribution on Π^* , then (Π^*, ν_∞) is the Poisson boundary of (Z_n) .*
- (b) *If $\delta(P_{\mathcal{T}}) < 0$, then (Z_n) converges almost surely to some random variable with values in ω^* , and (ω^*, ν_∞) is the Poisson boundary of (Z_n) , where ν_∞ is the limit distribution on ω^* .*
- (c) *If $\delta(P_{\mathcal{T}}) = 0$, then (Z_n) converges almost surely to some random variable in ω^* . Moreover, if (X_n) is a nearest neighbour random walk on \mathcal{T} then (ω^*, ν_∞) is again the Poisson boundary of (Z_n) .*

The convergence part follows basically from Theorem 3.2.3, and the description of the Poisson boundary in the case $\delta(P_{\mathcal{T}}) > 0$ (and $\delta(P_{\mathcal{T}}) < 0$, respectively) is an application of Theorem 4.2.1.

When $\delta(P_{\mathcal{T}}) = 0$, i.e., when the base walk (X_n) has zero modular drift on \mathcal{T} , the Poisson boundary of LRW (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$ is described in a completely different manner, and uses the existence of cutpoints for the random walk (X_n) on \mathcal{T} . Moreover, the correspondence between the tail σ -algebra of a random walk and its Poisson boundary, which in most cases coincide, will be used. This is the content of Section 5.4.

In *Chapter 6*, the base graph G will be replaced by a hyperbolic graph (in the sense of Gromov), and its boundary is the hyperbolic boundary $\partial_h G$. We are interested only in the case when the boundary is infinite. Then we can prove again the convergence of LRW to the boundary Π in Theorem 6.2.1 and describe the Poisson boundary in Theorem 6.2.4.

Finally, for the sake of completeness, in *Chapter 7* we show how to apply Theorem 3.2.3 and Theorem 4.2.1 to LRW over Euclidean lattices, that is, over base graphs $G = \mathbb{Z}^d$, with $d \geq 3$. The results in this chapter were earlier obtained by KAIMANOVICH [Kai01] (for non-zero drift on \mathbb{Z}^d) and for the zero-drift case, recently by ERSCHLER [Ers10].

In the last chapter of the first part several open problems and conjectures regarding lamplighter random walks are stated.

Concluding the overview of the first part of the thesis, let us remark that KAIMANOVICH AND VERSHIK [KV83] were the first to show that lamplighter groups and graphs are fascinating objects in the study of random walks. By now, there is a considerable amount of literature on this topic. The paper

of KAIMANOVICH [Kai91] may serve as a major source for earlier literature. LYONS, PEMANTLE AND PERES [LPP96] investigated the rate of escape of inward-biased random walks on lamplighter groups. The lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is one of the examples for which the entire spectrum for some random walks is known. GRIGORCHUK AND ZUK [GZ01] computed the complete spectrum for the random walk, corresponding to a specific generating set of the lamplighter group. ERSCHLER [Ers03] proved that the rate of escape of symmetric random walks on the wreath product $\mathbb{Z}_2 \wr A$, where A is a finitely generated group, is zero if and only if the random walk's projection onto A is recurrent. ERSCHLER [Ers10] investigated also the Poisson boundary of lamplighter random walks on $\mathbb{Z}_2 \wr \mathbb{Z}^d$, with $d \geq 5$, such that the projection on \mathbb{Z}^d has zero drift, and she proved that the Poisson boundary is isomorphic with the space of limit configurations.

Part II

In this part, we prove the entropy sensitivity of languages associated in a natural way with infinite labelled graphs X . The proof is based on considering Markov chains with forbidden transitions on X , and on investigating the spectral radius of such chains.

If Σ is a finite *alphabet* and Σ^* the set of all finite words over Σ , then a *language* L over Σ is a subset of Σ^* . The *growth* or *entropy* of L is

$$h(L) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |\{w \in L : |w| = n\}|.$$

The quantity $h(L)$ measures the parameter of exponential decay of L . For a finite, non-empty set $F \subset \Sigma^* \setminus \{\epsilon\}$ denote

$$L^F = \{w \in L : \text{no } v \in F \text{ is a subword of } w\},$$

where ϵ is the empty word.

Question: For which class of languages associated with infinite graphs, is $h(L^F) < h(L)$? If this holds for *any* set F of *forbidden subwords*, then the language L is called *growth sensitive* (or *entropy sensitive*).

Let (X, E, ℓ) be an infinite graph with vertex set X , edge set E and $l : E \rightarrow \Sigma$ a function which associates to each edge $e \in E$ its label $l(e) \in \Sigma$. With (X, E, ℓ) we associate the following languages

$$L_{x,y} = \{\text{the labels we read along all paths from } x \text{ to } y \text{ in } X\},$$

for $x, y \in X$. We write $h(X) = h(X, E, \ell) = \sup_{x,y \in X} h(L_{x,y})$ and call this the *entropy* of our oriented, labelled graph. Under general assumptions, we can prove the following results.

Theorem 9.1.5. *Suppose that (X, E, ℓ) is uniformly connected and deterministic with label alphabet Σ . Let $F \subset \Sigma^* \setminus \{\epsilon\}$ be a finite, non-empty set which is relatively dense in (X, E, ℓ) . Then*

$$\sup_{x,y \in V} h(L_{x,y}^F) < h(G).$$

Corollary 9.1.6. *If (X, E, ℓ) is uniformly connected and fully deterministic, then $L_{x,y}$ is growth sensitive for all $x, y \in X$.*

What is interesting here is that the proof of the previous results is based on Markov chains. We consider a Markov chain with state space X and transition matrix P , whose entries are induced by the labelled edges of (X, E, ℓ) . We then remark that the entropy $h(X)$ is in direct correspondence with the spectral radius $\rho(P)$ of the respective Markov chain. Moreover, to the restricted language L^F one can also associate a “restricted” Markov chain, that is, a Markov chain which is not allowed to cross edges with labels in F . Then the question of growth sensitivity can be interpreted in terms of Markov chains and its respective spectral radii on X .

Part II is completed with an example where one can apply the results developed previously: applications to pairs of groups and their Schreier graphs.

Chapter 1

Basic Facts and Preliminaries

This chapter is devoted to basic definitions and facts connected with the theory of Markov chains and random walks on graphs and groups. Moreover, we present here some basic tools which are useful for a better understanding of the results we are going to present throughout this work. We shall follow the notations from WOESS [Woe00].

A Markov chain on a state space G , which is adapted to the geometry of G , will be called a random walk throughout this thesis.

1.1 Markov Chains

A *Markov chain* is defined by a finite or countable *state space* G and a *transition matrix* (or *transition operator*) $P = (p(x, y))_{x, y \in G}$. In addition, one has to specify an initial position, that is, the position at time 0. The entry $p(x, y)$ of P represents the probability to move from x to y in one step. This defines a sequence of G -valued random variables X_0, X_1, \dots , called *Markov chain*, where X_n represents the random position in G at time n . One can imagine a Markov chain as a walker moving randomly in the state space G , according to the probabilities given by the transition matrix P .

In order to model the random variables (X_n) , one has to find a suitable probability space on which the random position after n steps can be described as the n -th random variable of a Markov chain. The usual choice of the probability space is the *trajectory space* $\Omega = G^{\mathbb{Z}^+}$, equipped with the product σ -algebra arising from the discrete one on G . An element $\omega = (x_0, x_1, x_2, \dots)$ of Ω represents a possible evolution (trajectory), that is, a possible sequence of points visited one after the other by the Markov chain. Then, X_n is the n -th projection from Ω to G . This describes the Markov chain starting at x , when Ω is equipped with the probability measure given via the Kolmogorov

extension theorem by

$$\mathbb{P}_x[X_0 = x_0, X_1 = x_1, \dots, X_n = x_n] = \delta_x(x_0)p(x_0, x_1) \cdots p(x_{n-1}, x_n).$$

The associated expectation is denoted by \mathbb{E}_x . Depending on the context, we shall call a Markov chain the pair (G, P) or the sequence of random variables (X_n) . We write

$$p^{(n)}(x, y) = \mathbb{P}_x[X_n = y],$$

which represents, on one hand the (x, y) -entry of the matrix power P^n , with $P^0 = I$ (I is the identity matrix over G), and on the other hand the n -step transition probability, that is, the probability the get from x to y in n steps.

Definition 1.1.1. *A Markov chain (G, P) is called irreducible, if for every $x, y \in G$, there is some $n \in \mathbb{N}$ such that $p^{(n)}(x, y) > 0$.*

This means that every state $y \in G$ can be reached from every other state $x \in G$ with positive probability. Throughout this thesis, we shall always require that the state space is infinite and all states communicate, i.e., the Markov chain is irreducible.

1.2 Random Walks on Graphs

Graphs. A *graph* consists of a finite or countable set of vertices (points) G , equipped with a symmetric *adjacency relation* \sim , which defines the set of edges $E(G)$ (as a subset of $G \times G$). We write (x, y) for the edge between the pair of neighbours x, y . For the sake of simplicity, we exclude loops, that is $(x, x) \notin E(G)$, for all $x \in G$.

In order to simplify the notation, instead of writing $(G, E(G))$ for a graph, we shall write only G . It will be clear from the context whether we are considering vertices or edges.

A *path* from x to y in G is a sequence $[x = x_0, x_1, \dots, x_n]$ of vertices, such that $x_i \sim x_{i-1}$, for all $i = 1, 2, \dots, n$. The number $n \geq 0$ is the *length* of the path. The graph G is called *connected* if every pair of vertices can be joined by a path. The usual *graph distance* $d(x, y)$ is the minimum among the lengths of all paths from x to y . A path is called *simple* if it has no repeated vertex, and *geodesic* if its length is $d(x, y)$. The degree $\deg(x)$ of a vertex x is the number of its neighbours.

The graph G is called *locally finite* if every vertex has finite degree. We say that G has *bounded geometry* if it is connected with bounded vertex degrees.

Let G and G' be two graphs, and d and d' the discrete graph metric on them, respectively. We say that G and G' are *quasi-isometric* if there exists

a mapping $\varphi : \mathbf{G} \rightarrow \mathbf{G}'$, such that

$$A^{-1}d(x, y) - A^{-1}B \leq d'(\varphi x, \varphi y) \leq Ad(x, y) + B,$$

for all $x, y \in \mathbf{G}$, and

$$d'(x', \varphi \mathbf{G}) \leq B,$$

for every $x' \in \mathbf{G}'$, where $A \geq 1$ and $B \geq 0$.

Graph Automorphisms. An *automorphism* of a graph \mathbf{G} is a self-isometry of \mathbf{G} with respect to the graph distance d , that is, a bijection $\varphi : \mathbf{G} \rightarrow \mathbf{G}$ with

$$d(x, y) = d(\varphi x, \varphi y), \quad \text{for all } x, y \in \mathbf{G}.$$

The set of all automorphisms of a graph \mathbf{G} forms a group denoted by $\text{AUT}(\mathbf{G})$.

The graph \mathbf{G} is called *(vertex)-transitive* if for every pair x, y of vertices in \mathbf{G} , there exists a graph automorphism φ with $\varphi x = y$. If \mathbf{G} is transitive, then all vertices have the same degree. If there is a subgroup Γ of $\text{AUT}(\mathbf{G})$, such that, for every $x, y \in \mathbf{G}$, there exists $\gamma \in \Gamma$, with $\gamma x = y$, then we say that Γ *acts transitively* on \mathbf{G} . Throughout this thesis, we shall only consider transitive graphs.

The Graph of a Markov Chain. Every Markov chain (\mathbf{G}, P) with state space \mathbf{G} and transition matrix $P = (p(x, y))_{x, y \in \mathbf{G}}$ defines a graph whose vertex set is the state space \mathbf{G} and the (oriented) set of edges is given by

$$E(\mathbf{G}) = \{(x, y) : p(x, y) > 0 \text{ with } x, y \in \mathbf{G}\}$$

When the transition matrix P is adapted to the structure of the underlying graph \mathbf{G} , then we shall speak of a *random walk* on \mathbf{G} (instead of a Markov chain).

The *simple random walk* (SRW) on a locally finite graph \mathbf{G} is the Markov chain whose state space is \mathbf{G} and the transition probabilities are given by

$$p(x, y) = \begin{cases} \frac{1}{\deg(x)}, & \text{if } y \sim x \\ 0, & \text{otherwise.} \end{cases} \quad (1.1)$$

This is the basic example of a Markov chain adapted to the underlying graph \mathbf{G} . Throughout this thesis, we shall consider more general types of adaptedness properties of the transition matrix P to the underlying structure, and we shall speak of random walks (instead of Markov chains). We define here some of these properties, which will be frequently used.

Definition 1.2.1. *The random walk (\mathbf{G}, P) is of nearest neighbour type, if $p(x, y) > 0$ occurs only when $d(x, y) \leq 1$.*

Definition 1.2.2. *The random walk (\mathbf{G}, P) is said to have k -th finite moment with respect to the usual graph distance d on the graph \mathbf{G} , if*

$$\sum_{x \in \mathbf{G}} d(o, x)^k p(o, x) < \infty, \quad \text{for all } x \in \mathbf{G},$$

for some fixed vertex o in \mathbf{G} .

Further adaptedness conditions of geometric type will be introduced later on.

Green Function and Spectral Radius. Assume that the random walk (\mathbf{G}, P) is irreducible. The *Green function* associated with (\mathbf{G}, P) is given by the power series

$$G(x, y|z) = \sum_{n=0}^{\infty} p^{(n)}(x, y)z^n, \quad x, y \in \mathbf{G}, z \in \mathbb{C}.$$

We write $G(x, y)$ for $G(x, y|1)$. This is the expected number of visits of (X_n) to y when $X_0 = x$.

Lemma 1.2.3. *For all $x, y \in \mathbf{G}$ the power series $G(x, y|z)$ has the same finite radius of convergence $\mathfrak{r}(P)$ given by*

$$\mathfrak{r}(P) := \left(\limsup_{n \rightarrow \infty} (p^{(n)}(x, y))^{1/n} \right)^{-1} < \infty.$$

Proof. The fact that the power series defining the functions $G(x, y|z)$ have all the same radius of convergence follows from a system of Harnack-type inequalities. Because of the irreducibility of (\mathbf{G}, P) , for all $x_1, x_2, y_1, y_2 \in \mathbf{G}$ there exist some $k, l \in \mathbb{N}$ such that,

$$p^{(k)}(x_1, x_2) > 0 \text{ and } p^{(l)}(y_2, y_1) > 0.$$

Therefore, for every $n \in \mathbb{N}$, we have

$$p^{(n+k+l)}(x_1, y_1) \geq p^{(k)}(x_1, x_2)p^{(n)}(x_2, y_2)p^{(l)}(y_2, y_1).$$

Consequently, for every positive argument of the Green function,

$$G(x_1, y_1|z) \geq p^{(k)}(x_1, x_2)p^{(l)}(y_2, y_1)z^{k+l}G(x_2, y_2|z).$$

It follows that the radius of convergence of $G(x_1, y_1|z)$ is at least as big as that of $G(x_2, y_2|z)$. The fact that $\mathfrak{r}(P) < \infty$ follows from the irreducibility of (\mathbf{G}, P) : let $k \in \mathbb{N}$ such that $p^{(k)}(x, x) = \varepsilon > 0$, then $p^{(nk)}(x, x) \geq \varepsilon^n > 0$ for every $n \geq 0$. \square

Hence, the Green function has the following important property: if the random walk on G is irreducible and z is a real number greater than zero, then the power series $G(x, y|z)$ either converges or diverges simultaneously for all $x, y \in G$. For more details, see WOESS [Woe00].

Definition 1.2.4. The spectral radius of the random walk (G, P) is

$$\rho(P) = \limsup_{n \rightarrow \infty} p^{(n)}(x, y)^{1/n} \in (0, 1].$$

Definition 1.2.5. The random walk (G, P) is called recurrent if $G(x, y) = \infty$ for some (\Leftrightarrow every) $x, y \in G$. Otherwise, it is called transient.

Proposition 1.2.6. Further characterizations of recurrence and transience:

(a) If $\rho(P) < 1$, then (G, P) is transient. The converse is not true.

(b) If (G, P) is recurrent then

$$\mathbb{P}_x[X_n = y \text{ for infinitely many } n] = 1, \quad \text{for all } x, y \in G.$$

(c) If (G, P) is transient, then for every finite $A \subset G$,

$$\mathbb{P}_x[X_n \in A \text{ for infinitely many } n] = 0, \quad \text{for all } x \in G.$$

In other words, a random walk (G, P) is *recurrent* if every element of the state space G is visited infinitely often with probability 1. Equivalently, in the transient case, each element is visited only finitely many times with probability 1. This is the same as saying that (X_n) leaves finite subset of G almost surely after a finite time.

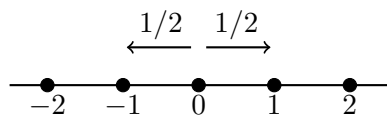
Example 1.2.7. The SRW on \mathbb{Z} is the Markov chain with state space $G = \mathbb{Z}$ and transition probabilities

$$p(x, x + 1) = p(x, x - 1) = \frac{1}{2}, \quad \text{for all } x \in \mathbb{Z}.$$

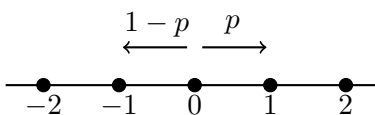
The random walk on \mathbb{Z} with drift to the right is the Markov chain on $G = \mathbb{Z}$ with

$$p(x, x + 1) = 1 - p(x, x - 1) = p, \quad \text{for all } x \in \mathbb{Z} \text{ and } p \in \left(\frac{1}{2}, 1\right).$$

The SRW on \mathbb{Z} is recurrent and the random walk with drift is transient.



Simple random walk on \mathbb{Z}



Random walk on \mathbb{Z} with drift

Rate of Escape

Proposition 1.2.8 (Rate of Escape, Drift). *If the random walk (G, P) has finite first moment with respect to d , then there exists a finite number $l = l(P)$ such that*

$$\frac{d(o, X_n)}{n} \rightarrow l, \quad \text{almost surely.}$$

The number $l = l(P)$ is called rate of escape or drift of the random walk (X_n) with transition matrix P .

The rate of escape is only of interest for transient random walks, since for the recurrent ones it is always zero. The existence of the rate of escape is a consequence of *Kingman's subadditive ergodic theorem* which we formulate now. See KINGMAN [Kin68] for details.

Theorem 1.2.9. *Consider the probability space $\Omega = \mathbb{G}^{\mathbb{Z}_+}$ and let T be the time shift on Ω with $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$. If (W_n) is a subadditive sequence of non-negative real-valued random variables, that is, for all $k, n \in \mathbb{Z}_+$*

$$W_{k+n} \leq W_n + W_k \circ T^n,$$

holds, and if W_1 is integrable, then there is a T -invariant real-valued integrable random variable W_∞ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} W_n = W_\infty, \quad \text{almost surely.}$$

Reversible Markov Chains

Definition 1.2.10. *A Markov chain (G, P) is called reversible if there exists a measure $m : G \rightarrow (0, \infty)$ such that*

$$m(x)p(x, y) = m(y)p(y, x), \quad \text{for all } x, y \in G.$$

Also, m is called the *reversible measure* for (G, P) . If (G, P) is the simple random walk on G , then $m(x) = \deg(x)$.

Definition 1.2.11. *A function $f \in \ell^\infty(G)$ is called P -harmonic if $Pf = f$ pointwise, where the Markov operator P acts on functions $f \in \ell^\infty(G)$ by*

$$Pf(x) = \sum_{y \in G} p(x, y)f(y).$$

We say that f is P -superharmonic if $Pf \leq f$ pointwise.

Reversibility is the same as saying that the transition matrix P acts on $l^2(\mathbf{G}, m)$ as a self-adjoint operator, that is, $(Pf_1, f_2) = (f_1, Pf_2)$, for all $f_1, f_2 \in l^2(\mathbf{G}, m)$, where the Hilbert space $l^2(\mathbf{G}, m)$ consists of functions $f : \mathbf{G} \rightarrow \mathbb{R}$ with

$$\sum_{x \in \mathbf{G}} |f(x)|^2 m(x) < \infty.$$

The inner product on $l^2(\mathbf{G}, m)$ is given by

$$(f_1, f_2) = \sum_{x \in \mathbf{G}} f_1(x) f_2(x) m(x).$$

There is another useful characterization of the recurrence of a random walk in terms of superharmonic functions, which we state now.

Theorem 1.2.12. *(\mathbf{G}, P) is recurrent if and only if all non-negative superharmonic functions are constant.*

Markov Chains and Reversed Markov Chains. If the Markov chain (\mathbf{G}, P) is reversible with respect to the measure m , then one can construct the *reversed Markov chain* (\mathbf{G}, \check{P}) on \mathbf{G} , whose transition probabilities are given by

$$\check{p}(x, y) = p(y, x) \frac{m(y)}{m(x)}.$$

The reversed Markov chain inherits the properties of the original one.

If \mathbf{G} is a transitive graph, P a transition matrix on \mathbf{G} and $\Gamma \subset \text{AUT}(\mathbf{G})$ is a group which acts transitively on \mathbf{G} , then one can construct a reversible measure m for the Markov chain (\mathbf{G}, P) . For doing this, let $o \in \mathbf{G}$ be a fixed reference vertex, whose choice is irrelevant by the transitivity assumption. Denote by Γ_x the *stabilizer* of x in Γ , that is,

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}, \quad \text{for } x \in \mathbf{G},$$

and by $\Gamma_o x$ the *orbit* of x under the action of Γ_o , i.e.

$$\Gamma_o x = \{\gamma x : \gamma \in \Gamma_o\}.$$

Then it is easy to check that

$$m(x) = \frac{|\Gamma_o x|}{|\Gamma_x o|}$$

is a reversible measure for (\mathbf{G}, P) . By $|\cdot|$ we denote the cardinality of the respective set. Note that if the group Γ is discrete, then the measure m is just the counting measure.

Homogeneous Markov Chains. Let (G, P) be an irreducible Markov chain on G . We denote by

$$\text{AUT}(G, P) = \{g \in \text{AUT}(G) : p(x, y) = p(gx, gy), \text{ for all } x, y \in G\}$$

the group of automorphisms (isometries) of G which leaves invariant the transition probabilities of P .

Definition 1.2.13. A Markov chain is called homogeneous or transitive if the group $\text{AUT}(G, P)$ acts on G transitively.

Throughout this thesis we shall consider transient Markov chains which are irreducible and homogeneous.

1.3 Random Walks on Finitely Generated Groups

Let Γ be a discrete group with unit element e , with the group operation written multiplicatively, unless Γ is abelian. Let also μ be a probability measure on Γ . The (*right*) random walk on Γ with law μ , denoted by (Γ, μ) , is the Markov chain with state space Γ and transition probabilities given by

$$p(x, y) = \mu(x^{-1}y), \quad \text{for all } x, y \in \Gamma. \quad (1.2)$$

In order to obtain an equivalent model of the random walk (Γ, μ) as a sequence of random variables (S_n) , we use the product space $(\Gamma, \mu)^\mathbb{N}$. For $n \geq 1$, the n -th projections Y_n of $\Gamma^\mathbb{N}$ onto Γ constitute a sequence of independent Γ -valued random variables with common distribution μ , and the right random walk starting at $x \in \Gamma$ is given as

$$S_n = xY_1 \cdots Y_n, \quad n \geq 1.$$

This is a generalization of the scheme of sums of i.i.d. random variables on the integers or on the reals. The n -step transition probabilities are obtained by

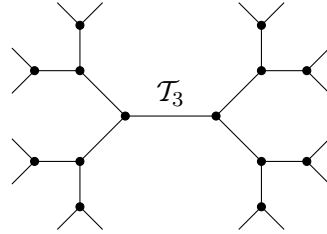
$$p^{(n)}(x, y) = \mu^{(n)}(x^{-1}y),$$

where $\mu^{(n)}$ is the n -fold convolution of μ with itself, with $\mu^0 = \delta_e$, the point mass at the group identity. We denote by \mathbb{P}_x the measure of the random walk and omit the subscript if the random walk starts at the identity e .

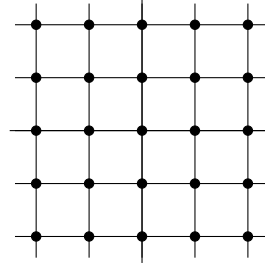
In order to relate random walks on groups with random walks on graphs, let us introduce the notion of *Cayley graphs*, that is, graphs that encode the structure of discrete groups. Suppose that the group Γ is finitely generated, and let S be a symmetric set of generators of Γ . The *Cayley graph* $\mathbb{G}(\Gamma, S)$ of Γ with respect to the generating set S has vertex set Γ , and two vertices

$x, y \in \Gamma$ are connected by an edge, if and only if $x^{-1}y \in S$. This graph is connected, locally finite, and regular (all vertices have the same degree $|S|$). Notice that Cayley graphs are transitive in the sense that they look the same from every vertex. If $e \in S$, then $G(\Gamma, S)$ has a loop at each vertex.

Example 1.3.1. The homogeneous tree \mathcal{T}_M of degree M is the Cayley graph of the group $\langle a_1, a_2, \dots, a_M \mid a_i^2 = e \rangle$ with respect to the generators $S = \{a_1, a_2, \dots, a_M\}$. This group is the free product of M copies of the two-element group \mathbb{Z}_2 . See Chapter II in WOESS [Woe00] for details on free products.



Example 1.3.2. Euclidean lattices are the most well-known Cayley graphs. In the abelian group \mathbb{Z}^d , we may choose the set of all elements with Euclidean length 1 as generating set S . The resulting Cayley graph is the usual grid. The group \mathbb{Z}^2 can be written as $\langle a, b \mid ab = ba \rangle$.



The *simple random walk* on $G = G(\Gamma, S)$ is the right random walk on Γ whose law μ is the equidistribution on S , i.e., $\mu(s) = 1/|S|$ for $s \in S$.

For an arbitrary distribution μ , we write $\text{supp}(\mu) = \{x \in \Gamma : \mu(x) > 0\}$. Then $\text{supp}(\mu^{(n)}) = (\text{supp}(\mu))^n$, and the random walk with law μ is irreducible if and only if

$$\bigcup_{n=1}^{\infty} (\text{supp}(\mu))^n = \Gamma,$$

i.e., the set $\text{supp}(\mu)$ generates Γ as a group.

Reversed Random Walk. If (S_n) is the right random walk on Γ with distribution law μ , then the *reversed random walk* (\check{S}_n) on Γ has the distribution law $\check{\mu}$ given by

$$\check{\mu}(\gamma) = \mu(\gamma^{-1}), \quad \text{for all } \gamma \in \Gamma. \quad (1.3)$$

Random Walks on G and on $\Gamma \subset \text{AUT}(G, P)$. Let Γ be a closed subgroup of $\text{AUT}(G, P)$ which acts transitively on G . The graph G should not be necessary a Cayley graph of Γ . One can then define random walks on Γ which are in direct correspondence with random walks (G, P) on G . Such random walks on Γ inherit the properties of (G, P) .

The group Γ carries a *left Haar measure* $d\gamma$, since $\Gamma \subset \text{AUT}(X, P)$. The measure $d\gamma$ has the following properties: every open set has positive measure,

every compact set has finite measure and $d\gamma$ is a left translation invariant measure. Moreover, as a Radon measure with these properties, $d\gamma$ is unique up to multiplication by constants. If Γ is discrete, the Haar measure is (a multiple of) the counting measure. For details concerning integration on locally compact groups and Haar measures on groups, the reader may have a look at the book of HEWITT AND ROSS [HR63].

Let us choose a left Haar measure $d\gamma$ on Γ , such that $\int_{\Gamma_o} d\gamma = 1$, where $o \in \mathbf{G}$ is a reference vertex. With the transition probabilities P of the random walk (X_n) on \mathbf{G} , one can associate a Borel measure μ on Γ by

$$\mu(d\gamma) = p(o, \gamma o) d\gamma. \tag{1.4}$$

The measure μ is absolutely continuous with respect to $d\gamma$. One can check that μ defines a probability measure on Γ , and induces the right random walk (S_n) on Γ . Then $(S_n o)$ is a model of the random walk (X_n) on \mathbf{G} starting at o . In other words, $(S_n o)$ is a homogeneous Markov chain with transition probabilities $p(x, y)$, for $x, y \in \mathbf{G}$. For more details, see also KAIMANOVICH AND WOESS [KW02, Section 2].

Therefore, whenever one has a Markov chain (\mathbf{G}, P) and a closed subgroup Γ of $\text{AUT}(\mathbf{G}, P)$, then one can construct a measure μ on Γ which is in direct correspondence with the transition probabilities P by equation (1.4). If \mathbf{G} is a Cayley graph of Γ with respect to some generating set, then the correspondence between μ and P is given by (1.2).

Part I

Behaviour at Infinity of Lamplighter Random Walks

Chapter 2

Lamplighter Random Walks

The aim of this chapter is to introduce a class of random walks on *wreath products* of groups and graphs. Wreath products of groups are the simplest non-trivial case of semi-direct products, because they essentially arise from the action of a group on itself by translation. Such groups are called *groups with dynamical configuration* in [KV83].

Random walks on wreath products are known in the literature as *lamplighter random walks*, because of the intuitive interpretation of such walks in terms of configuration of lamps. Such walks appear also in the paper of VAROPOULOS [Var83].

We first introduce lamplighter graphs and random walks on them, and afterwards we consider group actions on lamplighter graphs. Depending on the group actions, the random walks will inherit different behaviour at infinity, which will be studied in the sequel. For simplicity of notation, we shall mostly write LRW for lamplighter random walks.

2.1 Lamplighter Graphs

Let G be an infinite, locally finite, transitive, connected graph and let o be some reference vertex in G . Imagine that at each vertex of G sits a lamp, which can have different states of intensity, but only finitely many. For sake of simplicity, we shall consider the case when the lamp has only two states, encoded by the elements of the set $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, where the element 0 represents the state off (the lamp is switched off) and the element 1 represents the state on (the lamp is switched on). Anyway, instead of \mathbb{Z}_2 one can consider any finite set.

One can think of a person starting in o with all lamps switched off and

moving randomly in G , according to some given probability, and switching randomly lamps on or off. We investigate the following model: at each step the person may walk to some random vertex (situated in a bounded neighbourhood of his current position), and may change the state of some lamps in a bounded neighbourhood of his position. At every moment of time the lamplighter will leave behind a certain configuration of lamps. The configurations of lamps are encoded by functions

$$\eta : G \rightarrow \mathbb{Z}_2,$$

which give, for every $x \in G$, the state of the lamp sitting there. Denote by

$$\hat{\mathcal{C}} = \{\eta : G \rightarrow \mathbb{Z}_2\}$$

the set of all configurations, and let $\mathcal{C} \subset \hat{\mathcal{C}}$ be the set of all finitely supported configurations, where a configuration is said to have finite support if the set

$$\text{supp}(\eta) = \{x \in G : \eta(x) \neq 0\}$$

is finite. Denote by $\mathbf{0}$ the *zero* or *trivial configuration*, i.e. the configuration which corresponds to all lamps switched off, and by δ_x the configuration where only the lamp at $x \in G$ is on and all other lamps are off.

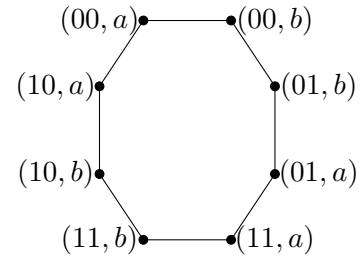
Definition 2.1.1. *The wreath product $\mathbb{Z}_2 \wr G$ of graphs \mathbb{Z}_2 and G is defined as the graph with vertex set $\mathcal{C} \times G$ and adjacency relation given by*

$$(\eta, x) \sim (\eta', x') : \Leftrightarrow \begin{cases} x \sim x' & \text{in } G \text{ and } \eta = \eta', \\ x = x' & \text{in } G \text{ and } \eta \triangle \eta' = \{x\}, \end{cases} \quad (2.1)$$

where $\eta \triangle \eta'$ represents the subset of G , where the configurations η and η' are different.

The wreath product $\mathbb{Z}_2 \wr G$ will be referred as the *lamplighter graph*. The vertices of $\mathbb{Z}_2 \wr G$ are pairs of the form (η, x) , where η represents a finitely supported \mathbb{Z}_2 -valued configuration of lamps and x some vertex in G . The graph G will be called the *base graph* or the *underlying graph* for the lamplighter graph $\mathbb{Z}_2 \wr G$.

Example 2.1.2. Consider the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2$: the base graph and the graph of lamp states are both \mathbb{Z}_2 . Denote the vertices of the base graph by $\{a, b\}$ and the state of lamps by $\{0, 1\}$. Then the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{Z}_2$ has 8 vertices and it can be represented as in the figure.



The wreath product of graphs of bounded geometry is a graph of bounded geometry, and the wreath product of regular graphs is also regular.

Let us now define a metric d on the graph $\mathbb{Z}_2 \wr \mathbf{G}$. If we denote by $l(x, x')$ the smallest length of a “travelling salesman” tour from x to x' that visits each element of the set $\eta \triangle \eta'$, then

$$d((\eta, x), (\eta', x')) = l(x, x') + |\eta' \triangle \eta| \quad (2.2)$$

defines a metric on $\mathbb{Z}_2 \wr \mathbf{G}$. Recall that the travelling salesman tour between two given points is the shortest possible tour that visits each point exactly once. Above, $|\cdot|$ represents the cardinality of the respective set. This metric will be called the *lamplighter metric* or *lamplighter distance*.

Remark 2.1.3. One can also consider a generalization of wreath products, which are called in ERSCHLER [Ers06] *wreath products of graphs with respect to a family of subsets* and, in particular, with respect to partitions.

2.2 Random Walks on Lamplighter Graphs

Consider the state space $\mathbb{Z}_2 \wr \mathbf{G}$ defined as in the previous section and an irreducible transition matrix P on it, which determines the walk $(\mathbb{Z}_2 \wr \mathbf{G}, P)$. This random walk will be called the *lamplighter random walk (LRW)*. The entries

$$p((\eta, x), (\eta', x')), \quad \text{with } (\eta, x), (\eta', x') \in \mathbb{Z}_2 \wr \mathbf{G},$$

of the transition matrix P are the one-step transition probabilities, while the corresponding n -step transition probabilities of the random walk are denoted by $p^{(n)}((\eta, x), (\eta', x'))$. Suppose that the starting point for the LRW is $(\mathbf{0}, o)$, where o is a fixed vertex in \mathbf{G} , and $\mathbf{0}$ is the trivial configuration.

The random walk on $\mathbb{Z}_2 \wr \mathbf{G}$ with transition matrix

$$P = \left(p((\eta, x), (\eta', x')) \right)$$

can also be described by a sequence of $(\mathbb{Z}_2 \wr \mathbf{G})$ -valued random variables (Z_n) . More precisely, we write $Z_n = (\eta_n, X_n)$, where η_n is the random configuration of lamps at time n , and X_n is the random vertex in \mathbf{G} where the lamplighter stands at time n . In the following, when referring to the lamplighter random walk, we shall use the sequence of random variables (Z_n) with $Z_n = (\eta_n, X_n)$, whose transitions are given by the matrix P .

Assume that the lamplighter random walk (Z_n) has finite first moment with respect to the lamplighter metric d defined in (2.2), that is,

$$\sum_{(\eta, x) \in \mathbb{Z}_2 \wr \mathbf{G}} d((\mathbf{0}, o), (\eta, x)) p((\mathbf{0}, o), (\eta, x)) < \infty.$$

The process (Z_n) on $\mathbb{Z}_2 \wr \mathbb{G}$ projects onto random processes (X_n) on \mathbb{G} and (η_n) on the space of configurations \mathcal{C} . Note that the stochastic process (η_n) is a sequence of random configurations of lamps, but not a Markov chain, since the configuration η_n at time n depends on the entire history of the process up to time n . The projection (X_n) is a random walk on the base graph \mathbb{G} with starting point $o \in \mathbb{G}$ and one-step transition probabilities given by

$$p_{\mathbb{G}}(x, x') = \sum_{\eta' \in \mathcal{C}} p((\mathbf{0}, x), (\eta', x')), \quad \text{for all } x, x' \in \mathbb{G}. \quad (2.3)$$

The corresponding n -step transition probabilities are denoted by $p_{\mathbb{G}}^{(n)}(x, x')$, and the transition matrix of (X_n) is $P_{\mathbb{G}} = (p_{\mathbb{G}}(x, y))$. The process (X_n) will be called *the base random walk* or *projected random walk* on \mathbb{G} .

Key assumption: throughout this thesis, we shall consider lamplighter random walks (Z_n) , such that the projection (X_n) on the transitive base graph \mathbb{G} is transient.

This implies that (X_n) leaves every finite subset of \mathbb{G} with probability 1 after a finite time. Transience of the base process is a key assumption, which leads to the transience of the lamplighter random walk (Z_n) . In other words, transience of the random walk (X_n) on \mathbb{G} implies that almost every path of the original random walk (Z_n) on $\mathbb{Z}_2 \wr \mathbb{G}$ will leave behind a certain limit configuration of lamps \mathbb{G} , which will not be necessarily finitely supported.

Question: Does the limit configuration of lamps completely describe the behaviour of the lamplighter random walk (Z_n) at “infinity”?

The behaviour of the lamplighter random walk at “infinity” is the main topic of the first part of the work. This comprises the study of the *convergence* and of the *Poisson boundary* for lamplighter random walks (Z_n) .

Remark 2.2.1. In the study of the behaviour of lamplighter random walks (Z_n) over lamplighter graphs $\mathbb{Z}_2 \wr \mathbb{G}$, with $Z_n = (\eta_n, X_n)$, the properties of the base random walk (X_n) and the geometry of the base graph \mathbb{G} play a crucial role.

2.2.1 Examples of Transition Matrices

There are different ways one can define transition matrices over $\mathbb{Z}_2 \wr \mathbb{G}$. One can start with transition matrices over \mathbb{Z}_2 and \mathbb{G} , and construct a new transition matrix on $\mathbb{Z}_2 \wr \mathbb{G}$ like in the sequel.

Consider $P_{\mathbb{G}}$ and $P_{\mathbb{Z}_2}$ transition matrices on \mathbb{G} and \mathbb{Z}_2 , respectively. One can

lift P_G on G to \bar{P}_G on $\mathbb{Z}_2 \wr G$ by setting

$$\bar{p}_G((\eta, x), (\eta', x')) = \begin{cases} p_G(x, x'), & \text{if } \eta = \eta' \\ 0, & \text{otherwise} \end{cases}.$$

One can also lift $P_{\mathbb{Z}_2}$ on \mathbb{Z}_2 to $\bar{P}_{\mathbb{Z}_2}$ on $\mathbb{Z}_2 \wr G$ by setting

$$\bar{p}_{\mathbb{Z}_2}((\eta, x), (\eta', x')) = \begin{cases} p_{\mathbb{Z}_2}(\eta(x), \eta'(x')), & \text{if } x = x' \text{ and } \eta \triangle \eta' = \{x\} \\ 0, & \text{otherwise} \end{cases}.$$

Using the embeddings of the transition matrices on \mathbb{Z}_2 and G into $\mathbb{Z}_2 \wr G$, one can construct different “types” of lamplighter random walks (transition matrices) on $\mathbb{Z}_2 \wr G$.

Walk or Switch Random Walk. Let a be a parameter with $0 < a < 1$. Define the transition matrix P_a on $\mathbb{Z}_2 \wr G$ by

$$P_a = a\bar{P}_G + (1 - a)\bar{P}_{\mathbb{Z}_2}.$$

The interpretation of P_a in lamplighter terms is as follows. If the lamplighter stands at x and the current configuration is η , then he first tosses a coin. If “head” comes up (with probability a) then he makes a random move according to the probability distribution $p_G(x, \cdot)$ while leaving the lamps unchanged. If “tail” comes up, then he makes no move in the graph G , but modifies the state of the lamp where he stands according to the distribution $p_{\mathbb{Z}_2}(\eta(x), \cdot)$.

Remark 2.2.2. If the graph G is regular (i.e. all vertices have the same degree) and P_G and $P_{\mathbb{Z}_2}$ are the transition matrices of the simple random walks on G and \mathbb{Z}_2 respectively, then the simple random walk on $\mathbb{Z}_2 \wr G$ is given by P_a with

$$a = \frac{\deg_G}{\deg_G + 2},$$

where \deg_G is the vertices degree in G .

Switch-Walk-Switch Random Walk. Define a transition matrix Q on $\mathbb{Z}_2 \wr G$ by the following matrix product

$$Q = \bar{P}_{\mathbb{Z}_2} \cdot \bar{P}_G \cdot \bar{P}_{\mathbb{Z}_2}.$$

The intuitive interpretation is: if the lamplighter stands at x and the current configuration of lamps is η , then he first changes the state of the lamp at x according to the probability distribution $p_{\mathbb{Z}_2}(\eta(x), \cdot)$. Then he makes a step to some point $x' \in G$ according to the probability distribution $p_G(x, \cdot)$, and

at last, he changes the state of the lamp at x' according to the probability distribution $p_{\mathbb{Z}_2}(\eta(x'), \cdot)$.

The *Switch-Walk-Switch* and *Walk or Switch* lamplighter random walks are two basic examples of random walks which are well studied in the literature. Nevertheless, we are not going to use this specific types of transition matrices, but we shall instead work with general irreducible transition matrices P over $\mathbb{Z}_2 \wr G$, such that its projection P_G onto G is a transient random walk.

2.3 Lamplighter Groups and Random Walks

Recall that the set \mathbb{Z}_2 encodes the intensities of lamps $\{0, 1\}$. If we endow this set with the operation of addition modulo 2, then it becomes a group. Consider now the group \mathbb{Z}_2 acting transitively on itself by left multiplication, and identify the set of intensities of lamps with the group \mathbb{Z}_2 , such that the state 0 corresponds to the group identity. For simplicity of notation we use \mathbb{Z}_2 for both the set of lamp intensities and the group with two elements acting on it.

The set \mathcal{C} of all finitely supported \mathbb{Z}_2 -valued configurations on G becomes then a group with the pointwise operation “ \oplus ”

$$(\eta \oplus \eta')(x) = \eta(x) \oplus \eta'(x),$$

taken in the group \mathbb{Z}_2 . The unit element of \mathcal{C} is the zero configuration $\mathbf{0}$, which corresponds to all lamps switched off.

Let Γ be a closed subgroup of $\text{AUT}(G)$ which acts transitively on G . We do not require that G is a Cayley graph of Γ . For instance, Γ can also be a non-discrete group like in Section 5.3.2.

Definition 2.3.1. *The wreath product of the groups \mathbb{Z}_2 and Γ , denoted by $\mathbb{Z}_2 \wr \Gamma$, is a semidirect product of Γ and the direct sum $\sum_{\gamma' \in \Gamma} \mathbb{Z}_2$ of copies of \mathbb{Z}_2 indexed by Γ , where every $\gamma \in \Gamma$ acts on $\sum_{\gamma' \in \Gamma} \mathbb{Z}_2$ by translation T_γ defined as*

$$(T_\gamma \phi)(\gamma') = \phi(\gamma^{-1} \gamma'), \quad \text{for all } \gamma' \in \Gamma, \phi \in \mathcal{C}.$$

The elements of $\mathbb{Z}_2 \wr \Gamma$ are pairs of the form $(\phi, \gamma) \in \mathcal{C} \times \Gamma$, where ϕ represents a finitely supported configuration of lamps and $\gamma \in \Gamma$. A group operation on $\mathbb{Z}_2 \wr \Gamma$, denoted by “ \cdot ” is given by

$$(\phi, \gamma) \cdot (\phi', \gamma') = (\phi \oplus T_\gamma \phi', \gamma \gamma'),$$

where $\gamma, \gamma' \in \Gamma$, $\phi, \phi' \in \mathcal{C}$, and \oplus is the componentwise addition modulo 2.

We shall call $\mathbb{Z}_2 \wr \Gamma$ together with this operation the *lamplighter group* over Γ . The group identity is $(\mathbf{0}, e)$, where $\mathbf{0}$ is the zero configuration and e is the unit element in Γ . Finally, define an action of elements $(\phi, \gamma) \in \mathbb{Z}_2 \wr \Gamma$ on $\mathbb{Z}_2 \wr \mathbf{G}$ by

$$(\phi, \gamma)(\eta, x) = (\phi \oplus T_\gamma \eta, \gamma x), \quad \text{for all } (\eta, x) \in \mathbb{Z}_2 \wr \mathbf{G}. \quad (2.4)$$

This action preserves the neighbourhood relation defined in (2.1). Therefore

$$(\phi, \gamma) \in \text{AUT}(\mathbb{Z}_2 \wr \mathbf{G}),$$

that is, the lamplighter group $\mathbb{Z}_2 \wr \Gamma$ is a subgroup of $\text{AUT}(\mathbb{Z}_2 \wr \mathbf{G})$. Now we have both an *underlying geometric structure* $\mathbb{Z}_2 \wr \mathbf{G}$ (the lamplighter graph) and an *action of the group* $\mathbb{Z}_2 \wr \Gamma$ (the lamplighter group) on it. When \mathbf{G} is a Cayley graph of Γ with respect to some finite generating set, then these two structures can be identified, but since we work with more general groups $\Gamma \subset \text{AUT}(\mathbf{G})$, it is important to distinguish between the lamplighter graph $\mathbb{Z}_2 \wr \mathbf{G}$ and the lamplighter group $\mathbb{Z}_2 \wr \Gamma$.

Since \mathbb{Z}_2 acts transitively on itself and Γ acts transitively on \mathbf{G} , it is straightforward to see that the wreath product $\mathbb{Z}_2 \wr \Gamma$ acts transitively on $\mathbb{Z}_2 \wr \mathbf{G}$, since it is by construction a subgroup of $\text{AUT}(\mathbb{Z}_2 \wr \mathbf{G})$. For more details, see also WOEISS [Woe05].

Assumption: suppose that $\Gamma \subset \text{AUT}(\mathbf{G})$ is chosen such that $\mathbb{Z}_2 \wr \Gamma$ is a subgroup of $(\text{AUT}(\mathbb{Z}_2 \wr \mathbf{G}), P)$, that is, the transition probabilities $p(\cdot, \cdot)$ of (Z_n) are invariant with respect to the action of the group $\mathbb{Z}_2 \wr \Gamma$. In other words, (Z_n) is a homogeneous random walk. This means that for all $g = (\phi, \gamma) \in \mathbb{Z}_2 \wr \Gamma$, we have

$$p(g(\eta, x), g(\eta', x')) = p((\eta, x), (\eta', x')), \quad \text{for all } (\eta, x), (\eta', x') \in \mathbb{Z}_2 \wr \mathbf{G}.$$

Corollary 2.3.2. *The factor chain (X_n) is also a homogeneous random walk on \mathbf{G} , i.e $\Gamma \subset \text{AUT}(\mathbf{G}, P_{\mathbf{G}})$.*

Proof. Definition (2.3) of the transition probabilities of the random walk (X_n) on \mathbf{G} implies that

$$p_{\mathbf{G}}(\gamma x, \gamma x') = \sum_{\eta \in \mathcal{C}} p((\mathbf{0}, \gamma x), (\eta, \gamma x')), \quad \text{for all } x, x' \in \mathbf{G}, \gamma \in \Gamma.$$

Since $\mathbb{Z}_2 \wr \Gamma$ acts transitively on $\mathbb{Z}_2 \wr \mathbf{G}$, it follows that there exists a configuration $\eta_1 \in \mathcal{C}$ such that

$$\eta = \mathbf{0} \oplus T_\gamma \eta_1.$$

Also, we can write $\mathbf{0} = \mathbf{0} \oplus T_\gamma \mathbf{0}$. Using the action (2.4) of the lamplighter group $\mathbb{Z}_2 \wr \Gamma$ on the lamplighter graph $\mathbb{Z}_2 \wr \mathbf{G}$, we can write

$$(\mathbf{0}, \gamma x) = (\mathbf{0}, \gamma)(\mathbf{0}, x) \text{ and } (\eta, \gamma x') = (\mathbf{0}, \gamma)(\eta_1, x').$$

Thus

$$p((\mathbf{0}, \gamma x), (\eta, \gamma x')) = p((\mathbf{0}, \gamma)(\mathbf{0}, x), (\mathbf{0}, \gamma)(\eta_1, x')).$$

This, together with the invariance of the transition probabilities of the lamplighter random walk yields

$$p_G(x, x') = p_G(\gamma x, \gamma x'), \quad \text{for all } x, x' \in G, \gamma \in \Gamma,$$

which proves the claim. □

Recall now a simple fact about the asymptotic configuration size of the lamplighter random walk, which will be needed later.

Lemma 2.3.3. *Let (Z_n) be a random walk with finite first moment on $\mathbb{Z}_2 \wr G$, with $Z_n = (\eta_n, X_n)$. Then there exists a constant $C \geq 0$, such that*

$$\lim_{n \rightarrow \infty} \frac{|\text{supp}(\eta_n)|}{n} = C.$$

In other words, the number of lamps which are turned on increases asymptotically at linear speed.

Proof. Kingman's subadditive ergodic theorem [Kin68] applied to the sequence $|\text{supp}(\eta_i)|$, which is subadditive, yields the desired result. □

Remark 2.3.4. The constant C was studied for a large class of lamplighter random walks over discrete graphs and groups. It is greater than zero if and only if the factor chain (X_n) is transient.

Induced Random Walks on $\mathbb{Z}_2 \wr \Gamma$. Let ν be a probability measure on $\mathbb{Z}_2 \wr \Gamma$, which determines the right random walk $(\mathbb{Z}_2 \wr \Gamma, \nu)$, and which is uniquely induced by the transition matrix P of (Z_n) like in equation (1.4).

The measure μ on Γ , which is induced by the transition matrix P_G of the random walk (X_n) on G , is then given by

$$\mu(x) = \sum_{\eta \in \mathcal{C}} \nu((\eta, x)).$$

Indeed, this follows from the fact that the transition probabilities of (X_n) on G are projections of the transition probabilities of (Z_n) on $\mathbb{Z}_2 \wr G$ as in equation (2.2).

For the correspondence between μ and P_G , and ν and P , we shall use the notation $\mu \leftrightarrow P_G$ and $\nu \leftrightarrow P$, respectively.

Chapter 3

Convergence to the Boundary

This chapter is devoted to the study of convergence (in a sense to be specified) of homogeneous lamplighter random walks (Z_n) on graphs $\mathbb{Z}_2 \wr G$, with $Z_n = (\eta_n, X_n)$, given that the base random walk (X_n) is transient on G . We emphasize that the geometry of G and the action of $\Gamma \subset \text{AUT}(G)$ play an important role in the study of the behaviour of (Z_n) , as n tends to infinity.

We are interested in transitive, infinite base graphs G endowed with a “rich” boundary ∂G . Using ∂G , we construct a boundary Π for the lamplighter graph $\mathbb{Z}_2 \wr G$. We then prove that (Z_n) converges to some random variable Z_∞ with values in the boundary Π almost surely, under some natural assumptions on the base random walk (X_n) . Finally, in the next chapters, the results obtained here will be applied to specific base graphs G : graphs with infinitely many ends, hyperbolic graphs, and Euclidean lattices.

For lamplighter random walks over Euclidean lattices, the results proved here were earlier obtained by KAIMANOVICH [Kai91]. For the sake of completeness, we just show how to apply our results in this case.

3.1 The Boundary of the Lamplighter Graph

The Boundary of the Base Graph. In order to “build” a boundary for the lamplighter graph $\mathbb{Z}_2 \wr G$, we start with the base graph G , which is assumed to be infinite, locally finite, connected and transitive. Let $d(\cdot, \cdot)$ be the discrete graph metric on G , and consider

$$\widehat{G} = G \cup \partial G$$

an extended space of G , not necessarily compact, with ideal *boundary* ∂G (the set of points at infinity), such that \widehat{G} is *compatible* with the group action Γ on G . Here, by compatibility between Γ and \widehat{G} we mean that the action of Γ on G extends to an action on \widehat{G} by homeomorphisms. Recall that $\Gamma \subset \text{AUT}(G)$.

Convergence to the Boundary. In order to introduce the notion of convergence of a random walk (X_n) on G to the boundary ∂G , recall first the definition of the trajectory space Ω of (X_n)

$$\Omega = G^{\mathbb{Z}^+} = \{\omega = (x_0, x_1, x_2, \dots) : x_n \in G \text{ for all } n \geq 0\}.$$

An element $\omega \in \Omega$ represents a possible evolution, that is, a possible sequence of points visited one after the other by (X_n) . For $\omega = (x_0, x_1, x_2, \dots) \in \Omega$ and $n \geq 0$, define the projections $X_n(\omega) = x_n$.

Suppose that we have the boundary ∂G and the extended space \widehat{G} of G defined in (3.1) and set

$$\Omega_\infty = \{\omega \in \Omega : X_\infty(\omega) = \lim_{n \rightarrow \infty} X_n(\omega) \in \partial G \text{ exists in the topology of } \widehat{G}\}.$$

Definition 3.1.1. *We say that the random walk (X_n) on G converges to the boundary ∂G if*

$$\mathbb{P}_x[\Omega_\infty] = 1, \quad \text{for every } x \in G.$$

For the convergence of (X_n) to $X_\infty \in \partial G$, the notation $X_n \rightarrow X_\infty$ will be used.

The random variable X_∞ is measurable with respect to the Borel σ -algebra of ∂G . If (X_n) converges to the boundary ∂G , the hitting distribution is the measure μ_∞ defined for Borel sets $B \subset \partial G$ by

$$\mu_\infty(B) = \mathbb{P}[X_\infty \in B | X_0 = o].$$

Definition 3.1.2. *The boundary ∂G is called projective, if the following holds for sequences $(x_n), (y_n)$ of vertices in G : if (x_n) converges to a boundary point $u \in \partial G$ and*

$$\sup_n d(x_n, y_n) < \infty,$$

then also (y_n) converges to u .

For our results, a somehow weaker property of the boundary ∂G is needed.

Definition 3.1.3. The boundary $\partial\mathbb{G}$ is called weakly projective if the following holds for sequences $(x_n), (y_n)$ of vertices in \mathbb{G} : if (x_n) converges to $\mathbf{u} \in \partial\mathbb{G}$ and

$$\frac{d(x_n, y_n)}{d(x_n, o)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

then (y_n) converges also to \mathbf{u} .

Remark 3.1.4. Note that a weakly projective boundary is also projective, but the other way round does not necessary hold.

Indeed, when the sequence of vertices (x_n) accumulates at some boundary point $\mathbf{u} \in \partial\mathbb{G}$ and

$$d(x_n, y_n) \approx \log n \text{ and } d(x_n, o) \approx cn, \text{ for some } c > 0,$$

then the requirements for $\partial\mathbb{G}$ to be a weakly projective boundary are satisfied, but not those for a projective boundary. We shall work with weakly projective boundaries.

The Boundary of the Lamplighter Graph $\mathbb{Z}_2 \wr \mathbb{G}$. The natural compactification of the set of finitely supported configurations \mathcal{C} in the topology of pointwise convergence is the set $\widehat{\mathcal{C}}$ of all, finitely or infinitely supported configurations.

Since the vertex set of the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{G}$ is $\mathcal{C} \times \mathbb{G}$, the space

$$\partial(\mathbb{Z}_2 \wr \mathbb{G}) = (\widehat{\mathcal{C}} \times \widehat{\mathbb{G}}) \setminus (\mathcal{C} \times \mathbb{G})$$

is a *natural boundary* at infinity for $\mathbb{Z}_2 \wr \mathbb{G}$. Let us write

$$\widehat{\mathbb{Z}_2 \wr \mathbb{G}} = \widehat{\mathcal{C}} \times \widehat{\mathbb{G}}.$$

The boundary $\partial(\mathbb{Z}_2 \wr \mathbb{G})$ contains all pairs (ζ, \mathbf{u}) , where $\mathbf{u} \in \partial\mathbb{G}$ and ζ is a finitely or infinitely supported configuration. This boundary is so “rich” that it gives us plenty of information on the behaviour of the lamplighter random walks at infinity.

3.2 Convergence of LRW

Convergence of LRW (Z_n) on $\mathbb{Z}_2 \wr \mathbb{G}$ follows mainly from the convergence of the base random walk (X_n) on \mathbb{G} . For the time being, we are still working with general random walks (X_n) on \mathbb{G} , where \mathbb{G} is a transitive graph. In order to get some information about the random walk (Z_n) on $\mathbb{Z}_2 \wr \mathbb{G}$, we have to know something about (X_n) . For this reason, some assumptions on (X_n) and \mathbb{G} are needed.

Assumption 3.2.1. *Assume that:*

- (A1) (X_n) has finite first moment on \mathbf{G} .
- (A2) (X_n) converges to $\partial\mathbf{G}$, with hitting distribution μ_∞ .
- (A3) $\partial\mathbf{G}$ is weakly projective.

These assumptions are not very restrictive. We shall give several examples of graphs \mathbf{G} and random walks on them where these assumptions hold.

Given that the base random walk (X_n) on \mathbf{G} converges to the boundary $\partial\mathbf{G}$, we prove that the lamplighter random walk (Z_n) on $\mathbb{Z}_2 \wr \mathbf{G}$ converges to random variables with values on the boundary $\partial(\mathbb{Z}_2 \wr \mathbf{G})$. This boundary is still too big for our purposes, that is, it contains many points towards (Z_n) converges with probability 0. For this reason, let us define a “smaller” boundary Π for the lamplighter graph, which is still dense in $\partial(\mathbb{Z}_2 \wr \mathbf{G})$, and we shall show that the random walk (Z_n) converges with probability 1 to a random variable with values in Π . Define the subset Π of $\partial(\mathbb{Z}_2 \wr \mathbf{G})$ by

$$\Pi = \bigcup_{\mathbf{u} \in \partial\mathbf{G}} \mathcal{C}_{\mathbf{u}} \times \{\mathbf{u}\}, \quad (3.1)$$

where the set $\mathcal{C}_{\mathbf{u}}$ consists of all configurations ζ , which are either finitely supported, or infinitely supported with $\text{supp}(\zeta)$ accumulating only at \mathbf{u} . The set $\mathcal{C}_{\mathbf{u}}$ is dense in $\widehat{\mathcal{C}}$ because $\mathcal{C} \subset \mathcal{C}_{\mathbf{u}}$ and \mathcal{C} is dense in $\widehat{\mathcal{C}}$. Hence, Π is also dense in $\partial(\mathbb{Z}_2 \wr \mathbf{G})$.

The action of the group $\mathbb{Z}_2 \wr \Gamma$ on the lamplighter graph $\mathbb{Z}_2 \wr \mathbf{G}$ extends to an action on $\widehat{\mathbb{Z}_2 \wr \mathbf{G}} = \widehat{\mathcal{C}} \times \widehat{\mathbf{G}}$ by homeomorphisms and leaves the Borel subset $\Pi \subset \partial(\mathbb{Z}_2 \wr \mathbf{G})$ invariant. If we take $(\phi, \gamma) \in \mathbb{Z}_2 \wr \Gamma$ and $(\zeta, \mathbf{u}) \in \Pi$, then

$$(\phi, \gamma)(\zeta, \mathbf{u}) = (\phi \oplus T_\gamma \zeta, \gamma \mathbf{u}).$$

If $\mathbf{u} \in \partial\mathbf{G}$ and ζ is finitely supported or accumulates only at \mathbf{u} , then $T_\gamma \zeta$ can accumulate at most at $\gamma \mathbf{u}$. Also the configuration $\eta \oplus T_\gamma \zeta$ accumulates again at most at $x\mathbf{u}$ because η is finitely supported, so that adding η modifies $T_\gamma \zeta$ only in finitely many points.

Definition 3.2.2. *We shall say that a sequence of lamp configurations η_n converges to a configuration η_∞ , if*

$$\lim_{n \rightarrow \infty} \eta_n(x) = \eta_\infty(x), \quad \text{for all } x \in \mathbf{G},$$

i.e., for all $x \in \mathbf{G}$ the sequence $\eta_n(x)$ stabilizes.

For a special case, where the lamplighter changes the lamps configuration only at the current vertex, it is clear that the lamplighter random walk converges to a point in Π . Indeed, by Assumption 3.2.1 (A2), the base random walk (X_n) converges to a random element $X_\infty \in \partial\mathbf{G}$. Since only the states of lamps which are visited can be modified, and by transience every vertex is visited only finitely many times, after some time every vertex is left forever and the state of the lamp sitting there remains unchanged. Therefore the random configuration η_n must converge pointwise to a random configuration which accumulates at X_∞ .

We shall prove the convergence for general homogeneous random walks on $\mathbb{Z}_2 \wr \mathbf{G}$, not only restricted to the situation when the configuration can be changed at the current position. This was also proved by KARLSSON AND WOESS [KW07] for a class of lamplighter random walks over homogeneous trees. The following result is a generalization of [KW07, Theorem 2.9] for lamplighter random walks (Z_n) over general transitive base graphs \mathbf{G} . In SAVA [Sav10] this was proved for lamplighter random walks on discrete groups.

Assume that the random walk (X_n) on the transitive base graphs \mathbf{G} satisfies Assumption 3.2.1. Then the following holds for homogeneous lamplighter random walks (Z_n) .

Theorem 3.2.3. *Let (Z_n) be an irreducible and homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathbf{G}$. Then there exists a Π -valued random variable $Z_\infty = (\eta_\infty, X_\infty)$, such that $Z_n \rightarrow Z_\infty$ almost surely in the topology of $\widehat{\mathbb{Z}_2 \wr \mathbf{G}}$, for every starting point (η_0, x_0) . Moreover, the distribution of Z_∞ is a continuous measure on Π .*

Proof. Without loss of generality, we may suppose that the starting point is $(\mathbf{0}, o)$, where $\mathbf{0}$ is the trivial (zero) lamps configuration and $o \in \mathbf{G}$ some reference vertex, whose choice is irrelevant by the transitivity of \mathbf{G} .

The random walk (Z_n) is homogeneous, and by Corollary 2.3.2 the factor chain (X_n) is also homogeneous. Also, the factor chain (X_n) on \mathbf{G} is transient, and it converges almost surely to a random variable $X_\infty \in \partial\mathbf{G}$ by Assumption 3.2.1.

Now, assume that (Z_n) has finite first moment on $\mathbb{Z}_2 \wr \mathbf{G}$. Then, the configuration η_i of lamps at time i can be obtained by modifying the previous configuration η_{i-1} in a finite number of vertices in \mathbf{G} . This implies that there exists a finitely supported configuration ϕ_i , such that

$$\eta_i = \phi_i \oplus \eta_{i-1}, \quad \text{for all } i = 1, 2, \dots, n.$$

The configuration ϕ_i is zero in all points which are not touched by the random walker. Let now (y_n) be an unbounded sequence of elements in \mathbf{G} ,

with $y_n \in \text{supp}(\phi_n)$. Thus, y_n is a sequence of vertices in \mathbf{G} where the lamp is switched on. Since (X_n) has finite first moment on \mathbf{G} , the following holds with probability 1:

$$\frac{d(y_n, X_n)}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Kingman's subadditive ergodic theorem 1.2.9 (see also KINGMAN [Kin68]) implies that there exists finite constant $l > 0$, such that

$$\frac{d(X_n, o)}{n} \rightarrow l, \quad \text{as } n \rightarrow \infty, \text{ almost surely.}$$

Making use of the previous two equations and the triangle inequality, we get

$$\frac{d(X_n, y_n)}{d(X_n, o)} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

Recall that by Assumption 3.2.1 we have $X_n \rightarrow X_\infty$. By the weakly projectivity of $\partial\mathbf{G}$ and from equation (3.2), it follows that (y_n) converges to X_∞ . Observe that

$$\text{supp}(\eta_n) \subset \bigcup_{i=1}^n \text{supp}(\phi_i),$$

which is a union of finite sets. Since the unbounded sequence $y_n \in \text{supp}(\phi_n)$ converges to X_∞ , it follows that $\text{supp}(\eta_n)$ must converge to X_∞ . That is, the random configuration η_n converges pointwise to a limit configuration η_∞ , which accumulates at X_∞ and $Z_n = (\eta_n, X_n)$ converges to a random element $Z_\infty = (\eta_\infty, X_\infty) \in \Pi$.

When the limit distribution of (X_n) is a continuous measure on $\partial\mathbf{G}$ (i.e., it carries no point mass), then the same is true for the limit distribution of (Z_n) on Π . Indeed, supposing that there exists some single point in Π with non-zero hitting probability measure, then a contradiction arises since one can find some single point in $\partial\mathbf{G}$ with non-zero measure. This is not possible because of the continuity of the limit distribution of (X_n) .

On the other side, when the limit distribution of (X_n) is not continuous on $\partial\mathbf{G}$, one can use Borel-Cantelli lemma in order to prove that the limit distribution of (Z_n) is still continuous. \square

Chapter 4

Poisson Boundary of LRW

The *Poisson boundary* of a random walk is a measure space which describes the stochastically significant behaviour of its paths at infinity. In this chapter we present a method to identify the Poisson boundary of lamplighter random walks over graphs $\mathbb{Z}_2 \wr G$, given that the base walk (X_n) over G is transient and satisfies some suitable assumptions. This method is called the *Half-Space Method*. The base graph G will be then replaced in the following chapters by some specific graphs, and the method described here will be applied.

The Poisson boundary of lamplighter random walks over groups $\mathbb{Z}_2 \wr \Gamma$, with Γ a discrete group endowed with a rich boundary, was determined in SAVA [Sav10].

For more information on the Poisson boundary, the reader is invited to have a look at the introductory and complex papers of KAIMANOVICH [Kai91], [Kai95] and [Kai00]. The description of the Poisson boundary of lamplighter random walks over Euclidean lattices \mathbb{Z}^d , with $d \geq 5$, such that the base walk has zero drift was an open problem for a long time, and has been recently solved by ERSCHLER [Ers10]. For the relation between the Poisson boundary and the linear drift of a random walk, see also KARLSSON AND LEDRAPPIER [KL07]. For other problems and methods related to the determination of the Poisson boundary, see BALLMANN AND LEDRAPPIER [BL94], and LEDRAPPIER [Led85].

4.1 Preliminaries

The *Poisson boundary* of a Markov chain (G, P) is defined as the space of ergodic components of the time shift in the path space. Under natural assumptions on the transition matrix P on G , there exists a measure space

(Λ, λ) , such that the *Poisson formula*

$$h_\varphi(x) = \int_\Lambda \varphi d\lambda_x$$

states an isometric isomorphism between the Banach space $H^\infty(\mathbf{G}, P)$ of bounded harmonic functions on \mathbf{G} with sup-norm and the space $L^\infty(\Lambda, \lambda)$ of λ -measurable functions on Λ . The space (Λ, λ) is called the *Poisson boundary* of the Markov chain (\mathbf{G}, P) . *Triviality of the Poisson boundary* is equivalent to the absence of non-constant bounded harmonic functions for the pair (\mathbf{G}, P) . This is the so-called *Liouville property*.

The Poisson formula characterizes the Poisson boundary up to a measure theoretical isomorphism. It also has a topological interpretation in terms of the Martin boundary, where it consists of the set of possible limits of the Markov chain at the boundary together with the family of corresponding harmonic hitting distributions. Nevertheless, we emphasize that the Poisson boundary is a measure-theoretical object and all objects connected with the Poisson boundary are defined modulo subsets of measure 0. For a detailed description, see KAIMANOVICH [Kai92].

Recall that if Γ is a group which acts transitively on \mathbf{G} and leaves the transition operator P on \mathbf{G} invariant, that is, if $\Gamma \subset \text{AUT}(\mathbf{G}, P)$, then there exists a measure μ on Γ , which determines the right random walk (Γ, μ) . Moreover, the measure μ is uniquely induced by the transition probabilities P of the pair (\mathbf{G}, P) as in equation (1.4). Recall the notation $P \leftrightarrow \mu$ for the respective correspondence.

Homogeneous Markov operators are intermediate between random walks on countable groups and random walks on general locally compact groups. Although the state space \mathbf{G} is countable, the Poisson boundary of the Markov chain (\mathbf{G}, P) is isomorphic with the Poisson boundary of the induced random walk (Γ, μ) on $\Gamma \subset \text{AUT}(\mathbf{G}, P)$, which is not necessarily discrete.

Proposition 4.1.1. *If $\Gamma \subset \text{AUT}(\mathbf{G}, P)$, then the Poisson boundary of the random walk (Γ, μ) coincides with the Poisson boundary of the pair (\mathbf{G}, P) .*

For the proof see KAIMANOVICH AND WOESS [KW02, Proposition 3.1].

Definition 4.1.2. *A μ -boundary for the random walk (Γ, μ) is a space (B, σ) with the following properties:*

- (a) *every path of the random walk converges to a limit in B with hitting distribution σ .*
- (b) *the measure σ is μ -harmonic, i.e. $\mu * \sigma = \sigma$.*
- (c) *Γ acts on B by measurable bijections.*

Due to the coincidence of the Poisson boundaries of (Γ, μ) and (G, P) , it follows that a μ -boundary for (Γ, μ) is also a μ -boundary for (G, P) .

If G is embedded into a topological space B , and every path of the Markov chain converges to a limit in B , then the space B with the hitting measure σ on it, is a *quotient of the Poisson boundary*. Such quotients are μ -boundaries. Moreover, the topology of B is irrelevant, since any projection from the path space $\Omega = G^{\mathbb{Z}^+}$ onto the space (B, σ) gives rise to a μ -boundary.

The Poisson boundary is the maximal μ -boundary. Here, we mean maximality in a measure theoretic sense, i.e., there is no way (up to measure 0) of further splitting the boundary points of this compactification. Therefore, the problem of identifying the Poisson boundary consists of two parts:

- (a) To find a μ -boundary (B, σ) . This space is *a priori* just a quotient of the Poisson boundary.
- (b) To show that this boundary is maximal, i.e., is isomorphic to the whole Poisson boundary.

The identification of a μ -boundary can be done in geometric or combinatorial terms. Throughout this thesis, we shall consider a geometric approach in order to prove the maximality of a μ -boundary. In Section 5.4, the Poisson boundary is described by a measure theoretical method, namely the correspondence between the tail σ -algebra of a random walk and its Poisson boundary is used. For the geometric approach, we shall use one very nice criterion called *Strip Criterion*, developed by KAIMANOVICH in [Kai00]. This “strip” or “bilateral” approximation is inspired by the use of bilateral geodesics in cocompact rank 1 Cartan-Hadamard manifolds by LEDRAPPIER AND BALLMANN [BL94].

There is a second criterion called *Ray Criterion*, due again to KAIMANOVICH [Kai00], which can also be used in the identification of the Poisson boundary, and which will be stated below for sake of completeness. These criteria are based on entropies of conditional random walks, and require an approximation of the sample paths of the random walk in terms of their limit behaviour.

These criteria allow to identify the Poisson boundary with natural boundaries for several classes of graphs and groups with *hyperbolic properties*: hyperbolic graphs (or more generally, Cayley graphs of groups of isometries of Gromov hyperbolic spaces), graphs with infinitely many ends, and some other semi-direct and wreath products. All these graphs are endowed with natural and nice rich geometric boundaries, which will be explained in what follows. Moreover, it is known that sample paths of the Markov chains on these graphs converge to natural boundaries.

Even if the determination of the Poisson boundary in this thesis is done by applying the *Strip Criterion*, it is instructive to state both criteria here.

Theorem 4.1.3. [Ray Criterion] *Let P be a homogeneous Markov operator with finite first moment on \mathbb{G} and let (B, σ) be a μ -boundary. If there exists a sequence of measurable maps $R_n : B \rightarrow \mathbb{G}$, such that*

$$d(X_n, R_n(X_\infty)) = o(n),$$

for almost every path of the random walk (X_n) (with transition operator P), then (B, σ) is the whole Poisson boundary of (X_n) .

For the second criterion, we shall assume that simultaneously with a μ -boundary (B_+, σ_+) we are also given a $\check{\mu}$ -boundary (B_-, σ_-) of the reversed Markov operator \check{P} . This criterion is symmetric with respect to the time reversal and leads to a simultaneous identification of the Poisson boundaries of (\mathbb{G}, P) and (\mathbb{G}, \check{P}) , respectively. For the definition of the transition probabilities of (\mathbb{G}, \check{P}) , see (1.3).

Theorem 4.1.4. [Strip Criterion] *Let P be a homogeneous Markov operator with finite first moment on \mathbb{G} and let (B_+, σ_+) , (B_-, σ_-) be a μ - and a $\check{\mu}$ -boundary, respectively. If there exists a measurable Γ -equivariant map S assigning to almost every pair of points $(b_-, b_+) \in B_- \times B_+$ a non-empty “strip” $S(b_-, b_+) \subset \mathbb{G}$, such that, for the ball $B(o, n)$ of radius n in the metric of \mathbb{G} ,*

$$\frac{1}{n} \log |S(b_-, b_+) \cap B(o, n)| \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (4.1)$$

for $(\sigma_- \times \sigma_+)$ -almost every $(b_-, b_+) \in B_- \times B_+$, then (B_+, σ_+) and (B_-, σ_-) are the Poisson boundaries of the Markov chains (\mathbb{G}, P) and (\mathbb{G}, \check{P}) , respectively.

Recall that $\Gamma \subset \text{AUT}(\mathbb{G}, P)$. Equivariance of the strip $S(b_-, b_+) \subset \mathbb{G}$ with respect to the group action Γ , means that, for all $\gamma \in \Gamma$,

$$\gamma S(b_-, b_+) = S(\gamma b_-, \gamma b_+).$$

In most of the cases, the equivariance of the strip is very easy to prove. The “harder” part of the theorem is to prove the subexponential growth of the chosen strip. The “thinner” the strips $S(b_-, b_+)$, the larger the class of Markov operators for which condition (4.1) is satisfied. This means that the sample paths of the random walk (X_n) on \mathbb{G} go to infinity “faster”.

In some cases, the existence of such strips is almost evident, whereas checking the *Ray Criterion* may be rather complicate, or in some cases it fails. However, there are also some situations where the *Ray Criterion* is more

helpful than the *Strip Criterion*. The ray criterion provides more information than the strip criterion about the behaviour of sample paths of the random walk, and can also be useful for other issues than the identification of the Poisson boundary.

See, for example, LEDRAPPIER [Led01] where it is used for estimating the Hausdorff dimension of the harmonic measure. On the other hand, for checking the ray criterion one often needs rather elaborate estimates, whereas existence of strips is easier.

We state here another result which will be useful in the following, and can be found in KAIMANOVICH AND WOESS [KW02].

Proposition 4.1.5. *Suppose that P is a homogeneous Markov operator with finite first moment on \mathbf{G} . Then the Poisson boundary of the random walk (X_n) with transition matrix P is trivial if*

- (a) \mathbf{G} has subexponential growth, or
- (b) the drift $l(P)$ vanishes.

4.2 Half-Space Method for LRW

Let us go back to our setting where a transient and irreducible random walk (X_n) with transition matrix $P_{\mathbf{G}}$ over the transitive graph \mathbf{G} is given. We also require Assumption 3.2.1 to hold. The corresponding lamplighter random walk on $\mathbb{Z}_2 \wr \mathbf{G}$ is (Z_n) , with transition matrix P and $Z_n = (\eta_n, X_n)$. It converges, by Theorem 3.2.3, to the geometric boundary Π defined in (3.1).

As before, μ_{∞}^x is the hitting distribution of the random walk (X_n) starting at $x \in \mathbf{G}$. For Borel sets $B \subset \partial\mathbf{G}$, we have

$$\mu_{\infty}^x(B) = \mathbb{P}[X_{\infty} \in B | X_0 = x].$$

Factorizing with respect to the first step, the Markov property yields

$$\mu_{\infty}^x = \sum_{y \in \mathbf{G}} p_{\mathbf{G}}(x, y) \mu_{\infty}^y.$$

The Borel probability measures family $\{\mu_{\infty}^x : x \in \mathbf{G}\}$, are called *harmonic measures*. In view of the irreducibility assumption, all harmonic measures μ_{∞}^x are equivalent to the measure $\mu_{\infty} = \mu_{\infty}^o$. Moreover, the space $(\partial\mathbf{G}, \mu_{\infty})$ is a factor space of the Poisson boundary of the random walk (X_n) , and the Poisson formula permits one to identify the space $L^{\infty}(\partial\mathbf{G}, \mu_{\infty})$ with a certain subspace of the space $H^{\infty}(\mathbf{G}, P_{\mathbf{G}})$ of bounded harmonic functions.

We are interested in describing the Poisson boundary of lamplighter random walks (Z_n) over $\mathbb{Z}_2 \wr \mathbf{G}$, with the base random walk (X_n) on \mathbf{G} being an irreducible, transient random walk which satisfies Assumption 3.2.1.

Under the assumptions of Theorem 3.2.3, let ν_∞ be the distribution of $Z_\infty = (\eta_\infty, X_\infty)$ on Π , given that the position of the random walk (Z_n) at time $n = 0$ is $(\mathbf{0}, o)$. This is a probability measure on Π defined for Borel sets $U \subset \Pi$ by

$$\nu_\infty(U) = \mathbb{P}[Z_\infty \in U | Z_0 = (\mathbf{0}, o)].$$

Let ν be the unique probability measure on $\mathbb{Z}_2 \wr \Gamma$ induced by P ($P \leftrightarrow \nu$) as in equation (1.4). Then the measure ν_∞ is a ν -harmonic measure for (Z_n) . This means that it satisfies the convolution equation $\nu * \nu_\infty = \nu_\infty$.

Since $\mathbb{Z}_2 \wr \Gamma \subset \text{AUT}(\mathbb{Z}_2 \wr \mathbf{G})$ acts on Π by measurable bijections and the measure ν_∞ is stationary with respect to ν , by Definition 4.1.2 the space (Π, ν_∞) is a ν -boundary for the random walk (Z_n) with transition matrix P . We want to prove that this ν -boundary is indeed the maximal one, that is, the Poisson boundary.

We state a general method to describe the Poisson boundary of LRW (Z_n) on $\mathbb{Z}_2 \wr \mathbf{G}$ under some reasonable assumptions on the base walk (X_n) .

4.2.1 The Half-Space Method

Assume that:

- (a) **Assumption 3.2.1** holds for (X_n) and (\check{X}_n) . Let μ_∞ and $\check{\mu}_\infty$ be the respective hitting distributions on $\partial\mathbf{G}$.
- (b) For $\mu_\infty \times \check{\mu}_\infty$ -almost every pair $(\mathbf{u}, \mathbf{v}) \in \partial\mathbf{G} \times \partial\mathbf{G}$, one has a strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$, which satisfies the conditions from Theorem 4.1.4: it is a subset of \mathbf{G} , it is Γ -equivariant, and it has subexponential growth, that is,

$$\frac{1}{n} \log |\mathfrak{s}(\mathbf{u}, \mathbf{v}) \cap B(o, n)| \rightarrow 0, \text{ as } n \rightarrow \infty, \quad (4.2)$$

where $B(o, n) = \{x \in \mathbf{G} : d(o, x) \leq n\}$ is the ball with center o and radius n in \mathbf{G} .

- (c) For every $x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})$, one can assign to the triple $(\mathbf{u}, \mathbf{v}, x)$ a partition of \mathbf{G} into *half-spaces* \mathbf{G}_\pm , such that \mathbf{G}_+ (respectively, \mathbf{G}_-) contains a neighbourhood of \mathbf{u} (respectively, \mathbf{v}), and the assignments

$$(\mathbf{u}, \mathbf{v}, x) \mapsto \mathbf{G}_\pm(x)$$

are Γ -equivariant.

As a matter of fact, one can partition G in more than two subsets and the method can still be applied. However, the relevant subsets are the ones containing a neighbourhood of \mathbf{u} (respectively, \mathbf{v}). Indeed, by Theorem 3.2.3, the LRW converges to the boundary Π , which is defined as the set of all pairs (ϕ, \mathbf{u}) with $\mathbf{u} \in \partial G$ and the only accumulation point of the configuration ϕ is \mathbf{u} . Therefore, only there may be infinitely many lamps switched on (because \mathbf{u} and \mathbf{v} are the respective boundary points toward the random walks (X_n) and (\check{X}_n) converge).

We want to build a finitely supported configuration associated to pairs (ϕ_+, ϕ_-) of limit configurations (of the lamplighter random walk and of the reversed random walk) accumulating at \mathbf{u} and \mathbf{v} , respectively. In order to do this, we restrict ϕ_+ and ϕ_- on G_- and G_+ , respectively, and then we “glue together” the restrictions. Since the new configuration depends on the partition of G , we cannot choose the same partition for all x , because we will have a constant configuration which is not equivariant. Therefore, the partition of G should depend on $x \in s(\mathbf{u}, \mathbf{v})$.

Let us now state one of the main results on this thesis, regarding the Poisson boundary of lamplighter random walks over general base graphs G . For discrete groups Γ , the result was published in SAVA [Sav10].

Theorem 4.2.1. *Let (Z_n) be an irreducible, homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr G$. If Π is defined as in (3.1) and the above assumptions are satisfied, then the measure space (Π, ν_∞) is the Poisson boundary of (Z_n) , where ν_∞ is the limit distribution on Π of Z_n .*

Proof. In order to apply the Strip Criterion (Theorem 4.1.4), we need to find ν - and $\check{\nu}$ -boundaries for the lamplighter random walk (Z_n) and the reversed lamplighter random walk (\check{Z}_n) , respectively. By Theorem 3.2.3 each of the random walks (Z_n) and (\check{Z}_n) starting at $(\mathbf{0}, o)$ converges almost surely to a Π -valued random variable. If ν_∞ and $\check{\nu}_\infty$ are their respective limit distributions on Π , then the spaces (Π, ν_∞) and $(\Pi, \check{\nu}_\infty)$ are ν - and $\check{\nu}$ -boundaries of the respective random walks.

Let us take $b_+ = (\phi_+, \mathbf{u})$, $b_- = (\phi_-, \mathbf{v}) \in \Pi$, where ϕ_+ and ϕ_- are the limit configurations of (Z_n) and (\check{Z}_n) , respectively, and $\mathbf{u}, \mathbf{v} \in \partial G$ are their only respective accumulation points. By the continuity of ν_∞ and $\check{\nu}_\infty$, the set

$$\{(b_+, b_-) \in \Pi \times \Pi : \mathbf{u} = \mathbf{v}\}$$

has $(\nu_\infty \times \check{\nu}_\infty)$ -measure 0, so that, in constructing the strip $S(b_+, b_-)$ we shall consider only the case $\mathbf{u} \neq \mathbf{v}$.

Use the third assumption in the *Half-Space Method*, and for each $x \in s(\mathbf{u}, \mathbf{v})$ consider a partition of G into $G_+(x)$, $G_-(x)$, and eventually $G \setminus (G_+ \cup G_-)$.

The set G_+ (respectively, G_-) contains a neighbourhood of \mathbf{u} (respectively, \mathbf{v}), and $G \setminus (G_+ \cup G_-)$ is the remaining subset (which may be empty). The set $G \setminus (G_+ \cup G_-)$ contains neither \mathbf{u} nor \mathbf{v} . The restriction of ϕ_+ on G_- (respectively, of ϕ_- on G_+) is finitely supported, since its only accumulation point is \mathbf{u} (respectively, \mathbf{v}), which is not in a neighbourhood of G_- (respectively, G_+). Now “put together” the restriction of ϕ_+ on G_- and of ϕ_- on G_+ in order to get the new configuration

$$\Phi(b_+, b_-, x) = \begin{cases} \phi_-, & \text{on } G_+ \\ \phi_+, & \text{on } G_- \\ 0, & \text{on } G \setminus (G_+ \cup G_-) \end{cases} \quad (4.3)$$

on G , which is, by construction, finitely supported. For a graphic visualisation of the above construction, see Figure 4.1.

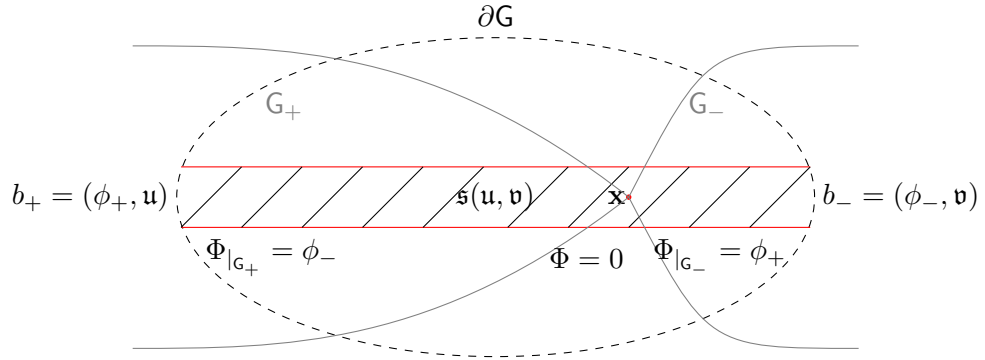


Figure 4.1: The construction of the Half-Spaces

The sought for the “bigger” strip $S(b_+, b_-) \subset \mathbb{Z}_2 \wr G$ is the set

$$S(b_+, b_-) = \{(\Phi, x) : x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})\} \quad (4.4)$$

of all pairs (Φ, x) , where $\Phi = \Phi(b_+, b_-, x)$ is the configuration defined above and x runs through the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ in G . This is a subset of $\mathbb{Z}_2 \wr G$. We prove that the map

$$(b_+, b_-) \mapsto S(b_+, b_-)$$

is $\mathbb{Z}_2 \wr \Gamma$ -equivariant, i.e., for $g = (\phi, \gamma) \in \mathbb{Z}_2 \wr \Gamma$

$$gS(b_+, b_-) = S(gb_+, gb_-).$$

Next,

$$gS(b_+, b_-) = (\phi, \gamma) \cdot \{(\Phi, x) : x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})\} = \{(\phi \oplus T_\gamma \Phi, \gamma x), x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})\}.$$

If $x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})$, then $\gamma x \in \mathfrak{s}(\gamma\mathbf{u}, \gamma\mathbf{v})$, since $\mathfrak{s}(\gamma\mathbf{u}, \gamma\mathbf{v})$ is Γ -equivariant. Also,

$$\phi \oplus T_\gamma \Phi = \begin{cases} \phi \oplus T_\gamma \phi_-, & \text{on } \gamma\mathbf{G}_+ \\ \phi \oplus T_\gamma \phi_+, & \text{on } \gamma\mathbf{G}_- \\ 0, & \text{on } \mathbf{G} \setminus (\gamma\mathbf{G}_+ \cup \gamma\mathbf{G}_-). \end{cases}$$

This means that

$$\phi \oplus T_\gamma \Phi(b_+, b_-, x) = \Phi(gb_+, gb_-, \gamma x),$$

for all $x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})$. On the other hand,

$$\begin{aligned} S(gb_+, gb_-) &= S((\phi \oplus T_\gamma \phi_+, \gamma\mathbf{u}), (\phi \oplus T_\gamma \phi_-, \gamma\mathbf{v})) \\ &= \{(\Phi(gb_+, gb_-, \gamma x), \gamma x), x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})\}, \end{aligned}$$

that is, $gS(b_+, b_-) = S(gb_+, gb_-)$, and this proves the $\mathbb{Z}_2 \wr \Gamma$ -equivariance of the strip $S(b_+, b_-)$.

Finally, let us prove that the strip $S(b_+, b_-)$ has subexponential growth. For this, let $(\eta, x) \in S(b_+, b_-)$ such that

$$d((\mathbf{0}, o), (\eta, x)) \leq n.$$

Definition 2.2 of the metric $d(\cdot, \cdot)$ on $\mathbb{Z}_2 \wr \mathbf{G}$ implies that $d(o, x) \leq n$. Therefore, if

$$(\eta, x) \in S(b_+, b_-) \cap B((\mathbf{0}, o), n), \quad \text{then } x \in \mathfrak{s}(\mathbf{u}, \mathbf{v}) \cap B(o, n), \quad (4.5)$$

where $B((\mathbf{0}, o), n)$ (respectively, $B(o, n)$) is the ball with center $(\mathbf{0}, o)$ (respectively, o) and of radius n in $\mathbb{Z}_2 \wr \mathbf{G}$ (respectively, \mathbf{G}). Since for every $x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})$ we associate only one configuration Φ in $S(b_+, b_-)$, equation (4.5) implies that

$$|S(b_+, b_-) \cap B((\mathbf{0}, o), n)| \leq |\mathfrak{s}(\mathbf{u}, \mathbf{v}) \cap B(o, n)|.$$

Finally, the assumption (4.2) that the $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ has subexponential growth leads to

$$\frac{\log |S(b_+, b_-) \cap B((\mathbf{0}, o), n)|}{n} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

and this proves the subexponential growth of the strip $S(b_+, b_-)$.

Since for almost every pair of points $(b_+, b_-) \in \Pi \times \Pi$, we have assigned a strip $S(b_+, b_-)$, which satisfies the conditions from Theorem 4.1.4, it follows that the measure space (Π, ν_∞) is the Poisson boundary of the lamplighter random walk (Z_n) . \square

As an application of the *Half Space Method*, we consider several classes of transitive base graphs G : graphs with infinitely many ends, hyperbolic graphs in the sense of Gromov and Euclidean lattices. For random walks (X_n) on these types of graphs Assumption 3.2.1 holds.

Remark 4.2.2. In principle, one can apply the construction from the previous proof to “iterated” lamplighter graphs, which are defined as follows. We consider $\mathbb{Z}_2 \wr G$ as our base graph, and construct the lamplighter graph $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr G)$ over $\mathbb{Z}_2 \wr G$, and so on. The “smaller” strip \mathfrak{s} will be in this case a subset of $\mathbb{Z}_2 \wr G$, which has subexponential growth, and using the Half Space Method, we lift it to a “bigger” strip which satisfies the conditions in Theorem 4.1.4.

Chapter 5

Graphs with Infinitely many Ends

In this chapter we introduce transitive graphs G with rich geometric boundaries, such as graphs with infinitely many ends, and we study the behaviour at infinity of random walks (X_n) on G and of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr G$, given that G is a graph with infinitely many ends.

The behaviour at infinity of random walks over graphs G with infinitely many ends depends on the action of the group $\Gamma \subset \text{AUT}(G, P)$. If Γ does not fix any element of ∂G , then it is easy to study the convergence and the Poisson boundary of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr G$ just using Theorem 3.2.3 and Theorem 4.2.1. On the other hand, if Γ fixes one end of G , which then has to be unique, then we encounter one interesting situation: when the base random walk (X_n) has zero drift on G , then one cannot describe the Poisson boundary of (Z_n) by making use of the *Half Space Method*. This case will be treated separately in Section 5.4, by considering the correspondence between the *tail-algebra* and the Poisson boundary of the respective walk. The proof makes also use of *cutpoints* and other results of JAMES AND PERES [JP97].

5.1 Ends of Graphs

The basic idea behind the concept of *an end* is to distinguish between different ways of going to infinity. Ends carry a natural topology which is often not mentioned explicitly. *Ends of graphs* and the *end compactification* were originally introduced by FREUDENTHAL [Fre44], who considered only locally finite graphs. HALIN [Hal64] was the first to consider ends of non-locally finite graphs.

Let G be an infinite, locally finite, connected graph. An *infinite path* or *ray* without self-intersections is a sequence $\pi = [x_0, x_1, \dots]$ of distinct vertices, such that $x_i \sim x_{i-1}$ for all i , where \sim denotes the neighbourhood relation. In F is a finite set of vertices and/or edges of G , then the (induced) graph $G \setminus F$ has finitely many connected components. Every ray π must have all but finitely many points in precisely one of them, and we say that π ends up in that component. Two rays are called *equivalent* if, for any finite set of edges F , they end up in the same component of $G \setminus F$. An *end* of G is an equivalence class of rays.

We write ∂G for the space of ends, and $\widehat{G} = G \cup \partial G$ for the *end compactification* of G . If C is a component of $G \setminus F$, then we write ∂C for the set of those ends whose rays end up in C and $\widehat{C} = C \cup \partial C$ for the resulting completion of C .

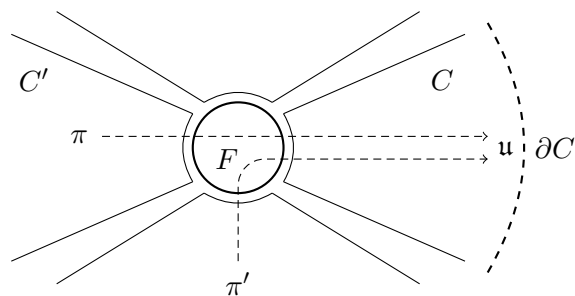


Figure 5.1: two equivalent rays π and π' ending up in the same component C .

Let us now explain the topology of \widehat{G} . If F is a finite set and $w \in \widehat{G}$, then there is exactly one component of $G \setminus F$ whose completion contains w . We denote the latter by $\widehat{C}(w, F)$. If we vary F (finite, with $w \notin F$), we obtain a neighbourhood base of w . If $x \in G$, we can take for F the finite set of neighbours of x to see that the topology is discrete on G . It has a countable base and it is Hausdorff. When $u \in \partial G$, we can find a *standard neighbourhood base*, that is, one of the form $\widehat{C}(u, F_k)$, $k \in \mathbb{N}$, where the finite sets $F_k \subset G$ are such that

$$F_k \cup \widehat{C}(u, F_k) \subset \widehat{C}(u, F_{k-1}), \text{ for all } k.$$

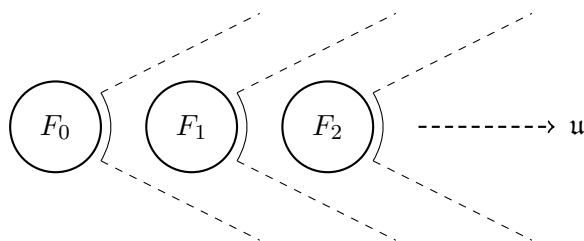


Figure 5.2: a standard neighbourhood base of u , with F_1, F_2, F_3 finite sets

As the number of components of $G \setminus F$ is finite, each $\widehat{C}(u, F)$ is open and compact, and \widehat{G} is totally disconnected. The space \widehat{G} is called the *end compactification* of G .

Definition 5.1.1. An end \mathbf{u} is called *thin*, if it has a standard neighbourhood base with $F_k \subset \mathbf{G}$, such that

$$\text{diam}(F_k) = m < \infty, \quad \text{for all } k,$$

where $\text{diam}(F_k)$ is the diameter of the set F_k in the graph metric. The minimal m with this property is the diameter of \mathbf{u} . Otherwise, \mathbf{u} is called *thick*.

Consider the ball $B(o, r)$ of center o (some fixed vertex of \mathbf{G}) and radius r in \mathbf{G} , with respect to the natural graph metric d . In order to prove that the space of ends $\partial\mathbf{G}$ is a weakly projective space (see Definition 3.1.3) we shall use the following result which can be found in WOESS [Woe00, page 231].

Lemma 5.1.2. Let $r \geq 1$ and $x, y \in \mathbf{G} \setminus B(o, r)$. If

$$d(o, x) + d(o, y) - d(x, y) > 2r,$$

then x and y belongs to the same component of $\mathbf{G} \setminus B(o, r)$.

In the results we are going to prove, it is enough for $\partial\mathbf{G}$ to be a weakly projective boundary.

Lemma 5.1.3. The space of ends $\partial\mathbf{G}$ of a locally finite graph \mathbf{G} is a weakly projective boundary.

Proof. Let (x_n) and (y_n) be sequences of vertices in \mathbf{G} , such that (x_n) converges to some end $\mathbf{u} \in \partial\mathbf{G}$ in the topology of $\widehat{\mathbf{G}}$, and

$$\frac{d(x_n, y_n)}{d(o, x_n)} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.1)$$

Assume that the sequence (y_n) converges to another end $\mathbf{v} \in \partial\mathbf{G}$, with $\mathbf{u} \neq \mathbf{v}$. Then there exists a $r \in \mathbb{N}$, such that, for all $n \geq n_r$,

$$x_n \in \widehat{C}(\mathbf{u}, B(o, r)) \text{ and } y_n \in \widehat{C}(\mathbf{v}, B(o, r)).$$

This means that for n big enough, the sequences (x_n) and (y_n) belong to different components of $\mathbf{G} \setminus B(o, r)$. From Lemma 5.1.2, it follows that

$$d(o, x_n) + d(o, y_n) - 2r < d(x_n, y_n).$$

Dividing both sides of the previous equation through $d(o, x_n)$, and using (5.1), we get

$$\frac{d(o, x_n) + d(o, y_n)}{d(o, x_n)} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

which is a contradiction. Therefore, the sequence (y_n) converges to the same end \mathbf{u} and $\partial\mathbf{G}$ is a weakly projective boundary. \square

Proposition 5.1.4. *The space of ends ∂G of a locally finite graph G is also projective space.*

For the proof see WOESS [Woe00, page 233].

5.2 The Structure Tree of a Graph

The theory of cuts and structure trees was first developed by Dunwoody; see the book by DICKS AND DUNWOODY [DD89], or for another detailed description see WOESS [Woe00] and THOMASSEN AND WOESS [TW93]. A detailed study of structure theory may be very fruitful for obtaining information on the behaviour of random walks.

A *cut* of a connected graph G is a set F of edges whose deletion disconnects G . If it disconnects G into precisely two connected components $A = A(F)$ and $A^* = A^*(F) = G \setminus A$, then we call F *tight*, and A, A^* are the *sides* of F . In [TW93], Thomassen has proved the following.

Lemma 5.2.1. *For any $k \in \mathbb{N}$, there are only finitely many tight cuts F with $|F| = k$ that contain a given edge of G .*

Two cuts F, F' are said to *cross* if all four sets

$$A(F) \cap A(F'), \quad A(F) \cap A^*(F'), \quad A^*(F) \cap A(F'), \quad A^*(F) \cap A^*(F')$$

are nonempty. DUNWOODY [Dun82] proved the following important theorem.

Theorem 5.2.2. *Every infinite, connected graph with more than one end has a finite tight cut F with infinite sides, such that F crosses no γF , where $\gamma \in \text{AUT}(G)$. A cut with these properties will be called a D -cut.*

Let F be a D -cut of the locally finite, connected graph G and Γ be a closed subgroup of $\text{AUT}(G)$. Define

$$\mathcal{E} = \{A(\gamma F), A^*(\gamma F) : \gamma \in \Gamma\}.$$

This set has the following properties:

- (a) All $A \in \mathcal{E}$ are infinite and connected.
- (b) If $A \in \mathcal{E}$, then $A^* = G \setminus A \in \mathcal{E}$.
- (c) If $A, B \in \mathcal{E}$ and $A \subset B$, then there are only finitely many $C \in \mathcal{E}$, such that $A \subset C \subset B$.

(d) If $A, B \in \mathcal{E}$, then one of $A \subset B$, $A \subset B^*$, $A^* \subset B$ or $A^* \subset B^*$ holds.

This properties can now be used to construct a tree \mathcal{T} , called the *structure tree* of G with respect to Γ and the D -cut F . One can think of an unoriented edge of \mathcal{T} as a pair of oriented edges, where the second edge points from the endpoint to the initial point of the first one. The oriented edge set of \mathcal{T} is \mathcal{E} . That is, if $A \in \mathcal{E}$, then (A, A^*) constitutes a pair of oppositely oriented edges between the same two vertices. If $A, B \in \mathcal{E}$ and $B \neq A^*$, then the endpoint of A is the initial point of B if $A \supset B$, and there is no $C \in \mathcal{E}$, such that $A \supset C \supset B$ properly.

In this way, we have defined \mathcal{T} in terms of its edges and their incidence, contrary to the usual approach of defining a graph by starting with its vertices. A vertex of \mathcal{T} is an equivalence class of edges “with the same endpoint”, that is, $A, B \in \mathcal{E}$ are *equivalent* in this sense, if $A = B$, or else, if $A \supset B^*$ properly and no $C \in \mathcal{E}$ satisfies $A \supset C \supset B^*$ properly. One can check that this is indeed an equivalence relation.

The vertex set is the set of all equivalence classes $[A]$, where $A \in \mathcal{E}$. Neighbourhood in \mathcal{T} is described by $[A] \sim [A^*]$. One can show that \mathcal{T} is a tree: it is connected because of the property (3), it has no cycles, since the neighbourhood relation is defined in terms of inclusion of sets.

The tree \mathcal{T} is countable by Lemma 5.2.1, but not necessarily locally finite. One can still define the set $\partial\mathcal{T}$ of ends of \mathcal{T} as equivalence classes of rays, as in Section 5.1, and $\widehat{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$. The group Γ acts by automorphisms on \mathcal{T} via $A \mapsto \gamma A$, where $\gamma \in \Gamma$ and $A \in \mathcal{E}$. The action has one or two orbits on \mathcal{E} according to wheather $\gamma A(F) = A^*(F)$ for some $\gamma \in \Gamma$ or not. Consequently, Γ acts transitively on \mathcal{T} or else acts transitively on each of the two bipartite classes of \mathcal{T} (that is, the sets of vertices at even/odd distance from a chosen origin).

Example 5.2.3. If G is the homogeneous tree \mathcal{T}_M of degree M , then any single edge constitutes a D -cut. Moreover, if Γ is the whole automorphism group of \mathcal{T}_M , then the structure tree is again \mathcal{T}_M . This is not the case, when $\Gamma = \mathbb{Z}_2 * \mathbb{Z}_2 \cdots * \mathbb{Z}_2$ (M times), or Γ is the free group.

Example 5.2.4. As an example with thick ends, consider the standard Cayley graph G of the free product

$$\Gamma = \mathbb{Z}^2 * \mathbb{Z}_2 = \langle a, b, c \mid ab = ba, c^2 = o \rangle,$$

acting by Γ on itself. Each copy $\gamma\mathbb{Z}^2$ of the square grid within G , where $\gamma \in \Gamma$, gives rise to a thick end (as an equivalence class of rays that end up in such a copy). The other ends are all thin, they have zero diameter. Let F consist of the single edge $[o, c]$. This is a D -cut, and the structure tree has infinite vertex degrees.

Next, let us introduce the *structure map* $\varphi : \widehat{\mathbf{G}} \rightarrow \widehat{\mathcal{T}}$. In order to understand it, let $x \in \widehat{\mathbf{G}}$. Then there is some $A_0 \in \mathcal{E}$ which contains x . If there is a minimal $A \in \mathcal{E}$ with this property, then we define $\varphi(x)$ as the end vertex of A as an edge of \mathcal{T} . If there is no minimal A with this property, then there must be a maximal strictly descending sequence $A_0 \supset A_1 \supset A_2 \supset \dots$ in \mathcal{E} , such that $x \in A_n$, for all n . As edges of \mathcal{T} , the A_n constitute a path which defines an end in $\partial\mathcal{T}$. This end is $\varphi(x)$. The image of x does not depend on the particular choice of the initial $A_0 \in \mathcal{E}$ containing x . Via $(\gamma, A) \mapsto \gamma A$ for $A \in \mathcal{E}$, the group Γ acts on \mathcal{T} and φ commutes with the actions of Γ on $\widehat{\mathbf{G}}$ and on $\widehat{\mathcal{T}}$.

If x is a vertex of \mathbf{G} , then $\varphi(x)$ is a vertex of \mathcal{T} . Given an end of \mathcal{T} , its preimage under φ consists of a single end of \mathbf{G} . However, there are ends of \mathbf{G} that are mapped to vertices of \mathcal{T} under φ . We write

$$\partial^{(0)}\mathbf{G} = \varphi^{-1}(\partial\mathcal{T}).$$

These are thin ends of \mathbf{G} , i.e., ends with finite diameter.

Group Actions. If \mathbf{G} is an infinite, locally finite, connected and transitive graph (i.e. there exists Γ subgroup of $\text{AUT}(\mathbf{G})$, which acts transitively on \mathbf{G}), then \mathbf{G} has *one, two or infinitely many ends*. For details, see WOESS [Woe89b].

If \mathbf{G} has one end, then the end compactification is not suitable for a good description of the structure of \mathbf{G} at infinity.

If \mathbf{G} has two ends, then it is isometric with the two-way-infinite path. Moreover, the behaviour at infinity of lamplighter random walks over the integer line (two-way-infinite path) is well-studied.

For the rest of this chapter we suppose that \mathbf{G} has infinitely many ends, since we want a base structure (as a base graph for the lamplighter graph) which is endowed with a rich boundary. The space $\widehat{\mathbf{G}}$ is the end compactification and the boundary $\partial\mathbf{G}$ is the space of ends of \mathbf{G} , with $|\partial\mathbf{G}| = \infty$.

The simplest examples of graphs with infinitely many ends are free products of graphs (with the exception of the Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_2$, which has two ends). More generally, the free product of two (or finitely many) rooted graphs has infinitely many ends, unless both have only two elements. The tree is the nicest example of a graph with infinitely many ends. Anyway, there are graphs with infinitely many ends which have a very complicated structure. For a detailed exposition, the reader is invited to have a look at the book of DICKS AND DUNWOODY [DD89].

Ends and Fixed Sets. Let G be a graph with infinitely many ends and $\Gamma \subset \text{AUT}(G)$ which acts transitively on G , and $\partial^{(0)}G$ the set of thin ends, i.e., those ends with finite diameter. From WOESS [Woe89b] it is known that the set of thin ends $\partial^{(0)}G$ is dense in ∂G .

Definition 5.2.5. A subset $B \subset \partial G$ is fixed by the action of Γ , if $\gamma B = B$ (pointwise action) for every $\gamma \in \Gamma$.

We also recall the following result, which will be relevant in the sequel. For a proof, see SOARDI AND WOESS [SW90].

Theorem 5.2.6. The group Γ cannot fix a finite subset of ∂G other than a singleton. This happens if and only if Γ is amenable.

Therefore, when studying random walks on graphs with infinitely many ends, one has to distinguish two substantially different cases: when $\Gamma \subset \text{AUT}(G)$ doesn't fix any end in ∂G and when one end in ∂G is fixed under the action of Γ .

5.3 LRW over Graphs with Infinitely many Ends

The goal of this section is to apply the results developed in Chapter 3 and Chapter 4 for lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr G$, where G is a transitive graph with infinitely many ends, i.e., $|\partial G| = \infty$.

Let us now recall the setting we are working on: we are interested in the behaviour at infinity of homogeneous random walks (Z_n) on $\mathbb{Z}_2 \wr G$, with $Z_n = (\eta_n, X_n)$. The factor chain (X_n) is a random walk over a transitive graph G with infinitely many ends. Note that when G has infinitely many ends, the lamplighter graph $\mathbb{Z}_2 \wr G$ over G has only one end. The group Γ acts transitively on G (with or without fixed end) and the wreath product $\mathbb{Z}_2 \wr \Gamma$ (the lamplighter group) acts transitively on the lamplighter graph $\mathbb{Z}_2 \wr G$.

5.3.1 No Fixed End

When $\Gamma \subset \text{AUT}(G, P)$ acts transitively on G and does not fix any end in ∂G , it is known from WOESS [Woe89a] that Γ is nonamenable.

The following holds for random walks (X_n) on transitive graphs G with infinitely many ends. For the proof, see WOESS [Woe89b].

Theorem 5.3.1. The random walk (X_n) over G converges almost surely in the end topology to a random end $X_\infty \in \partial G$. Denoting by μ_∞ the hitting distribution on ∂G we have:

- (a) The support of μ_∞ is the whole $\partial\mathbf{G}$.
- (b) μ_∞ is continuous on $\partial\mathbf{G}$, that is $\mu_\infty(\{\mathbf{u}\}) = 0$, for all $\mathbf{u} \in \partial\mathbf{G}$.
- (c) The mass of thick ends is zero, i.e., $\mu_\infty(\partial\mathbf{G} \setminus \partial^{(0)}\mathbf{G}) = 0$.

Recall that μ_∞ is the probability distribution defined for Borel subsets $B \subset \partial\mathbf{G}$ by

$$\mu_\infty(B) = \mathbb{P}[X_\infty \in B | X_0 = o]. \quad (5.2)$$

For convergence of (X_n) to $X_\infty \in \partial\mathbf{G}$ the finite first moment assumption is not needed. Suppose now that (X_n) has finite first moment on \mathbf{G} and recall that the space of ends $\partial\mathbf{G}$ is a weakly projective boundary. Then Assumption 3.2.1 holds.

Let us now state the result regarding the convergence of lamplighter random walks (Z_n) over $\mathbb{Z}_2 \wr \mathbf{G}$, whose projection is the random walk (X_n) over the graph with infinitely many ends \mathbf{G} . The boundary $\partial\mathbf{G}$ is the space of ends and the boundary Π of $\mathbb{Z}_2 \wr \mathbf{G}$ is defined in (3.1) as

$$\Pi = \bigcup_{\mathbf{u} \in \partial\mathbf{G}} \mathcal{C}_\mathbf{u} \times \{\mathbf{u}\},$$

where $\mathcal{C}_\mathbf{u}$ is the set of configurations which are either finitely supported or accumulate at \mathbf{u} .

Theorem 5.3.2. *Let (Z_n) be an irreducible, homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathbf{G}$, where \mathbf{G} is a graph with infinitely many ends and $\Gamma \subset \text{AUT}(\mathbf{G})$ does not fix any end in $\partial\mathbf{G}$. Then there exists a Π -valued random variable $Z_\infty = (\eta_\infty, X_\infty)$, such that $Z_n \rightarrow Z_\infty$ almost surely, in the topology of $\widehat{\mathbb{Z}_2 \wr \mathbf{G}}$ for every starting point. Moreover the distribution of Z_∞ is a continuous measure on Π .*

Proof. The proof of this result follows basically the proof of Theorem 3.2.3, which holds for general base graphs \mathbf{G} endowed with a rich boundary and such that Assumption 3.2.1 holds.

The general boundary $\partial\mathbf{G}$ of the base graph is here the space of ends and the boundary of the lamplighter graph $\mathbb{Z}_2 \wr \mathbf{G}$ is Π defined like in equation (3.1).

Since, by Theorem 5.3.1, the limit distribution μ_∞ is a continuous measure on $\partial\mathbf{G}$, the same is true (using a Borel-Cantelli argument) for the limit distribution of (Z_n) on Π . \square

The Poisson Boundary. First of all, let us apply the Strip criterion due to KAIMANOVICH [Kai00, Thm. 6.5 on p. 677] in order to determine the Poisson boundary of random walks (X_n) over graphs with infinitely many ends \mathbf{G} . This was done in KAIMANOVICH AND WOESS [KW02, Thm. 5.19]. Nevertheless, we give a complete proof here, since the geometric construction will be of interest for further results.

Theorem 5.3.3. *If Γ does not fix any end in $\partial\mathbf{G}$ and the random walk (X_n) has finite first moment on \mathbf{G} , then the Poisson boundary of (X_n) is the measure space $(\partial\mathbf{G}, \mu_\infty)$, where μ_∞ is the limit distribution of (X_n) on $\partial\mathbf{G}$.*

Proof. By Theorem 5.3.1 and Definition 4.1.2, the space $(\partial\mathbf{G}, \mu_\infty)$ is a μ -boundary for the random walk (X_n) with transition matrix $P_{\mathbf{G}}$, where μ is associated with $P_{\mathbf{G}}$ as in (1.4). Theorem 5.3.1 applies also to the reversed random walk (\check{X}_n) , which is the random walk on \mathbf{G} with transition matrix $\check{P}_{\mathbf{G}}$. If the corresponding limit distribution on $\partial\mathbf{G}$ is $\check{\mu}_\infty$, then $(\partial\mathbf{G}, \check{\mu}_\infty)$ is a $\check{\mu}$ -boundary for (\check{X}_n) .

Apply the Strip Criterion 4.1.4. By Theorem 5.2.2, there exists a D -cut $F \subset \mathbf{G}$, whose removal disconnects \mathbf{G} into finitely many infinite connected components. The D -cut was used for the construction of the structure tree \mathcal{T} of \mathbf{G} . Denote by F^0 the set of end vertices of F .

Let now \mathbf{u}, \mathbf{v} be ends of $\partial\mathbf{G}$. By continuity of μ_∞ and $\check{\mu}_\infty$, we have

$$\mu_\infty \times \check{\mu}_\infty(\{\mathbf{u}, \mathbf{v} \in \partial\mathbf{G} : \mathbf{u} = \mathbf{v}\}) = 0.$$

Therefore, we have to construct the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ only in the case $\mathbf{u} \neq \mathbf{v}$. Define the “small” strip

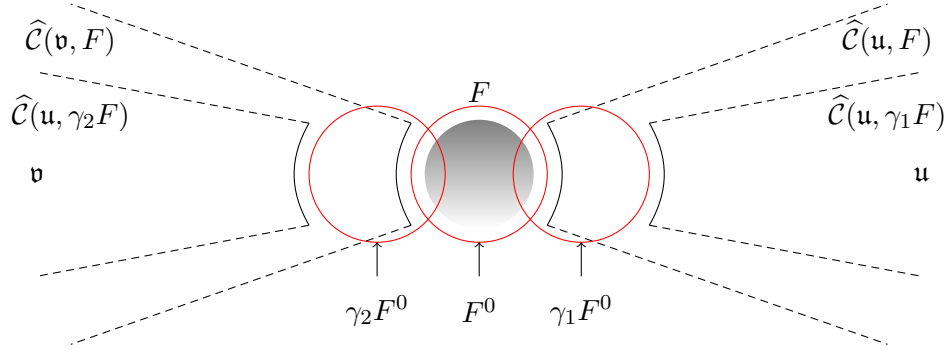
$$\mathfrak{s}(\mathbf{u}, \mathbf{v}) = \bigcup_{\gamma \in \Gamma} \{\gamma F^0 : \widehat{\mathcal{C}}(\mathbf{u}, \gamma F) \neq \widehat{\mathcal{C}}(\mathbf{v}, \gamma F)\}. \quad (5.3)$$

The set $\mathcal{C}(\mathbf{u}, F)$ is the connected component of \mathbf{G} which represents the end \mathbf{u} when we remove the finite set F from \mathbf{G} , and $\widehat{\mathcal{C}}(\mathbf{u}, F)$ its completion (which contains \mathbf{u}) in $\widehat{\mathbf{G}}$. The strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ is a subset of \mathbf{G} .

In other words, the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ is the union of all translates γF^0 , with the property that the components, which contain the ends \mathbf{u} and \mathbf{v} after removing the finite set F are different. See Figure 5.3 for the construction.

We have to check that the strip is Γ -equivariant and it is “thin” enough, i.e., it has subexponential growth.

Since Γ acts transitively on \mathbf{G} , it is clear that $\gamma \mathfrak{s}(\mathbf{u}, \mathbf{v}) = \mathfrak{s}(\gamma \mathbf{u}, \gamma \mathbf{v})$, for every $\gamma \in \Gamma$. The strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ is the union of all γF^0 , such that the sides of γF , seen as edges of the structure tree \mathcal{T} , lie on the geodesic between $\varphi \mathbf{u}$ and


 Figure 5.3: The construction of the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$

$\varphi\mathbf{v}$. Recall that φ is the function (the structure map) which maps $\widehat{\mathbf{G}}$ onto $\widehat{\mathcal{T}}$.

From Theorem 5.3.1, the random walk (X_n) on \mathbf{G} converges to a thin end in $\partial\mathbf{G}$. If \mathbf{u} and \mathbf{v} are thin ends in $\partial\mathbf{G}$, then $\varphi\mathbf{u}$ and $\varphi\mathbf{v}$ are ends in the structure tree \mathcal{T} and the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ is a two way infinite geodesic in \mathcal{T} . It can be empty in the case $\mathbf{u} = \mathbf{v}$, but we already excluded this case.

From Lemma 5.2.1 and the fact that F is a D -cut, there is an integer $k > 0$, such that the following holds: if $A_1, A_2, \dots, A_k \in \mathcal{E}$ are edges in \mathcal{T} and $A_0 \supset A_1 \supset \dots \supset A_k$ properly, then $d(A_k, A_0^*) \geq 2$. In other words, if A_k is one of the sides of γF , with $\gamma \in \Gamma$, then γF^0 is entirely contained in A_0 . The finiteness of F^0 implies the existence of a constant $c > 0$, such that, for the ball $B(o, n)$ of center o and radius n in the graph metric of \mathbf{G} ,

$$|\mathfrak{s}(\mathbf{u}, \mathbf{v}) \cap B(o, n)| \leq cn,$$

for all n and distinct $\mathbf{u}, \mathbf{v} \in \partial^{(0)}\mathbf{G}$. Applying the logarithm and dividing through n we get,

$$\frac{1}{n} \log |\mathfrak{s}(\mathbf{u}, \mathbf{v}) \cap B(o, n)| \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

and this proves the subexponential growth of the strip. Therefore, by Strip Criterion 4.1.4, the space $(\partial\mathbf{G}, \mu_\infty)$ is the Poisson boundary of the random walk (X_n) on \mathbf{G} . \square

Now we are able to apply the *Half Space Method* explained in Section 4.2 in order to describe the Poisson boundary of lamplighter random walks (Z_n) over $\mathbb{Z}_2 \wr \mathbf{G}$, where \mathbf{G} is a transitive graph with infinitely many ends. Assumption 3.2.1 holds for random walks (X_n) on \mathbf{G} by Theorem 5.3.1.

Similar results on the Poisson boundary of lamplighter random walks on groups with infinitely many ends were considered in SAVA [Sav10].

Theorem 5.3.4. *Let (Z_n) be an irreducible, homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathbf{G}$, where \mathbf{G} is a transitive graph with infinitely many ends. If $\Gamma \subset \text{AUT}(\mathbf{G})$ does not fix any end in $\partial\mathbf{G}$, then (Π, ν_∞) is the Poisson boundary of (Z_n) , starting at $(\mathbf{0}, o)$, where Π is defined in (3.1), and ν_∞ is the limit distribution on Π .*

Proof. In order for Theorem 4.2.1 to be applicable, we need the conditions required in the *Half Space Method* to be satisfied for the base graph \mathbf{G} and random walks (X_n) and (\check{X}_n) on it.

By assumption, (X_n) has finite first moment on \mathbf{G} , and Theorem 5.3.1 and Lemma 5.1.3 imply that Assumption 3.2.1 hold for (X_n) and also for (\check{X}_n) . As usual, μ_∞ and $\check{\mu}_\infty$ are the respective limit distributions on $\partial\mathbf{G}$.

Next, assign a strip $\mathfrak{s}(\mathbf{u}, \mathbf{v}) \subset \mathbf{G}$ to almost every pair of ends $(\mathbf{u}, \mathbf{v}) \in \partial\mathbf{G} \times \partial\mathbf{G}$. Consider the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v}) \subset \mathbf{G}$ defined in the proof of Theorem 5.3.4, by equation (5.3), which satisfies the Strip Criterion conditions in Theorem 4.1.4.

The next step is to partition \mathbf{G} into half-spaces. By construction of the “small” strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$, every $x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})$ is contained in some cut γF , for some $\gamma \in \Gamma$. Nevertheless, there are finitely many $\gamma \in \Gamma$, such that, for $x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})$, we have $x \in \gamma F$, since F is finite.

The partition of \mathbf{G} is done as follows: for each $x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})$, look at the D -cuts γF containing x , pick one of them and remove it from \mathbf{G} . Then the set $(\mathbf{G} \setminus \gamma F)$ contains finitely many connected components. This follows from the definition of a D -cut, and from the finiteness of the removed set F . Moreover, the connected components containing \mathbf{u} and \mathbf{v} are different, by the definition of the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$. Let $\mathbf{G}_+ = \mathbf{G}_+(x)$ be the connected component of $(\mathbf{G} \setminus \gamma F)$, which contains \mathbf{u} , and $\mathbf{G}_- = \mathbf{G}_-(x)$ be its complement in \mathbf{G} , which contains \mathbf{v} . One can see here that the partition of \mathbf{G} into the half-spaces \mathbf{G}_+ and \mathbf{G}_- depends on the cut γF containing x , that is, depends on x . The sets \mathbf{G}_+ and \mathbf{G}_- are Γ -equivariant.

From the above, it follows that all the assumptions needed in the *Half Space Method* hold in the case of a graph with infinitely many ends \mathbf{G} . Apply now Theorem 4.2.1.

By Theorem 3.2.3 each of the random walks (Z_n) and (\check{Z}_n) starting at $(\mathbf{0}, o)$ converges almost surely to a Π -valued random variable. If ν_∞ and $\check{\nu}_\infty$ are their respective limit distributions on Π , then the spaces (Π, ν_∞) and $(\Pi, \check{\nu}_\infty)$ are ν - and $\check{\nu}$ - boundaries of the respective walks. Take

$$b_+ = (\phi_+, \mathbf{u}) \in \Pi, \text{ and } b_- = (\phi_-, \mathbf{v}) \in \Pi$$

where ϕ_+ and ϕ_- are the limit configurations of (Z_n) and (\check{Z}_n) , respectively, and $\mathbf{u}, \mathbf{v} \in \partial\mathbf{G}$ are their only respective accumulation points. Define the

configuration $\Phi(b_+, b_-, x)$ like in (4.3), that is,

$$\Phi(b_+, b_-, x) = \begin{cases} \phi_-, & \text{on } \mathbf{G}_+ \\ \phi_+, & \text{on } \mathbf{G}_- \end{cases}$$

where $\mathbf{G} \setminus (\mathbf{G}_+ \cup \mathbf{G}_-)$ is the empty set. Consider the strip $S(b_+, b_-)$ exactly like in (4.4), i.e.,

$$S(b_+, b_-) = \{(\Phi, x) : x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})\}.$$

By Theorem 4.2.1, $S(b_+, b_-)$ satisfies the conditions from Strip Criterion 4.1.4, and this implies that the space (Π, ν_∞) is the Poisson boundary of the lamplighter random walk (Z_n) over $\mathbb{Z}_2 \wr \mathbf{G}$. \square

5.3.2 One Fixed End

Let $\xi \in \partial\mathbf{G}$ be an end of \mathbf{G} , which is fixed under the action of the group $\Gamma \subset \text{AUT}(\mathbf{G}, P)$, and $\partial^*\mathbf{G} = \partial\mathbf{G} \setminus \{\xi\}$ the set of the remaining ends. Recall a result which can be found in MÖLLER [Mö92] and WOESS [Woe89a], [Woe89b].

Theorem 5.3.5. *The following hold.*

- (a) Γ is amenable and acts transitively on $\partial^*\mathbf{G}$.
- (b) The structure tree \mathcal{T} of \mathbf{G} is a homogeneous tree with finite degree $q+1 \geq 3$.
- (c) The structure map $\varphi : \widehat{\mathbf{G}} \rightarrow \widehat{\mathcal{T}}$ is onto, and its restriction to $\partial\mathbf{G}$ is a homeomorphism $\partial\mathbf{G} \rightarrow \partial\mathcal{T}$. There is an integer $a > 0$, such that

$$d_{\mathcal{T}}(\varphi x, \varphi y) \leq d_{\mathbf{G}}(x, y) \leq a(d_{\mathcal{T}}(\varphi x, \varphi y) + 1).$$

Therefore, \mathbf{G} and \mathcal{T} are quasi-isometric graphs, whose ends are in bijection.

The interpretation of this result is that \mathbf{G} is described - up to small modifications encoded by the structure map φ - by its structure tree which looks like in Figure 5.4, with $\varphi(\xi) = \omega$.

Let \mathcal{T} be the homogeneous tree with a fixed end ω , like in Figure 5.4, with $\partial\mathcal{T}$ its space of ends and $\partial^*\mathcal{T} = \partial\mathcal{T} \setminus \{\omega\}$. Random walks on homogeneous trees with a fixed end are well studied by CARTWRIGHT, KAIMANOVICH AND WOESS [CKW94]. In this section we want to extend their results for lamplighter random walks on such graphs: we study the convergence and the Poisson boundary of homogeneous random walks (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$.

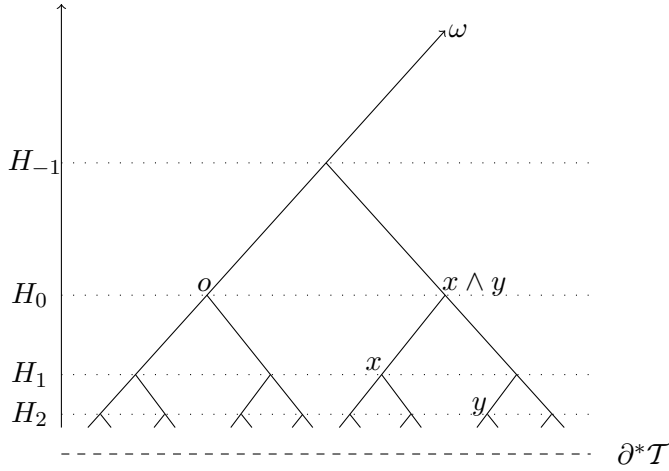


Figure 5.4: The homogeneous tree \mathcal{T} with a fixed end ω

Due to the quasi-isometry between G and its structure tree \mathcal{T} , and also to the fact that \mathcal{T} can be represented as a homogeneous tree with a fixed end, like in Figure 5.4, one can replace G with \mathcal{T} in $\mathbb{Z}_2 \wr G$. The geometry of \mathcal{T} is quite simple and the behaviour of random walks on such trees is completely understood.

Throughout this section, \mathcal{T} represents the base graph for the lamplighter graph $\mathbb{Z}_2 \wr \mathcal{T}$, and the goal is the study of random walks on $\mathbb{Z}_2 \wr \mathcal{T}$. We reconsider briefly the homogeneous tree and its affine group and recall the main known results.

Geometry of the Oriented tree. If ω is the fixed reference end in $\partial\mathcal{T}$, set for all $x \neq y$ in $\widehat{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$,

$$x \wedge y = \text{first common vertex of } \pi(x, \omega) \text{ and } \pi(y, \omega),$$

where $\pi(x, \omega)$ is the geodesic ray starting at x and ending at the fixed end ω . See once again Figure 5.4 for a graphic visualization.

Let o be a reference vertex in \mathcal{T} called *origin*. Define the *height function* $h : \mathcal{T} \rightarrow \mathbb{Z}$ by

$$h(x) = d(x, x \wedge o) - d(o, x \wedge o). \quad (5.4)$$

This function is known in the literature as the *Busemann function*, and represents the generation number of x . For $m \in \mathbb{Z}$, the *horocycle* at level m is the infinite set

$$H_m = \{x \in \mathcal{T} : h(x) = m\}.$$

One can imagine the oriented tree \mathcal{T} in Figure 5.4 as an infinite genealogical tree, where ω represents the “mythical ancestor” and every vertex $x \in H_m$

has an unique “father” x^- in H_{m-1} and $q \geq 2$ “sons” x_j , for $j = 1, \dots, q$ in H_{m+1} . Using the height function, one can express the distance between x and y in \mathcal{T} as

$$d(x, y) = h(x) + h(y) - 2h(x \wedge y).$$

Finally, define a bounded metric ρ on $\widehat{\mathcal{T}}$, which is an *ultrametric* (that is, $\rho(x, y) \leq \max\{\rho(x, z), \rho(z, y)\}$, for all $x, y, z \in \widehat{\mathcal{T}}$):

$$\rho(x, y) := \begin{cases} q^{-d(o, x \wedge y)}, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}.$$

Then a sequence (x_n) converges to x in $\widehat{\mathcal{T}}$ if $\rho(x_n, x)$ tends to zero as n tends to infinity.

The Affine Group $\text{AFF}(\mathcal{T})$ of a Tree. The *affine group* of a tree \mathcal{T} is the group $\text{AFF}(\mathcal{T})$ of all isometries γ which fix ω . Changing the reference end ω means passing to a conjugate of this group. The name is chosen because of the analogy with the Poincaré upper half plane, where the group of all isometries which fix the point at infinity coincides with the affine group of the real line. The affine group of an oriented tree and random walks on it was also studied by BROFFERIO [Bro04].

Random Walks on \mathcal{T} and on its Affine Group $\text{AFF}(\mathcal{T})$. Random walks on the oriented tree and on its affine group are well studied by CARTWRIGHT, KAIMANOVICH AND WOESS [CKW94]. For random walks on these type of trees, they have obtained the following results:

- Convergence to the boundary $\partial\mathcal{T}$, and hence, existence of a harmonic measure on $\partial\mathcal{T}$.
- The solution of the Dirichlet problem at infinity.
- Law of large numbers and central limit theorem, formulated with respect to two natural length functions on the affine group of \mathcal{T} .
- Identification of the Poisson boundary, that is, a description of bounded harmonic functions for random walks on \mathcal{T} .

In order to state the results on convergence and Poisson boundary of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$, with $Z_n = (\eta_n, X_n)$, similar results for random walks (X_n) on \mathcal{T} are needed. For complete proofs and more details see once again CARTWRIGHT, KAIMANOVICH AND WOESS [CKW94].

Consider an irreducible random walk (X_n) with transition matrix $P_{\mathcal{T}}$ on \mathcal{T} , and $\Gamma \subset \text{AFF}(\mathcal{T})$ which fixes the end $\omega \in \partial\mathcal{T}$. Like before, let μ be the

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probability measure on Γ , which is uniquely induced (recall the notation $P_{\mathcal{T}} \leftrightarrow \mu$) by the transition probabilities $P_{\mathcal{T}}$ of (X_n) , like in (1.4). The measure μ determines the right random walk (Γ, μ) on Γ .

Note that if (X_n) is a random walk on the oriented tree \mathcal{T} , then $h(X_n)$ is a random walk on the integer line \mathbb{Z} , where h is the height function defined in (5.4). Indeed, the mapping

$$\begin{aligned} \Psi : \mathcal{T} &\rightarrow \mathbb{Z} \\ x &\mapsto h(x) \end{aligned}$$

induces a projection of any random walk on \mathcal{T} onto a walk on \mathbb{Z} .

Define the *modular drift* of the random walk (X_n) on \mathcal{T} with transition matrix $P_{\mathcal{T}}$, as

$$\delta(P_{\mathcal{T}}) = \sum_{x \in \mathcal{T}} h(x) p_{\mathcal{T}}(o, x).$$

The reversed random walk (\check{X}_n) on \mathcal{T} has the transition matrix $\check{P}_{\mathcal{T}}$, whose entries are given by

$$\check{p}_{\mathcal{T}}(x, y) = p_{\mathcal{T}}(y, x) q^{h(x)-h(y)}.$$

Note that if $\delta(P_{\mathcal{T}}) < 0$, then $\delta(\check{P}_{\mathcal{T}}) > 0$ and the other way round. The following is known from CARTWRIGHT ET. AL [CKW94] and for the case $\delta(P_{\mathcal{T}}) = 0$ from BROFFERIO [Bro04].

Theorem 5.3.6. *If (X_n) is irreducible and has finite first moment on \mathcal{T} , then:*

- (a) *If $\delta(P_{\mathcal{T}}) > 0$, then (X_n) converges almost surely to a random end $X_{\infty} \in \partial^* \mathcal{T} = \partial \mathcal{T} \setminus \{\omega\}$. If μ_{∞} is the distribution of X_{∞} on $\partial^* \mathcal{T}$, we have that $\text{supp}(\mu_{\infty}) = \partial^* \mathcal{T}$ and $(\partial^* \mathcal{T}, \mu_{\infty})$ is the Poisson boundary of (X_n) .*
- (b) *If $\delta(P_{\mathcal{T}}) \leq 0$, then (X_n) converges to the fixed end ω almost surely, and the Poisson boundary of (X_n) is trivial.*

The case $\delta(P_{\mathcal{T}}) = 0$ is a special one, since the projection $h(X_n)$ of X_n is the simple random walk on \mathbb{Z} , which is recurrent. Nevertheless, the random walk (X_n) is transient and it converges to the fixed end ω , but its Poisson boundary is trivial.

The Poisson boundary in the previous result can be described by using the Strip Criterion 4.1.4, in the case of positive (negative, respectively) drift. For completeness, we shall give here the idea of the realization of the Poisson boundary for random walks on trees \mathcal{T} with a fixed end ω . For more details, see once again CARTWRIGHT, KAIMANOVICH AND WOESS [CKW94].

Idea of the proof of Theorem 5.3.6. If the modular drift $\delta(P_{\mathcal{T}}) = 0$, then also the drift (rate of escape) $l(P_{\mathcal{T}}) = 0$, and the triviality of the Poisson boundary follows from Proposition 4.1.5.

- (a) If $\delta(P_{\mathcal{T}}) > 0$, then $\delta(\check{P}_{\mathcal{T}}) < 0$. Also $X_n \rightarrow X_{\infty} \in \partial^*\mathcal{T}$ and $\check{X}_n \rightarrow \omega$, almost surely. Then $(\partial^*\mathcal{T}, \mu_{\infty})$ is a μ -boundary, where, as usual μ is associated with $P_{\mathcal{T}}$ like in 1.4, and $(\{\omega\}, \delta_{\omega})$ is a $\check{\mu}$ -boundary.

Thus, we can apply the Strip Criterion 4.1.4, and choose the geodesic lines between $\mathbf{u} \in \partial^*\mathcal{T}$ and ω as the strips $\mathfrak{s}(\mathbf{u}, \omega)$. Measurability of the map $\mathbf{u} \mapsto \mathfrak{s}(\mathbf{u}, \omega)$ is obvious, and

$$\gamma\mathfrak{s}(\mathbf{u}, \omega) = \mathfrak{s}(\gamma\mathbf{u}, \omega),$$

for every $\gamma \in \Gamma$ and $\mathbf{u} \in \partial^*\mathcal{T}$. Therefore the strips $\mathfrak{s}(\mathbf{u}, \omega)$ satisfy the conditions required in Theorem 4.1.4, and the measure space $(\partial^*\mathcal{T}, \mu_{\infty})$ is the Poisson boundary of the random walk (X_n) with transition matrix $P_{\mathcal{T}}$ over \mathcal{T} .

- (b) If $\delta(P_{\mathcal{T}}) < 0$, we exchange the roles of $P_{\mathcal{T}}$ and $\check{P}_{\mathcal{T}}$ in the first case and we get the triviality of the Poisson boundary of (X_n) .

□

Lamplighter Random Walks on $\mathbb{Z}_2 \wr \mathcal{T}$. Consider lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$, with $Z_n = (\eta_n, X_n)$ and \mathcal{T} is the tree with the fixed end ω represented in Figure 5.4. Assume that (X_n) has finite first moment on \mathcal{T} . The convergence of (Z_n) is a simple consequence of Theorem 3.2.3.

Recall the Definition 3.1 of the geometric boundary Π of the lamplighter graph $\mathbb{Z}_2 \wr \mathbb{G}$. We replace in this section the graph \mathbb{G} with the tree \mathcal{T} with a fixed end ω , and the boundary $\partial\mathbb{G}$ with $\partial\mathcal{T} = \partial^*\mathcal{T} \cup \{\omega\}$. We can then rewrite Π as

$$\Pi = \Pi^* \cup \omega^*,$$

where Π^* is the set of all pairs (ζ, \mathbf{u}) , that is,

$$\Pi^* = \bigcup_{\mathbf{u} \in \partial^*\mathcal{T}} \mathcal{C}_{\mathbf{u}} \times \{\mathbf{u}\}, \quad (5.5)$$

and $\mathcal{C}_{\mathbf{u}}$ is the set of all configurations ζ which are either finitely supported, or infinitely supported with $\text{supp}(\zeta)$ accumulating only at \mathbf{u} . Also

$$\omega^* = \{(\zeta, \omega) : \zeta \in \mathcal{C}_{\omega}\}. \quad (5.6)$$

Theorem 5.3.7. *Let (Z_n) be an irreducible, homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathcal{T}$, where \mathcal{T} is an homogeneous tree and $\Gamma \subset \text{AFF}(\mathcal{T})$.*

5.3. LRW OVER GRAPHS WITH INFINITELY MANY ENDS

- (a) If $\delta(P_{\mathcal{T}}) > 0$, then there exists a Π^* -valued random variable Z_{∞} , such that $Z_n \rightarrow Z_{\infty}$ almost surely, in the topology of $\widehat{\mathbb{Z}_2 \wr \mathcal{T}}$.
- (b) If $\delta(P_{\mathcal{T}}) \leq 0$, then (Z_n) converges almost surely to some ω^* -valued random variable, in the topology of $\widehat{\mathbb{Z}_2 \wr \mathcal{T}}$.

The distribution of Z_{∞} on Π^* (on ω^* respectively) is a continuous measure.

Proof. (a) The result is an application of Theorem 3.2.3, which holds for general transitive base graphs G endowed with a rich boundary and such that Assumption 3.2.1 holds. From Theorem 5.3.6 it follows that (X_n) satisfies Assumption 3.2.1, with the boundary $\partial G := \partial^* \mathcal{T}$. Then Π^* is the boundary for the lamplighter random walk and (Z_n) converges to Π^* almost surely.

The limit distribution of X_n is a continuous measure on $\partial^* \mathcal{T}$, and this implies the continuity of the limit distribution of Z_n on Π^* .

(b) This is again an application of Theorem 3.2.3 when $\delta(P_{\mathcal{T}}) < 0$, with ω in the place of ∂G and ω^* defined in (5.6) in the place of Π . Assumption 3.2.1 holds once again for (X_n) and the requirements of Theorem 3.2.3 are fulfilled. This proves the desired.

For $\delta(P_{\mathcal{T}}) = 0$, also the drift of the base random walk is zero, and Theorem 3.2.3 can be easily adapted to prove the convergence of the LRW (Z_n) to a random variable in ω^* .

When the base random walk (X_n) converges to ω almost surely, then the limit distribution is the point mass δ_{ω} at ω . Nevertheless, using Borel-Cantelli lemma, one can prove that the limit distribution ν_{∞} of (Z_n) on ω^* is a continuous measure. \square

The Poisson Boundary. The main goal of this subsection is to describe the Poisson boundary of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$, where \mathcal{T} is a homogeneous tree with a fixed end ω , like in Figure 5.4. This will be done by making use of *Half Space Method* in the case when the base random walk (X_n) has non-zero modular drift.

The case $\delta(P_{\mathcal{T}}) = 0$ is slightly different and is also the most difficult one, and it will be treated separately in the following section. We emphasize that the proof is completely different from all previous ones and is based on the existence of cutpoints for random walks. I am very grateful to Vadim Kaimanovich for useful discussions on this problem, and for the main idea of Section 5.4.

The correspondence between the tail σ - algebra of a random walk and its Poisson boundary will be used. Nevertheless, we are able to solve this problem only when the base random walk (X_n) is of nearest neighbour type. The general case of random walks over $\mathbb{Z}_2 \wr \mathcal{T}$ such that the projected random walk (X_n) on \mathcal{T} has zero drift and bounded range (not only range 1 like in our approach) is awaiting future work.

Remark 5.3.8. ERSCHLER [Ers10] proved recently that the Poisson boundary of lamplighter random walks over Euclidean lattices \mathbb{Z}^d , with $d \geq 5$, such that the projection on \mathbb{Z}^d has zero drift, is isomorphic with the space of infinite limit configurations of lamps. She uses a modified version of the Ray Criterion 4.1.3.

Her methods does not apply in our case, when the underlying tree \mathcal{T} has a fixed end, and the base walk has zero drift.

For the Poisson boundary of lamplighter random walks over homogeneous trees with the action of a transitive group without any fixed ends, see the paper of KARLSSON AND WOESS [KW07]. They proved that the Poisson boundary is the space of infinite limit configurations accumulating at boundary points (ends) of the tree.

Recall that $P_{\mathcal{T}}$ is the transition matrix of the base random walk (X_n) on \mathcal{T} , and $\delta(P_{\mathcal{T}})$ its modular drift.

Theorem 5.3.9. *Let (Z_n) be an irreducible, homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathcal{T}$, and $\Gamma \subset \text{AFF}(\mathcal{T})$.*

- (a) *If $\delta(P_{\mathcal{T}}) > 0$, then (Π^*, ν_{∞}) is the Poisson boundary of (Z_n) , where Π^* is given as in (5.5) and ν_{∞} is the limit distribution of (Z_n) on Π^* .*
- (b) *If $\delta(P_{\mathcal{T}}) < 0$, then (ω^*, ν_{∞}) is the Poisson boundary of (Z_n) , where ω^* is given as in (5.6) and ν_{∞} is the limit distribution of (Z_n) on ω^* .*

The case $\delta(P_{\mathcal{T}}) = 0$ is excluded here, and considered separately in Section 5.4.

Note that this result gives both the Poisson boundary of (Z_n) (with transition matrix P) and of the reversed random walk (\check{Z}_n) (with transition matrix \check{P}). The finite first moment condition, the irreducibility and the homogeneity hold for both (Z_n) and (\check{Z}_n) , simultaneously.

Proof of Theorem 5.3.9. Apply *Half Space Method* 4.2 and Theorem 4.2.1.

- (a) Case $\delta(P_{\mathcal{T}}) > 0$: In order to apply Theorem 4.2.1, we check again that the conditions required in *Half Space Method* 4.2 are satisfied for the oriented tree \mathcal{T} and random walks (X_n) and (\check{X}_n) on it.

5.3. LRW OVER GRAPHS WITH INFINITELY MANY ENDS

From the finite first moment assumption and Theorem 5.3.6 it follows that Assumption 3.2.1 hold for (X_n) , (\check{X}_n) respectively. By Theorem 5.3.6, (X_n) converges to a random end in $\partial^*\mathcal{T}$ and (\check{X}_n) converges to the fixed end ω . Denote by μ_∞ the limit distribution of (X_n) on $\partial^*\mathcal{T}$. The limit distribution of (\check{X}_n) is the point mass δ_ω at ω .

Next, assign a strip $\mathfrak{s}(\mathbf{u}, \omega) \subset \mathcal{T}$ to almost every end $\mathbf{u} \in \partial^*\mathcal{T}$. Consider the strip $\mathfrak{s}(\mathbf{u}, \omega) \subset \mathcal{T}$

$$\mathfrak{s}(\mathbf{u}, \omega) = \pi(\mathbf{u}, \omega),$$

where $\pi(\mathbf{u}, \omega)$ is the unique two-sided infinite geodesic between \mathbf{u} and ω , which has linear growth and is equivariant with respect to the action of Γ on \mathcal{T} . Therefore it satisfies the conditions required in Strip Criterion 4.1.4.

The partition of \mathcal{T} in half-spaces is done like this: for every $x \in \mathfrak{s}(\mathbf{u}, \omega)$ let $\mathcal{T}_+(x)$ be the unique connected component which contains the end \mathbf{u} after the removal of x from \mathcal{T} , and $\mathcal{T}_-(x)$ be its complement in \mathcal{T} . Then

$$\mathcal{T}_+ = \mathcal{T}_+(x) \text{ and } \mathcal{T}_- = \mathcal{T} \setminus \mathcal{T}_+,$$

are Γ -equivariant sets. Thus, all requirements for the *Half Space Method* 4.2 hold and we can apply Theorem 4.2.1.

By Theorem 5.3.7 the random walk (Z_n) ((\check{Z}_n) , respectively) converges to a random element in Π^* (ω^* , respectively) with limit distributions ν_∞ ($\check{\nu}_\infty$, respectively). Then (Π^*, ν_∞) and $(\omega^*, \check{\nu}_\infty)$ are ν - and $\check{\nu}$ -boundaries of the respective random walks. Take

$$b_+ = (\phi_+, \mathbf{u}) \in \Pi^*, \text{ and } b_- = (\phi_-, \omega) \in \omega^*,$$

where ϕ_+ and ϕ_- are the limit configurations of (Z_n) and (\check{Z}_n) , respectively, and $\mathbf{u} \in \partial^*\mathcal{T}$, ω are their only respective accumulation points. Define the configuration $\Phi(b_+, b_-, x)$ by

$$\Phi(b_+, b_-, x) = \begin{cases} \phi_-, & \text{on } \mathcal{T}_+ \\ \phi_+, & \text{on } \mathcal{T}_- \end{cases}$$

and the strip $S(b_+, b_-)$ like in equation (4.4):

$$S(b_+, b_-) = \{(\Phi, x) : x \in \mathfrak{s}(\mathbf{u}, \omega)\}.$$

By Theorem 4.2.1, $S(b_+, b_-)$ satisfies the conditions from Theorem 4.1.4, and it follows that the space (Π^*, ν_∞) is the Poisson boundary of (Z_n) and $(\omega^*, \check{\nu}_\infty)$ is the Poisson boundary of (\check{Z}_n) over $\mathbb{Z}_2 \wr \mathcal{T}$.

(b) Case $\delta(P_{\mathcal{T}}) < 0$: The proof is like before with the roles of $P_{\mathcal{T}}$ (respectively P) and $\check{P}_{\mathcal{T}}$ (respectively \check{P}) exchanged. \square

5.4 Zero-Drift Random Walks on the Oriented Tree

In this section we describe the Poisson boundary of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$, where \mathcal{T} is the oriented tree of degree $q + 1 \geq 3$ given in Figure 5.4 (with $q = 2$ in the picture), such that the base random walk (X_n) on \mathcal{T} has modular drift $\delta(P_{\mathcal{T}}) = 0$.

Suppose that (X_n) is a nearest neighbour random walk on \mathcal{T} with transition probabilities $P_{\mathcal{T}} = (p_{\mathcal{T}}(x, y))$ given by

$$p_{\mathcal{T}}(x, y) = \begin{cases} \frac{1}{2}, & \text{if } y = x^- \\ \frac{1}{2q}, & \text{if } y = x_j, \end{cases} \quad (5.7)$$

where x_j is one of the q ‘‘sons’’ of x , and x^- is the father of x . Then the horocyclic projection $h(X_n)$ of X_n on the integer line \mathbb{Z} is the simple random walk on \mathbb{Z} , which is recurrent. Nevertheless, the random walk (X_n) on \mathcal{T} is transient and converges to ω by Theorem 5.3.6. Also, the random walk (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$ converges to $(\zeta_{\infty}, \omega) \in \omega^*$ by Theorem 5.3.7, where ζ_{∞} is the limit configuration (not necessary with finite support) of the LRW.

We shall prove that the Poisson boundary of LRW (Z_n) , with $Z_n = (\eta_n, X_n)$, such that the base random walk (X_n) has transition probabilities given by (5.7), is described by the space of limit configurations ζ_{∞} accumulating at ω . For doing this, the Strip and Ray Criterion (which use in some sense the entropy of the conditional random walk) are not suitable and another approach is needed.

The proof will be done in several steps, and the first one is to give to the oriented tree in Figure 5.4 another geometric interpretation, which will make things easier.

Let us consider the reference point $o \in \mathcal{T}$ and the one-sided infinite geodesic $\pi(o, \omega)$ joining o with the fixed end ω . Then one can interpret the tree \mathcal{T} in Figure 5.4 as the infinite geodesic $\pi(o, \omega)$, which is isomorphic with \mathbb{Z}_+ , with a tree \mathcal{T}_k attached at each point $k \in \mathbb{Z}_+ \cong \pi(o, \omega)$, so that k is the origin of \mathcal{T}_k . For a graphic visualization, see Figure 5.5.

Consider now the following stopping times (exit times)

$$\tau_0 = 0 \text{ and } \tau_{m+1} = \min\{n > \tau_m : X_n \in \mathcal{T} \setminus \mathcal{T}_{\tau_m}\}, \text{ for } m \geq 1. \quad (5.8)$$

If $X_{\tau_m} = k$, this means that for $n \in [\tau_m, \tau_{m+1}]$, the random walk (X_n) will move only in the tree \mathcal{T}_k , and $X_{\tau_{m+1}} \in \{k - 1, k + 1\}$. In other words, the exit time τ_{m+1} represents the first time when the random walk (X_n) leaves the attached tree \mathcal{T}_k , with $X_{\tau_m} = k$.

The random walk (X_n) restricted to the stopping times τ_m is again a nearest neighbour random walk (Markov chain) on the positive integer line \mathbb{Z}_+ . For

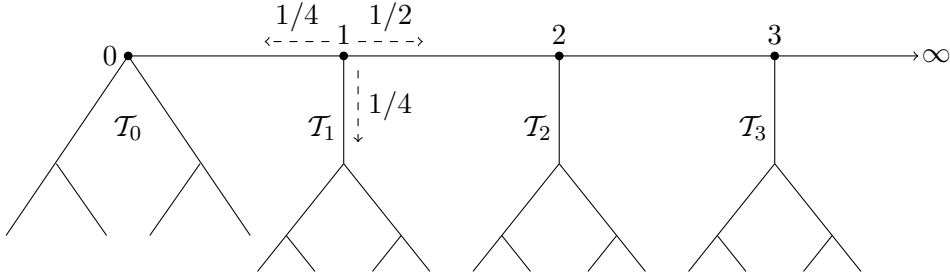


Figure 5.5: Another interpretation of the oriented tree

simplicity of notation, let us denote by $Y_n = X_{\tau_n}$ and by

$$Q = (q(i, j))_{i, j \in \mathbb{Z}_+}$$

the transition matrix of (Y_n) . Then the entries of Q are

$$q(i, j) = \mathbb{P}[Y_{n+1} = i | Y_n = j] = \mathbb{P}[X_{\tau_{n+1}} = i | X_{\tau_n} = j], \text{ for } i, j \in \mathbb{Z}_+.$$

We have $Y_0 = 0$ and $Y_1 = 1$, and the transition probabilities of (Y_n) can be easily computed as

$$q(i, i+1) = \frac{q}{q+1}, \text{ and } q(i, i-1) = \frac{1}{q+1}, \text{ for } i \geq 1 \quad (5.9)$$

where $q+1$ is the degree of \mathcal{T} , with $q \geq 2$, and

$$q(0, 1) = 1.$$

All the other entries $q(i, j)$ with $j \notin \{i-1, i+1\}$ are zero, since (X_n) is a nearest neighbour random walk. This random walk is transient and almost every path of (Y_n) converges almost surely to ∞ . This gives also another explanation of the transience of the random walk (X_n) on \mathcal{T} and of the convergence to the fixed end ω .

Next, we introduce some definitions and facts, which can be found in JAMES AND PERES [JP97].

Cut Points. Let (X_j) be a Markov chain on a countable state space \mathbf{G} . A *cut time* for the Markov chain (X_j) is an integer n with

$$X_{[0, n]} \cap X_{[n+1, \infty)} = \emptyset,$$

where $X_{[0, n]} = \{X_j : 0 \leq j \leq n\}$, and in this case the random variable X_n is called a *cut point*. A cutpoint is a point in the state space \mathbf{G} which is visited

exactly once by the random walk. If X_n is a cut point, then deleting X_n from the trajectory cuts the trajectory into two disjoint components, $X_{[0,n]}$ and $X_{[n+1,\infty)}$, with no possible transition from the first to the second.

The following result will be needed. For details, see JAMES AND PERES [JP97, Theorem 1.2 (a)].

Theorem 5.4.1. *Any transient random walk with bounded increments on the lattice \mathbb{Z}^d has infinitely many cut points with probability 1.*

For the random walk (Y_n) on \mathbb{Z}_+ with transition probabilities $q(i, j)$ given in (5.9), consider the random time

$$\xi_n = \inf\{j : |Y_j| \geq n\}.$$

Then one can show that with probability one ξ_n is a cut time for infinitely many n . Thus, the random walk (Y_n) on \mathbb{Z}_+ has infinitely many cut times, and implicitly also infinitely many cut points.

For the random walk (Y_n) , let I_n be the indicator function of the event $\{n \text{ is a cut time}\}$ and let

$$R_n = \sum_{j=0}^n I_j$$

be the number of cut times encountered up to time n . Then R_n is a stationary process and *Birkhoff Ergodic Theorem* implies that there exists a constant $p > 0$, such that

$$\lim_{n \rightarrow \infty} \frac{R_n}{n} = p > 0.$$

The constant p can be explicitly computed as $p = \mathbb{P}[Y_n \geq 1, \forall n \geq 1]$. The number $p > 0$ is called the *density of the cut points* for the random walk (Y_n) on \mathbb{Z}_+ .

Remark 5.4.2. The set of cut points of (Y_n) has strictly positive density p and this implies that for any integer $N > 1/p$, one can choose a set of trajectories, such that, in any collection of N trajectories, at least 2 have infinitely many common cut points.

The Poisson Boundary. Assume that the lamplighter random walk (Z_n) is of *nearest neighbour type* on $\mathbb{Z}_2 \wr \mathcal{T}$, that is, the following *local condition* is satisfied:

$$p((\eta, x)(\eta', x')) > 0, \tag{5.10}$$

only if x and x' are neighbours in \mathcal{T} and the configurations η, η' may only differ at the point x .

5.4. ZERO-DRIFT RANDOM WALKS ON THE ORIENTED TREE

Let us now state the main result on the description of the Poisson boundary of LRW (Z_n) , with $Z_n = (\eta_n, X_n)$, when the base random walk (X_n) on the tree \mathcal{T} with a fixed end ω has zero modular drift $\delta(P_{\mathcal{T}})$.

Theorem 5.4.3. *Let (Z_n) be an irreducible, nearest neighbour random walk on $\mathbb{Z}_2 \wr \mathcal{T}$, such that the base walk (X_n) on \mathcal{T} is of nearest neighbour type with $\delta(P_{\mathcal{T}}) = 0$. Then the Poisson boundary of (Z_n) is given by the space of limit configurations endowed with the corresponding hitting distribution ν_{∞} .*

The first step in the proof of this theorem is to construct another “lamplighter type” random walk (\tilde{Z}_n) - on a modified state space - with the same Poisson boundary as (Z_n) . Then, Theorem 5.4.3 is just a consequence of three other results which we will state and prove in what follows.

Extended “Lamplighter Type” Random Walk. Using the stopping times τ_m defined in (5.8), we construct another “lamplighter type” random walk (\tilde{Z}_n) , similar to (Z_n) .

For all $k \in \mathbb{Z}_+$, denote by \mathcal{C}_k the set of all \mathbb{Z}_2 -valued lamp configurations on the tree \mathcal{T}_k , rooted at k . Let also $\tilde{\mathcal{C}}$ be the space of all *finitely supported* “generalized” configurations Φ over \mathbb{Z}_+ , where Φ is given by

$$\begin{aligned} \Phi : \mathbb{Z}_+ &\rightarrow \mathcal{C}_k \\ k &\mapsto \Phi(k) \in \mathcal{C}_k. \end{aligned}$$

This means that the value $\Phi(k)$ of the configuration Φ at a point $k \in \mathbb{Z}_+$ is a configuration on the whole tree \mathcal{T}_k rooted at k , i.e., a configuration in \mathcal{C}_k .

Let the “new” state space $\tilde{\mathbb{Z}}_+$ be the product of the space $\tilde{\mathcal{C}}$ of “generalized” configurations Φ with \mathbb{Z}_+ , that is,

$$\tilde{\mathbb{Z}}_+ = \tilde{\mathcal{C}} \times \mathbb{Z}_+.$$

Consider now the random walk (\tilde{Z}_n) on $\tilde{\mathbb{Z}}_+$, with $\tilde{Z}_n = (\Phi_n, Y_n)$, such that Φ_n is a generalized configuration over \mathbb{Z}_+ , and the projection of (\tilde{Z}_n) on \mathbb{Z}_+ is the random walk (Y_n) on \mathbb{Z}_+ , with transition probabilities given as in (5.9). This description of the chain (\tilde{Z}_n) follows from the fact that in between of two consecutive stopping times τ_m, τ_{m+1} , the random walk (X_n) moves only in the tree \mathcal{T}_k , with

$$Y_m = X_{\tau_m} = k \quad \text{and} \quad \Phi_m = \eta_{\tau_m}.$$

Moreover, because of the local condition we have assumed in (5.10), the lamplighter configuration may change only in points of \mathcal{T}_k .

The walk (\tilde{Z}_n) , with $\tilde{Z}_n = (\Phi_n, Y_n)$, can be viewed as an “extended random walk” of (Y_n) . In other words, we add to the states of the chain (Y_n) the lamp

configuration Φ_n . The lamp configurations Φ_n are similar to the *occupation numbers* defined in JAMES AND PERES [JP97].

We denote by \tilde{Q} the transition matrix of (\tilde{Z}_n) . Its entries are of the form

$$\tilde{q}((\phi_1, k_1), (\phi_2, k_2)).$$

For $k_2 \notin \{k_1 - 1, k_1 + 1\}$, the entries are zero. Let n_1 and n_2 be such that

$$k_1 = Y_{n_1} = X_{\tau_{n_1}} \text{ and } k_2 = Y_{n_2} = X_{\tau_{n_2}}.$$

Then

$$\phi_1 = \eta_{n_1} \text{ and } \phi_2 = \eta_{n_2},$$

and the transition probabilities of (\tilde{Z}_n) are given by

$$\tilde{q}((\phi_1, y_1), (\phi_2, y_2)) = p((\eta_{n_1}, X_{\tau_{n_1}}), (\eta_{n_2}, X_{\tau_{n_2}})).$$

Remark 5.4.4. With this construction in hand, the identification problem for the Poisson boundary of (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$ reduces to the identification problem for the Poisson boundary of the new walk (\tilde{Z}_n) on $\tilde{Z}_+ = \tilde{\mathcal{C}} \times \mathbb{Z}_+$.

Recall that (Y_n) is a random walk with drift to the right on \mathbb{Z}_+ , and from Theorem 5.4.1, it has infinitely many cutpoints, with probability 1.

Another Description of the Poisson Boundary. For the random walk (X_n) on the state space \mathbb{G} , let like before $\Omega = \mathbb{G}^{\mathbb{Z}_+}$ be its path space. The shift $T : \Omega \rightarrow \Omega$ in the path space is defined as

$$T(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots).$$

Recall that the Poisson boundary is the space of ergodic components of the time shift in the path space $\mathbb{G}^{\mathbb{Z}_+}$, endowed with the probability measure \mathbb{P} . In a more detailed way, denote by \sim the *orbit equivalence relation* of the shift T in the path space:

$$(\mathbf{x}) \sim (\mathbf{x}') \iff \exists n, n' \geq 0 : T_n \mathbf{x} = T_{n'} \mathbf{x}', \text{ for } \mathbf{x}, \mathbf{x}' \in \Omega,$$

and by \mathcal{A}_T the σ -algebra of all measurable unions of \sim -classes (mod 0) in the space $(\mathbb{G}^{\mathbb{Z}_+}, \mathbb{P})$.

Then the Poisson boundary of (X_n) is the quotient of the path space $(\mathbb{G}^{\mathbb{Z}_+}, \mathbb{P})$ with respect to the equivalence relation \sim .

Tail σ -Algebra. The *tail boundary* is analogous to the definition of the Poisson boundary, with the equivalence relation \sim being now replaced with the *tail equivalence relation* \approx , which is defined as

$$(\mathbf{x}) \approx (\mathbf{x}') \Leftrightarrow \exists N \geq 0 : T_n \mathbf{x} = T_n \mathbf{x}', \text{ for } \mathbf{x}, \mathbf{x}' \in \Omega, \forall n \geq N.$$

An important difference is that unlike the σ -algebra \mathcal{A}_T from the definition of the Poisson boundary, the *tail σ -algebra* \mathcal{A}^∞ of all measurable unions of \approx -classes can be presented as the limit of the decreasing sequence of σ -algebras \mathcal{A}_n^∞ determined by the positions of sample paths of the random walk (X_n) at times $\geq n$. In other words,

$$\mathcal{A}^\infty = \bigcap_n \mathcal{A}_n^\infty,$$

where \mathcal{A}_n^∞ are the coordinate σ -algebras generated by the random variables $(X_k)_{k \geq n}$.

The Poisson and the tail boundaries are sometimes confused, and indeed they do coincide for “most common” random walks. Their coincidence for random walks on groups is a key ingredient of the entropy theory of random walks. See KAIMANOVICH [Kai92] for more details and for criteria of trivality of these boundaries and of their coincidence provided by 0–2 laws.

Consider the extended random walk $\tilde{Z}_n = (\Phi_n, Y_n)$ defined above on the state space \tilde{Z}_+ . Then the *tail σ -algebra* of (\tilde{Z}_n) can be described as the σ -algebra generated by the *tail equivalence relation* \approx :

$$((\phi_n, y_n)) \approx ((\phi'_n, y'_n)) \Leftrightarrow \exists N \geq 0 : y_n = y'_n, \phi_n = \phi'_n, \forall n \geq N, \quad (5.11)$$

where (ϕ_n, y_n) and (ϕ'_n, y'_n) are two trajectories of the random walk (\tilde{Z}_n) . The *tail boundary* of (\tilde{Z}_n) is the quotient of the path space with respect to the tail equivalence relation \approx .

Recall that (Y_n) is a transient random walk on \mathbb{Z}_+ , with transition probabilities given in (5.9), and (\tilde{Z}_n) is the extended “lamplighter type” random walk on \tilde{Z}_+ , which is also transient.

If $\Phi_n(k)$ is the random configuration of lamps at time n on the tree \mathcal{T}_k rooted at $k \in \mathbb{Z}_+$, let

$$\Phi_\infty(k) = \lim_{n \rightarrow \infty} \Phi_n(k), \text{ for every } k \in \mathbb{Z}_+.$$

The existence of the limit follows from the fact that a transient process visits every state infinitely often.

Lemma 5.4.5. *For $n \geq 1$ consider the event $C_n = \{n \text{ is a cut time}\}$. For $n \geq 1$ this event is in the σ -algebra $\sigma(\Phi_\infty, Y_0)$, and for trajectories in C_n the value of $\tilde{Z}_n = (\Phi_n, Y_n)$ can be approximated in terms of the limit configuration Φ_∞ only.*

Proof. Let (ϕ_n, y_n) be a trajectory of the walk $\tilde{Z}_n = (\Phi_n, Y_n)$ and ϕ_∞ the limit configuration of lamps for this trajectory. If $t \in \mathbb{Z}_+$ is a cut point of the trajectory (y_n) attained at time n (so that $y_n = t$), then ϕ_n is the restriction $\phi_\infty([0, t])$ of the limit configuration ϕ_∞ to the random subset $\{0, 1, 2, \dots, t-1\} \subset \mathbb{Z}_+$, because the random cut point t will be visited only once by (y_n) . That is, on the trees $\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_{t-1}$ the configuration will not change anymore after time n and on the tree \mathcal{T}_t the configuration will not be touched at all during the entire process. Therefore $\phi_n = \phi_\infty([0, t])$ and

$$\phi_\infty = \phi_\infty([0, t]) \oplus \phi_0(t) \oplus \phi'([t+1, \infty)), \quad (5.12)$$

where

$$\phi_\infty([0, t]) = \phi_\infty(0) \oplus \phi_\infty(1) \oplus \dots \oplus \phi_\infty(t-1), \text{ for } t \in \mathbb{Z}_+,$$

$\phi_0(t)$ is the initial configuration (all lamps are off) on the tree \mathcal{T}_t rooted at t , and $\phi'([t+1, \infty))$ is the configuration of lamps on the set $\{t+1, t+2, \dots\}$, which can still change at times $\{n+1, n+2, \dots\}$.

This proves that, under the condition that t is a cut point for the trajectory (y_n) , the lamp configuration is finalized on the random interval $\{0, 1, 2, \dots, t-1\}$. Moreover, its value is determined by the limit configuration ϕ_∞ , and a decomposition of the form (5.12) is given.

Let (ϕ'_n, y'_n) be another trajectory of the “lamplighter type” random walk (\tilde{Z}_n) such that the limit configurations ϕ_∞ and ϕ'_∞ coincide, and such that (y_n) and (y'_n) have infinitely many common cut points. The existence of such paths follows from Remark 5.4.2. If $t \in \mathbb{Z}_+$ is such a common cut point (which occurs at different cut time n' with $y'_{n'} = t$), then at time n' the lamp configuration $\phi'_{n'}$ is finalized and is again given by the restriction of the limit configuration ϕ_∞ on the random interval $\{0, 1, 2, \dots, t-1\}$. Moreover $\phi'_{n'} = \phi_n = \phi_\infty([0, t])$ and for ϕ_∞ a decomposition of the form (5.12) is given. This holds for all common cut points.

Thus C_n may be partitioned into subevents $C_n \cap \{\Phi_n = \Phi_\infty([0, t]), Y_n = t\}$ (with t a cut point of (Y_n)) which are in the σ -algebra generated by Φ_∞ and Y_0 , and the lemma is proved. \square

Proposition 5.4.6. *Let (\tilde{Z}_n) , with $Z_n = (\Phi_n, Y_n)$, be the extended “lamplighter type” random walk on \tilde{Z}_+ . Then the tail σ -algebra of (\tilde{Z}_n) is generated by the limit configurations of lamps Φ_∞ .*

Proof. We first describe the tail σ -algebra of the process (Φ_n, Y_n) conditioned on the limit configuration Φ_∞ and on $(\Phi_0, Y_0) = (\mathbf{0}, o)$. By Theorem 5.4.1, the process (Y_n) has infinitely many cut points, and implicitly also an infinite sequence $n_1 < n_2 < \dots$ of cut times. From Lemma 5.4.5, at

these cut times the values of Y_{n_1}, Y_{n_2}, \dots are determined, and the limit configuration is finalized, that is, it will not change anymore from now on. Let t_k, t_{k+1} be the respective cut points for two consecutive cut times n_k, n_{k+1} , i.e., $Y_{n_k} = t_k$.

Between times $\{n_k, n_k + 1, n_k + 2, \dots, n_{k+1} - 1\}$ the configuration of lamps can only be modified on the set $\{t_k, t_k + 1, \dots, t_{k+1} - 1\}$, that is, only on the trees $\mathcal{T}_{t_k}, \mathcal{T}_{t_k+1}, \dots, \mathcal{T}_{t_{k+1}-1}$. For different k , these sequences of modified configurations are mutually independent (conditionally on Φ_∞), and hence the tail σ -algebra of the process $(\Phi_n, Y_n)_n$, conditioned on Φ_∞ is trivial by Kolmogorov 0 – 1 law. Conditional triviality of a σ -algebra given Φ_∞ means that the σ -algebra is generated by Φ_∞ up to completion. That is, the tail σ -algebra of $(\Phi_n, Y_n)_n$ is generated by the limit configurations of lamps Φ_∞ . \square

Remark 5.4.7. Note that if for two sample trajectories (ϕ_n, y_n) and (ϕ'_n, y'_n) , the respective limit configurations coincide, and the paths (y_n) and (y'_n) have infinitely many common cut points, then they correspond to the same point of the Poisson boundary of (\tilde{Z}_n) .

Proposition 5.4.8. *For the extended “lamplighter type” random walk (\tilde{Z}_n) on the state space \tilde{Z}_+ , with $\tilde{Z}_n = (\Phi_n, Y_n)$, its tail boundary coincides with its Poisson boundary, up to sets of measure zero, for any starting point (Φ_0, Y_0) .*

Proof. Remark 5.4.2 (the density of the cut times) implies that the Poisson boundary of (\tilde{Z}_n) covers the space of limit configurations with finite fibers, that is, the tail boundary (the tail σ -algebra), which is, in view of Proposition 5.4.6, generated by the space of limit configurations Φ_∞ . This is possible, using the theory of covering Markov operators in KAIMANOVICH [Kai95] (see Theorem 4.3.3 and Example in 4.4.4), only when the cover is trivial. Thus, the Poisson boundary coincides with the tail boundary of (\tilde{Z}_n) . \square

Remark 5.4.9. The arguments in the previous two proofs are similar to the description of the exchangeable sigma-algebra of bounded range random walks on transient groups due to JAMES AND PERES [JP97]. In both cases the argument is based on the existence of cut points at which a certain part of the limit configuration is finalized.

In James and Peres [JP97], coincidence of the limit configurations for two trajectories implies that they have the same sequence of cut points and cut times. This is not the case in our setup, because of which we have to use a special argument dealing with different sequences of cut times associated to the same limit configuration.

Proof of Theorem 5.4.3. From the construction of the “lamplighter-type” random walk (\tilde{Z}_n) , it follows that the Poisson boundary of (Z_n) is reduced to the identification of the Poisson boundary for (\tilde{Z}_n) .

From Proposition 5.4.8 it follows that the Poisson boundary of (\tilde{Z}_n) coincides with its tail σ -algebra, which is by Proposition 5.4.6 generated by the space of limit configurations Φ_∞ . Consequently also the Poisson boundary of (Z_n) is isomorphic with the space of limit configurations with the respective hitting distribution ν_∞ . \square

The proof of the Theorem 5.4.3 is done only when the base random walk (X_n) is of nearest neighbour type and its associated LRW (Z_n) satisfies the local condition (5.10). A next step is to generalize this results to bounded range random walks (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$.

Chapter 6

Hyperbolic Graphs

In this chapter we study once more the behaviour at infinity of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr G$, when the transitive base graph G is a hyperbolic graph in the sense of Gromov. The method developed in Section 4.2 is again applied here. Hyperbolicity of G and its geometric properties will be of important use.

We first lay out the basic definitions and properties of the hyperbolic graphs and groups, and then briefly reconsider random walks (X_n) on such structures. Using the results known for random walks (X_n) on hyperbolic graphs G , we prove similar results for lamplighter random walks (Z_n) , with $Z_n = (\eta_n, X_n)$ on $\mathbb{Z}_2 \wr G$.

6.1 Preliminaries

We recall here the most important definitions of hyperbolic graphs and their hyperbolic boundary and compactification. There is a vast literature on hyperbolic spaces, in particular on *hyperbolic groups* (i.e., groups which have a hyperbolic Cayley graph). For details, the reader is invited to consult the texts by GROMOV [Gro], GHYS AND DE LA HARPE [GDLH90], COORNAERT, DELZANT AND PAPADOPOULUS [CDP90], or, for a presentation in the context of random walks on graphs, WOESS [Woe00, Section 22].

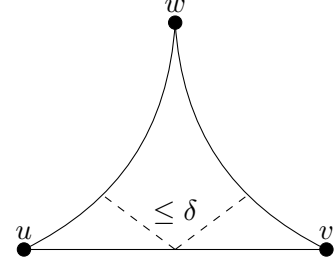
Let (G, d) be a *proper metric space*, that is, a space in which every closed ball $B(x, r) = \{y \in G : d(x, y) \leq r\}$ is compact. For $x, y \in G$, a *geodesic arc* $\pi(x, y)$ in G is the image of an isometric embedding of the real interval $[0, d(x, y)]$ into G which sends 0 to x and $d(x, y)$ to y . The geodesic arc may not be unique.

Suppose that G is also *geodesic*: for every pair of points $x, y \in G$, there is a

geodesic arc $\pi(x, y)$ in \mathbf{G} . A *geodesic triangle* consists of three points u, v, w together with the geodesic arcs $\pi(u, v), \pi(v, w), \pi(w, u)$, which are called the *sides* of the triangle.

Definition 6.1.1. A *geodesic triangle* is called δ -thin, with $\delta \geq 0$, if every point on any one of the sides is at distance at most δ from some point on one of the other two sides.

Definition 6.1.2. One says that the space \mathbf{G} is hyperbolic, if there is $\delta \geq 0$ such that every geodesic triangle in \mathbf{G} is δ -thin.



The most typical examples of a hyperbolic spaces are trees (where $\delta = 0$), the hyperbolic upper half-plane $\mathbb{H} = \mathbb{H}_2$ (where $\delta = \log(1 + \sqrt{2})$), and the Poincaré unit disc.

Gromov Metric. Let us choose a reference point o in \mathbf{G} and define, for $x, y \in \mathbf{G}$, the *Gromov inner product*

$$|x \wedge y| = \frac{1}{2} [|x| + |y| - d(x, y)],$$

where $|x| = d(o, x)$. If \mathbf{G} is a tree (which is 0-hyperbolic), then this is the usual graph distance between o and the geodesic $\pi(x, y)$. A good way to think of this is as follows. Using the thin triangles property of hyperbolic metric spaces, we see that two walkers moving from o to x and y respectively along suitable geodesics will remain close together (less than 2δ apart) for a certain distance, before beginning to diverge rapidly. The Gromov inner product measures approximately the length of time that the two walkers remain close together.

Hyperbolic Boundary and Compactification. Assume that \mathbf{G} is a locally finite, transitive δ -hyperbolic graph, with the natural metric (discrete graph metric) d on it. In order to describe the *hyperbolic boundary* $\partial_h \mathbf{G}$ and the *hyperbolic compactification* $\widehat{\mathbf{G}}$, we define a new metric on \mathbf{G} which can be extended on the “boundary”.

Choose a constant $a > 0$, such that $(e^{3\delta a} - 1) < \sqrt{2} - 1$, and define for $x, y \in \mathbf{G}$ and the fixed vertex o

$$\varrho_a(x, y) = \begin{cases} 0, & x = y \\ \exp(-a|x \wedge y|), & x \neq y. \end{cases}$$

This is not a metric unless G is a tree. We now define

$$\theta_a(x, y) = \inf \left\{ \sum_{i=1}^n \varrho_a(x_{i-1}, x_i) : n \geq 1, x = x_0, x_1, \dots, x_n = y \in G \right\}.$$

Then θ_a is a metric on G , and the graph G is discrete in this metric. We define \widehat{G} as the completion of G in the metric θ_a . For the hyperbolic graph G , the space \widehat{G} is compact. The space \widehat{G} is called the *hyperbolic compactification* of the graph G . Each isometry from $\text{AUT}(G)$ extends to a homeomorphism of \widehat{G} . A sequence (x_n) with $|x_n| \rightarrow \infty$ is Cauchy, if and only if

$$\lim_{m, n \rightarrow \infty} |x_m \wedge x_n| = \infty, \quad (6.1)$$

and another Cauchy sequence (y_n) will define the same boundary point, if and only if

$$\lim_{n \rightarrow \infty} |x_n \wedge y_n| = \infty. \quad (6.2)$$

Thus, one can also construct the *hyperbolic boundary* $\partial_h G$ by factoring the set of all sequences in $G^{\mathbb{Z}_+}$, which satisfy (6.1), with respect to the equivalence relation given by (6.2). The topology of \widehat{G} does not depend on the choice of a , and it is also independent of the choice of the base point o .

Similarly to trees, a third, equivalent way is to describe the hyperbolic boundary $\partial_h G$ via equivalence of geodesic rays. Two rays $\pi = [x_0, x_1, \dots]$ and $\pi' = [y_0, y_1, \dots]$ are equivalent if

$$\liminf_{n \rightarrow \infty} d(y_n, x_n) < \infty.$$

Hyperbolic and End Compactification. For a hyperbolic graph G , it is easy to understand how its hyperbolic boundary $\partial_h G$ is related to the space of ends ∂G . The hyperbolic boundary is finer, that is, the identity on G extends to a continuous surjection from the hyperbolic to the end compactification which maps $\partial_h G$ onto ∂G . For trees, the two compactifications are the same.

The hyperbolic compactification of a hyperbolic graph is a contractive compactification with respect to $\text{AUT}(G)$. For a proof, see WOESS [Woe00, Theorem 22.14]. Moreover, we can also prove even the weaker property of the hyperbolic boundary, namely the weak projectivity.

Recall the Definition 3.1.3 of the weak projectivity.

Lemma 6.1.3. *The hyperbolic boundary $\partial_h G$ of a hyperbolic graph G is a weakly projective space.*

Proof. Let $(x_n), (y_n)$ be sequences of vertices in G such that (x_n) converges to a hyperbolic boundary point $u \in \partial_h G$ and

$$\frac{d(x_n, y_n)}{|x_n|} \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (6.3)$$

In order to prove that (y_n) converges to the same boundary point \mathbf{u} , we show that equation (6.2) holds. Using the Gromov inner product

$$|x_n \wedge y_n| = \frac{1}{2} [|x_n| + |y_n| - d(x_n, y_n)],$$

we obtain

$$|x_n \wedge y_n| = \frac{1}{2} |x_n| \left[1 + \frac{|y_n|}{|x_n|} - \frac{d(x_n, y_n)}{|x_n|} \right].$$

Now equation (6.3) and $|x_n| \rightarrow \infty$ implies that $|x_n \wedge y_n| \rightarrow \infty$ as $n \rightarrow \infty$. Therefore the sequence (y_n) converges to the same boundary point $\mathbf{u} \in \partial_h \mathbf{G}$. \square

We shall also use the fact that for every two distinct hyperbolic boundary points $\mathbf{u}, \mathbf{v} \in \partial_h \mathbf{G}$, there is an infinite geodesic $\pi(\mathbf{u}, \mathbf{v})$ between them, which may not be unique. For the proof see again WOESS [Woe00].

The boundary $\partial_h \mathbf{G}$ of an infinite transitive hyperbolic graph \mathbf{G} is either infinite or has cardinality 2. In the latter case, it is a graph with two ends that is quasi-isometric with the two-way infinite path \mathbb{Z} , and the Poisson boundary of any homogeneous random walk with finite first moment is trivial. In the sequel, we consider hyperbolic graphs \mathbf{G} , which have infinite boundary.

6.2 LRW over Hyperbolic Graphs

Let \mathbf{G} be a transitive hyperbolic graph with infinite hyperbolic boundary $\partial_h \mathbf{G}$ and $\Gamma \subset \text{AUT}(\mathbf{G})$. Like in the previous sections, (Z_n) is a homogeneous lamplighter random walk with transition matrix P on $\mathbb{Z}_2 \wr \mathbf{G}$, such that the projection of (Z_n) on \mathbf{G} is the random walk (X_n) with transition matrix $P_{\mathbf{G}}$ on \mathbf{G} .

Recall that the boundary $\partial(\mathbb{Z}_2 \wr \mathbf{G})$ of $\mathbb{Z}_2 \wr \mathbf{G}$ is given by

$$\partial(\mathbb{Z}_2 \wr \mathbf{G}) = (\widehat{\mathcal{C}} \times \widehat{\mathbf{G}}) \setminus (\mathcal{C} \times \mathbf{G})$$

and the dense subset Π of it defined in (3.1) is now replaced by

$$\Pi_h = \bigcup_{\mathbf{u} \in \partial_h \mathbf{G}} \mathcal{C}_{\mathbf{u}} \times \{\mathbf{u}\}, \tag{6.4}$$

where the set $\mathcal{C}_{\mathbf{u}}$ consists of all configurations ζ which are either finitely supported, or infinitely supported with $\text{supp}(\zeta)$ accumulating only at $\mathbf{u} \in \partial_h \mathbf{G}$.

Taking into account the connection between the hyperbolic boundary $\partial_h \mathbf{G}$ and the space of ends $\partial \mathbf{G}$ of a transitive graph \mathbf{G} , we have to distinguish two different cases:

- (a) *Infinite hyperbolic boundary and infinitely many ends.*
- (b) *Infinite hyperbolic boundary and only one end.*

6.2.1 Infinite Hyperbolic Boundary and Infinitely Many Ends

From the fact that the identity on G extends to a continuous surjection from the hyperbolic to the end compactification, which maps $\partial_h G$ onto ∂G , it follows that we are in the case of Chapter 5, i.e., of a graph with infinitely many ends. The ends are the connected components of the hyperbolic boundary.

The convergence of (Z_n) is given by Theorem 5.3.2 and Theorem 5.3.7 respectively (with the hyperbolic boundary $\partial_h G$ instead of the space of ends ∂G), depending on whether an element of $\partial_h G$ is fixed under the action of Γ or not.

The Poisson boundary of the lamplighter random walk (Z_n) is given by:

- Theorem 5.3.4, with Π_h instead of Π , in the nondegenerate case when no hyperbolic element of $\partial_h G$ is fixed by Γ .
- In the “degenerate case”, when Γ fixes one point in $\partial_h G$ (which has to be unique, given that $\partial_h G$ is assumed to be infinite), then G “looks” like in Figure 5.4, since the ends are the connected components of the hyperbolic boundary by PAVONE [Pav89]. The convergence of *LRW* is given by Theorem 5.3.7 and the Poisson boundary is described in Theorem 5.3.9 for the non-zero modular drift case, and for zero modular drift in Theorem 5.4.3.

6.2.2 Infinite Hyperbolic Boundary and One End

The problem of the existence of an one-ended hyperbolic graph G with a transitive group Γ that fixes a boundary point in $\partial_h G$ is still unsolved. Experts believe that the answer is negative. See the remarks at the end of the paper KAIMANOVICH AND WOESS [KW02]. Since this situation is only hypothetical, we shall consider in this section the case when the graph G has infinite hyperbolic boundary $\partial_h G$ and only one end, and no element of $\partial_h G$ is fixed under the action of Γ .

For homogeneous random walks (X_n) on G with $|\partial_h G| = \infty$ the following holds. See WOESS [Woe93] for the proof.

Theorem 6.2.1. *If \mathbf{G} is a hyperbolic graph and $\Gamma \subset \text{AUT}(\mathbf{G})$ does not fix any element of $\partial_h \mathbf{G}$, then the random walk (X_n) converges almost surely in the hyperbolic topology to a random point $X_\infty \in \partial_h \mathbf{G}$. If μ_∞ is the limit distribution of (X_n) , then*

- (a) *The support of μ_∞ is the whole $\partial_h \mathbf{G}$.*
- (b) *The measure μ_∞ is continuous on $\partial \mathbf{G}$, that is $\mu_\infty(\{u\}) = 0$, $u \in \partial_h \mathbf{G}$.*

The convergence to the boundary of homogeneous random walks (X_n) on hyperbolic graphs \mathbf{G} holds without any need of the first moment assumption. In order to have similar convergence results for lamplighter random walks (Z_n) , we need this assumption on the base random walk (X_n) . From now on, we suppose that (X_n) has finite first moment on the transitive base graph \mathbf{G} . Recall also that the hyperbolic boundary $\partial_h \mathbf{G}$ is weakly projective, by Lemma 6.1.3.

Convergence of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathbf{G}$, with \mathbf{G} a hyperbolic graph, follows easily from Theorem 3.2.3, since Assumption 3.2.1 is satisfied. Recall the definition (6.4) of the boundary Π_h of the graph $\mathbb{Z}_2 \wr \mathbf{G}$.

Theorem 6.2.2. *Let (Z_n) be an irreducible, homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathbf{G}$, where \mathbf{G} is a hyperbolic graph and $\Gamma \subset \text{AUT}(\mathbf{G})$ does not fix any element of $\partial_h \mathbf{G}$. Then there exists a Π_h -valued random variable Z_∞ such that $Z_n \rightarrow Z_\infty$ almost surely, in the topology of $\widehat{\mathbb{Z}_2 \wr \mathbf{G}}$, for every starting point. The distribution of Z_∞ is a continuous measure on Π_h .*

Proof. The proof of this result follows basically the proof of Theorem 3.2.3 and Theorem 5.3.2, with the hyperbolic boundary $\partial_h \mathbf{G}$ instead of the space of ends $\partial \mathbf{G}$. □

Poisson Boundary

Let us first recall the description of the Poisson boundary of random walks (X_n) over transitive hyperbolic graphs \mathbf{G} , which will be used for the Poisson boundary of lamplighter random walks (Z_n) . For sake of completeness, we also give here the idea of the proof. For a detailed proof, see also KAIMANOVICH [Kai00].

Theorem 6.2.3. *Let \mathbf{G} be a hyperbolic graph with $|\partial_h \mathbf{G}| = \infty$, and $\Gamma \subset \text{AUT}(\mathbf{G})$ does not fix any element of $\partial_h \mathbf{G}$. If (X_n) is an homogeneous random walk with finite first moment on \mathbf{G} , then its Poisson boundary is $(\partial_h \mathbf{G}, \mu_\infty)$.*

Proof. The proof is very similar with the proof of Theorem 5.3.3. By Theorem 6.2.1 and Definition 4.1.2, the space $(\partial_h \mathbf{G}, \mu_\infty)$ is a μ -boundary for

the random walk (X_n) . Theorem 6.2.1 applies also to the reversed random walk (\check{X}_n) , with the limit distribution $\check{\mu}_\infty$, and $(\partial\mathbf{G}, \check{\mu}_\infty)$ is a μ -boundary for (\check{X}_n) . Recall that μ is the probability measure on Γ which is uniquely induced by the transition matrix P_G of (X_n) as in (1.4).

Apply the Strip Criterion 4.1.4 and define the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$, for $\mathbf{u}, \mathbf{v} \in \partial_h \mathbf{G}$. By continuity of μ_∞ and $\check{\mu}_\infty$, we have

$$\mu_\infty \times \check{\mu}_\infty(\{\mathbf{u}, \mathbf{v} \in \partial_h \mathbf{G} : \mathbf{u} = \mathbf{v}\}) = 0.$$

Therefore, we have to construct the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ only in the case $\mathbf{u} \neq \mathbf{v}$. Let

$$\mathfrak{s}(\mathbf{u}, \mathbf{v}) = \{x \in \mathbf{G} : x \text{ lies on a two way infinite geodesic between } \mathbf{u} \text{ and } \mathbf{v}\}. \quad (6.5)$$

The strip $\mathfrak{s}(\mathbf{u}, \mathbf{v})$ is set of all points x from all geodesics in \mathbf{G} joining \mathbf{u} and \mathbf{v} . This is a subset of \mathbf{G} , and

$$\gamma \mathfrak{s}(\mathbf{u}, \mathbf{v}) = \mathfrak{s}(\gamma \mathbf{u}, \gamma \mathbf{v}),$$

for every $\gamma \in \Gamma$. In a δ -hyperbolic graph any two geodesics with the same endpoints are within uniformly bounded distance at most 2δ one from another (see [GDLH90] for details), and the geodesics have linear growth. This implies that there exists a constant $c > 0$, such that

$$|\mathfrak{s}(\mathbf{u}, \mathbf{v}) \cap B(o, n)| \leq cn,$$

for all n and distinct $\mathbf{u}, \mathbf{v} \in \partial_h \mathbf{G}$. This proves the subexponential growth of $\mathfrak{s}(\mathbf{u}, \mathbf{v})$, which completes the proof. \square

In order to describe the Poisson boundary of lamplighter random walks (Z_n) over $\mathbb{Z}_2 \wr \mathbf{G}$, when \mathbf{G} is a δ -hyperbolic graph with infinite hyperbolic boundary and only one end, we need some additional facts.

Consider the hyperbolic graph \mathbf{G} and its hyperbolic boundary $\partial_h \mathbf{G}$ as being described by equivalence of geodesic rays. For $y \in \mathbf{G}$ and $\mathbf{u} \in \partial_h \mathbf{G}$ let $\pi = [y = y_0, y_1, \dots, \mathbf{u}]$ be a geodesic ray joining y with \mathbf{u} . For every $x \in \mathbf{G}$, let

$$\beta_{\mathbf{u}}(x, \pi) = \limsup_{i \rightarrow \infty} (d(x, y_i) - i).$$

Define the *Busemann function*

$$\beta_{\mathbf{u}} : \mathbf{G} \times \mathbf{G} \rightarrow \mathbb{R}, \text{ for } \mathbf{u} \in \partial_h \mathbf{G},$$

as follows:

$$\beta_{\mathbf{u}}(x, y) = \sup\{\beta_{\mathbf{u}}(x, \pi') : \pi' \text{ is a geodesic ray from } y \text{ to } \mathbf{u}\}.$$

The *horosphere* with the centre in \mathbf{u} and passing through $x \in \mathbf{G}$, denoted $H_x(\mathbf{u})$ is the set

$$H_x(\mathbf{u}) = \{y \in \mathbf{G} : \beta_{\mathbf{u}}(x, y) = 0\}.$$

For every $x, y \in \mathbf{G}$ and $\mathbf{u} \in \partial_h \mathbf{G}$ the distances $d(x, \mathbf{u})$ and $d(y, \mathbf{u})$ are not defined, but the Busemann function $\beta_{\mathbf{u}}(x, y)$ gives sense to the expression $d(x, \mathbf{u}) - d(y, \mathbf{u})$, which is of the type $\infty - \infty$. One can think of $\beta_{\mathbf{u}}(x, y)$ as being the distance between the horospheres $H_x(\mathbf{u})$ and $H_y(\mathbf{u})$, where $\beta_{\mathbf{u}}(x, y) > 0$ if x is at the exterior of the horoball limited by $H_y(\mathbf{u})$. For properties of the Busemann function, see [GDLH90, Chapter 8].

Theorem 6.2.4. *Let (Z_n) be an irreducible, homogeneous random walk with finite first moment on $\mathbb{Z}_2 \wr \mathbf{G}$, where \mathbf{G} is a hyperbolic transitive graph with $|\partial_e \mathbf{G}| = 1$ and $|\partial_h \mathbf{G}| = \infty$. If $\Gamma \subset \text{AUT}(\mathbf{G})$ acts transitively on \mathbf{G} and does not fix any element in $\partial_h \mathbf{G}$, and ν_∞ is the limit distribution on Π_h , with Π_h defined in (6.4), then (Π_h, ν_∞) is the Poisson boundary of (Z_n) .*

Proof. We apply Theorem 4.2.1. First of all, we check that the conditions required in the *Half Space Method* are satisfied for (X_n) and (\check{X}_n) on \mathbf{G} , which have finite first moments. From Theorem 6.2.1 and Lemma 6.1.3 it follows that the Assumption 3.2.1 holds.

For the second requirement in the *Half Space Method* consider the strip $s(\mathbf{u}, \mathbf{v}) \subset \mathbf{G}$ defined in (6.5), which has sub-exponential growth.

Finally, let us partition \mathbf{G} into half-spaces. Actually, this is one of the examples where the partition is made into two half-spaces and another “degenerate” set on which the lamplighter configuration will be set up to be 0. For every $x \in s(\mathbf{u}, \mathbf{v})$, let $H_x(\mathbf{u})$ (respectively, $H_x(\mathbf{v})$) be the horosphere with center \mathbf{u} (respectively, \mathbf{v}) and passing through x . Remark that the two horospheres may have non empty intersection. Consider the partition of \mathbf{G} into the subsets \mathbf{G}_+ , \mathbf{G}_- , and $\mathbf{G} \setminus (\mathbf{G}_+ \cup \mathbf{G}_-)$, where

$$\mathbf{G}_+(x) = H_x(\mathbf{u})$$

contains a neighbourhood of \mathbf{u} , and

$$\mathbf{G}_-(x) = H_x(\mathbf{v}) \setminus H_x(\mathbf{u}),$$

contains a neighbourhood of \mathbf{v} . This partition is Γ -equivariant.

Up to now, we have checked that the assumptions required in the *Half-Space Method* 4.2 are fulfilled, when \mathbf{G} is a hyperbolic graph. Apply now Theorem 4.2.1.

By Theorem 6.2.2 each of the random walks (Z_n) and (\check{Z}_n) starting at $(\mathbf{0}, o)$ converges almost surely to a Π_h -valued random variable, with Π_h given in

(6.4), with limit distributions ν_∞ and $\check{\nu}_\infty$ respectively. Then (Π_h, ν_∞) and $(\Pi_h, \check{\nu}_\infty)$ are ν - and $\check{\nu}$ - boundaries of the respective random walks. Take

$$b_+ = (\phi_+, \mathbf{u}), \text{ and } b_- = (\phi_-, \mathbf{v}) \in \Pi_h,$$

where ϕ_+ and ϕ_- are the limit configurations of (Z_n) and (\check{Z}_n) , respectively, and $\mathbf{u}, \mathbf{v} \in \partial\mathbf{G}$ are their only respective accumulation points.

Define the configuration $\Phi(b_+, b_-, x)$ like in (4.3), that is,

$$\Phi(b_+, b_-, x) = \begin{cases} \phi_-, & \text{on } H_x(\mathbf{u}) \\ \phi_+, & \text{on } H_x(\mathbf{v}) \setminus H_x(\mathbf{u}) \\ 0, & \text{on } \mathbf{G} \setminus (H_x(\mathbf{u}) \cup H_x(\mathbf{v})) \end{cases}$$

and the strip $S(b_+, b_-)$ exactly like in (4.4), i.e.,

$$S(b_+, b_-) = \{(\Phi, x) : x \in \mathfrak{s}(\mathbf{u}, \mathbf{v})\}.$$

For a graphic visualization of the above construction of the strip and lamps configuration Φ , see Figure 6.1.

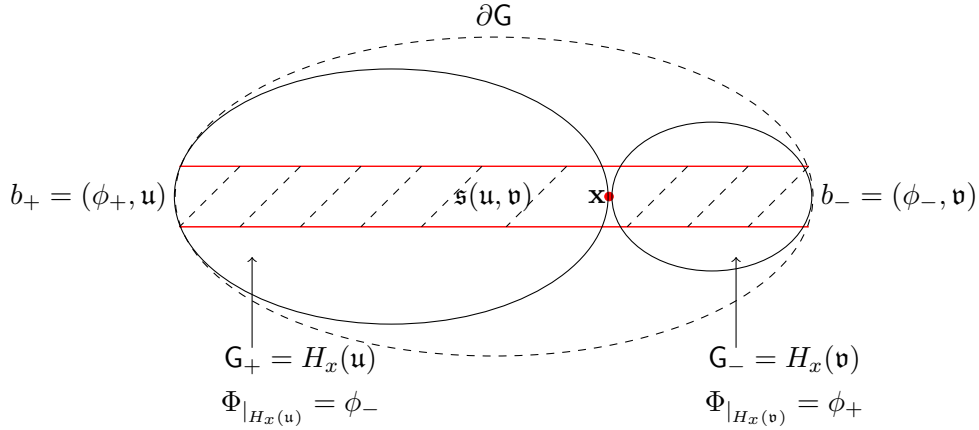


Figure 6.1: The strip construction using horospheres

From Theorem 4.2.1, $S(b_+, b_-)$ satisfies the conditions from Theorem 4.1.4, and it follows that the space (Π_h, ν_∞) is the Poisson boundary of the lamplighter random walk (Z_n) over $\mathbb{Z}_2 \wr \mathbf{G}$. \square

Chapter 7

LRW over Euclidean Lattices

The wreath product $\mathbb{Z}_2 \wr \mathbb{Z}^d$ was first considered in KAIMANOVICH AND VERSHIK [KV83] as a source of several examples and counterexamples illustrating the relationship between growth, amenability and the Poisson boundary for random walks on groups.

In this chapter we consider lamplighter random walks over Euclidean lattices \mathbb{Z}^d . The associated lamplighter graph is $\mathbb{Z}_2 \wr \mathbb{Z}^d$. We show how to apply the *Half Space Method* 4.2 in order to describe the Poisson boundary of lamplighter random walks (Z_n) , in the case when the random walk (X_n) of \mathbb{Z}^d has non-zero drift.

We emphasize that the results in this section were earlier obtained by Kaimanovich KAIMANOVICH [Kai01] in the non-zero drift case. Nevertheless, we still recall them, as another application of our methods. For the zero drift case, the description of the Poisson boundary was recently done in ERSCHLER [Ers10], by using a modified version of the Ray Criterion.

7.1 Random Walks on $\mathbb{Z}_2 \wr \mathbb{Z}^d$

Let now $G = \mathbb{Z}^d$, $d \geq 3$, be the d -dimensional lattice, with the Euclidean metric $|\cdot|$ on it. For \mathbb{Z}^d , there are also natural boundaries and compactifications. A nice example of compactification is obtained by embedding \mathbb{Z}^d into the d -dimensional unit disc via the map $x \mapsto x/(1 + |x|)$, and taking the closure. In this compactification, the boundary $\partial\mathbb{Z}^d$ is the unit sphere S_{d-1} in \mathbb{R}^d , and a sequence (x_n) in \mathbb{Z}^d converges to $\mathbf{u} \in S_{d-1}$ if and only if

$$|x_n| \rightarrow \infty \text{ and } \frac{x_n}{|x_n|} \rightarrow \mathbf{u} \text{ as } n \rightarrow \infty.$$

Recall first the Definition 3.1.3 of a weakly projective boundary.

Lemma 7.1.1. *The boundary S_{d-1} is a weakly projective boundary.*

Proof. Let (x_n) be a sequence which converges to $\mathbf{u} \in S_{d-1}$, and (y_n) be another sequence in \mathbb{Z}^d , such that

$$\frac{x_n}{|x_n|} \rightarrow \mathbf{u}, \text{ and } \frac{|x_n - y_n|}{|x_n|} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since

$$\left| \frac{x_n}{|x_n|} - \frac{y_n}{|x_n|} \right| = \frac{|x_n - y_n|}{|x_n|} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

it follows that $\frac{y_n}{|x_n|} \rightarrow \mathbf{u}$. Now

$$\frac{y_n}{|x_n|} = \frac{y_n}{|y_n|} \cdot \frac{|y_n|}{|x_n|}$$

and the sequence $|y_n|/|x_n|$ of real numbers converges to 1, since we can bound it from above and from below by two sequences both converging to 1. Therefore $y_n/|y_n| \rightarrow \mathbf{u}$, and this proves the desired. \square

Consider the random walk (X_n) with law μ on \mathbb{Z}^d and the lamplighter random walk (Z_n) with law ν on $\mathbb{Z}_2 \wr \mathbb{Z}^d$. If the law μ of (X_n) has non-zero first moment (drift)

$$m = \sum_x x\mu(x) \in \mathbb{R}^d,$$

then the law of large numbers implies that (X_n) converges to the boundary S_{d-1} in this compactification with deterministic limit $m/|m|$. In particular, the limit distribution μ_∞ is the Dirac mass at this point.

Let us now state the result on the Poisson boundary of lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathbb{Z}^d$ in the case of non-zero drift.

Theorem 7.1.2. *Let (Z_n) , with $Z_n = (\eta_n, X_n)$, be a random walk with law ν on $\mathbb{Z}_2 \wr \mathbb{Z}^d$, $d \geq 3$, such that $\text{supp}(\nu)$ generates $\mathbb{Z}_2 \wr \mathbb{Z}^d$, and (X_n) has non-zero drift on \mathbb{Z}^d . If ν has finite first moment, and Π is defined as in (3.1) with the unit sphere S_{d-1} instead of ∂G , then (Π, ν_∞) is the Poisson boundary of (Z_n) , where ν_∞ is the limit distribution of (Z_n) on Π .*

Thus, the Poisson boundary of lamplighter random walks is described by the space of infinite limit configurations of lamps.

Proof. Here it is easy to check the requirements in the *Half Space Method*.

The random walk (X_n) (respectively (\tilde{X}_n)) converges to the boundary S_{d-1} with deterministic limit $\mathbf{u} = m/|m|$ (respectively, $\mathbf{v} = -m/|m|$), in the case

of non-zero drift m . The limit distributions ν_∞ and $\check{\nu}_\infty$ are the Dirac-masses at the respective limit points.

Define the strip $\mathfrak{s}(\mathbf{u}, \mathbf{v}) = \mathbb{Z}^d$. It does not depend on the limit points, it is \mathbb{Z}^d -equivariant, and it has polynomial growth of order d , that is, also subexponential growth.

Next, let us partition \mathbb{Z}^d into half-spaces. Denote by $\pi(\mathbf{u}, \mathbf{v})$ the geodesic of S_{d-1} joining the two deterministic boundary points $\mathbf{u}, \mathbf{v} \in S_{d-1}$. This is exactly the diameter of the ball, since the points \mathbf{u} and \mathbf{v} are antipodal points, i.e., they are opposite through the centre. For every $x \in \mathfrak{s}(\mathbf{u}, \mathbf{v}) = \mathbb{Z}^d$, consider the hyperplane which passes through x and is orthogonal to $\pi(\mathbf{u}, \mathbf{v})$. This hyperplane cuts \mathbb{Z}^d into two disjoint spaces \mathbb{Z}_+ and \mathbb{Z}_- , containing \mathbf{u} and \mathbf{v} , respectively. The half-spaces \mathbb{Z}_+ and \mathbb{Z}_- are \mathbb{Z}^d -equivariant. Apply now Theorem 4.2.1.

By Theorem 3.2.3 each of the random walks (Z_n) and (\check{Z}_n) converges almost surely to a Π -valued random variable, where Π is defined as in (3.1), with S_{d-1} instead of ∂G . Nevertheless, the only ‘‘active’’ points of non-zero μ_∞ - and $\check{\mu}_\infty$ -measure on S_{d-1} are $\mathbf{u} = m/|m|$ and $\mathbf{v} = -m/|m|$, respectively. More precisely, Π can be written as

$$\Pi = \left(\mathcal{C}_u \times \{\mathbf{u}\} \right) \cup \left(\mathcal{C}_v \times \{\mathbf{v}\} \right),$$

where \mathcal{C}_u (respectively, \mathcal{C}_v) is the set of all configurations accumulating only at \mathbf{u} (respectively, \mathbf{v}).

If ν_∞ and $\check{\nu}_\infty$ are the limit distributions of (Z_n) and (\check{Z}_n) on Π , then the spaces (Π, ν_∞) and $(\Pi, \check{\nu}_\infty)$ are ν - and $\check{\nu}$ -boundaries of the respective random walks. Take

$$b_+ = (\phi_+, \mathbf{u}) \text{ and } b_- = (\phi_-, \mathbf{v}) \in \Pi,$$

where ϕ_+ and ϕ_- are the limit configurations of (Z_n) and (\check{Z}_n) , respectively, and \mathbf{u}, \mathbf{v} are their only respective accumulation points. Define the configuration $\Phi(b_+, b_-, x)$ like in (4.3), and the strip $S(b_+, b_-)$ exactly like in (4.4). From Theorem 4.2.1, $S(b_+, b_-)$ satisfies the conditions from Theorem 4.1.4, and it follows that the space (Π, ν_∞) is the Poisson boundary of the lamplighter random walk (Z_n) over $\mathbb{Z}_2 \wr \mathbb{Z}^d$. \square

Chapter 8

Open Problems on LRW

The goal of this chapter is to give a brief overview on some problems that are related to the first part of the thesis. This is only a small personal selection of the vast questionings concerning Lamplighter Random Walks.

8.1 Poisson Boundary of LRW

Let \mathcal{T} be the oriented tree in Figure 5.5 with a fixed end ω , and (X_n) be a random walk with zero modular drift $\delta(P)$ on \mathcal{T} . Consider the associated lamplighter random walk $Z_n = (\eta_n, X_n)$ on $\mathbb{Z}_2 \wr \mathcal{T}$. In Theorem 5.4.3, we have proved that the Poisson boundary of (Z_n) is the space of limit configurations of lamps together with the respective hitting distribution, only when (X_n) and (Z_n) are both of nearest neighbour type. This means that the configuration of the lamp can be changed only at the current position. This assumption cannot be avoided in our proof.

It will be interesting to generalize this result when the base random walk (X_n) has bounded range (not range 1 like in our case) and the lamp configuration can be changed in a bounded neighbourhood of the current position (not only at the current position like in our settings).

Conjecture 8.1.1. *For any random walk (Z_n) with bounded range on $\mathbb{Z}_2 \wr \mathcal{T}$, such that the projection (X_n) on \mathcal{T} has zero drift, its Poisson boundary is isomorphic with the space of infinite limit configurations of lamps, endowed with the respective hitting distribution.*

8.2 Return Probability Asymptotics of LRW

Let G be an infinite graph, \mathbb{Z}_2 the finite set of lamp states, and $\mathbb{Z}_2 \wr G$ the associated lamplighter graph. Consider the lamplighter random walk (Z_n) on $\mathbb{Z}_2 \wr G$. Recall first a known result due to VAROPOULOS [Var83] and PITTET AND SALOFF-COSTE [PSC01] on the return probabilities of random walks on $\mathbb{Z}_2 \wr G$.

Theorem 8.2.1. *If (Z_n) , with $Z_n = (\eta_n, X_n)$, is the Switch-Walk-Switch random walk on $\mathbb{Z}_2 \wr G$ and the transition matrix on \mathbb{Z}_2 is uniform, i.e., $p(\cdot, \cdot) = 1/2$, then the n -step return probabilities are*

$$q^{(n)}((\eta, x), (\eta, x)) = \mathbb{E}_x \left(2^{-R_n} \mathbf{1}_{\{X_n=x\}} \right),$$

where \mathbb{E}_x is the expectation on the trajectory space of (X_n) starting at x , and R_n represents the range of the random walk (X_n) on G .

The range R_n of a random walk (X_n) is defined as the number of distinct visited points up to time n by the random walk, that is

$$R_n = |\{X_1, X_2, \dots, X_n\}|.$$

So, in order to derive asymptotics for the return probabilities $q^{(n)}$ of (Z_n) on $\mathbb{Z}_2 \wr G$, as n goes to infinity, it is enough to study the asymptotics for the range R_n of its underlying walk (X_n) on G .

Asymptotics for $G = \mathbb{Z}^d$. There are several results on asymptotics of return probabilities of LRW over base groups which have polynomial growth, for instance on \mathbb{Z}^d , for all $d \geq 1$. We state here some of them.

For random walks on \mathbb{Z}^d , DONSKER AND VARADHAN [DV79] studied the asymptotic behaviour of the Laplace transform of the range R_n

$$\mathbb{E}[\exp\{-tR_n\}], \text{ as } n \rightarrow \infty$$

for $t > 0$. This behaviour depends on what is assumed about the one-step transition probabilities. They proved the following important theorem.

Theorem 8.2.2. *For simple random walks on \mathbb{Z}^d ,*

$$-\log \mathbb{E}[\exp\{-tR_n\}] \sim c(d) \cdot t^{d/(d+2)} \cdot n^{d/(d+2)}, \text{ as } n \rightarrow \infty,$$

where

$$c(d) = 2^{-1}(d+2)\omega_d^{2/(d+2)}(\lambda_d/d)^{d/(d+2)}$$

and λ_d is the lowest eigenvalue of the Laplacian with Dirichlet boundary condition in the Euclidean ball of radius 1, and $\omega_d = \pi^{d/2}/\Gamma(d/2 + 1)$ its volume.

Here, for sequences (a_n) and (b_n) of real numbers, we indicate by $a_n \sim b_n$ that their quotient tends to 1. We say that $a_n \preceq b_n$ if there are $C \geq c > 0$, such that for all sufficiently large n ,

$$a_n \leq C \sup\{b_k : cn \leq k \leq Cn\}.$$

If also $b_n \preceq a_n$, then we write $a_n \approx b_n$. An equivalence class of sequences under this relation is called an *asymptotic type*. Note that the asymptotic type of a sequence is not as sharp as asymptotic equivalence. For example, sequences of the form $(e^{-\lambda n} Q(n))$ (where $\lambda > 0$ and Q is a polynomial) are all of asymptotic type (e^{-n}) .

Using the previous Theorem, one gets asymptotic equivalence for the return probabilities $q^{(n)}$ of the LRW on $\mathbb{Z}_2 \wr \mathbb{Z}^d$.

REVELLE [Rev03], computed precise asymptotics for Switch-Walk-Switch lamplighter walks on $\mathbb{Z}_2 \wr \mathbb{Z}$. He obtained that

$$q^{(n)} \sim c_1 n^{1/6} \cdot \exp\{-c_2 n^{1/3}\},$$

using the relation with the one-dimensional trapping problem.

Asymptotics for $G = \mathcal{T}$. Consider now the underlying graph as being an oriented tree with a fixed end ω like in Figure 5.5 and (X_n) the random walk with transition probabilities $P = (p(x, y))$ given in (5.7), which has zero modular drift $\delta(P)$.

We are interested in asymptotics (asymptotic type or asymptotic equivalence) of the return probabilities $q^{(n)}$ for SWS lamplighter random walks (Z_n) on $\mathbb{Z}_2 \wr \mathcal{T}$, such that the base random walk (X_n) has zero drift on \mathcal{T} . Such a situation was not considered until now. The precise asymptotics for the range R_n are hard to determine.

An easy upper estimate can be obtained if we look at the horocyclic projection $h(X_n)$ of X_n on the integers, which is a simple random walk on \mathbb{Z} . Moreover, the range \tilde{R}_n of the horocyclic projection is much smaller than the range of (X_n) on \mathcal{T} . Using the large deviation estimate of Theorem 8.2.2 for \tilde{R}_n on \mathbb{Z} , one has

$$-\log \mathbb{E}[\exp\{-t\tilde{R}_n\}] \sim c \cdot t^{2/3} \cdot n^{1/3}, \text{ as } n \rightarrow \infty.$$

Since $\tilde{R}_n \leq R_n$, we get the following upper estimate for the return probabilities of LRW on $\mathbb{Z}_2 \wr \mathcal{T}$:

$$q^{(n)}((\mathbf{0}, x), (\mathbf{0}, x)) \leq C \cdot \rho(Q)^n \cdot \exp\{-cn^{1/3}\},$$

where Q is the transition matrix of the Switch-Walk-Switch random walk on $\mathbb{Z}_2 \wr \mathcal{T}$, and $\rho(Q)$ its spectral radius. Recall that $\mathbf{0}$ is the trivial configuration, where all lamps are off.

Question 1: How can one find a good lower bound for $q^{(n)}((\mathbf{0}, x), (\mathbf{0}, x))$? If a lower estimate of the same order $\exp\{-cn^{1/3}\}$ can be found, then one would have the asymptotic type for the return probabilities, which is weaker than the precise asymptotics.

Question 2: How can one find asymptotics for the range R_n of random walks on graphs with exponential growth, say trees? This problem was not considered up to now. Such estimates are well studied for random walks on groups and graphs with polynomial growth, where gaussian estimates are available.

Part II

Entropy-Sensitivity of Languages via Markov Chains

Chapter 9

Languages on Labelled Graphs

This part of the thesis is based on the paper by HUSS, SAVA AND WOESS [HSW10].

A language L over a finite alphabet Σ is called *growth sensitive* (or *entropy sensitive*) if forbidding any finite set of factors F yields a sublanguage L^F whose exponential growth rate (entropy) is smaller than that of L . Let (X, E, ℓ) be an infinite, oriented, edge-labelled graph with label alphabet Σ . Considering the edge-labelled graph as an (infinite) automaton, we associate with any pair of vertices $x, y \in X$ the language $L_{x,y}$ consisting of all words that can be read as labels along some path from x to y . Under suitable general assumptions we prove that these languages are growth sensitive. This is based on using Markov chains with forbidden transitions.

9.1 Introduction

Let Σ be a finite *alphabet* and Σ^* the set of all finite words over Σ , including the empty word ϵ . A *language* L over Σ is a subset of Σ^* . All our languages will be infinite. We denote by $|w|$ the length of the word w . A *factor* of a word $w = a_1 a_2 \dots a_n$ is a word of the form $a_i a_{i+1} \dots a_j$, with $1 \leq i \leq j \leq n$. The *growth* or *entropy* of L is

$$h(L) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |\{w \in L : |w| = n\}|.$$

For a finite, non-empty set $F \subset \Sigma^+ = \Sigma^* \setminus \{\epsilon\}$ consisting of factors of elements of L , we let

$$L^F = \{w \in L : \text{no } v \in F \text{ is a factor of } w\}.$$

The issue addressed here is to provide conditions under which, for a class of languages associated with infinite graphs, $h(L^F) < h(L)$. If this holds for *any* set F of *forbidden factors*, then the language L is called *growth sensitive* (or *entropy sensitive*).

Questions related to growth sensitivity have been considered in different contexts.

In *group theory* in relation to regular normal forms of finitely generated groups, the study of growth-sensitivity has been proposed by GRIGORCHUK AND DE LA HARPE [GdlH97] as a tool for proving the Hopfianity of a given group or class of groups; see also ARZHANTSEVA AND LYSENOK [AL02] and CECCHERINI-SILBERSTEIN AND SCARABOTTI [CSS04]. A group is called *Hopfian* if it is not isomorphic with a proper quotient of itself. The basic example where this tool applies is the free group.

In *symbolic dynamics*, the number $h(L)$ associated with a regular language accepted by a finite automaton with suitable properties appears as the *topological entropy* of a *sofic system*; see LIND AND MARCUS [LM95, Chapters 3 & 4]. Entropy sensitivity appears as the strict inequality between the entropies of an irreducible sofic shift and a proper subshift [LM95, Cor. 4.4.9].

Motivated by these bodies of work, CECCHERINI-SILBERSTEIN AND WOESS [CSW02], [CS07] have elaborated practicable criteria that guarantee the growth sensitivity of *context-free* languages.

The main result of this chapter can be seen as a direct extension of [LM95, Cor. 4.4.9] to the entropies of infinite sofic systems; see below for further comments and references. This will be done using a probabilistic approach, namely considering Markov chains with forbidden transitions.

Our basic object is an *infinite directed graph* (X, E, ℓ) whose edges are labelled by elements of a finite alphabet Σ . Each edge has the form $e = (x, a, y)$, where $e^- = x$ and $e^+ = y \in X$ are the initial and the terminal vertices of e , and $\ell(e) = a \in \Sigma$ is its label. We will also write $x \xrightarrow{a} y$ for the edge $e = (x, a, y)$, or just $x \rightarrow y$ in situations where we do not care about the label. Multiple edges and loops are allowed, but two edges with the same end vertices must have distinct labels.

A *path* of length n in (X, E, ℓ) is a sequence $\pi = e_1 e_2 \dots e_n$ of edges such that $e_i^+ = e_{i+1}^-$, for $i = 1, 2, \dots, n-1$. We say that it is a path from x to y , if $e_1^- = x$ and $e_n^+ = y$. The label $l(\pi)$ of π is the word $\ell(\pi) = \ell(e_1)\ell(e_2)\dots\ell(e_n) \in \Sigma^*$ that we read along the path. We also allow the empty path from x to x , whose label is the *empty word* $\epsilon \in \Sigma^*$. For $x, y \in X$, denote by $\Pi_{x,y}$ the set of all paths π from x to y in (X, E, ℓ) .

The languages which we consider here are

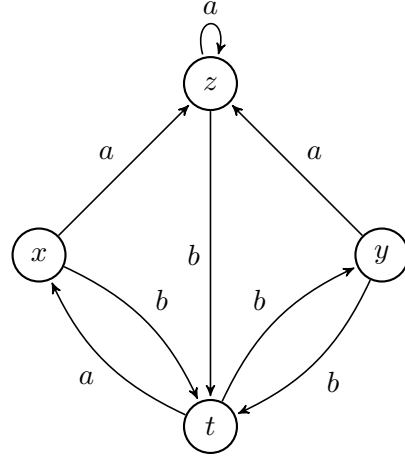
$$L_{x,y} = \{\ell(\pi) \in \Sigma^* : \pi \in \Pi_{x,y}\}, \text{ where } x, y \in X.$$

That is, we consider the edge-labelled graph (X, E, ℓ) as an infinite automaton (labelled digraph) with initial state x and terminal state y , so that $L_{x,y}$ is the language accepted by the automaton.

The languages under study will be infinite. Also, we shall require that the growth is very fast (exponential), since in the subexponential growth case the entropy h is zero.

Example 9.1.1. Let us consider the following finite labelled graph (X, E, ℓ) given in the Figure. For $x, y \in X$, the language $L_{x,y}$ is the set of all labels of paths from x to y , that is

$$L_{x,y} = \{a(a)^*bb, (ba)^*bb, bb, \dots\}$$



Definition 9.1.2. We say that (X, E, ℓ) is deterministic if, for every vertex x and every $a \in \Sigma$, there is at most one edge with initial point x and label a .

Any automaton (finite or infinite) can be transformed into a deterministic one that accepts the same language, by the well-known powerset construction. See, for example [BPR10, Prop. 1.4.1].

As in the finite case, we need an irreducibility assumption. The graph (X, E, ℓ) is called *strongly connected* if, for every pair of vertices x, y , there is an (oriented) path from x to y .

Definition 9.1.3. The graph (X, E, ℓ) is called *uniformly connected* if it is strongly connected and the following holds: there is a constant K , such that for every edge $x \rightarrow y$ there is a path from y to x with length at most K .

In the finite case, the two notions coincide as one can take $K = |X|$. The *forward distance* $d^+(x, y)$ of $x, y \in X$ is the minimum length of a path from x to y . We write

$$h(X) = h(X, E, \ell) = \sup_{x,y \in X} h(L_{x,y})$$

and call this the *entropy* of our oriented, labelled graph. It is a well-known and easy to prove fact, that for a strongly connected graph,

$$h(L_{x,y}) = h(X), \text{ for all } x, y \in X.$$

We also need a reasonable assumption on the set of forbidden factors.

Definition 9.1.4. *We say that a finite set $F \subset \Sigma^+$ is relatively dense in the graph (X, E, ℓ) if there is a constant D such that, for every $x \in X$, there are $y \in X$ and $w \in F$, such that $d^+(x, y) \leq D$ and there is a path starting at y which has label w .*

Note that the assumptions of uniformly connectedness and relatively dense-ness cannot be avoided, since they play an important role in the proof of the main result. This fails without these assumptions.

Theorem 9.1.5. *Suppose that (X, E, ℓ) is uniformly connected and deterministic with label alphabet Σ . Let $F \subset \Sigma^+$ be a finite, non-empty set which is relatively dense in (X, E, ℓ) . Then*

$$\sup_{x, y \in X} h(L_{x, y}^F) < h(X).$$

We say that (X, E, ℓ) is *fully deterministic* if, for every $x \in X$ and $a \in \Sigma$, there is precisely one edge with initial point x and label a . We remark that, in automata theory, the classical terminology is deterministic and complete, instead of fully deterministic. Since in graph theory a complete graph is one in which every pair a distinct vertices is connected by an unique edge, we shall use the notion of fully deterministic throughout this work.

As a consequence of Theorem 9.1.5 one can easily prove the following.

Corollary 9.1.6. *If (X, E, ℓ) is uniformly connected and fully deterministic then $L_{x, y}$ is growth sensitive for all $x, y \in X$.*

Indeed, in this case, for every $x \in X$ and every $w \in \Sigma^*$, there is precisely one path with label w starting at x .

With our edge-labelled graph (X, E, ℓ) , we can consider the *full shift space* which consists of all bi-infinite words over Σ that can be read along the edges of some bi-infinite path in (X, E, ℓ) . When (X, E, ℓ) is strongly connected, the entropy $h(L_{x, y})$ is independent of x and y and equals the topological entropy of the full shift space of the graph. See, for example, GUREVIČ [Gur69], PETERSEN [Pet86] or BOYLE, GUZZI AND GÓMEZ [BBG06] for a selection of related work and references, and also the discussion in [LM95, §13.9].

If we consider the shift space consisting of all those bi-infinite words as above that do not contain any factor in F , then the interpretation of Corollary 9.1.6 is that the associated entropy is strictly smaller than $h(X)$.

Theorem 9.1.5, once approached in the right way, is not hard to prove. It is based on a classical tool, a version of the Perron-Frobenius theorem

for infinite non-negative matrices; see, for example, SENETA [Sen06]. We shall first reformulate things in terms of Markov chains (random walks) and forbidden transitions.

9.2 Markov Chains and Forbidden Transitions

We now equip the oriented, edge-labelled graph (X, E, ℓ) with additional data: with each edge $e = (x, a, y)$, we associate a probability

$$p(e) = p(x, a, y) \geq \alpha > 0,$$

where α is a fixed constant, such that

$$\sum_{e \in E: e^- = x} p(e) \leq 1 \quad \text{for every } x \in X. \quad (9.1)$$

Our assumption to have the uniform lower bound $p(e) \geq \alpha$ for each edge implies that the outdegree (number of outgoing edges) of each vertex is bounded by $1/\alpha$. We interpret $p(e)$ as the probability that a particle with current position $x = e^-$ moves in one (discrete) time unit along e to its end vertex $y = e^+$. Observing the successive random positions of the particle at the time instants $0, 1, 2, \dots$, we obtain a Markov chain with state space X whose one-step transition probabilities are

$$p(x, y) = \sum_{a \in \Sigma: (x, a, y) \in E} p(x, a, y).$$

We shall also want to record the edges and their labels used in each step, which means considering a Markov chain on a somewhat larger state space, but we will not need to formalize it in detail. In (9.1), we admit the possibility that $1 - \sum_y p(x, y) > 0$ for some x . This number is then interpreted as the probability that a particle positioned at x dies at the next step.

We write $p^{(n)}(x, y)$ for the probability that the particle starting at x is at position y after n steps. This is the (x, y) -element of the n -power P^n of the transition matrix $P = (p(x, y))_{x, y \in X}$. If (X, E, ℓ) is strongly connected, then P is irreducible, and it is well-known that the number

$$\rho(P) = \limsup_{n \rightarrow \infty} p^{(n)}(x, y)^{1/n}$$

is independent of x and y . See once more [Sen06]. The quantity $\rho(P)$ is called the spectral radius of P . It is the parameter of exponential decay of the transition probabilities.

Let once more $F \subset \Sigma^+$ be finite. We interpret the elements of F as sequences of *forbidden transitions*. That is, we restrict the motion of the particle: at no

time is it allowed to traverse any path π with $\ell(\pi) \in F$ in k successive steps, where k is the length of π . The words in F are forbidden for the Markov chain. We write $p_F^{(n)}(x, y)$ for the probability that the particle starting at x is at position y after n steps, without having made any such sequence of forbidden transitions. Let

$$\rho_{x,y}(P_F) = \limsup_{n \rightarrow \infty} p_F^{(n)}(x, y)^{1/n}, \quad x, y \in X.$$

These numbers are not necessarily independent of x and y , and *they are not* the elements of the n -matrix power of some substochastic matrix.

Definition 9.2.1. A transition matrix $Q = (q(x, y))_{x, y \in X}$ on the state space X is called *substochastic* if there exists a constant $\varepsilon > 0$ such that, for all $x \in X$,

$$\sum_{y \in X} q(x, y) \leq 1 - \varepsilon.$$

That is, all row sums are bounded by $1 - \varepsilon$.

The restricted matrix P_F does not represent the transition matrix of a Markov chain. In order to give an upper bound for the restricted transition probabilities $p_F^{(n)}(x, y)$, we first show the following.

Theorem 9.2.2. Suppose that (X, E, ℓ) is strongly connected with label alphabet Σ and equipped with transition probabilities $p(e) \geq \alpha > 0$, $e \in E$. Let $F \subset \Sigma^+$ be a finite, non-empty set which is relatively dense in (X, E, ℓ) . Then there are $k \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that

$$\sum_{y \in X} p_F^{(k)}(x, y) \leq 1 - \varepsilon_0 \quad \text{for all } x \in X.$$

In other words, the transition matrix $Q = (p_F^{(k)}(x, y))_{x, y \in X}$ is strictly substochastic, with all row sums bounded by $1 - \varepsilon_0$.

Proof. Let $R = \max_{w \in F} |w|$, and let $D \in \mathbb{N}$ be the constant from the definition of relative denseness of F . Set

$$k = D + R.$$

For each $x \in X$, we can find a path π_1 from x to some $y \in X$ with length $d \leq D$ and a path π_2 starting at y which has label $w \in \Sigma^*$. Let z be the endpoint of π_2 , and choose any path π_3 that starts at z and has length $k - d - |w|$. Such a path exists by strong connectedness. Then let π be the path obtained by concatenating π_1 , π_2 and π_3 .

The probability that the Markov chain starting at x makes its first k steps along the edges of π is

$$\mathbb{P}(\pi) \geq \alpha^k = \varepsilon_0 > 0.$$

Hence

$$\sum_{y \in X} p_F^{(k)}(x, y) \leq \sum_{y \in X} p^{(k)}(x, y) - \mathbb{P}(\pi) \leq 1 - \varepsilon_0,$$

and this upper bound holds for every x . \square

Given that the transition matrix Q is substochastic, it is an easy exercise to prove that also its n -matrix power Q^n is also substochastic and the row sums of Q^n are bounded from above by $(1 - \varepsilon_0)^n$.

The matrix P acts on functions $h : X \rightarrow \mathbb{R}$ by $Ph(x) = \sum_y p(x, y)h(y)$. Next, we state two key results due to PRUITT [Pru64, Lemma 1] and [Pru64, Corollary to Theorem 2], which will be used in the proof of the main result.

Lemma 9.2.3. *If the transition matrix P is irreducible and $Ph \leq sh$ for some $s > 0$ and $h \neq 0$, then $h > 0$.*

Lemma 9.2.4. *If the transition matrix $P = (p(x, y))_{x, y \in X}$ is such that for every $x \in X$ the entries $p(x, y) = 0$ for all $y \in X$ except finitely many, then the equation*

$$Ph = sh$$

has a solution for all $s \geq \rho(P)$.

Based on these lemmatas, we prove the following result on sensitivity of the Markov chain with respect to forbidding the transitions in F .

Theorem 9.2.5. *Suppose that (X, E, ℓ) is uniformly connected with label alphabet Σ and equipped with transition probabilities $p(e) \geq \alpha > 0$, $e \in E$. Let $F \subset \Sigma^+$ be a finite, non-empty set which is relatively dense in (X, E, ℓ) . Then*

$$\sup_{x, y \in X} \rho_{x, y}(P_F) < \rho(P) \quad \text{strictly.}$$

Proof. We shall proceed in two steps.

Step 1. We assume that $P = (p(x, y))_{x, y \in X}$ is stochastic and that $\rho(P) = 1$.

Consider the matrix Q of Lemma 9.2.2. Let

$$Q^n = (q^{(n)}(x, y))_{x, y \in X}$$

be its n -th matrix power. The quantity $q^{(n)}(x, y)$ is the probability that the Markov chain starting at x is in y at time nk and does not make any forbidden sequence of transitions in each of the discrete time intervals

$$[(j-1)k, jk] \text{ for } j \in \{1, \dots, n\}.$$

Therefore

$$p_F^{(nk)}(x, y) \leq q^{(n)}(x, y),$$

and also, by the same reasoning, for $i = 0, \dots, k - 1$,

$$\begin{aligned} p_F^{(nk+i)}(x, y) &= \sum_{z \in X} p_F^{(nk)}(x, z) p_F^{(i)}(z, y) \\ &\leq \sum_{z \in X} q^{(n)}(x, z) p_F^{(i)}(z, y), \quad i = 0 \dots, k - 1. \end{aligned}$$

Therefore, for every $x \in X$ and $i = 0, \dots, k - 1$,

$$\sum_{y \in X} p_F^{(nk+i)}(x, y) \leq \sum_{z \in X} q^{(n)}(x, z) \underbrace{\sum_{y \in X} p_F^{(i)}(z, y)}_{\leq 1} \leq (1 - \varepsilon_0)^n,$$

since Lemma 9.2.2 implies that the row sums of the matrix power Q^n are bounded above by $(1 - \varepsilon_0)^n$. We conclude that

$$\limsup_{n \rightarrow \infty} p_F^{(nk+i)}(x, y)^{1/(nk+i)} \leq (1 - \varepsilon_0)^{1/k},$$

so $\rho_{x,y}(P_F) \leq (1 - \varepsilon_0)^{1/k} = 1 - \varepsilon$, where $\varepsilon > 0$.

Step 2. General case. We reduce this case to the previous one.

Since P is irreducible and every row of P has only finitely many non-zero entries, Lemma 9.2.3 and Lemma 9.2.4 guarantee the existence of a strictly positive solution $h : X \rightarrow \mathbb{R}$ for the equation

$$Ph = \rho(P) \cdot h,$$

that is, h is $\rho(P)$ -harmonic. Consider now the h -transform of the transition probabilities $p(e)$ of P , $e = (x, a, y) \in E$, given by

$$p^h(e) = p^h(x, a, y) = \frac{p(x, a, y)h(y)}{\rho(P)h(x)},$$

and the associated transition matrix P^h with entries

$$p^h(x, y) = \sum_{a : (x, a, y) \in E} p^h(x, a, y).$$

The Markov chain associated with P^h is called the h -process.

Then $\rho(P^h) = 1$. Using uniform connectedness, we show that there is a constant $\bar{\alpha} > 0$ such that $p^h(e) \geq \bar{\alpha}$ for each $e = (x, a, y) \in E$. Indeed, for such an edge, there is $k \leq K$ such that $d^+(y, x) = k$, whence

$$\rho(P)^k h(y) = \sum_{z \in X} p^{(k)}(y, z) h(z) \geq \alpha^k h(x),$$

so

$$p^h(x, a, y) \geq (\alpha/\rho(P))^{k+1}.$$

Recall that K is the constant used in the definition of the uniform connect-
edness. We can now choose

$$\bar{\alpha} = (\alpha/\rho(P))^{K+1}.$$

We see that with P^h we are now in the situation of *Step 1*. Thus, forbidding
the transitions of F for the Markov chain with transition matrix P^h , we get

$$\rho_{x,y}(P_F^h) \leq 1 - \varepsilon, \text{ for all } x, y \in X,$$

where $\varepsilon > 0$. We now show that

$$\rho_{x,y}(P_F^h) = \rho_{x,y}(P_F)/\rho(P),$$

which will conclude the proof.

For a path $\pi = e_1 \dots e_n$ from x to y , let (as above) $\mathbb{P}(\pi)$ be the probability
that the original Markov chain traverses the edges of π in n successive steps,
and let $\mathbb{P}^h(\pi)$ be the analogous probability with respect to the h -process.
Then

$$\mathbb{P}^h(\pi) = \frac{\mathbb{P}(\pi)h(y)}{\rho(P)^n h(x)}.$$

Let us write $\Pi_{x,y}^n(\neg F)$ for the set of all paths π from x to y with length n
for which $\ell(\pi)$ does not contain a factor in F . Then the n -step transition
probabilities of the h -process with the transitions in F forbidden are

$$p_F^{h(n)}(x, y) = \sum_{\pi \in \Pi_{x,y}^n(\neg F)} \mathbb{P}^h(\pi) = \sum_{\pi \in \Pi_{x,y}^n(\neg F)} \frac{\mathbb{P}(\pi)h(y)}{\rho(P)^n h(x)} = \frac{p_F^{(n)}(x, y)h(y)}{\rho(P)^n h(x)}.$$

Taking n -th roots and passing to the upper limit, we obtain the required
identity. \square

With this result, it is now easy to deduce Theorem 9.1.5.

Proof of Theorem 9.1.5. Since (X, E, l) is deterministic with label alphabet
 Σ , the outdegree of every $x \in X$ is at most $|\Sigma|$. Equip the edges of (X, E, l)
with the transition probabilities

$$p(x, a, y) = \frac{1}{|\Sigma|}, \text{ when } (x, a, y) \in E.$$

Then the n -step transition probabilities of the resulting Markov chain are
given by

$$p^{(n)}(x, y) = \frac{|\{w \in L_{x,y} : |w| = n\}|}{|\Sigma|^n}.$$

Therefore (because (X, E, ℓ) is uniformly connected)

$$\begin{aligned} h(X) &= h(L_{x,y}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log |\{w \in L_{x,y} : |w| = n\}| \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log(p^n(x, y) |\Sigma|^n) = \log(\rho(P) \cdot |\Sigma|). \end{aligned}$$

Analogously,

$$h(L_{x,y}^F) = \log(\rho_{x,y}(P_F) \cdot |\Sigma|).$$

By Theorem 9.2.5

$$\sup_{x,y \in X} \rho_{x,y}(P_F) < \rho(P),$$

and this implies that

$$\sup_{x,y \in X} h(L_{x,y}^F) < h(X)$$

strictly. □

9.3 Application to Schreier Graphs

Let G be a finitely generated group and K a (not necessary finitely generated) subgroup. Also, let Σ be a finite alphabet and let $\psi : \Sigma \rightarrow G$ be such that the set $\psi(\Sigma)$ generates G as a semigroup. We extend ψ to a monoid homomorphism from Σ^* to G by $\psi(w) = \psi(a_1) \cdots \psi(a_n)$ if $w = a_1 \dots a_n$ with $a_i \in \Sigma$ (and $\psi(\epsilon) = 1_G$). The mapping ψ is called a *semigroup presentation* of G in [CSW].

The *Schreier graph* $X = X(G, K, \psi)$ has vertex set

$$X = \{Kg : g \in G\},$$

the set of all right K -cosets in G , and the set of all labelled, directed edges E is given by

$$E = \{e = (x, a, y) : x = Kg, y = Kg\psi(a), \text{ where } g \in G, a \in \Sigma\}.$$

Note that the graph X is fully deterministic and uniformly connected.

The *word problem* of (G, K) with respect to ψ is the language

$$L(G, K, \psi) = \{w \in \Sigma^* : \psi(w) \in K\}.$$

The *word problem* for a recursively presented group G is the algorithmic problem of deciding whether two words represent the same element. Also, this terminology is used in the context of formal language theory and goes back at least to the seminal paper of MULLER AND SCHUPP [MS83]. For

additional information, see also MULLER AND SCHUPP [MS85]. In their work, for a finitely generated group G the *word problem* $W(G)$ is the set of all words on the generators and their inverses which represent the identity element of G .

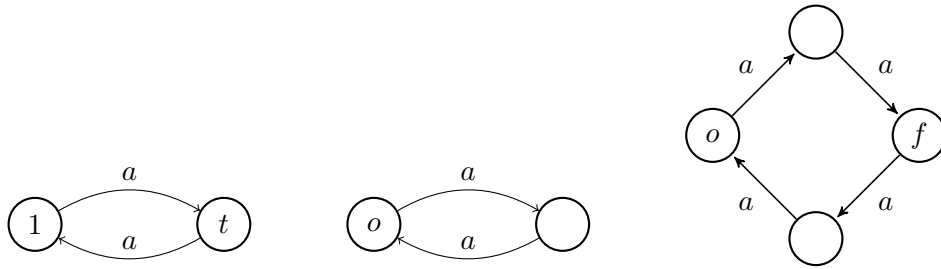
If we consider the “root” vertex $o = K$ of the Schreier graph, then in the notation of the introduction, we have $L(G, K, \psi) = L_{o,o}$; compare with [CSW, Lemma 2.4].

We can therefore apply Theorem 9.1.5 and Corollary 9.1.6 to the graph $X(G, K, \psi)$ in order to deduce the following.

Corollary 9.3.1. *The word problem of the pair (G, K) with respect to any semigroup presentation ψ is growth sensitive (with respect to forbidding an arbitrary non-empty finite subset $F \subset \Sigma^*$).*

Example 9.3.2. Let $G = \mathbb{Z}_2 = \{1, t\}$ be the group of order two and $K = \{1\}$ the trivial subgroup. Let $\Sigma = \{a\}$ and consider the presentations $\psi : \Sigma \rightarrow G$ such that $\psi(a) = t$. Then

$$L(G, K, \psi) = \{a^{2n} : n \geq 0\}.$$



In the figure above are represented in order, the Schreier graph $X(G, K, \psi)$ (which is the Cayley graph of G with respect to ψ), and two automata \mathcal{A}_1 and \mathcal{A}_2 . As usual o denoted the origin, while the set of final states are $F_1 = \{o\}$ and $F_2 = \{o, f\}$ respectively. We have

$$L(\mathcal{A}_1) = L(\mathcal{A}_2) = L(G, K, \psi).$$

For detailed information on properties of pairs of groups and their Schreier graphs, the reader may have a look at CECCHERINI AND WOESS [CSW].

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Index of Notation

Part I

G	locally finite, connected, infinite transitive graph
$d(\cdot, \cdot)$	graph metric on G
∂G	geometric boundary of G
$\widehat{G} = G \cup \partial G$	compactification of G
$\Omega = G^{\mathbb{Z}_+}$	the trajectory space of G
P_G	transition matrix on G
$AUT(G)$	the set of automorphisms (or isometries) of G
Γ	subgroup of $AUT(G)$ which acts transitively on G
μ	probability measure on Γ
(G, P_G)	Markov chain with state space G and transition P_G
(X_n)	random walk with transition matrix P_G on G
μ_∞	limit distribution of (X_n) on ∂G
$\rho(P_G)$	spectral radius of (G, P_G)
$\delta(P_G)$	modular drift of P_G
$l(P_G)$	rate of escape (drift) of P_G
\mathbb{Z}_2	cyclic group with two elements (or a set with two elements)
$\mathbb{Z}_2 \wr G$	lamplighter graph with base G
P	transition matrix on $\mathbb{Z}_2 \wr G$
$\partial(\mathbb{Z}_2 \wr G)$	geometric boundary of $\mathbb{Z}_2 \wr G$
$\mathbb{Z}_2 \wr \Gamma$	lamplighter group, subgroup of $AUT(\mathbb{Z}_2 \wr G)$
ν	probability measure on $\mathbb{Z}_2 \wr \Gamma$
η_n	random configuration of lamps at time n , with finite support
\mathcal{C}	set of finitely supported configurations over G
$\mathcal{C}_u, u \in \partial G$	set of lamps configurations accumulating at u
(Z_n)	lamplighter random walk (LRW) over $\mathbb{Z}_2 \wr G$ with transition matrix P , with $Z_n = (\eta_n, X_n)$ the position at time n , and (X_n) the base random walk on G
Π	dense subset of $\partial(\mathbb{Z}_2 \wr G)$ toward (Z_n) converges
ν_∞	limit distribution of (Z_n) on Π
\mathcal{T}_q	homogeneous tree of degree q

Part II

Σ	a finite alphabet
Σ^*	the set of all finite words over Σ
L	a language over Σ
(X, E, ℓ)	infinite, oriented, edge-labelled graph with label alphabet Σ
$h(L)$	the entropy of L
$L_{x,y}$	the set of all labels of paths from x to y
$p(e)$	the probability of the edge $e \in E$
P	transition matrix over (X, E, ℓ)
P^h	the h -transform of the matrix P
P^F	restricted matrix on (X, E, ℓ) , where $F \subset \Sigma^*$